Universidade de São Paulo
Instituto de Física

Formulação funtorial da Teoria Quântica de Campos Algébrica em espaços-tempos curvos e o Teorema de Reeh-Schlieder

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Dissertação de mestrado submetida ao Instituto de Física da Universidade de São Paulo, como requisito parcial para a obtenção do título de Mestre em Ciências.

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São Paulo
2021

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Formulação funtorial da Teoria Quântica de Campos Algébrica em espaços-tempos curvos e o Teorema de Reeh-Schlieder. / Functorial formulation of Algebraic Quantum Field Theory in curved spacetimes and the Reeh-Schlieder Theorem. São Paulo, 2021.

Dissertação (Mestrado) – Universidade de São Paulo, Instituto de Física, Depto. de Física Matemática.

Orientador: Prof. Dr. João Carlos Alves Barata

Área de Concentração: Teoria Quântica de Campos e Relatividade Geral.

Unitermos: 1. Teoria Quântica de Campos; 2. Relatividade (Física); 3. Teoria de Categorias; 4. Física Matemática;

USP/IF/SBI-019/2021

University of São Paulo
Physics Institute

Functorial formulation of Algebraic Quantum Field Theory in curved spacetimes and the Reeh-Schlieder Theorem

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Dissertation submitted to the Physics Institute of the University of São Paulo in partial fulfillment of the requirements for the degree of Master of Science.

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São Paulo
2021

Acknowledgements

First, I would like to thank my supervisor Prof. Dr. João Carlos Alves Barata for providing me with this project and for all the support and feedback given. His trust in me was essential for me to believe that I could finish this work.

The research group was essential for me to get an extra feedback on my presentations. The insights of my colleagues would always make me think about something that I had not thought of before and were crucial for the development of this project. So, here my thanks goes to Ricardo Corrêa, Victor Chabu, Gabriel Barbosa, Lissa Campos, Marcos Brum, and to all the undergraduate students that participated.

I thank all my friends in the Physics Institute for bringing joy to my day-to-day life and for sharing their insecurities regarding the academic life, which is not easy for anyone. So, thanks Caio, Gabriel, Giulio, Brenda, Carol, Felipe, Ivan, Lissa and Leandro for being part of my life. I admire you all, and spending time with you inspires me every day.

I want to thank Agnes, whose company in the first months of the masters made it much easier for me to adapt to São Paulo and my new routine. Also, thanks to Mari, Monah, Milena, Jake, Zetti, Ramon, Tatá, Priscila, Wilson, Daiana, Thassia, Matheus, Raissa, Flora, Gabs and Rodrigo, that are all very dear to me.

Thanks, Jean, for the unconditional support, for challenging me in my ways of thinking and thus making me open to different perspectives. Your company is the main reason the quarantine was not so difficult to me and it helped me get to the state of mind I needed in order to finish writing.

Last, but not least, thanks to my whole family, specially my parents and brother for the incentive and support given throughout my life.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001

Abstract

In this project we studied how Category Theory can be used in the formulation of Algebraic Quantum Field Theory in curved spacetimes and how the Reeh-Schlieder property translates to general curved spacetimes. Category Theory concepts such as functors, natural transformations and natural equivalences are used in the definition of a Locally Covariant Quantum Field Theory, that arose in a context in which it was of interest to generalize Axiomatic Quantum Field Theory to curved spacetimes taking into consideration the ideas of locality and covariance. In fact, a Locally Covariant Quantum Field Theory is defined as a covariant functor, which can be related to another Locally Covariant Quantum Field Theory by a natural transformation. The equivalence between theories then becomes clear if this natural transformation is an isomorphism. Furthermore, the Reeh-Schlieder theorem is of great significance in the realm of Quantum Field Theory, since it provides a great deal of properties for the vacuum state and it has relevance in justifying applications of Tomita-Takesaki modular theory in Quantum Field Theories. It has already been proven that states with a weak form of the Reeh-Schlieder property always exist in general curved spacetimes. This was accomplished using the spacetime deformation technique and assuming the time-slice axiom in a Locally Covariant Quantum Field Theory.

Keywords: Category Theory; Algebraic Quantum Field Theory; Locally Covariant Quantum Field Theory; Reeh-Schlieder theorem.

Resumo

Neste projeto estudamos como a Teoria de Categorias pode ser usada na formulação da Teoria Quântica de Campos Algébrica em espaços-tempos curvos e como a propriedade de Reeh-Schlieder é transportada para espaços-tempos curvos gerais. Conceitos da Teoria de Categorias como funtores, transformações naturais e equivalências naturais são usados na definição de uma Teoria Quântica de Campos Localmente Covariante (TQCLC), que surgiu em um contexto em que se tinha interesse em generalizar a Teoria Quântica de Campos Axiomática para espaços-tempos curvos levando em consideração as ideias de localidade e covariância. De fato, uma Teoria Quântica de Campos Localmente Covariante é definida como um funtor covariante, que pode ser relacionado com outra Teoria Quântica de Campos Localmente Covariante por meio de uma transformação natural. A equivalência entre teorias se torna clara se essa transformação natural é um isomorfismo. Ademais, o teorema de Reeh-Schlieder possui grande importância no contexto da Teoria Quântica de Campos, visto que ele fornece várias propriedades do estado de vácuo e possui relevância na justificativa para aplicar a teoria modular de Tomita-Takesaki em Teorias Quânticas de Campos. Já foi provado que estados com uma forma fraca da propriedade de Reeh-Schlieder sempre existem em espaços-tempos curvos gerais. Isso foi realizado por meio da técnica de deformação do espaço-tempo e assumindo o axioma da fatiação temporal em uma Teoria Quântica de Campos Localmente Covariante.

Palavras-chave: Teoria de Categorias; Teoria Quântica de Campos Algébrica; Teoria Quântica de Campos Localmente Covariante; Teorema de Reeh-Schlieder.

Contents

Chapter 1

Introduction

The idea that the laws of physics are the same for all observers is fundamental for the description of any theory. This is usually expressed by covariance properties with regard to spacetime symmetries and the current theory that makes this idea hold properly in curved spacetimes is General Relativity. In recent years, the interest has turned to finding a framework that provides a theory which is the same for all spacetimes simultaneously, which includes curved spacetimes that may not have any symmetries. Many properties of Quantum Field Theories (QFTs) rely on the existence of certain symmetries, like invariance under Poincaré transformations, so an approach which would make it easier to deal with any type of spacetime is necessary. In this scenario, Category Theory [1] was found to be very useful for providing such framework, as can be seen from its use to define a Locally Covariant Quantum Field Theory (LCQFT) [2,3]. This new approach to QFT turned out to be very powerful due to its rigour and generality, and among many applications, it was used to derive the Reeh-Schlieder property in curved spacetimes [4,5].

One of the main reasons why Category Theory is so useful is its level of abstraction. In this regard, we can make an analogy between Category and Group Theories, both of which can be used to describe a wide range of mathematical objects, and even ideas. This can be illustrated by a quote from James R. Newman, taken from [6]:

> "The Theory of Groups is a branch of mathematics in which one does something to something and then compares the result with the result obtained from doing the same thing to something else, or something else to the same thing."

This quote can be made clear if one thinks about the multiplication table of a group, which tells us how the elements of a group are related to one another. In fact, in group theory there are many groups that can be described by the same multiplication table and are, therefore, isomorphic to one another. Furthermore, a theorem by Cayley asserts that every group is isomorphic to a permutation group. These ideas can be easily translated to Category Theory, where instead of a multiplication table we can consider a diagram associated with the Category. We can actually talk about a category of groups and also prove Cayley's theorem in this approach, which already gives us a hint on how abstract this theory is. An isomorphism between categories would be expressed by the fact that diagrams for isomorphic categories are the same. In addition to that, we can talk about morphisms between categories that represent different types of mathematical objects, which is truly amazing, since this simplifies the description of many theories and provides a perspective "from the outside", allowing for different types of conclusions that would not be evident otherwise.

So, on one hand we have a powerful tool allowing for a general description of physical theories, and on the other we have the interest of constructing a Quantum Field Theory on curved spacetimes which is valid for all globally hyperbolic spacetimes simultaneously. This was realized with the local implementation of general covariance, by assigning local algebras to regions of spacetime which are related by isometric embeddings [3]. Needless to say, Category Theory provided the appropriate framework for this construction.

As an application of this formalism, we studied the Reeh-Schlider theorem, first in the algebraic approach [7] and then on the LCQFT setting. The Reeh-Schlieder theorem has a wide range of applicability and it is central in Algebraic Quantum Field Theory (AQFT). It has consequences that allow for the use of Tomita-Takesaki modular theory (KMS states), it is promising for recovering information lost in black holes [8] and even has uses on Quantum information [9, 10]. One example of its importance in Algebraic Quantum Field Theory lies in the fact that it is one of the conditions imposed on states. The Reeh-Schlieder theorem is the reason why the evolution of a state localized in a region $\mathcal{O}$ of spacetime at $t = 0$ and with finite energy, can evolve to any state of the Hilbert space [11]. That is, the result of operating on the vaccum with operators defined on $\mathcal{O}$ is not localized in $\mathcal{O}$.

The derivation of the Reeh-Schlieder property on general curved spacetimes [4] was done using the technique of spacetime deformation [12], where one assumes the existence of a property on a spacetime which has more symmetries, like Minkowski, and then goes on to derive it on a diffeomorphic spacetime. This derivation lies heavily on the study of the causal structure of spacetime and domain of dependence [13]. A lot of the conclusions taken via spacetime deformation are of geometric nature, but after that, the algebraic part turns out to be very straightforward.

Our aim in this work is to show how Category Theory can be used as a powerful tool in physics, while we also present the recent approach of Locally Covariant Quantum Field Theory. Specifically, we will discuss the Reeh-Schlieder property, and on curved spacetimes show how Category Theory and LCQFT are used in its derivation.

Chapter 2

Category Theory

Category Theory provides a powerful language for both mathematicians and physicists. Its formalism allows for great generalization, which can be illustrated by the fact that certain properties possessed by different mathematical structures can be represented diagrammatically in the same way. Within Category theory it is possible to connect different areas of mathematics, which are represented by the appropriate categories. This connection is performed by the functors, which are basically morphisms between categories. This feature of Category Theory is of high interest for physicists since a functor can act as a “bridge” between two different areas of Physics, such as General Relativity and Quantum Mechanics, which use different mathematical structures in their description, for example, manifolds and Hilbert spaces, respectively.

The formulation of Category Theory was first given by Eilenberg and MacLane in 1945 in their paper “General Theory of natural equivalences” [14]. They pointed out that the collection of mathematical objects of some type together with the morphisms between them should be studied as a structure itself, the category. These morphisms can be represented as arrows in a diagram, which allows one to visualize the differences and similarities in the structure of different categories. Furthermore, we are able to relate functors via natural transformations, which leads to important conclusions in Physics, as we will see when we discuss Locally Covariant Quantum Field Theory in section 4.2.

The construction of mathematics and its fields was done based on the structure of sets, since it is rather intuitive and also very flexible, which made it possible to describe all areas of mathematics within Set Theory. However, this does not mean that Set Theory is always the best approach. One of the main advocates of Category Theory today, John Baez proposed that some features of Quantum Mechanics, such as the failure of local realism and the impossibility of duplicating quantum information, can be better understood from the point of view of Category Theory [15].

There is some discussion regarding the foundations of Category Theory in Set Theory, since in this approach there are some problems when considering categories that are “too big” to be dealt with within set theory [16]. This led to an interest in constructing a theory of categories independently from Set Theory, which was done by Lawvere in [17].

One can also impose some conditions in order to obtain a "concrete category" that would not have such problems. In [1], it was first introduced the notion of a "metacategory", which does not require notions from set theory, and only then categories were defined as an interpretation of category axioms within Set Theory. This is the approach we will follow. We are not interested in presenting categories as a foundation for mathematics. We consider that categories exist in some "foundation framework", which will be Set Theory, but it is also possible to determine the structures and properties of sets within Category Theory. It is important to note that although we use Set Theory as background, everything else will be defined in terms of the axioms from Category Theory. So, we can use some notions from Set Theory, but they have to be appropriately translated to the language of categories.

In this chapter we will introduce the main definitions and features of Category Theory, discuss how it can be promising for the unification of Quantum Mechanics and General Relativity, and, as an example of its applicability in physics, present a simple way to show that second quantization is a functor. Most of the ideas presented in this chapter concerning definitions and properties of Category Theory are based on [1, 16].

2.1 Definition and examples

When dealing with categories, we are not concerned about the particularities of their objects. What is important here is how theses objects relate to each other. So, although we have to specify what kind of objects we are dealing with when defining a category, this is not enough and we must further specify what kind of morphisms exist between them [16]. This is an essential feature, since by looking at given objects from the "outside" we are able to connect different categories via functors, and different functors via natural transformations. Furthermore, in this formulation we are able to define, in a general way, certain notions that appear simultaneously in different areas of mathematics, such as isomorphism, duality, product, among others.

Now we will establish the axioms that define a category according to [1, 18]. A category $\mathfrak{C}$ consists of:

1. A class $\mathrm{Obj}(\mathfrak{C})$, whose elements are called objects;

2. A set $\hom_{\mathfrak{C}}(A, B)$ of morphisms (or arrows) from A to B, where A and B are any two objects in a specific order. For each arrow $f \in \hom_{\mathfrak{C}}(A, B)$ there must be a domain, $A = \mathrm{dom} f$, and a codomain, $B = \mathrm{cod} f$.

3. A rule which assigns for given objects A, B and C, and any arrows $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, a morphism $g \circ f : A \to C$ (Figure 2.1);

The morphism $g \circ f$ is called the composition of f with g and must satisfy the following conditions:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \arrow[dr, "h \circ g"] & \\ & C \arrow[r, "h"'] & D \end{tikzcd}$$

Figure 2.1: Diagrams illustrating the composition and associativity in a category.

(*i*) Associativity: If A, B, C and D are any four objects, and f, g and h are morphisms from A to B, from B to C, and from C to D, respectively, then

$$(h \circ g) \circ f = h \circ (g \circ f). \tag{2.1}$$

The second diagram in figure 2.1 illustrates this property.

(*ii*) Existence of identity: For each object A, there is a morphism 1_A from A to A, called the identity morphism, with the property that if f is any morphism from A to B, then

$$f \circ 1_A = f \tag{2.2}$$

and if g is any morphism from C to A, then

$$1_A \circ g = g. \tag{2.3}$$

We say that a diagram is commutative whenever the path to get from one object to another does not matter. For example, if we define in figure 2.1 an arrow $k : A \to C$, then it is clear that $k = g \circ f$.

The most basic example of a category is the category $\mathfrak{Sets}$, whose objects are sets and whose arrows are functions. From the properties of functions, it is easy to note that compositions exist and that the associativity of morphisms is satisfied for this category, since functions are always associative. Furthermore, the identity would take a set from $\mathfrak{Sets}$ to itself. This is the best example to start with, since the idea here is to describe how objects relate to one another while obeying the axioms, and functions necessarily obey all of them. The definition of a category depends only on the notions of composition and identity as defined in functions between sets, and the remaining properties and definitions within Category Theory will be obtained using only these ideas. Note, however, that not all categories have functions as arrows. A category can be anything, as long as it obeys the axioms [16].

By adding new structures to the objects of $\mathfrak{Sets}$ we can create categories of structured sets, i.e. categories of sets with certain structures whose morphisms preserve these structures. Some examples are the category of groups and group homomorphisms, differential manifolds and smooth functions, topological spaces and continuous mappings, and so on.

Two important examples of categories are a poset and a monoid. A poset P (partially ordered set) is naturally a category in which the objects are the elements of P, and there is a unique arrow: $a \to b \iff a \leq b$. A monoid (Figure 2.2) on the other hand is a category which has only one object and the arrows are the elements of the monoid. In this sense, a category can be regarded as a generalization of posets, of monoids, or of a combination of both. Furthermore, there are the categories $\mathfrak{Pos}$ of posets and monotone functions, and $\mathfrak{Mon}$ of monoids and functions that preserve the monoid structure. Note that while Posets and Monoids are not categories of sets and functions, $\mathfrak{Pos}$ and $\mathfrak{Mon}$ are indeed categories of structured sets with morphisms that preserve these structures.

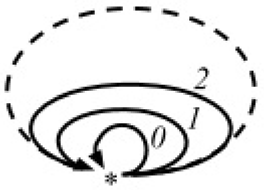

Figure 2.2: Ilustration of a monoid. Available at: http://eed3si9n.com/herding-cats/Monoid-as-categories.html

Although a group is a category by definition (as Posets and Monoids are), we can also construct another category, namely, $\mathfrak{Groups}$, that has groups as elements and homomorphisms as morphisms.

Before ending this section, we will present a few abstract notions from Category Theory that only depend on the axioms of the theory.

Definition 2.1. In a category $\mathfrak{C}$, an arrow $f : A \to B$ is an *isomorphism* if there is an arrow $g : B \to A$ such that

$$g \circ f = 1_A \quad \text{and} \quad f \circ g = 1_B,$$

i.e., if the arrow f is invertible. Whenever such arrow exists between two objects of a category, we say that they are *isomorphic* in $\mathfrak{C}$.

Definition 2.2. An arrow $m : A \to B$ is *monic* in $\mathfrak{C}$ when for any two parallel arrows (arrows that have the same domain and codomain) $f_1, f_2 : D \to A$ the equality $m \circ f_1 = m \circ f_2$ implies $f_1 = f_2$.

Definition 2.3. An arrow $h : A \to B$ is *epi* in $\mathfrak{C}$ when for any two arrows $g_1, g_2 : B \to C$, the equality $g_1 \circ h = g_2 \circ h$ implies $g_1 = g_2$.

When we consider these abstract notions in the context of $\mathfrak{Sets}$, its easy to show that the monic arrows (monomorphisms) are injective, and the epi arrows (epimorphisms) are surjective functions.

Definition 2.4. A *group* G is a monoid with an inverse g^{-1} for each element g. Therefore, G is a category with one object, in which every arrow is an isomorphism.

Definition 2.5. A *homomorphism* of groups $h : G \to H$ is just a homomorphism of monoids that also preserves inverses.

2.2 Functors

As we have discussed, functors play a fundamental role in Category Theory, and their definition is essential for the understanding of a Locally Covariant Quantum Field Theory. To put it simply, a functor is a homomorphism between categories, since it preserves the structure of the category by maintaining the relations between the objects and compositions of functions.

Definition 2.6 (Functor)**.** Given two categories $\mathfrak{C}$ and $\mathfrak{B}$, a (covariant) functor

$$\mathcal{F} : \mathfrak{C} \to \mathfrak{B}$$

consists of two functions: an object function, which assigns to each $C \in \mathfrak{C}$ an object $\mathcal{F}(C) \in \mathfrak{B}$, and an arrow function which assigns to each arrow $f \in \mathfrak{C}$ an arrow $\mathcal{F}(f) \in \mathfrak{B}$ such that

1. $\mathcal{F}(f : C \to D) := \mathcal{F}(f) : \mathcal{F}(C) \to \mathcal{F}(D)$,
2. $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$,
3. $\mathcal{F}(1_C) = 1_{\mathcal{F}(C)}$.

Then we have the diagrams from figure 2.3, which illustrate these conditions.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] & {} \arrow[r, "\mathcal{F}"] & {} & \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[dr, "\mathcal{F}(g \circ f)"'] & \mathcal{F}(B) \arrow[d, "\mathcal{F}(g)"] \\ & C & & & & \mathcal{F}(C) \end{tikzcd}$$

Figure 2.3: Diagrams of the action of a functor $\mathcal{F}$ on the objects and morphisms of a category.

We can compose functors in the following way: Given functors $\mathfrak{C} \xrightarrow{\mathcal{T}} \mathfrak{B} \xrightarrow{\mathcal{S}} \mathfrak{A}$ between categories $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$, the composite functions

$$C \mapsto \mathcal{S}(\mathcal{T}(C)) \quad f \mapsto \mathcal{S}(\mathcal{T}(f))$$

on objects C and arrows f of $\mathfrak{C}$ define a functor $\mathcal{S} \circ \mathcal{T} : \mathfrak{C} \to \mathfrak{A}$, which is the composition of $\mathcal{S}$ with $\mathcal{T}$. Since this composition is associative, and since there is an identity functor

$\mathcal{I}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ for each category $\mathfrak{A}$, which acts as an identity for composition, it is possible to form categories in which the objects are categories and the morphisms are functors, for example the category of all categories.

Definition 2.6 concerns a covariant functor, but there are also contravariant ones, which invert the direction of the morphisms. Usually, when functors are mentioned it is assumed that they are covariant, unless stated otherwise.

Definition 2.7 (Contravariant functor)**.** A functor $\mathcal{F} : \mathfrak{C} \to \mathfrak{B}$ is contravariant if it assigns to each object $C \in \mathfrak{C}$ an object $\mathcal{F}(C) \in \mathfrak{B}$, and to each arrow $f \in \mathfrak{C}$ an arrow $\mathcal{F}(f) \in \mathfrak{B}$ such that

1. $\mathcal{F}(f : C \to D) := \mathcal{F}(f) : \mathcal{F}(D) \to \mathcal{F}(C)$,
2. $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$,
3. $\mathcal{F}(1_C) = 1_{\mathcal{F}(C)}$.

An isomorphism $\mathcal{T} : \mathfrak{C} \to \mathfrak{B}$ of categories is a functor $\mathcal{T}$ from $\mathfrak{C}$ to $\mathfrak{B}$ which is a bijection on both objects and arrows. So, a functor $\mathcal{T} : \mathfrak{C} \to \mathfrak{B}$ is an isomorphism if and only if there is a functor $\mathcal{S} : \mathfrak{B} \to \mathfrak{C}$ so that both $\mathcal{S} \circ \mathcal{T} = \mathcal{I}_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}$ and $\mathcal{T} \circ \mathcal{S} = \mathcal{I}_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B}$ are identity functors.

There are certain types of functors that are worth mentioning:

1. The *forgetful* functor "forgets" some or all of the structure of an algebraic object;
2. A functor $\mathcal{T} : \mathfrak{C} \to \mathfrak{B}$ is *full* when to every pair C, C' of objects of $\mathfrak{C}$ and to every arrow $g : \mathcal{T}(C) \to \mathcal{T}(C')$ of $\mathfrak{B}$, there is an arrow $f : C \to C'$ of $\mathfrak{C}$ with $g = \mathcal{T}(f)$;
3. A functor $\mathcal{T} : \mathfrak{C} \to \mathfrak{B}$ is *faithful* when to every pair C, C' of objects of $\mathfrak{C}$ and to every pair $f_1, f_2 : C \to C'$ of parallel arrows of $\mathfrak{C}$, the equality $\mathcal{T}(f_1) = \mathcal{T}(f_2) : \mathcal{T}(C) \to \mathcal{T}(C')$ implies $f_1 = f_2$.

An example of a forgetful functor is $\mathcal{U} : \mathfrak{Groups} \to \mathfrak{Sets}$, which assigns to each group G the set $\mathcal{U}G$ of its elements and to each morphism $f : G \to G'$ the same function f, now taken to be a simple function of sets. This functor removes the group operation and its underlying properties. As for the full and faithful functors, they are the functor analogues of surjective and injective functions, respectively, when they are restricted to a specific set of morphisms with a given domain and codomain. For example, a functor $\mathcal{F} : \hom_{\mathfrak{C}}(A, B) \to \hom_{\mathfrak{B}}(\mathcal{F}(A), \mathcal{F}(B))$ is faithful if $\mathcal{F}$ is injective. But this is still true if there is a morphism which takes another object, say A', to B.

Definition 2.8. A subcategory $\mathfrak{S}$ of a category $\mathfrak{C}$ is a collection of some of the objects and some of the arrows of $\mathfrak{C}$ which satisfies:

1. for every arrow $f \in \mathfrak{S}$, the objects $\mathrm{dom} f$ and $\mathrm{cod} f$ are also in $\mathfrak{S}$;

2. for every object $S \in \mathfrak{S}$, its identity 1_S is in $\mathfrak{S}$;
3. for every pair of composable arrows $f, g \in \mathfrak{S}$, with $f \circ g \in \mathfrak{C}$, its composition $f \circ g$ must also be in $\mathfrak{S}$.

These properties guarantee that a subcategory is a category on its own. Therefore, there can be a faithful functor $\mathcal{S} : \mathfrak{S} \to \mathfrak{C}$, which we call inclusion functor, that maps

$$S \in \mathfrak{S} \mapsto S \in \mathfrak{C}$$

$$f \in \mathfrak{S} \mapsto f \in \mathfrak{C}.$$

$\mathfrak{S}$ will be a full subcategory whenever the inclusion functor is full.

2.3 Natural Transformations

Now that we have established the definition of functors, which are morphisms between categories, we can extend this idea and introduce the morphisms between functors, namely, natural transformations. The whole point of category theory is to have a framework in which natural transformations exist. The reason for that is that these morphisms allow us to study how different constructions of categories are related to one another, without having to investigate the actual objects of the category. And it is no surprise that this is very useful in physics too. As a matter of fact, we will see that equivalent Locally Covariant Quantum Field Theories are related by natural isomorphisms.

Definition 2.9 (Natural transformation)**.** Given two functors $\mathcal{S}, \mathcal{T} : \mathfrak{C} \to \mathfrak{B}$, a natural transformation $\tau : \mathcal{S} \to \mathcal{T}$ is a function which assigns to each object C of $\mathfrak{C}$ an arrow $\tau_C : \mathcal{S}(C) \to \mathcal{T}(C)$ of $\mathfrak{B}$ such that every arrow $f : C \to C'$ in $\mathfrak{C}$ yields a diagram as given in figure 2.4, which is commutative, i.e.

$$\tau_{C'} \circ \mathcal{S}(f) = \mathcal{T}(f) \circ \tau_C. \tag{2.4}$$

$$\begin{tikzcd} C \arrow[dd] & \mathcal{S}(C) \arrow[r, "\tau_C"] \arrow[dd, "\mathcal{S}(f)"'] & \mathcal{T}(C) \arrow[dd, "\mathcal{T}(f)"] \\ & & \\ C' & \mathcal{S}(C') \arrow[r, "\tau_{C'}"'] & \mathcal{T}(C') \end{tikzcd}$$

Figure 2.4: Diagrams associated with a natural transformation I.

A natural transformations is, therefore, a family of morphisms in $\mathfrak{B}$, indexed by the object C of the category $\mathfrak{C}$:

$$\tau = (\tau_C : \mathcal{S}(C) \to \mathcal{T}(C))_{C \in \mathrm{Obj}(\mathfrak{C})}.$$

Furthermore, whenever 2.4 is satisfied, we say that τ_C is natural in C.

We can interpret the functors $\mathcal{S}$ and $\mathcal{T}$ as morphisms that gives us illustrations of the category $\mathfrak{C}$ in $\mathfrak{B}$. In this sense, a natural transformation is a set of arrows that map the illustration made by $\mathcal{S}$ into the illustration made by $\mathcal{T}$. We would then obtain the commutative diagram given in figure 2.5, where the $\tau_A, \tau_B, \tau_C, \ldots$ are the components of the natural transformation τ.

$$
\begin{tikzcd}
A \arrow[dr, "f"] \arrow[dd, "h"'] & \\
& B \arrow[dl, "g"] \\
C &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{S}(A) \arrow[rr, "\tau_A"] \arrow[dr, "\mathcal{S}(f)"] \arrow[dd, "\mathcal{S}(h)"'] & & \mathcal{T}(A) \arrow[dr, "\mathcal{T}(f)"] \arrow[dd, "\mathcal{T}(h)"'] & \\
& \mathcal{S}(B) \arrow[rr, "\tau_B"] \arrow[dl, "\mathcal{S}(g)"] & & \mathcal{T}(B) \arrow[dl, "\mathcal{T}(g)"] \\
\mathcal{S}(C) \arrow[rr, "\tau_C"] & & \mathcal{T}(C) &
\end{tikzcd}
$$

Figure 2.5: Diagrams associated with a natural transformation II.

If we take functors to be the objects of a category, and the natural transformations to be the morphisms, we can define a functor category.

Definition 2.10. The functor category $\mathrm{Fun}(\mathfrak{C}, \mathfrak{B})$ consists of:

1. Objects: Functors $\mathcal{F}, \mathcal{G} : \mathfrak{C} \to \mathfrak{B}$ such that $\mathrm{dom}(\mathcal{F})$ is small (See definition 2.14);
2. Morphisms: Natural transformations $\tau : \mathcal{F} \to \mathcal{G}$;
3. Identity: For every object $\mathcal{F}$, the identity natural transformation $1_{\mathcal{F}}$ has components
$$(1_{\mathcal{F}})_C = 1_{\mathcal{F}(C)} : \mathcal{F}(C) \to \mathcal{F}(C);$$
4. Composition: The composite natural transformation $\phi \circ \tau : \mathcal{F} \xrightarrow{\tau} \mathcal{G} \xrightarrow{\phi} \mathcal{H}$ has components
$$(\phi \circ \tau)_C = \phi_C \circ \tau_C.$$

Definition 2.11. A natural isomorphism (or natural equivalence) is a natural transformation
$$\tau : \mathcal{F} \to \mathcal{G},$$
which is an isomorphism in $\mathrm{Fun}(\mathfrak{C}, \mathfrak{B})$. This means that every component of τ is invertible in $\mathfrak{B}$. We write $\mathcal{F} \cong \mathcal{G}$.

