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Equação de Korteweg-de Vries e Distribuição de Thomas-Fermi

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Ohana means family.
Family means nobody gets left behind, or forgotten.

— Lilo & Stitch

Every story has its beginning...

Dedicated to the loving memory of Adelma Mendes.
1915 – 2022

Dedicated to the loving memory of Otacílio Félix.
1937 – 2005

Dedicated to the loving memory of Manoel Marcelino.
1913 – 1980

Dedicated to Clara da Penha.

Verba volant,
scripta manent.

—

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Three is an underestimated number, but in physics it enumerates important laws, in geometry it appears as a condition to define a plane by non-aligned points, and three are the pillars necessary to constitute a solid foundation structure. That is the reason for dividing my acknowledgments into three classes.

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ABSTRACT

Plasma physics is generally associated with the treatment of regimes characterized by high temperature and low densities, where quantum mechanical effects do not have a significant impact. Recent studies, however, show that some systems can be studied from the perspective of dense plasmas, where the distance between the species is of the same order as the thermal de Broglie wavelength. In this way, the temperature associated with the thermal motion of the particles is lower than the Fermi temperature, i. e., the system is degenerate, and classical statistics must give way to the Pauli Exclusion principle. In this work, we construct a semiclassical fluid model from the consideration of a gas formed by degenerate electrons and singularly ionized ions, with the Thomas-Fermi distribution replacing the Maxwell-Boltzmann one in the description of the electrons. Thus, we discuss the possibility of the nonlinear oscillations evolution in the plasma to be described, through a reductive perturbation method, by the Korteweg-de Vries equation. Using the calculus of variations, it was possible to find the natural scales of the problem, as well as define the critical frame in which the nonlinear solution structures propagate. We also investigate the ion thermal effects and the consequences of applying a constant magnetic field to the system, in addition to looking at the solitonic pulses response to the introduction of these new parameters in the theory. We carefully show that the system is sensitive to normalization, allowing us to evaluate the results by introducing a control parameter. In general, we verified that it is possible to construct the KdV equation via a modified reductive perturbation method, with the inclusion of the control parameter, we characterized the subsonic reference frame $(M = 1/\sqrt{3})$ as the appropriate one to describe the propagation of solitons, which validates the perturbative description. We computed the effects of the temperature and magnetic field on the nonlinear and dispersive parameters, and the consequent modifications in the shape of the waves. Finally, having assumed the cold ions regime as the lower limit for all approaches carried out, we made use of the normalization control parameter (λ_0) to switch between expressions with different scales.

Key-words: Solitons, Korteweg-de Vries, Thomas-Fermi

RESUMO

A física de plasma é geralmente associada ao tratamento de regimes caracterizados por alta temperatura e baixas densidades, onde efeitos da mecânica quântica não possuem impacto significativo. Recentes estudos, no entanto, mostram que alguns ambientes podem ser estudados na perspectiva de plasmas densos, onde a distância entre as espécies é da mesma ordem que o comprimento de onda térmico de de Broglie. Desse modo, a temperatura associada ao movimento térmico das partículas é menor que a temperatura de Fermi, isto é, o sistema é degenerado, e a estatística clássica deve dar lugar ao princípio de Exclusão de Pauli. Neste trabalho construímos um modelo de fluido semiclássico, a partir da consideração de um gás formado por elétrons degenerados e íons singularmente ionizados, com a distribuição de Thomas-Fermi substituindo a distribuição de Maxwell-Boltzmann na descrição dos elétrons. Assim, discutimos a possibilidade de a evolução de oscilações não lineares no plasma ser descrita, através do método redutivo perturbativo, pela equação de Korteweg-de Vries. Utilizando do cálculo de variações foi possível encontrar as escalas naturais do problema, bem como definir o referencial crítico no qual as estruturas fornecidas como soluções da equação não linear se propagam. Também investigamos os efeitos térmicos dos íons e as consequências da aplicação de um campo magnético constante no sistema, além de examinar a resposta dos pulsos solitônicos à introdução desses novos parâmetros na teoria. Com cuidado, mostramos que o sistema é sensível à normalização, nos permitindo avaliar os resultados a partir da introdução de um parâmetro de controle. De modo geral, verificamos ser possível construir a equação de KdV via método redutivo perturbativo modificado, com a inclusão do parâmetro de controle, caracterizamos o referencial subsônico ($M = 1/\sqrt{3}$) como o adequado para descrever a propagação dos sólitons, o qual valida a descrição perturbativa. Avaliamos os efeitos da temperatura e do campo magnético nos índices não lineares e dispersivos, e as consequentes modificações na forma das ondas. Por fim, tendo assumido o regime de íons frios como limite inferior para todas as abordagens realizadas, utilizamos o parâmetro de controle (λ_0) da normalização para transitar entre as expressões com diferentes escalas.

Palavras-chave: Sólitons, Korteweg-de Vries, Thomas-Fermi

PUBLICATIONS

Some ideas have appeared previously in the following publication:

[1] F. E. M. Silveira, M. H. Benetti, I. L. Caldas, and K. N. M. M. Santos. "Description limit for soliton waves due to critical scaling of electrostatic potential." In: *Physics of Plasmas* 28 (9 Sept. 2021), p. 092115. ISSN: 1070-664X. DOI: 10.1063/5.0059437.

SYMBOLS

x	Position coordinate
t	Temporal coordinate
c	Phase speed
u	Wave amplitude
k	Wavenumber
ω	Angular frequency
c_G	Group Velocity
c_D	Conserved Density
v_i	i-Direction Velocity
k_B	Boltzmann constant
T	Temperature
$\bar{E}$	Average Kinetic Energy
n_i	Ion Density
n_e	Electron Density
n_0	Equilibrium Density
e	Particle Charge
ϵ_0	Vacuum Permittivity
ϕ	Electric Potential
λ_D	Debye Length
L	System Dimension
N_D	Number of Elements
τ	Mean Collisions Time
$\vec{E}$	Electric Field
$\vec{B}$	Magnetic Field
p	Pressure
γ	Polytropic Index
$\mathcal{N}$	Degrees of Freedom
ν_{eff}	Effective Collision Frequency
ω_p	Plasma Frequency
ρ	Mass Density
c_g	Neutral Gas Sound Velocity
c_w	Warm Plasma Sound Velocity
Ω	Cyclotron Frequency
c_c	Cold Plasma Sound Velocity
ω_{p_i}	Ion Plasma Frequency
M	Mach Number
ϵ	Excess Expansion Parameter
$\hbar$	Reduced Planck Constant
p_F	Fermi Momentum
K	Kinetic Energy
μ	Chemical Potential
ε_F	Fermi Energy
T_F	Fermi Temperature
λ_F	Linear TF Length
c_{TF}	Warm TF Plasma Sound Velocity
λ_e	Length Scale
φ	Potential Scale

l_i i-Direction Cosine

λ_0 Stretched Frame Phase Velocity

Ω_{p_i} Cyclotron-Ion Plasma Frequency Ratio

δ Isothermal Coefficient

β Adiabatic Coefficient

${}^{j}V_0$ j-Configuration Corrected Ion Sound Speed

m Elliptic Parameter

Θ Ion and Fermi Temperatures Ratio

ACRONYMS

SW Solitary Wave

PDE Partial Differential Equation

KdV Korteweg-de Vries

KZ Kruskal and Zabusky

FPU Fermi, Pasta and Ulam

TWS Traveling Wave Solution

JE Jacobian Elliptic

CW Cnoidal Wave

IST Inverse Scattering Transform

mKDV Modified Korteweg-de Vries

MB Maxwell-Boltzmann

e-i Electron-Ion

DS Debye Shielding

EM Electromagnetic

EOS Equation of State

TF Thomas-Fermi

RP Reductive Perturbation

CONTENTS

INTRODUCTION

In plasmas composed of electrons and ions, there is the possibility of the emergence of a wide variety of waves, a fact that has attracted and still attracts the attention of several scientists. Several works are available in the literature proposing the investigation of the influence of factors such as ion temperature [28, 38], presence of magnetic fields [49], presence of other ion species [39], ion inertia [11], electrons with different temperatures [20], and different geometries [29], among other things.

There are many differential equations that describe linear and nonlinear events in environments other than plasmas, such as oceans, water channels, Bose-Einstein condensates, and nonlinear optics. With regard to $(e-i)$ plasmas, the propagation of ion acoustic waves composes one of the main and most comprehensive fields of study, resulting in research involving the different structures, like solitons, cnoidal waves and envelopes, for example. The nonlinear investigation of these waves was initially carried out using the *Sagdeev pseudopotential* method [37, 44], where it was concluded that such oscillations exist in the form of solitary waves or periodic fluctuations, while Washimi and Taniuti [45] were responsible for the study of the system using the *reductive perturbation* technique, acting on the equations that govern the components dynamics.

Soliton is a nonlinear pulse, localized, shaped from the balance between nonlinearity and dispersive effects, and are characterized by the structure conservation after interaction, unless a relative displacement [21]. Korteweg and de Vries were responsible for obtaining for the first time the equation which has solitons and, under appropriate boundary conditions, cnoidal waves as solutions [25]. The expression is known as *KdV* equation.

Studies are mostly developed considering electrons governed by the *Maxwell-Boltzmann* density distribution. However, observations including astrophysical objects [48], laboratory experiments [34], laser confinement [10], and semiconductors [22] give evidence of a degenerate *Fermi* gas, which leads to the consideration of degenerate electrons governed by the *Thomas-Fermi* statistics, while ions are treated as a classical gas.

The fundamental aspects of Soliton Theory, especially its applied perspective, of major interest to the thesis content, are presented for completeness in Chap. 1, which began with a review of the linear and nonlinear process, as well as the historical remarks, providing the basis for the following investigation of the steady solutions construction and the Solitary Wave (SW) description from real bounded shapes analysis. The conservation laws of the Korteweg-de Vries (KdV) equation were then deduced, and the existence of infinite ones was verified, justifying the integrable character of the expression.

We provided then, in Chap. 2, a brief discussion of the basic plasma system, describing the classical electron-ion (e-i) gas from the three-dimensional Maxwell-Boltzmann distribution and fundamentally defining the fluid parameters. Furthermore, the cold and warm limits were introduced by the hydrodynamic description of the collisionless configuration. Afterward, using the linearization technique, we studied in Chap. 3 the plasma oscillations, first with an overview of sound waves theory and then considering the acoustic waves that propagate by vibration transmission to both non-magnetized and magnetized regimes. Continuing, we considered the finite-amplitude situation, in which some effects may arise from the nonlinearity, and used the Sagdeev potential model to determine the domain of validity of the Mach number in a Boltzmann plasma, define an expansion parameter for the amplitude of the perturbation fields and the space-time re-scaling, followed by the derivation of the system KdV equation.

Next, we reconsidered the previous scenarios, changing the electron description to the Thomas-Fermi (TF) approximation, introduced briefly. Therefore, the semiclassical plasma model returns new expressions for fluid quantities, and we showed how the oscillations implications to either the amplitude regime adopted are modified. To discuss the propagation of small-amplitude nonlinear dispersive ion sound waves, we proposed a modification to the Reductive Perturbation (RP) method, establishing a fundamental framework to control the system's dependence on free parameters, and then we determined the critical Mach number value from variational principles, excess disturbance variable, and stretched coordinates related to the propagation of traveling waves on a TF plasma. The combination of thermal/non-thermal and magnetized/unmagnetized regimes was made, resulting in interesting relations to both the linear and nonlinear limits, as well as in the configurations Mach number analysis. Thus, a complete description of the semiclassical gas was proceeded, noting that the dimensionless transformations were considered setting the cold TF plasma ion sound speed as a standard fluid quantity for all proposed compositions. To close Chap. 4, we re-discuss the linear and nonlinear results by adjusting the normalization to the characteristic regime ion sound velocity.

Moreover, in Chap. 5, to finish our nonlinear ion acoustic waves investigation, we summarized the main results and showed graphically the influence of the independent parameters on each proposed composition for the TF plasma, in special the temperature and magnetic field consequences in the shape of the KdV solutions, depicting and characterizing the solitons profile for several TF fluids configurations. We also justified the introduction of the control parameter in the RP method and used the resultant relation to link the initial and corrected KdV expressions. To complete, the expansion of the variables about the equilibrium state assumed for the magnetized gas is presented by completeness and self-containment in Appendix A, while in Appendix B we deduced, in a simple form, the well-known solution to KdV-like equations.

Part I

SOLITARY WAVES AND SOLITONS

1 SOLITON THEORY

Solitary waves are special solutions to nonlinear *partial differential equations* that arise due to the balance between nonlinear and dispersive effects, and propagate without any temporal change in shape or size when viewed in the reference frame moving with the wave group velocity. They are localized disturbances, i. e., the envelope has one global peak and decays far away from the central structure.

This chapter presents a general overview of the Soliton Theory major concepts, where we initially discuss the linear and nonlinear character of some equations, in addition to the dispersive or dissipative bias, being possible to introduce, under arguments, the KdV equation. Next, we approach some historical aspects of the study of solitary waves, showing for the first time in this work the characteristic one-soliton solution. Afterward, we briefly examine their interaction, which is responsible for naming the profile, as well as give a short explanation about traveling waves. We are then able to carry out a study on solitary solutions, followed by an analysis of the system bounded shapes. Finally, in order to show that the KdV equation is completely integrable, we demonstrate that it has infinite conservation laws.

In most of this material, *Solitary Waves* and *Solitons* are taken as synonyms, although the difference is reported throughout the text. The content is a bibliographic revision of these topics supported by [15] and [2].

[15]: Solitons: An Introduction
[2]: Nonlinear Dispersive Waves

1.1 LINEAR AND NONLINEAR PROCESS

Before describing a *Solitary Wave* (SW), or Soliton, let us give an overview of the background of these concepts.

The study of wave phenomena is a crucial topic in mathematical physics courses at the beginning of graduation, as such processes are ubiquitous in nature. Oscillations on a rope, on a stretched membrane, or on the free surface of fluid are usually studied, introducing the simplest model for one-dimensional wave propagation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1.1}$$

where c and $u(x,t)$ are commonly referred to as the *phase speed* and *amplitude* of the wave, respectively. The general solution of Eq. (1.1.1) can be expressed as

$$u(x,t) = f(x-ct) + g(x+ct), \tag{1.1.2}$$

where $f(x-ct)$ and $g(x+ct)$ are arbitrary functions. Eq. (1.1.2) is known as the *d'Alembert's solution* to Eq. (1.1.1), with $f(x-ct)$ describing a wave traveling to the right and $g(x+ct)$ a wave traveling to the left, both with speed c.

For our purpose, it is sufficient to restrict the discussion below to waves that propagate to the right. More specifically, by choosing $g=0$ in Eq. (1.1.2), redefining t as t/c, and taking $c=1$, we get

$$u(x,t) = f(x-t). \tag{1.1.3}$$

The deduction of a wave equation is usually made either from phenomenological observations or more general dynamics equations. Among important phenomena that should be described, we can mention dispersion, nonlinear phenomena, and dissipation. Let us illustrate the dispersion process of a wave. Consider the factorization of Eq. (1.1.1),

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u \equiv \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) u = 0. \tag{1.1.4}$$

Adopting $c=1$, we can restrict the discussion to the solution of the equation

$$\partial_t u + \partial_x u = 0. \tag{1.1.5}$$

Writing $u(x,t)$ as a *harmonic* function,

$$u(x,t) = \exp\left[i\left(kx - \omega t\right)\right], \tag{1.1.6}$$

we get the so-called *dispersion relation*, introducing the simplest dispersive term,

$$\partial_t u + \partial_x u + \partial_x^3 u = 0 \qquad \Rightarrow \qquad \omega(k) = k - k^3, \tag{1.1.7}$$

where k is commonly referred to as the *wavenumber* and ω as the *angular frequency*. From Eqs. (1.1.6) and (1.1.7), we then conclude that the wave propagates with speed

$$c = \frac{\omega}{k} = 1 - k^2. \tag{1.1.8}$$

As a characteristic of *dispersive* systems, it is straightforward to conclude from Eq. (1.1.8) that different wavenumbers imply different propagation velocities. The sum over all possible values of k return

$$u(x,t) = \int_{-\infty}^{\infty} A(k) \exp\left[i\left(kx - \omega(k)t\right)\right] dk \tag{1.1.9}$$

for a given $A(k)$ and the wave profile will *disperse* as time evolves. From Eq. (1.1.7) we can write the well-known *group velocity*,

$$c_G = \frac{d\omega}{dk} = 1 - 3k^2, \tag{1.1.10}$$

describing the *wave packet* speed.

To change the described system, we replace the odd derivative term with the even derivative case in Eq. (1.1.7), e.g.,

$$\partial_t u + \partial_x u - \partial_x^2 u = 0. \tag{1.1.11}$$

Assuming again a harmonic solution, Eq. (1.1.6), we get from Eq. (1.1.11)

$$\omega(k) = k - ik^2, \tag{1.1.12}$$

with a quite different expression for $u(x,t)$,

$$u(x,t) = \exp\left[-k^2 t + ik\left(x - t\right)\right]. \tag{1.1.13}$$

Eq. (1.1.13), influenced by the chosen sign in Eq. (1.1.11), describes a propagation that occurs at unity speed, regardless of the k value, which also exhibits an exponential decay, as $t \to \infty$, for any real $k \neq 0$. This decay is known as *dissipation*.

Once in a more accurate description of physical systems the small-amplitude limit becomes inappropriate, it is a better approximation the account of a nonlinear *partial differential equation* (PDE) like, in the simplest case,

$$\partial_t u + (1 + u)\,\partial_x u = 0. \tag{1.1.14}$$

Given by comparison with Eq. (1.1.2), the characteristic solution $u(x,t)$ for Eq. (1.1.14),

$$u(x,t) = f\left[x - (1+u)\,t\right], \tag{1.1.15}$$

will generate a single-valued solution for u only for a finite time interval, considering $f > 0$ for some x, and the change to a multi-valued solution indicates the modification in shape during the motion. While for linear equations is verified the *superposition principle*, for nonlinear equations, in general, this is not true.

Some assumptions can be made to consider more complex systems, such as nonlinear and dispersive or nonlinear and dissipative dynamical equations,

$$\partial_t u + (1+u)\,\partial_x u + \partial_x^3 u = 0, \tag{1.1.16}$$

$$\partial_t u + (1+u)\,\partial_x u - \partial_x^2 u = 0, \tag{1.1.17}$$

where Eq. (1.1.16) depicts nonlinear dispersive systems and is known as *Korteweg-de Vries* (KdV) form equation, while Eq. (1.1.17), called *Burgers equation*, describes some nonlinear dissipative process.

The properties of Eq. (1.1.17) are well known since *1906*, once the equation was reported in [18], had been discussed by [6] and worked extensively in [9]. Paving the way for the study of KdV equation properties, we can introduce the transformation

[18]: Theory of Differential Equations, Part IV. Partial Differential Equations

[6]: Some Recent Researches on the Motion of Fluids

[9]: A Mathematical Model Illustrating the Theory of Turbulence

$$1+u \to \alpha u, \qquad t \to \beta t, \qquad x \to \gamma x, \tag{1.1.18}$$

where α, β and γ are non-zero real constants. From Eqs. (1.1.16) and (1.1.18) we have

$$\partial_t u + \frac{\alpha\beta}{\gamma} u \partial_x u + \frac{\beta}{\gamma^3} \partial_x^3 u = 0, \tag{1.1.19}$$

usually called the general KdV, which makes us able to write

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \tag{1.1.20}$$

commonly referred to as the *KdV equation*.

1.2 HISTORICAL REMARKS

Single and localized entities, posteriorly named SW, were first observed by *J. Scott Russell* in 1834 on the Edinburgh-Glasgow Union Canal, Scotland, while riding on horseback beside the narrow [36]. *Russell* described it as follows:

[36]: Report of the 14th meeting of the British Association for the Advancement of Science

> I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

Performing experiments to analyze the supposed localized waves phenomena, *Russell* observed the existence of shallow and permanent form long waves and was able to deduce the speed of propagation of a SW in a channel as

$$c^2 = c_0^2\left(1 + \frac{A}{h}\right), \tag{1.2.1}$$

for $c_0^2 \equiv gh$, where g and h are the acceleration of gravity, and the undisturbed depth, properly, and A is the wave amplitude.

Knowing the work done by *Russell*, the fluid dynamicist *Ayri* concluded that this is a linear phenomenon [5]. Further investigations by *Boussinesq* [7, 8] and *Rayleigh* [35] confirmed *Russell's* result for the velocity, c, and also showed that the SW profile is given by

$$\zeta\,(x,t) = a\,\mathrm{sech}^2\,[\beta\,(x - ct - p_0)], \quad \beta^2 = \frac{3a}{4h^3}, \quad c = c_0\left(1 + \frac{a}{2h}\right), \tag{1.2.2}$$

where ζ is the wave height above h for any $a > 0$, $a \ll h$, and p_0 is an arbitrary phase. They, however, did not write a governing equation for $\zeta(x,t)$.

In *1895 Korteweg* and *de Vries* derived a nonlinear evolution equation to describe one dimensional long waves of small-amplitude propagating in water from a set of fundamental governing equations [25]. They have shown that

[5]: Tides and Waves

[7]: Théorie de l'intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire

[8]: Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal

[35]: On waves

[25]: On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves

$$\frac{1}{c_0}\partial_t\zeta + \partial_x\zeta + \frac{3}{2h}\zeta\partial_x\zeta + \frac{h^2}{2}\left(\frac{1}{3} - \hat{T}\right)\partial_x^3\zeta = 0, \tag{1.2.3}$$

with the parameter $\hat{T}$ being expressed as

$$\hat{T} = \frac{T}{\rho g h^2} \tag{1.2.4}$$

incorporating the surface tension, T, and the fluid density, ρ. *Russell's* SW is a solution of the KdV equation. *Korteweg* and *de Vries* also showed that Eq. (1.2.3) has periodic *Jacobian elliptic* function traveling wave solutions, termed by them as *cnoidal* function, cn, which return a SW solution when the elliptic modulus goes to unity.

The KdV equation, Eq. (1.2.3), can be written in a dimensionless form by transforming

$$\sigma = \hat{T} - \frac{1}{3}, \quad t' = \beta t, \quad x' = \frac{x - c_0 t}{h}, \quad \beta = -\frac{c_0\sigma}{2h}, \quad \zeta = 2h\sigma u, \tag{1.2.5}$$

where the derivative quantities are expressed in terms of the new coordinates x' and t' as

$$\partial_x = \frac{1}{h}\partial_{x'} \quad \partial_t = \beta\partial_{t'} - \frac{c_0}{h}\partial_{x'} \quad \partial_x^3 = \frac{1}{h^3}\partial_{x'}^3. \tag{1.2.6}$$

Dropping the primes and acting Eqs. (1.2.5) and (1.2.6) on Eq. (1.2.3), we have

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \tag{1.2.7}$$

which is exactly the *KdV equation* previously indicated by Eq. (1.1.20).

As mentioned, since the work of *Korteweg* and *de Vries*, it is known that Eq. (1.2.7) has the one soliton solution

$$u(x,t) = -2\kappa^2 \operatorname{sech}^2\left[\kappa\left(x - 4\kappa^2 t - \delta_0\right)\right], \tag{1.2.8}$$

with δ_0 and κ arbitrary constants, and so the wave velocity, $4\kappa^2$, is proportional to the amplitude, $2k^2$, and the *width* is inversely proportional to κ. Thus, taller waves are thinner and move faster.

Kruskal and *Zabusky* (KZ), in 1965, discovered that these SW profiles seems to be unaffected by passing through each other, although this could introduce a phase, and called them *solitons* for similarities to particles dynamics [50].

[50]: Interaction of "Solitons" in a Collisionless Plasma and the Recurrence of Initial States

1.3 SOLITON INTERACTIONS

Following the brief comment on the particle-like character of these waves in Sec. 1.2, let us take a more specific look at the discovery of such a characteristic. In a more careful analysis of *Russell's* work [36], we can see that he presents in *Plate XLVII - Genesis and Mechanism of the Wave of Translation* a first view of the compound wave behavior. In his description, the general profile approach individual SW in the limit of $t \to \infty$.

[36]: Report of the 14th meeting of the British Association for the Advancement of Science

Another interesting conclusion that follows *Russell's* result arises from the consideration of a system that starts with the tallest wave behind the shortest, considering the same propagation direction, so, in a two-wave case, if both waves travel to the right, the bigger is somewhat to the left of the smaller. In this configuration the taller one interacts with the shorter one, then passes and remains on its path, apparently, unbroken and undisturbed, which is expected from linear waves, as satisfy the *superposition principle*. Therefore, it suggests that this is a special kind of nonlinear process.

In 1955, *Fermi, Pasta, and Ulam* (FPU) worked on a numerical study of a one-dimensional anharmonic lattice [16]. Their model proved to be closely related to a discretization of the KdV equation. As mentioned before, KZ [50] investigated the problem by studying the model corresponding to the continuous limit of FPU considering

[16]: Studies of the Nonlinear Problems

[50]: Interaction of "Solitons" in a Collisionless Plasma and the Recurrence of Initial States

$$\partial_t u + u\partial_x u + \delta^2 \partial_x^3 u = 0, \tag{1.3.1}$$

as an initial value problem with periodic boundary conditions, noting that Eq. (1.3.1) is exactly the KdV equation with δ^2 as the *dispersive* parameter. KZ solved the KdV-like equation for small δ^2 with

$$u(x, 0) = \cos[\pi x], \qquad 0 \leqslant x \leqslant 2, \tag{1.3.2}$$

being u, $\partial_x u$, and $\partial_x^2 u$ periodic in the interval $[0, 2]$ for all t.

When $\delta = 0$, we recover the so-called *inviscid Burgers equation*

$$\partial_t u + u\partial_x u = 0, \tag{1.3.3}$$

which leads to a multi-valued solution, or *shock*, due to a wave steepening in a finite time. However, when $\delta^2 \ll 1$ the dispersive term makes a local balance between dispersion and nonlinearity, resulting at later times in a *train* of solitary type waves. Under propagation, the fastest waves, which are also the ones with the largest amplitude, catch up and overtake the slowest ones, with smaller amplitude, and after a long time, the initial solution profile returns to a state which resembles the initial conditions, as a *recurrence* phenomena.

It is the unexpected dynamics of these oscillations, characterized by *elastic* interaction, with speed and amplitude as invariants of motion, that led KZ to coin the name *soliton*, emphasizing the particle-like character retaining their identities. In fact, the only notable change implied by the interaction is the phase shift that occurs when compared to the behavior in the absence of interaction.

1.4 TRAVELING WAVE SOLUTION

A *traveling wave solution* (TWS) of a PDE in one-space and one-time dimensions, where $x \in \mathbb{R}$ and $t \in \mathbb{R}$ are independent spatial and temporal variables, respectively, and $u \in \mathbb{R}$ is the dependent variable, has the form

$$u(x,t) = w(x-ct) = w(z), \tag{1.4.1}$$

and the solitary wave is a special case of a traveling solution, which is bounded and has constant asymptotic states as $z \to \pm\infty$.

Unlike what happens with linear equations, nonlinear profiles can give some information about the system. The classical wave equation, Eq. (1.1.1), has the TWS given by Eq. (1.1.2), for *arbitrary* $f(x-ct)$ and $g(x+ct)$, while Eq. (1.1.14), which embodies the simplest type of *nonlinearity*, has a solution of Eq. (1.4.1) type only if it is satisfied

$$(1-c+w)\, d_z w = 0, \tag{1.4.2}$$

which implies a trivial non-wave-like solution.

In Sec. 1.1 we mentioned the general characteristic of Eq. (1.1.14) solutions. The expectation of no traveling solution is also held when only dispersion effects are considered, where the wave spreads out instead of steepening. To construct a steady solution by combining these configurations, let us consider a general method from

$$\partial_t u + (1+u)\partial_x u = \nu(u) \tag{1.4.3}$$

for some function $\nu(u)$. From Eqs. (1.4.1) and (1.4.3), we obtain

$$(1-c+w)\, d_z w = \nu(w), \tag{1.4.4}$$

where $z = x - ct$, and the integration of Eq. (1.4.4) gives

$$z = \int \frac{1-c+w}{\nu(w)} dw. \tag{1.4.5}$$

This is the general expression to get TWS for u(x,t). To exemplify, we can take $\nu(u)$ as

$$v(u) = u(1-u^2), \tag{1.4.6}$$

and then we can write from Eq. (1.4.5), for an *arbitrary* c,

$$z = \int \frac{1-c+w}{w(1-w^2)} dw, \tag{1.4.7}$$

which results in the expression, considering an integration constant c_1,

$$z = \frac{1}{2}(c-2)\log(1-w) - 2(c-1)\log(w) + c\log(w+1) + c_1, \tag{1.4.8}$$

and can be rewritten for an arbitrary constant A as

$$A\exp(2z) = \frac{w^2(1-w^2)^c}{w^{2c}(1-w)^2}. \tag{1.4.9}$$

If we take for simplicity $c = 1$, Eq. (1.4.9) is reduced to

$$A\exp(2z) = \frac{1+w}{1-w} \quad \Rightarrow \quad w = \frac{A\exp(2z)-1}{A\exp(2z)+1}, \tag{1.4.10}$$

and then Eq. (1.4.10) can be expressed more suitably as

$$u(x,t) = w(x-t) = \tanh(x-t-\delta_0), \tag{1.4.11}$$

where we write, without loss of generality,

$$A = \exp(-c_1) = \exp(-2\delta_0). \tag{1.4.12}$$

1.5 SOLITARY WAVE

Returning to the SW solution mentioned in Sec. 1.2, let us assume the KdV expressed by Eq. (1.2.7). Looking for TWS, we can write for the standard form of the KdV equation

$$-cd_z w - 6wd_z w + d_z^3 w = 0. \tag{1.5.1}$$

Thus, if we integrate Eq. (1.5.1) once, we get

$$-cw - 3w^2 + d_z^2 w = A, \tag{1.5.2}$$

where A is an arbitrary integration constant, and then we can multiply Eq. (1.5.2) by $d_z w$ to obtain

$$-cwd_z w - 3w^2 d_z w + d_z w d_z^2 w - Ad_z w = 0, \tag{1.5.3}$$

which enable us to write

$$(d_z w)\, d_z^2 w = d_z \left[\frac{1}{2}(d_z w)^2\right]. \tag{1.5.4}$$

So, by replacing Eq. (1.5.4) terms in Eq. (1.5.3) and integrating again, we get

$$\frac{1}{2}(d_z w)^2 = w^3 + \frac{c}{2}w^2 + Aw + B, \tag{1.5.5}$$

with another integration constant, B.

