UNIVERSIDADE DE SÃO PAULO
INSTITUTO DE FÍSICA

QUANTIZAÇÃO DA CORDA BOSÔNICA

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QUANTISATION OF THE BOSONIC STRING

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I dedicate this dissertation to my beloved
mother Marina and *бабушка* Elmira

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Resumo

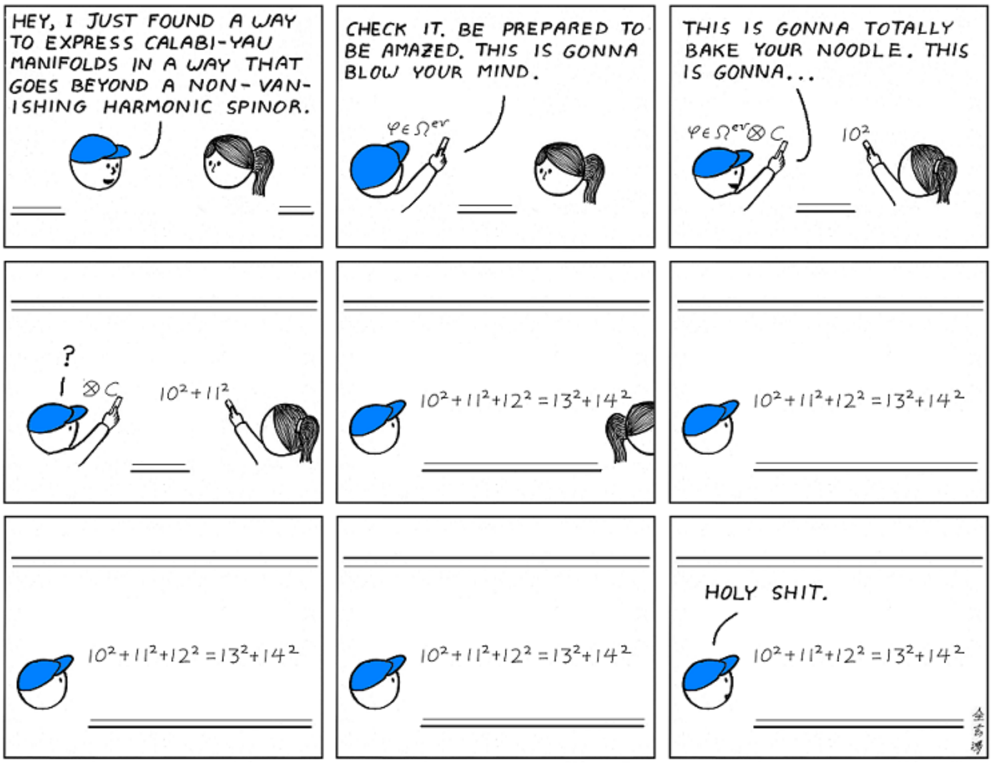
Neste trabalho fazemos uma revisão dos princípios básicos da teoria da corda bosônica relativística através do estudo dos funcionais ação de Nambu-Goto e de Polyakov e das técnicas necessárias para sua quantização canônica, no cone de luz e usando integrais de trajetória. Para tanto apresentamos uma pequena revisão das principais propriedades das simetrias de calibre a da teoria de campos conforme envolvidas nas técnicas estudadas.

Palavras-chave: Teoria de cordas; teoria de campos conforme; quantização BRST.

Abstract

In this work we review the basic principles of the theory of the relativistic bosonic string through the study of the action functionals of Nambu-Goto and Polyakov and the techniques required for their canonical, light-cone, and path-integral quantisation. For this purpose, we briefly review the main properties of the gauge symmetries and conformal field theory involved in the techniques studied.

Keywords: String theory; conformal field theory; BRST quantisation.



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1

Introduction

1.1 Historical remarks

String theory arose in the 1960s as a by-product of attempts at describing strong nuclear interactions. In 1968, G. Veneziano found a formula for the scattering amplitude of strongly interacting mesons that was at the same time simple and intruiguing [1]. A little after, Y. Nambu, H. B. Nielsen and L. Susskind realised, from different perspectives, that the dynamical object from which the Veneziano formula can be derived is a relativistic string—an extended one-dimensional object moving through spacetime sweeping a two-dimensional surface called the string worldsheet. Strings can be of two types: open (interval topology) or closed (circle topology). In 1971, P. Ramond (for fermions) and A. Neveu and J. H. Schwarz (for bosons) introduced what would become known as the Neveu-Ramond-Schwartz superstring.

Soon after the introduction of the string idea, it was realised that the theory sported spin-two massless particles that do not belong in the hadronic spectrum. That was a big flop for the budding theory. In 1974, however, J. Scherk and J. H. Schwarz came up with the idea that this spin-two particle could be interpreted as a graviton, the gauge boson of gravitation. This suggested, in turn, that string theory might be the key to the unification of fundamental forces at the quantum level by describing (all?) existing particles and their interactions as different vibrational modes of a single string (see figure 1.1).

So far, we believe that all interactions in our universe are mediated by four

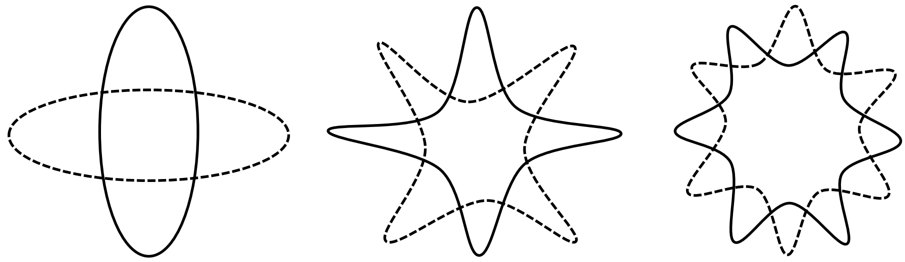

Figure 1.1: Different particles as different vibrational modes of a string

fundamental forces: the electromagnetic, weak, strong, and gravitational forces. The unification of the first three forces was achieved during the 1960s and 1970s by means of the "new" gauge field theory of G. 't Hooft and fellow travelers,(1) but the unification of all four forces continues to be a major conundrum of contemporary physics that string theory may help to solve. The string theory approach to unify all the fundamental forces and particles requires:

(*i*) The existence of gravity: gravitons appear in the closed string spectrum, and since any open string theory becomes a closed string theory when we connect the endpoints of the string, the presence of gravity in the theory is unavoidable;

(*ii*) The existence of the standard model: the Yang-Mills theory based on the $SU(3) \otimes SU(2) \otimes U(1)$ gauge group (that is, on quarks, leptons and gauge bosons) is, barring the dark sides of the Force, reasonably successful in describing the material contents of the universe;

(*iii*) Supersymmetry: introduced by J. Wess and B. Zumino in 1974 [2], supersymmetry relates the bosons and the fermions of the theory and is required to make the theory mathematically sound, but there is no proof that supersymmetries are realised in nature and there is no working theory that currently

(1)The saga that led to the proper renormalisation of gauge field theories, its *prima donnas* and unsung heroes is nicely told in F. Close, *The Infinity Puzzle* (New York: Perseus Books, 2011).

requires supersymmetry. There is, however, some hope that one day supersymmetry will show up in new phenomena or that we will need supersymmetry on the theoretical side of some problem;

(*iv*) Existence of extra dimensions: modern string theory predicts that spacetime must be 10 or 11-dimensional. Another possibility is provided by the bosonic string theory in 26 dimensions (critical dimension), but in this case the mass of the string squared is found to be negative, making the theory unstable;

(*v*) Compactification of extra dimentions: we can argue, by dimensional analysis, that the size of the string should be of the order of the Plank length

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-33}\,\text{cm},$$

and, accordingly, that its mass should be of the order of the Plank mass

$$m_P = \sqrt{\frac{\hbar c}{G}} = 1.2 \times 10^{19}\,\text{GeV}/c^2.$$

To detect an object of size ℓ_P we need energies like m_P, and this is why we cannot possibly hope to observe strings in the wild and, most likely, will never do it by any direct method. The theory must then provide means to compactify some extra tiny dimensions into non-observable corners of the spacetime at lower energies.

String theory was almost abandoned after the discovery that gravitational and gauge anomalies plagued the theory. But in 1984, M. Green and J. H. Schwarz showed that the open superstring is free of anomalies if the gauge group of the theory is $SO(32)$ [3]. Ten-dimensional supersymmetric Einstein-Yang-Mills field theory is also anomaly free with the $E_8 \times E_8$ gauge group. This group appeared first in string theory and later was used to formulate the theory of the heterotic string, a hybrid beast part 26-dimensional bosonic string, part 10-dimensional superstring advanced by D. Gross, J. Harvey, E. Martinec and R. Rohm in 1985 [4].

By 1994, the string theory programme counted on five different 10-dimensional contenders: the type I theory, two heterotic theories (the $SO(32)$ and the $E_8 \times E_8$ theories), and the types IIA and IIB theories. In 1995, E. Witten proposed a possible "unification" of these theories by means of dualities [5]. The new formulated theory lives in 11 dimensions, from which the former five 10-dimensional theories are recovered as different limits. The theory is called M-theory, although nobody is sure about what the "M" in its name stands for.(2). Later J. Polchinski found that the theories can bear additional degrees of freedom for the strings, effectively turning them into D-branes; strings are then just 1-branes.

The most significant recent development in the string theory programme came in 1998, when J. Maldacena formulated the AdS/CFT gauge-gravity duality conjecture [6, 7]. According to this conjecture, there is an holographic duality relating two theories with different dimensionalities: in the one side a 4-dimensional gauge theory and in the other side a 5-dimensional anti-de Sitter spacetime. Many have argued that the AdS/CFT correspondence is the most important development in all of theoretical physics since the establishment of general relativity in the beginning of the XX century.

Any further account of the string programme would require much deeper and detailed considerations that are way beyond our current degree of comprehension or interest, so we stop our historical perspective here and refer the reader to the literature for additional enlightening.(3)

(2) E. Witten refuses to tell what he had in mind when he coined the moniker; maybe the "M" is just an inverted "W," maybe it stands for "mother," maybe for "master"—maybe for "moniker!"

(3) At a nontechnical level, the books by B. Greene, *The Elegant Universe: Superstrings, Hidden Dimensions, and the Quest for the Ultimate Theory* (New York: W. W. Norton, 2003) and L. Randall, *Warped Passages: Unraveling the Mysteries of the Universe's Hidden Dimensions* (New York: Harper, 2006) provide good panoramas of the post "second revolution" status of string theory.

1.2 This dissertation

This dissertation summarises our modest attempt at understanding the basic ideas that build up the string theory approach to high-energy physics. Basically, we study how the bosonic string can be described and quantised in the most elementary terms.

The work is organised as follows. In Chapter 2, after briefly reviewing the dynamics of relativistic point particles, we introduce the Nambu-Goto action for the classical string and investigate its symmetries and the ensuing equations of motion. In the same chapter we also introduce the string sigma model action, analyse its symmetries, and introduce and solve the respective equations of motion. In Chapter 3 we present the covariant quantisation of the closed string, introduce the Virasoro algebra and the constraints for the closed string. Next comes the light-cone quantisation of the bosonic string. At the end of Chapter 3 we derive the spectrum of the closed bosonic string. Chapter 4 is reserved for an introduction to the conformal field theory (CFT) aspects and the Virasoro algebra implied in the study of the bosonic string. There we discuss the stress-energy tensor, Noether currents, operator product expansions, the Ward identities and the primary operators in the context of CFT. The Faddeev-Popov method for finding the Jacobian for the path integral of the Polyakov action is described in Chapter 5, where some aspects of the ghost's CFT and Virasoro algebra are also described and where we also finally introduce the BRST operators and discuss BRST symmetry for the bosonic string. Chapter 6 summarises our presentation, elaborate on further possible studies and conclude the work. A couple of technical derivations were relegated to appendices, four in total.

2

Relativistic string action

In this chapter we discuss the motion of a relativistic free point particle. We introduce the action and give some general aspects of its quantisation. We then introduce the Nambu-Goto and the Polyakov actions of the string. We consider for both actions basic aspects like symmetries and equations of motion. For the Polyakov action, that we will quantise later, we provide a more detailed review.

General references for the material on this chapter are the introductory books on string theory by Zwiebach [8] and by Becker, Becker and Schwartz [9], as well as the more advanced presentation by Green, Schwartz, and Witten [10]. The lecture notes by Tong [11] and Arutyunov [12] are also helpful.

2.1 Point particle action

In this section we will consider the motion of a free relativistic point particle in D-dimensional Minkowski spacetime. Its motion can be described by means of its action functional. The classical point particle moves along geodesics, which means that the action has to be proportional to the invariant length of the trajectory of the particle,

$$S_0 = -\alpha \int ds, \tag{2.1}$$

where α is a constant and s is the metric element. In a frame with fixed coordinates $X^\mu = (t, \vec{x})$ the action can be written as

$$S_0 = -\alpha \int \sqrt{dt^2 - d\vec{x}^2}.$$

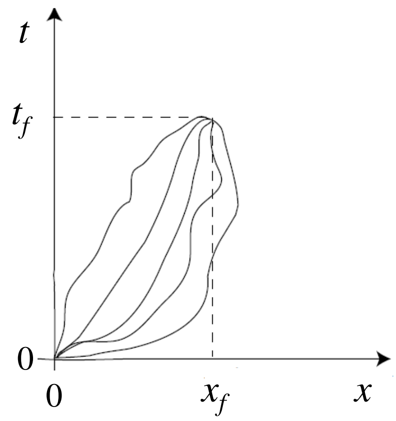

Figure 2.1: Possible spacetime trajectories for a free point particle between two fixed endpoints, where $(0,0)$ is the initial point and (x_f, t_f) is the final one.

If we define $\vec{v} = d\vec{x}/dt$, and go to the non-relativistic limit, action (2.1) becomes

$$S_0 = -\alpha \int dt \sqrt{1-\vec{v}^2} \approx -\alpha \int dt (1 - \frac{1}{2}\vec{v}^2 + \cdots).$$

Comparing with the action for a non-relativistic point particle, namely,

$$S_{non} = \int dt \frac{1}{2} m \vec{v}^2,$$

we can conclude that $\alpha = m$, where m is the mass of the point particle. In this way, the action for the free relativistic point particle can be written as

$$S_0 = -m \int ds. \tag{2.2}$$

The action is minimised for the classical trajectory, see figure 2.1 for the two dimensional case.

Let the metric element in D-dimensional flat spacetime be given by

$$ds^2 = -\eta_{\mu\nu}(X) dX^\mu dX^\nu,$$

where X^μ and X^ν are the coordinates of the D-dimensional spacetime and $\eta_{\mu\nu}$ describes the geometry of the problem, with $\mu, \nu = 0,\, 1,\, \cdots,\, D-1$. With the usual Minkowski signature $(-+\cdots+)$, we can write the action of the point particle as

$$S_0 = -m \int \sqrt{-\eta_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu}\, d\tau, \tag{2.3}$$

where we introduced the parameter τ, which gives the position along the world line of the particle, and the dot denotes derivative with respect to τ. The world line of a particle is the path of the particle in spacetime, tracing the history of its location in space at each instant in time.

Looking at the action (2.3) we could think that our physical system has D degrees of freedom, because the time direction is one of our dinamical variables, but this is not true. This is due to the τ reparameterisation invariance of the action, which is a gauge symmetry of the system. When we go to a new parametrisation of the world line $\tau \to \tilde{\tau} = \tilde{\tau}(\tau)$, we have

$$d\tau = d\tilde{\tau} \left| \frac{d\tau}{d\tilde{\tau}} \right| \quad \text{and} \quad \frac{dX^\mu}{d\tau} = \frac{dX^\mu}{d\tilde{\tau}} \frac{d\tilde{\tau}}{d\tau}. \tag{2.4}$$

Under these transformations the action becomes

$$S_0 = -m \int \left| \frac{d\tau}{d\tilde{\tau}} \right| d\tilde{\tau} \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\tilde{\tau}} \frac{d\tilde{\tau}}{d\tau} \frac{dX^\nu}{d\tilde{\tau}} \frac{d\tilde{\tau}}{d\tau}} = -m \int \sqrt{-\eta_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu}\, d\tilde{\tau}, \tag{2.5}$$

which has the same form as action (2.3). The consequence of this is that we actually have only $D-1$ degrees of freedom. For example, we find all X^μ in terms of τ, and then by using the reparameterisation invariance set

$$\tau = X^0(\tau) \equiv t.$$

If we plug this choice into action (2.3) we find

$$S_0 = -m \int dt \sqrt{1-\vec{v}^2}, \tag{2.6}$$

which has $D-1$ derees of freedom. So one of initial degrees of freedom is fake. We can check this also by computing the momenta

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{m\dot{X}^\nu \eta_{\mu\nu}}{\sqrt{-\eta_{\sigma\rho}(X)\dot{X}^\sigma \dot{X}^\rho}}, \tag{2.7}$$

Where L is the Lagrangian of the system. These momenta are not independent, since they satisfy the mass-shell constraint

$$m^2 + p_\mu p^\mu = 0 \tag{2.8}$$

for a relativistic particle with mass m. The mass-shell constraint tells us that in a Minkowski spacetime particles are always moving, at least in the time direction, with $(p^0)^2 \geqslant m^2$.

2.1.1 About quantisation

Action (2.3) contains a square root, which makes it difficult to quantise by the path integral approach. Also, it cannot describe the motion of massless particles. To avoid these problems, we introduce another action for the relativistic point particle which is equivalent to the former one at the classical level. First we introduce a new field $e(\tau)$ (as in [9]) on the world line and write the new action as

$$\tilde{S}_0 = \frac{1}{2}\int d\tau (e^{-1}\dot{X}^2 - m^2 e), \tag{2.9}$$

where $\dot{X}^2 = \eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu$. This action should again be invariant under τ reparametrisation. We can see this by considering an infinitesimal change of reparameterisation

$$\tau \to \tau' = f(\tau) = \tau - \varepsilon(\tau),$$

where $\varepsilon(\tau) \ll 1$ is an infinitesimal parameter. Under this reparameterisation the fields X^μ transform as a world line scalars, $X^{\mu'}(\tau') = X^\mu(\tau)$, so that to first order

$$\delta X^\mu = X^{\mu'}(\tau) - X^\mu(\tau) = \varepsilon(\tau)\dot{X}^\mu.$$

The additional field $e(\tau)$ should transform as $e'(\tau')d\tau' = e(\tau)d\tau$. Infinitesimally,

$$\delta e = e'(\tau) - e(\tau) = \frac{d}{d\tau}(\varepsilon e).$$

Now we vary the action (2.9) to obtain

$$\delta \tilde{S}_0 = \frac{1}{2}\int d\tau \left(\frac{2\dot{X}^\mu \delta \dot{X}_\mu}{e} - \frac{\dot{X}^\mu \dot{X}_\mu}{e^2}\delta e - m^2 \delta e \right) \tag{2.10}$$

Using the expressions for δe and δX^μ we find that

$$\delta \tilde{S}_0 = \frac{1}{2}\int d\tau \left(\frac{2\dot{X}^\mu}{e}(\dot{\varepsilon}\dot{X}_\mu + \varepsilon \ddot{X}_\mu) - \frac{\dot{X}^\mu \dot{X}_\mu}{e^2}(\dot{\varepsilon}e + \varepsilon \dot{e}) - m^2 \frac{d(\varepsilon e)}{d\tau} \right) \tag{2.11}$$

The last term drops out because it is a total derivative. The rest can be written as

$$\delta \tilde{S}_0 = \frac{1}{2}\int d\tau \frac{d}{d\tau}\left(\frac{\varepsilon}{e}\dot{X}^\mu \dot{X}_\mu \right), \tag{2.12}$$

which is also a total derivative. So $\delta \tilde{S}_0 = 0$ and $\tilde{S}_0$ is invariant under τ reparameterisations. The equation of motion for $e(\tau)$ follows from the principle of least action

$$\frac{\delta \tilde{S}_0}{\delta e} = -\frac{1}{2}(e^{-2}\dot{X}^\mu \dot{X}_\mu + m^2) = 0, \tag{2.13}$$

or

$$\dot{X}^\mu \dot{X}_\mu + e^2 m^2 = 0. \tag{2.14}$$

It may seem that our new action has one additional degree of freedom compared to the initial action, but this is not true, since $e(\tau)$ is completely fixed by its equation of motion. If we find $e(\tau)$ from (2.14) and plug it back into (2.9) we recover the action functional (2.3). Now we notice that $\tilde{S}_0$ can be rewritten as

$$\tilde{S}_0 = \frac{1}{2}\int \left(\eta_{\mu\nu}(X)dX^\mu \frac{dX^\nu}{e(\tau)d\tau} - m^2 e(\tau) \right) d\tau. \tag{2.15}$$

Since $e(\tau)$ always appears multiplied by $d\tau$ in the above expression and the action is τ-reparameterisation invariant, we can absorb $e(\tau)$ into $d\tau$, which is tantamount to choosing $e(\tau) = 1$ from the beginning, for this choice is largely irrelevant for the physics of the problem. Reparametrisation invariance thus allows us to choose a gauge with $e(\tau) = 1$, which gives

$$\dot{X}^\mu \dot{X}_\mu + m^2 = 0. \tag{2.16}$$

Identifying $\dot{X}^\mu = p^\mu$, the momentum conjugated to X^μ, we see that this equation is in fact just the mass-shell constraint $p^2 + m^2 = 0$. The variation of $\tilde{S}_0$ with respect to X^μ gives a second set of equations of motion,

$$\begin{aligned} \frac{\delta \tilde{S}_0}{\delta X^\mu} &= -\frac{d}{d\tau}(\eta_{\mu\nu}\dot{X}^\nu) + \frac{1}{2}\partial_\mu \eta_{\rho\lambda}\dot{X}^\rho \dot{X}^\lambda \\ &= -(\partial_\rho \eta_{\mu\nu})\dot{X}^\rho \dot{X}^\nu - \eta_{\mu\nu}\ddot{X}^\nu + \frac{1}{2}\partial_\mu \eta_{\rho\lambda}\dot{X}^\rho \dot{X}^\lambda. \end{aligned} \tag{2.17}$$

We can bring this equation to the form

$$\ddot{X}^\mu + \Gamma^\mu_{\rho\lambda}\dot{X}^\rho \dot{X}^\lambda = 0, \tag{2.18}$$

where

$$\Gamma^\mu_{\rho\lambda} = \frac{1}{2}\eta^{\mu\nu}\left(\partial_\rho \eta_{\lambda\nu} + \partial_\lambda \eta_{\rho\nu} - \partial_\nu \eta_{\rho\lambda}\right) \tag{2.19}$$

is the Levi-Civita connection. Equation (2.18) is the equation for the geodesics on the spacetime described by $g_{\mu\nu}$, or, equivalently, Newton's equation for a free particle in curved spacetime. For flat spacetime, $\Gamma^\mu_{\rho\lambda}$ vanishes and we recover the usual equation of motion for a pointlike free particle.

2.2 The Nambu-Goto action

Another important action is the Nambu-Goto string action, which describes the relativistic theory but, as we will see later, is not suited for the path integral quantisation of string theory.

Consider the motion of a relativistic string propagating in a D-dimensional Minkowski spacetime. While moving, the string draws a two dimensional surface called its worldsheet. We parametrise the worldsheet by the time-like coordinate $\sigma^0 = \tau$ and the space-like coordinate $\sigma^1 = \sigma$. Depending on σ, the string will be closed (periodic σ) or open (σ covering a finite interval). The action for the motion of the relativistic string is then taken to be proportional to the proper area of its worldsheet and is given by

$$S_{NG} = -T \int d^2\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}, \tag{2.20}$$

where T is the tension of the string (with dimension of mass per length) and

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu = \frac{\partial X^\mu}{\partial \sigma}. \tag{2.21}$$

S_{NG} in (2.20) is called the Nambu-Goto action of a relativistic string. From this action we can see that the classical string motion minimises the worldsheet area.

Now we define an induced metric $\gamma_{\alpha\beta}$ on the worldsheet as

$$\gamma_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta}. \tag{2.22}$$

Accordingly, the Nambu-Goto action becomes

$$S_{NG} = -T \int d^2\sigma \sqrt{-\gamma} \tag{2.23}$$

where $\gamma = \det \gamma_{\alpha\beta}$. Written in this form, S_{NG} is manifestly reparameterisation invariant.

2.2.1 Symmetries

The Nambu-Goto action has two types of global symmetries:

Poincaré invariance of the spacetime: $X^\mu \to \Lambda^\mu_\nu X^\nu + c^\mu$, where Λ is a Lorentz transformation satisfying $\Lambda^\mu_\nu \eta^{\nu\rho}\Lambda^\sigma_\rho = \eta^{\mu\sigma}$ and c^μ is a constant translation. The symmetry is global from the perspective of the worldsheet, this means that Λ^μ_ν and c^μ are constants and do not depend on worldsheet coordinates σ^α.