Finally, we can define an equivalence between categories $\mathfrak{C}, \mathfrak{B}$ as a pair of functors $\mathcal{S} : \mathfrak{C} \to \mathfrak{B}$, $\mathcal{T} : \mathfrak{B} \to \mathfrak{C}$ together with the natural isomorphisms
$$\mathcal{T} \circ \mathcal{S} \cong \mathcal{I}_{\mathfrak{C}}, \quad \mathcal{S} \circ \mathcal{T} \cong \mathcal{I}_{\mathfrak{B}}.$$

2.4 Cayley's theorem

Cayley's theorem is an example of how abstract group theory is, and here we will show how we can translate this theorem to the language of category theory, so that we have an analogue for categories. This way, we emphasize the power that category theory has to provide an abstract framework, which in turn can be used to describe theories that have a general aspect. Furthermore, some notions regarding the size of a category will be introduced, since in our definition of Category Theory we are considering locally small categories, which can be dealt with within Set Theory without many problems. Here we will follow [16].

Theorem 2.12 (Cayley)**.** *Every group G is isomorphic to a permutation group.*

Proof. In order to prove this, we will be using only the axioms of category theory and its abstract notions, such as isomorphisms, inverse, etc.

Let G be a group with elements $g, h, x, ...$. We will define Cayley's representation $\bar{G}$ of G as the following permutation group:

- The associated set is G
- For each $g \in G$, the permutation $\bar{g}$, defined for every $h \in G$ is

$$\bar{g}(h) = g \cdot h.$$

Now,

$$\bar{g}(x) = g \cdot x, \qquad \bar{h}(x) = h \cdot x,$$

so that if $\bar{g} = \bar{h}$, then

$$g \cdot x = h \cdot x.$$

Since G is a group, there is an element x^{-1} such that $x \cdot x^{-1} = e$, where e is the identity element. So,

$$g \cdot x \cdot x^{-1} = h \cdot x \cdot x^{-1} \Leftrightarrow g \cdot e = h \cdot e \Leftrightarrow g = h.$$

Let's define the homomorphism $\mathcal{I} : G \to \bar{G}$ by $\mathcal{I}(g) = \bar{g}$ and $\mathcal{J} : \bar{G} \to G$ by $\mathcal{J}(\bar{g}) = g$. We then obtain,

$$\mathcal{I} \circ \mathcal{J}(\bar{g}) = \mathcal{I}(g) = \bar{g} \Rightarrow \mathcal{I} \circ \mathcal{J} = 1_{\bar{G}}$$

and

$$\mathcal{J} \circ \mathcal{I}(g) = \mathcal{J}(\bar{g}) = g \Rightarrow \mathcal{I} \circ \mathcal{I} = 1_G.$$

Therefore, $\mathcal{I}$ and $\mathcal{J}$ are isomorphisms and they are each other's inverse.

$\square$

In this proof we dealt with two levels of isomorphisms. Since we have permutations for each element of G, this corresponds to an isomorphism in $\mathfrak{Sets}$, in which a set of elements G is mapped to a set of permutations of this set. But we also have an isomorphism between G and $\bar{G}$, which are in the category $\mathfrak{Groups}$ and in this case we have a map $i : (G, \cdot) \to (G, \bullet)$.

What this theorem tells us is that any abstract group can be represented as a "concrete" group, the permutation group. This statement can be generalized to categories and we will show next how an abstract category can be represented as a "concrete" category. However, this can only be done to "small" categories, and we will present now the definitions and conditions necessary to make this happen.

Definition 2.13. A concrete category is a category equipped with a faithful functor between it and the category $\mathfrak{Sets}$. In other words, It is a pair $(\mathfrak{C}, \mathcal{U})$, so that $\mathfrak{C}$ is a category and $\mathcal{U} \to \mathfrak{Sets}$ is a faithful functor.

Definition 2.14. A category $\mathfrak{C}$ is said to be small if both its collection of objects, $\mathrm{Obj}(\mathfrak{C})$, and its collection of morphisms, $\hom(\mathfrak{C})$, are sets. Otherwise, it is a big category.

Theorem 2.15. *Every small category $\mathfrak{C}$ with a set of arrows is isomorphic to one in which the objects are sets and the arrows are functions.*

Proof. First, define the Cayley representation as the following concrete category:

- Objects: Sets of the form

$$\overline{C} = \{f \in \mathfrak{C} | \quad \mathrm{cod}(f) = C\}$$

for every $C \in \mathfrak{C}$.

- Morphisms: functions

$$\bar{g} : \bar{C} \to \bar{D}$$

for $g : C \to D$ in $\mathfrak{C}$ defined as

$$\bar{g}(f) = g \circ f.$$

Now we show that if $\bar{g}(f) = \bar{h}(f)$ then $g = h$:

$$\bar{g}(f) = \bar{h}(f) \Rightarrow g \circ f = h \circ f \Rightarrow g \circ f(a) = h \circ f(a) \Rightarrow g(f(a)) = h(f(a)),$$

which, by the definition of functions, implies that $g = h$.

Define a homomorphism between categories $\mathcal{I} : \mathfrak{C} \to \overline{\mathfrak{C}}$ by $\mathcal{I}(C) = \bar{C}$, so that $\mathcal{I}(g) = \bar{g}$, and another homomorphism $\mathcal{J} : \overline{\mathfrak{C}} \to \mathfrak{C}$ by $\mathcal{J}(\bar{C}) = C$, with $\mathcal{J}(\bar{g}) = g$. Then we can show that

$$\mathcal{I} \circ \mathcal{J}(\bar{g}) = \mathcal{I}(g) = \bar{g} \quad \mathcal{I} \circ \mathcal{J} = 1_{\overline{\mathfrak{C}}}$$

$$\mathcal{J} \circ \mathcal{I}(g) = \mathcal{J}(\bar{g}) = g \quad \mathcal{J} \circ \mathcal{I} = 1_{\mathfrak{C}}.$$

Then, there is an isomorphism between each locally small category and some category of sets and functions (which is a subcategory of $\mathfrak{Sets}$).

□

2.5 Some constructions: Product and opposite categories

It is possible to construct new categories from other ones. This can be done by taking the product of categories, "slicing" the category with respect to an object, or considering a category with the same objects but with inverted arrows, among other ways. We will present here the product construction, as well as the opposite category, since it enables us to discuss duality within category theory [1, 16].

Example 2.1 (Product category)**.** The product of two categories, $\mathfrak{C}$ and $\mathfrak{D}$, written as $\mathfrak{C} \times \mathfrak{D}$, consists of:

1. Objects: (C, D), where $C \in \mathfrak{C}$ and $D \in \mathfrak{D}$;
2. Morphisms: $(f, g) : (C, D) \to (C', D')$, where $f : C \to C' \in \mathfrak{C}$ and $g : D \to D' \in \mathfrak{D}$;
3. Composition: $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$;
4. Identity: $1_{(C,D)} = (1_C, 1_D)$.

$$\begin{tikzcd}
(C,D) \arrow[r, "{(f,g)}"] \arrow[dr, "{(f',g')\circ(f,g)}"'] & (C',D') \arrow[d, "{(f',g')}"] \\
 & (C'',D'')
\end{tikzcd}$$

Figure 2.6: Diagram of the composition of morphisms in the product category.

From the product category, it is possible to obtain the individual categories via the projection functors

$$\mathfrak{C} \xleftarrow{\mathcal{P}} \mathfrak{C} \times \mathfrak{D} \xrightarrow{\mathcal{Q}} \mathfrak{D}, \tag{2.5}$$

defined by

$$\mathcal{P}(C, D) = C \quad \text{and} \quad \mathcal{P}(f, g) = f \tag{2.6}$$

$$\mathcal{Q}(C, D) = D \quad \text{and} \quad \mathcal{Q}(f, g) = g \tag{2.7}$$

Analogously, the product construction is itself a functor. Furthermore, in order to relate different product categories, we must have a product functor. Given two functors, $\mathcal{U} : \mathfrak{B} \to \mathfrak{B}'$ and $\mathcal{V} : \mathfrak{C} \to \mathfrak{C}'$, the product functor $\mathcal{U} \times \mathcal{V} : \mathfrak{B} \times \mathfrak{C} \to \mathfrak{B}' \times \mathfrak{C}'$ acts on the objects and arrows in the following way:

$$(\mathcal{U} \times \mathcal{V})(B, C) = (\mathcal{U}B, \mathcal{V}C), \quad (\mathcal{U} \times \mathcal{V})(f, g) = (\mathcal{U}f, \mathcal{V}g).$$

One important concept in mathematics is that of duality. Not only in mathematics actually, in physics we see it all the time, especially in Quantum Mechanics. The idea of duality is that given a valid statement within some background theory, its dual is also valid. This is the principle of duality. The most basic example are the ones in logic, where one can invert the logical quantifiers, for example $\exists$ by $\forall$, which are dual. In set theory one can obtain dual statements by interchanging sets by their complements, union by intersections, reversing inclusions, and so on [19]. If one statement is valid, then the other should also be. This makes proving theorems much easier, because if we already have the proof for one theorem, then we do not need to prove its dual. It is not a surprise that this also works in Category Theory. We can construct categories and their duals by simply inverting the arrows [1]. Other dual concepts are monic and epi arrows, domain and codomain, left and right inverse, morphisms $f \circ g$ and $g \circ f$, among others. So, in the context of category theory it is also possible to use the principle of duality to obtain certain properties by proving their dual.

Example 2.2. We can then associate to each category $\mathfrak{C}$ the opposite (or dual) category $\mathfrak{C}^{op}$, which has the same objects as $\mathfrak{C}$, and an arrow $f : C \to D$ in $\mathfrak{C}^{op}$ is an arrow $f : D \to C$ in $\mathfrak{C}$. Equivalently, $\mathfrak{C}^{op}$ is $\mathfrak{C}$ with all its arrows inverted.

We write

$$\bar{f} : \bar{D} \to \bar{C} \tag{2.8}$$

in $\mathfrak{C}^{\mathrm{op}}$ for

$$f : C \to D \tag{2.9}$$

in $\mathfrak{C}$.

With this notation we can define the composition and identity in $\mathfrak{C}^{op}$ in terms of the corresponding operations in $\mathfrak{C}$:

Identity: $1_{\bar{C}} = \overline{1_C}$;

Composition:$\bar{f} \circ \bar{g} = g \bar{\circ} f$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\
 & C
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{A} & \bar{B} \arrow[l, "\bar{f}"'] \\
 & \bar{C} \arrow[u, "\bar{g}"'] \arrow[ul, "\bar{f} \circ \bar{g}"]
\end{tikzcd}$$

Figure 2.7: Diagrams of a category and its associated dual category. Available at: Steve Awodey

The bar over the morphisms indicate that the arrow has the opposite direction. Although the arrows are inverted when we take the dual, the same does not occur to functors.

If $\mathcal{T} : \mathfrak{C} \to \mathfrak{B}$ is a functor, its object function is $C \mapsto \mathcal{T}(C)$ and its morphism function is $f \mapsto \mathcal{T}(f)$, which, when we take the dual, can be rewritten as

$$f^{op} \mapsto (\mathcal{T}(f))^{op}.$$

They define a functor from $\mathfrak{C}^{op}$ to $\mathfrak{B}^{op}$, which we write as

$$\mathcal{T}^{op} : \mathfrak{C}^{op} \to \mathfrak{B}^{op}.$$

The assignments

$$\mathfrak{C} \mapsto \mathfrak{C}^{op} \quad \text{and} \quad \mathcal{T} \mapsto \mathcal{T}^{op}$$

define a covariant functor $\mathfrak{Cat} \to \mathfrak{Cat}$.

If we have a contravariant functor $\overline{\mathcal{F}} : \mathfrak{C} \to \mathfrak{B}$, we can write it as a covariant functor $\mathcal{F} : \mathfrak{C}^{op} \to \mathfrak{B}$ by making

$$\mathcal{F}(f^{op}) = \overline{\mathcal{F}}(f).$$

2.6 Diagrammatic properties of categories

Many statements in category theory can be formulated by asserting that some diagram commutes, and different areas of mathematics can use the same diagram to represent a given property that is shared by different mathematical structures. So the diagrams play a central role in category theory as it is one of the ways in which the generality of this theory is presented. In this section we will present a few examples of mathematical notions that can be illustrated as a commutative diagram [1, 16].

Example 2.3 (Cartesian product)**.** The cartesian product $X \times Y$ of two sets, consists of all the ordered pairs (x, y) of elements $x \in X$ and $y \in Y$. This example is similar to the product of any two categories, but it is a special case in which we consider sets.

The projections $(x, y) \mapsto x$, $(x, y) \mapsto y$ of the cartesian product in its X and Y "axes" are funtions

$$p : X \times Y \to X,$$

$$q : X \times Y \to Y.$$

Any function $h : W \to X \times Y$ of a third set W is uniquely determined by the compositions $p \circ h$ and $q \circ h$. Alternatively, given a set W and two functions f and g, as in the diagram of figure 2.8, there is a unique function h which makes the diagram commutes. Namely,

$$h(w) = (f(w), g(w)) = (x, y). \tag{2.10}$$

And in order for the diagram to commute, we also must have

$$p \circ h = f \quad \text{e} \quad q \circ h = g.$$

$$\begin{tikzcd}
& W \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "p"] \arrow[r, "q"'] & Y
\end{tikzcd}$$

Figure 2.8: Diagram of the function from an arbitrary set to a cartesian product.

Here we have the property that (p, q) is universal among pairs of functions from some set to X and Y. Any pair (f, g) is uniquely factored through the pair (p, q) and the function h. This universal property describes the cartesian product uniquely, up to bijection, since a bijection would lead to a different "representation" of this product.

If we add more structure into this set, we can use the same diagram to describe, for example, the product of spaces in the category of topological spaces, the direct product of groups in the category of groups, and the same for the categories of modules, manifolds, and so on.

We can also define a one-point set $1 = \{0\}$, which acts as an identity under a "cartesian product" operation, taking into account the bijections

$$1 \times X \xrightarrow{\lambda} X \xleftarrow{\rho} X \times 1 \tag{2.11}$$

given by

$$\lambda(0, x) = x \quad , \quad \rho(x, 0) = x. \tag{2.12}$$

Example 2.4 (Associativity and identity in monoids)**.** Consider a monoid M, which in Category Theory can be described as a set M along with two functions

$$\mu : M \times M \to M, \quad \eta : 1 \to M, \tag{2.13}$$

where η maps the element 0 of the set $1 = \{0\}$ to the identity element u of M, such that the diagrams from figure 2.9 commute.

$$\begin{tikzcd}
M \times M \times M \arrow[r, "1 \times \mu"] \arrow[d, "\mu \times 1"'] & M \times M \arrow[d, "\mu"] \\
M \times M \arrow[r, "\mu"'] & M
\end{tikzcd}$$

(Associativity)

$$\begin{tikzcd}
1 \times M \arrow[r, "\eta \times 1"] \arrow[d, "\lambda"'] & M \times M \arrow[d, "\mu"] & M \times 1 \arrow[l, "1 \times \eta"'] \arrow[d, "\rho"] \\
M \arrow[r, phantom, "="] & M \arrow[r, phantom, "="] & M
\end{tikzcd}$$

(Identity)

Figure 2.9: Diagrams representing associativity and identity in a monoid.

In the diagrams, 1 in $1 \times \mu$ is the identity function $M \to M$ and 1 in $1 \times M$ is the one-point set $1 = \{0\}$.

If the two diagrams commute, we have

$$\mu \circ (1 \times \mu) = \mu \circ (\mu \times 1) \tag{2.14}$$

and

$$\mu(\eta \times 1) = \lambda, \quad \mu \circ (1 \times \eta) = \rho. \tag{2.15}$$

If we rewrite these diagrams in terms of the elements of the monoids, using $\mu(x, y) = xy$ for $x, y \in M$ and $\eta(0) = u$ we will find the diagrams in figure 2.10, which make it clear that the commutativity of the diagrams imply that the multiplication in the monoid is associative and that the identity element u is right and left identity.

$$\begin{tikzcd}
\langle x,y,z\rangle \arrow[r, mapsto] \arrow[d, mapsto] & \langle x,yz\rangle \arrow[d, mapsto] \\
\langle xy,z\rangle \arrow[r, mapsto] & (xy)z = x(yz)
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle 0,x\rangle \arrow[r, mapsto] \arrow[d, mapsto] & \langle u,x\rangle \arrow[d, mapsto] & \langle x,u\rangle \arrow[r, mapsto] \arrow[d, mapsto] & \langle x,0\rangle \arrow[d, mapsto] \\
x \arrow[r, phantom, "="] & ux & xu \arrow[r, phantom, "="] & x
\end{tikzcd}$$

(Associativity) (Identity)

Figure 2.10: Diagrams representing associativity and identity in a monoid in terms of its elements.

Example 2.5 (Inverses in groups)**.** We can define a group as a monoid M equipped with a function $\zeta : M \to M$, which is the function $x \mapsto x^{-1}$, such that the diagrams in figure 2.11 commute. Here, $\delta : M \to M \times M$ is the diagonal function $x \mapsto (x, x)$ for $x \in M$, while $M \to 1 = \{0\}$ is the function from M to the one-point set. Following the same logic, it can also be shown that x^{-1} is a left inverse.

$$\begin{tikzcd}
M \arrow[r, "\delta"] \arrow[d] & M \times M \arrow[r, "1\times\zeta"] & M \times M \arrow[d, "\mu"] \\
1 \arrow[rr] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & \langle x,x\rangle \arrow[r, mapsto] & \langle x,x^{-1}\rangle \arrow[d, mapsto] \\
0 \arrow[r, mapsto] & u \arrow[r, phantom, "="] & xx^{-1}
\end{tikzcd}$$

Figure 2.11: Diagrams representing an inverse in a group.

The same diagrams used to define a group, figures 2.9 and 2.11, can be used to define other groups, such as topological groups and Lie groups. This shows the power of generalization possessed by Category Theory, since this approach makes it easier to see similarities between different structures.

Example 2.6 (Homomorphism)**.** The homomorphisms of algebraic structures can also be described by diagrams. If (M, μ, ν) and (M', μ', η') are two monoids, then a homomorphism from the first to the second one can be described as a function

$$f : M \to M',$$

so that the diagrams from figure 2.12 commute. If they commute, then for the diagram in the middle, we must have

$$\mu' \circ (f \times f) = f \circ \mu \tag{2.16}$$

and for the third one,

$$\eta' = f \circ \eta, \tag{2.17}$$

where

$$\eta : 1 \to M \quad \text{and} \quad \mu : M \times M \to M.$$

$$\begin{tikzcd}
M \arrow[d, "f"] \\
M'
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times M \arrow[r, "\mu"] \arrow[d, "f \times f"'] & M \arrow[d, "f"] \\
M' \times M' \arrow[r, "\mu'"'] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\eta"] \arrow[d, equal] & M \arrow[d, "f"] \\
1 \arrow[r, "\eta'"'] & M'
\end{tikzcd}$$

Figure 2.12: Diagrams representing a homomorphism.

The diagrams in terms of the elements can be seen on figure 2.13. From them it is clear to see the property of a homomorphism, that is, the fact that compositions and identities are preserved.

$$\begin{tikzcd}
(x, y) \arrow[r, mapsto] \arrow[d, mapsto] & xy \arrow[d, mapsto] \\
(x), f(y)) \arrow[r, mapsto] & f(x)f(y) = f(xy)
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[d, mapsto] & u \arrow[d, mapsto] \\
0 \arrow[r, mapsto] & u' = f(u)
\end{tikzcd}$$

Figure 2.13: Diagrams representing homomorphisms in terms of its elements.

Example 2.7 (Action of a monoid)**.** As a last example, we will show how the action of a monoid can be represented diagrammatically. Let M be a monoid. Then a (left) action of a monoid is a set S together with an operation $\cdot : M \times S \to S$, which is compatible with the monoid operation $*$ in the following way:

- for all $s, t \in M$ and $x \in S$, $s \cdot (t \cdot x) = (s * t) \cdot x$;
- for all $x \in S$, $u \cdot x = x$.

Now, we give the formulation of this definition in terms of diagrams. An action of a monoid (M, μ, η) in a set S is defined as a function $\nu : M \times S \to S$, so that the diagrams in figure 2.14 commute. Writing $\nu(x, s) = x \cdot s$ for the action of an element of the monoid in the element $s \in S$, these diagrams affirm that

$$x \cdot (y \cdot s) = (xy) \cdot s, \quad u \cdot s = s,$$

which agrees with our definition of monoid.

Once again, these diagrams can be used to illustrate the action of a group in a set, of topological monoids in topological spaces (in the category of topological spaces), and so on.

$$\begin{tikzcd} M \times M \times S \arrow[r, "1 \times \nu"] \arrow[d, "\mu \times 1"'] & M \times S \arrow[d, "\nu"] \\ M \times S \arrow[r, "\nu"'] & S \end{tikzcd} \qquad \begin{tikzcd} 1 \times S \arrow[r, "\eta \times 1"] \arrow[dr, "\lambda"'] & M \times S \arrow[d, "\nu"] \\ & S \end{tikzcd}$$

Figure 2.14: Diagrams representing the action of a monoid.

2.7 Second Quantization

"First Quantization is a mistery, but second quantization is a functor"
Edward Nelson[1]

We have already discussed how one of the main advantages of Category Theory is the fact that it allows for a global view of mathematical structures, therefore providing ways of connecting some structures to others. So in this section we would like to present an example which illustrates a concept of physics that has the structure of a functor. The above quote by Edward Nelson depicts the fact that when considering first quantization, one would expect to translate the first quantization procedure into a functorial language. But this attempt has led to certain problems in the past, while second quantization is very easily proven to behave exactly as a functor.

Let us recall that first quantization is the process of assigning operators to observers, so in order to describe it via categories one could expect to have a functor between the phase space with a Hamiltonian and the Hilbert space with a Hamiltonian operator. So, a functor which performs the following maps:

Object function: Phase space $\longrightarrow$ Hilbert space,

Arrow function: Canonical transformations $\longrightarrow$ Unitary operators.

However, this raises some issues, like the fact that this would not be a positivity preserving functor, in the sense that, if we had a positive Hamiltonian, it would not be

[1] This is a famous quote by Edward Nelson that was taken from [20]. There seems to be no written register of it by Edward Nelson himself and it is argued that it was actually spoken.

mapped into positive generators [21]. A way in which this could be done correctly is via geometrical quantization.

Geometrical quantization has been studied for some years now [22,23], and it seems to shed some light on certain inconsistencies of quantum mechanics, like quantization itself, that turns out to be a functor in this approach, as discussed by Baez [24]. We are not going to get into details here, but we find important to mention such an important discovery. As Baez said, "A space of quantum states is an example of a space of classical states equipped with precisely all the complicated extra structure that lets us geometrically quantize it." [2] As it turns out, this procedure allows one to get a functor from the classical system to the quantum one, and yet another functor which reverses this procedure.

Next we will show how second quantization [20,25] can be proven to be a functor. But in order to do that, we would like to give a precise definition of a Hilbert space[3], which is essential for studying quantum systems.

Definition 2.16. A *Hilbert space* $\mathcal{H}$ is a complex vector space, whose elements are called vectors, endowed with an inner product and complete with respect to the norm

$$||\Psi|| = \sqrt{(\Psi, \Psi)} \geq 0, \tag{2.18}$$

which is induced from the inner product.

To clarify the above definition, recall that in order for $\mathcal{H}$ to be a vector space, we must have for vectors Ψ_1 and Ψ_2 in $\mathcal{H}$ and complex numbers c_1, c_2, a linear combination

$$c_1\Psi_1 + c_2\Psi_2$$

which is itself a vector in $\mathcal{H}$. Additionally, all the axioms of vectors spaces must be satisfied, such as associativity and commutativity of addition, existence of identity, among others. The inner product between vectors Ψ and Φ, written as (Ψ, Φ), satisfies

1. Linearity in the second slot: $(\Psi, c_1\Phi_1 + c_2\Phi_2) = c_1(\Psi, \Phi_1) + c_2(\Psi, \Phi_2)$,

2. Hermiticity: $\overline{(\Psi, \Phi)} = (\Phi, \Psi)$,

3. Positivity and non-degeneracy: $(\Psi, \Psi) \geq 0$. If $(\Psi, \Psi) = 0$, then $\Psi = 0$.

[2]In this procedure, one must admit that quantum states are not vectors in a Hilbert space H, but instead they are 1-dimensional subspaces of H. Then, the set of quantum states would be PH, a projective space that is a symplectic manifold of H. When this PH is finite-dimensional, it is actually the simplest example of a Kähler manifold equipped with a holomorphic hermitian line bundle whose curvature is the imaginary part of the Kähler structure. This structure added to it is exactly the one that it must have in order fot it to be quantized.

[3]For further information on functional analysis, the reader is referred to [26].

Antilinearity in the first slot follows from 1 and 2:

$$(c_1\Psi_1 + c_2\Psi_2, \Phi) = \bar{c}_1(\Psi_1, \Phi) + \bar{c}_2(\Psi_2, \Phi), \tag{2.19}$$

where $\bar{c}$ is the complex conjugate of c.

Furthermore, from 3 we can obtain for any Ψ, Φ the Cauchy-Schwarz inequality

$$|(\Psi, \Phi)|^2 \leq (\Psi, \Psi)(\Phi, \Phi), \tag{2.20}$$

which together with the inner product properties allows us to define the norm as given by equation 2.18, which respects:

1. $||\Psi|| \geq 0$,
2. $||\Psi|| = 0$ iff $\Psi = 0$,
3. $||c\Psi|| = |c|||\Psi||$,
4. Triangle inequality: $||\Psi_1 + \Psi_2|| \leq ||\Psi_1|| + ||\Psi_2||$.

It is important to mention that any vector space that has a norm is also a metric space[4], whose metric

$$d(\Psi, \Phi) = \|\Psi - \Phi\|$$

is induced by the norm.

Last, but not least, we define convergence of sequences so that we are able to give meaning to the "completeness" of Hilbert space [7].

Definition 2.17. Let Ψ_n be a sequence of vectors.

1. If Ψ_n satisfies

$$\lim ||\Psi_n - \Psi|| = 0, \tag{2.21}$$

then the sequence *converges strongly* to Ψ, and we write $\Psi_n = \Psi$.

2. If for any vector Φ

$$\lim(\Phi, \Psi_n - \Psi) = 0 \tag{2.22}$$

holds, then the sequence *converges weakly* to Ψ, and we write

$$w - \lim \Psi_n = \Psi.$$

3. If Ψ_n satisfies

$$||\Psi_m - \Psi_n|| \to 0 \quad (m, n \to \infty). \tag{2.23}$$

then it is a *Cauchy sequence*.

[4]For more information on metric spaces, we refer the reader to [27].

A sequence which converges strongly is a Cauchy sequence, but the converse is not always true. We then say that a vector space is *complete* if every Cauchy sequence in it is strongly convergent. However, if we have a *pre-Hilbert space* $\mathcal{H}$ (a Hilbert space with all the above requirements except for completeness), it is possible to complete it by extending it to a Hilbert space which contains limit vectors for all the Cauchy surfaces[5] in $\mathcal{H}$ [7].

We would like to get from a quantum description of a single particle to a quantum description of a many-particle system by the introduction of field operators. We will carry out the following steps:

(*i*) Start with a Hilbert Space $\mathcal{H}$ for the single particle system.

(*ii*) Form the symmetric (or antisymmetric) tensor algebra over $\mathcal{H}$.

(*iii*) Complete it to form a Hilbert space $\mathscr{F}_{\pm}(\mathcal{H})$ called the bosonic (fermionic) Fock space over $\mathcal{H}$.

(*iv*) Extend unitary operators in $\mathcal{H}$ to unitary operators in $\mathscr{F}(\mathcal{H})$.

We will then have a functor F called second quantization:
F: (Hilbert space) $\mapsto$ (Fock Space)
$F(f)$: Unitary operator in $\mathcal{H} \mapsto$ Unitary operator in $\mathscr{F}(\mathcal{H})$

In this construction we must take into account the fact that the particles are indistinguishable, so for an element Ψ of $\mathscr{F}(\mathcal{H})$, where

$$\Psi = \{\Psi^{(n)}\}_{n\geq 0}, \tag{2.24}$$

the components $\Psi^{(n)}$ of each Ψ are either symmetric (bosons) or antissymetric (fermions) under interchange of coordinates. This leads to two subspaces of Fock space, $\mathscr{F}(\mathcal{H})_+$ and $\mathscr{F}(\mathcal{H})_-$, respectively.

