Universidade de São Paulo
Instituto de Física

# Coexistência de fases magnéticas e supercondutoras em modelos do tipo BCS induzida por repulsão local

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Assinado de forma digital por Walter de Siqueira Pedra
Dados: 2021.03.11 16:24:16 -03'00'

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Dissertação apresentada ao Instituto de Física da Universidade de São Paulo como requisito parcial para a obtenção do título de Mestre em Ciências.

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São Paulo
2021

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Alves, Kauê Rodrigues

Coexistência de fases magnéticas e supercondutoras em modelos do tipo BCS induzida por repulsão local / Magnetic-superconducting phase coexistence driven by on-site repulsions in BCS-like models. São Paulo, 2021.

Dissertação (Mestrado) – Universidade de São Paulo, Instituto de Física, Depto. de Física Matemática.

Orientador: Prof. Dr. Walter Alberto de Siqueira Pedra

Área de Concentração: Física Matemática

Unitermos: 1. Mecânica estatística quântica; 2. C* álgebras; 3 Mudança de fase; 4. Teoria das catástrofes

USP/IF/SBI-011/2021

University of São Paulo
Physics Institute

# Magnetic-superconducting phase coexistence driven by on-site repulsions in BCS-like models

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Thesis submitted to the Institute of Physics of the University of São Paulo in partial fulfillment of the requirements for the degree of Master of Science.

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São Paulo
2021

# Acknowledgment

I would like to express my gratitude to my supervisor Prof. Walter de Siqueira Pedra, for his patience and guidance throughout all the time of the research.

I also would like to thank CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) for the financial support.

And last but not least, I would like to thank my mom and dad, Regina Celia and José Alves, for all the support throughout my entire life.

# Abstract

This thesis focuses on the study of the thermodynamics of fermionic systems on lattices, from the point of view of the C*-algebraic formulation of quantum mechanics, and on the application of the so-called “catastrophe theory” to the analysis of the phase diagram of an explicit fermionic model. The first chapter is devoted to the introduction of some of the basic and most important properties of C*-algebras, and also to the study of its representations and its set of states. The second chapter is devoted to the development of some important results of catastrophe theory. The results derived in the chapter allows one to analyze the behavior of the minima of members of a family of functions around a degenerate critical point, and they will be used to study the behavior of the thermodynamic pressure for a given fermionic lattice model. The third chapter presents some of the formalism developed in [4], that concerns the existence of the thermodynamics and equilibrium states of lattice fermi systems subject to a suitable set of long-range interactions (i.e., interactions containing mean-field terms). Finally, in the last chapter, using all the formalisms and results already studied in the previous chapters, the thermodynamics of an explicit BCS-like model is analyzed. With the help of catastrophe theory, it is shown that such model exhibits a coexistence of magnetic and superconducting phases for a suitable choice of parameters, and through a perturbative analysis it is also shown that the coexistence still holds when a small kinetic term is added into the model.

*Keywords:* coexistence of phases; fermi lattice; C*-algebraic formalism; quantum thermodynamics; catastrophe theory.

# Resumo

Esta dissertação se concentra no estudo da termodinâmica de sistemas fermiônicos em rede, pelo ponto de vista da formulação de álgebra C* da mecânica quântica, e na aplicação da chamada "teoria de catástrofe" para a análise do diagrama de fases de um modelo fermiônico explicito. O primeiro capítulo é dedicado à introdução de algumas propriedades básicas e mais importantes das álgebras C*, e ao estudo de suas representações e de seu conjunto de estados. O segundo capítulo é dedicado ao desenvolvimento de alguns resultados importantes relacionados à teoria de catástrofe. Os resultados obtidos no capítulo permitem a análise do comportamento dos mínimos de membros de uma família de funções em torno de um ponto crítico degenerado, e eles serão utilizados para o estudo do comportamento da pressão termodinâmica para um dado sistema fermiônico. O terceiro capítulo apresenta o formalismo desenvolvido em [4], que está relacionado à existência da termodinâmica e de estados de equilíbrio em sistemas fermiônicos em rede sujeitos à certas interações de longo-alcance (i.e., interações contendo termos de campo médio). Por fim, no último capítulo, utilizando-se de todos os resultados estudados nos capítulos anteriores, a termodinâmica de um modelo explícito do tipo BCS é analizada. Com a ajuda da teoria de catástrofe, é provado que tal modelo apresenta uma coexistência de fases magnéticas e supercondutoras para uma escolha adequada de parâmetros, e através de uma análise perturbativa é mostrado também que a coexistência ainda persiste quando um pequeno termo cinético é adicionado no modelo.

*Palavras-chave:* coexistência de fases; redes fermiônicas; formalismo de álgebra C*; termodinâmica quântica; teoria de catástrofe.

# Contents

# Introduction

The mathematical formalism of what is known today as "quantum mechanics" has its roots in the mid-1920s, when the theories of Werner Heisenberg and Erwin Schrödinger emerged; both proposing different postulates from the firmly stablished classical mechanics. In the 1930s the formalism was finally unified, mainly due to the efforts of the physicists Paul Dirac and John von Neumann. The axioms that were stablished for a one-particle system, known as Dirac-von Neumann axioms, are as follows:

- For a one-particle system, the possible "states" that a particle can be at should be represented by normalized vectors of a separable Hilbert space $\mathfrak{h}$.
- The "observables" correspond to self-adjoint operators acting on $\mathfrak{h}$, and the expectation value of an observable $A$ in the state $\psi$ is given by $(\psi, A\psi)$

The above formalism can also be extended to systems with more than one particle. The possible states of a system with $n$ particles are now vectors in the tensor product $\mathfrak{h}_1 \otimes \cdots \otimes \mathfrak{h}_n$, where $\mathfrak{h}_i$ is the Hilbert space of the $i^{th}$ particle. However, in the scenario of $n$ particles of the same type, two more properties play a role: the indistinguishability of quantum particles and the Pauli exclusion principle, if the particles are fermions. The former implies that only symmetric or anti-symmetric vectors[1] can represent the state of the particles, and the latter states that for fermionic particles, only the anti-symmetric vector states can be occupied. A straightforward generalization of the formalism for an arbitrary number of particles is to assume that the Hilbert space representing the states of the system is the so-called fermionic Fock space

$$\mathfrak{F}_-(\mathfrak{h}) = \bigoplus_{n\geq 0} P_- \mathfrak{h}^n,$$

where $\mathfrak{h}$ is the Hilbert space of the single-particle system and $P_-$ is the projection operator of the subspace of anti-symmetric vectors. I that case, the algebra of operators acting on $\mathfrak{F}_-(\mathfrak{h})$ is generated by the so-called *creation and annihilation operators* $a^*(f)$ and $a(f)$, that creates (annihilates) a particle in a state $f \in \mathfrak{h}$. Moreover, due to Pauli exclusion principle, the operators $a^*(f)$ and $a(f)$ must satisfy the *Canonical Anti-commutation Relations* (CARs)

$$\begin{aligned} &\{a(f), a(g)\} = \{a^*(f), a^*(g)\} = 0, \quad f, g \in \mathfrak{h} \\ &\{a(f), a^*(g)\} = (f, g)\mathbb{1}, \end{aligned} \tag{1}$$

[1] Given a basis $\{\psi_1, \ldots, \psi_m\}$ of $\mathfrak{h}$, a vector $\psi \in \mathfrak{h}^n$ can always be expressed as $\psi = \sum_{i_1,\ldots,i_n=1}^{m} T_{i_1,\ldots,i_n} \psi_{i_1} \otimes \cdots \otimes \psi_{i_n}$. $\psi$ is symmetric (anti-symmetric) if $T_{i_1,\ldots,i_n}$ is a symmetric (anti-symmetric) tensor

where $\{A, B\} = AB + BA$. Now, it is important to note that the CARs of eq. 1 allow one define a more general algebra (a C*-algebra, to be more specific), usually known as the *CAR algebra*, whose elements also satisfy the CARs. Such algebra also allows *representations*, (i.e., a *-morphism into the algebra of operators of a Hilbert space), and hence the algebra of operators acting on $\mathfrak{F}_-(\mathfrak{h})$ can be seen as one specific representation of the CAR algebra. When the Hilbert space $\mathfrak{h}$ of a single-particle system is finite, it follows that all of the (irreducible) representations of its corresponding CAR algebra are unitarily equivalent. Therefore, in that case, choosing the Fock space $\mathfrak{F}_-(\mathfrak{h})$ as the state space of the system is no different then choosing any other Hilbert space where the algebra of its operators is a (irreducible) representation of the CAR algebra. However, when $\mathfrak{h}$ is infinite-dimensional, the unitary equivalence of the representations of its CAR algebra does not hold anymore. Moreover, when studying the thermodynamic properties of quantum systems, specially phase transitions, it is necessary to consider an infinite-dimensional $\mathfrak{h}$ (corresponding to a infinite-volume limit of the system). The argument for the necessity of such idealization to study phase transitions in a quantum system, which is the goal of this thesis, is the same as in the case of a classical system. A phase transition corresponds mathematically to a discontinuity in the derivatives of the thermodynamic potentials. But if the system is finite, then the thermodynamic potentials are always smooth (since the partition function is simply a finite sum of smooth terms) and hence there is no phase transition present in such system. Therefore, in that case the Fock space may not be the only Hilbert space such that the CARs can be represented up to unitary equivalence. In fact, for systems with phase transitions, it can be shown that *different phases of the system are connected to different representations of the CAR algebra*. Therefore, in that scenario, the above axioms of quantum mechanics that sees the observables of a quantum system as operators over a fixed Hilbert space is not useful anymore.

A solution to adequate the presented formalism to the study of phase transitions is not to fix the Hilbert space of the states; but rather to fix the C*-algebra of the observables, and talk about (generally unequivalent) *representations* of this algebra. This approach is usually called *the C* algebra formalism of quantum mechanics*. Then, the axioms can be rephrased as follows:

- The "observables" of a quantum system correspond to self-adjoint elements of a C*-algebra.
- The "states" of a quantum system are normalized positive linear functionals over a C*-algebra, and the expectation value of an observable $A$ in the state $\omega$ is given by $\omega(A)$.

Moreover, since the algebra of bounded operators of any Hilbert space is always a $C^*$-algebra, and any $C^*$-algebra can be represented as the algebra of bounded operators on a Hilbert space (this is known as the Gelfand-Naimark theorem), one may always recover the previous formalism, while also admitting unequivalent representations that can be related to different phases of the system. There is an extensive literature on the $C^*$-algebra formalism of quantum mechanics, and it is widely used in quantum statistical mechanics (see [1] and [2], for example) and quantum field theory.

Under that formalism, a quantum $d$-dimensional lattice fermion system can be modeled from the CAR algebra associated with $\mathfrak{h} = l^2(\mathbb{Z}^d) \otimes \mathfrak{H}_S$, where $\mathfrak{H}_S$ is the Hilbert space of the spin of the particle considered. The interactions acting on the particles of the system are thus represented by families of self-adjoint elements of the CAR algebra. However, for studying the thermodynamics of such system, one still needs to define and prove the existence of equilibrium states and thermodynamic potentials. In chapter three, a set of interactions called *long-range models* (i.e., models containing mean-field interaction terms) is defined, and it follows that for these interactions the thermodynamics of the lattice fermion system can be defined. To prove this, some sets of states possessing useful symmetries are studied in detail, and it is shown that for states that are translation-invariant, a *free-energy density functional* exists and is well-behaved, which allows the definition of equilibrium states (in fact, in this thesis the equilibrium states are only defined for *purely attractive* long-range models, since this is the case of the interaction studied in the final chapter, and the generalization for arbitrary long-range models adds some unnecessary complications). Moreover, it also shown that for the long-range models the thermodynamic pressure of the infinite system, defined as the limit of the finite-volume pressures, exists.

In chapter four, it is studied the thermodynamics of an interaction containing a BCS term (that corresponds to a long-range interaction) and an on-site repulsive term (that corresponds to a short-range interaction), whose strengths are denoted by $\gamma$ and $\lambda$, respectively, with the final goal of showing that there exists some parameters $(\bar{\gamma}, \bar{\lambda})$ for which a magnetic and superconducting phase coexist in the model, and that the coexistence also holds if a small kinetic hopping term is also added to the interaction. For this, one needs to study the behavior of the thermodynamic pressure of the model, and this is done with the help of a mathematical theory called "catastrophe theory", that is studied in chapter two. One of the main objectives of catastrophe theory is to study the qualitative behavior of a family of functions around a critical point: for example, let $F(x, u)$ be a function from $\mathbb{R} \times \mathbb{R}^r$ to $\mathbb{R}$ ($F(x, u)$ is usually seen as a family of functions $F_u(x)$ that maps $x \mapsto F(x, u)$, and $u \in \mathbb{R}^r$ are seen as the parameters of the family), let $f(x) = F(x, 0)$ and suppose that $f$ has a critical point at $0$. Then, catastrophe theory tries to answer what happens with the functions $F_u(x)$ near $u = 0$ and $x = 0$ (i.e., if the critical point at $0$ bifurcates into other critical points, if $0$ is not a critical point anymore, if some local minima appear, etc.). It follows that, for certain families of functions, their behavior is qualitatively identical to the behavior of a family of polynomial functions of the type

$$P(x, u) = \pm x^{k+1} + u_1 x + \dots u_k x^k,$$

depending on the degeneracy of the critical point (that is encoded in a variable called the *determinacy* of $f$) and on the degeneracy of the family (that is encoded in the derivatives $\frac{\partial F(x,0)}{\partial u_i}$). Therefore, since this family of polynomial functions can be easily analyzed, from that it is possible to extract relevant information about bifurcations and appearances of other minima in the family $F(x, u)$. It is important to note that the catastrophe theory developed in chapter two is an adaptation of the general situation described above, but for families of functions $F(x, u)$ such that $F_u(x)$ is *even* for any parameter $u$, since this is the case that one finds useful when analyzing the pressure of the model studied.

Another very important use of catastrophe theory is to prove that some families of functions are *stable*. In this scenario, the stability of a family $F(x, u)$ means that for any small perturbation $p(x, u)$ – small in a suitable sense, that takes into account also the derivatives of $p$ – there exists a small parameter $\bar{u}$ where the perturbed family $F(x, u) +$

$p(x, u)$ behaves the same way around $u = \bar{u}$ as $F(x, u)$ does around $u = 0$.

In the final sections, using the results developed in chapters two and three, it is shown the coexistence of phases for the unperturbed model, and with the aid of the so-called *cluster expansions*, which is a technique to write the thermodynamic pressure of the perturbed model in a convenient way that allows one to prove that its derivatives are well-behaved, it is shown that the coexistence of phases still holds for the perturbed model. This provides a concrete application of catastrophe theory to the study of phase transitions in a fermionic lattice system.

# Chapter 1

# C*-Algebras

## 1.1 Introduction

This section aims to present some of the main and most important properties of a $C^*$-algebra, many of them which will be used extensively throughout the thesis. It is based on section 2.1 of [1], and also follows the same notation convention. Before going to the definition of a $C^*$-algebra, it is useful to begin with other concepts and notions that will be used in its definition and later on in the text.

### 1.1.1 Basic definitions

**Definition 1.1.1.** An *algebra* $\mathfrak{A}$ is a vector space over a field (which in this thesis will be always assumed to be the field $\mathbb{C}$ of the complex numbers, unless stated otherwise), equipped with a multiplication law that is associative and distributive; i.e., every $A$, $B$, $C$ $\in \mathfrak{A}$, and every $\alpha$, $\beta \in \mathbb{C}$ satisfies the following:

(a) $A(BC) = (AB)C$,

(b) $A(B + C) = AB + AC$,

(c) $\alpha\beta(AB) = (\alpha A)(\beta B)$.

Moreover, an algebra $\mathfrak{A}$ is said to be a $*$-*algebra* if it possesses a mapping $A \in \mathfrak{A} \to A^* \in \mathfrak{A}$ (usually called an *involution*), satisfying the following properties:

1. $A^{**} = A$,
2. $(AB)^* = B^*A^*$,
3. $(\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*$.

The element $A^*$ is called the *adjoint of* $A$. If $A^* = A$ then $A$ is said to be *self-adjoint*. Moreover, a set $\mathfrak{M} \subset \mathfrak{A}$ is called self-adjoint when $A \in \mathfrak{M}$ implies $A^* \in \mathfrak{M}$, and the set of all self-adjoint elements of a $*$-algebra $\mathfrak{A}$ is denoted by $\mathfrak{A}^{\mathbb{R}}$.

Another useful notion needed for the definiton of a $C^*$-algebra is the notion of a *normed algebra*:

**Definition 1.1.2.** A *normed algebra* $\mathfrak{A}$ is an algebra equipped with a norm $\|\ \|$: $\mathfrak{A} \to \mathbb{R}$, satisfying, for all $A, B \in \mathfrak{A}$:

(a) $\|A\| \geq 0$ and $\|A\| = 0 \Leftrightarrow A = 0$,

(b) $\|\lambda A\| = |\lambda|\|A\|$ for all $\lambda \in \mathbb{C}$,

(c) $\|A + B\| \leq \|A\| + \|B\|$,

(d) $\|AB\| \leq \|A\|\|B\|$ for all A, B $\in \mathfrak{A}$,

where the last condition is called the *product inequality*. If a normed algebra is complete with respect to its norm, it is also called a *Banach algebra*.

**Definition 1.1.3.** A *$C^*$-algebra* is a Banach *-algebra $\mathfrak{A}$ with the property

$$\|A^*A\| = \|A\|^2 \tag{1.1}$$

for all $A \in \mathfrak{A}$.

The above property is usually referred to as the *$C^*$-property*. Combined with the product inequality, the $C^*$-property also yelds the condition $\|A\| = \|A^*\|$ for all $A \in \mathfrak{A}$.

**Proposition 1.1.1.** *For any Hilbert space $\mathfrak{H}$, the algebra $\mathfrak{L}(\mathfrak{H})$ of the bounded linear operators acting on $\mathfrak{H}$ is a $C^*$-algebra with respect to the operator norm.*

*Proof.* As a reminder, the operator norm is given by

$$\|A\|_{op} = \sup_{v\in\mathfrak{H}\ |\ \|v\|=1} \|Av\|, \text{ where } \|Av\| = \sqrt{(Av, Av)}.$$

Clearly $\mathfrak{L}(\mathfrak{H})$ together with the operator norm is a normed $*$-algebra, and its completeness easily follows from the completeness of $\mathfrak{H}$ (remember that every Hilbert space is complete by definition). The less trivial property to show is the $C^*$-property.

For $A = 0$, clearly the $C^*$-property holds, so let $A \neq 0$. From the definition of the operator norm and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}\|A^*\|_{op}^2 = \sup_{v\in\mathfrak{H}\ |\ \|v\|=1} (A^*v, A^*v) \implies \|A^*\|_{op} &= \sup_{v\in\mathfrak{H}\ |\ \|v\|=1,\ v\neq\mathrm{Ker}(A^*)} \frac{(A^*v, A^*v)}{\|A^*v\|} \\ &\leq \sup_{\substack{v\in\mathfrak{H}\ |\ \|v\|=1,\\ w\in\mathfrak{H}\ |\ \|w\|=1}} |(w, A^*v)| \\ &= \sup_{\substack{v\in\mathfrak{H}\ |\ \|v\|=1,\\ w\in\mathfrak{H}\ |\ \|w\|=1}} |(Aw, v)| \leq \|A\|_{op}.\end{aligned} \tag{1.2}$$

Also, from the Cauchy-Schwarz inequality and the product inequality, it follows that

$$\|A\|_{op}^2 = \sup_{v\in\mathfrak{H}\ |\ \|v\|=1} (Av, Av) = \sup_{v\in\mathfrak{H}\ |\ \|v\|=1} (A^*Av, v) \leq \|A^*A\|_{op} \leq \|A^*\|_{op}\|A\|_{op} \tag{1.3}$$

From 1.2 and 1.3, one deduces that $\|A\|_{op} = \|A^*\|_{op}$. But applying it to 1.3, one finally arrives at

$$\|A\|_{op}^2 \leq \|A^*A\|_{op} \leq \|A^*\|_{op}\|A\|_{op} = \|A\|_{op}^2 \implies \|A\|_{op}^2 = \|A^*A\|_{op}. \tag{1.4}$$

□

Some algebras possess an *identity* element $\mathbb{1}$, i.e., an element of the algebra $\mathfrak{A}$ such that

$$A = \mathbb{1}A = A\mathbb{1}$$

for all $A \in \mathfrak{A}$. Also, it is not hard to check that, if the identity exists, then it is unique and self-adjoint. In fact, a general algebra may not have an identity, and those who have are called *unital* algebras. This thesis will only deal with unital algebras. Moreover, note that for a unital $C^*$-algebra, the $C^*$-property implies

$$\|\mathbb{1}\|^2 = \|\mathbb{1}^*\mathbb{1}\| = \|\mathbb{1}\|,$$

and hence $\|\mathbb{1}\| =$ either 1 or 0. But if $\|\mathbb{1}\| = 0$, from the product inequality, then $\|A\| \leq \|\mathbb{1}\|\|A\| = 0 \implies \|A\| = 0$ for all $A \in \mathfrak{A}$, which means that the algebra is identically zero. Hence, in this thesis, the latter case will be ignored and it will always be assumed that $\|\mathbb{1}\| = 1$.

### 1.1.2 Spectra and spectral radius

A very important notion in the theory of algebras, and specially $C^*$-algebras, is the notion of the spectrum of an element. In the case of finite-dimensional matrix algebras, the spectrum of a matrix corresponds to the set of its eigenvalues, and if the matrix is self-adjoint, then its eigenvalues are real numbers. This is of physical relevance: since the observables are represented by self-adjoint elements, and the possible outcomes of a measure correspond to their eigenvalues, then their eigenvalues must be real. As it shall be seen, such property extends also for arbitrary $C^*$-algebras; for a self-adjoint element, its spectrum is always a subset of the real line. This is also important because it can be used to define a partial order in a $C^*$-algebras, as it shall be seen later. Another important property of finite-dimensional matrix algebras is that, for a self-adjoint matrix $M$, the direction of "maximum growth" of the mapping $v \mapsto Mv$ is achieved when $v$ is an eigenvector of $M$. Hence, it follows that the operator norm of $M$ corresponds to the highest eigenvalue, in module, of $M$; i.e., the highest number, in module, of its spectrum. This property also has an analogue for the case of arbitrary $C^*$-algebras, and proving it is one of the main goals of this section.

**Definition 1.1.4.** Let $\mathfrak{A}$ be a unital algebra and $A \in \mathfrak{A}$. $A$ is said to be *invertible* if there exists an element $A^{-1} \in \mathfrak{A}$, such that $AA^{-1} = A^{-1}A = \mathbb{1}$. The element $A^{-1} \in \mathfrak{A}$ is called the *inverse* of $A$

**Definition 1.1.5.** Let $\mathfrak{A}$ be a unital algebra and $A \in \mathfrak{A}$. The *spectrum* $\sigma_{\mathfrak{A}}(A)$ of $A$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda\mathbb{1} - A$ is *not* invertible. The *resolvent set* $r_{\mathfrak{A}}(A)$ is defined as the complement of $\sigma_{\mathfrak{A}}(A)$ in $\mathbb{C}$; i.e., the set of all $\lambda \in \mathbb{C}$ such that $\lambda\mathbb{1} - A$ is invertible. For $\lambda \in r_{\mathfrak{A}}(A)$, the element $(\lambda\mathbb{1} - A)^{-1}$ is called the *resolvent of $A$ at $\lambda$*.

**Proposition 1.1.2.** *Let $\mathfrak{A}$ be a unital Banach algebra and $A \in \mathfrak{A}$. Then, for every $\lambda \in \mathbb{C}$ such that $|\lambda| > \|A\|$, the series*

$$\lambda^{-1} \sum_{i=0}^{\infty} \left(\frac{A}{\lambda}\right)^i \tag{1.5}$$

*converges to the resolvent of $A$ at $\lambda$.*

*Proof.* First, note that for any $0 \leq c < 1$, the series $\sum\limits_{i=0}^{\infty} c^i$ converges (and in particular, is Cauchy). Taking $c = \|\frac{A}{\lambda}\| < 1$, it follows that

$$\left\|\sum_{i=n}^{m} \left(\frac{A}{\lambda}\right)^i\right\| \leq \sum_{i=n}^{m} \left\|\left(\frac{A}{\lambda}\right)^i\right\| \leq \sum_{i=n}^{m} \left\|\frac{A}{\lambda}\right\|^i = \sum_{i=n}^{m} c^i. \tag{1.6}$$

Hence, 1.5 is Cauchy, and since $\mathfrak{A}$ is complete by definition, it follows that 1.5 converges. Now, note that for all $N \in \mathbb{N}$:

$$\begin{aligned} &\left\|(\lambda\mathbb{1} - A).\lambda^{-1} \sum_{i=0}^{N} \left(\frac{A}{\lambda}\right)^i - \mathbb{1}\right\| = \left\|\lambda^{-1} \sum_{i=0}^{N} \left(\frac{A}{\lambda}\right)^i .(\lambda\mathbb{1} - A) - \mathbb{1}\right\| = \\ &\left\|-\sum_{i=0}^{N} \left(\left(\frac{A}{\lambda}\right)^{i+1} - \left(\frac{A}{\lambda}\right)^i\right) - \mathbb{1}\right\| = \left\|\left(\frac{A}{\lambda}\right)^{N+1}\right\| \leq \left\|\left(\frac{A}{\lambda}\right)\right\|^{N+1}, \end{aligned} \tag{1.7}$$

which goes to 0 as $N \to \infty$. Therefore, it follows that

$$\lambda^{-1} \sum_{i=0}^{\infty} \left(\frac{A}{\lambda}\right)^i = (\lambda\mathbb{1} - A)^{-1} \tag{1.8}$$

□

**Proposition 1.1.3.** *Let $\mathfrak{A}$ be a unital Banach algebra. Then, for all $A \in \mathfrak{A}$, the resolvent set $r_{\mathfrak{A}}(A)$ is open and the function $R_A(\lambda) = (\lambda\mathbb{1} - A)^{-1}$ is analytic on $r_{\mathfrak{A}}(A)$.*

*Proof.* Let $A \in \mathfrak{A}$ and $\lambda_0 \in r_{\mathfrak{A}}(A)$. Then, it follows that

$$R_A(\lambda) = (\lambda\mathbb{1} - A)^{-1} = (\lambda_0\mathbb{1} - A)^{-1} \sum_{i=0}^{\infty} \left(\frac{\lambda_0 - \lambda}{\lambda_0\mathbb{1} - A}\right)^i \tag{1.9}$$

for all $\lambda$ such that $|\lambda - \lambda_0| < \|(\lambda_0\mathbb{1} - A)^{-1}\|$, where the proof of this statement is analogous to the proof of proposition 1.1.2. Hence, $r_{\mathfrak{A}}(A)$ is open, and since $R_A(\lambda)$ can be expressed as a Neumann series in every point of $r_{\mathfrak{A}}(A)$, it follows that it is analytic on $r_{\mathfrak{A}}(A)$. □

**Definition 1.1.6.** Let $\mathfrak{A}$ be a unital Banach algebra and $A \in \mathfrak{A}$. The *spectral radius* $\rho(A)$ of $A$ is defined as

$$\rho(A) = \sup\{|\lambda|; \lambda \in \sigma_{\mathfrak{A}}(A)\}. \tag{1.10}$$

**Proposition 1.1.4.** *Let $\mathfrak{A}$ be a unital Banach algebra and $A \in \mathfrak{A}$. Then, it follows that*

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \inf \|A^n\|^{\frac{1}{n}} \leq \|A\|. \tag{1.11}$$

*In particular, the limit exists. Thus the spectrum is a nonempty compact set.*

*Proof.* Since $\sigma_{\mathfrak{A}}(A) = \mathbb{C} \setminus r_{\mathfrak{A}}(A)$, and by proposition 1.1.2 $r_{\mathfrak{A}}(A)$ is open, it follows that $\sigma_{\mathfrak{A}}(A)$ is closed. Futhermore, by proposition 1.1.2, $\sigma_{\mathfrak{A}}(A)$ is bounded. Hence, it is compact. Also, suppose that $\sigma_{\mathfrak{A}}(A)$ is empty. Then, $r_{\mathfrak{A}}(A) = \mathbb{C}$, and therefore $R_A(\lambda)$ would be an entire function, since by proposition 1.1.5 it is analytic on $r_{\mathfrak{A}}(A)$. But

$$\lim_{|\lambda|\to\infty} \|R_A(\lambda)\| = \lim_{|\lambda|\to\infty} \|(\lambda\mathbb{1} - A)^{-1}\| = \lim_{|\lambda|\to\infty} \frac{\|(\mathbb{1} - \frac{A}{\lambda})^{-1}\|}{|\lambda|} = 0,$$

and by Liouville's theorem, it follows that $R_A(\lambda) = (\lambda\mathbb{1} - A)^{-1} = 0$ for all $\lambda \in \mathbb{C}$. Absurd, hence the spectrum $\sigma_{\mathfrak{A}}(A)$ is nonempty. To prove eq. 1.11, first consider the equality

$$\lambda^n\mathbb{1} - A^n = (\lambda\mathbb{1} - A)(\lambda^{n-1}\mathbb{1} + \lambda^{n-1}\mathbb{1}A + \cdots + \lambda\mathbb{1}A^{n-1} + A^{n-1}). \tag{1.12}$$

Note that, if $\lambda \in \sigma_{\mathfrak{A}}(A)$, then $\lambda^n \in \sigma_{\mathfrak{A}}(A^n)$ for all $n \in \mathbb{N}$ and hence, by proposition 1.1.2, $|\lambda^n| \leq \|A^n\|$ for all $n \in \mathbb{N} \implies |\lambda| \leq \|A^n\|^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. Therefore,

$$\rho(A) \leq \inf \|A^n\|^{\frac{1}{n}} \leq \liminf_{n\to\infty} \|A^n\|^{\frac{1}{n}}. \tag{1.13}$$

Now, let $\Delta$ be the open disc in $\mathbb{C}$ centered at $0$ of radius $\frac{1}{\rho(A)}$ (where $\frac{1}{\rho(A)} = +\infty$ if $\rho(A) = 0$) and, for $\lambda \in \Delta$, define

$$\tilde{R}_A(\lambda) = \begin{cases} R_A\left(\frac{1}{\lambda}\right), & \text{if } \lambda \neq 0 \\ \lim_{z\to 0} R_A\left(\frac{1}{z}\right) = 0, & \text{if } \lambda = 0 \end{cases}. \tag{1.14}$$

Note that, for $\lambda \in \Delta$ such that $|\lambda| < \frac{1}{\|A\|} \leq \frac{1}{\rho(A)}$, one has from proposition 1.1.2 that $\tilde{R}_A$ is given by

$$\tilde{R}_A(\lambda) = \lambda \sum_{i=0}^{\infty} (\lambda A)^n. \tag{1.15}$$

Moreover, by proposition 1.1.3 and the definition of $\rho(A)$, it follows that $\tilde{R}_A$ is analytic in $\Delta$, and hence, the power series in eq. 1.15 can be extended to all $\lambda \in \Delta$. The convergence of the power series implies that, for any $\lambda \in \Delta$, there must be some positive number $M$ such that

$$\|(\lambda A)^n\| < M \text{ for all } n \in \mathbb{N} \implies \|A^n\|^{\frac{1}{n}} < \frac{M^{\frac{1}{n}}}{|\lambda|} \text{ for all } n \in \mathbb{N} \implies$$
$$\limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leq \frac{1}{|\lambda|}. \tag{1.16}$$

Note that eq. 1.16 must hold for any $\lambda \in \Delta$, and hence it follows that $\limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leq \rho(A)$. This, together with eq. 1.13, implies

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \inf \|A^n\|^{\frac{1}{n}} \leq \|A\|.$$

□

**Definition 1.1.7.** Let $p : \mathbb{C} \to \mathbb{C}$ be some polynomial. Then, $p$ is (uniquely) given by

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

for some $n \in \mathbb{N}$ and some $a_1, \ldots, a_n \in \mathbb{C}$. If $\mathfrak{A}$ is an algebra and $A \in \mathfrak{A}$, one denotes by $p(A)$ the element

$$p(A) = \sum_{i=0}^{n} a_i A^i \quad \in \quad \mathfrak{A}.$$

**Theorem 1.1.1.** *Let $\mathfrak{A}$ be a unital Banach algebra, $A \in \mathfrak{A}$ and $p$ a polynomial. Then,*

$$\sigma_{\mathfrak{A}}(p(A)) = p(\sigma_{\mathfrak{A}}(A))$$

*Proof.* By the fundamental theorem of algebra, $\lambda - p(z)$, for any $\lambda \in \mathbb{C}$, can be factorized as

$$\lambda - p(z) = c \prod_{i=1}^{n} (\lambda_i - z)$$

and, analogously, $\lambda \mathbb{1} - p(A)$ can be factorized as

$$\lambda \mathbb{1} - p(A) = c \prod_{i=1}^{n} (\lambda_i \mathbb{1} - A).$$

Since all terms commute, $\lambda \in \sigma_{\mathfrak{A}}(p(A))$ if and only if $\lambda_j \in \sigma_{\mathfrak{A}}(A)$ for some $j = 1, ..., n$. Hence, Let $\lambda \in \sigma_{\mathfrak{A}}(p(A))$. Then, $\lambda_j \in \sigma_{\mathfrak{A}}(A)$ for some $j = 1, ..., n$. But note that $p(\lambda_j) - \lambda = 0 \iff p(\lambda_j) = \lambda$, i.e., $\lambda \in p(\sigma_{\mathfrak{A}}(A))$. Now, let $\lambda \in p(\sigma_{\mathfrak{A}}(A))$. then, $\exists\ \alpha \in \sigma_{\mathfrak{A}}(A)$ such that $p(\alpha) = \lambda \iff \lambda - p(\alpha) = 0$. By the above decomposition of $p(z)$, $\alpha = \lambda_j$ for some $j = 1, ..., n$. Therefore, since $\alpha = \lambda_j \in \sigma_{\mathfrak{A}}(A)$, it follows that $\lambda \in \sigma_{\mathfrak{A}}(p(A))$.

□

**Proposition 1.1.5.** *Let $\mathfrak{A}$ be a unital *-algebra. For $A \in \mathfrak{A}$:*

*(a)* $\sigma_{\mathfrak{A}}(A^*) = \overline{\sigma_{\mathfrak{A}}(A)}$,

*(b) if $A$ is invertible, then $\sigma_{\mathfrak{A}}(A^{-1}) = \sigma_{\mathfrak{A}}(A)^{-1}$,*

*(c)* $\sigma_{\mathfrak{A}}(AB) \cup \{0\} = \sigma_{\mathfrak{A}}(BA) \cup \{0\}$ *for all $A, B \in \mathfrak{A}$.*

*Proof.* $(a)$ It is easier (but of course equivalent) to prove that $r_{\mathfrak{A}}(A^*) = \overline{r_{\mathfrak{A}}(A)}$. $\lambda \in r_{\mathfrak{A}}(A^*) \iff \exists\ B \in \mathfrak{A}$ such that $B(\lambda\mathbb{1} - A^*) = (\lambda\mathbb{1} - A^*)B = \mathbb{1} \iff (\lambda\mathbb{1} - A^*)^* B^* = B^*(\lambda\mathbb{1} - A^*)^* = \mathbb{1}^* \iff (\overline{\lambda}\mathbb{1} - A)B^* = B^*(\overline{\lambda}\mathbb{1} - A) = \mathbb{1} \iff \overline{\lambda} \in r_{\mathfrak{A}}(A^*)$.

$(b)$ If $\lambda \neq 0$, then

$$(\lambda^{-1}\mathbb{1} - A) = -\lambda A(\lambda\mathbb{1} - A^{-1}).$$

Hence, for $A$ invertible $(\lambda^{-1}\mathbb{1} - A)$ is invertible $\iff$ $(\lambda\mathbb{1} - A^{-1})$ is invertible, i.e., $\lambda \in \sigma_{\mathfrak{A}}(A^{-1}) \iff \lambda^{-1} \in \sigma_{\mathfrak{A}}(A) \iff \lambda \in \sigma_{\mathfrak{A}}(A)^{-1}$.

$(c)$ Let $\lambda \in r_{\mathfrak{A}}(BA)$. Then $(\lambda\mathbb{1} - BA)^{-1}$ exists and it is not hard to see that

$$(\lambda\mathbb{1} - AB)(\mathbb{1} + A(\lambda\mathbb{1} - BA)^{-1}B) = (\mathbb{1} + A(\lambda\mathbb{1} - BA)^{-1}B)(\lambda\mathbb{1} - AB) = \lambda\mathbb{1}.$$

Therefore, $(\lambda\mathbb{1} - AB)$ is invertible, with the possible exception of $\lambda = 0$. Hence, $\sigma_{\mathfrak{A}}(AB) \cup \{0\} \subset \sigma_{\mathfrak{A}}(BA) \cup \{0\}$,and interchanging $A$ and $B$ gives $\sigma_{\mathfrak{A}}(AB) \cup \{0\} \supset \sigma_{\mathfrak{A}}(BA) \cup \{0\}$. $\square$

**Definition 1.1.8.** Let $\mathfrak{A}$ be a Banach *-algebra. An element $A \in \mathfrak{A}$ is said to be *normal* if

$$AA^* = A^*A.$$

If $\mathfrak{A}$ is unital then $A$ is said to be *isometric* whenever

$$A^*A = \mathbb{1},$$

and *unitary* if

$$A^*A = \mathbb{1} = AA^*.$$

**Theorem 1.1.2.** *Let $\mathfrak{A}$ be a unital C*-algebra and $A \in \mathfrak{A}$.*

*(a) If $A$ is normal then the spectral radius $\rho(A)$ of $A$ is given by*

$$\rho(A) = \|A\|,$$

*(b) if $A$ is isometric then*

$$\rho(A) = 1,$$

*(c) if $A$ is unitary then*

$$\sigma_{\mathfrak{A}}(A) \subset \{\lambda \in \mathbb{C}; |\lambda| = 1\},$$

*(d) and if $A$ is self-adjoint,*

$$\sigma_{\mathfrak{A}}(A) \subset [-\|A\|, \|A\|] \quad \textit{and} \quad \sigma_{\mathfrak{A}}(A^2) \subset [0, \|A\|^2].$$

*Proof.* $(a)$ Using the normality of $A$ and the $C^*$-norm identity, one can prove by induction that

$$(A^*)^n(A)^n = (A^*A)^n,$$

and

$$\|(A^*A)^{2^n}\| = \|(A^*A)^{2^{(n-1)}}(A^*A)^{2^{(n-1)}}\| = \|(A^*A)^{2^{(n-1)}}\|^2 = ... = \|A^*A\|^{2^n} = \|A\|^{2^{(n+1)}}.$$

Then,

$$\|A^{2^n}\| = \|(A^*)^{2^n}(A)^{2^n}\|^{1/2} = \|(A^*A)^{2^n}\|^{1/2} = \|A\|,^{2^n}$$

but

$$\rho(A) = \lim_{n\to\infty} \|A^{2^n}\|^{1/2^n} = (\|A\|^{2^n})^{1/2^n} = \|A\|.$$

$(b)$ Again by induction, one has for an isometry $A$

$$\|A^n\|^2 = \|(A^*)^n A^n\| = \|(A^*)^{(n-1)}(A^*A)A^{(n-1)} = \|(A^*)^{(n-1)}A^{(n-1)}\| = ... = \|\mathbb{1}\| = 1.$$

Therefore $\rho(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \lim_{n\to\infty} 1^{\frac{1}{n}} = 1$.

$(c)$ Since any unitary element $A$ is also isometric, it follows from $(b)$ that $\sigma_{\mathfrak{A}}(A) \subset \{\lambda \in \mathbb{C}; |\lambda| \leq 1\}$. Suppose $\exists\ \lambda \in \sigma_{\mathfrak{A}}(A)$ such that $|\lambda| < 1$. From proposition 1.1.5, $\sigma_{\mathfrak{A}}(A) = \overline{\sigma_{\mathfrak{A}}(A^*)}$. Since $A$ is unitary, then

$$\sigma_{\mathfrak{A}}(A) = \overline{\sigma_{\mathfrak{A}}(A^{-1})} = (\overline{\sigma_{\mathfrak{A}}(A)})^{-1}.$$

Therefore, one must have $(\overline{\lambda})^{-1} \in \sigma_{\mathfrak{A}}(A)$, but $|(\overline{\lambda})^{-1}| = \frac{1}{|\lambda|} > 1$, absurd.

$(d)$ Since any self-adjoint element $A$ is also normal, by $(a)$ it folows that $\rho(A) = \|A\|$, and hence $\sigma_{\mathfrak{A}}(A) \subset \{\lambda \in \mathbb{C}; |\lambda| \leq \|A\|\}$. Given $\lambda \in \mathbb{C}$ such that $|\lambda^{-1}| > \|A\|$, from proposition 1.1.2 it follows that $\mathbb{1} + i|\lambda|A$ is invertible. Hence, define $U \in \mathfrak{A}$ by

$$U = (\mathbb{1} - i|\lambda|A)(\mathbb{1} + i|\lambda|A)^{-1}.$$

Since $A$ is self-adjoint, it is straightforward to see that

$$U^* = ((\mathbb{1} - i|\lambda|A)(\mathbb{1} + i|\lambda|A)^{-1})^* = (\mathbb{1} + i|\lambda|A)(\mathbb{1} - i|\lambda|A)^{-1} = U^{-1}.$$

Thus, $U$ is unitary. From $(c)$ it follows that that $(1 - i|\lambda|\alpha)(1 + i|\lambda|\alpha)^{-1}\mathbb{1} - U$ is invertible whenever

$$|(1 - i|\lambda|\alpha)(1 + i|\lambda|\alpha)^{-1}| \neq 1 \implies |1 - i|\lambda|\alpha| \neq |1 + i|\lambda|\alpha|,$$

which is the case when $\mathrm{Im}(\alpha) \neq 0$, but

$$(1 - i|\lambda|\alpha)(1 + i|\lambda|\alpha)^{-1}\mathbb{1} - U = -2i|\lambda|(1 + i|\lambda|\alpha)^{-1}(\alpha\mathbb{1} - A)(\mathbb{1} + i|\lambda|A)^{-1}.$$

Hence, $\alpha\mathbb{1} - A$ is invertible for all $\alpha$ such that $\mathrm{Im}(\alpha) \neq 0$. Therefore, $\sigma_{\mathfrak{A}}(A) \subset \{\lambda \in \mathbb{C}; |\lambda| \leq \|A\|\} \cap \mathbb{R} = [-\|A\|, \|A\|]$. Taking $p(z) = z^2$, from Theorem 1.1.1, one has $\sigma_{\mathfrak{A}}(p(A)) = \sigma_{\mathfrak{A}}(A^2) = (\sigma_{\mathfrak{A}}(A))^2 \subset [0, \|A\|^2]$. □

Note that the above proposition contains a very important property of a $C^*$-algebra $\mathfrak{A}$: for a self-adjoint element $A$ of $\mathfrak{A}$, its norm is equal to the spectral radius, which is an algebraic property of $A$; i.e., it solely depends on the the algebraic structure of $\mathfrak{A}$, and not on some additional structure defined a posteriori, for example a topology. This readily implies that all norms on $\mathfrak{A}$ that also make it a $C^*$-algebra must agree in the self-adjoint elements of $\mathfrak{A}$, and with the $C^*$-norm property, such observation can also be extended to any element $A$ of $\mathfrak{A}$, as shown by the corollary below:

**Corollary 1.1.1.** *Let $\mathfrak{A}$ be a unital *-algebra. If there exists a norm on $\mathfrak{A}$ with the $C^*$-norm property and for which $\mathfrak{A}$ is complete, then this norm is unique.*

*Proof.* Let $\|\ \|$ be a norm on $\mathfrak{A}$ that makes it a $C^*$-algebra. For any $A \in \mathfrak{A}$, $A^*A$ is self-adjoint. Then, $\|A^*A\| = \rho(A^*A)$. But from the $C^*$-norm property,

$$\|A\| = \|A^*A\|^{\frac{1}{2}} = \rho(A^*A)^{\frac{1}{2}}.$$

Since the spectral radius depends only on the algebraic structure of $\mathfrak{A}$, it follows that such norm must be unique.

□

**Proposition 1.1.6.** *Let $\mathfrak{B}$ be a unital $C^*$-sub-algebra of a $C^*$-algebra $\mathfrak{A}$. If $A \in \mathfrak{B}$ then*

$$\sigma_{\mathfrak{A}}(A) = \sigma_{\mathfrak{B}}(A).$$

*Proof.* The goal is to show that if $\lambda\mathbb{1} - A$ is invertible in $\mathfrak{A}$, then it is invertible in the smallest unital $C^*$-sub-algebra that contains $A$, i.e., the $C^*$-sub-algebra $\mathfrak{C}$ generated by $\mathbb{1}$, $A$ and $A^*$. But since the the $C^*$-sub-algebra $\mathfrak{C}'$ generated by $\mathbb{1}$, $\lambda\mathbb{1}-A$ and $\overline{\lambda}\mathbb{1}-A^*$ is equal to the sub-algebra $\mathfrak{C}$ defined above, it is sufficient to show that if $A \in \mathfrak{A}$ is invertible, then $A^{-1} \in \mathfrak{C}$. Changing $A \to \lambda\mathbb{1} - A$, then $\lambda\mathbb{1} - A$ invertible implies $(\lambda\mathbb{1} - A)^{-1} \in \mathfrak{C}' = \mathfrak{C}$.

Let $\mathfrak{C}$ be as defined above, with $A$ invertible. Suppose first that $A \in \mathfrak{A}^{\mathbb{R}}$. Then, $R_A(\lambda)$ is analytic for all $\lambda$ such that $\mathrm{Im}(\lambda) \neq 0$ and also in some neighborhood of $0$. Let $\lambda_0$ be some purely imaginary number such that $|\lambda_0| > \|A\|$. From proposition 1.1.2, $(A - \lambda_0\mathbb{1})$ is invertible and hence $-R_A$ can be given as (see proposition 1.1.3)

$$-R_A(\lambda) = (A - \lambda\mathbb{1})^{-1} = (A - \lambda_0\mathbb{1})^{-1} \sum_{i=0}^{\infty} \left(\frac{\lambda - \lambda_0}{A - \lambda_0\mathbb{1}}\right)^i \tag{1.17}$$

for all $\lambda$ such that $|\lambda - \lambda_0| < \|(A - \lambda_0\mathbb{1})^{-1}\|$. By propositions 1.1.1 and 1.1.5, one has $\sigma_{\mathfrak{A}}(A - \lambda\mathbb{1})^{-1} = (\sigma_{\mathfrak{A}}(A) - \lambda)^{-1}$, and thus

$$\|(A - \lambda_0\mathbb{1})^{-1}\| = \sup|\sigma_{\mathfrak{A}}(A - \lambda_0\mathbb{1})^{-1}| = \sup|(\sigma_{\mathfrak{A}}(A) - \lambda_0)^{-1}| = \inf|\sigma_{\mathfrak{A}}(A) - \lambda_0|.$$

But remember that $\sigma_{\mathfrak{A}}(A) \subset \mathbb{R}$ and $\lambda_0$ is purely imaginary. Then,

$$\inf|\sigma_{\mathfrak{A}}(A) - \lambda_0| = \sqrt{\inf|\sigma_{\mathfrak{A}}(A)|^2 - |\lambda_0|^2} > |\lambda_0|,$$

since $0 \notin \sigma_{\mathfrak{A}}(A)$, $\sigma_{\mathfrak{A}}(A)$ is closed. Therefore, eq. 1.17 converges for $\lambda = 0$, and hence

$$A^{-1} = (A - \lambda_0\mathbb{1})^{-1} \sum_{i=0}^{\infty} \left(\frac{-\lambda_0}{A - \lambda_0\mathbb{1}}\right)^i \in \mathfrak{C}.$$

Now, let $A$ be invertible but not necessarily self-adjoint. Then $A^*A$ is invertible (where $(A^*A)^{-1} = A^{-1}(A^{-1})^*$) and self-adjoint, and hence by the above argument $(A^*A)^{-1}$ is contained in $\mathfrak{C}$. Define

$$X = (A^*A)^{-1}A^* \in \mathfrak{C}. \tag{1.18}$$

Since $XA = \mathbb{1}$, right multiplying by $A^{-1}$ gives $X = A^{-1}$. □

### 1.1.3 Continuous functional calculus

For a normed $*$-algebra $\mathfrak{A}$ to be a $C^*$-algebra, apart from satisfying the $C^*$-property, it must also be a Banach algebra, i.e., $\mathfrak{A}$ must also be complete with respect to its norm. In a naive sense, this means that $\mathfrak{A}$ cannot be "too small", otherwise its completeness would not hold. In fact, it can be shown that its completeness ensures that the algebra is "big enough" so that, for all self-adjoint element $A \in \mathfrak{A}$, the Banach sub-algebra generated by $\{\mathbb{1}, A\}$ is homomorphic to the $C^*$-algebra of continuous functions on a compact. Such homomorphism is a very useful tool in the study of $C^*$-algebras, since it allows one to make sense of applying continuous functions on self-adjoint elements of $\mathfrak{A}$, and therefore to compute, for example, the square root $\sqrt{A}$, or the exponentiation $e^A$, of any $A \in \mathfrak{A}^{\mathbb{R}}$. The idea behind proving the existence of such homomorphism is to show that it is contained in the closure of the algebra of polynomials of $A$, and the first step for this is to prove that the space of polynomials is dense in the algebra of continuous functions. This result is a corollary of the so-called Stone-Weierstrass theorem:

**Theorem 1.1.3** (Stone-Weierstrass)**.** *Let $K$ be a compact metric space, $\mathcal{C}(K;\mathbb{C})$ the algebra of continuous functions from $K$ to $\mathbb{C}$ with the supremum norm, and $S \subset \mathcal{C}(K;\mathbb{C})$ a unital sub-algebra which separates points of $K$. Then, $S$ is dense in $\mathcal{C}(K;\mathbb{C})$.*

**Definition 1.1.9.** Let $\mathfrak{A}$ be a unital $C^*$-algebra and $A \in \mathfrak{A}^{\mathbb{R}}$. Define:

$$\mathcal{C}_A \doteq \mathcal{C}(\sigma(A);\mathbb{C}), \quad \text{and} \quad \mathcal{P}_A \doteq \{f \in \mathcal{C}_A \mid \exists \text{ a polynomial } p \text{ where } f = p \text{ on } \sigma(A)\}.$$

**Corollary 1.1.2.** *For any $A \in \mathfrak{A}^{\mathbb{R}}$, $\mathcal{P}_A$ is dense in $\mathcal{C}_A$.*

*Proof.* This follows by direct application of the Stone-Weierstrass theorem, with $K = \sigma(A)$ and $S = \mathcal{P}_A$. □

**Definition 1.1.10.** Let $\mathfrak{A}$ be a unital $C^*$-algebra and $A \in \mathfrak{A}^{\mathbb{R}}$. Define $\Phi_A : \mathcal{P}_A \to \mathfrak{A}$ by

$$\Phi_A(f) = p(A), \quad \text{where } p : \mathbb{C} \to \mathbb{C} \text{ is any polynomial such that } p = f \text{ on } \sigma(A).$$

Note that, in general, it is possible to have two distinct polynomials $p_1, p_2$ such that $p_1 = f = p_2$ on $\sigma(A)$. But in this case, them $(p_1 - p_2)(\sigma(A)) = \{0\}$, and by proposition 1.1.1, it follows that

$$\sigma((p_1 - p_2)(A)) = (p_1 - p_2)(\sigma(A)) = \{0\},$$

and since $\Phi_A$ is defined only for self-adjoint $A$, then

$$\|p_1(A) - p_2(A)\| = \sup\{|\lambda|; \lambda \in \sigma((p_1 - p_2)(A))\} = 0.$$

Therefore, $\Phi_A$ is well-defined for all $A \in \mathfrak{A}^{\mathbb{R}}$. The goal now is to extend the domain of $\Phi_A$ to $\mathcal{C}_A$, but before that it is convenient to prove some useful properties of $\Phi_A$.

**Proposition 1.1.7.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra. Then, for all $A \in \mathfrak{A}^{\mathbb{R}}$, the map $\Phi_A : \mathcal{P}_A \to \mathfrak{A}$ defined above is a unital isometric $*$-homomorphism, where in $\mathcal{P}_A$ the norm is given by*

$$\|p\| = \sup\{|\lambda|; \lambda \in p(\sigma(A))\}.$$

*Proof.* Let $A \in \mathfrak{A}$. Clearly, by the definition of $\Phi_A$, $\Phi_A(1) = \mathbb{1}$. Moreover, for any polynomials $p_1, p_2$, it is true that

(a) $p_1(A) + p_2(A) = (p_1 + p_2)(A)$,

(b) $p_1(A) \cdot p_2(A) = (p_1 \cdot p_2)(A)$,

(c) $\overline{p}_1(A) = p_1(A)^*$,

hence, it follows that $\Phi_A$ is a $*$-homomorphism. Finally, for any polynomial $p \in \mathcal{P}_A$:

$$\|\Phi_A(p)\| = \|p(A)\| = \sup\{|\lambda|; \lambda \in \sigma(p(A))\} = \sup\{|\lambda|; \lambda \in p(\sigma(A))\} = \|p\|,$$

and thus $\Phi_A$ is isometric. □

In adition, note that, for any $A \in \mathfrak{A}^{\mathbb{R}}$, $\Phi_A$ is also the *unique* $*$-homomorphism from $\mathcal{P}_A$ to $\mathfrak{A}$ satisfying

$$\Phi_A(\mathrm{id}) = A$$

where $\mathrm{id}$ is the identity function, $\mathrm{id}(x) = x$. This is true because any polynomial $p \in \mathcal{P}$ can be uniquely written as a linear combination of the powers of $\mathrm{id}$, and hence, any $*$-homomorphism $\phi$ acting on such $p$ will result in

$$\phi(p) = \phi\left(\sum_{i=0}^{n} a_i \,\mathrm{id}^i\right) = \sum_{i=0}^{n} a_i \phi(\mathrm{id})^i$$

Therefore, if $\phi(\mathrm{id}) = A$, then clearly $\phi(p) = p(A) = \Phi_A(p)$.

**Theorem 1.1.4.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra. For all $A \in \mathfrak{A}^{\mathbb{R}}$, there exists a unique $*$-homomorphism $\tilde{\Phi}_A : \mathcal{C}_A \to \mathfrak{A}$ such that $\tilde{\Phi}_A(\mathrm{id}) = A$. Moreover, $\tilde{\Phi}_A$ is isometric.*

*Proof.* Corollary 1.1.2 states that $\mathcal{P}_A$ is dense in $\mathcal{C}_A$. Hence, since the map $\Phi_A : \mathcal{P}_A \to \mathfrak{A}$ is bounded, define $\tilde{\Phi}_A$ as the unique linear extension in $\mathcal{C}_A$ of $\Phi_A$. From the continuity of the norm, it easily follows that $\tilde{\Phi}_A$ is isometric, and from the continuity of the involution "$*$" and the property that $\lim_n x_n \lim_n y_n = \lim_n x_n y_n$ holds for convergent sequences, it is also not hard to show that $\tilde{\Phi}_A$ is a $*$-homomorphism. Finally, the uniqueness of $\tilde{\Phi}_A$ is a consequence of the uniqueness of $\Phi_A$ on $\mathcal{P}_A$ and the fact that $\mathcal{P}_A$ is dense in $\mathcal{C}_A$. □

**Definition 1.1.11.** For a unital $C^*$-algebra $\mathfrak{A}$, the family of unital $*$-homomorphisms $\Phi_A : \mathcal{C}_A \to \mathfrak{A}$, $A \in \mathfrak{A}^{\mathbb{R}}$, satisfying $\Phi_A(\mathrm{id}) = A$, is called the *continuous functional calculus of* $A$, and for any $A \in \mathfrak{A}^{\mathbb{R}}$, the algebra generated by $\{\Phi_A(f); f \in \mathcal{C}_A\}$ wil be denoted by $\mathfrak{F}(A)$. To simplify the notation, from now on the element $\Phi_A(f) \in \mathfrak{A}$ will be denoted by $f(A)$.

Note that, for any $A \in \mathfrak{A}^{\mathbb{R}}$, $\mathfrak{F}(A)$ is the smallest unital $C^*$-sub-algebra of $\mathfrak{A}$ that contains $A$ (since it is the closure of the unital sub-algebra of polynomials of $A$). This remark is important to prove the following proposition:

**Proposition 1.1.8.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $A \in \mathfrak{A}^{\mathbb{R}}$. For all $f \in \mathcal{C}_A$, $\sigma(f(A)) = f(\sigma(A))$*

*Proof.* $(\sigma(f(A)) \subset f(\sigma(A)))$. Suppose that $\lambda \notin f(\sigma(A))$ for some $f \in \mathcal{C}_A$. Then, $\lambda - f(x) \neq 0$ for all $x \in \sigma(A)$. Therefore, the function $g : \sigma(A) \to \mathbb{C}$ given by

$$g(x) = \frac{1}{\lambda - f(x)}$$

is well-defined, continuous and $((\lambda - f).g)(x) = 1$. Therefore,

$$g(A)(\lambda\mathbb{1} - f(A)) = (\lambda\mathbb{1} - f)g(A) = ((\lambda - f).g)(A) = \mathbb{1},$$

and hence $\lambda \notin \sigma(f(A))$.

$(f(\sigma(A)) \subset \sigma(f(A)))$. Now, suppose that $\lambda \notin \sigma(f(A))$. Then, $\lambda\mathbb{1} - f(A)$ is invertible. By proposition 1.1.6 and since $\mathfrak{F}(A)$ is a $C^*$-algebra which contains $A$, it follows that $(f(A) - \lambda\mathbb{1})^{-1} \in \mathfrak{F}(A)$, that is, there exists some function $g \in \mathcal{C}_A$ such that

$$g(A)(f(A) - \lambda\mathbb{1}) = (f(A) - \lambda\mathbb{1})g(A) = \mathbb{1} \quad \Longrightarrow \quad (g(f - \lambda))(A) = \mathbb{1}.$$

Hence, $g(x)(f(x) - \lambda) = 1$ on $\sigma(A)$, which implies that $f(x) - \lambda \neq 0$ on $\sigma(A)$, i.e., $\lambda \neq f(\sigma(A))$. □

**Proposition 1.1.9.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra, $A \in \mathfrak{A}^{\mathbb{R}}$ and $\tau$ any *-automorphism on $\mathfrak{A}$. For all $f \in \mathcal{C}_A$, $\tau(f(A)) = f(\tau(A))$.*

*Proof.* If $\tau$ is a *-automorphism on $\mathfrak{A}$, and $A \in \mathfrak{A}^{\mathbb{R}}$, then clearly $\tau(A) \in \mathfrak{A}^{\mathbb{R}}$. Moreover, for any $B \in \mathfrak{A}$,

$$\tau(\mathbb{1})B = B\tau(\mathbb{1}) = \tau(\tau^{-1}(B)) = B \implies \tau(\mathbb{1}) = \mathbb{1}.$$

In particular, this implies that $\sigma(\tau(A)) = \sigma(A)$. Since any *-automorphism on $C^*$-algebras is continuous (this is proven later in the text, see proposition 1.2.1) and for any polynomial $p \in \mathcal{P}_A$, one has $\tau(p(A)) = p(\tau(A))$, by the density of $\mathcal{P}_A$ on $\mathcal{C}_A$, the proposition follows. □

### 1.1.4 Positive elements

The existence of a continuous functional calculus in a $C^*$-algebra allows one to explore some decomposition properties of the algebra. One important decomposition that is studied in this section is the decomposition into *positive* elements. Roughly speaking, the positive elements of a $C^*$-algebra resemble the positive numbers in the real line: for example, an element of a $C^*$-algebra can be decomposed in positive elements in a similar way that a complex number can have its real and imaginary parts decomposed as the subtraction of positive numbers. Such decomposition allows one to define a partial order in a $C^*$-algebra, and turns out to be useful when studying, for example, the linear functionals on a $C^*$-algebra, as it shall be seen in the next section.

**Definition 1.1.12.** Let $\mathfrak{A}$ be a *-algebra and $A \in \mathfrak{A}$. $A$ is defined to be *positive* if it is self-adjoint and $\sigma(A) \subset [0, +\infty)$. The set of all positive elements of $\mathfrak{A}$ is denoted by $\mathfrak{A}_+$.

**Lemma 1.1.1.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $A \in \mathfrak{A}^{\mathbb{R}}$. Then $A$ is positive if, and only if, $\left\|\mathbb{1} - \frac{A}{\|A\|}\right\| \leq 1$. Moreover, if $\|A\| \leq 1$ and $\|\mathbb{1} - A\| \leq 1$, then $A$ is positive.*

*Proof.* If $A$ is positive then $\sigma(A) \subset [0, \|A\|]$. By theorem 1.1.1, one has $\sigma\left(\mathbb{1} - \frac{A}{\|A\|}\right) = \sigma(\mathbb{1}) - \frac{1}{\|A\|}\sigma(A) \subset [0, 1]$, and therefore $\left\|\mathbb{1} - \frac{A}{\|A\|}\right\| = \sup \left|\sigma\left(\mathbb{1} - \frac{A}{\|A\|}\right)\right| \leq 1$. Conversely, since $A$ is self-adjoint by hypotesis, $\left\|\mathbb{1} - \frac{A}{\|A\|}\right\| \leq 1$ implies

$$\sigma\left(\mathbb{1} - \frac{A}{\|A\|}\right) \subset [-1, 1] \implies \sigma(\mathbb{1}) - \sigma(A) \subset [-\|A\|, \|A\|] \implies$$
$$\implies \sigma(A) \subset [-\|A\|, \|A\|] + \|A\|\sigma(\mathbb{1}) = [0, 2\|A\|].$$

Hence, $A$ is positive.

Now if $A \in \mathfrak{A}^{\mathbb{R}}$, then $\mathbb{1} - A \in \mathfrak{A}^{\mathbb{R}}$. Thus $\|\mathbb{1} - A\| \leq 1$ implies $\sigma(\mathbb{1} - A) = \sigma(\mathbb{1}) - \sigma(A) \subset [-1, 1] \implies \sigma(A) \subset [0, 2]$ and hence $A$ is positive. □

The next proposition allows one to define a partial order in a unital $C^*$-algebra:

**Proposition 1.1.10.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra. Then, $\mathfrak{A}_+$ is a closed convex cone with the property*

$$\mathfrak{A}_+ \cap (-\mathfrak{A}_+) = \{0\},$$

*Proof.* Let $A \in \mathfrak{A}_+ \cap (-\mathfrak{A}_+)$. Then clearly one must have $\sigma(A) = \|A\| = 0$, and hence $A = 0$. If $A \in \mathfrak{A}_+$ and $\lambda \geq 0$, then clearly $\lambda A \in \mathfrak{A}_+$, since $\sigma(\lambda A) = \lambda\sigma(A) \subset [0, \lambda\|A\|]$. To conclude it will be shown that if $A, B \in \mathfrak{A}_+$ then $\frac{A+B}{2} \in \mathfrak{A}_+$. Clearly, it is sufficient to consider only the case where $\|A\| = 1$ and $\|B\| \leq 1$. Then, from the triangle inequality, one has

$$\left\|\frac{A + B}{2}\right\| \leq 1, \tag{1.19}$$

and

$$\left\|\mathbb{1} - \frac{A+B}{2}\right\| \leq \frac{\|\mathbb{1} - A\|}{2} + \frac{\|\mathbb{1} - B\|}{2}.$$

From Lemma 1.1.1, since $A$ is positive and $\|A\| = 1$, it follows that $\|\mathbb{1} - A\| \leq 1$. Futhermore, $\|\mathbb{1} - B\| = \sigma(\mathbb{1} - B) = \sigma(\mathbb{1}) - \sigma(B) \subset [1 - \|B\|, 1] \subset [0, 1]$, since $\|B\| \leq 1$. Therefore, $\|\mathbb{1} - B\| \leq 1$, and hence,

$$\left\|\mathbb{1} - \frac{A+B}{2}\right\| \leq \frac{\|\mathbb{1} - A\|}{2} + \frac{\|\mathbb{1} - B\|}{2} \leq 1. \tag{1.20}$$

From eq.s 1.19, 1.20 and lemma 1.1.1 it follows that $\frac{A+B}{2} \in \mathfrak{A}_+$. □

**Corollary 1.1.3.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra, and $\succeq$ a binary relation over $\mathfrak{A}$ defined by*

$$A \succeq B \iff A - B \in \mathfrak{A}_+. \tag{1.21}$$

*Then, $\succeq$ is a partial order in $\mathfrak{A}$. Moreover, note that $\mathfrak{A}_+ = \{A \in \mathfrak{A}; A \succeq 0\}$.*

*Proof.* From proposition 1.1.10, it is straightforward to see that, for all $A, B, C \in \mathfrak{A}$, the binary relation $\succeq$ satisfies

(a) $A \succeq A$,

(b) if $A \succeq B$ and $B \succeq C$, then $A \succeq C$,

(c) if $A \succeq B$ and $B \succeq A$, then $A = B$.