To discuss the limit of SW, we admit as boundary conditions

$$\left(w, d_z w, d_z^2 w\right) \to 0 \quad \text{as} \quad z \to \pm\infty, \tag{1.5.6}$$

which implies that the integration constants A and B are both nulls. Thus,

$$(d_z w)^2 = w^2 (2w + c)\,, \tag{1.5.7}$$

and we can conclude that real solutions are possible only if it is satisfied

$$(d_z w)^2 \geqslant 0 \quad \text{so} \quad 2w + c \geqslant 0. \tag{1.5.8}$$

To express $w(z) = w(x - ct)$, we integrate Eq. (1.5.7) using the substitution $w = -(c/2)\,\text{sech}^2\,\theta$, with $c \geqslant 0$, and so

$$w(x - ct) = -\frac{c}{2}\,\text{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct - \delta_0)\right], \tag{1.5.9}$$

with a *phase* δ_0 that arises from the integration constant. We can note that Eq. (1.5.9) exists without an upper limit for c, and as a consequence of a negative nonlinear parameter in Eq. (1.2.7), we have $w \leqslant 0$.

1.6 GENERAL TRAVELING WAVE ANALYSIS

In Sec. 1.5, we adopted the boundary conditions of Eq. (1.5.6) to describe the SW solution. We can, however, investigate $w(z)$ for arbitrary values of A, B, and c looking for *real bounded* shapes from Eq. (1.5.5). Defining

$$w^3 + \frac{c}{2}w^2 + Aw + B \equiv W(w), \tag{1.6.1}$$

which enables us to write

$$d_z w = \pm\sqrt{2W(w)}, \tag{1.6.2}$$

and requiring condition $(d_z w)^2 \geqslant 0$ to be satisfied, from Eq. (1.6.1) we can see that $W(w)$ has at least one real zero and that w will vary monotonically until $d_z w$ vanishes. The roots of Eq. (1.6.1), then, carry important system properties.

Since we are interested in obtaining real bounded solutions for the KdV equation, we require that $W(w) \geqslant 0$ and this leads to the study of the zeros of $W(w)$. There are two different cases to consider, being the one real zero and the three real root cases, for which there are sub-cases. First, if we consider $w = \alpha$ as Eq. (1.6.1) *simple* zero, and expand in *Taylor* $W(w)$ about α, we have from Eq. (1.5.5)

$$(d_z w)^2 = 2\left[d_z W(\alpha)\right](w-\alpha) + \mathcal{O}\left[(w-\alpha)^2\right], \tag{1.6.3}$$

where we used $W(\alpha) = 0$, and then

$$d_z w\big|_\alpha = 0 \quad \text{and} \quad d_z^2 w\big|_\alpha = d_z W(\alpha). \tag{1.6.4}$$

So, as $z \to z_1$, where $w(z_1) = \alpha$, we can write using Eq. (1.6.4),

$$w(z) = \alpha + \frac{1}{2}\left[d_z W(\alpha)\right](z-z_1)^2 + \mathcal{O}\left[(z-z_1)^3\right], \tag{1.6.5}$$

and then the function w has a local maximum (minimum) at $z = z_1$, as $d_z W(\alpha)$ is negative (positive). If at some point $z = z_0$ the slope is such that $d_z w > 0$, for all $z > z_0$ we have $W > 0$ and, consequently, as $z \to \infty$ it is also valid that $w \to \infty$. However, if $d_z w < 0$ we have that w decreases until it reaches α, the largest real root of W, at z_1. Thus, $d_z w$ changes sign and $w \to \infty$ as $z \to \infty$. Therefore, there are no bounded solutions in this case.

On the other hand, if $W(w)$ has three real zeros, β, δ, and γ, assuming that $\beta \leqslant \delta \leqslant \gamma$, we may write

$$W(w) = (w-\beta)\,(w-\delta)\,(w-\gamma)\,, \tag{1.6.6}$$

where looking at Eq. (1.6.1) and comparing it with Eq. (1.6.6), we verify the relations

$$c = -2\,(\beta+\delta+\gamma) \qquad A = (\delta\gamma+\beta\gamma+\beta\delta) \qquad B = -\beta\delta\gamma. \tag{1.6.7}$$

Supposing distinct β, δ, and γ, the solution of the KdV equation can be expressed in terms of the *Jacobian Elliptic* (JE) function, $\mathrm{cn}\,(z; m)$,

$$w(z) = \delta - (\gamma-\delta)\,\mathrm{cn}^2\left[\left(\frac{1}{2}\,(\gamma-\beta)\right)^{\frac{1}{2}}(z-\delta_0)\,; m\right], \tag{1.6.8}$$

where δ_0 is a constant. The parameter m, referred to as the *elliptic parameter*, denotes the modulus of the JE function and is expressed as [47]

[47]: A Course of Modern Analysis

$$m = \frac{\gamma-\delta}{\gamma-\beta} \quad \text{and} \quad 0 < m < 1. \tag{1.6.9}$$

As mentioned, the well-known *cnoidal wave* (CW) was found and appears in the paper of *Korteweg* and *de Vries*, who coined the CW term.

However, considering $\beta = \delta \neq \gamma$, we take the limit of $m \to 1$, i. e., we have $\delta \to \beta$, and then Eq. (1.6.8) reduces to

$$w(z) = \beta - (\gamma-\beta)\,\mathrm{sech}^2\left[\left(\frac{1}{2}\,(\gamma-\beta)\right)^{\frac{1}{2}}(z-\delta_0)\,; m\right], \tag{1.6.10}$$

since $\mathrm{cn}\,(z; 1) = \mathrm{sech}\,(z)$ [4]. In this case we also have

[4]: Handbook of Mathematical Functions.

$$c = -2\,(2\beta+\gamma)\,. \tag{1.6.11}$$

Note that if we write $\beta = w_0$ and $\kappa^2 = \frac{1}{2}\,(\gamma-\beta)$, we obtain

$$w(z) = w_0 - 2\kappa^2\,\mathrm{sech}^2\,[\kappa\,(z-\delta_0)], \tag{1.6.12}$$

which exhibits, as expected, the *speed*, *shape*, and *wavelength* dependence on the *amplitude*, with

$$c = -2\left(2\kappa^2+3w_0\right), \tag{1.6.13}$$

and, if $w_0 = 0$ we get exactly Eq. (1.2.8).

Now, taking $\beta \neq \delta = \gamma$ we have

$$w(z) = \delta + (\delta - \beta)\sec^2\left[\left[\frac{1}{2}(\delta - \beta)\right]^{\frac{1}{2}}(z - \delta_0)\right], \quad (1.6.14)$$

and the solution is unbounded unless $\delta = \gamma$, that is $m \to 0$, in which case the only bounded real solution is the constant $w(z) = \delta$. Looking for Eq. (1.6.8) and defining $\kappa^2 \equiv \frac{1}{2}(\gamma - \beta)$, as for small m we have [4]

[4]: Handbook of Mathematical Functions.

$$\mathrm{cn}\,(z; m) = \cos(z) + \mathcal{O}(m)\,, \quad (1.6.15)$$

it follows that $\gamma \to \delta$ implies

$$w(z) \sim \frac{1}{2}(\delta + \gamma) - \kappa^2 m \cos\left[2\kappa(z - \delta_0)\right], \quad (1.6.16)$$

and this is a limiting case of a *sinusoidal* wave.

Finally, considering $\beta = \delta = \gamma$ we can integrate directly to find

$$w(z) = \beta + \frac{2}{(z - \delta_0)^2}, \quad (1.6.17)$$

where δ_0 is a constant again. Thus, the solution in Eq. (1.6.17) is unbounded at $z = \delta_0$, and the only real bounded solution is the constant $w(z) = \beta$, which is obtained by taking the limit $\gamma \to \beta$ in Eq. (1.6.10).

1.7 KDV CONSERVATION LAWS

The interest in the KDV equation, as well as in the development of a general method of solution, led to the discovery of an infinite number of independent conservation laws, being referred to as a *completely integrable* or *exactly solvable* equation. For an initial-value problem, the configuration can be linearized by employing a method termed *inverse scattering transform* (IST) [1, 19] .

[1]: Solitons, Nonlinear Evolution Equations and Inverse Scattering

[19]: Method for Solving the Korteweg-deVries Equation

To show that the KdV has another remarkable property beyond the solitonic solution, let us denote the PDE as

$$\mathcal{W}(x, t, u(x, t)) = 0, \quad (1.7.1)$$

where, again, $x \in \mathbb{R}$, $t \in \mathbb{R}$ are, respectively, the independent space and time variables, $u(x, t) \in \mathbb{R}$ is the dependent variable, and the possible dependence on partial derivatives of u is also understood in Eq. (1.7.1). A *conservation law* is an equation of the form

$$\partial_t \mathcal{T}(u) + \partial_x \mathcal{X}(u) = 0, \quad (1.7.2)$$

which is satisfied for all solutions of Eq. (1.7.1). The variables $\mathcal{T}(x,t,u)$, associated with the *conserved density*, and $\mathcal{X}$, *associated flux*, are functions of x, t, u, and also of partial derivatives, in general.

We can integrate Eq. (1.7.2) to obtain

$$\int_{-\infty}^{\infty} \partial_t \mathcal{T}(x,t,u)dx = -\int_{-\infty}^{\infty} \partial_x \mathcal{X}(x,t,u)dx, \tag{1.7.3}$$

and if $u \to 0$ as $x \to \pm\infty$ sufficiently rapidly, the right-hand side of Eq. (1.7.3) will be zero. So,

$$d_t \int_{-\infty}^{\infty} \mathcal{T}(x,t,u)dx = 0, \tag{1.7.4}$$

and we have that in this process the quantity

$$\int_{-\infty}^{\infty} \mathcal{T}(x,t,u)dx = c_D \tag{1.7.5}$$

must be conserved, where the constant c_D is the *conserved density*.

Looking at the KdV equation expressed by Eq. (1.1.20), we can see that it is written in conservation form as

$$\partial_t u + \partial_x \left(-3u^2 + \partial_x^2 u\right) = 0, \tag{1.7.6}$$

and this conservation law, generally corresponding to the conserved mass or momentum, expresses the conservation of

$$c_{D_1} \equiv \int_{-\infty}^{\infty} u dx. \tag{1.7.7}$$

Following the same process, if we multiply Eq. (1.1.20) by $2u$ and rearrange the terms, we can write

$$\partial_t \left(u^2\right) + \partial_x \left(-4u^3 + 2u\partial_x^2 u - [\partial_x u]^2\right) = 0. \tag{1.7.8}$$

Multiplying Eq. (1.1.20) by $3u^2 - \partial_x^2 u$ and performing some manipulations, we obtain

$$\partial_t \left(u^3 + \frac{1}{2}[\partial_x u]^2\right) + \partial_x \left(-\frac{9}{2}u^4 + 3u^2\partial_x^2 u - \frac{1}{2}\left[\partial_x^2 u\right]^2 - \partial_x u \partial_t u\right) = 0, \tag{1.7.9}$$

with Eq. (1.7.8) being interpreted as the energy conservation law for several systems and Eq. (1.7.9) corresponding to the Hamiltonian [46].

[46]:Non-linear dispersive waves

Kruskal and *Zabusky* [26] were responsible for discovering the fourth and fifth conservation laws, and later another four were discovered before *Miura conjectured* the existence of infinite conserved densities. Studying the special aspects of a more general class of equations,

[26]: Korteweg-deVries Equation and Generalizations. V. Uniqueness and Nonexistence of Polynomial Conservation Laws

$$\partial_t v - 6v^p \partial_x v + \partial_x^3 v = 0, \quad \text{with} \quad p = 1, 2, \dots, \tag{1.7.10}$$

where we can verify that the first equation is the KdV, and the second is

$$\partial_t v - 6v^2 \partial_x v + \partial_x^3 v = 0, \tag{1.7.11}$$

called the *modified KDV equation* (mKDV), *Miura* discovered a transformation, now known as *Miura's transformation*, which states that if v is a solution of the mKDV equation, Eq. (1.7.11), then

$$u = v^2 + \partial_x v \tag{1.7.12}$$

is a solution of the KdV equation, Eq. (1.1.20). This is directly seen by substituting Eq. (1.7.12) into Eq. (1.1.20) to obtain the relation

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = (2v + \partial_x)\left(\partial_t v - 6v^2 \partial_x v + \partial_x^3 v\right). \tag{1.7.13}$$

We can note that the solution of the mKdV equation leads to a solution of the KdV, but not every solution of the KdV is obtained from a mKdV solution [3]. *Miura's transformation* is an important *tool* to prove that Eqs. (1.1.20) and (1.7.11) have an infinite number of conserved densities [33] but it is also fundamental for the development of the IST.

[3]: A note on Miura's transformation

[33]: Korteweg-de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion

To show the existence of an infinite number of conservation laws, we can define ω by

$$u = \omega + \epsilon \partial_x \omega + \epsilon^2 \omega^2, \tag{1.7.14}$$

which is the *Gardner's transform*, a generalization of Eq. (1.7.12). Then, the Eq. (1.7.13) equivalent is

$$\begin{aligned}\partial_t u - 6u\partial_x u + \partial_x^3 u \\ = \left(1 + \epsilon\partial_x + 2\epsilon^2 \omega\right)\left[\partial_t \omega - 6\left(\omega + \epsilon^2\omega^2\right)\partial_x \omega + \partial_x^3 \omega\right],\end{aligned} \tag{1.7.15}$$

and the *variable* u, defined by Eq. (1.7.14), is a solution of the KdV equation provided that ω is a solution of

$$\partial_t \omega - 6\left(\omega + \epsilon^2 \omega^2\right) \partial_x \omega + \partial_x^3 \omega = 0. \quad (1.7.16)$$

Observing that u is independent of ϵ, while ω is a function of x, t, and ϵ, let us assume ω as a power series of ϵ. So,

$$\omega\,(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \omega_n\,(x, t) = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \ldots, \quad (1.7.17)$$

and since Eq. (1.7.16) can be written in a conservation form, we have

$$\int_{-\infty}^{\infty} \omega\,(x, t; \epsilon) = \tilde{c}. \quad (1.7.18)$$

From Eqs. (1.7.17) and (1.7.18), we can simplify the relation to

$$\int_{-\infty}^{\infty} \omega_n\,(x, t) = \tilde{c}_n \quad \text{for each} \quad n = 0, 1, 2, \ldots \quad (1.7.19)$$

So, substituting Eq. (1.7.17) into Eq. (1.7.14) and determining ω recursively by equaling in powers of ϵ, we get

$$\begin{aligned}
&\omega_0 = u, \\
&\omega_1 = -\partial_x \omega_0 = -\partial_x u, \\
&\omega_2 = -\left[\partial_x \omega_1 + (\omega_0)^2\right] = \partial_x^2 u - u^2, \\
&\omega_3 = -\left[\partial_x \omega_2 + 2\omega_0 \omega_1\right] = -\partial_x^3 u + 4u\partial_x u, \\
&\omega_4 = -\left[\partial_x \omega_3 + (\omega_1)^2 + 2\omega_0 \omega_2\right] = \partial_x^4 - 5\left(\partial_x u\right)^2 - 6u\partial_x^2 u + 2u^3, \\
&\ldots
\end{aligned} \quad (1.7.20)$$

and this gives us an infinite number of *conserved densities* associated with each power of ϵ.

Part II

FUNDAMENTALS OF PLASMA PHYSICS

2 BASIC PLASMA THEORY

Plasma, to a good approximation, is usually an assembly of ions and electrons obtained by heating a gas. It is seen as a fully, or almost fully, ionized gas in quantities larger than one so-called *Debye sphere*, which corresponds to the extent of an electric charge influence on other charged particles.

This chapter is divided into five sections. First, we present a review of what is generally termed classical plasma, introducing the collective and quasi-neutral characteristics. Subsequently, we make a brief comment on the conditions of the system classification as a plasma. In the third part, we introduce the fluid description indicating the equations that govern the dynamics of the configuration. Then, we allocate two parts to differentiate the absence and presence, respectively, of ion thermal effects and their consequences in the plasma dynamics.

Several important ideas for structuring the basis of future computations in this work appear here. These consolidated topics are a literature review following [12] and [42].

[12]:Introduction to Plasma Physics and Controlled Fusion
[42]:Plasma Waves

2.1 CLASSICAL PLASMA

Sometimes called the *fourth state of matter* and characterized as a state where long-range electromagnetic interactions dominate, among particles, the short-range inter-atomic and inter-molecular forces, i. e., a system of a large number of particles with behavior mediated by electromagnetic forces, a *plasma* has as a short and useful definition;

> *A plasma is a quasi-neutral gas of charged and neutral particles which exhibits collective behavior*

so it is interesting for us to define the meaning of *collective behavior* and *quasi-neutral* in this case.

First, if we look for an arbitrary ideal gas, where the force of gravity is negligible and there is no net electromagnetic force, the neutral molecule moves freely until it collides with another, and the successive collisions are responsible to map the motion of the particles. Then, a force applied to the gas is transmitted by collisions. In a plasma, however, once we are dealing with charged particles, the dynamics can generate concentrations of positive and negative charges and result in electric fields, which affect the motion of other more distant particles. It is the long-range electromagnetic forces that give rise to the system complexity and makes possible the existence of interesting phenomena even in an analysis of the *collisionless* plasma, in which the forces due to *local* collisions are suppressed by the *global* electromagnetic forces. So, the *collective behavior* refers to the non-local character, as the dynamic is not only influenced by the neighbor but also by charged particles far away.

To describe the *quasi-neutral* character of the matter, it is interesting for us to make a small aside on the *temperature* concept in that system. As in the gas in thermodynamic equilibrium we have particles of all velocities, the statistical distribution of speeds is given by the *Maxwell-Boltzmann* (MB) distribution [30, 31]. The three-dimensional Maxwellian distribution assumes the form

[30]: II. Illustrations of the dynamical theory of gases

[31]: V. Illustrations of the dynamical theory of gases

$$f(v_x, v_y, v_z) = A \exp\left(-\frac{1}{2}\frac{m}{k_B T}\left(v_x^2 + v_y^2 + v_z^2\right)\right) \tag{2.1.1}$$

where the quantity (fd^3v) is the number of particles with velocity in the narrow range v_i to $v_i + dv_i$, for $i = 1, 2, 3$, since $v^2 = v_x^2 + v_y^2 + v_z^2$ and $d^3v = dv_x dv_y dv_z$ define an element of volume in velocity space. The constant k_B is known as *Boltzmann's constant*, and A is given by

$$A = n\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}}, \tag{2.1.2}$$

being n the particle density.

We can see that the distribution width is related to T, a constant *temperature*. Computing the average kinetic energy of particles in MB distribution, $\bar{E}$, we have

$$\bar{E} = \frac{\int_{-\infty}^{\infty}\left(\frac{1}{2}mv^2\right) A \exp\left(-\frac{1}{2}\frac{m}{k_B T}v^2\right) d^3v}{\int_{-\infty}^{\infty} A \exp\left(-\frac{1}{2}\frac{m}{k_B T}v^2\right) d^3v} \tag{2.1.3}$$

which is symmetric in the three directions, and each of the velocity space integrals can be evaluated apart. Then, we can obtain for Eq. (2.1.3)

$$\bar{E} = \frac{3}{2}k_B T, \tag{2.1.4}$$

since we have $k_B T/2$ per degree of freedom, and we have a close relation between $\bar{E}$ and T. It is important to cite that a plasma can be characterized by several temperatures. Different components, for example, can be described by MB distributions with different temperatures, like T_i for ions and T_e for electrons in an *electron-ion* (e-i) plasma, once the collision rate between ions or electrons is greater than collision among ions and electrons.

Let us now consider the introduction of a charged ball inside the fluid. As a direct response to the insertion of the ball, oppositely charged particles will be attracted and immediately a cloud of ions would surround the ball. In the case of a cold plasma, where no thermal motion occurs, the amount of total charge in the cloud would be the same as the charge of the ball, then we verify a perfect shielding and no electric field exist in the system outside the cloud. However, if we are dealing with a thermal plasma, with finite temperature, particles on the edge of the cloud have enough thermal energy to escape from the electrostatic potential, once the electric field is weak. So, the limit at which the shielding is incomplete characterizes the configuration *radius*, and this is characterized by the equality of the potential and particle thermal energies.

Assuming an (e-i) plasma, where the ion-electron mass ratio is large enough that we can consider fixed ions, forming a uniform background of positive charge, due to the inertia, the *Poisson's* equation takes the form

$$\nabla^2\phi = -\frac{e}{\epsilon_0}\left(n_i - n_e\right), \tag{2.1.5}$$

where n_i and n_e are the ion and electron density, respectively, $e > 0$ the particle charge, ϕ the electric potential, and ϵ_0 the *vacuum permittivity*. Considering the potential energy, we have for the electron distribution

$$f(v) = A\exp\left(-\frac{1}{k_B T_e}\left(\frac{m}{2}v^2 - e\phi\right)\right) \tag{2.1.6}$$

and integrating Eq. (2.1.6) over v, we get for the electron density

$$n_e = n_0\exp\left(\frac{e\phi}{k_B T_e}\right), \tag{2.1.7}$$

noting that in the limit of $\phi \to 0$, $n_e = n_0$. The ion density is

$$n_i = n_0. \tag{2.1.8}$$

Then, if we substitute Eqs. (2.1.7) and (2.1.8) into Eq. (2.1.5), we obtain

$$\nabla^2\phi = -\frac{e}{\epsilon_0}n_0\left[1 - \exp\left(\frac{e\phi}{k_B T_e}\right)\right], \tag{2.1.9}$$

so, expanding Eq. (2.1.9) in the region where $e\phi \ll k_B T_e$,

$$\nabla^2\phi = \frac{e}{\epsilon_0}n_0\left[\frac{e\phi}{k_B T_e} + \frac{1}{2}\left(\frac{e\phi}{k_B T_e}\right)^2 + \cdots\right] \tag{2.1.10}$$

and keeping the linear term,

$$\nabla^2 \phi = \frac{n_0 e^2}{\epsilon_0 k_B T_e} \phi. \tag{2.1.11}$$

If we look for the solution of Eq. (2.1.11), adopting spherical geometry and requiring $\phi \to 0$ as $r \to \infty$, for $r^2 = x^2 + y^2 + z^2$,

$$\phi(r) = \frac{\bar{\phi}_0}{r} \exp\left(-\frac{r}{\lambda_D}\right) \tag{2.1.12}$$

where we have defined the *Debye length* quantity, λ_D, which is a measure of the shielding distance and is expressed as

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T_e}{n_0 e^2}}. \tag{2.1.13}$$

Understanding the influences of the temperature parameter, T, and the *Debye Shielding* (DS) effect on the system, and considering the system dimension L much larger than λ_D, the external charge perturbations introduced into the plasma are shielded out in a short distance when compared to the dimension, and we do not verify electric potentials, or fields, in the fluid bulk. Thus, $n_i \approx n_e$, and the system is *quasineutral*.

2.2 PLASMA PARAMETERS

Sometimes it can be difficult to distinguish a plasma from an ordinary ionized gas, for example, and it becomes interesting to define some relations that form a set of criteria to characterize the system as plasma. As we saw in Sec. 2.1, a central idea in the matter concept is that the fluid must be dense enough such that L is much larger than λ_D, so we can write as a first *plasma criterion*

$$\lambda_D \ll L. \tag{2.2.1}$$

Another conclusion can be made from the analysis of the DS process. To enable the cloud of charged particles shaping, the configuration must have enough particles for the statistical validity of the concept. Being N_D the number of elements in a *Debye* sphere, the *collective behavior* requires

$$N_D \gg 1. \tag{2.2.2}$$

Finally, following previous comments, a plasma is a fluid in which the particle's motion is controlled by electromagnetic forces.

So, we can distinguish the species dynamics in a fluid governed by collision or by charge with the introduction of a plasma oscillation characteristic frequency parameter, ω, and the mean time between collisions with neutral fragments, τ. Thus, for the gas to behave like a plasma, we require

$$\omega > \frac{1}{\tau}. \tag{2.2.3}$$

However, Eqs. (2.2.1), (2.2.2), and (2.2.3) must be satisfied to be considered as a plasma.

2.3 FLUID DESCRIPTION

We can study plasma dynamics using a fluid mechanics model, in which only the analysis of fluid elements is considered, simplifying the process because we can neglect the individual particle identities. However, new considerations have to be made since the plasma components are charged, unlike what occurs in an ordinary fluid, in which, as we said, has the motion explained by collisions. So even in the description of a collisionless plasma, the fluid approximation works.

As we are dealing with a self-consistent problem, with the support of *Maxwell's* equation [32], we must have an expression that dictates the system's response to EM fields, so we will consider *interpenetrating distinct fluids*. Neglecting collisions and thermal motions, the moving equation of a single particle is

[32]: VIII. A dynamical theory of the electromagnetic field

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\left(\vec{E} + \vec{v} \times \vec{B}\right), \tag{2.3.1}$$

where $\vec{v}$, q, and m are the particle velocity, charge, and mass, respectively. Once we consider that all particles in an element of fluid move together with *average* velocity $\vec{u}$, and noting that to a frame moving with the fluid $\frac{d}{dt}$ is the material derivative, we can write in a more convenient form

$$mn\left[\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u}\right] = qn\left(\vec{E} + \vec{u} \times \vec{B}\right), \tag{2.3.2}$$

and Eq. (2.3.2) is called the *momentum equation*. If we consider the addition of thermal motion effects in the system, a pressure gradient force needs to be added in Eq. (2.3.2), and we get

$$mn\left[\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u}\right] = qn\left(\vec{E} + \vec{u} \times \vec{B}\right) - \vec{\nabla}p, \tag{2.3.3}$$

being p the *scalar* pressure parameter. As we are dealing with the fluid motion of a single particle type, of specific m/q, then mn and qn are the *mass* and *charge* densities, respectively.

The conservation of matter is expressed by the well-known *continuity equation*,

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{u}) = 0, \tag{2.3.4}$$

where $n\vec{u}$ is the *particle flux density*. In a multi-fluid analysis, each species must satisfy Eq. (2.3.4) independently. To close the system we need an *equation of state* (EOS) for the fluid. In some circumstances, we can start from the *ideal gas law*,

$$p = nk_B T, \tag{2.3.5}$$

and then assumes the EOS in a standard form, *statistically* expressed as

$$p = p_0 \left(\frac{n}{n_0}\right)^{\gamma} \quad \text{or} \quad T = T_0 \left(\frac{n}{n_0}\right)^{\gamma-1} \tag{2.3.6}$$

where $\gamma = C_v/C_p$ is the *polytropic index*, and C_v and C_p are the *specific heats* at constant volume and pressure, respectively. Considering $\mathcal{N}$ the number of degrees of freedom, for a negligible heat flow we can write

$$\gamma = \frac{\mathcal{N}+2}{\mathcal{N}}. \tag{2.3.7}$$

Otherwise, if we are dealing with an *isothermal* system, we have $\gamma = 1$, and for an *isobaric* process, i. e., constant pressure, $\gamma = 0$.

2.4 COLD PLASMA MODEL

It is interesting here to distinguish two plasma models from the fluid approach set of equations obtained in the previous section, Sec. 2.3. The main difference between the two limits lies in the thermal consideration and gives rise to important consequences in several plasma phenomena.

By *cold plasma* we mean the system in which the ion's thermal motion can be neglected, then we can obtain a simpler set of macroscopic equations by simplifying the momentum equation, Eq. (2.3.3). So,

$$p = 0, \tag{2.4.1}$$

and pressure effects are null. In that case, the particle density, n, and fluid velocity, $\vec{u}$, are the remaining macroscopic variables, which are fully described by

$$mn\left[\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\right] = qn\left(\vec{E} + \vec{u}\times\vec{B}\right) \tag{2.4.2}$$

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{u}) = 0. \tag{2.4.3}$$

In addition, for completeness, the condition given by Eq. (2.4.1) is also useful in the analysis of collisional plasma dynamics. For instance, in some systems the collision is taken as an *impediment to motion*, causing a rate of decrease in the moment,

$$\vec{S} = -mn\nu_{\text{eff}}\vec{u}, \tag{2.4.4}$$

where ν_{eff} is the *effective collision frequency*.

2.5 WARM PLASMA MODEL

Different from the cold plasma approach, at the *warm plasma* limit the thermal effects are not neglected, and the configuration becomes described by a distinct set of macroscopic equations. As we are neglecting viscous forces, we have

$$mn\left[\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\right] = qn\left(\vec{E} + \vec{u}\times\vec{B}\right) - \vec{\nabla}p, \tag{2.5.1}$$

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{u}) = 0, \tag{2.5.2}$$

but now the system is not closed, as we have a new variable to consider. We can use the above-mentioned EOS to make the configuration solvable since it relates p to n.

For an isothermal configuration, assuming T_0 as the constant temperature, the EOS can be expressed as

$$p = nk_B T_0, \tag{2.5.3}$$

so that the gradient of Eq. (2.5.3) is given by

$$\vec{\nabla}p = k_B T_0 \vec{\nabla}n. \tag{2.5.4}$$

Alternatively, if the plasma does not exchange energy with its surroundings, we can use the *adiabatic* EOS, given by

$$pn^{-\gamma} = C, \tag{2.5.5}$$

where C is a constant, and then we can equate to the pressure

$$\vec{\nabla} p = \gamma C n^{\gamma - 1} \vec{\nabla} n. \tag{2.5.6}$$

We can rewrite Eq. (2.5.6) using, again, the condition presented in Eq. (2.5.5), to get

$$\vec{\nabla} p = \gamma \frac{p}{n} \vec{\nabla} n \tag{2.5.7}$$

and we obtain in a more suitable form the $\vec{\nabla} p$ expressed as

$$\frac{\vec{\nabla} p}{p} = \gamma \frac{\vec{\nabla} n}{n}. \tag{2.5.8}$$

Moreover, as in an adiabatic compression, for example, T also changes, we have from Eq. (2.5.8)

$$\frac{\vec{\nabla} n}{n} + \frac{\vec{\nabla} T}{T} = \gamma \frac{\vec{\nabla} n}{n}, \tag{2.5.9}$$

and so, relating the fluid parameters n and T,

$$\frac{\vec{\nabla} T}{T} = (\gamma - 1) \frac{\vec{\nabla} n}{n}. \tag{2.5.10}$$

3

WAVES IN PLASMA

Plasmas show a wide range of phenomena basically because they are composed of two or more components and also can be made strongly anisotropic by the introduction of magnetic fields. Due to the interconnected behavior of particles and fields, the system supports a vast variety of waves, which can propagate periodically through the medium and determine the plasma dynamics.

We start this chapter by recapping, in the first section, the background of oscillations, allowing the determination of the characteristic frequency of the fluid. In the following section, we investigate, in the unmagnetized and magnetized regimes, the linear (vanishing amplitude) limit of ion acoustic waves from a parallel with pressure ones in a neutral gas. To complete, in the third part we study the small, finite, amplitude case for a non-magnetized, cold, classical configuration. We use the *Sagdeev's* pseudopotential method to demonstrate that, under certain constraints, ion sound waves are *KdV* solitons when considering one higher order than the linear approach.