So, we assume that the states of each particle form a complex Hilbert space $\mathcal{H}$ and if we let

$$\mathcal{H}^n = \mathcal{H} \otimes ... \otimes \mathcal{H} \tag{2.25}$$

be the tensor product of n Hilbert spaces, we then introduce the Fock space by

$$\mathscr{F}(\mathcal{H}) = \bigoplus_{n\geq 0} \mathcal{H}^n, \tag{2.26}$$

where $\mathcal{H}^0 = \mathbb{C}$. So, a vector $\Psi \in \mathscr{F}(\mathcal{H})$ is a sequence $\{\Psi^{(n)}\}_{n\geq 0}$ of vectors $\Psi^{(n)} \in \mathcal{H}^n$.

Now let us find the permutation operators. Let P_n be the permutation group in n elements and let f_k be a base for $\mathcal{H}$. For each $\pi \in P_n$ we define an operator in the base elements of $\mathcal{H}^n$ by

$$\pi(f_{k_1} \otimes f_{k_2} \otimes ... \otimes f_{k_n}) = f_{k_{\pi(1)}} \otimes f_{k_{\pi(2)}} \otimes ... \otimes f_{k_{\pi(n)}}. \tag{2.27}$$

[5] We define Cauchy surfaces is section 4.1. See definition 4.6

We can extend π by linearity to a bounded operator[6] of norm 1 in $\mathcal{H}^n$. But we know that if we have a certain bounded linear operator A on a Banach space X, we can define an operator of the form

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n,$$

where e^A is also a bounded linear operator with $||e^A|| \leq e^{||A||}$. So, we can define the bounded operator

$$P_+ = \frac{1}{n!} \sum_{\pi \in P_n} \pi. \tag{2.28}$$

The image of P_+ is the n-fold *symmetric* tensor product of the basis elements of $\mathcal{H}^n$. Similarly, we define

$$P_- = \frac{1}{n!} \sum_{\pi \in P_n} \varepsilon_\pi \pi, \tag{2.29}$$

where $\varepsilon_\pi = 1$ if π is even, and $\varepsilon_\pi = -1$ if π is odd.

So, when $P_\pm^n$ acts on Fock space, it will yield

$$P_\pm \mathscr{F}(\mathcal{H}) = \bigoplus_{n \geq 0} P_\pm^n \mathcal{H}^n. \tag{2.30}$$

In order to introduce the desired subspaces, we will first define the permutation operators on $\mathcal{H}^n$ by

$$P_+(f_1 \otimes ... \otimes f_n) = (n!)^{-1} \sum_\pi f_{\pi_1} \otimes ... \otimes f_{\pi_n} \tag{2.31}$$

and

$$P_-(f_1 \otimes ... \otimes f_n) = (n!)^{-1} \sum_\pi \varepsilon_\pi f_{\pi_1} \otimes ... \otimes f_{\pi_n}, \tag{2.32}$$

for all $f_1, ..., f_n \in \mathcal{H}$. The sum here is over all the permutations $\pi(1, 2, ..., n) \mapsto (\pi_1, ..., \pi_n)$.

Once again, we extend the $P_\pm$ by linearity to all Fock space by completing $\mathcal{H}^n$. This will result in two densely defined bounded operators $P_\pm$ of norm $||P_\pm|| = 1$.

It is straightforward to see that the $P_\pm$ are orthogonal projections that when acting on the Fock space will yield the corresponding subspace, which is (anti)symmetric and defined by

$$\mathscr{F}_\pm(\mathcal{H}) = P_\pm \mathscr{F}(\mathcal{H}). \tag{2.33}$$

In turn, the n-particle subspaces are defined by

$$\mathcal{H}_\pm^n = P_\pm \mathcal{H}^n. \tag{2.34}$$

[6]An operator U is *bounded* if there exists a real number λ satisfying

$$||U\Psi|| \leq \lambda ||\Psi||$$

for any vector Ψ. The infimum of λ satisfying this equation is denoted as $||U||$ and is called the *norm* of U.

We can see that the same operator acts on both Fock space and tensor product of Hilbert spaces, due to its linearity.

The structure of Fock space enables us to extend the operators of $\mathcal{H}$ to the spaces $\mathscr{F}_{\pm}(\mathcal{H})$ by the so called *second quantization* method.

Self-adjoint operators

If H is a bounded self-adjoint operator on $\mathcal{H}$, we can define H_n in $\mathcal{H}_{\pm}^{n}$ by making $H_0 = 0$ and

$$H_n(P_{\pm}(f_1 \otimes ... \otimes f_n)) = P_{\pm}\left(\sum_{i=1}^{n} f_1 \otimes ... \otimes Hf_i \otimes ... \otimes f_n\right) \tag{2.35}$$

for all $f_i \in D(H)$, where

$$D(N) = \left\{\Psi; \Psi = \{\Psi^{(n)}\}_{n\geq 0}, \sum_{n\geq 0} n^2 ||\Psi^{(n)}||^2 < +\infty\right\}$$

and

$$N\Psi = \{n\Psi^{(n)}\}_{n\geq 0}$$

which the define a number operator N on $\mathscr{F}(\mathcal{H})$ for each $\Psi \in D(N)$. We then extend it by continuity. We have that $\bigoplus H_n$ is self-adjoint, since it is symmetric, closable, and yields a dense set of analytical vectors of H.

The self-adjoint closure,

$$d\Gamma(H) = \overline{\bigoplus_{n\geq 0} H_n}, \tag{2.36}$$

is called *second quantization* of H and is also self-adjoint. It maps all the self-adjoint operators of $\mathcal{H}$ into self-adjoint operators of $\mathscr{F}_{\pm}(\mathcal{H})$.

Unitary operators

If U is a bounded unitary operator, we can define U_n by setting $U_0 = \mathbb{1}$ and

$$U_n(P_{\pm}(f_1 \otimes ... \otimes f_n)) = P_{\pm}(Uf_1 \otimes Uf_2 \otimes ... \otimes Uf_n). \tag{2.37}$$

Finally, we extend it by continuity. The second quantization of U is denoted by

$$\Gamma(U) = \bigoplus_{n\geq 0} U_n \tag{2.38}$$

and is also unitary. Well, if $U_t = e^{itH}$ is a strongly continuous one-parameter unitary group, then we can write

$$\Gamma(U_t) = e^{itd\Gamma(H)}. \tag{2.39}$$

Proof that second quantization is a functor

Now we will show that second quantization is, in fact, a functor. This is a way of giving a clear example of how category theory can be used to describe physical concepts. Naturally, we would like to prove that $d\Gamma(H)$, as well as $\Gamma(H)$ are functors. But, as it turns out only the second quantization of unitary operators is indeed a functor. $d\Gamma$ is not really the second quantization we are looking for, it is just a name we gave to extending self-adjoint operators to Fock space.

Statement: Second quantization is a functor.

Proof. We would like to prove that

$$\Gamma : \mathscr{H} \to \mathscr{F},$$

$$\mathcal{H}_1 \mapsto \mathscr{F}(\mathcal{H}_1),$$
$$U \mapsto \Gamma(U) = \bigoplus_{n\geq 0} U_n$$

satisfy:

1. $\Gamma(U)(U : \mathcal{H}_1 \to \mathcal{H}_2) := \bigoplus_{n\geq 0} U_n : \mathscr{F}(\mathcal{H}_1) \to \mathscr{F}(\mathcal{H}_2)$
2. $\Gamma(U)(1_{\mathcal{H}}) = 1_{\mathscr{F}(\mathcal{H})}$
3. $\Gamma(S \circ U) = \Gamma(S) \circ \Gamma(U)$

1. Since $\mathscr{F}(\mathcal{H}_1) = \bigoplus_{n\geq 0} \mathcal{H}_1^n$ and $\mathscr{F}(\mathcal{H}_2) = \bigoplus_{n\geq 0} \mathcal{H}_2^n$, we find that

$$\begin{aligned}\Gamma(U)\mathscr{F}(\mathcal{H}_1) = \bigoplus_{n\geq 0} U_n\mathcal{H}_1^n &= U_0\mathcal{H}_1^0 \oplus U_1\mathcal{H}_1^1 \oplus ... \oplus U_n\mathcal{H}_1^n\\ &= \mathbb{1}\mathbb{C} \oplus \mathcal{H}_2^1 \oplus ... \oplus \mathcal{H}_2^n = \oplus_{n\geq 0}\mathcal{H}_2^n\\ &= \mathscr{F}(\mathcal{H}_2)\end{aligned}$$

2. We have that $1_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ and $\Gamma(U) = \bigoplus_{n\geq 0} U_n$, so,

$$\begin{aligned}\mathbb{1}_n(P_\pm(f_1 \otimes ... \otimes f_n)) &= P_\pm(\mathbb{1}f_1 \otimes ... \otimes \mathbb{1}f_n)\\ &= P_\pm(f_1 \otimes ... \otimes f_n)\end{aligned}$$

$$\begin{aligned}\oplus_{n\geq 0}\mathbb{1}_n \oplus_{n\geq 0} P_\pm\mathcal{H}^n &= \oplus_{n\geq 0}\mathbb{1}_n \oplus_{n\geq 0} P_\pm(f_1 \otimes ... \otimes f_n)\\ &= \mathbb{1}_1(P_\pm f_1) \oplus \mathbb{1}_2(P_\pm(f_1 \otimes f_2)) \oplus ... \oplus \mathbb{1}_n[P_\pm(f_1 \otimes ... \otimes f_n)]\\ &= P_\pm f_1 \oplus P_\pm(f_1 \otimes f_2) \oplus ... \oplus P_\pm(f_1 \otimes ... \otimes f_n)\\ &= \oplus_{n\geq 0}P_\pm(f_1 \otimes ... \otimes f_n)\\ &= \oplus_{n\geq 0}P_\pm\mathcal{H}^n = \mathscr{F}_\pm(\mathcal{H}).\end{aligned}$$

3. We have that

$$\begin{aligned} S_n \circ U_n[P_\pm(f_1 \otimes ... \otimes f_n)] &= S_n[P_\pm(Uf_1 \otimes ... \otimes Uf_n)] \\ &= P_\pm(SUf_1 \otimes ... \otimes SUf_n), \end{aligned}$$

therefore,

$$\begin{aligned} \Gamma(S \circ U)\mathscr{F}_\pm(\mathcal{H}) &= \oplus_{n\geq 0}(S_n \circ U_n) \oplus_{n\geq 0} P_\pm(f_1 \otimes ... \otimes f_n) \\ &= P_\pm[SUf_1 \oplus SU(f_1 \otimes f_2) \oplus ... \oplus SU(f_1 \otimes ... \otimes f_n)] \\ &= \oplus_{n\geq 0} P_\pm[SUf_1 \otimes ... \otimes SUf_n]. \end{aligned}$$

But,

$$\Gamma(S)\mathscr{F}_\pm(\mathcal{H}) = \oplus_{n\geq 0} P_\pm[Sf_1 \otimes ... \otimes Sf_n]$$

and

$$\begin{aligned} \Gamma(S) \circ \Gamma(U)\mathscr{F}_\pm(\mathcal{H}) &= \Gamma(S)[\oplus_{n\geq 0} P_\pm(Uf_1 \otimes ... \otimes Uf_n)] \\ &= \oplus_{n\geq 0} P_\pm[SUf_1 \otimes ... \otimes SUf_n] \end{aligned}$$

Which enables us to affirm that

$$\Gamma(S \circ U) = \Gamma(S) \circ \Gamma(U,)$$

and along with the fact that the other properties are satisfied, we can finally conclude that second quantization is a functor.

□

2.8 Category Theory and Quantum Gravity: A motivation

Among the many advantages of using the language of Category Theory in Physics, a notable one is the possibility of unifying General Relativity and Quantum Mechanics. One of the difficulties of unifying these areas lies in the fact that they use different mathematical structures. But if we study these structures within Category Theory, it is possible to define functors that relate categories useful in Quantum Mechanics to categories useful in General Relativity.

In this scenario, there are two functors worthy of notice: Topological Quantum Field Theory and Locally Covariant Quantum Field Theory. Both theories can be defined as functors from a Quantum Mechanics category to a General Relativity Category.

In the case of topological QFT, we have a functor

$$\mathscr{Z} : \mathbf{n\mathfrak{Cob}} \to \mathfrak{Hilb},$$

where $\mathbf{n\mathfrak{Cob}}$ is a category of $(n-1)$-dimensional manifolds whose morphisms are cobordisms, and $\mathfrak{Hilb}$ is a category of Hilbert spaces and bounded linear operators as morphisms (See table 2.1). The objects of $\mathbf{n\mathfrak{Cob}}$ actually represent possible choices of space, while the cobordisms represent possible choices of spacetime [15]. Then, one can interpret these cobordisms as the passage of time, which can (but not necessarily) lead to a change in the topology of the space. These cobordisms are manifolds whose boundaries must have the same metric as the spaces that they connect (Figure 2.15).

Both Quantum Mechanics and General Relativity are usually described within Set Theory, via the addition of structures to sets in order for them to have the desired features. So, one could think that their associated categories would have many similarities with the category $\mathfrak{Sets}$ of sets and functions. However, it turns out that $\mathbf{n\mathfrak{Cob}}$ and $\mathfrak{Hilb}$ have more similarities between them, than either one of them have with $\mathfrak{Sets}$. In [15], Baez shows that they are $*-$categories with a noncartesian monoidal structure, which are not properties that they share with $\mathfrak{Sets}$. He argues that this result suggests that Quantum Mechanics could be better understood if it was seen as part of a theory of spacetime and that features like the failure of local realism and impossibility of duplicating quantum information are easier to make sense of in the context of Category Theory. These similarities between $\mathbf{n\mathfrak{Cob}}$ and $\mathfrak{Hilb}$ show great promise into the unification of quantum theory and general relativity, and Category Theory played a crucial role here, which suggests that it can be a useful approach to Quantum Gravity.

	$\mathbf{n\mathfrak{Cob}}$	$\mathfrak{Hilb}$
Objects	$(n-1)$-dimensional manifolds (space)	Hilbert spaces (states)
Morphisms	cobordisms: n-dimensional manifolds (spacetime)	bounded linear operators (processes)
Composition	composition of cobordisms	composition of operators
Identity	identity cobordism	identity operator

Table 2.1: Description of the categories $\mathbf{n\mathfrak{Cob}}$ and $\mathfrak{Hilb}$.

In this work we will address in more detail only Locally Covariant Quantum Field Theory, which will be useful in the derivation of the Reeh-Schlieder property in curved spacetimes. It will be defined as a functor $\mathscr{A}$,

$$\mathscr{A} : \mathfrak{Man} \to \mathfrak{Alg},$$

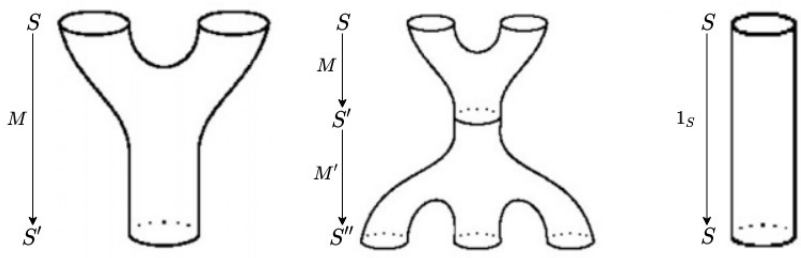

Figure 2.15: Cobordism, composition of cobordisms and identity cobordism, respectively. In this case, the first cobordism represents a process in which two separated spaces collide to form one space [15].

between a category of four-dimensional globally hyperbolic manifolds whose morphisms are isometric embeddings and a category of C^*-algebras whose morphisms are faithful $*$-homomorphisms. In a recent paper by Brunetti, Fredenhagen and Rejzner [28], perturbative quantum gravity is constructed in this framework of LCQFT and they claim that using local covariance many ideas can be made rigorous in this approach.

The nature of the categories from these theories is actually quite different since a cobordism can be thought of as the process of time passing from from a moment S to a moment S' (figure 2.15) and this is translated into a process which takes states of a system into states of another system. Meanwhile, in LCQFT the idea is that each region of spacetime is assigned to a local algebra of observables and the isometric embeddings and $*$-homomorphisms (which in $\mathfrak{Alg}$ always takes an algebra $\mathcal{A}$ into an algebra $\mathcal{B}$ which contains $\mathcal{A}$) implement general covariance in the local algebras, as well as make sure that isotony holds, which is an important axiom of Algebraic Quantum Field Theory. However, both theories suggest that using Category Theory in Quantum Field Theory can be promising to quantum gravity.

Chapter 3

Algebraic Quantum Field Theory

Algebraic quantum field theory is an approach of Axiomatic Quantum Field Theory [29] which aims to describe a theory in a well founded mathematical framework and which provides precise definitions for the concepts of Quantum Field Theory. The idea is to study the algebras of observables localized in a bounded region of spacetime and that satisfy certain axioms, that are due to Haag and Kastler [30]. It is a very general approach, and it allows for the recovery of quantum fields from these algebras of observables. AQFT is a very successful theory and it will be used as the foundation for Locally Covariant Quantum Field Theory.

The theory is formulated on top of some basic assumptions like relativistic causality, stability of the vaccum state, the notion of a relativistic particle in quantum mechanics, as well as special relativity, where the theory of unitary irreducible representations of the Poincaré group, as developed by Wigner [31, 32], is used. So we consider an abstract algebra of observables which respect *spacetime locality* (Quantum fields are dependent of the local structure of spacetime) and *causality* (Causally separated quantum fields (anti)commute). It is important to point out that the Haag-Kastler axioms are specific for Minkowski spacetime, so this chapter only concerns QFT in flat spacetimes. Then, when we discuss LCQFT, AQFT will be generalized to all globally hyperbolic spacetimes.

The reason why we use it to study Quantum Field Theory in curved spacetime is that on general spacetimes one cannot require the existence of a preferred state, such as the vacuum state, due to the spacetimes' lack of symmetries, and there is also no unique particle interpretation. The solution is to formulate QFT without the use of such preffered state, which is possible in a theory that is focused on local observables. Therefore, Algebraic Quantum Field Theory arises as the most general and mathematically precise framework to formulate such theory.

In this chapter, we will present the ideas necessary to introduce the axioms of Algebraic Quantum Field Theory in Minkowski spacetime, as well as define the vacuum state, based on [7, 33, 34]. The aim here is to provide the framework on which LCQFT is grounded, namely, a QFT described by the algebra of local observables, and also all the tools needed for understanding and proving the Reeh-Schlieder theorem.

3.1 Preliminaries on operator algebras

We will now set the mathematical foundation that gives meaning to measurements in quantum mechanics. In a laboratory, a physical measurement consists of a physical system that is measured by the appropriate tools. In addition, there is an observer which performs the measurement and the environment, both of which should not interfere in the process [7]. Loosely speaking, observables describe the measuring instruments, while the states describe the physical systems. We use the assumption from quantum mechanics that observables are represented by linear operators on a Hilbert space [35], and that we can consider algebras of linear operators, where the self-adjoint elements are the observables.

The aim of this section is then to present some definitions and establish the notation regarding algebras of operators, which underlies the algebraic viewpoint of quantum theory together with the GNS construction and Wigner's theory of unitary irreducible representations of the Poincaré group, that has a fundamental role in describing the symmetries of quantum mechanics [7]. With this in mind, we will consider a Hilbert space $\mathcal{H}$ and define the algebras that can be represented on it, as well as the GNS construction, which provides a representation associated with each state.

So we turn our attention to the operators defined on a Hilbert space $\mathcal{H}$ and their associated algebras. We will be dealing with algebras of bounded operators, and we will write $\mathcal{B}(\mathcal{H})$ for the set of all bounded operators in $\mathcal{H}$. Here we are considering abstract algebras, but later we will present representations of this algebras that account for the symmetries of Minkowski spacetime, i.e. representations of the Poincaré group.

Definition 3.1. A *normed algebra* $\mathcal{A}$ is an algebra over $\mathbb{C}$ equipped with a norm $||A|| \in \mathbb{R}$, $A \in \mathcal{A}$, so that $\mathcal{A}$ is also a normed space which satisfies properties 1-4 of the norm, together with

$$\|A_1 A_2\| \leq \|A_1\|\|A_2\|.$$

Definition 3.2. A *Banach algebra* is a normed algebra which is complete.

Definition 3.3. An algebra is *unital* if it has a unit $\mathbf{1} \in \mathcal{A}$ so that for all $A \in \mathcal{A}$ we have

$$\mathbf{1}A = A\mathbf{1} = A.$$

Definition 3.4. An *involution* is a unitary map $* : A \mapsto A^*$, $A, A^* \in \mathcal{A}$, satisfying

1. $(c_1 A_1 + c_2 A_2)^* = \bar{c}_1 A_1^* + \bar{c}_2 A_2^*$,
2. $(A_1 A_2)^* = A_2^* A_1^*$,
3. $(A^*)^* = A$,

where $A_i \in \mathcal{A}$ and $c_j \in \mathbb{C}$.

Definition 3.5. A *Banach $*$-algebra* $\mathcal{A}$ is an algebra endowed with an involution, so that for all $A \in \mathcal{A}$, we have

$$\|A^*\| = \|A\|.$$

Definition 3.6. A C^*-algebra $\mathcal{A}$ is a $*$-Banach algebra with a C^* norm that satisfies

$$\|A^* A\| = \|A\|^2.$$

Furthermore, if these algebras are unital, then $\|\mathbf{1}\| = 1$.

The requirements of convergence for algebras of operators are essential, since we would like them to be closed under the algebraic calculus and in the appropriate topology. So we can give the following additional definitions [7]:

Definition 3.7. A set $\mathcal{A}$ of bounded linear operators on $\mathcal{H}$, with the C^*-norm

$$\|B\| = \sup\Big\{ \frac{\|B\Psi\|}{\|\Psi\|}; \Psi \in \mathcal{H}, \Psi \neq 0 \Big\}, \tag{3.1}$$

is a *concrete C^*-algebra* if:

1. $\mathcal{A}$ is a $*$-algebra,
2. $\mathcal{A}$ is closed in the *norm topology*, which means that if a sequence $A_n \in \mathcal{A}$ has a limit operator A satisfying

$$\lim \|A_n - A\| = 0, \tag{3.2}$$

then $A \in \mathcal{A}$.

Definition 3.8. A set of bounded linear operators $\mathcal{M}$ on $\mathcal{H}$ is a *von Neumann algebra* if it satisfies the following conditions:

1. $\mathcal{M}$ is a $*$-algebra,
2. $\mathcal{M}$ is closed in the *weak operator topology*, which means that if a net[1] of operators $A_\nu \in \mathcal{A}$ has a weak limit A satisfying

$$\lim(\Psi, A_\nu \Phi) = (\Psi, A\Phi) \tag{3.3}$$

for any vector $\Psi, \Phi \in \mathcal{H}$, then $A \in \mathcal{M}$ and we write

$$A = w - \lim A_\nu. \tag{3.4}$$

3. $\mathbf{1} \in \mathcal{M}$.

[1] See definition B.8.

Additionally, a W^*-algebra meets all the above requirements for a von Neumann algebra, except for the existence of unit. Therefore, a concrete C^*-algebra, a von Neumann algebra and a W^*-algebra are all $*$-subalgebras of $\mathcal{B}(\mathcal{H})$. Since the norm topology is stronger than the weak operator topology, a von Neumann algebra is also a C^*-algebra.

For a self-adjoint set of operators in $\mathcal{H}$ the commutant algebra S' of S is given by

$$S' = \{Q \in \mathcal{B}(\mathcal{H}); \forall Q_1 \in S \Rightarrow [Q_1, Q] = 0\}. \tag{3.5}$$

S', as well as $S'' = (S')'$, the bicommutant of S, are von Neumann algebras, and the latter is the smallest von Neumann algebra that contains S. Therefore, a von Neumann algebra $\mathcal{M}$ is equivalent to its bicommutant, $\mathcal{M} = \mathcal{M}''$.

Given a von Neumann algebra $\mathcal{M}$ and a vector Ω in $\mathcal{H}$, we say that Ω is a *cyclic vector* if $\mathcal{M}\Omega$ is dense in $\mathcal{H}$, and that it is separating if $A\Omega = B\Omega$, for $A, B \in \mathcal{M}$, implies $A = B$. It is a property of von Neumann algebras that if Ω is cyclic for $\mathcal{M}$, then it is separating for $\mathcal{M}'$ (and vice versa). If $\mathcal{M}$ possesses a cyclic vector and a separating vector, than it possesses a cyclic and separating vector. When we discuss the Reeh-Schlieder theorem we will see that the vacuum vector satisfies these properties in its GNS representation.

Definition 3.9. Given a C^*-algebra and a Hilbert space $\mathcal{H}$, a *representation* of $\mathcal{A}$ in $\mathcal{H}$ is a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, which satisfies:

1. $\pi(c_1 A_1 + c_2 A_2) = c_1 \pi(A_1) + c_2 \pi(A_2)$
2. $\pi(A_1 A_2) = \pi(A_1)\pi(A_2)$
3. $\pi(A^*) = \pi(A)^*$

where $A_i \in \mathcal{A}$ and $c_i \in \mathbb{C}$.

The observables in this framework are then represented by self-adjoint elements of an abstract algebra and different representations of this algebra account for different physical situations. We will see that each state can induce a representation of the algebra in a Hilbert space.

We can represent a state φ as the expectation value

$$\varphi(A) = (\Psi_\varphi, A\Psi_\varphi) \tag{3.6}$$

of an operator A with respect to a vector state[2] $\Psi_\varphi \in \mathcal{H}$. Equation 3.6 actually defines a *pure state* over $\mathcal{B}(\mathcal{H})$ for any unit vector in $\mathcal{H}$. We now introduce a precise definition of a quantum mechanical state [7]:

Definition 3.10. A *state* is a complex valued function over $\mathcal{B}(\mathcal{H})$ that satisfies:

[2]A vector state is a unit vector satisfying $\|\Psi_\varphi\| = 1$.

1. Linearity: for $A_1, A_2 \in \mathcal{B}(\mathcal{H})$ and $c_1, c_2 \in \mathbb{C}$
$$\varphi(c_1A_1 + c_2A_2) = c_1\varphi(A_1) + c_2\varphi(A_2);$$

2. Positivity: for $A \in \mathcal{B}(\mathcal{H})$
$$\varphi(A^*A) \geq 0;$$

3. Normalization: $\varphi(\mathbf{1}) = 1$.

Definition 3.11. A state φ over $\mathcal{B}(\mathcal{H})$ is *normal* if for any bounded increasing net of positive operators A_ν over $\mathcal{B}(\mathcal{H})$,

$$\varphi(\sup A_\nu) = \sup \varphi(A_\nu). \tag{3.7}$$

This means that $\varphi(A_\nu)$ has a limit so that $\varphi(\lim A_\nu) = \lim \varphi(A_\nu)$ is satisfied. For more details, see [7].

The states in quantum mechanics can be either pure states, which we have defined, and mixed states, that are described by density matrices.

Definition 3.12. Let $\{e_\nu\}$ be an orthonormal basis in $\mathcal{H}$ and ρ a positive operator with trace 1,

$$\mathrm{Tr}\rho \equiv \sum_\nu (e_\nu, \rho e_\nu) = 1. \tag{3.8}$$

For any $A \in \mathcal{B}(\mathcal{H})$, we have that

$$\mathrm{Tr}(\rho A) \equiv \sum_\nu (e_\nu, \rho A e_\nu) = \sum_\nu (\rho e_\nu, A e_\nu) \tag{3.9}$$

is absolutely convergent and its value is independent of the choice of basis. In addition, any normal state $\varphi \in \mathcal{B}(\mathcal{H})$ is of the form of equation 3.9. So we can define the *density matrix* as the functional

$$\rho(A) = \mathrm{Tr}(\rho A) \tag{3.10}$$

over $\mathcal{B}(\mathcal{H})$, which is a normal state over $\mathcal{B}(\mathcal{H})$ and which is uniquely determined by φ.

Now we are able to present in theorem 3.13 the construction of the GNS representation, which resulted from a joint paper by Gelfand and Naimark [36], and another paper by Segal [37]. For the proof we refer the reader to [7, 38].

Theorem 3.13. *For any state φ over a C^*-algebra $\mathcal{A}$ there is a Hilbert space $\mathcal{H}_\varphi$, a representation π_φ of $\mathcal{A}$ in $\mathcal{H}_\varphi$ and a unit vector Ω_φ in $\mathcal{H}$ satisfying:*

1. *For any $A \in \mathcal{A}$, $\varphi(A) = (\Omega_\varphi, \pi_\varphi(A), \Omega_\varphi)$.*

2. *Ω_φ is a cyclic vector of the representation π_φ, that is,*
$$\pi_\varphi(\mathcal{A})\Omega_\varphi \equiv \{\pi_\varphi(A)\Omega_\varphi; A \in \mathcal{A}\} \tag{3.11}$$
is dense in $\mathcal{H}_\varphi$.