□

**Lemma 1.1.2.** *Let $\mathfrak{A}$ be a *-algebra. Then, any element $A \in \mathfrak{A}$ has a unique decomposition in terms self-adjoint elements $A_1, A_2$, of the form*

$$A = A_1 + iA_2$$

*Proof.* Taking the conjugate of the above equation, one has

$$A = A_1 + iA_2, \qquad (1)$$
$$A^* = A_1 - iA_2. \qquad (2)$$

$(1) + (2) \implies A_1 = (A + A^*)/2$, and $(1) - (2) \implies A_2 = (A - A^*)/2i$. It is easy to check that $A_1, A_2$ are self-adjoints. □

**Proposition 1.1.11.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra. Then,*

*(a) for all $A \in \mathfrak{A}_+$, there is a unique element $\sqrt{A} \in \mathfrak{A}_+$ such that $\sqrt{A}.\sqrt{A} = A$,*

*(b) for all $A \in \mathfrak{A}^{\mathbb{R}}$ there are elements $A^+, A^- \in \mathfrak{A}_+$ such that $A = A^+ - A^-$, $\|A^\pm\| \leq \|A\|$, and $A^+.A^- = 0$.*

*Proof.* $(a)$ Consider the continuous function $f(x) = \sqrt{x}$. Since $A \in \mathfrak{A}_+$, then $f$ is continuous on $\sigma(A)$ and hence define $\sqrt{A} \doteq f(A)$. By proposition 1.1.8, $\sqrt{A} \in \mathfrak{A}_+$, and by theorem 1.1.4, it follows that $\sqrt{A}.\sqrt{A} = A$. To show that $\sqrt{A}$ is unique, suppose there is another $B \in \mathfrak{A}_+$ such that $B^2 = A$. Then, since $A \in \mathfrak{F}(B)$, it follows that $\mathfrak{F}(A) \subset \mathfrak{F}(B)$. Therefore, $\sqrt{A} \in \mathfrak{F}(B)$, i.e., $\sqrt{A} = f(B)$ for some $f \in \mathcal{C}_B$. Taking the square on both sides, one has:

$$\sqrt{A}^2 = B^2 = (f(B))^2 = f^2(B) \iff (\mathrm{id}^2 - f^2)(B) = 0 \iff f = \pm\,\mathrm{id} \text{ or } B = 0.$$

If $B = 0$, then $A = B = \sqrt{A} = 0$. Now, suppose $B \neq 0$. Then, If $f = -\,\mathrm{id}$, one has $B = -\sqrt{A} \notin \mathfrak{A}_+$. Hence, $f = \mathrm{id}$ and $\sqrt{A} = B$.

$(b)$ Now, consider the continuous functions $f^{\pm}(x) = (|x| \pm x)/2$. Then, for all $A \in \mathfrak{A}^{\mathbb{R}}$, it follows from proposition 1.1.8 that $A^{\pm} \doteq f^{\pm}(A) \in \mathfrak{A}_+$ and $\|A^{\pm}\| = \sup\left|f^{\pm}([-\|A\|, \|A\|])\right| \leq \|A\|$. Moreover, from theorem 1.1.4 it follows that $A = A^+ - A^-$ and $A^+.A^- = 0$. □

**Corollary 1.1.4.** *Let $\mathfrak{A}$ be a unital $C^*$algebra and $A \in \mathfrak{A}$. Then, there exists some $A^+_{re}, A^-_{re}, A^+_{im}, A^-_{im} \in \mathfrak{A}_+$ such that*

*(a)* $A = {A_{re}}^+ - A^-_{re} + i(A^+_{im} - A^-_{im})$,

*(b)* $\|A^{\pm}_{re(im)}\| \leq \|A\|$,

*(c)* $A^+_{re}.A^-_{re} = A^+_{im}.A^-_{im} = 0$.

*Proof.* This easily follows from applying 1.1.2 on $A$ and 1.1.11 on $A_1$ and $A_2$. □

The above decomposition of an element of a $C^*$-algebra into positive elements is called the *orthogonal decomposition*. Another useful decomposition of a $C^*$-algebra element is the following:

**Proposition 1.1.12.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra, Then, any element $A \in \mathfrak{A}$ can be written as a linear combination of unitary elements.*

*Proof.* If $A = 0$, then the proposition is obvious, Let $A \neq 0$. By the previous lemma, it suffices to prove it only for $A$ self-adjoint. And if $A$ is self-adjoint and $A \neq 0$, then $\sigma(A) \subset [-\|A\|, \|A\|]$, and on can write

$$A = \|A\|B, \quad \text{where} \quad B = \frac{A}{\|A\|}$$

and $\sigma(B) \subset [-1, 1]$. Hence, $\sigma(\mathbb{1} - B^2) \subset [0, 1]$, and thus $\sqrt{\mathbb{1} - B^2}$ is well-defined. Consider the elements

$$\begin{aligned} B_1 &= B + i\sqrt{\mathbb{1} - B^2}, \\ B_2 &= B - i\sqrt{\mathbb{1} - B^2}. \end{aligned}$$

It is easy to see that $B_1, B_2$ are unitary, and $B = (B_1 + B_2)/2$. Hence,

$$A = \frac{\|A\|}{2}B_1 + \frac{\|A\|}{2}B_2.$$

□

To finish, it is shown another useful property of the positive elements of a $C^*$-algebra:

**Proposition 1.1.13.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $A \in \mathfrak{A}$. Then, the following conditions are equivalent:*

*(a)* $A \in \mathfrak{A}_+$,

*(b)* $A = B^*B$ *for some* $B \in \mathfrak{A}$.

*Proof.* $(a) \implies (b)$ This easily follows from proposition 1.1.11, with $B = \sqrt{A}$.

$(c) \implies (a)$ By proposition 1.1.11, $B^*B$ can be decomposed as

$$B^*B = C^+ - C^-,$$

where $C^\pm \in \mathfrak{A}_+$ and $C^+C^- = 0$. First, note that

$$(BC^-)^*(BC^-) = C^-(C^+ - C^-)C^- = -(C^-)^3 \in (-\mathfrak{A}_+). \tag{1.22}$$

Moreover, by lemma 1.1.2, one has

$$BC^- = F + iG,$$

where $F, G \in \mathfrak{A}^{\mathbb{R}}$. Then, it follows that

$$\begin{aligned}(BC^-)(BC^-)^* + (BC^-)^*(BC^-) = 2(F^2 + S^2) \implies \\ (BC^-)(BC^-)^* = (C^-)^3 + 2(F^2 + S^2) \in \mathfrak{A}_+.\end{aligned} \tag{1.23}$$

Proposition 1.1.5 implies $\sigma((BC^-)^*(BC^-)) \cup \{0\} = \sigma((BC^-)(BC^-)^*) \cup \{0\}$. Hence, by eq. 1.23, one has $(BC^-)^*(BC^-) \in \mathfrak{A}_+$, but by eq. 1.22, one has $(BC^-)^*(BC^-) \in (-\mathfrak{A}_+)$. Therefore, $(BC^-)^*(BC^-) = -(C^-)^3 = 0$ and by proposition 1.1.8 it follows that $C^- = 0$, which implies $B^*B = C^+ \in \mathfrak{A}_+$. □

## 1.2 Representations and States

Another important topic with physical relevance in the theory of $C^*$-algebras is the study of their representations. As already said in the introduction, in order to recover, to some extent, the Hilbert space formalism of quantum mechanics described in the early theories of Heisenberg and Schrödinger, it should be possible to represent a $C^*$-algebra as linear operators acting on some Hilbert space. Moreover, for the theory to be capable of dealing with phenomena emerging from the infinite-particle system idealization, such as phase transitions, there should also exist non-equivalent representations of the same $C^*$-algebra. If fact, both of these assertions holds, and the existence and non-equivalence of $C^*$-algebra representations is closely related to the study of its states, which will be discussed in the next section. Here it is presented a brief introduction to some elementary properties of $C^*$-algebras representations, that will be useful when talking about representations associated with states.

### 1.2.1 Representations

**Definition 1.2.1.** Let $\mathfrak{A}, \mathfrak{B}$ be two $*$-algebras. A **-morphism* between $\mathfrak{A}$ and $\mathfrak{B}$ is a mapping $\pi$: $A \in \mathfrak{A} \mapsto \pi(A) \in \mathfrak{B}$ satisfying

(a) $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$,

(b) $\pi(AB) = \pi(A)\pi(B)$,

(c) $\pi(A^*) = \pi(A)^*$,

for all $A, B \in \mathfrak{A}$ and $\alpha, \beta \in \mathbb{C}$. Moreover, the *range of* $\pi$, denoted by $\mathfrak{B}_\pi$, is defined as $\mathfrak{B}_\pi = \{\pi(A) \mid A \in \mathfrak{A}\}$.

**Lemma 1.2.1.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $P \in \mathfrak{A}$ a nonzero projection (i.e., a nonzero self-adjoint element such that $P^2 = P$). Then, $\|P\| = 1$ and $P\mathfrak{A}P$ is a unital $C^*$-algebra, with $\mathbb{1}_{P\mathfrak{A}P} = P$.*

*Proof.* Note that $P^2 = P$ implies $\|P\|^2 = \|P^2\| = \|P\|$, and hence, $\|P\| = 0$ or $1$. Since $P$ is nonzero by hypothesis it follows that $\|P\| = 1$. Now, clearly $P\mathfrak{A}P$ is a normed $*$-algebra with the norm inherited from $\mathfrak{A}$ – that satisfies the $C^*$-property – and $\mathbb{1}_{P\mathfrak{A}P} = P$. Therefore, to show that $P\mathfrak{A}P$ is a $C^*$-algebra it suffices to show that $P\mathfrak{A}P$ is closed in $\mathfrak{A}$. Let $\{PA_nP\}$ be a sequence in $P\mathfrak{A}P$ that converges to $A \in \mathfrak{A}$, and $m \in \mathbb{N}$ be such that $\|PA_mP - A\| < \frac{\epsilon}{2}$. Then,

$$\begin{aligned}\|A - PAP\| =& \|A - PA_mP + PA_mP - PAP\| \leq \|A - PA_mP\| + \\ & \|P(PA_mP - A)P\| \leq \|A - PA_mP\|(1 + \|P\|^2) < \epsilon.\end{aligned}$$

Since $\epsilon$ can be arbitrarily small, it follows that $A = PAP \in P\mathfrak{A}P$. □

The next lemma is provides a useful result for the proof of the next proposition. However, since proving it requires some properties of $C^*$-algebras that are unimportant to the rest of this thesis, its proof is omitted here, but it can be found in ([1], section 2.2.3).

**Lemma 1.2.2.** *Let $\mathfrak{A}, \mathfrak{B}$ be a unital $C^*$-algebras and $\pi$ a *-morphism between $\mathfrak{A}$ and $\mathfrak{B}$. Then, the quotient algebra $\mathfrak{A}_\pi = \mathfrak{A}/\ker(\pi)$ is a $C^*$-algebra.*

**Proposition 1.2.1.** *Let $\mathfrak{A}, \mathfrak{B}$ be unital *-algebras with the $C^*$-norm property and $\pi$ a *-morphism between $\mathfrak{A}$ and $\mathfrak{B}$. Then, $\pi$ is continuous with $\|\pi(A)\| \leq \|A\|$. Moreover, if $\mathfrak{A}$ and $\mathfrak{B}$ are $C^*$-algebras, $\pi$ maps positive elements of $\mathfrak{A}$ into positive elements of $\mathfrak{B}$, and its range $\mathfrak{B}_\pi$ is a $C^*$-sub-algebra of $\mathfrak{B}$.*

*Proof.* If $\mathfrak{A}, \mathfrak{B}$ are unital *-algebras with the $C^*$-norm property, then clearly $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ are $C^*$-algebras. Moreover, note that $\pi(\mathbb{1})$ is a projection in $\overline{\mathfrak{B}}$. Hence, by lemma 1.2.1, $\tilde{\mathfrak{B}} = \pi(\mathbb{1})\overline{\mathfrak{B}}\pi(\mathbb{1})$ is a unital $C^*$-algebra with the norm inherited by $\overline{\mathfrak{B}}$ and $\mathbb{1}_{\tilde{\mathfrak{B}}} = \pi(\mathbb{1})$. Since $\pi(\mathfrak{A}) \subset \tilde{\mathfrak{B}}$, consider $\pi$ as a $*$-morphism between $\mathfrak{A}$ and $\tilde{\mathfrak{B}}$. Given $A \in \mathfrak{A}$, suppose $\lambda \in r_{\overline{\mathfrak{A}}}(A)$. Then,

$$\pi(\lambda\mathbb{1} - A)\pi((\lambda\mathbb{1} - A)^{-1}) = \pi((\lambda\mathbb{1} - A)(\lambda\mathbb{1} - A)^{-1}) = \pi(\mathbb{1}) = \mathbb{1}_{\tilde{\mathfrak{B}}},$$

that is, $\pi((\lambda\mathbb{1}-A)^{-1})$ is the inverse of $\pi(\lambda\mathbb{1}-A)$ in $\tilde{\mathfrak{B}}$. Thus, one has $r_{\overline{\mathfrak{A}}}(A) \subset r_{\mathfrak{B}}(\pi(A))$ or, equivalently, $\sigma_{\tilde{\mathfrak{B}}}(\pi(A)) \subset \sigma_{\overline{\mathfrak{A}}}(A)$. Therefore,

$$\|\pi(A)\|^2 = \|\pi(A^*A)\| = \rho_{\tilde{\mathfrak{B}}}(\pi(A^*A))^{\frac{1}{2}} \leq \rho_{\overline{\mathfrak{A}}}(A^*A)^{\frac{1}{2}} = \|A^*A\| = \|A\|^2.$$

Hence, $\|\pi(A)\| \leq \|A\|$ for all $A \in \mathfrak{A}$ and $\pi$ is continuous.

Now, assume that $\mathfrak{A}, \mathfrak{B}$ are $C^*$-algebras. Let $A \in \mathfrak{A}_+$. By proposition 1.1.13, $A = B^*B$ for some $B \in \mathfrak{A}$. Hence, $\pi(A) = \pi(B^*B) = \pi(B^*)\pi(B) = \pi(B)^*\pi(B) \in \mathfrak{B}_+$.

Finally, to prove that $\mathfrak{B}_\pi = \{\pi(A) \mid A \in \mathfrak{A}\}$ is a $C^*$-sub-algebra of $\mathfrak{B}$, it suffices to show that $\mathfrak{B}_\pi$ is closed in $\mathfrak{B}$, since $\mathfrak{B}_\pi$ is clearly a *-sub-algebra of $\mathfrak{B}$. Let $\hat{\pi}$ be the $*$-isomorphism between $\mathfrak{A}_\pi = \mathfrak{A}/\ker(\pi)$ and $\mathfrak{B}_\pi$ given by

$$\hat{\pi} : \mathfrak{A}_\pi \to \mathfrak{B}_\pi, \qquad \hat{\pi}(\hat{A}) = \pi(A),$$

where $\hat{A} = \{A + I \mid I \in \ker(\pi)\}$. Since from lemma 1.2.2 it follows that $\mathfrak{A}_\pi$ is a $C^*$-algebra, applying the first statement of the proposition to both $\hat{\pi}$ and its inverse $\hat{\pi}^{-1}$, it follows that

$$\|\hat{\pi}(\hat{A})\| \leq \|\hat{A}\|, \quad \text{and} \quad \|\hat{\pi}(\hat{A})\| \leq \|\hat{\pi}^{-1}(\hat{\pi}(\hat{A}))\| = \|\hat{A}\|.$$

Hence, $\|\hat{\pi}(\hat{A})\| = \|\hat{A}\|$ and since $\mathfrak{A}_\pi$ is closed, it is not hard to see that $\mathfrak{B}_\pi$ is also closed. □

**Definition 1.2.2.** Let $\mathfrak{A}$ be a $C^*$-algebra. A *representation of* $\mathfrak{A}$ is defined to be a pair $(\mathfrak{H}, \pi)$, where $\mathfrak{H}$ is a complex Hilbert space and $\pi$ is a *-morphism of $\mathfrak{A}$ into $\mathfrak{L}(\mathfrak{H})$, the algebra of bounded operators acting on $\mathfrak{H}$. The representation is said to be *faithful* if $\pi$ is a *-isomorphism between $\mathfrak{A}$ and $\pi(\mathfrak{A})$. Moreover, two representations $(\mathfrak{H}_1, \pi_1)$, $(\mathfrak{H}_2, \pi_2)$ of the same algebra $\mathfrak{A}$ are said to be *equivalent* if there exists a unitary operator $U : \mathfrak{H}_1 \to \mathfrak{H}_2$ such that

$$\pi_1(A) = U^{-1}\pi_2(A)U \quad \text{for all } A \in \mathfrak{A}.$$

**Proposition 1.2.2.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $(\mathfrak{H}, \pi)$ be a representation of $\mathfrak{A}$. Then the following conditions are equivalent:*

*(a)* *ker* $\pi = \{0\}$,

*(b)* $\|\pi(A)\| = \|A\|$ *for all* $A \in \mathfrak{A}$,

*(c)* $\pi(A) \succ 0$ *for all* $A \succ 0$.

*Proof.* $(a) \implies (b)$ Since $\ker(\pi) = \{0\}$, then $\pi$ is a $*$-isomorphism, and since $\mathfrak{L}(\mathfrak{H})$ is also a $C^*$-algebra, one can apply proposition 1.2.1 to $\pi$ and $\pi^{-1}$ to obtain

$$\|\pi^{-1}(\pi(A))\| = \|A\| \leq \|\pi(A)\| \leq \|A\|.$$

for any $A \in \mathfrak{A}$. Hence, $(b)$ follows.

$(b) \implies (c)$ If $A \succ 0$ then $\|A\| > 0$ and by $(b)$ $\|\pi(A)\| > 0$. But by proposition 1.2.1 $\pi(A)$ is positive, and hence $\pi(A) \succ 0$.

$(c) \implies (a)$ If $(a)$ is false then $\exists\, B \in \ker \pi$ with $B \neq 0$. Since $\mathfrak{A}$ a $C^*$-algebra, one has $\|B^*B\| = \|B\|^2 > 0 \implies B^*B \neq 0$. Also, by theorem proposition 1.1.13, $B^*B$ is positive, hence $B^*B \succ 0$. But $\pi(B^*B) = \pi(B^*)\pi(B) = 0$, therefore $(c)$ is false. □

**Corollary 1.2.1.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $\tau$ a $*$-automorphism on $\mathfrak{A}$. Then, $\tau$ is isometric.*

*Proof.* Since every automorphism is an isomorphism, it follows that $\ker(\tau) = \{0\}$, and hence, by the above proposition, $\|\tau(A)\| = \|A\|$ for all $A \in \mathfrak{A}$. □

Sometimes, the representation $(\mathfrak{H}, \pi)$ of a $C^*$-algebra $\mathfrak{A}$ can be "split" into other representations: if $\mathfrak{H}_1$ is a closed subspace of $\mathfrak{H}$ that is *invariant under* $\pi$, i.e., $\pi(A)\mathfrak{H}_1 \subset \mathfrak{H}_1$ for all $A \in \mathfrak{A}$, then it is not hard to see that, defining $\pi_1 : \mathfrak{A} \to \mathfrak{L}(\mathfrak{H}_1)$, $\pi_{1_\perp} : \mathfrak{A} \to \mathfrak{L}(\mathfrak{H}_{1_\perp})$ by $\pi_1(A) = P_{\mathfrak{H}_1}\pi(A)P_{\mathfrak{H}_1}$ and $\pi_{1_\perp}(A) = P_{\mathfrak{H}_{1_\perp}}\pi(A)P_{\mathfrak{H}_{1_\perp}}$, the pairs $(\pi_1, \mathfrak{H}_1)$ and $(\pi_{1_\perp}, \mathfrak{H}_{1_\perp})$ are also representations of $\mathfrak{A}$, with $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_{1_\perp}$ and $\pi = \pi_1 \oplus \pi_{1_\perp}$. Note that if a representation $(\pi, \mathfrak{H})$ of $\mathfrak{A}$ can be decomposed in such a way for a nontrivial $\mathfrak{H}_1$, then clearly the orbit of any nonzero vector in $\mathfrak{H}_1$ under $\pi(\mathfrak{A})$ can never reach a nonzero vector in $\mathfrak{H}_{1_\perp}$, and vice-versa. The next definitions formalize the observations discussed above:

**Definition 1.2.3.** Let $\mathfrak{H}$ be a Hilbert space, $\Omega \in \mathfrak{H}$ and $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$. $\Omega$ is said to be *cyclic* for $\mathfrak{M}$ if the set $\{A\Omega; A \in \mathfrak{M}\}$ is dense in $\mathfrak{H}$. A *cyclic representation* of a $C^*$-algebra $\mathfrak{A}$ is defined to be triple $(\mathfrak{H}, \pi, \Omega)$, where $(\mathfrak{H}, \pi)$ is a representation of $\mathfrak{A}$ and $\Omega$ is a vector in $\mathfrak{H}$ which is cyclic for $\pi$ in $\mathfrak{H}$.

**Definition 1.2.4.** Let $\mathfrak{H}$ be a Hilbert space and $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$. $\mathfrak{M}$ is said to be *irreducible* when the only closed subspaces $V \subset \mathfrak{H}$ which are invariant under the action of $\mathfrak{M}$ are $\{0\}$ and $\mathfrak{H}$. A representation $(\pi, \mathfrak{H})$ of a $C^*$ -algebra $\mathfrak{A}$ is said to be *irreducible* if $\pi(\mathfrak{A})$ is irreducible on $\mathfrak{H}$.

It is also possible for a representation $(\mathfrak{H}, \pi)$ of a $C^*$-algebra $\mathfrak{A}$ to be nontrivial but nonetheless possess a trivial part. This is the case when $\overline{\pi(\mathfrak{A})\mathfrak{H}} \subsetneq \mathfrak{H}$. It is not hard to see that this holds if and only if the set $\mathfrak{H}_0 \doteq \{\psi \in \mathfrak{H} \mid \pi(A)\psi = 0 \text{ for all } A \in \mathfrak{A}\}$ is nontrivial. Since $\mathfrak{H}_0$ is clearly a closed subspace of $\mathfrak{H}$, then $\pi$ can be decomposed as $\pi = \pi_0 \oplus \pi_{0_\perp}$, where $\pi_0(A) = 0$ for all $A \in \mathfrak{A}$. Hence, if $\mathfrak{H}_0 \neq \{0\}$, it follows that $\overline{\pi(\mathfrak{A})\mathfrak{H}} = \overline{\pi_{0_\perp}(\mathfrak{A})\mathfrak{H}_0^\perp} = \mathfrak{H}_0^\perp \subsetneq \mathfrak{H}$. The representations with nontrivial $\mathfrak{H}_0$ are called *nondegenerate*, as properly defined below:

**Definition 1.2.5.** Let $\mathfrak{A}$ be a $C^*$-algebra and $(\mathfrak{H}, \pi)$ a representation of $\mathfrak{A}$. Consider the set

$$\mathfrak{H}_0 \doteq \{\psi \in \mathfrak{H} \mid \pi(A)\psi = 0 \text{ for all } A \in \mathfrak{A}\}.$$

If $\mathfrak{H}_0 = \{0\}$, then the representation $(\mathfrak{H}, \pi)$ is said to be *nondegenerate*. otherwise, $(\mathfrak{H}, \pi)$ is said to be *degenerate*.

**Proposition 1.2.3.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $(\mathfrak{H}, \pi)$ a representation of $\mathfrak{A}$. Then $(\mathfrak{H}, \pi)$ is nondegenerate if, and only if, $\pi(\mathbb{1}) = \mathbb{1}_{\mathfrak{L}(\mathfrak{H})}$.*

*Proof.* As already stated before, $\pi(\mathbb{1})$ is a projection in $\mathfrak{L}(\mathfrak{H})$. If $\pi(\mathbb{1}) \neq \mathbb{1}_{\mathfrak{L}(\mathfrak{H})}$, then $(\pi(\mathbb{1})\mathfrak{H})^\perp \neq \{0\}$, and for all $\psi \in (\pi(\mathbb{1})\mathfrak{H})^\perp$, it follows that

$$\pi(A)\psi = \pi(A\mathbb{1})\psi = \pi(A)\pi(\mathbb{1})\psi = 0 \quad \text{for all } A \in \mathfrak{A}.$$

Hence, $(\pi(\mathbb{1})\mathfrak{H})^\perp \subset \mathfrak{H}_0 \neq \{0\}$ and $(\mathfrak{H}, \pi)$ is degenerate. Now, assume $\pi(\mathbb{1}) = \mathbb{1}_{\mathfrak{L}(\mathfrak{H})}$ and suppose that $(\mathfrak{H}, \pi)$ is degenerate. Then, $\exists\, \psi \in \mathfrak{H}_0$ with $\psi \neq 0$. But $\pi(\mathbb{1})\psi = \psi \neq 0$, and hence $\psi \notin \mathfrak{H}_0$. Absurd, thus $(\mathfrak{H}, \pi)$ is nondegenerate. □

**Proposition 1.2.4.** *Let $\mathfrak{A}$ be a $C^*$-algebra and $(\mathfrak{H}, \pi, \Omega)$ a cyclic representation of $\mathfrak{A}$. Then, $(\mathfrak{H}, \pi)$ is nondegenerate.*

*Proof.* Suppose that $(\mathfrak{H}, \pi)$ is degenerate and let $\psi \in \mathfrak{H}_0$, $\psi \neq 0$. Since $\Omega$ is cyclic for $\pi(\mathfrak{A})$, $\exists\, A \in \mathfrak{A}$ such that $(\pi(A)\Omega, \psi) \neq 0$. Then, it follows that

$$(\pi(A)\Omega, \psi) = (\Omega, \pi(A^*)\psi) \neq 0,$$

and hence, $\pi(A^*)\psi \neq 0$. Absurd, since $\psi \in \mathfrak{H}_0$. Thus, $(\mathfrak{H}, \pi)$ is nondegenerate. □

**Corollary 1.2.2.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $(\mathfrak{H}, \pi, \Omega)$ a cyclic representation of $\mathfrak{A}$. Then, $\pi(\mathbb{1}) = \mathbb{1}_{\mathfrak{L}(\mathfrak{H})}$.*

*Proof.* This easily follows from the last two propositions. □

Some relevant properties of a representation $(\mathfrak{H}, \pi)$ of a $C^*$-algebra $\mathfrak{A}$ (for example its irreducibility, as it will be shown soon) can be obtained by looking at the set of all operators in $\mathfrak{L}(\mathfrak{H})$ that commutes with $\pi(\mathfrak{A})$. This set, called the *commutant of* $\pi(\mathfrak{A})$, will be denoted by $\pi(\mathfrak{A})'$, as the next definition implies:

**Definition 1.2.6.** Given a set $\mathfrak{M}$ of bounded operators on a Hilbert space $\mathfrak{H}$, the set of all bounded operators on $\mathfrak{H}$ which commute with every operator in $\mathfrak{M}$ is called the *commutant of* $\mathfrak{M}$ and will be denoted by $\mathfrak{M}'$. The *double commutant of* $\mathfrak{M}$, i.e., $(\mathfrak{M}')'$, will be denoted by $\mathfrak{M}''$, and so on.

**Lemma 1.2.3.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $\mathfrak{M} \subset \mathfrak{A}$ a self-adjoint set. Then $\mathfrak{M}'$ is a unital $C^*$-algebra.*

*Proof.* Clearly $\mathfrak{M}'$ is an algebra with the $C^*$-norm property and $\mathbb{1} \in \mathfrak{M}'$. Moreover, since $\mathfrak{M}$ is a self-adjoint set, it is not hard to check that $\mathfrak{M}'$ is also a $*$-algebra. From the continuity of the product, it also easily follows that $\mathfrak{M}'$ is closed in $\mathfrak{A}$. Hence, $\mathfrak{M}'$ is a $C^*$-algebra. □

**Proposition 1.2.5.** *Let $\mathfrak{M}$ be a subset of an arbitrary algebra. It follows that $\mathfrak{M}' = \mathfrak{M}'''$.*

*Proof.* Let $A \in \mathfrak{M}'$. Then, by definition of $\mathfrak{M}''$, $A$ commutes with every element of $\mathfrak{M}''$. Hence, $A \in \mathfrak{M}'''$. Now let $A \in \mathfrak{M}'''$. Then, $A$ commutes with every element of $\mathfrak{M}''$. Moreover, by the definition of $\mathfrak{M}'$, it follows that all elements of $\mathfrak{M}$ commutes with elements of $\mathfrak{M}'$. Hence, $\mathfrak{M} \subset \mathfrak{M}''$, and thus $A$ also commutes with every element of $\mathfrak{M}$; that is, $A \in \mathfrak{M}'$. □

In the context of bounded operators on a Hilbert space $\mathfrak{H}$, the double commutant has some important properties. It can be shown that for a self-adjont $*$-algebra $\mathfrak{M}$ on $\mathfrak{L}(\mathfrak{H})$, its double commutant $\mathfrak{M}''$ is the closure of $\mathfrak{M}$ in the strong (or weak; in this scenario they coincide) operator topology. The $*$-sub-algebras $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$ such that $\mathfrak{M} = \mathfrak{M}''$ are called *von Neumann algebras*. From the above proposition, it clearly follows that for any self-adjoint set $\mathfrak{M}$ of $\mathfrak{L}(\mathfrak{H})$, $\mathfrak{M}'$ is a von Neumann algebra. von Neumann algebras commonly appear when dealing with representations of $C^*$-algebras (in fact, von Neumann algebras are also $C^*$-algebras), and one of the main results for them is that one can extend the functional calculus of continuous functions to measurable functions. However, since they are not fundamentally needed for the development of the results presented here, its discussion shall be omitted.

**Proposition 1.2.6.** *Let $\mathfrak{H}$ be a Hilbert space and $\mathfrak{M}$ a self-adjoint set of bounded operators on $\mathfrak{H}$. Consider the following conditions:*

*(a) every nonzero vector $\psi \in \mathfrak{M}$ is cyclic for $\mathfrak{M} \in \mathfrak{H}$, or $\mathfrak{M} = \{0\}$ and $\mathfrak{H} = \mathbb{C}$,*

*(b) $\mathfrak{M}'$ consists of multiples of the identity operator,*

*(c) $\mathfrak{M}$ is irreducible.*

*It follows that $(a) \implies (b) \iff (c)$. Moreover, if $\mathfrak{M}$ is also an algebra, then all three conditions are equivalent.*

*Proof.* $(a) \implies (b)$. If $\mathfrak{M} = \{0\}$ and $\mathfrak{H} = \mathbb{C}$, then clearly $(b)$ follows. Now, suppose there is some $A \in \mathfrak{M}'$ that is not a multiple of the identity operator. By lemma 1.2.3, $\mathfrak{M}'$ is, in particular, a $*$-algebra, so $\frac{A+A^*}{2}$ and $\frac{A-A^*}{2i}$ belong to $\mathfrak{M}'$. But if $A$ is not a multiple of the identity, then $\frac{A+A^*}{2}$ or $\frac{A-A^*}{2i}$ (which are self-adjoint) must also not be a multiple of identity. Hence, without loss of generality, assume that $A$ is self-adjoint. Also, note that $\sigma(A)$ must have at least two distinct elements; otherwise, if $\sigma(A)$ had only one element, say $\lambda$, then $\sigma(A - \lambda\mathbb{1}) = \{0\} \implies A = \lambda\mathbb{1}$. Let $\lambda_1$, $\lambda_2$ be two elements of $\sigma(A)$ such that $\lambda_1 < \lambda_2$. Since $\mathfrak{M}'$ is a $C^*$-algebra, $\mathfrak{F}(A) \subset \mathfrak{M}'$. Define

$$f_1 : \sigma(A) \to \mathbb{C}, \qquad f_1(x) = \begin{cases} 1, \text{ if } 0 \leq x \leq \lambda_1 \\ 1 - \left( \frac{x-\lambda_1}{\frac{\lambda_1+\lambda_2}{2} - \lambda_1} \right), \text{ if } \lambda_1 < x \leq \frac{\lambda_1+\lambda_2}{2} \\ 0, \text{ if } x > \frac{\lambda_1+\lambda_2}{2} \end{cases}$$

and

$$f_2 : \sigma(A) \to \mathbb{C}, \qquad f_2(x) = \begin{cases} 0, \text{ if } 0 \leq x \leq \frac{\lambda_1+\lambda_2}{2} \\ \left( \frac{x-\frac{\lambda_1+\lambda_2}{2}}{\lambda_2 - \frac{\lambda_1+\lambda_2}{2}} \right), \text{ if } \frac{\lambda_1+\lambda_2}{2} < x \leq \lambda_2 \\ 1, \text{ if } x > \lambda_2 \end{cases}.$$

It is not hard to see that $f_{1,2} \in \mathcal{C}_A$, $(f_1.f_2)(x) = 0$, and $\|f_{1,2}(A)\| = \|f_{1,2}\|_\infty = 1$. For $i = 1, 2$, let $v_i \in f_i(A)(\mathfrak{H}) \setminus \{0\}$. Then, $v_i = f_i(A)\psi_i$ for $\psi_i \in \mathfrak{H} \setminus \{0\}$. Also,

$$f_1(A)v_2 = f_1(A)f_2(A)\psi_2 = (f_1.f_2)(A)\psi_2 = 0,$$

and thus $v_2 \in \mathrm{Ker}(f_1(A))$. For any $B \in \mathfrak{M}$, it follows that

$$(v_2, Bv_1) = (v_2, Bf_1(A)\psi_1) = (v_2, f_1(A)B\psi_1) = (f_1(A)v_2, B\psi_1) = 0.$$

Hence, $\mathfrak{M}$ is not cyclic for $v_1$.

$(c) \implies (b)$ Again, assume $(b)$ is false and consider the same operators $f_{1,2}(A) \in \mathfrak{M}'$ as defined before. From the above discussion it follows that $\{0\} \neq f_2(A)(\mathfrak{H}) \subset \mathrm{Ker}(f_1(A)) \neq \mathfrak{H}$. Since $\mathrm{Ker}(f_1(A))$ is closed, then $\overline{f_2(A)(\mathfrak{H})}$ is a closed subspace of $\mathfrak{H}$ different from $\{0\}$ and $\mathfrak{H}$, and for any $v \in \overline{f_2(A)(\mathfrak{H})}$ and any $B \in \mathfrak{M}$, it follows that

$$Bv = Bf_2(A)\psi = f_2(A)(B\psi) \in \overline{f_2(A)(\mathfrak{H})}.$$

Therefore, $\overline{f_2(A)(\mathfrak{H})}$ is invariant under the action of $\mathfrak{M}$ and hence $\mathfrak{M}$ is not irreducible.

$(b) \implies (c)$. Suppose that $\mathfrak{M}$ is not irreducible, and let $V$ be a closed subspace of $\mathfrak{H}$ which is invariant under $\mathfrak{M}$. Note that, for any $v \in V$, any $\omega \in V^\perp$ and any $B \in \mathfrak{M}$,

$$(B\omega, v) = (\omega, B^*v) = 0.$$

Hence, $V^\perp$ is also invariant under $\mathfrak{M}$. From this, it can easily be checked that $P_V$, the orthogonal projector of $V$, commutes with every element of $\mathfrak{M}$, and therefore $(c)$ is false.

$(c) \implies (a)$. Now, assume further that $\mathfrak{M}$ is also an algebra. Suppose that $(a)$ is false. For the case $\mathfrak{M} = \{0\}$ and $\mathfrak{H} \neq \mathbb{C}$, it clearly follows that $\mathfrak{M}$ is not irreducible. Hence, let $\mathfrak{M} \neq \{0\}$ and $\psi \in \mathfrak{H}$ be a nonzero vector that is not cyclic for $\mathfrak{M}$. Note that those two conditions imply $\mathfrak{H} \neq \mathbb{C}$. Define $V = \overline{\{A\psi; A \in \mathfrak{M}\}}$ if $\{A\psi; A \in \mathfrak{M}\} \neq \{0\}$ or $V = \langle\psi\rangle_{\mathbb{C}}$ otherwise. In both cases, $V$ is a closed subspace of $\mathfrak{H}$, different from $\{0\}$ and $\mathfrak{H}$, and since $\mathfrak{M}$ is an algebra, $V$ is invariant under $\mathfrak{M}$. Hence $\mathfrak{M}$ is not irreducible. □

**Corollary 1.2.3.** *Let $\mathfrak{H}$ be a Hilbert space and $\mathfrak{M}$ a self-adjoint set of bounded operators on $\mathfrak{H}$. Then, $\mathfrak{M}$ is irreducible if and only if $\mathfrak{M}''$ is irreducible.*

*Proof.* Proposition 1.2.5 affirms that $\mathfrak{M}' = \mathfrak{M}''$. Hence, equivalence $(b) \iff (c)$ of proposition 1.2.6 gives the desired result. □

### 1.2.2 States

In the Heisenberg and Schrödinger picture of quantum mechanics, the state of a quantum system is represented by a normalized vector $\psi$ of a Hilbert space $\mathfrak{H}$. In order to include a classical statistical uncertainty in the determination of a state, they can be defined as *density matrices*: positive operators in $\mathfrak{L}(\mathfrak{H})$ with trace equal to one. Given a density matrix $\rho$, the expected value of any observable $A \in \mathfrak{L}(\mathfrak{H})^{\mathbb{R}}$ at the state $\rho$ is given by $\langle A\rangle_\rho = \mathrm{Tr}(\rho A)$. Note that, since by definition $\rho$ is positive and with trace one, the linear functional $A \mapsto \mathrm{Tr}(\rho A)$ has the property of having norm one (i.e., $\sup\{|\mathrm{Tr}(\rho A)| \mid \|A\| = 1\} = 1$) and also being a positive number when computed in positive elements of $\mathfrak{L}(\mathfrak{H})$. In the $C^*$-algebra formalism, the states of a system are defined as precisely the linear functionals which are positive in positive elements and have norm one, where the expected value of an observable at a state $\omega$ is given by $\langle A\rangle_\omega = \omega(A)$ (For finite-dimensional $C^*$-algebras, there is a one-to-one correspondence between density matrices and linear functionals with norm one that are positive in positive elements; however for infinite-dimensional $C^*$-algebras, the latter definition is more general than the former). In this thesis the set of states of a $C^*$-algebra will be studied extensively. This section aims to present and prove the basic topological and decomposition properties of this set, as well as its connection with the theory of $C^*$-algebra representations. Later, it will be studied the properties of certain states invariant under some specific symmetries, that are present in the specific model analyzed here.

**Definition 1.2.7.** Let $\mathfrak{A}$ be a topological vector space. The space of all continuous linear functions over $\mathfrak{A}$ is denoted by $\mathfrak{A}^*$. If $\mathfrak{A}$ is a normed space, the norm on $\mathfrak{A}$ induces a norm on $\mathfrak{A}^*$ defined by

$$\|f\| = \sup\{|f(A)| \mid \|A\| = 1, A \in \mathfrak{A}\}.$$

Moreover if $\mathfrak{A}$ is a $C^*$-algebra, the partial order $\succeq$ defined in corollary 1.1.3 induces a partial order in $\mathfrak{A}^*$, given by

$$\omega_1 \succeq \omega_2 \iff \omega_1(A) \geq \omega_2(A) \tag{1.24}$$

for all $A \in \mathfrak{A}_+$. A linear functional $\omega \in \mathfrak{A}^*$ is called *positive* if

$$\omega(A) \succeq 0,$$

and the set of all positive elements of $\mathfrak{A}^*$ is denoted by $\mathfrak{A}^*_+$, and $\mathfrak{A}^*_+$ is a closed convex cone such that $\mathfrak{A}^*_+ \cap (-\mathfrak{A}^*_+) = \{0\}$. If $\mathfrak{A}$ is a $C^*$-algebra, a positive linear functional $\omega \in \mathfrak{A}^*_+$ is called a *state* if $\|\omega\| = 1$. The set of all states of a $C^*$-algebra $\mathfrak{A}$ shall be denoted by $E_{\mathfrak{A}}$.

*Remark*: In fact, the partial order $\succeq$ in $\mathfrak{A}^*$ can also be defined for the set of *all* linear functionals over $\mathfrak{A}$, not necessarily continuous. However, it can be shown that every linear functional $\omega$ satisfying $\omega(A) \geq 0$ for all $A \succeq 0$ must also be continuous. Hence, the set of positive elements is the same whether $\succeq$ is defined in $\mathfrak{A}^*$ or in the more general set of all linear functionals over $\mathfrak{A}$.

**Lemma 1.2.4.** *(Cauchy-Schwarz inequality). Let $\mathfrak{A}$ be a $C^*$-algebra and $\omega \in \mathfrak{A}^*_+$. Then, it follows that*

*(a)* $\omega(A^*B) = \overline{\omega(B^*A)}$

*(b)* $|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)$ *for all* $A, B \in \mathfrak{A}$

*Proof.* For $A, B \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, the positivity of $\omega$ implies that

$$\omega((\lambda A + B)^*(\lambda A + B)) \geq 0.$$

By linearity this becomes

$$|\lambda|^2\omega(A^*A) + \overline{\lambda}\omega(A^*B) + \lambda\omega(B^*A) + \omega(B^*B) \geq 0.$$

The necessary and sufficient conditions for the positivity of this quadratic form in $\lambda$ are exactly the two conditions of the lemma. □

**Proposition 1.2.7.** *Let* $\mathfrak{A}$ *be a* $C^*$*-algebra and* $\omega \in \mathfrak{A}^*$*. Then, the following conditions are equivalent:*

*(a)* $\omega \in \mathfrak{A}^*_+$,

*(b)* $\|\omega\| = \omega(\mathbb{1})$.

*Proof.* $(a) \implies (b)$ If $\omega \in \mathfrak{A}^*_+$, from lemma 1.2.4 it follows that

$$|\omega(A)|^2 \leq \omega(A^*A)\omega(\mathbb{1}) \quad \text{for all } A \in \mathfrak{A}. \tag{1.25}$$

Now, note that by lemma 1.1.1 $\|A\|^2\mathbb{1} - A^*A$ is positive for all $A \in \mathfrak{A}$. Hence,

$$\begin{aligned} \omega(\|A\|^2\mathbb{1} - A^*A) = \|A\|^2\omega(\mathbb{1}) - \omega(A^*A) \geq 0 \implies \\ \implies \omega(A^*A) \leq \|A\|^2\omega(\mathbb{1}) \quad \text{for all } A \in \mathfrak{A}. \end{aligned} \tag{1.26}$$

Eq. 1.25 together with eq. 1.26 yelds

$$|\omega(A)| \leq \|A\|\omega(\mathbb{1}) \quad \text{for all } A \in \mathfrak{A}.$$

This establishes that $\|\omega\| = \sup\{|\omega(A)|; \|A\| = 1\} = \omega(\mathbb{1})$.

$(b) \implies (a)$ First note that, by the definition of the norm on $\mathfrak{A}^*_+$, one has

$$|\omega(A)| \leq \|\omega\|\|A\| \tag{1.27}$$

for all $\omega \in \mathfrak{A}^*_+$ and $A \in \mathfrak{A}$. If $\|\omega\| = 0$, then $\omega = 0 \in \mathfrak{A}^*_+$. If $\|\omega\| \neq 0$, Asssume, without loss of generality, that $\|\omega\| = 1$ (one can always re-scale $\omega$ by $\frac{1}{\|\omega\|}$ and if $\frac{\omega}{\|\omega\|}$ is positive, then $\omega$ also is). The first step is to show that for all $A \in \mathfrak{A}^{\mathbb{R}}$, $\omega(A) \in \mathbb{R}$. Given $A \in \mathfrak{A}^{\mathbb{R}}$, set

$$\omega(A) = \alpha + i\beta \qquad \alpha, \beta \in \mathbb{R}.$$

Then, for any real $\gamma$ it follows that

$$\omega(A + i\gamma\mathbb{1}) = \alpha + i(\beta + \gamma).$$

Since $\sigma(A + i\gamma\mathbb{1}) = \sigma(A) + i\gamma\sigma(\mathbb{1})$ and $A \in \mathfrak{A}^{\mathbb{R}}$, one has

$$\rho(A + i\gamma\mathbb{1}) = |\rho(A) + i\gamma| = \sqrt{\|A\|^2 + \gamma^2}.$$

But $A + i\gamma\mathbb{1}$ is normal and hence, by theorem 1.1.2, its norm is given by

$$\|A + i\gamma\mathbb{1}\| = \rho(A + i\gamma\mathbb{1}) = \sqrt{\|A\|^2 + \gamma^2}. \tag{1.28}$$

Therefore, combining eq. 1.28 with eq. 1.27 and the fact that $|\beta + \gamma| \leq |\omega(A + i\gamma\mathbb{1})|$, one has

$$|\beta + \gamma| \leq \sqrt{\|A\|^2 + \gamma^2} \implies \beta^2 + 2\beta\gamma \leq \|A\|^2. \tag{1.29}$$

Since eq. 1.29 must hold for all $\gamma \in \mathbb{R}$, it follows that $\beta = 0$, and hence $\omega(A) \in \mathbb{R}$. Finally, note that

$$\left\| \mathbb{1} - \frac{A^*A}{\|A\|^2} \right\| \leq 1$$

for any $A \in \mathfrak{A}$, by lemma 1.1.1. Thus, by eq. 1.27

$$\left| \omega(\mathbb{1}) - \frac{\omega(A^*A)}{\|A\|^2} \right| \leq 1 \tag{1.30}$$

for all $A \in \mathfrak{A}$. But since $\omega(\mathbb{1}) = 1$ and $\omega(A^*A)$ is real, for eq. 1.30 to hold it is necessary that

$$\omega(A^*A) \geq 0.$$

Thus, $\omega$ is positive. □

**Corollary 1.2.4.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra. Then, the set $E_{\mathfrak{A}}$ of states over $\mathfrak{A}$ is nonempty and convex.*

*Proof.* Let $\mathfrak{B} = \mathbb{C}.\mathbb{1} \subset \mathfrak{A}$ and define in $\mathfrak{B}$ the linear functional

$$f(\lambda\mathbb{1}) = \lambda.$$

Clearly $f$ is bounded and $\|f\|_{\mathfrak{B}} = 1$. By the Hahn-Banach theorem, there exists a bounded linear extension $\omega$ of $f$ on $\mathfrak{A}$ such that $\|\omega\| = \|f\|_{\mathfrak{B}} = 1$. Moreover, $\omega(\mathbb{1}) = f(\mathbb{1}) = 1$, and hence by proposition 1.2.7, $\omega \in E_{\mathfrak{A}}$. Let $\omega_1, \omega_2 \in E_{\mathfrak{A}}$. Then, for any $\lambda \in [0, 1]$, clearly $\lambda\omega_1 + (1-\lambda)\omega_2$ is positive and by proposition 1.2.7

$$\|\lambda\omega_1 + (1-\lambda)\omega_2\| = \lambda\omega_1(\mathbb{1}) + (1-\lambda)\omega_2(\mathbb{1}) = \lambda\|\omega_1\| + (1-\lambda)\|\omega_2\| = 1.$$

Hence, $\lambda\omega_1 + (1-\lambda)\omega_2 \in E_{\mathfrak{A}}$. □

**Definition 1.2.8.** Let $\mathfrak{A}$ be a unital $C^*$-algebra and $\omega \in E_{\mathfrak{A}}$. $\omega$ is said to be *pure* if the only positive linear functionals $\omega' \in \mathfrak{A}^*_+$ such that $\omega \succeq \omega'$ are given by $\omega' = \alpha\omega$ for $0 \leq \alpha \leq 1$.

**Proposition 1.2.8.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $\omega \in E_{\mathfrak{A}}$. Then, $\omega$ is pure if, and only if, $\omega$ is an extreme point of the convex set $E_{\mathfrak{A}}$.*

*Proof.* Suppose $\omega$ is pure and let $\omega_1, \omega_2 \in E_{\mathfrak{A}}$ be such that $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ for some $\lambda \in [0, 1]$. Then, clearly $\omega \succeq \lambda\omega_1$ and $\omega \succeq (1-\lambda)\omega_2$. Since $\omega$ is pure, it follows that $\omega \succeq c_1\omega_1$ and $\omega \succeq c_2\omega_2$ for some $c_1, c_2 \in [0, 1]$, but as $\omega, \omega_1$ and $\omega_2$ are normalized, one has $c_1 = c_2 = 1$ and hence $\omega_1 = \omega_2 = \omega$. Thus, $\omega$ is an extreme point. Now suppose $\omega$ is not pure. Then, $\exists\, \omega_1 \in \mathfrak{A}^*_+$ with $\omega_1 \neq \alpha\omega$ for any $\alpha \in [0, 1]$, such that

$$\omega \succeq \omega_1 = \|\omega_1\|.\frac{\omega_1}{\|\omega_1\|} = \|\omega_1\|\tilde{\omega}_1,$$

where $\tilde{\omega}_1$ is a state. Note also that $\|\omega_1\| \leq 1$, since $\omega \succeq \omega_1$ implies $\omega(1) \geq \omega_1(1) \implies 1 = \|\omega\| \geq \|\omega_1\|$. Define $\tilde{\omega}_2 \in \mathfrak{A}^*$ by

$$\tilde{\omega}_2 = \frac{1}{1 - \|\omega_1\|}(\omega - \|\omega_1\|\tilde{\omega}_1).$$

It is not hard to see that $\tilde{\omega}_2$ is a state and $\omega = \|\omega_1\|\tilde{\omega}_1 + (1 - \|\omega_1\|)\tilde{\omega}_2$, where $\|\omega_1\| \leq 1$ and $\omega \neq \tilde{\omega}_1$. Hence, $\omega$ is not an extreme point.

□

**Definition 1.2.9.** Let $\mathfrak{A}$ be a unital $C^*$-algebra, and $E_{\mathfrak{A}}$ the set of states over $\mathfrak{A}$. The subset of extreme points (or equivalently, of pure states) of $E_{\mathfrak{A}}$ will be denoted by $\mathcal{E}_{\mathfrak{A}}$.

### 1.2.3 States and representations

Given a representation $(\mathfrak{H}, \pi)$ of a $C^*$-algebra $\mathfrak{A}$ and a vector $\Omega \in \mathfrak{H}$, one can define a linear functional $\omega_\Omega \in \mathfrak{A}^*$ by

$$\omega_\Omega(A) \doteq (\Omega, \pi(A)\Omega) \quad \text{for all } A \in \mathfrak{A}. \tag{1.31}$$

$\omega_\Omega$ is also positive, since

$$\omega_\Omega(A^*A) = (\Omega, \pi(A^*A)\Omega) = (\Omega, \pi(A)^*\pi(A)\Omega) = (\pi(A)\Omega, \pi(A)\Omega) = \|(\pi(A)\Omega\|^2 \geq 0.$$

Moreover, if $(\mathfrak{H}, \pi)$ is nondegenerate and $\|\Omega\| = 1$, then with the help of proposition 1.2.3 it is easy to see that $\omega_\Omega$ is also a state. States given by a scalar product as in eq. 1.31 are usully refered to as *vector states*. In fact, it is true that, for any state $\omega$ of a $C^*$-algebra $\mathfrak{A}$, there exists a cyclic representation $(\mathfrak{H}, \pi, \Omega)$ of $\mathfrak{A}$ such that $\omega$ is a vector state given by $\omega(A) = (\Omega, \pi(A)\Omega)$ as in eq. 1.31. The construction of such representation is called the *GNS construction*:

**Theorem 1.2.1** (GNS construction)**.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $\omega \in E_{\mathfrak{A}}$. Then, there exists a cyclic representation $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathfrak{A}$ such that*

$$\omega(A) = (\Omega_\omega, \pi_\omega(A), \Omega_\omega)$$

*for all $A \in \mathfrak{A}$ and $\|\Omega_\omega\|^2 = \|\omega\| = 1$.*

*Proof.* First note that, for any $A \in \mathfrak{A}$, $\|A^*A\|\mathbb{1} - A^*A = \|A\|^2\mathbb{1} - A^*A$ is positive, and hence by proposition 1.1.13 $\|A\|^2\mathbb{1} - A^*A = B^*B$ for some $B \in \mathfrak{A}$. Therefore, for any $C \in \mathfrak{A}$, one as

$$\|A\|^2C^*C - (AC)^*AC = C^*(\|A\|^2 - A^*A)C = (BC)^*BC \succeq 0 \implies$$
$$\|A\|^2C^*C \succeq (AC)^*AC.$$

Now, let $\mathfrak{I}_\omega \subset \mathfrak{A}$ be the complex vector subspace of $\mathfrak{A}$ given by

$$\mathfrak{I}_\omega \doteq \{A \in \mathfrak{A} \mid \omega(A^*A) = 0\}.$$

In particular, for any $I \in \mathfrak{I}_\omega$, by the Cauchy-Schwarz inequality for states, it is easy to see that $\omega(I) = 0$. Let $A \in \mathfrak{A}$ and $I \in \mathfrak{I}_\omega$. Then, by the above observation, it follows that

$$0 \leq \omega((AI)^*AI) \leq \omega(\|A\|^2 I^* I) = 0 \implies \omega((AI)^*AI) = 0,$$

i.e., $\mathfrak{I}_\omega$ is also a left ideal of $\mathfrak{A}$. Hence, for any $A, B \in \mathfrak{A}$ and any $I_1, I_2 \in \mathfrak{I}_\omega$,

$$\omega((A+I_1)^*(B+I_2)) = \omega(A^*B) + \omega(A^*I_2) + \overline{\omega(B^*I_1)} + \omega(I_1^*I_2) = \omega(A^*B). \quad (1.32)$$

Let $\tilde{H}_\omega = \mathfrak{A}/\mathfrak{I}_\omega$, and for $\hat{A}, \hat{B} \in \tilde{H}_\omega$, define a sesquilinear form on $\tilde{H}_\omega$ by

$$(\hat{A}, \hat{B}) \doteq \omega(A^*B).$$

The well-definiteness of such form is a consequence of eq. 1.32. Moreover, by the definition of $\mathfrak{I}_\omega$, it is straightfoward to see that this is actually a scalar product on $\tilde{\mathfrak{H}}_\omega$. Let $\mathfrak{H}_\omega$ be the completion of $\tilde{\mathfrak{H}}_\omega$, and define a function $\tilde{\pi}_\omega : \mathfrak{A} \to \tilde{\mathfrak{H}}_\omega$ by

$$\tilde{\pi}_\omega(A)\hat{B} = \widehat{AB}, \text{ for all } A, B \in \mathfrak{A}.$$

it is easy to see that $\tilde{\pi}_\omega$ is linear, and since $\mathfrak{I}_\omega$ is a left ideal of $\mathfrak{A}$, $\tilde{\pi}_\omega$ is well-defined. Moreover for any $A, B, C \in \mathfrak{A}$, $\tilde{\pi}_\omega$ satisfies

$$\tilde{\pi}_\omega(AB)\hat{C} = \widehat{ABC} = \tilde{\pi}_\omega(A)\widehat{BC} = \tilde{\pi}_\omega(A)\tilde{\pi}_\omega(B)\hat{C},$$

i.e., $\tilde{\pi}_\omega(AB) = \tilde{\pi}_\omega(A)\tilde{\pi}_\omega(B)$. Moreover, for any $A, B \in \mathfrak{A}$,

$$\|\tilde{\pi}_\omega(A)\hat{B}\|^2 = \omega((AB)^*AB) \leq \|A\|^2 \omega(B^*B) = \|A\|^2 \|\hat{B}\|^2_{\tilde{\mathfrak{H}}_\omega}.$$

Therefore, for any $A \in \mathfrak{A}$, $\tilde{\pi}_\omega(A)$ is continuous on $\tilde{\mathfrak{H}}_\omega$, and hence there exists a unique linear extension $\pi_\omega(A)$ of $\tilde{\pi}_\omega(A)$ on $\mathfrak{H}_\omega$. By continuity, it also follows that $\pi_\omega(AB) = \pi_\omega(A)\pi_\omega(B)$ for any $A, B \in \mathfrak{A}$. Finally, define $\Omega_\omega \doteq \hat{\mathbb{1}}$. Then, it is easy to see that for any $A \in \mathfrak{A}$

$$(\Omega_\omega, \pi_\omega(A)\Omega_\omega) = (\Omega_\omega, \tilde{\pi}_\omega(A)\Omega_\omega) = \omega(A).$$

Hence, the triple $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ satisfies the hypothesis of the theorem. □

*Remark:* The GNS construction also holds even if $\mathfrak{A}$ is not unital, with some modifications on how to choose the vector $\Omega_\omega$. Moreover, if $\mathfrak{A}$ is separable, then it can easily be seen that $\mathfrak{H}_\omega$ will also be separable.