This stage is concerned with the propagation of ion waves, both linear and nonlinear, but concentrates on the latter. The main ideas introduced here are a review of topics present in [12] and [42].

[12]:Introduction to Plasma Physics and Controlled Fusion
[42]:Plasma Waves

3.1 OSCILLATIONS IN PLASMA

In view of the study of oscillations in plasma, we assume the electrons in an ion uniform background. If we displace the electrons from their equilibrium position, due to the *neutrality* character of the fluid, electric fields are induced to return them to the initial configuration. However, the particle inertia results in an oscillation around the equilibrium point.

To characterize the system, let us assume an *infinite, unmagnetized, cold* plasma with *uniformly distributed fixed* ions, such that

$$\vec{\nabla} = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z \qquad\qquad \vec{E} = -\vec{\nabla}\phi, \tag{3.1.1}$$

where $\vec{E}$ is the electric field associated with the electrostatic potential ϕ and, following the discussion presented in Sec. 2.1, the dynamics of negatively charged particles is described by

$$\frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(n_e\vec{v}_e) = 0, \tag{3.1.2}$$

$$\frac{\partial \vec{v}_e}{\partial t} + \left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e = \frac{e}{m_e}\vec{\nabla}\phi, \tag{3.1.3}$$

with the e subscripts referencing electron quantities. To complete the description, we need the support of *Poisson's* equation

$$\nabla^2\phi = -\frac{e}{\epsilon_0}\left(n_i - n_e\right). \tag{3.1.4}$$

Once the oscillations are so fast that we consider fixed ions, not responding to field oscillations, and of small amplitude, the set of Eqs. (3.1.2), (3.1.3), and (3.1.4) can be solved by *linearization* over the equilibrium quantities, i. e., considering a physical parameter A, scalar or vector, it becomes possible to write $A = A_0 + A_1$, where A_0 and A_1 are the representation of the quantities in the equilibrium and perturbed states, respectively. Thus,

$$n_e = n_0^e + n_1^e, \tag{3.1.5}$$
$$\vec{v}_e = \vec{v}_0^e + \vec{v}_1^e, \tag{3.1.6}$$
$$\phi = \phi_0 + \phi_1, \tag{3.1.7}$$

and the initial assumption of a *resting neutral* plasma implies

$$\vec{v}_0^e = \phi_0 = 0, \tag{3.1.8}$$
$$\vec{\nabla} n_0^e = 0, \tag{3.1.9}$$
$$\partial_t \vec{v}_0^e = \partial_t \phi_0 = \partial_t n_0^e = 0. \tag{3.1.10}$$

We know that any fluid periodic motion can be decomposed by *Fourier analysis*. Decomposing the perturbative quantities into *Fourier* modes

$$A_1 \to A_1 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right], \tag{3.1.11}$$

denoting the oscillation frequency as ω and the propagation vector as $\vec{k}$, where

$$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z, \tag{3.1.12}$$

and neglecting higher-order terms, combining Eqs. (3.1.5), (3.1.6), and (3.1.7) to Eqs. (3.1.2), (3.1.3), and (3.1.4) and applying Eqs. (3.1.8), (3.1.9), and (3.1.10), we have

$$-i\omega n_1^e = -i n_0\left(\vec{k}\cdot\vec{v}_1^e\right) \tag{3.1.13}$$
$$-i m_e \omega \vec{v}_1^e = i e \vec{k}\phi_1 \tag{3.1.14}$$
$$-k^2 \epsilon_0 \phi_1 = e n_1^e \tag{3.1.15}$$

taking into account the ion density conditions,

$$n_0^i = n_0^e \quad \text{and} \quad n_1^i = 0. \tag{3.1.16}$$

Therefore, solving for v_1^e in Eqs. (3.1.13), (3.1.14), and (3.1.15) and manipulating, we get

$$-im_e\omega\vec{v}_1^e = -i\frac{n_0 e^2}{\epsilon_0\omega}\vec{v}_1^e, \tag{3.1.17}$$

and if $\vec{v}_1^e \neq 0$, we can write the *plasma frequency*, ω_p,

$$\omega_p \equiv \omega = \left(\frac{n_0 e^2}{\epsilon_0 m_e}\right)^{\frac{1}{2}}. \tag{3.1.18}$$

3.2 LINEAR ION WAVES

Before describing the propagation of ion waves in plasma, it is interesting to give an overview of the theory of sound waves. Neglecting viscosity, ordinary fluids obey

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right] = -\gamma\frac{p}{\rho}\vec{\nabla}\rho, \tag{3.2.1}$$

which is the well-known *Navies-Stokes* equation and is the same as the plasma equation, Eq. (2.3.3), in the absence of electromagnetic forces and collisions. Writing the continuity equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0, \tag{3.2.2}$$

taking harmonic dependence in the perturbed quantities, and linearizing about the equilibrium, we get

$$-i\omega\rho_0\vec{v}_1 = -i\gamma\frac{p_0}{\rho_0}\vec{k}\rho_1 \tag{3.2.3}$$

$$-i\omega\rho_1 + i\rho_0\left(\vec{k}\cdot\vec{v}_1\right) = 0, \tag{3.2.4}$$

where p_0 and ρ_0 are the uniform fluid pressure and mass density, properly. From Eqs. (3.2.3) and (3.2.4), we obtain

$$\omega^2 = \gamma\frac{p_0}{\rho_0}k^2 \tag{3.2.5}$$

which enables us to express the velocity of a sound wave in neutral gas as

$$c_g \equiv \frac{\omega}{k} = \left(\gamma \frac{k_B T}{m}.\right)^{\frac{1}{2}} \tag{3.2.6}$$

In an unmagnetized collisionless plasma, since pressure waves cannot propagate by particles collision, a different mechanism is verified. First, let us write the warm ion fluid equation as

$$m_i n_i \left[\frac{\partial \vec{v}_i}{\partial t} + \left(\vec{v}_i \cdot \vec{\nabla}\right) \vec{v}_i\right] = -e n_i \vec{\nabla}\phi - \gamma_i k_B T_i \vec{\nabla} n_i \tag{3.2.7}$$

where the subscript i refers to *ion* quantities, and we have used the EOS expressed as

$$\vec{\nabla} p_i = \gamma_i k_B T_i \vec{\nabla} n_i, \tag{3.2.8}$$

following Eq. (2.5.8), where, for instance, isothermal compression implies $\gamma_i = 1$. If we linearize Eq. (3.2.7) by assuming oscillatory perturbation parameters, we get

$$-i\omega \vec{v}_1^i = -i \frac{e}{m_i} \phi_1 \vec{k} - i\gamma_i \frac{k_B T_i}{m_i} \frac{n_1^i}{n_0} \vec{k}, \tag{3.2.9}$$

and from the continuity equation, Eq. (2.5.2),

$$i\omega n_1^i = i n_0 \left(\vec{k} \cdot \vec{v}_1^i\right), \tag{3.2.10}$$

where Eqs. (3.2.9) and (3.2.10) form a set of linearized governing equations. As the system is quasineutral, let us adopt, momentarily, the $n_i = n_e$ approximation. So, expanding Eq. (2.1.7) we have

$$n \equiv n_e = n_0 \left(1 + \frac{e\phi}{k_B T_e} + \ldots\right), \tag{3.2.11}$$

once we are considering a *MB* plasma, where n_0 also stands for the equilibrium density and we choose $\phi_0 = 0$. Then,

$$n_1 = n_0 \frac{e\phi_1}{k_B T_e} \tag{3.2.12}$$

which makes us able to write,

$$\omega^2 = \frac{k_B}{m_i} \left(T_e + \gamma_i T_i\right) k^2. \tag{3.2.13}$$

Therefore, the sound speed in a warm classical plasma is expressed as

$$c_w \equiv \frac{\omega}{k} = \left[\frac{k_B\left(T_e + \gamma_i T_i\right)}{m_i}\right]^{\frac{1}{2}}, \tag{3.2.14}$$

characterizing the acoustic waves that propagate by vibration transmission due to the particle's charge.

In a more accurate approach, linearizing the *Poisson* equation, Eq. (2.1.5), we obtain

$$k^2\phi_1 = \frac{e}{\epsilon_0}\left(n_1^i - n_1\right), \tag{3.2.15}$$

and it is possible to write from Eqs. (3.2.9) and (3.2.10)

$$\vec{v}_1^i = \frac{\vec{k}}{\omega m_i}\left(e\phi_1 + \gamma_i \frac{k_B T_i}{n_0} n_1^i\right), \tag{3.2.16}$$

$$n_1^i = \frac{n_0}{\omega}\left(\vec{k}\cdot\vec{v}_1^i\right), \tag{3.2.17}$$

which together express the *first order* ion perturbed density term as

$$n_1^i = \frac{e\phi_1}{\left(\frac{m_i\omega^2}{n_0 k^2} - \gamma_i \frac{k_B T_i}{n_0}\right)}. \tag{3.2.18}$$

Inserting Eqs. (3.2.12) and (3.2.18) into Eq. (3.2.15),

$$\omega^2 = \frac{k^2}{m_i}\left[\frac{n_0}{\left(\frac{\epsilon_0 k^2}{e^2} + \frac{n_0}{k_B T_e}\right)} + \gamma_i k_B T_i\right], \tag{3.2.19}$$

and with the support of the *Debye length*, Eq. (2.1.13), we find

$$\frac{\omega^2}{k^2} = \frac{k_B T_e}{m_i}\frac{1}{\left(k^2\lambda_D^2 + 1\right)} + \gamma_i \frac{k_B T_i}{m_i}. \tag{3.2.20}$$

So, the result obtained from the linearized *Poisson* equation differs from the approximation density approach by a factor of $k^2\lambda_D^2$, which justifies the validity of considering $n_i \approx n_e$ for low-frequency oscillations.

Up to now, we have assumed $\vec{B} = 0$. Considering then a magnetized collisionless plasma, we have the momentum equation for ions expressed as

$$m_i n_i\left[\frac{\partial \vec{v}_i}{\partial t} + \left(\vec{v}_i\cdot\vec{\nabla}\right)\vec{v}_i\right] = e n_i \vec{E} - \gamma_i k_B T_i \vec{\nabla} n_i + e n_i\left(\vec{v}_i \times \vec{B}\right),$$

(3.2.21)

and for simplicity we can choose $\vec{B} = B_0\hat{z}$, so

$$m_i n_i \left[\frac{\partial \vec{v}_i}{\partial t} + \left(\vec{v}_i \cdot \vec{\nabla}\right)\vec{v}_i\right] = e n_i \vec{E} - \gamma_i k_B T_i \vec{\nabla} n_i + e n_i B_0 \left(\vec{v}_i \times \hat{z}\right). \tag{3.2.22}$$

As ion acoustic waves are electrostatic ones we have that only *longitudinal* waves are possible, with $\vec{k} \parallel \vec{E}_1$, where $\vec{E}_1$ is the fluctuating electric field. Linearizing Eq. (3.2.22)

$$-i\omega\vec{v}_1^i = \frac{e}{m_i}\vec{E}_1 - i\gamma_i \frac{k_B T_i}{m_i}\frac{n_1^i}{n_0}\vec{k} + \frac{e}{m_i}B_0\left(\vec{v}_1^i \times \hat{z}\right), \tag{3.2.23}$$

since the equilibrium plasma has constant and uniform n_0 and $\vec{B}_0$, and zero $\vec{E}_0$ and $\vec{v}_0$. From Eq. (3.2.23) we can write for each spatial component

$$-i\omega v_{1x}^i = \frac{e}{m_i}E_{1x} - i\gamma_i \frac{k_B T_i}{m_i}\frac{n_1^i}{n_0}k_x + \frac{e}{m_i}B_0 v_{1y}^i, \tag{3.2.24}$$

$$-i\omega v_{1y}^i = \frac{e}{m_i}E_{1y} - i\gamma_i \frac{k_B T_i}{m_i}\frac{n_1^i}{n_0}k_y - \frac{e}{m_i}B_0 v_{1x}^i, \tag{3.2.25}$$

$$-i\omega v_{1z}^i = \frac{e}{m_i}E_{1z} - i\gamma_i \frac{k_B T_i}{m_i}\frac{n_1^i}{n_0}k_z, \tag{3.2.26}$$

and identifying the *cyclotron frequency*

$$\Omega \equiv \frac{eB_0}{m_i}, \tag{3.2.27}$$

we can write for Eqs. (3.2.24) and (3.2.25)

$$-i\omega v_{1x}^i = \frac{e}{m_i}E_{1x} - i\gamma_i \frac{k_B T_i}{m_i}\frac{n_1^i}{n_0}k_x + \Omega v_{1y}^i, \tag{3.2.28}$$

$$-i\omega v_{1y}^i = \frac{e}{m_i}E_{1y} - i\gamma_i \frac{k_B T_i}{m_i}\frac{n_1^i}{n_0}k_y - \Omega v_{1x}^i. \tag{3.2.29}$$

If we look at the linearized continuity equation, Eq. (3.2.10), and considering the low-frequency approximation, we can conclude that a *pure* sound wave needs to cancel the cyclotron frequency dependence. So, $\vec{k}$ is such that

$$\vec{k} \parallel \vec{B} \quad \text{and} \quad \vec{k} \parallel \vec{E}_1, \tag{3.2.30}$$

for an arbitrary perturbed velocity $\vec{v}_1^i$, and we have from the continuity condition for the proposed configuration

$$i\omega n_1^i = i n_0 k_z v_{1z}^i. \quad (3.2.31)$$

Therefore, assuming $n_i \approx n_e$, writing the disturbed electron density as expressed in Eq. (3.2.12), after combining Eqs. (3.2.26) and (3.2.31) to get

$$\frac{\omega^2}{k_z^2} = \frac{n_0}{n_1^i}\left(\frac{e}{m_i}\phi_1 + \gamma_i \frac{k_B T_i}{m_i}\frac{n_1^i}{n_0}\right), \quad (3.2.32)$$

where we used $\vec{E} = -\vec{\nabla}\phi$, and performing some manipulations, we obtain

$$\frac{\omega^2}{k^2} = \frac{k_B T_e + \gamma_i k_B T_i}{m_i}, \quad (3.2.33)$$

as $k^2 = k_z^2$. Thus, we have for the magnetized plasma the same sound speed that we deduce previously for the unmagnetized system, c_w. To complete the analysis, since Eq. (3.2.32) returns n_1^i as

$$n_1^i = \frac{e\phi_1}{\left(\frac{m_i\omega^2}{n_0 k_z^2} - \gamma_i \frac{k_B T_i}{n_0}\right)}, \quad (3.2.34)$$

we obtain from *Poisson's* equation an equal expression to quantify the error associated with the density equivalence assumption, Eq. (3.2.20).

3.3 NONLINEAR ION WAVES

Once we have introduced the linear limit of ion waves, it is interesting to mention that when the wave amplitude is not small enough, several effects may result due to the finite-amplitude aspect. Let us consider only the dispersive character and neglect the possible coupling to other waves.

Assuming a cold plasma, instead of starting the analysis from energy concepts, we can follow Sec. 2.1. The ions governing equations are,

$$\frac{\partial \vec{v}_i}{\partial t} + \left(\vec{v}_i \cdot \vec{\nabla}\right)\vec{v}_i = -\frac{e}{m_i}\vec{\nabla}\phi, \quad (3.3.1)$$

$$\frac{\partial n_i}{\partial t} + \vec{\nabla}\cdot(n_i\vec{v}_i) = 0, \quad (3.3.2)$$

$$\nabla^2\phi = -\frac{e}{\epsilon_0}(n_i - n_e), \quad (3.3.3)$$

and we have a closed system description considering an *MB* distribution for electrons. For simplicity, for a one dimensional field-free plasma, Eqs. (3.3.1), (3.3.2), and (3.3.3) return

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial z} = -\frac{e}{m_i}\frac{\partial \phi}{\partial z}, \tag{3.3.4}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial z}(n_i v_i) = 0, \tag{3.3.5}$$

$$\frac{\partial^2 \phi}{\partial z^2} = -\frac{e}{\epsilon_0}(n_i - n_e). \tag{3.3.6}$$

Conveniently, we can introduce the transformation of the variables

$$z \to \frac{c_c}{\omega_{p_i}} z,\ t \to \frac{t}{\omega_{p_i}},\ \phi \to \frac{k_B T_e}{e}\phi,\ n_i \to n_0 n,\ n_e \to n_0 n_e,\ v_i \to c_c v, \tag{3.3.7}$$

where the parameter c_c is the sound speed in a cold plasma, which can be obtained from Eq. (3.2.14) in the limit of absence of thermal effects on ions,

$$c_c = \left(\frac{k_B T_e}{m_i}\right)^{\frac{1}{2}}, \tag{3.3.8}$$

and ω_{p_i} is the characteristic ion frequency, similar to Eq.(3.1.18),

$$\omega_{p_i} = \left(\frac{n_0 e^2}{\epsilon_0 m_i}\right)^{\frac{1}{2}}. \tag{3.3.9}$$

Note that the transformations given by Eq. (3.3.7), introducing a short notation for the derivatives, imply

$$\partial_z \to \frac{\omega_{p_i}}{c_c}\partial_z, \qquad \partial_t \to \omega_{p_i}\partial_t. \tag{3.3.10}$$

Then, if we substitute Eq. (3.3.7) into Eqs. (3.3.4), (3.3.5), and (3.3.6), we may write

$$\partial_t v + v\partial_z v = -\partial_z \phi, \tag{3.3.11}$$

$$\partial_t n + \partial_z (nv) = 0, \tag{3.3.12}$$

$$\partial_z^2 \phi = n_e - n, \tag{3.3.13}$$

and supposing now a wave-like solution

$$\eta = z - Mt, \tag{3.3.14}$$

where M is a constant, normalized by c_c, with

$$\partial_z = d_\eta \qquad \partial_t = -M d_\eta, \tag{3.3.15}$$

we get for the dynamic and Poisson equations,

$$-M d_\eta v + v d_\eta v = -d_\eta \phi, \tag{3.3.16}$$
$$-M d_\eta n + d_\eta (nv) = 0, \tag{3.3.17}$$
$$d_\eta^2 \phi = n_e - n. \tag{3.3.18}$$

Taking as system boundary conditions

$$\left(\phi, \phi', v, v', n'\right) \to 0, \;\; n \to 1 \quad \text{as} \quad |z - Mt| \to \infty, \tag{3.3.19}$$

where the primes denote the derivative with respect to η, and integrating the continuity equation, Eq. (3.3.17), we obtain the relation

$$n (M - v) = M. \tag{3.3.20}$$

After that, integrating the momentum equation, Eq. (3.3.16), and applying the asymptotic conditions

$$-Mv + \frac{v^2}{2} = -\phi, \tag{3.3.21}$$

and then we have from Eqs. (3.3.20) and (3.3.21)

$$n = \frac{M}{(M - v)}, \tag{3.3.22}$$
$$(M - v)^2 = M^2 - 2\phi. \tag{3.3.23}$$

Remembering that we are analyzing the classical cold plasma, with the electron distribution presented in Eq. (2.1.7), which after Eq. (3.3.7) transformations becomes

$$n_e = \exp(\phi), \tag{3.3.24}$$

we are able to investigate the system from the normalized Poisson's equation, Eq. (3.3.18). Thus, we can write

$$d_\eta^2 \phi = \exp(\phi) - \frac{M}{(M - v)}, \tag{3.3.25}$$

and introducing the potential dependence for the ion density term by using Eq. (3.3.23), we obtain

$$d_\eta^2\phi = \exp(\phi) - \frac{M}{(M^2 - 2\phi)^{\frac{1}{2}}}. \tag{3.3.26}$$

Multiplying Eq. (3.3.26) by $d_\eta\phi$, an integration can be done, returning

$$\frac{1}{2}(d_\eta\phi)^2 = \exp(\phi) + M\left(M^2 - 2\phi\right)^{\frac{1}{2}} - 1 - M^2, \tag{3.3.27}$$

and now we have that traveling wave solutions do not exist for all values of M and ϕ. It is straightforward from Eq. (3.3.27) that

$$\phi \leqslant \frac{M^2}{2}. \tag{3.3.28}$$

The solutions behavior of Eq. (3.3.27) was studied by *Sagdeev*, using an analogy with a unitary mass oscillator in a potential well [37]. Being ϕ the pseudo-coordinate and η the pseudo-time, we can define the *pseudopotential*

[37]:*Reviews of plasma physics.*

$$V(\phi) = M^2 + 1 - \exp(\phi) - M\left(M^2 - 2\phi\right)^{\frac{1}{2}}. \tag{3.3.29}$$

For a bounded localized solution, we need $V(\phi)$ with two zero crossings, and this implies M lying in a certain range. Looking for the $\phi << 1$ limit, we need $V(\phi << 1) < 0$ to be a potential well rather than a hill. So,

$$V(\phi << 1) \approx M^2 + 1 - \left[1 + \phi + \frac{\phi^2}{2}\right] - M\left[M - \frac{\phi}{M} - \frac{\phi^2}{2M^3}\right], \tag{3.3.30}$$

and then we have as a condition

$$\frac{\phi^2}{2}\left(\frac{1}{M^2} - 1\right) < 0, \tag{3.3.31}$$

which returns as the lower limit for M,

$$M^2 > 1 \quad \Rightarrow \quad M > 1. \tag{3.3.32}$$

The upper limit of M is imposed by the condition that $V(\phi)$ must cross the ϕ axis for $\phi > 0$. Since we have from Eq. (3.3.28) that the maximum value of ϕ, $\phi_{máx}$, is

$$\phi_{máx} = \frac{M^2}{2}, \tag{3.3.33}$$

we insert Eq. (3.3.33) into Eq. (3.3.29), getting

$$V\left(\phi = \frac{M^2}{2}\right) = M^2 + 1 - \exp\left(\frac{M^2}{2}\right). \tag{3.3.34}$$

We have then that M must satisfy

$$M^2 + 1 - \exp\left(\frac{M^2}{2}\right) > 0, \tag{3.3.35}$$

and the solution of Eq. (3.3.35) gives us that

$$M \lesssim 1.585 \quad \Rightarrow \quad M < 1.6. \tag{3.3.36}$$

Localized bounded traveling waves in a cold-ion plasma, therefore, exist only for

$$1 < M < 1.6. \tag{3.3.37}$$

The parameter M is usually identified as the *Mach number*.

To show that ion acoustic waves are KdV solitons, i. e., the KdV equation describes "large-amplitude" ion waves, knowing that this is true only for a range of *Mach numbers*, we may expand the amplitude, which is related to speed, in terms of the M excess, defined as

$$\epsilon \equiv |M - 1| \quad \Rightarrow \quad 0 < \epsilon < 0.6, \tag{3.3.38}$$

being ϵ a legitimate expansion parameter. We then write,

$$n = 1 + \epsilon n_1 + \epsilon^2 n_2 + \cdots, \tag{3.3.39}$$
$$v = \epsilon v_1 + \epsilon^2 v_2 + \cdots, \tag{3.3.40}$$
$$\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots. \tag{3.3.41}$$

It is convenient to rescale the variables according to the *reductive perturbation method* [43],

[43]: Reductive Perturbation Method in Nonlinear Wave Propagation. I

$$\xi = \epsilon^{\frac{1}{2}}(z - t), \quad \tau = \epsilon^{\frac{3}{2}} t, \tag{3.3.42}$$

and these changes of variables lead to

$$\partial_t = \epsilon^{\frac{3}{2}} \partial_\tau - \epsilon^{\frac{1}{2}} \partial_\xi \tag{3.3.43}$$
$$\partial_z = \epsilon^{\frac{1}{2}} \partial_\xi \tag{3.3.44}$$
$$\partial_z^2 = \epsilon \partial_\xi^2. \tag{3.3.45}$$

Once we know the space-time rescalings and the governing expressions of the ion fluid, we are able to derive the KdV equation. Substituting Eqs. (3.3.39) - (3.3.41) and (3.3.43) - (3.3.45) into Eqs. (3.3.11) - (3.3.13), we find for the lowest orders in ϵ

$$\phi_1 = n_1, \quad \partial_\xi v_1 = \partial_\xi \phi_1 = \partial_\xi n_1, \tag{3.3.46}$$

and after integration, we can conclude that all linear perturbations are equal

$$\phi_1 = n_1 = v_1 \equiv \varphi. \tag{3.3.47}$$

For the higher-order terms, we obtain

$$\partial_\tau v_1 + v_1 \partial_\xi v_1 + \partial_\xi \phi_2 - \partial_\xi v_2 = 0, \tag{3.3.48}$$
$$\partial_\tau n_1 + \partial_\xi v_2 + \partial_\xi (n_1 v_1) - \partial_\xi n_2 = 0, \tag{3.3.49}$$
$$\partial_\xi^2 \phi_1 + n_2 - \phi_2 - \frac{1}{2}\phi_1^2 = 0, \tag{3.3.50}$$

so, we can eliminate the second-order parameters by differentiating Eq. (3.3.50) with respect to ξ and adding the result with Eqs. (3.3.48) and (3.3.49). Thus, we get

$$\partial_\tau \varphi + \varphi \partial_\xi \varphi + \frac{1}{2}\partial_\xi^3 \varphi = 0, \tag{3.3.51}$$

where we have used the equality relation obtained for the first-order quantities, Eq. (3.3.47). Then, Eq. (3.3.51) is the KdV equation with unitary nonlinear coefficient and one-half dispersive factor, and we have that the *Korteweg-de Vries* equation describes ion acoustic waves of amplitude one order higher than in the linear regime.

Part III

THOMAS-FERMI APPROXIMATION AND KDV EQUATION

4

THOMAS-FERMI MODEL

Many systems can be seen as dense plasmas, whose characterization in a first approximation can be made from cold ions and degenerate electrons. Thus, we consider the *Thomas-Fermi* distribution instead of the MB one in the description of electrons. Our interest here is therefore to investigate whether ion acoustic waves are, in the nonlinear limit, KdV solitons for this plasma model, i. e., we wish to verify if the evolution of nonlinear ion oscillations in a semiclassical configuration is governed by the KdV equation. It is also interesting to study the response of the system to additional free parameters, like the ion temperature and an external magnetic field.

The chapter has two sections, the second being segmented. We start with a quick review of the concepts that permeate the *Thomas-Fermi* approach. From the second part, we move on to the plasma description, where we revisit topics already covered for a classical fluid. We analyze the system in its linear and nonlinear regimes for both magnetization status, taking into account the thermal state of the ion gas.

Most of the second part results are original, highlighting the determination of the natural scales from the calculus of variations, instead of the pseudopotential method, as well as the non-magnetized cold plasma nonlinear analysis, whose results are presented in [40]. We also compute the magnetized case for a cold ion plasma. Another original proposal is to modify the reductive perturbation method to demonstrate the normalization control character of the introduced parameter. By defining a structural model to determine the natural scales for the warm configurations, we can also describe their solitonic dynamics, with the proper normalization. These are results to be further explored in future works, and eventually published.

[40]:Description limit for soliton waves due to critical scaling of electrostatic potential

We are dealing with a semiclassical approach since certain ideas are borrowed from quantum mechanics, but on the other hand we operate with normal functions instead of quantum mechanical operators. The first part content is a short review of topics from [23] and [27].

[23]:Introduction to Solid State Physics

[27]:Course of Theoretical Physics - Statistical Physics

4.1 THOMAS-FERMI APPROXIMATION

At high temperatures, the *Thomas-Fermi* (TF) *approximation* is the easiest and best dense matter model to implement [17]. It is based on *Fermi-Dirac statistics* and the semiclassical approximation for elec-

[17]: Equations of State of Elements Based on the Generalized Fermi-Thomas Theory

trons, considering a gas of electrons continuously distributed in phase space according to *Fermi-Dirac* concepts [24].

[24]:Statistical Laws

Let us start the description by the relation between the number of electrons, N, in a uniform electron gas and the integral over the phase space density,

$$\frac{2}{(2\pi)^3\hbar^3}\int d^3p d^3r = N, \tag{4.1.1}$$

where $\hbar$ is the *reduced Planck constant*, and this allows us to write

$$n(\vec{r}) = \frac{2}{(2\pi)^3\hbar^3}4\pi\int^{p_F(\vec{r})} p^2 dp \tag{4.1.2}$$

being $n(\vec{r})$ the density of particles at position $\vec{r}$, and using that

$$N = \int n(\vec{r})\, d^3r. \tag{4.1.3}$$

We defined the local *Fermi momentum*, $p_F(\vec{r})$, so that the integral over the momentum up to the *Fermi momentum* gives the local density. As the volume of occupied momentum space is

$$\int d^3p = \frac{4\pi}{3}p_F^3(\vec{r}), \tag{4.1.4}$$

we can write for the local density of particles, Eq. (4.1.2),

$$n(\vec{r}) = \frac{1}{3\pi^2\hbar^3}p_F^3(\vec{r}). \tag{4.1.5}$$

Since in the ground state all the momentum space states will be filled out to the maximum momentum p_F, the kinetic energy of the *Fermi* gas is given by integrating the kinetic energy of the particles over all momentum states up to the p_F,

$$K(\vec{r}) = \frac{2}{(2\pi)^3\hbar^3}\int\frac{p^2}{2m_e}d^3p \tag{4.1.6}$$

and then it is straightforward that

$$K(\vec{r}) = \frac{2}{(2\pi)^3\hbar^3}\frac{4\pi}{2m_e}\frac{p_F^5(\vec{r})}{5}. \tag{4.1.7}$$

Thus, we can write, substituting Eq. (4.1.5) into Eq. (4.1.7),

$$K(\vec{r}) = \frac{2}{(2\pi)^3 \hbar^3} \frac{4\pi}{10m_e} \left(3\pi^2\hbar^3\right)^{\frac{5}{3}} n^{\frac{5}{3}}(\vec{r}) \tag{4.1.8}$$

which results for the local kinetic energy of the ideal Fermi gas,

$$K(\vec{r}) = \frac{3\hbar^2}{10m_e} \left(3\pi^2\right)^{\frac{2}{3}} n^{\frac{5}{3}}(\vec{r}). \tag{4.1.9}$$

According to *Thomas* and *Fermi*, the ground-state energy of electrons in a region where the electrostatic potential is ϕ, when the potential varies slowly and the particles move slowly, is classically given by

$$\mu \equiv \varepsilon_F = \frac{p_F^2}{2m_e} - e\phi, \tag{4.1.10}$$

where μ is the *chemical potential*, which is constant in thermal equilibrium and is the so-called *Fermi Energy*, ε_F, at $T = 0$. Therefore, $p_F(\vec{r})$ has to vary in space along with the spatial variation of $\phi(\vec{r})$. It then follows that

$$p_F = (2m_e\varepsilon_F)^{\frac{1}{2}} \left[1 + \frac{e\phi}{\varepsilon_F}\right]^{\frac{1}{2}}, \tag{4.1.11}$$

and the coupled system of equations, Eqs. (4.1.5) and (4.1.11), corresponds to the TF approximation. Finally, we can write the Fermi gas electron density as

$$n = n_0 \left(1 + \frac{e\phi}{\varepsilon_F}\right)^{\frac{3}{2}}, \tag{4.1.12}$$

where n_0 is the constant Fermi gas density in the absence of the potential,

$$n_0 = \frac{(2m_e\varepsilon_F)^{\frac{3}{2}}}{3\pi^2\hbar^3}. \tag{4.1.13}$$

4.2 THOMAS-FERMI PLASMA

First, to define semiclassical plasma model parameters, we will consider a two-component ideal cold fluid, composed of electrons and ions, unmagnetized, collisionless, and infinitely extended, in order to disregard the edge effects that may arise. From the electron-ion mass relation, $m_i \gg m_e$, we will again consider initially uniformly distributed fixed ions.