Additionally, the triple $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ *satisfying* 1 *and* 2 *is unique up to unitary equivalence.*

This theorem allows one to obtain a representation of an algebra for each state over a C^*-algebra. $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ is called the GNS triplet, where $\mathcal{H}_\varphi$ is the cyclic representation space, π_φ is the cyclic representation and Ω_φ is the cyclic vector associated with φ. The GNS construction will be very useful when considering the Reeh-Schlieder theorem, since we will use the cyclic representation associated with the vacuum state.

3.2 Axioms of Algebraic Quantum Field Theory

The objects of study of Algebraic Quantum Field Theory are local observables, that is, observables measurable in a local region of Minkowski spacetime $\mathbb{M}$. We refer the reader to appendix A for a few details regarding $\mathbb{M}$. This means that we not only specify the spatial region where they are measured, but also the time interval in which the measurement occurred. The difference between the frameworks of Algebraic Quantum Field Theory and the one formulated by Wightman[3] lies in the fact that here our main focus are the algebras of bounded operators associated with observables, whereas Wightman formulated a theory which studies quantum fields as operator valued distributions, which are generally unbounded. However, it is still possible to recover Wightman's axioms from the algebraic viewpoint [7].

Let D be a bounded open domain of spacetime. We call $\mathcal{O}(D)$ the set of all the observables which can be measured in each $D \subset \mathbb{M}$, i.e., a set of self-adjoint operators. Then, $\mathcal{O}(D)$ is a subset of the C^*-algebra $\mathcal{A}$ (over $\mathbb{M}$) generated by the observables. In fact, $\mathcal{O}(D)$ can also generate a C^*-algebra $\mathcal{A}(D)$ [7].

The simplest choice for the bounded domain D is the double cone

$$V_q^p \equiv \{x; p - x \in V_+, x - q \in V_+\}. \tag{3.12}$$

In fact, every bounded set is contained in a double cone (figure 3.1).

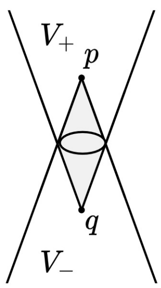

Figure 3.1: Double cone.

Since AQFT is formulated in flat spacetime, the relativistic symmetry is given by the proper orthocronous Poincaré group $\mathscr{P}_+^\uparrow$ (See section A.1), and the elements

[3]Wightman's axioms are presented in appendix C.

$g = (a, \Lambda) \in \mathscr{P}_+^\uparrow$ can be represented by a $*$-automorphism[4] $\alpha_g = \alpha_{(a,\Lambda)}$ of $\mathcal{A}$ satisfying

$$\alpha_g \mathcal{O}(D) = \mathcal{O}(gD), \quad gD = \{\Lambda x + a; x \in D\}. \tag{3.13}$$

So if we consider only translations, we will have automorphisms $\alpha_{(a,1)}$, for example. This formulation of automorphisms is actually applied to symmetries in general and Araki presents a wide discussion on it [7]. For us, it suffices to know that this is true for the Poincaré group, and that we can represent a Poincaré transformation g by a unitary operator $U(g)$ on $\mathcal{H}$ as

$$\varphi(A) = (\Phi, A\Phi) \to g\varphi(A) = (U(g)\Phi, AU(g)\Phi),$$

$$gQ = U(g)QU^*(g).$$

The following results regarding invariance of a state under a symmetry are useful [7]:

Theorem 3.14. *If φ is an invariant state with respect to the symmetry g, i.e.*

$$\varphi(\alpha_g Q) = \varphi(Q),$$

then there is a unique unitary operator $U(g)$ in the cyclic representation space $\mathcal{H}_\varphi$ satisfying

$$U(g)\Omega_\varphi = \Omega_\varphi, \quad U(g)\pi_\varphi(Q)U^*(g) = \pi_\varphi(\alpha_g(Q)).$$

Proof. The operator $U(g)$ is defined as follows. For vectors of the form $\pi_\varphi(Q)\Omega_\varphi$ with $Q \in \mathcal{A}$, define

$$U(g)\pi_\varphi(Q)\Omega_\varphi \ := \ \pi_\varphi\big(\alpha_g(Q)\big)\Omega_\varphi \ . \tag{3.14}$$

Notice that for the vector in the r.h.s., we have

$$\begin{aligned}\left\|\pi_\varphi\big(\alpha_g(Q)\big)\Omega_\varphi\right\|^2 &= \Big(\pi_\varphi\big(\alpha_g(Q)\big)\Omega_\varphi,\ \pi_\varphi\big(\alpha_g(Q)\big)\Omega_\varphi\Big) = \Big(\Omega_\varphi,\ \pi_\varphi\big(\alpha_g(Q^*Q)\big)\Omega_\varphi\Big)\\ &= \varphi\big(\alpha_g(Q^*Q)\big) = \varphi(Q^*Q) \ .\end{aligned}$$

Hence, according to the definition (3.14), we have

$$\|U(g)\pi_\varphi(Q)\Omega_\varphi\|^2 \ = \ \varphi(Q^*Q) \ . \tag{3.15}$$

Now, observe that, by the same computation,

$$\|\pi_\varphi(Q)\Omega_\varphi\|^2 \ = \ \varphi(Q^*Q) \ . \tag{3.16}$$

These observations show that $U(g)$, defined in (3.14), is an isometry and, therefore, defines indeed a linear operador, for the image of $U(g)$ on vectors with $\pi_\varphi(Q)\Omega_\varphi = 0$ is indeed the null vector, as it should be.

[4] A $*$-automorphism α of $\mathcal{A}$ is a bijection $\alpha : \mathcal{A} \to \mathcal{A}$ which preserves linear combinations, product and the involution [7].

They also show that $U(g)$ is a bounded operador acting on the dense set

$$\{\pi_\varphi(Q)\Omega_\varphi,\ Q \in \mathcal{A}\}$$

and, therefore, can be extended to the whole Hilbert space as a bounded operator (and as an isometry).

It follows from (3.14) that

$$U(g^{-1})\,(U(g)\pi_\varphi(Q)\Omega_\varphi) \;=\; U(g^{-1})\pi_\varphi\big(\alpha_g(Q)\big)\Omega_\varphi \;=\; \pi_\varphi\big(Q\big)\Omega_\varphi\ , \qquad (3.17)$$

showing that $U(g)$ is invertible, with $U(g^{-1}) = U(g)^{-1}$. This also implies that $U(g)$ is unitary and $U(g)^* = U(g^{-1}) = U(g)^{-1}$.

Moreover, one has, for any $g_1,\ g_2$ in the symmetry group,

$$\begin{aligned} U(g_1)U(g_2)\pi_\varphi(Q)\Omega_\varphi \;&=\; U(g_1)\pi_\varphi(\alpha_{g_2}Q)\Omega_\varphi \\ &=\; \pi_\varphi(\alpha_{g_1 g_2}Q)\Omega_\varphi \\ &=\; U(g_1 g_2)\pi_\varphi(Q)\Omega_\varphi \end{aligned}$$

which implies

$$U(g_1)U(g_2) \;=\; U(g_1 g_2)\ , \qquad (3.18)$$

since the set $\{\pi_\varphi(Q)\Omega_\varphi,\ Q \in \mathcal{A}\}$ is dense in $\mathcal{H}$. Eq. (3.18) shows that the operators $U(g)$ are indeed a *unitary representation* of the symmetry group.

It follows from (3.14) that

$$U(g)\Omega_\varphi \;=\; \Omega_\varphi \qquad (3.19)$$

and that, for arbitrary $Q_1,\ Q_2 \in \mathcal{A}$,

$$\begin{aligned} &\Big(U(g)\pi_\varphi(Q_1)U(g)^*\Big)\pi_\varphi(Q_2)\Omega_\varphi \;=\; U(g)\pi_\varphi(Q_1)\pi_\varphi\big(\alpha_{g^{-1}}(Q_2)\big)\Omega_\varphi \;= \\ &\quad U(g)\pi_\varphi\big(Q_1\alpha_{g^{-1}}(Q_2)\big)\Omega_\varphi \;=\; \pi_\varphi\big(\alpha_g(Q_1)Q_2\big)\Omega_\varphi \;=\; \pi_\varphi(\alpha_g(Q_1))\pi_\varphi(Q_2)\Omega_\varphi\ , \end{aligned} \qquad (3.20)$$

which implies

$$U(g)\pi_\varphi(Q_1)U(g)^* \;=\; \pi_\varphi(\alpha_g(Q_1)) \qquad (3.21)$$

since the set $\{\pi_\varphi(Q_2)\Omega_\varphi,\ Q_2 \in \mathcal{A}\}$ is dense in $\mathcal{H}$.

$\square$

Corollary 3.15. *Given a state φ which is invariant with respect to α_g, the continuity of $\psi(\alpha_g Q)$ for all the $Q \in \mathcal{A}$ and $\psi \in S_0(\varphi)$,*

$$S_0(\varphi) \equiv \{\psi \in S(\mathcal{A});\quad \exists Q \in \mathcal{A}, \psi(\cdot) = \varphi(Q^*Q)\},$$

is equivalent to the condition that $U(g)$ is continuous in the strong operator topology.

Now we are in the position to establish the axioms of AQFT, i.e. the requirements that must be satisfied by local observables on Minkowski spacetime [7, 33]. The algebras of observables which meet these conditions are often called *local nets of observables* and the axioms are usually referred to as the Haag-Kastler axioms [30].

Axioms for local observables:

(*i*) **Existence of local algebras**: There is a unital C^*-algebra $\mathcal{A}$ over Minkowski spacetime $\mathbb{M}$, and to each domain $D \subset \mathbb{M}$ the correspondent algebra $\mathcal{A}(D)$. Furthermore, $\bigcup_D \mathcal{A}(D)$ is dense in $\mathcal{A}$.

(*ii*) **Isotony**: For every inclusion of spacetime domains $D_1 \subset D_2$, there is an associated inclusion of algebras $\mathcal{A}(D_1) \subset \mathcal{A}(D_2)$.

(*iii*) **Einstein causality:** If D_1 and D_2 are spacelike separated, then

$$[\mathcal{A}(D_1), \mathcal{A}(D_2)] = 0.$$

(*iv*) **Poincaré covariance:** To every transformation $g \in \mathscr{P}_+^\uparrow$, we have

$$\alpha_g : \mathcal{A}(D) \to \mathcal{A}(g^{-1}D)$$

and

$$\alpha(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}, \quad \alpha_{g_1} \circ \alpha_{g_2} = \alpha_{(g_1 \circ g_2)}$$

for every $g_i \in \mathscr{P}_+^\uparrow$.

Note that the axioms above are just the minimum requirements, and additional ones can be imposed if need be, which is the case of the time-slice axiom:

(*v*) **Existence of dynamics:** If $D_1 \subset D_2$ and D_1 contains a Cauchy surface of D_2, then

$$\mathcal{A}(D_2) = \mathcal{A}(D_1).$$

These axioms provide us with a theory based on algebras of local observables for any bounded region of spacetime and that does not require fields in its formulation, relying purely on algebraic relations.

It is often useful to restrict to von Neumann algebras (which are also C^*) [7]. They can be obtained once an appropriate representation is considered, like the GNS representation of the vacuum state.

A crucial ingredient in the proof of the Reeh-Schlieder theorem is the property of weak additivity. It was actually proven by Borchers [39] that weak additivity is not only sufficient, but necessary for the derivation of the Reeh-Schlieder property. He presented a general version of the Reeh-Schlieder theorem and also a theorem which is a converse (although not an exact one) of the Reeh-Schlieder theorem. Next we define the weak additivity property as done in [7]:

Definition 3.16. Let π_φ be the GNS representation of a state φ. If for any double cone K and any of its open coverings $K = \cup_i D_i$ the equation

$$\pi_\varphi(\mathcal{A}(K))'' = \left(\bigcup_i \pi_\varphi(\mathcal{A}(D_i))\right)'' \tag{3.22}$$

is satisfied, then we have additivity. If

$$\pi_\varphi(\mathcal{A})'' = \left(\bigcup_x \pi_\varphi(D+x)\right)'' \tag{3.23}$$

is satisfied for any domain D, then we have weak additivity.

3.3 The vacuum state

The study of the vacuum state is of central importance in Quantum Field Theory, since in its GNS representation, local observables acting on it yield a dense set of states. As we will see, the Reeh-Schlieder theorem asserts that this is true for local observables restricted to any domain of spacetime. So, in order to investigate this assertions, we must first define a vacuum state in the context of Algebraic Quantum Field Theory, as well as discuss some of its properties and establish some notation. The concepts introduced here are based on [7].

We would like to define energy-increasing and energy-decreasing operators, since we know from QFT that the latter annihilates the vacuum state. So, let us first take the Fourier transform of a C^∞-function $\tilde{g}$ to be

$$g(x) = (2\pi)^{-4} \int \tilde{g}(p) e^{-i(p,x)} d^4x. \tag{3.24}$$

We will define for each four-dimensional bounded closed set Δ an operator which increases the energy momentum by Δ as

$$Q(g) = \int \alpha_{(x,1)}(Q) g(x) d^4x, \tag{3.25}$$

which is constructed from any $Q \in \mathcal{A}$ and any C^∞-function $\tilde{g}$ with support in Δ.

If the relativistic symmetry[5] $\tilde{\mathscr{P}}_+^\uparrow$ is represented by a unitary transformation $U(a, A)$, in a representation π of the algebra, an operator would be transformed as

$$\pi(\alpha_{(a,\Lambda)}(Q)) = U(a, A)\pi(Q)U(a, A)^*, \tag{3.26}$$

where $\Lambda = \Lambda(A)$. It is assumed that the generator of translations $U(a, \mathbf{1})$ is interpreted as the energy-momentum,

$$\Pi^\mu = \lambda P^\mu, \tag{3.27}$$

where λ is a proportionality constant that we take to be 1.

[5]The symmetry is given by $\tilde{\mathscr{P}}_+^\uparrow$, the universal covering group of $\mathscr{P}_+^\uparrow$, and elements $U(a, A)$ of $\tilde{\mathscr{P}}_+^\uparrow$ correspond to elements $U(a, \Lambda)$ of $\mathscr{P}_+^\uparrow$ (See [7] p.64).

Definition 3.17. A state φ is called a vacuum state if $Q \in \ker\varphi$,

$$\ker\varphi \equiv \{Q \in \mathcal{A}, \varphi(Q^*Q) = 0\},$$

for any operator $Q \in \mathcal{A}$ which decreases the energy in some coordinate system.

Furthermore, any vacuum state is translation invariant, i.e. for any $Q \in \mathcal{A}$

$$\varphi(\alpha_{(a,1)}Q) = \varphi(Q). \tag{3.28}$$

The above definition means that the vacuum is the state of lowest energy, and in order to guarantee its stability (in the sense that there is no state with energy lower than that of the vacuum), we must further require that it satisfies the spectrum condition, which restricts the spectrum of the momentum operators to the closed future cone. The precise statement is the following [7]:

Theorem 3.18 (Spectrum condition)**.** *On the GNS representation space of the vacuum state φ, a unitary representation T_φ of the translation group satisfying*

$$T_\varphi(a)\pi_\varphi(Q)\Omega_\varphi \equiv \pi_\varphi(\alpha_{(a,1)}Q)\Omega_\varphi \tag{3.29}$$

can be defined and the spectral measure E_φ defined by

$$T_\varphi(a) = \int e^{i(p,a)} E_\varphi(d^4p) \tag{3.30}$$

has its support contained in $\bar{V}_+$.

We establish the following notation for an operator $Q \in \mathcal{A}$:

$$Q(x) = \alpha_{(x,1)}Q, \quad \pi_\varphi(Q(x)) = T_\varphi(x)\pi_\varphi(Q)T_\varphi(x)^*, \tag{3.31}$$

and

$$Q_\varphi(x) = \pi_\varphi(\alpha_{(\lambda x,1)}Q). \tag{3.32}$$

Another important feature of the vacuum state is the cluster property [7]. It states that

$$\lim_{\lambda\to\infty} \varphi(Q_1\alpha_{(\lambda x,1)}Q_2) = \varphi(Q_1)\varphi(Q_2), \tag{3.33}$$

where x is spacelike and Q_1, Q_2 can be elements in $\mathcal{A}$. This means that correlation between Q_1 and Q_2 decays with increasing distance, and that to a good approximation, any irreducible vacuum state is independent for regions that are far enough.

Chapter 4

QFT in Curved Spacetime: The locally covariant approach

Since the development of General Relativity and the discovery that the universe does not lie in a Minkowski spacetime, attempts have been made to extend QFTs to curved spacetimes. With this comes the question of how to formulate a given physical theory in arbitrary spacetimes while preserving its characteristics, which is the main point of QFT in curved spacetimes [40].

One of the main motivations for its study is the development of a quantum gravity theory. It would also be very beneficial to formulate QFT on curved spacetimes in such a way that it is reduced to standard QFT on flat spacetimes [41]. As it stands, it offers an approach to the 'low' energy and 'small' curvatures regime, i.e. situations in which quantum gravity effects are small, and it has made important predictions, such as the Fulling-Davies-Unruh effect, the Hawking effect and generation of curvature fluctuations during inflation, which shows that it has many applications in cosmology and on the investigation of the early universe. However, problems are found when dealing with the axioms related to the Poincaré group, since arbitrary spacetimes are not required to have any symmetries.

We face the issue that it is necessary to find a way to describe a system such that the formulation of the theory is the same in all space-times, which is the principle of *covariance*. But it is also important that we are able to make assessments about such system in a way that is independent of its background, that is, we should be able to neglect things that are too far of our reach, and that are, sometimes, impossible to know, for example, due to the limit imposed to us by the light speed. This leads to the idea of *locality*. The need for constructing theories which combine both spacetime locality and covariance in a curved background framework is what motivated the formulation of Locally Covariant Quantum Field Theories in [2, 3].

It will be shown that in this framework models of Quantum Field Theories have the structure of covariant functors between a category that has spacetimes as objects and the allowed spacetime embeddings as morphisms, and a category whose objects describe the algebra of observables and whose morphisms $f : P \to Q$ can be interpreted as functions that embed the physical system P as a subsystem of Q. Of course alterations can be made in these categories, so that they are very specific to a certain situation, for example using categories for the classical case, adding structure to account for spin, etc. For our purposes, i.e. to address QFT in curved spacetimes, we are going to use a category of four-dimensional oriented and time-oriented globally hyperbolic space-times and a category whose objects are C^*-algebras, with their respective morphisms, as will be shown in section 4.2.

As we have discussed in chapter 3, AQFT is the best framework to study QFT in curved spacetimes, and Locally Covariant Quantum Field Theory arises as a modern approach to it. So far, it has been crucial in the development of Renormalization and Perturbation on AQFT [42–44], of superselection sectors on curved spacetimes [45] and it has also provided concrete and abstract results on gauge theories [46–51].

Dimock was one of the first to argue that AQFT is the best suited approach to study quantum fields in curved spacetimes. In [52] he established a set of axioms which takes into consideration a net of observables as in the Haag-Kastler approach. Additionally, since he was interested in studying general spacetimes, he could not consider Poincaré covariance. Instead, he imposed a general covariance associated with isometries between spacetimes. So, if there is an isometry between spacetimes, there should be an isomorphism between the algebras for these spacetimes. This would be sufficient as a covariance axiom, since isometric spacetimes have the same physical information, which guarantees that the dynamics is preserved.

Hence, the idea of Locally Covariant Quantum Field Theory to use isometric embeddings as morphisms in the category of manifolds is justified, as these would be sufficient to guarantee that the theory obeys general covariance, thus satisfying one of the most important principles of general relativity.

An important condition that must be imposed on states is that of stability. On Minkowski spacetime, stability is guaranteed by the positivity of energy, that is, the spectrum condition. But on curved spacetimes, this was initially a problem [53]. Brunetti, Fredenhagen and Köhler stated that the formulation of a specific model for quantum fields on curved spacetimes is realized in two steps:

(i) The construction of the algebra of observables in terms of commutation and anticommutation relations.

(ii) A class of states with an appropriate stability condition must be found.

Regarding (i), Dimock has constructed the algebra of observables for the free scalar [52], Dirac [54] and electromagnetic [55] fields on globally hyperbolic spacetimes. As for (ii), on [53] it is proven that the microlocal spectrum condition is valid for the Wick polynomials of the free scalar field. This conditions acts as a generalization for manifolds of the spectrum condition in Minkowski spacetime. Later, in [3] the free scalar field theory is constructed as a Locally Covariant Quantum Field Theory.

The goal of this chapter is to introduce the main ideas of causal structure and of Locally Covariant Quantum Field Theory. These will be fundamental for deriving the Reeh-Schlieder property in curved spacetime.

4.1 Causal structure of spacetime

Before delving into LCQFT, we must first establish the geometry of the spacetime we will consider. It turns out that the main tools used to study the global properties of spacetime are the domain of dependence and causal structure [13], so we will work our way through them. Although the Reeh-Schlieder theorem is not necessarily a global property, it certainly has consequences on a global scale, which is why these tools will be central in its derivation in curved spacetime.

We consider a spacetime $M = (\mathcal{M}, g)$, which we regard as a smooth four-dimensional manifold with a Lorentzian metric g of signature $(+,-,-,-)$ and that is time and spatially oriented. By smooth, we mean that it is C^∞, Hausdorff, paracompact, and connected. Having a time orientation in the spacetime means that a choice was made to differentiate past from future. We refer the reader to appendix B for some prerequisites on general relativity and topological spaces, as well as the textbooks [56, 57].

The causal structure between different events can be illustrated by a light cone (Figure 4.1), which is given with respect to the event at its origin, which we will call p. Then, the interior of the future lightcone gather the events which can be reached by a material particle starting at p. This is the chronological future of p. If we add to that the events lying on the curve itself, we get the causal future of p. We get similar statements for the past lightcone by replacing "future" with "past", and we say that events which are neither in the causal past nor future of p are causally separated from p.

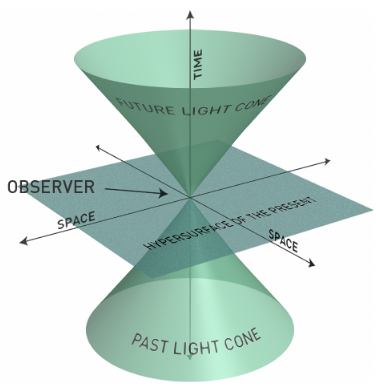

Figure 4.1: Lightcone. Available at: https://en.wiktionary.org/wiki/light_cone.

It is important to point out that although there are significant global differences between the causal structure of flat and curved spacetime, locally, they have the same qualitative nature. In fact, at each $p \in M$, the tangent space V_p is isomorphic to Minkowski spacetime, and the light cone of p is a subset of V_p, passing through the origin of V_p.

Furthermore, the tangent vectors in the tangent space of a point p can be divided into two classes, which are the future directed and past directed timelike vectors.

Now that we have given an idea of what we mean by causal structure, we will give some proper definitions regarding a time orientable spacetime $(\mathcal{M}, g)$ [3, 56, 57].

Definition 4.1. A differentiable curve $\lambda(t)$ is a *future directed timelike curve* if at each $p \in \lambda$ the tangent t^a is a *future directed timelike vector.* λ is a *future directed causal curve* if at each $p \in \lambda$, t^a is either a future directed timelike or null vector.

Definition 4.2. The *chronological future* of $p \in M$, $I^+(p)$, is the set of events that can be reached by a future directed timelike curve starting from p. For any subset $S \subset M$, $I^+(S)$ is defined by

$$I^+(S) = \bigcup_{p \in S} I^+(p). \tag{4.1}$$

Definition 4.3. The *causal future* of $p \in M$, $J^+(p)$, is the set of events that can be reached by a future directed causal curve starting from p. Similarly, we have

$$J^+(S) = \bigcup_{p \in S} J^+(p). \tag{4.2}$$

Definition 4.4. An *achronal set* is a set such that no two of its points may be joined by a timelike curve. More precisely, a subset $S \subset M$ is said to be *achronal* if there do not exist $p, q \in S$ such that $q \in I^+(p)$, i. e. if $I^+(S) \cap S = \emptyset$.

Definition 4.5. Let S be a closed and achronal set. The *future domain of dependence* of S, $D^+(S)$, is given by points p such that every past inextendible causal curve through p intersects S.

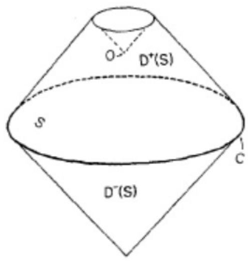

Figure 4.2: Domains of dependence of a disk in Minkowski spacetime, with the point O removed. Source: [13].

The *past domain of dependence* is defined in a similar way by exchanging “past” with “future”, and vice versa, and the *domain of dependence* is given by

$$D(S) = D^+(S) \cup D^-(S).$$

In other words, a point p is in the domain of dependence if the state of any system in p can be specified by the initial conditions given in S [13]. Figure 4.2 is an example of the domains of dependence of a set S.

Definition 4.6. A *Cauchy surface* is a closed achronal set Σ for which $D(\Sigma) = \mathcal{M}$.

Definition 4.6 gives us a hint on why the domain of dependence is also called Cauchy developments and why it is so useful when investigating global properties in general relativity. As we can see, a Cauchy surface gives us the initial conditions for the whole spacetime, which allows us to uniquely determine the state of any system in the given spacetime.

Definition 4.7. An open set $O \in \mathcal{M}$ is *causally convex* if a causal curve which intercepts O cannot leave the set and enter it again (See figure 4.3). Alternatively, O is causally convex iff for all $x, y \in O$ all causal curves from x to y lie entirely in O.

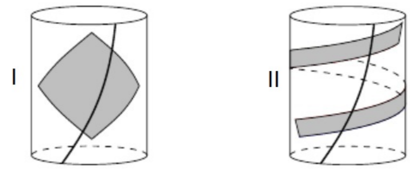

Figure 4.3: Einstein's bidimensional cylinder. I is convex, while II is not. Source: [58]

Now we can invoke a theorem by Geroch [13] to define a globally hyperbolic spacetime.

Theorem 4.8. *A spacetime* $(\mathcal{M}, g)$ *is globally hyperbolic if and only if it admits a Cauchy surface.*

Usually, in order to define a globally hyperbolic spacetime one requires that it satisfy *strong causality*, that is, for all neighbourhoods V of $x \in M$ there is a neighbourhood $U \subset V$ which is causally convex. However the definition by Geroch is equivalent to any other. Note that both strong causality and global hyperbolicity are causal stability conditions for spacetimes, but global hyperbolicity is at the top of the hierarchy [59]. This is why this is the condition that we will impose on the spacetimes we will consider, which is necessary to avoid some causality paradoxes.

Next we list some other details regarding causally convex sets that will be needed when deriving the Reeh-Schlieder property [4]. We may, sometimes, refer to causally convex sets as cc-regions.

Definition 4.9. A *cc-region* is a non-empty open set which is connected and causally convex.

Definition 4.10. A *bounded cc-region* is a cc-region whose closure is compact.

The relevance of causally convex sets is that the causality structure of a spacetime M_1 coincides with the one of $\Psi(M_1)$, obtained by an isometric embedding Ψ (Figures 4.4 and 4.5). So,

$$\Psi(J^{\pm}_{M_1}(x)) = J^{\pm}_{M_2}(\Psi(x)) \cap \Psi(M_1), \quad x \in M_1, \tag{4.3}$$

which means that the system in M_1 does not depend on the supersystem, and that the smaller system has enough information to allow one to study it as a subsystem on its own.

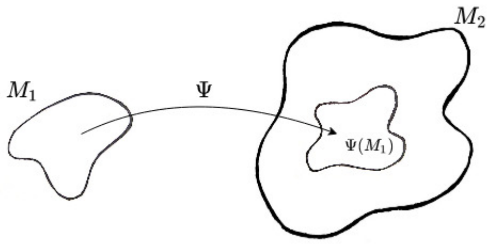

Figure 4.4: Isometric embedding.

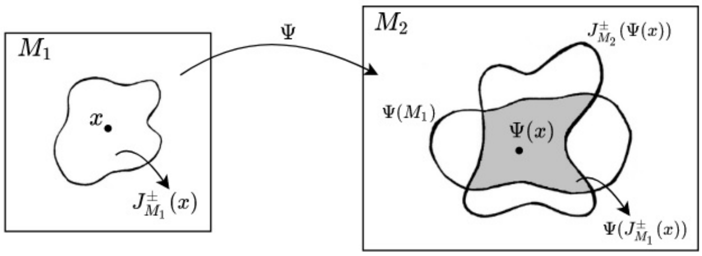

Figure 4.5: Causal structure of causally convex sets.