Reparametrisation invariance: Also known as diffeomorphisms, reparametrisations are gauge symmetries on the worldsheet. The transformations

$$\sigma^\alpha \to f^\alpha(\sigma) = \sigma'^\alpha \quad \text{and} \quad g_{\alpha\beta}(\sigma) = \frac{\partial f^\lambda}{\partial \sigma^\alpha}\frac{\partial f^\rho}{\partial \sigma^\beta} g_{\lambda\rho}(\sigma') \tag{2.24}$$

keep the action invariant. This type of symmetry implies that the transformations and their inverses are infinitely differentiable. Sometimes these transformations are used in infinitesimal form. If we change coordinates $\sigma^\alpha \to \sigma'^\alpha = \sigma^\alpha - \eta^\alpha(\sigma)$ for some small η, the transformations of the fields become

$$\begin{aligned} \delta X^\mu(\sigma) &= \eta^\alpha \partial_\alpha X^\mu \\ \delta g_{\alpha\beta}(\sigma) &= \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha, \end{aligned} \tag{2.25}$$

where the covariant derivative is defined by $\nabla_\alpha \eta_\beta = \partial_\alpha \eta_\beta - \Gamma^\sigma_{\alpha\beta}\eta_\sigma$ with the Levi-Civita connection given by

$$\Gamma^\sigma_{\alpha\beta} = \frac{1}{2} g^{\sigma\rho}(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}).$$

2.2.2 Equations of motion

Let us obtain the equations of motion for the Nambu-Goto action. First we rewrite the action in terms of a Lagrangian density,

$$S_{NG} = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma\, \mathscr{L}_{NG}(\dot{X}^\mu, X'^\mu), \tag{2.26}$$

where

$$\mathscr{L}_{NG}(\dot{X}^\mu, X'^\mu) = -T\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}. \tag{2.27}$$

To obtain the equations of motion we calculate the first variation of the action (2.26),

$$\delta S_{NG} = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial \mathscr{L}_{NG}}{\partial \dot{X}^\mu}\frac{\partial(\delta X^\mu)}{\partial \tau} + \frac{\partial \mathscr{L}_{NG}}{\partial X'^\mu}\frac{\partial(\delta X^\mu)}{\partial \sigma}\right], \tag{2.28}$$

and set it to zero. For brevity of notation we introduce the quantities

$$\Pi^\tau_\mu = \frac{\partial \mathscr{L}_{NG}}{\partial \dot{X}^\mu} = -T\frac{(\dot{X}\cdot X')X'_\mu - X'^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}}, \tag{2.29}$$

$$\Pi^\sigma_\mu = \frac{\partial \mathscr{L}_{NG}}{\partial X'^\mu} = -T\frac{(\dot{X}\cdot X')\dot{X}_\mu - \dot{X}^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}}. \tag{2.30}$$

Using this notation (2.28) becomes

$$\delta S_{NG} = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial}{\partial \tau}\left(\delta X^\mu \Pi^\tau_\mu\right) + \frac{\partial}{\partial \sigma}\left(\delta X^\mu \Pi^\sigma_\mu\right) - \delta X^\mu\left(\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma}\right)\right]. \tag{2.31}$$

For initial and final moments of time we put $\delta X^\mu(\tau_f,\sigma) = \delta X^\mu(\tau_i,\sigma) = 0$. The variation of the action with respect to X^μ then becomes

$$\delta S_{NG} = \int_{\tau_i}^{\tau_f} d\tau \left[\delta X^\mu \Pi^\sigma_\mu\right]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \delta X^\mu \left(\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma}\right). \tag{2.32}$$

Since the second term in the right hand side of (2.32) should vanish for all variations δX^μ of the motion, we obtain

$$\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma} = 0. \tag{2.33}$$

This is an equation of motion for both open and closed relativistic strings. The first term in the right hand side can be written explicitly as

$$\begin{aligned}\int_{\tau_i}^{\tau_f} d\tau \big[&\delta X^0(\tau,\sigma_1)\Pi_0^\sigma(\tau,\sigma_1) - \delta X^0(\tau,0)\Pi_0^\sigma(\tau,0) \\ &+\delta X^1(\tau,\sigma_1)\Pi_1^\sigma(\tau,\sigma_1) - \delta X^1(\tau,0)\Pi_1^\sigma(\tau,0) \\ &\vdots \\ &+\delta X^{D-1}(\tau,\sigma_1)\Pi_{D-1}^\sigma(\tau,\sigma_1) - \delta X^{D-1}(\tau,0)\Pi_{D-1}^\sigma(\tau,0)\big].\end{aligned} \tag{2.34}$$

We see that we need a total of $2D$ boundary conditions, one for each term in (2.34). For the open string we can impose two types of boundary conditions:

Dirichlet boundary condition: In this case the spatial coordinates of the endpoint are fixed, say, at σ', and we impose, for every $\mu \neq 0$, that

$$\frac{\partial X^\mu}{\partial \tau}(\tau,\sigma') = 0. \tag{2.35}$$

Since time varies as τ varies, $\dfrac{\partial X^0}{\partial \tau} \neq 0$, Dirichlet boundary conditions are applicable only to spatial coordinates.

Neumann boundary condition: In this case we impose

$$\Pi_\mu^\sigma(\tau,\sigma') = 0 \tag{2.36}$$

for every $\mu = 0, 1, \ldots, D-1$. This boundary condition is also called "free endpoint boundary condition."

Any one of these two boundary conditions make (2.34) vanish.

For the closed string we pick σ_1 to be an integer multiple of π, in which case the first term in the equation (2.32) vanishes identically.

We can write the equations of motion (2.33) in a leaner fashion. From the definition of $\gamma_{\alpha\beta}$ and Jacobi's formula for differentiating a determinant,

$$\delta\sqrt{-\gamma} = -\frac{1}{2\sqrt{-\gamma}}\delta\gamma = -\frac{1}{2}\sqrt{-\gamma}\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta} = \frac{1}{2}\sqrt{-\gamma}\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta}, \tag{2.37}$$

variation of the action (2.23) provides the equations

$$\partial_\alpha\left(\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\beta X^\mu\right) = 0. \tag{2.38}$$

Even though the equations of motion look better in this notation, they are still the same nasty equations!

2.3 The string sigma model action

The Nambu-Goto action contains a square-root that makes it troublesome to quantise by the path integral approach. There is, however, another action which is more convenient to quantise and that recovers the Nambu-Goto action in the classical limit. We can get rid of the square root in S_{NG} by introducing a new field $g_{\alpha\beta}$ to the action and by rewriting it as

$$S = -\frac{1}{2}T\int d^2\sigma\sqrt{-g}\,g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}, \tag{2.39}$$

where $g = \det g_{\alpha\beta}$, and $g^{\alpha\beta} = (g_{\alpha\beta})^{-1}$. This action is called the Polyakov or the sigma model action. The new field $g_{\alpha\beta}$ is a dynamical metric on the worldsheet. From the point of view of the worldsheet, the sigma model action describes a bunch of scalar fields X^μ coupled to two-dimensional gravity $g_{\alpha\beta}$.

The Polyakov action gives rise to the same equations of motion (2.38) as the Nambu-Goto action but with $g_{\alpha\beta}$ in the place of $\gamma_{\alpha\beta}$,

$$\partial_\alpha\left(\sqrt{-g}\,g^{\alpha\beta}\partial_\beta X^\mu\right) = 0. \tag{2.40}$$

However, $g_{\alpha\beta}$ is now an independent variable which should be fixed by its own equations of motion. To determine the equivalence of the equations of motion we

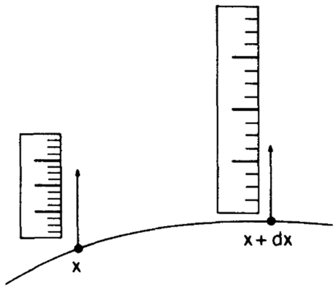

Figure 2.2: Weyl's gauge transformation is illustrated by the change in length of metre stick from X to $X+dX$. (Source: Moriyasu [13])

vary the action with respect to $g_{\alpha\beta}$ and obtain

$$\delta S = -\frac{T}{2}\int d^2\sigma\,\sqrt{-g}\,\delta g^{\alpha\beta}\left(\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2} g_{\alpha\beta} g^{\lambda\rho}\partial_\lambda X^\mu \partial_\rho X^\nu\right)\eta_{\mu\nu}, \quad (2.41)$$

where we have made use of Jacobi's formula (2.37). Putting $\delta S = 0$ implies that the worldsheet metric should be of the form

$$g_{\alpha\beta} = 2f(\tau,\sigma)\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu, \quad \text{with} \quad f^{-1}(\tau,\sigma) = \eta_{\mu\nu} g^{\lambda\rho}\partial_\lambda X^\mu \partial_\rho X^\nu. \quad (2.42)$$

We see that the difference between $g_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ is the factor $2f$. But this factor drops out of the equations of motion (2.40), because the $\sqrt{-g}$ term has a scale of f, and the inverse metric $g^{\alpha\beta}$ has a scale of f^{-1}. So the Nambu-Goto and the string sigma model actions provide the same equations of motion for the X^μ.

2.3.1 Symmetries

The action of the bosonic string sigma model in Minkowski spacetime has a number of symmetries. Poincare invariance of the space time and reparameterisation invariance are also hold for the Nambu-Goto action, and were discussed earlier in 2.2.1

There is yet a third type of symmetry for the sigma model action:

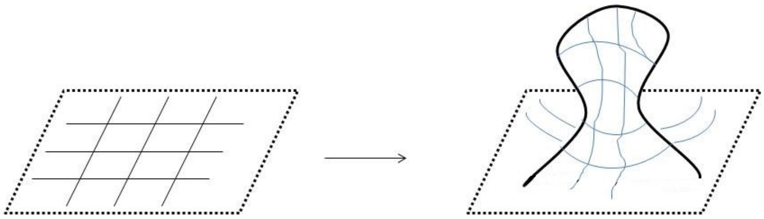

Figure 2.3: An example of a Weyl transformation.

Weyl invariance: The Polyakov action is invariant under the rescaling

$$g_{\alpha\beta} \to \Omega^2(\sigma) g_{\alpha\beta} \quad \text{and} \quad \delta X^\mu = 0. \tag{2.43}$$

Putting $\Omega^2(\sigma) = e^{2\phi(\sigma)}$ for some small $\phi(\sigma)$, we can write infinitesimally that

$$\delta g_{\alpha\beta}(\sigma) = 2\phi(\sigma) g_{\alpha\beta}(\sigma). \tag{2.44}$$

The Polyakov action with different metrics related by a Weyl transformation describes the same physical state. Weyl invariance is not invariance under a coordinate change, it is the invariance of the theory under a local change of scale preserving the angles between all lines; see figure 2.2. The two worldsheet metrics in figure 2.3 are viewed by the Polyakov string as equivalent. Weyl invariance is special to two dimensions.

2.3.2 Gauge fixing

We can fix the gauge of $g_{\alpha\beta}$ by analogy with what we did with the gauge field $e(\tau)$ in the case of the point particle. The gauge field

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} \tag{2.45}$$

has three independent components, since $g_{10} = g_{01}$. Using the symmetries of the model we can fix $g_{\alpha\beta}$ completely. The choice of a conformal gauge fixes two components by the reparametrisation invariance. Reparameterisation invariance is used to choose coordinates such that locally $g_{\alpha\beta} = \Omega^2(\sigma,\tau)\eta_{\alpha\beta}$ with $\eta_{\alpha\beta}$ the two dimensional Minkowski metric defined by $ds^2 = -d\tau^2 + d\sigma^2$. Let us show that this can be always done. For any 2-dimensional metric $g_{\alpha\beta}$, we consider two null vectors at each point. Like this we get two vector fields and their integral curves, which we label by σ^+ and σ^-. Then $ds^2 = -\Omega^2 d\sigma^+ d\sigma^-$ and $g_{++} = g_{--} = 0$ since the curves are null. Now let

$$\sigma^{\pm} = \tau \pm \sigma,$$

from which $ds^2 = \Omega^2(-d\tau^2 + d\sigma^2)$. So we really always can make a choice $g_{\alpha\beta} = \Omega^2(\sigma,\tau)\eta_{\alpha\beta}$. A choice of coordinate system in which the 2-dimensional metric is conformally flat, i.e. in which

$$ds^2 = \Omega^2(-d\tau^2 + d\sigma^2) = -\omega^2 d\sigma^+ d\sigma^-, \tag{2.46}$$

is called a conformal gauge. $\sigma^{\pm}$ are called light-cone or conformal coordinates. And now we can use the Weyl rescaling to fix the remaining component,

$$\Omega^2(\sigma,\tau)\eta_{\alpha\beta} = \eta_{\alpha\beta}.$$

As a result the gauge field can be chosen as

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.47}$$

where $\eta_{\alpha\beta}$ is a flat worldsheet metric. This choice is possible only if there are no topological obstructions, and if it is allowed the action becomes

$$S = \frac{T}{2}\int d^2\sigma\, \partial_\alpha X \partial^\alpha X. \tag{2.48}$$

For a flat Minkowski spacetime we could have two more renormalisable and Poincaré invariant terms in the action, namely,

$$\lambda_1 \int d^2\sigma \sqrt{-g} \quad \text{and} \quad \lambda_2 \int d^2\sigma \sqrt{-g} R^{(2)}(g). \tag{2.49}$$

The first term is a cosmological constant term in the worldsheet that is not allowed by the equations of motion.[(1)] The second term, due to the scalar curvature $R^{(2)}(g)$ of the worldsheet geometry, is allowed but will not be discussed here.

2.3.3 Equations of motion

When $g_{\alpha\beta} = \eta_{\alpha\beta}$, the Polyakov action becomes the theory of D free scalar fields with action (2.48). In this case we can rewrite the equations of motion as

$$\partial_\alpha \partial^\alpha X^\mu = 0. \tag{2.50}$$

These equations of motion are not really equal to the initial ones. We must ensure that the equation of motion for $g_{\alpha\beta}$ is also satisfied. The variation of the action with

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[(1)] When we add a cosmological constant term to the string sigma model action (2.39), the first variation of $S' = S + \lambda_1 \int d^2\sigma \sqrt{-g}$ gives, using (2.41) and Jacobi's formula (2.37),

$$\delta S' = -\frac{T}{2} \int d^2\sigma\, \delta g^{\alpha\beta} \Big(\sqrt{-g}\, \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2} \sqrt{-g}\, g_{\alpha\beta} g^{\rho\sigma} \partial_\rho X^\mu \partial_\sigma X^\nu \Big) \eta_{\mu\nu} - \frac{1}{2} \lambda_1 \sqrt{-g}\, g_{\alpha\beta} \delta g^{\alpha\beta},$$

and the equation of motion for the worldsheet metric becomes

$$\frac{2}{\sqrt{-g}} \frac{\delta S_\sigma}{\delta g^{\alpha\beta}} = -T \Big[(\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2} g_{\alpha\beta} g^{\rho\sigma} \partial_\rho X^\mu \partial_\sigma X^\nu) \eta_{\mu\nu} \Big] - \lambda_1 g_{\alpha\beta} = 0.$$

Multiplying the last equation by $g^{\alpha\beta}$ furnishes

$$g_{\alpha\beta} g^{\alpha\beta} \lambda_1 = T \Big(\frac{1}{2} g_{\alpha\beta} g^{\alpha\beta} - 1 \Big) g^{\alpha\beta} \partial_\rho X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = 0,$$

because $g_{\alpha\beta} g^{\alpha\beta} = 2$. Consistency of the equations of motion thus requires that $\lambda_1 = 0$.

respect to the metric gives rise to the stress-energy tensor

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\partial S}{\partial g^{\alpha\beta}}. \tag{2.51}$$

If we set $g_{\alpha\beta} = \eta_{\alpha\beta}$ in equations (2.41) and (2.51), the stress-energy tensor becomes

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X_\mu. \tag{2.52}$$

The equation of motion for $g_{\alpha\beta}$ is just $T_{\alpha\beta} = 0$. Explicitly,

$$T_{01} = \dot{X}\cdot X' = 0 \quad \text{and} \quad T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0. \tag{2.53}$$

The equations of motion for the string are thus the free wave equations (2.50) with the conditions (2.53) as constraints. The first constraint in (2.53) means that we have to choose our parametrisation such that the lines of constant σ are perpendicular to the lines of constant τ. Under the reparameterisation invariance we can go to static gauge

$$X^0 \equiv t = R\tau \quad \Rightarrow \quad X'^0 = 0 \quad \text{and} \quad \dot{X}^0 = R,$$

where R is constant. If we put $X^\mu = (t, \vec{X})$, the equation of motion for the spatial components becomes

$$\ddot{\vec{X}} - \vec{X}'' = 0, \tag{2.54}$$

the free wave equation, and the constraints (2.53) become

$$\begin{aligned} \dot{\vec{X}}\cdot\vec{X}' &= 0, \\ \dot{\vec{X}}^2 + \vec{X}'^2 &= R^2. \end{aligned} \tag{2.55}$$

From this constraints we see that motion of the string should be perpendicular to the string, so that it can only have transverse oscillations.

2.3.4 Boundary conditions

To completely define the problem we have to specify the boundary conditions. Let us take $0 \leqslant \sigma \leqslant \pi$ for the space coordinate in the worldsheet. Varying the action (2.48) with respect to X^μ with $\delta X^\mu(\tau_i) = \delta X^\mu(\tau_f) = 0$ we obtain

$$\delta S = T \int d^2\sigma \, \delta X^\mu \partial_\alpha \partial^\alpha X^\mu - T \int d\tau \left[X'_\mu \delta X^\mu \right]_{\sigma=0}^{\sigma=\pi}. \tag{2.56}$$

The second term on the right hand side of (2.56) has to vanish if δS is to vanish. We can achieve this by choosing appropriate boundary conditions, which depend on wether the string is open or closed. The available options are:

Closed strings: In this case the endpoints of the string coincide, $X^\mu(\sigma,\tau) = X^\mu(\sigma + \pi,\tau)$, and the second term on the right hand side of (2.56) vanishes identically;

Open strings: In this case we can adopt either Dirichlet or Neumann boundary conditions for the string coordinates. Under Dirichlet boundary conditions the endpoints of the string are fixed and all $\delta X^\mu = 0$ for $\mu \neq 0$. Under Neumann boundary conditions $X'_\mu = 0$ at $\sigma = 0, \pi$ for all $\mu = 0, 1, \ldots, D-1$. Neumann boundary conditions respect the D-dimensional Poincaré invariance of the system, since no momentum flows through the ends of the string.

Note that we can also impose mixed boundary consitions, with some coordinates observing Dirichlet boundary conditions (except $\mu = 0$) and others observing Neumann boundary conditions. In any case, for all choices given above the second term on the right hand side of (2.56) vanishes identically, as desired.

2.3.5 Solution of the equations of motion

It is convenient to solve the equations of motion in light-cone coordinates. As we mentioned before light-cone coordinates on the worldsheet are

$$\sigma^\pm = \tau \pm \sigma. \tag{2.57}$$

The derivatives and the two-dimensional Lorentz metric in these coordinates are given by

$$\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \quad \text{and} \quad \begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.58}$$

The equation of motion for X^μ now reads

$$\partial_+ \partial_- X^\mu = 0. \tag{2.59}$$

Let us consider solutions for this equation in open and closed string cases.

Closed strings: The most general solution for this equation is

$$X^\mu(\sigma,\tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \tag{2.60}$$

for arbitrary functions X^μ_L and X^μ_R, which we call left and right moving waves for reasons that will be clear in the following. Solution $X^\mu(\sigma,\tau)$ must respect the constraints (2.53) and be periodic,

$$X^\mu(\sigma,\tau) = X^\mu(\sigma+\pi,\tau). \tag{2.61}$$

We then expand each of the terms of the solution (2.60) in Fourier modes,

$$\begin{aligned} X^\mu_L(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\bar{\alpha}^\mu_n e^{-2in\sigma^+}, \\ X^\mu_R(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n e^{-2in\sigma^-}. \end{aligned} \tag{2.62}$$

The mode expansion is important for quantisation. The following observations about the above mode expansions apply:

- X^μ_L and X^μ_R are not individually π periodic but their sum and difference are;

- The variables x^μ and p^μ are the position and the momentum of the centre of mass of the string;

- Since X^μ is real, $\alpha_n^{\mu *} = \alpha_{-n}^\mu$ and similarly $\bar{\alpha}_n^{\mu *} = \bar{\alpha}_{-n}^\mu$.

In the light-cone coordinates $\sigma^\pm$, the two constraints (2.53) are expressed by

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0. \tag{2.63}$$

In terms of the Fourier mode expansions (2.62), the first of these conditions imply

$$\partial_- X^\mu = \partial_- X_R^\mu = \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-2in\sigma^-} = \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-2in\sigma^-} \tag{2.64}$$

where in the last sum we define $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$ and let $n \in \mathbb{Z}$. Then

$$\begin{aligned}(\partial_- X)^2 &= \frac{\alpha'}{2} \sum_{m,n} \alpha_m^\mu \alpha_n^\mu \, e^{-2i(m+n)\sigma^-} = \\ &= \frac{\alpha'}{2} \sum_{m,n} \alpha_m^\mu \alpha_{n-m}^\mu \, e^{-2in\sigma^-} = \alpha' \sum_n L_n \, e^{-2in\sigma^-} = 0,\end{aligned} \tag{2.65}$$

where $L_n = \frac{1}{2} \sum_m \alpha_m^\mu \alpha_{n-m}^\mu$. For left-moving modes, by analogy, $\bar{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$ and $\bar{L}_n = \frac{1}{2} \sum_m \bar{\alpha}_m^\mu \bar{\alpha}_{n-m}^\mu$. L_n and $\bar{L}_n$ are the Fourier modes of the constraints. Any classical solution of the form (2.62) for the closed string must obey the infinite number of constraints

$$L_n = \bar{L}_n = 0, \quad n \in \mathbb{Z}. \tag{2.66}$$

These constraints give rise to an infinite-dimensional algebra known as the classical Virasoro algebra. Now remember that $p_\mu p^\mu = -M^2$, where M is the rest mass of the string. Using the L_0 and $\bar{L}_0$ constraints we obtain the squared mass of the closed string in terms of the excited modes of the oscillator as

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n^\mu \alpha_{-n}^\mu = \frac{4}{\alpha'} \sum_{n>0} \bar{\alpha}_n^\mu \bar{\alpha}_{-n}^\mu. \tag{2.67}$$

Since $\alpha_0^\mu = \bar{\alpha}_0^\mu$, we have two expressions for M^2: one in terms of right-moving oscillators and the other in terms of left-moving oscillators. These two expressions must be equal, a condition known as level matching. This condition is important in the quantisation of the strings and will be discussed later in this dissertation.

Let the canonical momentum conjugate to X^μ be defined as $\Pi^\mu(\sigma,\tau) = \frac{1}{2\pi\alpha'}\dot{X}^\mu$. With this definition the classical Poisson brackets become

$$\begin{aligned} \left\{\Pi^\mu(\sigma,\tau),\Pi^\nu(\sigma',\tau)\right\} &= \left\{X^\mu(\sigma,\tau),X^\nu(\sigma',\tau)\right\} = 0, \\ \left\{\Pi^\mu(\sigma,\tau),X^\nu(\sigma',\tau)\right\} &= 2\pi\alpha'\eta^{\mu\nu}\delta(\sigma-\sigma'). \end{aligned} \tag{2.68}$$

Using equations (2.62) we find that the classical Poisson brackets for the Fourier modes read

$$\left\{\alpha_m^\mu, \alpha_n^\nu\right\} = \left\{\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu\right\} = im\eta^{\mu\nu}\delta_{m+n} \quad \text{and} \quad \left\{\alpha_m^\mu, \bar{\alpha}_n^\nu\right\} = 0. \tag{2.69}$$

Now we will determine the classical Virasoro algebra of the generators L_n and $\bar{L}_n$, i. e., the Poisson bracket $\{L_m, L_n\}$. We find that

$$\begin{aligned} \left\{\alpha_m^\nu, L_n\right\} &= \frac{1}{2}\sum_{p=-\infty}^{+\infty}\left\{\alpha_m^\nu, \alpha_{n-p}^\mu\alpha_p^\mu\right\} = \\ &= \frac{1}{2}\sum_{p=-\infty}^{+\infty}\left(\left\{\alpha_m^\nu, \alpha_{n-p}^\mu\right\}\alpha_p^\mu + \alpha_{n-p}^\mu\left\{\alpha_m^\nu, \alpha_p^\mu\right\}\right) = \\ &= \frac{1}{2}\sum_{p=-\infty}^{+\infty}\left(im\eta^{\mu\nu}\delta_{n-p+m}\alpha_p^\mu + \alpha_{n-p}^\mu im\eta^{\mu\nu}\delta_{m+p}\right) = \\ &= im\eta^{\mu\nu}\alpha_{m+n}^\mu, \end{aligned} \tag{2.70}$$

from which follows

$$\begin{aligned}\{L_m,L_n\} &= \frac{1}{2}\sum_{p=-\infty}^{+\infty}\{\alpha^{\mu}_{m-p}\alpha^{\mu}_{p},L_n\} = \\ &= \frac{1}{2}\sum_{p=-\infty}^{+\infty}\Big(\{\alpha^{\mu}_{m-p},L_n\}\alpha^{\mu}_{p}+\alpha^{\mu}_{m-p}\{\alpha^{\mu}_{p},L_n\}\Big) = \\ &= \frac{1}{2}im\eta^{\mu\nu}\sum_{p=-\infty}^{+\infty}(\alpha^{\nu}_{m-p+n}\alpha^{\mu}_{p}+\alpha^{\mu}_{m-p}\alpha^{\nu}_{p+n}).\end{aligned} \tag{2.71}$$

Changing $m\to n$ in the first term and $p+n\to q$ in the second term of the summation on the right-hand side of equation (2.71) provides

$$\begin{aligned}\{L_m,L_n\} &= \frac{1}{2}in\eta^{\nu\mu}\sum_{p=-\infty}^{+\infty}\alpha^{\mu}_{m+n-p}\alpha^{\nu}_{p}+\frac{1}{2}im\eta^{\mu\nu}\sum_{q=-\infty}^{+\infty}\alpha^{\mu}_{m+n-q}\alpha^{\nu}_{q} = \\ &= i(m-n)L_{m+n},\end{aligned} \tag{2.72}$$

which is the defining commutation relation for the classical Virasoro algebra.