Once we have characterized the fluid, we can see from Sec. 3.1 that the oscillations of electrons in the plasma can be obtained from the linearization of the cold plasma governing equations presented in Sec. 2.1 without previous assumptions of the form of the electron distribution. So, it is useful to reference again the *plasma frequency*,

$$\omega_p = \left(\frac{n_0 e^2}{\epsilon_0 m_e} \right)^{\frac{1}{2}} . \tag{4.2.1}$$

4.2.1 *Linear Ion Waves*

As we saw in Sec. 4.1, the electron distribution in a TF plasma is given by Eq. (4.1.12), and from the Poisson equation, Eq. (2.1.5), we have

$$\nabla^2 \phi = -\frac{e}{\epsilon_0} n_0 \left[1 - \left(1 + \frac{e\phi}{\varepsilon_F} \right)^{\frac{3}{2}} \right] , \tag{4.2.2}$$

where at the equilibrium limit, with $\phi \to 0$, we verify the quasi-neutrality condition $n_e = n_i = n_0$. Note that the *Fermi temperature*, T_F, definition according to the *Fermi energy*,

$$\varepsilon_F = k_B T_F , \tag{4.2.3}$$

allows us to write the electron density in terms of T_F,

$$n_e = n_0 \left(1 + \frac{e\phi}{k_B T_F} \right)^{\frac{3}{2}} , \tag{4.2.4}$$

and the Poisson equation returns

$$\nabla^2 \phi = -\frac{e}{\epsilon_0} n_0 \left[1 - \left(1 + \frac{e\phi}{k_B T_F} \right)^{\frac{3}{2}} \right] . \tag{4.2.5}$$

So, for $e\phi \ll k_B T_F$ we can expand Eq. (4.2.4), such

$$\nabla^2 \phi = \frac{e}{\epsilon_0} n_0 \left[\frac{3}{2} \frac{e\phi}{k_B T_F} + \frac{3}{8} \left(\frac{e\phi}{k_B T_F} \right)^2 + \cdots \right] \tag{4.2.6}$$

and keeping only the first-order term for a linear analysis we get

$$\nabla^2 \phi = \frac{3}{2} \frac{n_0 e^2}{\epsilon_0 k_B T_F} \phi . \tag{4.2.7}$$

Similarly to the procedure adopted earlier, in spherical coordinates, requiring $\phi \to 0$ as $r \to \infty$, the solution of Eq. (4.2.7) may be expressed as

$$\phi(r) = \frac{\bar{\phi}_0}{r} \exp\left(-\frac{r}{\lambda_F}\right) \tag{4.2.8}$$

where we identify the λ_F quantity,

$$\lambda_F = \sqrt{\frac{2}{3}\frac{\epsilon_0 k_B T_F}{n_0 e^2}}, \tag{4.2.9}$$

defining the *linear* electron screening length in a TF gas of ions and electrons. Conveniently, we will refer to λ_F as the short notation of the *linear TF length*.

4.2.1.1 *Unmagnetized Plasma Ion Waves*

Having initially described some parameters of the suggested plasma model, it becomes valuable to re-investigate the linear limit of the propagation of ion waves in a TF plasma. Thus, following Sec. 3.2, we can start by considering in a more general way the momentum equation of the warm ion plasma configuration,

$$m_i n_i \left[\frac{\partial \vec{v}_i}{\partial t} + \left(\vec{v}_i \cdot \vec{\nabla}\right)\vec{v}_i\right] = -e n_i \vec{\nabla}\phi - \gamma_i k_B T_i \vec{\nabla} n_i;\ \vec{\nabla} p_i = \gamma_i k_B T_i \vec{\nabla} n_i, \tag{4.2.10}$$

and as expressed by Eq. (3.2.9), the linearization of Eq. (4.2.10) gives

$$-i\omega \vec{v}_1^i = -i\frac{e}{m_i}\phi_1 \vec{k} - i\gamma_i \frac{k_B T_i}{m_i}\frac{n_1^i}{n_0}\vec{k}. \tag{4.2.11}$$

Linearizing the continuity equation, Eq. (2.5.2), we obtain

$$i\omega n_1^i = i n_0 \left(\vec{k} \cdot \vec{v}_1^i\right). \tag{4.2.12}$$

Now, different from the previous analysis, we must consider the TF electron distribution instead of the MB distribution. The Taylor expansion of Eq. (4.1.12) about $\phi = 0$ is

$$n_e = n_0 \left[1 + \frac{3}{2}\frac{e\phi}{k_B T_F} + \frac{3}{8}\left(\frac{e\phi}{k_B T_F}\right)^2 + \cdots\right], \tag{4.2.13}$$

and as the equilibrium potential is null, $\phi_0 = 0$, we have for the first-order perturbation term

$$n_1^e = \frac{3}{2}\frac{e\phi_1}{k_B T_F} n_0. \tag{4.2.14}$$

It is then possible to write, since the linearized governing equations combine to

$$\omega^2 = \left(\frac{n_0}{n_1^i}\frac{e\phi_1}{m_i} + \gamma_i \frac{k_B T_i}{m_i}\right) k^2, \tag{4.2.15}$$

using the quasi-neutrality condition, assuming $n_i = n_e$,

$$\omega^2 = \frac{k_B}{m_i}\left(\frac{2}{3}T_F + \gamma_i T_i\right) k^2. \tag{4.2.16}$$

So, from Eq. (4.2.16) we have that the acoustic waves propagating in a warm TF plasma have as characteristic sound speed, c_{TF},

$$c_{TF} \equiv \frac{\omega}{k} = \left[\frac{k_B}{m_i}\left(\frac{2}{3}T_F + \gamma_i T_i\right)\right]^{\frac{1}{2}}. \tag{4.2.17}$$

To verify the validity of the approximation in the model, let us consider again the linear Poisson equation,

$$k^2 \phi_1 = \frac{e}{\epsilon_0}\left(n_1^i - n_1^e\right) \tag{4.2.18}$$

and since we can express n_i from Eq. (4.2.15) as

$$n_1^i = \frac{e\phi_1}{\left(\frac{m_i \omega^2}{n_0 k^2} - \gamma_i \frac{k_B T_i}{n_0}\right)}, \tag{4.2.19}$$

we obtain by substituting Eqs. (4.2.14) and (4.2.19) into Eq. (4.2.18),

$$\omega^2 = \frac{k^2}{m_i}\left[\frac{n_0}{\left(\frac{\epsilon_0 k^2}{e^2} + \frac{3}{2}\frac{n_0}{k_B T_F}\right)} + \gamma_i k_B T_i\right]. \tag{4.2.20}$$

Using the expression obtained for the TF length, Eq. (4.2.9), we are able to write the complete dispersion relation from Eq. (4.2.20) as

$$\frac{\omega^2}{k^2} = \frac{2}{3}\frac{k_B T_F}{m_i}\frac{1}{(k^2\lambda_F^2 + 1)} + \gamma_i \frac{k_B T_i}{m_i} \tag{4.2.21}$$

and comparing with Eq. (4.2.16) we can see that the considered limits are distinguished from each other by a factor of $(k^2\lambda_F^2)$ as is expected since the same analysis taken from the classic plasma model showed that the approximation differs from the full linear approach on this same scale when we adjust the characteristic length of the fluid.

4.2.1.2 *Magnetized Plasma Ion Waves*

As we concluded in Sec. 3.2, when we assume $\vec{B} \neq 0$, a pure ion acoustic wave is verified only for

$$\vec{k} \parallel \vec{B} \quad \text{and} \quad \vec{k} \parallel \vec{E}_1. \tag{4.2.22}$$

Stating $\vec{B} = B_0\hat{z}$ we have then that $\vec{k} = k_z\hat{z}$ and $\vec{E}_1 = E_{1z}\hat{z}$, and from the momentum equation for the magnetized warm fluid,

$$\left[\frac{\partial \vec{v}_i}{\partial t} + \left(\vec{v}_i \cdot \vec{\nabla}\right)\vec{v}_i\right] = \frac{e}{m_i}\vec{E} - \gamma_i \frac{k_B T_i}{m_i}\frac{\vec{\nabla} n_i}{n_i} + \frac{eB_0}{m_i}\left(\vec{v}_i \times \hat{z}\right) \tag{4.2.23}$$

we can write after the linearization,

$$-i\omega v^i_{1x} = \Omega v^i_{1y}, \tag{4.2.24}$$

$$-i\omega v^i_{1y} = -\Omega v^i_{1x}, \tag{4.2.25}$$

$$-i\omega v^i_{1z} = \frac{e}{m_i}E_{1z} - i\gamma_i \frac{k_B T_i}{m_i}\frac{n^i_1}{n_0}k_z. \tag{4.2.26}$$

From Eq. (4.2.26), writing $\vec{E} = -\vec{\nabla}\phi$, we get

$$-i\omega v^i_{1z} = -i\frac{e\phi_1}{m_i}k_z - i\gamma_i \frac{k_B T_i}{m_i}\frac{n^i_1}{n_0}k_z, \tag{4.2.27}$$

and considering the linear continuity equation we can combine Eq. (3.2.31) and Eq. (4.2.27) to obtain

$$\omega^2 = \frac{k_z^2}{m_i}\left(\frac{e\phi_1}{n^i_1}n_0 + \gamma_i k_B T_i\right). \tag{4.2.28}$$

When we apply the linear perturbation term of the TF approximation expressed by Eq. (4.2.14), which is possible under the consideration of $n_i = n_e$, the dispersion relation, Eq. (4.2.28), assumes the form

$$\frac{\omega^2}{k^2} = \frac{2}{3}\frac{k_B T_F}{m_i} + \gamma_i \frac{k_B T_i}{m_i} \tag{4.2.29}$$

and this essentially characterizes sound waves propagating in a magnetized warm TF plasma with the same speed c_{TF} derived in the unmagnetized case, as also occurs in the classical fluid model.

Furthermore, as Eq. (4.2.28) returns to n^i_1 the same expression that we obtained in Eq. (4.2.19), we have that the consideration of Poisson's equation as the relation that closes the set of governing equations, instead of the particle densities equality, implies an additional factor $k^2\lambda_F^2$, as expected. We then have the unmagnetized and magnetized approaches to the TF plasma being valid for the same scenario when we are considering the quasi-neutrality condition.

4.2.2 *Nonlinear Ion Waves*

It is interesting now, following the main idea presented by Silveira et al. [40], to investigate the propagation of nonlinear ion sound waves in TF plasmas. Conveniently, let us start by considering an unmagnetized cold plasma, completely described by

[40]: Description limit for soliton waves due to critical scaling of electrostatic potential

$$\frac{\partial \vec{v}_i}{\partial t} + \left(\vec{v}_i \cdot \vec{\nabla}\right) \vec{v}_i = -\frac{e}{m_i} \vec{\nabla} \phi, \tag{4.2.30}$$

$$\frac{\partial n_i}{\partial t} + \vec{\nabla} \cdot (n_i \vec{v}_i) = 0, \tag{4.2.31}$$

$$\nabla^2 \phi = -\frac{e}{\epsilon_0} (n_i - n_e), \tag{4.2.32}$$

with the electron density given by the assumption of the *Thomas-Fermi distribution*,

$$n_e = n_0 \left(1 + \frac{e\phi}{k_B T_F}\right)^{\frac{3}{2}}, \tag{4.2.33}$$

which allows us to determine specific parameters for our system. First, we can set the dimensionless variables transformations,

$$x_j \to \lambda_e x_j, \ t \to \frac{t}{\omega_{p_i}}, \ \phi \to \varphi\phi, \ n_i \to n_0 n, \ n_e \to n_0 n_e, \ v_j \to \lambda_e \omega_{p_i} v, \tag{4.2.34}$$

where λ_e and φ are, respectively, the system length scale and the natural potential scale, to be clarified later, and ω_{p_i} is the ion plasma frequency obtained considering the ions oscillations,

$$\omega_{p_i} = \left(\frac{n_0 e^2}{\epsilon_0 m_i}\right)^{\frac{1}{2}}. \tag{4.2.35}$$

We are also considering the normalization of the particle's density distribution to the equilibrium density, n_0, and the velocity normalized according to the length and frequency parameters. Finally, note that the dimensionless definition, Eq. (4.2.34), transforms the derivatives quantities,

$$\partial_{x_j} \to \frac{1}{\lambda_e} \partial_{x_j}, \quad \partial_t \to \omega_{p_i} \partial_t. \tag{4.2.36}$$

4.2.2.1 *Natural Scales by Variational Principle*

Starting our scale analysis, we can study the case of cold TF plasma in a dimensionless form by introducing the transformed variables. So, acting Eq. (4.2.34) on the fluid equations, Eqs. (4.2.30) - (4.2.32), we get

$$\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\frac{e\varphi}{m_i \lambda_e^2 \omega_{p_i}^2} \vec{\nabla}\phi, \tag{4.2.37}$$

$$\partial_t n + \vec{\nabla} \cdot (n\vec{v}) = 0, \tag{4.2.38}$$

$$\vec{\nabla}^2 \phi = -\frac{e n_0 \lambda_e^2}{\epsilon_0 \varphi} (n - n_e). \tag{4.2.39}$$

Since we are investigating the propagation of traveling waves, we can look at wave-like solutions of the form

$$\eta = (l_x x + l_y y + l_z z - Mt), \tag{4.2.40}$$

where M is a constant, due to a velocity normalization by $\lambda_e \omega_{p_i}$,

$$M = \frac{v_0}{\lambda_e \omega_{p_i}}, \tag{4.2.41}$$

and we have introduced the direction cosines, l_x, l_y, and l_z, of the wave vector, $\vec{k}$, along x, y, and z axes, respectively, such that

$$l_x^2 + l_y^2 + l_z^2 = 1, \tag{4.2.42}$$

adopting as boundary conditions

$$\left(\phi, \phi', \vec{v}, \vec{v}', n'\right) \to 0; \quad n \to 1 \quad \text{as} \quad |l_x x + l_y y + l_z z - Mt| \to \infty, \tag{4.2.43}$$

where $(')$ refers to η derivatives.

So, we can write for λ_e, from Eq. (4.2.41),

$$\lambda_e = \frac{v_0}{M} \left(\frac{\epsilon_0 m_i}{n_0 e^2}\right)^{\frac{1}{2}}, \tag{4.2.44}$$

and a dimensional analysis of Eq. (4.2.44) provides us

$$\lambda_e = \left(\frac{\epsilon_0}{n_0 e}\right)^{\frac{1}{2}} \varphi^{\frac{1}{2}}, \tag{4.2.45}$$

which gives us, comparing Eqs. (4.2.44) and (4.2.45),

$$\varphi = \frac{m_i v_0^2}{eM^2}. \tag{4.2.46}$$

We then have, from Eqs. (4.2.37) - (4.2.39),

$$\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\vec{\nabla}\phi, \tag{4.2.47}$$

$$\partial_t n + \vec{\nabla} \cdot (n\vec{v}) = 0, \tag{4.2.48}$$

$$\vec{\nabla}^2 \phi = -(n - n_e). \tag{4.2.49}$$

From the definitions presented in Eq. (4.2.40), we can see that

$$\partial_{x_j} = l_j d_\eta \qquad \partial_t = -M d_\eta \tag{4.2.50}$$

and then we can write from Eqs. (4.2.47) - (4.2.49)

$$-M d_\eta \vec{v} + [\vec{v} \cdot (\hat{x} l_x + \hat{y} l_y + \hat{z} l_z)] \, d_\eta \vec{v} = -(\hat{x} l_x + \hat{y} l_y + \hat{z} l_z) \, d_\eta \phi, \tag{4.2.51}$$

$$-M d_\eta n + (\hat{x} l_x d_\eta + \hat{y} l_y d_\eta + \hat{z} l_z d_\eta) \cdot (n\vec{v}) = 0, \tag{4.2.52}$$

$$d_\eta^2 \phi = -(n - n_e). \tag{4.2.53}$$

Simplifying terms in Eqs. (4.2.51) and (4.2.52), we obtain more compactly

$$-M d_\eta v_j + (l_x v_x + l_y v_y + l_z v_z) \, d_\eta v_j = -l_j d_\eta \phi, \tag{4.2.54}$$

$$-M d_\eta n + l_x d_\eta (n v_x) + l_y d_\eta (n v_y) + l_z d_\eta (n v_z) = 0, \tag{4.2.55}$$

with $j = 1, 2, 3$ identifying the x, y, and z velocity components in the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions, respectively.

Integrating Eq. (4.2.55) and imposing the conditions expressed by Eq. (4.2.43), we have

$$l_x v_x + l_y v_y + l_x v_z = M\left(1 - \frac{1}{n}\right), \tag{4.2.56}$$

and by rearranging the terms it is possible to express the density n as

$$n = \frac{M}{M - (l_x v_x + l_y v_y + l_z v_z)}. \tag{4.2.57}$$

We can use Eq. (4.2.50) to obtain the following relation,

$$v_x \partial_x + v_y \partial_y + v_z \partial_z = (l_x v_x + l_y v_y + l_z v_z) \, d_\eta, \tag{4.2.58}$$

and then it is verified from Eqs. (4.2.56) and (4.2.58),

$$v_x \partial_x + v_y \partial_y + v_z \partial_z = M\left(1 - \frac{1}{n}\right) d_\eta. \tag{4.2.59}$$

Thus, it is possible to write for the momentum equation, Eq. (4.2.54),

$$-M d_\eta v_j + M\left(1 - \frac{1}{n}\right) d_\eta v_j = -l_j d_\eta \phi \tag{4.2.60}$$

which implies for the simplified momentum equation in terms of the parameter M,

$$\frac{M}{n} d_\eta v_j = l_j d_\eta \phi. \tag{4.2.61}$$

Differentiating Eq. (4.2.56) on η,

$$l_x d_\eta v_x + l_y d_\eta v_y + l_z d_\eta v_z = \frac{M}{n^2} d_\eta n, \tag{4.2.62}$$

while the product of the set of equations represented by Eq. (4.2.61) by their respective directional cosines, l_j, implies

$$\frac{M}{n} l_x d_\eta v_x = l_x^2 d_\eta \phi, \tag{4.2.63}$$

$$\frac{M}{n} l_y d_\eta v_y = l_y^2 d_\eta \phi, \tag{4.2.64}$$

$$\frac{M}{n} l_z d_\eta v_z = l_z^2 d_\eta \phi, \tag{4.2.65}$$

and so we have from the association of Eq. (4.2.62) with Eqs. (4.2.63) - (4.2.65),

$$\frac{n}{M}\left(l_x^2 + l_y^2 + l_z^2\right) d_\eta \phi = \frac{M}{n^2} d_\eta n, \tag{4.2.66}$$

or isolating the term that quantifies the electrostatic potential variation, using the directional cosine property,

$$d_\eta \phi = \frac{M^2}{n^3} d_\eta n. \tag{4.2.67}$$

Once we have obtained an expression to $d_\eta \phi$ depending only on the fluid ion density, we can do an implicit integration of Eq. (4.2.67), applying the boundary conditions, to get

$$\phi = \frac{M^2}{2n^2}\left(n^2 - 1\right), \tag{4.2.68}$$

so that we can use Eq. (4.2.68) to express the density n as a function of ϕ and M,

$$n = \frac{M}{(M^2 - 2\phi)^{\frac{1}{2}}}. \tag{4.2.69}$$

Remembering that we are studying the semiclassical plasma model, the electron distribution function is presented in Eq. (4.2.33), which after applying the transformations indicated in Eq. (4.2.34) takes the form

$$n_e = (1 + 2\phi)^{\frac{3}{2}}, \tag{4.2.70}$$

and now we have all the tools necessary to investigate the system from the normalized Poisson equation. The substitution of Eqs. (4.2.69) and (4.2.70) into Eq. (4.2.53) gives us

$$\phi'' = -\left[M\left(M^2 - 2\phi\right)^{-\frac{1}{2}} - (1 + 2\phi)^{\frac{3}{2}}\right], \tag{4.2.71}$$

where again we are using the $(')$ notation to refer to η derivative terms. From Eq. (4.2.71) it is straightforward that we have a maximum and a minimum value to ϕ, i. e., ϕ is completely limited,

$$\phi < \frac{M^2}{2} \quad \text{and} \quad \phi \geqslant -\frac{1}{2}. \tag{4.2.72}$$

Thus, the lower and upper bounds of ϕ can be presented as

$$-\frac{1}{2} \leqslant \phi < \frac{M^2}{2}. \tag{4.2.73}$$

To follow with our analysis, let us consider again the untransformed variables to write the dimensional form of Poisson's equation, Eq. (4.2.32). So, we can express the ion density, n, from Eqs. (4.2.34), (4.2.41), and (4.2.69) as

$$n_i = n_0\left(1 - \frac{2e}{m_i v_0^2}\phi\right)^{-\frac{1}{2}}, \tag{4.2.74}$$

and being the *TF distribution* given by Eq. (4.2.33), we have

$$\nabla^2\phi = -\frac{en_0}{\epsilon_0}\left[\left(1 - \frac{2e}{m_i v_0^2}\phi\right)^{-\frac{1}{2}} - \left(1 + \frac{e\phi}{k_B T_F}\right)^{\frac{3}{2}}\right]. \tag{4.2.75}$$

Taylor-expanding both terms related to the component densities around their equilibrium values,

$$\left(1-\frac{2e}{m_i v_0^2}\phi\right)^{-\frac{1}{2}} = 1+\frac{e}{m_i v_0^2}\phi+\frac{3}{2}\frac{e^2}{m_i^2 v_0^4}\phi^2+\cdots \tag{4.2.76}$$

$$\left(1+\frac{e}{k_B T_F}\phi\right)^{\frac{3}{2}} = 1+\frac{3}{2}\frac{e}{k_B T_F}\phi+\frac{3}{8}\frac{e^2}{k_B^2 T_F^2}\phi^2+\cdots \tag{4.2.77}$$

and we can write for our purposes Eq. (4.2.75) in the simplified form up to the first nonlinear term,

$$\nabla^2\phi = -\frac{en_0}{\epsilon_0}\left[1+\frac{e}{m_i v_0^2}\phi+\frac{3}{2}\frac{e^2}{m_i^2 v_0^4}\phi^2-\left(1+\frac{3}{2}\frac{e}{k_B T_F}\phi+\frac{3}{8}\frac{e^2}{k_B^2 T_F^2}\phi^2\right)\right], \tag{4.2.78}$$

which can be rearranged to

$$\nabla^2\phi = -\frac{en_0}{\epsilon_0}\left\{\left[\frac{e}{m_i v_0^2}-\frac{3}{2}\frac{e}{k_B T_F}\right]\phi+\left[\frac{3}{2}\frac{e^2}{m_i^2 v_0^4}-\frac{3}{8}\frac{e^2}{k_B^2 T_F^2}\right]\phi^2\right\}, \tag{4.2.79}$$

or, conveniently, using the potential scale expression, Eq. (4.2.46), we can write

$$\nabla^2\phi = -\frac{en_0}{\epsilon_0}\left\{\left[\frac{1}{\varphi M^2}-\frac{3}{2}\frac{e}{k_B T_F}\right]\phi+\left[\frac{3}{2}\frac{1}{\varphi^2 M^4}-\frac{3}{8}\frac{e^2}{k_B^2 T_F^2}\right]\phi^2\right\}. \tag{4.2.80}$$

To explicitly determine the natural parameters of the problem, we can note that all scales can be written from the potential one. So, to scale the potential by variational method, let us suppose that φ satisfies [40]

[40]: Description limit for soliton waves due to critical scaling of electrostatic potential

$$\frac{1}{\varphi M^2}-\frac{3}{2}\frac{e}{k_B T_F} = \varphi\left(\frac{3}{2}\frac{1}{\varphi^2 M^4}-\frac{3}{8}\frac{e^2}{k_B^2 T_F^2}\right). \tag{4.2.81}$$

First, Eq. (4.2.81) is dimensionally consistent with Eq. (4.2.80). In addition, the variational problem emerges from the condition

$$\Lambda\left(M^2\right) \equiv \left(\frac{3}{4}\frac{e^2}{k_B^2 T_F^2}\varphi-3\frac{e}{k_B T_F}\right)\varphi M^4+2M^2-3=0, \tag{4.2.82}$$

where a stationary analysis with respect to the Mach number variation returns after an M^2 differentiation of Eq. (4.2.82),

$$\Lambda'\left(M^2\right) = 2\left(\frac{3}{4}\frac{e^2}{k_B^2 T_F^2}\varphi - 3\frac{e}{k_B T_F}\right)\varphi M^2 + 2 = 0. \tag{4.2.83}$$

Therefore, from Eq. (4.2.83) we have

$$\frac{1}{\varphi M^2} = 3\frac{e}{k_B T_F} - \frac{3}{4}\frac{e^2}{k_B^2 T_F^2}\varphi, \tag{4.2.84}$$

which means that those M^2-values that satisfy Eq. (4.2.84) define a class of configurations that keep Eq. (4.2.82) stationary after small variations of the Mach number.

To complete, the association of Eqs. (4.2.81) and (4.2.84) falls back into a variational problem, following

$$\bar{\Lambda}\left(\varphi\right) \equiv \frac{9}{4}\frac{e^2}{k_B^2 T_F^2}\varphi^2 - 9\frac{e}{k_B T_F}\varphi + 1 = 0, \tag{4.2.85}$$

which determines the value of φ that keeps $\bar{\Lambda}$ stationary after a small variation of the potential scale. Differentiating then Eq. (4.2.85) with respect to φ, we get

$$\bar{\Lambda}'\left(\varphi\right) = \frac{9}{2}\frac{e^2}{k_B^2 T_F^2}\varphi - 9\frac{e}{k_B T_F} = 0, \tag{4.2.86}$$

and finally the solution of Eq. (4.2.86) returns as the natural potential scale by variational method

$$\varphi = 2\frac{k_B T_F}{e}. \tag{4.2.87}$$

The length scale, λ_e, velocity, v_0, and consequently the Mach number, M, can be fully represented using Eq. (4.2.87), so that Eqs. (4.2.45), (4.2.46), with (4.2.84), and (4.2.41) can be rewritten, in that order, as

$$\lambda_e = \left(2\frac{\epsilon_0 k_B T_F}{n_0 e^2}\right)^{\frac{1}{2}}; \quad v_0 = \left(\frac{2}{3}\frac{k_B T_F}{m_i}\right)^{\frac{1}{2}}; \quad M = \left(\frac{1}{3}\right)^{\frac{1}{2}}. \tag{4.2.88}$$

4.2.2.2 *Modified Reductive Perturbation Method*

From the beginning, we have been interested in investigating the propagation of traveling waves, more specifically the propagation of dispersive nonlinear small-amplitude ion acoustic waves, described by the KdV equation in a TF plasma. As mentioned in Sec. 3.3, we can use the *reductive perturbation* (RP) technique to re-scale the system variables when we are deriving the KdV equation. We will, however, introduce a new approach to RP that applies to more general configurations and will be important for the next steps.

To characterize our schematization, interested in finite amplitude SW, we write the expansion parameter related to the Mach number excess,

$$\epsilon \equiv |M-1| \ll 1. \tag{4.2.89}$$

Altogether, we can investigate the existence and shape of SW in a TF plasma in the nonlinear regime by writing the stretched variables

$$\xi = \epsilon^{\frac{1}{2}} \left(l_x x + l_y y + l_z z - \lambda_0 t\right) \qquad \tau = \epsilon^{\frac{3}{2}} t, \tag{4.2.90}$$

where the parameter λ_0 is the normalized phase velocity and, consequently, implies that the partial derivatives transform as

$$\partial_{x_j} = \epsilon^{\frac{1}{2}} l_j \partial_\xi \qquad \partial_t = \epsilon^{\frac{3}{2}} \partial_\tau - \epsilon^{\frac{1}{2}} \lambda_0 \partial_\xi \qquad \partial^2_{x_j} = \epsilon l_j^2 \partial^2_\xi. \tag{4.2.91}$$

Finally, as usual, to study the behavior of small-amplitude solutions, we introduce the perturbation expansion of n, v_j, and ϕ as

$$n = 1 + \epsilon n_1 + \epsilon^2 n_2 + \cdots \tag{4.2.92}$$

$$v_j = \epsilon v_{1j} + \epsilon^2 v_{2j} + \cdots \tag{4.2.93}$$

$$\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots, \tag{4.2.94}$$

and now we can investigate each of the fluid configurations in the nonlinear limit.

4.2.2.3 *Unmagnetized Cold Plasma*

To verify all the background involved in the analysis of the nonlinear limit of cold TF fluid in the absence of a magnetic field, we will reintroduce the linear regime.