Next we introduce a lemma which lists some useful properties regarding causally convex regions [4].

Lemma 4.11. *Let $M = (M, g)$ be a globally hyperbolic spacetime, $O \subset M$ an open subset, and $A \subset M$ and achronal set. Then:*

1. *The intersection of two causally convex sets is causally convex;*
2. *For any subset $S \subset M$ the sets $I^{\pm}(S)$ are causally convex;*
3. *$O^{\perp}$ is causally convex;*
4. *O is causally convex iff $O = J^+(O) \cap J^-(O)$;*
5. *$\mathrm{int}(D(A))$ and $\mathrm{int}(D^{\pm}(A))$ are causally convex;*
6. *if O is a cc-region, then $D(O)$ is a cc-region;*
7. *if $S \subset M$ is an acausal continuous hypersurface then $D(S)$, $D(S) \cap I^{(}S)$ and $D(S) \cap I^-(S)$ are open and causally convex.*

A corollary by Bär [60] will also be useful:

Corollary 4.12. *Let C be a Cauchy hypersurface in a globally hyperbolic Lorentzian manifold $\mathcal{M}$ and let $K \subset \mathcal{M}$ be compact. Then $J^{\mathcal{M}}_{\pm}(K) \cap C$ and $J^{\mathcal{M}}_{\pm}(K) \cap J^{\mathcal{M}}_{\mp}(C)$ are compact.*

The following definitions regarding spacetimes might be of interest [5, 56]:

Definition 4.13. A spacetime is *analytic* if it is endowed with an analytic structure in which the metric is analytic. In other words, all component functions $g_{\mu\nu}$ of the metric are analytic in any choice of coordinates on the analytic manifold $\mathcal{M}$.

Definition 4.14. A spacetime is said to be *stationary* if there exists a one-parameter group of isometries, ϕ_t, whose orbit are timelike curves, and which expresses the "time translation symmetry" of spacetime.

Definition 4.15. A spacetime is said to be *static* if it is stationary and if there exists a spacelike hypersurface Σ which is orthogonal to the orbits of the isometry.

4.2 Locally Covariant Quantum Field Theory

Now we will address Locally Covariant Quantum Field theory, which as we have mentioned, is a functor between a category of spacetimes and a category of observables [2,3,61]. It will soon become clear the role that category theory plays here, as many of its structures can be used to define certain physical notions, such as state spaces, quantum fields, and the theory itself.

Quantum Field Theory incorporates to quantum physics the principles of locality and covariance (There are no preferred coordinates and we must take the points of spacetime simultaneously as elements of various locally diffeomorphic spacetimes). So, in order to bring QFT to curved spacetimes, we will consider the notions of AQFT in order to construct the algebras of observables and we will impose general covariance in a local sense, that is, associating to each isometric embedding between spacetimes an inclusion of algebras, and we can further impose causality and the time-slice axiom. This is the idea of *local covariance*, which is the foundation for the construction of LCQFT [3]. This approach provides a useful framework for the construction of a semi-classical approximation to quantum gravity, which is one of the main interests of physics today.

According to the principle of general covariance, the physical laws should be the same in every coordinate system, even in the ones that are not inertial. In this sense, the functors play an important role, since they are "descriptions" of a category that we can actually transform into other "descriptions" via natural transformations. With this is mind, we could say that observables are functorials, since this is precisely the characteristic that we need them to have in order to satisfy general covariance. In fact, this is the reason why Locally Covariant Quantum Field Theories are functors, since the aim is to obtain a theory that works for every possible spacetime simultaneously.

Category theory will be used to implement local covariance in AQFT by introducing two categories, $\mathfrak{Man}$ and $\mathfrak{Alg}$, which will be necessary for the definition of a Locally Covariant Quantum Field Theory as a functor.

The simplest case for a category of spacetimes is the one which takes globally hyperbolic spacetime $M = (\mathcal{M}, g)$ as objects. So, we have a category $\mathfrak{Man}$ of spacetimes [3, 4], such that:

(i) $\mathrm{Obj}(\mathfrak{Man})$ has as elements all four-dimensional globally hyperbolic spacetimes $M = (\mathcal{M}, g)$ that are oriented and time-oriented.

(ii) Given any two objects $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$, the morphisms are given by all maps $\psi : (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, g_2) \in \hom_{\mathfrak{Man}}(M_1, M_2)$ which are smooth isometric embeddings (i. e. $\psi : \mathcal{M}_1 \to \psi(\mathcal{M}_1)$ is a diffeomorphism and $\psi_* g_1 = g_2|_{\psi(\mathcal{M}_1)}$) such that orientation and time-orientation are preserved and $\psi(\mathcal{M}_1)$ is causally convex.

(iii) The composition of any $\psi \in \hom_{\mathfrak{Man}}(M_1, M_2)$ and $\psi' \in \hom_{\mathfrak{Man}}(M_2, M_3)$, $\psi' \circ \psi : M_1 \to M_3$, is a well defined map, which is also an isometric diffeomorphism.

(iv) Associativity of the composition rule follows from the associativity of composition of maps.

(v) Each $\hom_{\mathfrak{Man}}(M, M)$ has a unit element, which is the identity map $\mathrm{id}_{\mathcal{M}} : x \mapsto x$, $x \in \mathcal{M}$.

The fact that $\psi(\mathcal{M}_1)$ is causally convex means that if there is a causal curve $\gamma : [a, b] \to \mathcal{M}_2$, and $\gamma(a), \gamma(b) \in \psi(\mathcal{M}_1)$, then $\gamma(t) \in \psi(\mathcal{M}_1)$ for all $t \in]a, b[$ (Figure 4.6).

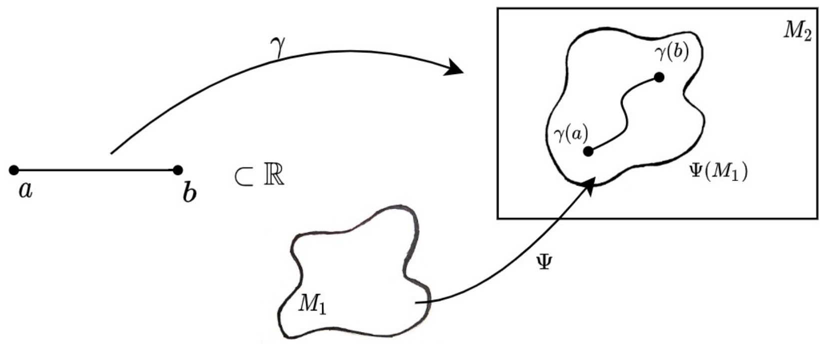

Figure 4.6: Causal curves.

The language of Category Theory allows one to consider a whole class of possible systems, instead of just one. So, we have a category $\mathfrak{Alg}$ of algebras [3, 4], such that:

(i) The class of objects $\mathrm{Obj}(\mathfrak{Alg})$ has as elements all unital C^*-algebras $\mathcal{A}$.

(ii) Given any two objects $\mathcal{A}_1, \mathcal{A}_2$, the morphisms are faithfull $*$-homomorphisms $\alpha : \mathcal{A}_1 \to \mathcal{A}_2 \in \hom_{\mathfrak{Alg}}(\mathcal{A}_1, \mathcal{A}_2)$ that preserve the unit, i.e $\alpha(I) = I$.

(iii) Given two morphisms $\alpha \in \hom_{\mathfrak{Alg}}(\mathcal{A}_1, \mathcal{A}_2)$ and $\alpha' \in \hom_{\mathfrak{Alg}}(\mathcal{A}_2, \mathcal{A}_3)$, the composition $\alpha' \circ \alpha$ is the composition of maps, and is an element of $\hom_{\mathfrak{Alg}}(\mathcal{A}_1, \mathcal{A}_3)$.

(*iv*) Associativity again follows from the associativity of composition of maps.

(*v*) For any element $\mathcal{A} \in \mathfrak{Alg}$ there is a unit element in $\hom_{\mathfrak{Alg}}(\mathcal{A}, \mathcal{A})$, such that for any $A \in \mathcal{A}$ we have $\mathrm{id}_{\mathcal{A}} : A \mapsto A$.

According to the Haag-Kastler axioms for Minkowski spacetime, a local net of observables consists of the algebra of observables, together with the inclusion of algebras and Poincaré symmetries. Since in general curved spacetimes we cannot assume Poincaré invariance, we must think of another way to guarantee general covariance. In LCQFT this covariance is imposed via the isometric embeddings. Since isometric spacetimes share the same physical information, the isometric embeddings discussed here preserve said information and allows us to consider spacetimes as subregions of a fixed spacetime, thus giving us all the requirement for a Haag-Kastler net of observables. This way, all the local observables measurable in a general region of spacetime meet the requirements for them to be also studied in subregions of spacetime.

Now we have all the elements required to define a Locally Covariant Quantum Field Theory [3]:

Definition 4.16. A *locally Covariant Quantum Field Theory* is a covariant functor $\mathscr{A}$ between the categories $\mathfrak{Man}$ and $\mathfrak{Alg}$. If we write α_ψ for $\mathscr{A}(\psi)$, we have the diagram in figure 4.7 together with the covariance properties

$$\alpha_{\psi'} \circ \alpha_\psi = \alpha_{\psi' \circ \psi}, \quad \alpha_{id_M} = id_{\mathscr{A}(M,g)},$$

for all morphisms $\psi \in \hom_{\mathfrak{Man}}((M_1, g_1), (M_2, g_2))$, all $\psi' \in \hom_{\mathfrak{Man}}((M_2, g_2), (M_3, g_3))$ and all $(M, g) \in \mathrm{Obj}(\mathfrak{Man})$.

$$\begin{tikzcd}
(M, \mathbf{g}) \arrow[r, "\Psi"] \arrow[d, "\mathscr{A}"'] & (M', \mathbf{g}') \arrow[d, "\mathscr{A}"] \\
\mathscr{A}(M, \mathbf{g}) \arrow[r, "\alpha_\Psi"'] & \mathscr{A}(M', \mathbf{g}')
\end{tikzcd}$$

Figure 4.7: Diagram of a Locally Covariant Quantum Field Theory.

Definition 4.17. A Locally Covariant Quantum Field Theory described by a covariant functor $\mathscr{A}$ is called *causal* if whenever there are morphisms $\psi_j \in \hom_{\mathfrak{Man}}((M_j, g_j), (M, g)), j = 1, 2$, so that the sets $\psi_1(M_1)$ and $\psi_2(M_2)$ are causally separated in (M, g), then one has

$$[\alpha_{\psi_1}(\mathscr{A}(M_1, g_1)), \alpha_{\psi_2}\mathscr{A}(M_2, g_2))] = \{0\},$$

where $[\mathcal{A}, \mathcal{B}] = \{AB - BA : A \in \mathcal{A}, B \in \mathcal{B}\}$ for subsets $\mathcal{A}$ and $\mathcal{B}$ of an algebra.

Definition 4.18. A Locally Covariant Quantum Field Theory described by a functor $\mathscr{A}$ satisfies the *time-slice axiom* if

$$\alpha_\psi(\mathscr{A}(M,g)) = \mathscr{A}(M',g')$$

holds for all $\psi \in \hom_{\mathfrak{Man}}((M,g),(M',g'))$ such that $\psi(M)$ contains a Cauchy surface for (M',g').

These last two definitions account for causality properties and definition 4.18 is also called *strong Einstein causality* or *existence of a causal dynamical law.*

The concept of natural transformation from Category Theory can be used to relate Locally Covariant Quantum Field Theories:

Definition 4.19. Let $\mathscr{A}$ and $\mathscr{A}'$ be two Locally Covariant Quantum Field Theories. Then, a *natural transformation* between $\mathscr{A}$ and $\mathscr{A}'$ is a family $\{\beta_{(M,g)}\}_{(M,g)\in\mathfrak{Man}}$ of $*$-monomorphisms $\beta_{(M,g)} : \mathscr{A}(M,g) \to \mathscr{A}'(M,g)$ such that the commutative diagram in figure 4.8 is valid whenever ψ is a morphism in $\hom_{\mathfrak{Man}}((M_1,g_1),(M_2,g_2))$.

$$\begin{tikzcd}
\mathscr{A}(M_1,\mathbf{g}_1) \arrow[r, "\beta(M_1,\mathbf{g}_1)"] \arrow[d, "\alpha_\Psi"'] & \mathscr{A}'(M_1,\mathbf{g}_1) \arrow[d, "\alpha'_\Psi"] \\
\mathscr{A}(M_2,\mathbf{g}_2) \arrow[r, "\beta(M_2,\mathbf{g}_2)"'] & \mathscr{A}'(M_2,\mathbf{g}_2)
\end{tikzcd}$$

Figure 4.8: Diagram of a natural transformation between Locally Covariant Quantum Field Theories.

Two Locally Covariant Quantum Field Theories are equivalent whenever there is a natural transformation between them which is an isomorphism, which means they are physically indistinguishable theories.

The definition of a locally covariant QFT is not enough to guarantee that we would have states that are physically realistic and that would lead to a reasonable interpretation of our theory. This is an important issue since the states are the entities necessary to relate the theory with the measurement outcomes of experiments. It is then necessary to impose some regularity properties on them, in order to select physical states [62]. First, we will define a category of states so that we have a restriction on the set of all states. And we will then impose some conditions on the states. It is worth mentioning that the Reeh-Schlieder theorem is in fact one of these conditions, which shows its relevance on the definition of physical theories.

Now, we will make use of the GNS construction, that allows us to form the triple $(\mathcal{H}_\omega,\pi_\omega,\Omega_\omega)$ for each state ω, and introduce some definitions regarding the states and representations of the algebra of the theory [3,4].

Definition 4.20 (Folium of a representation)**.** Let $\mathcal{A}$ be a C^*-algebra and $\pi : \mathcal{A} \to B(\mathcal{H})$ a $*$-representation of $\mathcal{A}$ by bounded linear operators on a Hilbert space $\mathcal{H}$. The *folium* of π, denoted by $\mathbf{F}(\pi)$, is the set of all states ω' on $\mathcal{A}$ which can be written as

$$\omega'(A) = \mathrm{tr}(\rho \cdot \pi(A)), \quad A \in \mathscr{A}(\mathcal{M}, g).$$

That is, the folium of a representation is the set of all density matrix states in that representation.

Consider an object $(\mathcal{M}, g)$ of $\mathfrak{Man}$. Denote by $\mathcal{K}(\mathcal{M}, g)$ the set of all open subsets in M which are relatively compact and which contain for each pair of points x and y all g-causal curves in M that connect them. Now we are able to define the following [3]:

Definition 4.21 (Local quasi-equivalence and local normality)**.** Let $\mathscr{A}$ be a Locally Covariant Quantum Field Theory, and for a given $M = (\mathcal{M}, g)$, let ω, ω' be two states on $\mathscr{A}$. These states (or their GNS-representations) are *locally quasi-equivalent* if for all $O \in \mathcal{K}(\mathcal{M}, g)$, the relation

$$\mathbf{F}(\pi \circ \alpha_{M,O}) = \mathbf{F}(\pi' \circ \alpha_{M,O}) \tag{4.4}$$

is valid, where $\alpha_{M,O} = \alpha_{\iota_{M,O}}$ and $\iota_{M,O} : (O, g_O) \to (M, g)$ is the natural embedding.

Definition 4.22. Let $\mathcal{A}_1^{*+}$ be the set of all states on $\mathcal{A}$. The category $\mathfrak{States}$ has as its objects all subsets $S \subset \mathcal{A}_1^{*+}$, for all unital C^*-algebras $\mathcal{A}$ in $\mathfrak{Alg}$ and as its morphisms all maps $\alpha^* : S_1 \to S_2$ for which $S_i \subset (\mathcal{A}_i)_1^{*+}, i = 1, 2$, and α^* is the restriction of the dual of a morphism $\alpha : \mathcal{A}_2 \to \mathcal{A}_1$ in $\mathfrak{Alg}$, i.e. $\alpha^*(\omega) = \omega \circ \alpha$ for all $\omega \in S_1$. The product of morphisms is given by the composition of maps and the identity map id_S on a given object serves as an identity morphism.

Definition 4.23. A *state space* for a Locally Covariant Quantum Field Theory $\mathscr{A}$ is a contravariant functor $\mathbf{S} : \mathfrak{Man} \to \mathfrak{States}$ such that for all objects M we have $M \mapsto S_M \subset (\mathscr{A}_M)_1^{*+}$ and for all morphisms $\Psi : M_1 \to M_2$ we have $\Psi \mapsto \alpha_\Psi^*|_{S_{M_2}}$. The set S_M is called the *state space* for M.

Definition 4.24. A state space $\mathbf{S}$ for a Locally Covariant Quantum Field Theory $\mathscr{A}$ is called *locally quasi-equivalent* iff for every morphism $\Psi : M_1 \to M_2$ such that $\psi(\mathcal{M}_1) \subset \mathcal{M}_2$ is bounded, and for every pair of states $\omega, \omega' \in S_{M_2}$ the GNS-representations $\pi_\omega, \pi_{\omega'}$ of $\mathcal{A}_{M_2}$ are quasi-equivalent on $\alpha_\Psi(\mathcal{A}_{M_1})$. Then, the local von Neumann algebras $\mathcal{R}_{M_1}^\omega := \pi_\omega(\alpha_\Psi(\mathcal{A}_{M_1}))''$ are $*$-isomorphic for all $\omega \in S_{M_2}$.

In his Ph.D thesis, Sanders [5] makes a contribution to LCQFT by introducing the new concept of nowhere-classicality, motivated by the fact that classical theories are described by commutative algebras. And he also gives a refined version of the time-slice axiom (Definition 4.26), which adds a restriction on the state space. Sanders based his work on the original paper of LCQFT [3], and also on [63], where LCQFT is introduced with some changes.

Definition 4.25. A Locally Covariant Quantum Field Theory $\mathscr{A}$ with a state space $\mathbf{S}$ is called *nowhere classical* iff for every morphism $\Psi : M_1 \to M_2$ and for every state $\omega \in S_{M_2}$ the local von Neumann algebra $\mathcal{R}^{\omega}_{M_1}$ is not commutative.

Definition 4.26. A Locally Covariant Quantum Field Theory $\mathscr{A}$ with state space $\mathbf{S}$ satisfies the time-slice axiom iff for all morphisms $\Psi : M_1 \to M_2$ such that $\psi(M_1)$ contains a Cauchy surface for M_2 we have $\alpha_\Psi(\mathscr{A}_{M_1}) = \mathscr{A}_{M_2}$ and $\alpha^*_\Psi(S_{M_2}) = S_{M_1}$.

Now we present an important lemma that will be used in the next chapter [4].

Lemma 4.27. *For a Locally Covariant Quantum Field Theory $\mathscr{A}$ with a state space* $\mathbf{S}$ *satisfying the time-slice axiom, an object $(\mathcal{M}, g) \in \mathfrak{Man}$ and a causally convex region $O \subset M$, we have*

$$\mathcal{A}_O = \mathcal{A}_{D(O)} \quad \text{and} \quad S_O = S_{D(O)}.$$

If O contains a Cauchy surface for $\mathcal{M}$, then

$$\mathcal{A}_O = \mathcal{A}_{\mathcal{M}} \quad \text{and} \quad S_O = S_M.$$

Many other concepts of Algebraic Quantum Field Theory can be brought to this approach, such as quantum fields and representations. Furthermore, we can also study the dynamics of a Locally Covariant Quantum Field Theory via *relative Cauchy revolutions*, if it obeys the time-slice axiom [3].

4.3 Some remarks

In [3] it was shown that the Klein-Gordon field theory is a LCQFT and that the fields of this theory are natural transformations, as expected. The Klein-Gordon field theories with different masses do not possess a natural isomorphism between them, and, therefore, are not equivalent theories. Furthermore it is possible to recover the algebraic version of this QFT from this approach, which shows that LCQFT is indeed more general.

It is also possible to obtain the LCQFT in spacetimes with a spin structure, i.e. for Dirac fields. In this case, it is necessary to introduce more geometric structure into the spacetime, and objects such as frame bundles and tangent bundles are used. Sanders points out that every globally hyperbolic spacetime admits a spin structure, which does not need to be unique [4].

These examples come to show that Locally Covariant Quantum Field Theory is, in fact, a very useful framework for dealing with quantum fields in general curved spacetimes, since it adds to these theories a precise notion of Einstein's general covariance principle. The embeddings of the theory guarantee the local aspect, since subsystems and their relations are accounted for when dealing with subregions of spacetime. Furthermore, the use of causally convex regions ensure that the causality structure of a subregion coincides with that of the embedded subregion.

Sanders also argues that LCQFT can provide an appropriate framework for a semi-classical approximation of quantum gravity. According to [64], LCQFT opens the way for a perturbative quantization of gravity that is independent of the background.

So, in this framework we start from a region of spacetime where we can measure certain observables, and via isometric embeddings we get to other regions of spacetime in which these observables can also be measured. Then, through methods of Algebraic Quantum Field Theory we associate an algebra to these regions of spacetime. Since there is an isometric embedding between them, there must also be an inclusion map which takes one algebra into the other, where the latter is the supersystem. In this manner, covariance is introduced locally.

Chapter 5

The Reeh-Schlieder Theorem

The Reeh-Schlieder theorem is an important result that can be used to impose some conditions on the states in axiomatic QFT. It claims that for any bounded open region, the vaccum state is a cyclic vector for an algebra $\mathcal{A}(D)$ of local observables in the appropriate representation, which means that the result of the action of this operators on the vacuum vector is a set which is dense in the corresponding Hilbert space. It also states that the vacuum vector is separating for the von Neumann algebra $\mathcal{A}(D)''$.

This theorem is due to Reeh and Schlieder [65] and it was first proven in Whightman's approach to QFT, but it is also valid for the Haag-Kastler vacuum representation. Since this theorem is of crucial importance in the mathematical structure of AQFT, it is not a surprise that an investigation was made to find if it also holds in curved spacetimes. The analytic continuation arguments of Reeh and Schlieder's proof was extended to analytic spacetimes via the introduction of a microlocal spectrum condition [66] and there was also a contribution for stationary spacetimes [67]. But the study of the Reeh-Schlieder property for general curved spacetimes (which accounts for spacetimes that may not be stationary or analytic) was carried out by Sanders [4, 5] following the general framework of LCQFT.

Although we are dealing with local operators, the Reeh-Schlieder theorem says that the effects can be in regions which are causally separated from the initial bounded region. This is an important result, since it suggests that vector states are highly entangled in the vacuum representation, and that states have non-local properties.

It is important to note that although the original arguments used in the derivation of the Reeh-Schlieder theorem use global symmetry properties of the vacuum state, in general curved spacetimes there is no guarantee that such symmetries exist [68]. Therefore, it was an actual challenge to obtain this results in curved spacetimes, which makes Sanders' work even more impressive. The technique of spacetime deformation along with LCQFT provided us with an elegant way of deriving the Reeh-Schlider property in general curved spacetimes. However, this result is still limited as Sanders himself stated, primarily due to the fact that full Reeh-Schlieder states are not guaranteed to be of physical relevance.

In the main text we will present the Algebraic Quantum Field Theory approach to the proof of the theorem, as done in [7], but Wightman's proof [69] is discussed in Appendix C. Then, following [4], we will see how this property translates to general curved spacetimes using the technique of spacetime deformation [12]. We may sometimes refer to a Reeh-Schlieder property of states, which means that we are taking the Reeh-Schlieder theorem as an axiom of the theory.

5.1 The Reeh-Schlieder theorem in Algebraic QFT

In this section we will state and prove the Reeh-Schlieder theorem in the framework of AQFT [7]. Then, we will discuss how this result is closely related to entanglement.

Theorem 5.1 (Reeh-Schlieder). *Suppose weak additivity for the vacum state φ. Then the vacuum vector Ω_φ is a cyclic vector of $\pi_\varphi(\mathcal{A}(D))$ for any domain D, i.e.*

$$\overline{\pi_\varphi(\mathcal{A}(D))\Omega_\varphi} = \mathcal{H}_\varphi,$$

and is a separating vector of $\pi_\varphi(\mathcal{A}(D))''$ for any bounded domain D (which has a non-empty causal complement), i.e.

$$A\Omega_\varphi = 0 \Rightarrow A = 0, \quad A \in \pi_\varphi(\mathcal{A}(D))''.$$

Proof. We must prove the cyclic and separating properties of the vacuum vector. For the first it is sufficient to show that if $\Psi \perp \pi_\varphi(\mathcal{A}(D))\Omega_\varphi$ then $\Psi = 0$. And for the latter, we must show that for $A, B \in \pi_\varphi(\mathcal{A}(D))''$, if $A\Omega_\varphi = B\Omega_\varphi$, then $A = B$.

Proof of the cyclic property

The idea of the proof is to show that if there is a vector Ψ which is orthogonal to $\pi_\varphi(\mathcal{A}(D))\Omega_\varphi$, then it is also orthogonal to $\pi_\varphi(\mathfrak{A})\Omega_\varphi$, that is, we eliminate the restriction on the small open domain D. Then we can show that this is true only if $\Psi = 0$.

First, we will take a domain D_1 satisfying $\bar{D}_1 \subset D$. For any x in a sufficiently small neighbourhood N of 0 we get $D_1 + x \subset D$. That is, we are taking small translations of the domain D_1 which are still in D (Figure 5.1).

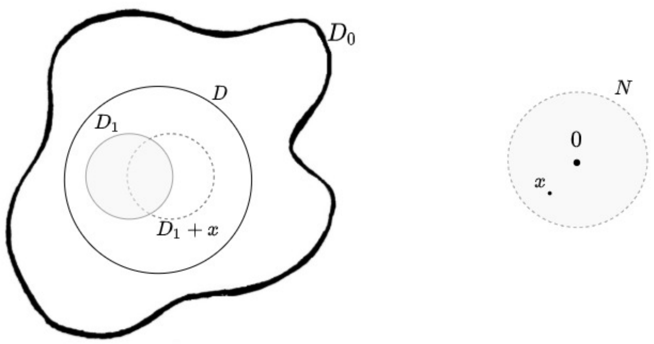

Figure 5.1: Translation of domain.

Since $D_1 + x_j \subset D$, by isotony we have

$$Q_j(x_j) \in \mathcal{A}(D_1 + x_j) \subset \mathcal{A}(D)$$

for elements $Q_1, ..., Q_n$ in $\mathcal{A}(D_1)$ and points $x_1, ..., x_n$ in N.

If we assume that the vacuum vector is not cyclic, then there should be a state Ψ in the GNS representation of φ such that

$$f(x_1, ..., x_n) = (\Psi, Q_{1\varphi}(x_1)...Q_{n\varphi}(x_n)\Omega_\varphi) = 0. \tag{5.1}$$

We will show that

$$f(x_1, ..., x_n) = 0 \quad \forall x_j \in N \Rightarrow f(x_1, ..., x_n) = 0 \quad \forall x_j$$

by using a representation of the translation group in $\mathcal{H}_\varphi$ which satisfies

$$Q_{j\varphi}(x_j) = T_\varphi(x_j) Q_{j\varphi} T_\varphi(x_j)^*, \quad Q_{j\varphi} \equiv \pi_\varphi(Q_j)$$

and with spectral measure E_φ defined by

$$T_\varphi(x) = \int e^{i(x,p)} E_\varphi(d^4p),$$

whose support is contained in $\overline{V}_+$, due to the spectrum condition.

Then we can define the function

$$T_\varphi(\xi) = \int e^{i(\xi,p)} E_\varphi(d^4p)$$

for complex vectors ξ which satisfy Im $\xi \in \overline{V}_+$. $T_\varphi(\xi)$ is continuous in the strong operator topology and is holomorphic in ξ for Im $\xi \in V_+$ (See corollary 3.15).

For Im $\xi_j \in \overline{V}_+$ we can define the continuous function

$$g(\xi_1, ..., \xi_n) = (\Psi, T_\varphi(\xi_1) Q_{1\varphi} T_\varphi(\xi_2) Q_{2\varphi} ... T_\varphi(\xi_n) Q_{n\varphi} \Omega_\varphi), \tag{5.2}$$

which is holomorphic for $\xi_j \in V_+$.

Theorem C.2 tells us that if $f(x_1, ..., x_n)$ and $g(\xi_1, ..., \xi_n)$ are holomorphic functions which coincide on the real axis, then they are analytic continuations of one another. We will show that this is true.

We take the limit Im $\xi \to 0$, perform the change of variables

$$\xi = x_1, \xi_k = x_k - x_{k-1}, ..., \xi_n = x_n - x_{n-1},$$

and substitute these on equation 5.2.