Open strings: The Fourier mode expansion for the open string reads

$$\begin{aligned}X^{\mu}_{L}(\sigma^{+}) &= \frac{1}{2}x^{\mu}+\alpha' p^{\mu}\sigma^{+}+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}^{\mu}_{n}e^{-in\sigma^{+}}, \\ X^{\mu}_{R}(\sigma^{-}) &= \frac{1}{2}x^{\mu}+\alpha' p^{\mu}\sigma^{-}+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^{\mu}_{n}e^{-in\sigma^{-}}.\end{aligned} \tag{2.73}$$

Boundary conditions impose relations on the modes of the open string:

- Neumann boundary conditions $\partial_{\sigma}X^{a}=0$, at the endpoints require that $\alpha^{a}_{n}=\tilde{\alpha}^{a}_{n}$, which gives

$$X^{\mu}(\sigma,\tau)=X^{\mu}+2\alpha' p^{\mu}\tau+i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha^{\mu}_{n}e^{-in\tau}\cos n\sigma. \tag{2.74}$$

For open strings there is only one set of modes α^{μ}_{n}. Instead of right and left-moving modes we have just standing waves.

- Direchlet boundary conditions, $X^I = c^I$, at the end point srequire that $x^I = c^I$, $p^I = 0$ and $\alpha_n^I = -\tilde{\alpha}_n^I$. which gives

$$X^\mu(\sigma,\tau) = c^I + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \sin n\sigma. \tag{2.75}$$

For the open string, constraints (2.53) are also like in equation (2.63). The Fourier mode expansion of these conditions imply

$$2\partial_\pm X^\mu = \dot{X}^\mu \pm X'^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in(\tau\pm\sigma)}, \tag{2.76}$$

where $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$, as before. By analogy with the closed string, we find the Fourier modes of the constraints, just in this case we have only one set of L_n. So any classical solution of the form (2.74) for the open string must obey the infinite number of constraints

$$L_n = 0, \quad n \in \mathbb{Z}. \tag{2.77}$$

For the open string we find, that the squared mass is given by

$$M^2 = \frac{1}{\alpha'} \left(\sum_{i=1}^{p-1} \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{i=p+1}^{D-1} \sum_{n>0} \alpha_{-n}^i \alpha_n^i \right). \tag{2.78}$$

The first sum is over modes parallel to the brane where the end points of the open string are, the second over perpendicular to the brane.

3

Quantisation of the bosonic string

In this chapter we will consider two ways to quantise the bosonic string:

Covariant quantisation: To quantise the string covariantly we first quantise the system and then impose the constraints that arise from the gauge fixing.

Light-cone gauge quantisation: In light-cone gauge quantisation we first solve all the constraints of the system to determine the space of all classical solutions and then quantise these solutions.

Our main references for this chapter are the book by Becker, Becker and Schwartz [9] and the lecture notes by Tong [11].

3.1 Covariant quantisation of the closed string

In this section we will quantise D free scalar fields X^μ, with dynamics governed by the action (2.48). For the canonical quantisation we first change Poisson brackets by commutators

$$\{\ ,\ \} \longrightarrow \frac{1}{i}[\ ,\ ]. \tag{3.1}$$

We define momentum conjugated with X^μ as $\Pi^\mu = \frac{1}{2\pi\alpha'}\dot{X}^\mu$, where $\mu = 0, 1, \ldots, D-1$, quantum mechanical operators obey the following commutation relations

$$\begin{aligned}
&[X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = i\delta(\sigma-\sigma')\delta^\mu_\nu, \\
&[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [\Pi_\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = 0.
\end{aligned} \tag{3.2}$$

These commutation relations can be used to find the commutation relations for the Fourier modes $x^\mu, p^\mu, \alpha_n^\mu$ and $\bar{\alpha}_n^\mu$. Using (2.62) we find for the closed string

$$[x^\mu, p_\nu] = i\delta_\nu^\mu, \tag{3.3}$$

which is the expected commutation relation for the operators of the center of mass of the string. The remaining commutation relations are

$$[\alpha_m^\mu, \bar{\alpha}_n^\nu] = 0 \quad \text{and} \quad [\alpha_n^\mu, \alpha_m^\nu] = [\bar{\alpha}_n^\mu, \bar{\alpha}_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0}. \tag{3.4}$$

We see that α_n^μ and $\bar{\alpha}_n^\mu$ behave like creation and annihilation operators for a set of harmonic oscillators. Using the rescaling

$$a_n^\mu = \frac{1}{\sqrt{n}}\alpha_n^\mu, \quad \text{for } n > 0 \tag{3.5}$$

and hermicity

$$a_n^{\mu\dagger} = \frac{1}{\sqrt{n}}\alpha_{-n}^\mu, \quad \text{for } n > 0, \tag{3.6}$$

we find that

$$\begin{aligned} [a_m^\mu, a_n^{\nu\dagger}] &= \eta^{\mu\nu}\delta_{m,n} \\ [\bar{a}_m^\mu, \bar{a}_n^{\nu\dagger}] &= \eta^{\mu\nu}\delta_{m,n}, \end{aligned} \tag{3.7}$$

which coincide with the standard commutation relations for two infinite sets of independent quantum harmonic oscillators. For every integer n we introduce a number operator

$$N_n = \alpha_n^\mu \alpha_{-n,\mu}. \tag{3.8}$$

Using equations (3.4) and (3.8) we find for every $n > 0$

$$\begin{aligned} [N_n, \alpha_n] &= -n\alpha_n, \\ [N_n, \alpha_{-n}] &= n\alpha_{-n}. \end{aligned} \tag{3.9}$$

From (3.9) we conclude that

- Modes with $n > 0$ are identified with the lowering operators,
- Modes with $n < 0$ are identified with the rasing operators.

To build the so-called Fock ("multiparticle") space for our theory, we first introduce the ground state $|0\rangle$ of the string obeying the relations

$$\alpha_n^\mu |0\rangle = \bar{\alpha}_n^\mu |0\rangle = 0, \quad \text{for } n > 0; \tag{3.10}$$

$|0\rangle$ is just the vacuum ground state of a single string. The true ground state of the string is $|0\rangle$ tensored with a spatial wave function $\Psi(x)$. In momentum space, the vacuum carries another quantum number p^μ, which is an eigenvalue of $\hat{p}$, so our vacuum changes to $|0;p\rangle$. The $|0,p\rangle$ vacuum obeys the relation

$$\hat{p}^\mu |0,p\rangle = p^\mu |0;p\rangle. \tag{3.11}$$

The infinite-dimensional Fock space can be built from the ground state by means of the creation operators in the usual manner,

$$(\alpha_{-1}^{\mu_1})^{n_{\mu_1}} (\alpha_{-2}^{\mu_2})^{n_{\mu_2}} \dots (\bar{\alpha}_{-1}^{\nu_1})^{n_{\nu_1}} (\bar{\alpha}_{-2}^{\nu_2})^{n_{\nu_2}} \dots |0;p\rangle. \tag{3.12}$$

Each state in the Fock space is an excited state of the string. This construction brings us to the major problem of canonical quantisation. Let us consider for $n > 0$ the following commutation relations

$$[\alpha_n^0, \alpha_{-n}^0] = [\alpha_n^0, \alpha_n^{0\dagger}] = n\eta^{00} = -n. \tag{3.13}$$

Using (3.13)for some states we obtain the norm given by

$$\langle 0|[\alpha_n^0, \alpha_n^{0\dagger}]|0\rangle = \langle 0|\alpha_n^0 \alpha_n^{0\dagger}|0\rangle = ||\alpha_n^{0\dagger}|0\rangle||^2 = -n < 0, \tag{3.14}$$

that is, we get states with negative norm in the Hilbert space. This states are called ghosts and do not allow the probabilistic interpretation of the corresponding quantum-mechanical system. We got ghosts because we ignored the so called Virasoro constraints, which we will discuss later in this chapter. To make sense of the theory, we have to make sure that ghosts do not appear in any physical process.

3.1.1 *The constraints*

We saw in Chapter 2, eqs. (2.66)–(2.77), that in the classical string theory a certain set of constraints $L_m = \bar{L}_m = 0$ must hold for the case of closed string. In this section we will discuss the analogous constraints in the quantum theory. On the quantum level, the expressions for L_m and $\bar{L}_m$ are quadratic in the oscillator operators and might have operators that do not commute with each other. For example, the constraint

$$L_0 = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \alpha_n^\mu \alpha_{-n,\mu}, \tag{3.15}$$

would suffer, in the quantum theory, from ordering problems, since the quantum versions of α_n^μ and $\alpha_{-n,\mu}$ do not commute. To deal with this problem we use the normal ordering prescription

$$:\alpha_1,\ldots,\alpha_k: = \underbrace{\alpha_{cr_1}\ldots\alpha_{cr_p}}_{creation}\underbrace{\alpha_{an_1}\ldots\alpha_{an_{k-p}}}_{annihilation}, \tag{3.16}$$

with all creation operators to the left of all annihilation operators. Since creation (annihilation) operators commute among themselves, their order inside each of the groups does not matter. Using the normal ordering prescription, we rewrite the quantum version of the classical constraints L_m in terms of the normal-ordered operators as

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} :\alpha_{m-n}^\mu \alpha_{n,\mu}: . \tag{3.17}$$

In particular, the L_0 constraint becomes

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{+\infty} \alpha_{-n}^{\mu}\alpha_{n,\mu} - a, \tag{3.18}$$

where a is a normal ordering constant.

Since the zero-modes are given in terms of p^μ, we also need a prescription for the normal-ordering of momentum operators, which reads

$$: p^\mu x^\nu := x^\nu p^\mu, \tag{3.19}$$

The zero state $|0;p\rangle$ satisfies $\hat{p}^\mu|0;p\rangle = p^\mu|0;p\rangle$ and can be considered the usual quantum mechanical eigenstate of the momentum operator $\hat{p}^\mu = -i\frac{\partial}{\partial x^\mu}$. In momentum representation, the zero state is then given by a plane wave

$$|0;p\rangle = e^{ip^\mu x_\mu}|0;0\rangle. \tag{3.20}$$

Plane waves are not square-integrable but form the basis of a generalised Hilbert space if normalised as $\langle 0;p|0;p'\rangle = \delta(p-p')$.

3.1.2 The Virasoro algebra

Classical case

As we saw in the classical theory, the Virasoro generators L_m satisfy the algebra

$$\{L_m, L_n\} = i(m-n)L_{m+n}. \tag{3.21}$$

The Virasoro constraints appear because the gauge choice

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.22}$$

does not fully fix the reparametrisation gauge symmetry. If ξ^α is an infinitesimal parameter of the reparametrisation and Λ is an infinitesimal parameter of the Weyl rescaling, then residual reparametrisation symmetries, which satisfy

$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda \eta^{\alpha\beta}, \tag{3.23}$$

remain in the theory. These reparameterisations are also Weyl rescalings, discussed earlier in section 2.2.1. If we put

$$\xi^\pm = \xi^0 \pm \xi^1 \quad \text{and} \quad \sigma^\pm = \sigma^0 \pm \sigma^1,$$

we find that

$$\xi^+ = \xi^+(\sigma^+) \quad \text{and} \quad \xi^- = \xi^-(\sigma^-)$$

are solutions of equation (3.23). The infinitesimal generators for the transformations $\delta\sigma^\pm = \xi^\pm$ are given by

$$V^\pm = \frac{1}{2}\xi^\pm(\sigma^\pm)\frac{\partial}{\partial \sigma^\pm}. \tag{3.24}$$

The complete basis for the transformations is

$$\xi_n^\pm(\sigma^\pm) = e^{2in\sigma^\pm}, \quad n \in \mathbb{Z}. \tag{3.25}$$

The generators $V_n^\pm$ give two copies of the Virasoro algebra. In the same way we can find that for the open string there is just one Virasoro algebra, with infinitesimal generators

$$V_n = e^{in\sigma^+}\frac{\partial}{\partial \sigma^+} + e^{in\sigma^-}\frac{\partial}{\partial \sigma^-}, \quad n \in \mathbb{Z}. \tag{3.26}$$

Quantum case

Now let us consider the algebra of the operators L_m in the quantum case. We first take $a = 0$ in equation (3.15). Our goal is to compute the commutator $[L_m, L_n]$.

Using equations (3.8) and (3.9) we find

$$\begin{aligned}[\alpha_m^\mu, L_n] &= \frac{1}{2}\sum_{p=-\infty}^{+\infty}[\alpha_m^\mu, :\alpha_p^\nu \alpha_{n-p,\nu}:] = \\ &= \frac{1}{2}\sum_{p=-\infty}^{+\infty}(m\delta_{m+p}\alpha_{n-p}^\mu + m\alpha_p^\mu \delta_{m+n-p}) = m\alpha_{n+m}^\mu.\end{aligned} \tag{3.27}$$

Using the definition (3.17) for the operators L_m we write

$$[L_m, L_n] = \frac{1}{2}\sum_{p=-\infty}^{+\infty}[:\alpha_p^\mu \alpha_{m-p,\mu}:, L_n], \tag{3.28}$$

or, writing out the normal ordering explicitly,

$$\begin{aligned}[L_m, L_n] &= \frac{1}{2}\sum_{p=-\infty}^{0}[\alpha_p^\mu \alpha_{m-p}^\mu, L_n] + \frac{1}{2}\sum_{p=1}^{+\infty}[\alpha_{m-p}^\mu \alpha_p^\mu, L_n] = \\ &= \frac{1}{2}\sum_{p=-\infty}^{0}\left(p\alpha_{p+n}^\mu \alpha_{m-p}^\mu + (m-p)\alpha_p^\mu \alpha_{m-p+n}^\mu\right) + \\ &+ \frac{1}{2}\sum_{p=1}^{+\infty}\left((m-p)\alpha_{m-p+n}^\mu \alpha_p^\mu + p\alpha_{m-p}^\mu \alpha_{n+p}^\mu\right).\end{aligned} \tag{3.29}$$

Changing the summation indexes $p \to q-n$ in the first and fourth terms and $p \to q$ in the second and third terms in the right-hand side of the above equation gives

$$\begin{aligned}[L_m, L_n] = \frac{1}{2}\Bigg(&\sum_{q=-\infty}^{0}(m-q)\alpha_q^\mu \alpha_{m+n-q}^\mu + \sum_{q=-\infty}^{n}(q-n)\alpha_q^\mu \alpha_{m+n-q}^\mu + \\ &+ \sum_{q=1}^{+\infty}(m-q)\alpha_{m+n-q}^\mu \alpha_q^\mu + \sum_{q=n+1}^{+\infty}(q-n)\alpha_{m+n-q}^\mu \alpha_q^\mu\Bigg).\end{aligned} \tag{3.30}$$

Now, splitting the second and third terms in the right-hand side of the above equation into two parts gives

$$
\begin{aligned}
[L_m, L_n] = \frac{1}{2}\Bigg(& \sum_{q=-\infty}^{0} (m-q)\alpha_q^\mu \alpha_{m+n-q}^\mu + \sum_{q=-\infty}^{0} (q-n)\alpha_q^\mu \alpha_{m+n-q}^\mu + \\
& + \sum_{q=1}^{n} (q-n)\alpha_q^\mu \alpha_{m+n-q}^\mu + \sum_{q=1}^{n} (m-q)\alpha_{m+n-q}^\mu \alpha_q^\mu + \\
& + \sum_{q=n+1}^{+\infty} (m-q)\alpha_{m+n-q}^\mu \alpha_q^\mu + \sum_{q=n+1}^{+\infty} (q-n)\alpha_{m+n-q}^\mu \alpha_q^\mu \Bigg),
\end{aligned}
\tag{3.31}
$$

and the summation of similar terms eventually leads to

$$
\begin{aligned}
[L_m, L_n] = \frac{1}{2}\Bigg(& \sum_{q=-\infty}^{0} (m-n)\alpha_q^\mu \alpha_{m+n-q}^\mu + \sum_{q=1}^{n} (q-n) \underbrace{\alpha_q^\mu \alpha_{m+n-q}^\mu}_{\textit{not ordered}!} + \\
& + \sum_{q=1}^{n} (m-n)\alpha_{m+n-q}^\mu \alpha_q^\mu + \sum_{q=n+1}^{+\infty} (m-q)\alpha_{m+n-q}^\mu \alpha_q^\mu \Bigg).
\end{aligned}
\tag{3.32}
$$

If we assume that $n > 0$, all terms in the above expression except the second one are normal-ordered. If in the second term we use that

$$
\alpha_q^\mu \alpha_{m+n-q}^\mu = \alpha_{m+n-q}^\mu \alpha_q^\mu + q\delta_{m+n}\delta_\mu^\mu = \alpha_{m+n-q}^\mu \alpha_q^\mu + qD\delta_{m+n},
$$

where D is the dimension of the (target) Minkowski spacetime where the string propagates, we obtain the following relation for the algebra,(1)

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{2} \sum_{q=-\infty}^{+\infty} (m-n) : \alpha_q^\mu \alpha_{m+n-q}^\mu : + \frac{D}{2}\delta_{n+m} \sum_{q=1}^{n} (q^2 - nq) = \\
&= (m-n)L_{m+n} + \frac{D}{12} m(m^2-1)\delta_{m+n}.
\end{aligned}
\tag{3.33}
$$

Here D can be replaced by the so-called central charge of the system c. The

(1) We use the elementary identities $\sum_{q=1}^{n} q = \frac{1}{2}n(n+1)$ and $\sum_{q=1}^{n} q^2 = \frac{1}{6}n(n+1)(2n+1)$.

algebra (3.33) is different from the classical case by the presence of the central term. To introduce the normal ordering constant a, we shift L_m to $L_m - \delta_{m,0}$ and then the algebra commutator becomes

$$[L_m, L_n] = (m-n)L_{m+n} + \left(\frac{c}{12}m^3 + (2a - \frac{c}{12})m\right)\delta_{m+n}. \tag{3.34}$$

We see that the central term cannot be removed by adjusting the normal ordering constant a.

3.1.3 The Virasoro constraints for the closed string

Due to the appearence of normal ordering in the definition of L_0 in the quantum theory, the classical conditions $L_0 = \bar{L}_0 = 0$ are replaced by

$$(L_0 - a)|\Phi\rangle = 0, \quad (\bar{L}_0 - \bar{a})|\Phi\rangle = 0, \tag{3.35}$$

where L_0 and $\bar{L}_0$ are the normal-ordered generators of the Virasoro algebra and $|\Phi\rangle$ is any physical state. The constraint or level-matching condition

$$(L_0 - \bar{L}_0)|\Phi\rangle = 0, \tag{3.36}$$

discussed at the end of Chapter 2, gives $a = \bar{a}$. Equations (3.35) are not satisfied for all n, otherwise we would have

$$[L_n, L_{-n}]|\Phi\rangle = 2nL_0|\Phi\rangle + \frac{D}{12}n(n^2-1)|\Phi\rangle \tag{3.37}$$

which would imply that

$$\left(2na + \frac{D}{12}n(n^2-1)\right)|\Phi\rangle = 0 \quad \text{for any } n, \tag{3.38}$$

while this is true only for $|\Phi\rangle = 0$. We conclude that it is impossible to impose in the quantum theory the same constraints as in the classical case.

We can try to impose only half of the Virasoro constraints to the physical states,

$$L_n|\Phi\rangle = 0 \quad \text{for } n > 0, \quad (L_0 - a)|\Phi\rangle = 0. \tag{3.39}$$

For the conjugate state $L_n^\dagger = L_{-n}$ we have

$$\langle\Phi|L_{-n} = 0 \quad \text{for } n > 0. \tag{3.40}$$

This means that the expectation values of L_n vanish for all nonnegative n.

Now let us obtain the mass operator from the constraint $L_0 - a = 0$ using equations (2.67) (where the second term depending on $\bar{\alpha}$ drops out) and (3.15)

$$M^2 = -p^2 = \frac{4}{\alpha'}(-a+N) \tag{3.41}$$

where $N = \sum_{n=1}^{+\infty} \alpha_{-n}^{\mu}\alpha_{n,\mu}$ is a number operator

$$N = \sum_{n=1}^{+\infty}\Big(-\alpha_{-n}^0\alpha_n^0 + \sum_{i=1}^{D-1}\alpha_{-n}^i\alpha_n^i\Big). \tag{3.42}$$

We can state that all eigenvalues of the number operator N are nonnegative, and the sign of the first term comes from the time-like oscillations. But time-like operators at the end provide only nonnegative contribution to N, because for any $n > 0$, using (3.8) and (3.9) we find that

$$[N, a_{-m}^0] = -\sum_{n=1}^{+\infty}[\alpha_{-n}^0\alpha_n^0, \alpha_{-m}^0] = -\sum_{n=1}^{+\infty}\alpha_{-n}^0[\alpha_n^0, a_{-m}^0] = m\alpha_{-m}^0, \tag{3.43}$$

since the commutator of two time-like oscillators contributes with the negative sign.

Now we can find the spectrum of the negative-norm free states. In the quantum theory a and D are not arbitrary. To find the allowed values for a and D, an effective strategy is to search for zero-norm states satisfying physical-state conditions. A state $|\Psi\rangle$ is called spurious if it satisfies the mass-shell condition and is orthogonal to all the physical states

$$(L_0 - a)|\Psi\rangle = 0 \quad \text{and} \quad \langle\Phi|\Psi\rangle = 0, \tag{3.44}$$

where $|\Phi\rangle$ is any physical state in the theory. An example of a spurious state is

$$|\Psi\rangle = \sum_{n=1}^{+\infty} L_{-n}|\xi\rangle \tag{3.45}$$

with $(L_0 - a + n)|\xi_n\rangle = 0$. In fact, due to the Virasoro algebra relation $L_{-3} = [L_{-1}, L_{-2}]$, any state like the above one can be represented as

$$|\Psi\rangle = L_{-1}|\xi_1\rangle + L_{-2}|\xi_2\rangle. \tag{3.46}$$

Moreover, any spurious state can be put in this form. A state $|\Psi\rangle$ defined like this is orthogonal to every physical state, since

$$\langle\Phi|\Psi\rangle = \sum_{n=1}^{+\infty} \langle\Phi|L_{-n}|\xi_n\rangle = \sum_{n=1}^{+\infty} \langle\xi_n|L_n|\Phi\rangle^* = 0. \tag{3.47}$$

If a state $|\Psi\rangle$ is spurious and physical, then it is orthogonal to all physical states including itself

$$\langle\Psi|\Psi\rangle = \sum_{n=1}^{+\infty} \langle\xi_n|L_n|\Psi\rangle = 0, \tag{3.48}$$

so such a state has zero norm. One class of zero-norm spurious states has the form

$$|\Psi\rangle = L_{-1}|\xi_1\rangle, \tag{3.49}$$

with

$$(L_0 - a + 1)|\xi_1\rangle = 0 \quad \text{and} \quad L_m|\xi_1\rangle = 0, \quad m > 0. \tag{3.50}$$

If $|\Psi\rangle$ is physical then

$$L_m|\Psi\rangle = (L_0 - a)|\Psi\rangle = 0 \quad \text{for } m = 1, 2, \ldots \tag{3.51}$$

The Virasoro algebra identity

$$L_1 L_{-1} = 2L_0 + L_{-1}L_1 \tag{3.52}$$

gives

$$L_1|\Psi\rangle = L_1 L_{-1}|\xi_1\rangle = (2L_0 + L_{-1}L_1)|\xi_1\rangle = 2(a-1)|\xi_1\rangle. \tag{3.53}$$

The value $a = 1$ can then be viewed as a "threshold" separating positive-norm and negative-norm physical states. The number of zero-norm spurious states can be maximised if, besides taking $a = 1$, we choose the appropriate spacetime dimension. To see how this can be done, we construct zero-norm spurious states of the form

$$|\Psi\rangle = (L_{-2} + \gamma L_{-1}^2)|\xi\rangle, \tag{3.54}$$

which have zero-norm for a certain γ. $|\Psi\rangle$ is spurious if $|\bar{\xi}\rangle$ satisfies

$$(L_0 + 1)|\bar{\xi}\rangle = L_m|\bar{\xi}\rangle = 0 \quad \text{for } m = 1, 2, \ldots \tag{3.55}$$

If $|\Psi\rangle$ is physical then $L_1|\Psi\rangle = L_2|\Psi\rangle = 0$, which as consequence of the Virasoro algebra gives $L_m|\Psi\rangle = 0$ for $m \geqslant 3$. We have

$$\begin{aligned}[L_1, L_{-2} + \gamma L_{-1}^2] &= 3L_{-1} + 2\gamma L_0 L_{-1} + 2\gamma L_{-1} L_0 \\ &= (3 - 2\gamma)L_{-1} + 4\gamma L_0 L_{-1}\end{aligned} \tag{3.56}$$

and then we find

$$L_1|\Psi\rangle = L_1(L_{-2} + \gamma L_{-1}^2)|\bar{\xi}\rangle = \big[(3 - 2\gamma)L_{-1} + 4\gamma L_0 L_{-1}\big]|\bar{\xi}\rangle. \tag{3.57}$$

The first term on the right-hand side of (3.57) is zero for $\gamma = \frac{3}{2}$ and the second term is always zero because

$$L_0 L_{-1}|\bar{\xi}\rangle = L_{-1}(L_0+1)|\bar{\xi}\rangle = 0. \tag{3.58}$$

So, as a result we get $\gamma = \frac{3}{2}$. Now let us consider the $L_2|\Psi\rangle = 0$ condition. Making use of the relation

$$[L_2, L_{-2} + \tfrac{3}{2}L_{-1}^2] = 13L_0 + 9L_{-1}L_1 + \tfrac{1}{2}D \tag{3.59}$$

we obtain that

$$L_2|\Psi\rangle = L_2(L_{-2} + \tfrac{3}{2}L_{-1}^2)|\bar{\xi}\rangle = (-13 + \tfrac{1}{2}D)|\bar{\xi}\rangle. \tag{3.60}$$

Thus the spacetime dimension that maximises the number of zero-norm spurious states is $D = 26$. The zero-norm spurious states are non-physical, but they decouple from all physical processes, together with the negative-norm states. So with the conditions $a = 1$ and $D = 26$ we have a spectrum of physical states only with positive norm. The $a = 1$ and $D = 26$ bosonic string theory is called critical theory. When $a \leqslant 1$ and $D \leqslant 25$ the spectrum is also negative-norm free; in these cases, the theories are called non-critical.