Linear Regime

The ion momentum and mass conservations and the Poisson equation are, respectively, in the initial representation of the variables

$$\partial_t \vec{v}_i + \left(\vec{v}_i \cdot \vec{\nabla}\right) \vec{v}_i = -\frac{e}{m_i} \vec{\nabla}\phi, \tag{4.2.95}$$

$$\partial_t n_i + \vec{\nabla} \cdot \left(n_i \vec{v}_i\right) = 0, \tag{4.2.96}$$

$$\nabla^2 \phi = -\frac{e}{\epsilon_0} \left(n_i - n_e\right), \tag{4.2.97}$$

which linearization of Eqs. (4.2.95) and (4.2.96) returns

$$-i\omega\vec{v}_1 = -i\frac{e\phi_1}{m_i}\vec{k}, \tag{4.2.98}$$

$$-i\omega n_1 + in_0\left(\vec{k}\cdot\vec{v}_1\right) = 0. \tag{4.2.99}$$

So we can combine Eqs. (4.2.98) and (4.2.99) to write the perturbed ion density term as

$$n_1^i = \frac{en_0k^2}{m_i\omega^2}\phi_1 \tag{4.2.100}$$

and, once the linearization of Eq. (4.2.97) gives

$$-k^2\phi_1 = -\frac{e}{\epsilon_0}\left(n_1^i - n_1^e\right), \tag{4.2.101}$$

where n_1^i and n_1^e are the perturbed ion and electron densities, respectively, with

$$n_e^1 = \frac{3}{2}\frac{n_0e}{k_BT_F}\phi_1, \tag{4.2.102}$$

we can write from Eqs. (4.2.100) - (4.2.102),

$$k^2 = \frac{n_0e^2}{\epsilon_0}\left(\frac{k^2}{m_i\omega^2} - \frac{3}{2}\frac{1}{k_BT_F}\right). \tag{4.2.103}$$

We know that the ion plasma frequency can be expressed in terms of the unperturbed density n_0, presented in Eq. (4.2.35), and then identifying ω_{p_i} in Eq. (4.2.103), we obtain, after some manipulations, to the dispersion relation

$$\omega^2 = \frac{\omega_{p_i}^2k^2}{\left(k^2 + \frac{3}{2}\frac{n_0e^2}{\epsilon_0k_BT_F}\right)}. \tag{4.2.104}$$

From Eq. (4.2.104) we can conclude that, in an unmagnetized plasma with degenerate electrons, the ion sound speed is equal to

$$c_S = \left(\frac{2}{3}\frac{k_BT_F}{m_i}\right)^{\frac{1}{2}}, \tag{4.2.105}$$

which is essentially the second of Eq. (4.2.88) and can be expressed as the $T_i \to 0$ limit of Eq. (4.2.17), corroborating previous derivations.

In Sec. 4.2.2.1 we investigated the natural scales of the system, such as λ_e, ν_0, and M, when we studied the propagation of traveling waves. Here we will deal with the nonlinear regime using those results.

Nonlinear Regime

The action of the dimensionless variables transformations, Eq. (4.2.34), in the unmagnetized cold TF plasma governing equations, Eqs. (4.2.95) - (4.2.97), gives rise to the set of normalized hydrodynamic expressions, Eqs. (4.2.47) - (4.2.49). Moreover, using the stretched variables and their respective partial derivative transformations, Eqs. (4.2.90) and (4.2.91), we get for Eq. (4.2.48)

$$\left(-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi + \epsilon^{\frac{3}{2}} \partial_\tau\right) n + \epsilon^{\frac{1}{2}} \vec{l}_j \cdot \partial_\xi \left(n \vec{v}_j\right) = 0, \tag{4.2.106}$$

where we use $\vec{l}_j = \hat{x} l_x + \hat{y} l_y + \hat{z} l_z$, and then we have explicitly from Eq. (4.2.106)

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi n + \epsilon^{\frac{3}{2}} \partial_\tau n + \epsilon^{\frac{1}{2}} \left[l_x \partial_\xi \left(n v_x\right) + l_y \partial_\xi \left(n v_y\right) + l_z \partial_\xi \left(n v_z\right)\right] = 0. \tag{4.2.107}$$

If we do the same to Eqs. (4.2.47) and (4.2.49), it is obtained

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_j + \epsilon^{\frac{3}{2}} \partial_\tau v_j + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} v_k l_k\right) \partial_\xi v_j = -l_j \epsilon^{\frac{1}{2}} \partial_\xi \phi \tag{4.2.108}$$

$$\epsilon \partial_\xi^2 \phi = -\left(n - n_e\right) \tag{4.2.109}$$

where we used the unitary relation for the directional cosines. Eqs. (4.2.107) - (4.2.109) will be our starting point for the perturbative analysis.

Applying Eqs. (4.2.92) - (4.2.94) to the relations obtained above, for the lowest order of perturbation is valid

$$-\lambda_0 \partial_\xi n_1 + l_x \partial_\xi v_{1x} + l_y \partial_\xi v_{1y} + l_z \partial_\xi v_{1z} = 0, \tag{4.2.110}$$

$$-\lambda_0 \partial_\xi v_{1j} + l_j \partial_\xi \phi_1 = 0, \tag{4.2.111}$$

$$\left(n_1 - 3\phi_1\right) = 0, \tag{4.2.112}$$

where we can conclude from Eq. (4.2.112)

$$n_1 = 3\phi_1, \tag{4.2.113}$$

and the Eq. (4.2.111) results

$$\partial_\xi v_{1j} = \frac{l_j}{\lambda_0} \partial_\xi \phi_1, \tag{4.2.114}$$

whose integration returns,

$$v_{1j} = \frac{l_j}{\lambda_0}\phi_1, \tag{4.2.115}$$

or yet, using the relation obtained for n_1, Eq. (4.2.113), it becomes possible to write

$$n_1 = 3\frac{\lambda_0}{l_j}v_{1j}. \tag{4.2.116}$$

It is interesting to associate Eqs. (4.2.110) and (4.2.111) to get the value of λ_0. Thus,

$$-3\lambda_0\partial_\xi\phi_1 + \frac{1}{\lambda_0}\partial_\xi\phi_1 = 0 \quad \Rightarrow \quad \lambda_0^2 = \frac{1}{3} \tag{4.2.117}$$

where we have used the relation expressed in Eq. (4.2.113). For now, however, it is feasible to maintain the explicit dependence on the variable λ_0.

Analyzing a higher order of perturbation, we have as results

$$-\lambda_0\partial_\xi n_2 + \partial_\tau n_1 + \sum_{k=1}^{3}\left\{l_k\left[\partial_\xi\left(n_1 v_{1k}\right) + \partial_\xi v_{2k}\right]\right\} = 0, \tag{4.2.118}$$

$$-\lambda_0\partial_\xi v_{2j} + \partial_\tau v_{1j} + \left(\sum_{k=1}^{3} v_{1k}l_k\right)\partial_\xi v_{1j} + l_j\partial_\xi\phi_2 = 0, \tag{4.2.119}$$

$$\partial_\xi^2\phi_1 + \left(n_2 - 3\phi_2 - \frac{3}{2}\phi_1^2\right) = 0, \tag{4.2.120}$$

and once we know how to write n_1 and v_{1j} in terms of ϕ_1, Eqs. (4.2.118) - (4.2.120) returns

$$-\lambda_0\partial_\xi n_2 + 3\partial_\tau\phi_1 + \frac{6}{\lambda_0}\phi_1\partial_\xi\phi_1 + l_x\partial_\xi v_{2x} + l_y\partial_\xi v_{2y} + l_z\partial_\xi v_{2z} = 0, \tag{4.2.121}$$

$$-\lambda_0\partial_\xi v_{2j} + \frac{l_j}{\lambda_0}\partial_\tau\phi_1 + \frac{l_j}{\lambda_0^2}\phi_1\partial_\xi\phi_1 + l_j\partial_\xi\phi_2 = 0, \tag{4.2.122}$$

$$\partial_\xi^3\phi_1 + \partial_\xi n_2 - 3\partial_\xi\phi_2 - 3\phi_1\partial_\xi\phi_1 = 0. \tag{4.2.123}$$

Associating Eqs. (4.2.121), (4.2.122) and (4.2.123),

$$
\begin{aligned}
&-\lambda_0\partial_\xi n_2 + 3\partial_\tau\phi_1 + \frac{6}{\lambda_0}\phi_1\partial_\xi\phi_1 \\
&\qquad\qquad + \frac{1}{\lambda_0}\left(\frac{1}{\lambda_0}\partial_\tau\phi_1 + \frac{1}{\lambda_0^2}\phi_1\partial_\xi\phi_1 + \partial_\xi\phi_2\right) = 0 \\
&-\lambda_0\partial_\xi n_2 + \left(3 + \frac{1}{\lambda_0^2}\right)\partial_\tau\phi_1 + \frac{1}{\lambda_0}\left(6 + \frac{1}{\lambda_0^2}\right)\phi_1\partial_\xi\phi_1 + \frac{1}{\lambda_0}\partial_\xi\phi_2 = 0 \\
&\frac{1}{\lambda_0}\left(\partial_\xi\phi_2 - \lambda_0^2\partial_\xi n_2\right) + \left(3 + \frac{1}{\lambda_0^2}\right)\partial_\tau\phi_1 + \frac{1}{\lambda_0}\left(6 + \frac{1}{\lambda_0^2}\right)\phi_1\partial_\xi\phi_1 = 0 \\
&\frac{1}{3\lambda_0}\left(\partial_\xi^3\phi_1 - 3\phi_1\partial_\xi\phi_1\right) + 6\partial_\tau\phi_1 + \frac{9}{\lambda_0}\phi_1\partial_\xi\phi_1 = 0,
\end{aligned}
\tag{4.2.124}
$$

where we have used the $\lambda_0^2 = \frac{1}{3}$ relation. So,

$$
6\partial_\tau\phi_1 + \frac{8}{\lambda_0}\phi_1\partial_\xi\phi_1 + \frac{1}{3\lambda_0}\partial_\xi^3\phi_1 = 0 \tag{4.2.125}
$$

or yet,

$$
\partial_\tau\phi_1 + \frac{4}{3\lambda_0}\phi_1\partial_\xi\phi_1 + \frac{1}{18\lambda_0}\partial_\xi^3\phi_1 = 0. \tag{4.2.126}
$$

We can then conclude that Eq. (4.2.126) is a KdV-like equation

$$
\partial_\tau\phi_1 + A\phi_1\partial_\xi\phi_1 + B\partial_\xi^3\phi_1 = 0 \quad \text{with} \quad A = \frac{4\sqrt{3}}{3}, \;\; B = \frac{\sqrt{3}}{18}, \;\; \lambda_0 = \frac{1}{\sqrt{3}}. \tag{4.2.127}
$$

4.2.2.4 *Unmagnetized Isothermal Plasma*

To construct the nonlinear theory that describes steady-state ion sound waves in an isothermal TF plasma, in which electrons are assumed degenerate and ions are considered classical, we will take an overview of the linear approximation.

Linear Regime

Let us start from the hydrodynamic equations of an isothermal gas,

$$
\partial_t\vec{v}_i + \left(\vec{v}_i\cdot\vec{\nabla}\right)\vec{v}_i = -\frac{e}{m_i}\vec{\nabla}\phi - \frac{k_B T_i}{m_i n_i}\vec{\nabla}n_i, \tag{4.2.128}
$$

$$
\partial_t n_i + \vec{\nabla}\cdot\left(n_i\vec{v}_i\right) = 0, \tag{4.2.129}
$$

$$
\nabla^2\phi = -\frac{e}{\epsilon_0}\left(n_i - n_e\right), \tag{4.2.130}
$$

where we take the limit $\gamma_i = 1$ in Eq. (3.2.8), and linearizing

$$-i\omega\vec{v}_1 = -i\frac{e\phi_1}{m_i}\vec{k} - i\frac{k_B T_i}{m_i}\frac{n_1}{n_0}\vec{k}, \tag{4.2.131}$$

$$-i\omega n_1 + i n_0 \left(\vec{k}\cdot\vec{v}_1\right) = 0, \tag{4.2.132}$$

$$-k^2\phi_1 = -\frac{e}{\epsilon}\left(n_1^i - n_1^e\right). \tag{4.2.133}$$

From Eq. (4.2.131) we can write for the perturbed velocity term

$$\vec{v}_1 = \frac{\vec{k}}{\omega m_i}\left(e\phi_1 + \frac{k_B T_i}{n_0}n_1\right), \tag{4.2.134}$$

and then associating Eqs. (4.2.132) and (4.2.134) we obtain

$$n_1 = n_0\frac{e\phi_1}{\left(\frac{m_i\omega^2}{k^2} - k_B T_i\right)}. \tag{4.2.135}$$

First, assuming the quasi-neutrality approximation, $n_i \approx n_e$, and considering the linear term of the TF electron density,

$$n_0\frac{e\phi_1}{\left(\frac{m_i\omega^2}{k^2} - k_B T_i\right)} = \frac{3}{2}\frac{e\phi_1}{k_B T_F}n_0, \tag{4.2.136}$$

after some manipulations, we have the system dispersion relation

$$\omega^2 = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{k_B T_i}{m_i}\right)k^2, \tag{4.2.137}$$

and the configuration ion sound speed is

$$c_{SI} \equiv \frac{\omega}{k} = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{k_B T_i}{m_i}\right)^{\frac{1}{2}}. \tag{4.2.138}$$

Note that Eq. (4.2.138) is essentially the warm TF plasma sound velocity, expressed in Eq. (4.2.17), with $\gamma_i \to 1$, as expected.

Nonetheless, we can look at the complete linear theory using the linear Poisson equation, Eq. (4.2.133), so

$$k^2\phi_1 = \frac{e^2 n_0}{\epsilon_0}\left[\frac{k^2}{(m_i\omega^2 - k^2 k_B T_i)} - \frac{3}{2}\frac{1}{k_B T_F}\right]\phi_1 \tag{4.2.139}$$

and then we are able to write the complete dispersion relation,

$$\omega^2 = \frac{\omega_{p_i}k^2}{\left(k^2 + \frac{\omega_{p_i}^2}{c_S^2}\right)} + k^2\frac{k_B T_i}{m_i}, \tag{4.2.140}$$

where we have written in terms of the ion plasma frequency, Eq. (4.2.35), and the cold plasma ion acoustic speed, Eq. (4.2.105), which will be useful later.

Nonlinear Regime

We will start the investigation by transforming the governing equations of an unmagnetized isothermal fluid into their dimensionless forms. Conveniently, applying the transformations stated by Eq. (4.2.34), we obtain for Eqs. (4.2.128) - (4.2.130)

$$\partial_t\vec{v}+\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}=-\vec{\nabla}\phi-\frac{1}{2}\frac{\Theta}{n}\vec{\nabla}n, \tag{4.2.141}$$

$$\partial_t n+\vec{\nabla}\cdot(n\vec{v})=0, \tag{4.2.142}$$

$$\vec{\nabla}^2\phi=-\left(n-n_e\right), \tag{4.2.143}$$

where $\Theta=\frac{T_i}{T_F}$ is the ion and Fermi temperatures ratio, and introducing the stretched coordinates defined in Eq. (4.2.90), we have

$$\begin{aligned}-\lambda_0\epsilon^{\frac{1}{2}}\partial_\xi v_j+\epsilon^{\frac{3}{2}}\partial_\tau v_j+\epsilon^{\frac{1}{2}}\left(\sum_{k=1}^{3}v_k l_k\right)\partial_\xi v_j&\\=-l_j\epsilon^{\frac{1}{2}}\partial_\xi\phi-\frac{1}{2}\frac{\Theta}{n}l_j\epsilon^{\frac{1}{2}}\partial_\xi n,&\end{aligned} \tag{4.2.144}$$

$$-\lambda_0\epsilon^{\frac{1}{2}}\partial_\xi n+\epsilon^{\frac{3}{2}}\partial_\tau n+\epsilon^{\frac{1}{2}}\left[\sum_{k=1}^{3}l_k\partial_\xi\left(nv_k\right)\right]=0, \tag{4.2.145}$$

$$\epsilon\partial_\xi^2\phi=-\left(n-n_e\right). \tag{4.2.146}$$

So, proceeding with the expansion of the variables n, v_j, and ϕ, stated in Eqs. (4.2.92) - (4.2.94), and applying to Eqs. (4.2.144) - (4.2.146), it is valid in the lower perturbative order,

$$-\lambda_0\partial_\xi v_{1j}+l_j\partial_\xi\phi_1+\frac{1}{2}\Theta l_j\partial_\xi n_1=0, \tag{4.2.147}$$

$$-\lambda_0\partial_\xi n_1+l_x\partial_\xi v_{1x}+l_y\partial_\xi v_{1y}+l_z\partial_\xi v_{1z}=0, \tag{4.2.148}$$

$$-\left(n_1-3\phi_1\right)=0, \tag{4.2.149}$$

and then it is again straight from Eq. (4.2.149) that

$$n_1=3\phi_1, \tag{4.2.150}$$

which can be combined with Eq. (4.2.148), resulting

$$-3\lambda_0\partial_\xi\phi_1+l_x\partial_\xi v_{1x}+l_y\partial_\xi v_{1y}+l_z\partial_\xi v_{1z}=0. \tag{4.2.151}$$

From Eq. (4.2.147), we can get

$$l_j\partial_\xi v_{1j}=\frac{l_j^2}{\lambda_0}\left(1+\frac{3}{2}\Theta\right)\partial_\xi\phi_1, \tag{4.2.152}$$

which together with Eq. (4.2.151) gives

$$-3\lambda_0 \partial_\xi \phi_1 + \frac{1}{\lambda_0}\left(1 + \frac{3}{2}\Theta\right)\partial_\xi \phi_1 = 0 \quad \Rightarrow \quad \lambda_0^2 = \frac{1}{3} + \frac{1}{2}\Theta. \tag{4.2.153}$$

Moreover, it is possible to write from Eqs. (4.2.150) and (4.2.152),

$$n_1 = \frac{1}{l_j \lambda_0} v_{1j} \quad \text{and} \quad v_{1j} = 3\lambda_0 l_j \phi_1. \tag{4.2.154}$$

We start the analysis of the next perturbation order from the expressions

$$\begin{aligned} -\lambda_0 \partial_\xi v_{2j} + 3\lambda_0 l_j \partial_\tau \phi_1 + 9\left(\lambda_0^2 - \frac{1}{2}\Theta\right) l_j \phi_1 \partial_\xi \phi_1 & \\ + l_j \partial_\xi \phi_2 + \frac{1}{2}\Theta l_j \partial_\xi n_2 = 0, & \end{aligned} \tag{4.2.155}$$

$$-\lambda_0 \partial_\xi n_2 + 3\partial_\tau \phi_1 + 18\lambda_0 \phi_1 \partial_\xi \phi_1 + \left(\sum_{k=1}^{3} l_k \partial_\xi v_{2k}\right) = 0, \tag{4.2.156}$$

$$\partial_\xi^3 \phi_1 + \partial_\xi n_2 - 3\partial_\xi \phi_2 - 3\phi_1 \partial_\xi \phi_1 = 0, \tag{4.2.157}$$

and so, from the association of Eqs. (4.2.155), (4.2.156), and (4.2.157),

$$\begin{aligned} &-\lambda_0 \partial_\xi n_2 + 3\partial_\tau \phi_1 + 18\lambda_0 \phi_1 \partial_\xi \phi_1 \\ &\qquad + \frac{1}{\lambda_0}\left[3\lambda_0 \partial_\tau \phi_1 + 9\left(\lambda_0^2 - \frac{1}{2}\Theta\right)\phi_1 \partial_\xi \phi_1 + \partial_\xi \phi_2 + \frac{1}{2}\Theta \partial_\xi n_2\right] = 0 \\ &\frac{1}{\lambda_0}\left(\frac{1}{2}\Theta - \lambda_0^2\right)\partial_\xi n_2 + 6\partial_\tau \phi_1 + \frac{27}{\lambda_0}\left(\lambda_0^2 - \frac{\Theta}{6}\right)\phi_1 \partial_\xi \phi_1 + \frac{1}{\lambda_0}\partial_\xi \phi_2 = 0 \\ &\frac{1}{3\lambda_0}\left(3\partial_\xi \phi_2 - \partial_\xi n_2\right) + 6\partial_\tau \phi_1 + \frac{27}{\lambda_0}\left(\frac{1}{3} + \frac{1}{3}\Theta\right)\phi_1 \partial_\xi \phi_1 = 0 \\ &\frac{1}{3\lambda_0}\left(\partial_\xi^3 \phi_1 - 3\phi_1 \partial_\xi \phi_1\right) + 6\partial_\tau \phi_1 + \frac{9}{\lambda_0}\left(1 + \Theta\right)\phi_1 \partial_\xi \phi_1 = 0, \end{aligned} \tag{4.2.158}$$

where we have used the relation obtained to λ_0^2, expressed in Eq. (4.2.153). Thus,

$$6\partial_\tau \phi_1 + \frac{1}{\lambda_0}\left(8 + 9\Theta\right)\phi_1 \partial_\xi \phi_1 + \frac{1}{3\lambda_0}\partial_\xi^3 \phi_1 = 0, \tag{4.2.159}$$

which implies,

$$\partial_\tau \phi_1 + \frac{(8 + 9\Theta)}{6\lambda_0}\phi_1 \partial_\xi \phi_1 + \frac{1}{18\lambda_0}\partial_\xi^3 \phi_1 = 0. \tag{4.2.160}$$

Finally, we have that Eq. (4.2.160) is a KdV-like equation

$$\begin{gathered}\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with} \\ A = \frac{(8+9\Theta)}{6\lambda_0}, \quad B = \frac{1}{18\lambda_0}, \quad \lambda_0 = \left(\frac{1}{3} + \frac{\Theta}{2}\right)^{\frac{1}{2}}.\end{gathered} \tag{4.2.161}$$

4.2.2.5 *Unmagnetized Adiabatic Plasma*

To study the adiabatic ion system, let us consider the unmagnetized warm TF plasma governing equations, which allow us to write the polytropic index γ_i, Eq. (2.3.7), as

$$\gamma_i = \frac{5}{3}. \tag{4.2.162}$$

Linear Regime

With the consideration of Eq. (4.2.162), the hydrodynamic equations for an adiabatic fluid take the form

$$\partial_t \vec{v}_i + \left(\vec{v}_i \cdot \vec{\nabla}\right) \vec{v}_i = -\frac{e}{m_i}\vec{\nabla}\phi - \frac{1}{m_i n_i}\vec{\nabla}\left[n_0 k_B T_i \left(\frac{n_i}{n_0}\right)^{\frac{5}{3}}\right], \tag{4.2.163}$$

$$\frac{\partial n_i}{\partial t} + \vec{\nabla}\cdot(n_i \vec{v}_i) = 0, \tag{4.2.164}$$

$$\nabla^2 \phi = -\frac{e}{\epsilon_0}(n_i - n_e), \tag{4.2.165}$$

and proceeding with a linearization of Eqs. (4.2.163) - (4.2.165), we have

$$-i\omega \vec{v}_1 = -i\frac{e\phi}{m_i}\vec{k} - i\frac{5}{3}\frac{k_B T_i}{m_i}\frac{n_1}{n_0}\vec{k}, \tag{4.2.166}$$

$$-i\omega n_1 + i n_0 \left(\vec{k}\cdot\vec{v}_1\right) = 0, \tag{4.2.167}$$

$$-k^2 \phi_1 = -\frac{e}{\epsilon}\left(n_1^i - n_1^e\right). \tag{4.2.168}$$

It is then possible to write an expression for $\vec{v}_1$ from Eq. (4.2.166),

$$\vec{v}_1 = \frac{k}{\omega m_i}\left(e\phi + \frac{5}{3}\frac{K_B T_i}{n_0} n_1\right), \tag{4.2.169}$$

and by the combination of the linearized continuity equation and the perturbed velocity, Eqs. (4.2.167) and (4.2.169), we obtain

$$n_1 = n_0 \frac{e\phi_1}{\left(\frac{m_i \omega^2}{k^2} - \frac{5}{3}\frac{K_B T_i}{n_0}\right)}. \tag{4.2.170}$$

The approximation $n_i \approx n_e$ enables us to associate the perturbative density term for ions and degenerate electrons, being

$$n_0 \frac{e\phi_1}{\left(\frac{m_i\omega^2}{k^2} - \frac{5}{3}\frac{K_B T_i}{n_0}\right)} = \frac{3}{2}\frac{e\phi_1}{k_B T_F} n_0, \tag{4.2.171}$$

which returns as the dispersion relation for the configuration

$$\omega^2 = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{5}{3}\frac{k_B T_i}{m_i}\right) k^2, \tag{4.2.172}$$

and we can find the ion acoustic velocity,

$$c_{SA} \equiv \frac{\omega}{k} = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{5}{3}\frac{k_B T_i}{m_i}\right)^{\frac{1}{2}}. \tag{4.2.173}$$

Eq. (4.2.173) is exactly the acoustic speed of a warm TF plasma presented in Eq. (4.2.17), with $\gamma_i \to \frac{5}{3}$.

If we now consider the linearized Poisson equation, Eq. (4.2.168), for a complete analysis, we get

$$k^2\phi_1 = \frac{e^2 n_0}{\epsilon_0}\left[\frac{k^2}{\left(m_i\omega^2 - \frac{5}{3}k^2 k_B T_i\right)} - \frac{3}{2}\frac{1}{k_B T_F}\right]\phi_1 \tag{4.2.174}$$

which implies in the dispersion relation assuming the form

$$\omega^2 = \frac{\omega_{p_i} k^2}{\left(k^2 + \frac{\omega_{p_i}^2}{c_S^2}\right)} + \frac{5}{3}k^2\frac{k_B T_i}{m_i}, \tag{4.2.175}$$

where again we write in terms of the ion plasma frequency and the cold TF plasma ion sound speed, Eqs. (4.2.35) and (4.2.105), respectively.

Nonlinear Regime

To study the nonlinear limit of traveling waves propagation in unmagnetized adiabatic TF plasma, we apply the transformations presented in Eq. (4.2.34) to the fluid governing equations, so

$$\partial_t \vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}\phi - \frac{5}{6}\frac{\Theta}{n^{\frac{1}{3}}}\vec{\nabla}n, \tag{4.2.176}$$

$$\partial_t n + \vec{\nabla}\cdot(n\vec{v}) = 0, \tag{4.2.177}$$

$$\vec{\nabla}^2\phi = -(n - n_e), \tag{4.2.178}$$

which using the stretched coordinates, Eq. (4.2.90), give

$$\begin{aligned}
&-\lambda_0\epsilon^{\frac{1}{2}}\partial_\xi v_j+\epsilon^{\frac{3}{2}}\partial_\tau v_j+\epsilon^{\frac{1}{2}}\left(\sum_{k=1}^{3}v_k l_k\right)\partial_\xi v_j \\
&\qquad=-l_j\epsilon^{\frac{1}{2}}\partial_\xi\phi-\frac{5}{6}\frac{\Theta}{n^{\frac{1}{3}}}l_j\epsilon^{\frac{1}{2}}\partial_\xi n,
\end{aligned} \tag{4.2.179}$$

$$-\lambda_0\epsilon^{\frac{1}{2}}\partial_\xi n+\epsilon^{\frac{3}{2}}\partial_\tau n+\epsilon^{\frac{1}{2}}\left[\sum_{k=1}^{3}l_k\partial_\xi\left(nv_k\right)\right]=0, \tag{4.2.180}$$

$$\epsilon\partial_\xi^2\phi=-\left(n-n_e\right), \tag{4.2.181}$$

and then expanding the variables according to Eqs. (4.2.92) - (4.2.94), we get for the lowest order of Eqs. (4.2.179) - (4.2.181)

$$-\lambda_0\partial_\xi v_{1j}+l_j\partial_\xi\phi_1+\frac{5}{6}\Theta l_j\partial_\xi n_1=0, \tag{4.2.182}$$

$$-\lambda_0\partial_\xi n_1+l_x\partial_\xi v_{1x}+l_y\partial_\xi v_{1y}+l_z\partial_\xi v_{1z}=0, \tag{4.2.183}$$

$$-\left(n_1-3\phi_1\right)=0, \tag{4.2.184}$$

where, as in the prior analysis, we have

$$n_1=3\phi_1. \tag{4.2.185}$$

It is possible to write from Eqs. (4.2.182) and (4.2.185),

$$\partial_\xi v_{1j}=\frac{l_j}{\lambda_0}\left[\frac{1}{3}+\frac{5}{6}\Theta\right]\partial_\xi n_1, \tag{4.2.186}$$

thus, substituting Eq. (4.2.186) into Eq. (4.2.183),

$$\partial_\xi n_1=\frac{1}{\lambda_0^2}\left[\frac{1}{3}+\frac{5}{6}\Theta\right]\partial_\xi n_1, \tag{4.2.187}$$

which enables us to write for the parameter λ_0,

$$\lambda_0^2=\frac{1}{3}+\frac{5}{6}\Theta. \tag{4.2.188}$$

So, it is straightforward from the link of Eqs. (4.2.186) and (4.2.188) that

$$v_{1j}=l_j\lambda_0 n_1. \tag{4.2.189}$$

Looking at terms of higher order, we have for a singular direction

$$\begin{aligned} -\lambda_0 \partial_\xi v_{2j} + \partial_\tau v_{1j} + \left(\sum_{k=1}^{3} l_k v_{1k}\right) \partial_\xi v_{1j} + l_j \partial_\xi \phi_2 \\ + \frac{5}{6}\Theta l_j \partial_\xi n_2 - \frac{5}{6}\Theta l_j \left(\frac{n_1}{3}\right) \partial_\xi n_1 = 0, \end{aligned} \tag{4.2.190}$$

$$-\lambda_0 \partial_\xi n_2 + \partial_\tau n_1 + \sum_{k=1}^{3} l_j \partial_\xi v_{2j} + \sum_{k=1}^{3} l_j \partial_\xi \left(n_1 v_{1j}\right) = 0, \tag{4.2.191}$$

$$\partial_\xi^2 \phi_1 + n_2 - 3\phi_2 - \frac{3}{2}\phi_1^2 = 0, \tag{4.2.192}$$

and once Eq. (4.2.190) can be rearranged to express

$$\begin{aligned} \partial_\xi v_{2j} = \frac{l_j}{\lambda_0}\Bigg[3\lambda_0 \partial_\tau \phi_1 + 3\lambda_0 \left(\sum_{k=1}^{3} l_k v_{1k}\right) \partial_\xi \phi_1 \\ + \partial_\xi \phi_2 + \frac{5}{6}\Theta \partial_\xi n_2 - \frac{5}{2}\Theta \phi_1 \partial_\xi \phi_1 \Bigg], \end{aligned} \tag{4.2.193}$$

where we have used Eqs. (4.2.185) and (4.2.189) to rewrite the variables v_{1j} and n_1, substituting Eq. (4.2.193) into Eq. (4.2.191), we get, using the unitary relation for direction cosines when considering all directions,

$$\begin{aligned} 6\partial_\tau \phi_1 + \frac{1}{\lambda_0}\partial_\xi \phi_2 + \frac{1}{\lambda_0}\left(\frac{5}{6}\Theta - \lambda_0^2\right) \partial_\xi n_2 \\ + \frac{1}{\lambda_0}\left(27\lambda_0^2 - \frac{5}{2}\Theta\right) \phi_1 \partial_\xi \phi_1 = 0, \end{aligned} \tag{4.2.194}$$

which can be expressed with the support of Eq. (4.2.188) and the derivative of Eq. (4.2.192) as

$$6\partial_\tau \phi_1 + \frac{1}{3\lambda_0}\left(\partial_\xi^3 \phi_1 - 3\phi_1 \partial_\xi \phi_1\right) + \frac{1}{\lambda_0}\left(9 + 20\Theta\right) \phi_1 \partial_\xi \phi_1 = 0. \tag{4.2.195}$$

Therefore, we can easily reduce Eq. (4.2.195) to

$$6\partial_\tau \phi_1 + \frac{4}{\lambda_0}\left(2 + 5\Theta\right) \phi_1 \partial_\xi \phi_1 + \frac{1}{3\lambda_0}\partial_\xi^3 \phi_1 = 0, \tag{4.2.196}$$

and simplifying Eq. (4.2.196), we obtain

$$\partial_\tau \phi_1 + \frac{2}{3\lambda_0}\left(2 + 5\Theta\right) \phi_1 \partial_\xi \phi_1 + \frac{1}{18\lambda_0}\partial_\xi^3 \phi_1 = 0, \tag{4.2.197}$$

which is a KdV-like equation,

$$\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with}$$
$$A = \frac{2\,(2+5\Theta)}{3\lambda_0}, \; B = \frac{1}{18\lambda_0}, \; \lambda_0 = \left(\frac{1}{3} + \frac{5\Theta}{6}\right)^{\frac{1}{2}}. \tag{4.2.198}$$

4.2.2.6 *Magnetized Cold Plasma*

As in the previous sections, we will go directly from the linear to the nonlinear regime considering the variables transformations stated before.