**Theorem 1.2.2.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $\omega \in E_{\mathfrak{A}}$. Then, there exists a cyclic representation $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathfrak{A}$ such that*

$$\omega(A) = (\Omega_\omega, \pi_\omega(A), \Omega_\omega)$$

*for all $A \in \mathfrak{A}$ and $\|\Omega_\omega\|^2 = \|\omega\| = 1$. Moreover, the representation is unique up to unitary equivalence.*

*Proof.* The existence some representation satisfying the hypothesis of the theorem is guaranteed by the GNS construction. Suppose that $(\tilde{\mathfrak{H}}_\omega, \tilde{\pi}_\omega, \tilde{\Omega}_\omega)$ is a second cyclic representation such that

$$\omega(A) = (\tilde{\Omega}_\omega, \tilde{\pi}_\omega(A), \tilde{\Omega}_\omega).$$

Define, on the dense set $\mathfrak{D}_\omega \doteq \{\pi_\omega(A)\Omega_\omega; A \in \mathfrak{A}\}$ of $\mathfrak{H}_\omega$, an operator $U$ as

$$U(\pi_\omega(A)\Omega_\omega) = \tilde{\pi}_\omega(A)\tilde{\Omega}_\omega$$

By corollary 1.2.2, clearly $U\Omega_\omega = \tilde{\Omega}_\omega$, and since $\pi, \tilde{\pi}$ are morphisms, $U$ is linear. Moreover, for any $\pi_\omega(A)\Omega_\omega \in \mathfrak{D}_\omega$

$$\|U\pi_\omega(A)\Omega_\omega\|^2 = \|\tilde{\pi}_\omega(A)\tilde{\Omega}_\omega\|^2 = \omega(A^*A) = \|\pi_\omega(A)\Omega_\omega\|^2.$$

Hence, $U$ is bounded on $\mathfrak{D}_\omega$ and can be linearly extended to $\mathfrak{H}_\omega$. Moreover, for any $v_1 = \lim_{n\to\infty} \pi_\omega(A_n)\Omega_\omega \in \mathfrak{H}_\omega$, $v_2 = \lim_{n\to\infty} \pi_\omega(B_n)\Omega_\omega \in \mathfrak{H}_\omega$, it follows that

$$\begin{aligned}
(Uv_1, Uv_2) &= (\lim_n U\pi_\omega(A_n)\Omega_\omega, \lim_n U\pi_\omega(B_n)\Omega_\omega) = (\lim_n \tilde{\pi}_\omega(A_n)\tilde{\Omega}_\omega, \lim_n \tilde{\pi}_\omega(B_n)\tilde{\Omega}_\omega) \\
&= \lim_n \left(\lim_m (\tilde{\pi}_\omega(A_n)\tilde{\Omega}_\omega, \tilde{\pi}_\omega(B_m)\tilde{\Omega}_\omega)\right) = \lim_n \left(\lim_m \omega(A_n^* B_m)\right) \\
&= \lim_n \left(\lim_m (\pi_\omega(A_n)\Omega_\omega, \pi_\omega(B_m)\Omega_\omega)\right) = (\lim_n \pi_\omega(A_n)\Omega_\omega, \lim_n \pi_\omega(B_n)\Omega_\omega) \\
&= (v_1, v_2).
\end{aligned}$$

Hence, $U$ is unitary. Also,

$$\begin{aligned}
U^{-1}\tilde{\pi}_\omega(B)Uv_1 &= \lim_n U^{-1}\tilde{\pi}_\omega(B)U\pi_\omega(A_n)\Omega_\omega = \lim_n U^{-1}\tilde{\pi}_\omega(B)\tilde{\pi}_\omega(A_n)\tilde{\Omega}_\omega \\
&= \lim_n U^{-1}\tilde{\pi}_\omega(BA_n)\tilde{\Omega}_\omega = \lim_n \pi_\omega(BA_n)\Omega_\omega = \pi_\omega(B)\lim_n \pi_\omega(A_n)\Omega_\omega \\
&= \pi_\omega(B)v_1,
\end{aligned}$$

thus the cyclic representations $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ and $(\tilde{\mathfrak{H}}_\omega, \tilde{\pi}_\omega, \tilde{\Omega}_\omega)$ are unitarily equivalent. Now it is proven the uniqueness of $U$. Suppose that there are two unitary operators $U_1, U_2$ such that

$$U_{1,2}^{-1}\tilde{\pi}_\omega(A)U_{1,2} = \pi_\omega(A)$$

for all $A \in \mathfrak{A}$, and

$$U_{1,2}\Omega_\omega = \Omega_\omega.$$

Then, it follows that

$$U_1^{-1}U_2\pi_\omega(A)U_2^{-1}U_1 = \pi_\omega(A) \quad \Longrightarrow \quad U_2^{-1}U_1\pi_\omega(A) = \pi_\omega(A)U_1^{-1}U_2$$

for all $A \in \mathfrak{A}$.Hence, for any $v = \pi_\omega(B)\Omega_\omega \in \mathfrak{D}_\omega$, one has

$$U_2^{-1}U_1 v = U_2^{-1}U_1\pi_\omega(B)\Omega_\omega = \pi_\omega(B)U_1^{-1}U_2\Omega_\omega = \pi_\omega(B)\Omega_\omega = v.$$

That is, $U_2^{-1}U_1$ is the identity operator in $\mathfrak{D}_\omega$, and since $\mathfrak{D}_\omega$ is dense in $\mathfrak{H}_\omega$, it follows that $U_2^{-1}U_1$ is the identity in $\mathfrak{H}_\omega$. Therefore $U_1 = U_2$. □

**Proposition 1.2.9.** *Let $\omega$ be a state over a $C^*$-algebra $\mathfrak{A}$ and $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ a cyclic representation associated with $\omega$. Then, for every $\rho \in \mathfrak{A}^*_+$ such that $\rho \preceq \omega$, there exists a unique positive $T \in \pi_\omega(\mathfrak{A})'$ with $\|T\|_{op} \leq 1$, where*

$$\rho(A) = (\Omega_\omega, T\pi_\omega(A)\Omega_\omega).$$

*Conversely, all positive $T \in \pi_\omega(\mathfrak{A})'$ with $\|T\|_{op} \leq 1$ defines a unique positive linear functional*

$$\rho_T(A) = (\Omega_\omega, T\pi_\omega(A)\Omega_\omega)$$

*which is majorized by $\omega$.*

*Proof.* Let $\rho \in \mathfrak{A}^*_+$ with $\rho \preceq \omega$. Consider the set $\mathfrak{D} \doteq \{\pi_\omega(B)\Omega_\omega \times \pi_\omega(A)\Omega_\omega \,|\, A, B \in \mathfrak{A}\}$, and define in $\mathfrak{D}$ the sesquilinear form

$$\tilde{\rho}_{\mathfrak{D}}(\pi_\omega(B)\Omega_\omega \times \pi_\omega(A)\Omega_\omega) = \rho(B^*A).$$

From Cauchy-Schwartz inequality for states, it follows that

$$|\rho(B^*A)|^2 \leq \rho(B^*B)\rho(A^*A) \leq \omega(B^*B)\omega(A^*A) = \|\pi_\omega(B)\Omega_\omega\|^2\|\pi_\omega(A)\Omega_\omega\|^2. \quad (1.33)$$

Hence, $\tilde{\rho}_{\mathfrak{D}}$ is bounded in $\mathfrak{D}_\omega$, and since, by the cyclicity of $\Omega_\omega$, $\mathfrak{D}$ is dense in $\mathfrak{H}_\omega \times \mathfrak{H}_\omega$, $\tilde{\rho}_{\mathfrak{D}}$ has a bounded sesquilinear extension $\tilde{\rho}$ in $\mathfrak{H}_\omega \times \mathfrak{H}_\omega$. By the Riesz representation theorem, there exists a unique bounded operator $T$ on $\mathfrak{H}_\omega$ such that

$$\tilde{\rho}(\psi_1, \psi_2) = (\psi_1, T\psi_2) \quad \text{for all } \psi_1 \times \psi_2 \in \mathfrak{H}_\omega \times \mathfrak{H}_\omega.$$

But as $\tilde{\rho}(\pi_\omega(B)\Omega_\omega \times \pi_\omega(A)\Omega_\omega) = \rho(B^*A)$, it follows that

$$\rho(B^*A) = (\pi_\omega(B)\Omega_\omega, T\pi_\omega(A)\Omega_\omega).$$

Moreover, eq. 1.33 implies

$$\|T\| \leq \sup_{\psi_1,\psi_2 \in \mathfrak{H}_\omega \,|\, \|\psi_1\|=\|\psi_2\|=1} |(\psi_1, T\psi_2)| = \sup_{\psi_1,\psi_2 \in \mathfrak{D} \,|\, \|\psi_1\|=\|\psi_2\|=1} |\tilde{\rho}_{\mathfrak{D}}(\psi_1, \psi_2)| \leq 1.$$

To prove that $T$ is positive, note that for any Hilbert space $\mathfrak{H}$ an element $A \in \mathfrak{L}(\mathfrak{H})$ is positive if, and only if, $(\psi, A\psi) \geq 0$ for all $\psi \in \mathfrak{H}$. Since $\rho$ is positive, it follows that

$$(\pi_\omega(A)\Omega_\omega, T\pi_\omega(A)\Omega_\omega) = \rho(A^*A) \geq 0 \quad \text{for all } A \in \mathfrak{A},$$

and using again the cyclicity of $\Omega_\omega$, one has that $T$ is positive. The equality $\rho(B^*(AB)) - \rho((A^*B)B) = 0$ for all $A, B \in \mathfrak{A}$ implies

$$(\pi_\omega(B)\Omega_\omega, (T\pi_\omega(A) - \pi_\omega(A)T)\pi_\omega(B)\Omega_\omega) = 0,$$

and once again by cyclicity of $\Omega_\omega$, it follows that $T\pi_\omega(A) = \pi_\omega(A)T$ for all $A \in \mathfrak{A}$. Hence $T \in \pi_\omega(\mathfrak{A})'$. Now, take any positive $T \in \pi_\omega(\mathfrak{A})'$ with $\|T\|_{op} \leq 1$. Then,

$$\begin{aligned}\rho_T(A^*A) &= (\Omega_\omega, T\pi_\omega(A^*A)\Omega\omega) = (\Omega_\omega, T\pi_\omega(A)^*\pi_\omega(A)\Omega_\omega) = (\Omega_\omega, \pi_\omega(A)^*T\pi_\omega(A)\Omega_\omega)\\ &= (\pi_\omega(A)\Omega_\omega, T\pi_\omega(A)\Omega_\omega) \geq 0,\end{aligned}$$

and hence $\rho_T$ is positive. Moreover, for any $A \in \mathfrak{A}$,

$$\begin{aligned}\omega(A^*A) - \rho_T(A^*A) =&(\pi_\omega(A)\Omega_\omega, \pi_\omega(A)\Omega_\omega) - (\pi_\omega(A)\Omega_\omega, T\pi_\omega(A)\Omega_\omega)\\ &= \|\pi_\omega(A)\Omega_\omega\|^2 - (\pi_\omega(A)\Omega_\omega, T\pi_\omega(A)\Omega_\omega)\\ &\geq \|\pi_\omega(A)\Omega_\omega\|^2 - \|T\|\|\pi_\omega(A)\Omega_\omega\|^2 \geq 0,\end{aligned}$$

since $\|T\| \leq 1$. Hence, $\rho_T \leq \omega$. Finally, let $T_{1,2}$ be two positive operators in $\pi_\omega(\mathfrak{A})'$ with $\|T_{1,2}\| \leq 1$ and $\rho_{T_1} - \rho_{T_2} = 0$. Then, for all $A \in \mathfrak{A}$,

$$(\pi_\omega(A)\Omega_\omega, (T_1 - T_2)\pi_\omega(A)\Omega_\omega) = 0.$$

Since $\Omega_\omega$ is cyclic and $T_1 - T_2$ is self-adjoint, one has $\|T_1 - T_2\| = 0 \implies T_1 = T_2$. □

**Theorem 1.2.3.** *Let $\omega$ be a state over a $C^*$-algebra $\mathfrak{A}$ and $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ a cyclic representation associated with $\omega$. Then, $\omega$ is pure if and only if $(\mathfrak{H}_\omega, \pi_\omega)$ is irreducible.*

*Proof.* Suppose that $(\mathfrak{H}_\omega, \pi_\omega)$ is irreducible and let $\rho \in \mathfrak{A}^*_+$ be any positive linear functional such that $\rho \succeq \omega$. From proposition 1.2.6, $\pi_\omega(\mathfrak{A})' = \mathbb{C}.\mathbb{1}$, and hence last proposition implies that $\rho(A) = (\Omega_\omega, \alpha.\mathbb{1}\pi_\omega(A)\Omega_\omega) = \alpha(\Omega_\omega, \pi_\omega(A)\Omega_\omega) = \alpha\omega(A)$, where $\alpha \in [0, 1]$. Thus, $\omega$ is pure. Now, suppose that $(\mathfrak{H}_\omega, \pi_\omega)$ is not irreducible. Then, there exists some $B \in \pi_\omega(\mathfrak{A})'$ such that $B \notin \mathbb{C}.\mathbb{1}$. Since $\pi_\omega(\mathfrak{A})'$ is a unital $C^*$-sub-algebra of $\mathcal{B}(\mathfrak{H})$, there are positive elements

$$B_R^+, B_R^-, B_I^+, B_I^- \in \pi_\omega(\mathfrak{A})'$$

such that $B = (B_R^+ - B_R^-) + i(B_I^+ - B_R^-)$. Clearly, at least one of those four elements must not be in $\mathbb{C}.\mathbb{1}$, and hence, $\pi_\omega(\mathfrak{A})'$ contains some positive element $T \notin \mathbb{C}.\mathbb{1}$. Re-scaling $T$ by some convenient positive constant, it can be assumed that $\|T\|_{op} \leq 1$. By the last proposition, $\rho_T(A) = (\Omega_\omega, T\pi_\omega(A)\Omega_\omega)$ is a positive linear functional such that $\rho_T \preceq \omega$ but, since $T \neq \alpha\mathbb{1}$, for any $\alpha \in [0, 1]$, it follows from the uniqueness of $T$ that $\rho_T \neq \alpha\omega$ for any $\alpha \in [0, 1]$. Hence, $\omega$ is not pure. □

**Corollary 1.2.5.** *Let $\mathfrak{A}$ be a $C^*$-algebra, $\omega \in E_{\mathfrak{A}}$ a state and $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ a cyclic representation associated with $\omega$. If $\tau$ is a $^*$-automorphism of $\mathfrak{A}$ which leaves $\omega$ invariant, i.e., if*

$$\omega(\tau(A)) = \omega(A) \quad \textit{for all } A \in \mathfrak{A},$$

*then there exists a unique unitary operator $U_\omega : \mathfrak{H}_\omega \to \mathfrak{H}_\omega$ such that*

$$\begin{aligned}&U_\omega\pi_\omega(A)U_\omega^{-1} = \pi_\omega(\tau(A)) \quad \textit{for all } A \in \mathfrak{A}, \textit{ and}\\ &U_\omega\Omega_\omega = \Omega_\omega\end{aligned}$$

*Proof.* Since $\tau$ is a $^*$-automorphism, and hence bijective by definition, it can be easily checked that $(\mathfrak{H}_\omega, \pi_\omega \circ \tau, \Omega_\omega)$ is also a cyclic representation of $\mathfrak{A}$. Moreover, $(\Omega_\omega, (\pi_\omega \circ \tau)(A), \Omega_\omega) = \omega(\tau(A)) = \omega(A)$. Hence, the desired result follows by applying the above theorem to $(\mathfrak{H}_\omega, \pi_\omega \circ \tau, \Omega_\omega)$. □

## 1.3 The state space of $C^*$-algebras

This section is devoted to a more profound study of the state space $E_{\mathfrak{A}}$; more specifically, the decomposition properties and topological aspects of the set. For any infinite-dimensional $C^*$-algebra $\mathfrak{A}$, the state space $E_{\mathfrak{A}}$ is always non-compact in the norm topology of $\mathfrak{A}^*$. Therefore, it is not convenient work in the norm topology when looking for maxima and minima of functionals in $E_{\mathfrak{A}}$, as it will be necessary later in the text. A clever way to contour this problem is to consider the weak*-topology in $E_{\mathfrak{A}}$. The weak*-topology, besides making $E_{\mathfrak{A}}$ a compact set, keeps $\mathfrak{A}^*$ as locally convex vector space, that is also metrizable if $\mathfrak{A}$ is separable. These three properties assure that any state can be represented by a Choquet integral in the pure states, a very useful decomposition property of $E_{\mathfrak{A}}$, that will also be explored later in the text.

### 1.3.1 The weak*-topology

**Definition 1.3.1.** Let $\mathfrak{A}$ be a set, $\mathcal{F}$ a nonempty family of mappings $f : X \to Y_f$, where each $Y_f$ is a topological space, and $\tau$ the collection of all unions of finite intersections of the sets $f^{-1}(V)$, where $f \in \mathcal{F}$ and $V$ is open in $Y_f$. Then, $(\mathfrak{A}, \tau)$ is a topological space and $\tau$ is called the *initial topology of* $\mathcal{F}$.

**Definition 1.3.2.** Let $\mathfrak{A}$ be a topological vector space. Every element $A \in \mathfrak{A}$ induces a linear functional $f_A : \mathfrak{A}^* \to \mathbb{C}$ on $\mathfrak{A}^*$, given by

$$f_A(\omega) = \omega(A) \quad \text{for all } \omega \in \mathfrak{A}^*.$$

Let $\mathcal{F}_{\mathfrak{A}} = \{f_A \mid A \in \mathfrak{A}\}$, and $\tau_{w^*}$ be the initial topology of $\mathcal{F}_{\mathfrak{A}}$. $\tau_{w^*}$ is called the *weak*-topology of* $\mathfrak{A}^*$.

The following theorems are well-established results in the theory of topological vector spaces. Their proofs can be found in standard textbooks of the subject, such as [11].

**Theorem 1.3.1.** *Let $X$ be a vector space and $X'$ a separating vector space of linear functionals on $X$. Then the initial topology $\tau_{x'}$ of $\mathcal{F}_{X'}$ turns $X$ into a locally convex space whose dual space is $X'$.*

**Theorem 1.3.2** (Banach-Alaoglu)**.** *Let $\mathfrak{A}$ be a normed space. Then, the unit ball*

$$B_1 \doteq \{\omega \in \mathfrak{A}^* \mid \|\omega\| \leq 1\} \subset \mathfrak{A}^*$$

*is weakly*-compact.*

**Proposition 1.3.1.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra. Then, the weak*-topology makes $\mathfrak{A}^*$ into a locally convex vector space whose dual space is $\mathcal{F}_{\mathfrak{A}}$. Moreover, the set $E_{\mathfrak{A}}$ of states over $\mathfrak{A}$ is weakly*-compact.*

*Proof.* The first part of the proposition is a consequence of theorem 1.3.1. Defining $S \doteq \{\omega \in \mathfrak{A}^* \mid \omega(\mathbb{1}) = 1\}$ and $B_1 \doteq \{\omega \in \mathfrak{A}^* \mid \|\omega\| \leq 1\} \subset \mathfrak{A}^*$, it follows that $E_{\mathfrak{A}} = B_1 \cap S \cap \mathfrak{A}^*_+$. Since, by the Banach-Alaoglu theorem, $B_1$ is weakly*-compact, to prove that $E_{\mathfrak{A}}$ is weakly*-compact it suffices to show that $S$ and $\mathfrak{A}^*_+$ are weakly*-closed. For this, note that for any $A \in \mathfrak{A}$ the linear functional $f_A \in \mathcal{F}_{\mathfrak{A}}$ is, by definition, weakly*-continuous.

Hence, for any closed set $C \subset \mathbb{C}$, the preimage $f_A^{-1}(C) = \{\omega \in \mathfrak{A}^* \mid \omega(A) \in C\}$ is weakly*-closed in $\mathfrak{A}^*$. Since $\mathfrak{A}^*_+$ can be written as

$$\mathfrak{A}^*_+ = \bigcap_{A \in \mathfrak{A}_+} \{\omega \in \mathfrak{A}^* \mid \omega(A) \geq 0\} = \bigcap_{A \in \mathfrak{A}_+} f_A^{-1}([0, +\infty)),$$

it follows that $\mathfrak{A}^*_+$ is weakly*-closed. Moreover, $S$ can also be written as

$$S = \{\omega \in \mathfrak{A}^* \mid \omega(\mathbb{1}) = 1\} = f_{\mathbb{1}}^{-1}(\{1\}),$$

and hence $S$ is also weakly*-closed. □

**Proposition 1.3.2.** *Let $\mathfrak{A}$ be a separable normed space. Then, the unit ball*

$$B_1 \doteq \{\omega \in \mathfrak{A}^* \mid \|\omega\| \leq 1\} \subset \mathfrak{A}^*$$

*is metrizable in the weak*-topology.*

*Proof.* Since $\mathfrak{A}$ is separable, let $\{x_n\}$ be a dense sequence in $B_1$, and $d : B_1 \times B_1 \to \mathbb{R}$ be given by

$$d(\omega_1, \omega_2) = \sum_{n=1}^{\infty} 2^{-n} |\omega_1(x_n) - \omega_2(x_n)| = \sum_{n=1}^{\infty} 2^{-n} |f_{x_n}(\omega_1 - \omega_2)|.$$

By the triangle inequality and since $\{x_n\}$ is dense, it is not hard to see that $d$ is indeed a metric on $B_1$. Then, for any $\omega \in B_1$ and $\epsilon > 0$, the sets

$$D_\epsilon(\omega) \doteq \{\omega_1 \in B_1 \mid d(\omega, \omega_1) < \epsilon\}$$

form a base for the topology $\tau_d$ of $B_1$ induced by the metric $d$. For a given $\omega \in B_1$ and $\epsilon > 0$, let $\Delta_\epsilon$ be the disk of radius $\epsilon$ in $\mathbb{C}$, and $V_{\epsilon,n,\omega} \doteq \Delta_\epsilon + f_{x_n}(\omega) \subset \mathbb{C}$. Then, it clearly follows that

$$f_{x_n}^{-1}(V_{\epsilon,n,\omega}) = \{\omega_1 \in \mathfrak{A}^* \mid \omega_1(x_n) \in \Delta_\epsilon + \omega(x_n)\} = \{\omega_1 \in \mathfrak{A}^* \mid |\omega_1(x_n) - \omega(x_n)| < \epsilon\}.$$

Moreover, since the sequence $\sum_{n=1}^{\infty} 2^{-n}$ converges to $1$, there exist some $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} 2^{-n} < \frac{\epsilon}{4}$. Now, consider the set

$$W = B_1 \cap \left( \bigcap_{n=1}^{N} f_{x_n}^{-1} \left( V_{\frac{\epsilon}{2N}, n, \omega} \right) \right).$$

Clearly $W$ is open in the weak*-topology of $B_1$ and $\omega \in W$. Moreover, for any $\omega_1 \in W$, it follows that

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{-n} |\omega(x_n) - \omega_1(x_n)| &\leq \sum_{n=1}^{N} 2^{-n} |\omega(x_n) - \omega_1(x_n)| + \sum_{n=N}^{\infty} 2^{-n} |\omega(x_n)| + \\ &\sum_{n=N}^{\infty} 2^{-n} |\omega_1(x_n)| \leq \sum_{n=1}^{N} 2^{-n} |\omega(x_n) - \omega_1(x_n)| + \\ &2 \sum_{n=N}^{\infty} 2^{-n} < \frac{\epsilon}{2N} \sum_{n=1}^{N} 2^{-n} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Therefore, $W \subset D_\epsilon(\omega)$ and hence $\tau_d \subset \tau_{w^*}$ in $B_1$. To prove the converse, let $A \subset B_1$ be a closed set in the weak* topology of $B_1$, and let $\{O_i\}_{i\in I}$ be an open cover of $A$ in the metric topology of $B_1$. Since $B_1$ is weak* compact, it follows that $A$ is also weak* compact. Moreover, since $\tau_d \subset \tau_{w^*}$, $\{O_i\}_{i\in I}$ is also an open cover of $A$ in the weak* topology. Therefore, there exists a finite subcover $\{O_i\}_{i\in F}$ such that $A \subset \bigcup_{i\in F} O_i$, and hence, $A$ is also compact in the topology $\tau_d$. Since any compact set is closed in a metric space, it follows that $A$ is closed in the same topology, and thus $\tau_{w^*} \subset \tau_d$. □

**Proposition 1.3.3.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra. Then, the set $\mathcal{E}_\mathfrak{A}$ of extreme points of $E_\mathfrak{A}$ is a Borel set in the weak*-topology of $E_\mathfrak{A}$.*

*Proof.* Since $B_1 \doteq \{\omega \in \mathfrak{A}^* \mid \|\omega\| \leq 1\}$ is metrizable in the weak*-topology by proposition 1.3.2, it follows that $E_\mathfrak{A} \subset B_1$ is also metrizable in the same topology. Let $d$ be a metric associated with the weak*-topology on $E_\mathfrak{A}$ and define

$$F_n \doteq \left\{\omega \in E_\mathfrak{A} \mid \omega = \frac{\omega_1 + \omega_2}{2} \text{ for some } \omega_1, \omega_2 \in E_\mathfrak{A} \text{ such that } d(\omega_1, \omega_2) \geq \frac{1}{n}\right\}, \quad \text{and}$$

$$F \doteq \bigcap_{n\in\mathbb{N}} F_n.$$

The goal now is to prove that $F_n$ is closed for all $n \in \mathbb{N}$. Let $\{\omega_m\}$ be a sequence in $F_n$ converging to $\omega \in E_\mathfrak{A}$. and let $\{\omega_{1_m}\}$ and $\{\omega_{2_m}\}$ be sequences of elements in $E_\mathfrak{A}$ such that

$$\omega_m = \frac{\omega_{1_m} + \omega_{2_m}}{2} \quad \text{and} \quad d(\omega_{1_m}, \omega_{2_m}) \geq \frac{1}{n} \quad \text{for all } m \in \mathbb{N}.$$

Since $E_\mathfrak{A}$ is weakly*-compact and$(E_\mathfrak{A}, \tau_{w^*})$ is metrizable, it follows that both $\{\omega_{1_m}\}$ and $\{\omega_{2_m}\}$ have convergent subsequences. Let $\omega_1$ and $\omega_2$ be the limit of a convergent subsequence in $\{\omega_{1_m}\}$ and $\{\omega_{2_m}\}$, respectively. Then, it is not hard to show that $\omega = \frac{\omega_1+\omega_2}{2}$ and $d(\omega_1, \omega_2) \geq \frac{1}{n}$. Hence, $F_n$ is closed. Therefore, $F$ is a Borel set. Now, note that, if $\omega \in F$, clearly $\omega \notin \mathcal{E}_\mathfrak{A}$. Conversely, assume that $\omega \in E_\mathfrak{A}$ and $\omega \notin \mathcal{E}_\mathfrak{A}$. Then, $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ for some $0 < \lambda < 1$ and some $\omega_1, \omega_2 \in E_\mathfrak{A}$, with $\omega_1 \neq \omega_2$. Without loss of generality, assume $\lambda \leq \frac{1}{2}$. Then, defining

$$\omega_1' \doteq \frac{3\lambda}{2}\omega_1 + (1 - \frac{3\lambda}{2}\omega_2) \quad \text{and} \quad \omega_2' \doteq \frac{\lambda}{2}\omega_1 + (1 - \frac{\lambda}{2})\omega_2,$$

it is not hard to see that $\omega_1', \omega_2' \in E_\mathfrak{A}$, $\omega = \frac{\omega_1'+\omega_2'}{2}$ and $\omega_1' \neq \omega_2'$. Hence, $\omega \in F$. Therefore, $F = \mathcal{E}_\mathfrak{A}^c$ and $\mathcal{E}_\mathfrak{A} = F^c$ is a Borel set. □

**Definition 1.3.3.** Let $\mathfrak{A}$ be a compact metric space and $\mu$ a measure on $\mathfrak{A}$. Then, $\mu$ is said to be a *probability measure on* $\mathfrak{A}$ if $\mu$ is a Borel measure and $\mu(\mathfrak{A}) = 1$. Moreover, if $S$ is a Borel subset of $\mathfrak{A}$, $\mu$ is said to be *supported by* $S$ if $\mu(S^c) = 0$.

*Remark*: Usually, a probability measure is defined as a *regular* Borel measure. However, for the case of a compact metric space, every Borel measure is also regular Borel.

**Definition 1.3.4.** Let $K$ be a nonempty compact set of a locally convex vector space $\mathfrak{A}$ and $\mu$ a probability measure on $K$. A point $x \in K$ is said to be *represented by* $\mu$ if, for any continuous linear functional $f$ on $\mathfrak{A}$,

$$f(x) = \int_K f d\mu.$$

The next theorem, known as Choquet's theorem, is also a well-established result in the theory of topological vector spaces, and its proof can be found at [10].

**Theorem 1.3.3** (Choquet's theorem)**.** *Let $S$ be a metrizable compact convex subset of a locally convex topological space $\mathfrak{A}$. Then, for all $x \in S$, there exists a probability measure $\mu$ on $S$ which represents $x$ and is supported by the extreme points of $S$.*

**Corollary 1.3.1.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra, and $E_{\mathfrak{A}} \subset \mathfrak{A}^*$ the set of states over $\mathfrak{A}$, endowed with the weak*-topology. Then, for any $\omega \in E_{\mathfrak{A}}$, there exists a probability measure $\mu$ on $E_{\mathfrak{A}}$ supported by the extreme points $\mathcal{E}_{\mathfrak{A}}$ of $E_{\mathfrak{A}}$ such that*

$$\Lambda(\omega) = \int_{E_{\mathfrak{A}}} \Lambda d\mu = \int_{\mathcal{E}_{\mathfrak{A}}} \Lambda d\mu$$

*for any weakly*-continuous linear functional on $\mathfrak{A}^*$.*

*Proof.* Propositions 1.3.1 and 1.3.2 state that $\mathfrak{A}^*$ is locally compact in the weak*-topology and $E_{\mathfrak{A}} \subset \mathfrak{A}^*$ is compact and metrizable in the same topology. Hence, the corollary follows from the application of Choquet's theorem to $E_{\mathfrak{A}}$. □

## 1.4 G-invariant states

This section is devoted to the study of states which are invariant under the action of a group. Since the interactions present in the model studied here are invariant under the action of certain symmetry groups, such invariant states appear naturally when analyzing the equilibrium states of the system. At first, some general properties are derived for a broad class of symmetry groups, and later the study will be focused on the group of translations and permutations in the $d$-dimensional square lattice.

**Definition 1.4.1.** Let $\mathfrak{A}$ be a $C^*$-algebra, $G$ a group, and $g \mapsto \tau_g$ a homomorphism from $G$ to the group of $*$-automorphisms of $\mathfrak{A}$. A state $\omega \in E_{\mathfrak{A}}$ is said to be *$G$-invariant* if

$$\omega(\tau_g(A)) = \omega(A)$$

for all $g \in G$ and $A \in \mathfrak{A}$. The set of all $G$-invariant states will be denoted by $E_{\mathfrak{A}}^G$.

**Proposition 1.4.1.** *For any group $G$, $E_{\mathfrak{A}}^G$ is a convex weakly*-compact subset of $\mathfrak{A}$.*

*Proof.* Since $E_{\mathfrak{A}}^G \subset E_{\mathfrak{A}}$, and $E_{\mathfrak{A}}$ is weakly*-compact (see proposition 1.3.1), it suffices to show that $E_{\mathfrak{A}}^G$ is convex and weakly*-closed. The convexity of $E_{\mathfrak{A}}^G$ is clear; for $\omega_1, \omega_2 \in E_{\mathfrak{A}}^G$, then $\lambda\omega_1 + (1-\lambda)\omega_2$ is also a state if $\lambda \in [0,1]$, and for any $g \in G$,

$(\lambda\omega_1 + (1-\lambda)\omega_2)(A) = \lambda\omega_1(A) + (1-\lambda)\omega_2(A) = \lambda\omega_1(\tau_g(A)) + (1-\lambda)\omega_2(\tau_g(A)) = (\lambda\omega_1 + (1-\lambda)\omega_2)(\tau_g(A))$.

Now, given $g \in G$, define

$$E_{\mathfrak{A}}^g \doteq \bigcap_{A\in\mathfrak{A}} f^{-1}_{\tau_g(A)-A}(\{0\}).$$

$E_{\mathfrak{A}}^g$ is clearly weakly*-closed. Moreover, since $\omega \in E_{\mathfrak{A}}^g \iff \omega(\tau_g(A)) = \omega(A)$ for all $A \in \mathfrak{A}$, it follows that

$$E_{\mathfrak{A}}^G = \bigcap_{g\in G} E_{\mathfrak{A}}^g,$$

and hence $E_{\mathfrak{A}}^G$ is weakly*-closed. □

For any group $G$, since the set $E_{\mathfrak{A}}^G$ of $G$-invariant states is also weakly*-compact, it is straightforward to generalize the results already obtained for the set of states $E_{\mathfrak{A}}$ to $E_{\mathfrak{A}}^G$. The following definitions and propositions presented below are merely generalizations of concepts already proved and discussed for $E_{\mathfrak{A}}$ to the set of $G$-invariant states $E_{\mathfrak{A}}^G$, and since the proofs of these propositions are almost identical to the proofs for the most general set $E_{\mathfrak{A}}$, they will be omitted.

**Definition 1.4.2.** Let $\mathfrak{A}$ be a unital $C^*$-algebra and $G$ a group acting as *-automorphisms $\tau$. The set of extreme points of the set $E_{\mathfrak{A}}^G$ of $G$-invariant states will be denoted by $\mathcal{E}_{\mathfrak{A}}^G$, and a state $\omega \in \mathcal{E}_{\mathfrak{A}}^G$ is called *G-ergodic*.

**Definition 1.4.3.** Let $\mathfrak{A}$ be a unital $C^*$-algebra, $G$ a group acting as *-automorphisms $\tau$ of $\mathfrak{A}$ and $\omega \in E_{\mathfrak{A}}^G$. $\omega$ is said to be *G-pure* if the only positive $G$-invariants linear functionals $\omega' \in \mathfrak{A}^*_+$ such that $\omega \succeq \omega'$ are given by $\omega' = \alpha\omega$ for $0 \leq \alpha \leq 1$.

**Proposition 1.4.2.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra, $G$ a group acting as *-automorphisms $\tau$ of $\mathfrak{A}$ and $\omega \in E_{\mathfrak{A}}^G$. Then, $\omega$ is pure if, and only if, $\omega$ is an extreme point of the convex set $E_{\mathfrak{A}}^G$.*

**Proposition 1.4.3.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $G$ a group acting as *-automorphisms $\tau$ of $\mathfrak{A}$. Then, the set $\mathcal{E}_{\mathfrak{A}}^G$ of extreme points of $E_{\mathfrak{A}}^G$ is a Borel set in the weak*-topology of $E_{\mathfrak{A}}^G$.*

**Corollary 1.4.1.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra, and $E_{\mathfrak{A}}^G \subset \mathfrak{A}^*$ the set of $G$-invariant states over $\mathfrak{A}$, endowed with the weak*-topology. Then, for any $\omega \in E_{\mathfrak{A}}^G$, there exists a probability measure $\mu_\omega$ on $E_{\mathfrak{A}}^G$ supported by the extreme points $\mathcal{E}_{\mathfrak{A}}^G$ of $E_{\mathfrak{A}}^G$ such that*

$$\Lambda(\omega) = \int_{E_{\mathfrak{A}}^G} \Lambda(\omega')\mu_\omega(d\omega') = \int_{\mathcal{E}_{\mathfrak{A}}^G} \Lambda(\omega')\mu_\omega(d\omega')$$

*for any weakly*-continuous functional on $\mathfrak{A}^*$.*

**Proposition 1.4.4.** *Let $\mathfrak{A}$ be a $C^*$-algebra, $G$ a group acting as $^*$-automorphisms $\tau$ of $\mathfrak{A}$, $\omega \in E_{\mathfrak{A}}^G$ a $G$-invariant state and $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ a representation associated with $\omega$. Then, there exists a representation of $G$ by unitary operators acting on $\mathfrak{H}_\omega$, satisfying*

$$\begin{aligned}&U_g^{-1}\pi_\omega(\tau_g(A))U_g = \pi_\omega(A) \quad \textit{for all } A \in \mathfrak{A}\textit{, and}\\ &U_g\Omega_\omega = \Omega_\omega.\end{aligned}$$

*Proof.* For any $g \in G$, define $U_g$ as the unique unitary operator on $\mathfrak{H}_\omega$ such that

$$\begin{aligned}&U_g^{-1}\pi_\omega(\tau_g(A))U_g = \pi_\omega(A) \quad \text{for all } A \in \mathfrak{A}\text{, and}\\ &U_g\Omega_\omega = \Omega_\omega.\end{aligned}$$

The existence and uniqueness of $U_g$ is guaranteed by corollary 1.2.5. Given $g_1, g_2 \in G$, then for all $A \in \mathfrak{A}$,

$$\begin{aligned}\pi_\omega(A) &= U_{g_1\circ g_2}^{-1}\pi_\omega(\tau_{g_1\circ g_2}(A))U_{g_1\circ g_2} = U_{g_1\circ g_2}^{-1}\pi_\omega(\tau_{g_1}(\tau_{g_2}(A)))U_{g_1\circ g_2}\\ &= U_{g_1\circ g_2}^{-1}U_{g_1}\pi_\omega(\tau_{g_2}(A))U_{g_1}^{-1}U_{g_1\circ g_2},\end{aligned}$$

and clearly

$$U_{g_1}^{-1}U_{g_1\circ g_2}\Omega_\omega = \Omega_\omega.$$

By the uniqueness of $U_{g_2}$, it follows that

$$U_{g_1}^{-1}U_{g_1\circ g_2} = U_{g_2} \implies U_{g_1\circ g_2} = U_{g_1}U_{g_2},$$

and hence, $\{U_g \mid g \in G\}$ is a representation of $G$ in $\mathfrak{H}_\omega$. □

**Definition 1.4.4.** Let $\mathfrak{A}$ be a $C^*$-algebra, $G$ a group acting as $^*$-automorphisms $\tau$ of $\mathfrak{A}$, $\omega \in E_{\mathfrak{A}}^G$ a $G$-invariant state and $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ a cyclic representation associated with $\omega$. The projection operator of the subspace of all $G$-invariant vectors of $\mathfrak{H}_\omega$, i.e., the subspace given by

$$\{\psi \in \mathfrak{H}_\omega; U_g\psi = \psi \text{ for all } U_g \in U_G\}$$

will be denoted by $E_\omega$.

The next theorem, known as the von Neumann ergodic theorem, is a very important tool in the study of representations of $G$-invariant states, shows that the projection operator $E_\omega$ may be obtained as a limit of combinations of the $U_g's$. However, the convergence of such limit is to be understood in the context of the *strong operator topology* (SOT), a topology in $\mathfrak{L}(\mathfrak{H}_\omega)$ where a sequence $T_n$ of operators converge to $T$ if and only if the convergence is *pointwise*. Note that pointwise convergence is weaker than the uniform convergence induced by the operator norm in $\mathfrak{L}(\mathfrak{H}_\omega)$, and hence a sequence converging in the SOT may not converge in the norm topology. Moreover, in its more general form, the von Neumann's ergodic theorem does not explicitly give one the sequence of combinations but rather only states that it exists, where its existence is connected the so-called minimum principle of a Hilbert space. To find the explicit sequence, it is necessary to assume additional structures of the group analyzed, as it will be done later.

**Definition 1.4.5.** Let $\mathfrak{H}$ be a Hilbert space. Every pair of elements $(T, v)$, where $v \in \mathfrak{H}$ and $T \in \mathfrak{L}(\mathfrak{H})$ induces a mapping $f_{(T,v)} : \mathfrak{L}(\mathfrak{H}) \to \mathbb{R}$ given by

$$f_{(T,v)}(L) = \|(T - L)v\|.$$

Let $F_{SOT} = \{f_{(T,v)} \mid T \in \mathfrak{L}(\mathfrak{H}) \text{ and } v \in \mathfrak{H}\}$. The *strong operator topology* $\tau_{SOT}$ *of* $\mathfrak{L}(\mathfrak{H})$ is the initial topology of $F_{SOT}$.

**Proposition 1.4.5.** *Let $\mathfrak{H}$ be a Hilbert space. Then, a sequence $\{T_n\}$ of bounded operators in $\mathfrak{L}(\mathfrak{H})$ converges to some $T \in \mathfrak{L}(\mathfrak{H})$ in the strong operator topology if and only if $\{T_n v\}$ converges to $Tv$ for any $v \in \mathfrak{H}$.*

*Proof.* If $\{T_n\}$ converges to $T$ in the SOT, then for a given $\epsilon > 0$ and $v \in \mathfrak{H}$ there exists some $N \in \mathbb{N}$ such that for all $n > N$,

$$T_n \in f^{-1}_{T,v}((-\epsilon, \epsilon)) \implies \|Tv - T_n v\| < \epsilon,$$

i.e., $\{T_n v\}$ converges to $Tv$. Now, assume that $\{T_n v\}$ converges to $Tv$ for any $v \in \mathfrak{H}$, and let $U$ be a neighborhood of $T$ in the SOT. Then, there exists some $\epsilon > 0$ such that

$$U = \bigcap_{i=1}^{n} f^{-1}_{(A_i,v_i)}((f_{(A_i,v_i)}(T) - \epsilon, f_{(A_i,v_i)}(T) + \epsilon)),$$

for some $A_1, \ldots, A_n \in \mathfrak{L}(\mathfrak{H})$ and some $v_1, \ldots, v_n \in \mathfrak{H}$. Let $N \in \mathbb{N}$ be such that

$$\|T_n v_i - T v_i\| < \epsilon \text{ for all } i = 1, \ldots, n.$$

Then, for all $m > N$ and all $i = 1, \ldots, n$, one has

$$\begin{aligned} &\|(T_m - A_i)v_i\| \le \|(T_m - T)v_i\| + \|(T - A_i)v_i\| < \|(T - A_i)v_i\| + \epsilon, \text{ and} \\ &\|(T - A_i)v_i\| \le \|(T_m - T)v_i\| + \|(T_m - A_i)v_i\| < \|(T_m - A_i)v_i\| + \epsilon. \end{aligned} \tag{1.34}$$

The two inequalities of eq. 1.34 imply that, for all $m > \mathbb{N}$ and all $i = 1, \ldots, n$,

$$\begin{aligned} &\|(T - A_i)v_i\| - \epsilon < \|(T_m - A_i)v_i\| < \|(T - A_i)v_i\| + \epsilon \implies \\ &f_{(A_i,v_i)}(T) - \epsilon < f_{(A_i,v_i)}(T_m) < f_{(A_i,v_i)}(T) + \epsilon, \end{aligned}$$

and hence, $T_m \in U$. Therefore, $\{T_n\}$ converges to $T$ in the SOT. □

**Proposition 1.4.6** (Minimum principle)**.** *Let $\mathfrak{H}$ be a Hilbert space and $C \subset \mathfrak{H}$ a closed convex set. Then there exists a unique element of $C$ with minimal norm, i.e., an element $\psi \in C$ satisfying*

$$\|\psi\| < \|v\| \textit{ for all } v \in C \mid v \neq \psi.$$

*Proof.* Let $k = \inf_{v \in C} \|v\|$, and let $\{\psi_n\}$ be a sequence of elements in $C$ such that $\{\|\psi_n\|\}$ converges to $k$. Given $\epsilon > 0$, let $N \in \mathbb{N}$ be such that

$$\|\psi_n\| < k + \epsilon \text{ for all } n > N.$$

Since $C$ is convex, for any $v, w \in C$, $\frac{v+w}{2} \in C$. Therefore,

$$\|v+w\|^2 = 4\left\|\frac{v+w}{2}\right\|^2 \geq 4k^2.$$

Hence, for any $n, m > N$, since $\psi_n, \psi_m$ must satisfy the parallelogram identity, one has

$$\|\psi_n - \psi_m\|^2 = 2\|\psi_n\|^2 + 2\|\psi_m\|^2 - \|\psi_n + \psi_m\|^2 < 2(k+\epsilon)^2 + 2(k+\epsilon)^2 - 4k^2 = 4\epsilon(2k+\epsilon).$$

Therefore, the sequence $\{\psi_n\}$ is Cauchy, and since any Hilbert space is complete and $C$ is closed, it converges to some $\psi \in C$. Now, let $\{\phi_n\}$ be some other sequence in $C$ such that $\{\|\phi_n\|\}$ converges to $\|\psi\|$. Let $N \in \mathbb{N}$ be such that

$$\|\phi_n\| < k + \epsilon \text{ for all } n > N,$$

and let $m \in \mathbb{N}$ be such that $\|\psi_m\| < k + \epsilon$ and $\|\psi_m - \psi\| < \epsilon$. Then, for any $n > N$, one has

$$\begin{aligned}\|\phi_n - \psi\|^2 \leq & \|\phi_n - \psi_m\|^2 + \|\psi_m - \psi\|^2 = 2\|\psi_n\|^2 + 2\|\psi_m\|^2 - \|\psi_n + \psi_m\|^2 + \\ & \|\psi_m - \psi\|^2 < 4\epsilon(2k+\epsilon) + \epsilon^2.\end{aligned}$$

Therefore, $\{\phi_n\}$ also converges to $\psi$, and hence $\psi$ is unique. □

**Theorem 1.4.1** (von Neumann's ergodic theorem)**.** *Let $\mathfrak{H}$ be a Hilbert space, G a group, $U_G \doteq \{U_g \mid g \in G\} \subset \mathfrak{L}(\mathfrak{H})$ a unitary representation of G, and E the projection operator of the subspace*

$$\{\psi \in \mathfrak{H}; U_g\psi = \psi \textit{ for all } U_g \in U_G\}.$$

*Then, $E$ lies in the strong closure of $Co(U_G)$, where $Co(U_G)$ denotes the convex hull of $U_G$ (i.e., the set of all convex combinations of elements in $U_G$).*

*Proof.* Let $W \subset \mathfrak{L}(\mathfrak{H})$ be a neighborhood of $E$ in the strong operator topology. Then, there exists a strongly-open subset $V \subset W$ such that

$$V = \bigcap_{i=1}^{n} \{T \in \mathfrak{L}(\mathfrak{H}) \mid \|T\psi_i - E\psi_i\| < \epsilon\}. \tag{1.35}$$

for some $\psi_i \in \mathfrak{H}$, $i = 1, \ldots n$, and some $\epsilon > 0$. Let

$$\tilde{\mathfrak{H}} \doteq \bigoplus_{i=1}^{n} \mathfrak{H}, \quad \tilde{U}_G \doteq \left\{\bigoplus_{i=1}^{n} U \in \mathfrak{L}(\tilde{\mathfrak{H}}) \mid U \in U_G\right\}, \quad \text{and} \quad \tilde{E} = \bigoplus_{i=1}^{n} E \in \mathfrak{L}(\tilde{\mathfrak{H}}).$$

Note that, for any $v \in E\mathfrak{H}$ and $w \in (E\mathfrak{H})^{\perp}$,

$$(U_g w, v) = (w, U_{g^{-1}} v) = (w, v) = 0 \text{ for all } U_g \in U_G.$$

Hence, $(E\mathfrak{H})^\perp$ is invariant under $U_G$, and a straightforward calculation shows similarly that $(\tilde{E}\tilde{\mathfrak{H}})^\perp$ is invariant under $\tilde{U}_G$. Now, consider the vectors $\psi_i$ appearing in 1.35 as a vector in $\tilde{\mathfrak{H}}$, i.e., define $\tilde{\psi} \doteq \bigoplus_{i=1}^n \psi_i \in \tilde{\mathfrak{H}}$. Since $\tilde{E}$ is a projection on $\tilde{\mathfrak{H}}$, $\tilde{\psi}$ may be decomposed as

$$\tilde{\psi} = \tilde{\psi}_\| + \tilde{\psi}_\perp, \quad \text{where} \quad \tilde{\psi}_\| \in \tilde{E}\tilde{\mathfrak{H}} \text{ and } \tilde{\psi}_\perp = (\tilde{E}\tilde{\mathfrak{H}})^\perp.$$

Define

$$C_{\tilde{\psi}_\perp} \doteq Co(\tilde{U}_G)\tilde{\psi}_\perp = \{\tilde{U}\tilde{\psi}_\perp \mid \tilde{U} \in Co(\tilde{U}_G)\} \subset (\tilde{E}\tilde{\mathfrak{H}})^\perp.$$

$\overline{C}_{\tilde{\psi}_\perp}$ (the closure of $C_{\tilde{\psi}_\perp}$ in the norm topology) is clearly nonempty, closed and convex, and hence by proposition 1.4.6 it contains an element $\tilde{\phi}$ of minimal norm. Moreover, it is not hard to see that $\tilde{U}_G\overline{C}_{\tilde{\psi}_\perp} \subset \overline{C}_{\tilde{\psi}_\perp}$. Therefore, for any $\tilde{U}_g = \bigoplus_{i=1}^n U_g \in \tilde{U}_G$,

$$\|\tilde{U}_g\tilde{\phi}\| = \left\|\bigoplus_{i=1}^n U_g\phi_i\right\| = \bigoplus_{i=1}^n \|U_g\phi_i\| = \bigoplus_{i=1}^n \|\phi_i\| = \|\tilde{\phi}\|.$$

Hence by the uniqueness of $\tilde{\phi}$, it must satisfy

$$\tilde{U}_g\tilde{\phi} = \tilde{\phi} \text{ for all } g \in G \implies \tilde{\phi} \in \tilde{E}\tilde{\mathfrak{H}}.$$

But since also $\tilde{\phi} \in \overline{C}_{\tilde{\psi}_\perp} \subset (E\mathfrak{H})^\perp$, it follows that $\tilde{\phi} = 0$. Let $\tilde{U} \in Co(\tilde{U}_g)$ be such that $\|\tilde{U}\tilde{\psi}_\perp\| < \epsilon$. $\tilde{U}$ may be decomposed as $\tilde{U} = \bigoplus_{i=1}^n U$, where $U \in Co(U_g)$, and it follows that

$$\sum_{i=1}^n \|U\psi_i - E\psi_i\| = \|\tilde{U}\tilde{\psi} - \tilde{E}\tilde{\psi}\| = \|\tilde{U}(\tilde{\psi}_\| + \tilde{\psi}_\perp) - \tilde{E}(\tilde{\psi}_\| + \tilde{\psi}_\perp)\| = \|\tilde{U}\tilde{\psi}_\perp\| < \epsilon.$$

Therefore, $U \in V$, and thus $E$ lies in the strong closure of $Co(U_G)$. □

For an arbitrary quantum system, the algebra of operators representing the observables of the system are, in general, non-abelian. This becomes obvious when analyzing, for example, a fermionic or bosonic system, where the commutation relations readily imply the non-abelianness of the algebra. In fact, such non-abelianness is crucial for the description of inumerous quantum phenomena in the light of the usual formalism of quantum mechanics. However, it is still possible for the operators of the algebra to "commute asymptotically", when considered the action of a group $G$: for example, it is possible for any two operators $A, B$ of a system to satisfy

$$\inf_{g \in G} \|[\tau_g(A), B]\| = 0. \tag{1.36}$$

This "asymptotic abelianness" is sometimes related to very useful properties of the $G$-invariant states of the algebra. Some of these properties are explored below, in an even more general situation than the one showed in eq. 1.36, with the definition of $G$-abelian states:

**Definition 1.4.6.** Let $\mathfrak{A}$ be a $C^*$-algebra, $G$ a group, $g \mapsto \tau_g$ a homomorphism from $G$ to the group of $*$-automorphisms of $\mathfrak{A}$ and $\omega \in E_{\mathfrak{A}}^G$ a $G$-invariant state. For each $A \in \mathfrak{A}$, let $Co(\tau_G(A))$ denote the convex hull of $\{\tau_g(A); g \in G\}$. The pair $(\mathfrak{A}, \omega)$ is said to be *$G$-abelian* if

$$\inf_{A' \in Co(\tau_G(A))} |\omega'([A', B])| = 0$$

for all $A, B \in \mathfrak{A}$ and all $G$-invariant vector states $\omega'$ of $\mathfrak{H}_\omega$.

**Proposition 1.4.7.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra and $G, G'$ two groups acting as *-automorphisms of $\mathfrak{A}$ such that, for all $A \in \mathfrak{A}$ and all $g \in G$,*

$$\inf_{g' \in G'} \|\tau'_{g'}(A) - \tau_g(A)\| = 0.$$

*Then, $E_{\mathfrak{A}}^{G'} \subset E_{\mathfrak{A}}^G$. Moreover, if a pair $(\mathfrak{A}, \omega)$ is $G$-abelian and $\omega \in E_{\mathfrak{A}}^{G'}$, then the pair $(\mathfrak{A}, \omega)$ is also $G'$-abelian.*

*Proof.* Let $g \in G$, $A \in \mathfrak{A}$, and given $\epsilon > 0$, let $g' \in G'$ be such that

$$\|\tau'_{g'}(A) - \tau_g(A)\| < \epsilon.$$

For any $\omega \in E_{\mathfrak{A}}^{G'}$, one has

$$|\omega(\tau_g(A)) - \omega(A)| = |\omega(\tau_g(A)) - \omega(\tau'_{g'}(A))| \leq \|\tau'_{g'}(A) - \tau_g(A)\| < \epsilon.$$

Since $\epsilon$ is arbitrary, it follows that $\omega(\tau_g(A)) = \omega(A)$ and hence $\omega \in E_{\mathfrak{A}}^G$. Now, let $(\mathfrak{A}, \omega)$ be a G-abelian pair with $\omega \in E_{\mathfrak{A}}^{G'}$. Moreover, let $A, B \in \mathfrak{A}$, let $S_\lambda$ be an arbitrary convex combination of $\tau_G$:

$$S_\lambda(A) = \sum_{i=1}^{n} \lambda_i \tau_{g_i}(A),$$

and let $S'_\lambda$ be the corresponding convex combination of $\tau_{G'}$:

$$S'_\lambda(A) = \sum_{i=1}^{n} \lambda_i \tau'_{g'_i}(A),$$

with $\tau'_{g'_i}$ satisfying

$$\|\tau'_{g'_i}(A) - \tau_{g_i}(A)\| < \frac{\epsilon}{2\|B\|} \text{ for all } i = 1, \ldots, n.$$

Note that for any $\omega' \in E_{\mathfrak{A}}$ and any $B \in \mathfrak{A}$,

$$|\omega'([S'_\lambda(A), B]) - \omega'([S_\lambda(A), B])| \leq \|[S'_\lambda(A) - S_\lambda(A), B]\| \leq 2\|S'_\lambda(A) - S_\lambda(A)\|\|B\| < \epsilon. \tag{1.37}$$

Since $S_\lambda(A) \in \tau_G(A)$ is arbitrary, eq. 1.37 implies that, for any $\omega' \in E_{\mathfrak{A}}$ and any $A, B \in \mathfrak{A}$,

$$\inf_{A' \in Co(\tau_{G'}(A))} |\omega'([A', B])| \leq \inf_{A' \in Co(\tau_G(A))} |\omega'([A', B])|. \tag{1.38}$$

Hence, if $\omega'$ is a $G'$-invariant vector state of $\mathfrak{H}_\omega$, then $\omega'$ is also G-invariant and the $G$-abelianness of $(\mathfrak{A}, \omega)$, together with eq. 1.38, imply that, for any $A, B \in \mathfrak{A}$,

$$\inf_{A' \in Co(\tau_{G'}(A))} |\omega'([A', B])| = \inf_{A' \in Co(\tau_G(A))} |\omega'([A', B])| = 0.$$

Thus, $(\mathfrak{A}, \omega)$ is $G'$-abelian.

□

**Proposition 1.4.8.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra, $G$ a group, $g \mapsto \tau_g$ a homomorphism from $G$ to the group of $*$-automorphisms of $\mathfrak{A}$, $\omega \in E_{\mathfrak{A}}^G$ a $G$-invariant state and $E_\omega$ the orthogonal projection on the subspace of vectors invariant under $U_\omega(G)$. The following conditions are equivalent:*

*(a) the pair $(\mathfrak{A}, \omega)$ is $G$-abelian,*

*(b) $E_\omega \pi_\omega(\mathfrak{A}) E_\omega$ is abelian.*

*Proof.* $(a) \implies (b)$. Any $E_\omega \pi_\omega(A) E_\omega \in E_\omega \pi_\omega(\mathfrak{A}) E_\omega$ can be decomposed as

$$E_\omega \pi_\omega(A) E_\omega = E_\omega \operatorname{Re}(\pi_\omega(A)) E_\omega + i E_\omega \operatorname{Im}(\pi_\omega(A)) E_\omega,$$

with $E_\omega \operatorname{Re}(\pi_\omega(A)) E_\omega$ and $E_\omega \operatorname{Im}(\pi_\omega(A)) E_\omega$ self-adjoints. Hence, it suffices to show $(b)$ only for self-adjoint elements. Let $\pi_\omega(A), \phi_\omega(B) \in \pi_\omega(\mathfrak{A})^{\mathbb{R}}$, $\psi \in E_\omega \mathfrak{H}$ and $S_\lambda$ denote some convex combination of $\tau_G$:

$$S_\lambda(A) = \sum_{i=1}^n \lambda_i \tau_{g_i}(A).$$

Define also $\tilde{S}_\lambda$ as the respective convex combination in terms of the operators $U_g$:

$$\tilde{S}_\lambda = \sum_{i=1}^n \lambda_i U_{g_i}.$$

From theorem 1.4.1, one may choose a convenient $S_\lambda$ such that its corresponding $\tilde{S}_\lambda$ satisfies

$$\|(\tilde{S}_\lambda - E_\omega)\pi_\omega(A)\psi\| < \frac{\epsilon}{2\|\pi_\omega(B)\psi\|}$$

Let $S_\mu$ be another convex combination of $\tau_G$. Since

$$\pi_\omega(S_\mu(A))\psi = \sum_{i=1}^n \lambda_i U_{g_i} \pi_\omega(A) U_{g_i}^{-1} \psi = \tilde{S}_\mu \pi_\omega(A)\psi,$$

it follows that

$$\|E_\omega \pi_\omega(A)\psi - \pi_\omega(S_\mu(S_\lambda(A))\psi\| = \|\tilde{S}_\mu(E_\omega - \tilde{S}_\lambda)\pi_\omega(A)\psi\| < \frac{\epsilon}{2\|\pi_\omega(B)\psi\|}.$$

Hence,

$$|(\psi,[E_\omega\pi_\omega(A)E_\omega,E_\omega\pi_\omega(B)E_\omega]\psi)| = |(\psi,\pi_\omega(A)E_\omega\pi_\omega(B)\psi)-(\psi,\pi_\omega(B)E_\omega\pi_\omega(A)\psi)|$$
$$= 2\,|\mathrm{Im}\,((E_\omega\pi_\omega(A)\psi,\pi_\omega(B)\psi))| = 2|\,\mathrm{Im}\,((E_\omega\pi_\omega(A)\psi-\pi_\omega(S_\mu(S_\lambda(A))\psi,\pi_\omega(B)\psi))$$
$$+\,\mathrm{Im}\,((\pi_\omega(S_\mu(S_\lambda(A)),\pi_\omega(B)\psi))\,| \le 2\|E_\omega\pi_\omega(A)\psi-\pi_\omega(S_\mu(S_\lambda(A))\psi\|\|\pi_\omega(B)\psi\|$$
$$+\,2|\,\mathrm{Im}\,((\pi_\omega(S_\mu(S_\lambda(A)),\pi_\omega(B)\psi))\,|$$
$$<\epsilon+2|\,\mathrm{Im}\,((\pi_\omega(S_\mu(S_\lambda(A)),\pi_\omega(B)\psi))\,|. \tag{1.39}$$

But

$$|\,\mathrm{Im}((\pi_\omega(S_\mu(S_\lambda(A)),\pi_\omega(B)\psi))| = |(\pi_\omega(B)\tilde S_\mu\tilde S_\lambda\pi_\omega(A)\psi,\psi)-(\psi,\pi_\omega(B)\tilde S_\mu\tilde S_\lambda\pi_\omega(A)\psi)|$$
$$= |(\pi_\omega(B)\tilde S_\mu\tilde S_\lambda\pi_\omega(A)\psi,\psi)-(\tilde S_\mu\tilde S_\lambda\pi_\omega(A)\pi_\omega(B)\psi,\psi)$$
$$+(\psi,\tilde S_\mu\tilde S_\lambda\pi_\omega(A)\pi_\omega(B)\psi)-(\psi,\pi_\omega(B)\tilde S_\mu\tilde S_\lambda\pi_\omega(A)\psi)|$$
$$= |(\pi_\omega([S_\mu(S_\lambda(A)),B])\psi,\psi)+(\psi,\pi_\omega([S_\mu(S_\lambda(A)),B])\psi)|$$
$$\le 2|(\psi,\pi_\omega([S_\mu(S_\lambda(A)),B])\psi)|. \tag{1.40}$$

Hence, combining eq. 1.40 into eq. 1.39, one has

$$|(\psi,[E_\omega\pi_\omega(A)E_\omega,E_\omega\pi_\omega(B)E_\omega]\psi)| < \epsilon+4|(\psi,\pi_\omega([S_\mu(S_\lambda(A)),B])\psi)|,$$

and since $S_\mu$ is arbitrary, using the $G$-abeliannes of $(\mathfrak{A},\omega)$, it follows that

$$|(\psi,[E_\omega\pi_\omega(A)E_\omega,E_\omega\pi_\omega(B)E_\omega]\psi)| = 0.$$

As $E_\omega = E_\omega^2$, one finally arrives at

$$\|[E_\omega\pi_\omega(A)E_\omega,E_\omega\pi_\omega(B)E_\omega]\| = \sup_{v\in\mathfrak{H}}|(v,[E_\omega\pi_\omega(A)E_\omega,E_\omega\pi_\omega(B)E_\omega]v)|$$
$$= \sup_{\psi\in E_\omega\mathfrak{H}}|(\psi,[E_\omega\pi_\omega(A)E_\omega,E_\omega\pi_\omega(B)E_\omega]\psi)| = 0.$$

$(b) \implies (a)$. If $E_\omega\pi_\omega(\mathfrak{A})E_\omega$ is abelian, then for any $\psi\in E_\omega\mathfrak{H}$ and any $A,B\in\mathfrak{A}$,

$$(\psi,[E_\omega\pi_\omega(A)E_\omega,E_\omega\pi_\omega(B)E_\omega]\psi) = (\psi,\pi_\omega(A)E_\omega\pi_\omega(B)\psi)-(\psi,\pi_\omega(B)E_\omega\pi_\omega(A)\psi)$$
$$= (\psi,\pi_\omega(B^*)E_\omega\pi_\omega(A^*)\psi)-(\psi,\pi_\omega(B)E_\omega\pi_\omega(A)\psi) = 0.$$

Thus, for any convex combination $S_\lambda$ of $\tau_G$,

$$|(\psi,\pi_\omega([S_\lambda(A),B])\psi)| = |(\psi,\pi_\omega(S_\lambda(A))\pi_\omega(B)\psi)-(\psi,\pi_\omega(B)\pi_\omega(S_\lambda(A))\psi|$$
$$= |(\pi_\omega(B^*)\tilde S_\lambda\pi_\omega(A^*)\psi,\psi)-(\psi,\pi_\omega(B)\tilde S_\lambda\pi_\omega(A)\psi)|$$
$$= |(\pi_\omega(B^*)E_\omega-\tilde S_\lambda)\pi_\omega(A^*)\psi,\psi)+(\psi,\pi_\omega(A)(\tilde S_\lambda-E_\omega)\pi_\omega(B)\psi)|$$
$$\le \|(E_\omega-\tilde S_\lambda)\pi_\omega(A^*)\psi\|\|\pi_\omega(B^*)\psi\|+\|\pi_\omega(A)(\tilde S_\lambda-E_\omega)\psi\|\|\pi_\omega(B)\psi\|.$$

Finally, applying theorem 1.4.1, it follows that

$$\inf_{S_\lambda(A)\in co(\tau_G(A))} |(\psi, \pi_\omega([S_\lambda(A), B])\psi)| = 0,$$

and hence, $(\mathfrak{A}, \omega)$ is $G$-abelian. □

**Theorem 1.4.2.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra, $G$ a group, $g \mapsto \tau_g$ a homomorphism from $G$ to the group of $*$-automorphisms of $\mathfrak{A}$ and $\omega \in E^G_{\mathfrak{A}}$ a $G$-invariant state. Consider the following:*

*(a) $E_\omega$ has rank one,*

*(b) $\omega$ is $G$-ergodic, i.e., $\omega \in \mathcal{E}^G_{\mathfrak{A}}$,*

*(c) $\{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}$ is irreducible on $\mathfrak{H}_\omega$.*

*It follows that $(a) \implies (b) \iff (c)$. Moreover, if $(\mathfrak{A}, \omega)$ is $G$-abelian, then all three conditions are equivalent.*

*Proof.* $(b) \implies (c)$. Suppose $(c)$ is false. Then, as shown in the proof of proposition 1.2.6, there exists a self-adjoint $T \in \{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}'$ which is not a multiple of the identity. Re-scaling $T$ by some positive constant, it can be assumed that $\|T\|_{op} \le 1$. By proposition 1.2.9, the positive linear functional

$$\rho_T(A) = (\Omega_\omega, T\pi_\omega(A)\Omega_\omega)$$

is majorized by $\omega$, and since $T$ is not a multiple of the identity, the uniqueness of $\rho_T$ assures that $\rho_T \neq \lambda\omega$ for any $0 \le \lambda \le 1$. Hence, $\omega \notin \mathcal{E}_{\mathfrak{A}}$, and to prove that $\omega \notin \mathcal{E}^G_{\mathfrak{A}}$, it suffices to show that $\rho_T$ is $G$-invariant. For any $g \in G$ and $A \in \mathfrak{A}$,

$$\begin{aligned}\rho_T(A) - \rho_T(\tau_g(A)) =& (\Omega_\omega, T\pi_\omega(A)\Omega_\omega) - (\Omega_\omega, TU_g\pi_\omega(A)U_g^{-1}\Omega_\omega)\\ =& (\Omega_\omega, U_gT\pi_\omega(A)\Omega_\omega) - (\Omega_\omega, TU_g\pi_\omega(A)\Omega_\omega)\\ =& (\Omega_\omega, [T, U_g]\pi_\omega(A)\Omega_\omega) = 0.\end{aligned}$$

Thus, $\rho_T$ is $G$-invariant and $\omega \notin \mathcal{E}^G_{\mathfrak{A}}$.

$(c) \implies (b)$. Now suppose $(b)$ is false. Then there exists a $G$-invariant positive $\rho_T \in \mathfrak{A}^*_+$ such that $\rho_T \preceq \omega$, where $\rho_T \neq \lambda\omega$ for any $0 \le \lambda \le 1$. By proposition 1.2.9,

$$\rho_T(A) = (\omega_\omega, T\pi_\omega(A)\Omega_\omega)$$

for some $T \in \pi_\omega(\mathfrak{A})'$, where $T$ is not a multiple of the identity and $\|T\|_{op} \le 1$. From the $G$-invariance of $\rho_T$, it follows that, for any $A \in \mathfrak{A}$ and any $g \in G$,

$$\begin{aligned}\rho_T(A^*A) - \rho_T(\tau_g(A^*A)) =& (\omega_\omega, T\pi_\omega(A^*A)\Omega_\omega) - (\omega_\omega, T\pi_\omega(\tau_g(A^*A))\Omega_\omega)\\ =& (\pi_\omega(A)\Omega_\omega, T\pi_\omega(A)\Omega_\omega) - (\pi_\omega(\tau_g(A))\omega_\omega, T\pi_\omega(\tau_g(A))\Omega_\omega)\\ =& (\pi_\omega(A)\Omega_\omega, [T - U_gTU_g^{-1}]\pi_\omega(A)\Omega_\omega) = 0,\end{aligned}$$

and from the ciclicity of $\Omega_\omega$, it follows that $T - U_g T U_g^{-1} = 0 \implies TU_g = U_g T$. Hence, $T \in \{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}'$ and $\{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}$ is not irreducible.

$(a) \implies (c)$. Suppose $\{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}$ is not irreducible. Then, by corollary 1.2.3, $\{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}''$ is irreducible. Since $\{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}''$ is a self-adjoint algebra, from proposition 1.2.6 it follows that there is a nonzero $\psi \in \mathfrak{H}$ which is not cyclic for $\{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}''$. By the cyclicity of $\Omega_\omega$, there exists some $A \in \mathfrak{A}$ such that

$$|(\psi, \pi_\omega(A)\Omega_\omega)| = |(E_\omega \pi_\omega(A^*)\psi, \Omega_\omega)| > \epsilon \implies E_\omega \pi_\omega(A^*)\psi \neq 0.$$

If $E_\omega \pi_\omega(A^*)\psi = \lambda \Omega_\omega$ for some $\lambda \in \mathbb{C}$, then $\psi$ would be cyclic for $\{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}''$ since $\{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\} \subset \{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}''$ and, by theorem 1.4.1, it is not hard to see that $E_\omega \in \{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}''$. Therefore, $E_\omega$ must have rank at least two.