3.2 Light-cone gauge quantisation

By fixing the gauge we set the worldsheet metric to

$$g_{\alpha\beta} = \eta_{\alpha\beta}. \tag{3.61}$$

But even after this we still have a gauge transformation which preserves the choice of the metric. By making a particular non-covariant gauge choice, we can describe a Fock space that is manifestly free of negative-norm states, and solve explicitly all the Virasoro conditions. The coordinate transformation $\sigma \to \bar{\sigma}(\sigma)$, which changes the metric like in (2.43), and can be undone by Weyl transformations. To find these kind of transformations we will go to worldsheet light-cone coordinates

$$\sigma^{\pm} = \tau \pm \sigma, \tag{3.62}$$

where the flat metric is

$$ds^2 = -d\sigma^+ d\sigma^-. \tag{3.63}$$

In this coordinates we see that any transformation of the form

$$\sigma^+ \to \bar{\sigma}^+(\sigma^+), \quad \sigma^- \to \bar{\sigma}^-(\sigma^-) \tag{3.64}$$

just multiplies the flat metric by an $\Omega^2(\sigma)$ factor and can be undone by a compensating Weyl transformation. The best way to fix the remaining reparametrisation invariance is called light-cone gauge. At this point we introduce the spacetime light-cone coordinates as

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}). \tag{3.65}$$

In the light-cone coordinates the Lorentz invariance is no longer manifest, since two coordinates are treated differently than the others. In light-cone coordinates the Minkowski spacetime metric is given by

$$ds^2 = -2dX^+ dX^- + \sum_{i=1}^{D-2} dX^i dX^i. \tag{3.66}$$

The indexes of operators are raised and lowered like

$$A_+ = -A^-, \quad A_- = -A^+ \quad \text{and} \quad A_i = A^i, \tag{3.67}$$

and the product of spacetime vectors is

$$A \cdot B = -A^+ B^- - A^- B^+ + \sum_i A^i B^i. \tag{3.68}$$

The equation of motion for X^+ then becomes

$$\partial_+\partial_- X^+ = 0, \tag{3.69}$$

the general solution of which is

$$X^+ = X_L^+(\sigma^+) + X_R^+(\sigma^-). \tag{3.70}$$

Now we must fix the gauge. Because of reparametrisation invariance, we can choose the coordinates as

$$X_L^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^+, \quad X_R^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^-. \tag{3.71}$$

The outcome of this choice of gauge is that

$$X^+ = x^+ + \alpha' p^+ \tau. \tag{3.72}$$

This gauge is called the light-cone gauge. Equation (3.72) fixes the reparametrisation invariance, but may give rise to new conditions in addition to the already existing ones, namely,

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0. \tag{3.73}$$

Now let us look at equation of motion for X^-:

$$\partial_+\partial_- X^- = 0. \tag{3.74}$$

The general solution is given by

$$X^- = X_L^-(\sigma^+) + X_R^-(\sigma^-). \tag{3.75}$$

This solution is completely determined by constraints (3.73). The first constraint is

$$2\partial_+X^-\partial_+X^+ = \sum_{i=1}^{D-2}\partial_+X^i\partial_+X^i, \tag{3.76}$$

which together with $\partial_+X^+ = \alpha' p^+$ gives

$$\partial_+X_L^- = \frac{1}{\alpha' p^+}\sum_{i=1}^{D-2}\partial_+X^i\partial_+X^i, \tag{3.77}$$

and

$$\partial_-X_R^- = \frac{1}{\alpha' p^+}\sum_{i=1}^{D-2}\partial_-X^i\partial_-X^i. \tag{3.78}$$

So $X^-(\sigma^+,\sigma^-)$ is determined in terms of other fields up to an integration constant. Now we write the mode expansion for $X_{L,R}^-$ as

$$X_L^-(\sigma^+) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^-\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^- e^{-in\sigma^+}, \tag{3.79}$$

$$X_R^-(\sigma^-) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^-\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\sigma^-}, \tag{3.80}$$

where x^- is an integration constant and p_-, α_n^-, and $\bar{\alpha}_n^-$ are found from (3.77) and (3.78). The oscillator modes are

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p'}\sum_{n=-\infty}^{+\infty}\sum_{i=1}^{D-2}\alpha_{n-m}^i\alpha_m^i. \tag{3.81}$$

A special case of this is

$$\frac{\alpha'}{2}p^- = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0}\alpha_n^i\alpha_{-n}^i\right) \tag{3.82}$$

We also have an equation for p^- in terms of $\bar{\alpha}_0^-$,

$$\frac{\alpha'}{2}p^- = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0}\bar{\alpha}_n^i\bar{\alpha}_{-n}^i\right) \tag{3.83}$$

from here we can write the string mass-shell condition

$$\begin{aligned} M^2 = -p_\mu p^\mu = 2p^+p^- + \sum_{i=1}^{D-2} p^i p^i \\ = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\bar{\alpha}_{-n}^i\bar{\alpha}_n^i. \end{aligned} \tag{3.84}$$

3.2.1 Quantisation

To quantise the closed string we first have to impose commutation relations. Some are

$$[\hat{x}^i,\hat{p}^j] = i\delta^{ij} \quad \text{and} \quad [\alpha_n^i,\alpha_m^j] = [\bar{\alpha}_n^i,\bar{\alpha}_m^j] = n\delta^{ij}\delta_{n+m,0}, \tag{3.85}$$

which follow from (3.3) and (3.4). These commutation relations hold only when we are quantising physical degrees of freedom. For $\hat{x}^+$ and $\hat{p}_-$ the commutation relation is

$$[\hat{x}^+,\hat{p}^-] = -i. \tag{3.86}$$

This relation means that we can choose states to be eigenstates of $\hat{p}^\mu$, with $\mu = 0,1,\ldots,D-1$, but the constraints (3.82) and (3.83) must hold. We define the vacuum state $|0;p\rangle$ such that

$$\hat{p}^\mu|0;p\rangle = p^\mu|0;p\rangle \quad \text{and} \quad \alpha_n^i|0;p\rangle = \bar{\alpha}_n^i|0;p\rangle = 0 \quad \text{for } n>0, \tag{3.87}$$

and the Fock space is built by acting on $|0;p\rangle$ with the operators α_{-n}^i and $\bar{\alpha}_{-n}^i$ for $n>0$. The difference from covariant quantisation is that we act only with transverse oscillators which carry a spatial index $i = 1,\ldots,D-2$. So the Hilbert space is positive defined and we do not have ghosts.

3.2.2 *The constraints*

p^- is not an independent variable of the theory, and we have to impose constraints (3.82) and (3.83) to define physical states. In the classical theory this constaints are the mass-shell conditions. But on the quantum level we have the problem of normal ordering. If we choose all operators to be normal-ordered we will gain a constant a. In the same way as in previous sections we find the mass in the light-cone gauge as

$$M^2 = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\bar{\alpha}^i_{-n}\bar{\alpha}^i_n - a\right). \tag{3.88}$$

Now we introduce the operators

$$N = \sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n \quad \text{and} \quad \bar{N} = \sum_{i=1}^{D-2}\sum_{n>0}\bar{\alpha}^i_{-n}\bar{\alpha}^i_n, \tag{3.89}$$

which resemble the number operators except that the summations in their definitions exclude the time coordinate and one of the space coordinates $(D-1)$. With these operators we can write the mass of the string as

$$M^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\bar{N}-a). \tag{3.90}$$

This expression looks similar to the expression (3.41) that we got before for the mass, just lucking some of the coordinates . Now the open question is how to fix a. We will try to answer this question in the next section.

3.2.3 *The zeta function regularization*

In this section our goal is to find a sensible value for the normal ordering constant a that appears in the expresions (3.41) and (3.90) for the mass of the string. We will do this in a heuristic way.

Let us look at the following classical result without normal ordering,

$$\frac{1}{2}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n = \frac{1}{2}\sum_{n<0}\alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n, \tag{3.91}$$

where we left the summation over $i = 1,\dots,D-2$ implicit. Now we try to normal-order the above expressions by moving all α^i_n, the annihilation operator, to the right for $n > 0$ and to the left for $n < 0$. The second term on the right-hand side of equation (3.91) is already normal-ordered and we just have to adjust its first term, which gives

$$\frac{1}{2}\sum_{n<0}[\alpha^i_n\alpha^i_{-n} - n(D-2)] + \frac{1}{2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n = \sum_{n>0}\alpha^i_{-n}\alpha^i_n + \frac{1}{2}(D-2)\sum_{n>0} n. \tag{3.92}$$

The last term in this expression diverges (badly!), but this divergence has a physical interpretation: it is the sum of the zero-point energies of an infinite number of harmonic oscillators (which may or may not be observable). We may try to "sum" that term by by extension of so called zeta-function (see, e. g., [11]). The zeta-function is defined, for $Re(s) > 1$, by the sum

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

But $\zeta(s)$ has a unique analytic continuation to all values of s. In particular

$$\zeta(-1) = -\frac{1}{12}.$$

Due to this result sum in equation (3.92) is[(2)]

$$\sum_{n=1}^{+\infty} n = -\frac{1}{12}. \tag{3.93}$$

[(2)] This "result" is clearly outragious out of its context and should not be taken at face value!

In agreement with the last result we must have $a = -\frac{1}{24}(D-2)$, which together with equation (3.90) gives

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\bar{N} - \frac{D-2}{24}\right) \tag{3.94}$$

for the mass of the closed relativistic bosonic string.

3.3 The string spectrum

In this section we will analyse the spectrum of a free closed single string.

3.3.1 The tachyon

We want to find the mass-shell constraint. Let us start with the ground state $|0;p\rangle$ defined in equation (3.87). If we do not have excited oscillators, $N = 0$ and

$$M^2 = -\frac{1}{\alpha'}\frac{D-2}{6}. \tag{3.95}$$

We got for any $D > 2$ a negative mass-squared. We call these kind of hypothetical particles with negative mass-squared tachyons. In the special theory of relativity, tachyons can be interpreted as particles that travel faster than light. Let us briefly discuss the QFT interpretation for the tachyons. Suppose that we have a field $T(X)$ in spacetime whose quanta give rise to tachyons. In this case, the mass-squared of the particle can be expressed as

$$M^2 = \left.\frac{\partial^2 V(T)}{\partial T^2}\right|_{T=0}. \tag{3.96}$$

The negative mass-squared for the tachyon shows that we are expanding around a maximum of the potential $V(T)$ for the tachyon field. The string theory thus sits on an unstable point, and we do not know if the potential has a good minimum point elsewhere. So the real behaviour of the tachyon in the bosonic string theory is tricky. But tachyons do not cause problems in fermionic string theory, which will not be discussed.

3.3.2 First excited states

Let us now look at the first excited states of the theory. To get them we first act on $|0;p\rangle$ with a creation operator α^{j}_{-1}, then equation (3.90) tells us that we need also to act with the operator $\bar{\alpha}^{i}_{-1}$. So we have $(D-2)^2$ particles in the first excited state given by

$$\bar{\alpha}^{i}_{-1}\alpha^{j}_{-1}|0;p\rangle, \tag{3.97}$$

each one with mass-squared

$$M^2 = \frac{4}{\alpha'}\Big(1-\frac{D-2}{24}\Big). \tag{3.98}$$

Here we seem to be in trouble. The operators α^i and $\bar{\alpha}^i$ each transform in the vector representation of $SO(D-2) \subset SO(1,D-1)$ which is manifest in light-cone gauge. But we want these states to fix into some full representation of the Lorentz group $SO(1,D-1)$. As we already mentioned in Sec. 3.2, it is not easy to see Lorentz invariance in the light-cone gauge. To continue, we will use Wigner's classification of representations of the Poincaré group. First we have massive particles in $R^{1,D-1}$, where $R^{1,D-1}$ is the group of translations. When we go to the rest frame of the particle by setting $p^\mu = (p,0,\dots,0)$, we can watch how any internal index transforms under the little group $SO(D-1)$ of spatial rotations. As a consequence any massive particle must form a representation of $SO(D-1)$. But particles described by equation (3.97) have $(D-2)^2$ states. We cannot pack $(D-2)^2$ states into the $SO(D-1)$ representation, so the first excited states of the string cannot form a massive representation of the D-dimensional Poincaré group. The way out from this situation is to pick massless states, but then we cannot go to the rest frame. We can just set a spacetime momentum for the particle to the form $p^\mu = (p,0,\dots,0,p)$. In this case particles represent the little group $SO(D-2)$, so massless particles have less internal states than the massive ones. The first excited states (3.97) sit in a representation of $SO(D-2)$. And if we want quantum theory to preserve the initial $SO(1,D-1)$ Lorentz symmetry, the first excited states should be massless. Using $M^2 = 0$ and equation (3.98) we get

$$D = 26,$$

which is the critical dimension of the bosonic string.

3.3.3 Higher excited states

We kept the Lorentz invariance of the first excited states by choosing $D = 26$, to make the states massless. For the higher excited states we still have indexes in the range $i, j = 1, \ldots, D-2 = 24$. From equation (3.94) all the higher excited states are massive, and must form representation of $SO(D-1)$. Let us see what we can do this time. Let us look at the string at level $N = \bar{N} = 2$. In the right-moving sector we have two states: $\alpha^i_{-1}\alpha^j_{-1}|0\rangle$ and $\alpha^i_{-2}|0\rangle$. The same holds in the left-moving sector, so we have in total

$$\left(\alpha^i_{-1}\alpha^j_{-1} \oplus \alpha^i_{-2}\right) \otimes \left(\bar{\alpha}^i_{-1}\bar{\alpha}^j_{-1} \oplus \bar{\alpha}^i_{-2}\right)|0;p\rangle \tag{3.99}$$

states. These states have mass $M^2 = 4/\alpha'$. In the left-moving sector we have

$$\frac{1}{2}(D-2)(D-1) + (D-2) = \frac{1}{2}D(D-1) - 1 \tag{3.100}$$

states, which fit into the $SO(D-1)$ representation. So after we fix $D = 26$, the Lorentz invariance is preserved for all the excited states. We conclude that to quantise the bosonic string in the light-cone gauge we have to set

$$a = 1 \quad \text{and} \quad D = 26. \tag{3.101}$$

4

Conformal field theory and strings

In this chapter we introduce the two dimensional conformal field theories and their connection with strings. The ideas and techniques presented here are used in the BRST quantisation of the string. We draw our notes mostly from the books by Becker, Becker and Schwartz [9] and by Polchinsky [14].

4.1 Introduction to conformal field theory

A conformal field theory is a theory that is invariant under conformal transformations. We introduce conformal transformations as changes of coordinates $\sigma^\alpha(\sigma) \to \bar{\sigma}^\alpha(\sigma)$ under which the metric transforms as

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma) g_{\alpha\beta}(\sigma). \tag{4.1}$$

We already encountered a particular case of these transformations earlier; see (2.43). Looking at the transformations we conclude that the theory is rescaling invariant but is sensitive to the change of angles. The transformations (4.1) can have different interpretations. If $g_{\alpha\beta}(\sigma)$ is a dynamical metric, we have a gauge symmetry. If $g_{\alpha\beta}(\sigma)$ is fixed, the transformation is physical, and we have a global symmetry with conserved currents. For the Polyakov action the metric is dynamical and the transformations (4.1) are residual gauge transformations, which can be undone by applying the Weyl transformations. Conformal field theory is a subset of QFT, but since the theory does not have a preferred length scale, it is true only for massless excitations.

4.1.1 Euclidean worldsheet

It is simpler to deal with CFT on Euclidean worldsheets. To go from a Minkowki worldsheet to an Euclidean worldsheet we use a Wick rotation $\sigma^0 \to i\sigma^0$. On an Euclidean worldsheet, the metric $g_{\alpha\beta}$ is positive definite. The Euclidean worldsheet coordinates are defined as $(\sigma^1, \sigma^2) = (\sigma^1, i\sigma^0)$. We also have complex worldsheet coordinates defined as

$$z = \sigma^1 + i\sigma^2 \quad \text{and} \quad \bar{z} = \sigma^1 - i\sigma^2; \tag{4.2}$$

they are analogue of the light-cone coordinates. The holomorphic and antiholomorphic derivatives are given, respectively, by

$$\partial_z = \partial = \frac{1}{2}(\partial_1 - i\partial_2) \quad \text{and} \quad \partial_{\bar{z}} = \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2). \tag{4.3}$$

We will usually work in flat Euclidean space with the metric

$$ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dz d\bar{z}. \tag{4.4}$$

In components the metric tensor $g_{\alpha\beta}$ can be written as [(1)]

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \quad \text{and} \quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}. \tag{4.5}$$

On the Euclidean worldsheet the delta function is defined as

$$\int d^2z \, \delta(z, \bar{z}) = 1, \tag{4.6}$$

[(1)] We use (4.2) and (4.3) to calculate the components of the metric tensor:

$$g_{\bar{z}z} = g_{z\bar{z}} = \partial_z \sigma^\mu \partial_{\bar{z}} \sigma^\nu g_{\mu\nu} = \partial_z \sigma^1 \partial_{\bar{z}} \sigma^1 + \partial_z \sigma^2 \partial_{\bar{z}} \sigma^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

By the same token, $g_{zz} = g_{\bar{z}\bar{z}} = 0$.

and the expression $\int d^2\sigma\delta(\sigma) = 1$ is also satisfied. Vectors in this space are defined as

$$v^z = (v^1 + iv^2) \quad \text{and} \quad v^{\bar{z}} = (v^1 - iv^2), \tag{4.7}$$

for the upper indices, and

$$v_z = \frac{1}{2}(v^1 - iv^2) \quad \text{and} \quad v_{\bar{z}} = \frac{1}{2}(v^1 + iv^2), \tag{4.8}$$

for the lower indices.

4.1.2 The conformal group in D dimensions

A D-dimensional manifold is called conformally flat if the metric element reads

$$ds^2 = e^{\omega(x)} dx \cdot dx, \tag{4.9}$$

where $\omega(x)$ is allowed to be x-dependent. The conformal group is the subgroup of the group of general transformations that preserves the conformal flatness of the metric. Conformal transformations preserve the angles and distort the lengths. The conformal group contains four types of transformations:

Translations: $x^\mu \to x^\mu + a^\mu$, with a^μ constant. For the infinitesimal case $\delta x^\mu = a^\mu$, where a^μ is an infinitesimal constant.

Rotations: $x^\mu \to \omega^\mu_\nu x^\mu$. This transformation is the same for finite and infinitesimal cases, but for infinitesimal rotations ω^μ_ν is an infinitesimal constant obeying $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

Scale transformations: $x^\mu \to \lambda x^\mu$. This transformation is also the same for finite and infinitesimal cases; in the infinitesimal case λ is an infinitesimal constant.

Special conformal trasformations: The conformal group includes inversions,

$$x^\mu \to \frac{x^\mu}{x^2}, \tag{4.10}$$

which maps

$$dx \cdot dx \to \frac{dx \cdot dx}{(x^2)^2}, \tag{4.11}$$

so that the metric remains conformally flat. If we perform a sequence of inversion, translation, and inversion again we then obtain the special conformal transformation

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2bx + b^2 x^2}. \tag{4.12}$$

If b^μ is infinitesimal, we have

$$\delta x^\mu = b^\mu x^2 - 2x^\mu bx. \tag{4.13}$$

Taken together, these transformations mean that the following infinitesimal transformations are conformal,

$$\delta x^\mu = a^\mu + \omega^\mu_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu (b \cdot x), \tag{4.14}$$

with the parameters $a^\mu, \omega^\mu_\nu, \lambda$ and b^μ infinitesimal constants. Altogether we have

$$D + \frac{1}{2}D(D-1) + 1 + D = \frac{1}{2}(D+2)(D+1) \tag{4.15}$$

linearly independent infinitesimal conformal transformations, so this is the number of generators of the conformal group. The conformal group for $D > 2$ Euclidean dimensions is the indefinite special orthogonal group $SO(D+1,1)$.

4.1.3 Conformal transformations in two dimensions

Conformal transformations of the flat space are

$$z \to z' = f(z) \quad \text{and} \quad \bar{z} \to \bar{z}' = \bar{f}(\bar{z}), \tag{4.16}$$

which are any holomorphic change of the coordinates. Under this transformations the metric transforms as

$$ds^2 = dzd\bar{z} \to \left|\frac{df}{dz}\right|^2 dzd\bar{z} \tag{4.17}$$

which is of form (4.1). The infinitesimal conformal trasformations are of the form

$$z \to z' = z - \varepsilon z^{n+1} \quad \text{and} \quad \bar{z} \to \bar{z}' = \bar{z} - \bar{\varepsilon}_n \bar{z}^{n+1}, \tag{4.18}$$

such that for each $n \in \mathbb{Z}$ we have the following infinitesimal generators

$$l_n = -z^{n+1}\partial \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1}\bar{\partial}. \tag{4.19}$$

Since $n \in \mathbb{Z}$ we have an infinite number of conformal transformations, which is typical for two dimensions. In higher dimensions, the space of conformal transformations is a finite dimensional group. These generators satisfy the classical Virasoro algebra

$$\begin{aligned}
[l_m, l_n] = z^{m+1}\partial_z(z^{n+1}\partial_z) - z^{n+1}\partial_z(z^{m+1}\partial_z) = \\
= (n+1)z^{m+n+1}\partial_z - (m+1)z^{n+m+1}\partial_z = \\
= -(m-n)z^{m+n+1}\partial_z = (m-n)l_{m+n},
\end{aligned} \tag{4.20}$$

and in the same way

$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \quad \text{and} \quad [l_m, \bar{l}_n] = 0. \tag{4.21}$$

As already had been told in previous chapters, in the quantum case the Virasoro algebra acquire a central extension

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{4.22}$$

with c the central charge. In two dimensions the Virasoro operators are the modes of the energy-momentum tensor, which generates the conformal transformations. Central extension means that the constant term is multiplying the unit operator, which is adjoined to the Lie algebra. The infinite-dimensional conformal group in two dimensions contains a finite-dimensional subgroup generated by $l_{0,\pm1}$ and $\bar{l}_{0,\pm1}$, infinitesimally from (4.18) the transformations are

$$\begin{aligned} l_{-1}: &\quad z \to z - \varepsilon, \\ l_0: &\quad z \to z - \varepsilon z, \\ l_1: &\quad z \to z - \varepsilon z^2. \end{aligned} \tag{4.23}$$

The finite form of the group transformations is

$$z \to \frac{az+b}{cz+d} \quad \text{with} \quad a,b,c,d \in \mathbb{C}, \quad ad - bc = 1. \tag{4.24}$$

This transformation is invariant under the change of signs of all parameters, so this is the group of projective conformal transformations $SL(2,\mathbb{C})/\mathbb{Z}_2 = SO(3,1)$, which is the two-dimensional case of $SO(D+1,1)$, where $D > 2$. The finite-dimensional subgroup of the two-dimensional infinite conformal group is called the restricted conformal group.

4.2 Classical aspects of CFT

We will see some aspects of classical theories that are invariant under conformal transformations.

4.2.1 The stress-energy tensor

The stress-energy (or energy-momentum) tensor is very important for any field theory. The energy-momentum tensor is a matrix of conserved currents arising from translational invariance $\delta\sigma^\alpha = \varepsilon^\alpha$, with ε^α an infinitesimal constant.

Let us derive the stress-energy tensor in the general case, supposing that we are in a flat space, with $g_{\alpha\beta} = \eta_{\alpha\beta}$. To derive conserved currents we take $\varepsilon = \varepsilon(\sigma)$ a function of the spacetime coordinates. Then the change in the action must be

$$\delta S = \int d^2\sigma J^\alpha \partial_\alpha \varepsilon \tag{4.25}$$

for some function J^α, which depends on the fields involved. δS should vanish for constant ε. If the equations of motion are satisfied, we must have $\delta S = 0$ for any $\varepsilon(\sigma)$. This means that

$$\partial_\alpha J^\alpha = 0, \tag{4.26}$$

so that the J^α are conserved.