Linear Regime

Let us admit a constant external magnetic field directed along the z-axis, i.e., $\vec{B} = B_0\hat{z}$, applied to the two-component TF plasma, with degenerate electrons obeying the Thomas-Fermi distribution and ions treated as cold fluid, and then, using Eq. (3.2.22), we have for the ion momentum equation

$$\partial_t \vec{v}_i + \left(\vec{v}_i \cdot \vec{\nabla}\right) \vec{v}_i = -\frac{e}{m_i} \vec{\nabla}\phi + \Omega\left(\vec{v}_i \times \hat{z}\right), \tag{4.2.199}$$

where Ω is the cyclotron frequency expressed in Eq. (3.2.27), and proceeding with a linearization, we obtain

$$-i\omega\vec{v}_1^i = -i\frac{e\phi_1}{m_i}\vec{k} + \Omega\left(\vec{v}_1^i \times \hat{z}\right). \tag{4.2.200}$$

So, as a consequence of the magnetic term, we have for each direction a specific relation, being

$$-i\omega v_{1x}^i = -i\frac{e\phi_1}{m_i}k_x + \Omega v_{1y}, \tag{4.2.201}$$
$$-i\omega v_{1y}^i = -i\frac{e\phi_1}{m_i}k_y - \Omega v_{1x}, \tag{4.2.202}$$
$$-i\omega v_{1z}^i = -i\frac{e\phi_1}{m_i}k_z, \tag{4.2.203}$$

then Eqs. (4.2.201) - (4.2.203) imply

$$v_{1x} = \frac{e}{m_i}\frac{k_x}{\omega}\phi_1 + i\frac{\Omega}{\omega}v_{1y}, \tag{4.2.204}$$
$$v_{1y} = \frac{e}{m_i}\frac{k_y}{\omega}\phi_1 - i\frac{\Omega}{\omega}v_{1x}, \tag{4.2.205}$$
$$v_{1z} = \frac{e}{m_i}\frac{k_z}{\omega}\phi_1, \tag{4.2.206}$$

where we have suppressed the ion identification index, and the association of Eqs. (4.2.204) and (4.2.205) allows us to write for v_{1x} and v_{1y},

$$v_{1x} = \frac{e}{m_i}\frac{(\omega k_x + i\Omega k_y)}{\omega^2 - \Omega^2}\phi_1, \tag{4.2.207}$$

$$v_{1y} = \frac{e}{m_i}\frac{(\omega k_y - i\Omega k_x)}{\omega^2 - \Omega^2}\phi_1. \tag{4.2.208}$$

Since we know that the linearized continuity and Poisson equations assume the form, respectively,

$$n_1 = \frac{n_0}{\omega}\left(\vec{k}\cdot\vec{v}_1\right), \tag{4.2.209}$$

$$k^2\phi_1 = \frac{e}{\epsilon_0}\left(n_1^i - n_1^e\right), \tag{4.2.210}$$

the substitution of Eqs. (4.2.206) - (4.2.208) into Eq. (4.2.209) returns

$$n_1 = \frac{n_0 e}{\omega m_i}\left[\frac{(\omega k_x^2 + i\Omega k_x k_y)}{\omega^2 - \Omega^2} + \frac{(\omega k_y^2 - i\Omega k_x k_y)}{\omega^2 - \Omega^2} + \frac{k_z^2}{\omega}\right]\phi_1, \tag{4.2.211}$$

which we can manipulate to obtain

$$n_1 = \frac{n_0 e}{m_i}\left[\frac{(k_x^2 + k_y^2)}{\omega^2 - \Omega^2} + \frac{k_z^2}{\omega^2}\right]\phi_1, \tag{4.2.212}$$

and writing $k^2 = k_x^2 + k_y^2 + k_z^2$ we get

$$n_1 = \frac{n_0 e}{m_i}\left[\frac{k^2}{\omega^2 - \Omega^2} - \frac{\Omega^2 k_z^2}{\omega^2\left(\omega^2 - \Omega^2\right)}\right]\phi_1. \tag{4.2.213}$$

The linear term of the electron density, as presented in Eq. (4.2.102), is

$$n_e^1 = \frac{3}{2}\frac{n_0 e}{k_B T_F}\phi_1, \tag{4.2.214}$$

then assuming the quasi-neutrality approximation, we have from Eqs. (4.2.213) and (4.2.214)

$$\omega^2 = \frac{2}{3}\frac{k_B T_F}{m_i}k^2\left\{\left(1 - \frac{\Omega^2}{\omega^2}\right)^{-1} + \frac{k_z^2}{k^2}\left[1 - \left(1 - \frac{\Omega^2}{\omega^2}\right)^{-1}\right]\right\} \tag{4.2.215}$$

and as we had concluded in Sec. 3.2, pure acoustic waves exist only for $\vec{k} \parallel \vec{B}$, with sound speed corresponding to

$$c_S = \left(\frac{2}{3}\frac{k_B T_F}{m_i}\right)^{\frac{1}{2}}. \tag{4.2.216}$$

Finally, instead of the approximation, we can adopt the linear limit of the Poisson Equation, Eq. (4.2.210), so

$$k^2 = \frac{n_0 e^2}{m_i \epsilon_0}\left\{\frac{k^2}{\omega^2 - \Omega^2} - k_z^2 \frac{\Omega^2}{\omega^2\left(\omega^2 - \Omega^2\right)} - \frac{3}{2}\frac{m_i}{k_B T_F}\right\}, \tag{4.2.217}$$

which returns as the dispersion relation, after manipulations,

$$\omega^2 = \frac{2}{3}\frac{k_B T_F}{m_i} k^2 \left[\frac{\omega^2}{\omega^2 - \Omega^2} - \frac{\omega^2}{\omega_{p_i}^2} - \frac{k_z^2}{k^2}\frac{\Omega^2}{\left(\omega^2 - \Omega^2\right)}\right], \tag{4.2.218}$$

where we have used Eq. (4.2.35), and we can verify that in the limit of $k_z^2 \to k^2$ Eq. (4.2.218) is expressed as

$$\omega^2 = \frac{k^2}{\left(\frac{k^2}{\omega_{p_i}^2} + \frac{3}{2}\frac{m_i}{k_B T_F}\right)}, \tag{4.2.219}$$

corroborating with the system acoustic speed derived before, Eq. (4.2.216), since

$$c_S = \lim_{k\to 0}\frac{\omega}{k} = \left(\frac{2}{3}\frac{k_B T_F}{m_i}\right)^{\frac{1}{2}}. \tag{4.2.220}$$

Nonlinear Regime

The nonlinear dynamics of low-frequency ion acoustic waves in the cold *e-i* plasma is governed by the dimensionless equations

$$\partial_t \vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}\phi + \frac{\Omega}{\omega_{p_i}}\left(\vec{v}\times\hat{z}\right), \tag{4.2.221}$$

$$\partial_t n + \vec{\nabla}\cdot\left(n\vec{v}\right) = 0, \tag{4.2.222}$$

$$\nabla^2\phi = -\left(n - n_e\right), \tag{4.2.223}$$

where we have used the transformation presented in Eq. (4.2.34). Using the stretched coordinates defined in Eq. (4.2.90) we get from Eqs. (4.2.221) - (4.2.223)

$$\begin{aligned}
-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_j + \epsilon^{\frac{3}{2}} \partial_\tau v_j + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} l_k v_k \right) \partial_\xi v_j & \\
= -l_j \epsilon^{\frac{1}{2}} \partial_\xi \phi + \Omega_{p_i} (\vec{v} \times \hat{z})_j \,, &
\end{aligned} \tag{4.2.224}$$

$$- \lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi n + \epsilon^{\frac{3}{2}} \partial_\tau n + \epsilon^{\frac{1}{2}} \left[\sum_{k=1}^{3} l_k \partial_\xi (n v_k) \right] = 0, \tag{4.2.225}$$

$$\epsilon \partial_\xi^2 \phi = -(n - n_e)\,, \tag{4.2.226}$$

identifying Ω_{p_i} as

$$\Omega_{p_i} = \frac{\Omega}{\omega_{p_i}}, \tag{4.2.227}$$

and we need to be more cautious in the analysis of Eq. (4.2.224), once the three different directions result

$$\begin{aligned}
-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_x + \epsilon^{\frac{3}{2}} \partial_\tau v_x + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} l_k v_k \right) \partial_\xi v_x & \\
= -l_x \epsilon^{\frac{1}{2}} \partial_\xi \phi + \Omega_{p_i} v_y, &
\end{aligned} \tag{4.2.228}$$

$$\begin{aligned}
-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_y + \epsilon^{\frac{3}{2}} \partial_\tau v_y + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} l_k v_k \right) \partial_\xi v_y & \\
= -l_y \epsilon^{\frac{1}{2}} \partial_\xi \phi - \Omega_{p_i} v_x, &
\end{aligned} \tag{4.2.229}$$

$$- \lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_z + \epsilon^{\frac{3}{2}} \partial_\tau v_z + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} l_k v_k \right) \partial_\xi v_z = -l_z \epsilon^{\frac{1}{2}} \partial_\xi \phi, \tag{4.2.230}$$

introducing an inhomogeneity in the description of the dynamics. As justified in Appendix A, to proceed with the nonlinear investigation in the presence of a constant magnetic field, we expand the perturbed quantities about the equilibrium values in powers of ϵ as

$$n = 1 + \epsilon n_1 + \epsilon^2 n_2 + \cdots\,, \tag{4.2.231}$$

$$\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots\,, \tag{4.2.232}$$

$$v_{x,y} = \epsilon^{\frac{3}{2}} v_{1(x,y)} + \epsilon^2 v_{2(x,y)} + \cdots\,, \tag{4.2.233}$$

$$v_z = \epsilon v_{1z} + \epsilon^2 v_{2z} + \cdots\,. \tag{4.2.234}$$

Thus, if we use Eqs. (4.2.231) - (4.2.234) in Eqs. (4.2.228) - (4.2.230), we get in the lowest order of ϵ

$$- l_x \partial_\xi \phi_1 + \Omega_{p_i} v_{1y} = 0, \tag{4.2.235}$$

$$- l_y \partial_\xi \phi_1 - \Omega_{p_i} v_{1x} = 0, \tag{4.2.236}$$

$$- l_z \partial_\xi \phi_1 + \lambda_0 \partial_\xi v_{1z} = 0, \tag{4.2.237}$$

while the results of the substitution of Eqs. (4.2.231) - (4.2.234) into continuity and Poisson equations, Eqs. (4.2.225) and (4.2.226), are, respectively,

$$-\lambda_0 \partial_\xi n_1 + l_z \partial_\xi \nu_{1z} = 0, \tag{4.2.238}$$

$$-(n_1 - 3\phi_1) = 0, \tag{4.2.239}$$

and then we can write for the first-order components of the momentum equation,

$$\nu_{1y} = \frac{l_x}{\Omega_{p_i}} \partial_\xi \phi_1, \tag{4.2.240}$$

$$\nu_{1x} = -\frac{l_y}{\Omega_{p_i}} \partial_\xi \phi_1, \tag{4.2.241}$$

$$\nu_{1z} = \frac{l_z}{\lambda_0} \phi_1. \tag{4.2.242}$$

From the Poisson and continuity terms we obtain that

$$n_1 = 3\phi_1, \tag{4.2.243}$$

$$n_1 = \frac{l_z}{\lambda_0} \nu_{1z}, \tag{4.2.244}$$

so the association of Eqs. (4.2.242) - (4.2.244) returns

$$n_1 = \frac{l_z^2}{\lambda_0^2} \phi_1 \qquad \Rightarrow \qquad \lambda_0^2 = \frac{l_z^2}{3}. \tag{4.2.245}$$

If we look at the next order, the following relations are obtained from the x and y components of the momentum equation and from the continuity equation, respectively,

$$\nu_{2x} = \frac{\lambda_0}{\Omega_{p_i}} \partial_\xi \nu_{1y}, \tag{4.2.246}$$

$$\nu_{2y} = -\frac{\lambda_0}{\Omega_{p_i}} \partial_\xi \nu_{1x}, \tag{4.2.247}$$

$$\nu_{1x} = -\frac{l_y}{l_x} \nu_{1y}, \tag{4.2.248}$$

where we can verify that the condition Eq. (4.2.248) is satisfied by Eqs. (4.2.240) and (4.2.241), and from the Poisson expression we can write,

$$\partial_\xi^3 \phi_1 - 3\phi_1 \partial_\xi \phi_1 = 3\partial_\xi \phi_2 - \partial_\xi n_2. \tag{4.2.249}$$

From the analysis of the next higher order of the z-component momentum relation, we get

$$\partial_\xi v_{2z} = \frac{1}{\lambda_0}\left(\partial_\tau v_{1z} + l_z v_{1z}\partial_\xi v_{1z} + l_z \partial_\xi \phi_2\right), \tag{4.2.250}$$

while the continuity equation gives

$$-\lambda_0 \partial_\xi n_2 + \partial_\tau n_1 + l_x \partial_\xi v_{2x} + l_y \partial_\xi v_{2y} + l_z \partial_\xi \left(n_1 v_{1z}\right) + l_z \partial_\xi v_{2z} = 0. \tag{4.2.251}$$

Thus, using the results obtained above, we can write from Eq. (4.2.251)

$$\begin{aligned}
&-\lambda_0 \partial_\xi n_2 + 3\partial_\tau \phi_1 + l_x \frac{\lambda_0}{\Omega_{p_i}} \partial_\xi^2 v_{1y} - l_y \frac{\lambda_0}{\Omega_{p_i}} \partial_\xi^2 v_{1x} \\
&\qquad\qquad + 3\frac{l_z^2}{\lambda_0}\partial_\xi\left(\phi_1^2\right) + l_z \partial_\xi v_{2z} = 0 \\
&-\lambda_0 \partial_\xi n_2 + 3\partial_\tau \phi_1 + \left(l_x^2 + l_y^2\right)\frac{\lambda_0}{\Omega_{p_i}^2}\partial_\xi^3 \phi_1 \\
&\qquad\qquad + 6\frac{l_z^2}{\lambda_0}\phi_1 \partial_\xi \phi_1 + l_z \partial_\xi v_{2z} = 0 \\
&-\lambda_0 \partial_\xi n_2 + 3\partial_\tau \phi_1 + \left(1 - l_z^2\right)\frac{\lambda_0}{\Omega_{p_i}^2}\partial_\xi^3 \phi_1 + 6\frac{l_z^2}{\lambda_0}\phi_1 \partial_\xi \phi_1 + l_z \partial_\xi v_{2z} = 0,
\end{aligned} \tag{4.2.252}$$

and once we have from Eq. (4.2.250)

$$l_z \partial_\xi v_{2z} = 3\partial_\tau \phi_1 + 9\lambda_0 \phi_1 \partial_\xi \phi_1 + 3\lambda_0 \partial_\xi \phi_2, \tag{4.2.253}$$

Eq. (4.2.252) becomes

$$\begin{aligned}
&6\partial_\tau \phi_1 - \lambda_0 \partial_\xi n_2 + \frac{\lambda_0}{\Omega_{p_i}^2}\left(1 - l_z^2\right)\partial_\xi^3 \phi_1 + 27\lambda_0 \phi_1 \partial_\xi \phi_1 + 3\lambda_0 \partial_\xi \phi_2 = 0 \\
&6\partial_\tau \phi_1 + \lambda_0\left(3\partial_\xi \phi_2 - \partial_\xi n_2\right) + \frac{\lambda_0}{\Omega_{p_i}^2}\left(1 - l_z^2\right)\partial_\xi^3 \phi_1 + 27\lambda_0 \phi_1 \partial_\xi \phi_1 = 0 \\
&6\partial_\tau \phi_1 + \lambda_0\left[1 + \frac{\left(1 - l_z^2\right)}{\Omega_{p_i}^2}\right]\partial_\xi^3 \phi_1 + 24\lambda_0 \phi_1 \partial_\xi \phi_1 = 0.
\end{aligned} \tag{4.2.254}$$

So we can write for a magnetized TF cold plasma

$$\partial_\tau \phi_1 + 4\lambda_0 \phi_1 \partial_\xi \phi_1 + \frac{\lambda_0}{6}\left[1 + \frac{\left(1 - l_z^2\right)}{\Omega_{p_i}^2}\right]\partial_\xi^3 \phi_1 = 0, \tag{4.2.255}$$

which is a KdV-like equation with

$$\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with}$$
$$A = 4\lambda_0, \; B = \frac{\lambda_0}{6}\left[1 + \frac{(1 - l_z^2)}{\Omega_{p_i}^2}\right], \; \lambda_0^2 = \frac{l_z^2}{3}. \tag{4.2.256}$$

4.2.2.7 *Magnetized Isothermal Plasma*

Linear Regime

We will start with a system configuration similar to the one discussed above, but now we will consider thermal ion effects, in particular the isothermal limit model. In this way, the isothermal magnetized fluid dynamic equations are

$$\partial_t \vec{v}_i + \left(\vec{v}_i \cdot \vec{\nabla}\right)\vec{v}_i = -\frac{e}{m_i}\vec{\nabla}\phi - \frac{k_B T_i}{m_i n_i}\vec{\nabla} n_i + \Omega\left(\vec{v}_i \times \hat{z}\right), \tag{4.2.257}$$

$$\partial_t n_i + \vec{\nabla} \cdot (n_i \vec{v}_i) = 0, \tag{4.2.258}$$

$$\nabla^2 \phi = -\frac{e}{\epsilon_0}(n_i - n_e), \tag{4.2.259}$$

and the linearization of Eq. (4.2.257) results

$$-i\omega \vec{v}_1^i = -i\frac{e\phi_1}{m_i}\vec{k} - i\frac{k_B T_i}{m_i}\frac{n_1}{n_0}\vec{k} + \Omega\left(\vec{v}_1^i \times \hat{z}\right), \tag{4.2.260}$$

which gives a characteristic expression for each direction, being

$$-i\omega v_{1x}^i = -i\frac{e\phi_1}{m_i}k_x - i\frac{k_B T_i}{m_i}\frac{n_1}{n_0}k_x + \Omega v_{1y}, \tag{4.2.261}$$

$$-i\omega v_{1y}^i = -i\frac{e\phi_1}{m_i}k_y - i\frac{k_B T_i}{m_i}\frac{n_1}{n_0}k_y - \Omega v_{1x}, \tag{4.2.262}$$

$$-i\omega v_{1z}^i = -i\frac{e\phi_1}{m_i}k_z - i\frac{k_B T_i}{m_i}\frac{n_1}{n_0}k_z. \tag{4.2.263}$$

Therefore, from Eqs. (4.2.261) - (4.2.263), we can write for the perturbed velocity quantities,

$$v_{1x} = \frac{k_x}{\omega m_i}\left(e\phi_1 + \frac{k_B T_i}{n_0}n_1\right) + i\frac{\Omega}{\omega}v_{1y}, \tag{4.2.264}$$

$$v_{1y} = \frac{k_y}{\omega m_i}\left(e\phi_1 + \frac{k_B T_i}{n_0}n_1\right) - i\frac{\Omega}{\omega}v_{1x}, \tag{4.2.265}$$

$$v_{1z} = \frac{k_z}{\omega m_i}\left(e\phi_1 + \frac{k_B T_i}{n_0}n_1\right), \tag{4.2.266}$$

where we have dropped the index notation, and so the combination of Eqs. (4.2.264) and (4.2.265) gives

$$v_{1x} = \frac{(n_0 e\phi_1 + k_B T_i n_1)}{n_0 m_i} \frac{(\omega k_x + i\Omega k_y)}{\omega^2 - \Omega^2}, \tag{4.2.267}$$

$$v_{1y} = \frac{(n_0 e\phi_1 + k_B T_i n_1)}{n_0 m_i} \frac{(\omega k_y - i\Omega k_x)}{\omega^2 - \Omega^2}. \tag{4.2.268}$$

The linearized continuity equation is expressed as

$$n_1 = \frac{n_0}{\omega}\left(\vec{k}\cdot\vec{v}_1\right), \tag{4.2.269}$$

and the application of Eqs. (4.2.266) - (4.2.268) into Eq. (4.2.269) implies

$$n_1 = \left[\frac{(k_x^2 + k_y^2)}{\omega^2 - \Omega^2} + \frac{k_z^2}{\omega^2}\right] \frac{(n_0 e\phi_1 + k_B T_i n_1)}{m_i}, \tag{4.2.270}$$

which can be rearranged to

$$n_1 = \left[\frac{k^2}{\omega^2 - \Omega^2} - \frac{\Omega^2 k_z^2}{\omega^2(\omega^2 - \Omega^2)}\right] \times \frac{n_0 e\phi_1}{m_i}\left[1 - \frac{k_B T_i}{m_i}\left(\frac{k^2}{\omega^2 - \Omega^2} - \frac{\Omega^2 k_z^2}{\omega^2(\omega^2 - \Omega^2)}\right)\right]^{-1}. \tag{4.2.271}$$

So, rewriting the following expressions as

$$\frac{k^2}{\omega^2 - \Omega^2} - \frac{\Omega^2 k_z^2}{\omega^2(\omega^2 - \Omega^2)} = \frac{(k^2\omega^2 - \Omega^2 k_z^2)}{\omega^2(\omega^2 - \Omega^2)}, \tag{4.2.272}$$

$$1 - \frac{k_B T_i}{m_i}\left(\frac{k^2}{\omega^2 - \Omega^2} - \frac{\Omega^2 k_z^2}{\omega^2(\omega^2 - \Omega^2)}\right) = \frac{m_i \omega^2(\omega^2 - \Omega^2) - k_B T_i (k^2\omega^2 - \Omega^2 k_z^2)}{m_i \omega^2 (\omega^2 - \Omega^2)}, \tag{4.2.273}$$

and being the electron density linear term

$$n_1^e = \frac{3}{2}\frac{n_0 e}{k_B T_F}\phi_1, \tag{4.2.274}$$

we can assume the quasi-neutral approximation to obtain from Eqs. (4.2.271) - (4.2.274)

$$\frac{3}{2}\frac{1}{k_B T_F} = \frac{(k^2\omega^2 - \Omega^2 k_z^2)}{m_i \omega^2(\omega^2 - \Omega^2) - k_B T_i (k^2\omega^2 - \Omega^2 k_z^2)}, \tag{4.2.275}$$

which can be manipulated to

$$\omega^2 = \frac{2}{3}\frac{k_B T_F}{m_i}\frac{\left(k^2\omega^2 - \Omega^2 k_z^2\right)}{\left[(\omega^2 - \Omega^2) - \frac{k_B T_i}{m_i\omega^2}\left(k^2\omega^2 - \Omega^2 k_z^2\right)\right]}. \tag{4.2.276}$$

When we adopt the limit of $k_z^2 \to k^2$ in Eq. (4.2.276), we have that

$$c_{SI} \equiv \frac{\omega}{k} = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{k_B T_i}{m_i}\right)^{\frac{1}{2}} \tag{4.2.277}$$

which is the same expression for the ion sound speed obtained in the unmagnetized regime, Eq. (4.2.138).

To complete, we can use the linear Poisson equation,

$$k^2\phi_1 = \frac{e}{\epsilon_0}\left(n_1^i - n_1^e\right), \tag{4.2.278}$$

and now the substitution of Eqs. (4.2.271) and (4.2.274) into Eq. (4.2.278) returns

$$k^2 = \frac{n_0 e^2}{\epsilon_0 m_i}\left\{\frac{k^2\omega^2 - k_z^2\Omega^2}{\omega^2\left(\omega^2 - \Omega^2\right) - \frac{k_B T_i}{m_i}\left(k^2\omega^2 - k_z^2\Omega^2\right)} - \frac{3}{2}\frac{m_i}{k_B T_F}\right\} \tag{4.2.279}$$

where we identify the ion plasma frequency term, ω_{p_i}, given by Eq. (4.2.35) and manipulate Eq. (4.2.279) to write the dispersion relation as

$$\omega^2 = \frac{2}{3}\frac{k_B T_F}{m_i}k^2\left[\frac{\left(\omega^2 - \frac{k_z^2}{k^2}\Omega^2\right)}{\left[(\omega^2 - \Omega^2) - \frac{k_B T_i}{m_i}\frac{k^2}{\omega^2}\left(\omega^2 - \frac{k_z^2}{k^2}\Omega^2\right)\right]} - \frac{\omega^2}{\omega_{p_i}^2}\right], \tag{4.2.280}$$

and then, taking the limit of $k_z^2 \to k^2$ in Eq. (4.2.280), we have

$$\omega^2 = \frac{k^2}{\left(\frac{3}{2}\frac{m_i}{k_B T_F} + \frac{k^2}{\omega_{p_i}^2}\right)} + \frac{k_B T_i}{m_i}k^2, \tag{4.2.281}$$

which confirms the configuration sound speed, Eq. (4.2.277), as

$$c_{SI} = \lim_{k\to 0}\frac{\omega}{k} = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{k_B T_i}{m_i}\right)^{\frac{1}{2}}. \tag{4.2.282}$$

Nonlinear Regime

The dimensionless form of the hydrodynamic equations for an isothermal fluid, under consideration of the transformation stated in Eq. (4.2.34),

$$\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\vec{\nabla}\phi - \frac{1}{2}\frac{\Theta}{n}\vec{\nabla}n + \Omega_{p_i}\left(\vec{v} \times \hat{z}\right), \tag{4.2.283}$$

$$\partial_t n + \vec{\nabla} \cdot (n\vec{v}) = 0, \tag{4.2.284}$$

$$\nabla^2 \phi = -\left(n - n_e\right), \tag{4.2.285}$$

along with the stretched coordinates definitions, Eqs. (4.2.90) and (4.2.91), result

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_j + \epsilon^{\frac{3}{2}} \partial_\tau v_j + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} l_k v_k\right) \partial_\xi v_j = -l_j \epsilon^{\frac{1}{2}} \partial_\xi \phi - \frac{1}{2}\frac{\Theta}{n} l_j \epsilon^{\frac{1}{2}} \partial_\xi n + \Omega_{p_i}\left(\vec{v} \times \hat{z}\right)_j, \tag{4.2.286}$$

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi n + \epsilon^{\frac{3}{2}} \partial_\tau n + \epsilon^{\frac{1}{2}} \left[\sum_{k=1}^{3} l_k \partial_\xi \left(n v_k\right)\right] = 0, \tag{4.2.287}$$

$$\epsilon \partial_\xi^2 \phi = -\left(n - n_e\right), \tag{4.2.288}$$

and as expected we have an irregular behavior when considering the momentum terms, compiled by Eq. (4.2.286), with

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_x + \epsilon^{\frac{3}{2}} \partial_\tau v_x + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} l_k v_k\right) \partial_\xi v_x = -l_x \epsilon^{\frac{1}{2}} \partial_\xi \phi - \frac{1}{2}\frac{\Theta}{n} l_x \epsilon^{\frac{1}{2}} \partial_\xi n + \Omega_{p_i} v_y, \tag{4.2.289}$$

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_y + \epsilon^{\frac{3}{2}} \partial_\tau v_y + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} l_k v_k\right) \partial_\xi v_y = -l_y \epsilon^{\frac{1}{2}} \partial_\xi \phi - \frac{1}{2}\frac{\Theta}{n} l_y \epsilon^{\frac{1}{2}} \partial_\xi n - \Omega_{p_i} v_x, \tag{4.2.290}$$

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_z + \epsilon^{\frac{3}{2}} \partial_\tau v_z + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} l_k v_k\right) \partial_\xi v_z = -l_z \epsilon^{\frac{1}{2}} \partial_\xi \phi - \frac{1}{2}\frac{\Theta}{n} l_z \epsilon^{\frac{1}{2}} \partial_\xi n. \tag{4.2.291}$$

Considering, therefore, the variables expansion as presented in Eqs. (4.2.231) - (4.2.234) and applying them to Eqs. (4.2.289) - (4.2.291), we have for the lowest perturbation order

$$-l_x\partial_\xi\phi_1 - \frac{1}{2}\Theta l_x\partial_\xi n_1 + \Omega_{p_i}v_{1y} = 0, \tag{4.2.292}$$
$$-l_y\partial_\xi\phi_1 - \frac{1}{2}\Theta l_y\partial_\xi n_1 - \Omega_{p_i}v_{1x} = 0, \tag{4.2.293}$$
$$-l_z\partial_\xi\phi_1 - \frac{1}{2}\Theta l_z\partial_\xi n_1 + \lambda_0\partial_\xi v_{1z} = 0, \tag{4.2.294}$$

which allows us to write

$$v_{1y} = \frac{l_x}{\Omega_{p_i}}\left(1+\frac{3}{2}\Theta\right)\partial_\xi\phi_1, \tag{4.2.295}$$
$$v_{1x} = -\frac{l_y}{\Omega_{p_i}}\left(1+\frac{3}{2}\Theta\right)\partial_\xi\phi_1, \tag{4.2.296}$$
$$v_{1z} = \frac{l_z}{\lambda_0}\left(1+\frac{3}{2}\Theta\right)\phi_1, \tag{4.2.297}$$

once the lowest order of ϵ in Eqs. (4.2.287) and (4.2.288) implies, respectively,

$$n_1 = \frac{l_z}{\lambda_0}v_{1z}, \tag{4.2.298}$$
$$n_1 = 3\phi_1, \tag{4.2.299}$$

and then combining Eqs. (4.2.297) - (4.2.299) we get

$$n_1 = \frac{l_z^2}{\lambda_0^2}\left(\frac{1}{3}+\frac{1}{2}\Theta\right)n_1 \quad\Rightarrow\quad \lambda_0^2 = l_z^2\left(\frac{1}{3}+\frac{1}{2}\Theta\right). \tag{4.2.300}$$