$(c) \implies (a)$. Now assume further that $(\mathfrak{A}, \omega)$ is $G$-abelian. The idea of the proof is to show that on the Hilbert subspace $E_\omega \mathfrak{H}$,

$$(E_\omega \pi_\omega(\mathfrak{A}) E_\omega)' = E_\omega \{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}' E_\omega. \tag{1.41}$$

Assuming eq. 1.41 holds, then since $\{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}$ is irreducible, $(E_\omega \pi_\omega(\mathfrak{A}) E_\omega)' = \mathbb{C} E_\omega$, and by proposition 1.4.8, the $G$-abelianness of $(\mathfrak{A}, \omega)$ implies

$$E_\omega \pi_\omega(\mathfrak{A}) E_\omega \subset (E_\omega \pi_\omega(\mathfrak{A}) E_\omega)' = \mathbb{C} E_\omega.$$

Hence, for any $A \in \mathfrak{A}$, $E_\omega \pi_\omega(A) E_\omega \Omega_\omega = \lambda_A \Omega_\omega$ for some $\lambda_A \in \mathbb{C}$ and from the cyclicity of $\Omega_\omega$, it follows that $E_\omega$ can only have rank one. The inclusion $E_\omega \{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}' E_\omega \subset (E_\omega \pi_\omega(\mathfrak{A}) E_\omega)'$ is not hard to see: for $A \in \{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}'$, by theorem 1.4.1 $A$ also commutes with $E_\omega$, and for any $B \in \pi_\omega(\mathfrak{A})$

$$E_\omega A E_\omega E_\omega B E_\omega = E_\omega A B E_\omega = E_\omega B A E_\omega = E_\omega B E_\omega E_\omega A E_\omega.$$

Hence, $E_\omega \{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}' E_\omega \subset (E_\omega \pi_\omega(\mathfrak{A}) E_\omega)'$. Now, to prove the converse, take $T \in (E_\omega \pi_\omega(\mathfrak{A}) E_\omega)'$, *considering $T$ as an operator in $E_\omega \mathfrak{H}$*. Since $(E_\omega \pi_\omega(\mathfrak{A}) E_\omega)'$ is a unital $C^*$-sub-algebra of $\mathfrak{L}(E_\omega \mathfrak{H})$, and by proposition 1.2.5 any element of a $C^*$-algebra is a linear combination of unitary elements, it suffices to consider only the case where $T$ is unitary. Now, note that for any $A \in \mathfrak{A}$,

$$\begin{aligned}
\|\pi_\omega(A) T \Omega_\omega\|^2 =& (\pi_\omega(A) T \Omega_\omega, \pi_\omega(A) T \Omega_\omega) = (E_\omega \pi_\omega(A^* A) E_\omega T \Omega_\omega, T \Omega_\omega) \\
=& (T E_\omega \pi_\omega(A^* A) E_\omega \Omega_\omega, T \Omega_\omega) = (E_\omega \pi_\omega(A^* A) E_\omega \Omega_\omega, \Omega_\omega) \\
=& \|\pi_\omega(A) \Omega_\omega\|^2 .
\end{aligned}$$

Hence, the linear mapping $\pi_\omega(A)\Omega_\omega \mapsto \pi_\omega(A) T \Omega_\omega$, from $\mathfrak{D}_\omega \doteq \{\pi_\omega(A)\Omega_\omega; A \in \mathfrak{A}\}$ to $\mathfrak{H}$, is isometric, and extends uniquely to a bounded linear operator $S$ on $\overline{\mathfrak{D}_\omega} = \mathfrak{H}$. For any $A, B \in \mathfrak{H}$, $S$ satisfies

$$\begin{aligned}
S \pi_\omega(B) \pi_\omega(A) \Omega_\omega =& S \pi_\omega(BA) \Omega_\omega = \pi_\omega(BA) T \Omega_\omega = \pi_\omega(B) \pi_\omega(A) T \Omega_\omega \\
=& \pi_\omega(B) S \pi_\omega(A) \Omega_\omega
\end{aligned}$$

and, for any $U_g \in U_\omega(G)$,

$$\begin{aligned} SU_g\pi_\omega(A)\Omega_\omega =& S\pi_\omega(\tau_g(A))\Omega_\omega = \pi_\omega(\tau_g(A))T\Omega_\omega = U_g\pi_\omega(A)U_g^{-1}T\Omega_\omega \\ =& U_g\pi_\omega(A)U_g^{-1}E_\omega T\Omega_\omega = U_g\pi_\omega(A)T\Omega_\omega = U_gS\pi_\omega(A)\Omega_\omega. \end{aligned}$$

Therefore, by the cyclicity of $\Omega_\omega$ it is not hard to see that $S \in \{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}$. Moreover, for any $\psi \in E_\omega\mathfrak{H}$,

$$\begin{aligned} E_\omega S\psi =& E_\omega S \lim_{n\to\infty} \pi_\omega(A_n)\Omega_\omega = E_\omega \lim_{n\to\infty} \pi_\omega(A_n)T\Omega_\omega = \lim_{n\to\infty} E_\omega\pi_\omega(A_n)E_\omega T\Omega_\omega \\ =& \lim_{n\to\infty} TE_\omega\pi_\omega(A_n)E_\omega\Omega_\omega = TE_\omega \lim_{n\to\infty} \pi_\omega(A_n)\Omega_\omega = T\psi. \end{aligned}$$

Hence, on $E_\omega\mathfrak{H}$, $T = E_\omega SE_\omega \in E_\omega\{\pi_\omega(\mathfrak{A}) \cup U_\omega(G)\}'E_\omega$. □

**Proposition 1.4.9.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra, $G$ a group, $g \mapsto \tau_g$ a homomorphism from $G$ to the group of $*$-automorphisms of $\mathfrak{A}$ and $\omega \in E_{\mathfrak{A}}^G$ a $G$-invariant state. The following are equivalent:*

*(a) $E_\omega$ has rank one,*

*(b) For all $B \in \mathfrak{A}$ there exists a net $\{B_\alpha\}$ in $Co(\tau_G(B))$ satisfying*

$$\lim_\alpha |\omega(A\tau_g(B_\alpha)) - \omega(A)\omega(B)| = 0 \text{ for all } A \in \mathfrak{A} \text{ and } g \in G,$$

*where the convergence is uniform in g,*

*(c)*

$$\inf_{B' \in Co(\tau_G(B))} |\omega(AB') - \omega(A)\omega(B)| = 0 \quad \text{for all } A, B \in \mathfrak{A}.$$

*Proof.* $(a) \implies (b)$. Given any $A, B \in \mathfrak{A}$ and any $\epsilon > 0$, let $\{S_{\lambda^\alpha}\}_{\alpha\in I}$ be a net of convex combinations $S_{\lambda^\alpha} = \sum_{i=1}^n \lambda_i^\alpha \tau_{g_i^\alpha}$, where the respective net $\{\tilde{S}_{\lambda^\alpha}\}_{\alpha\in I}$ of convex combinations $\tilde{S}_{\lambda^\alpha} = \sum_{i=1}^n \lambda_i^\alpha U_{g_i^\alpha}$ in terms of the $U_g'$s converges to $E_\omega$ strongly. Note that, if $E_\omega$ has rank one, then for any $\psi \in \mathfrak{H}_\omega$,

$$E_\omega\psi = (\Omega_\omega, \psi)\Omega_\omega.$$

Hence,

$$\begin{aligned} (\Omega_\omega, \pi_\omega(A)\Omega_\omega)(\Omega_\omega, \pi_\omega(B)\Omega_\omega) =& (\Omega_\omega, \pi_\omega(A)(\Omega_\omega, \pi_\omega(B)\Omega_\omega)\Omega_\omega) \\ =& (\Omega_\omega, \pi_\omega(A)E_\omega\pi_\omega(B)\Omega_\omega). \end{aligned}$$

Let $j \in I$ be such that

$$\|(\tilde{S}_{\lambda^\alpha} - E_\omega)\pi_\omega(B)\Omega_\omega\| < \frac{\epsilon}{\|\pi_\omega(A)\|} \text{ for all } \alpha \geq j.$$

Then, for any $\alpha \geq j$ and any $g \in G$, one has

$$\begin{aligned}|\omega(A\tau_g(S_{\lambda^\alpha}(B))) - \omega(A)\omega(B)| =&|(\Omega_\omega, \pi_\omega(A)U_g\tilde{S}_{\lambda^\alpha}\pi_\omega(B)\Omega_\omega) \\ &- (\Omega_\omega, \pi_\omega(A)\Omega_\omega)(\Omega_\omega, \pi_\omega(B)\Omega_\omega)| \\ =&|\Omega_\omega, \pi_\omega(A)U_g(\tilde{S}_\lambda - E_\omega)\pi_\omega(B)\Omega_\omega)| \\ \leq&\|\pi_\omega(A)\|\|(\tilde{S}_\lambda - E_\omega)\pi_\omega(B)\Omega_\omega\| < \epsilon.\end{aligned}$$

$(b) \implies (c)$. This is obvious by taking $g$ as the identity element of $G$.

$(c) \implies (a)$. Given any $A, B \in \mathfrak{A}$ and any $\epsilon > 0$, let $S_\lambda$ be as before and let $S_\mu$ be another convex combination of $\tau_G$ such that

$$|\omega(S^*_\lambda(B)S^*_\mu(A)) - \omega(S^*_\lambda(B))\omega(A^*)| < \epsilon, \quad \text{where } S^*_\lambda \doteq \sum_{i=1}^n \lambda\tau_{g_i^{-1}}, S^*_\mu \doteq \sum_{i=1}^m \mu\tau_{g_i^{-1}}.$$

Since $(b)$ is true, such $S_\mu$ exists. Then,

$$\begin{aligned}&|(\Omega_\omega, \pi_\omega(A)E_\omega\pi_\omega(B)\Omega_\omega) - (\Omega_\omega, \pi_\omega(A)\Omega_\omega)(\Omega_\omega, \pi_\omega(B)\Omega_\omega)| = \\ &|(\Omega_\omega, \tilde{S}_\mu\pi_\omega(A)E_\omega\pi_\omega(B)\Omega_\omega) - (\Omega_\omega, \pi_\omega(A)\Omega_\omega)(\Omega_\omega, \tilde{S}_\lambda\pi_\omega(B)\Omega_\omega)| = \\ &|(\Omega_\omega, \pi_\omega(S_\mu(A))E_\omega\pi_\omega(B)\Omega_\omega) - (\Omega_\omega, \pi_\omega(S_\mu(A))\pi_\omega(S_\lambda(B))\Omega_\omega) \\ &\quad + (\Omega_\omega, \pi_\omega(S_\mu(A))\pi_\omega(S_\lambda(B))\Omega_\omega) - (\Omega_\omega, \pi_\omega(A)\Omega_\omega)(\Omega_\omega, \pi_\omega(S_\lambda(B))\Omega_\omega)| \leq \\ &|(\Omega_\omega, \pi_\omega(S_\mu(A))(E_\omega - \tilde{S}_\lambda)\pi_\omega(B)\Omega_\omega)| + |\omega(S_\mu(A)S_\lambda(B)) - \omega(A)\omega(S_\lambda(B))| \leq \\ &\|\pi_\omega(A)\|\|(E_\omega - \tilde{S}_\lambda)\pi_\omega(B)\Omega_\omega\| + |\omega(S^*_\lambda(B)S^*_\mu(A)) - \omega(S^*_\lambda(B))\omega(A^*)| < 2\epsilon.\end{aligned}$$

Since $\epsilon$ is arbitrary, it follows that

$$\begin{aligned}&(\Omega_\omega, \pi_\omega(A)E_\omega\pi_\omega(B)\Omega_\omega) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)(\Omega_\omega, \pi_\omega(B)\Omega_\omega) \text{ for all } A, B \in \mathfrak{A} \implies \\ &|(\pi_\omega(A)\Omega_\omega, E_\omega\pi_\omega(B)\Omega_\omega)| = |(\Omega_\omega, \pi_\omega(A)\Omega_\omega)(\Omega_\omega, \pi_\omega(B)\Omega_\omega)| \text{ for all } A, B \in \mathfrak{A}. \quad (1.42)\end{aligned}$$

If $E_\omega$ has rank more than one, let $\psi \in E_\omega\mathfrak{H}_\omega$ be such that $\|\psi\| = 1$ and $(\Omega_\omega, \psi) = 0$. Since $\Omega_\omega$ is cyclic for $\pi_\omega(\mathfrak{A})$, choose $A \in \mathfrak{A}$ such that $\|\pi_\omega(A)\Omega_\omega - \psi\| < \epsilon$ for some $\epsilon > 0$. Then, one has

$$\begin{aligned}|(\pi_\omega(A)\Omega_\omega, E_\omega\pi_\omega(A)\Omega_\omega)| =&|(E_\omega\pi_\omega(A)\Omega_\omega, E_\omega\pi_\omega(A)\Omega_\omega)| \\ =&\|E_\omega\pi_\omega(A)\Omega_\omega\| \geq \|\psi\| - \|E_\omega(\pi_\omega(A)\Omega_\omega - \psi)\| > 1 - \epsilon\end{aligned}$$

and

$$|(\Omega_\omega, \pi_\omega(A)\Omega_\omega)| = |(\Omega_\omega, \pi_\omega(A)\Omega_\omega - \psi + \psi)| = \|\pi_\omega(A)\Omega_\omega - \psi\| < \epsilon.$$

For $\epsilon$ small enough, clearly eq. 1.42 is not satisfied. Hence, $E_\omega$ must have rank one. $\square$

Some of the symmetries that will be explored in the models analyzed here (translational and permutation symmetries) are described by groups that have the property of being *amenable*. The amenability of a group is related to a certain invariance property of its measures (or, equivalently, means in the countable case). The main property of amenable groups that will be explored here is the existence of what is called a *Følner sequence*, which allows one to find an explicit sequence of the convex combinations of $U_g's$ that converges to $E_\omega$, thus improving the result of theorem 1.4.1 (which only shows the existence of such sequence) for this class of groups. Here the discussion is restricted to only countable amenable groups, however the results may also be extended to the uncountable case.

**Definition 1.4.7.** Let $G$ be a countable group and $l^\infty(G)$ the Banach space of all bounded real functions on $G$ with the supremum norm. For any $g \in G$, the *left-translation of* $g$ is a mapping $\tau_g : l^\infty(G) \to l^\infty(G)$ given by

$$\tau_g(f)(x) \doteq f(g^{-1}x).$$

**Definition 1.4.8.** Let $G$ be a countable set. A *finite mean* on $G$ is a non-negative function $\mu \in l^\infty(G)$ of finite support such that $\sum_{g\in G} \mu(g) = 1$.

The next proposition is central to the study of the so-called *amenable groups*. It will not be proven here, but its proof can be found in ([6], chapter four).

**Proposition 1.4.10.** *Let $G$ be a countable group. The following are equivalent:*

*(a) There exists a non-negative linear functional $\lambda : l^\infty(G) \to \mathbb{R}$ with $\lambda(\mathbb{1}) = 1$ such that $\lambda(\tau_g(f)) = \lambda(f)$ for all $g \in G$,*

*(b) for any finite set $S \in G$, and any $\epsilon > 0$, there exists a finite mean $\mu$ such that*

$$\sum_{g\in G} |(\mu - \tau_s(\mu))(g)| \leq \epsilon \textit{ for all } s \in S,$$

*(c) there exists a sequence $\{\Phi_n\}$ of non-empty finite subsets of $G$ such that*

$$\lim_{n\to\infty} \frac{|(g.\Phi_n)\Delta\Phi_n|}{|\Phi_n|} = 0 \textit{ for all } g \in G,$$

*where $g.\Phi_n = \{g\phi; \phi \in \Phi_n\}$ and $A\Delta B$ denotes $(A \setminus B) \cup (B \setminus A)$.*

**Definition 1.4.9.** Let $G$ be a countable group. If $G$ satisfies one (and hence all) of the properties in the above proposition, $G$ is said to be *amenable*. Moreover, if a sequence $\{\Phi_n\}$ of non-empty subsets of $G$ satisfies property $(c)$ of the above proposition, $\{\Phi_n\}$ is said to be a *Følner sequence of* $G$.

**Proposition 1.4.11.** *Let $\mathfrak{H}$ be a Hilbert space, $G$ be an amenable group, $U_G \doteq \{U_g \mid g \in G\} \subset \mathfrak{L}(\mathfrak{H})$ a unitary representation of $G$, $\{\Phi_n\}$ a Følner sequence of $G$ and $E$ the orthogonal projection of the subspace*

$$\{\psi \in \mathfrak{H}; U_g\psi = \psi \textit{ for all } U_g \in U_G\}.$$

*Then, for all $\psi \in \mathfrak{H}$,*

$$\lim_{n\to\infty} \frac{1}{|\Phi_n|} \sum_{g\in\Phi_n} U_g\psi = E\psi.$$

*Proof.* First note that since $U_{g^{-1}} = U_g^{-1} = U_g^*$, $E = E^*$, and, by definition, $U_g E = E$ for all $g \in G$, it is easy to see that $EU_g = (U_{g^{-1}}E)^* = E$ for all $g \in G$. Note also that, for any two sets $A, B$, $(A \setminus B) \cap (B \setminus A) = \varnothing$, and hence $|A\Delta B| = |(A \setminus B) \cup (B \setminus A)| = |(A \setminus B)| + |(B \setminus A)|$. Moreover, since $|A \setminus B| = |A| - |A \cap B|$ and clearly $|g.\Phi_n| = |\Phi_n|$,

$$|(g.\Phi_n) \setminus \Phi_n| = |\Phi_n \setminus (g.\Phi_n)| = \frac{|(g.\Phi_n)\Delta\Phi_n|}{2}.$$

Let $\psi \in \mathfrak{H}$. By theorem 1.4.1, there exists a convex combination $S_\lambda = \sum_{i=1}^n \lambda_i U_{g_i}$ such that

$$\|S_\lambda\psi - E_G\psi\| < \frac{\epsilon}{2}. \tag{1.43}$$

Since $\{\Phi_n\}$ is a Følner sequence, let $N \in \mathbb{N}$ be such that

$$\frac{|(g.\Phi_n)\Delta\Phi_n|}{|\Phi_n|} < \frac{\epsilon}{4\|\psi\|} \text{ for all } n > N \text{ and all } g \in G.$$

For all $n > N$, it follows that

$$\begin{aligned}
\left\| \frac{1}{|\Phi_n|} \left( \sum_{h\in\Phi_n} U_h\psi - U_g \sum_{h\in\Phi_n} U_h\psi \right) \right\| &= \frac{1}{|\Phi_n|} \left\| \sum_{h\in\Phi_n} U_h\psi - \sum_{h\in\Phi_n} U_{g\circ h}\psi \right\| \\
&= \frac{1}{|\Phi_n|} \left\| \sum_{h\in\Phi_n} U_h\psi - \sum_{h\in g.\Phi_n} U_h\psi \right\| \\
&= \frac{1}{|\Phi_n|} \Bigg\| \sum_{h\in\Phi_n\setminus(g.\Phi_n)} U_h\psi + \sum_{h\in\Phi_n\cap(g.\Phi_n)} U_h\psi \\
&\qquad - \left( \sum_{h\in\Phi_n\setminus g.\Phi_n} U_h\psi + \sum_{h\in g.\Phi_n\cap\Phi_n} U_h\psi \right) \Bigg\| \\
&= \frac{1}{|\Phi_n|} \left\| \sum_{h\in\Phi_n\setminus(g.\Phi_n)} U_h\psi - \sum_{h\in(g.\Phi_n)\setminus\Phi_n} U_h\psi \right\| \\
&\leq \frac{1}{|\Phi_n|} (|\Phi_n \setminus (g.\Phi_n)| + |(g.\Phi_n) \setminus \Phi_n|)\|\psi\| \\
&= \frac{|(g.\Phi_n)\Delta\Phi_n|}{|\Phi_n|} \|\psi\| < \frac{\epsilon}{2}.
\end{aligned} \tag{1.44}$$

Hence, eq.s 1.43 and 1.44 imply, for all $n > N$,

$$\begin{aligned}\left\| \frac{1}{|\Phi_n|} \sum_{h\in\Phi_n} U_h\psi - E\psi \right\| &= \left\| \frac{1}{|\Phi_n|} \left( \sum_{h\in\Phi_n} U_h\psi - \sum_{h\in\Phi_n} U_h S_\lambda\psi + \sum_{h\in\Phi_n} U_h S_\lambda \psi \right) - E\psi \right\| \\ &= \left\| \sum_{i=1}^{k} \frac{\lambda_i}{|\Phi_n|} \left( \sum_{h\in\Phi_n} U_h\psi - U_{g_i} \sum_{h\in\Phi_n} U_h\psi \right) \right. \\ &\qquad \left. + \frac{1}{|\Phi_n|} \sum_{h\in\Phi_n} U_h(S_\lambda - E)\psi \right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

□

**Proposition 1.4.12.** *Let $\mathfrak{A}$ be a unital $C^*$-algebra, $G$ a discrete amenable group acting as *-automorphisms $\tau$ of $\mathfrak{A}$, $\{\Phi_n\}$ a Følner sequence of $G$, $\omega \in E_{\mathfrak{A}}^G$ a $G$-invariant state and assume that $(\mathfrak{A}, \omega)$ is $G$-abelian. Then, the following are equivalent:*

*(a)* $\omega \in \mathcal{E}_{\mathfrak{A}}^G$,

*(b) For any* $A \in \mathfrak{A}$,

$$|\omega(A)|^2 = \lim_{n\to\infty} \omega(A_{\Phi_n}^* A_{\Phi_n}), \quad \textit{where} \quad A_{\Phi_n} \doteq \frac{1}{|\Phi_n|} \sum_{g\in\Phi_n} \tau_g(A).$$

*Proof.* Let $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ be a cyclic representation of $\mathfrak{A}$ associated with $\omega$ and $\{\Phi_n\}$ a Følner sequence of $G$. First, note that for any $A \in \mathfrak{A}$ and any $n \in \mathbb{N}$,

$$\omega(A_{\Phi_n}^* A_{\Phi_n}) = \|\pi_\omega(A_{\Phi_n})\Omega_\omega\|^2 = \left\| \frac{1}{|\Phi_n|} \sum_{g\in\Phi_n} U_g \pi_\omega(A)\Omega_\omega \right\|^2.$$

Hence, by proposition 1.4.11 it follows that for any $A \in \mathfrak{A}$,

$$\lim_{n\to\infty} \omega(A_{\Phi_n}^* A_{\Phi_n}) = \|E_\omega \pi_\omega(A)\Omega_\omega\|^2.$$

Moreover, by the Cauchy-Schwarz inequality, for any $A \in \mathfrak{A}$,

$$\begin{aligned}|\omega(A)|^2 =& |(\Omega_\omega, \pi_\omega(A)\Omega_\omega)|^2 = |(\Omega_\omega, E_\omega\pi_\omega(A)\Omega_\omega)|^2 \\ \leq& \|E_\omega\pi_\omega(A)\Omega_\omega\|^2 = \lim_{n\to\infty} \omega(A_{\Phi_n}^* A_{\Phi_n}),\end{aligned}$$

where the equality is valid if and only if $E_\omega\pi_\omega(A)\Omega_\omega \in \mathbb{C}.\Omega_\omega$. Assuming $(a)$ is true, by theorem 1.4.2, $E_\omega$ has rank one, since $(\mathfrak{A}, \omega)$ is $G$-abelian. Hence, $E_\omega\pi_\omega(A)\Omega_\omega \in \mathbb{C}.\Omega_\omega$ for all $A \in \mathfrak{A}$ and therefore $(b)$ holds. If $(a)$ is false, then $E_\omega$ has rank at least two, and by the cyclicity of $\Omega_\omega$ there exists some $A \in \mathfrak{A}$ such that $E_\omega\pi_\omega(A)\Omega_\omega \notin \mathbb{C}.\Omega_\omega$. Hence, $(b)$ is false. □

## 1.5 The CAR algebra

In the so-called second quantization formalism of quantum statistical mechanics, the Hilbert space of a fermionic system is given by the fermionic Fock space $\mathfrak{F}_-(\mathfrak{h})$, constructed from the respective one-particle Hilbert space $\mathfrak{h}$, as already discussed in the introduction. The algebra of observables, in turn, is generated by the creation and annihilation operators $a^*(f), a(f)$, acting on $\mathfrak{F}_-(\mathfrak{h})$, which satisfy, for all $f \in \mathfrak{h}$, the *Canonical Anti-commutation Relations* (CARs)

$$\begin{aligned} \{a(f), a(g)\} &= \{a^*(f), a^*(g)\} = 0, \quad f, g \in \mathfrak{h} \\ \{a(f), a^*(g)\} &= (f, g)\mathbb{1}, \end{aligned} \tag{1.45}$$

where $\{A, B\} = AB + BA$. In order to go beyond the usual Fock space formalism, the $C^*$-algebraic approach starts by defining the algebra of observables as an abstract $C^*$-algebra generated by elements $\{a(f) \mid f \in \mathfrak{h}\}$ satisfying the CARs. Indeed, the existence of such $C^*$-algebra follows from the existence of the annihilation and creation operators acting on $\mathfrak{F}_-(\mathfrak{h})$, and its uniqueness will be proven in the next theorem. In this way, when the one-particle Hilbert space $\mathfrak{h}$ is finite-dimensional, the $C^*$-algebra formalism and the Fock space formalism are equivalent, since in this case any irreducible representation of the algebra is equivalent to the Fock space representation. However, when $\mathfrak{h}$ is infinite-dimensional, then the representation of the algebra acting on the fermionic Fock space $\mathfrak{F}_-(\mathfrak{h})$ is only one of infinitely many inequivalent irreducible representations of the $C^*$-algebra.

**Theorem 1.5.1.** *Let $\mathfrak{h}$ be a separable Hilbert space and let $\mathfrak{A}_i, i = 1, 2,$ be two $C^*$-algebras generated by the identity $\mathbb{1}$ and the elements $\{a_i(f) \mid f \in \mathfrak{h}\}$, satisfying for all $f, g \in \mathfrak{h}$ and $i = 1, 2,$*

*(a)* $f \mapsto a_i(f)$ *is antilinear,*

*(b)* $\{a_i(f), a_i(g)\} = 0,$

*(c)* $\{a_i(f), a_i^*(g)\} = (f, g)\mathbb{1}.$

*Then, there exists a unique *-isomorphism $\alpha : \mathfrak{A}_1 \mapsto \mathfrak{A}_2$ such that*

$$\alpha(a_1(f)) = a_2(f) \qquad \text{for all } f \in \mathfrak{h}.$$

*Furthermore,*

1. $\|a(f)\| = \|f\|$ *for all* $f \in \mathfrak{h}$,
2. *if $\mathfrak{h}$ is $n$ dimensional, where $n \leq +\infty$, then $\mathfrak{A}(\mathfrak{h})$ is isomorphic with the $C^*$-algebra of $2^n \times 2^n$ complex matrices,*
3. $\mathfrak{A}(\mathfrak{h})$ *is separable.*

*Proof.* 1. From the $C^*$-property, one has

$$\|a(f)\|^4 = \|a^*(f)a(f)\|^2 = \|(a^*(f)a(f))^2\|.$$

Since, from $(b)$, $a(f)a(f) = 0$, one can write $(a^*(f)a(f))^2$ as

$$(a^*(f)a(f))^2 = a^*(f)\{a(f), a^*(f)\}a(f) = (f, f)a^*(f)a(f) = \|f\|^2 a^*(f)a(f).$$

Thus

$$\|a(f)\|^4 = \|\|f\|^2 a^*(f)a(f)\| = \|f\|^2\|a(f)\|^2$$

If $f = 0$, then $a(f) = 0$ by $(a)$, and hence $\|a(f)\| = \|f\| = 0$. If $f \neq 0$, then $a(f) \neq 0$ by $(c)$. Therefore, $\|a(f)\| \neq 0$ and the above equation implies $\|a(f)\| = \|f\|$.

2. Assume that $\mathfrak{h}$ is finite dimensional, and let $\{f_1, ..., f_n\}$ be an orthonormal basis for $\mathfrak{h}$. For $k = 1, \ldots, n$, define

$$\begin{aligned} e_{11}^{(k)} &= a(f_k)a^*(f_k) & e_{12}^{(k)} &= V_{k-1}a(f_k) \\ e_{21}^{(k)} &= V_{k-1}a^*(f_k) & e_{22}^{(k)} &= a^*(f_k)a(f_k), \end{aligned} \tag{1.46}$$

where

$$V_k = \prod_{l=1}^{k}(a(f_l)a^*(f_l) - a^*(f_l)a(f_l)) = \prod_{l=1}^{k}(\mathbb{1} - 2a^*(f_l)a(f_l)).$$

Note that, for any $f \in \mathfrak{h}$,

$$\begin{aligned} (\mathbb{1} - 2a^*(f)a(f))(\mathbb{1} - 2a^*(f)a(f)) =& \mathbb{1} - 4a^*(f)a(f) + 4a^*(f)a(f)a^*(f)a(f) \\ =& \mathbb{1} - 4a^*(f)a(f) + 4a^*(f)a(f)(\mathbb{1} - a(f)^*a(f)) \\ =& \mathbb{1}. \end{aligned}$$

Hence, the elements defined in eq. 1.46 generate the same algebra as the $a(f)$, since for any $k = 1, \ldots, n$,

$$\begin{aligned} \left(\prod_{l=1}^{k-1}(e_{11}^{(k-l)} - e_{22}^{(k-l)})\right) e_{12}^{(k)} =& \prod_{l=1}^{k}(\mathbb{1} - 2a^*(f_{k-l})a(f_{k-l})) \prod_{l=1}^{k}(\mathbb{1} - 2a^*(f_l)a(f_l))a(f_k) \\ =& a(f_k). \end{aligned}$$

Moreover, using the CARs, it is not hard to see that each family $\{e_{11}^{(k)}, e_{12}^{(k)}, e_{21}^{(k)}, e_{22}^{(k)}\}$ satisfies

$$e_{ij}^{(k)} e_{lm}^{(k)} = \delta_{jl} e_{im}^{(k)}.$$

Thus, one may construct an isomorphism between the algebra $\mathfrak{M}_k$ generated by $e_k = \{e_{11}^{(k)}, e_{12}^{(k)}, e_{21}^{(k)}, e_{22}^{(k)}\}$ and the algebra of $2 \times 2$ complex matrices, by the map $\pi : \mathfrak{M}_k \to M_{2\times 2}(\mathbb{C})$ defined by

$$\pi(e_{11}^{(k)}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \pi(e_{12}^{(k)}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$\pi(e_{21}^{(k)}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \pi(e_{22}^{(k)}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, since the members of different families commute; that is

$$k \neq g \implies e_{ij}^{(k)} e_{lm}^{(g)} = e_{lm}^{(g)} e_{ij}^{(k)},$$

it follows that the algebra $\mathfrak{A}(\mathfrak{h})$ which is (as mentioned above) generated by $E_n = \cup_{k=1}^n e_k$, is isomorphic to the tensor product $M_{2\times 2}(\mathbb{C})^{\otimes n}$, which in turn is isomorphic to $M_{2^n\times 2^n}(\mathbb{C})$. This establishes property 2 of the theorem and also the existence of the *-isomorphism $\alpha$ when $\mathfrak{h}$ is finite dimensional. When $\mathfrak{h}$ is infinite dimensional but separable, let $\{f_1, \dots, f_n, f_{n+1}, \dots\}$ be an orthonormal basis for $\mathfrak{h}$. For each $n \in \mathbb{N}$, let $\mathfrak{h}_n$ be the Hilbert space with base $\{f_1, \dots, f_n\}$, and let $\mathfrak{A}_0(\mathfrak{h})$ be the *-algebra given by

$$\mathfrak{A}_0(\mathfrak{h}) = \bigcup_{n=1}^{\infty} \mathfrak{A}(\mathfrak{h}_n).$$

Note that for any $n \in \mathbb{N}$, the $C^*$-sub-algebras $\mathfrak{A}(\mathfrak{h}_n)$ are *-isomorphic to $M_{2^n\times 2^n}(\mathbb{C})$. Hence, $\mathfrak{A}_0(\mathfrak{h})$ is *-isomorphic to the direct limit of the $C^*$-algebras $M_{2^n\times 2^n}(\mathbb{C})$, that shall be denoted by $M_{2^\infty\times 2^\infty}(\mathbb{C})$. Since $M_{2^\infty\times 2^\infty}(\mathbb{C})$ is the direct limit of separable $C^*$-algebras, it follows that $M_{2^\infty\times 2^\infty}(\mathbb{C})$ is also separable and its elements satisfy the $C^*$-norm property. Moreover, by the continuity and linearity of $f \mapsto a(f)$, it is easy to see that $a(f) \in \overline{\mathfrak{A}}_0(\mathfrak{h})$ for all $f \in \mathfrak{h}$, and consequently $\overline{\mathfrak{A}}_0(\mathfrak{h}) = \mathfrak{A}(\mathfrak{h})$. Therefore, the *-isomorphism between $\mathfrak{A}_0(\mathfrak{h})$ and $M_{2^\infty\times 2^\infty}(\mathbb{C})$ can be extended to a *-isomorphism between $\mathfrak{A}(\mathfrak{h})$ and the $C^*$-algebra $\overline{M_{2^\infty\times 2^\infty}}(\mathbb{C})$. This establishes the existence of the *-isomorphism when $\mathfrak{h}$ is infinite dimensional and the separability of $\mathfrak{A}(\mathfrak{h})$.

To show that such $\alpha$ is unique, let $\alpha_1, \alpha_2$ be two *-isomorphisms from $\mathfrak{A}_1$ to $\mathfrak{A}_2$ such that

$$\alpha_{1,2}(a_1(f)) = a_2(f) \qquad \text{for all } f \in \mathfrak{h}.$$

Then, the *-isomorphism $\beta : \mathfrak{A}_1 \mapsto \mathfrak{A}_2$ given by $\beta = \alpha_1 - \alpha_2$ satisfies

$$\beta(a_1(f)) = 0 \qquad \text{for all } f \in \mathfrak{h}.$$

This implies that $\beta = 0$ in the *-algebra of polynomials of $\{a_1(f), a_1^*(f) \mid f \in \mathfrak{h}\}$, and since $\mathfrak{A}_1$ is the closure of such algebra, the continuity of $\beta$ (see proposition 1.2.1) implies that $\beta = 0$ on $\mathfrak{A}_1$, i.e., $\alpha_1 = \alpha_2$.

□

**Definition 1.5.1.** Let $\mathfrak{h}$ be a Hilbert space. The $C^*$-algebra $\mathfrak{A}$ generated by elements $\{a(f) \mid f \in \mathfrak{h}\}$ satisfying

(a) $f \mapsto a(f)$ is antilinear,

(b) $\{a_i(f), a(g)\} = 0$,

(c) $\{a_i(f), a^*(g)\} = (f, g)\mathbb{1}$,

is called the *CAR algebra of* $\mathfrak{h}$.

# Chapter 2

# Catastrophe Theory

This chapter presents a mathematical tool required for the analysis of the thermodynamics of the model studied in this thesis. The tool presented here is an adaptation of the so-called *catastrophe theory*, a mathematical theory that allows one to study the bifurcations of critical points in a parametric family of functions. To give a more clear picture of the idea behind the theory, let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and $F : \mathbb{R}^{n+r} \to \mathbb{R}$ another smooth function such that $F(x, 0, \dots, 0) = f(x)$, $x \in \mathbb{R}^n$. $F$ is to be seen as a family of functions $F_u(x) \doteq F(x, u)$, depending on the parameters $u \in \mathbb{R}^r$. Suppose that $0$ is a *degenerate critical point* of $f$ (i.e., $\vec{\nabla} f(0) = 0$ and the hessian of $f$ is singular at $0$), then catastrophe theory tries to answer whether this critical point in $F_0(x)$ bifurcates into other critical points of $F_u(x)$, for $u$ small, and how they behave. One of the most important results of catastrophe theory is the so-called *Thom's Theorem*; it states that, if $r \leq 5$ and $F$ is *transversal* (a property that will be defined later), then, in a neighborhood of $0 \in \mathbb{R}^{n+r}$, $F$ is equivalent (via some suitable diffeomorphisms) to one of the eleven *elementary catastrophes*, which are polynomial functions, and therefore the behavior of their critical points can be easily analyzed and extended to the critical points of $F$.

One of the uses of catastrophe theory in this thesis will be to show that a specific family of one variable functions (intimately related to the pressure of the physical system) is equivalent to a certain elementary polynomial, and conclude from this that, for a specific choice of parameters (which are related to the strength of the interactions considered in the model) the corresponding function has two minima, where each one is related to a different thermodynamic phase of the system. However, due to the parity symmetry of the specific family of functions of interest, it fails to fit in the usual formalism of the theory. Therefore, in this chapter it is derived a version of the theory in order to include the elementary catastrophes of families of functions with parity symmetry. This will be done only for families of one variable functions (which is the case of our function of interest), since the generalization for families of functions with an arbitrary number of variables adds a lot more technical difficulties.

Another important result of catastrophe theory that will be explored in this thesis is the fact that some transversal families $F$ are "stable", i.e., $F + p$ is equivalent to $F$ when $p$ is "small" (in a suitable topology). This will be important to show that when a kinetic term is added into the model, the phase diagram still remains qualitatively unchanged, provided that the hopping term is small. Some good references for an introduction to the more general formalism of catastrophe theory are [5] and [13].

## 2.1 Basic definitions

The results obtained in catastrophe theory are only local, i.e., they are valid only in a neighborhood of the degenerate critical point. Therefore, for the development of the theory it is more convenient to deal with a certain equivalence class of functions, called *germs*:

**Definition 2.1.1.** Let $X, Y$ be two topological spaces and $C \subset F(X, Y)$, where $F(X, Y)$ is the space of all continuous functions from $X$ to $Y$. For any $x \in X$, there is an equivalence relation $\sim_x$ on $C$, where $f \sim_x g$ if and only if $f$ and $g$ agree in some neighborhood of $x$. The equivalence class of $f$ under the equivalence relation $\sim_x$ is called the *germ of $f$ at $x$*.

Throughout the text, the topological spaces $X, Y$ will always be open subsets of $\mathbb{R}^n$, for some $n \in \mathbb{N}$, and $C$ will always denote the space of infinitely differentiable functions. From now on, any "germ" mentioned in the text is to be considered as a germ at $0$, unless stated otherwise. Moreover, to keep the notation simpler, there will be no distinction between a germ and its representative if there is no room for ambiguity.

**Definition 2.1.2.** Let $C^\infty(U, \mathbb{R}^p)$ be the space of all infinitely differentiable functions from $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$, where $U$ is a neighborhood of $0$. The set of all germs (at $0$) of functions in $C^\infty(U, \mathbb{R}^p)$ will be denoted by $\epsilon(n, p)$, the subset of all germs of functions that are even (odd) on $U$ will be denoted by $\epsilon_{e(o)}(n, p) \subset \epsilon(n, p)$. Moreover, the subset of all germs whose $k-1$ first derivatives vanish at 0 will be denoted by $m^k(n, p) \subset \epsilon(n, p)$ , and the sets $\epsilon_{e(o)}(n, p) \cap m^k(n, p)$ by $m^k_{e(o)}(n, p)$. To simplify the notation, denote $m^1(n, p)$ (or $m^1_{e(o)}(n, p)$) by simply $m(n, p)$ (or $m_{e(o)}(n, p)$). Also, denote $\epsilon(n, 1)$ (or $\epsilon_{e(o)}(n, 1)$) by $\epsilon(n)$ (or $\epsilon_{e(o)}(n)$), and $m^k(n, 1)$ (or $m^k_{e(o)}(n, 1)$) by $m^k(n)$ (or $m^k_{e(o)}(n)$), respectively.

**Definition 2.1.3.** A germ $f \in \epsilon(n, n)$ is said to be a *germ diffeomorphism* if $f(0) = 0$ and $f$ is the germ of a local diffeomorphism at $0$. (i.e., if $f \in m(n, n)$ and $df(0) \neq 0$.)

Note that $\epsilon(n)$ ($\epsilon_{e(o)}(n)$) is a vector space and also an abelian ring (except for $\epsilon_o(n)$). Hence, the following notation introduced here will be useful later on.

**Definition 2.1.4.** Let $a_1, a_2, \ldots, a_n$ be elements of an abelian ring $B$. the ideal generated by $\{a_1, \ldots, a_n\}$ will be denoted by $\langle a_1, \ldots, a_n \rangle_B$. If $a_1, a_2, \ldots, a_n$ are elements of a $K$-vector space, the $K$-linear span of $\{a_1, a_2, \ldots, a_n\}$ will be denoted by $\langle a_1, a_2, \ldots, a_n \rangle_K$.

**Proposition 2.1.1.** *For any* $n \in \mathbb{N}$, $\epsilon(n) = \epsilon_e(n) \oplus \epsilon_o(n)$.

*Proof.* Let $f \in \epsilon(n)$. $f$ can always be decomposed as

$$f(x) = f_e(x) + f_o(x), \quad \text{where} \quad f_e(x) = \left( \frac{f(x) + f(-x)}{2} \right) \in \epsilon_e(n)$$
$$\text{and} \quad f_o(x) = \left( \frac{f(x) - f(-x)}{2} \right) \in \epsilon_o(n).$$

Suppose that $f(x) = f_{e'}(x) + f_{o'}(x)$ is another decomposition of $f$ in terms of even and odd germs. Then, one has

$$f_e(x) - f_{e'}(x) = f_{o'}(x) - f_o(x).$$

But the left side is even and the right side is odd. Hence, $f_{e(o')} - f_{e'(o)}$ must be even and odd germs. However, this implies

$$(f_{e(o')} - f_{e'(o)})(x) = (f_{e(o')} - f_{e'(o)})(-x) = -(f_{e(o')} - f_{e'(o)})(x),$$

and hence, $(f_{e(o')} - f_{e'(o)})(x) = 0$. □

## 2.2 One variable germs

As already stated in the beginning of the chapter, the development and the application of catastrophe theory in this thesis will be restricted to the case of families of one variable germs. This section focuses on the study of this class of germs, in order to derive properties and definitions that will be useful later on. One of the main tools to analyze one variable germs is Taylor's theorem, since most of the important results follows from it:

**Theorem 2.2.1** (Taylor's theorem). *Let $f \in \epsilon(1)$. Then, for every $k \in \mathbb{N}$, there exists a germ $g \in \epsilon(1)$ such that*

$$f(x) = f(0) + \frac{df(0)}{dx}x + \cdots + \frac{1}{(k-1)!}\frac{d^{k-1}f(0)}{dx^{k-1}}x^{k-1} + g(x)x^k,$$
$$\textit{where} \;\; g(0) = \frac{1}{k!}\frac{d^k f(0)}{dx^k}.$$

The next lemma, a well-known result of single-variable calculus, leads to a useful corollary of Taylor's Theorem when the germ has a parity symmetry:

**Lemma 2.2.1.** *If $f \in \epsilon_{e(o)}(1)$, then $\frac{d^k f(0)}{dx^k} = 0$ for all odd (even) $k$.*

**Corollary 2.2.1.** *If $f \in \epsilon_e(1)$, then, for every $k \in \mathbb{N}$, there exists a germ $g \in \epsilon_e(1)$ such that*

$$f(x) = f(0) + \frac{d^2 f(0)}{dx^2}x^2 + \cdots + \frac{1}{(2(k-1))!}\frac{d^{2(k-1)}f(0)}{dx^{2(k-1)}}x^{2(k-1)} + g(x)x^{2k},$$
$$\textit{where} \;\; g(0) = \frac{1}{(2k)!}\frac{d^{2k}f(0)}{dx^{2k}},$$

*and if $f \in \epsilon_o(1)$, then, for every $k \in \mathbb{N}$, there exists a germ $g \in \epsilon_e(1)$ such that*

$$f(x) = \frac{df(0)}{dx}x + \frac{1}{3!}\frac{d^3 f(0)}{dx^3}x^3 + \cdots + \frac{1}{(2k-1)!}\frac{d^{2k-1}f(0)}{dx^{2k-1}}x^{2k-1} + g(x)x^{2k+1},$$
$$\textit{where} \;\; g(0) = \frac{1}{(2k+1)!}\frac{d^{2k+1}f(0)}{dx^{2k+1}}.$$

*Proof.* This follows directly from the application of lemma 2.2.1 and Taylor's theorem of even and odd germs. □

**Definition 2.2.1.** Let $f \in \epsilon(1)$. Denote by $j^k f$ the germ in $\epsilon(1)$ of the function

$$j^k f(x) \doteq f(0) + \frac{df(0)}{dx}x + \cdots + \frac{1}{k!}\frac{d^k f(0)}{dx^k}x^k.$$

That is, $j^k f$ is the germ of the truncated Taylor expansion of $f$ up to order $k$. $j^k f$ is also called the $k-$ *jet* of $f$.

**Definition 2.2.2.** Let $f, g \in \epsilon(1)$. $f$ and $g$ are said to be *equivalent* if there exists a germ diffeomorphism $\phi \in m(1)$ such that $f = g \circ \phi$.

**Proposition 2.2.1.** *Let $f, g \in \epsilon_e(1)$ or $f, g \in \epsilon_o(1)$ be two equivalent germs. Then, the germ diffeomorphism $\phi \in m(1)$ such that $f = g \circ \phi$ can be chosen to be odd.*

*Proof.* Let $\phi \in m(1)$ be a germ diffeomorphism such that $f = g \circ \phi$. Define the germ diffeomorphism $\tilde{\phi}$ given by

$$\tilde{\phi}(x) = \begin{cases} \phi(x), & \text{if } x \geq 0, \\ -\phi(-x), & \text{if } x < 0. \end{cases}$$

It is not hard to see that $\phi \in m_o(1)$ and that $f = g \circ \phi = g \circ \tilde{\phi}$, since $f$ and $g$ are either even or odd.

□

The next definition is of fundamental importance for the development of catastrophe theory:

**Definition 2.2.3.** Let $f \in m(1)$. The *determinacy of* $f$, denoted by $\sigma(f)$, is the smallest integer $k$ for which $f$ is equivalent to $j^k f$, when it exists. If $f$ is not equivalent to $j^k$ for any $k \in \mathbb{R}$, then define $\sigma(f) = \infty$.

**Theorem 2.2.2.** *Let $f \in \epsilon(1)$ be such that*

$$f(0) = \frac{df(0)}{dx} = \cdots = \frac{d^{k-1} f(0)}{dx^{k-1}} = 0, \quad \textit{and} \quad \frac{d^k f(0)}{dx^k} \neq 0.$$

*Then, $f$ is equivalent to $x^k$ if k is odd, or to $\pm x^k$ if k is even, where in the latter case the sign is that of $\frac{d^k f(0)}{dx^k}$.*

*Proof.* By Taylor's theorem, it follows that

$$f(x) = x^k g(x),$$

where $g \in \epsilon(1)$ and $g(0) = \frac{d^k f(0)}{dx^k} \neq 0$. Now, assume that $k$ is odd. Then, $g(x)^{\frac{1}{k}}$ well-defined and smooth (in a suitable neighborhood of 0). Define $\phi \in \epsilon(1)$ by

$$\phi(x) = x g(x)^{\frac{1}{k}}.$$

Note that $\phi(0) = 0$ and $\frac{d\phi(0)}{dx} = g(0)^{\frac{1}{k}} \neq 0$. Thus, $\phi$ is a germ diffeomorphism in $m(1)$. Moreover, note also that

$$f(x) = (\phi(x))^k.$$

Hence, $(f \circ \phi^{-1})(x) = x^k$.

If $k$ is even, $g(x)^{\frac{1}{k}}$ may not exist in $\mathbb{R}$ if $g(0)$ turns out to be negative. Hence, define

$$\phi(x) = x|g(x)|^{\frac{1}{k}}.$$

As in the previous case, $\phi(0) = 0$, and $\phi$ is smooth, since $|g(x)|^{\frac{1}{k}}$ well-defined and smooth in some suitable neighborhood of 0. Moreover, $\frac{d\phi(0)}{dx} = |g(0)|^{\frac{1}{k}} \neq 0$, and thus $\phi$ is also a germ diffeomorphism. However, now one has

$$f(x) = \pm(\phi(x))^k,$$

where the plus sign is for when $g(0) > 0$ and the minus sign is for when $g(0) < 0$. Therefore, it follows that

$$(f \circ \phi^{-1})(x) = \begin{cases} x^k, & \text{if } \frac{d^k f(0)}{dx^k} > 0, \\ -x^k, & \text{if } \frac{d^k f(0)}{dx^k} < 0. \end{cases}$$

□

**Corollary 2.2.2.** *Let $f \in m(1)$ be such that*

$$f(0) = \frac{df(0)}{dx} = \cdots = \frac{d^{k-1}f(0)}{dx^{k-1}} = 0, \quad \text{and} \quad \frac{d^k f(0)}{dx^k} \neq 0.$$

*Then, $\sigma(f) = k$.*

*Proof.* By theorem 2.2.2, both $f$ and $j^k f$ are equivalent to either $x^k$ or $-x^k$, and hence they are equivalent to each other. For any $k' < k$, $j^{k'} f$ will be zero, and hence it will not be equivalent to $f$. □

Theorem 2.2.2 and corollary 2.2.2 are the main results of this section. Note that they provide a straightforward way to find the determinacy of a one variable germ, as well as an equivalent polynomial of such germ. Their generalizations to several variables, however, are not trivial, and this is one of the reasons for restricting the theory only to the case of one variable germs.

The next set propositions will be relevant in a future section, for the discussion of the *transversality* of a family of germs.

**Proposition 2.2.2.** *Let $f \in m_{e(o)}(1)$. If $\sigma(f) < \infty$, then $\sigma(f)$ is even (odd).*

*Proof.* This proposition trivially follows from lemma 2.2.1 and corollary 2.2.2. □

**Lemma 2.2.2.** *$f \in m^k(1)$ if and only if $f(x) = g(x)x^k$ for some $g \in \epsilon(1)$.*

*Proof.* If $f \in m^k(1)$, then by Taylor's theorem $f(x) = g(x)x^k$ for some $g \in \epsilon(1)$. Now let $f(x) = g(x)x^k$. If $k = 1$, the lemma is trivial. Suppose $k > 1$. Then,

$$g(x)x^k = (g(x)x^{k-1})x = l(x)x,$$

where $l(x) = g(x)x^{k-1}$. Assuming the lemma is true for $k-1$, it follows that $\frac{d^{k-2}l(0)}{dx^{k-2}} = 0$. But the $(k-1)$-th derivative of $g(x)x^k$ at 0 is given by

$$\left.\frac{d^{k-1}(g(x)x^k)}{dx^{k-1}}\right|_0 = \left.\left(x\frac{d^{k-1}l(x)}{dx^{k-1}}\right)\right|_0 + (k-1)\frac{d^{k-2}l(0)}{dx^{k-2}} = 0.$$

Hence, $g(x)x^k \in m^k(1)$. □

**Proposition 2.2.3.** *Let $f \in m(1)$ with $\sigma(f) < \infty$. Then, $f\epsilon(1) = m^{\sigma(f)}(1)$.*

*Proof.* Let $k = \sigma(f)$. By Taylor's Theorem it follows that

$$f(x) = h(x)x^k,$$

where $h(x) \in \epsilon(1)$ and $h(0) \neq 0$. Hence, for any $s \in \epsilon(1)$,

$$f(x)s(x) = h(x)s(x)x^k = \tilde{s}(x)x^k,$$

where $\tilde{s}(x) \in \epsilon(1)$, and by lemma 2.2.2, $f(x)s(x) \in m^k(1)$. Now, let $g \in m^k(1)$. Then, $g(x) = \tilde{h}(x)x^k$ for some $\tilde{h} \in \epsilon(1)$. But since $h(0) \neq 0$, as said before, the germ $\frac{1}{h}$ is well-defined, and hence it follows that

$$g(x) = \frac{\tilde{h}(x)}{h(x)}h(x)x^k = \frac{\tilde{h}(x)}{h(x)}f(x) \in f\epsilon(1).$$

□

**Corollary 2.2.3.** *Let $f \in m_o(1)$ with $\sigma(f) < \infty$. Then, $f\epsilon_o(1) = m_e^{\sigma(f)+1}(1)$.*

*Proof.* Let $g \in f\epsilon_o(1)$ and $k = \sigma(f)$. Note that since $f \in m_o(1)$, $k$ is odd and $g$ is even. By proposition 2.2.3, $g \in m^k(1)$, and hence, by lemma 2.2.2, $g(x) = h(x)x^k$. But for $g$ to be even, $h$ must be odd, since $x^k$ is odd for odd $k$. Therefore, from corollary 2.2.1, $h(x) = s(x)x$, where $s \in \epsilon_e(1)$. Hence,

$$g(x) = s(x)xx^k = s(x)x^{k+1} \in m_e^{k+1}(1).$$

Now let $g \in m_e^{k+1}(1)$. Then, by proposition 2.2.3, $g(x) = f(x)h(x)$ for some $h \in \epsilon(1)$. But since $g$ is even and $f$ is odd, $h$ must be odd, and thus $g \in f\epsilon_o(1)$. □

**Proposition 2.2.4.** *For $k$ even, $\epsilon_e(1)/m_e^k(1) = \langle 1, x^2, \ldots, x^{k-2}\rangle_{\mathbb{R}}$*

*Proof.* Clearly any nonzero $f \in \langle 1, x^2, \ldots, x^{k-2}\rangle_{\mathbb{R}}$ is in $\epsilon_e(1)$ but not in $m_e^k(1)$. Now, let $f \in \epsilon_e(1)$. Then, by corollary 2.2.1,

$$f(x) = f(0) + \frac{d^2f(0)}{dx^2}x^2 + \cdots + \frac{1}{(k-2)!}\frac{d^{k-2}f(0)}{dx^{k-2}}x^{k-2} + g(x)x^k, \text{ where}$$

$$f(0) + \frac{d^2f(0)}{dx^2}x^2 + \cdots + \frac{1}{(k-1)!}\frac{d^{k-2}f(0)}{dx^{k-2}}x^{k-2} \in \langle 1, x^2, \ldots, x^{k-2}\rangle_{\mathbb{R}} \text{ and}$$

$$g(x)x^k \in m_e^k(1).$$

□

## 2.3 Unfoldings

In catastrophe theory, the parametric families of germs recieve the name of *unfoldings*. This section presents some basic definitions and results about them.

**Definition 2.3.1.** Given a germ $f \in \epsilon(n)$, an *$r$-parameter unfolding of $f$* is a germ $F \in \epsilon(n+r)$ such that $F(x,0) = f(x)$. The unfolding is said to be *even (odd)* if $x \mapsto F(x,u) \in \epsilon_e(n)$ for all $u$ in some neighborhood of $0 \in \mathbb{R}^r$. The space of all $r$-unfoldings will be detoned by $\mathcal{U}(n,r)$, and the space of all even (odd) $r$-unfoldings will be denoted by $\mathcal{U}_{e(o)}(n,r)$.

**Proposition 2.3.1.** $\mathcal{U}(n,r) = \mathcal{U}_e(n,r) \oplus \mathcal{U}_o(n,r)$.

*Proof.* Any $F \in \mathcal{U}(n,r)$ can be decomposed as

$$F(x,u) = F_e(x,u) + F_o(x,u), \quad \text{where } F_e(x,u) = \left(\frac{F(x,u)+F(-x,u)}{2}\right) \in \mathcal{U}_e(n,r)$$
$$\text{and } F_o(x,u) = \left(\frac{F(x,u)-F(-x,u)}{2}\right) \in \mathcal{U}_o(n,r)$$

By proposition 2.1.1, it follows that for each $u$, the above decomposition is unique. Hence, $\mathcal{U}(n,r) = \mathcal{U}_e(n,r) \oplus \mathcal{U}_o(n,r)$.

□

*Remark:* To be more consistent with the references mentioned in the text, in most part of the propositions here, the germ $f(x) = F(x,0)$ of an unfolding $F(x,u) \in \mathcal{U}(n,r)$ will usually be in $m(n)$.

**Definition 2.3.2.** Let $F$ be an $r$-unfolding of $f \in m(n)$ and $G$ a $d$-unfolding of $g \in m(n)$. $G$ is said to be *induced from* $F$ if there exists:

(a) a germ $\phi \in m(n+d,n)$, where $\phi_0(x) \doteq \phi(x,0)$ is a germ diffeomorphism,

(b) a germ $\psi \in m(d,r)$, and

(c) a germ $\gamma \in m(d)$,

such that

$$G(x,u) = F(\phi(x,u),\psi(u)) + \gamma(u).$$

Moreover, if $d = r$ and $\psi$ is a germ diffeomorphism, then $F$ and $G$ are said to be *equivalent*.

Note that for any germ $\phi \in m(n+d,n)$, if $\phi_0(x) \doteq \phi(x,0)$ is a germ diffeomorphism, then in a suitable neighborhood of $0 \in \mathbb{R}^d$, the functions $\phi_u(x) \doteq \phi(x,u)$ are also diffeomorphisms. Note also that, defining $G_u(x) \doteq G(x,u)$, $F_u(x) \doteq F(x,u)$, $\gamma_u \doteq \gamma(u)$, and assuming $G$ is equivalent to $F$, then for $u$ small there is a one-to-one correspondence between the members $G_u$ of the unfolding $G$ and the members $F_{\psi(u)}$ of the unfolding $F$. Moreover, for a fixed $u$, one has

$$G_u(x) = F_{\psi(u)}(\phi_u(x)) + \gamma_u. \tag{2.1}$$

If $\phi_u$ is a diffeomorphism, then by differentiating eq. 2.1, it is easy to see that any critical point of $G_u$ must match another critical point of $F_{\psi(u)}$, and vice-versa. Hence, one may study the critical points of the members $G_u$ of the unfolding $G$ by looking at the critical points of the functions in the unfolding $F$.

**Proposition 2.3.2.** *Let $F$ be an even $r$-unfolding of $f \in m_e(1)$, with $\sigma(f) < \infty$, and $G$ an even $d$-unfolding of $g \in m_e(1)$, where $G$ is induced from $F$. Then, the germ $\phi \in m(n+d, n)$ of definition 2.3.2 may be chosen such that, in a suitable neighborhood of $0 \in \mathbb{R}^d$, the germs of the functions $\phi_u(x) \doteq \phi(x, u)$ are odd in $x$.*

*Proof.* If $\sigma(f) = \sigma(F(\cdot, 0)) < \infty$, then there exists some $\delta > 0$ such that $F(x, 0)$ has only one critical point in $(-\delta, \delta)$, at $x = 0$. Let $\epsilon > 0$ be such that for some neighborhood $U$ of $0 \in \mathbb{R}^d$, $\phi((-\delta, \delta) \times U) \subset (-\epsilon, \epsilon)$, and suppose that for any neighborhood of $0 \in \mathbb{R}^d$ there is a point $u$ such that $\phi(0, u) \neq 0$. In that case, by the symmetry of $F$ and $G$, $\phi(0, u)$ must be a critical point of $F_{\psi(u)}(x)$. Let $W \subset U$ be a neighborhood of $0 \in \mathbb{R}^d$ such that for all $u \in W$, the critical points of $F_{\psi(u)}(x)$ are in $(-\epsilon, \epsilon)$. Since $F_{\psi(u)}(x)$ is even, the critical points are equally distributed around $0 \in (-\epsilon, \epsilon)$. But if there is some $\bar{u} \in W$ such that $\phi(0, \bar{u}) \neq 0$, then since $\phi(0, \bar{u})$ is a critical point of $F_{\psi(\bar{u})}(x)$, the function $F_{\psi(\bar{u})}(\phi_{\bar{u}}(x)))$ will not be symmetric in $(-\delta, \delta)$, since there will be a different number of critical points to the right and to the left of $\phi(0, \bar{u})$. Therefore, there exists a neighborhood $V$ of $0 \in \mathbb{R}^d$ such that $\phi(0, V) = 0$. Now, define

$$\tilde{\phi}(x, u) = \begin{cases} \phi(x, u), & \text{if } x \geq 0, \\ -\phi(-x, u), & \text{if } x < 0. \end{cases}$$

Clearly $\tilde{\phi}$ satisfies the requirements of the proposition in a suitable neighborhood $K \subset V$ of $0 \in \mathbb{R}^d$. □

**Definition 2.3.3.** Let $F$ be an $r$-unfolding of a germ $f$. If any other unfolding of $f$ is induced from $F$, $F$ is said to be *versal*. If, additionally, for any other versal $d$-unfolding of $f$, one has $d \geq r$, then $F$ is said to be *universal*.

These are the usual definitions of versal and universal unfoldings that appear in catastrophe theory. However, when restricting the attention to only even unfoldings of a germ, they become of little use since, for example, any $f \in m^3(1)$, even unfoldings of $f$ cannot be versal. Therefore, a more suitable definition for even unfoldings is the following:

**Definition 2.3.4.** Let $F$ be an even $r$-unfolding of a germ $f$. If any other even unfolding of $f$ is induced from $F$, $F$ will be called *e-versal*. If, additionally, for any other e-versal $d$-unfolding of $f$, one has $d \geq r$, then $F$ will be called *e-universal*.

**Proposition 2.3.3.** *Let $F_1, F_2, F_3$ be $r_1$-, $r_2$-, $r_3$-unfoldings, respectively, such that $F_1$ is induced from $F_2$ and $F_2$ is induced from $F_3$. Then, $F_1$ is induced from $F_3$.*

*Proof.* Let $\phi_i$, $\psi_i$ and $\gamma_i$, $i = 1, 2$, be the corresponding germs of definition 2.3.2, satisfying, for $u_i \in \mathbb{R}^{r_i}$,

$$F_i(x,u_i) = F_{i+1}(\phi_i(x,u_i)),\psi_i(u_i)) + \gamma_i(u_i), \quad i = 1,2.$$

Defining

$$\phi_3(x,u_1) = \phi_2(\phi_1(x,u_1),\psi_1(u_1)), \quad \psi_3(u_1) = \psi_2(\psi_1(u_1)), \text{ and}$$
$$\gamma_3(u_1) = \gamma_1(u_1) + \gamma_2(\psi(u_1)),$$

it follows that $\phi_3, \psi_3, \gamma_3$ satisfy the conditions (a), (b) and (c), of definition 2.3.2, with $d = r_1$ and $r = r_3$. Moreover, one has

$$F_1(x,u_1) = F_3(\phi_3(x,u_1),\psi_3(u_1)) + \gamma_3(u_1).$$

Hence, $F_1$ is induced from $F_3$. □

**Proposition 2.3.4.** *Let $F_1, F_2, F_3$ be $r$-unfoldings such that $F_1$ is equivalent to $F_2$ and $F_2$ equivalent to $F_3$. Then, $F_1$ equivalent to $F_3$.*

*Proof.* Defining $\phi_3, \psi_3$ and $\gamma_3$ as in proposition 2.3.3, then they satisfy the conditions (a), (b) and (c), of definition 2.3.2 and, since $\psi_1, \psi_2 \in m(1)$ are invertible, $\psi_3$ will be invertible in a suitable neighborhood of 0. Therefore, $F_1$ is equivalent to $F_3$. □

**Proposition 2.3.5.** *Let $F, G$ be two even unfoldings such that $G$ is e-versal and $G$ is induced from $F$. Then, $F$ is e-versal.*

*Proof.* Let $f$ be the germ that $F$ unfolds, and $g$ be the germ that $G$ unfolds. Moreover, let $\phi, \psi, \gamma$ be the germs of definition 2.3.2, satisfying

$$G(x,u) = F(\phi(x,u),\psi(u)) + \gamma(u).$$

Note that $g = f \circ \phi_0$, where $\phi_0(x) \doteq \phi(x,0)$. Consider another arbitrary even unfolding $\tilde{F}$ of $f$. Then, the germ $\tilde{G}(x,u) = \tilde{F}(\phi_0(x),u)$ is an even unfolding of $g$, and is clearly equivalent to $\tilde{F}$. But since $G$ is e-versal, $\tilde{G}$ is induced from $G$, that is induced from $F$. Hence, by proposition 2.3.3, $\tilde{F}$ is induced from $F$, and thus $F$ is e-versal. □

**Lemma 2.3.1.** *Let $f \in \mathcal{U}_e(n,r)$ be such that, for some $k \leq r$, $f$ satisfies*

$$f(x,0,\ldots,0,u_{k+1},\ldots,u_r) = 0,$$

*for all $x \in \mathbb{R}^n$ in some neighborhood of 0 and all $(u_{k+1},\ldots,u_r) \in \mathbb{R}^{r-k}$ in some neighborhood of 0. Then, there exists germs $g_1,\ldots,g_k \in \mathcal{U}_e(n,r)$ such that*

$$f(x,u) = \sum_{i=1}^{k} u_i g_i(x,u).$$

*Proof.* Note that

$$f(x,u) = \int_0^1 \frac{d}{dt}(f(x,tu_1,\ldots,tu_k,u_{k+1},\ldots,u_r))dt.$$

Computing the derivative on the right side of the equality, one has

$$\begin{aligned} f(x,u) &= \int_0^1 \sum_{i=1}^k \frac{\partial f}{\partial u_i}\bigg|_{(x,tu_1,\ldots,tu_k,u_{k+1},\ldots,u_r)} \cdot u_i dt \\ &= \sum_{i=1}^k \int_0^1 \frac{\partial f}{\partial u_i}\bigg|_{(x,tu_1,\ldots,tu_k,u_{k+1},\ldots,u_r)} dt \cdot u_i. \end{aligned}$$

For each $i = 1, \ldots, k$, define

$$g_i(x,u) \doteq \int_0^1 \frac{\partial f}{\partial u_i}\bigg|_{(x,tu_1,\ldots,tu_k,u_{k+1},\ldots,u_r)} dt.$$

Since $f \in \mathcal{U}_e(n,r)$, it follows that $g_i \in \mathcal{U}_e(n,r)$ for all $i = 1, \ldots, k$, and hence, the lemma follows. □

The following theorem, known as *Mather division theorem*, or also as *Malgrange preparation theorem*, is of fundamental importance for the proof of the central theorem in catastrophe theory regarding the equivalence of unfoldings, as it will be seen later in the text. Despite its simple appearance, this theorem is very deep and there is no known elementary way of proving it. Therefore, its proof is omitted here, but the reader may find it at ([9], chapter four).

**Theorem 2.3.1** (Mather division theorem)**.** *Let $F \in \mathcal{U}(1,r)$ be such that $\sigma(F_0) = \sigma(F(\cdot,0)) = k < \infty$. Then,*

$$\mathcal{U}(1,r) = F\mathcal{U}(1,r) + \langle 1, x, \ldots, x^{k-1}\rangle_{\epsilon(r)},$$

*where the functions on $\epsilon(r)$ do not depend on $x$.*

**Corollary 2.3.1.** *Let $F \in \mathcal{U}_o(1,r)$ be such that $\sigma(F(\cdot,0)) = 2k-1 < \infty$. Then,*

$$\mathcal{U}_e(1,r) = F\mathcal{U}_o(1,r) + \langle 1, x^2, \ldots, x^{2k-2}\rangle_{\epsilon(r)}.$$

*Proof.* Applying Mather division theorem to $F$ and decomposing the right side into even and odd parts in $x$, one has

$$\begin{aligned} \mathcal{U}(1,r) =& F\mathcal{U}_o(1,r) + \langle 1, x^2, \ldots, x^{2k-2}\rangle_{\epsilon(r)} + \\ & F\mathcal{U}_e(1,r) + \langle x, x^3, \ldots, x^{2k-1}\rangle_{\epsilon(r)}. \end{aligned}$$

Thus, since $\mathcal{U}(1,r) = \mathcal{U}_e(1,r) \oplus \mathcal{U}_o(1,r)$ by proposition 2.3.1, it follows that

$$\mathcal{U}_e(1,r) = F\mathcal{U}_o(1,r) + \langle 1, x^2, \ldots, x^{2k-2}\rangle_{\epsilon(r)}.$$

□

**Definition 2.3.5.** Let $F \in \mathcal{U}(n,r)$. Denote by $\alpha_i(F)$, $i = 1, \ldots, r$, the germs in $\epsilon(n)$ of the functions

$$\alpha_i(F)(x) \doteq \frac{\partial F(x,0)}{\partial u_i}.$$

Note that, for an even $r$-unfolding, since $x \mapsto F(x,u)$ is even for all $u$ in some neighborhood of $0 \in \mathbb{R}^r$, $\alpha_i(F)(x) = \frac{\partial F(x,0)}{\partial u_i}$ is also even.