Now we will look at translational invariance. If ε becomes a function of the worldsheet coordinates, the action changes as in (4.25). To find what J^α is in this case, we consider a coupling to a dynamical background metric $g_{\alpha\beta}(\sigma)$. We know that in this case the theory is invariant under the transformation

$$\delta\sigma^\alpha = \varepsilon^\alpha(\sigma), \tag{4.27}$$

as long as the metric also changes simultaneously by

$$\delta g_{\alpha\beta} = \partial_\alpha \varepsilon_\beta + \partial_\beta \varepsilon_\alpha. \tag{4.28}$$

Due to the last statement we have

$$\delta S = -\int d^2\sigma \frac{\partial S}{\partial g_{\alpha\beta}} = -2\int d^2\sigma \frac{\partial S}{\partial g_{\alpha\beta}} \partial_\alpha \varepsilon_\beta. \tag{4.29}$$

Now we have a new translational constant, which with some coefficient will be called stress-energy tensor

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\partial S}{\partial g^{\alpha\beta}}. \tag{4.30}$$

On a curved worldsheet the energy-momentum tensor is covariantly conserved

$$\nabla^{\alpha} T_{\alpha\beta} = 0; \tag{4.31}$$

the proof of (4.31) is given in Appendix A. S is invariant under infinitesimal Weyl transformations, which transform only the metric, $\delta g^{\alpha\beta} = -\delta\lambda g^{\alpha\beta}$. From here we can prove that the stress-energy tensor is traceless,

$$0 = -\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta\lambda} = -\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\alpha\beta}}\frac{\delta g^{\alpha\beta}}{\delta\lambda} = T_{\alpha\beta} g^{\alpha\beta} = T^{\alpha}_{\alpha}. \tag{4.32}$$

We can see the same in the complex coordinates:

$$\begin{aligned} T_{z\bar{z}} &= \frac{\partial\sigma^{\alpha}}{\partial z}\frac{\partial\sigma^{\beta}}{\partial\bar{z}} T_{\alpha\beta} = \\ &= \frac{\partial\sigma^{1}}{\partial z}\frac{\partial\sigma^{1}}{\partial\bar{z}} T_{11} + \frac{\partial\sigma^{2}}{\partial z}\frac{\partial\sigma^{1}}{\partial\bar{z}} T_{21} + \frac{\partial\sigma^{1}}{\partial z}\frac{\partial\sigma^{2}}{\partial\bar{z}} T_{12} + \frac{\partial\sigma^{2}}{\partial z}\frac{\partial\sigma^{2}}{\partial\bar{z}} T_{22} = \\ &= \frac{1}{4}(T_{11} + T_{22}) = 0. \end{aligned} \tag{4.33}$$

The conservation conditions (4.31) can be written as

$$\bar{\partial} T_{zz} = 0 \quad \text{and} \quad \partial T_{\bar{z}\bar{z}} = 0. \tag{4.34}$$

So, there are just two non-vanishing components of the stress-energy tensor, one holomorphic $T_{zz} = T_{zz}(z)$ and one antiholomorphic $T_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}}(\bar{z})$. In the new notation we can write the mode expansion for the closed string as

$$X^{\mu}_{R}(\sigma,\tau) \to X^{\mu}_{R}(z) = \frac{1}{2}x^{\mu} - \frac{i}{4}p^{\mu}\ln z + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^{\mu}_{n} z^{-n}, \tag{4.35}$$

and

$$X^{\mu}_{L}(\sigma,\tau) \to X^{\mu}_{L}(\bar{z}) = \frac{1}{2}x^{\mu} - \frac{i}{4}p^{\mu}\ln\bar{z} + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^{\mu}_{n}\bar{z}^{-n}; \tag{4.36}$$

for the holomorphic derivatives we have

$$\partial X^\mu(z,\bar{z}) = -\frac{i}{2}\sum_{n=-\infty}^{+\infty} \alpha_n^\mu z^{-n-1}, \tag{4.37}$$

and

$$\bar{\partial} X^\mu(z,\bar{z}) = -\frac{i}{2}\sum_{n=-\infty}^{+\infty} \tilde{\alpha}_n^\mu \bar{z}^{-n-1}. \tag{4.38}$$

We can find the holomorphic component of the energy-momentum tensor as

$$T_X(z) = -2 : \partial X \cdot \partial X := \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}}. \tag{4.39}$$

while for the antiholomorphic component we find

$$\tilde{T}_X(\bar{z}) = -2 : \bar{\partial} X \cdot \bar{\partial} X := \sum_{n=-\infty}^{\infty} \frac{\tilde{L}_n}{\bar{z}^{n+2}}. \tag{4.40}$$

4.2.2 Noether currents

The stress-energy tensor provides the Noether currents for translation. Now we need to see how they look for other conformal transformations. Let us suppose that we have an infinitesimal change

$$z' = z + \varepsilon(z), \quad \bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z}); \tag{4.41}$$

when ε is constant we have translations, while $\varepsilon(z) \sim z$ corresponds to rotations and dilatations. We again go from $\varepsilon(z) \to \varepsilon(z,\bar{z})$. In this case the action changes like

$$\delta S = -\int d^2\sigma \frac{\partial S}{\partial g^{\alpha\beta}} \delta g^{\alpha\beta} = \frac{1}{2\pi}\int d^2\sigma T_{\alpha\beta}(\partial^\alpha \delta\sigma^\beta). \tag{4.42}$$

Since in a conformal theory $T_{z\bar{z}} = 0$,

$$\delta S = \frac{1}{2\pi}\int d^2z \frac{1}{2}\left[T_{zz}(\partial^z \delta z) + T_{\bar{z}\bar{z}}(\partial^{\bar{z}} \delta\bar{z})\right] = \frac{1}{2\pi}\int d^2z \left[T_{zz}\partial_{\bar{z}}\varepsilon + T_{\bar{z}\bar{z}}\partial_z\bar{\varepsilon}\right]. \tag{4.43}$$

If ε is holomorphic and $\bar{\varepsilon}$ is antiholomorphic then $\delta S = 0$, which means that we have a symmetry. Let us treat z and $\bar{z}$ as independent variables. First we look at symmetry

$$\delta z = \varepsilon(z), \quad \delta \bar{z} = 0. \tag{4.44}$$

Since ε already depends on z, we go from $\varepsilon(z) \to \varepsilon(z) f(\bar{z})$ for some function f. In this case

$$\delta S = \frac{1}{2\pi} \int d^2 z [T_{zz} \varepsilon(z) \partial_{\bar{z}} f(\bar{z})], \tag{4.45}$$

so that

$$J^z = 0 \quad \text{and} \quad J^{\bar{z}} = T_{zz} \varepsilon(z) \tag{4.46}$$

and we see that J is holomorphic. For the case of symmetry

$$\delta \bar{z} = \bar{\varepsilon}(\bar{z}), \quad \delta z = 0 \tag{4.47}$$

we get

$$\bar{J}^z = T_{\bar{z}\bar{z}}(\bar{z}) \bar{\varepsilon}(\bar{z}) \quad \text{and} \quad \bar{J}^{\bar{z}}, \tag{4.48}$$

with an antiholomorphic current $\bar{J}$.

4.3 Quantum aspects of CFT

In this section we will discuss some aspects of quantum theories that are preserved under conformal transformations.

4.3.1 Operator product expansion

In CFT the term field refers to any local expression that we can write down. This includes Φ, but also $\partial^n \Phi$ and composite operators such as $e^{i\Phi}$. All these are different fields in a CFT. The number of fields in CFT is infinite.

Let $\{O_i\}$ be a set of all local operators in a CFT. Two local operators inserted at nearby points can be closely approximated by a string of operators at one of these points. We call such an approximation as an operator product expansion, or OPE. The OPE for two operators is given by

$$O_i(z,\bar{z})O_j(\omega,\bar{\omega}) = \sum_k c_{ij}^k(z-\omega,\bar{z}-\bar{\omega})O_k(\omega,\bar{\omega}), \tag{4.49}$$

where $c_{ij}^k(z-\omega,\bar{z}-\bar{\omega})$ are a set of functions which depend only on the separation between the two operators. Equations of type (4.49) are statements which hold as operator insertions inside time-ordered corelation functions,

$$\langle O_i(z,\bar{z})O_j(\omega,\bar{\omega})\cdots\rangle = \sum_k c_{ij}^k(z-\omega,\bar{z}-\bar{\omega})\langle O_k(\omega,\bar{\omega})\cdots\rangle \tag{4.50}$$

where the $\cdots$ can be any operator insertion that we choose. Important aspects of OPEs are:

- The correlation functions are always time-ordered—as far as the OPE is concerned, everything commutes, $O_i(z,\bar{z})O_j(\omega,\bar{\omega}) = O_j(\omega,\bar{\omega})O_i(z,\bar{z})$ (except for Grassmann objects, which get a minus sign).
- Operator insertions are arbitrary, but they should be a distance large compared to $|z-\omega|$. So in a CFT the OPEs are exact statements and have a radius of covergence equal to the distance to the nearest insertion. OPEs have singular behaviour as $z \to \omega$.
- OPEs contain the same information as the commutation relations, and also tell us the way operators transform under symmetries.

4.3.2 Ward identities

Ward identities are operator analogs to Noether's theorem in quantum field theories. Let us derive them in a general theory with a symmetry. We will use the partition function of the field theory written as a path integral,

$$Z = \int \mathscr{D}\phi e^{-S[\phi]}, \tag{4.51}$$

where ϕ stands for all fields in a path integral sense. The quantum symmetry should be such that an infinitesimal translation

$$\phi' = \phi + \varepsilon\delta\phi \tag{4.52}$$

keeps the action and the measure invariant

$$S[\phi'] = S[\phi] \quad \text{and} \quad \mathscr{D}\phi' = \mathscr{D}\phi. \tag{4.53}$$

If we promote $\varepsilon \to \varepsilon(\sigma)$, the action and the measure do not stay invariant anymore, but to leading order in ε, the change is proportional to $\partial\varepsilon$. In this case

$$\begin{aligned} Z &\to \int \mathscr{D}\phi' \exp(-S[\phi']) \\ &= \int \mathscr{D}\phi \exp(-S[\phi'] - \frac{1}{2\pi}\int d^2\sigma\sqrt{g}J^\alpha\partial_\alpha\varepsilon) \\ &= \int \mathscr{D}\phi e^{-S[\phi]}(1 - \frac{1}{2\pi}\int d^2\sigma\sqrt{g}J^\alpha\partial_\alpha\varepsilon); \end{aligned} \tag{4.54}$$

here J^α can have contributions from the measure transformation. The integral for the partition function changed but the function itself could not change, so we must ensure that

$$Z' - Z = 0. \tag{4.55}$$

More explicitly,

$$\int \mathscr{D}\phi e^{-S[\phi]]}\left(\int d^2\sigma\sqrt{g}J^\alpha\partial_\alpha\varepsilon\right) = 0. \tag{4.56}$$

Since this condition must hold for all ε, the vacuum expectation value of the divergence of the current must vanish

$$\langle\partial_\alpha J^\alpha\rangle = 0. \tag{4.57}$$

Let us see the result when we have other insertions in the path integral. Let O_i be any general expressions formed from the ϕ fields. Under a given symmetry transformation they will change like

$$O_i \to O_i + \varepsilon \delta O_i. \tag{4.58}$$

We again go from $\varepsilon \to \varepsilon(\sigma)$. Let us pick $\varepsilon(\sigma)$ such as all $O_i(\sigma_i)$ are far from the operator insertions, in this case

$$\delta O_i(\sigma_i) = 0. \tag{4.59}$$

And the derivation is the same as in the first case, which gives

$$\langle \partial_\alpha J^\alpha(\sigma) O_1(\sigma_1) \cdots O_n(\sigma_n) \rangle = 0 \quad \text{for} \quad \sigma \neq \sigma_i. \tag{4.60}$$

Because this is true for operator insertions far from σ, we have

$$\partial_\alpha J^\alpha = 0. \tag{4.61}$$

Now let us see the case when some of the operators are in the same point as J^α. Let us suppose that $\varepsilon(\sigma)$ is in the same region only with σ_1. We put $\varepsilon(\sigma)$ constant in this region, and zero in the outside of it. Now the original correlation function is

$$\frac{1}{Z} \int \mathscr{D}\phi e^{-S[\phi]} \left(1 - \frac{1}{2\pi} \int d^2\sigma \sqrt{g} J^\alpha \partial_\alpha \varepsilon \right) (O_1 + \varepsilon \delta O_1) O_2 \cdots O_n. \tag{4.62}$$

In leading order in ε it gives

$$-\frac{1}{2\pi} \int_\varepsilon d^2\sigma \sqrt{g} \partial_\alpha \langle J^\alpha(\sigma) O_1(\sigma_1) \cdots \rangle = \langle \delta O_1(\sigma_1) \cdots \rangle \tag{4.63}$$

with the integral over non-zero ε. This is the Ward identity.

Ward identities for conformal transformations

Ward identities hold for all types of symmetries. Let us see what happens for conformal transformation in two dimensions. Let $\hat{n}^\alpha$ be the unit vector normal to the boundary. Then for any J^α, we have, by Stokes' theorem [15],

$$\int_\varepsilon \partial_\alpha J^\alpha = \oint_{\partial\varepsilon} J_\alpha \hat{n}^\alpha = \oint_{\partial\varepsilon} (J_1 d\sigma^2 - J_2 d\sigma^1) = -i \oint_{\partial\varepsilon} (J_z dz - J_{\bar{z}} d\bar{z}), \tag{4.64}$$

both in Cartesian and complex coordinates. For complex coordinates we use the notation

$$J_z = \frac{1}{2}(J_1 - iJ_2) \quad \text{and} \quad J_{\bar{z}} = \frac{1}{2}(J_1 + iJ_2). \tag{4.65}$$

Using (4.64) for Ward identities we get

$$\frac{i}{2\pi} \oint_{\partial\varepsilon} dz \langle J_z(z,\bar{z}) O_1(\sigma_1) \cdots \rangle - \frac{i}{2\pi} \oint_{\partial\varepsilon} d\bar{z} \langle J_{\bar{z}}(z,\bar{z}) O_1(\sigma_1) \cdots \rangle = \langle \delta O_1(\sigma_1) \cdots \rangle. \tag{4.66}$$

This holds for any J in two dimensions. Now we go to the currents (4.46, 4.48) found for conformal transformations. J_z is holomorphic and $J_{\bar{z}}$ is antiholomorphic, so by the residue theorem we get

$$\frac{i}{2\pi} \oint_{\partial\varepsilon} dz J_z(z) O_1(\sigma_1) = -\mathrm{Res}\,[J_z O_1]. \tag{4.67}$$

Using last equation we can write the OPE of the two operators as

$$J_z(z) O_1(\omega.\bar{\omega}) = \cdots + \frac{\mathrm{Res}\,[J_z O_1(\omega,\bar{\omega})]}{z-\omega} + \cdots. \tag{4.68}$$

Now if we treat z and $\bar{z}$ as independent variables, we will have a Ward identity in two parts. For $\delta z = \varepsilon(z)$,

$$\delta O_1(\sigma_1) = -\mathrm{Res}\,[J_z(z) O_1(\sigma_1)] = -\mathrm{Res}\,[\varepsilon(z) T_{zz} O_1(\sigma_1)], \tag{4.69}$$

and for $\varepsilon\bar{z} = \bar{\varepsilon}(\bar{z})$

$$\delta O_1(\sigma_1) = -\mathrm{Res}\,[\bar{J}_{\bar{z}}(\bar{z})O_1(\sigma_1)] = -\mathrm{Res}\,[\bar{\varepsilon}(\bar{z})T_{\bar{z}\bar{z}}(\bar{z})O_1(\sigma_1)]. \tag{4.70}$$

So if we know the OPE between an operator and the stress-energy tensor $T(z)$ and $T_{\bar{z}\bar{z}}(\bar{z})$, we know how the operator transforms under the conformal symmetry.

4.3.3 Primary operators

In this section we will discuss some tipes of OPEs. Let us start from some simple cases of conformal symmetries.

Translations: $\delta z = \varepsilon = constant$. For infinitesimal translation all operators transform as

$$O(z-\varepsilon) = O(z) - \varepsilon\partial O(z) + \cdots. \tag{4.71}$$

The Noether current $J_z(z)$ for the translation is the stress-energy tensor T. Using equation (4.68) and analogues of equations (4.69,4.70) with $J_z(z) = T$, we conclude that OPE of T with any operator O must be

$$T(z)O(\omega,\bar{\omega}) = \cdots + \frac{\partial O(\omega,\bar{\omega})}{z-\omega} + \cdots, \tag{4.72}$$

and for $\bar{T}$ by analogy

$$\bar{T}(\bar{z})O(\omega,\bar{\omega}) = \cdots + \frac{\bar{\partial} O(\omega,\bar{\omega})}{\bar{z}-\bar{\omega}} + \cdots. \tag{4.73}$$

Rotation and scaling: This are the transformations of form

$$z \to z + \varepsilon z \quad \text{and} \quad \bar{z} \to \bar{z} + \bar{\varepsilon}\bar{z}, \tag{4.74}$$

which present rotations for imaginary ε and dilatations for real ε. Not all the operators have good transformation properties under these transformations. We state that we always can choose a basis of local operators that have good transformation properties under rotations and dilatations.

It is said that an operator O has weight $(h, \tilde{h})$ if, under $\delta z = \varepsilon z$ and $\delta \bar{z} = \bar{\varepsilon}\bar{z}$ it transforms as

$$\delta O = -\varepsilon(hO + z\partial O) - \bar{\varepsilon}(\tilde{h}O + \bar{z}\bar{\partial}O). \tag{4.75}$$

The terms with ∂O are general terms, and terms hO and $\tilde{h}O$ are special to operators which are eigenstates of dilatations and rotations.

Let us make some comments about the last definition.

- h and $\tilde{h}$ are real numbers; for all operators in a unitary CFT $h, \tilde{h} \geqslant 0$.
- The eigenvalue under rotation is the "old friend" spin given by $s = h - \tilde{h}$. The scaling dimension Δ of an operator is $\Delta = h + \tilde{h}$.
- Rotation acts on the underlying coordinates as

  $$L = i(\sigma^1\partial_2 - \sigma^2\partial_1) = z\partial + \bar{z}\bar{\partial}.$$

  Dilatation acts on the underlying coordinates as the Dilaton operator

  $$D = \sigma^\alpha\partial_\alpha = z\partial + \bar{z}\bar{\partial}.$$

- The rescaling dimension Δ is the common one found for the fields and operators by the use of dimensional analysis—for example, $\Delta[\partial] = +1$—but we have to remember that the quantum dimension is not always the same as the classical one.

The Noether current $J_z(z)$ for the translation is zT. Using equation (4.68) and analogues of equations (4.69,4.70) with $J_z(z) = zT$, we conclude that OPE of T with any operator O must be

$$T(z)O(\omega, \bar{\omega}) = \cdots + h\frac{O(\omega, \bar{\omega})}{(z-\omega)^2} + \frac{\partial O(\omega, \bar{\omega})}{z-\omega} + \cdots, \tag{4.76}$$

and for $\bar{T}$ by the analogy

$$\bar{T}(\bar{z})O(\omega,\bar{\omega}) = \cdots + \bar{h}\frac{O(\omega,\bar{\omega})}{(\bar{z}-\bar{\omega})^2} + \frac{\bar{\partial}O(\omega,\bar{\omega})}{\bar{z}-\bar{\omega}} + \cdots . \tag{4.77}$$

Primary operators

A primary operator is one that has OPE that does not have singularities in higher order than $(z-\omega)^{-2}$ for T and $(\bar{z}-\bar{\omega})^{-2}$ for $\bar{T}$. This means we know all the singularities in the OPE of TO, such that we can see the transformation under all conformal transformations. For $\delta z = \varepsilon(z)$ transformation

$$\begin{aligned} \delta O(\omega,\bar{\omega}) &= -\mathrm{Res}\left[\varepsilon z T(z) O(\omega,\bar{\omega})\right] \\ &= -\mathrm{Res}\left[\varepsilon(z)\left(h\frac{O(\omega,\bar{\omega})}{(z-\omega)^2} + \frac{\partial O(\omega,\bar{\omega})}{z-\omega} + \cdots\right)\right] \end{aligned} \tag{4.78}$$

We require $\varepsilon(z)$ not to have singularities at $z=\omega$. Then we can Taylor expand it

$$\varepsilon(z) = \varepsilon(\omega) + \varepsilon'(\omega)(z-\omega) + \cdots . \tag{4.79}$$

Using the expansion we find the infinitesimal change of a primary operator under a general conformal transformation $\delta z = \varepsilon(z)$

$$\delta O(\omega,\bar{\omega}) = -h\varepsilon'(\omega)O(\omega,\bar{\omega}) - \varepsilon(\omega)\partial O(\omega,\bar{\omega}), \tag{4.80}$$

and the simulat equation for the antiholomorphic transformations $\delta\bar{z} = \bar{\varepsilon}(\bar{z})$. The mentioned equations hold for infinitesimal conformal transformations. For general transformation $z \to \tilde{z}(z)$ and $\bar{z} \to \tilde{\bar{z}}(\bar{z})$ of a primary operator we have

$$O(z,\bar{z}) \to \tilde{O}(\tilde{z},\tilde{\bar{z}}) = \left(\frac{\partial\tilde{z}}{\partial z}\right)^{-h}\left(\frac{\partial\tilde{\bar{z}}}{\partial\bar{z}}\right)^{-\bar{h}} O(z,\bar{z}). \tag{4.81}$$

The spectrum of weights $(h,\bar{h})$ of primary fields is one of main interests of CFT.

4.4 The central charge

The stress-energy tensor is not a primary operator. T is an operator of weight $(h,\bar{h}) = (2,0)$ in any CFT, because $T_{\alpha\beta}$ has dimension $\Delta = h+\bar{h} = 2$, because we obtain the energy by integrating over space, and it has spin $s = h-\bar{h} = 2$ because it is asymmetric 2-tensor. Similarly $\bar{T}$ has weight $(0,2)$. So the TT OPE is

$$T(z)T(\omega) = \cdots + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} + \cdots . \tag{4.82}$$

The same for $\bar{T}\bar{T}$. Since each term in the equation has dimension $\Delta = 4$, we can only have operators of the form

$$\frac{O_n}{(z-\omega)^n},$$

where $\Delta[O_m] = 4-n$. But as we stated before in unitary CFT there are no operators with $h,\bar{h} < 0$. So we can have only singular term of order $(z-\omega)^{-4}$, it gets into the OPE with some constant coefisient. So the TT OPE can be written as

$$T(z)T(\omega) = \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} + \cdots , \tag{4.83}$$

and the same for $\bar{T}\bar{T}$

$$\bar{T}(\bar{z})\bar{T}(\bar{\omega}) = \frac{\bar{c}/2}{(\bar{z}-\bar{\omega})^4} + \frac{2\bar{T}(\bar{\omega})}{(\bar{z}-\bar{\omega})^2} + \frac{\partial \bar{T}(\bar{\omega})}{\bar{z}-\bar{\omega}} + \cdots . \tag{4.84}$$

c and $\bar{c}$ are the central charges, we can guess that they measure the number of degrees of freedom of the theory. Let us now see why we do not have singular term of order $(z-\omega)^{-3}$. The reason is that for the OPE $T(z)T(\omega) = T(\omega)T(z)$, because TT is included in the correlation functions, and they are time ordered. But the term proportional to $(z-\omega)^{-3}$ is not invariant under $z \leftrightarrow \omega$. Now we will see why $(z-\omega)^{-1}$ term is invariant under $z \leftrightarrow \omega$. We Taylor expand $T(z) = T(\omega) + (z-\omega)\partial T(\omega) + \cdots$ and $\partial T(z) = \partial T(\omega) + \cdots$, and put in equation (4.83), which gives

$$T(\omega)T(z) = \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)+2(z-\omega)\partial T(\omega)}{(z-\omega)^2} - \frac{\partial T(\omega)}{z-\omega} = T(z)T(\omega). \quad (4.85)$$

We saw that equation (4.83) is indeed the full equation fot TT OPE.