The next ϵ power of the momentum and continuity equations gives

$$v_{2x} = \frac{\lambda_0}{\Omega_{p_i}}\partial_\xi v_{1y}, \tag{4.2.301}$$
$$v_{2y} = -\frac{\lambda_0}{\Omega_{p_i}}\partial_\xi v_{1x}, \tag{4.2.302}$$
$$v_{1x} = -\frac{l_y}{l_x}v_{1y}, \tag{4.2.303}$$

and from the Poisson result we can write

$$\partial_\xi^3\phi_1 - 3\phi_1\partial_\xi\phi_1 = 3\partial_\xi\phi_2 - \partial_\xi n_2. \tag{4.2.304}$$

In addiction, the next higher order of the momentum equation returns to the z-component

$$
\begin{aligned}
\partial_\xi v_{2z} = \frac{1}{\lambda_0}\Bigg(& 3\frac{\lambda_0}{l_z}\partial_\tau\phi_1 + 9\frac{\lambda_0^2}{l_z}\phi_1\partial_\xi\phi_1 + l_z\partial_\xi\phi_2 \\
& + \frac{1}{2}\Theta l_z\partial_\xi n_2 - \frac{9}{2}\Theta l_z\phi_1\partial_\xi\phi_1\Bigg),
\end{aligned} \tag{4.2.305}
$$

and from the continuity equation we have

$$
-\lambda_0\partial_\xi n_2 + 3\partial_\tau\phi_1 + l_x\partial_\xi v_{2x} + l_y\partial_\xi v_{2y} + 9\lambda_0\partial_\xi\left(\phi_1^2\right) + l_z\partial_\xi v_{2z} = 0, \tag{4.2.306}
$$

so we can proceed from the previous results

$$
\begin{aligned}
& -\lambda_0\partial_\xi n_2 + 3\partial_\tau\phi_1 + \lambda_0\frac{(l_x^2+l_y^2)}{\Omega_{p_i}^2}\left(1+\frac{3}{2}\Theta\right)\partial_\xi^3\phi_1 \\
& \qquad + 18\lambda_0\phi_1\partial_\xi\phi_1 + l_z\partial_\xi v_{2z} = 0 \\
& -\lambda_0\partial_\xi n_2 + 3\partial_\tau\phi_1 + 3\frac{\lambda_0^3}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\partial_\xi^3\phi_1 \\
& \qquad + 18\lambda_0\phi_1\partial_\xi\phi_1 + l_z\partial_\xi v_{2z} = 0 \\
& -\lambda_0\partial_\xi n_2 + 6\partial_\tau\phi_1 + 3\frac{\lambda_0^3}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\partial_\xi^3\phi_1 + 9\frac{l_z^2}{\lambda_0}\left(3\frac{\lambda_0^2}{l_z^2}-\frac{\Theta}{2}\right)\phi_1\partial_\xi\phi_1 \\
& \qquad + \frac{l_z^2}{\lambda_0}\partial_\xi\phi_2 + \frac{1}{2}\Theta\frac{l_z^2}{\lambda_0}\partial_\xi n_2 = 0,
\end{aligned} \tag{4.2.307}
$$

where we have used Eq. (4.2.305) in the last term. Thus, manipulating Eq. (4.2.307),

$$
\begin{aligned}
& 6\partial_\tau\phi_1 + \frac{l_z^2}{\lambda_0}\left(\frac{1}{2}\Theta - \frac{\lambda_0^2}{l_z^2}\right)\partial_\xi n_2 + 3\frac{\lambda_0^3}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\partial_\xi^3\phi_1 \\
& \qquad + 9\frac{l_z^2}{\lambda_0}(1+\Theta)\phi_1\partial_\xi\phi_1 + \frac{l_z^2}{\lambda_0}\partial_\xi\phi_2 = 0 \\
& 6\partial_\tau\phi_1 + \frac{l_z^2}{3\lambda_0}(3\partial_\xi\phi_2 - \partial_\xi n_2) + 3\frac{\lambda_0^3}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\partial_\xi^3\phi_1 \\
& \qquad + 9\frac{l_z^2}{\lambda_0}(1+\Theta)\phi_1\partial_\xi\phi_1 = 0 \\
& 6\partial_\tau\phi_1 + \frac{1}{3\lambda_0}\left(l_z^2 + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\right)\partial_\xi^3\phi_1 + \frac{l_z^2}{\lambda_0}(9\Theta+8)\phi_1\partial_\xi\phi_1 = 0,
\end{aligned} \tag{4.2.308}
$$

and we are able to write for an isothermal TF plasma

$$\partial_\tau\phi_1 + \frac{l_z^2}{6\lambda_0}(9\Theta+8)\,\phi_1\partial_\xi\phi_1 + \frac{1}{18\lambda_0}\left(l_z^2 + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{\left(1-l_z^2\right)}{l_z^2}\right)\partial_\xi^3\phi_1 = 0, \tag{4.2.309}$$

being a KdV-like equation

$$\begin{aligned} &\partial_\tau\phi_1 + A\phi_1\partial_\xi\phi_1 + B\partial_\xi^3\phi_1 = 0 \quad \text{with} \\ &A = \frac{l_z^2}{6\lambda_0}(9\Theta+8)\,,\;\; B = \frac{1}{18\lambda_0}\left(l_z^2 + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{\left(1-l_z^2\right)}{l_z^2}\right), \\ &\lambda_0^2 = l_z^2\left(\frac{1}{3}+\frac{1}{2}\Theta\right). \end{aligned} \tag{4.2.310}$$

4.2.2.8 *Magnetized Adiabatic Plasma*

Linear Regime

To investigate the adiabatic TF plasma in the presence of an external magnetic field, $\vec{B} = B_0\hat{z}$, we consider the three-dimensional system hydrodynamic equations,

$$\partial_t\vec{v}_i + \left(\vec{v}_i\cdot\vec{\nabla}\right)\vec{v}_i = -\frac{e}{m_i}\vec{\nabla}\phi - \frac{1}{m_i n_i}\vec{\nabla}\left[n_0 k_B T_i\left(\frac{n_i}{n_0}\right)^{\frac{5}{3}}\right] + \Omega\left(\vec{v}_i\times\hat{z}\right), \tag{4.2.311}$$

$$\partial_t n_i + \vec{\nabla}\cdot\left(n_i\vec{v}_i\right) = 0, \tag{4.2.312}$$

$$\nabla^2\phi = -\frac{e}{\epsilon_0}\left(n_i - n_e\right). \tag{4.2.313}$$

The linearization of Eq. (4.2.311) gives

$$-i\omega\vec{v}_1^i = -i\frac{e\phi_1}{m_i}\vec{k} - i\frac{5}{3}\frac{k_B T_i}{m_i}\frac{n_1}{n_0}\vec{k} + \Omega\left(\vec{v}_1^i\times\hat{z}\right), \tag{4.2.314}$$

which implies for each singular direction

$$-\,i\omega v_{1x}^i = -i\frac{e\phi_1}{m_i}k_x - i\frac{5}{3}\frac{k_B T_i}{m_i}\frac{n_1}{n_0}k_x + \Omega v_{1y}, \tag{4.2.315}$$

$$-\,i\omega v_{1y}^i = -i\frac{e\phi_1}{m_i}k_y - i\frac{5}{3}\frac{k_B T_i}{m_i}\frac{n_1}{n_0}k_y - \Omega v_{1x}, \tag{4.2.316}$$

$$-\,i\omega v_{1z}^i = -i\frac{e\phi_1}{m_i}k_z - i\frac{5}{3}\frac{k_B T_i}{m_i}\frac{n_1}{n_0}k_z, \tag{4.2.317}$$

and then we can write for the perturbed velocity terms

$$v_{1x} = \frac{k_x}{\omega m_i}\left(e\phi_1 + \frac{5}{3}\frac{k_B T_i}{n_0}n_1\right) + i\frac{\Omega}{\omega}v_{1y}, \tag{4.2.318}$$

$$v_{1y} = \frac{k_y}{\omega m_i}\left(e\phi_1 + \frac{5}{3}\frac{k_B T_i}{n_0}n_1\right) - i\frac{\Omega}{\omega}v_{1x}, \tag{4.2.319}$$

$$v_{1z} = \frac{k_z}{\omega m_i}\left(e\phi_1 + \frac{5}{3}\frac{k_B T_i}{n_0}n_1\right). \tag{4.2.320}$$

If we associate Eqs. (4.2.318) and (4.2.319),

$$v_{1x} = \frac{\left(n_0 e\phi_1 + \frac{5}{3}k_B T_i n_1\right)}{n_0 m_i}\frac{\left(\omega k_x + i\Omega k_y\right)}{\omega^2 - \Omega^2}, \tag{4.2.321}$$

$$v_{1y} = \frac{\left(n_0 e\phi_1 + \frac{5}{3}k_B T_i n_1\right)}{n_0 m_i}\frac{\left(\omega k_y - i\Omega k_x\right)}{\omega^2 - \Omega^2}, \tag{4.2.322}$$

and make use of the linear term of the continuity equation,

$$n_1 = \frac{n_0}{\omega}\left(\vec{k}\cdot\vec{v}_1\right), \tag{4.2.323}$$

we get from Eqs. (4.2.320) - (4.2.323),

$$n_1 = \left[\frac{\left(k^2 - k_z^2\right)}{\omega^2 - \Omega^2} + \frac{k_z^2}{\omega^2}\right]\frac{\left(n_0 e\phi_1 + \frac{5}{3}k_B T_i n_1\right)}{m_i}, \tag{4.2.324}$$

or yet, after some manipulations,

$$n_1 = \frac{n_0 e\phi_1}{m_i}\left[\frac{k^2}{\omega^2 - \Omega^2} - \frac{\Omega^2 k_z^2}{\omega^2\left(\omega^2 - \Omega^2\right)}\right]\times \left[1 - \frac{5}{3}\frac{k_B T_i}{m_i}\left(\frac{k^2}{\omega^2 - \Omega^2} - \frac{\Omega^2 k_z^2}{\omega^2\left(\omega^2 - \Omega^2\right)}\right)\right]^{-1}. \tag{4.2.325}$$

It is possible to simplify Eq. (4.2.325) since

$$\frac{k^2}{\omega^2 - \Omega^2} - \frac{\Omega^2 k_z^2}{\omega^2\left(\omega^2 - \Omega^2\right)} = \frac{\left(k^2\omega^2 - \Omega^2 k_z^2\right)}{\omega^2\left(\omega^2 - \Omega^2\right)}, \tag{4.2.326}$$

$$1 - \frac{5}{3}\frac{k_B T_i}{m_i}\left(\frac{k^2}{\omega^2 - \Omega^2} - \frac{\Omega^2 k_z^2}{\omega^2\left(\omega^2 - \Omega^2\right)}\right) = \frac{3m_i\omega^2\left(\omega^2 - \Omega^2\right) - 5k_B T_i\left(k^2\omega^2 - \Omega^2 k_z^2\right)}{3m_i\omega^2\left(\omega^2 - \Omega^2\right)}, \tag{4.2.327}$$

so we are able to write for the perturbed ion density

$$n_1 = \frac{3n_0 e\left(\omega^2 k^2 - \Omega^2 k_z^2\right)}{3m_i\omega^2\left(\omega^2-\Omega^2\right) - 5k_B T_i\left(\omega^2 k^2 - \Omega^2 k_z^2\right)}\phi_1. \tag{4.2.328}$$

Once we have obtained Eq. (4.2.328) and we know that the linear electron density term is written as

$$n_1^e = \frac{3}{2}\frac{n_0 e}{k_B T_F}\phi_1, \tag{4.2.329}$$

we can get, assuming the quasi-neutrality condition,

$$\frac{3}{2}\frac{1}{k_B T_F} = \frac{3\left(k^2\omega^2 - \Omega^2 k_z^2\right)}{3m_i\omega^2\left(\omega^2-\Omega^2\right) - 5k_B T_i\left(k^2\omega^2 - \Omega^2 k_z^2\right)}, \tag{4.2.330}$$

which allows us to write

$$\omega^2 = \frac{2}{3}\frac{k_B T_F}{m_i}\frac{\left(k^2\omega^2 - \Omega^2 k_z^2\right)}{\left[\left(\omega^2-\Omega^2\right) - \frac{5}{3}\frac{k_B T_i}{m_i\omega^2}\left(k^2\omega^2 - \Omega^2 k_z^2\right)\right]}. \tag{4.2.331}$$

Thus, taking the limit of $k_z^2 \to k^2$ in Eq. (4.2.331), we obtain

$$c_{SA} \equiv \frac{\omega}{k} = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{5}{3}\frac{k_B T_i}{m_i}\right)^{\frac{1}{2}} \tag{4.2.332}$$

being Eq. (4.2.332) the same result found before for the unmagnetized regime, Eq. (4.2.173).

Nonetheless, using the linearized Poisson equation,

$$k^2\phi_1 = \frac{e}{\epsilon_0}\left(n_1^i - n_1^e\right), \tag{4.2.333}$$

and introducing Eqs. (4.2.328) and (4.2.329) into Eq. (4.2.333),

$$k^2 = \frac{n_0 e^2}{\epsilon_0 m_i}\left\{\frac{k^2\omega^2 - k_z^2\Omega^2}{\omega^2\left(\omega^2-\Omega^2\right) - \frac{5}{3}\frac{k_B T_i}{m_i}\left(k^2\omega^2 - k_z^2\Omega^2\right)} - \frac{3}{2}\frac{m_i}{k_B T_F}\right\}, \tag{4.2.334}$$

which after manipulations results in the dispersion relation as follows

$$\omega^2 = \frac{2}{3}\frac{k_B T_F}{m_i}k^2\left[\frac{\left(\omega^2 - \frac{k_z^2}{k^2}\Omega^2\right)}{\left[\left(\omega^2-\Omega^2\right) - \frac{5}{3}\frac{k_B T_i}{m_i}\frac{k^2}{\omega^2}\left(\omega^2 - \frac{k_z^2}{k^2}\Omega^2\right)\right]} - \frac{\omega^2}{\omega_{p_i}^2}\right], \tag{4.2.335}$$

where we have introduced the ion plasma frequency notation, ω_{p_i}, by the Eq. (4.2.35) definition. Finally, to $k_z^2 \rightarrow k^2$ we have from Eq. (4.2.335)

$$\omega^2 = \frac{k^2}{\left(\frac{3}{2}\frac{m_i}{k_B T_F} + \frac{k^2}{\omega_{p_i}^2}\right)} + \frac{5}{3}\frac{k_B T_i}{m_i}k^2, \tag{4.2.336}$$

and, in accordance with Eq. (4.2.332), we confirm that

$$c_{SA} = \lim_{k\to 0}\frac{\omega}{k} = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{5}{3}\frac{k_B T_i}{m_i}\right)^{\frac{1}{2}}. \tag{4.2.337}$$

Nonlinear Regime

Considering again the variables transformations presented in Eq. (4.2.34), we have the hydrodynamic equations for a magnetized adiabatic TF fluid into its dimensionless form as

$$\partial_t \vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}\phi - \frac{5}{6}\frac{\Theta}{n^{\frac{1}{3}}}\vec{\nabla}n + \Omega_{p_i}\left(\vec{v}\times\hat{z}\right), \tag{4.2.338}$$

$$\partial_t n + \vec{\nabla}\cdot\left(n\vec{v}\right) = 0, \tag{4.2.339}$$

$$\nabla^2\phi = -\left(n - n_e\right), \tag{4.2.340}$$

and changing to the stretched coordinates defined by Eq. (4.2.90), we get

$$\begin{aligned}-\lambda_0\epsilon^{\frac{1}{2}}\partial_\xi v_j + \epsilon^{\frac{3}{2}}\partial_\tau v_j + \epsilon^{\frac{1}{2}}\left(\sum_{k=1}^{3} v_k l_k\right)\partial_\xi v_j \\ = -l_j\epsilon^{\frac{1}{2}}\partial_\xi\phi - \frac{5}{6}\frac{\Theta}{n^{\frac{1}{3}}}l_j\epsilon^{\frac{1}{2}}\partial_\xi n + \Omega_{p_i}\left(\vec{v}\times\hat{z}\right)_j,\end{aligned} \tag{4.2.341}$$

$$-\lambda_0\epsilon^{\frac{1}{2}}\partial_\xi n + \epsilon^{\frac{3}{2}}\partial_\tau n + \epsilon^{\frac{1}{2}}\left[\sum_{k=1}^{3} l_k\partial_\xi\left(n v_k\right)\right] = 0, \tag{4.2.342}$$

$$\epsilon\partial_\xi^2\phi = -\left(n - n_e\right). \tag{4.2.343}$$

As we know, the momentum equation, Eq. (4.2.341), has a different contribution to each direction, being

$$
\begin{aligned}
&-\lambda_0\epsilon^{\frac{1}{2}}\partial_\xi v_x+\epsilon^{\frac{3}{2}}\partial_\tau v_x+\epsilon^{\frac{1}{2}}\left(\sum_{k=1}^{3}l_k v_k\right)\partial_\xi v_x\\
&\qquad=-l_x\epsilon^{\frac{1}{2}}\partial_\xi\phi-\frac{5}{6}\frac{\Theta}{n^{\frac{1}{3}}}l_x\epsilon^{\frac{1}{2}}\partial_\xi n+\Omega_{p_i}v_y,
\end{aligned}
\tag{4.2.344}
$$

$$
\begin{aligned}
&-\lambda_0\epsilon^{\frac{1}{2}}\partial_\xi v_y+\epsilon^{\frac{3}{2}}\partial_\tau v_y+\epsilon^{\frac{1}{2}}\left(\sum_{k=1}^{3}l_k v_k\right)\partial_\xi v_y\\
&\qquad=-l_y\epsilon^{\frac{1}{2}}\partial_\xi\phi-\frac{5}{6}\frac{\Theta}{n^{\frac{1}{3}}}l_y\epsilon^{\frac{1}{2}}\partial_\xi n-\Omega_{p_i}v_x,
\end{aligned}
\tag{4.2.345}
$$

$$
\begin{aligned}
&-\lambda_0\epsilon^{\frac{1}{2}}\partial_\xi v_z+\epsilon^{\frac{3}{2}}\partial_\tau v_z+\epsilon^{\frac{1}{2}}\left(\sum_{k=1}^{3}l_k v_k\right)\partial_\xi v_z\\
&\qquad=-l_z\epsilon^{\frac{1}{2}}\partial_\xi\phi-\frac{5}{6}\frac{\Theta}{n^{\frac{1}{3}}}l_z\epsilon^{\frac{1}{2}}\partial_\xi n,
\end{aligned}
\tag{4.2.346}
$$

and expanding the quantities about the equilibrium, Eqs. (4.2.231) - (4.2.234), we obtain in the lowest order of ϵ,

$$-l_x\partial_\xi\phi_1-\frac{5}{6}\Theta l_x\partial_\xi n_1+\Omega_{p_i}v_{1y}=0, \tag{4.2.347}$$

$$-l_y\partial_\xi\phi_1-\frac{5}{6}\Theta l_y\partial_\xi n_1-\Omega_{p_i}v_{1x}=0, \tag{4.2.348}$$

$$-l_z\partial_\xi\phi_1-\frac{5}{6}\Theta l_z\partial_\xi n_1+\lambda_0\partial_\xi v_{1z}=0, \tag{4.2.349}$$

which directly imply

$$v_{1y}=\frac{l_x}{\Omega_{p_i}}\left(1+\frac{5}{2}\Theta\right)\partial_\xi\phi_1, \tag{4.2.350}$$

$$v_{1x}=-\frac{l_y}{\Omega_{p_i}}\left(1+\frac{5}{2}\Theta\right)\partial_\xi\phi_1, \tag{4.2.351}$$

$$v_{1z}=\frac{l_z}{\lambda_0}\left(1+\frac{5}{2}\Theta\right)\phi_1, \tag{4.2.352}$$

where we have used that in the smallest order of perturbation the Poisson equation returns

$$n_1=3\phi_1. \tag{4.2.353}$$

Moreover, at this point the continuity equation results

$$n_1=\frac{l_z}{\lambda_0}v_{1z}, \tag{4.2.354}$$

enabling us to write from Eqs. (4.2.352) - (4.2.354)

$$n_1 = \frac{l_z^2}{\lambda_0^2}\left(\frac{1}{3}+\frac{5}{6}\Theta\right)n_1 \quad \Rightarrow \quad \lambda_0^2 = l_z^2\left(\frac{1}{3}+\frac{5}{6}\Theta\right). \tag{4.2.355}$$

Analyzing the next order of perturbation, from the momentum and continuity equations we have

$$v_{2x} = \frac{\lambda_0}{\Omega_{p_i}}\partial_\xi v_{1y}, \tag{4.2.356}$$

$$v_{2y} = -\frac{\lambda_0}{\Omega_{p_i}}\partial_\xi v_{1x}, \tag{4.2.357}$$

$$v_{1x} = -\frac{l_y}{l_x}v_{1y}, \tag{4.2.358}$$

and the Poisson equation result can be presented as

$$\partial_\xi^3\phi_1 - 3\phi_1\partial_\xi\phi_1 = 3\partial_\xi\phi_2 - \partial_\xi n_2, \tag{4.2.359}$$

while in the next order of ϵ we have for the z-component of the momentum equation

$$\begin{aligned}\partial_\xi v_{2z} = \frac{1}{\lambda_0}\Bigg(3\frac{\lambda_0}{l_z}\partial_\tau\phi_1 + 9\frac{\lambda_0^2}{l_z}\phi_1\partial_\xi\phi_1 + l_z\partial_\xi\phi_2 \\ + \frac{5}{6}\Theta l_z\partial_\xi n_2 - \frac{5}{2}\Theta l_z\phi_1\partial_\xi\phi_1\Bigg),\end{aligned} \tag{4.2.360}$$

and the continuity equation gives

$$-\lambda_0\partial_\xi n_2 + 3\partial_\tau\phi_1 + l_x\partial_\xi v_{2x} + l_y\partial_\xi v_{2y} + 18\lambda_0\phi_1\partial_\xi\phi_1 + l_z\partial_\xi v_{2z} = 0. \tag{4.2.361}$$

Therefore, applying the results obtained above, we can write from Eq. (4.2.361)

$$\begin{aligned}&-\lambda_0\partial_\xi n_2 + 3\partial_\tau\phi_1 + \lambda_0\frac{(l_x^2+l_y^2)}{\Omega_{p_i}^2}\left(1+\frac{5}{2}\Theta\right)\partial_\xi^3\phi_1 \\ &\qquad + 18\lambda_0\phi_1\partial_\xi\phi_1 + l_z\partial_\xi v_{2z} = 0 \\ &-\lambda_0\partial_\xi n_2 + 3\partial_\tau\phi_1 + 3\frac{\lambda_0^3}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\partial_\xi^3\phi_1 \\ &\qquad + 18\lambda_0\phi_1\partial_\xi\phi_1 + l_z\partial_\xi v_{2z} = 0 \\ &-\lambda_0\partial_\xi n_2 + 6\partial_\tau\phi_1 + 3\frac{\lambda_0^3}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\partial_\xi^3\phi_1 \\ &\qquad + 3\frac{l_z^2}{\lambda_0}\left(9\frac{\lambda_0^2}{l_z^2}-\frac{5}{6}\Theta\right)\phi_1\partial_\xi\phi_1 + \frac{5}{6}\frac{\Theta}{\lambda_0}l_z^2\partial_\xi n_2 + \frac{l_z^2}{\lambda_0}\partial_\xi\phi_2 = 0,\end{aligned}$$

(4.2.362)

since we can express $\partial_\xi v_{2z}$ from Eq. (4.2.360), and then

$$\begin{aligned}
&6\partial_\tau\phi_1 + \frac{l_z^2}{\lambda_0}\left(\frac{5}{6}\Theta - \frac{\lambda_0^2}{l_z^2}\right)\partial_\xi n_2 + 3\frac{\lambda_0^3}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\partial_\xi^3\phi_1 \\
&\qquad + 3\frac{l_z^2}{\lambda_0}\left(3+\frac{20}{3}\Theta\right)\phi_1\partial_\xi\phi_1 + \frac{l_z^2}{\lambda_0}\partial_\xi\phi_2 = 0 \\
&6\partial_\tau\phi_1 + \frac{l_z^2}{3\lambda_0}\Big(3\partial_\xi\phi_2 - \partial_\xi n_2\Big) + 3\frac{\lambda_0^3}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\partial_\xi^3\phi_1 \\
&\qquad + 3\frac{l_z^2}{\lambda_0}\left(3+\frac{20}{3}\Theta\right)\phi_1\partial_\xi\phi_1 = 0 \\
&6\partial_\tau\phi_1 + \frac{1}{3\lambda_0}\left(l_z^2 + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\right)\partial_\xi^3\phi_1 \\
&\qquad + \frac{l_z^2}{\lambda_0}\left(20\Theta+8\right)\phi_1\partial_\xi\phi_1 = 0,
\end{aligned} \tag{4.2.363}$$

which implies that for an adiabatic magnetized TF plasma we have

$$\partial_\tau\phi_1 + \frac{2}{3}\frac{l_z^2}{\lambda_0}\left(5\Theta+2\right)\phi_1\partial_\xi\phi_1 + \frac{1}{18\lambda_0}\left(l_z^2 + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\right)\partial_\xi^3\phi_1 = 0, \tag{4.2.364}$$

or yet, using Eq. (4.2.355) again,

$$\partial_\tau\phi_1 + 4\lambda_0\phi_1\partial_\xi\phi_1 + \frac{1}{18\lambda_0}\left(l_z^2 + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\right)\partial_\xi^3\phi_1 = 0. \tag{4.2.365}$$

Eq. (4.2.365) is also a KdV-like equation, so

$$\begin{aligned}
&\partial_\tau\phi_1 + A\phi_1\partial_\xi\phi_1 + B\partial_\xi^3\phi_1 = 0 \quad \text{with} \\
&A = 4\lambda_0,\ \ B = \frac{1}{18\lambda_0}\left(l_z^2 + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\right),\ \ \lambda_0^2 = l_z^2\left(\frac{1}{3}+\frac{5}{6}\Theta\right).
\end{aligned} \tag{4.2.366}$$

4.2.3 *Nonlinear Warm Plasma Normalization*

In the previous analysis, we have considered, for convenience, the variables transformations presented in Eq. (4.2.34), where the position and velocity quantities are weighted by a velocity factor,

$$x_j \to \lambda_e x_j = \frac{V_0}{\omega_{p_i}} x_j; \quad v_j \to \lambda_e \omega_{p_i} v = V_0 v; \quad V_0 = \frac{v_0}{M} = \left(2\frac{k_B T_F}{m_i}\right)^{\frac{1}{2}}, \tag{4.2.367}$$

obtained from the cold TF plasma study. Although this is a lower limit for the warm plasma configuration, taking $T_i \to 0$, the natural scales of the these systems are characterized by the specific acoustic speed in the medium. We will, therefore, revisit the nonlinear regime of warm plasmas.

To follow the investigation, assuming the same steps as in the unmagnetized cold case, we need to write the ion density for the thermal configurations as a function of the potential and then use the Poisson equation to obtain, under specific considerations, the natural scales of warm cases. However, it is possible to express the isothermal ion density, ${}^{Iso}n_i(\phi)$, with the support of the *Lambert W function* [13], while the adiabatic one, ${}^{Adb}n_i(\phi)$, cannot be written compactly from the usual approach. So, we will present here, in first consideration, an alternative proposal.

[13]:On the Lambert W Function

4.2.3.1 *Structure Model for Natural Scales*

Initially, all cases under consideration have the cold plasma as their lower limit. Supported by previous results, Eq. (4.2.88), with Eq. (4.2.35), and using the configuration linear regime characteristic ion sound speed, we can verify from the cold fluid

$$c_s \equiv v_0 = M\lambda_e \omega_{p_i}. \tag{4.2.368}$$

We then propose to describe the other systems in the same terms, being, for instance,

$${}^{Iso}v_0 = {}^{Iso}\lambda_e \omega_{p_i} M; \tag{4.2.369}$$
$${}^{Adb}v_0 = {}^{Adb}\lambda_e \omega_{p_i} M, \tag{4.2.370}$$

for the isothermal and adiabatic plasmas, respectively, where we adopt the ions plasma frequency common to the fluids, the same critical Mach number, and we assume that the entire thermal characteristic of the plasma is carried by the natural length scale of the systems, ensuring the reduction to the cold case with the maintenance of M and the explicit thermal dependence in the characterization.