## 2.4 Transversality

The following sections contain the main results of catastrophe theory that will be used for the analysis of the quantum system presented here, and all of them revolve around the concept of *transversality* of an unfolding. In short, catastrophe theory says that the *transversal* unfoldings – a special class of unfoldings – are all equivalent to certain polynomial unfoldings, and that every transversal unfolding is universal, and vice-versa. Moreover, it also says that the transversal unfoldings are *stable*, a very important concept that will be precisely defined in the next section. In this thesis, similar results are obtained regarding the even unfoldings.

**Definition 2.4.1.** Let $f \in m(1)$ and $F$ be an $r$-unfolding of $f$. $F$ is said to be *transversal* if

$$\frac{df}{dx}\epsilon(1) + \langle 1, \alpha_1(F), \ldots, \alpha_r(F)\rangle_{\mathbb{R}} = \epsilon(1)$$

Similar to the definitions of versal and universal unfoldings, this definition of transversality needs to be adapted when dealing only with even unfoldings. The following definition is more suitable for that purpose:

**Definition 2.4.2.** Let $f \in m_e(1)$ and $F$ be an even $r$-unfolding of $f$. $F$ will be called *e-transversal* if

$$\frac{df}{dx}\epsilon_o(1) + \langle 1, \alpha_1(F), \ldots, \alpha_r(F)\rangle_{\mathbb{R}} = \epsilon_e(1)$$

The goal now is to find an equivalent, but more useful, condition for the *e-tranversality* of an unfolding as defined above.

**Proposition 2.4.1.** *Let $f \in m_e(1)$ with $k = \sigma(f) < \infty$ and let $F$ be an even $r$-unfolding of $f$. Then, $F$ is e-transversal if and only if*

$$\langle 1, x^2, \ldots, x^{k-2}\rangle_{\mathbb{R}} = \langle 1, j^{k-2}(\alpha_1(F)), \ldots, j^{k-2}(\alpha_r(F))\rangle_{\mathbb{R}}.$$

*Proof.* If $\sigma(f) = k$, then by corollary 2.2.2 it is easy to see that $\sigma\left(\frac{df}{dx}\right) = k-1$. By proposition 2.2.3, $\frac{df}{dx}\epsilon_o(1) = m_e^k(1)$, and by proposition 2.2.4, $\epsilon_e(1)/m_e^k(1) = \langle 1, x^2, \ldots, x^{k-2}\rangle_{\mathbb{R}}$. Hence, the e-transversality condition is equivalent to

$$\langle 1, x^2, \ldots, x^{k-2}\rangle_{\mathbb{R}} \subset \langle 1, \alpha_1(F), \ldots, \alpha_r(F)\rangle_{\mathbb{R}}.$$

But note that, for even $k$, $j^k$ acting on $\epsilon_e(1)$ can be regarded as the projection operator of the subspace $\langle 1, x^2, \ldots, x^k\rangle_{\mathbb{R}}$. Thus, the above relation can be rephrased as

$$\begin{aligned}\langle 1, x^2, \ldots, x^{k-2}\rangle_{\mathbb{R}} =& j^{k-2}\left(\langle 1, \alpha_1(F), \ldots, \alpha_r(F)\rangle_{\mathbb{R}}\right)\\ =& \langle 1, j^{k-2}(\alpha_1(F)), \ldots, j^{k-2}(\alpha_r(F))\rangle_{\mathbb{R}}.\end{aligned}$$

□

**Corollary 2.4.1.** *Let $f \in m_e(1)$ be any germ with $k = \sigma(f) < \infty$. Then, the unfolding*

$$F(x,u) = f(x) + u_1 x^2 + u_2 x^4 + \cdots + u_r x^{k-2}$$

*is an e-transversal $r$-unfolding of $f$, with $r = (k-2)/2$.*

*Proof.* This is obvious from proposition 2.4.1. □

**Proposition 2.4.2.** *Let $f \in m_e(1)$ be any germ with $\sigma(f) < \infty$, $F$ an e-versal $r$-unfolding of $f$. Then, $F$ is e-transversal.*

*Proof.* Let $G$ be an e-transversal $d$-unfolding of $f$ (by corollary 2.4.1, such $G$ exists). Since $F$ is e-versal, then $G$ is induced by $F$; i,.e., there exists some germs $\phi \in m_o(1+d)$, $\psi \in m(d,r)$ and $\gamma \in m(d)$ such that

$$G(x,u) = F(\phi(x,u), \psi(u)) + \gamma(u).$$

Computing the derivative on both sides, one has

$$\alpha_i(G) = \frac{df}{dx}\frac{\partial \phi(x,0)}{\partial u_i} + \sum_{j=1}^{r} \alpha_j(F)\frac{\partial \psi(0)}{\partial u_i}, \quad \text{for all } i = 1,\ldots,d.$$

Note that the above equation, together with the parity of $\phi$, $F$ and $G$, imply

$$\langle 1, \alpha_1(G), \ldots, \alpha_d(G)\rangle_{\mathbb{R}} \subset \frac{df}{dx}\epsilon_o(1) + \langle 1, \alpha_1(F), \ldots, \alpha_r(F)\rangle_{\mathbb{R}} \subset \epsilon_e(1).$$

From the e-transversality of $G$ and since $\frac{df}{dx}\epsilon_o(1) + \frac{df}{dx}\epsilon_o(1) = \frac{df}{dx}\epsilon_o(1)$, it follows that

$$\epsilon_e(1) = \frac{df}{dx}\epsilon_o(1) + \langle 1, \alpha_1(G), \ldots, \alpha_d(G)\rangle_{\mathbb{R}} = \frac{df}{dx}\epsilon_o(1) + \langle 1, \alpha_1(F), \ldots, \alpha_r(F)\rangle_{\mathbb{R}}.$$

Hence, $F$ is e-transversal. □

**Definition 2.4.3.** Let $F, G \in \mathcal{U}_e(1, 1+r)$ be two e-transversal unfoldings of the same germ $f \in m_e(1)$. $F$ and $G$ are said to be *elementary homotopic* when, for all $s \in [0,1]$, the unfolding

$$H_s = sG + (1-s)F$$

is e-transversal. Moreover, $F$ and $G$ are said to be *homotopic* when there exists a finite sequence of even unfoldings $F = F_0, F_1, \ldots, F_{n-1}, F_n = G$ such that $F_{i-1}$ and $F_i$ are elementary homotopic for all $i = 1, \ldots, n$.

The proof of the next theorem, regarding the equivalence of e-transversal unfoldings, relies on the existence of certain germs satisfying a differential equation. Their existence are guaranteed by the following proposition, whose proof is contained in appendix A.

**Proposition 2.4.3.** *Let $F \in m(n+1)$. Suppose that there exist $\xi \in \epsilon(n+1, n)$ and $\eta \in \epsilon(n+2)$ such that for any $x \in \mathbb{R}^n$ near $0$ and any $t \in \mathbb{R}$ near $0$, the following equation holds:*

$$\frac{\partial F(x,t)}{\partial t} = \sum_{i=1}^{n} \frac{\partial F(x,t)}{\partial x_i}\xi_i(x,t) + \eta(F(x,t),x,t) \tag{2.2}$$

*Then, there exist $\phi \in \epsilon(n+1,n)$ and $\lambda \in \epsilon(n+2)$ such that for any $x \in \mathbb{R}^n$ near $0$ and any $t, s \in \mathbb{R}$ near $0$, the following holds:*

*(a)* $\phi(x,0) = x$ *and* $\lambda(s,x,0) = s$,

*(b)*

$$\frac{\partial \phi_i(x,t)}{\partial t} = -\epsilon_i(\phi(x,t),t) \text{ for all } i = 1,\dots,n, \text{ and}$$
$$\frac{\partial \lambda(s,x,t)}{\partial t} = \eta(\lambda(s,x,t),\phi(x,t),t),$$

*(c)* $F(\phi(x,t),t) = \lambda(F(x,0),x,t)$.

**Lemma 2.4.1.** *Let $f \in \epsilon(n+r+1,p)$ be such that $f(x,y,0)$ does not depend on $x$, and that $f$ satisfies*

$$\frac{\partial f_i}{\partial t}(x,y,t) = g(f(x,y,t),t) + h(x,y,t),$$

*where $g \in \epsilon(p+1,p)$, $h \in \epsilon(n+r+1,p)$, and $h$ does not depend on $x$. Then $f(x,y,t)$ does not depend on $x$.*

*Proof.* For all $x \in \mathbb{R}^n$, define $f_x \in \epsilon(r+1,p)$ by $f_x(y,t) = f(x,y,t)$. Then, it follows that for any $x \in \mathbb{R}^n$, $f_x(y,t)$ satisfies

$$\frac{\partial (f_x)_i}{\partial t}(y,t) = g(f_x(y,t),t) + h(0,y,t), \text{ and } f_x(y,0) = f(0,y,0).$$

But by the uniqueness of the solutions to the above equations, it follows that $f_x(y,t) = f_{x'}(y,t)$ for all $x, x' \in \mathbb{R}^n$. □

**Theorem 2.4.1.** *Let $f \in m_e(1)$ be any germ such that $\sigma(f) < \infty$, and $F, G \in \mathcal{U}_e(1,r)$ be e-transversal $r$-unfoldings of $f$. Then, $F$ and $G$ are equivalent.*

*Proof.* First, it shall be shown that it is enough to prove the theorem for the special case where $F$ and $G$ are elementary homotopic. Note that, for an e-transversal unfolding $F$ of $f$, one has

$$\langle 1, x^2, \dots, x^{2n} \rangle_{\mathbb{R}} = \langle 1, j^{2n}(\alpha_1(F)), \dots, j^{2n}(\alpha_r(F)) \rangle_{\mathbb{R}},$$

where $n = (\sigma(f) - 2)/2 \in \mathbb{N}$. The above equation means, in particular, that there are constants $a_0, \dots, a_r \in \mathbb{R}$ such that

$$x^2 = a_0 + \sum_{j=1}^{r} a_j j^{2n}(\alpha_j(F)). \tag{2.3}$$

Clearly, there exists some $j_1 > 0$ for which $a_{j_1} \neq 0$ in the above equation. Hence, define a new unfolding $F_1$ given by

$$F_1(x,u) = f(x) + u_1\alpha_1(F) + \cdots + u_{j_1}x^2 + \cdots + u_r\alpha_r(F),$$

if $a_{j_1} > 0$, or

$$F_1(x,u) = f(x) + u_1\alpha_1(F) + \cdots - u_{j_1}x^2 + \cdots + u_r\alpha_r(F),$$

if $a_{j_1} < 0$. Note that $F_1$ is also an even $r$-unfolding of $f$. Moreover, $F$ and $F_1$ are elementary homotopic. Indeed, for an unfolding $H_s = sF_1 + (1-s)F$ where $s \in [0,1]$, $j^{2n}(\alpha_j(H_s)) = j^{2n}(\alpha_j(F))$ for all $j \neq j_1$, and

$$j^{2n}(\alpha_{j_1}(H_s)) = sj^{2n}(\alpha_{j_1}(F)) + sgn(a_{j_1})(1-s)x^2.$$

By eq. 2.3, $j^{2n}(\alpha_{j_1}(F))$ can be expressed as

$$\begin{aligned} j^{2n}(\alpha_{j_1}(F)) =& \frac{j^{2n}(\alpha_{j_1}(H_s)) - sgn(a_{j_1})\left(a_0 + \sum_{\substack{j=1\\ j\neq j_1}}^{r} a_{j_1}j^{2n}(\alpha_j(F))\right)}{s+(1-s)|a_{j_1}|} \\ =& \frac{j^{2n}(\alpha_{j_1}(H_s)) - sgn(a_{j_1})\left(a_0 + \sum_{\substack{j=1\\ j\neq j_1}}^{r} a_{j_1}j^{2n}(\alpha_j(H_s))\right)}{s+(1-s)|a_{j_1}|}. \end{aligned}$$

Hence, $\langle 1, j^{2n}(\alpha_1(H_s)), \ldots, j^{2n}(\alpha_r(H_s))\rangle_{\mathbb{R}} = \langle 1, j^{2n}(\alpha_1(F)), \ldots, j^{2n}(\alpha_r(F))\rangle_{\mathbb{R}}$ and therefore it follows that $H_s$ is e-transversal. Moreover, by a diffeomorphic change of parameters $e \in \epsilon(r,r)$ given by

$$e(u_1, \ldots, u_{j_1}, \ldots, u_r) = (u_{j_1}, u_2, \ldots, sgn(a_{j_1})u_1, u_{j_1+1}, \ldots, u_r),$$

one can obtain another e-transversal even $r$-unfolding $\tilde{F}_1$ of $f$ given by

$$\begin{aligned} \tilde{F}_1(x,u) = F_1(x,e(u)) =& f(x) + u_1x^2 + u_2\alpha_2(F) + \cdots + u_{j_1}\alpha_1(F) + u_{j_1+1}\alpha_{j_1+1}(F) \\ & + \cdots + u_r\alpha_r(F), \end{aligned}$$

where $\tilde{F}_1$ is clearly equivalent to $F_1$. Now, consider an e-transversal $r$-unfolding $F_m$ of $f$, where $m < n$, such that $\alpha_j(F_m) = x^{2j}$ for $j \leq m$, and $\alpha_j(F_m) = \alpha_j(F)$ for $j > m$. Then, since $F_m$ is e-transversal, it follows that, similar to the above case, there are constants $a_0, \ldots, a_r \in \mathbb{R}$ such that

$$x^{2(m+1)} = a_0 + \sum_{j=1}^{m} a_jx^{2j} + \sum_{j=m+1}^{r} a_jj^{2n}(\alpha_j(F))$$

Again, clearly $a_{j'} \neq 0$ for some $j' > m$. Defining a new unfolding $F_{m+1}$ of $f$ by

$$\begin{aligned} F_{m+1}(x,u) =& f(x) + u_1x^2 + \cdots + u_mx^{2m} + u_{m+1}\alpha_{m+1}(F_m) + \cdots + sgn(a_{j'})u_{j'}x^{2(m+1)} \\ & + u_{j'+1}\alpha_{j'+1}(F_m) + \cdots + u_r\alpha_r(F_m), \end{aligned}$$

and proceeding in a similar way to the above discussion, it is not hard to see that $F_{m+1}$ is also e-transversal and elementary homotopic to $F_m$. Moreover, as before, by the germ diffeomorphism $e \in \epsilon(r,r)$

$$e(u_1,\dots,u_r) = (u_1,\dots,u_m,u_{j'},u_{m+2},\dots,u_{j'-1},sgn(a_{j'})u_{m+1},u_{j'+1},\dots,u_r),$$

one can find another unfolding $\tilde{F}_{m+1}$ of $f$ given by

$$\begin{aligned}\tilde{F}_{m+1}(x,u) = F_{m+1}(x,e(u)) =& f(x) + u_1x^2 + \cdots + u_{m+1}x^{2(m+1)} + u_{m+2}\alpha_{m+2}(F_m) \\ &+ \cdots + u_{j'}\alpha_{m+1}(F_m) + u_{j'+1}\alpha_{j'+1}(F_m) + \cdots + u_r\alpha_r(F_m),\end{aligned}$$

where $\tilde{F}_{m+1}$ is equivalent to $F_{m+1}$. By induction, it follows that there exists a finite sequence of e-transversal unfoldings of $f$, $F, F_1, \tilde{F}_n, \dots, F_n, \tilde{F}_n$, where every unfolding in the sequence is either equivalent or elementary homotopic to the next, and $\tilde{F}_n$ is given by

$$\tilde{F}_n(x,u) = f(x) + u_1x^2 + \cdots + u_nx^{2n} + h(x,u_{n+1},\dots,u_r)$$

By a straightforward calculation it can be shown that $\tilde{F}_n$ is elementary homotopic to the unfolding

$$Y(x,u) = f(x) + u_1x^2 + \cdots + u_nx^{2n}$$

Hence, for any transversal unfolding $F, G$ of $f$ with $r$ parameters, there exists a finite sequence of unfoldings $F, F_1, \tilde{F}_1, \dots, F_n, \tilde{F}_n, Y, \tilde{G}_n, G_n, \dots, \tilde{G}_1, G_1, G$ such that every unfolding in the sequence is either equivalent or elementary homotopic to the next. Hence, If one proves that elementary homotopic unfoldings are also equivalent, one proves that $F$ is equivalent to $G$.

The idea for the proof is to show that, when $F$ and $G$ are elementary homotopic, then for every $s \in [0,1]$, there is an open interval $I_s$ of $[0,1]$ with $s \in I_s$ such that for all $s' \in I_s$, the unfolding

$$H_{s'} = s'G + (1-s')F$$

is isomorphic to $H_s$. Assuming that this holds, then the union $\cup_{s\in[0,1]} I_s$ is an open cover of $[0,1]$. But since $[0,1]$ is compact, it follows that there exists a finite subcover $\cup_{i\in\{1,\dots,N\}} I_i$ of $[0,1]$. If discarding the intervals $I_i$ that are just subsets of other bigger intervals in the finite subcover, one can conveniently label them to satisfy $\inf\{I_i\} < \inf\{I_j\}$ if $i < j$, and obtain another finite subcover $\cup_{i\in\{1,\dots,M\}} I_i$. Is this case, the intervals also must satisfy $I_i \cap I_{i+1} \neq \varnothing$ for all $i = 1,\dots,M-1$. Let $s_i$ denote an element in $I_i \cap I_{i+1}$. Then, from the above remarks, it follows that $H_0, H_{s_1}, \dots, H_{s_{M-1}}, H_1$ are all equivalent. But since $H_0 = F$ and $H_1 = G$, this implies that $F$ and $G$ are equivalent. Hence, let $F, G$ be two elementary homotopic even unfoldings of $f$, and $s \in (0,1)$. Define

$$H(x,u,t) = (s+t)G + (1-s-t)F.$$

The goal is to show that for some sufficiently small $t \in \mathbb{R}$, $H(x,u,t)$ is equivalent to $H(x,u,0)$. Note that there exits a neighborhood $V$ of $0 \in \mathbb{R}$ such that if $t \in V$, $H(x,u,t)$

is e-transversal. Note also that $\frac{\partial H(x,u,t)}{\partial x} \in \mathcal{U}_o(1, r+1)$, and hence, defining $H_t(x,u) = H(x,u,t)$, by corollary 2.3.1 one has

$$\mathcal{U}_e(1, r+1) = \frac{\partial H_t}{\partial x}\mathcal{U}_o(1, r+1) + \langle 1, x^2 \ldots, x^{2n}\rangle_{\epsilon(r+1)}.$$

But if $t \in V$, $\langle 1, x^2 \ldots, x^{2n}\rangle_{\epsilon(r+1)} \subset \langle 1, \alpha_1(H_t) \ldots, \alpha_r(H_t)\rangle_{\epsilon(r+1)}$. Hence, for $t \in V$ one may write

$$\mathcal{U}_e(1, r+1) = \frac{\partial H_t}{\partial x}\mathcal{U}_o(1, r+1) + \langle 1, \alpha_1(H_t) \ldots, \alpha_r(H_t)\rangle_{\epsilon(r+1)}.$$

Multiplying both sides by $m(r)$ and taking the linear span, one finally arrives at

$$\begin{aligned}\langle m(r)\mathcal{U}_e(1, r+1)\rangle_{\mathbb{R}} = \left\langle m(r)\frac{\partial H_t}{\partial x}\mathcal{U}_o(1, r+1)\right\rangle_{\mathbb{R}} + \\ \langle m(r)\{1, \alpha_1(H_t) \ldots, \alpha_r(H_t)\}\epsilon(r+1)\rangle_{\mathbb{R}}.\end{aligned}$$

Now, note that $H(x,u,t) \in \mathcal{U}_e(1, r+1)$ and $\frac{\partial H}{\partial t}(x, 0, t) = \frac{\partial (f(x))}{\partial t} = 0$. Hence, by lemma 2.3.1, $\frac{\partial H}{\partial t} \in \langle m(r)\mathcal{U}_e(1, r+1)\rangle_{\mathbb{R}}$ and by the above equation, it follows that there exists some germ $\zeta \in \langle m(r)\mathcal{U}_o(1, r+1)\rangle_{\mathbb{R}}$ and some germs $\chi_0, \chi_1, \ldots, \chi_r \in \langle m(r)\epsilon(r+1)\rangle_{\mathbb{R}}$ such that

$$\frac{\partial H}{\partial t}(x,u,t) = \frac{\partial H}{\partial x}(x,u,t)\zeta(x,u,t) + \sum_{i=1}^{r}\frac{\partial H}{\partial u_i}(x,u,t)\chi_i(u,t) + \chi_0(u,t)$$

Applying proposition 2.4.3 for $H \in \epsilon(n+r+1)$, where $\xi(x,u,t) = (\zeta(x,u,t), \chi_1(u,t), \ldots, \chi_r(u,t))$ and $\eta(x,u,t) = \chi_0(u,t)$, it follows that there is a germ $\Phi = (\phi, \psi_1, \ldots, \psi_r) \in \epsilon(1+r+1, 1+r)$ and a germ $\lambda \in \epsilon(1+1+r+1)$ such that

(a)

$$\begin{aligned}\phi(x,u,0) &= x,\\ \psi(x,u,0) &= u,\\ \lambda(\tau,x,u,0) &= \tau,\end{aligned}$$

(b) $H(\Phi(x,u,t),t) = \lambda(H(x,u,0),x,u,t)$,

(c)

$$\begin{aligned}\frac{\partial \phi}{\partial t}(x,u,t) &= -\zeta(\phi(x,u,t), \psi(x,u,t), t),\\ \frac{\partial \psi_i}{\partial t}(x,u,t) &= -\chi_i(\psi(x,u,t), t),\\ \frac{\partial \lambda}{\partial t}(\tau,x,u,t) &= \chi_0(\psi(x,u,t), t).\end{aligned}$$

Note that items $(a)$ and $(c)$, together with lemma 2.4.1, imply that $\phi$ and $\lambda$ do not depend on $x$. Hence, from now on, the $x$ variable will be ommited in both functions, considering $\psi$ as a germ in $\epsilon(r+1,r)$ and $\lambda$ as a germ in $\epsilon(1+r+1)$. Note also that $\psi(0,t)$ satisfies

$$\frac{\partial \psi_i}{\partial t}(0,t) = -\chi_i(\psi(0,t),t), \quad \text{and} \quad \psi(0,0)=0.$$

But the function $g(t)=0 \in \epsilon(1,r)$ also satisfies the above equation, since $\chi_i \in \langle m(r)\epsilon(r+1)\rangle_{\mathbb{R}}$. Hence, by uniqueness, $\psi(0,t)=0$. Moreover, one also has

$$\begin{aligned}\frac{\partial \phi}{\partial t}(x,0,t) &= -\zeta(\phi((x,0,t),0,t) = 0, \quad \text{and}\\ \frac{\partial \lambda}{\partial t}(\tau,0,t) &= \chi_0(0,t) = 0,\end{aligned}$$

since $\zeta \in \langle m(r)\epsilon(1+r+1)\rangle_{\mathbb{R}}$ and $\chi_0 \in \langle m(r)\epsilon(r+1)\rangle_{\mathbb{R}}$. Hence, $\phi(x,0,t)=x$ and $\lambda(\tau,0,t)=\tau$.

From $(c)$, it follows that

$$\frac{\partial}{\partial t}\left(\frac{\partial \lambda}{\partial \tau}(\tau,u,t)\right) = 0, \quad \text{and} \quad \frac{\partial \lambda}{\partial \tau}(\tau,u,0) = 1$$

Therefore, $\frac{\partial \lambda}{\partial \tau}(\tau,u,t) = 1$ for all $t$, and hence $\lambda(\tau,u,t) = \tau + \alpha(u,t)$ for some $\alpha \in \epsilon(r+1)$, with $\alpha(0,t)=0$. Now, denoting $\phi_t(x,u) = \phi(x,u,t)$, $\psi_t(u) = \psi(u,t)$ and $\alpha_t(u) = \alpha(u,t)$, from $(b)$ one has

$$H(x,u,0) = H(\phi_t(x,u),\psi_t(u),t) - \alpha_t(u),$$

where $\phi_t(x,0)=x$, $\psi_t(0)=0$, $\alpha_t(0)=0$ and, from $(a)$, it follows that for any sufficiently small $t' \neq 0$, the mappings $x \mapsto \phi_{t'}(x,u)$ and $u \mapsto \psi_{t'}(u)$ are invertible. Hence, $H(x,u,0)$ is equivalent to $H(x,u,t')$. □

**Proposition 2.4.4.** *An even unfolding $F$ of a germ $f \in m_e(1)$ with $\sigma(f) < \infty$ is e-versal if and only if it is e-transversal.*

*Proof.* By proposition 2.4.2, any e-versal unfolding of a germ $f \in m_e(1)$ with $\sigma(f) < \infty$ is e-transversal. Hence, it suffices to prove the converse. Let $F$ be an e-transversal $r$-unfolding of a germ $f \in m_e(1)$ with $\sigma(f) < \infty$, and let $G$ be an arbitrary even $d$-unfolding of $f$. Define another two even $(r+d)$-unfoldings of $f$ by

$$H(x,u,v) = F(x,u) + G(x,v) - f(x), \text{ and } K(x,u,v) = F(x,u).$$

Clearly, $H$ and $K$ are e-transversal, as $F$ is e-transversal. Hence, by proposition 2.4.1, $H$ and $K$ are equivalent. Moreover, $G$ is induced from $H$, by the germs $\phi(x,u)=x$, $\psi(u,v)=u$, $\gamma(u,v)=0$, and $K$ is induced from $F$, by the germs $\phi(x,u)=x$, $\psi(v) = (0,v)$, $\gamma(v)=0$. Therefore, by proposition 2.3.3, $G$ is induced from $F$. □

**Corollary 2.4.2.** *An e-versal $r$-unfolding of a germ $f \in m_e(1)$, with $\sigma(f) = k < \infty$, is e-universal if and only if $r = (k-2)/2$.*

*Proof.* Clearly, the even $d$-unfolding $F$ of $f$ given by

$$F(x,u) = f + u_1x^2 + u_2x^4 + \cdots + u_dx^{k-2},$$

where $d = (k-2)/2$, is e-transversal and hence e-versal. Therefore, any e-universal $r$-unfolding of $f$ must satisfy $r \le (k-2)/2$. Let $G$ be an even unfolding of $f$ with $r < (k-2)/2$. Then $\langle 1, x^2, \ldots, x^{k-2}\rangle_{\mathbb{R}}$ will never be contained in $\langle 1, \alpha_1(G), \ldots, \alpha_r(G)\rangle_{\mathbb{R}}$, since the former has dimension $1+(k-2)/2$ and the latter has dimension $1+r < 1+(k-2)/2$. Hence, $G$ is not e-transversal and thus not e-versal.

□

**Theorem 2.4.2** (Thom's theorem for even unfoldings of one variable germs)**.** *Let $F$ be an e-transversal $r$-unfolding of a germ $f \in m_e(1)$, with $k = \sigma(f) < \infty$. Then, the even polynomial $d$-unfolding*

$$P(x,u) = \pm x^k + u_1x^2 + u_2x^4 + \cdots + u_dx^{k-2},$$

*where $d = (k-2)/2$ and the sign is that of $\frac{d^k f(0)}{dx}$, is induced from $F$. Moreover, if $r = d$, $F$ and $P$ are equivalent.*

*Proof.* Since $k = \sigma(f) < \infty$, by theorem 2.2.2 and proposition 2.2.1, there exists a germ diffeomorphism $\phi \in m_o(1)$ such that $\pm x^k = f(\phi(x))$, where the sign is that of $\frac{d^k f(0)}{dx}$. Let $\tilde{F}(x,u) = F(\phi(x,u),u)$. Then, $\tilde{F}$ is clearly an even unfolding of $\pm x^k$. Moreover, since it is equivalent to $F$, by proposition 2.3.3 $\tilde{F}$ is e-transversal, and hence by proposition 2.4.1 $\tilde{F}$ is e-versal. Therefore, $P$ is induced from $\tilde{F}$. If $r = d$, since $P$ and $\tilde{F}$ are e-transversal and unfold the same germ, by theorem 2.4.1 $P$ is equivalent to $\tilde{F}$, and hence to $F$. □

**Corollary 2.4.3.** *Let $F$ and $G$ be two e-universal $r$-unfoldings of germs in $m_e(1)$ with finite determinacy. Then, $F$ and $G$ are equivalent.*

*Proof.* By corollary 2.4.2, both $F$ and $G$ are even unfoldings of germs with determinacy equals to $2r+2$. Hence by Thom's theorem, they both are equivalent to the same polynomial unfolding, and hence they are equivalent to each other. □

## 2.5 Stability of unfoldings

The last important property of e-transversal unfoldings that will be presented here is their *e-stability*, i.e., a small even perturbation added to an e-transversal unfolding does not break its e-transversality, and the perturbed unfolding is equivalent to the non-perturbed. In order to properly define the concept of stability for unfoldings, it is necessary to generalize the previous definition of equivalence of unfoldings.

**Definition 2.5.1.** Let $U, V$ be open subsets of $\mathbb{R}^{n+r}$, and $F : U \to \mathbb{R}$, $G : V \to \mathbb{R}$ two smooth functions. Then, *$G$ at $(x,u)$ is equivalent to $F$ at $(y,v)$* if, for some neighborhood $U_1$ of $\mathbb{R}^n$ and some neighborhood $U_2$ of $\mathbb{R}^r$, such that $U_1 \times U_2 \subset U$, there exist smooth functions $\phi : U_1 \times U_2 :\to \mathbb{R}^n$, $\psi : U_2 \to \mathbb{R}^r$ and $\gamma : U_2 \to \mathbb{R}$, satisfying

(a) $(\phi(x',u'), \psi(u')) \in V$ for all $(x',u') \in U_1 \times U_2$, with $(\phi(x,u), \psi(u)) = (y,v)$,

(b) $\phi_u(x') \doteq \phi(x', u)$ and $\psi(u')$ are diffeomorphisms,

such that, for all $(x', u') \in U_1 \times U_2$, one has

$$G(x', u') = F(\phi(x', u'), \psi(u')) + \gamma(u').$$

Also, in order to proceed with the definition of stability, one must choose a convenient topology in $C^\infty(U, \mathbb{R})$. For the situation presented here, a good topology to work with is the topology associated with the following norm:

**Definition 2.5.2.** Let $U$ be an open subset of $\mathbb{R}^{1+r}$, let $\mathbf{v} = (x, u_1, \dots, u_r)$, and for some $(r+1)$-index $\alpha = (\alpha_0, \dots, \alpha_r)$, let

$$\frac{\partial^{|\alpha|}}{\partial \mathbf{v}^\alpha} \doteq \frac{\partial^{|\alpha|}}{\partial x^{\alpha_0} \partial {u_1}^{\alpha_1} \dots \partial {u_r}^{\alpha_r}}.$$

For any $n \in \mathbb{N}$, define the norm $\| \cdot \|_n$ in $C^\infty(U, \mathbb{R})$ by

$$\|F\|_n \doteq \sum_{|\alpha| \le n} \frac{1}{\alpha!} \sup_{(x,u) \in U} \left| \frac{\partial^{|\alpha|} F(x, u)}{\partial \mathbf{v}^\alpha} \right|.$$

**Definition 2.5.3.** Let $F \in \mathcal{U}_e(1, r)$. $F$ is said to be *e-stable* if, for every neighborhood $U$ of 0 in $\mathbb{R}^{1+r}$ and every representative $F'$ of $F$ defined on $U$, there exists some $n \in \mathbb{N}$ and some neighborhood $V$ of 0 in $C^\infty(U, \mathbb{R})$, with respect to the norm $\| \cdot \|_n$, such that for all functions $G' \in V$ where $x \mapsto G'(x, u)$ is even for $u \in U$, there is a point $(0, u) \in U$ such that $F'$ at $(0, 0)$ is equivalent to $F' + G'$ at $(0, u)$.

The following two results are required for the proof of the main theorem of this section.

**Theorem 2.5.1** (Brouwer fixed-point theorem)**.** *Every continuous function from a closed ball in $\mathbb{R}^n$ to itself has a fixed point.*

**Proposition 2.5.1.** *Let $\{u_1, \dots, u_n\}$ be an orthonormal basis of a Hilbert space $H$, and $\{v_1, \dots, v_n\}$ a set of vectors in $H$ such that*

$$\sum_{i=1}^n \|u_i - v_i\| < 1.$$

*Then, $\{v_1, \dots, v_n\}$ is a basis for $H$.*

*Proof.* Let $f \in H$, and define $\rho \doteq \sum_{i=1}^n \|u_i - v_i\| < 1$. Since $\{u_1, \dots, u_n\}$ is an orthonormal basis, then $f = \sum_{i=1}^n c_i u_i$, where $c_i = (f, u_i)$. Define $f_1 = \sum_{i=1}^n c_i v_i$. Then, one has

$$\|f - f_1\| = \left\| \sum_{i=1}^n c_i (u_i - v_i) \right\| \le \sum_{i=1}^n |c_i| \, \|u_i - v_i\| = \sum_{i=1}^n |(f, u_i)| \, \|u_i - v_i\| \, .$$

By Cauchy-Schwarz inequality, $|(f, u_i)| \le \|f\| \|u_i\| = \|f\|$ for all $i = 1, \dots, n$. Hence, it follows that

$$\|f - f_1\| \leq \|f\| \sum_{i=1}^{n} \|u_i - v_i\| = \rho\|f\|.$$

Proceeding in exactly the same way, one may find some $f_2 \in H$, also spanned by the $v_i's$, such that

$$\|f - f_1 - f_2\| \leq \rho\|f - f_1\| \leq \rho^2\|f\|.$$

Therefore, by induction, one may construct a sequence $\{\tilde{f}_n\}$ of elements spanned by the $v_i's$ satisfying, for all $n \in N$,

$$\|f - \tilde{f}_n\| \leq \rho^n\|f\|.$$

Hence, since $\rho < 1$, $\{\tilde{f}\}$ converges to $f$, and since any finite-dimensional subspace of a Hilbert space is closed, it follows that $f$ is also spanned by the $v_i's$. Thus, $\{v_1, \ldots, v_n\}$ is a basis for $H$. □

**Theorem 2.5.2.** *Every e-universal unfolding of a germ $f \in m_e(1)$ with $\sigma(f) < \infty$ is e-stable.*

*Proof.* Let $U$ be an open neighborhood of $0 \in \mathbb{R}^{1+r}$, $F : U \to \mathbb{R}$ be a representative of an e-universal $r$-unfolding of a germ $f \in m_e(1)$, with $k = \sigma(f) = 2r + 2 < \infty$, and $G \in C^\infty(U, \mathbb{R})$ an even smooth function. By Thom's theorem, there exist some germs $\phi, \psi, \gamma$ such that in a neighborhood $U_1 \subset U$ of 0, one has

$$\tilde{F}(x, u) \doteq F(\phi(x, u), \psi(u)) + \gamma(u) = x^{2r+2} + u_1 x^2 + \cdots + u_r x^{2r}.$$

Let $\tilde{G}(x, u) = G(\phi(x, u), \psi(u))$, and define $\tilde{H}(x, u) \doteq \tilde{F}(x, u) + \tilde{G}(x, u) - \tilde{G}(0, u)$. Expanding $\tilde{G}(x, u)$ into its Taylor series for $x = 0$, one has

$$\tilde{H}(x, u) = K(x, u)x^{2r+4} + (1 - g_{r+1}(u))x^{2r+2} + (u_1 - g_1(u))x^2 + \cdots + (u_r - g_r(u))x^{2r},$$

where $g_i(u) = -\dfrac{\partial^{2i}\tilde{G}(0, u)}{\partial x^{2i}}$.

let $g$ be given by $g(u) \doteq (g_1(u), \ldots, g_r(u))$, let $W_1 \in \mathbb{R}$, $W_2 \in \mathbb{R}^r$, with $0 \in W_1 \times W_2 \subset U_1$, and let $B_2 \subset W_2$ be a closed ball centered at 0. Note that, since $g_i$ depends on the derivatives of $G$ up to order $2i$, then taking $n = 2(r + 1) + 1$, it follows that there exists a neighborhood $V$ of 0 in $C^\infty(U, \mathbb{R})$ with respect to the norm $\|\cdot\|_n$ such that, for any even $G \in V$, one has

(a) $g(B_2) \subset B_2$,

(b) $|1 - g_{r+1}(u)| > 0$ for all $u \in B_2$, and

(c)
$$\sum_{i=1}^{r} \left\| \left( \frac{\partial g_1(u)}{\partial u_i}, \ldots, \frac{\partial g_r(u)}{\partial u_i} \right) \right\| < 1 \text{ for all } u \in B_2.$$

Now, assume that $G \in V$. From item (a), one may apply the Brouwer fixed-point theorem for $g$. Thus, there exists a point $\overline{u} \in B_2$ such that $g(\overline{u}) = \overline{u}$. Define the diffeomorphism $\overline{\psi}(u) \doteq u + \overline{u}$, and define $\tilde{H}_1(x,u) \doteq \tilde{H}(x, \overline{\psi}(u))$. Clearly $\tilde{H}$ at $(0,\overline{u})$ is equivalent to $\tilde{H}_1$ at $(0,0)$. Moreover, $j^n(\tilde{H}_1(x,0)) = 0$ if $n < 2r$, and $j^{2r+2}(\tilde{H}_1(x,0)) = 1 - g_{r+1}(\overline{u})$. Thus, by item (b), $\tilde{H}_1$ can be regarded as an even unfolding of a germ with determinacy $k = 2r + 2$. Now, define an inner product on $\langle 1, x^2, \ldots, x^{2r} \rangle_{\mathbb{R}}$ as the sesquilinear form defined in the basis $\{1, x^2, \ldots, x^{2r}\}$ by

$$(x^i, x^j) = \delta_{i,j}, \quad \text{for all } i, j = 0, 2, \ldots, 2r.$$

Since $j^{2r}(\alpha_i(\tilde{H}_1))$ is given by

$$j^{2r}(\alpha_i(\tilde{H}_1))(x) = x^{2i} - \frac{\partial g_1(\overline{u})}{\partial u_i} x^2 - \cdots - \frac{\partial g_r(\overline{u})}{\partial u_i} x^{2r}, \quad \text{for all } i = 1, \ldots, r,$$

item (c) implies

$$\sum_{i=1}^{r} \| x^{2i} - j^{2r}(\alpha_i(\tilde{H})) \| < 1,$$

and by proposition 2.5.1, it follows that

$$\langle 1, j^{2r}(\alpha_1(\tilde{H}_1)), \ldots, j^{2r}(\alpha_r(\tilde{H}_1)) \rangle_{\mathbb{R}} = \langle 1, x^2, \ldots, x^{2r} \rangle_{\mathbb{R}}.$$

Hence, $\tilde{H}_1$ is e-universal as an $r$-unfolding and by corollary 2.4.3, $\tilde{H}_1$ is equivalent to $F$. From this, it follows that $F + G$ at $(0, \overline{u})$ (which is equivalent to $\tilde{H}$ at $(0, \overline{u})$) is equivalent to $F$ at $(0, 0)$.

□

# Chapter 3

# Fermi systems on Lattices

This chapter is devoted to showing the existence of thermodynamics for the quantum system analyzed here, and it is based on the formalism developed in [4]. The system considered here is modeled on a CAR algebra over an infinite lattice, where each point of the lattice is supposed to represent a vacant orbital – with a two-fold spin degeneracy – of the atoms in the crystal lattice of a solid material. Throughout the chapter is is shown that, for a suitable set of interactions and a suitable set of states, the thermodynamic behavior of the system can be extracted.

**Definition 3.0.1.** Let $\mathbb{Z}^d$ denote the $d$-dimensional cubic lattice and let $\mathfrak{h}$ be the Hilbert space given by $\mathfrak{h} \doteq l^2(\mathbb{Z}^d) \otimes \mathfrak{H}_S$, where $\mathfrak{H}_S \cong \mathbb{C}^2$ denotes the spin space, with basis $S = \{\uparrow, \downarrow\}$. From now on, $\mathfrak{A}$ will denote the CAR algebra associated with $\mathfrak{h}$. Note that a convenient orthonormal basis for $l^2(\mathbb{Z}^d)$ is given by $\{f_x \mid x \in \mathbb{Z}^d\}$, where

$$f_x(y) \doteq \begin{cases} 1, & \text{if } y = x \\ 0, & \text{otherwise.} \end{cases}$$

To simplify the notation, the generators $a(f_x \otimes s)$ of the $C^*$-algebra $\mathfrak{A}$, where $x \in \mathbb{Z}^d$ and $s \in S = \{\uparrow, \downarrow\}$, will be denoted by $a_{x,s}$.

*Remark:* It is important to note that all of the formalism developed throughout this chapter can be easily generalized for any spin set $S$, as long as it is finite.

**Definition 3.0.2.** Let $\mathcal{P}_f(\mathbb{Z}^d)$ be the set of all finite subsets of $\mathbb{Z}^d$, and let $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$. The CAR algebra associated with $\mathfrak{h}_\Lambda \doteq l^2(\Lambda) \otimes \mathfrak{H}_S$ will be denoted by $\mathfrak{A}_\Lambda$. Moreover, for all $l \in \mathbb{N}$, define

$$\Lambda_l \doteq \{x \in \mathbb{Z}^d \mid |x_1|, \ldots, |x_d| \leq l\} \in \mathcal{P}_f(\mathbb{Z}^d),$$

and denote by $\mathfrak{A}_0$ the *-algebra given by

$$\mathfrak{A}_0 \doteq \bigcup_{l=1}^{\infty} \mathfrak{A}_{\Lambda_l} = \bigcup_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)} \mathfrak{A}_\Lambda.$$

Note that for any $l \in \mathbb{N}$, the algebra $\mathfrak{A}_{\Lambda_l}$ is finite-dimensional, and $\mathfrak{A}_0$ is the smallest *-algebra generated by the elements $\{a_{x,s} \mid x \in \mathbb{Z}^d, s \in S\}$. Moreover, as already discussed in the proof of theorem 1.5.1, $\mathfrak{A}_0$ is dense in $\mathfrak{A}$.

## 3.1 States of Fermi systems on lattices

For an infinite-dimensional CAR algebra, such as the algebra $\mathfrak{A}$ studied here, some important quantities with physical meaning (e.g., the entropy density) are not well-defined for all states in $E_{\mathfrak{A}}$. Therefore, in this case one must restrict the thermodynamical analysis the system to a convenient subset of states, where the relevant thermodynamical variables can be calculated. When the interactions in the model are translation invariant, a convenient subset of states is the set of *translation invariant states*, that will be presented in this section. Moreover, since the interactions studied here are also permutation invariant in the so-called strong-coupling limit, it will also be important to analyze the set of *permutation invariant states*.

**Lemma 3.1.1.** *Let $\mathfrak{H}$ be a separable Hilbert space with basis $V = \{e_1, \dots, e_n, e_{n+1}, \dots\}$, $\mathfrak{A}(\mathfrak{H})$ the CAR algebra associated with $\mathfrak{H}$ and $\tau_1, \tau_2$ two *-automorphisms of $\mathfrak{A}$. If $\tau_1 = \tau_2$ on $a(V)$, then $\tau_1 = \tau_2$ on $\mathfrak{A}(\mathfrak{H})$.*

*Proof.* If $\mathfrak{H}$ is finite-dimensional, by the anti-linearity of the mapping $f \mapsto a(f)$, the lemma is obvious. If $\mathfrak{H}$ is infinite-dimensional, define

$$\mathfrak{A}_0(\mathfrak{H}) = \bigcup_{n=1}^{\infty} \mathfrak{A}(\mathfrak{H}_n),$$

where $\mathfrak{H}_n$ is the finite-dimensional Hilbert space with basis $\{e_1, \dots, e_n\}$. By the anti-linearity of $f \mapsto a(f)$, any element is $\mathfrak{A}_0(\mathfrak{H})$ can be written as a finite linear combination of $a(e_i)$, $e_i \in V$, and hence $\tau_1(a(f)) = \tau_2(a(f))$ on $\mathfrak{A}_0(\mathfrak{H})$. But $\mathfrak{A}(\mathfrak{H}) = \overline{\mathfrak{A}_0}(\mathfrak{H})$, and the continuity of $\tau_1, \tau_2$ (see proposition 1.2.1) implies that $\tau_1 = \tau_2$ on $\mathfrak{A}(\mathfrak{H})$. □

*Remark*: Let $U$ be an unitary operator acting on $\mathfrak{h}$, and let $\tilde{\mathfrak{A}}$ be the $C^*$-algebra generated by the operators $\{\tilde{a}(f) \mid f \in \mathfrak{h}\}$, where

$$\tilde{a}_y(f) \doteq a(Uf) \text{ for all } f \in \mathfrak{h}.$$

Since $U$ is unitary, it is not hard to see that $\tilde{\mathfrak{A}} = \mathfrak{A}$ and that the operators $\tilde{a}(f)$ satisfy the CARs. By theorem 1.5.1, it follows that there exists a unique *-automorphism $\tau$ on $\mathfrak{A}$ such that

$$\tau(a(f)) = a(Uf) \text{ for all } f \in \mathfrak{h}.$$

Moreover, by lemma 3.1.1, if $V$ is a basis for $\mathfrak{h}$, $\tau$ is defined by its action on $a(V)$.

Let $(\mathbb{Z}^d, +)$ be the group of the $d$-dimensional cubic lattice under the addition operation. $(\mathbb{Z}^d, +)$ acts on $\mathfrak{h} = l^2(\mathbb{Z}^d) \otimes \mathfrak{H}_S$ via the unitary operators $\{U_y \mid y \in \mathbb{Z}^d\}$, defined as

$$U_y f(x) \otimes s = f(x-y) \otimes s, \quad \text{for all } f \in l^2(\mathbb{Z}^d) \text{ and } s \in S.$$

Since $U_y$ acts on the basis elements $f_x \otimes s$ of $\mathfrak{h}$ as $U_y f_x \otimes s = f_{x+y} \otimes s$ for all $x, y \in \mathbb{Z}^d$ and $s \in S$, there is a unique *-automorphism $\tau_y$ on $\mathfrak{A}$ satisfying

$$\tau_y(a_{x,s}) = a_{x+y,s}, \quad \text{for all } x \in \mathbb{Z}^d \text{ and } s \in S.$$

Moreover, for any $x, y, z \in \mathbb{Z}^d$ and $s \in S$,

$$\tau_z \circ \tau_y(a_{x,s}) = a_{x+y+z,s} = \tau_{z+y}(a_{x,s}).$$

Thus, the family of *-automorphisms $\{\tau_x \mid x \in \mathbb{Z}^d\}$ over $\mathfrak{A}$ is a representation of the group $(\mathbb{Z}^d, +)$.

**Definition 3.1.1.** Let $(\mathbb{Z}^d, +)$ be the group of the $d$-dimensional cubic lattice under the addition operation. $(\mathbb{Z}^d, +)$ acts on $\mathfrak{A}$ via the family of *-automorphisms $\{\tau_x \mid x \in \mathbb{Z}^d\}$, where

$$\tau_x(a_{y,s}) = a_{x+y,s}, \quad \text{for all } y \in \mathbb{Z}^d \text{ and } s \in \{\uparrow, \downarrow\}.$$

The set of all states $\omega \in E_{\mathfrak{A}}$ satisfying

$$\omega(\tau_x(A)) = \omega(A) \;\text{ for all } x \in \mathbb{Z}^{\mathrm{d}} \text{ and } A \in \mathfrak{A} \tag{3.1}$$

will be denoted by $E_{\mathbb{Z}^d}$. The states $\omega \in E_{\mathbb{Z}^d}$ are said to be *translation invariant (t.i.)*. More generally, for $l \in \mathbb{N}$, the set of all states $\omega \in E_{\mathfrak{A}}$ satisfying

$$\omega(\tau_{l \cdot x}(A)) = \omega(A) \;\text{ for all } x \in \mathbb{Z}^{\mathrm{d}} \text{ and } A \in \mathfrak{A},$$

where $l \cdot x = (lx_1, \ldots, lx_d)$, will be denoted by $E_{l.\mathbb{Z}^d}$, and the states $\omega \in E_{l.\mathbb{Z}^d}$ are said to be *$l$-periodic*.

Given $\phi \in [0, 2\pi)$, let $U_\phi$ be the unitary operator acting on $\mathfrak{h}$ satisfying $U_\phi f = e^{-i\phi} f$ for all $f \in \mathfrak{h}$. It follows that there exists a unique *-automorphism $\sigma_\phi$ on $\mathfrak{A}$ such that

$$\sigma_\phi(a(f)) = e^{-i\phi} a(f) \;\text{ for all } f \in \mathfrak{h}.$$

An important case is when $\phi = \pi$, as it will be seen later. Note that, in particular, one has $\sigma_\pi \circ \sigma_\pi = \mathrm{id}$.

**Definition 3.1.2.** Let $\sigma_\pi$ be the unique *-automorphism of $\mathfrak{A}$ satisfying

$$\sigma_\pi(a(f)) = -a(f) \;\text{ for all } f \in \mathfrak{h},$$

and let $\mathfrak{A}^+$, $\mathfrak{A}^-$ be given by

$$\mathfrak{A}^+ \doteq \{A \in \mathfrak{A} \mid \sigma_\pi(A) = A\}, \quad \mathfrak{A}^- \doteq \{A \in \mathfrak{A} \mid \sigma_\pi(A) = -A\}.$$

Any element $A \in \mathfrak{A}^+$ is said to be *even*, and any element $A \in \mathfrak{A}^-$ is said to be *odd*. Moreover, the set of all states $\omega \in E_{\mathfrak{A}}$ satisfying

$$\omega(\sigma_\pi(A)) = \omega(A) \;\text{ for all } A \in \mathfrak{A}$$

will be denoted by $E_+$. The states $\omega \in E_+$ are said to be *even*.

*Remark*: Note that for any $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$, if $A \in \mathfrak{A}_\Lambda$, then $\sigma_\pi(A) \in \mathfrak{A}_\Lambda$. Therefore, the notions of evenness and oddness are also well-defined for a state $\omega \in E_{\mathfrak{A}_\Lambda}$.

**Proposition 3.1.1.** $\mathfrak{A} = \mathfrak{A}^+ \oplus \mathfrak{A}^-$. *Moreover,* $\mathfrak{A}_0^{+(-)} \doteq \mathfrak{A}_0 \cap \mathfrak{A}^{+(-)}$ *is dense in* $\mathfrak{A}^{+(-)}$, $\mathfrak{A}^+$ *is a $C^*$-sub-algebra of* $\mathfrak{A}$, $\mathfrak{A}^+.\mathfrak{A}^- \subset \mathfrak{A}^-$, $\mathfrak{A}^-.\mathfrak{A}^+ \subset \mathfrak{A}^-$ *and* $\mathfrak{A}^-.\mathfrak{A}^- \subset \mathfrak{A}^+$.

*Proof.* Clearly $\mathfrak{A}^+, \mathfrak{A}^-$ are vector spaces, with $\mathfrak{A}^+ \cap \mathfrak{A}^- = \{0\}$. For any element $A \in \mathfrak{A}$, define

$$A^+ = \frac{A + \sigma_\pi(A)}{2}, \qquad A^- = \frac{A - \sigma_\pi(A)}{2}.$$

Clearly $A = A^+ + A^-$, and since $\sigma_\pi \circ \sigma_\pi = \mathrm{id}$, it follows that $A^+ \in \mathfrak{A}^+$ and $A^- \in \mathfrak{A}^-$. Hence, $\mathfrak{A} = \mathfrak{A}^+ \oplus \mathfrak{A}^-$.

Let $A \in \mathfrak{A}^+$. Given $\epsilon > 0$, there exists some element $A_\epsilon \in \mathfrak{A}_0$ such that $\|A_\epsilon - A\| < \epsilon$. Then, for $A_\epsilon^+ = (A_\epsilon + \sigma_\pi(A_\epsilon))/2 \in \mathfrak{A}_0^+$, one has

$$\|A_\epsilon^+ - A\| = \left\| \frac{1}{2}(A_\epsilon - A) + \frac{1}{2}\sigma_\pi(A_\epsilon - A) \right\| \leq \frac{1}{2}(\|A_\epsilon - A\| + \|\sigma_\pi(A_\epsilon - A)\|) < \epsilon.$$

Hence, $\mathfrak{A}_0^+$ is dense in $\mathfrak{A}^+$. An analogous calculation shows that $\mathfrak{A}_0^-$ is dense in $\mathfrak{A}^-$.

Since $\sigma_\pi$ is a *-automorphism, it is easy to see that $\mathfrak{A}^+$ is a *-sub-algebra of $\mathfrak{A}$, and since any *-automorphism over a $C^*$-algebra is automatically continuous, it is also easy to see that $\mathfrak{A}^+$ is closed. The other statements of the proposition are trivial. □

Note that, since any element $A \in \mathfrak{A}_0$ is a sum of polynomials of $\{a_{x,s}, a^*_{x,s} \mid x \in \Lambda, s \in S\}$, then it is not hard to see that $A \in \mathfrak{A}_0^+ \iff A$ is a sum of *even* polynomials of $\{a_{x,s}, a^*_{x,s} \mid x \in \Lambda, s \in S\}$, and $A \in \mathfrak{A}_0^- \iff A$ is a sum of *odd* polynomials of $\{a_{x,s}, a^*_{x,s} \mid x \in \Lambda, s \in S\}$.

**Proposition 3.1.2.** *If $A \in \mathfrak{A}^+$, then*

$$\lim_{|x|\to\infty} [A, \tau_x(B)] = 0 \ \textit{for all } B \in \mathfrak{A},$$

*and if $A, \in \mathfrak{A}^-$, then*

$$\lim_{|x|\to\infty} \{A, \tau_x(B)\} = 0 \ \textit{for all } B \in \mathfrak{A}^-.$$

*Proof.* Let $\Lambda, \Lambda' \in \mathcal{P}_f(\mathbb{Z}^d)$ be such that $\Lambda \cap \Lambda' = \varnothing$. If $A \in \mathfrak{A}_\Lambda^+ \doteq \mathfrak{A}^+ \cap \mathfrak{A}_\Lambda$, then $A$ is a linear combination of even polynomials of the local generators $\{a_{x,s}, a^*_{x,s} \mid x \in \Lambda, s \in S\}$, and from the CARs it is easy to see that $A$ commutes with any $a_{x,s}, a^*_{x,s}$ when $x \in \Lambda'$. Hence, it commutes with every element of $\mathfrak{A}_{\Lambda'}$. Similarly, if $A \in \mathfrak{A}_\Lambda^- \doteq \mathfrak{A}^- \cap \mathfrak{A}_\Lambda$, then $A$ is a linear combination of odd polynomials of the local generators $\{a_{x,s}, a^*_{x,s} \mid x \in \Lambda, s \in S\}$, and from the CARs it is easy to see that any odd polynomial in $\mathfrak{A}_\Lambda^-$ anti-commutes with any odd polynomial in $\mathfrak{A}_{\Lambda'}^-$. Therefore, by linearity, $A$ anti-commutes with every element of $\mathfrak{A}_{\Lambda'}^-$.

Now, let $A \in \mathfrak{A}^+$, $B \in \mathfrak{A}$ and $\epsilon > 0$. By proposition 3.1.1, for some $l \in \mathbb{N}$, there exists some $A_l \in \mathfrak{A}_{\Lambda_l}^+$ and some $B_l \in \mathfrak{A}_{\Lambda_l}$ such that

$$\|R_{A_l}\| = \|A - A_l\| < \min\left\{\frac{\epsilon}{\|A\|}, \frac{\epsilon}{\|B\|}, \epsilon, 1\right\}, \quad \text{and}$$
$$\|R_{B_l}\| = \|B - B_l\| < \min\left\{\frac{\epsilon}{\|A\|}, \frac{\epsilon}{\|B\|}, \epsilon, 1\right\}.$$

For any $x \in \mathbb{Z}^d$ with $|x| > l\sqrt{d}$, $\Lambda_l \cap \Lambda_{x+l} = \varnothing$ and hence $A_l$ commutes with $\tau_x(B_l)$. Moreover, since $\tau_x$ is always isometric, it is not hard to see that for any $x \in \mathbb{Z}^d$ with $|x| > l\sqrt{d}$,

$$\|[A, \tau_x(B)]\| = \|[A_l, \tau_x(B_l)] + [R_{A_l}, \tau_x(B)] + [A, \tau_x(R_{B_l})] - [R_{A_l}, \tau_x(R_{B_l})]\| < \epsilon.$$

Hence,

$$\lim_{|x|\to\infty} [A, \tau_x(B)] = 0.$$

Now, let $A, B \in \mathfrak{A}^-$ and $\epsilon > 0$. Similarly, for some $l \in \mathbb{N}$, there exists some $A_l \in \mathfrak{A}^-_{\Lambda_l}$ and some $B_l \in \mathfrak{A}^-_{\Lambda_l}$ such that

$$\begin{aligned}\|R_{A_l}\| &= \|A - A_l\| < \min\left\{\frac{\epsilon}{\|A\|}, \frac{\epsilon}{\|B\|}, \epsilon, 1\right\}, \quad \text{and} \\ \|R_{B_l}\| &= \|B - B_l\| < \min\left\{\frac{\epsilon}{\|A\|}, \frac{\epsilon}{\|B\|}, \epsilon, 1\right\}.\end{aligned}$$

Therefore, for any $x \in \mathbb{Z}^d$ with $|x| > l\sqrt{d}$, $\{A_l, \tau_x(B_l)\} = 0$. Thus, for any $x \in \mathbb{Z}^d$ with $|x| > l\sqrt{d}$,

$$\|\{A, \tau_x(B)\}\| = \|\{A_l, \tau_x(B_l)\} + \{R_{A_l}, \tau_x(B)\} + \{A, \tau_x(R_{B_l})\} - \{R_{A_l}, \tau_x(R_{B_l})\}\| < \epsilon.$$

Hence,

$$\lim_{|x|\to\infty} \{A, \tau_x(B)\} = 0.$$

□

**Proposition 3.1.3.** *The group $(\mathbb{Z}^d, +)$ is amenable, and the sequence $\{\Lambda_l\}$ is a Følner sequence of $(\mathbb{Z}^d, +)$, where $\Lambda_l$ are the sets defined in definition 3.0.2.*

*Proof.* Let $y = (y_1, \dots, y_d) \in \mathbb{Z}^d$. Note that, for any $l \in \mathbb{N}$, one has

$$x \in (y.\Lambda_l) \cap \Lambda_l \iff x_i \in [l - y_i, l + y_i] \cap [-l, l] \quad \text{for all } i = 1, \dots, d.$$

It is not hard to see that the above relation defines a box in $\mathbb{Z}^d$, with

$$|(y.\Lambda_l) \cap \Lambda_l| = \prod_{i=1}^{d} \max\{2l + 1 - y_i, 0\}.$$

Assuming $l$ large enough such that $2l + 1 - y_i > 0$ for all $i = 1, \dots, d$ and denoting $\tilde{y} = \max_{i=1,\dots,d}\{y_i\}$, it follows that

$$|(y.\Lambda_l) \cap \Lambda_l| \geq (2l + 1 - \tilde{y})^d.$$

Hence,

$$\frac{|(y.\Lambda_l)\Delta\Lambda_l|}{|\Lambda_l|} = \frac{|(y.\Lambda_l)| + |\Lambda_l| - 2|(y.\Lambda_l)\cap\Lambda_l|}{|\Lambda_l|} \leq \frac{(2l+1)^d + (2l+1)^d - 2(2l+1-\tilde{y})^d}{(2l+1)^d}$$
$$= 2 - 2(1-\alpha_l)^d, \quad \text{where } \alpha_l = \frac{\tilde{y}}{2l+1}.$$

Since $\alpha_l \to 0$ as $l \to \infty$, clearly

$$\lim_{l\to\infty} \frac{|(y.\Lambda_l)\Delta\Lambda_l|}{|\Lambda_l|} = 0.$$

Thus, $\{\Lambda_l\}$ is a Følner sequence of $(\mathbb{Z}^d, +)$, and by proposition 1.4.10, $(\mathbb{Z}^d, +)$ is amenable. □

**Proposition 3.1.4.** *For any t.i. state $\omega \in E_{\mathbb{Z}^d}$, the pair $(\mathfrak{A}, \omega)$ is $(\mathbb{Z}^d, +)$-abelian.*

*Proof.* By proposition 1.4.10, in the strong operator topology of $\mathfrak{B}(\mathfrak{H}_\omega)$, one has

$$E_\omega = \lim_{l\to\infty} \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l} U_x,$$

and since $\sum_{x\in\Lambda_l} U_x = \sum_{x\in\Lambda_l} U_{-x} = \sum_{x\in\Lambda_l} U_x^{-1}$, for any $A \in \mathfrak{A}^+$ and any $B \in \mathfrak{A}$, it follows that, in the strong operator topology,

$$[E_\omega\pi_\omega(A)E_\omega, E_\omega\pi_\omega(B)E_\omega] = E_\omega\pi_\omega(A)E_\omega\pi_\omega(B)E_\omega - E_\omega\pi_\omega(B)E_\omega\pi_\omega(A)E_\omega =$$
$$\lim_{l\to\infty} \frac{1}{|\Lambda_l|} \left( E_\omega\pi_\omega(A) \sum_{x\in\Lambda_l} U_x\pi_\omega(B)E_\omega - E_\omega\pi_\omega(B) \sum_{x\in\Lambda_l} U_x\pi_\omega(A)E_\omega \right) =$$
$$\lim_{l\to\infty} \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l} E_\omega \left(\pi_\omega(A)U_x\pi_\omega(B)U_x^{-1} - U_x\pi_\omega(B)U_x^{-1}\pi_\omega(A)\right) E_\omega =$$
$$\lim_{l\to\infty} \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l} E_\omega\pi_\omega([A, \tau_x(B)])E_\omega = \lim_{l\to\infty} E_\omega\pi_\omega \left( \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l} [A, \tau_x(B)] \right) E_\omega.$$

Given $\epsilon > 0$, let $L \in \mathbb{N}$ be such that for all $x \in \mathbb{Z}^d$ with $|x| > L$,

$$\|[A, \tau_x(B)]\| < \frac{\epsilon}{2},$$

and let $l' > L$ be such that

$$\frac{1}{|\Lambda_{l'}|} \sum_{|x|\leq L} \|[A, \tau_x(B)]\| < \frac{\epsilon}{2}.$$

Then, for all $l > l'$,

$$\left\| \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l} [A, \tau_x(B)] \right\| \leq \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l \mid |x|\leq L} \|[A, \tau_x(B)]\| + \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l \mid |x|> L} \|[A, \tau_x(B)]\| < \epsilon.$$

Hence, one has

$$\begin{aligned}[E_\omega\pi_\omega(A)E_\omega, E_\omega\pi_\omega(B)E_\omega] =& \lim_{l\to\infty} \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l} E_\omega\pi_\omega([A, \tau_x(B)])E_\omega \\ =& E_\omega\pi_\omega \left( \lim_{l\to\infty} \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l} [A, \tau_x(B)] \right) E_\omega = 0. \end{aligned} \tag{3.2}$$

Similarly, for any $A \in \mathfrak{A}^-$, one may show that

$$\begin{aligned}\{E_\omega\pi_\omega(A^*)E_\omega, E_\omega\pi_\omega(A)E_\omega\} =& \lim_{l\to\infty} \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l} E_\omega\pi_\omega(\{A^*, \tau_x(A)\})E_\omega \\ =& E_\omega\pi_\omega \left( \lim_{l\to\infty} \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l} \{A^*, \tau_x(A)\} \right) E_\omega = 0.\end{aligned}$$

Clearly, the above equation implies

$$(E_\omega\pi_\omega(A)E_\omega)^*(E_\omega\pi_\omega(A)E_\omega) = -(E_\omega\pi_\omega(A)E_\omega)(E_\omega\pi_\omega(A)E_\omega)^* \tag{3.3}$$

Note that the element in the l.h.s of eq. 3.3 is positive, while the element in the r.h.s is the negative of a positive element. Hence, by proposition 1.1.10,

$$(E_\omega\pi_\omega(A)E_\omega)^*(E_\omega\pi_\omega(A)E_\omega) = 0,$$

and by the $C^*$-norm property, it follows that

$$\|E_\omega\pi_\omega(A)E_\omega\|^2 = \|(E_\omega\pi_\omega(A)E_\omega)^*(E_\omega\pi_\omega(A)E_\omega)\| = 0 \implies E_\omega\pi_\omega(A)E_\omega = 0. \tag{3.4}$$

This, together with eq. 3.2, implies that $E_\omega\pi_\omega(\mathfrak{A})E_\omega$ is abelian, and hence, by proposition 1.4.8, $(\mathfrak{A}, \omega)$ is $(\mathbb{Z}^d, +)$-abelian.