The transformation of energy

Let us see how T transforms.

$$\begin{aligned}\delta T(\omega) &= -\mathrm{Res}\left[\varepsilon(z)T(z)T(\omega)\right] \\ &= -\mathrm{Res}\left[\varepsilon(z)\left(\frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} + \cdots\right)\right].\end{aligned} \quad (4.86)$$

If $\varepsilon(z)$ has no singularities, we expand

$$\varepsilon(z) = \varepsilon(\omega) + \varepsilon'(\omega)(z-\omega) + \frac{1}{2}\varepsilon''(\omega)(z-\omega)^2 + \frac{1}{6}\varepsilon'''(\omega)(z-\omega)^3 + \cdots, \quad (4.87)$$

from which we find infinitesimal transformation of T

$$\delta T(\omega) = -\varepsilon(\omega)\partial T(\omega) - 2\varepsilon'(\omega)T(\omega) - \frac{c}{12}\varepsilon'''(\omega). \quad (4.88)$$

And for finite conformal transformation $z \to \bar{z}(z)$ T transforms as

$$\bar{T}(\bar{z}) = \left(\frac{\partial \bar{z}}{\partial z}\right)^{-2}\left[T(z) - \frac{c}{12}S(\bar{z},z)\right], \quad (4.89)$$

where $S(\bar{z},z)$ is known as the Schwarzian derivative, given by

$$S(\bar{z},z) = \left(\frac{\partial^3 \bar{z}}{\partial z^3}\right)\left(\frac{\partial \bar{z}}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 \bar{z}}{\partial z^2}\right)^2\left(\frac{\partial \bar{z}}{\partial z}\right)^{-2}. \quad (4.90)$$

4.4.1 The Weyl anomaly

In the classical theory $T_\alpha^\alpha = 0$. The story is different in the quantum case. On the flat space $\langle T_\alpha^\alpha \rangle$ also vanishes, but on a curved background it does not. We will try to show that in curved space

$$\langle T_\alpha^\alpha \rangle = -\frac{c}{12} R, \tag{4.91}$$

where R is the Ricci scalar of the 2-dimensional worldsheet. Before derivation let us make some coments:

- Equation (4.91) holds for any state of the theory. This comes from the fact that $\langle T_\alpha^\alpha \rangle$ regulates short distance divergences in the theory, and on short distance all finite energies are basically the same.
- Since (4.91) is true for any state it should be something that depends only on the background metric. This something should be local and with $\Delta = 2$. The only candidate is the Ricci scalar R. But we do not know with which coeficient.
- Any 2-dimensional metric can be written as $g_{\alpha\beta} = e^{2\omega}\delta_{\alpha\beta}$. In these coordinates $R = -2e^{-2\omega}\partial^2\omega$. So R depends on ω explicitely. Equation (4.91) tells that any conformal theory with $c \neq 0$ has at least one physical observable $\langle T_\alpha^\alpha \rangle$, which takes different values on background related by Weyl transformation ω. This result is called Weyl anomaly.
- In 4-dimensions there is also a Weyl anomaly

  $$\langle T_\mu^\mu \rangle_{4d} = \frac{c}{16\pi^2} c_{\rho\sigma\kappa\lambda} c^{\rho\sigma\kappa\lambda} - \frac{a}{16\pi^2} \bar{R}_{\rho\sigma\kappa\lambda} \bar{R}^{\rho\sigma\kappa\lambda}, \tag{4.92}$$

  with c the Weyl tensor and $\bar{R}$ the dual of the Riemann tensor.
- Equation (4.91) is in the left-moving sector, but we also have the equation $\langle T_\alpha^\alpha = -\frac{\bar{c}}{12} R \rangle$ in the right-moving sector. But for consistency in fixed curved

backgrounds $c = \bar{c}$. This is a gravitational anomaly.

Now let us prove equation (4.91). First we need to derive $T_{z\bar{z}} - T_{\omega\bar{\omega}}$ OPE. From energy conservation

$$\partial T_{z\bar{z}} = -\bar{\partial} T_{zz}$$

Using this

$$\partial_z T_{z\bar{z}}(z,\bar{z}) \partial_\omega T_{\omega\bar{\omega}}(\omega,\bar{\omega}) = \bar{\partial}_{\bar{z}} T_{zz}(z,\bar{z}) \bar{\partial}_{\bar{\omega}} T_{\omega\omega}(\omega,\bar{\omega}) = \bar{\partial}_{\bar{z}} \bar{\partial}_{\bar{\omega}} \left[\frac{c/2}{(z-\omega)^4} + \cdots \right]. \tag{4.93}$$

In the last equation we have an antiholomorphic derivative of a holomorphic function, which is supposed to be zero, but since we have singularities in T it is not. Using the fact that $\bar{\partial}_{\bar{z}} \partial_z |z-\omega|^2 = \bar{\partial}_{\bar{z}} \frac{1}{z-\omega} = 2\pi\delta(z-\omega, \bar{z}-\bar{\omega})$[(2)]

$$\bar{\partial}_{\bar{z}} \bar{\partial}_{\bar{\omega}} \frac{1}{(z-\omega)^4} = \frac{1}{6} \bar{\partial}_{\bar{z}} \bar{\partial}_{\bar{\omega}} \left(\partial_z^2 \partial_\omega \frac{1}{z-\omega} \right) = \frac{\pi}{3} \partial_z^2 \partial_\omega \bar{\partial}_{\bar{\omega}} \delta(z-\omega, \bar{z}-\bar{\omega}). \tag{4.94}$$

Putting the last result into equation (4.93) we get rid of derivatives $\partial_z \partial_\omega$ on both sides. And finally we get

$$T_{z\bar{z}}(z,\bar{z}) T_{\omega,\bar{\omega}}(\omega,\bar{\omega}) = \frac{c\pi}{6} \partial_z \bar{\partial}_{\bar{\omega}} \delta(z-\omega, \bar{z}-\bar{\omega}), \tag{4.95}$$

---

(2) The proof comes from the fact that $\partial^2 \ln(\sigma - \sigma')^2 = 4\pi\delta(\sigma - \sigma')$, just with a correction factor of 2. If we set $\sigma' = 0$ and integrate over $d^2\sigma$ on the right-hand side we get

$$\int d^2\sigma 4\pi\delta(\sigma) = 4\pi,$$

and applying Stokes' theorem on the left-hand side we find

$$\int d^2\sigma \partial^2 \ln(\sigma_1^2 + \sigma_2^2) = \int d^2\sigma \partial^\alpha \left(\frac{2\sigma_\alpha}{\sigma_1^2 + \sigma_2^2} \right) = 2 \oint \frac{\sigma_1 d\sigma_2 - \sigma_2 d\sigma 1}{\sigma_1^2 + \sigma_2^2}.$$

In polar coordinates $\sigma_1 + i\sigma_2 = re^{i\theta}$ and

$$\int d^2\sigma 4\pi\delta(\sigma) = 2 \int \frac{r^2 d\theta}{r^2} = 4\pi,$$

and our statement is proved.

which is zero except for the singularity $z \to \omega$. It is a contact term between operators. Let us now see how $\langle T^{\alpha}_{\alpha} \rangle$ changes under ageneral shift of the metrioc $\delta g_{\alpha\beta}$ Using the deffinition of the energy-momentum tensor (4.30), we have

$$\begin{aligned}\delta\langle T^{\alpha}_{\alpha}(\sigma)\rangle &= \delta\int \mathscr{D}\phi e^{-S} T^{\alpha}_{\alpha}(\sigma) = \\ &= \frac{1}{4\pi}\int \mathscr{D}\phi e^{-S}(T^{\alpha}_{\alpha}(\sigma)\int d^2\sigma'\sqrt{g}\delta g^{\beta\gamma} T_{\beta\gamma}(\sigma')).\end{aligned} \tag{4.96}$$

If we change to a Weyl transformation, the change to flat metric is $\delta g_{\alpha\beta} = 2\omega\delta_{\alpha\beta}$, so the change in the inverse metric is $\delta g^{\alpha\beta} = -2\omega\delta^{\alpha\beta}$, which gives

$$\delta\langle T^{\alpha}_{\alpha}(\sigma\rangle) = -\frac{1}{2\pi}\int \mathscr{D}\phi e^{-S}(T^{\alpha}_{\alpha}(\sigma)\int d^2\sigma'\omega(\sigma')T^{\beta}_{\beta}(\sigma')). \tag{4.97}$$

Now using equation (4.95) we will determine the Weyl anomaly. To do so we change from complex to Cartesian coordinates. We have

$$T^{\alpha}_{\alpha}(\sigma)T^{\beta}_{\beta}(\sigma) = 16 T_{z\bar{z}}(z,\bar{z})\, T_{\omega\bar{\omega}}(\omega\bar{\omega}),$$

and

$$8\partial_z\bar{\partial}_{\bar{\omega}}\delta(z-\omega,\bar{z}-\bar{\omega}) = -\partial^2\delta(\sigma-\sigma').$$

Together with (4.95) this gives

$$T^{\alpha}_{\alpha}(\sigma)T^{\beta}_{\beta}(\sigma') = -\frac{c\pi}{3}\partial^2\delta(\sigma-\sigma') \tag{4.98}$$

this OPE in cartesian coordinates. Now putting it into equation (4.97) and integrating by parts gives

$$\delta\langle T^{\alpha}_{\alpha}\rangle = \frac{c}{6}\partial^2\omega.$$

using that $R = -2e^{-2\omega}\partial^2\omega$ and putting $e^{-2\omega} \approx 1$ since we are discussing an infinitesimal case, we get

$$\langle T^{\alpha}_{\alpha}\rangle = -\frac{c}{12}R. \tag{4.99}$$

The result is called the Weyl anomaly.

4.5 The Virasoro algebra

4.5.1 Primary field and the radial quantisation

The fields of CFT are characterized by their conformal dimensions, which specify their scale transformations. Φ is a conformal field of conformal dimension $(h,\tilde{h})$ if

$$\Phi(z,\bar{z}) \to \left(\frac{\partial\omega}{\partial z}\right)^{h}\left(\frac{\partial\bar{\omega}}{\partial\bar{z}}\right)^{\tilde{h}}\Phi(\omega,\bar{\omega}), \tag{4.100}$$

when we have $z\to\omega(z)$ finite conformal transformation. Under an infinitesimal coordinate transformation $z\to z+\varepsilon(z)$ (4.100) gives

$$\delta\Phi(z,\bar{z}) = (h\partial\varepsilon+\varepsilon\partial+\bar{h}\bar{\partial}\bar{\varepsilon}+\bar{\varepsilon}\bar{\partial})\Phi(z,\bar{z}). \tag{4.101}$$

The two-dimensional conformal algebra is infinite-dimensional, so we have an infinite number of conserved charges, which are the Virasoro generators. And the Virasoro generators are the modes of the energy-momentum tensor. If we have an infinitesimal conformal transformation

$$\delta z = \varepsilon(z) \quad \text{and} \quad \delta\bar{z} = \tilde{\varepsilon}(\bar{z}), \tag{4.102}$$

the conserved charge that generates this transformation is

$$Q = Q_{\varepsilon}+Q_{\tilde{\varepsilon}} = \frac{1}{2\pi i}\oint_C \left[T(z)\varepsilon(z)dz+\tilde{T}(\bar{z})\tilde{\varepsilon}(\bar{z})d\bar{z}\right] \tag{4.103}$$

The integral is over some contour C. The variation of a field $\Phi(z,\bar{z})$ under a conformal transformation is given by

$$\delta_{\varepsilon}\Phi(z,\bar{z}) = [Q_{\varepsilon},\Phi(z,\bar{z})] \quad \text{and} \quad \delta_{\tilde{\varepsilon}}\Phi(z,\bar{z}) = [Q_{\tilde{\varepsilon}},\Phi(z,\bar{z})]. \tag{4.104}$$

From (4.103) and (4.104) we have

$$
\begin{aligned}
\delta_\varepsilon \Phi(\omega,\bar{\omega}) &= [Q_\varepsilon, \Phi(\omega,\bar{\omega})] \\
&= \left[\frac{1}{2\pi i}\oint_C T(z)\varepsilon(z)dz, \Phi(\omega,\bar{\omega})\right] \\
&= \frac{1}{2\pi i}\oint_C dz\,\varepsilon(z)[T(z),\Phi(\omega,\bar{\omega})] \\
&= \frac{1}{2\pi i}\oint_{|z|>|\omega|} dz\,\varepsilon(z)T(z)\Phi(\omega,\bar{\omega}) \\
&- \frac{1}{2\pi i}\oint_{|z|<|\omega|} dz\,\varepsilon(z)\Phi(\omega,\bar{\omega})T(z).
\end{aligned}
\tag{4.105}
$$

The operator products $T(z)\Phi(\omega,\bar{\omega})$ and $\Phi(\omega,\bar{\omega})T(z)$ have convergent series expansions only for radially ordered operators. This means that the integral

$$\oint dz\,\varepsilon(z)T(z)\Phi(\omega,\bar{\omega})$$

should be evaluated along a contour with $|z| > |\omega|$, and similarly the second integral in (4.105) should be evaluated along a contour with $|z| < |\omega|$. Difference of these two contours gives a contour z that encircles the point ω, see fig. 4.1. The radial ordering goes as follows

$$
: R(A(z)B(\omega)) := \begin{cases} A(z)B(\omega), & \text{for} \quad |z| > |\omega|; \\ B(\omega)A(z), & \text{for} \quad |z| < |\omega|. \end{cases}
\tag{4.106}
$$

With this notation we get that

$$
\delta_\varepsilon \Phi(\omega,\bar{\omega}) = \frac{1}{2\pi i}\oint_{C(\omega)} dz R(T(z)\varepsilon(z)\Phi(\omega)).
\tag{4.107}
$$

Conformal transformations take into account only the short distance behaviour of energy-momentum tensor close to an operator insertion $\Phi(\omega,\bar{\omega})$. So to evaluate this contour integral we need to know singular terms in operator product expansion

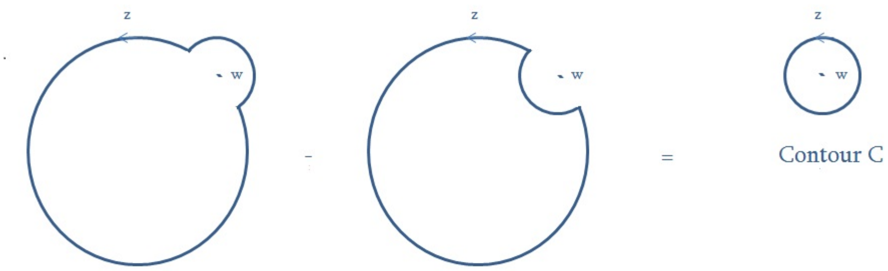

Figure 4.1: Integration contour for the z variable in eqs. (4.105) and (4.107).

for $z \to \omega$

$$A_i(z)B_j(\omega) \sim \sum_k c_{ij}^k(z-\omega)O_k(\omega), \tag{4.108}$$

where O_k is a complete set of local operators and c_{ij}^k are singular coefficients. The singular part of the OPE between $T(z)$ and $\Phi(\omega,\bar{\omega})$ is given in equation (4.76), just changing $O(\omega,\bar{\omega})$ for $\Phi(\omega,\bar{\omega})$.

4.5.2 The Virasoro algebra

In this section we obtain the Virasoro algebra for the CFT. The modes L_n and $\bar{L}_n$ of energy-momentum tensor are

$$L_n = \oint \frac{dz}{2i\pi} T(z) z^{n+1} \quad , \quad \bar{L}_n = \oint \frac{d\bar{z}}{2i\pi} \bar{T}(\bar{z}) \bar{z}^{n+1} \tag{4.109}$$

If we use radial ordering and equation (4.83), we will obtain the following algebra

$$
\begin{aligned}
[L_n, L_m] &= \oint_{c_0} \frac{d\omega}{2i\pi} \oint_{c_\omega} \frac{dz}{2i\pi} \omega^{n+1} z^{m+1} R(T(z)T(\omega)) = \\
&= \oint_{c_0} \frac{d\omega}{2i\pi} \oint_{c_\omega} \frac{dz}{2i\pi} \omega^{n+1} z^{m+1} \left(\frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{(z-\omega)} \right) \\
&= \oint_{c_0} \frac{d\omega}{2i\pi} \omega^{n+1} \left(\oint_{c_\omega} \frac{dz}{2i\pi} \frac{z^{m+1} c/2}{(z-\omega)^4} + \oint_{c_\omega} \frac{dz}{2i\pi} \frac{z^{m+1} 2T(\omega)}{(z-\omega)^2} \right. \\
&\left. + \oint_{c_\omega} \frac{dz}{2i\pi} \frac{z^{m+1} \partial T(\omega)}{z-\omega} \right) \\
&= \oint_{c_0} \frac{d\omega}{2i\pi} \omega^{n+1} \left(\frac{c}{12}(m+1)m(m-1)\omega^{m-2} + 2(m+1)\omega^m T(\omega) \right. \\
&\left. + \omega^{m+1} \partial_\omega T(\omega) \right) \\
&= \frac{c}{12}(m^3 - m) \oint_{c_0} \frac{\omega^{m+n-1} d\omega}{2i\pi} + 2(m+1) \oint_{c_0} \frac{d\omega}{2i\pi} T_\omega \omega^{m+n+1} \\
&+ \oint_{c_0} \frac{d\omega}{2i\pi} \omega^{m+n+2} \partial_\omega T(\omega) \\
&= \frac{c}{12}(m^3 - m)\delta_{m+n} + 2(m+1)L_{m+n} \\
&- \oint_{c_0} \frac{d\omega}{2i\pi} (m+n+2)\omega^{m+n+1} T(\omega) \\
&= \frac{c}{12}(m^3 - m)\delta_{m+n} + 2(m+1)L_{m+n} - (m+n+2)L_{m+n} \\
&= \frac{c}{12}(m^3 - m)\delta_{m+n} + (m-n)L_{m+n}.
\end{aligned}
\tag{4.110}
$$

$\bar{L}_n$ modes satisfy the same algebra, and $[L_n, \bar{L}_n] = 0$. This is the Virasoro algebra. And every CFT defines a representation of this algebra, depending on central charge c. The central charge part is due to the Weyl rescaling, not to reparametrisation as we could think.

4.5.3 Representation of the Virasoro algebra

Let us suppose we have a state $|\psi\rangle$, that is an eigenstate of L_0 and $\bar{L}_0$

$$L_0|\psi\rangle = h|\psi\rangle, \quad \bar{L}_0|\psi\rangle = \bar{h}|\psi\rangle. \tag{4.111}$$

By acting with the L_n operator we can get further states with eigenvalues

$$L_0 L_n|\psi\rangle = (L_n L_0 - nL_n)|\psi\rangle = (h-n)L_n|\psi\rangle, \tag{4.112}$$

so the L_n are rasing and lowering operators depending on the sign of n. Then $n > 0$, L_n is lowering and L_{-n} rasing operator. If the spectrum is bounded below where should be some states annihilated by all L_n and $\bar{L}_n$ for $n > 0$. Such states are called primary or highest weight states

$$L_n|\psi\rangle = \bar{L}_n|\psi\rangle = 0 \quad \text{for all} \quad n > 0. \tag{4.113}$$

This states are the states of lowest energy. So we can now built the Virasoro algebra by acting on the primary states with raising operators L_{-n} with $n > 0$. The states obtained like this are called descendants. The whole set of states is called a Verna module. They are the irreducible representations of the Virasoro algebra. So knowing the primary states spectrum brings us to knowing the spectrum of whole theory. For the vacuum state $|0\rangle = h = 0$

$$L_n|0\rangle = 0 \quad \text{for all} \quad n \geqslant 0. \tag{4.114}$$

This state preserves the maximum number of symmetries. Central charge term in Virasoro algebra does not allow us to have $L_n|0\rangle = 0$ for all n. The states in the Verma module do not need to be all independent. Some linear combinations of them can vanish. These linear combinations are called null states. They appear depending on values of h and c.

4.5.4 Unitarity of the theory

To be consistent the theory should be unitary. This means that probabilities are conserved in Minkovski signature space-time. Let us take a glance on Hamiltonian density

$$\mathscr{H} = T_{\omega\omega} + T_{\bar{\omega}\bar{\omega}} = \sum_n L_n e^{-in\sigma^+} + \bar{L}_n e^{-in\sigma^-}. \tag{4.115}$$

So for the Hamiltonian to be Hermitian, we require

$$L_n = L^+_{-n}. \tag{4.116}$$

We get some strong constraints on the structure of CFTs. Here are some of them

- $h \geqslant 0$: to prove this we write down the norm

  $$|L_{-1}|\psi\rangle|^2 = \langle\psi|L_{+1}L_{-1}|\psi\rangle = \langle\psi|[L_{+1}, L_{-1}]|\psi\rangle = 2h\langle\psi|\psi\rangle \geqslant 0 \tag{4.117}$$

  the only state with $h = 0$ is the vacuum state $|\psi\rangle$.

- $c > 0$: to see this we look at

  $$|L_{-n}|0\rangle|^2 = \langle 0|[L_n, L_{-n}]|0\rangle = \frac{c}{12}n(n^2-1) \geqslant 0 \tag{4.118}$$

  this gives $c \geqslant 0$. If $c = 0$ the only state in the vacuum module is the vacuum itself. The vacuum is also the only state in the whole theory.

5

The Fadeev-Popov method and the BRST symmetry

In this chapter we describe the Fadeev-Popov method devised to quantise the Polyakov bosonic action by the path integral method and the BRST (Becchi-Rouet-Stora-Tyutin) symmetry that can also be used to quantise the bosonic string. We limit ourselves to the description of the BRST symmetry; we do not quantise the bosonic string by the BRST method. References for this chapter are [11, 12, 14].

5.1 The path integral

In Euclidean space, the Polyakov action, first introduced by Polyakov in 1981 [16], is given by

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}\,g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\delta_{\mu\nu}, \tag{5.1}$$

where the integration is over all worldsheet metrics $g_{\alpha\beta}$ and all embedding coordinates X^μ. The corresponding path integral is

$$Z = \int \mathscr{D}g_{\alpha\beta}(\sigma,\tau)\mathscr{D}X^\mu(\sigma,\tau)\exp\left(iS_P[X,g]\right). \tag{5.2}$$

Because of the gauge invariance, the integral is not well defined. This comes from integrating infinitely many times over field configurations related by gauge transformations, which are physically equivalent configurations. To turn (5.2) into a well defined path integral we have to split the integration over all field configurations into two pieces corresponding, respectively, to physically distinct configurations and to gauge transformations. More explicitly, if we divide the original path integral (5.2) by the infinite volumes of its symmetry groups, we remove the degeneracy factor due to the gauge transformations. For the Polyakov action there are two types of gauge symmetries:

Diffeomorphisms

$$\sigma^\alpha \to \sigma'^\alpha(\sigma^\alpha), \tag{5.3}$$

under which

$$g_{\alpha\beta}(\sigma) \to g'_{\alpha\beta}(\sigma') = \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\beta} g_{\gamma\delta}(\sigma), \tag{5.4}$$

Weyl transformations

$$\sigma^\alpha \to \sigma^\alpha, \tag{5.5}$$

under which

$$g_{\alpha\beta}(\sigma) \to \Omega(\sigma) g_{\alpha\beta}(\sigma). \tag{5.6}$$

We have thus two symmetry groups, and so two infinite volumes V_{Diff} and V_{Weyl}. With the last argument the path integral becomes

$$Z' = \frac{1}{V_{Diff}V_{Weyl}} \int \mathscr{D}g_{\alpha\beta}(\sigma,\tau)\mathscr{D}X^\mu(\sigma,\tau)\exp\left(iS_P[X,g]\right). \tag{5.7}$$

To decompose our integration variables into physical fields and gauge orbits we need to find the associated Jacobian of the change of variables. The standard method to do this was first introduced by Fadeev and Popov in a nearly unintelligible 2 pages paper [17]. The method works for all types of gauge symmetries.

5.1.1 The Faddeev-Popov method

To describe the so called Faddeev-Popov method of finding the Jacobian, we schematically denote both gauge symmetries mentioned before by ζ. The change of the metric under a general transformation is $g \to g^\zeta$. This is shorthand for

$$g_{\alpha\beta}(\sigma) \to g^\zeta_{\alpha\beta}(\sigma') = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\beta} g_{\gamma\delta}(\sigma). \tag{5.8}$$

In two dimensions we can put the metric into any form, let us say $\hat{g}$, which represents our choice of gauge fixing. But for $2d$ metric we have some problems:

- We cannot put any $2d$ metric into the $\hat{g}$ form we want, we can do this only locally. Globally, it is true only for the worldsheet that has the topology of a cylinder or a sphere, but not for higher genus surfaces.
- Fixing the metric locally does not fix all the gauge symmetries—conformal symmetries are still present.

Our goal is to integrate over all physically inequivalent configurations. Let us first consider the integral over the gauge orbit of $\hat{g}$. For some value of ζ, the configuration g^{ζ} will be the same as the original metric g. If we put a delta function in the integral we will have

$$\int \mathscr{D}\zeta\,\delta(g-\hat{g}^{\zeta}) = \frac{1}{\Delta_{FP}[g]}, \tag{5.9}$$

which is not equal to one because of the Jacobian factor. The Jacobian factor is the $\Delta_{FP}^{-1}[g]$, and its inverse Δ_{FP} is called the Faddeev-Popov determinant. There are some facts that we should remember. First, all this procedure is formal and possess the same difficulties of the path integral approach. Second, we assume that our gauge fixing is good, and we do not have physically equivalent configurations. The integral over gauge transformations $\mathscr{D}\zeta$ meets only once with the delta function (means that we integrate only ones through every distinct configuration), such that we do not have discrete ambiguities (copies of QCD). Finally, the measure is analogous to the Haar measure for Lie groups, namely, it is invariant under left and right actions

$$\mathscr{D}\zeta = \mathscr{D}(\zeta'\zeta) = \mathscr{D}(\zeta\zeta'). \tag{5.10}$$

Let us carry out the Faddeev-Popov procedure. We first insert the factor of unity into the path integral

$$1 = \Delta_{FP}[g]\int \mathscr{D}\zeta\,\delta(g-\hat{g}^{\zeta}). \tag{5.11}$$

We call the new path integral $Z[\hat{g}]$ since it depends on the choice of $\hat{g}$:

$$\begin{aligned} Z[\hat{g}] &= \frac{1}{V_{Diff}V_{Weyl}} \int \mathscr{D}\zeta \mathscr{D}X \mathscr{D}g \Delta_{FP}[g] \delta(g-\hat{g}^{\zeta}) \exp\left(-S_P[X,g]\right) \\ &= \frac{1}{V_{Diff}V_{Weyl}} \int \mathscr{D}\zeta \mathscr{D}X \Delta_{FP}[\hat{g}^{\zeta}] \exp\left(-S_P[X,\hat{g}^{\zeta}]\right) \\ &= \frac{1}{V_{Diff}V_{Weyl}} \int \mathscr{D}\zeta \mathscr{D}X \Delta_{FP}[\hat{g}] \exp\left(-S_P[X,\hat{g}]\right). \end{aligned} \tag{5.12}$$

In the last line we use that the action and the Faddeev-Popov determinant are gauge invariant.[(1)] Now nothing depends on ζ, so the integral over it cancels with the volumes and we get

$$Z[\hat{g}] = \int \mathscr{D}X \Delta_{FP}[\hat{g}] \exp\left(-S_P[X,\hat{g}]\right). \tag{5.13}$$

This is the integral over physically inequivalent configurations. The Faddeev-Popov determinant is indeed the Jacobian factor that we need.