Due to the properties of translation operators, we have that

$$T_\varphi(x_2 - x_1) = T(x_2)T(-x_1) = T(x_2)T^*(x_1) = T^*(x_1)T(x_2).$$

So,

$$T_\varphi(x_1)Q_{1\varphi}T_\varphi(x_2-x_1)Q_{2\varphi}... = T_\varphi(x_1)Q_{1\varphi}T^*_\varphi(x_1)T(x_2)Q_{2\varphi}...$$

And we know that $T_\varphi(x_1)Q_{1\varphi}T^*_\varphi(x_1) = Q_{1\varphi}(x_1)$. Analogously,

$$T_\varphi(x_n-x_{n-1})Q_{n\varphi}\Omega_\varphi = T^*(x_{n-1})T_\varphi(x_n)Q_{n\varphi}\Omega_\varphi,$$

and since the vacuum is translation invariant, we have that

$$T_\varphi(x_n)\pi_\varphi Q_n\Omega_\varphi \equiv \pi_\varphi(\alpha_{(x_n,1)}Q)\Omega_\varphi = Q_\varphi(x_n)\Omega_\varphi.$$

Therefore, for Im $\xi \to 0$, we find that

$$g = (\Psi, Q_{1\varphi}(x_1)...Q_{n\varphi}(x_n)\Omega_\varphi) = 0 \quad \forall x_j \in N. \tag{5.3}$$

Then, we invoke theorem C.3 to state that $g = 0$ for any x_j, thus eliminating the restriction that it had to be inside N.

Now, let $\mathcal{B}$ be the $*$-algebra generated by

$$\bigcup_x \pi_\varphi(\mathcal{A}(D_1+x)).$$

The elements of $\mathcal{B}$ are linear combinations of monomials of the form $Q_{1\varphi}(x_1)...Q_{n\varphi}$. Since 5.3 is now true for all x_j, we should have $\Psi \perp \mathcal{B}\Omega_\varphi$, and we can invoke weak additivity in order to state that

$$\overline{B} = \left(\bigcup_x \pi_\varphi(\mathcal{A}(D_1+x))\right)'' = \pi_\varphi(\mathfrak{A})''. \tag{5.4}$$

Due to the GNS construction, $\pi_\varphi(\mathfrak{A})\Omega_\varphi$ is dense in $\mathcal{H}_\varphi$, so $\mathcal{B}\Omega_\varphi$ is also dense in $\mathcal{H}_\varphi$. In thise case,

$$\Psi \perp \mathcal{B}\Omega_\varphi \Rightarrow \Psi = 0. \tag{5.5}$$

Thus, we have proven that Ω_φ must be cyclic.

Proof of the separating property

If D is a bounded domain, then it has a nontrivial causal complement. So, for D_1 spacelike separated from D,

$$[\mathcal{A}(D_1), (\mathcal{A}(D)''] = 0.$$

Let $C \in \pi_\varphi(\mathcal{A}(D_1))$, $\Psi = C\Omega_\varphi$, and Φ any vector in $\mathcal{H}_\varphi$. For any A, we have

$$(\Psi, A^*\Phi) = (A\Psi, \Phi) = (AC\Omega_\varphi, \Phi) = (CA\Omega_\varphi, \Phi).$$

So, if $A\Omega_\varphi = 0$, then $(\Psi, A^*\Phi) = 0$. This means that $A^*\Phi = 0$, since Ψ spans $\mathcal{H}_\varphi$. Additionally, $(A\Psi, \Phi) = 0$ implies $A\Psi = 0$, since Φ is arbitrary. Therefore, $A = 0$. This proves that the vacuum vector is separating for the von Neumann algebra $\pi_\varphi(\mathcal{A}(D))''$.

$\square$

This illustrates the fact that a vector that is cyclic for a von Neumann algebra is separating for its commutant, since we could have taken $\mathcal{A}(D_1)$ to be the commutant of $\mathcal{A}(D)''$.

An idea that always comes up when discussing the Reeh-Schlieder theorem is correlation. The reason is that the theorem implies that the vacuum is an entangled state, even for spatially separated regions. Recall that observables A_1 and A_2 are correlated in a state Ψ if

$$(\Psi, A_1 A_2 \Psi) \neq (\Psi, A_1 \Psi)(\Psi, A_2 \Psi). \tag{5.6}$$

Then the Reeh-Schlieder theorem immediately gives us the following corollary [33]:

Corollary 5.2. *For every pair of local regions O_1 and O_2, there are vacuum correlation between $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$.*

Proof. If we suppose that these correlations do not exist, then there are states ω_i on $\mathcal{A}(O_i)$ such that

$$(\Omega, \pi(A_1)\pi(A_2)\Omega) = \omega_1(A_1)\omega_2(A_2), \quad A_i \in \mathcal{A}(O_i).$$

If we set $A_1 = \mathbb{1}$, we get

$$(\Omega, \pi(A_2)\Omega) = \omega(A_2),$$

and if we set $A_2 = \mathbb{1}$, we get

$$(\Omega, \pi(A_1)\Omega) = \omega(A_1),$$

so that

$$(\Omega, \pi(A_1)\pi(A_2)\Omega) = (\Omega, \pi(A_1)\Omega)(\Omega, \pi(A_2)\Omega), \quad A_i \in \mathcal{A}(O_i).$$

Then, by fixing A_2 and letting A_1 vary in $\mathcal{A}(O_1)$, it yields a dense set of states, due to the Reeh-Schlieder theorem. So we find

$$\pi(A_2)\Omega = (\Omega, \pi(A_2)\Omega)\Omega, \quad A_2 \in \mathcal{A}(O_2).$$

But since $\pi(A_2)\Omega$ can be any vector in $\mathcal{H}$, this can only be true if the Hilbert space has dimension 1, which contradicts the Reeh-Schlieder theorem (specifically the cyclic property of the vacuum vector). Thus, the existence of uncorrelated algebras in the vacuum sector would not apply to Hilbert spaces with $\dim\mathcal{H} > 1$.

□

It is then established that vacuum correlations between any two regions must always exist, no matter how far apart they are. Wald, in his awarded essay [70], argued that correlations between observables over all spacetime is a fundamental feature of QFT and that these effects should not be neglected unless a proper justification is given. He claimed that these correlations "beyond the horizons", i.e. between events whose pasts have no intersection, could have an important role in phenomena which took place in the early universe.

However, the fact that these correlations exist and must be accounted for does not mean that we can use it to create objects in places incredibly far away from us. The cluster property makes this situation very unlikely to happen, due to the approximation that vacuum states are independent at large distances. What the theorem says is that these correlations exist, and that it is a feature of the theory that the vacuum is highly entangled, even if these effects are very small. This does not mean that this result is any less relevant, though. As a matter of fact, it is central in AQFT since it provides an important aspect of states that must be considered when formulating a Quantum Field Theory. Furthermore, it has a wide range of applicability, one of them being a recent paper by Yonekura [8], which suggests that the Reeh-Schlieder theorem, together with the time-slice axiom, may be used to recover information lost in black holes.

5.1.1 Some consequences and interpretation

We know from QFT that we can obtain a dense set of states in the Hilbert space by acting on the vacuum with local operators. But the claim from the Reeh-Schlieder theorem is even more powerful, because this idea is still valid if we restrict to operators measurable in any given open region. In other words, by acting on the vacuum with local observables in a certain region, it is possible to obtain effects even in regions spacelike to it, which is why we can say that states are highly entangled in the vacuum representation. This may not sound very intuitive, since one could expect that the action of an observable in $\mathcal{A}(D)$ would result in a vector in D, however it has a strong foundation and it suggests that states are not localized, although observables are.

The physical interpretation of the Reeh-Schlieder theorem might bring the question on whether it violates causality, but we will see now that this is not the case. In order to show that, we will consider an example given by Witten [71]. Take the universe to be at an initial time slice, where it looks like the vacuum at a region U and it contains a planet at a region V spacelike to U. Next, define an operator P, so that $\Phi(P) = 1$ if Φ contains said planet, and $\Phi(P) = 0$ otherwise.

The Reeh-Schlieder theorem tells us that there is an operator X in U so that $X\Omega$ contains the planet in V, which is very surprising. But at the same time, one might think that this leads to a contradiction, since

$$(\Omega, P\Omega) \cong 0$$

and

$$(X\Omega, PX\Omega) \cong 1 \Leftrightarrow (\Omega, X^\dagger PX\Omega) = (\Omega, PX^\dagger X\Omega) \cong 1.$$

This would in fact be a contradiction if the X were unitary operators, and causality would be violated. But the Reeh-Schlider theorem does not require them to be unitary. Due to equation 5.6, we can say that there is a correlation between the operators P and $X^\dagger X$ in the vacuum in spacelike separated regions. This type of correlation is quite common in QFT, so the Reeh-Schlider theorem does not really deviates from the theory and it is rather a fundamental aspect of it. Once again, it is crucial to note how the Reeh-Schlider theorem is always bringing up the question of correlation, thus suggesting that it is more common than what was initially imagined.

Another apparent issue regards the principle of locality, since it claims that observables are local, i.e. their algebra acts in local regions of spacetime, but states exhibit these non-local correlations, i.e. entanglement. However, the Reeh-Schlieder theorem shows us that this characteristic of states is well founded in QFT, and entanglement should be expected to occur. This suggests that separating the construction of observables from that of states can shed some light over this apparent problem.

So far we have discussed consequences regarding the cyclic property of the vacuum vector, however, the separating property also has important implications. It basically states that if O has a non-empty causal complement, then $\mathcal{A}(O)$ does not have any operator that annihilates the vaccum [72]. So every non trivial positive local operator has a strictly positive vacuum expectation value, and the Reeh-Schlieder theorem makes it impossible to create, annihilate or count particles using local operators [5]. This creates a problem in Quantum Field Theory regarding the idea of *localized particles*, which is an issue that cannot be easily avoided, and adds to the idea of entanglement being a rule, instead of an exception. A paper by Fleming [73] claimed that they could avoid the Reeh-Schlieder theorem and its "counterintuitive" consequences by introducing a "Newton Wigner" scheme of localization (which we will not discuss here), but Halvorson later argued that the Reeh-Schlieder theorem still has the final word and this issue is not that simple to solve [74].

There are also mathematical consequences regarding the Reeh-Schlieder theorem which are mentioned in [4], namely its use to determine the type of von Neumann algebras, and it also provides a framework which allows for the use of Tomita-Takesaki modular theory.

5.2 The Reeh-Schlieder property in curved spacetimes

We will now present a way to investigate whether there are states with the Reeh-Schlieder property in general curved spacetimes as done by Sanders [4]. The idea is to suppose the existence of a state ω_1 with the property in M_1, and then try to derive the existence of an ω_2 in a diffeomorphic (but not isometric) spacetime which also has the property (Figure 5.2). This is done with the aid of an "intermediate" spacetime M', which exists if M_1 and M_2 possess diffeomorphic Cauchy surfaces (Figure 5.3). It is actually a feature of globally hyperbolic spacetimes that they can be deformed to more symmetric ones [62]. If we add to that the time-slice axiom and local covariance, then we can transfer a property from one spacetime to the other. In our case, we will deform a general globally hyperbolic spacetime to Minkowski spacetime, where we assume the existence of a state with the Reeh-Schlieder property. Then, by deriving some relations between the algebras for regions of these spacetimes, we are able to transfer the property from one spacetime to the other.

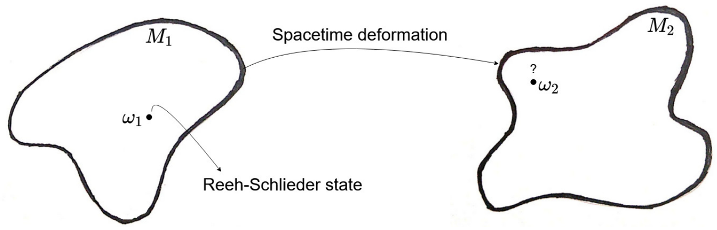

Figure 5.2: Idea of the derivation of the Reeh-Schlieder property in an arbitrary curved spacetime.

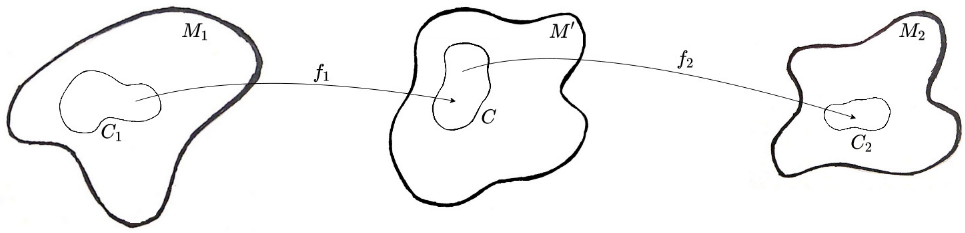

Figure 5.3: Spacetime deformation of M_1 into M_2 via an auxiliary spacetime M'.

We have stated the Reeh-Schlieder theorem previously in the algebraic approach to QFT. Now we would like to formulate it in the context of LCQFT, which is better suited for curved spacetimes. We present now Sander's definition of the Reeh-Schlieder property:

Definition 5.3. Consider a Locally Covariant Quantum Field Theory $\mathscr{A}$ with a state space $\mathbf{S}$. A state $\omega \in S_M$ has the *Reeh-Schlieder property* for a cc-region $O \subset \mathcal{M}$ iff

$$\overline{\pi_\omega(\mathcal{A}_O)\Omega_\omega} = \mathcal{H}_\omega,$$

where $(\pi_\omega, \Omega_\omega, \mathcal{H}_\omega)$ is the *GNS-representation* of $\mathcal{A}_M$ in the state ω. We then say that ω is a *Reeh-Schlieder state* for O. We say that ω is a *full* Reeh-Schlieder state iff it is a Reeh-Schlieder state for *all* cc-regions in $\mathcal{M}$.

According to this definition, the vacuum state on Minkowski spacetime has the full Reeh-Schlieder property, but on curved spacetimes the existence of such states is still controversial [4,5], as we will discuss below. Nevertheless, states of physical interest with the weak form of the property are guaranteed to exist, and, even in this case, non-local correlations exist between O and any region causally separated from it.

5.2.1 Spacetime deformation

The spacetime deformation technique was developed by Fulling and Narcowich in [12], where they proved the existence of Hadamard states of the free scalar fields in curved spacetimes. It has also been used in the proof of a spin-statistics theorem for Locally Covariant Quantum Field Theories [2]. This procedure allows one to derive, in a general curved spacetime, a property that is known to be valid in flat spacetimes. The result is a spacetime which contains algebras that are isomorphic to the algebras of the undeformed spacetimes, and because of that, it is possible to take some conclusion regarding the states.

In [62], Fewster and Verch systematized this results of deformation of globally hyperbolic spacetimes as a "rigidity argument" for Locally Covariant Quantum Field Theory. They show that this argument can carry over to LCQFTs Einsein Causality, the Reeh-Schlieder property and extended locality. To put it simply, this argument asserts that any property that a Locally Covariant Quantum Field theory has in a certain spacetime can be shown to be valid for any spacetime.

In the algebraic approach, the Reeh-Schlieder theorem was proved using the fact that the vacuum is translation invariant, as well as some analyticity results concerning holomorphic functions. However, this invariance of the vacuum vector is not guaranteed to hold in curved spacetimes, as we are not necessarily dealing with spacetimes that have any type of symmetries. This imposes a problem on how to derive certain properties on general curved spacetimes. In this scenario, the use of the spacetime deformation procedure in the framework of Locally Covariant Quantum Field Theory is an elegant and general solution to this issue, since this formulation is more general and does not require too many details regarding the spacetimes in order for it to describe physical theories.

Before delving into the details of spacetime deformation, let us first recall a few results regarding globally hyperbolic spacetimes and Cauchy surfaces. It was Geroch [13] who first saw the intrinsic relation between these two, as stated in Theorem 4.8. He also proved the existence of a time function $t : M \to \mathbb{R}$ such that every level $t =$ constant is a (topological) Cauchy surface and M is homeomorphic to $\mathbb{R} \times S$.

These results, however, were on a topological level, and it was Bernal and Sanchez [75–77] who adapted them to smooth Cauchy surfaces, so that there is a global diffeomorphism between $\mathbb{R} \times S$ and M.

Fulling [12] then showed that a spacetime (M_1, g_1, C_1) can be deformed into a spacetime (M_2, g_2, C_2) (C_1 and C_2 Cauchy surfaces) while preserving global hyperbolicity (See figure 5.4). He proved the following result:

Proposition 5.4. *Consider two globally hyperbolic spacetimes $M_i, i = 1, 2$, with spacelike Cauchy surfaces C_i both diffeomorphic to C. Then there exists a globally hyperbolic spacetime $M' = (\mathbb{R} \times C, g')$ with spacelike Cauchy surfaces $C'_i, i = 1, 2$, such that C'_i is isometrically diffeomorphic to C_i and an open neighbourhood of C'_i is isometrically diffeomorphic to an open neighbourhood of C_i.*

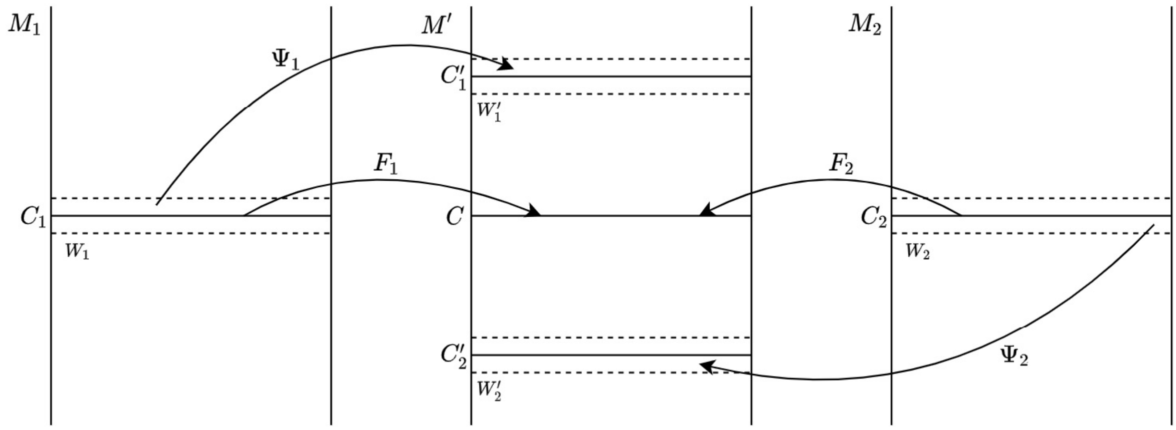

Figure 5.4: Idea of proposition 5.4. The Ψ_i are isometric diffeomorphisms, while the f_i are diffeomorphisms.

When we apply this result to a Locally Covariant Quantum Field Theory, we have the following corollary regarding the algebras [3]:

Corollary 5.5. *Two globally hyperbolic spacetimes M_i with diffeomorphic Cauchy surfaces are mapped to isomorphic C^*-algebras $\mathcal{A}_{M_i}$ by any Locally Covariant Quantum Field Theory $\mathscr{A}$ satisfying the time-slice axiom (with some state space $\mathbf{S}$).*

Proof. Let M_i, $i = 1, 2$, be two globally hyperbolic spacetimes and M' the auxiliary spacetime. Let $W_i \subset M_i$ be open neighbourhoods of $C_i \subset M_i$ which are isometrically diffeomorphic through ψ_i to the open neighbourhoods $W_i' \subset M'$ of $C_i' \subset M'$. We take the W_i and W_i' to be causally convex regions, which makes the ψ_i isomorphisms in $\mathfrak{Man}$, since causally convex regions are objects of this category.

From Lemma 4.27 we have that

$$\mathcal{A}_{M_1} = \mathcal{A}_{W_1},$$

as W_1 has a Cauchy surface of M_1. From $\psi_1 : W_1 \to W_1'$, we can rewrite

$$\mathcal{A}_{M_1} = \mathcal{A}_{\psi_1^{-1}}(W_1') = \alpha_{\psi_1}^{-1}(\mathcal{A}_{W_1'}),$$

since $\alpha_{\psi_1} : \mathcal{A}_{W_1} \to \mathcal{A}_{W_1'}$. Now we use the fact that W_1' has a Cauchy surface of M', so that

$$\mathcal{A}_{M_1} = \alpha_{\psi_1}^{-1}(\mathcal{A}_{M_1'}) = \alpha_{\psi_1}^{-1} \circ \alpha_{\psi_2}(\mathcal{A}_{M_2}),$$

since

$$\mathcal{A}_{M'} = \mathcal{A}_{W_2'} = \alpha_{\psi_2}(\mathcal{A}_{W_2}) = \alpha_{\psi_2}(\mathcal{A}_{M_2}).$$

Here the α_{ψ_i} are $*$-isomorphisms, so we can conclude that the algebras $\mathcal{A}_{M_1}$ and $\mathcal{A}_{M_2}$ are isomorphic.

$\square$

Corollary 5.5 tells us that if the C_i are isomorphic to C, then the algebras in (M_i, g_i) are isomorphic to the algebras in $(\mathbb{R} \times C, g)$, as well as isomorphic to one another. This result is essential for what we are interested in studying, since by preserving the algebras, we are preserving the observables of the theory. Furthermore, since the GNS representations are related by unitary transformations, the expectation values of the observables remain the same. Therefore, deforming a general spacetime into an easier one to study actually gives us the same physical information, and we can then move on to deriving properties in curved spacetimes.

There are still some geometrical results by Sanders [4] regarding spacetime deformation that need to be discussed in order for us to apply the technique to the Reeh-Schlieder property, namely propositions 5.6 and 5.7. It is important to note that the Reeh-Schlieder property is valid for operators acting on a given bounded region of spacetime, so it is necessary to achieve, via spacetime deformation, the same configuration that allows us to take conclusions regarding a Reeh-Schlieder state. This means that we are looking for a bounded cc-region in a curved spacetime that can be deformed into a bounded cc-region on Minkowski spacetime, where we know that the Reeh-Schlieder property is valid.

Proposition 5.6. *Consider two globally hyperbolic spacetimes $M_i, i = 1, 2$, with diffeomorphic Cauchy surfaces and a bounded cc-region $O_2 \subset \mathcal{M}_2$ with non-empty causal complement, $O_2^{\perp} \neq \emptyset$. Then there are a globally hyperbolic spacetime $M' = (\mathcal{M}', g')$, spacelike Cauchy surfaces $C_i \subset \mathcal{M}_i$ and $C_1', C_2' \in \mathcal{M}'$, and bounded cc-regions $U_2, V_2 \subset \mathcal{M}_2$ and $U_1, V_1 \subset \mathcal{M}_1$ such that the following hold:*

(i) There are isometric diffeomorphisms $\psi_i : W_i \to W_i'$, where $W_1 := I^-(C_1), W_1' := I^-(C_1')$, $W_2 := I^+(C_2)$ and $W_2' := I^+(C_2')$,

(ii) $U_2, V_2 \subset W_2$, $U_2 \subset D(O_2)$, $O_2 \subset D(V_2)$,

(iii) $U_1, V_1 \subset W_1$, $U_1 \neq \emptyset$, $V_1^{\perp} \neq \emptyset$, $\psi(U_1) \subset D(\psi_2(U_2))$ and $\psi_2(V_2) \subset D(\psi_1(V_1))$.

Sketch of the proof. Our first assumption is that there are two spacetimes M_1 and M_2 with diffeomorphic Cauchy surfaces. We already know that any globally hyperbolic spacetime admits a time function $t : \mathbb{R} \times C \to \mathbb{R}$, so that there are diffeomorphisms

$$F_i : \mathcal{M}_i \to \mathbb{R} \times C = \mathcal{M}', \tag{5.7}$$

responsible for splitting the manifold $\mathcal{M}'$. Here, C is a smooth three-dimensional manifold so that each $t \in \mathbb{R}$ yields a spacelike Cauchy surface $F_i^{-1}(\{t\} \times C)$.

The metric of the spacetime $M' = (\mathbb{R} \times C, g')$ is given by

$$g'_{\mu\nu} = \beta dt_\mu dt_\nu - h_{\mu\nu},$$

where $g' := F_{i*}g_i$ is the pushed-forward metric, $\beta : \mathbb{R} \times C \to (0, \infty)$ is a strictly positive smooth function, and $h_{\mu\nu}$ is a Riemannian metric on C. In addition, orientation and time orientation in $\mathbb{R} \times C$ are induced by the ones in the $\mathcal{M}_i$ via F_i. Since we assume that the M_i have diffeomorphic Cauchy surfaces, we can take C to be the same for both $i = 1, 2$.

Derivation of the bounded cc-regions U_2 and V_2

Our second assumption is the existence of the bounded cc-region $O_2 \in \mathcal{M}_2$ with $O_2^\perp \neq \emptyset$. We can use the diffeomorphism of equation 5.7 to define the bounded region

$$O_2' := F_2(O_2) \tag{5.8}$$

in $\mathcal{M}'$ (see figure 5.5). Additionally, we take $t_{\min}$ and $t_{\max}$ to be the minimum and maximum values that t has on the compact set $\overline{O_2}$. Then it can be proven that [4]

$$F_2^{-1}((t_{\min}, t_{\max}) \times C)) \cap O_2^\perp \neq \emptyset. \tag{5.9}$$

This means that we can define a Cauchy surface on M_2 which intersects O_2 but does not lie entirely inside it. So we can choose a $t_2 \in (t_{\min}, t_{\max})$ such that we have the Cauchy surface

$$C_2 := F_2^{-1}(\{t_2\} \times C)$$

satisfying

$$C_2 \cap O_2 \neq \emptyset \quad \text{and} \quad C_2 \cap O_2^\perp \neq \emptyset.$$

We then use F_2 to define a Cauchy surface in M' associated with C_2,

$$C_2' := F_2(C_2), \tag{5.10}$$

as well as the regions

$$W_2 := I^+(C_2) \quad \text{and} \quad W_2' := (t_2, \infty) \times C\,. \tag{5.11}$$

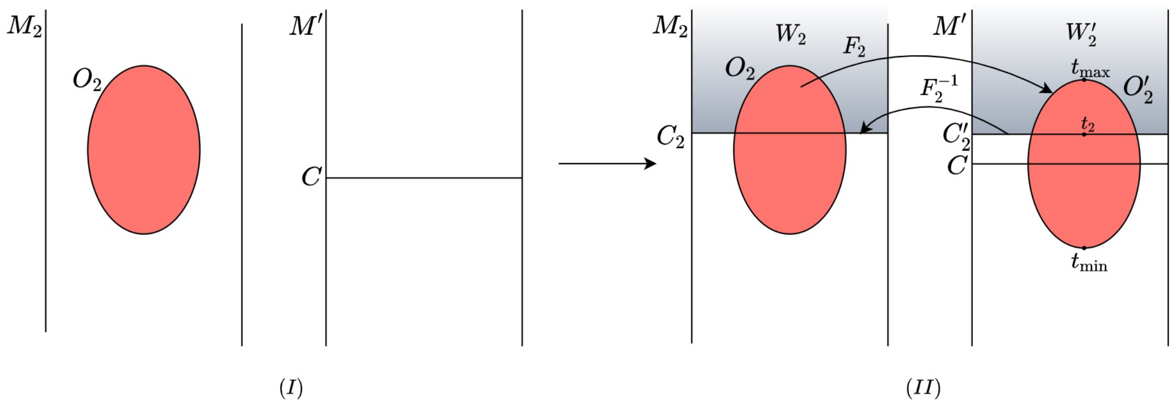

Figure 5.5: In (I) it is illustrated the assumptions we start with. In (II) we can see the Cauchy surface in M_2 and the region O_2' in M'.

Next, we use corollary 4.12 to state that $C_2 \cap J(\overline{O_2})$ is compact, which allows us to find relatively compact open sets $K, N \subset C$ so that we can define

$$K_2' := \{t_2\} \times K, \quad K_2 := F_2^{-1}(K_2'), \quad N_2' := \{t_2\} \times N \quad \text{and} \quad N_2 := F_2^{-1}(N_2'), \tag{5.12}$$

which satisfy $K \neq \emptyset$, $\overline{N} \neq C, \overline{K_2} \subset O_2$ and $C_2 \cap J(\overline{O_2}) \subset N_2$.

We can also define the Cauchy surface

$$C_{\max} := F_2^{-1}(\{t_{\max}\} \times C), \tag{5.13}$$

which is not a Cauchy surface for the whole manifold $\mathcal{M}_2$ and which must intersect $\overline{O_2}$. Then we can define the regions

$$U_2 := D(K_2) \cap I^+(K_2) \cap I_-(C_{\max}), \tag{5.14}$$

and

$$V_2 := D(N_2) \cap I^+(N_2) \cap I^-(C_{\max}), \tag{5.15}$$

which are guaranteed to be bounded cc-regions in M_2 by lemma 4.11. Furthermore, U_2 and V_2 satisfy

$$U_2, V_2 \subset W_2\ , \quad U_2 \subset D(O_2)\ , \quad O_2 \subset D(V_2) \quad \text{and} \quad V_2^{\perp} \neq \emptyset$$

and can be seen in figure 5.6.