□

**Proposition 3.1.5.** *Any t.i. state is also even.*

*Proof.* Let $\omega$ be a t.i. state. Note that eq. 3.4 in the proof of proposition 3.1.4 implies that, for any $A^- \in \mathfrak{A}^-$,

$$\omega(A) = (\Omega_\omega, \pi_\omega(A^-)\Omega_\omega) = (\Omega_\omega, E_\omega\pi_\omega(A^-)E_\omega\Omega_\omega) = 0.$$

Hence, for any $A \in \mathfrak{A}$,

$$\omega(A) = \omega(A^+) + \omega(A^-) = \omega(A^+), \quad \text{and}$$
$$\omega(\sigma_\pi(A)) = \omega(A^+) - \omega(A^-) = \omega(A^+) \implies \omega(A) = \omega(\sigma_\pi(A)).$$

□

**Definition 3.1.3.** Let $A \in \mathfrak{A}$, and for any $l \in \mathbb{N}$, define

$$\bar{A}_l \doteq \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} \tau_x(A).$$

The functional $\Delta_A : E_{\mathbb{Z}^d} \to \mathbb{R}$, given by

$$\Delta_A(\omega) \doteq \lim_{l \to \infty} \omega(\bar{A}_l^* \bar{A}_l) = \|E_\omega \pi_\omega(A)\Omega_\omega\|^2,$$

is called the *space-averaging functional associated with* $A$. To simplify the notation, the element $\bar{A}_l^* \bar{A}_l$ will also be denoted by $|\bar{A}|_l^2$.

**Corollary 3.1.1.** *A t.i. state* $\omega \in E_{\mathbb{Z}^d}$ *is* $(\mathbb{Z}^d, +)$*-ergodic if, and only if,*

$$\Delta_A(\omega) = |\omega(A)|^2 \ \textit{for all } A \in \mathfrak{A}.$$

*Proof.* By propositions 3.1.3 and 3.1.4, $\{\Lambda_l\}$ is a Følner sequence of $\mathbb{Z}^d$ and the pair $(\mathfrak{A}, \omega)$ is $(\mathbb{Z}^d, +)$-abelian. Therefore, this corollary is a consequence of proposition 1.4.12.
□

**Proposition 3.1.6.** *For any* $A \in \mathfrak{A}$, $\Delta_A : E_{\mathbb{Z}^d} \to \mathbb{R}$ *is affine and weak* upper semi-continuous. Moreover, for fixed* $\omega \in E_{\mathbb{Z}^d}$, $\Delta_A(\omega)$ *is continuous on* $A$ *and satisfies*

$$|\Delta_A(\omega) - \Delta_B(\omega)| \leq (\|A\| + \|B\|)\|A - B\|, \ \textit{for all } A, B \in \mathfrak{A}.$$

*Proof.* The affinity of $\Delta_A$ follows directly from its definition and from the convexity of the set $E_{\mathbb{Z}^d}$. Now, note that for any $l \in \mathbb{N}$,

$$\omega(\bar{A}_l^* \bar{A}_l) = \|\pi_\omega(\bar{A}_l)\Omega_\omega\|^2 = \left\| \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} U_x \pi_\omega(A)\Omega_\omega \right\|^2.$$

Therefore, for any $l \in \mathbb{N}$, one has

$$\lim_{l' \to \infty} \omega(\bar{A}_{l'}^* \bar{A}_{l'}) = \|E_\omega \pi_\omega(A)\Omega_\omega\|^2 = \left\| E_\omega \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} U_x \pi_\omega(A)\Omega_\omega \right\|^2$$
$$\leq \left\| \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} U_g \pi_\omega(A)\Omega_\omega \right\|^2 = \omega(\bar{A}_l^* \bar{A}_l),$$

and hence, this implies

$$\Delta_A(\omega) = \lim_{l \to \infty} \omega(\bar{A}_l^* \bar{A}_l) = \inf_{l \in \mathbb{N}} \omega(\bar{A}_l^* \bar{A}_l).$$

But since $\omega \mapsto \omega(\bar{A}_l^* \bar{A}_l)$ is clearly weak* continuous for any $A \in \mathfrak{A}$ and any $l \in \mathbb{N}$, it follows that $\Delta_A$ is weak* upper semi-continuous. For the last statement, first note that for any $l \in \mathbb{N}$ and any $A \in \mathfrak{A}$,

$$\|\bar{A}_l\| = \left\| \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} \tau_x(A) \right\| \leq \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} \|\tau_x(A)\| = \|A\|.$$

Let $A, B \in \mathfrak{A}$, $l \in \mathbb{N}$ and $\omega \in E_{\mathbb{Z}^d}$. Then, one has

$$\begin{aligned} |\omega(\bar{A}_l^* \bar{A}_l) - \omega(\bar{B}_l^* \bar{B}_l)| \leq & \|\bar{A}_l^* \bar{A}_l - \bar{B}_l^* \bar{B}_l\| = \|\bar{A}_l^* \bar{A}_l - \bar{A}_l^* \bar{B}_l + \bar{A}_l^* \bar{B}_l - \bar{B}_l^* \bar{B}_l\| \\ \leq & \|\bar{A}_l^*\| \|(\overline{A-B})_l\| + \|\bar{B}_l\| \|(\overline{A-B})_l^*\| \\ \leq & (\|A\| + \|B\|) \|A - B\|. \end{aligned}$$

Hence, the statement follows. □

Let $\Pi(\mathbb{Z}^d)$ be the set of all bijections $\pi : \mathbb{Z}^d \to \mathbb{Z}^d$ that leave invariant all but finitely many elements of $\mathbb{Z}^d$. $\Pi(\mathbb{Z}^d)$ is a group under the composition operation, and it is usually called the *permutation group of* $\mathbb{Z}^d$. It also acts on $\mathfrak{h}$, via the family of unitary operators $\{U_\pi \mid \pi \in \Pi(\mathbb{Z}^d\}$ given by

$$U_\pi f(x) \otimes s = f \circ \pi^{-1}(x) \otimes s, \quad \text{for all } x \in \mathbb{Z} \text{ and } s \in S.$$

Hence, for any $\pi \in \Pi(\mathbb{Z}^d)$, there exists a unique *-automorphism $\alpha_\pi$ on $\mathfrak{A}$ satisfying

$$\alpha_\pi(a_{x,s}) = a_{\pi(x),s} \ \text{ for all } x \in \mathbb{Z}^d \text{ and } s \in \{\uparrow, \downarrow\}.$$

It is not hard to see that the family of *-automorphisms $\{\alpha_\pi \mid \pi \in \Pi(\mathbb{Z}^d)\}$ is a representation of $\Pi(\mathbb{Z}^d)$.

**Definition 3.1.4.** Let $\Pi(\mathbb{Z}^d)$ be the group of all bijections $\pi : \mathbb{Z}^d \to \mathbb{Z}^d$ that leave invariant all but finitely many elements of $\mathbb{Z}^d$, under the composition operation. $\Pi(\mathbb{Z}^d)$ acts on $\mathfrak{A}$ via the family of *-automorphisms $\{\alpha_\pi \mid \pi \in \Pi(\mathbb{Z}^d)\}$, where

$$\alpha_\pi(a_{x,s}) = a_{\pi(x),s} \ \text{ for all } x \in \mathbb{Z}^d \text{ and } s \in \{\uparrow, \downarrow\}.$$

The set of all states $\omega$ satisfying

$$\omega(\alpha_\pi(A)) = \omega(A) \ \text{ for all } \pi \in \Pi(\mathbb{Z}^d) \text{ and } A \in \mathfrak{A}$$

will be denoted by $E_\Pi$. The states $\omega \in E_\Pi$ are said to be *permutation invariant (p.i.)*.

**Proposition 3.1.7.** *For any $x \in \mathbb{Z}^d$, there exists a sequence $\{\pi_{x,n}\}$ of elements $\pi_{x,n} \in \Pi(\mathbb{Z}^d)$ such that for all $A \in \mathfrak{A}$,*

$$\lim_{n \to \infty} \|\alpha_{\pi_{x,n}}(A) - \tau_x(A)\| = 0. \tag{3.5}$$

*Proof.* Given $x \in \mathbb{Z}^d$ and $l \in \mathbb{N}$, consider the sets $\Lambda_l \cap (x + \Lambda_l)^c$ and $(x + \Lambda_l) \cap \Lambda_l^c$. Note that both of them are finite and have the same number of elements. Hence, let

$s_{x,l} : (x + \Lambda_l) \cap \Lambda_l^c \to \Lambda_l \cap (x + \Lambda_l)^c$ be some arbitrary bijective function, and define $\pi_{x,l} : \mathbb{Z}^d \to \mathbb{Z}^d$ by

$$\pi_{x,l}(y) = \begin{cases} y + x, \text{ if } y \in \Lambda_l, \\ s_{x,l}(y), \text{ if } y \in (x + \Lambda_l) \cap \Lambda_l^c, \\ y, \text{ otherwise.} \end{cases}$$

It is not hard to see that $\pi_{x,l} \in \Pi(\mathbb{Z}^d)$. Moreover, $\alpha_{\pi_{x,l'}}(A_l) = \tau_x(A_l)$ for all $l' \geq l$. Given $\epsilon > 0$, let $l \in \mathbb{N}$ be such that

$$\|A_l - A\| < \frac{\epsilon}{2}.$$

Then, for all $l' \geq l$,

$$\|\alpha_{\pi_{x,l'}}(A) - \tau_x(A)\| = \|\alpha_{\pi_{x,l'}}(A) - \alpha_{\pi_{x,l'}}(A_l) + \tau_x(A_l) - \tau_x(A)\| \leq 2\|A_l - A\| < \epsilon.$$

□

**Corollary 3.1.2.** *Every p.i. state is also t.i. Moreover, every pair* $(\mathfrak{A}, \omega)$*, where* $\omega$ *is a p.i. state, is* $\Pi(\mathbb{Z}^d)$*-abelian.*

*Proof.* This follows directly from propositions 3.1.7 and 1.4.7 □

**Theorem 3.1.1** (Størmer's Theorem)**.** *Let* $E_\otimes$ *be the set of all even states* $\omega \in E_{\mathfrak{A}}^{\sigma_\pi}$ *satisfying*

$$\omega(\tau_{x_1}(A_1) \dots \tau_{x_n}(A_n)) = \omega(A_1) \dots \omega(A_n)$$

*for all* $A_1, \dots, A_n \in \mathfrak{A}_{\{0\}}$ *and all* $x_1, \dots, x_n \in \mathbb{Z}^d$ *different from each other. Then,* $\mathcal{E}_\Pi = E_\otimes \subset \mathcal{E}_{\mathbb{Z}^d}$.

*Proof.* Note that, since $(\mathfrak{A}, \omega)$ is $\Pi(\mathbb{Z}^d)$-abelian for any $\omega \in E_\Pi$, theorem 1.4.2 and proposition 1.4.9 imply that showing $E_\otimes = \mathcal{E}_\Pi$ is equivalent to showing that any $\omega \in E_\otimes$ is $\Pi(\mathbb{Z}^d)$-invariant and satisfies $(c)$ of proposition 1.4.9, and vice-versa.

Let $A \in \mathfrak{A}_0$. Then, it follows from the CARs that $A$ can be written as

$$A = \sum_{i=1}^{M_A} \prod_{j=1}^{N_i} \tau_{x_{i,j}}(A_{i,j})$$

for $M_A, \in \mathbb{N}$, where $x_{i,j} \in \mathbb{Z}^d$, $x_{i,1}, \dots, x_{i,N_i}$ are different from each other for all $i = 1, \dots, M_A$, and $A_{i,j} = a_{0,\uparrow},\ a_{0,\downarrow},\ a_{0,\uparrow}^*,\ a_{0,\downarrow}^*$ or some nonzero binomial of these terms. Note that $A_{i,j} \in \mathfrak{A}_{\{0\}}$. Let $s \in \Pi(\mathbb{Z}^d)$. Then, by the injectivity of $s$, $s(x_{i,1}), \dots, s(x_{i,N_i})$ are also different from each other for all $i = 1, \dots, M_A$. Therefore, for any $\omega \in E_\otimes$, one has

$$\omega(\alpha_s(A)) = \sum_{i=1}^{M_A} \omega\left(\prod_{j=1}^{N_A} \tau_{s(x_{i,j})}(A_{i,j})\right) = \sum_{i=1}^{M_A} \prod_{j=1}^{N_A} \omega(A_{i,j}) = \omega(A).$$

Since $\mathfrak{A}_0$ is dense in $\mathfrak{A}$, by continuity it follows that $\omega(\alpha_s(A)) = \omega(A)$ for all $A \in \mathfrak{A}$, and thus $\omega \in E_\Pi$. Let also $B \in \mathfrak{A}_0$ be given by

$$B = \sum_{i=1}^{M_B} \prod_{j=1}^{N_B} \tau_{y_{i,j}}(B_{i,j}),$$

where $y_{i,j} \in \mathbb{Z}^d$ are different from each other and $B_{i,j} = a_{0,\uparrow},\ a_{0,\downarrow},\ a^*_{0,\uparrow},\ a^*_{0,\downarrow}$ or some nonzero binomial of these terms, and let $s \in \Pi(\mathbb{Z}^d)$ be such that for all $i, j$, $s(y_{i,j})$ is different from $x_{1,1}, x_{1,2}, \dots, x_{M_A,N_{M_A}}$. Then, for any $\omega \in E_\otimes$, one has

$$\begin{aligned}\omega(A\alpha_s(B)) &= \sum_{i=1,i'=1}^{M_A,M_B} \omega\left(\prod_{j=1}^{N_A} \tau_{x_{i,j}}(A_{i,j}) \prod_{j'=1}^{N_B} \tau_{s(y_{i',j'})}(B_{i',j'})\right) = \sum_{i=1,i'=1}^{M_A,M_B} \prod_{j=1}^{N_A} \omega(A_{i,j}) \prod_{j'=1}^{N_B} \omega(B_{i',j'}) \\ &= \left(\sum_{i=1}^{M_A} \prod_{j=1}^{N_A} \omega(A_{i,j})\right)\left(\sum_{i'=1}^{M_B} \prod_{j'=1}^{N_B} \omega(B_{i',j'})\right) = \omega(A)\omega(B).\end{aligned}$$

Again by the continuity of $\omega$ and the density of $\mathfrak{A}_0$, it follows that $\inf_{s\in\Pi(\mathbb{Z}^d)} |\omega(A\alpha_s(B)) - \omega(A)\omega(B)| = 0$ for all $A, B \in \mathfrak{A}$, and by proposition 1.4.9, $\omega \in \mathcal{E}_\Pi$.

Now, let $\omega \in \mathcal{E}_\Pi$. The idea is to prove by induction that

$$\omega(\tau_{x_1}(A_1) \dots \tau_{x_n}(A_n)) = \omega(A_1) \dots \omega(A_n)$$

for all $A_1, \dots, A_n \in \mathfrak{A}_{\{0\}}$ and all $x_1, \dots, x_n \in \mathbb{Z}^d$ different from each other. If $n = 1$, the equality is obvious, since $E_\Pi \subset E_{\mathbb{Z}^d}$. Assume it holds for some arbitrary $n$. Note that, by proposition 1.4.9, given $\epsilon > 0$, for any $A_1, \dots, A_{n+1} \in \mathfrak{A}_{\{0\}}$ and any $x_1, \dots, x_{n+1} \in \mathbb{Z}^d$, there exists some convex combination

$$S_\lambda = \sum_{i=1}^{k} \lambda_i \alpha_{s_i}$$

such that for any $s \in \Pi(\mathbb{Z}^d)$,

$$\begin{aligned}\Big|\sum_{i=1}^{k} \lambda_i \omega(\tau_{x_1}(A_1) \dots \tau_{x_n}(A_n)\alpha_{s\circ s_i}(\tau_{x_{n+1}}(A_{n+1}))) \\ - \omega(\tau_{x_1}(A_1) \dots \tau_{x_n}(A_n))\omega(\tau_{x_{n+1}}(A_{n+1}))\Big| < \epsilon.\end{aligned} \tag{3.6}$$

Let $s$ be some permutation satisfying $s(s_i(x_{n+1})) \neq x_j$ for all $i, j = 1, \dots, n$, and for each $i = 1, \dots, k$ define

$$s'_i(x) = \begin{cases} x_{n+1}, & \text{if } x = s(s_i(x_{n+1})), \\ s(s_i(x_{n+1})), & \text{if } x = x_{n+1}, \\ x, & \text{otherwise.} \end{cases}$$

If $x_1, \dots, x_{n+1}$ are different from each other, then by the permutation invariance of $\omega$ it follows that

$$\omega(\tau_{x_1}(A_1)\dots\tau_{x_n}(A_n)\alpha_{sos_i}(\tau_{x_{n+1}}(A_{n+1}))) = \omega(\alpha_{s_i'}(\tau_{x_1}(A_1)\dots\tau_{x_n}(A_n)\alpha_{sos_i}(\tau_{x_{n+1}}(A_{n+1})))) \\ = \omega(\tau_{x_1}(A_1)\dots\tau_{x_n}(A_n)\tau_{x_{n+1}}(A_{n+1})).$$

This, together with eq. 3.6 and the induction hypothesis implies that

$$\left|\omega(\tau_{x_1}(A_1)\dots\tau_{x_n}(A_n)\tau_{x_{n+1}}(A_{n+1})) - \omega(\tau_{x_1}(A_1)\dots\tau_{x_n}(A_n))\omega(\tau_{x_{n+1}}(A_{n+1}))\right| = \\ \left|\omega(\tau_{x_1}(A_1)\dots\tau_{x_n}(A_n)\tau_{x_{n+1}}(A_{n+1})) - \omega(\tau_{x_1}(A_1))\dots\omega(\tau_{x_n}(A_n))\omega(\tau_{x_{n+1}}(A_{n+1}))\right| < \epsilon.$$

Since $\epsilon$ is arbitrary, one finally has

$$\omega(\tau_{x_1}(A_1)\dots\tau_{x_n}(A_n)\tau_{x_{n+1}}(A_{n+1})) = \omega(\tau_{x_1}(A_1))\dots\omega(\tau_{x_n}(A_n))\omega(\tau_{x_{n+1}}(A_{n+1})).$$

The inclusion $\mathcal{E}_\Pi \subset \mathcal{E}_{\mathbb{Z}^d}$ is a straightforward consequence of proposition 3.4.3 – that will be shown later – and corollary 3.1.1, together with the equality $E_\otimes = \mathcal{E}_\Pi$ shown above. □

### 3.1.1 The tracial state and periodic extension of states

**Definition 3.1.5.** A *tracial state* over a C*-algebra $\mathfrak{A}$ is state $\mathrm{tr} \in E_{\mathfrak{A}}$ satisfying

$$\mathrm{tr}(AB) = \mathrm{tr}(BA), \quad \text{for all } A, B \in \mathfrak{A}.$$

**Proposition 3.1.8.** *Every CAR algebra associated with a separable Hilbert space has a unique tracial state. Moreover, it is invariant under any $*$-automorphism.*

*Proof.* This result actually holds for any *uniformly hyperfinite algebra*, i.e., any $C^*$-algebra that is a direct limit of finite-dimensional full matrix algebras (for a proof, see [7], chapter four). Hence, since this is the case of a CAR algebra associated with a separable Hilbert space, as it is shown in the proof of theorem 1.5.1, the proposition follows. □

*Remark:* The above proposition ensures that the algebra $\mathfrak{A}$ has a tracial state, and from now on it will be denoted by "tr". Moreover, by the uniqueness of the tracial state, it is easy to see that, for all $A \in \mathfrak{A}_\Lambda$ where $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$,

$$\mathrm{tr}(A) = \frac{\mathrm{Tr}_{\mathfrak{A}_\Lambda}(A)}{\mathrm{Dim}(\mathfrak{A}_\Lambda)/2} = \frac{\mathrm{Tr}_{\mathfrak{A}_\Lambda}(A)}{2^{2|\Lambda|}}$$

**Proposition 3.1.9.** *Let $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ be two mutually disjoint sets. Then, for any $A \in \mathfrak{A}_{\Lambda_1}$ and $B \in \mathfrak{A}_{\Lambda_2}$,*

$$\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B).$$

*Proof.* First note that for any $\Lambda \subset \mathbb{Z}^d$, the linear span of terms given by

$$A_{x_1} \dots A_{x_k},$$

where $x_1, \dots, x_k \in \Lambda$ and $A_{x_i} = a_{x_i,\uparrow},\ a_{x_i,\downarrow},\ a^*_{x_i,\uparrow},\ a^*_{x_i,\downarrow}$ or some nonzero binomial of these elements, is dense in $\mathfrak{A}_\Lambda$. Therefore, it suffices to prove the equality for those kind of elements. Hence, let

$$A = A_{x_1} \dots A_{x_k} \text{ and } B = B_{y_1} \dots B_{y_j},$$

where $x_1, \dots, x_k \in \Lambda_1$ and $y_1, \dots, y_j \in \Lambda_2$, and let $A' = A_{x_2} \dots A_{x_k}$. Note that $A'B$ is either even or odd.

Case 1: $A'B$ is even. Then, if $A_{x_1}$ is odd, $AB$ is also odd and by the invariance of the trace under $\sigma_\pi$, one has

$$\mathrm{tr}(AB) = \mathrm{tr}(A_{x_1})\,\mathrm{tr}(A'B) = 0.$$

Now assume that $A_{x_1}$ is even. If $A_{x_1} = a_{x_1,\uparrow}a_{x_1,\downarrow}$, then by the CARs and the cyclic property of the trace, one has

$$\mathrm{tr}(A_{x_1}) = \mathrm{tr}(a_{x_1,\uparrow}a_{x_1,\downarrow}) = \frac{1}{2}\,\mathrm{tr}(a_{x_1,\uparrow}a_{x_1,\downarrow} + a_{x_1,\downarrow}a_{x_1,\uparrow}) = 0.$$

Similarly, since $A_{x_1}$ commutes with $A'B$ as they are both even and belong to local algebras of disjoint sets, one has

$$\mathrm{tr}(AB) = \frac{1}{2}\,\mathrm{tr}((a_{x_1,\uparrow}a_{x_1,\downarrow} + a_{x_1,\downarrow}a_{x_1,\uparrow})A'B) = 0,$$

i.e.,

$$\mathrm{tr}(AB) = \mathrm{tr}(A_{x_1})\,\mathrm{tr}(A'B) = 0.$$

The same holds if $A_{x_1} = a_{x_1,\downarrow}a_{x_1,\uparrow},\ a^*_{x_1,\uparrow}a^*_{x_1,\downarrow},\ a^*_{x_1,\downarrow}a^*_{x_1,\uparrow},\ a^*_{x_1,\uparrow}a_{x_1,\downarrow},\ a^*_{x_1,\downarrow}a_{x_1,\uparrow},\ a_{x_1,\uparrow}a^*_{x_1,\downarrow}$ or $a_{x_1,\downarrow}a^*_{x_1,\uparrow}$. Now if $A_{x_1} = a^*_{x_1,\uparrow}a_{x_1,\uparrow}$, then

$$\mathrm{tr}(A_{x_1}) = \mathrm{tr}(a^*_{x_1,\uparrow}a_{x_1,\uparrow}) = \frac{1}{2}\,\mathrm{tr}(a^*_{x_1,\uparrow}a_{x_1,\uparrow} + a_{x_1,\uparrow}a^*_{x_1,\uparrow}) = \frac{1}{2},$$

and

$$\mathrm{tr}(AB) = \frac{1}{2}\,\mathrm{tr}((a^*_{x_1,\uparrow}a_{x_1,\uparrow} + a_{x_1,\uparrow}a^*_{x_1,\uparrow})A'B) = \mathrm{tr}(A_{x_1})\,\mathrm{tr}(A'B).$$

Case 2: $A'B$ is odd. Then, one has $\mathrm{tr}(A'B) = 0$. If $A_{x_1}$ is even then $AB$ is odd and also $\mathrm{tr}(AB) = 0$. If $A_{x_1}$ is odd then $\mathrm{tr}(A_{x_1}) = 0$.

Note that, in any case, one has

$$\mathrm{tr}(AB) = \mathrm{tr}(A_{x_1})\,\mathrm{tr}(A'B). \tag{3.7}$$

Applying eq. 3.7 recursively, it follows that

$$\mathrm{tr}(AB) = \mathrm{tr}(A_{x_1})\dots\mathrm{tr}(A_{x_k})\,\mathrm{tr}(B_{y_1})\dots\mathrm{tr}(B_{y_j}) = \mathrm{tr}(A)\,\mathrm{tr}(B).$$

□

**Proposition 3.1.10.** *Let $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$ and $\omega \in E_{\mathfrak{A}_\Lambda}$. Then, $\omega$ is even if and only if its density matrix $\rho_\omega$ is even.*

*Proof.* Let $\omega \in E_{\mathfrak{A}_\Lambda}$. Note that, by the uniqueness of the trace and the tracial state, one has

$$\omega(A) = \mathrm{tr}(\rho_\omega A), \quad \text{for all } A \in \mathfrak{A}_\Lambda.$$

If $\rho_\omega \in \mathfrak{A}_\Lambda^+$, then since $\mathrm{tr}$ is invariant under any *-automorphism, for any $A \in \mathfrak{A}_\Lambda$ one has

$$\omega(\sigma_\pi(A)) = \mathrm{tr}(\rho_\omega \sigma_\pi(A)) = \mathrm{tr}(\sigma_\pi(\rho_\omega A)) = \mathrm{tr}(\rho_\omega A) = \omega(A),$$

i.e., $\omega$ is even. Now if $\omega$ is even, again by the invariance of $\mathrm{tr}$ under $\sigma_\pi$, for any $A \in \mathfrak{A}$ one has

$$\begin{aligned}\omega(\sigma_\pi(A)) = \omega(A) \implies \mathrm{tr}(\rho_\omega\sigma_\pi(A)) &= \mathrm{tr}(\sigma_\pi(\rho_\omega\sigma_\pi(A))) = \mathrm{tr}(\sigma_\pi(\rho_\omega)\sigma_\pi \circ \sigma_\pi(A)) \\ &= \mathrm{tr}(\sigma_\pi(\rho_\omega)A) = \mathrm{tr}(\rho_\omega A),\end{aligned}$$

and the uniqueness of the density matrix implies that $\sigma_\pi(\rho_\omega) = \rho_\omega$, i.e., $\rho_\omega$ is even. □

**Definition 3.1.6.** For any $l, n \in \mathbb{N}$ define $\Lambda_l^{(n)} \in \mathcal{P}_f(\mathbb{Z}^d)$ as the set given by

$$\Lambda_l^{(n)} \doteq \Lambda_{l+(2l+1)n} = \bigcup_{x\in\Lambda_n} \{(2l+1)x + \Lambda_l\}.$$

**Proposition 3.1.11.** *Let $\omega \in E_{\mathfrak{A}_{\Lambda_l}}$ be an even state over $\mathfrak{A}_{\Lambda_l}$ for some $l \in \mathbb{N}$. Then, there exists a unique extension of $\omega$ to the algebra $\mathfrak{A}$, denoted by $\hat{\omega}$, that satisfies*

$$\hat{\omega}(\tau_{(2l+1)x_1}(A_1)\dots\tau_{(2l+1)x_n}(A_n)) = \omega(A_1)\dots\omega(A_n) \tag{3.8}$$

*for all $A_1, \dots, A_n \in \mathfrak{A}_{\Lambda_l}$ and all $x_1, \dots, x_n \in \mathbb{Z}^d$ different from each other.*

*Proof.* Let $\rho_\omega$ be the density matrix associated with $\omega$. For each $n \in \mathbb{N}$, define the state $\omega_n \in E_{\mathfrak{A}_{\Lambda_l^{(n)}}}$ by

$$\omega_n(A) = \mathrm{tr}\left(A \prod_{x\in\Lambda_n} \tau_{(2l+1)x}(\rho_\omega)\right).$$

Note that since $\omega$ is even, $\rho_\omega$ is even, and since $\rho_\omega \in \mathfrak{A}_{\Lambda_l}$, if $x \neq y$, then $\tau_{(2l+1)x}(\rho_\omega)$ and $\tau_{(2l+1)y}(\rho_\omega)$ belong to disjoint local algebras. Therefore, they commute with each other. Let $A = \tau_{(2l+1)x_1}(A_1)\dots\tau_{(2l+1)x_m}(A_m)$, where $A_1, \dots, A_m \in \Lambda_l$ and $x_1, \dots, x_m \in \Lambda_n$ are different from each other. Then, by proposition 3.1.9 one has

$$\begin{aligned}\omega_n(A) &= \mathrm{tr}\left(\tau_{(2l+1)x_1}(A_1)\dots\tau_{(2l+1)x_m}(A_m)\tau_{(2l+1)x_m}(\rho_\omega)\dots\tau_{(2l+1)x_1}(\rho_\omega)\right) \\ &= \mathrm{tr}\left(\tau_{(2l+1)x_1}(\rho_\omega A_1)\tau_{(2l+1)x_2}(A_2)\dots\tau_{(2l+1)x_m}(A_m)\tau_{(2l+1)x_m}(\rho_\omega)\dots\tau_{(2l+1)x_2}(\rho_\omega)\right) \\ &= \mathrm{tr}(\rho_\omega A_1)\,\mathrm{tr}(\tau_{(2l+1)x_2}(A_2)\dots\tau_{(2l+1)x_m}(A_m)\tau_{(2l+1)x_m}(\rho_\omega)\dots\tau_{(2l+1)x_2}(\rho_\omega)).\end{aligned}$$

By recursion, it follows that

$$\begin{aligned}\omega_n(A) = \omega_n(\tau_{(2l+1)x_1}(A_1)\dots\tau_{(2l+1)x_m}(A_m)) =& \operatorname{tr}(\rho_\omega A_1)\dots\operatorname{tr}(\rho_\omega A_m)\\ =&\omega(A_1)\dots\omega(A_m).\end{aligned}$$

Moreover, since the linear span of elements

$$\tau_{(2l+1)x_1}(A_1)\dots\tau_{(2l+1)x_m}(A_m),$$

where $A_1,\dots,A_m \in \Lambda_l$ and $x_1,\dots,x_m \in \Lambda_n$ are different from each other, generates the local algebra $\mathfrak{A}_{\Lambda_l^{(n)}}$, it follows that for any $n' > n$, $\omega_{n'} = \omega_n$ on $E_{\mathfrak{A}_{\Lambda_l^{(n)}}}$. Hence, one can define a state $\omega_\infty$ on $\mathfrak{A}_0$ by

$$\omega_\infty(A) = \omega_n(A),$$

where $n$ is any natural such that $A \in \Lambda_l^{(n)}$. By continuity, $\omega_\infty$ can also be extended to a state $\hat{\omega}$ on $\mathfrak{A}$. Obviously $\hat{\omega}$ satisfies eq. 3.8, and its uniqueness follows from the fact that the linear span of elements

$$\tau_{(2l+1)x_1}(A_1)\dots\tau_{(2l+1)x_m}(A_m),$$

where $A_1,\dots,A_m \in \Lambda_l$ and $x_1,\dots,x_m \in \mathbb{Z}$ are different from each other, is dense in $\mathfrak{A}$. □

**Proposition 3.1.12.** *Let $\omega \in E_{\mathfrak{A}_{\Lambda_l}}$ be an even state over $\mathfrak{A}_{\Lambda_l}$ for some $l \in \mathbb{N}$, and let $\hat{\omega} \in E_{\mathfrak{A}}$ be its unique periodic extension to the algebra $\mathfrak{A}$ that satisfies eq. 3.8. Then, the state $\tilde{\omega} \in E_{\mathfrak{A}}$ defined as*

$$\tilde{\omega} \doteq \frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l}\hat{\omega}\circ\tau_x$$

*is $\mathbb{Z}^d$-ergodic.*

*Proof.* Let $A \in \mathfrak{A}_0$. Then, by eq. 3.8 there exists some $C > 0$ such that

$$\hat{\omega}(\tau_x(A^*)\tau_y(A)) = \hat{\omega}(\tau_x(A^*))\hat{\omega}(\tau_y(A))$$

for any $x, y \in \mathbb{Z}^d$ such that $|x-y| > C$. Let $n \in \mathbb{N}$. Then it follows that

$$\begin{aligned}\hat{\omega}(|A|_n^2) =&\frac{1}{|\Lambda_n|^2}\sum_{x,y\in\Lambda_n \,|\, |x-y|>C}\hat{\omega}(\tau_x(A^*))\hat{\omega}(\tau_y(A)) + \frac{1}{|\Lambda_n|^2}\sum_{x,y\in\Lambda_n \,|\, |x-y|\le C}\hat{\omega}(\tau_x(A^*)\tau_y(A))\\ =&\frac{1}{|\Lambda_n|^2}\sum_{x,y\in\Lambda_n}\hat{\omega}(\tau_x(A^*))\hat{\omega}(\tau_y(A))\\ &+\frac{1}{|\Lambda_n|^2}\sum_{x,y\in\Lambda_n \,|\, |x-y|\le C}(\hat{\omega}(\tau_x(A^*)\tau_y(A)) - \hat{\omega}(\tau_x(A^*))\hat{\omega}(\tau_y(A))).\end{aligned} \tag{3.9}$$

Note also that the last term of eq. 3.9 goes to 0 as $n \to \infty$. Moreover, let $n' \in \mathbb{N}$ be the largest natural such that $\Lambda_l^{(n')} \subset \Lambda_n$. Then, by the $(2l+1)$-periodicity of $\hat{\omega}$, one has

$$
\begin{aligned}
\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\hat{\omega}(\tau_x(A)) =&\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_l^{(n')}}\hat{\omega}(\tau_x(A))+\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n\backslash\Lambda_l^{(n')}}\hat{\omega}(\tau_x(A))\\
=&\frac{|\Lambda_l^{(n')}|}{|\Lambda_n|}\left(\frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l}\hat{\omega}(\tau_x(A))\right)+\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n\backslash\Lambda_l^{(n')}}\hat{\omega}(\tau_x(A))\\
=&\frac{|\Lambda_l^{(n')}|}{|\Lambda_n|}\tilde{\omega}(A)+\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n\backslash\Lambda_l^{(n')}}\hat{\omega}(\tau_x(A)). \qquad (3.10)
\end{aligned}
$$

Since the last term of eq. 3.10 goes to $0$ and $\frac{|\Lambda_l^{(n')}|}{|\Lambda_n|}$ goes to $1$ as $n\to\infty$, eq.s 3.9 and 3.10 imply that

$$\lim_{n\to\infty}\hat{\omega}(|A|_n^2)=|\tilde{\omega}(A)|^2.$$

Hence, one finally has

$$
\begin{aligned}
\Delta_A(\tilde{\omega}) = \lim_{n\to\infty}\frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l}\hat{\omega}(\tau_x(|A|_n^2)) &= \lim_{n\to\infty}\frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l}\hat{\omega}(|\tau_x(A)|_n^2)\\
=&\frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l}|\tilde{\omega}(\tau_x(A))|^2=|\tilde{\omega}(A)|^2.
\end{aligned}
$$

Thus, by corollary 3.1.1, the proposition follows.

□

## 3.2 Entropy density

In the analysis of systems in thermodynamical equilibrium, the entropy density plays an important role in the definition of equilibrium states. Here it is defined the entropy density for periodic states of the algebra $\mathfrak{A}$, as a straightforward extension of the von Neummann entropy for matrix densities. Moreover, its most important properties – affinity and weak* upper-semicontinuity, that will be useful later in the text – are also derived.

**Definition 3.2.1.** Let $\Lambda\in\mathcal{P}_f(\mathbb{Z}^d)$ and $\omega\in E_{\mathfrak{A}_\Lambda}$. Define the function $S_\Lambda : E_{\mathfrak{A}_\Lambda}\to\mathbb{R}_0^+$ as the von Neumann entropy of $\omega$, i.e.,

$$S_\Lambda(\omega)=-\operatorname{Tr}_{\mathfrak{A}_\Lambda}(\rho_\omega\ln\rho_\omega),$$

where $\rho_\omega$ is the density matrix associated with the state $\omega\in E_{\mathfrak{A}_\Lambda}$ and $\operatorname{Tr}_{\mathfrak{A}_\Lambda}$ is the trace operator in the finite-dimensional algebra $\mathfrak{A}_\Lambda\cong M_{2^{2|\Lambda|}\times 2^{2|\Lambda|}}(\mathbb{C})$.

*Remark:* Note that, since for finite-dimensional matrix algebras the trace operator is invariant under *-isomorphisms, $S_\Lambda$ does not depend on the choice of the *-isomorphism used to identify $\mathfrak{A}_\Lambda$ with $M_{2^{2|\Lambda|}\times 2^{2|\Lambda|}}(\mathbb{C})$.

The next proposition is a standard result in quantum statistical mechanics:

**Proposition 3.2.1.** *For any $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$, $S_\Lambda$ satisfies:*

*(a) $S_\Lambda$ is concave, i.e., for any $\omega_1, \omega_2 \in E_{\mathbb{Z}^d}$ and any $\lambda \in [0,1]$,*

$$S_\Lambda(\lambda\omega_1 + (1-\lambda)\omega_2) \geq \lambda S_\Lambda(\omega_1) + (1-\lambda)S_\Lambda(\omega_2),$$

*(b) $S_\Lambda$ is approximately convex, i.e., for any $\omega_1, \omega_2 \in E_{\mathbb{Z}^d}$ and any $\lambda \in [0,1]$,*

$$S_\Lambda(\lambda\omega_1 + (1-\lambda)\omega_2) \leq \lambda S_\Lambda(\omega_1) + (1-\lambda)S_\Lambda(\omega_2) - \lambda \ln \lambda - (1-\lambda)\ln(1-\lambda),$$

*(c) $S_\Lambda$ is strongly subadditive, i.e., for any $\Lambda_1, \Lambda_2 \subset \Lambda$ with $\Lambda = \Lambda_1 \cup \Lambda_2$,*

$$S_\Lambda(\omega) + S_{\Lambda_1\cap\Lambda_2}(\omega|_{\mathfrak{A}_{\Lambda_1\cap\Lambda_2}}) \leq S_{\Lambda_1}(\omega|_{\mathfrak{A}_{\Lambda_1}}) + S_{\Lambda_2}(\omega|_{\mathfrak{A}_{\Lambda_2}}),$$

*(d) $S_\Lambda$ is additive for product states, i.e., if $\Lambda_1, \ldots, \Lambda_n$ is a collection of mutually disjoint sets such that $\bigcup_{k=1}^n \Lambda_k = \Lambda$ and $\omega \in E_{\mathfrak{A}_\Lambda}$ satisfies*

$$\omega(A_{i_1} \ldots A_{i_n}) = \omega_{i_1}(A_{i_1}) \ldots \omega_{i_n}(A_{i_n})$$

*for any set $(i_1, \ldots, i_n)$ of distinct indexes, where $A_{i_j}$ are arbitrary elements of $\mathfrak{A}_{\Lambda_{i_j}}$ and $\omega_{i_j}$ are states over $\mathfrak{A}_{\Lambda_{i_j}}$, then*

$$S_\Lambda(\omega) = \sum_{j=1}^n S_{\Lambda_{i_j}}(\omega_{i_j}),$$

*(e) $S_\Lambda$ is continuous.*

**Definition 3.2.2.** The *mean entropy density* is a mapping $s : E_{l\mathbb{Z}^d} \to \mathbb{R}_0^+$ defined as

$$s(\omega) \doteq \lim_{l\to\infty} \frac{S_{\Lambda_l}(\omega|_{\mathfrak{A}_{\Lambda_l}})}{|\Lambda_l|}.$$

A very important property that the thermodynamic functions studied here must hold is the upper (or lower) semi-continuity. As it will be seen later, this property assures that the functions achieve a maximum (or minimum) in the set of t.i. states.

**Definition 3.2.3.** Let $X$ be a topological space, and $f : X \to \mathbb{R}$ a functional on $X$. Then, $f$ is said to be *upper semi-continuous* if for any $a \in \mathbb{R}$, the set $f^{-1}((-\infty, a))$ is open in $X$, and $f$ is said to be *lower semi-continuous* if for any $a \in \mathbb{R}$, the set $f^{-1}((-a, \infty))$ is open in $X$.

**Proposition 3.2.2.** *Let $X$ be a topological space. The following holds:*

*(a) the space of all upper semi-continuous functionals on $X$ is a convex cone,*

*(b) a functional $f$ on $X$ is upper semi-continuous if and only if $-f$ is lower semi-continuous,*

*(c) if $f : X \to \mathbb{R}$ is given by*

$$f(x) = \inf_{i\in I} f_i(x), \quad \textit{for all } x \in X,$$

*where $\{f_i\}_{i\in I}$ is family of upper semi-continuous functionals on $X$, then $f$ is upper semi-continuous.*

*Proof.* Items $(a)$ and $(b)$ are obvious from the above definitions. Let $f$ be as in $(c)$. Then, given $a \in \mathbb{R}$, for any $x \in X$, one has

$$f(x) < a \iff f_i(x) < a, \quad \text{for some } i \in I.$$

Therefore, it follows that

$$f^{-1}((-\infty, a)) = \bigcup_{i\in I} f_i^{-1}((-\infty, a)),$$

but the set on the r.h.s. is clearly open, since it is an union of open sets. Hence, $f^{-1}((-\infty, a))$ is open, and thus $f$ is upper semi-continuous. □

**Proposition 3.2.3.** *The mean entropy density is a well-defined affine functional, weak* upper semi-continuous, and satisfies:*

*(a)*

$$s(\omega) = \inf_{l\in\mathbb{N}} \frac{S_{\Lambda_l}(\omega|_{\mathfrak{A}_{\Lambda_l}})}{|\Lambda_l|},$$

*(b)*

$$s(\omega \circ \tau_x) = s(\omega), \quad \textit{for all } x \in \mathbb{Z}^d.$$

*Proof.* The proof of properties $(a)$ and $(b)$ are extensive and can be found in ([2], [4]). The concavity of $s$ follows from $(a)$ of proposition 3.2.1, and the convexity follows from $(b)$, noting that $\lim_{l\to\infty} |\Lambda_l| = \infty$. To prove that $s$ is weak* upper semi-continuous, by proposition 3.2.2 it suffices to show that $\omega \mapsto S_{\Lambda_l}(\omega|_{\mathfrak{A}_{\Lambda_l}})$ is weak* continuous. But by $(e)$ of proposition 3.2.1, $\omega \mapsto -\operatorname{Tr}(\rho_\omega \ln \rho_\omega)$, from $E_{\mathfrak{A}_\Lambda}$ to $\mathbb{R}_0^+$, is continuous. Hence, it suffices to prove that the mapping $\omega \to \omega|_{\mathfrak{A}_\Lambda}$ from $E_{l\mathbb{Z}^d}$ to $E_{\mathfrak{A}_\Lambda}$ is weak* continuous.

Let $\{A_1, \dots, A_n\}$ be an orthonormal basis for $\mathfrak{A}_\Lambda$. Given $\omega \in E_{l\mathbb{Z}^d}$ and $\epsilon > 0$, let $B_{\Lambda,\omega}(\epsilon)$ be the weak* neighborhood of $\omega$ defined as

$$B_{\Lambda,\omega}(\epsilon) \doteq \{\omega' \in E_{l\mathbb{Z}^d} \mid |\omega'(A_i) - \omega(A_i)| < \epsilon, \quad \text{for all } i = 1, \dots, n\}.$$

Since any $A \in \mathfrak{A}_\Lambda$ such that $\|A\|^2 = 1$ is uniquely written as $A = \sum_{i=1}^n c_{A,i} A_i$ for some $c_{A,i}$'s with $\sum_{i=1}^n |c_{A,i}|^2 = 1$, for any $\omega' \in B_{\Lambda,\omega}(\epsilon)$, one has

$$\begin{aligned}\|\omega'|_{\mathfrak{A}_\Lambda} - \omega|_{\mathfrak{A}_\Lambda}\|^2 = \sup_{A\in\mathfrak{A}_\Lambda \mid \|A\|=1} |\omega'|_{\mathfrak{A}_\Lambda}(A) - \omega|_{\mathfrak{A}_\Lambda}(A)|^2 &\leq \sum_{i=1}^n |c_{A,i}|^2 |\omega'(A_i) - \omega(A_i)|^2 \\ &< \epsilon^2.\end{aligned}$$

Therefore, $\omega \mapsto S_{\Lambda_l}(\omega|_{\mathfrak{A}_{\Lambda_l}})$ is weak* continuous.

□

## 3.3 Interactions and long-range models

In the usual Hilbert space formalism of quantum mechanics, the interactions of a system are described by an observable, usually called the *Hamiltonian*. In the $C^*$-algebra formalism, when the CAR algebra is infinite-dimensional, not every self-adjoint element of it is suitable to describe an interaction, since, similarly to what happens to general states of the algebra, some thermodynamical quantities are not well-defined for arbitrary self-adjoint elements of $\mathfrak{A}$. Therefore, this section is devoted to the definition of a convenient space of interactions, where all the necessary thermodynamical functions can be computed. To do this, first one starts with the space of *short-range interactions*, and then expand it to include more general terms, forming the so-called *long-range models*, that also include mean-field interactions. However, it is worth noting that, if a long-range model possesses a *repulsive* long-range component, the analysis of the equilibrium states of the system presents some difficulties that require some additional work to be overcome. Therefore, here the analysis is restricted only to models with a purely attractive long-range component.

**Definition 3.3.1.** An *interaction* is a family $\{\Phi_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)}$ of even local elements $\Phi_\Lambda \in \mathfrak{A}_\Lambda^+$, with $\Phi_\varnothing = 0$. If $\Phi_\Lambda$ is self-adjoint for all $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$, the interaction is said to be *self-adjoint*. Moreover, if it satisfies

$$\Phi_{x+\Lambda} = \tau_x(\Phi_\Lambda) \text{ for all } \Lambda \in \mathcal{P}_f(\mathbb{Z}^d) \text{ and all } x \in \mathbb{Z}^d,$$

the interaction is said to be *translation invariant (t.i.)*, and if it satisfies

$$\Phi_\Lambda = 0 \text{ for all } \Lambda \in \mathcal{P}_f(\mathbb{Z}^d) \text{ such that } |\Lambda| \neq 1,$$

the interaction is said to be *permutation invariant (p.i.)*.

**Definition 3.3.2.** Let $\Phi = \{\Phi_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)}$ be an interaction, and let $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$. The *internal energy of* $\Phi$ *on* $\Lambda$ is the local element

$$U_\Lambda^\Phi \doteq \sum_{\Lambda' \in \mathcal{P}_f(\mathbb{Z}^d), \Lambda' \subset \Lambda} \Phi_{\Lambda'} \in \mathfrak{A}_\Lambda^+.$$

**Definition 3.3.3.** The space $\mathcal{W}$ of *short-range interactions* is the set of all t.i. interactions $\Phi$ such that the norm

$$\|\Phi\|_{\mathcal{W}} \doteq \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L}), 0 \in \Lambda} \frac{\|\Phi_\Lambda\|}{|\Lambda|} \tag{3.11}$$

is finite. The subset of all *self-adjoint short-range interactions* will bedenoted by $\mathcal{W}^\mathbb{R}$.

**Proposition 3.3.1.** *$\mathcal{W}$ and $\mathcal{W}^\mathbb{R}$ are real Banach spaces.*

*Proof.* Clearly $\mathcal{W}$ and $\mathcal{W}^\mathbb{R}$ are real vector spaces, with sum and scalar multiplication defined by

$$(\Phi + \lambda\Psi)_\Lambda \doteq \Phi_\Lambda + \lambda\Psi_\Lambda,$$

for $\Phi, \Psi \in \mathcal{W}$ or $\mathcal{W}^\mathbb{R}$ and $\lambda \in \mathbb{R}$. Moreover, it is not hard to see that $\|\cdot\|_{\mathcal{W}}$ is indeed a norm on $\mathcal{W}$, and consequently on $\mathcal{W}^\mathbb{R}$. Let $\{\Phi_n\}$ be a Cauchy sequence in $\mathcal{W}$, where

$\Phi_n = \{\Phi_{n,\Lambda}\}_{\Lambda\in\mathcal{P}_f(\mathbb{Z}^d)}$. Note that, given $\epsilon > 0$, for any $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$ there exists some $N_\Lambda \in \mathbb{N}$ such that, for all $n, m > N_\Lambda$,

$$\|\Phi_{n,\Lambda} - \Phi_{m,\Lambda}\| \leq |\Lambda| \sum_{\Lambda'\in\mathcal{P}_f(\mathfrak{L})\,|\,0\in\Lambda'} \frac{\|\Phi_{n,\Lambda'} - \Phi_{m,\Lambda'}\|}{|\Lambda'|} < |\Lambda| \cdot \frac{\epsilon}{|\Lambda|} = \epsilon.$$

Thus, for any $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$, the sequence $\{\Phi_{n,\Lambda}\}$ is a Cauchy sequence in $\mathfrak{A}^+$, and since $\mathfrak{A}^+$ is complete, they converge in $\mathfrak{A}^+$, and it is not hard to see that the limit is also self-adjoint. Let $\Phi = \{\Phi_\Lambda\}_{\Lambda\in\mathcal{P}_f(\mathbb{Z}^d)}$ be the interaction whose elements $\Phi_\Lambda$ are the limit of the sequences $\{\Phi_{n,\Lambda}\}$, and let $\{\Phi_{n_k}\}$ be a subsequence of $\{\Phi_n\}$ satisfying

$$\|\Phi_{n_k} - \Phi_{n_{k+1}}\|_{\mathcal{W}} < 2^{-k}, \quad \text{for all } k \in \mathbb{N}.$$

Given $\epsilon > 0$, let $K \in \mathbb{N}$ be such that

$$2^{1-K} < \epsilon,$$

and let $\{\Lambda_i\}$ be an enumeration of the set $\{\Lambda \mid \Lambda \in \mathcal{P}_f(\mathfrak{L}) \mid 0 \in \Lambda\}$. Then for any $k \in \mathbb{N}$, $k > K$, one has

$$\begin{aligned}
\|\Phi_{n_k} - \Phi\|_{\mathcal{W}} &= \sum_{i=1}^{\infty} \frac{\left\|\sum_{j=1}^{\infty}(\Phi_{n_{k+j-1},\Lambda_i} - \Phi_{n_{k+j},\Lambda_i})\right\|}{|\Lambda_i|} \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\|\Phi_{n_{k+j-1},\Lambda_i} - \Phi_{n_{k+j},\Lambda_i}\|}{|\Lambda_i|} \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \frac{\|\Phi_{n_{k+j-1},\Lambda_i} - \Phi_{n_{k+j},\Lambda_i}\|}{|\Lambda_i|} = \sum_{j=1}^{\infty} \|\Phi_{n_{k+j-1}} - \Phi_{n_{k+j}}\|_{\mathcal{W}} \leq 2^{1-k} \\
&< 2^{1-K} < \epsilon.
\end{aligned}$$

Hence, $\{\Phi_n\}$ converges to $\Phi$, and thus $\mathcal{W}$ is complete. Now, if $\{\Phi_n\}$ is a sequence in $\mathcal{W}^{\mathbb{R}}$ converging to $\Phi \in \mathcal{W}$, then by the above analysis one has

$$\Phi_\Lambda = \lim_{n\to\infty} \Phi_{n,\Lambda} \quad \text{for all } \Lambda \in \mathcal{P}_f(\mathbb{Z}^d).$$

Therefore, since $\mathfrak{A}^{\mathbb{R}}_\Lambda$ is closed for all $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$, it follows that $\mathcal{W}^{\mathbb{R}}$ is closed, and consequently, also complete. □

**Definition 3.3.4.** Let $\Phi \in \mathcal{W}$ be any short-range interaction and $l \in \mathbb{N}$. The *energy density functional* associated with $\Phi$ is a functional $e_\Phi : E_{l.\mathbb{Z}^d} \to \mathbb{R}$ given by

$$e_\Phi(\omega) \doteq \lim_{l\to\infty} \frac{\omega(U^\Phi_{\Lambda_l})}{|\Lambda_l|}.$$

**Proposition 3.3.2.** *For any $\Phi \in \mathcal{W}$, and any $l \in \mathbb{N}$, the functional $e_\Phi : E_{l.\mathbb{Z}^d} \to \mathbb{R}$ is affine, weak* continuous and translation invariant (i.e., $e_\Phi(\omega \circ \tau_x) = e_\Phi(\omega)$ for any $\omega \in E_{l.\mathbb{Z}^d}$ and any $x \in \mathbb{Z}^d$). Moreover, for any $\omega \in E_{l.\mathbb{Z}^d}$, $e_\Phi(\omega) = \omega(\mathfrak{e}_{\Phi,l})$, where*

$$\mathfrak{e}_{\Phi,l} \doteq \frac{1}{l^d} \sum_{x\,|\,x_i\in\{0,\ldots,l-1\}} \sum_{\Lambda\in\mathcal{P}_f(\mathbb{Z}^d)\,|\,0\in\Lambda} \frac{\tau_x(\Phi_\Lambda)}{|\Lambda|} \in \mathfrak{A}^+.$$

*Proof.* Let $\Phi \in \mathcal{W}$ and $l \in \mathbb{N}$. Note that, by the definition of $\mathcal{W}$, the series

$$\sum_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d) \mid 0 \in \Lambda} \frac{\tau_x(\Phi_\Lambda)}{|\Lambda|}$$

is absolutely convergent for any $x \in \mathbb{Z}^d$. Hence, since $\mathfrak{A}^+$ is complete, it converges in $\mathfrak{A}^+$ for any $x \in \mathbb{Z}^d$. Therefore, $\mathfrak{e}_{\Phi,l}$ is well-defined. Now, note that for any $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$, one has

$$U_\Lambda^\Phi = \sum_{x \mid x_i \in \{0,\ldots,l-1\}} \sum_{\Lambda' \in \mathcal{P}_f(\mathfrak{L}) \mid 0 \in \Lambda'} \sum_{\substack{y \in \Lambda \cap l.\mathbb{Z}^d, \\ x+y \in \Lambda, \\ x+y+\Lambda' \subset \Lambda}} \frac{\tau_y(\tau_x(\Phi_{\Lambda'}))}{|\Lambda'|}$$

Therefore, from the definition of $e_\Phi$ and the translation invariance of $\Phi$, for any $\omega \in E_{l.\mathbb{Z}^d}$ it follows that

$$e_\Phi(\omega) = \lim_{n \to \infty} \sum_{x \mid x_i \in \{0,\ldots,l-1\}} \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L}) \mid 0 \in \Lambda} \omega\left(\frac{\tau_x(\Phi_\Lambda)}{|\Lambda|}\right) \sum_{\substack{y \in \Lambda_n \cap l.\mathbb{Z}^d, \\ x+y \in \Lambda_n, \\ x+y+\Lambda \subset \Lambda_n}} \frac{1}{|\Lambda_n|}.$$

But since $\sum_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d) \mid 0 \in \Lambda} \omega\left(\frac{\tau_x(\Phi_\Lambda)}{|\Lambda|}\right)$ converges absolutely for any $x \in \mathbb{Z}^d$, and

$$\sum_{\substack{y \in \Lambda_n \cap l.\mathbb{Z}^d, \\ x+y \in \Lambda_n, \\ x+y+\Lambda \subset \Lambda_n}} \frac{1}{|\Lambda_n|} \leq 1, \text{ with } \quad \lim_{n \to \infty} \sum_{\substack{y \in \Lambda_n \cap l.\mathbb{Z}^d, \\ x+y \in \Lambda_n, \\ x+y+\Lambda \subset \Lambda_n}} \frac{1}{|\Lambda_n|} = \frac{1}{l^d},$$

from the dominated convergence theorem, it follows that

$$\begin{aligned} e_\Phi(\omega) = & \frac{1}{l^d} \sum_{x \mid x_i \in \{0,\ldots,l-1\}} \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L}) \mid 0 \in \Lambda} \omega\left(\frac{\tau_x(\Phi_\Lambda)}{|\Lambda|}\right) \\ = & \omega\left(\frac{1}{l^d} \sum_{x \mid x_i \in \{0,\ldots,l-1\}} \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L}) \mid 0 \in \Lambda} \frac{\tau_x(\Phi_\Lambda)}{|\Lambda|}\right) = \omega(\mathfrak{e}_{\Phi,l}), \end{aligned} \tag{3.12}$$

and from eq. 3.12, it is not hard to see that for any $x \in \mathbb{Z}^d$, $e_\Phi(\omega \circ \tau_x) = \omega(\tau_x(\mathfrak{e}_{\Phi,l})) = \omega(\mathfrak{e}_{\Phi,l}) = e_\Phi(\omega)$. For any net $\{\omega_\alpha\}$ converging to $\omega$ in the weak* topology, one has

$$\lim_\alpha e_\Phi(\omega_\alpha) = \lim_\alpha \omega_\alpha(\mathfrak{e}_{\Phi,l}) = \omega(\mathfrak{e}_{\Phi,l}) = e_\Phi(\omega).$$

Hence, $e_\Phi$ is weak* continuous. The affinity of $e_\Phi$ is obvious. □

*Remark:* From now on, for any $\Phi \in \mathcal{W}$, the element $\mathfrak{e}_{\Phi,1} \in \mathfrak{A}^+$ given by

$$\mathfrak{e}_{\Phi,1} = \sum_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d) \mid 0 \in \Lambda} \frac{\Phi_\Lambda}{|\Lambda|} \in \mathfrak{A}^+$$

will be denoted by $\mathfrak{e}_\Phi$.

### 3.3.1 Long-Range models

**Theorem 3.3.1** (Hahn decomposition theorem)**.** *Let $(X, \mathcal{A})$ be a measurable space and $\mu$ a signed measure on $(X, \mathcal{A})$. Then, there exists some measurable set $P \in \mathcal{A}$, satisfying*

*(a) $\mu(A) \geq 0$ for all $A \subset P$ measurable,*

*(b) $\mu(A) \leq 0$ for all $A \subset P^c$ measurable.*

**Theorem 3.3.2** (Jordan decomposition)**.** *Let $(X, \mathcal{A})$ be a measurable space and $\mu$ a signed measure on $(X, \mathcal{A})$. Then, there exists a unique pair $(\mu^+, \mu^-)$ of non-negative measures on $(X, \mathcal{A})$ such that*

*(a) $\mu = \mu^+ - \mu^-$,*

*(b) $\mu^+(P^c) = 0$ for any $P$ satisfying the conditions of theorem 3.3.1,*

*(c) $\mu^-(P) = 0$ for for any $P$ satisfying the conditions of theorem 3.3.1.*

*The pair $(\mu^+, \mu^-)$ is called the Jordan decomposition of $\mu$.*

The existence of a Jordan decomposition for a measure allows one to define the *variation* of a signed measure:

**Definition 3.3.5.** Let $(X, \mathcal{A})$ be a measurable space. For any signed measure $\mu$ on $(X, \mathcal{A})$, the quantity

$$|\mu| \doteq \mu^+(X) + \mu^-(X),$$

where $(\mu^+, \mu^-)$ is the Jordan decomposition of $\mu$, is called the *variation* of $\mu$.

**Proposition 3.3.3.** *Let $(X, \mathcal{A})$ be a measurable space and $\mathbf{M}$ the real vector space of all signed measures on $(X, \mathcal{A})$ with finite variation. For any $\mu \in \mathbf{M}$, define*

$$\|\mu\| \doteq |\mu|(X) = \mu^+(X) + \mu^-(X).$$

*Then, $\|\cdot\|$ is a norm on $\mathbf{M}$, that turns $\mathbf{M}$ into a real Banach space.*

*Proof.* Let $\mu \in \mathbf{M}$. By the uniqueness of the Jordan decomposition, it is easy to see that $\|c\mu\| = |c|\|\mu\|$ for any $c \in \mathbb{R}$ and that $\mu = 0 \implies \|\mu\| = 0$. If $\|\mu\| = 0$, then $\mu^+(X) = \mu^-(X) = 0$ and by the subadditivity of non-negative measures it follows that $\mu^+ = \mu^- = 0$. Hence, $\mu = 0$. Let $\mu_1, \mu_2 \in \mathbf{M}$, and let $P \in \mathcal{A}$ be as in theorem 3.3.1. Then, one has

$$\begin{aligned}
\|\mu_1 + \mu_2\| =& (\mu_1 + \mu_2)^+(X) + (\mu_1 + \mu_2)^-(X) = (\mu_1 + \mu_2)(P) - (\mu_1 + \mu_2)(P^c) \\
=& \mu_1(P) + \mu_2(P) - \mu_1(P^c) - \mu_2(P^c) \\
\leq& |\mu_1|(P) + |\mu_1|(P^c) + |\mu_2|(P) + |\mu_2|(P^c) \\
=& |\mu_1|(X) + |\mu_2|(X) = \|\mu_1\| + \|\mu_2\|.
\end{aligned}$$

Now, let $\{\mu_n\}$ be a Cauchy sequence on $\mathbf{M}$. Note that, by the subadditivity of non-negative measures, for any $n, m \in \mathbb{N}$ one has

$$|\mu_n(A) - \mu_m(A)| \leq |\mu_n - \mu_m|(A) \leq |\mu_n - \mu_m|(X) = \|\mu_n - \mu_m\|, \quad \text{for all } A \in \mathcal{A}. \tag{3.13}$$

Hence, for all $A \in \mathcal{A}$, the sequence $\{\mu_n(A)\}$ is a Cauchy sequence in $\mathbb{R}$, and since $\mathbb{R}$ is complete, it converges. Define $\mu : \mathcal{A} \to \mathbb{R}$ by

$$\mu(A) = \lim_{n\to\infty} \mu_n(A), \quad \text{for all } A \in \mathcal{A}.$$

Note also that, by eq. 3.13, the measures $\{\mu_n\}$ converge uniformly to $\mu$. Clearly $\mu(\varnothing) = 0$. Let $\{A_n\}$ be a sequence of pairwise disjoint sets in $\mathcal{A}$ and let $\tilde{A} \doteq \bigcup_{n=1}^{\infty} A_n$. Given $\epsilon > 0$, let $N \in \mathbb{N}$ be such that

$$|\mu(A) - \mu_N(A)| < \frac{\epsilon}{3}, \quad \text{for all } A \in \mathcal{A}.$$

Since $\mu_N$ is a finite signed measure, there exists some $M \in \mathbb{N}$ such that

$$\left|\mu_N(\tilde{A}) - \mu_N\left(\bigcup_{n=1}^{m} A_n\right)\right| = \left|\mu_N - \sum_{n=1}^{m} \mu_N\left(A_n\right)\right| < \frac{\epsilon}{3}, \text{ for all } m > M$$

Hence, for all $m > M$, it follows that

$$\left|\mu(\tilde{A}) - \sum_{n=1}^{m} \mu\left(A_n\right)\right| = \left|\mu(\tilde{A}) - \mu\left(\bigcup_{n=1}^{m} A_n\right)\right| \leq \left|\mu(\tilde{A}) - \mu_N(\tilde{A})\right|$$
$$+ \left|\mu_N(\tilde{A}) - \mu_N\left(\bigcup_{n=1}^{m} A_n\right)\right| + \left|\mu_N\left(\bigcup_{n=1}^{m} A_n\right) - \mu\left(\bigcup_{n=1}^{m} A_n\right)\right| = \epsilon.$$

Therefore, $\mu(\tilde{A}) = \sum_{n=1}^{\infty} \mu(A_n)$, and hence $\mu \in \mathbf{M}$. Moreover, again given $\epsilon > 0$, let $N \in \mathbb{N}$ be such that

$$|\mu(A) - \mu_n(A)| < \frac{\epsilon}{2}, \quad \text{for all } A \in \mathcal{A} \text{ and all } n > N,$$

and let $P_n$ be as in 3.3.1. Then, for all $n > N$, one has

$$\|\mu_n - \mu\| = (\mu_n - \mu)^+(X) + (\mu_n - \mu)^-(X) = (\mu_n - \mu)(P_n) + (\mu_n - \mu)(P_n^c) < \epsilon,$$

and hence $\{\mu_n\}$ converges to $\mu$ in $\mathbf{M}$. □

*Remark:* Let $\mathbb{S}$ denote the unit sphere of $\mathcal{W}$, and $\mathcal{S}$ denote the space of all signed Borel measures on $\mathbb{S}$ with finite variation. In the following, the measures $\mathfrak{a}$ in $\mathcal{S}$, together with the space-averaging functional, will be used to define an extension of the space of short-range interactions, known as *long-range models*, that also includes some mean-field type of interactions.