5.1.2 The Faddeev-Popov determinant

Now we need to compute $\Delta_{FP}[\hat{g}]$ defined as

$$\int \mathscr{D}\zeta \delta(g-\hat{g}^{\zeta}) = \Delta_{FP}^{-1}[g]. \tag{5.14}$$

Let us assume that the delta function $\delta(\hat{g}-\hat{g}^{\zeta})$ is non-zero when $\zeta = 0$. We parameterize the infinitesimal Weyl transformation by $\omega(\sigma)$ and the infinitesimal diffeomorphism by $\delta\sigma^{\alpha} = v^{\alpha}(0)$. With these definitions the change in the metric is written as

$$\hat{g}_{\alpha\beta} = 2\omega\hat{h}_{\alpha\beta} + \nabla_{\alpha}v_{\beta} + \nabla_{\beta}v_{\alpha}. \tag{5.15}$$

---

(1) Proof: $\Delta_{FP}^{-1}[g^{\zeta}] = \int \zeta' \delta(g^{\zeta} - \hat{g}^{\zeta'}) = \int \zeta' \delta(g - \hat{g}^{\zeta^{-1}\zeta'}) = \int \zeta'' \delta(g - \hat{g}^{\zeta''}) = \Delta_{FP}^{-1}[g]$, where we use the fact that the measure is invariant.

Using the last formulae we get

$$\Delta_{FP}^{-1}[\hat{g}] = \int \mathscr{D}\zeta(2\omega\hat{g}_{\alpha\beta} + \nabla_\alpha v_\beta + \nabla_\beta v_\alpha). \tag{5.16}$$

Since we are near the identity we can replace the integral $\int \mathscr{D}\zeta$ over the gauge group with the integral $\int \mathscr{D}\omega\mathscr{D}v$ over the Lie algebra group and get

$$\Delta_{FP}^{-1}[\hat{g}] = \int \mathscr{D}\omega\mathscr{D}v(2\omega\hat{g}_{\alpha\beta} + \nabla_\alpha v_\beta + \nabla_\beta v_\alpha). \tag{5.17}$$

Now, by analogy with the Fourier representation of the delta function,

$$\delta(x) = \int dp e^{2\pi i p x},$$

we can insert a “delta functional” into the former functional integral to get

$$\Delta_{FP}^{-1}[\hat{g}] = \int \mathscr{D}\omega\mathscr{D}v\mathscr{D}\beta \exp\left(2\pi i \int d^2\sigma \sqrt{\hat{g}}\, \beta^{\alpha\beta}[2\omega\hat{g}_{\alpha\beta} + \nabla_\alpha v_\beta + \nabla_\beta v_\alpha]\right), \tag{5.18}$$

where $\beta^{\alpha\beta}$ is a symmetric traceless 2-tensor field on the worldsheet, $\beta^{\alpha\beta}\hat{g}_{\alpha\beta} = 0$, since the integral over ω has to be non-zero (it comes with derivatives). Finally we get

$$\Delta_{FP}^{-1}[\hat{g}] = \int \mathscr{D}v\mathscr{D}\beta \exp\left(4\pi i \int d^2\sigma \sqrt{\hat{g}}\, \beta^{\alpha\beta}\nabla_\alpha v_\beta\right). \tag{5.19}$$

We got the Δ_{FP}^{-1}, now we want to get Δ_{FP}. To invert the path integral we just need to go from all bosonic variables to fermionic (Grassmann) variables

$$\begin{aligned} \beta_{\alpha\beta} &\to b_{\alpha\beta}, \\ v^\alpha &\to c^\alpha, \end{aligned} \tag{5.20}$$

where b and c are anticommuting Grassmann-valued fields. These fields are called ghost fields. With these variables we get for the Faddeev-Popov determinant the expression

$$\Delta_{FP}[g] = \int \mathscr{D}b\mathscr{D}c \exp\left(4\pi i \int d^2\sigma \sqrt{g}\, b_{\alpha\beta}\nabla^\alpha c^\beta\right). \tag{5.21}$$

We define the ghost action as

$$S_{ghost} = \frac{1}{2\pi}\int d^2\sigma \sqrt{g}\, b_{\alpha\beta}\nabla^{\alpha} c^{\beta}, \tag{5.22}$$

with this definition and rescaling of c and b, we get

$$\Delta_{FP}[g] = \int \mathscr{D}b\mathscr{D}c \exp[iS_{ghost}]. \tag{5.23}$$

In Euclidean space

$$\Delta_{FP}[g] = \int \mathscr{D}b\mathscr{D}c \exp[-S_{ghost}]. \tag{5.24}$$

By putting equation (5.24) in equation (5.13) we find

$$Z[\hat{g}] = \int \mathscr{D}X\mathscr{D}b\mathscr{D}c \exp(-S_P[X,\hat{g}] - S_{ghost}[b,c,\hat{g}]). \tag{5.25}$$

So from gauge fixing we got extra ghost fields. This ghost fields are to cancel the unphysical gauge degrees of freedom, and we are left with $D-2$ transverse modes of X^{μ}. Lorentz invariance is preserved in this type of quantisation. Now let us simplify the ghost action. In the conformal gauge

$$\hat{g}_{\alpha\beta} = e^{2\omega}\delta_{\alpha\beta}, \tag{5.26}$$

and the determinant is

$$\sqrt{\hat{g}} = e^{2\omega}. \tag{5.27}$$

In complex coordinates, the measure is

$$d^2\sigma = \frac{1}{2}d^2z \tag{5.28}$$

and

$$\nabla^z = g^{z\bar{z}}\nabla_{\bar{z}} = 2e^{-2\omega}\nabla_z. \tag{5.29}$$

$b_{z\bar{z}} = 0$ because $b_{\alpha\beta}$ is traceless. So the ghost action is

$$\begin{aligned} S_{ghost} &= \frac{1}{2\pi}\int \frac{1}{2} d^2 z e^{2\omega}(b_{zz} + b_{\bar{z}\bar{z}})(\nabla^z + \nabla^{\bar{z}})(c^z + c^{\bar{z}}) = \\ &= \frac{1}{2\pi}\int d^2 z (b_{zz} + b_{\bar{z}\bar{z}})(\nabla^z + \nabla^{\bar{z}})(c^z + c^{\bar{z}}) = \\ &= \frac{1}{2\pi}\int d^2 z (b_{zz}\nabla_{\bar{z}} c^z + b_{\bar{z}\bar{z}}\nabla_z c^{\bar{z}}). \end{aligned} \tag{5.30}$$

We know that

$$\nabla_{\bar{z}} c^z = \partial_{\bar{z}} c^z + \Gamma^z_{\bar{z}\alpha} c^\alpha, \tag{5.31}$$

where the Christoffel symbols are

$$\Gamma^z_{\bar{z}\alpha} = \frac{1}{2} g^{z\bar{z}}(\partial_{\bar{z}} g_{\alpha\bar{z}} + \partial_\alpha g_{\bar{z}\bar{z}} - \partial_{\bar{z}} g_{\bar{z}\alpha}) = 0 \quad \text{for} \quad \alpha = z, \bar{z}. \tag{5.32}$$

So the covariant derivative is just an ordinary derivative

$$\nabla_{\bar{z}} c^z = \partial_{\bar{z}} c^z, \tag{5.33}$$

which gives

$$S_{ghost} = \frac{1}{2\pi}\int d^2 z (b_{zz}\partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}). \tag{5.34}$$

At the end we get two free theories. The ghosts are Weyl invariant since they do not depend on ω.

5.2 The ghost CFT

b and c are anticommuting Grassmann variables. Their dynamics is a subject of CFT, we will prove this later. If we define them as

$$\begin{aligned} b = b_{zz}, \quad \bar{b} = b_{\bar{z}\bar{z}} \\ c = c^z, \quad \bar{c} = c^{\bar{z}}. \end{aligned} \tag{5.35}$$

The ghost action will be

$$S_{ghost} = \frac{1}{2\pi} \int d^2z (b\bar{\partial}c + \bar{b}\partial\bar{c}). \tag{5.36}$$

For the equations of motion we get

$$\bar{\partial}b = \partial\bar{b} = \bar{\partial}c = \partial\bar{c} = 0 \tag{5.37}$$

so b and c are holomorphic, but $\bar{b}$ and $\bar{c}$ are antiholomorphic. In addition we have the following conditions:

- periodicity condition for the closed string,

$$b(\sigma + 2\pi) = b(\sigma), \quad c(\sigma + 2\pi) = c(\sigma);$$

- boundary conditions for the open string,

$$b(\sigma) = \bar{b}(\sigma), \quad c = \bar{c}(\sigma) \quad \text{for} \quad \sigma = 0, \pi,$$

  which come from the vanishing of boundaries upon the derivation of the equations of motion.

Now we need to find the stress tensor for the bc ghosts. The derivation is given in the appendix B.

$$T = 2(\partial c)b + c\partial b, \tag{5.38}$$

and for antiholomorphic part

$$\bar{T} = 2(\bar{\partial}\bar{c})\bar{b} + \bar{c}\bar{\partial}\bar{b}. \tag{5.39}$$

5.2.1 Operator product expansion

Now we will find the OPEs by the path integral method. We will do this just for the holomorphic piece of CFT. Using the fact that the integral of a total derivative vanishes for the path integral, we can write

$$\int \mathscr{D}b\mathscr{D}c \frac{\delta}{\delta b(\sigma)}[e^{-S_{ghost}} b(\sigma')] = \int \mathscr{D}b\mathscr{D}c e^{-S_{ghost}} \left[-\frac{1}{2\pi}\bar{\partial}c(\sigma)b(\sigma') + \delta(\sigma - \sigma') \right] = 0. \tag{5.40}$$

Since this is true for any b and c, we have

$$\bar{\partial}c(\sigma)b(\sigma') = 2\pi\delta(\sigma - \sigma'). \tag{5.41}$$

From

$$\int \mathscr{D}b\mathscr{D}c \frac{\delta}{\delta c(\sigma)}[e^{-S_{ghost}} c(\sigma')] = \int \mathscr{D}b\mathscr{D}c e^{-S_{ghost}} \left[-\frac{1}{2\pi}\bar{\partial}b(\sigma)c(\sigma') + \delta(\sigma - \sigma') \right] = 0. \tag{5.42}$$

in the same way we get

$$\bar{\partial}b(\sigma)c(\sigma') = 2\pi\delta(\sigma - \sigma'). \tag{5.43}$$

Integrating both equations using the formula $\bar{\partial}(1/z) = 2\pi\delta(z,\bar{z})$, we find OPEs between the ghost fields

$$\begin{aligned} b(z)c(\omega) &= \frac{1}{z-\omega} + \cdots \\ c(\omega)b(z) &= \frac{1}{\omega - z} + \cdots \end{aligned} \tag{5.44}$$

The OPEs of $b(z)b(\omega)$ and $c(z)c(\omega)$ have no singular parts and vanish as $z \to \omega$. The normal ordered stress-energy tensor of the theory is

$$T(z) = 2 : \partial c(z)b(z) : + : c(z)\partial b(z) :$$

5.2.2 Primary fields

Here we show that b and c are primary fields, with weights $h = 2$ and $h = -1$ respectively. To see this we compute OPEs with the stress-energy tensor. For the c

ghost field

$$\begin{aligned} T(z)c(\omega) &= 2 : \partial c(z) b(z) : c(\omega) + : c(z) \partial b(z) : c(\omega) = \\ &= \frac{2\partial c(z)}{z-\omega} - \frac{c(z)}{(z-\omega)^2} + \cdots = -\frac{c(\omega)}{(z-\omega)^2} + \frac{\partial c(\omega)}{z-\omega} + \cdots , \end{aligned} \tag{5.45}$$

and we see that c has weight -1. For the b ghost field we have

$$\begin{aligned} T(z)b(\omega) &= 2 : \partial c(z) b(z) : b(\omega) + : c(z) \partial b(z) : b(\omega) = \\ &= -\frac{\partial b(z)}{z-\omega} + \frac{2b(z)}{(z-\omega)^2} + \cdots = \frac{2b(\omega)}{(z-\omega)^2} + \frac{\partial b(\omega)}{z-\omega} + \cdots , \end{aligned} \tag{5.46}$$

giving weight 2 for b.

5.2.3 The central charge

To find the central charge for the bc system we compute the TT OPE

$$\begin{aligned} T(z)T(\omega) = 4 &: \partial c(z) b(z) :: \partial c(\omega) b(\omega) : +2 : \partial c(z) b(z) :: c(\omega) \partial b(\omega) : \\ &+ 2 : c(z) \partial b(z) :: \partial c(\omega) b(\omega) : + : c(z) \partial b(z) :: c(\omega) \partial b(\omega) : \end{aligned} \tag{5.47}$$

Using equations (5.44) we find

$$\begin{aligned} T(z)T(\omega) = & \frac{-4}{(z-\omega)^4} + \frac{4 : \partial c(z) b(\omega) :}{(z-\omega)^2} - \frac{4 : b(z) \partial c(\omega) :}{(z-\omega)^2} \\ & - \frac{4}{(z-\omega)^4} + \frac{2 : \partial c(z) \partial b(\omega) :}{z-\omega} - \frac{4 : b(z) c(\omega) :}{(z-\omega)^3} \\ & - \frac{4}{(z-\omega)^4} - \frac{4 : c(z) b(\omega) :}{(z-\omega)^3} + \frac{2 : \partial b(z) \partial c(\omega) :}{z-\omega} \\ & - \frac{1}{(z-\omega)^4} - \frac{: c(z) \partial b(\omega) :}{(z-\omega)^2} + \frac{: \partial b(z) c(\omega) :}{(z-\omega)^2} + \cdots \end{aligned}$$

After Taylor-expanding the terms depending on z by ω we find

$$T(z)T(\omega) = \frac{-13}{(z-\omega)^4} + \frac{2t(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} + \cdots$$

We see that the form of the TT OPE is the same as in equation (4.83), and the central charge for the bc ghost system is $c = -26$. We also learned before that the Weyl symmetry does not have anomalies only when $c = 0$. So we are forced to add new degrees of freedom to the string to cancel the contribution from the ghosts. One of the possibilities is to add D free scalar fields. Each scalar field contributes $c = 1$ to the central charge, so the procedure works if we pick $D = 26$, a well known result for the critical dimension of string theory, cf. Sec. 3.3.

5.3 The Virasoro algebra for ghost fields

For closed strings, the ghost fields can be expanded in Fourier modes in the following way:

$$\begin{aligned} c(\sigma,\tau) &= \sum_{n=-\infty}^{+\infty} \bar{c}_n e^{-in(\tau+\sigma)}, \quad \bar{c}(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} c_n e^{-in(\tau-\sigma)}, \\ b(\sigma,\tau) &= \sum_{n=-\infty}^{+\infty} \bar{b}_n e^{-in(\tau+\sigma)}, \quad \bar{b}(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} b_n e^{-in(\tau-\sigma)}. \end{aligned} \tag{5.48}$$

Using these expansions, the coefficients anticommute like

$$\{b_m, c_n\} = \delta_{m,-n} \quad \text{and} \quad \{b_m, b_n\} = \{c_m, c_n\} = 0, \tag{5.49}$$

and the same holds for the conjugated ones. The Virasoro generators of the theory are

$$\begin{aligned} L_m^{gh} &= \sum_{n=-\infty}^{\infty} (m-n) : b_{m+n} c_{-n} : \\ \bar{L}_m^{gh} &= \sum_{n=-\infty}^{\infty} (m-n) : \bar{b}_{m+n} \bar{c}_{-n} : \end{aligned} \tag{5.50}$$

Using the Virasoro generators we can prove that the ghosts and antighosts are conformal fields, cf. appendix C). The Virasoro algebra for the ghost operators is (detailed calculations are given in appendix C)

$$[L_m^{gh}, L_n^{gh}] = (m-n)L_{m+n}^{gh} + c^{gh}(m)\delta_{m+n}, \tag{5.51}$$

where the central charge is

$$c^{gh}(m) = \frac{1}{12}(2m - 26m^3).$$

For the total Virasoro generators introduced by

$$L_m = L_m^X + L_m^{gh} - a\delta_{m,0} \tag{5.52}$$

we have the algebra

$$[L_m, L_n] = (m-n)L_{m+n} + c(m)\delta_{m+n}, \tag{5.53}$$

where the central charge is

$$c(m) = \frac{d}{12}(m^3 - m) + \frac{1}{2}(2m - 26m^3) + 2am. \tag{5.54}$$

We see that central charge vanishes for $d = 26$ and $a = 1$, the same result as we got in the light-cone quantisation.

5.4 BRST operators

5.4.1 Lie algebra

We can assosiate the so-called BRST operator to any given Lie algebra defined by generators K_i satisfying the commutation relations

$$[K_i, K_j] = f_{ij}^k K_k. \tag{5.55}$$

The ghost and antighost fields c_i and b_i satisfy the anticommutation relations

$$\{c^i, b_j\} = \delta^i_j, \quad i, j = 1, \ldots, \dim K. \tag{5.56}$$

We introduce the ghost number operator U as

$$U = \sum_i c^i b_i. \tag{5.57}$$

The eigenvalues of U are integers from 0 to $\dim K$. The BRST operator for the Lie algebra is defined by

$$Q = c^i K_i - \frac{1}{2} f^k_{ij} c^i c^j b_k. \tag{5.58}$$

The BRST and the ghost number operator satisfy the following commutation relation (cf. appendix D)

$$[U, Q] = Q. \tag{5.59}$$

As the result of this commutation relation, the ghost number operator increases the ghost number by one

$$UQ|\chi\rangle = (QU + Q)|\chi\rangle = (N_{gh} + 1)Q|\chi\rangle. \tag{5.60}$$

In appendix D we show that

$$\{Q, Q\} = 0, \tag{5.61}$$

such that the BRST operator is nilpotent, $Q^2 = \frac{1}{2}\{Q, Q\} = 0$. We assume that Q is a hermitian operator, i. e., that $Q^\dagger = Q$. In the Hilbert space $\mathscr{H}^k$ with fixed ghost number $U = k$, an element $|\chi\rangle \in \mathscr{H}^k$ is called BRST-invariant if

$$Q|\chi\rangle = 0. \tag{5.62}$$

Any state of form $Q|\lambda\rangle$, where λ is any state with ghost number $k-1$, is BRST-invariant since

$$Q(Q|\lambda\rangle) = Q^2|\lambda\rangle = 0. \tag{5.63}$$

We can also see that $Q|\lambda\rangle$ has zero norm,

$$\langle\lambda|Q\dagger Q|\lambda\rangle = \langle\lambda|Q^2|\lambda\rangle = 0. \tag{5.64}$$

Next let us discus states which cannot be presented in the form $|\chi\rangle = Q|\lambda\rangle$. We say that two solutions of equation (5.62) are equivalent if

$$|\chi'\rangle - |\chi\rangle = Q|\lambda\rangle \tag{5.65}$$

for some λ. The states with zero ghost charge should be annihilated by all b_k. For such states the BRST operator is

$$Q = c^i K_i, \quad \text{so } Q|\chi\rangle = c^i K_i|\chi\rangle = 0, \tag{5.66}$$

which gives

$$K_i|\chi\rangle = 0. \tag{5.67}$$

This last expression means that $|\chi\rangle$ is invariant under the action of the Lie algebra. This state cannot be represented as $|\chi\rangle = Q|\lambda\rangle$ for some λ, because in that case the ghost number had to be -1, which is impossible.

5.4.2 The string case

For strings, we take the Lie algebra to be the Virasoro algebra. In addition, we also have the ghosts c_m and antighosts b_m. In this case the BRST operator is given by

$$Q = \sum_{-\infty}^{+\infty} L^X_{-m} c_m - \frac{1}{2}\sum_{-\infty}^{+\infty}(m-n) : c_{-m}c_{-n}b_{m+n} : -ac_0, \tag{5.68}$$

where a is the normal ordering constant for L_0. This expression can be written as

$$Q = \sum_{-\infty}^{=\infty} : (L^X_{-m} + \frac{1}{2}L^{gh}_{-m} - a\delta_{m,0})c_m : \tag{5.69}$$

The ghost number becomes

$$U = \sum_{-\infty}^{+\infty} : c_{-m} b_m : \tag{5.70}$$

Now let us see what happens with the relation $Q^2 = 0$. We find

$$Q^2 = \frac{1}{2}\{Q,Q\} = \sum_{n,m=-\infty}^{+\infty} ([L_m, L_n] - (m-n)L_{m+n})c_{-m}c_{-n}, \tag{5.71}$$

where $L_m = L_m^X + L_m^{gh} - a\delta_{m,0}$ is a total Virasoro operator. Since the total central charge vanishes $Q^2 = 0$ for $d = 26$ and $a = 1$. The inverse is also true: from $Q^2 = 0$ follows that the central charge of the Virasoro algebra vanishes. First, using (5.49), we see that

$$L_m = \{Q, b_m\}. \tag{5.72}$$

Next

$$[L_m, Q] = [\{Q, b_m\}, Q] = (Qb_m + b_m Q)Q - Q(Qb_m + b_m Q) = b_m Q^2 - Q^2 b_m = [b_m, Q^2], \tag{5.73}$$

as Q^2 goes to zero $[L_m, Q]$ also goes to zero. Therefore

$$\begin{aligned}
[L_m, L_n] = [L_m, \{Q, b_n\}] &= L_m Q b_n + L_m b_n Q - Q b_n L_m - b_n Q L_m \\
&+ L_m Q b_n - Q L_m b_n + L_m b_n Q - b_n L_m Q + Q L_m b_n - Q b_n L_+ b_n L_m Q - b_n Q L_m \\
&= [L_m, Q] b_n + [L_m, b_n] Q + Q[L_m, b_n] + b_n [L_m, Q] = \{[L_m, Q], b_n\} + \{Q, [L_m, b_n]\} \\
&= \{Q, [L_m, b_n]\} = \{Q, (m-n) b_{m+n}\} = (m-n)\{Q, b_{m+n}\} = (m-n) L_{m+n}.
\end{aligned} \tag{5.74}$$

In the theory with ghosts we have the new conserved currents

$$J^B = 2c(T^X + \frac{1}{2}T^{gh}) \quad \text{and} \quad J = cb.$$

There is a new fermionic symmetry present, it is a BRST symmetry. The BRST transformation is defined as

$$\delta Y = [\lambda Q, Y],$$

where λ is a Grassman (anticomutating) parameter. The explicit transformations are

$$\delta X^\mu = \lambda c \partial X^\mu + \lambda \bar{c} \bar{\partial} X^\mu, \quad \delta c = \lambda c \partial c, \quad \delta b = 2\lambda T, \quad \text{and} \quad \delta T = 0. \quad (5.75)$$

The ghost number operator is

$$U = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty} (c_{-n} b_n - b_{-n} c_n),$$

where c_n, b_n for $n > 0$ are annihilation operators. For zero-modes we have

$$c_0^2 = b_0^2 = 0, \quad \{c_0, b_0\} = 1.$$

Or in two dimensions

$$\begin{aligned} c_0|\downarrow\rangle &= |\uparrow\rangle \qquad b_0|\uparrow\rangle = |\downarrow\rangle \\ c_0|\uparrow\rangle &= 0 \qquad b_0|\downarrow\rangle = 0. \end{aligned}$$

The ghost numbers are $U_\downarrow = -\frac{1}{2}$ and $U_\uparrow = \frac{1}{2}$. Physical states should have the ghost number $-\frac{1}{2}$, so they are annihilated by b_0. Indeed $c_n|\xi\rangle = b_n|\xi\rangle = 0$ when $n > 0$ and $b_0|\xi\rangle = 0$. The condition of BRST invariance becomes

$$0 = Q|\xi\rangle = \left(c_0(L_0 - 1) + \sum_{n>0} c_{-n} L_n \right) |\xi\rangle,$$

the same as in the case of canonical quantisation. Physical states of the bosonic string are cohmology classes of the BRST operator with the ghost number $-\frac{1}{2}$.