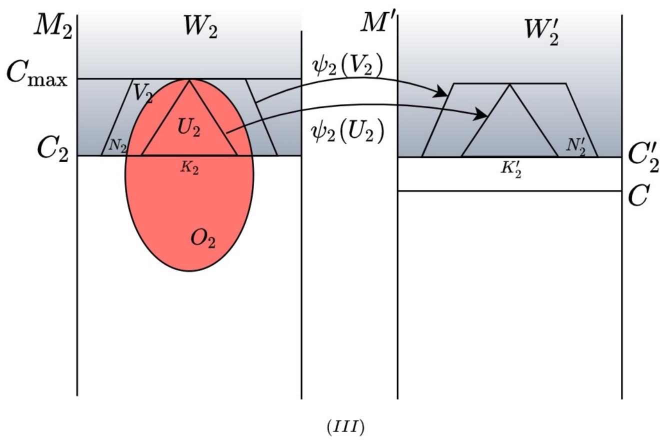

Figure 5.6: Illustration of the bounded regions U_2 and V_1 in M_2.

Derivation of the bounded cc-regions U_1 and V_1

Now we follow a similar method to derive the regions U_1 and V_1. We start by choosing $t_1 \in (t_{\min}, t_2)$ and then we define the Cauchy surfaces

$$C_1' := \{t_1\} \times C \quad \text{and} \quad C_1 := F_1^{-1}(C_1') \tag{5.16}$$

and the regions

$$W_1 := I^-(C_1) \quad \text{and} \quad W_1' := (-\infty, t_1) \times C. \tag{5.17}$$

Now, let $N', K' \subset C$ be relatively compact connected open sets satisfying

$$K' \neq \emptyset, \quad \overline{N'} \neq C, \quad \overline{K'} \subset K \quad \text{and} \quad \overline{N} \subset N'.$$

Then we can define

$$N_1' := \{t_1\} \times N', \quad K_1' := \{t_1\} \times K', \quad N_1 := F_1^{-1}(N_1'), \quad \text{and} \quad K_1 := F_1^{-1}(K_1'), \quad (5.18)$$

as well as the Cauchy surface

$$C_{\min} := F_1^{-1}(\{t_{\min}\} \times C). \tag{5.19}$$

Finally, we define the bounded regions (figure 5.7)

$$U_1 := D(K_1) \cap I^-(K_1) \cap I^+(C_{\min}) \tag{5.20}$$

and

$$V_1 := D(N_1) \cap I^-(N_1) \cap I^+(C_{\min}), \tag{5.21}$$

which are, again, causally convex due to lemma 4.11.

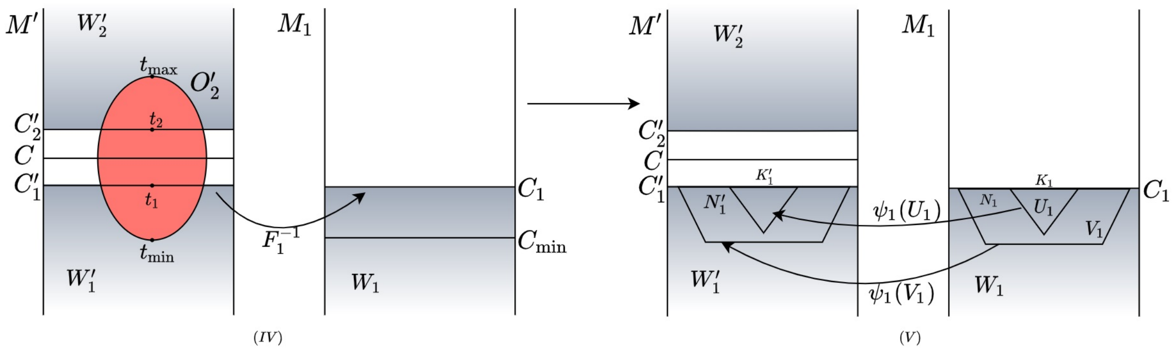

Figure 5.7: In (IV) we can see the Cauchy surface C_1 in M_1. In (V) we see the bounded regions U_1 and V_1.

We then choose the metric g' of the spacetime M' so that the metrics of W_i' coincide with those of W_i. That is, we take

$$((F_i)_* g_i)_{\mu\nu} = \beta_i dt_\mu dt_\nu - (h_i)_{\mu\nu} \tag{5.22}$$

in order to define

$$g'_{\mu\nu} := \beta dt_\mu dt_\nu - f \cdot (h_1)_{\mu\nu} - (1-f) \cdot (h_2)_{\mu\nu}, \tag{5.23}$$

where f is a smooth function on $\mathcal{M}'$ which has the values

$$f = \begin{cases} 1 & \text{on } W_1' \\ 0 & \text{on } W_2' \\ 0 < f < 1 & \text{on } (t_1, t_2) \times C \end{cases}, \tag{5.24}$$

and β is a positive smooth function so that

$$\beta = \begin{cases} \beta_1 & \text{on } W_1 \\ \beta_2 & \text{on } W_2 \end{cases}. \tag{5.25}$$

Therefore, the maps F_i restrict to isometric diffeomorphisms $\psi_i : W_i \to W_i'$ [4].

It is then argued that by choosing β small enough on $(t_1, t_2) \times C$, we can make $(\mathcal{M}', g')$ globally hyperbolic [4, 12].

Since $(t_1, t_2) \times N'$ is compact and $(h_i)_{\mu\nu}$ is continuous, by choosing β small enough on this set we guarantee that causal curves through $\overline{K'}_1$ must intersect K_2', and causal curves through $\overline{N'}_2$ must intersect N_1' [4]. As a result,

$$\overline{K'}_1 \subset D(K_2') \quad \text{and} \quad \overline{N'}_2 \subset D(N_1'),$$

therefore

$$\psi_1(U_1) \subset D(\psi_2(U_2)) \quad \text{and} \quad \psi_2(V_2) \subset D(\psi_1(V_1)), \tag{5.26}$$

as can be seen from figure 5.8.

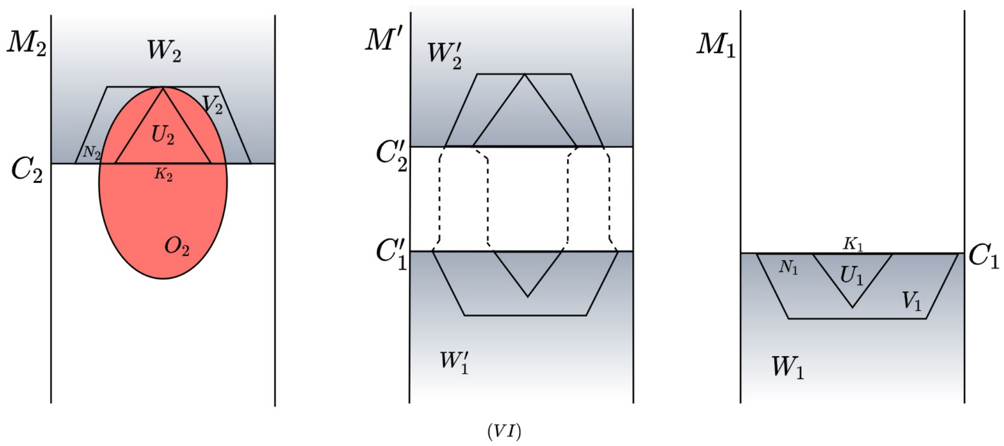

Figure 5.8: Final result of the deformation process.

□

Now that we have found a way to construct the bounded regions that we are interested in, we can consider the relations between the algebras of observables localized in such regions:

Proposition 5.7. *Consider a Locally Covariant Quantum Field Theory $\mathscr{A}$ with a state space* $\mathbf{S}$ *satisfying the time-slice axiom, and two globally hyperbolic spacetimes $M_i, i = 1, 2$ with diffeomorphic Cauchy surfaces.*

(i) *For any bounded cc-region $O_2 \subset \mathcal{M}_2$ with non-empty causal complement there are bounded cc-regions $U_1, V_1 \subset \mathcal{M}_1$ and a $*$-isomorphism $\alpha : \mathcal{A}_{M_2} \to \mathcal{A}_{M_1}$ such that $V_1^{\perp} \neq \emptyset$ and*

$$\mathcal{A}_{U_1} \subset \alpha(\mathcal{A}_{O_2}) \subset \mathcal{A}_{V_1}. \tag{5.27}$$

(ii) *Moreover, if the spacelike Cauchy surfaces of the M_i are non-compact and $P_2 \subset \mathcal{M}_2$ is any bounded cc-region, then there are bounded cc-regions $Q_2 \subset \mathcal{M}_2$ and $P_1, Q_1 \subset$*

$\mathcal{M}_1$ such that $Q_i \subset P_i^\perp$ for $i = 1, 2$ and

$$\alpha(\mathcal{A}_{P_2}) \subset \mathcal{A}_{P_1}, \quad \mathcal{A}_{Q_1} \subset \alpha(\mathcal{A}_{Q_2}), \tag{5.28}$$

where α is the same as in (i).

Proof. We will first prove statement (i), which is quite straightforward, and then we consider the situation of a bounded cc-region in M_2, as stated in (ii). In this case, the proof is illustrated in figures 5.9 and 5.10 and can be summarized in the following way:

1. Start from a bounded cc-region P_2 in M_2;
2. From P_2, obtain a bounded cc-region R in M_2;
3. From R, obtain a bounded cc-region R' in M';
4. From R', obtain a bounded cc-region P_1' in M';
5. From P_1', obtain a bounded cc-region P_1 in M_1.
6. Obtain regions Q_2, Q_2', Q_1', and Q_1 causally separated from the ones we have obtained before, since the Cauchy surfaces are not compact;
7. Take conclusions regarding the algebras.

Proof of statement (i)

Using proposition 5.6 we obtain the sets U_1, V_1 and isomorphisms $\psi_i : W_i \to W_i'$ in $\mathfrak{Man}$. We have seen that the ψ_i gives us $*$-isomorphisms α_{ψ_i} and

$$\alpha := \alpha_{\psi_1^{-1}} \circ \alpha_{\psi_2} : \mathcal{A}_{M_2} \to \mathcal{A}_{M_1}.$$

Using the properties of the U_i and V_i, we have

$$\mathcal{A}_{U_1} = \alpha_{\psi_1}^{-1}(\mathcal{A}_{U_1'}) \subset \alpha_{\psi_1}^{-1}(\mathcal{A}_{D(U_2')}),$$

since $\psi_1 : U_1 \to U_1'$ and $U_1' \subset D(\psi_2(U_2)) = D(U_2')$. Using lemma 4.27, we find

$$\mathcal{A}_{U_1} \subset \alpha_{\psi_1}^{-1}(\mathcal{A}_{U_2'}) = \alpha_{\psi_1}^{-1} \circ \alpha_{\psi_2}(\mathcal{A}_{U_2}) = \alpha(\mathcal{A}_{U_2}) \subset \alpha(\mathcal{A}_{O_2}) \subset \alpha(\mathcal{A}_{V_2}),$$

since $U_2 \subset O_2 \subset V_2$. But as

$$\alpha(\mathcal{A}_{V_2}) = \alpha_{\psi_1}^{-1}(\mathcal{A}_{V_2'}) \subset \alpha_{\psi_1}^{-1}(\mathcal{A}_{D(V_1')}) = \alpha_{\psi_1}^{-1}(\mathcal{A}_{V_1'}) = \mathcal{A}_{V_1},$$

we have that

$$\mathcal{A}_{U_1} \subset \alpha(\mathcal{A}_{O_2}) \subset \mathcal{A}_{V_1}, \tag{5.29}$$

as we intended to show.

Proof of statement (ii)

We now assume that the Cauchy surfaces are not compact, as required in (ii). Then we choose Cauchy surfaces $T_2, T_+ \subset W_2$ such that $T_+ \subset I^+(T_2)$.

It is clear that $J(\overline{P_2}) \cap T_2$ is compact, therefore, there is a relatively compact connected open neighbourhood $N_2 \subset T_2$. A proper choice of T_+ allows us to define

$$R := D(N_2) \cap I^+(N_2) \cap I^-(T_+).$$

Since R is the intersection of cc-regions, we invoke lemma 4.11 to claim that it is itself a cc-region. Then we are able to define a region R' (Figure 5.9) on the spacetime M' as

$$R' := \psi_2(R).$$

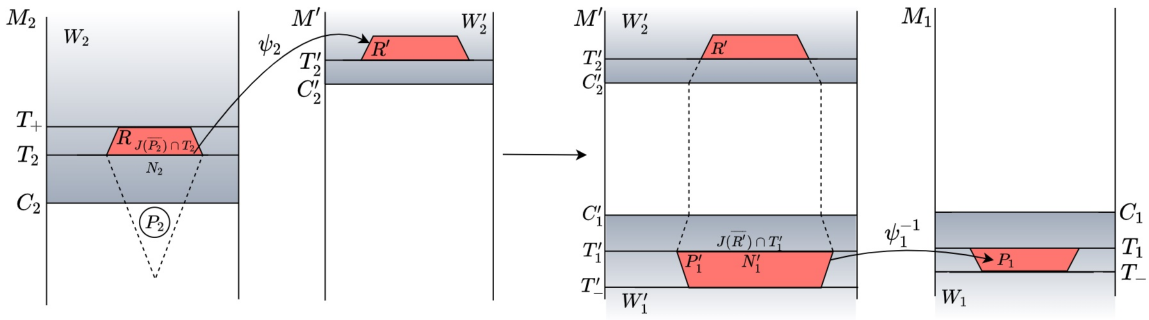

Figure 5.9: Steps of the deformation process I.

Now we bring our attention to M', where we let T'_-, $T'_1 \subset W'_1$ be Cauchy surfaces such that $T'_- \subset I^-(T'_1)$. Again, we have a compact set $J(\overline{R'}) \cap T'_1$, so that we also have a relatively compact connected open neighbourhood on M', namely, $N'_1 \subset T'_1$.

Once again, lemma 4.11 allows us to define a bounded cc-region (Figure 5.9)

$$P'_1 := D(N'_1) \cap I^-(N'_1) \cap I^+(T'_-),$$

and we also define

$$P_1 := \psi_1^{-1}(P'_1),$$

on M_1.

Since the Cauchy surfaces are not compact, there is a set $L'_1 \subset T'_1$, which is connected and relatively compact, such that $L'_1 \cap N'_1 = \emptyset$. Then, we define the bounded cc-regions

$$Q'_1 := D(L'_1) \cap I^-(L'_1) \cap I^+(T'_-),$$

and

$$Q_1 := \psi_1^{-1}(Q'_1), \quad Q_1 \subset P_1^{\perp}$$

in M' and M_1, respectively. We note that

$$Q'_1 \subset D(\psi_2(L_2)),$$

where $L_2 \subset T_2 \backslash N_2$ is a relatively compact open set, since $D(\psi_2(L_2))$ has a Cauchy surface for the region where Q'_1 is located.

Now we define the bounded cc-region Q_2 in M_2 as (See figure 5.10)

$$Q_2 := D(L_2) \cap I^+(L_2) \cap I^-(T_+),$$

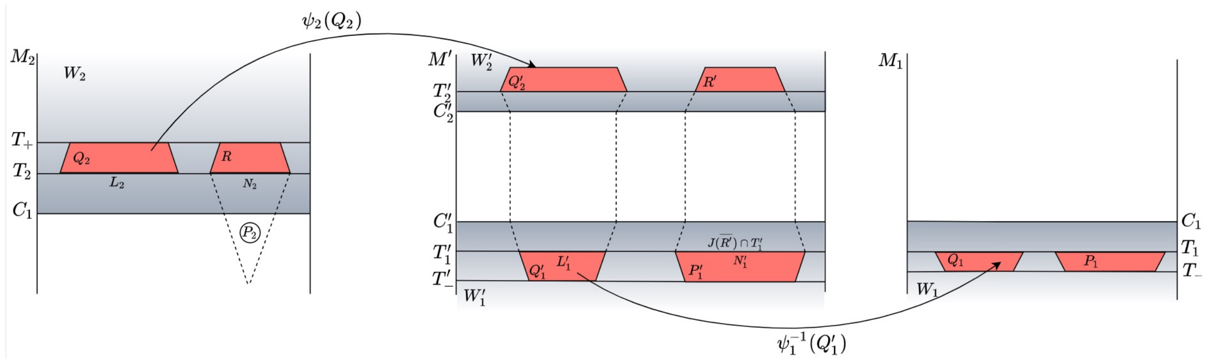

Figure 5.10: Steps of the deformation process II.

and the cc-region

$$Q_2' := \psi_2(Q_2)$$

in M' such that

$$Q_2 \subset P_2^{\perp} \quad \text{and} \quad Q_1' \subset D(Q_2').$$

We have finished the geometric part of the proof, so we can move on to the algebras for these cc-regions. Since R contains a Cauchy surface for the region which contains P_2, by lemma 4.27 we have that

$$\mathcal{A}_{P_2} \subset \mathcal{A}_R.$$

Besides,

$$\mathcal{A}_{R'} = \alpha_{\psi_2}(\mathcal{A}_R).$$

Using lemma 4.27 on $D(N_1')$ we see that

$$\mathcal{A}_{R'} \subset \mathcal{A}_{P_1'} \quad \text{and} \quad \mathcal{A}_{P_1} = \alpha_{\psi}^{-1}(\mathcal{A}_{P_1'}).$$

Bringing these results together, we get

$$\alpha(\mathcal{A}_{P_2}) \subset \alpha(\mathcal{A}_R) = \alpha_{\psi_1}^{-1}(\mathcal{A}_{R'}) \subset \alpha_{\psi_1}^{-1}(\mathcal{A}_{P_1'}) = \mathcal{A}_{P_1},$$

i.e., the inclusion of algebras

$$\alpha(\mathcal{A}_{P_2}) \subset \mathcal{A}_{P_1}.$$

Similarly,

$$\mathcal{A}_{Q_1} = \alpha_{\psi_1}^{-1}(\mathcal{A}_{Q_1'}),$$
$$\mathcal{A}_{Q_2'} = \alpha_{\psi_2}(\mathcal{A}_{Q_2}),$$

and $\mathcal{A}_{Q_1'} \subset \mathcal{A}_{Q_2'}$ by lemma 4.27. Then we find

$$\alpha(\mathcal{A}_{Q_2}) = \alpha_{\psi_1}^{-1} \circ \alpha_{\psi_2}(\mathcal{A}_{Q_2}) = \alpha_{\psi_1}^{-1}(\mathcal{A}_{Q_2'}) \supset \alpha_{\psi_1}^{-1}(\mathcal{A}_{Q_1'}) = \mathcal{A}_{Q_1},$$

i.e.,

$$\alpha(\mathcal{A}_{Q_2}) \supset \mathcal{A}_{Q_1},$$

which ends our proof.

□

5.2.2 Results in curved spacetimes

Here we show the conclusions we can take after using the spacetime deformation results in order to obtain a Reeh-Schlieder state in curved spacetime.

Theorem 5.8. *Consider a Locally Covariant Quantum Field Theory $\mathscr{A}$ with state space* $\mathbf{S}$ *which satisfies the time-slice axiom. Let M_i be two globally hyperbolic spacetimes with diffeomorphic Cauchy surfaces and suppose that $\omega_1 \in S_{M_1}$ is a Reeh-Schlieder state. Then given any bounded cc-region $O_2 \subset \mathcal{M}_2$ with non-empty causal complement, $O_2^\perp \neq \emptyset$, there is a *-isomorphism $\alpha : \mathcal{A}_{M_2} \to \mathcal{A}_{M_1}$ such that $\omega_2 = \alpha^*(\omega_1)$ has the Reeh-Schlieder property for O_2. Moreover, if the Cauchy surfaces of the M_i are non-compact and $P_2 \subset \mathcal{M}_2$ is a bounded cc-region, then there is a bounded cc-region $Q_2 \subset P_2^\perp$ for which ω_2 has the Reeh-Schlieder property.*

Proof. Let $U_1 \subset M_1$ be a bounded cc-region and let $\alpha : \mathcal{A}_{M_2} \to \mathcal{A}_{M_1}$ be a $*$-isomorphism such that $\mathcal{A}_{U_1} \subset \alpha(\mathcal{A}_{O_2})$. This α gives us a unitary map

$$U_\alpha : \mathcal{H}_{\omega_2} \to \mathcal{H}_{\omega_1},$$

and we know that GNS representations are unique as a unitary class of equivalence. So,

$$U_\alpha \Omega_{\omega_2} = \Omega_{\omega_1} \quad \text{and} \quad U_\alpha \pi_{\omega_2} U_\alpha^* = \pi_{\omega_1} \circ \alpha.$$

Since $\mathcal{A}_{U_1} \subset \alpha(\mathcal{A}_{O_2})$, we get $\alpha(\mathcal{A}_{O_2}) \supset \mathcal{A}_{U_1}$, and

$$\pi_{\omega_1} \circ \alpha(\mathcal{A}_{O_2}) \supset \pi_{\omega_1}(\mathcal{A}_{U_1}),$$

i.e.,

$$U_\alpha \pi_{\omega_2}(\mathcal{A}_{O_2}) U_\alpha^* \supset \pi_{\omega_1}(\mathcal{A}_{U_1}).$$

Hence,

$$\overline{U_\alpha^* U_\alpha \pi_{\omega_2}(\mathcal{A}_{O_2}) U_\alpha^* U_\alpha \Omega_{\omega_2}} \supset \overline{U_\alpha^* \pi_{\omega_1}(\mathcal{A}_{U_1}) U_\alpha \Omega_{\omega_2}}$$
$$\overline{\pi_{\omega_2}(\mathcal{A}_{O_2})\Omega_{\omega_2}} \supset \overline{U_\alpha^* \pi_{\omega_1}(\mathcal{A}_{U_1})\Omega_{\omega_1}} = U_\alpha^* \mathcal{H}_{\omega_1} = \mathcal{H}_{\omega_2}$$

Now we consider the situation of the bounded cc-regions P_2 in M_2. From proposition 5.7 we know that there are cc-regions Q_1 and Q_2, so we have

$$U_\alpha \pi_{\omega_2}(\mathcal{A}_{Q_2}) U_\alpha^* \supset \pi_{\omega_1}(\mathcal{A}_{Q_1}),$$

$$\overline{\pi_{\omega_2}(\mathcal{A}_{Q_2})\Omega_{\omega_2}} \supset \overline{U_\alpha^* \pi_{\omega_1}(\mathcal{A}_{Q_1})\Omega_{\omega_1}} = U_\alpha^* \mathcal{H}_{\omega_1} = \mathcal{H}_{\omega_2},$$

i.e.,

$$\overline{\pi_{\omega_2}(\mathcal{A}_{Q_2})\Omega_{\omega_2}} \supset \mathcal{H}_{\omega_2}. \tag{5.30}$$

□

Corollary 5.9. *In the situation of the previous theorem if $\mathscr{A}$ is causal then Ω_{ω_2} is a cyclic and separating vector for $\mathcal{R}_{O_2}^{\omega_2}$. If the Cauchy surfaces are non-compact Ω_{ω_2} is a separating vector for all $\mathcal{R}_{P_2}^{\omega_2}$, where P_2 is a bounded cc-region.*

Proof. For this proof, it is important the fact that a vector is separating for a von Neumann algebra $\mathcal{R}$ if and only if it is cyclic for its commutant, $\mathcal{R}'$. We consider a region V_1 (as given in the previous discussions). Then, the result by proposition 5.7 allows us to state that

$$U_\alpha \pi_{\omega_2}(\mathcal{A}_{O_2})U_\alpha^* \subset \pi_{\omega_1}(\mathcal{A}_{V_1}).$$

It is straightforward that

$$(U_\alpha \mathcal{R}_{O_2}^{\omega_2} U_\alpha^*)' \supset (\mathcal{R}_{V_1}^{\omega_1})'. \tag{5.31}$$

But $V_1^\perp \neq 0$, so $(\mathcal{R}_{V_1}^{\omega_1})'$ contains the local algebra for some cc-region for which Ω_{ω_1} is cyclic. This is also true for $(U_\alpha \mathcal{R}_{O_2}^{\omega_2} U_\alpha^*)'$, due to Equation 5.31. This means that $(\mathcal{R}_{O_2}^{\omega_2})'$ also contains the algebra for which Ω_{ω_1} is cyclic. So we can conclude that Ω_{ω_1} is separating for $\mathcal{R}_{V_1}^{\omega_1}$ and Ω_{ω_2} is separating for $\mathcal{R}_{O_2}^{\omega_2}$.

Now let us consider the situation in which the Cauchy surfaces are not compact. Let P_2 be bounded and $Q_2 \subset P_2^\perp$, as we have found in proposition 5.7. Then

$$(\mathcal{R}_{P_2}^{\omega_2})' \supset \pi_{\omega_2}(\mathcal{A}_{Q_2}).$$

But we can use theorem 5.8 to claim that Ω_{ω_2} is cyclic for $\pi_{\omega_2}(\mathcal{A}_{Q_2})$, which means that it must be separating for $\mathcal{R}_{P_2}^{\omega_2}$.

□

Corollary 5.9 allows for the use of Tomita-Takesaki modular theory, since the existence of a cyclic and separating vector on a von Neumann algebra is the minimum requirement for it [33]. Additionally, theorem 5.8 has applications regarding the type of von Neumann algebras, as we can see from the following corollary:

Corollary 5.10. *Consider a nowhere classical causal Locally Covariant Quantum Field Theory $\mathscr{A}$ with a locally quasi-equivalent state space* $\mathbf{S}$ *which satisfy the time-slice axiom. Let M_i be two globally hyperbolic spacetimes with diffeomorphic Cauchy surfaces and let $\omega_1 \in S_{M_1}$ be a Reeh-Schlieder state. Then for any state $\omega \in S_{M_i}$ and any cc-region $O \subset \mathcal{M}_i$ the local von Neumann algebra $\mathcal{R}_O^\omega$ is not finite.*

Sanders attempted to find full Reeh-Schlieder states (theorem 5.11) by considering a state space which is locally quasi-equivalent (See definition 4.24) and large enough. However, this large state space may contain singular states, and the existence of a state space of physical interest with the full property needs further investigation [4].

Theorem 5.11. *Consider a Locally Covariant Quantum Field Theory $\mathscr{A}$ with a locally-quasi-equivalent state space* $\mathbf{S}$ *which is causal and satisfies the time-slice axiom. Assume that S is maximal in the sense that for any state ω on some $\mathcal{A}_M$ which is locally quasi-equivalent to a state in S_M we have $\omega \in S_M$. Let $M_i, i = 1, 2,$ be two globally hyperbolic spacetimes with diffeomorphic non-compact Cauchy surfaces and assume that ω_1 is a Reeh-Schlieder state on M_1. Then S_{M_2} contains a (full) Reeh-Schlieder state.*

Chapter 6

Conclusions and perspectives

Locally Covariant Quantum Field Theory has been shown to be a very useful approach for studying Quantum Field Theory in curved spacetime. Its great power of generalization is in part due to the use of Category Theory, which has been (fortunately) gaining space in Physics recently. But it is also a general theory due to its foundation relying on Algebraic Quantum Field Theory, and the latter can in fact be recovered from it [3]. Furthermore, it seems to provide an appropriate framework for a semi-classical approximation to quantum gravitation. And in the same way, category theory provides a new perspective towards quantum theory and general relativity, since the categories Hilb and nCob have similar properties, which suggests a possible unification of these theories [15].

The main result we intended to portray is the one proven by Sanders [4, 5], which approached the problem of finding Reeh-Schlieder states in general globally hyperbolic spacetimes (which can be neither analytic nor stationary). The Reeh-Schlieder theorem has been known for a while as a fundamental aspect of Axiomatic Quantum Field Theory, and it makes a huge statement: States are naturally entangled in the vacuum sector. Therefore, it is a big deal to prove if it is valid in general curved spacetimes.

Using the spacetime deformation method along with the time-slice axiom, it was possible to 'transport' the Reeh-Schlieder property from Minkowski spacetime to a diffeomorphic globally hyperbolic curved spacetime. The power that spacetime deformation has in translating properties from one spacetime to the other leaves the question of which other properties of Quantum Field Theory can be transferred to curved spacetimes in this manner. Is is truly amazing that by considering certain geometrical properties of spacetimes we can later take such powerful conclusions regarding the algebras for these regions. This last part becomes almost trivial after we have finished the deformation process.

Even though the result by Sanders is limited, it still gives us important conclusions regarding the type of von Neumann algebras, as well as the possibility of using Tomita-Takesaki modular theory. While Hadamard states with the weak form of the Reeh-Schlieder property exist [4, 5], one of the remaining open questions is whether they also have the full form of property in a general curved spacetime.

We hope that the present work contributes in making Category Theory more present in Physics, since its abstract framework allows for connecting different areas, as well as provides a general way for defining certain properties that are common for different mathematical structures.