**Definition 3.3.6.** A *long-range model* is a pair $(\Phi, \mathfrak{a})$, where $\Phi \in \mathcal{W}^{\mathbb{R}}$ and $\mathfrak{a} \in \mathcal{S}$. The *space of long-range models* $\mathcal{M}$ is defined as $\mathcal{M} \doteq \mathcal{W}^{\mathbb{R}} \times \mathcal{S}$, with norm given by

$$\|\mathfrak{m}\|_{\mathcal{M}} \doteq \|\Phi\|_{\mathcal{W}} + |\mathfrak{a}|(\mathbb{S}), \quad \text{for all } \mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}.$$

$\mathfrak{a}^+$ and $\mathfrak{a}^-$ are called, respectively, the *repulsive* and *attractive* long-range component of the long-range model $\mathfrak{m} = (\Phi, \mathfrak{a})$. If $\mathfrak{a}^+ = 0$, then $\mathfrak{m}$ is said to be a *purely attractive* long-range model. Moreover, if $\Phi$ is a permutation invariant interaction and $\mathfrak{a}$ is supported by the set of permutation invariant interactions in $\mathcal{W}$, $\mathfrak{m}$ is said to be a *permutation invariant* long-range model.

Since $\mathcal{W}^{\mathbb{R}}$ and $\mathcal{S}$ are Banach spaces, it easily follows that $\mathcal{M}$ is also Banach.

**Definition 3.3.7.** Let $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}$ be a long-range model, and let $l \in \mathbb{N}$. The *internal long-range energy of* $\mathfrak{m}$ *on* $\Lambda_l$ is given by

$$U_{\Lambda_l}^{\mathfrak{m}} \doteq U_{\Lambda_l}^{\Phi} + \frac{1}{|\Lambda_l|} \int_{\mathbb{S}} \left(U_{\Lambda_l}^{\Psi}\right)^* U_{\Lambda_l}^{\Psi} \, \mathfrak{a}(d\Psi).$$

**Definition 3.3.8.** For any $\mathfrak{a} \in \mathcal{S}$, define the *long-range space-averaging functional* $\Delta_{\mathfrak{a}} : E_{\mathbb{Z}^d} \to \mathbb{R}$ by

$$\Delta_{\mathfrak{a}}(\omega) \doteq \int_{\mathbb{S}} \Delta_{\mathfrak{e}_\Psi}(\omega) \, \mathfrak{a}(d\Psi).$$

Note that, since the mapping $\Phi \mapsto \mathfrak{e}_\Phi$ is clearly continuous, and for any $\omega \in E_{\mathbb{Z}^d}$ the mapping $A \mapsto \Delta_A(\omega)$ is also continuous by proposition 3.1.6, $\Delta_{\mathfrak{a}}$ is well-defined for all $\mathfrak{a} \in \mathcal{S}$.

**Proposition 3.3.4.** *For any* $\mathfrak{a} \in \mathcal{S}$*,* $\Delta_{\mathfrak{a}}$ *is affine. Moreover, if* $\mathfrak{a}^+ = 0$*,* $\Delta_{\mathfrak{a}}$ *is weak* lower semi-continuous.*

*Proof.* The affinity of $\Delta_{\mathfrak{a}}$ easily follows from the affinity of $\omega \mapsto \Delta_A(\omega)$ (see propisition 3.1.6). Now, note that the mapping

$$\omega \mapsto \int_{\mathbb{S}} \Delta_{\mathfrak{e}_\Psi}(\omega) \mathfrak{a}^-(d\Psi)$$

is weak* upper semi-continuous, since $\omega \mapsto \Delta_A(\omega)$ is weak* upper semi-continuous for any $A \in \mathfrak{A}$ and $\mathfrak{a}^-$ is a non-negative measure. If $\mathfrak{a}^+ = 0$, then

$$\Delta_{\mathfrak{a}}(\omega) = -\int_{\mathbb{S}} \Delta_{\mathfrak{e}_\Psi}(\omega) \mathfrak{a}^-(d\Psi),$$

and hence, by Fatou's lemma it follows that $\Delta_{\mathfrak{a}}$ is weak* lower semi-continuous. □

## 3.4 Free energy density and equilibrium states

Given a long-range model as defined in the previous section, the goal now is to define and prove the existence of *equilibrium states* for such model. In this text, the equilibrium states will be defined as the minimizers of the so-called *free energy density functional*. For finite-dimensional CAR algebras, it can be shown that an equilibrium state always exist and it is unique — usually known as the Gibbs state. For infinite-dimensional CAR algebras, the existence of equilibrium states is guaranteed as long as the long-range model is purely attractive, but its uniqueness, in general, does not hold anymore. This is the scenario where a coexistence of different phases might appear, as it will be seen later.

**Definition 3.4.1.** Given $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}$, for any $l \in \mathbb{N}$, the *Gibbs equilibrium state* $\omega_{l,G} \in E_{\mathfrak{A}_{\Lambda_l}}$ defined as

$$\omega_{l,G}(A) \doteq \mathrm{tr}(A \cdot e^{-\beta U^{\mathfrak{m}}_{\Lambda_l}}) = \frac{\mathrm{Tr}_{\mathfrak{A}_{\Lambda_l}}(A \cdot e^{-\beta U^{\mathfrak{m}}_{\Lambda_l}})}{Z_{\Lambda_l}},$$
$$\text{where } Z_{\Lambda_l} \doteq \mathrm{Tr}_{\mathfrak{A}_{\Lambda_l}}(e^{-\beta U^{\mathfrak{m}}_{\Lambda_l}}),$$

and the *finite-volume pressure* is defined as

$$p_{l,\mathfrak{m}} = \frac{\ln Z_{\Lambda_l}}{\beta|\Lambda_l|}.$$

**Theorem 3.4.1** (Global stability of the Gibbs state)**.** *For any $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}$ and any $l \in \mathbb{N}$, the Gibbs state $\omega_{l,G} \in E_{\mathfrak{A}_{\Lambda_l}}$ is the unique minimizer of the functional $f_{l,\mathfrak{m}} : E_{\mathfrak{A}_{\Lambda_l}} \to \mathbb{R}$ given by*

$$f_{l,\mathfrak{m}}(\omega) \doteq \frac{\omega(U^{\mathfrak{m}}_{\Lambda_l})}{|\Lambda_l|} - \frac{S_{\Lambda_l}(\omega)}{\beta|\Lambda_l|} = \frac{\omega(U^{\Phi}_{\Lambda_l})}{|\Lambda_l|} + \frac{1}{|\Lambda_l|^2}\int_{\mathbb{S}} \omega((U^{\Psi}_{\Lambda_l})^* U^{\Psi}_{\Lambda_l})\mathfrak{a}(d\Psi) - \frac{S_{\Lambda_l}(\omega)}{\beta|\Lambda_l|}.$$

*Moreover, one has*

$$p_{l,\mathfrak{m}} = -\inf_{\omega \in E_{\mathfrak{A}_{\Lambda_l}}} f_{l,\mathfrak{m}}(\omega) = -f_{l,\mathfrak{m}}(\omega_{l,G}).$$

**Definition 3.4.2.** Given a long-range model $\mathfrak{m} = (\Phi, \mathfrak{a})$, the equilibrium states of the infinite-dimensional lattice system are the minimizers of the *free-energy density functional* $f_{\mathfrak{m}} : E_{\mathbb{Z}^d} \to \mathbb{R}$ defined as

$$f_{\mathfrak{m}}(\omega) \doteq \Delta_{\mathfrak{a}}(\omega) + e_{\Phi}(\omega) - \beta^{-1}s(\omega).$$

The set of all equilibrium states for a given long-range model $\mathfrak{m}$ will be denoted by $\Omega_{\mathfrak{m}}$. Moreover, the *thermodynamic pressure* $P_{\mathfrak{m}}$ of the system is defined as

$$P_{\mathfrak{m}} \doteq \lim_{l\to\infty} p_{l,\mathfrak{m}}.$$

**Proposition 3.4.1.** *For any long-range model $\mathfrak{m} = (\Phi, \mathfrak{a})$ The free-energy density $f_{\mathfrak{m}}$ is affine, and given by*

$$f_{\mathfrak{m}}(\omega) = \int_{\mathcal{E}_{\mathbb{Z}^d}} g_{\mathfrak{m}}(\omega')\mu_{\omega}(d\omega'),$$

*where $\mu_{\omega}$ is the measure representing $\omega$ of corollary 1.4.1 and $g_{\mathfrak{m}} : E_{\mathbb{Z}^d} \to \mathbb{R}$ is defined as*

$$g_{\mathfrak{m}}(\omega) \doteq \int_{\mathbb{S}} |\omega(\mathfrak{e}_\Psi)|^2 \mathfrak{a}(d\Psi) + e_\Phi(\omega) - \beta^{-1} s(\omega).$$

*Moreover, if* $\mathfrak{m}$ *is purely attractive,* $f_{\mathfrak{m}}$ *is weak* lower semi-continuous.*

*Proof.* The affinity of $f_{\mathfrak{m}}$ and its weak* lower semi-continuity when $\mathfrak{a}^+ = 0$ follows from propositions 3.2.3, 3.3.2 and 3.3.4, and the decomposition

$$f_{\mathfrak{m}}(\omega) = \int_{\mathcal{E}_{\mathbb{Z}^d}} \left( \int_{\mathbb{S}} |\omega'(\mathfrak{e}_\Psi)|^2 \mathfrak{a}(d\Psi) + e_\Phi(\omega') - \beta^{-1} s(\omega') \right) \mu_\omega(d\omega')$$

follows from corollary 1.4.1, noting that $\Delta_A(\omega) = |\omega(A)|^2$ when $\omega \in \mathcal{E}_{\mathbb{Z}^d}$. □

**Theorem 3.4.2** (Bauer maximum principle)**.** *Let* $X$ *be a compact convex set and* $f : X \to \mathbb{R}$ *a concave and upper semi-continuous function. Then,* $f$ *has a maximum and*

$$\max_{x \in X} f(x) = \max_{x \in \mathcal{E}(X)} f(x),$$

*where* $\mathcal{E}(X)$ *is the set of extreme points of* $X$*.*

**Corollary 3.4.1.** *For any purely attractive long-range model* $\mathfrak{m}$*,* $f_{\mathfrak{m}}$ *has a minimum and*

$$\min_{\omega \in E_{\mathbb{Z}^d}} f_{\mathfrak{m}}(\omega) = \min_{\omega \in \mathcal{E}_{\mathbb{Z}^d}} f_{\mathfrak{m}}(\omega) = \min_{\omega \in \mathcal{E}_{\mathbb{Z}^d}} g_{\mathfrak{m}}(\omega).$$

*Proof.* Since $f_{\mathfrak{m}}$ is affine, then in particular, $-f$ is concave. If $\mathfrak{m}$ is a purely attractive long-range model, then by proposition 3.4.1 $f_{\mathfrak{m}}$ is weak* lower semi-continuous, and hence $-f_{\mathfrak{m}}$ is weak* upper semi-continuous. Since $E_{\mathbb{Z}^d}$ is weak* compact, by the Bauer maximum principle $-f_{\mathfrak{m}}$ attains a maximum on $\mathcal{E}_{\mathbb{Z}^d}$, which is equivalent to $f_{\mathfrak{m}}$ attaining a minimum on $\mathcal{E}_{\mathbb{Z}^d}$. Therefore, since $f_{\mathfrak{m}} = g_{\mathfrak{m}}$ on $\mathcal{E}_{\mathbb{Z}^d}$, the corollary follows. □

**Proposition 3.4.2.** *For any purely attractive long-range model* $\mathfrak{m}$*,* $\Omega_{\mathfrak{m}} \subset E_{\mathbb{Z}^d}$ *is non-empty, convex and weakly* compact.*

*Proof.* By corollary 3.4.1, $\Omega_{\mathfrak{m}}$ is non-empty. The convexity of $\Omega_{\mathfrak{m}}$ follows from the affinity of $f_{\mathfrak{m}}$. Now, note that $\Omega_{\mathfrak{m}}^c = f_{\mathfrak{m}}^{-1}((\min_{\omega \in E_{\mathbb{Z}^d}} f_{\mathfrak{m}}(\omega), \infty))$. Since $f_{\mathfrak{m}}$ is weak* lower semi-continuous, $\Omega_{\mathfrak{m}}^c = E_{\mathbb{Z}^d} \setminus \Omega_{\mathfrak{m}}$ is weak* open, and since $E_{\mathbb{Z}^d}$ is weak* compact, $\Omega_{\mathfrak{m}}$ is also weak* compact. □

*Remark:* For a general long-range model it can also be proven, with a generalization of the Bauer maximum principle, that

$$\inf_{\omega \in E_{\mathbb{Z}^d}} f_{\mathfrak{m}}(\omega) = \inf_{\omega \in \mathcal{E}_{\mathbb{Z}^d}} f_{\mathfrak{m}}(\omega) = \inf_{\omega \in \mathcal{E}_{\mathbb{Z}^d}} g_{\mathfrak{m}}(\omega).$$

However, since $f$ may not be weak* lower semi-continuous anymore, it may not achieve its infimum at any point. Therefore, one is required to define the so-called *generalized* equilibrium states, as it is done in [4].

The next lemma will be useful to prove the existence of the thermodynamic pressure:

**Lemma 3.4.1.** *For any long-range model* $\mathfrak{m}$ *and any* $\omega \in E_{\mathbb{Z}^d}$*,*

$$\liminf_{l\to\infty} p_{l,\mathfrak{m}} \geq -f_{\mathfrak{m}}(\omega).$$

*Proof.* Let $\mathfrak{m} = (\Phi, \mathfrak{a})$ be a long-range model and $\omega \in E_{\mathbb{Z}^d}$. By theorem 3.4.1, for any $l \in \mathbb{N}$ one has

$$p_{l,\mathfrak{m}} \geq -\left(\frac{\omega(U^{\Phi}_{\Lambda_l})}{|\Lambda_l|} + \frac{1}{|\Lambda_l|^2}\int_{\mathbb{S}} \omega(\left(U^{\Psi}_{\Lambda_l}\right)^* U^{\Psi}_{\Lambda_l})\mathfrak{a}(d\Psi) - \frac{S_{\Lambda_l}(\omega|_{\mathfrak{A}_{\Lambda_l}})}{\beta|\Lambda_l|}\right).$$

Since $\tau_x$ is continuous for every $x \in \mathbb{Z}^d$, for any $\Psi \in \mathcal{W}$ one has

$$\begin{aligned}\left\|\bar{\mathfrak{e}}_{\Psi,l} - \frac{U^{\Psi}_{\Lambda_l}}{|\Lambda_l|}\right\| =& \frac{1}{|\Lambda_l|}\left\|\sum_{x\in\Lambda_l}\tau_x\left(\sum_{\Lambda\in\mathcal{P}_f(\mathbb{Z}^d)}\frac{\Psi_\Lambda}{|\Lambda|}\right) - \sum_{\Lambda\in\mathcal{P}_f(\mathbb{Z}^d)}\ \sum_{x\in\Lambda_l\ |\ \Lambda+x\subset\Lambda_l}\frac{\tau_x(\Psi_\Lambda)}{|\Lambda|}\right\| \\ \leq& \sum_{\Lambda\in\mathcal{P}_f(\mathbb{Z}^d)}\frac{\|\Psi_\Lambda\|}{|\Lambda|}\sum_{x\in\Lambda_l\ |\ \Lambda+x\not\subseteq\Lambda_l}\frac{1}{|\Lambda_l|}.\end{aligned}$$

By the dominated convergence theorem it is not hard to see that

$$\lim_{l\to\infty}\left\|\bar{\mathfrak{e}}_{\Psi,l} - \frac{U^{\Psi}_{\Lambda_l}}{|\Lambda_l|}\right\| = 0, \quad \text{for all } \Psi \in \mathcal{W}. \tag{3.14}$$

In particular,

$$\lim_{l\to\infty}\omega\left(\left(\frac{U^{\Psi}_{\Lambda_l}}{|\Lambda_l|}\right)^*\frac{U^{\Psi}_{\Lambda_l}}{|\Lambda_l|}\right) = \lim_{l\to\infty}\omega\left(|\bar{\mathfrak{e}}_{\Psi}|^2_l\right) = \Delta_{\mathfrak{e}_\Psi}(\omega),$$

and again by the dominated convergence theorem, one has

$$\begin{aligned}\liminf_{l\to\infty} p_{l,\mathfrak{m}} \geq& -\lim_{l\to\infty}\left(\frac{\omega(U^{\Phi}_{\Lambda_l})}{|\Lambda_l|} + \frac{1}{|\Lambda_l|^2}\int_{\mathbb{S}} \omega(\left(U^{\Psi}_{\Lambda_l}\right)^* U^{\Psi}_{\Lambda_l})\mathfrak{a}(d\Psi) - \frac{S_{\Lambda_l}(\omega|_{\mathfrak{A}_{\Lambda_l}})}{\beta|\Lambda_l|}\right) \\ =& -\left(e_\Phi(\omega) + \int_{\mathbb{S}}\Delta_{\mathfrak{e}_\Psi}(\omega)\mathfrak{a}(d\Psi) - s(\omega)\right) = -f_{\mathfrak{m}}(\omega).\end{aligned}$$

□

### 3.4.1 Permutation invariant systems

In order to prove the existence of the thermodynamic pressure for arbitrary long-range models, it is easier to begin with the specific situation of permutation invariant interactions. Furthermore, due to Størmer's theorem (theorem 3.1.1), the problem of finding the thermodynamic pressure for permutation invariant systems can be reduced to a minimization problem over a one-site CAR algebra, as it will be seen in this section.

**Proposition 3.4.3.** *For any p.i. state* $\omega \in E_\Pi$ *and any* $A \in \mathfrak{A}_{\{0\}}$*, one has*

$$\Delta_A(\omega) = \omega(A^*\tau_x(A)),$$

*where* $x$ *is any element in* $\mathbb{Z}^d \setminus \{0\}$*. Moreover, for any* $A \in \mathfrak{A}$*,* $\Delta_A$ *is weak*-continuous on* $E_\Pi$*.*

*Proof.* For any $y, y' \in \mathbb{Z}^d$, let $s_{y,y'} : \mathbb{Z}^d \to \mathbb{Z}^d$ be the permutation given by

$$s_{y,y'}(z) = \begin{cases} y', & \text{if } z = y, \\ y, & \text{if } z = y', \\ z, & \text{otherwise.} \end{cases}$$

Fix $x \in \mathbb{Z}^d \setminus \{0\}$. For any $l \in \mathbb{N}$, any $\omega \in E_\Pi$ and any $A \in \mathfrak{A}_{\{0\}}$, one has

$$\begin{aligned}
\omega(|A|_l^2) =& \frac{1}{|\Lambda_l|^2}\Bigg( \sum_{y\in\Lambda_l} \omega(\tau_y(A)^*\tau_y(A)) + \sum_{y\in\Lambda_l \mid y\neq 0} \omega(\tau_y(A^*)A) + \\
& \qquad \sum_{y,y'\in\Lambda_l \mid y\neq y', y'\neq 0} \omega(\tau_y(A^*)\tau_{y'}(A)) \Bigg) \\
=& \frac{1}{|\Lambda_l|^2}\Bigg( \sum_{y\in\Lambda_l} \omega(\alpha_{s_{y,0}}(\tau_y(A)^*\tau_y(A))) + \sum_{y\in\Lambda_l \mid y\neq 0} \omega(\alpha_{s_{y,0}}(\tau_y(A^*)A)) + \\
& \qquad \sum_{y,y'\in\Lambda_l \mid y\neq y', y'\neq 0} \omega(\alpha_{s_{y,0}}(\tau_y(A^*)\tau_{y'}(A))) \Bigg) \\
=& \frac{1}{|\Lambda_l|}\omega(A^*A) + \frac{1}{|\Lambda_l|^2}\Bigg( \sum_{y\in\Lambda_l \mid y\neq 0} \omega(A^*\tau_y(A)) + \sum_{y,y'\in\Lambda_l \mid y\neq y', y'\neq 0} \omega(A^*\tau_{y'}(A)) \Bigg) \\
=& \frac{1}{|\Lambda_l|}\omega(A^*A) + \frac{1}{|\Lambda_l|^2}\Bigg( \sum_{y\in\Lambda_l \mid y\neq 0} \omega(\alpha_{y,x}(A^*\tau_y(A))) + \\
& \qquad \sum_{y,y'\in\Lambda_l \mid y\neq y', y'\neq 0} \omega(\alpha_{y'',x}(A^*\tau_{y'}(A))) \Bigg) \\
=& \frac{1}{|\Lambda_l|}\omega(A^*A) + \frac{|\Lambda_l| - 1}{|\Lambda_l|^2}\omega(A^*\tau_x(A)) + \frac{|\Lambda_l|^2 - 2|\Lambda_l| + 1}{|\Lambda_l|^2}\omega(A^*\tau_x(A))
\end{aligned}$$

Therefore, taking the limit $l \to \infty$, one has

$$\Delta_A(\omega) = \lim_{l\to\infty} \omega(|A|_l^2) = \omega(A^*\tau_x(A)). \tag{3.15}$$

The weak* continuity of $\Delta_A$ in $E_\Pi$ is obvious from eq. 3.15. □

**Theorem 3.4.3.** *For any p.i. long-range model* $\mathfrak{m}$*,* $P_\mathfrak{m}$ *is well-defined and*

$$P_\mathfrak{m} = -\inf_{\omega\in E_\Pi} f_\mathfrak{m}(\omega) = -\inf_{\omega\in\mathcal{E}_\Pi} f_\mathfrak{m}(\omega).$$

*Proof.* By lemma 3.4.1, one already has

$$\liminf_{l\to\infty} p_{l,\mathfrak{m}} \geq -\inf_{\omega\in E_\Pi} f_\mathfrak{m}(\omega).$$

Hence, it suffices to show that

$$\limsup_{l\to\infty} p_{l,\mathfrak{m}} \leq -\inf_{\omega\in E_\Pi} f_{\mathfrak{m}}(\omega).$$

For any p.i. interaction $\Psi \in \mathcal{W}$, $U^\Psi_{\Lambda_l}$ is invariant under any permutation $\pi \in \Pi(\mathbb{Z}^d)$ such that $\pi|_{\mathbb{Z}^d\setminus\Lambda_l} = \text{id}$. Therefore, using proposition 3.1.8 it is easy to see that the Gibbs state $\omega_{l,G}$ associated with $U^\Psi_{\Lambda_l}$, and consequently its periodic extension $\hat{\omega}_{l,G}$ (see proposition 3.1.11), are also invariant under these permutations. Moreover, since $E_{\mathfrak{A}}$ is weakly* compact and metrizable in the weak* topology, the sequence $\{\hat{\omega}_{l,G}\}$ has a subsequence $\{\hat{\omega}_{l_k,G}\}$ that converges in the weak* topology. Let $\omega_\infty$ be this limit. By the invariance of $\hat{\omega}_{l,G}$ under permutations that leave $\mathbb{Z}^d \setminus \Lambda_l$ invariant, it follows that $\omega_\infty \in E_\Pi$. As in proposition 3.4.3, for $y, y' \in \mathbb{Z}^d$, let $s_{y,y'} : \mathbb{Z}^d \to \mathbb{Z}^d$ be the permutation given by

$$s_{y,y'}(z) = \begin{cases} y', & \text{if } z = y, \\ y, & \text{if } z = y', \\ z, & \text{otherwise.} \end{cases}$$

Note that if $y, y' \in \Lambda_l$, then $s_{y,y'}|_{\mathbb{Z}^d\setminus\Lambda_l} = \text{id}$. Moreover, since one has

$$\frac{U^\Psi_{\Lambda_l}}{|\Lambda_l|} = \frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l}\tau_x(\Psi_{\{0\}}) = \frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l}\tau_x(\mathfrak{e}_\Psi) = \frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l}\alpha_{s_{0,x}}(\mathfrak{e}_\Psi),$$

it follows that

$$\begin{aligned}\lim_{k\to\infty}\omega_{l_k,G}\left(\frac{U^\Psi_{\Lambda_{l_k}}}{|\Lambda_{l_k}|}\right) &= \lim_{k\to\infty}\frac{1}{|\Lambda_{l_k}|}\sum_{x\in\Lambda_{l_k}}\omega_{l_k,G}(\alpha_{s_{0,x}}(\mathfrak{e}_\Psi)) = \lim_{k\to\infty}\omega_{l_k,G}(\mathfrak{e}_\Psi) \\ &= \omega_\infty(\mathfrak{e}_\Psi) = e_\Psi(\omega_\infty). \end{aligned} \tag{3.16}$$

Moreover, using eq. 3.14 and following the same steps as in proposition 3.4.3, one has

$$\begin{aligned}\lim_{k\to\infty}\omega_{l_k,G}\left(\left(\frac{U^\Psi_{\Lambda_{l_k}}}{|\Lambda_{l_k}|}\right)^*\frac{U^\Psi_{\Lambda_{l_k}}}{|\Lambda_{l_k}|}\right) &= \lim_{k\to\infty}\omega_{l_k,G}\left(|\bar{\mathfrak{e}}_\Psi|^2_{l_k}\right) = \lim_{k\to\infty}\omega_{l_k,G}\left(\mathfrak{e}^*_\Psi\tau_x(\mathfrak{e}_\Psi)\right) \\ &= \omega_\infty(\mathfrak{e}^*_\Psi\tau_x(\mathfrak{e}_\Psi)). \end{aligned} \tag{3.17}$$

Now, define the state $\tilde{\omega}_{l,G}$ as

$$\tilde{\omega}_{l,G}(A) \doteq \frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l}\hat{\omega}_{l,G}(\tau_x(A)).$$

It is easy to see that $\tilde{\omega}_{l,G} \in E_{\mathbb{Z}^d}$. Let $n < l$. Then, for $x \in \Lambda_{l-n}$, there exists a permutation $\pi_{x,n}$, as defined in proposition 3.1.7, such that $\pi_{x,n} = \tau_x$. Therefore, for any $A \in \mathfrak{A}$, one has

$$\begin{aligned}\left|\frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l}\hat{\omega}_{l,G}(\tau_x(A))-\hat{\omega}_{l,G}(A)\right| &= \left|\frac{1}{|\Lambda_l|}\left((|\Lambda_{l-n}|-|\Lambda_l|)\hat{\omega}_{l,G}(A)+\sum_{x\in\Lambda_l\backslash\Lambda_{l-n}}\hat{\omega}_{l,G}(\tau_x(A))\right)\right| \\ &\leq\left|\frac{|\Lambda_{l-n}|-|\Lambda_l|}{|\Lambda_l|}\right|\|A\|+\left|\frac{|\Lambda_l|-|\Lambda_{l-n}|}{|\Lambda_l|}\right|\|A\| \\ &=2\left|\left(1-\frac{n}{l}\right)^d-1\right|\|A\|.\end{aligned}$$

Therefore, the sequence $\{\tilde{\omega}_{l_k,G}\}$ also converges to $\omega_\infty$ in the weak* topology. Moreover, by the periodicity of $\hat{\omega}_{l,G}$ and the translation invariance of the entropy density, one has

$$s(\tilde{\omega}_{l,G})=s(\hat{\omega}_{l,G})=\frac{S_{\mathfrak{A}_{\Lambda_l}}(\omega_{l,G})}{|\Lambda_l|}. \tag{3.18}$$

Hence, from eq.s 3.16, 3.17 and 3.18, together with the weak* upper semi-continuity of $s$ on $E_{\mathbb{Z}^d}$ and the fact that $\{\tilde{\omega}_{l_k,G}\}$ converges to $\omega_\infty$ in the weak* topology, one has

$$\limsup_{l\to\infty} p_l\leq -f_{\mathfrak{m}}(\omega_\infty)\leq -\inf_{\omega\in E_{\mathbb{Z}^d}} f_{\mathfrak{m}}(\omega).$$

Therefore, the result follows. □

**Corollary 3.4.2.** *For any p.i. long-range model* $\mathfrak{m}=(\Phi,\mathfrak{a})$*,* $P_{\mathfrak{m}}$ *is given by*

$$P_{\mathfrak{m}}=-\inf_{\omega\in E_{\mathfrak{A}_{\{0\}}}}\left\{\omega(\mathfrak{e}_\Psi)+\int_{\mathbb{S}}|\omega(\mathfrak{e}_\Psi)|^2d\mathfrak{a}(\Psi)-\beta^{-1}S_{\{0\}}(\omega)\right\}.$$

*Proof.* This follows from theorem 3.4.3 and Størmer's theorem, noting that $\mathfrak{e}_\Psi\in\mathfrak{A}_{\{0\}}$ for any p.i. interaction $\Psi\in\mathcal{W}$ and $s(\omega)=S_{\{0\}}(\omega)$ for a product state $\omega\in\mathcal{E}_\Pi$. □

## 3.4.2 Translation invariant systems

To prove the existence of the thermodynamic pressure $P_{\mathfrak{m}}$ for a general long-range model $\mathfrak{m}$, it is convenient to instead work with the so-called discrete finite long-range models:

**Definition 3.4.3.** Let $\mathfrak{m}=(\Phi,\mathfrak{a})\in\mathcal{W}$ be a long-range model, and suppose that there exists some interactions $\Phi_1,\ldots,\Phi_N\in\mathbb{S}$ such that the measure $\mathfrak{a}$ can be written as

$$\mathfrak{a}=\sum_{k=1}^N\zeta_k\delta_{\Phi_k},$$

where $\zeta_k=\pm1$ and $\delta_{\Phi_k}$ is the Dirac measure of $\Phi_k$. In that case, $\mathfrak{m}$ is said to be a *discrete finite long-range model*. The set of all discrete finite long-range models will be defined by $\mathcal{M}_{df}$.

*Remark:* The measure $\mathfrak{a}$ of a discrete finite long-range model $\mathfrak{m}=(\Phi,\mathfrak{a})$ can also be identified with a finite sequence $\{(\zeta_k,\Phi_k)\}_{k=1}^N$, and its internal long-range energy is given by

$$U_{\Lambda_l}^{\mathfrak{m}} = U_{\Lambda_l}^{\Phi} + \frac{1}{|\Lambda_l|}\sum_{k=1}^{N}\zeta_k\left(U_{\Lambda_l}^{\Phi_k}\right)^* U_{\Lambda_l}^{\Phi_k}.$$

Then, after proving the existence of the pressure for the finite discrete long-range models, the proposition below – that is proven in ([4], section 6.1) – allows one to generalize the result to any long-range model:

**Proposition 3.4.4.** *For any long-range model $\mathfrak{m} \in \mathcal{M}$, there exists a sequence $\{\mathfrak{m}_n\}$ of finite discrete long-range models converging to $\mathfrak{m}$ such that*

$$P_{\mathfrak{m}} = \lim_{n\to\infty} P_{\mathfrak{m}_n} \quad \textit{and} \quad \inf_{\omega\in E_{\mathbb{Z}^d}} f_{\mathfrak{m}}(\omega) = \lim_{n\to\infty}\inf_{\omega\in E_{\mathbb{Z}^d}} f_{\mathfrak{m}_n}(\omega).$$

**Lemma 3.4.2.** *For any $\Phi \in \mathcal{W}$ and $n \in \mathbb{N}$,*

$$\lim_{l\to\infty}\frac{1}{|\Lambda_l^{(n)}|}\left\|U_{\Lambda_l^{(n)}}^{\Phi} - \sum_{x\in\Lambda_n} U_{\Lambda_l+(2l+1)x}^{\Phi}\right\| = 0,$$

*where the convergence is uniform in $n$.*

*Proof.* Note that

$$\frac{1}{|\Lambda_l^{(n)}|}\left\|U_{\Lambda_l^{(n)}}^{\Phi} - \sum_{x\in\Lambda_n} U_{\Lambda_l+(2l+1)x}^{\Phi}\right\| \leq \sum_{\Lambda\in\mathcal{P}_f(\mathbb{Z}^d)\,|\,0\in\Lambda}\frac{\|\Phi_\Lambda\|}{|\Lambda|}\sum_{x\in\Lambda_l^{(n)}}{}' \frac{1}{|\Lambda_l^{(n)}|},$$

where the primed sum is only over those $x \in \Lambda_l^{(n)}$ such that $x + \Lambda \cap \Lambda_{l+(2l+1)y} \neq \varnothing$ and $x + \Lambda \cap \Lambda_{l+(2l+1)z} \neq \varnothing$ for some $y, z \in \Lambda_l^{(n)}$, $y \neq z$. For $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$, define

$$m_\Lambda \doteq \max_{x,x'\in\Lambda}\{d(x,x')\}, \quad \text{where} \;\; d(x,x') = \sqrt{|x_1 - x_1'|^2 + \cdots + |x_d - x_d'|^2}.$$

Then,

$$\sum_{x\in\Lambda_l^{(n)}}{}' \frac{1}{|\Lambda_l^{(n)}|} \leq \frac{|\Lambda_n|(|\Lambda_l| - |\Lambda_{l-m_\Lambda}|)}{|\Lambda_l^{(n)}|} = \frac{n^d(l^d - (l-m_\Lambda)^d)}{((2l+1)n)^d} = \frac{1-\left(1-\frac{m_\Lambda}{l}\right)^d}{\left(2-\frac{1}{l}\right)^d}.$$

By the dominated convergence theorem, the lemma follows. □

**Lemma 3.4.3.** *Let $\Phi \in \mathcal{W}$ be an arbitrary interaction, $\omega_{l_G}$ be the Gibbs states associated with $U_{\Lambda_l}^{\Phi}$ and let $\hat{\omega}_{l,G}$, $\tilde{\omega}_{l,G}$ be as in proposition 3.1.12. Then, it follows that*

$$\left|e_\Phi(\tilde{\omega}_{l,G}) - \frac{\hat{\omega}_{l,G}(U_{\Lambda_l}^{\Phi})}{|\Lambda_l|}\right| = 0.$$

*Proof.* By the periodicity of $\hat{\omega}_{l,G}$, one has

$$\hat{\omega}_{l,G}(U^{\Phi}_{\Lambda_l}) = \frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_n} \hat{\omega}_{l,G}(U^{\Phi}_{\Lambda_l+(2l+1)x}).$$

Therefore, by lemma 3.4.2, it follows that

$$\lim_{l\to\infty} \left| e_{\Phi}(\hat{\omega}_{l,G}) - \frac{\hat{\omega}_{l,G}(U^{\Phi}_{\Lambda_l})}{|\Lambda_l|} \right| = 0,$$

and by the affinity and translation invariance of $e_\Phi$, the lemma follows. □

The next lemma will not be proven here, but it is a well-known result and it is necessary to the proof of the following theorem.

**Lemma 3.4.4.** *For any $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$ and any $A, B \in \mathfrak{A}_\Lambda$, it follows that*

$$\left|\ln(\mathrm{Tr}_{\mathfrak{A}_\Lambda}(e^A)) - \ln(\mathrm{Tr}_{\mathfrak{A}_\Lambda}(e^B))\right| \leq \|A - B\|.$$

**Theorem 3.4.4.** *For any long-range model $\mathfrak{m}$, $P_\mathfrak{m}$ is well-defined and*

$$P_\mathfrak{m} = -\inf_{\omega\in E_{\mathbb{Z}^d}} f_\mathfrak{m}(\omega) = -\inf_{\omega\in \mathcal{E}_{\mathbb{Z}^d}} f_\mathfrak{m}(\omega).$$

*Proof.* As in the case of p.i. long-range models, lemma 3.4.1 ensures that

$$\liminf_{l\to\infty} p_{l,\mathfrak{m}} \geq -\inf_{\omega\in E_{\mathbb{Z}^d}} f_\mathfrak{m}(\omega).$$

Hence, it suffices to show that

$$\limsup_{l\to\infty} p_{l,\mathfrak{m}} \leq -\inf_{\omega\in E_{\mathbb{Z}^d}} f_\mathfrak{m}(\omega).$$

Moreover, by proposition 3.4.4, it suffices to consider only the case of finite discrete long-range models. For any $l, n \in \mathbb{N}$ and any $\Psi \in \mathcal{W}$, define

$$U^{\Psi}_{l,n} \doteq \sum_{x\in\Lambda_n} \tau_{(2l+1)x}(U^{\Phi}_{\Lambda_l}),$$

and given a finite discrete long-range model $\mathfrak{m} = \left(\Phi, \mathfrak{a} = \sum_{k=1}^N \zeta_k \delta_{\Phi_k}\right)$, define

$$U^{\mathfrak{m}}_{l,n} \doteq U^{\Phi}_{l,n} + \frac{1}{|\Lambda^{(n)}_l|} \int_{\mathbb{S}} (U^{\Psi}_{l,n})^* U^{\Psi}_{l,n} \mathfrak{a}(d\Psi) = U^{\Phi}_{l,n} + \frac{1}{|\Lambda^{(n)}_l|} = \sum_{k=1}^N \zeta_k \left(U^{\Phi_k}_{l,n}\right)^* U^{\Phi_k}_{l,n}.$$

The pressure associated with $U^{\mathfrak{m}}_{l,n}$ is given by

$$p_{l,\mathfrak{m}}(n,\beta) \doteq \frac{1}{\beta|\Lambda^{(n)}_l|} \ln \mathrm{Tr}_{\mathfrak{A}_{\Lambda^{(n)}_l}}(e^{-\beta U^{\mathfrak{m}}_{l,n}}).$$

By lemmata 3.4.2 and 3.4.4, it follows that

$$\lim_{l\to\infty}\left(\limsup_{n\to\infty}|p_{l,\mathfrak{m}}(n,\beta)-p_{2ln+n+l,\mathfrak{m}}|\right)=0. \tag{3.19}$$

Now, note that the pressure $p_{l,\mathfrak{m}}(n,\beta)$ can be seen as the thermodynamic pressure for some p.i. long-range model $\tilde{\mathfrak{m}}_l$, defined as follows: let $\tilde{\mathfrak{A}}_l$ be the CAR algebra generated by the Hilbert space $\tilde{\mathfrak{h}}_l = l^2(\mathbb{Z}^d)\otimes\mathfrak{H}_{\tilde{S}}$, where the spin set $\tilde{S}$ is given by $\tilde{S}=S\times\Lambda_l$. Then, let $\tilde{\mathcal{M}}$ denote the space of long-range models of $\tilde{\mathfrak{A}}$, and define $\tilde{\mathfrak{m}}_l$ as the p.i. long-range model $\tilde{\mathfrak{m}}_l \doteq \left(\tilde{\Phi}^{(l)}, \tilde{\mathfrak{a}}_l = \sum_{k=1}^N \zeta_k \delta_{\Phi_k^{(l)}}\right)\in\tilde{\mathcal{M}}_{df}$ satisfying

$$\tilde{\Phi}^{(l)}_{\{0\}} = \frac{U^{\Phi}_{\Lambda_l}}{|\Lambda_l|},\quad \tilde{\Phi}^{(l)}_{k,\{0\}} = \frac{U^{\Phi_k}_{\Lambda_l}}{|\Lambda_l|}.$$

Then, it follows that

$$p_{l,\mathfrak{m}}(n,\beta) = p_{n,\tilde{\mathfrak{m}}_l}(0,\beta_l),\quad \text{where } \beta_l = |\Lambda_l|\beta.$$

Hence, since $\tilde{\mathfrak{m}}_l$ is permutation invariant, by corollary 3.4.2 and from the fact that $(\tilde{\mathfrak{A}}_l)_{\{0\}}$ can be canonically identified with $\mathfrak{A}_{\Lambda_l}$, one has

$$P_{\tilde{\mathfrak{m}}_l} = \lim_{n\to\infty} p_{l,\mathfrak{m}}(n,\beta) = -\inf_{\omega\in E_{\Lambda_l}}\left\{\frac{\omega(U^{\Phi}_{\Lambda_l})}{|\Lambda_l|} + \sum_{k=1}^N \zeta_k\left|\frac{\omega(U^{\Phi_k}_{\Lambda_l})}{|\Lambda_l|}\right|^2 - \frac{S_{\Lambda_l}(\omega)}{\beta|\Lambda_l|}\right\}. \tag{3.20}$$

The right-hand side of eq. 3.20 can be seen as a minimization problem of a weak* continuous functional over the set $E_{\Lambda_l}$. Hence, it achieves a minimum. Let $\hat{\omega}_l \in E_{l.\mathbb{Z}^d}$ be the periodic extension of its minimizer, and let

$$\tilde{\omega}_l = \frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l}\hat{\omega}\circ\tau_x.$$

By proposition 3.1.12, $\tilde{\omega}_l \in \mathcal{E}_{\mathbb{Z}^d}$ for any $l\in\mathbb{N}$. In particular, $\Delta_A(\tilde{\omega}) = |\tilde{\omega}(A)|^2$ for any $A\in\mathfrak{A}$ and any $l\in\mathbb{N}$. Therefore, from lemma 3.4.3, eq. 3.20, and the translation invariance of the entropy density, it follows that

$$\lim_{l\to\infty}\lim_{n\to\infty} p_{l,\mathfrak{m}}(n,\beta) = -\lim_{l\to\infty} g_{\mathfrak{m}}(\tilde{\omega}_l) = -\lim_{l\to\infty} f_{\mathfrak{m}}(\tilde{\omega}_l),$$

and from eq. 3.19, one finally has

$$\limsup_{l\to\infty} p_{l,\mathfrak{m}} = -\lim_{l\to\infty} g_{\mathfrak{m}}(\tilde{\omega}_l) = -\lim_{l\to\infty} f_{\mathfrak{m}}(\tilde{\omega}_l) \leq -\inf_{\omega\in E_{\mathbb{Z}^d}} f_{\mathfrak{m}}(\omega).$$

□

## 3.5 Approximating interactions for purely attractive long-range models

In this section, it is shown that any purely attractive long-range model $\mathfrak{m}$ can be "approximated" – in a certain sense –, by a suitable interaction with no long-range component.

With this approximation, the thermodynamic pressure of $\mathfrak{m}$ can be obtained from a model with a simpler interaction. Moreover, its ergodic equilibrium states are intimately related to the equilibrium states of its approximate interaction, as it will be seen next. It is worth remembering that, when the long-range model is purely attractive, the free-energy density $f_{\mathfrak{m}}$ *always* achieves its minimum, and at some extreme point $\omega \in \mathcal{E}_{\mathbb{Z}^d}$. These observations are important for the development of the results presented in this section.

**Definition 3.5.1.** Let $\mathfrak{m} = (\Phi, \mathfrak{a})$ be a purely attractive long-range model (i.e., $\mathfrak{a}^+ = 0$). For $c \in L^2(\mathbb{S}, \mathbb{C}, \mathfrak{a}^-)$, define $\mathfrak{m}(c)$ as the long-range model given by $\mathfrak{m}(c) = (\Phi_{\mathfrak{m}(c)}, 0)$, where

$$\Phi_{\mathfrak{m}(c)} \doteq \Phi + \int_{\mathbb{S}} 2\,\mathrm{Re}[\overline{c}(\Psi)\Psi]\mathfrak{a}(d\Psi).$$

**Lemma 3.5.1.** *For any $\omega \in E_{\mathbb{Z}^d}$ and any measure $\mathfrak{a} \in \mathcal{S}$, one has*

$$\int_{\mathbb{S}} |\omega(\mathfrak{e}_\Psi)|^2 \mathfrak{a}^-(d\Psi) = \sup_{c \in L^2(\mathbb{S},\mathbb{C},\mathfrak{a}^-)} \left\{ -\|c\|_2^2 + \int_{\mathbb{S}} 2\,\mathrm{Re}[\overline{c}(\Psi) e_\Psi(\omega)]\mathfrak{a}_-(d\Psi) \right\},$$

*where the r.h.s. has a unique maximizer $d_\omega \in L^2(\mathbb{S}, \mathbb{C}, \mathfrak{a}^-)$ given by $d_\omega(\Psi) = \omega(\mathfrak{e}_\Psi)$.*

*Proof.* Note that, for any $\omega \in E_{\mathbb{Z}^d}$, any $A \in \mathfrak{A}$ and any $c \in \mathbb{C}$, one has

$$-|c|^2 + 2\,\mathrm{Re}[\overline{c}\omega(A)] = |\omega(A)|^2 - |\omega(A) - c|^2.$$

Therefore,

$$\sup_{c \in \mathbb{C}} \{-|c|^2 + 2\,\mathrm{Re}[\overline{c}\omega(A)]\} = |\omega(A)|^2,$$

with unique maximizer $d = \omega(A)$. Hence, since $d_\omega(\Psi) = \omega(\mathfrak{e}_\Psi)$ clearly belongs to $L^2(\mathbb{S}, \mathbb{C}, \mathfrak{a}^-)$, as $\Psi \mapsto \mathfrak{e}_\Psi$ and $\omega$ are continuous, it follows that

$$\begin{aligned}\int_{\mathbb{S}} |\omega(\mathfrak{e}_\Psi)|^2 \mathfrak{a}^-(d\Psi) &= \int_{\mathbb{S}} \left( \sup_{c \in \mathbb{C}} \{-|c|^2 + 2\,\mathrm{Re}[\overline{c} e_\Psi(\mathfrak{e}_\Psi)]\} \right) \mathfrak{a}^-(d\Psi) \\ &= \sup_{c \in L^2(\mathbb{S},\mathbb{C},\mathfrak{a}^-)} \left\{ -\|c\|_2^2 + \int_{\mathbb{S}} 2\,\mathrm{Re}[\overline{c}(\Psi) e_\Psi(\omega)]\mathfrak{a}^-(d\Psi) \right\},\end{aligned}$$

with unique maximizer $d_\omega(\Psi) = \omega(\mathfrak{e}_\Psi)$. □

**Proposition 3.5.1.** *Let $\mathfrak{m} = (\Phi, \mathfrak{a})$ be a purely attractive long-range model. Then, it follows that*

$$P_{\mathfrak{m}} = -\inf_{c \in L^2(\mathbb{S},\mathbb{C},\mathfrak{a}^-)} h_{\mathfrak{m}}(c) = -\inf_{c \in B_R} h_{\mathfrak{m}}(c),$$

*where*

$$h_{\mathfrak{m}}(c) \doteq \|c\|_2^2 - P_{\mathfrak{m}(c)}$$

*and $B_R$ is a closed ball at $0$ in $L^2(\mathbb{S}, \mathbb{C}, \mathfrak{a}^-)$ of sufficiently large radius $R > 0$.*

*Proof.* By theorem 3.4.4 and lemma 3.5.1, and since for an attractive long rang model $\mathfrak{m} = (\Phi, \mathfrak{a})$ $\mathfrak{a} = -\mathfrak{a}^-$, it follows that

$$\begin{aligned}
P_{\mathfrak{m}} &= -\inf_{\omega \in \mathcal{E}_{\mathbb{Z}^d}} \left\{ e_\Phi(\omega) + \int_{\mathbb{S}} |\omega(\mathfrak{e}_\Psi)|^2 \mathfrak{a}(d\Psi) - \beta^{-1} s(\omega) \right\} \\
&= -\inf_{\omega \in \mathcal{E}_{\mathbb{Z}^d}} \left( e_\Phi(\omega) - \sup_{c \in L^2(\mathbb{S},\mathbb{C},\mathfrak{a}_-)} \left\{ -\|c\|_2^2 - \int_{\mathbb{S}} 2\operatorname{Re}[\overline{c} e_\Psi(\mathfrak{e}_\Psi)] \mathfrak{a}(d\Psi) \right\} - \beta^{-1} s(\omega) \right) \\
&= -\inf_{\omega \in \mathcal{E}_{\mathbb{Z}^d}} \left( e_\Phi(\omega) + \inf_{c \in L^2(\mathbb{S},\mathbb{C},\mathfrak{a}_-)} \left\{ \|c\|_2^2 + \int_{\mathbb{S}} 2\operatorname{Re}[\overline{c} e_\Psi(\mathfrak{e}_\Psi)] \mathfrak{a}(d\Psi) \right\} - \beta^{-1} s(\omega) \right) \\
&= -\inf_{\omega \in \mathcal{E}_{\mathbb{Z}^d}} \inf_{c \in L^2(\mathbb{S},\mathbb{C},\mathfrak{a}_-)} \left\{ \|c\|_2^2 + e_\Phi(\omega) + \int_{\mathbb{S}} 2\operatorname{Re}[\overline{c} e_\Psi(\mathfrak{e}_\Psi)] \mathfrak{a}(d\Psi) - \beta^{-1} s(\omega) \right\} \\
&= -\inf_{\omega \in \mathcal{E}_{\mathbb{Z}^d}} \inf_{c \in L^2(\mathbb{S},\mathbb{C},\mathfrak{a}_-)} \left\{ \|c\|_2^2 + f_{\Phi_{\mathfrak{m}(c)}}(\omega) \right\} \qquad (3.21) \\
&= -\inf_{c \in L^2(\mathbb{S},\mathbb{C},\mathfrak{a}_-)} \left\{ \|c\|_2^2 - P_{\mathfrak{m}(c)} \right\} \\
&= -\inf_{L^2(\mathbb{S},\mathbb{C},\mathfrak{a}_-)} h_{\mathfrak{m}}(c).
\end{aligned}$$

Now, note that for any $\omega \in E_{\mathbb{Z}^d}$, $f_{\mathfrak{m}(c)}(\omega)$ is given by

$$\begin{aligned}
f_{\mathfrak{m}(c)}(\omega) =& e_\Phi(\omega) + 2\operatorname{Re}\left[ \int_{\mathbb{S}} \overline{c}(\Psi) e_\Psi(\omega) \mathfrak{a}(d\Psi) \right] - \beta s(\omega) \\
=& e_\Phi(\omega) - 2\operatorname{Re}\left[ (c, e_\Psi(\omega))_{L^2(\mathbb{S},\mathbb{C},\mathfrak{a}^-)} \right] - \beta s(\omega) \\
\leq& e_\Phi(\omega) - 2\operatorname{Re}\left[ (c, e_\Psi(\omega))_{L^2(\mathbb{S},\mathbb{C},\mathfrak{a}^-)} \right].
\end{aligned}$$

Therefore, using Cauchy-Schwarz inequality and other straight-forward estimates, it is not hard to see that

$$|P_{\mathfrak{m}(c)}| \leq 2\|\mathfrak{m}\|_{\mathcal{M}} \|c\|_2.$$

In particular, one has

$$\|c\|_2^2 - 2\|\mathfrak{m}\|_{\mathcal{M}}\|c\|_2 \leq h_{\mathfrak{m}}(c) \leq \|c\|_2^2 + 2\|\mathfrak{m}\|_{\mathcal{M}}\|c\|_2,$$

and hence since $(\|c\|_2^2 - 2\|\mathfrak{m}\|_{\mathcal{M}}\|c\|_2) \to \infty$ as $\|c\|_2 \to \infty$, the proposition follows. □

**Definition 3.5.2.** For any purely attractive long-range model $\mathfrak{m}$, define the set $\mathcal{C}_{\mathfrak{m}} \subset L^2(\mathbb{S}, \mathbb{C}, \mathfrak{a}^-)$ as

$$\mathcal{C}_{\mathfrak{m}} = \left\{ d \in L^2(\mathbb{S}, \mathbb{C}, \mathfrak{a}^-) \mid h_{\mathfrak{m}}(d) = \inf_{c \in B_R} h_{\mathfrak{m}}(c) = P_{\mathfrak{m}} \right\}.$$

**Definition 3.5.3.** Let $\mathfrak{H}$ be a Hilbert space. Define the *weak topology on* $\mathfrak{H}$ as the initial topology of its dual space $\mathfrak{H}^*$.

*Remark:* By the Riesz-Fréchet theorem, any Hilbert space $\mathfrak{H}$ can be canonically identified with its dual space $\mathfrak{H}^*$. Therefore, it follows that the weak topology on $\mathfrak{H}$ is equivalent to the weak* topology on $\mathfrak{H}^*$.

**Proposition 3.5.2.** *For any purely attractive long-range model* $\mathfrak{m}$*,* $h_{\mathfrak{m}}$ *is weak lower semi-continuous.*

*Proof.* For any $c \in \mathfrak{H} \doteq L^2(\mathbb{S}, \mathbb{C}, \mathfrak{a}^-)$, let $\phi_c$ denote the respective linear functional of Riesz-Fréchet theorem associated with $c$. Note that

$$\|c\|_2^2 = \sup_{x \in \mathfrak{H}} |\phi_c(x)|^2 = \sup_{x \in \mathfrak{H}} |f_x(\phi_c)|^2,$$

where $f_x : \mathfrak{H}^* \to \mathbb{C}$ is the linear functional given by

$$f_x(\phi) = \phi(x).$$

Clearly $f_x$ is weak* continuous, and by proposition 3.2.2 $\phi \mapsto \sup_{x \in \mathfrak{H}} |f_x(\phi)|^2$ is weak* lower semi-continuous. Therefore, by the above remark, it follows that the mapping

$$c \mapsto \|c\|_2^2$$

is weak lower semi-continuous. Now, note that the map

$$c \mapsto f_{\mathfrak{m}(c)}(\omega) = e_\Phi(\omega) - 2\,\mathrm{Re}\left[(c, e_\Psi(\omega))_{L^2(\mathbb{S},\mathbb{C},\mathfrak{a}^-)}\right] - \beta s(\omega)$$

is weak continuous. For any $\omega \in E_{\mathbb{Z}^d}$. In fact, it follows that the family of mappings

$$\{c \mapsto f_{\mathfrak{m}(c)}(\omega)\}_{\omega \in E_{\mathbb{Z}^d}}$$

is weak equicontinuous. Hence, it is not hard to see that this implies that

$$c \mapsto P_{\mathfrak{m}(c)} = -\min_{\omega \in E_{\mathbb{Z}^d}} \{f_{\mathfrak{m}(c)}(\omega)\}$$

is weak continuous. Thus, the proposition follows. □

**Proposition 3.5.3.** *For any purely attractive long-range model* $\mathfrak{m}$*,* $\mathcal{C}_{\mathfrak{m}}$ *is non-empty, norm-bounded and weakly compact.*

*Proof.* From the estimate

$$\|c\|_2^2 - 2\|\mathfrak{m}\|_{\mathcal{M}}\|c\|_2 \leq h_{\mathfrak{m}}(c)$$

it follows that $\mathcal{C}_{\mathfrak{m}}$ is norm-bounded. Now, note that from the above remark on the weak topology of a Hilbert space and from the Banach-Alaoglu theorem, it follows that any closed ball of finite radius is weak compact on a Hilbert space. Hence, from the weak lower semi-continuity of $h_{\mathfrak{m}}$, the proposition follows. □

**Proposition 3.5.4.** *For any purely attractive long-range model* $\mathfrak{m}$*, it follows that*

*(a) for any* $\omega \in \Omega_{\mathfrak{m}} \cap \mathcal{E}_{\mathbb{Z}^d}$*,*

$$d = \Psi \mapsto e_\Psi(\omega) \in \mathcal{C}_{\mathfrak{m}},$$

*and* $\omega \in \Omega_{\mathfrak{m}(d)}$*,* *(b) For any* $d \in \mathcal{C}_{\mathfrak{m}}$, $\Omega_{\mathfrak{m}(d)} \cap \mathcal{E}_{\mathbb{Z}^d} \subset \Omega_{\mathfrak{m}} \cap \mathcal{E}_{\mathbb{Z}^d}$ *and any* $\omega \in \Omega_{\mathfrak{m}(d)}$ *satisfy*

$$d = \Psi \mapsto e_{\Psi}(\omega). \tag{3.22}$$

*Proof.* $(a)$ Let $\omega \in \Omega_{\mathfrak{m}} \cap \mathcal{E}_{\mathbb{Z}^d}$. By lemma 3.5.1, $d$ is a solution of

$$\inf_{c \in L^2(\mathbb{S},\mathbb{C},\mathfrak{a}^-)} \{\|c\|_2^2 + f_{\mathfrak{m}(c)}(\omega)\}.$$

Moreover, since $\mathfrak{m}$ is purely attactive, $\omega$ is the solution of the first infinum of eq. 3.21, $d \in \mathcal{C}_{\mathfrak{m}}$, and since the two infima of 3.21 commute, $\omega \in \Omega_{\mathfrak{m}(d)}$.

$(b)$ If $d \in \mathcal{C}_{\mathfrak{m}}$, then $d$ is a solution of

$$\inf_{c \in L^2(\mathbb{S},\mathbb{C},\mathfrak{a}^-)} \left\{ \|c\|_2^2 + \inf_{\omega \in E_{\mathbb{Z}^d}} f_{\mathfrak{m}(c)}(\omega) \right\}, \tag{3.23}$$

and since the two infima in eq. 3.23 commute, any $\omega \in \Omega_{\mathfrak{m}(d)}$ must satisfy eq. 3.22 due to lemma 3.5.1. Moreover, $\Omega_{\mathfrak{m}(d)} \cap \mathcal{E}_{\mathbb{Z}^d} \subset \Omega_{\mathfrak{m}} \cap \mathcal{E}_{\mathbb{Z}^d}$ from eq. 3.21. □

# Chapter 4

# The model

In the last chapter of the thesis, the thermodynamics of a specific purely attractive long-range model is analyzed. The model possesses a BCS interaction term, of strength $\gamma$, and a locally repulsive term between electrons occupying the same site, of strength $\lambda$. First, the model is analyzed in the so-called *strong-coupling limit*: when the kinetic energy term is ignored, and hence it becomes permutation invariant. In this scenario, it is shown that for a certain choice of parameters $(\gamma, \lambda)$ there exists a coexistence of magnetic and superconducting phases. Then, the kinetic term added to the model, and it is shown that if the term is small, the coexistence also holds true.

## 4.1 Permutation-invariant approximation

**Definition 4.1.1.** Let $\tilde{\Psi} \in \mathbb{S}$ be the p.i. interaction given by

$$\tilde{\Psi}_{\{x\}} = a_{x,\downarrow} a_{x,\uparrow},$$

and let $\tilde{\mathfrak{m}}_0 = (\tilde{\Phi}^{(0)}, \tilde{\mathfrak{a}}) \in \mathcal{M}$ be the p.i. purely attractive long-range model defined as

$$\tilde{\Phi}^{(0)}_{\{x\}} = -\mu(n_{x,\uparrow} + n_{x,\downarrow}) - h(n_{x,\uparrow} - n_{x,\downarrow}) + 2\lambda n_{x,\uparrow} n_{x,\downarrow}, \quad \mu, h, \lambda \in \mathbb{R},$$

where $n_{x,s} = a^*_{x,s} a_{x,s}$, and

$$\tilde{a}(X) = \begin{cases} -\gamma, & \text{if } \tilde{\Psi} \in X, \\ 0, & \text{otherwise,} \end{cases} \qquad \gamma \geq 0.$$

Note that, for $l \in \mathbb{N}$, the internal long-range energy of $\tilde{\mathfrak{m}}_0$ on $\Lambda_l$ is

$$U^{\tilde{\mathfrak{m}}_0}_{\Lambda_l} = \sum_{x \in \Lambda_l} \left(-\mu(n_{x,\uparrow} + n_{x,\downarrow}) - h(n_{x,\uparrow} - n_{x,\downarrow}) + 2\lambda n_{x,\uparrow} n_{x,\downarrow}\right) - \frac{\gamma}{|\Lambda_l|} \sum_{x,y \in \Lambda_l} a^*_{x,\uparrow} a^*_{x,\downarrow} a_{y,\downarrow} a_{y,\uparrow},$$

i.e., it is the BCS-Hubbard model in the so-called strong coupling limit (no kinetic energy). Since this model is p.i., its thermodynamic pressure can be easily calculated from the formalism developed in the previous chapter.

**Proposition 4.1.1.** *The pressure* $P_{\tilde{\mathfrak{m}}_0}$ *of the long-range model* $\tilde{\mathfrak{m}}_0$ *is given by*

$$P_{\tilde{\mathfrak{m}}_0} = \beta^{-1} \ln 2 + \mu - \inf_{c \in \mathbb{C}} f(c),$$

*where*

$$f(c) \doteq \gamma |c|^2 - \frac{1}{\beta} \ln \left( \cosh(\beta h) + e^{-\lambda\beta} \cosh\left(\beta g_c\right) \right), \quad \textit{and} \quad g_c = \sqrt{(\mu - \lambda)^2 + \gamma^2 |c|^2}.$$

*Moreover, for any* $d \in \mathbb{C}$ *minimizer of* $f(c)$ *there exists an ergodic equilibrium state* $\omega \in \Omega_{\tilde{\mathfrak{m}}_0} \cap \mathcal{E}_{\mathbb{Z}^d}$ *satisfying* $\omega(a_{x,\downarrow} a_{x,\uparrow}) = d$.

*Proof.* First, note that $L^2(\mathbb{S}, \mathbb{C}, \tilde{\mathfrak{a}}^-) \cong \mathbb{C}$. By proposition 3.5.1, it follows that

$$P_{\tilde{\mathfrak{m}}_0} = - \inf_{c \in \mathbb{C}} \left\{ \gamma |c|^2 - P_{\tilde{\mathfrak{m}}_0(c)} \right\},$$

where for any $c \in \mathbb{C}$, $P_{\tilde{\mathfrak{m}}_0(c)}$ is the thermodynamic pressure of the p.i. long-range model $\tilde{\mathfrak{m}}_0(c) = (\tilde{\Phi}_c^{(0)}, 0)$ satisfying

$$\tilde{\Phi}_{c,\{x\}}^{(0)} = \tilde{\Phi}_{\{x\}}^{(0)} - \gamma(\bar{c} a_{x,\downarrow} a_{x,\uparrow} + c a_{x,\uparrow}^* a_{x,\downarrow}^*).$$

But since $\tilde{\mathfrak{m}}_0(c)$ is p.i. and has no long-range component, its thermodynamic pressure can be explicitly calculated. By propositions 3.1.9 and 3.1.8, it follows that

$$Z_{\Lambda_l} = \mathrm{Tr}_{\mathfrak{A}_{\Lambda_l}} \left( e^{-\beta U_{\Lambda_l}^{\tilde{\mathfrak{m}}_0(c)}} \right) = \mathrm{Tr}_{\mathfrak{A}_{\{0\}}} \left( e^{-\beta U_{\{0\}}^{\tilde{\mathfrak{m}}_0(c)}} \right)^{|\Lambda_l|},$$

and hence, for any $l \in \mathbb{N}$, the finite-volume pressures $p_{l,\tilde{\mathfrak{m}}_0(c)}$ are unchanged and given by

$$p_{l,\tilde{\mathfrak{m}}_0(c)} = \frac{1}{\beta} \ln \left( \mathrm{Tr}_{\mathfrak{A}_{\{0\}}} \left( e^{-\beta U_{\{0\}}^{\tilde{\mathfrak{m}}_0(c)}} \right) \right).$$

Since $\mathfrak{A}_{\{0\}} \cong M_{4\times 4}(\mathbb{C})$, $p_{l,\tilde{\mathfrak{m}}_0(c)}$ can be computed, and one has

$$P_{\tilde{\mathfrak{m}}_0(c)} = p_{l,\tilde{\mathfrak{m}}_0(c)} = -\beta^{-1} \ln 2 - \mu + \frac{1}{\beta} \ln \left( \cosh(\beta h) + e^{-\lambda\beta} \cosh\left(\beta g_c\right) \right),$$
$$\text{where } g_c = \sqrt{(\mu - \lambda)^2 + \gamma^2 |c|^2}.$$

Therefore, the first part of the proposition follows. The second part is a straightforward consequence of proposition 3.5.4. □

*Remark:* note that since $f(c)$ depends only on the modulus of $c$, it follows that $\inf_{c\in\mathbb{C}} f(c) = \inf_{x\in\mathbb{R}} f(x)$. From now on, $f$ will be seen as a function over the real numbers.

A fermionic system where the expected value of $a_{x,\downarrow} a_{x,\uparrow}$ (usually called the *superconducting gap* or *pairing amplitude*) is different from zero is said to be in a *superconducting phase*.

**Definition 4.1.2.** The *magnetic susceptibility* $\chi$ of $P_{\tilde{\mathfrak{m}}_0}$ is defined as

$$\chi = \frac{\partial^2 P_{\tilde{\mathfrak{m}}_0}}{dh^2}.$$

**Proposition 4.1.2.** *For $\mu = h = 0$, in a neighborhood of a point $(\lambda, \gamma)$ where $f(x)$ has a unique minimizer $d = 0$, the magnetic susceptibility $\chi$ is given by*

$$\frac{\beta}{1 + (1 + e^{-2\beta\lambda})/2}.$$

*Proof.* The proposition follows from a straightforward evaluation of $\frac{\partial^2 P_{\tilde{\mathfrak{m}}_0}}{dh^2}$. □

Note that, under the conditions of proposition 4.1.2 and if $\lambda > 0$, when $\beta \to \infty$, the magnetic susceptibility goes to $\infty$. Therefore, when 0 is the unique minimizer of $f$, the system is said to be in a *magnetic phase*.