6

Conclusions

In this dissertation we reviewed the Nambu-Goto and Polyakov actions for the bosonic string and quantised these actions covariantly and in the light-cone. We derived the Virasoro algebra and the Virasoro constraints for the quantum bosonic string. From the constraints of the theory we could verify that the critical dimension for the bosonic string is $D = 26$. We then analysed the spectrum of a free closed string and, once again, found that the critical dimension of the bosonic string is $D = 26$. Some important classical and quantum aspects of two-dimensional conformal field theories (CFT) that were used to quantise the string by the BRST method were also analysed, and the Virasoro algebra for the CFT was derived. This allowed us to treat the bosonic string as a two-dimensional CFT. In order to introduce the BRST quantisation procedure for the bosonic string we were led to explore, at a pedagogical level, the Faddeev-Popov method of "finding the Jacobian," the role of ghosts in the theory and their CFT and Virasoro algebra. The introduction of the BRST operators then allowed us to conclude that the physical states should have ghost number $-1/2$.

The next steps in the study of basic string theory would be to complete the BRST quantisation programme for the bosonic string to the end and then proceed with the treatment of the fermionic string and superstrings following, for example, the presentations in [10, 14, 18]. Our final goal would be, as per our original project, to understand integrable models related with the quantisation of superstrings on semisymmetric superspaces [19, 20].

String theory is a vast subject that employs a diverse set of modern mathematical ideas and techniques. In this dissertation we were able only to start the study of the techniques that now are classic and belong to the toolbox of almost every theoretical physicist. We were particularly pleased to learn the basics of CFT and Virasoro algebras. On the other hand, we were not able to start studying anything about superstrings, quantum gravity, or the recent developments founded by the AdS/CFT correspondence. Unfortunately, we are not going to pursue this research programme anymore, since we decided to move into another subject.

A

Conservation of the stress-energy tensor

Let us prove that the covariant stress-energy tensor is conserved,

$$\nabla^{\alpha} T_{\alpha\beta} = 0. \tag{A.1}$$

The stress-energy tensor is given by

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X_\mu. \tag{A.2}$$

Plugging this expression in (A.1) we get

$$\nabla^\alpha T_{\alpha\beta} = \nabla^\alpha \partial_\alpha X^\mu \partial_\beta X_\mu + \partial_\alpha X^\mu \nabla^\alpha \partial_\beta X_\mu - \frac{1}{2}\nabla_\beta \eta^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X_\mu. \tag{A.3}$$

The equation of motion for X^μ reads $\nabla^\alpha \partial_\alpha X^\mu = 0$, so that

$$\begin{aligned}
\nabla^\alpha T_{\alpha\beta} &= \partial_\alpha X^\mu \nabla^\alpha \partial_\beta X_\mu - \frac{1}{2}\partial_\beta \eta^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X_\mu = \\
&= \partial_\alpha X^\mu \eta^{\alpha\sigma}\nabla_\sigma \partial_\beta X_\mu - \frac{1}{2}\partial_\beta \eta^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X_\mu - \eta^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma \partial_\beta X_\mu = \\
&= \partial_\alpha X^\mu \eta^{\alpha\sigma}\partial_\sigma \partial_\beta X_\mu - \partial_\alpha X^\mu \partial_\lambda X_\mu \eta^{\alpha\sigma}\Gamma^\lambda_{\sigma\beta} \\
&- \frac{1}{2}\partial_\beta \eta^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X_\mu - \eta^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma \partial_\beta X_\mu.
\end{aligned} \tag{A.4}$$

The first and last terms on the right-hand side of the above equation cancel each other and we are left with

$$\nabla^\alpha T_{\alpha\beta} = -\partial_\alpha X^\mu \partial_\lambda X_\mu \eta^{\alpha\sigma}\Gamma^\lambda_{\sigma\beta} - \frac{1}{2}\partial_\beta \eta^{\rho\sigma}\partial_\sigma X^\mu \partial_\sigma X_\mu. \tag{A.5}$$

Since

$$\eta^{\alpha\sigma}\Gamma^\lambda_{\sigma\beta} = \frac{1}{2}\eta^{\alpha\sigma}\eta^{\lambda\theta}\partial_\sigma \eta_{\theta\beta} + \partial_\beta \eta_{\theta\sigma} - \partial_\theta \eta_{\beta\sigma}, \tag{A.6}$$

the only terms of the above expression that survive the derivatives in (A.5) are the ones symmetric in α and λ. Plugging them back into (A.5) furnishes

$$\nabla^\alpha T_{\alpha\beta} = -\frac{1}{2}\eta^{\alpha\sigma}\partial_\beta \eta_{\theta\sigma}\eta^{\lambda\theta}\partial_\alpha X^\mu \partial_\lambda X_\mu - \frac{1}{2}\partial_\beta \eta^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X_\mu = 0, \tag{A.7}$$

that is what we wanted to prove.

B

The stress-energy tensor for the *bc* ghost fields

The stress-energy tensor for the *bc* ghost fields is given by

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S_{ghost}}{\delta g^{\alpha\beta}}, \tag{B.1}$$

with S_{ghost} given by

$$S_{ghost} = -\frac{1}{2\pi} \int d^2\sigma \sqrt{g}\, g^{\alpha\mu} b_{\alpha\beta} \nabla_\mu c^\beta. \tag{B.2}$$

The first variation of S_{ghost} is

$$\begin{aligned} \delta S_{ghost} = &- \frac{1}{2\pi} \int d^2\sigma (\delta\sqrt{g}) g^{\alpha\mu} b_{\alpha\beta} \nabla_\mu c^\beta - \frac{1}{2\pi} \int d^2 \sqrt{g}\, (\delta g^{\alpha\mu}) b_{\alpha\beta} \nabla_\mu c^\beta + \\ &- \frac{1}{2\pi} \int d^2 \sqrt{g}\, g^{\alpha\mu} (\delta b_{\alpha\beta}) \nabla_\mu c^\beta - \frac{1}{2\pi} \int d^2 \sqrt{g}\, g^{\alpha\mu} b_{\alpha\beta} \delta(\nabla_\mu c^\beta). \end{aligned} \tag{B.3}$$

Using the facts that $\nabla_\mu c^\beta = \partial_\mu c^\beta + \Gamma^\beta_{\mu\lambda} c^\lambda$ and $\delta b_{\alpha\beta} = 0$ we get

$$\begin{aligned} \delta S_{ghost} = &- \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \left[b_{\alpha\mu} \nabla_\beta c^\mu + b_{\beta\mu} \nabla_\alpha c^\mu - g_{\alpha\beta} b_{\rho\sigma} \nabla^\rho c^\sigma \right] \delta g^{\alpha\beta} + \\ &- \frac{1}{2\pi} \int d^2\sigma \sqrt{g}\, g^{\alpha\mu} b_{\alpha\beta} c^\lambda \delta\Gamma^\beta_{\mu\lambda} \end{aligned} \tag{B.4}$$

Now, since

$$\delta\Gamma^\beta_{\mu\lambda} = \frac{1}{2} g^{\beta\rho} \Big(\nabla_\lambda \delta g_{\rho\mu} + \nabla_\mu \delta g_{\rho\lambda} - \nabla_\rho \delta g_{\mu\lambda} \Big), \tag{B.5}$$

the last term on the right-hand side of (B.4) becomes (note that $\delta\Gamma^{\beta}_{\mu\lambda}$ is a tensor)

$$\begin{aligned}
&-\frac{1}{2\pi}\int d^2\sigma\sqrt{g}\,g^{\alpha\mu}b_{\alpha\beta}c^{\lambda}\delta\Gamma^{\beta}_{\mu\lambda} = \\
&= -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,b^{\mu\rho}c^{\lambda}\left(\nabla_{\lambda}\delta g_{\rho\mu}+\nabla_{\mu}\delta g_{\rho\lambda}-\nabla_{\rho}\delta g_{\rho\lambda}\right) = \\
&= -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,b^{\mu\rho}c^{\lambda}\nabla_{\lambda}\delta g_{\rho\mu} = \\
&= -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,\nabla_{\lambda}\left(b_{\alpha\beta}c^{\lambda}\right)\delta g^{\alpha\beta}.
\end{aligned} \tag{B.6}$$

On the other hand, in a flat frame

$$\nabla_{\lambda}\left(b_{\alpha\beta}c^{\lambda}\right)\delta g^{\alpha\beta} = c^{\lambda}\partial_{\lambda}\left(b_{\alpha\beta}\right)\delta g^{\alpha\beta}+\partial_{\lambda}c^{\lambda}\left(b_{\alpha\beta}\delta g^{\alpha\beta}\right), \tag{B.7}$$

and since $b_{\alpha\beta}$ is traceless the second term on the righ-hand side of the above expression vanishes, $\partial_{\lambda}c^{\lambda}\left(b_{\alpha\beta}\delta g^{\alpha\beta}\right)=0$. The variation of S_{ghost} then becomes

$$\begin{aligned}
\delta S_{ghost} = -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,\Big[&b_{\alpha\mu}\nabla_{\beta}c^{\mu}+b_{\beta\mu}\nabla_{\alpha}c^{\mu} \\
&-g_{\alpha\beta}b_{\rho\sigma}\nabla^{\rho}c^{\sigma}+c^{\lambda}\partial_{\lambda}\left(b_{\alpha\beta}\right)\delta g^{\alpha\beta}\Big]\delta g^{\alpha\beta},
\end{aligned} \tag{B.8}$$

and from the definition of the stress-energy tensor we find

$$T_{\alpha\beta} = b_{\alpha\mu}\nabla_{\beta}c^{\mu}+b_{\beta\mu}\nabla_{\alpha}c^{\mu}-g_{\alpha\beta}b_{\rho\sigma}\nabla^{\rho}c^{\sigma}+c^{\lambda}\partial_{\lambda}\left(b_{\alpha\beta}\right)\delta g^{\alpha\beta}. \tag{B.9}$$

Using that $b_{\alpha\beta}g^{\alpha\beta}=0$ we get

$$T = 2(\partial c)b + c(\partial b), \tag{B.10}$$

while the anti-holomorphic part reads

$$\bar{T} = 2(\bar{\partial}\bar{c})\bar{b} + \bar{c}(\bar{\partial}\bar{b}). \tag{B.11}$$

C

The algebra of ghost generators

We are going to prove that ghosts and anti-ghosts are conformal fields. To do so we just compute the commutation relations between the fields and the Virasoro generators (in the proof we use (5.49))

$$
\begin{aligned}
[L_m^{gh}, b_n] &= \sum_{p=-\infty}^{+\infty} (m-p)[: b_{m+p} c_{-p} :, b_n] = \sum_{p=-\infty}^{+\infty} (m-p)(b_{m+p} c_{-p} b_n - b_n b_{m+p} c_{-p}) \\
&= \sum_{p=-\infty}^{+\infty} (m-p)(b_{m+p} c_{-p} b_n + b_{m+p} b_n c_{-p}) = \sum_{p=-\infty}^{+\infty} (m-p) b_{m+p} c_{-p}, b_n \\
&= \sum_{p=-\infty}^{+\infty} (m-p)(b_{m+p} \delta_{n,-p}) = (m-n) b_{m+n},
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
[L_m^{gh}, c_n] &= \sum_{p=-\infty}^{+\infty} (m-p)[: b_{m+p} c_{-p} :, c_n] = \sum_{p=-\infty}^{+\infty} (m-p)(b_{m+p} c_{-p} c_n - c_n b_{m+p} c_{-p}) \\
&= \sum_{p=-\infty}^{+\infty} (m-p)(-b_{m+p} c_n c_{-p} - c_n b_{m+p} c_{-p}) = \sum_{p=-\infty}^{+\infty} (m-p) b_{m+p}, c_n c_{-p} \\
&= \sum_{p=-\infty}^{+\infty} (m-p)(\delta_{m+p,-n}) c_{-p} = -(2m+n) c_{m+n}.
\end{aligned}
\tag{C.2}
$$

On the other hand, the transformation rule of the modes of a conformal operator of dimension Δ is given by [12]

$$[L_m, A_n] = (m(\Delta-1)-n)A_{m+n}.$$

Comparing the last three formulas we conclude that b and c are conformal operators of dimension $\Delta = 2$ and $\Delta = -1$ respectively.

To compute the Virasoro algebra for the ghost fields we write the commutator between the generators of the algebra, which reads

$$\begin{aligned}
[L_m^{gh}, L_n^{gh}] &= \sum_{p=-\infty}^{+\infty} (m-p)[: b_{m+p}c_{-p} :, L_n^{gh}] = \\
&= \sum_{p=-\infty}^{0} (m-p)[b_{m+p}c_{-p}, L_n^{gh}] + \sum_{p=1}^{+\infty} (m-p)[c_{-p}b_{m+p}, L_n^{gh}] = \\
&= \sum_{p=-\infty}^{0} (p-m)\big([L_n^{gh}, b_{m+p}]c_{-p} + b_{m+p}[L_n^{gh}, c_{-p}]\big) \\
&+ \sum_{p=1}^{+\infty} (p-m)\big([L_n^{gh}, c_{-p}]b_{m+p} + c_{-p}[L_n^{gh}, b_{m+p}]\big) = \\
&= \sum_{p=-\infty}^{0} (p-m)(n-p-m)b_{n+m+p}c_{-p} \\
&+ \sum_{p=-\infty}^{0} (p-m)b_{m+p}(-(2n-p)c_{n-p}) \\
&+ \sum_{p=1}^{+\infty} (p-m)(-(2n-p)c_{n-p}b_{m+p}) \\
&+ \sum_{p=1}^{+\infty} (p-m)c_{-p}(n-m-p)b_{n+m+p},
\end{aligned} \tag{C.3}$$

where in last step we used equations (C.1) and (C.2). If we put $p = q - n$ in the first and fourth terms and let $p \to q$ in the second and third terms on the righ-hand side of (C.3) we obtain

$$
\begin{aligned}
[L_m^{gh}, L_n^{gh}] = & \sum_{q=-\infty}^{n} (q-n-m)(2n-q-m) b_{q+m} c_{n-q} + \\
& + \sum_{q=-\infty}^{0} (q-m)(q-2n) b_{m+q} c_{n-q} \\
& + \sum_{q=1}^{+\infty} (q-m)(q-2n) c_{n-q} b_{m+q} \\
& + \sum_{q=n+1}^{+\infty} (q-n-m)(2n-m-q) c_{n-q} b_{q+m} = \\
& = \sum_{q=-\infty}^{0} (m^2 - mq + nq + mn - 2n^2) b_{q+m} c_{n-q} \\
& + \sum_{q=1}^{n} (q-n-m)(2n-q-m) b_{m+q} c_{n-q} \\
& + \sum_{q=n+1}^{+\infty} (m^2 - mq + nq + mn - 2n^2) c_{n-q} b_{m+q} \\
& + \sum_{q=1}^{n} (q-m)(q-2n) c_{n-q} b_{q+m}.
\end{aligned} \tag{C.4}
$$

Supposing that $n > 0$, then all terms except the second one on the righ-hand side of (C.4) are normal ordered. That term can be written as

$$
\sum_{q=1}^{n} (q-n-m)(2n-q-m) c_{n-q} b_{m+q} + \sum_{q=1}^{n} (q-n-m)(2n-q-m) \delta_{m+n},
$$

which plugged into (C.4) furnishes

$$
\begin{aligned}
[L_m^{gh}, L_n^{gh}] = \sum_{q=-\infty}^{+\infty} (m^2 - mq + nq + mn - 2n^2) : b_{m+q} c_{n-q} : \\
+ \sum_{q=1}^{n} (2n - q - m)(q - n - m)\delta_{m+n} = \\
= \sum_{q=-\infty}^{+\infty} (m-n)(m+n+n-q) : b_{m+n-n+q} c_{-(q-n)} : \\
+ \sum_{q=1}^{n} (2n - q - m)(q - n - m)\delta_{m+n} = \\
= (m-n) L_{m+n}^{gh} + \sum_{q=1}^{n} (2n - q - m)(q - n - m)\delta_{m+n},
\end{aligned}
\tag{C.5}
$$

that can then be summed to provide the desired result.

D

Commutation relations between BRST and ghost number operators

In this appendix we derive some commutation relations between Q, the BRST operator, and U, the ghost number operator, that were used in this work.

By their definitions,

$$\begin{aligned}
[Q,U] &= [c^i K_i - \frac{1}{2} f_{ij}^k c^i C^j B_k, C^m b_m] = \\
&= c^i K_i c^m b_m - c^m b_m c^i K_i - \frac{1}{2} f_{ij}^k c^i c^j b_k c^m b_m + \frac{1}{2} f_{ij}^k c^m b_m c^i c^j b_k = \\
&= c^i c^m b_m K_i - c^m b_m c^i K_i - \frac{1}{2} f_{ij}^k c^i c^j b_k c^m b_m \\
&- \frac{1}{2} f_{ij}^k c^i c^j c^m b_k b_m + \frac{1}{2} f_{ij}^k c^i c^j c^m b_k b_m + \frac{1}{2} f_{ij}^k c^m b_m c^i c^j b_k = \\
&= -c^m \{c^i, b_m\} K_i - \frac{1}{2} f_{ij}^k c^i c^j \{b_k, c^m\} b_m - \frac{1}{2} f_{ij}^k c^m c^i c^j b_m b_k + \frac{1}{2} f_{ij}^k c^m b_m c^i c^j b_k = \\
&= -c^m \{c^i, b_m\} K_i - \frac{1}{2} f_{ij}^k c^i c^j \{b_k, c^m\} b_m - \frac{1}{2} f_{ij}^k c^m [c^i c^j, b_m] b_k = \\
&= -c^m \delta_m^i K_i - \frac{1}{2} f_{ij}^k c^i c^j \delta_k^m b_m - \frac{1}{2} f_{ij}^k c^m (c^i \{c^j, b_m\} - \{c^i, b_m\} c^j) b_k = \\
&= -c_i K^i - \frac{1}{2} f_{ij}^k c^i c^j b_k - \frac{1}{2} f_{ij}^k c^m (c^i \delta_j^m - \delta_i^m c^j) b_k = \\
&= -c_i K^i - \frac{1}{2} f_{ij}^k c^i c^j b_k - \frac{1}{2} f_{ij}^k c^j c^i b_k + \frac{1}{2} f_{ij}^k c^i c^j b_k = \\
&= -c_i K^i + \frac{1}{2} f_{ij}^k c^i c^j b_k = -Q
\end{aligned} \tag{D.1}$$

Now let us calculate $\{Q,Q\}$:

$$
\begin{aligned}
\{Q,Q\} &= \{c^iK_i - \frac{1}{2}f_{ij}^k c^i c^j b_k, c^sK_s - \frac{1}{2}f_{mn}^p c^m c^n b_p\} = \\
&= c^iK_ic^sK_s - \frac{1}{2}c^iK_if_{mn}^p c^m c^n b_p - \frac{1}{2}f_{ij}c^ic^jb_kc^sK_s + \frac{1}{4}f_{ij}^k c^ic^jb_kf_{mn}^p c^mc^nb_p \\
&+ c^sK_sc^iK_i - \frac{1}{2}c^sK_sf_{ij}^k c^ic^jb_k - \frac{1}{2}f_{mn}^p c^mc^nb_pc^iK_i + \frac{1}{4}f_{mn}^p c^mc^nb_pf_{ij}^k c^ic^jb_k = \\
&= c^ic^sK_iK_s + c^sc^iK_sK_i - \frac{1}{2}f_{mn}^p c^mc^nc^ib_pK_i \\
&- \frac{1}{2}f_{mn}^p c^mc^nb_pc^iK_i - \frac{1}{2}f_{ij}^k c^ic^jb_kc^sK_s - \frac{1}{2}f_{ij}^k c^ic^jc^sb_kK_s \\
&+ \frac{1}{4}f_{ij}^k f_{mn}^p c^ic^jb_kc^mc^nb_p + \frac{1}{4}f_{ij}^k f_{mn}^p c^mc^nb_pc^ic^jb_k = \\
&= c^ic^sK_iK_s + c^sc^iK_sK_i - \frac{1}{2}f_{mn}^p c^mc^n\{c^i,b_p\}K_i \\
&- \frac{1}{2}f_{ij}^k c^ic^j\{b_k,c^s\}K_s + \frac{1}{4}f_{ij}^k f_{mn}^p\{c^ic^jb_k, c^mc^nb_p\}
\end{aligned} \tag{D.2}
$$

The last term is calculated as follows:

$$
\begin{aligned}
f_{ij}^{k}f_{mn}^{p}\{c^{i}c^{j}b_{k},c^{m}c^{n}b_{p}\} &= f_{ij}^{k}f_{mn}^{p}(c^{i}c^{j}\{b_{k},c^{m}c^{n}\}b_{p}-c^{i}c^{j}c^{m}c^{n}\{b_{k},b_{p}\}\\
&+c^{m}c^{n}[b_{p},c^{i}c^{j}]b_{k}+\{c^{m}c^{n},c^{i}c^{j}\}b_{p}b_{k})=\\
&=f_{ij}^{k}f_{mn}^{p}c^{i}c^{j}c^{m}[c^{n},b_{k}]b_{p}+f_{ij}^{k}f_{mn}^{p}c^{i}c^{j}\{c^{m},b_{k}\}c^{n}b_{p}\\
&+f_{ij}^{k}f_{mn}^{p}c^{m}c^{n}\{b_{p},c^{i}\}c^{j}b_{k}-f_{ij}^{k}f_{mn}^{p}c^{m}c^{n}c^{i}\{b_{p},c^{j}\}b_{k})\\
&+f_{ij}^{k}f_{mn}^{p}c^{m}\{c^{n},c^{i}\}c^{j}b_{p}b_{k}-f_{ij}^{k}f_{mn}^{p}c^{m}c^{i}\{c^{n},c^{j}\}b_{p}b_{k}\\
&+f_{ij}^{k}f_{mn}^{p}c^{i}[c^{j},c^{m}]c^{n}b_{p}b_{k}+f_{ij}^{k}f_{mn}^{p}\{c^{i},c^{m}\}c^{j}c^{n}b_{p}b_{k}=\\
&=f_{ij}^{k}f_{mn}^{p}c^{i}c^{j}\delta_{m}^{k}c^{n}b_{p}+f_{ij}^{k}f_{mn}^{p}c^{m}c^{n}\delta_{p}^{i}c^{j}b_{k}-f_{ij}^{k}f_{mn}^{p}c^{m}c^{n}c^{i}\delta_{p}^{j}b_{k}\\
&+f_{ij}^{k}f_{mn}^{p}c^{i}c^{j}c^{m}[c^{n},b_{k}]b_{p}+f_{ij}^{k}f_{mn}^{p}c^{i}[c^{j},c^{m}]c^{n}b_{p}b_{k}\\
&=f_{ij}^{k}f_{mn}^{p}(c^{i}c^{j}c^{m}c^{n}b_{k}b_{p}-c^{i}c^{j}c^{m}b_{k}c^{n}b_{p}+c^{i}c^{j}c^{m}c^{n}b_{p}b_{k}\\
&-c^{i}c^{m}c^{j}c^{n}b_{p}b_{k}+c^{i}c^{j}\delta_{m}^{k}c^{n}b_{p}+c^{m}c^{n}\delta_{p}^{i}c^{j}b_{k}-c^{m}c^{n}c^{i}\delta_{p}^{j}b_{k}=\\
&=-f_{ij}^{k}f_{mn}^{p}c^{i}c^{j}c^{m}\delta_{k}^{n}b_{p}+f_{ij}^{k}f_{mn}^{p}c^{i}c^{j}\delta_{m}^{k}c^{n}b_{p}\\
&+f_{ij}^{k}f_{mn}^{p}c^{m}c^{n}\delta_{p}^{i}c^{j}b_{k}-f_{ij}^{k}f_{mn}^{p}c^{m}c^{n}c^{i}\delta_{p}^{j}b_{k}=\\
&=-f_{ij}^{k}f_{mk}^{p}c^{i}c^{j}c^{m}b_{p}+f_{ij}^{k}f_{kn}^{p}c^{i}c^{j}c^{n}b_{p}\\
&+f_{ij}^{k}f_{mn}^{i}c^{m}c^{n}c^{j}b_{k}-f_{ij}^{k}f_{mn}^{j}c^{m}c^{n}c^{i}b_{k}=\\
&=4f_{ij}^{k}f_{km}^{p}c^{i}c^{j}c^{m}b_{p}.
\end{aligned}
\tag{D.3}
$$

Using the above result we obtain

$$
\{Q,Q\}=c^{i}c^{j}[K_{i},K_{j}]-\frac{1}{2}f_{ij}^{k}c^{i}c^{j}K_{k}-\frac{1}{2}f_{ij}^{k}c^{i}c^{j}K_{k}+f_{ij}^{k}f_{km}^{p}c^{i}c^{j}c^{m}b_{p}, \tag{D.4}
$$

and from the commutation relations for the K_i generators we get

$$
\{Q,Q\}=f_{ij}^{k}f_{km}^{p}c^{i}c^{j}c^{m}b_{p}. \tag{D.5}
$$

Due to the anti-commutativity of the ghosts we can write

$$\{Q,Q\} = \frac{1}{3}\big(f^k_{ij}f^p_{km} + f^k_{mi}f^p_{kj} + f^k_{jm}f^p_{ki}\big)c^i c^j c^m b_p, \tag{D.6}$$

where the expression inside the parentheses is the Jacobi identity, which is satisfied by the structure constants of the Lie algebra. This last statement implies that

$$\{Q,Q\} = 0. \tag{D.7}$$

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