Appendix A

Minkowski spacetime and Poincaré covariance

In this chapter we will present the characteristics of a Minkowski spacetime and its symmetry under special relativity's group of transformations, namely, the Poincaré group. The use of this group in Quantum Field Theory in Minkowski spacetime is fundamental [32], since its irreducible unitary representations can be used to classify particles, which is due to Wigner [31].

Special relativity [78] regards time and space as equals in Minkowski spacetime, which is a four-dimensional space $\mathbb{M}$ that consists of points

$$x = x^{\mu} = (x^0, \mathbf{x}), \quad \mu = 0, 1, 2, 3, \tag{A.1}$$

where x^0 corresponds to the time dimension and $\mathbf{x}$ are the three-dimensional vectors associated with a spatial point.

If we take the speed of light to be $c = 1$, the scalar product between spacetime points x and y is given by

$$(x, y) = x^0 y^0 - \mathbf{x} \cdot \mathbf{y} = g_{\mu\nu} x^{\mu} y^{\nu}, \tag{A.2}$$

where the metric tensor is

$$g = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A.3}$$

and we assume Einstein's convention (the sum over repeated indices is implicit).

This allows us to classify the four-vectors x^{μ} in the following way:

(i) Timelike if $(x, x) > 0$. Positive and negative timelile if $x^0 > 0$ and $x^0 < 0$, respectively;

(ii) Lightlike if $(x, x) = 0$;

(*iii*) Spacelike if $(x, x) < 0$.

Then, the lightcone (figure 4.1) constitutes of the following sets of vectors:

1. Open future cone: $V_+ = \{x \in \mathbb{M}; (x, y) > 0, x^0 > 0\}$;
2. Closed future cone: $\overline{V}_+ = \{x \in \mathbb{M}; (x, y) \geq 0, x^0 > 0\}$;
3. Open past cone: $V_- = \{x \in \mathbb{M}; (x, y) > 0, x^0 < 0\}$;
4. Closed past cone: $\overline{V}_- = \{x \in \mathbb{M}; (x, y) \geq 0, x^0 < 0\}$;

Therefore, if two points x and y are spacelike separated, that is $(x, y) < 0$, they are not in each other's light cones and one event happening in x is entirely independent from an event happening in y. So, if we consider a a region $S \subset \mathbb{M}$, its causal complement is

$$S' \equiv \{x \in \mathbb{M}; (x, y) < 0 \quad \forall y \in S\}, \tag{A.4}$$

and we can say that observables measured in S commute with all the observables measured in S'.

A.1 The Poincaré group

In order to guarantee the equivalence of physical theories described by different observers, it is necessary to preserve the causal conditions illustrated by the light cone. This means that the quantities given in equation A.2 must be invariant under transformations of coordinates. The following theorem [7] determines the type of transformation that preserves the inner product:

Theorem A.1. *Let $\kappa : \mathbb{M} \to \mathbb{M}$ be a bijection. In order for κ to satisfy*

$$(x, y) \in \overline{V}_+ \Leftrightarrow (\kappa x, \kappa y) \in \overline{V}_+, \tag{A.5}$$

it is enough and sufficient that it is a linear transformation of the form

$$\kappa x = \lambda(\Lambda x + a), \quad (\Lambda x)^\mu = \sum_\nu \Lambda^\mu_\nu x^\nu, \tag{A.6}$$

where λ is a positive real number, $a \in \mathbb{M}$, and $\Lambda = (\Lambda^\mu_\nu)$ is a 4×4 matrix satisfying

$$\Lambda^T g \Lambda = g \quad \text{and} \quad \Lambda^0_0 > 0, \tag{A.7}$$

where Λ^T is the transposed matrix of Λ, i.e. $(\Lambda^T)^\mu_\nu = \Lambda^\nu_\mu$.

The linear transformation which satisfies equation A.7 is called a (homogeneous) Lorentz transformation, and the collection of these forms the *full (homogeneous) Lorentz group $\mathscr{L}$*. It is not of our interest to consider time and spatial inversions, so we restrict to the case of the proper orthocronous Lorentz group (or restricted Lorentz group), $\mathscr{L}_+^\uparrow$, for which $\det(\Lambda) = 1$ and $\Lambda_0^0 \geq 1$. Lorentz transformations can be regarded as 3-dimensional rotations with respect to a fixed axis. We can, for example, fix the time axis and rotate only the spatial coordinates, or fix one of the spatial axis and rotate the rest. However, another important symmetry of spacetime is not encoded in this, the invariance under translations. The Poincaré group, or inhomogeneous Lorentz group, accounts for this translation symmetry and is the collection of transformations of the form

$$x \to \Lambda x + a,$$

where $a \in \mathbb{M}$. Again, we restrict to the proper orthocronous Poincaré group, $\mathscr{P}_+^\uparrow$. If we consider only translations, then the elements of the Poincaré group of the type $U(a, \mathbf{1})$ will be the elements of a translation subgroup T.

Due to Wigner's theorem (See [7] p.45), there is a unique unitary representation of the Poincaré group $U(g)$, $g \in \mathscr{P}_+^\uparrow$, which acts on states and observables in the following way:

$$\Psi_{g\alpha} = U(g)\Psi_\alpha, \quad gQ = U(g)QU(g)^*, \tag{A.8}$$

where Ψ_α is a unitary ray corresponding to a pure state α, and Q is the operator associated with the observable. Furthermore, if we write an element of the Poincaré group as $g = (a, \Lambda)$, we write its unitary representation as $U(a, \Lambda)$.

Appendix B

Some preliminaries on General Relativity

General relativity [79] has a central role in this work, since we are studying properties of Quantum Field Theory in curved spacetimes. From the physical point of view, this theory relates the gravitation effect of matter with the curvature of spacetime, so it brings together the study of gravity with the study of matter, changing our perspective of acceleration as we know it.

The main notion that needs to be addressed here is that of spacetime, not only because it is essential in any study of General Relativity, but also because it will form the class whose elements will be the objects of the category of manifolds, which was used to define a Locally Covariant Quantum Field Theory.

For our study, we considered a spacetime defined as follows:

Definition B.1. The spacetime $M = (\mathcal{M}, g)$ is a smooth four-dimensional manifold with a Lorentzian metric g of signature $(+,-,-,-)$ and that is time and spatially oriented.

So, the main goal of this chapter is to present some notions which will hopefully clarify this definition and what we mean when we consider some regions of spacetime that were used throughout the text. The notions defined here are mainly based on [56].

B.1 Topological spaces

It is important to note that in general relativity a spacetime is nothing but a set that needs some additional properties in order for it to make sense as the "environment" where physical events occur. So, the first property that we must be able to define on this set is that of continuity, after all, the trajectory of a particle, which is represented by a curve, must be continuous. It turns out that the easiest way to talk about continuity is by defining a topology on this set, and a spacetime is indeed a topological space. Therefore, this section will be devoted to establishing some definitions and properties regarding topological spaces [56].

Definition B.2. A *topological space* $(X, \mathscr{T})$ consists of a set X together with a collection $\mathscr{T}$ of subsets O_α of X satisfying the following properties:

(i) If $O_\alpha \in \mathscr{T}$ for all α, then

$$\bigcup_\alpha O_\alpha \in \mathscr{T}.$$

(ii) If $O_1, ..., O_n \in \mathscr{T}$, then

$$\bigcap_{i=1}^{n} O_i \in \mathscr{T}.$$

(iii) $\emptyset, X \in \mathscr{T}$.

We say that $\mathscr{T}$ is a *topology* on X, and the subsets O_α are called *open sets.*

Definition B.3. Let $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ be topological spaces. A map $f : X \to Y$ is *continuous* if the inverse image $f^{-1}[O] \equiv \{x \in X | f(x) \in O\}$ of every open set O in Y is an open set in X.

Definition B.4. If f is continuous, one-to-one, onto, and its inverse is continuous, f is called a *homeomorphism* and $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ are said to be *homeomorphic.*

Definition B.5. A topological space is said to be

(i) *connected* if the only subsets which are both open and closed are the entire space X and the empty set $\emptyset$;

(ii) *Hausdorff* if for each pair of distinct points $p, q \in X$, $p \neq q$, one can find open sets $O_p, O_q \in \mathscr{T}$ such that $p \in O_p$, $q \in O_q$, and $O_p \cap O_q = \emptyset$.

Definition B.6. Let $(X, \mathscr{T})$ be a topological space and A a subset of X. Then

(i) A is *closed* if its complement $X - A \equiv \{x \in X | x \notin X\}$ is open;

(ii) the *closure*, $\bar{A}$, of A is defined as the intersection of all closed sets containing A. So, $\bar{A}$ is closed, contains A, and equals A if and only if A is closed;

(iii) the interior of A is the union of all open sets contained within A;

(iv) the *boundary* of A, denoted $\dot{A}$, consists of all points which lie in $\bar{A}$, but not in the interior of A;

(v) if $(X, \mathscr{T})$ is a topological space and A is a subset of X, a collection $\{O_\alpha\}$ of open sets is said to be an *open cover* of A if the union of these sets contains A;

(vi) a subcollection of the sets $\{O_\alpha\}$ which also covers A is referred to as a *subcover*;

(vii) the set A is *compact* if every open cover of A has a finite subcover.

From two topological spaces $(X_1, \mathscr{T}_1)$ and $(X_2, \mathscr{T}_2)$, we can turn the product space $X_1 \times X_2 \equiv \{(x_1, x_2)|x_1 \in X_1, x_2 \in X_2\}$ into a topological space $(X_1 \times X_2, \mathscr{T})$, where $\mathscr{T}$ is called the *product topology*, and consists of all subsets of $X_1 \times X_2$ which can be expressed as unions of sets of the form $O_1 \times O_2$ with $O_1 \in \mathscr{T}_1$ and $O_2 \in \mathscr{T}_2$. This allows for the definition of a topology on $\mathbb{R}^n$. Then, Tychonoff's theorem (See [56] p.425) tells us that if $(X_1, \mathscr{T}_1)$ and $(X_2, \mathscr{T}_2)$ are compact, so is the product space in the product topology. Furthermore, a subset A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Definition B.7. Let $(X, \mathscr{T})$ be a topological space and $\{O_\alpha\}$ an open cover of X.

(i) An open cover $\{V_\beta\}$ is said to be a refinement of $\{O_\alpha\}$ if for each V_β there exists an O_α such that $V_\beta \subset O_\alpha$.

(ii) The cover $\{V_\beta\}$ is said to be *locally finite* if each $x \in X$ has an open neighbourhood W such that only finitely many V_β satisfy $W \cap V_\beta \neq \emptyset$.

(iii) The topological space $(X, \mathscr{T})$ is said to be *paracompact* if every open cover $\{O_\alpha\}$ of X has a locally finite refinement $\{V_\beta\}$.

Finally, we aim to define the notion of net, which is a very important concept for us, since Algebraic Quantum Field Theory provides a description of the theory in terms of nets of local observables. We will first define a pre-order, a directed set and then a net in a general context [80].

Definition B.8. Let X be a non-empty set.

(i) A relation $R \subset X \times X$ is a *preorder* in X if the following conditions are met:

1. Reflexivity: for all $a \in X$, $(a, a) \in R$;
2. Transitivity: If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

We write $a \prec b$ for $(a, b) \in R$.

(ii) A set I is called a *directed set* if it is endowed with a preorder relation $\prec$, and if for any two elements a and b of I there is at least a third element $c \in I$ such that $a \prec c$ and $b \prec c$.

(iii) If M is a non-empty set, a function $f : I \to M$ is a *net* in M based on I with respect to $\prec$ (We can also say that f is a net in M).

(iv) A *sequence* in M is a net $f : \mathbb{N} \to M$ based in the directed set $\mathbb{N}$.

Nets generalize the notion of sequence, and in topological spaces they have an important role in the definition of convergence, in the same way that sequences act in metric spaces. Therefore, nets are very important in the study of continuous functions in general topological spaces, and are central in the framework of AQFT.

B.2 Manifolds

We are now able to define a topological manifold, which is a mathematical structure that is locally similar to Minkowski spacetime. This means that at each point of the manifold we are able to construct a vector space where we know how to make calculations. We present the following definition by Wald [56]:

Definition B.9. An *n-dimensional, C^∞, real manifold* $\mathcal{M}$ is a set with a collection of subsets $\{O_\alpha\}$ satisfying the following properties:

(*i*) Each $p \in \mathcal{M}$ lies in at least one O_α, i.e., the O_α cover $\mathcal{M}$.

(*ii*) For each α, there is a one-to-one, onto, map $\psi_\alpha : O_\alpha \to U_\alpha$, where U_α is an open subset of $\mathbb{R}^n$. These maps are called charts or coordinate systems.

(*iii*) If any two sets O_α and O_β overlap, $O_\alpha \cap O_\beta \neq \emptyset$, we can consider the map $\psi_\beta \circ \psi_\alpha^{-1}$ which takes points in $\psi_\alpha[O_\alpha \cap O_\beta] \subset U_\alpha \subset \mathbb{R}^n$ to points in $\psi_\beta[O_\alpha \cap O_\beta] \subset U_\beta \subset \mathbb{R}^n$ (See figure B.1). Furthermore, we require that this map is C^∞, i.e., infinitely continuously differentiable.

In addition, the cover $\{O_\alpha\}$ and chart family $\{\psi_\alpha\}$ are maximal, i.e., we consider all coordinate systems compatible with (*ii*) and (*iii*).

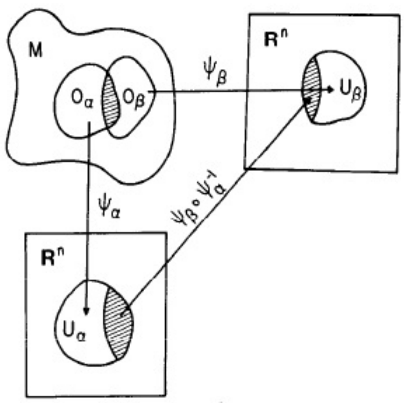

Figure B.1: Illustration of the map $\psi_\beta \circ \psi_\alpha^{-1}$. Available at [56].

Another definition of a manifold can be given where we start from a topological space and require that the charts respect certain properties, such as being continuous and having continuous inverses. However, the above definition still allows us to define a topology on $\mathcal{M}$ by demanding that all maps ψ_α are homeomorphisms. In this case, the topological manifolds that we are interested in are Hausdorff and paracompact.

We can also use two manifolds $\mathcal{M}$ and $\mathcal{M}'$ of dimension n and n' to construct the product space $\mathcal{M} \times \mathcal{M}'$, which consists of all pairs (p, p'), where $p \in \mathcal{M}$ and $p \in \mathcal{M}'$ (See [56] p.13). The resulting product space then has dimension $(n + n')$.

Definition B.10. Let $\mathcal{M}$ and $\mathcal{M}'$ be manifolds, $\{\psi_\alpha\}$ and $\{\psi'_\beta\}$ the chart maps, and let $f : \mathcal{M} \to \mathcal{M}'$. Then

(*i*) f is C^∞ (*infinitely continuously differentiable*) if for each α and β, the map $\psi'_\beta \circ f \circ \psi_\alpha^{-1}$ taking $U_\alpha \subset \mathbb{R}^n$ into $U'_\beta \subset \mathbb{R}^{n'}$ is C^∞;

(*ii*) if f is C^∞, one-to-one, onto, and has C^∞ inverse, then it is a *diffeomorphism* and $\mathcal{M}$ and $\mathcal{M}'$ are said to be *diffeomorphic*. This means that $\mathcal{M}$ and $\mathcal{M}'$ have identical manifold structure.

Note that when considering curved backgrounds we lose the vector space structure, but as mentioned before, we are able to recover it locally in the tangent space V_p, which is the collection of tangent vectors at p. These tangent vectors are defined as a directional derivatives, so that a vector $v = (v^1, ..., v^n)$ defines the directional derivative operator

$$\sum_\mu v^\mu \left(\frac{\partial}{\partial x^\mu}\right)$$

and vice versa [56].

Definition B.11. Let $\mathscr{F}$ be a collection of C^∞ functions from $\mathcal{M}$ into $\mathbb{R}$,

$$\mathscr{F} = \{f, f : \mathcal{M} \to \mathbb{R}\}.$$

A *tangent vector* v at point $p \in M$ is a map $v : \mathscr{F} \to \mathbb{R}$ which satisfies the following properties:

(*i*) $v(af + bg) = av(f) + bv(g)$, for all $f, g \in \mathscr{F}$ and $a, b \in \mathbb{R}$ (Linearity);

(*ii*) $v(fg) = f(p)v(g) + g(p)v(f)$ (Leibniz rule).

It turns out that every tangent space associated with a point $p \in \mathcal{M}$ has the same dimension of $\mathcal{M}$.

A *smooth curve*, C, on a manifold $\mathcal{M}$ is defined as a C^∞ map of $\mathbb{R}$ into M, i.e. '$C : \mathbb{R} \to \mathcal{M}$.

Next we will define tensors, which will allow us to present the definition of a metric. This is important since the existence of symmetry transformations between metrics, that is, isometries, allows us to relate spacetimes which have the same physical information.

Definition B.12. Let V be any finite-dimensional vector space over the real numbers and let V^* be the collection of linear maps $f : V \to \mathbb{R}$. Then we can construct a vector space structure on V^* by defining addition and scalar multiplication on these f. This vector space is then the *dual vector space* to V, and we call its elements *dual vectors*.

From the basis $v_1, ..., v_n$ of V, we obtain the *dual basis* $v^{1^*}, ..., v^{n^*}$ of V^* by

$$v^{\mu^*}(v_\nu) = \delta^\mu{}_\nu, \tag{B.1}$$

where $\delta^\mu{}_\nu = 1$ if $\mu = \nu$ and 0 otherwise. Furthermore, $\dim V^* = \dim V$.

Definition B.13. A *tensor* T of type (k, l) over V is a multilinear map

$$T : V^* \times ... \times V^* \times V \times ... \times V \to \mathbb{R}, \tag{B.2}$$

where k and l are the number of times V^* and V appear in B.2, respectively.

Definition B.14. At each point p, a *metric* g is a map $V_p \times V_p \to \mathbb{R}$, i.e., a tensor of type $(0, 2)$. Additionally, it must be:

(i) Symmetric: For all $v_1, v_2 \in V_p$ we have $g(v_1, v_2) = g(v_2, v_1)$;

(ii) Nondegenerate: If $g(v, v_1) = 0$ for all $v \in V_p$ then $v_1 = 0$.

Now, let $\mathcal{M}$ and $\mathcal{N}$ be manifolds and define a C^∞ map $\phi : \mathcal{M} \to \mathcal{N}$ which "pulls back" a function $f : \mathcal{N} \to \mathbb{R}$ to the functions

$$f \circ \phi : \mathcal{M} \to \mathbb{R}.$$

Additionally, ϕ takes tangent vectors at $p \in \mathcal{M}$ to tangent vectors at $\phi(p) \in \mathcal{N}$. This is done by defining a map $\phi^* : V_p \to V_{\phi(p)}$ in the following way: For $v \in V_p$ we define $\phi^* v \in V_{\phi(p)}$ as

$$(\phi^* v)(f) = v(f \circ \phi) \tag{B.3}$$

for all smooth $f : \mathcal{N} \to \mathbb{R}$.

In a similar way, we can use ϕ to "pull back" dual vectors at $\phi(p)$ to dual vectors at p. This is done by defining a map $\phi_* : V^*_{\phi(p)} \to V^*_p$, demanding that for all $v^a \in V_p$ we have

$$(\phi_* \mu)_a v^a = \mu_a (\phi^* v)^a, \tag{B.4}$$

where the μ_a are the dual vectors in $V^*_{\phi(p)}$.

Then, if $\phi : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, T is a tensor field on $\mathcal{M}$ and $\phi^* T = T$, then ϕ is a *symmetry transformation* for the tensor field T. In the case of the metric g_{ab}, this symmetry transformation is called an *isometry*.

Appendix C

The Reeh-Schlieder theorem in Wightman's approach to QFT

Both Wightman's approach to QFT and Algebraic Quantum Field Theory are part of what is called Axiomatic Quantum Field Theory. These frameworks are known for their mathematical rigour, but they present different points of views regarding Quantum Field Theory. On chapter 3 we have discussed how algebraic quantum field theory focuses on local nets of observables, which correspond to bounded self-adjoint operators acting on vectors of a Hilbert space $\mathcal{H}$. But here the emphasis is on the quantum fields, which are regarded as field operators smeared with test functions of compact support, and which may sometimes be unbounded.

In this chapter we'll make a brief exposure of the assumptions necessary to define a Quantum Field Theory according to Wightman, so that we can enunciate and sketch the proof of the Reeh-Schlieder theorem in this approach. On [69] the reader will find a rich discussion regarding distributions, which will be omitted here.

The test functions discussed here are elements of the set $\mathscr{S}$ of all infinitely differentiable functions f such that $\|f\|_{r,s} < \infty$ for all integers r, s.

We will now present Wightman axioms for a Quantum Field Theory [69]. Note that some of these assumptions regard the same physical content as some of the ones from Algebraic Quantum Field Theory, such as invariance under Poincaré transformation and commutativity of operators defined on causally separated regions. Furthermore, this is a theory that is only valid for Minkowski spacetime.

1. Assumptions of Relativistic Quantum Theory

- (i) The states of the theory are described by unit rays in a separable Hilbert space $\mathcal{H}$.
- (ii) The relativistic transformation law of the states is given as a continuous unitary representation $U(a, A)$ of the the group $\tilde{\mathscr{P}}_+^\uparrow$.
- (iii) There is a unique vacuum state Ψ_0 (up to a constant phase), which is an invariant

state, i.e.

$$U(a, A)\Psi_0 = \Psi_0. \tag{C.1}$$

2. Assumptions regarding the Domain and Continuity of the Field

(*i*) For each test function $f \in \mathscr{S}$ defined on spacetime, there is a set $\phi_1(f), ..., \phi_n(f)$ of operators, which together with their adjoints $\phi_1(f)*, ..., \phi_n(f)*$ are defined on a domain D of vectors, which is dense in $\mathcal{H}$.

(*ii*) D is a linear set containing the vacuum vector Ψ_0 and the following holds:

$$U(a, A)D \subset D \qquad \phi_j(f)D \subset D \qquad \phi_j(f) * D \subset D, \quad j = 1, 2, ...n \; .$$

(*iii*) If $\Phi, \Psi \in D$, then $(\Phi, \phi_j(f)\Psi)$ is a tempered distribution, taken as functional of f.

Furthermore, we take $D_0 \subset D$ to be the domain which contains the vectors obtained from the vacuum state by applying polynomials in the smeared fields.

3. Transformation Law of the Field

The transformation of the fields under the Poincaré group, given by

$$U(a, A)\phi_j(f)U(a, A)^{-1} = \sum S_{jk}(A^{-1})\phi_k(\{a, A\}f), \tag{C.2}$$

where S_{jk} is a matrix representation of $\tilde{\mathscr{P}}_+^\uparrow$ and

$$\{a, A\}f(x) = f(A^{-1}(x - a)), \tag{C.3}$$

is valid when each side is applied to any vector in D.

4. Local Commutativity

If the supports of f and g are causally separated, then one or the other of

$$[\phi_j(f), \phi_k(g)] \equiv \phi_j(f)\phi_k(g) \pm \phi_k(g)\phi_j(f) = 0 \tag{C.4}$$

holds for all j, k when the left-hand side is applied to any vector in D.

Furthermore, the spectrum of the energy-momentum operator P lies in the closed future cone $\overline{V^+}$.

These assumptions lead to the following definition of a Quantum Field Theory by Wightman [69]:

Definition C.1. A relativistic quantum theory satisfying axiom 1 with a field ϕ_j, $j = 1, ..., n$ satisfying axioms 2, 3 and 4, is a Field Theory if the vacuum state is cyclic for the smeared fields, i.e. if polynomials in the smeared field components $P(\phi_1(f), \phi_2(g), ...)$, when applied to the vacuum state, yield a set D_0 of vectors dense in the Hilbert space of states.

C.1 The Reeh-Schlieder theorem

Before stating the Reeh-Schlieder theorem, we will present a few results regarding analyticity properties of holomorphic functions in order to provide some tools for the proof of the Reeh-Schlieder theorem in the axiomatic approach to Quantum Field Theory. There is a wide discussion regarding such properties in [69] and it is not our intention to discuss them in detail. So, we will present now a version of the Edge of the wedge theorem for general regions of spacetime.

Theorem C.2. *Let $\mathcal{O}$ be an open set of $\mathbb{C}^n$ which contains a real environment, E, where E is some open set of $\mathbb{R}^n$. Let $\mathscr{C}$ be an open convex cone of $\mathbb{R}^n$. Suppose F_1 is a holomorphic function in*

$$D_1 = (\mathbb{R}^n + i\mathscr{C}) \cap \mathcal{O}$$

and F_2 in

$$D_2 = (\mathbb{R}^n - i\mathscr{C}) \cap \mathcal{O},$$

where $\mathbb{R}^n \pm i\mathscr{C}$ is the set of all vectors $x \pm iy$ where x and y are real and $y \in \mathscr{C}$. Suppose the limits

$$\lim_{y\to 0} F_1(x+iy) = F_1(x) \quad \textit{and} \quad \lim_{y\to 0} F_2(x-iy) = F_2(x),$$

for $x \in E$, exist and are equal in E. Then there is a complex neighbourhood N of E and a holomorphic function G which coincides with F_1 in D_1 and F_2 in D_2 and is holomorphic in N.

Theorem C.3. *Let $\mathcal{O}$ be an open set of $\mathbb{C}$ with a real environment E which is an open set of $\mathbb{R}^n$. Suppose F is a holomorphic function in $\mathscr{B} = (\mathbb{R}^n + i\mathscr{C}) \cap \mathcal{O}$. Furthermore, suppose*

$$\lim_{y\to 0} F(x+iy) = 0 \quad \text{for} \quad x \in E.$$

Then $F = 0$ throughout $\mathscr{B}$.

Now we state the Reeh-Schlieder theorem as presented in [69].

Reeh-Schlieder Theorem: Suppose O is an open set of space-time. Then Ψ_0 is a cyclic vector for $\mathscr{P}(O)$, if it is a cyclic vector for $\mathscr{P}(\mathbb{R}^4)$. That is, vectors of the form

$$\sum_{j=0}^{N} \phi(f_1^{(j)})...\phi(f_j^{(j)})\Psi_0 \tag{C.5}$$

with $\mathrm{supp} f_j^{(k)} \subset O$ are dense in $\mathcal{H}$.

Here, $\mathscr{P}(O)$ is the set of all polynomials of the form

$$c + \sum_{j=1}^{N} \phi(f_1^{(j)})...\phi(f_j^{(j)}) \tag{C.6}$$

where f_k^j are the test functions whose support is in the open set O of space-time, and c is any complex constant.

The proof of the Reeh-Schlieder theorem in this approach is done by showing that a vector Ψ orthogonal to all the vectors of the form of equation C.5 must also be orthogonal to all the polynomials in the smeared field which do not have a condition in the support of the test functions. In other words, it can be shown that if Ψ is orthogonal to

$$\mathscr{P}(\mathbf{R}^4)\Psi_0 = D_0, \tag{C.7}$$

then $\Psi = 0$, since D_0 spans the Hilbert space.

By noting that

$$(\Psi, \phi(f_1)...\phi(f_n)\Psi_0) \tag{C.8}$$

is a separately continuous multilinear functional of the test functions $f_1...f_n$, it is possible to use Schwartz Theorem (See [69]) to extend it to a tempered distribution in all the variables, which we define as

$$F(-x_1, x_1 - x_2, ...x_{n-1} - x_n) = (\Psi, \phi(x_1)...\phi(x_n)\Psi_0). \tag{C.9}$$

It is then argued [69] that F can be shown to be zero by taking into consideration that its Fourier transform goes to zero unless each variable of the four-momentum is in the physical spectrum.

So there is a function $\mathbf{F}$, which is holomorphic in the tube $\mathscr{T}_n$ in the variables

$$(-x_1) - i\eta_1, (x_1 - x_2) - i\eta_2, ...(x_{n-1} - x_n) - i\eta_n,$$

whose boundary value is F as $\eta_1...\eta_n \to 0$ in V_+ due to theorem C.2. Here $\mathscr{T}_n$ is the tube $\mathbb{R} - i\Gamma$, where $\Gamma = (a_1, ..., a_n)$ with $a_j \in V_+, j = 1, ...n$.

Now, the boundary value F goes to zero for $-x_1, x_1 - x_2, ..., x_{n-1} - x_n$ in an open set determined by $x_1...x_n \in O$, and due to theorem C.3 so does $\mathbf{F}$. Then, from equation C.9 we can see that Ψ is orthogonal to D_0, which means that Ψ should be zero, since D_0 spans $\mathcal{H}$.

As expected, we can conclude that from the vacuum state Ψ_0 it is possible to obtain any arbitrary state in $\mathcal{H}$.

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