**Proposition 4.1.3.** *For $\beta = 1$ and $\mu = h = 0$, there exists some parameters $\bar{\lambda}, \bar{\gamma} > 0$ such that the unfolding $F(x, \lambda, \gamma) = f(x; \bar{\lambda} + \lambda, \bar{\gamma} + \gamma) - f(0; \bar{\lambda}, \bar{\gamma})$ is e-transversal, with $\sigma(F(x, 0, 0)) = \sigma(f(x; \bar{\lambda}, \bar{\gamma})) = 6$.*

*Proof.* By the parity symmetry of $f$ it follows that $\frac{\partial^k f(0,\lambda,\gamma)}{\partial x^k} = 0$ for any odd $k$ and any $\lambda, \gamma > 0$. Moreover, by direct calculations, it can be seen that the system of equations

$$\begin{cases} \frac{\partial^2 f(0;\lambda,\gamma)}{\partial x^2} = 0 \\ \frac{\partial^4 f(0;\lambda,\gamma)}{\partial x^4} = 0 \end{cases} \qquad \lambda, \gamma > 0,$$

has a unique solution $(\bar{\lambda}, \bar{\gamma})$, given by

$$(\bar{\lambda}, \bar{\gamma}) =\simeq (0.75, 6.22),$$

that also satisfies $\frac{\partial^6 f(0,\bar{\lambda},\bar{\gamma})}{\partial x^6} \neq 0$. Hence, defining $F(x, \lambda, \gamma) = f(x; \bar{\lambda} + \lambda, \bar{\gamma} + \gamma) - f(0; \bar{\lambda}, \bar{\gamma})$, it follows that $\sigma(F(x, 0, 0)) = \sigma(f(x; \bar{\lambda}, \bar{\gamma})) = 6$. Furthermore, calculating $j^4(\alpha_1(F)), j^4(\alpha_2(F))$, one gets

$$j^4(\alpha_1(F)) \simeq 0.14 + 3.86x^2 - 10.2x^4, \text{ and } j^4(\alpha_2(F)) = -x^2 \implies \\ \langle 1, j^4(\alpha_1(F)), j^4(\alpha_2(F))\rangle_{\mathbb{R}} = \langle 1, x^2, x^4\rangle_{\mathbb{R}},$$

i.e., $F(x, \lambda, \gamma)$ is e-transversal. □

**Proposition 4.1.4.** *For $\beta = 1$, $\mu = h = 0$ and any small enough neighborhood $U$ of $(\bar{\lambda}, \bar{\gamma}) \in \mathbb{R}^2$, it follows that*

*(a) there exists an open subset of $U$ where $x = 0$ is the unique minimizer of $f$,*

*(b) there exists an open subset of $U$ where $f$ has only two symmetric minimizers $x = \pm d$ for some $d > 0$,*

*(c) there exists a curve in $U$ where $f$ has three distinct minimizers: $x = 0$, and $x = \pm d$ for some $d > 0$.*

*Proof.* Since, by proposition 4.1.3, the 2-unfolding $F(x, \lambda, \gamma) = f(x; \bar{\lambda} + \lambda, \bar{\gamma} + \gamma)$ is e-transversal, by Thom's theorem (theorem 2.4.2) it follows that $F$ is equivalent to the unfolding $G(x, u_1, u_2) = x^6 + u_1 x^2 + u_2 x^4$, i.e., there exists some small neighborhood of $V$ of $0 \in \mathbb{R}^2$, some small neighborhood $I$ of $0 \in \mathbb{R}$, and some smooth functions:

(a) $\phi : I \times V \to \mathbb{R}$, where $\phi(0, 0, 0) = 0$ and $x \mapsto \phi(x, u_1, u_2)$ is a diffeomorphism for all $(u_1, u_2) \in U$,

(b) $\psi : V \to \mathbb{R}^2$, where $\psi(0, 0) = (0, 0)$ and $\psi$ is a diffeomorphism,

(c) $\kappa : V \to \mathbb{R}$,

such that in $I \times V$,

$$F(x, u_1, u_2) = G(\phi(x, u_1, u_2), \psi(u_1, u_2)) + \kappa(u_1, u_2).$$

Therefore, studying the critical points of $F$ in $I \times V$ is equivalent to studying the critical points of $G$. Figure 4.1 shows the behavior of the critical points of $G$.

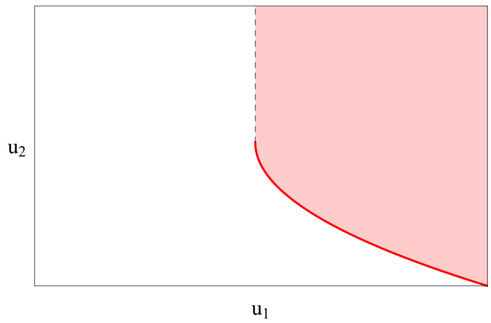


Figure 4.1: Distribution of the minima of $x^6 + u_1 x^2 + u_2 x^4$. In the red region, the only minimum is $x = 0$. In the white region, $x = 0$ is not a minimum, and in the red line $\left(u_2 = -2\sqrt{u_1}\right)$ the minima are in $x = 0, \pm u_1^{1/4}$.

To extend the results for any $x \in \mathbb{R}$ possibly outside of $I$, it suffices to note that, at $(\lambda, \gamma) = (\bar{\lambda}, \bar{\gamma})$, $x = 0$ is the unique critical point of $f$. Hence, there exists a neighborhood $U \subset (\bar{\lambda}, \bar{\gamma}) + V$ of $(\bar{\lambda}, \bar{\gamma})$ where the only critical points of $f$ are inside $I$. $\square$

Propositions 4.1.1 and 4.1.4 imply that for any $(\lambda, \gamma)$ that lies in the curve mentioned in $(c)$ of proposition 4.1.4 (i.e., any $(\lambda, \gamma)$ such that $\psi(\lambda - \bar{\lambda}, \gamma - \bar{\gamma})$ lies in the red line of figure 4.1), the correspondent fermionic system is in a coexistence of magnetic and superconducting phases. In order to establish the correct proportions of each phase, an analysis must be made fixing the electron density of the system, as it is done in [3].

## 4.2 Cluster expansions

In this last section, the goal now is to prove that the coexistence of magnetic and superconducting phases also holds with the introduction of a small kinetic term in the interaction analyzed above. However, with the introduction of a kinetic term, the new interaction fails to be permutation invariant, and thus its thermodynamic pressure cannot be easily obtained as before. Therefore, to analyze the new perturbed thermodynamic pressure, it is used here a technique known as "cluster expansions". This technique allows one to prove that the derivatives of the perturbed pressure is "well-behaved", and hence by arguments coming from catastrophe theory, it can be shown that ,for a small perturbation, the minima of the new pressure has the same behavior as the minima of the unperturbed pressure.

**Definition 4.2.1.** Let $\mathcal{G}_n$ denote the set of all unoriented graphs with $n$ vertices, and $\mathcal{C}_n \subset \mathcal{G}_n$ denote the set of all connected unoriented graphs with $n$ vertices. Define a function $\phi$ on the space of finite sequences $(x_1, \ldots, x_n)$ of $X$ by:

$$\phi(x_1, \ldots, x_n) \doteq \begin{cases} 1, & \text{if } n = 1, \\ \frac{1}{n!} \sum_{G \in \mathcal{C}_n} \prod_{(i,j) \in G} \zeta(x_i, x_j) & \text{if } n \geq 2, \end{cases}$$

where $\prod_{(i,j) \in G}$ is to be understood as a product over the edges of $G$. A finite sequence $(x_1, \ldots, x_n)$ in $X$ is said to be a *cluster* if the graph with $n$ vertices and edges between $i$ and $j$ whenever $\zeta(x_i, x_j) \neq 0$ is connected.

The following theorem will not be proven here, but its proof can be found at [12].

**Theorem 4.2.1.** *Let $(X, \mathcal{A}, \mu)$ be a measurable space, where $\mu$ is a signed measure with finite variation, $\zeta$ a measurable symmetric function on $X \times X$, and $Z$ given by*

$$Z = \sum_{n \geq 0} \frac{1}{n!} \int d\mu(x_1) \cdots \int d\mu(x_n) \prod_{1 \leq i < j \leq n} (1 + \zeta(x_i, x_j)).$$

*Assuming that $|1 + \zeta(x, x')| \leq 1$ for all $x, x' \in X$, and that there exists a nonnegative function $a : X \to \mathbb{R}$ such that*

$$\int d|\mu|(x')|\zeta(x, x')|e^{a(x')} \leq a(x) \textit{ for all } x \in X, \quad \textit{and} \quad \int d|\mu|(x)e^{a(x)} < \infty,$$

*then*

$$Z = \exp\left(\sum_{n \geq 1} \int d\mu(x_1) \cdots \int d\mu(x_n) \phi(x_1, \ldots, x_n)\right),$$

*where the combined sum and integral converges absolutely. Furthermore, for all $x_1 \in X$,*

$$1 + \sum_{n \geq 2} n \int d|\mu|(x_2) \cdots \int d|\mu|(x_n)|\phi(x_1, \ldots, x_n)| \leq e^{a(x_1)}.$$

**Definition 4.2.2.** Let $A_1, \ldots A_m \in \mathcal{P}_f(\mathbb{Z}^d)$. Define $\mathcal{G}_{A_1, \ldots, A_m}$ as the unoriented graph of $m$ vertices and edges between $i$ and $j$ whenever $A_i \cap A_j \neq \varnothing$. Moreover, a set $A \in \mathcal{P}_f(\mathbb{Z}^d)$ is said to be a *polymer* if $A$ us connected.

**Definition 4.2.3.** Let $\Phi^{(0)}, \Phi^{(\epsilon)} \in \mathcal{W}$, such that $\Phi^{(0)}$ is p.i. and $\Phi^{(\epsilon)}_\Lambda = 0$ if $\Lambda$ is not connected, and let $\Phi = \Phi^{(0)} + \Phi^{(\epsilon)}$. For any polymer $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$, define

$$\rho(\Lambda) = \frac{1}{Z_{0,\Lambda}} \sum_{m=1}^{\infty} (-1)^m \sum_{A_1,\ldots,A_m}{}' \int_0^\beta \int_0^{\tau_m} \cdots \int_0^{\tau_2} \mathrm{Tr}_{\mathfrak{A}_\Lambda} \left( \Phi^{(\epsilon)}_{A_1}(\tau_1) \ldots \Phi^{(\epsilon)}_{A_m}(\tau_m) e^{-\beta U^{\Phi^{(0)}}_\Lambda} \right) \times d\tau_1 \ldots d\tau_m,$$

where

$$\Phi^{(\epsilon)}_{A_i}(\tau) = e^{-\tau U^{\Phi^{(0)}}_\Lambda} \Phi^{(\epsilon)}_{A_1} e^{\tau U^{\Phi^{(0)}}_\Lambda}, \quad Z_{0,\Lambda} = \mathrm{Tr}_{\mathfrak{A}_\Lambda} \left( e^{-\beta U^{\Phi^{(0)}}_\Lambda} \right) = \mathrm{Tr}_{\mathfrak{A}_{\{0\}}} \left( e^{-\beta \Phi^{(0)}_{\{0\}}} \right)^{|\Lambda|},$$

and the sum $\sum{}'$ is over polymers $A_1, \ldots A_m \subset \Lambda$ such that $\bigcup_{i=1}^m A_i = \Lambda$ and the graph $\mathcal{G}_{A_1,\ldots,A_m}$ is connected.

The proof of the following proposition is extensive, and hence it is left to appendix B.

**Proposition 4.2.1.** *Let $\Phi^{(0)}, \Phi^{(\epsilon)} \in \mathcal{W}$, such that $\Phi^{(0)}$ is p.i. and $\Phi^{(\epsilon)}_\Lambda = 0$ if $\Lambda$ is not connected, and let $\Phi = \Phi^{(0)} + \Phi^{(\epsilon)}$. Then, for any $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$, one has*

$$Z_\Lambda = Z_0^{|\Lambda|} \left( \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{A_1} \rho(A_1) \cdots \sum_{A_n} \rho(A_n) \prod_{1 \le i < j \le n} (1 + \zeta(A_i, A_j)) \right),$$

*where $Z_\Lambda$ is the partition function of $U^\Phi_\Lambda$, $Z_0$ is the partition function of $U^{\Phi^{(0)}}_{\{0\}}$,*

$$\zeta(A_i, A_j) = \begin{cases} -1, & \textit{if } A_i \cap A_j \neq 0, \\ 0, & \textit{otherwise}, \end{cases}$$

*and the sum $\sum_{A_i}$ is over all polymers of $\Lambda$.*

**Definition 4.2.4.** For any $\epsilon > 0$, define $\tilde{\Phi}^{(\epsilon)}$ as the short-range interaction given by

$$\tilde{\Phi}^{(\epsilon)}_\Lambda = \begin{cases} \epsilon(a^*_{x,\uparrow} a_{x+y,\uparrow} + a^*_{x,\downarrow} a_{x+y,\downarrow} + a^*_{x+y,\uparrow} a_{x,\uparrow} + a^*_{x+y,\downarrow} a_{x,\downarrow}), & \text{if } |y| = 1, y_i \geq 0 \text{ for all } i = 1, \ldots d \text{ and } \Lambda = \{x, x+y\} \\ 0, & \text{otherwise}, \end{cases}$$

and let $\tilde{\mathfrak{m}}$ be the purely attractive long-range model given by

$$\tilde{\mathfrak{m}} = \tilde{\mathfrak{m}}_0 + (\tilde{\Phi}^{(\epsilon)}, 0) = (\tilde{\Phi}^{(0)} + \tilde{\Phi}^{(\epsilon)}, \tilde{\mathfrak{a}}) = (\tilde{\Phi}, \tilde{\mathfrak{a}}),$$

with $\tilde{\Phi}^{(0)}$ and $\tilde{\mathfrak{a}}$ as in definition 4.1.1.

*Remark:* It is easy to see that $\|\tilde{\Phi}^{(\epsilon)}\|_{\mathcal{W}}$ can be estimated by

$$\|\tilde{\Phi}^{(\epsilon)}\|_{\mathcal{W}} \leq 2d\epsilon,$$

and that its internal energy $U^{\tilde{\Phi}^{(\epsilon)}}_{\Lambda_l}$ is given by

$$U_{\Lambda_l}^{\tilde{\Phi}^{(\epsilon)}} = \sum_{x\in\Lambda_l} \sum_{|y|=1, x+y\in\Lambda_l} \epsilon(a_{x,\uparrow}^* a_{x+y,\uparrow} + a_{x,\downarrow}^* a_{x+y,\downarrow}).$$

Hence, $\tilde{\Phi}^{(\epsilon)}$ can be seen as a perturbative kinetic energy term.

Note that the interaction $\tilde{\Phi}_c$ of $\tilde{\mathfrak{m}}(c) = (\tilde{\Phi}_c, 0)$ is given by $\tilde{\Phi}_c = \tilde{\Phi}_c^{(0)} + \tilde{\Phi}^{(\epsilon)}$, since the long-range component of $\tilde{\mathfrak{m}}$ and $\tilde{\mathfrak{m}}_0$ are the same. Moreover, for any $\theta \in [0, 2\pi)$, let $\sigma_\theta$ be the *-automorphism satisfying $\sigma_\theta(a_{x,s}^*) = e^{i\theta/2} a_{x,s}^*$ for any $x \in \mathbb{Z}^d$ and $s \in S$. It is easy to see that taking $\sigma_\theta(U_{\Lambda_l}^{\tilde{\Phi}})$ corresponds to the change $c \mapsto ce^{i\theta}$. Thus, by the invariance of the trace under *-automorphisms, it follows that the pressure $P_{\tilde{\mathfrak{m}}(c)}$ is invariant under the change $c \mapsto ce^{i\theta}$. Therefore, as in the previous section, $c$ will be regarded as a real number.

**Definition 4.2.5.** Let $D \subset \mathbb{R}^6$ be an open domain, let $\mathbf{v} = (c, \gamma, \lambda, h, \mu, \epsilon)$, and given a six-index $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$, let

$$\frac{\partial^{|\alpha|}}{\partial \mathbf{v}^\alpha} = \frac{\partial^{|\alpha|}}{\partial c^{\alpha_1} \partial \gamma^{\alpha_2} \partial \lambda^{\alpha_3} \partial h^{\alpha_4} \partial \mu^{\alpha_5} \partial \epsilon^{\alpha_6}}.$$

For any $r > 0$, define $\mathcal{RA}(D, r, \mathbb{R})$ as the subset of $C^\infty(D, \mathbb{R})$ where the norm

$$\|f\|_{D,r} \doteq \sum_\alpha \frac{r^{|\alpha|}}{\alpha!} \sup_{\boldsymbol{x}\in D} \left| \frac{\partial^{|\alpha|} f(\boldsymbol{x})}{\partial \mathbf{v}^\alpha} \right|$$

is finite.

*Remark:* Note that every function in $\mathcal{RA}(D, r, \mathbb{R})$ is, in particular, real analytic on $D$. Moreover, it is a well-known fact that $\mathcal{RA}(D, r, \mathbb{R})$ is complete with respect to the norm $\|\cdot\|_{D,r}$.

**Proposition 4.2.2.** *If the product $\beta\epsilon$ satisfies*

$$\beta\epsilon \le \frac{e^{-2k}}{16d}, \quad \text{where} \quad k = 3\ln 2 + 2\ln(2d\phi) + \phi^{-1} + 1 \quad \text{and} \quad \phi = \frac{\sqrt{5}+1}{2}, \tag{4.1}$$

*then, it follows that $p_{l,\tilde{\mathfrak{m}}(c)} = p_{l,\tilde{\mathfrak{m}}_0(c)} + p_{l,\epsilon}$, where*

$$p_{l,\epsilon} = \frac{1}{\beta|\Lambda_l|} \sum_{n=1}^\infty \sum_{A_1} \rho(A_1) \cdots \sum_{A_n} \rho(A_n) \phi(A_1, \ldots, A_n),$$

*where the sums $\sum_{A_i}$ are over all polymers of $\Lambda_l$. Moreover, for any bounded open set $D \subset \mathbb{R}^6$, there exists some $r > 0$ such that $\|p_{l,\epsilon}\|_{D,r} < \infty$.*

*Proof.* First, note that

$$\left\| \frac{\partial^{|\alpha|} e^{-\tau U_\Lambda^{\tilde{\Phi}^{(0)}}}}{\partial \mathbf{v}^\alpha} \right\| \le (4\tau|\Lambda|)^{|\alpha|} \left\| e^{-\tau U_\Lambda^{\tilde{\Phi}^{(0)}}} \right\|,$$

and

$$\left\| \frac{\partial^{|\alpha|} \tilde{\Phi}^{(\epsilon)}_{A_i}}{\partial \boldsymbol{\nu}^{\alpha}} \right\| \leq 4, \quad \text{with} \quad \|\tilde{\Phi}^{(\epsilon)}_{A_i}\| \leq 4\epsilon.$$

Therefore, let $D$ some bounded open set in $\mathbb{R}^6$ and let $r > 0$ be chosen such that

$$\left\| e^{-\beta \tilde{\Phi}^{(0)}_{\{0\}}} \right\|_{D,r} \leq \sqrt{2} \left\| e^{-\beta \tilde{\Phi}^{(0)}_{\{0\}}} \right\|, \quad \left\| \tilde{\Phi}^{(\epsilon)}_{A_i} \right\|_{D,r} \leq 2\|\tilde{\Phi}^{(\epsilon)}_{A_i}\|, \quad \text{and} \quad \|Z^{-1}_{0,\{0\}}\|_{D,r} \leq \sqrt{2} Z^{-1}_{0,\{0\}}.$$

Note that $r$ does not depend on $\Lambda$. Let $\lambda_0$ be the lowest eigenvalue of $\tilde{\Phi}^{(0)}_{\{0\}} \in \mathfrak{A}_{\{0\}} \cong M_{4\times 4}(\mathbb{C})$. Since $\tilde{\Phi}^{(0)}_{\{0\}}$ is self-adjoint, one has

$$\left\| e^{-\beta \tilde{\Phi}^{(0)}_{\{0\}}} \right\| = e^{-\beta\lambda_0} \leq \mathrm{Tr}_{\mathfrak{A}_{\{0\}}} \left( e^{-\beta \tilde{\Phi}^{(0)}_{\{0\}}} \right) \implies \left\| e^{-\beta \tilde{\Phi}^{(0)}_{\{0\}}} \right\|^{|\Lambda|} Z^{-1}_{0,\Lambda} \leq 1.$$

Therefore, one may estimate $\|\rho(\Lambda)\|_{D,r}$ by

$$\begin{aligned} \|\rho(\Lambda)\|_{D,r} \leq & 2^{2|\Lambda|} 2^{|\Lambda|} \left\| e^{-\beta \tilde{\Phi}^{(0)}_{\{0\}}} \right\|^{|\Lambda|} Z^{-1}_{0,\Lambda} \sum_{m=1}^{\infty} \frac{\beta^m}{m!} \sum_{A_1,\ldots,A_m} {}' \prod_{i=1}^{m} 2\|\tilde{\Phi}^{(\epsilon)}_{A_i}\| \\ \leq & e^{3|\Lambda| \ln 2} e^{-k|\Lambda|} \sum_{m=1}^{\infty} \frac{\beta^m}{m!} \sum_{A_1,\ldots,A_m} {}' \prod_{i=1}^{m} \left( 2\|\tilde{\Phi}^{(\epsilon)}_{A_i}\| e^{k|A_i|} \right). \end{aligned}$$

Now, note that there are at most $d|\Lambda|$ subsets of $\Lambda$ for which $\tilde{\Phi}^{(\epsilon)}_{A} \neq 0$, and those subsets also satisfy $|A| = 2$. Therefore, the sum $\sum_{A_1,\ldots,A_m}{}'$ has at most $(d|A|)^m$ elements, and if $\beta\epsilon$ satisfies the bound 4.1, one has

$$\|\rho(\Lambda)\|_{D,r} \leq e^{(3\ln 2 - k)|\Lambda|} \sum_{m=1}^{\infty} \frac{1}{m!} \left( \beta\epsilon 16 d e^{2k} |\Lambda| \right)^m \leq e^{(3\ln 2 - k + 1)|\Lambda|} = e^{-(2\ln(2d\phi) + \phi^{-1})|\Lambda|}.$$

Hence, since $\mathcal{RA}(D, r, \mathbb{R})$ is complete with respect to the norm $\| \cdot \|_{D,r}$, it follows that $\|\rho(\Lambda)\|_{D,r}$ is indeed well-defined and $\rho(\Lambda) \in \mathcal{RA}(D, r, \mathbb{R})$. Moreover, following the same steps as in ([12], 4.2), choosing the functional $a$ over the polymers of $\Lambda_l$ given by $a(A) = \phi^{-1}|A|$, it follows that

$$\sum_{A'} \|\rho(A')\|_{D,r} |\zeta(A, A')| e^{a(A')} \leq a(A),$$

where the sum is over the polymers of $\Lambda_l$. Therefore, by theorem 4.2.1, the sum

$$\sum_{n=1}^{\infty} \sum_{A_1} \|\rho(A_1)\|_{D,r} \cdots \sum_{A_n} \|\rho(A_n)\|_{D,r} |\phi(A_1, \ldots, A_n)|$$

converges. Hence, again since $\mathcal{RA}(D, r, \mathbb{R})$ is complete with respect to the norm $\| \cdot \|_{D,r}$, it follows that $p_{l,\epsilon} \in \mathcal{RA}(D, r, \mathbb{R})$, and this proves the second part of the proposition. The first part follows by noting that $|\rho(\Lambda)| \leq \|\rho(\Lambda)\|_{D,r}$, and applying proposition 4.2.1 and theorem 4.2.1 for the signed weights $\rho(\Lambda)$. □

**Proposition 4.2.3.** *The pressure $P_{\tilde{\mathfrak{m}}}$ of the long-range model $\tilde{\mathfrak{m}}$ is given by*

$$P_{\tilde{\mathfrak{m}}} = \beta^{-1} \ln 2 + \mu - \inf_{c \in \mathbb{R}} \{f(c) + p_\epsilon\},$$

*where $p_\epsilon = \lim_{l\to\infty} p_{l,\epsilon}$ is real analytic with respect to $c, \gamma, \lambda, h, \mu$ and $\epsilon$. Moreover, for any $d \in \mathbb{C}$ minimizer of $f(c) + p_\epsilon$ there exists an ergodic equilibrium state $\omega \in \Omega_{\tilde{\mathfrak{m}}} \cap \mathcal{E}_{\mathbb{Z}^d}$ satisfying $\omega(a_{x,\downarrow} a_{x,\uparrow}) = d$.*

*Proof.* First note that since $p_{l,\tilde{\mathfrak{m}}(c)} = p_{l,\tilde{\mathfrak{m}}_0(c)} + p_{l,\epsilon}$ and $P_{\tilde{\mathfrak{m}}(c)} = \lim_{l\to\infty} p_{l,\tilde{\mathfrak{m}}(c)}$, $P_{\tilde{\mathfrak{m}}_0} = \lim_{l\to\infty} p_{l,\tilde{\mathfrak{m}}_0(c)}$ are well-defined, it follows that $p_\epsilon = \lim_{l\to\infty} p_{l,\epsilon}$ is also well-defined and $P_{\tilde{\mathfrak{m}}(c)} = P_{\tilde{\mathfrak{m}}_0(c)} + p_\epsilon$. Moreover, by proposition 3.5.1, one has

$$P_{\tilde{\mathfrak{m}}} = -\inf_{c\in\mathbb{C}} \left\{\gamma|c|^2 - P_{\tilde{\mathfrak{m}}_0(c)} + p_\epsilon\right\} = \beta^{-1} \ln 2 + \mu - \inf_{c\in\mathbb{R}} \{f(c) + p_\epsilon\}.$$

Now, let $K \subset \mathbb{R}^6$ be some compact set, and let $D$ be some open bounded set such that $K \subset D$. By proposition 4.2.2, there exists some $r > 0$ such that $\|p_{l,\epsilon}\|_{D,r} < \infty$. In particular, for every six-index $\alpha$, $\frac{\partial^{|\alpha|} p_{l,\epsilon}}{\partial \mathbf{v}^\alpha}$ is a sequence of functions which is uniformly equicontinuous and bounded w.r.t. the supremum norm on $K$. Therefore, by applying the Arzelà-Ascoli theorem, it follows that there exists a subsequence $p_{l_k,\epsilon}$ such that, for all six-indexes $\alpha$, $\frac{\partial^{|\alpha|} p_{l_k,\epsilon}}{\partial \mathbf{v}^\alpha}$ uniformly converges on $K$. As the derivatives commute with uniform limits, one concludes that the pointwise limit $p_\epsilon$ of $p_{l_k,\epsilon}$, is a smooth function such that $\left\|\frac{\partial^{|\alpha|} p_\epsilon}{\partial \mathbf{v}^\alpha}\right\|_\infty \leq \alpha! r^{-|\alpha|}$, i.e., $p_\epsilon$ is real analytic on $K$, and by the arbitrariness of $K$ the second part of the proposition follows. The last part of the proposition is a consequence of proposition 3.5.4. □

**Proposition 4.2.4.** *For $\beta = 1$, $\mu = h = 0$ and any $\epsilon > 0$ small enough, the long-range model $\tilde{\mathfrak{m}}$ also shows a coexistence of magnetic and superconducting phases.*

*Proof.* Let $U$ be a bounded open neighborhood of $(0, 0)$ such that $\lambda, \gamma > 0$ for any $(\lambda, \gamma) \in (\bar{\lambda}, \bar{\gamma}) + U$, and let $F'_\epsilon(x, \lambda, \gamma) = p_\epsilon(x; \bar{\lambda} + \lambda, \bar{\gamma} + \gamma)$. Note that, in particular, $p_\epsilon(x; \bar{\lambda}+\lambda, \bar{\gamma}+\gamma)$ is real analytic with respect to $x, \gamma, \lambda, \epsilon$, and since $p_0(x; \bar{\lambda}+\lambda, \bar{\gamma}+\gamma) = 0$, it follows that its derivatives with respect to $x, \gamma, \lambda$ go to 0 as $\epsilon \to 0$. Moreover, note that $F(x, \lambda, \gamma) = f(x; \bar{\lambda} + \lambda, \bar{\gamma} + \gamma) - f(0, \bar{\lambda}, \bar{\gamma})$ is e-transversal and thus clearly e-universal (see proposition 2.4.4 and definition 2.3.4). Hence, $F(x, \lambda, \gamma)$ is e-stable by theorem 2.5.2. Therefore, by the e-stability of $F$ (see definition 2.5.3), it follows that there exists an interval $I$ of $0 \in \mathbb{R}$ such that if $\epsilon \in I$, $F$ at $(0, 0, 0)$ is equivalent to $F + F'_\epsilon$ at $(0, \lambda_\epsilon, \gamma_\epsilon)$ for some $(\lambda_\epsilon, \gamma_\epsilon) \in U$. Therefore, an analogous analysis to that done in proposition 4.1.4 holds for the unfolding $F + F'_\epsilon$, and hence, the proposition follows.

□

# Chapter 5

# Conclusion

In this thesis, the goal of showing a coexistence of magnetic and superconducting phases in a quantum lattice fermi system is achieved, through the study of the formalism presented in chapter 3, which is based on the solid mathematical ground of the C*-algebraic formulation of quantum mechanics, and through the application of some results coming from catastrophe theory.

As it could be seen, the formalism in chapter 3, that was developed in [4], provides some very important and relevant results associated with the existence and the analysis of the thermodynamics for systems that can be modeled by a “long-range model”, as it is defined in this thesis. Moreover, by means of the approximating interactions, the formalism also provides a way to find the thermodynamic pressure of a purely attractive long-range model from the pressure of a simpler interaction, and also shows that there exists an equivalence between the equilibrium states of the exact system and of the approximating system. Showing this equivalence of equilibrium states was actually an open problem (that was proposed by J. Ginibre in [8], regarding the equivalence of equilibrium states for the so-called Bogoliubov approximation), prior to the publication of [4].

In fact, the results obtained in [4] go beyond of what is presented here: in particular, it is shown that when a repulsive long-range term is present in the long-range model, an approximating interaction can still be defined, and the relation between the exact pressure and the approximate pressure is associated with the equilibrium of a two-person zero-sum game, instead of just a minimization problem, as it is for the purely attractive long-range models. It is important to note, however, that although the long-range models defined here can be used to model some important physical mean-field interactions such as the BCS interaction, other important physical interactions, such as the Coulomb potential, or any interaction decaying with a fixed power of the distance, are still out of the scope of the formalism.

This thesis also shows that some of the results provided by catastrophe theory, related to the bifurcations and the stability of unfoldings, are very interesting mathematical tools for the study of phase transitions, and for obtaining a qualitative analysis of the behavior of the thermodynamic pressure of a quantum system. In fact, another analysis that can possibly be done, with the aid of catastrophe theory, to the model studied here is to show the breakdown of the superconducting phase above some critical magnetic field $\bar{h}$, by fixing some $\lambda \geq 0$ and considering the pressure as an unfolding of the parameters $(\gamma, h)$.

# Appendix A

# Catastrophe Theory

In the following it is shown the proof of proposition 2.4.3, but for that one first needs this preliminary proposition:

**Proposition A.0.1.** *Let $F \in m(n+1,p)$. Suppose that there exist $\xi \in \epsilon(n+1,n)$ and $\eta \in \epsilon(p+1,p)$ such that for any $x \in \mathbb{R}^n$ near $0$, any $t \in \mathbb{R}$ near $0$ and any $i = 1,\dots,p$, the following equation holds:*

$$\frac{\partial F_i(x,t)}{\partial t} = \sum_{j=1}^{n} \frac{\partial F_i(x,t)}{\partial x_j} \xi_j(x,t) + \eta_i(F(x,t),t). \tag{A.1}$$

*Then, there exist $\phi \in \epsilon(n+1,n)$ and $\lambda \in \epsilon(p+1,p)$ such that for any $x \in \mathbb{R}^n$ near $0$, any $y \in \mathbb{R}^p$ near $0$ and any $t \in \mathbb{R}$ near $0$, the following holds:*

*(a)* $\phi(x,0) = x$ *and* $\lambda(y,0) = y$,

*(b)*

$$\frac{\partial \phi_i(x,t)}{\partial t} = -\epsilon_i(\phi(x,t),t) \text{ for all } i = 1,\dots,n, \text{ and}$$
$$\frac{\partial \lambda_j(y,t)}{\partial t} = \eta_j(\lambda(y,t),t) \text{ for all } j = 1,\dots,p,$$

*(c)* $F(\phi(x,t),t) = \lambda(F(x,0),t)$.

*Proof.* Let $\epsilon_j, j = 1,\dots,n$ and $\eta$ be as stated in the proposition. By the fundamental existence theorem for solutions of ordinary differential equations, there are unique smooth germs $\phi \in \epsilon(n+1,n)$ and $\lambda \in \epsilon(p+1,p)$ which solve the differential equations in $(b)$ and satisfy the initial conditions in $(a)$. Condition $(a)$, together with the smoothness of $\phi$ and $\lambda$, imply that the germs

$$\phi_t : x \mapsto \phi(x,t), \quad \text{and} \quad \lambda_t : y \mapsto \lambda(y,t)$$

are germ diffeomorphisms for $t$ sufficiently close to $0$. Hence, they are invertible, and $(c)$ can be rewritten as

$$(\lambda_t^{-1} \circ F_t \circ \phi_t)(x) = F_0(x),$$

where $F_t(x) = F(x,t)$. Clearly, by $(a)$, the equation holds for $t = 0$. Hence, to prove $(c)$, it suffices to show that for $t$ close to 0 in $\mathbb{R}$ and $x$ close to 0 in $\mathbb{R}^n$ one has

$$\frac{\partial}{\partial t}(\lambda_t^{-1} \circ F_t \circ \phi_t)(x) = \frac{\partial}{\partial t} F_0(x) = 0. \tag{A.2}$$

Expanding the left-hand side of eq. A.2 one has

$$\begin{aligned}\frac{\partial}{\partial t}(\lambda_t^{-1} \circ F_t \circ \phi_t)(x) =& \frac{\partial \lambda_t^{-1}}{\partial t}((F_t \circ \phi_t)(x)) + \sum_{i=1}^{p} \frac{\partial \lambda_t^{-1}}{\partial y_i}((F_t \circ \phi_t)(x)) \Bigg[ \frac{\partial F_{i_t}}{\partial t}(\phi_t(x)) + \\ & \sum_{j=1}^{n} \frac{\partial F_{i_t}}{\partial x_j}(\phi_t(x)) \frac{\partial \phi_j}{\partial t}(x,t) \Bigg]. \end{aligned} \tag{A.3}$$

But note that, for all $y, t$ near 0 in $\mathbb{R}^p \times \mathbb{R}$,

$$\lambda_t^{-1}(\lambda_t(y)) = y,$$

and differentiating the above equation with respect to $t$, one has

$$\frac{\partial \lambda_t^{-1}}{\partial t}(\lambda_t^{-1}(y)) + \sum_{i=1}^{p} \frac{\partial \lambda_t^{-1}}{\partial y_i}(\lambda_t(y)) \frac{\partial \lambda_{i_t}}{\partial t}(y) = 0.$$

Substituting $y$ for $\lambda_t^{-1}(F_t \circ \phi_t)(x)$, one gets an equation for $\frac{\partial \lambda_t^{-1}}{\partial t}((F_t \circ \phi_t)(x))$, and substituting further in eq. A.3, one arrives at

$$\begin{aligned}\frac{\partial}{\partial t}(\lambda_t^{-1} \circ F_t \circ \phi_t)(x) =& \sum_{i=1}^{p} \frac{\partial \lambda_t^{-1}}{\partial y_i}((F_t \circ \phi_t)(x)) \Bigg[ \frac{\partial F_{i_t}}{\partial t}(\phi_t(x)) - \frac{\partial \lambda_{i_t}}{\partial t}((\lambda_t^{-1} \circ F_t \circ \phi_t)(x)) + \\ & \sum_{j=1}^{n} \frac{\partial F_{i_t}}{\partial x_j}(\phi_t(x)) \frac{\partial \phi_j}{\partial t}(x,t) \Bigg]. \end{aligned}$$

Now, the goal is to show that the expression in square brackets is 0. Since $\phi(0) = 0$, one may replace $x$ in eq. A.1 by $\phi(x,t)$ for $(x,t)$ close to 0 in $\mathbb{R}^{n+1}$, and obtain

$$\frac{\partial F_{i_t}}{\partial t}(\phi_t(x)) - \sum_{j=1}^{n} \frac{\partial F_{i_t}}{\partial x_j}(\phi_t(x)) \xi_j(\phi(x,t),t) + \eta_i((F_t \circ \phi_t)(x),t) = 0. \tag{A.4}$$

But by $(b)$, it follows that

$$\begin{aligned}\xi_i(\phi(x,t),t) = \frac{\partial \phi_i}{\partial t}(x,t), \quad \text{and} \quad \eta_i((F_t \circ \phi_t)(x),t) =& \eta_i(\lambda_t((\lambda_t^{-1} \circ F_t \circ \phi_t)(x))) \\ =& \frac{\partial \lambda_{i_t}}{\partial t}((\lambda_t^{-1} \circ F_t \circ \phi_t)(x)) \end{aligned} \tag{A.5}$$

Substituting A.5 into A.4 the desired result follows. □

**Proposition A.0.2.** *Let $F \in m(n+1)$. Suppose that there exist $\xi \in \epsilon(n+1,n)$ and $\eta \in \epsilon(n+2)$ such that for any $x \in \mathbb{R}^n$ near 0 and any $t \in \mathbb{R}$ near 0, the following equation holds:*

$$\frac{\partial F(x,t)}{\partial t} = \sum_{i=1}^{n} \frac{\partial F(x,t)}{\partial x_i} \xi_i(x,t) + \eta(F(x,t),x,t). \tag{A.6}$$

*Then, there exist $\phi \in \epsilon(n+1,n)$ and $\lambda \in \epsilon(n+2)$ such that for any $x \in \mathbb{R}^n$ near 0 and any $t,s \in \mathbb{R}$ near 0, the following holds:*

*(a)* $\phi(x,0) = x$ *and* $\lambda(s,x,0) = s$,

*(b)*

$$\frac{\partial \phi_i(x,t)}{\partial t} = -\epsilon_i(\phi(x,t),t) \text{ for all } i = 1,\ldots,n, \text{ and}$$
$$\frac{\partial \lambda(s,x,t)}{\partial t} = \eta(\lambda(s,x,t),\phi(x,t),t),$$

*(c)* $F(\phi(x,t),t) = \lambda(F(x,0),x,t)$.

*Proof.* Define $\tilde{F} \in \epsilon(n+1,n+1)$ by setting $\tilde{F}(x,t) = (F(x,t),x)$, and $\mu \in \epsilon(n+2,n+1)$ by setting $\mu(s,x,t) = (\eta(s,x,t), -\xi_1(x,t),\ldots,-\xi_n(x,t))$. Then, one has

$$\frac{\partial \tilde{F}_i}{\partial t}(x,t) = \sum_{j=1}^{n} \frac{\partial \tilde{F}_i}{\partial x_j}(x,t)\xi_j(x,t) + \mu_i(\tilde{F}(x,t),t). \tag{A.7}$$

For $i = 1$, eq. A.7 is just A.6. For $i > 1$, the left-hand side is clearly 0, and the right-hand side reduces to $\xi_{i-1}(x,t) - \xi_{i-1}(x,t) = 0$. Hence, applying proposition A.0.1 to eq. A.7, it follows that there exists $\phi \in \epsilon(n+1,n)$ and $\Lambda \in \epsilon(n+2,n+1)$ such that

(a') $\phi(x,0) = x$ and $\Lambda(s,x,0) = (s,x)$, $s \in \mathbb{R}$,

(b')

$$\frac{\partial \phi_i(x,t)}{\partial t} = -\xi_i(\phi(x,t),t) \text{ for all } i = 1,\ldots,n, \text{ and}$$
$$\frac{\partial \Lambda_j(s,x,t)}{\partial t} = \mu_j(\Lambda(s,x,t),t) \text{ for all } j = 1,\ldots,n+1,$$

(c') $F(\phi(x,t),t) = \Lambda(F(x,0),x,t)$.

Let $\lambda = \Lambda_1$. Since $F = F'_1$, clearly the items $(a)$ and $(c)$ of the proposition follow from $(a')$ and $(c')$ above. Now, note that the germs $(\Lambda_2,\ldots,\Lambda_{n+1})$ satisfy the same differential equations that $\phi = (\phi_1,\ldots,\phi_n)$ satisfies, since from $(b')$ one has

$$\frac{\partial \Lambda_{j+1}}{\partial t}(s,x,t) = \mu_{j+1}(\Lambda(s,x,t),t) = -\xi_j(\Lambda_2(s,x,t),\ldots,\Lambda_{n+1}(s,x,t),t).$$

Moreover, they also satisfy the same initial conditions, since from $(a')$ one has

$$(\Lambda_2(s,x,0),\ldots,\Lambda_{n+1}(s,x,0)) = x.$$

Hence, it follows that $(\Lambda_2(s,x,t),\ldots,\Lambda_{n+1}(s,x,t)) = (\phi_1(x,t),\ldots,\phi_n(x,t))$ and therefore, by $(b')$ again, one has

$$\frac{\partial\lambda}{\partial t}(s,x,t) = \frac{\partial\Lambda_1}{\partial t}(s,x,t) = \mu_1(\Lambda(s,x,t),t) = \eta(\lambda(s,x,t),\phi(x,t),t).$$

□

# Appendix B

# Cluster expansions

This appendix is dedicated to the proof of proposition 4.2.1:

**Proposition B.0.1.** *Let $\Phi^{(0)}, \Phi^{(\epsilon)} \in \mathcal{W}$, such that $\Phi^{(0)}$ is p.i. and $\Phi^{(\epsilon)}_\Lambda = 0$ if $\Lambda$ is not connected, and let $\Phi = \Phi^{(0)} + \Phi^{(\epsilon)}$. Then, for any $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$, one has*

$$Z_\Lambda = Z_0^{|\Lambda|} \left( \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{A_1} \rho(A_1) \cdots \sum_{A_n} \rho(A_n) \prod_{1 \le i < j \le n} (1 + \zeta(A_i, A_j)) \right),$$

*where $Z_\Lambda$ is the partition function of $U^\Phi_\Lambda$, $Z_0$ is the partition function of $U^{\Phi^{(0)}}_{\{0\}}$,*

$$\zeta(A_i, A_j) = \begin{cases} -1, & \text{if } A_i \cap A_j \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

*and the sum $\sum_{A_i}$ is over all polymers of $\Lambda$.*

*Proof.* The proof relies on a perturbative expansion derived from the so-called *Duhamel formula*. In general, The Duhamel formula provides an expansion of the quantity $e^{-\beta(H_0+V)}$, when $H_0, V$ are matrices and $V$ is small. Therefore, it can be used to expand $e^{-\beta U^\Phi_{\Lambda_l}} = e^{-\beta(U^{\Phi^{(0)}}_{\Lambda_l} + U^{\Phi^{(\epsilon)}}_{\Lambda_l})}$. Start with the identity

$$e^{-\beta(H_0+V)} - e^{-\beta H_0} = \int_0^\beta \frac{d}{d\tau} \left( e^{-\tau(H_0+V)} e^{-(\beta-\tau)H_0} \right) d\tau.$$

Computing the derivative and isolating $e^{-\beta(H_0+V)}$, one obtains the Duhamel formula:

$$e^{-\beta(H_0+V)} = e^{-\beta H_0} - \int_0^\beta e^{-\tau(H_0+V)} V e^{-(\beta-\tau)H_0} d\tau.$$

Applying it recursively, a perturbative series expansion for $e^{-\beta(H_0+V)}$ can be found:

$$e^{-\beta(H_0+V)} = e^{-\beta H_0} + \sum_{m=1}^{\infty}(-1)^m \int_0^\beta \int_0^{\tau_m} \cdots \int_0^{\tau_2} e^{-\tau_1 H_0} V e^{-(\tau_2-\tau_1)H_0} \ldots V e^{-(\beta-\tau_m)H_0} \times d\tau_1 \ldots d\tau_m$$

$$= e^{-\beta H_0} + \left( \sum_{m=1}^{\infty}(-1)^m \int_0^\beta \int_0^{\tau_m} \cdots \int_0^{\tau_2} e^{-\tau_1 H_0} V e^{\tau_1 H_0} \ldots e^{-\tau_m H_0} V e^{\tau_m H_0} \times d\tau_1 \ldots d\tau_m \right) e^{-\beta H_0}$$

For the specific case of proposition 4.2.1, the interaction $V$ is given by

$$V = \sum_{A\subset\Lambda}{}' \Phi_A^{(\epsilon)},$$

where $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$ and the primed sum is over the polymers of $\Lambda$. The goal now is to express the term

$$\sum_{m=1}^{\infty}(-1)^m \int_0^\beta \int_0^{\tau_m} \cdots \int_0^{\tau_2} \left( \sum_{A\subset\Lambda}{}' \Phi_A^{(\epsilon)}(\tau_1) \right) \ldots \left( \sum_{A\subset\Lambda}{}' \Phi_A^{(\epsilon)}(\tau_m) \right) d\tau_1 \ldots d\tau_m, \quad \text{(B.1)}$$

where

$$\Phi_A^{(\epsilon)}(\tau_i) = e^{-\tau_i H_0} \Phi_A^{(\epsilon)} e^{\tau_i H_0},$$

as a product of cluster terms. The first step is to re-write the above sum in a convenient way. For this, consider the following definition

**Definition B.0.1.** Given a finite set $\{A_1, \ldots, A_n\}$ and positive numbers $k_1, \ldots, k_n \in \mathbb{N}$, denote by $F(\{(A_1, k_1), \ldots, (A_n, k_n)\})$ the set of all sequences that possess $k_1$ elements $A_1$, $k_2$ elements $A_2$, ... $k_n$ elements $A_n$.

Then, it follows that

$$(-1)^m \left( \sum_{A\subset\Lambda}{}' \Phi_A^{(\epsilon)}(\tau_1) \right) \ldots \left( \sum_{A\subset\Lambda}{}' \Phi_A^{(\epsilon)}(\tau_m) \right) = \sum_{\substack{\{A_1,\ldots,A_n\}\subset\Lambda; \\ n\le m}}{}' \sum_{\substack{k_1,\ldots,k_n; \\ k_1+\cdots+k_n=m}} \sum_{B\in F(\{(A_1,k_1),\ldots,(A_n,k_n)\})} (-1)^{k_1+\cdots+k_n} \Phi_{B_1}^{(\epsilon)}(\tau_1) \ldots \Phi_{B_m}^{(\epsilon)}(\tau_m),$$

and applying the above equality to expression B.1, one has

$$\sum_{m=1}^{\infty}(-1)^m \int_0^\beta \int_0^{\tau_m} \cdots \int_0^{\tau_2} \left( \sum_{A\subset\Lambda}{}' \Phi_A^{(\epsilon)}(\tau_1) \right) \ldots \left( \sum_{A\subset\Lambda}{}' \Phi_A^{(\epsilon)}(\tau_m) \right) d\tau_1 \ldots d\tau_m = \sum_{\{A_1,\ldots,A_n\}\subset\Lambda}{}' \sum_{k_1,\ldots,k_n} \sum_{B\in F(\{(A_1,k_1),\ldots,(A_n,k_n)\})} (-1)^{k_1+\cdots+k_n} \int_0^\beta \int_0^{\tau_N} \cdots \int_0^{\tau_2} \Phi_{B_1}^{(\epsilon)}(\tau_1) \ldots \ldots \Phi_{B_N}^{(\epsilon)}(\tau_N) d\tau_1 \ldots d\tau_N \quad \text{(B.2)}$$

**Definition B.0.2.** For each $i = 1, \dots, n$ let $f_i : \mathbb{R} \to \mathfrak{A}$ be arbitrary functions of one variable (usually interpreted as the time variable) to the algebra $\mathfrak{A}$. Then, define $\tilde{\mathcal{T}}$ by

$$\tilde{\mathcal{T}}(f_1(t_1)\dots f_n(t_n)) \doteq \sum_{\pi\in\Pi(n)} \Theta(t_{\pi_n} - t_{\pi_{n-1}})\dots\Theta(t_{\pi_2} - t_{\pi_1}) f_{\pi_1}(t_{\pi_1})\dots f_{\pi_n}(t_{\pi_n}),$$

where $\Pi(n)$ is the set of all permutations of $(1, \dots, n)$.

Note that a term in the sum will be different from zero only if

$$t_{\pi_1} \le t_{\pi_2} \le \dots \le t_{\pi_n}.$$

A straightforward consequence of the above definition is the following equality:

$$\int_0^a \cdots \int_0^a \tilde{\mathcal{T}}(f_1(t_1)\dots f_n(t_n)) dt_1 \dots dt_n =$$
$$\sum_{\pi\in P_n} \int_0^a \int_0^{\tau_n} \cdots \int_0^{\tau_2} f_{\pi_1}(t_{\pi_1})\dots f_{\pi_n}(t_{\pi_n}) dt_{\pi_1}\dots dt_{\pi_n}.$$

Now note that, if there are repeating terms in $f_1(t_1), \dots, f_n(t_n)$, e.g., if $f_1$ appears $k_1$ times in this sequence, ..., and $f_n$ appears $k_n$ times, then there will be repeated terms in the sum

$$\sum_{\pi\in P_n} \int_0^a \int_0^{\tau_n} \cdots \int_0^{\tau_2} f_{\pi_1}(t_{\pi_1})\dots f_{\pi_n}(t_{\pi_n}) dt_{\pi_1}\dots dt_{\pi_n}.$$

More exactly, for some fixed permutation, there are $k_1!$ other redundant permutations, that only permute the elements with $f_1$, ..., and $k_n!$ redundant permutations that only permute the elements with $f_n$. Hence, with that in mind one may arrive at the following equality

$$\sum_{B\in F(\{(A_1,k_1),\dots,(A_n,k_n)\})} \int_0^\beta \int_0^{\tau_N} \cdots \int_0^{\tau_2} \Phi^{(\epsilon)}_{B_1}(\tau_1)\dots\Phi^{(\epsilon)}_{B_N}(\tau_N) d\tau_1\dots d\tau_N =$$
$$\frac{1}{k_1!\cdots k_n!} \int_0^\beta \cdots \int_0^\beta \tilde{\mathcal{T}}(\underbrace{\Phi^{(\epsilon)}_{A_1}(\tau_1)\dots\Phi^{(\epsilon)}_{A_1}(\tau_{k_1})}_{k_1 \text{ times}}\dots\underbrace{\Phi^{(\epsilon)}_{A_n}(\tau_i)\dots\Phi^{(\epsilon)}_{A_n}(\tau_N)}_{k_n \text{ times}}) d\tau_1\dots d\tau_N$$

Hence, substituting into eq. B.2, one gets

$$\sum_{m=1}^{\infty} (-1)^m \int_0^\beta \int_0^{\tau_m} \cdots \int_0^{\tau_2} \left(\sum_{A\subset\Lambda}{}' \Phi^{(\epsilon)}_A(\tau_1)\right)\dots\left(\sum_{A\subset\Lambda}{}' \Phi^{(\epsilon)}_A(\tau_m)\right) d\tau_1\dots d\tau_m =$$
$$\sum_{\{A_1,\dots,A_n\}\subset\Lambda}{}' \sum_{k_1,\dots,k_n} \frac{(-1)^{k_1+\dots+k_n}}{k_1!\cdots k_n!} \int_0^\beta \cdots \int_0^\beta \tilde{\mathcal{T}}(\underbrace{\Phi^{(\epsilon)}_{A_1}(\tau_1)\dots\Phi^{(\epsilon)}_{A_1}(\tau_{k_1})}_{k_1 \text{ times}}\dots\underbrace{\Phi^{(\epsilon)}_{A_n}(\tau_i)\dots\Phi^{(\epsilon)}_{A_n}(\tau_N)}_{k_n \text{ times}})\times$$
$$d\tau_1\dots d\tau_N \tag{B.3}$$

Now, for every set of polymers $\{A_1, \ldots, A_N\}$ of $\Lambda$, construct a graph with $N$ vertices and an edge between $i$ and $j$ whenever $A_i \cap A_j \neq \varnothing$. This graph can be uniquely decomposed into connected subgraphs, and this decomposition induces a partition $\{\mathcal{A}_1, \ldots, \mathcal{A}_l\}$ on $\{A_1, \ldots, A_N\}$. To each set $\{A_1, \ldots, A_N\}$ of polymers, there corresponds a unique partition $\{\mathcal{A}_1, \ldots, \mathcal{A}_l\}$, which satisfies

$$\cup_{i=1}^{N} A_i = \cup_{i=1}^{l} \mathcal{A}_i, \quad \text{and} \quad \mathcal{A}_i \cap \mathcal{A}_j = \varnothing \text{ if } i \neq j.$$

Let $\mathcal{D}(\Lambda)$ be the set of all the finite sequences of mutually disjoint polymers of $\Lambda$, and for $\{\mathcal{A}_1, \ldots, \mathcal{A}_l\} \in \mathcal{D}(\Lambda)$, let $\mathcal{S}(\{\mathcal{A}_1, \ldots, \mathcal{A}_l\})$ be the set of all sets $\{A_1, \ldots, A_m\}$ that generate the polymers $\{\mathcal{A}_1, \ldots, \mathcal{A}_l\}$ in the way mentioned above. With this notation, the r.h.s. of eq. B.3 can be re-written as

$$\sum_{\{\mathcal{A}_1,\ldots,\mathcal{A}_l\}\in\mathcal{D}(\Lambda)} \sum_{A\in S(\{\mathcal{A}_1,\ldots,\mathcal{A}_l\})} \sum_{k_1,\ldots,k_n} \frac{(-1)^{k_1+\cdots+k_n}}{k_1!\ldots k_n!} \times$$

$$\int_0^\beta \cdots \int_0^\beta \tilde{\mathcal{T}}(\underbrace{\Phi^{(\epsilon)}_{A_1}(\tau_1)\ldots\Phi(\epsilon)_{A_1}(\tau_{k_1})}_{k_1 \text{ times}}\ldots\underbrace{\Phi^{(\epsilon)}_{A_n}(\tau_i)\ldots\Phi^{(\epsilon)}_{A_n}(\tau_N)}_{k_n \text{ times}})d\tau_1\ldots d\tau_N \tag{B.4}$$

Suppose, without loss of generality, that $A_1, A_2, \ldots, A_{m_1} \in S(\mathcal{A}_1)$, and in general, $A_{m_{i-1}+1}, \ldots, A_{m_i} \in S(\mathcal{A}_i)$. Since $\mathcal{A}_i \cap \mathcal{A}_j = \varnothing$ if $i \neq j$ and the interactions are, by definition, even, it follows that elements that generate distinct polymers $\mathcal{A}_i$ commute with each other. Hence, one has

$$\tilde{\mathcal{T}}(\underbrace{\Phi^{(\epsilon)}_{A_1}(\tau_1)\ldots\Phi^{(\epsilon)}_{A_1}(\tau_a)}_{k_1 \text{ times}}\ldots\underbrace{\Phi^{(\epsilon)}_{A_{m_1}}(\tau_b)\ldots\Phi^{(\epsilon)}_{A_{m_1}}(\tau_c)}_{k_{m_1} \text{ times}}\ldots\underbrace{\Phi^{(\epsilon)}_{A_{m_{i-1}+1}}(\tau_d)\ldots\Phi^{(\epsilon)}_{A_{m_{i-1}+1}}(\tau_e)}_{k_{m_{i-1}+1} \text{ times}}\ldots$$

$$\ldots\underbrace{\Phi^{(\epsilon)}_{A_{m_i}}(\tau_f)\ldots\Phi^{(\epsilon)}_{A_{m_i}}(\tau_g)}_{k_{m_i} \text{ times}}\ldots\underbrace{\Phi^{(\epsilon)}_{A_{m_{l-1}+1}}(\tau_h)\ldots\Phi^{(\epsilon)}_{A_{m_{l-1}+1}}(\tau_i)}_{k_{m_{l-1}+1} \text{ times}}\ldots\underbrace{\Phi^{(\epsilon)}_{A_n}(\tau_j)\ldots\Phi^{(\epsilon)}_{A_n}(\tau_N)}_{k_n \text{ times}}) =$$

$$\tilde{\mathcal{T}}(\underbrace{\Phi^{(\epsilon)}_{A_1}(\tau_1)\ldots\Phi^{(\epsilon)}_{A_1}(\tau_a)}_{k_1 \text{ times}}\ldots\underbrace{\Phi^{(\epsilon)}_{A_{m_1}}(\tau_b)\ldots\Phi^{(\epsilon)}_{A_{m_1}}(\tau_c)}_{k_{m_1} \text{ times}})\ldots\tilde{\mathcal{T}}(\underbrace{\Phi^{(\epsilon)}_{A_{m_{i-1}+1}}(\tau_d)\ldots\Phi^{(\epsilon)}_{A_{m_{i-1}+1}}(\tau_e)}_{k_{m_{i-1}+1} \text{ times}}\ldots$$

$$\ldots\underbrace{\Phi^{(\epsilon)}_{A_{m_i}}(\tau_f)\ldots\Phi^{(\epsilon)}_{A_{m_i}}(\tau_g)}_{k_{m_i} \text{ times}})\ldots\tilde{\mathcal{T}}(\underbrace{\Phi^{(\epsilon)}_{A_{m_{l-1}+1}}(\tau_h)\ldots\Phi^{(\epsilon)}_{A_{m_{l-1}+1}}(\tau_i)}_{k_{m_{l-1}+1} \text{ times}}\ldots\underbrace{\Phi^{(\epsilon)}_{A_n}(\tau_j)\ldots\Phi^{(\epsilon)}_{A_n}(\tau_N)}_{k_n \text{ times}})$$

Therefore, integral in B.4 can be factorized into the product of terms that generate the same polymer $\mathcal{A}_i$, and expression B.4 can be re-written as

$$\sum_{\{\mathcal{A}_1,\ldots,\mathcal{A}_l\}\in\mathcal{D}(\Lambda)} \prod_{i=1}^{l} \Bigg( \sum_{A\in S(\mathcal{A}_i)} \sum_{k_1,\ldots,k_m} (-1)^{k_1+\cdots+k_m} \frac{1}{k_1!\cdots k_m!} \times$$

$$\int_0^\beta \cdots \int_0^\beta \tilde{\mathcal{T}}(\underbrace{\Phi^{(\epsilon)}_{A_1}(\tau_1)\ldots\Phi^{(\epsilon)}_{A_1}(\tau_{k_1})}_{k_1 \text{ times}}\ldots\underbrace{\Phi^{(\epsilon)}_{A_m}(\tau_i)\ldots\Phi^{(\epsilon)}_{A_m}(\tau_M)}_{k_m \text{ times}})d\tau_1\ldots d\tau_M \Bigg). \tag{B.5}$$

Note that the sum $\sum_{A\in S(\mathcal{A}_i)}$ is equal to the primed sum $\sum'$ of definition 4.2.3, where the sum is over polymers $A_1,\dots A_m \subset \mathcal{A}_i$ such that $\bigcup_{i=1}^m A_i = \mathcal{A}_i$ and the graph $\mathcal{G}_{A_1,\dots,A_m}$ is connected. Moreover, applying eq. B.3, it follows that the term in parenthesis of expression B.5 is equal to

$$\sum_{m=1}^{\infty}(-1)^m \sum_{A_1,\dots,A_m\subset\mathcal{A}_i}{}' \int_0^{\beta}\int_0^{\tau_m}\cdots\int_0^{\tau_2} \Phi^{(\epsilon)}_{A_1}(\tau_1)\dots\Phi^{(\epsilon)}_{A_m}(\tau_m)d\tau_1\dots d\tau_m$$

Hence, if $H_0$ is simply a sum of on-site interactions:

$$H_0 = \sum_{x\in\Lambda}\Phi^{(0)}_{\{x\}}$$

then, noting that $\mathcal{A}_1,\dots,\mathcal{A}_l$ are disjoint sets, $e^{-\beta H_0}$ can be factorized into terms in $\mathcal{A}_1,\dots,\mathcal{A}_l$, and one may arrive at

$$e^{-\beta(H_0+V)} = e^{-\beta H_0} + \sum_{\{\mathcal{A}_1,\dots,\mathcal{A}_l\}\in\mathcal{D}(\Lambda)} e^{-\beta\sum_{x\in\Lambda\setminus\cup_{j=1}^l\mathcal{A}_j}\Phi^{(0)}_{\{x\}}}\left[\prod_{i=1}^{l}\left(\sum_{m=1}^{\infty}(-1)^m\sum_{A_1,\dots,A_m\subset\mathcal{A}_i}{}'\times\right.\right.$$
$$\left.\left.\int_0^{\beta}\int_0^{\tau_m}\cdots\int_0^{\tau_2}\Phi^{(\epsilon)}_{A_1}(\tau_1)\dots\Phi^{(\epsilon)}_{A_m}(\tau_m)e^{-\beta\sum_{x\in\mathcal{A}_i}\Phi^{(0)}_{\{x\}}}d\tau_1\dots d\tau_m,\right)\right]$$

and finally, taking the trace, it follows that

$$\mathrm{Tr}_{\mathfrak{A}_\Lambda}\left(e^{-\beta(H_0+V)}\right) = \mathrm{Tr}_{\mathfrak{A}_\Lambda}\left(e^{-\beta H_0}\right)\left(1+\sum_{\{\mathcal{A}_1,\dots,\mathcal{A}_l\}\in\mathcal{D}(\Lambda)}\frac{1}{\mathrm{Tr}_{\mathfrak{A}_\Lambda}\left(e^{-\beta\sum_{x\in\cup_{j=1}^l\mathcal{A}_j}\Phi^{(0)}_{\{x\}}}\right)}\times\right.$$
$$\left[\prod_{i=1}^{l}\left(\sum_{m=1}^{\infty}(-1)^m\sum_{A_1,\dots,A_m\subset\mathcal{A}_i}{}'\int_0^{\beta}\int_0^{\tau_m}\cdots\int_0^{\tau_2}\mathrm{Tr}_{\mathfrak{A}_\Lambda}\left(\Phi^{(\epsilon)}_{A_1}(\tau_1)\dots\Phi^{(\epsilon)}_{A_m}(\tau_m)e^{-\beta\sum_{x\in\mathcal{A}_i}\Phi^{(0)}_{\{x\}}}\right)\times\right.\right.$$
$$\left.\left.\left. d\tau_1\dots d\tau_m\right)\right]\right)$$
$$= Z_0^{|\Lambda|}\left(1+\sum_{\{\mathcal{A}_1,\dots,\mathcal{A}_l\}\in\mathcal{D}(\Lambda)}\prod_{i=1}^{l}\rho(\mathcal{A}_i)\right)$$
$$= Z_0^{|\Lambda|}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{A_1}\rho(A_1)\cdots\sum_{A_n}\rho(A_n)\prod_{1\le i<j\le n}(1+\zeta(A_i,A_j))\right).$$

□

# Bibliography


[1] Ola Bratteli and Derek W. Robinson. *Operator Algebras and Quantum Statistical Mechanics, vol. 1*. Springer-Verlag, 1987.

[2] Ola Bratteli and Derek W. Robinson. *Operator Algebras and Quantum Statistical Mechanics, vol. 2*. Springer-Verlag, 1987.

[3] Jean-Bernard Bru and Walter de Siqueira Pedra. Effect of a locally repulsive interaction on s-wave superconductors. *Reviews in Mathematical Physics*, 22:233–303, 2010.

[4] Jean-Bernard Bru and Walter de Siqueira Pedra. Non-cooperative equilibria of fermi systems with long range interactions. *Memoirs of the American Mathematical Society*, 224, 2013.

[5] Domencio Castrigiano. *Catastrophe theory*. CRC Press, 2019.

[6] Tullio Ceccherini-Silberstein and Michel Coornaert. *Amenable Groups*. Springer Berlin Heidelberg, 2010.

[7] Ilijas Farah. *Combinatorial Set Theory of C*-algebras*. Springer, 2019.

[8] J Ginibre. On the asymptotic exactness of the bogoliubov approximation for many boson systems. *Communications in Mathematical Physics*, 8(1):26–51, 1968.

[9] Martin Golubitsky and Victor Guillemin. *The Malgrange Preparation Theorem*. Springer US, 1973.

[10] Robert R. Phelps. *Lectures on Choquet's Theorem*. Springer, 2000.

[11] Walter Rudin. *Functional Analysis*. McGraw-Hill Book Company, 2015.

[12] Daniel Ueltschi. Cluster expansions and correlation functions. *Moscow Mathematical Journal*, 4:511–522, 2004.

[13] Gordon Wassermann. *Stability of unfoldings*, volume 393. Springer, 2006.