From the linear regime discussion of the isothermal gas, we have the ion sound speed presented in Eq. (4.2.277), so

$$c_{SI} = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{k_B T_i}{m_i}\right)^{\frac{1}{2}} \equiv {}^{Iso}v_0, \tag{4.2.371}$$

definition which allows us to write

$$
\begin{aligned}
{}^{\mathrm{Iso}}\lambda_e &= \frac{{}^{\mathrm{Iso}}v_0}{M\omega_{p_i}} && (4.2.372)\\
&= \sqrt{3}\left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{k_B T_i}{m_i}\right)^{\frac{1}{2}}\left(\frac{\epsilon_0 m_i}{n_0 e^2}\right)^{\frac{1}{2}} && (4.2.373)\\
&= \left(2\frac{\epsilon_0 k_B T_F}{n_0 e^2} + 3\frac{\epsilon_0 k_B T_i}{n_0 e^2}\right)^{\frac{1}{2}}. && (4.2.374)
\end{aligned}
$$

We know, however, the dimensional validity of

$$
\lambda_e = \left(\frac{\epsilon_0}{n_0 e}\right)^{\frac{1}{2}} \varphi^{\frac{1}{2}}, \tag{4.2.375}
$$

and then we can write the potential natural scale as

$$
\begin{aligned}
{}^{\mathrm{Iso}}\varphi &= {}^{\mathrm{Iso}}\lambda_e^2 \frac{n_0 e}{\epsilon_0} && (4.2.376)\\
&= \left[2\frac{\epsilon_0 k_B T_F}{n_0 e^2} + 3\frac{\epsilon_0 k_B T_i}{n_0 e^2}\right]\frac{n_0 e}{\epsilon_0} && (4.2.377)\\
&= 2\frac{k_B TF}{e} + 3\frac{k_B T_i}{e}, && (4.2.378)
\end{aligned}
$$

where we have the reduction to the cold case in the limit of $T_i \to 0$. Condensing,

$$
\begin{aligned}
{}^{\mathrm{Iso}}\varphi &= 2\frac{k_B TF}{e} + 3\frac{k_B T_i}{e}; && (4.2.379)\\
{}^{\mathrm{Iso}}\lambda_e &= \left(2\frac{\epsilon_0 k_B T_F}{n_0 e^2} + 3\frac{\epsilon_0 k_B T_i}{n_0 e^2}\right)^{\frac{1}{2}}; && (4.2.380)\\
{}^{\mathrm{Iso}}v_0 &= \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{k_B T_i}{m_i}\right)^{\frac{1}{2}}; && (4.2.381)\\
M &= \left(\frac{1}{3}\right)^{\frac{1}{2}}. && (4.2.382)
\end{aligned}
$$

For the adiabatic system, from Eq. (4.2.332) we define

$$
c_{SA} = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{5}{3}\frac{k_B T_i}{m_i}\right)^{\frac{1}{2}} \equiv {}^{\mathrm{Adb}}v_0, \tag{4.2.383}
$$

which enables us to write as of Eq. (4.2.370)

$$
{}^{\mathrm{Adb}}\lambda_e = \left(2\frac{\epsilon_0 k_B T_F}{n_0 e^2} + 5\frac{\epsilon_0 k_B T_i}{n_0 e^2}\right)^{\frac{1}{2}}, \tag{4.2.384}
$$

and under the argument of Eq. (4.2.375), we get

$$^{\mathrm{Adb}}\varphi = 2\frac{k_B T_F}{e} + 5\frac{k_B T_i}{e}. \tag{4.2.385}$$

So, summarizing the results

$$^{\mathrm{Adb}}\varphi = 2\frac{k_B T_F}{e} + 5\frac{k_B T_i}{e}; \tag{4.2.386}$$

$$^{\mathrm{Adb}}\lambda_e = \left(2\frac{\epsilon_0 k_B T_F}{n_0 e^2} + 5\frac{\epsilon_0 k_B T_i}{n_0 e^2}\right)^{\frac{1}{2}}; \tag{4.2.387}$$

$$^{\mathrm{Adb}}v_0 = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{5}{3}\frac{k_B T_i}{m_i}\right)^{\frac{1}{2}}; \tag{4.2.388}$$

$$M = \left(\frac{1}{3}\right)^{\frac{1}{2}}. \tag{4.2.389}$$

The considerations mentioned above are based on the fixed Mach number condition. In addition to having the limit $T_i \to 0$ delivering the expected result, the study of the unmagnetized warm configurations from the cold natural scales returns, as verified in Eqs. (4.2.153) and (4.2.188),

$$^{\mathrm{Iso}}M = \left[\frac{1}{3}\left(1 + \frac{3}{2}\Theta\right)\right]^{\frac{1}{2}} = \left(\frac{1}{3}\right)^{\frac{1}{2}} \delta^{\frac{1}{2}}; \tag{4.2.390}$$

$$^{\mathrm{Adb}}M = \left[\frac{1}{3}\left(1 + \frac{5}{2}\Theta\right)\right]^{\frac{1}{2}} = \left(\frac{1}{3}\right)^{\frac{1}{2}} \beta^{\frac{1}{2}}, \tag{4.2.391}$$

identifying $\lambda_0 = M$ and introducing the auxiliary variables

$$\delta = 1 + \frac{3}{2}\Theta; \tag{4.2.392}$$

$$\beta = 1 + \frac{5}{2}\Theta. \tag{4.2.393}$$

Looking carefully at the velocity expressions, we have

$$^{\mathrm{Iso}}v_0 = \left(\frac{2}{3}\frac{k_B T_F}{m_i}\right)^{\frac{1}{2}} \left(1 + \frac{3}{2}\Theta\right)^{\frac{1}{2}} = v_0 \delta^{\frac{1}{2}}; \tag{4.2.394}$$

$$^{\mathrm{Adb}}v_0 = \left(\frac{2}{3}\frac{k_B T_F}{m_i}\right)^{\frac{1}{2}} \left(1 + \frac{5}{2}\Theta\right)^{\frac{1}{2}} = v_0 \beta^{\frac{1}{2}}, \tag{4.2.395}$$

where the velocity parameter without superscription is the cold one, and the additional terms, δ and β, ponder the normalization of the fluid governing equations when we consider the ion sound speed of the cold and warm plasmas. Thus, the normalization correction adjusts the critical value of M to

$$M = \left(\frac{1}{3}\right)^{\frac{1}{2}} \tag{4.2.396}$$

in both cases. The phenomenological viability of the obtained results can be verified later.

The previous approach can be naturally extended to magnetized cases by taking the cold configuration as a starting point. As we can see from Eq. (4.2.245), the critical M, in this case, is given by

$$_{\vec{B}}M = l_z \left(\frac{1}{3}\right)^{\frac{1}{2}}, \tag{4.2.397}$$

and acting in a similar way as described above, we can write Eqs. (4.2.300) and (4.2.355) as

$$^{Iso}_{\vec{B}}M = l_z \left[\frac{1}{3}\left(1 + \frac{3}{2}\Theta\right)\right]^{\frac{1}{2}}; \tag{4.2.398}$$

$$^{Adb}_{\vec{B}}M = l_z \left[\frac{1}{3}\left(1 + \frac{5}{2}\Theta\right)\right]^{\frac{1}{2}}, \tag{4.2.399}$$

and the same auxiliary variables δ and β are responsible for the reduction to Eq. (4.2.397).

Finally, before starting the case-by-case renormalization, we can look at the continuity equation since its terms do not change under the consideration of different normalizations. From the continuity equation, considering a general system sound velocity, c_G,

$$\omega_{p_i} n_0 \left[\partial_t n + \vec{\nabla} \cdot (n\vec{v})\right] = 0, \tag{4.2.400}$$

and then we recover to the generalized dimensionless form

$$\partial_t n + \vec{\nabla} \cdot (n\vec{v}) = 0, \tag{4.2.401}$$

which will be considered in the next sections when mentioned.

4.2.3.2 *Unmagnetized Isothermal Plasma*

As we have verified in Sec. 4.2.2.4, the linear regime returns, in this limit, the configuration ion sound speed presented in Eq. (4.2.138), allowing us to write

$$c_{SI} = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{k_B T_i}{m_i}\right)^{\frac{1}{2}} = c_S\left(1 + \frac{3}{2}\Theta\right)^{\frac{1}{2}}, \qquad (4.2.402)$$

and since we have identified $c_{SI} = {}^{Iso}v_0$, it is interesting to rewrite from the variables transformations, Eq. (4.2.34), the normalization factor

$$\begin{aligned} {}^{Iso}V_0 \equiv \frac{{}^{Iso}v_0}{M} &= \left(2\frac{k_B T_F}{m_i} + 3\frac{k_B T_i}{m_i}\right)^{\frac{1}{2}} && (4.2.403)\\ &= \left(2\frac{k_B T_F}{m_i}\right)^{\frac{1}{2}}\left(1 + \frac{3}{2}\Theta\right)^{\frac{1}{2}} && (4.2.404)\\ &= \frac{v_0}{M}\delta^{\frac{1}{2}} && (4.2.405)\\ &= V_0\delta^{\frac{1}{2}}, && (4.2.406) \end{aligned}$$

with the support of the cold parameter

$$V_0 \equiv \frac{v_0}{M}. \qquad (4.2.407)$$

So, here we will mostly consider the transformations constructed before, although

$$x_j \to \frac{{}^{Iso}V_0}{\omega_{p_i}}x_j, \quad v_j \to {}^{Iso}V_0 v \quad \text{with} \quad \partial_{x_j} \to \frac{\omega_{p_i}}{{}^{Iso}V_0}\partial_{x_j}. \qquad (4.2.408)$$

We can then write the fluid momentum equation, Eq. (4.2.128), in the dimensionless form

$$\partial_t\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\frac{1}{\delta}\vec{\nabla}\phi - \frac{1}{2\delta}\frac{\Theta}{n}\vec{\nabla}n, \qquad (4.2.409)$$

where we identify the parameter δ

$$\delta = 1 + \frac{3}{2}\Theta, \qquad (4.2.410)$$

and the Poisson equation, Eq. (4.2.130), becomes expressed as

$$\nabla^2\phi = -\delta\left(n - n_e\right). \qquad (4.2.411)$$

Transforming now to the stretched coordinates space, Eq. (4.2.90), we obtain for Eqs. (4.2.401), (4.2.409), and (4.2.411), respectively,

$$-\epsilon^{\frac{1}{2}}\lambda_0\partial_\xi n+\epsilon^{\frac{3}{2}}\partial_\tau n+\epsilon^{\frac{1}{2}}\left[\sum_{k=1}^{3} l_k\partial_\xi\left(n v_k\right)\right]=0, \tag{4.2.412}$$

$$\begin{aligned}-\epsilon^{\frac{1}{2}}\lambda_0\partial_\xi v_j+\epsilon^{\frac{3}{2}}\partial_\tau v_j+\epsilon^{\frac{1}{2}}\left(\sum_{k=1}^{3} v_k l_k\right)\partial_\xi v_j&\\ =-\frac{1}{\delta}l_j\epsilon^{\frac{1}{2}}\partial_\xi\phi-\frac{1}{2\delta}\frac{\Theta}{n}l_j\epsilon^{\frac{1}{2}}\partial_\xi n,&\end{aligned} \tag{4.2.413}$$

$$\epsilon\partial_\xi^2\phi=-\delta\left(n-n_e\right), \tag{4.2.414}$$

and since we are in the unmagnetized limit, the variables can be expanded following Eqs. (4.2.92) - (4.2.94), which application to Eqs. (4.2.412) - (4.2.414) implies for the lowest order of ϵ

$$-\lambda_0\partial_\xi n_1+l_x\partial_\xi v_{1x}+l_y\partial_\xi v_{1y}+l_z\partial_\xi v_{1z}=0, \tag{4.2.415}$$

$$\lambda_0\partial_{xi}v_{1j}-\frac{1}{\delta}l_j\partial_\xi\phi_1-\frac{1}{2\delta}\Theta l_j\partial_\xi n_1=0, \tag{4.2.416}$$

$$-\delta\left(n_1-3\phi_1\right)=0, \tag{4.2.417}$$

allowing us to conclude directly from Eq. (4.2.417) that

$$n_1=3\phi_1. \tag{4.2.418}$$

It is then possible to write from Eqs. (4.2.416) and (4.2.418),

$$l_j\partial_\xi v_{1j}=\frac{l_j^2}{\lambda_0}\partial_\xi\phi_1, \tag{4.2.419}$$

and the combination of Eqs. (4.2.415) and (4.2.419) gives

$$-3\lambda_0\partial_\xi\phi_1+\frac{1}{\lambda_0}\partial_\xi\phi_1=0 \quad\Rightarrow\quad \lambda_0^2=\frac{1}{3}. \tag{4.2.420}$$

We can also conclude from Eqs. (4.2.418) and (4.2.419) that

$$v_{1j}=\frac{l_j}{3\lambda_0}n_1. \tag{4.2.421}$$

By the analysis of the next perturbation order for the governing equations, we get

$$-\lambda_0\partial_\xi n_2 + 3\partial_\tau\phi_1 + \frac{6}{\lambda_0}\phi_1\partial_\xi\phi_1 + \left(\sum_{k=1}^{3} l_k\partial_\xi v_{2k}\right) = 0, \tag{4.2.422}$$

$$\begin{aligned} -\lambda_0\partial_\xi v_{2j} + \partial_\tau v_{1j} + \left(\sum_{k=1}^{3} v_{1k}l_k\right)\partial_\xi v_{1j} \\ + \frac{1}{\delta}l_j\partial_\xi\phi_2 + \frac{\Theta}{2\delta}l_j\partial_\xi n_2 - \frac{\Theta}{2\delta}l_j n_1\partial_\xi n_1 = 0, \end{aligned} \tag{4.2.423}$$

$$\partial_\xi^3\phi_1 - 3\delta\phi_1\partial_\xi\phi_1 = 3\delta\partial_\xi\phi_2 - \delta\partial_\xi n_2, \tag{4.2.424}$$

and so, associating Eqs. (4.2.422) - (4.2.424),

$$\begin{aligned} &-\lambda_0\partial_\xi n_2 + 6\partial_\tau\phi_1 + \frac{9}{\lambda_0}\phi_1\partial_\xi\phi_1 + \frac{1}{\lambda_0\delta}\partial_\xi\phi_2 \\ &\qquad + \frac{1}{2}\frac{\Theta}{\lambda_0\delta}\partial_\xi n_2 - \frac{9}{2}\frac{\Theta}{\lambda_0\delta}\phi_1\partial_\xi\phi_1 = 0 \\ &\frac{1}{\lambda_0\delta}\left(\frac{1}{2}\Theta - \lambda_0^2\delta\right)\partial_\xi n_2 + 6\partial_\tau\phi_1 + \frac{9}{\lambda_0\delta}\left(\delta - \frac{1}{2}\Theta\right)\phi_1\partial_\xi\phi_1 \\ &\qquad + \frac{1}{\lambda_0\delta}\partial_\xi\phi_2 = 0 \\ &\frac{1}{3\lambda_0\delta}\left(3\partial_\xi\phi_2 - \partial_\xi n_2\right) + 6\partial_\tau\phi_1 + \frac{9}{\lambda_0\delta}\left(\Theta+1\right)\phi_1\partial_\xi\phi_1 = 0 \\ &\frac{1}{3}\frac{1}{\lambda_0\delta^2}\left(\partial_\xi^3\phi_1 - 3\delta\phi_1\partial_\xi\phi_1\right) + 6\partial_\tau\phi_1 + \frac{9}{\lambda_0\delta}\left(\Theta+1\right)\phi_1\partial_\xi\phi_1 = 0, \end{aligned} \tag{4.2.425}$$

where we have used also some results obtained from the study of the lowest order terms. Then, we are able to write from Eq. (4.2.425)

$$6\partial_\tau\phi_1 + \frac{1}{3}\frac{1}{\lambda_0\delta^2}\partial_\xi^3\phi_1 + \frac{1}{\lambda_0\delta}\left(9\Theta+8\right)\phi_1\partial_\xi\phi_1 = 0, \tag{4.2.426}$$

or yet, manipulating Eq. (4.2.426),

$$\partial_\tau\phi_1 + \frac{1}{3\lambda_0}\frac{(9\Theta+8)}{(3\Theta+2)}\phi_1\partial_\xi\phi_1 + \frac{2}{9\lambda_0}\frac{1}{(3\Theta+2)^2}\partial_\xi^3\phi_1 = 0. \tag{4.2.427}$$

Thus, revising the normalization factor in the parameters transformations, we got a corrected KdV-like equation for the unmagnetized isothermal TF plasma,

$$\begin{aligned} &\partial_\tau\phi_1 + A\phi_1\partial_\xi\phi_1 + B\partial_\xi^3\phi_1 = 0 \quad \text{with} \\ &\quad A = \frac{1}{3\lambda_0}\frac{(9\Theta+8)}{(3\Theta+2)}, \quad B = \frac{2}{9\lambda_0}\frac{1}{(3\Theta+2)^2}, \quad \lambda_0 = \frac{1}{\sqrt{3}}. \end{aligned} \tag{4.2.428}$$

4.2.3.3 *Unmagnetized Adiabatic Plasma*

Similarly to the procedure adopted before, since the linear regime ion acoustic speed in an adiabatic TF plasma takes the form

$$c_{SA} = \left(\frac{2}{3}\frac{k_B T_F}{m_i} + \frac{5}{3}\frac{k_B T_i}{m_i}\right)^{\frac{1}{2}} = c_S\left(1 + \frac{5}{2}\Theta\right)^{\frac{1}{2}}, \tag{4.2.429}$$

with the normalization factor assuming the form

$$^{Adb}V_0 = V_0 \beta^{\frac{1}{2}}, \tag{4.2.430}$$

we can transform the fluid momentum and Poisson equations using Eq. (4.2.408), replacing $^{Iso}V_0$ by $^{Adb}V_0$, as

$$\partial_t \vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\frac{1}{\beta}\vec{\nabla}\phi - \frac{5}{6}\frac{\Theta}{n^{\frac{1}{3}}}\frac{1}{\beta}\vec{\nabla}n, \tag{4.2.431}$$

$$\nabla^2\phi = -\beta\left(n - n_e\right), \tag{4.2.432}$$

where we have identified β before as

$$\beta = 1 + \frac{5}{2}\Theta. \tag{4.2.433}$$

So, if we transform to coordinates ξ and τ, following Eq. (4.2.90), proceed with the variables expansion, Eqs. (4.2.92) - (4.2.94), and select the lowest order of the ϵ factor in the dynamic equations, we get to the continuity, momentum, and Poisson equations, relatively,

$$-\lambda_0\partial_\xi n_1 + l_x\partial_\xi v_{1x} + l_y\partial_\xi v_{1y} + l_z\partial_\xi v_{1z} = 0, \tag{4.2.434}$$

$$-\lambda_0\partial_\xi v_{1j} + \frac{1}{\beta}l_j\partial_\xi\phi_1 + \frac{5}{6}\frac{\Theta}{\beta}l_j\partial_\xi n_1 = 0, \tag{4.2.435}$$

$$-\beta\left(n_1 - 3\phi_1\right) = 0, \tag{4.2.436}$$

and once more we have from Eq. (4.2.436)

$$n_1 = 3\phi_1, \tag{4.2.437}$$

while Eq. (4.2.435) returns

$$l_j\partial_\xi v_{1j} = \frac{l_j^2}{3\lambda_0}\partial_\xi n_1, \tag{4.2.438}$$

which, combined with Eq. (4.2.434), yields

$$\lambda_0 \partial_\xi n_1 = \frac{1}{3\lambda_0} \partial_\xi n_1 \quad \Rightarrow \quad \lambda_0^2 = \frac{1}{3}. \tag{4.2.439}$$

To complete, we can write for the first-order perturbed velocity terms

$$v_{1j} = \frac{l_j}{3\lambda_0} n_1. \tag{4.2.440}$$

From the next perturbation order of the governing equations, we have

$$-\lambda_0 \partial_\xi n_2 + 3\partial_\tau \phi_1 + \frac{6}{\lambda_0} \phi_1 \partial_\xi \phi_1 + \left(\sum_{k=1}^{3} l_k \partial_\xi v_{2k} \right) = 0, \tag{4.2.441}$$

$$\begin{aligned} -\lambda_0 \partial_\xi v_{2j} + \partial_\tau v_{1j} + \left(\sum_{k=1}^{3} v_{1k} l_k \right) \partial_\xi v_{1j} + \frac{l_j}{\beta} \partial_\xi \phi_2 & \\ + \frac{5}{6} \frac{\Theta l_j}{\beta} \partial_\xi n_2 - \frac{5}{6} \frac{\Theta l_j}{\beta} \frac{n_1}{3} \partial_\xi n_1 = 0, & \end{aligned} \tag{4.2.442}$$

$$\partial_\xi^3 \phi_1 - 3\beta \phi_1 \partial_\xi \phi_1 = 3\beta \partial_\xi \phi_2 - \beta \partial_\xi n_2, \tag{4.2.443}$$

and as Eq. (4.2.442) allows us to write

$$\begin{aligned} \partial_\xi v_{2j} = \frac{l_j}{\lambda_0^2} \partial_\tau \phi_1 + \frac{3}{\lambda_0} l_j \phi_1 \partial_\xi \phi_1 + \frac{1}{\beta} \frac{l_j}{\lambda_0} \partial_\xi \phi_2 & \\ + \frac{5}{6} \frac{\Theta}{\beta} \frac{l_j}{\lambda_0} \partial_\xi n_2 - \frac{5}{2} \frac{\Theta}{\lambda_0 \beta} l_j \phi_1 \partial_\xi \phi_1, & \end{aligned} \tag{4.2.444}$$

we obtain from Eqs. (4.2.441) and (4.2.444),

$$\begin{aligned} -\lambda_0 \partial_\xi n_2 + 6\partial_\xi \phi_1 + \frac{9}{\lambda_0} \phi_1 \partial_\xi \phi_1 + \frac{1}{\lambda_0 \beta} \partial_\xi \phi_2 & \\ + \frac{5}{6} \frac{\Theta}{\lambda_0 \beta} \partial_\xi n_2 - \frac{5}{2} \frac{\Theta}{\lambda_0 \beta} \phi_1 \partial_\xi \phi_1 = 0. & \end{aligned} \tag{4.2.445}$$

Thus, manipulating Eq. (4.2.445), we get

$$
\begin{aligned}
6\partial_\tau\phi_1 &+ \frac{1}{\lambda_0\beta}\left(\frac{5}{6}\Theta - \lambda_0^2\beta\right)\partial_\xi n_2 + \frac{1}{\lambda_0\beta}\partial_\xi\phi_2 \\
&\qquad\qquad + \frac{3}{\lambda_0\beta}\left(3\beta - \frac{5}{6}\Theta\right)\phi_1\partial_\xi\phi_1 = 0 \\
6\partial_\tau\phi_1 &+ \frac{1}{3\lambda_0\beta}\left(3\partial_\xi\phi_2 - \partial_\xi n_2\right) + \frac{1}{\lambda_0\beta}\left(9 + 20\Theta\right)\phi_1\partial_\xi\phi_1 = 0 \\
6\partial_\tau\phi_1 &+ \frac{4}{3\lambda_0}\frac{1}{(2+5\Theta)^2}\partial_\xi^3\phi_1 + \frac{4}{\lambda_0\beta}\left(2 + 5\Theta\right)\phi_1\partial_\xi\phi_1 = 0,
\end{aligned}
\tag{4.2.446}
$$

where we have used explicitly Eq. (4.2.443), and so we can write

$$
6\partial_\tau\phi_1 + \frac{4}{3\lambda_0}\frac{1}{(2+5\Theta)^2}\partial_\xi^3\phi_1 + \frac{8}{\lambda_0}\phi_1\partial_\xi\phi_1 = 0, \tag{4.2.447}
$$

being straight that we can rewrite Eq. (4.2.447) as

$$
\partial_\tau\phi_1 + \frac{4}{3\lambda_0}\phi_1\partial_\xi\phi_1 + \frac{2}{9\lambda_0}\frac{1}{(2+5\Theta)^2}\partial_\xi^3\phi_1 = 0. \tag{4.2.448}
$$

Eq. (4.2.448) is a corrected KdV-like equation for the unmagnetized adiabatic TF fluid,

$$
\begin{aligned}
&\partial_\tau\phi_1 + A\phi_1\partial_\xi\phi_1 + B\partial_\xi^3\phi_1 = 0 \quad \text{with} \\
&\qquad A = \frac{4}{3\lambda_0}, \quad B = \frac{2}{9\lambda_0}\frac{1}{(2+5\Theta)^2}, \quad \lambda_0 = \frac{1}{\sqrt{3}}.
\end{aligned}
\tag{4.2.449}
$$

4.2.3.4 *Magnetized Isothermal Plasma*

We have concluded earlier that pure ion acoustic waves in magnetized TF plasma have the same dispersion relation as if we consider the same configuration for an unmagnetized TF fluid. Knowing this, we will take here the c_{SI} ion sound speed, as specified in Eq. (4.2.402), and the modifications in the variables transformations as shown by Eq. (4.2.408). It is interesting to remember that in the presence of an external magnetic field, the quantities expansion follows Eqs. (4.2.231) - (4.2.234).

Let us start with the dimensionless momentum equation. From the previous analysis, it is straightforward that

$$
\partial_t\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\frac{1}{\delta}\vec{\nabla}\phi - \frac{1}{2\delta}\frac{\Theta}{n}\vec{\nabla}n + \Omega_{p_i}\left(\vec{v}\times\hat{z}\right), \tag{4.2.450}
$$

where δ and Ω_{p_i} are given by

$$\delta = 1 + \frac{3}{2}\Theta, \tag{4.2.451}$$

$$\Omega_{p_i} = \frac{\Omega}{\omega_{p_i}}, \tag{4.2.452}$$

Ω and ω_{p_i} written as presented in Eqs. (3.2.27) and (3.3.9). So, we have from Eq. (4.2.450), when applied the stretched coordinates transformation,

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_x + \epsilon^{\frac{3}{2}} \partial_\tau v_x + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} v_k l_k \right) \partial_\xi v_x + \frac{1}{\delta} \epsilon^{\frac{1}{2}} l_x \partial_\xi \phi + \frac{1}{2} \frac{\Theta}{n} \frac{l_x}{\delta} \epsilon^{\frac{1}{2}} \partial_\xi n - \Omega_{p_i} v_y = 0, \tag{4.2.453}$$

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_y + \epsilon^{\frac{3}{2}} \partial_\tau v_y + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} v_k l_k \right) \partial_\xi v_y + \frac{1}{\delta} \epsilon^{\frac{1}{2}} l_y \partial_\xi \phi + \frac{1}{2} \frac{\Theta}{n} \frac{l_y}{\delta} \epsilon^{\frac{1}{2}} \partial_\xi n + \Omega_{p_i} v_x = 0, \tag{4.2.454}$$

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_z + \epsilon^{\frac{3}{2}} \partial_\tau v_z + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} v_k l_k \right) \partial_\xi v_z + \frac{1}{\delta} \epsilon^{\frac{1}{2}} l_z \partial_\xi \phi + \frac{1}{2} \frac{\Theta}{n} \frac{l_z}{\delta} \epsilon^{\frac{1}{2}} \partial_\xi n = 0, \tag{4.2.455}$$

for each spatial direction. The continuity and Poisson equations, as before, are

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi n + \epsilon^{\frac{3}{2}} \partial_\tau n + \epsilon^{\frac{1}{2}} \left[\sum_{k=1}^{3} l_k \partial_\xi (n v_k) \right] = 0, \tag{4.2.456}$$

$$\epsilon \partial_\xi^2 \phi = -\delta (n - n_e). \tag{4.2.457}$$

We can now expand the fluid parameters about the equilibrium, and for the lowest perturbation order of Eqs. (4.2.453) - (4.2.455), we have

$$-\frac{1}{\delta} l_x \partial_\xi \phi_1 - \frac{1}{2} \frac{\Theta}{\delta} l_x \partial_\xi n_1 + \Omega_{p_i} v_{1y} = 0, \tag{4.2.458}$$

$$-\frac{1}{\delta} l_y \partial_\xi \phi_1 - \frac{1}{2} \frac{\Theta}{\delta} l_y \partial_\xi n_1 - \Omega_{p_i} v_{1x} = 0, \tag{4.2.459}$$

$$-\frac{1}{\delta} l_z \partial_\xi \phi_1 - \frac{1}{2} \frac{\Theta}{\delta} l_z \partial_\xi n_1 + \lambda_0 \partial_\xi v_{1z} = 0, \tag{4.2.460}$$

while Eqs. (4.2.456) and (4.2.457) result, respectively,

$$n_1 = \frac{l_z}{\lambda_0} v_{1z}, \tag{4.2.461}$$

$$n_1 = 3\phi_1. \tag{4.2.462}$$

Thus, Eqs. (4.2.458) - (4.2.460) return to the first-order perturbed velocity terms,

$$v_{1y} = \frac{l_x}{\Omega_{p_i}} \partial_\xi \phi_1, \tag{4.2.463}$$

$$v_{1x} = -\frac{l_y}{\Omega_{p_i}} \partial_\xi \phi_1, \tag{4.2.464}$$

$$v_{1z} = \frac{l_z}{\lambda_0} \phi_1, \tag{4.2.465}$$

and combining Eqs. (4.2.461), (4.2.462) and (4.2.465), we get

$$3\frac{\lambda_0}{l_z} = \frac{l_z}{\lambda_0} \quad \Rightarrow \quad \lambda_0^2 = \frac{l_z^2}{3}. \tag{4.2.466}$$

The next perturbation order terms of the momentum and continuity equations allow us to write for the perturbed x and y-component velocity parameters

$$v_{2x} = \frac{\lambda_0}{\Omega_{p_i}} \partial_\xi v_{1y}, \tag{4.2.467}$$

$$v_{2y} = -\frac{\lambda_0}{\Omega_{p_i}} \partial_\xi v_{1x}, \tag{4.2.468}$$

$$v_{1x} = -\frac{l_y}{l_x} v_{1y}, \tag{4.2.469}$$

while the Poisson equation implies

$$\partial_\xi^3 \phi_1 - 3\delta\phi_1 \partial_\xi \phi_1 = 3\delta \partial_\xi \phi_2 - \delta \partial_\xi n_2, \tag{4.2.470}$$

and analyzing the next higher order of the z-component momentum and continuity relations, we obtain, respectively,

$$-\lambda_0 \partial_\xi v_{2z} + \partial_\tau v_{1z} + l_z v_{1z} \partial_\xi v_{1z} + \frac{1}{\delta} l_z \partial_\xi \phi_2 + \frac{1}{2}\frac{\Theta}{\delta} l_z \partial_\xi n_2 - \frac{1}{2}\frac{\Theta}{\delta} l_z n_1 \partial_\xi n_1 = 0, \tag{4.2.471}$$

$$-\lambda_0 \partial_\xi n_2 + \partial_\tau n_1 + l_x \partial_\xi v_{2x} + l_y \partial_\xi v_{2y} + l_z \partial_\xi (n_1 v_{1z}) + l_z \partial_\xi v_{2z} = 0. \tag{4.2.472}$$

Since Eq. (4.2.471) can be rewritten as

$$\begin{aligned} l_z\partial_\xi v_{2z} &= 3\partial_\xi\phi_1 + 9\lambda_0\phi_1\partial_\xi\phi_1 + \frac{1}{\delta}\frac{l_z^2}{\lambda_0}\partial_\xi\phi_2 \\ &\quad + \frac{\Theta}{2\delta}\frac{l_z^2}{\lambda_0}\partial_\xi n_2 - \frac{9}{2}\frac{\Theta}{\delta}\frac{l_z^2}{\lambda_0}\phi_1\partial_\xi\phi_1, \end{aligned} \tag{4.2.473}$$

we combine the previous results to obtain from Eq. (4.2.472)

$$\begin{aligned} &-\lambda_0\partial_\xi n_2 + 3\partial_\tau\phi_1 + 18\lambda_0\phi_1\partial_\xi\partial_1 + \frac{l_x\lambda_0}{\Omega_{p_i}}\partial_\xi^2 v_{1y} \\ &\qquad - \frac{l_y\lambda_0}{\Omega_{p_i}}\partial_\xi^2 v_{1x} + l_z\partial_\xi v_{2z} = 0 \\ &-\lambda_0\partial_\xi n_2 + 6\partial_\tau\phi_1 + 27\lambda_0\phi_1\partial_\xi\phi_1 + \frac{\lambda_0}{\Omega_{p_i}^2}\left(l_x^2 + l_y^2\right)\partial_\xi^3\phi_1 \\ &\qquad + \frac{1}{\delta}\frac{l_z^2}{\lambda_0}\partial_\xi\phi_2 + \frac{\Theta}{2\delta}\frac{l_z^2}{\lambda_0}\partial_\xi n_2 - \frac{9\Theta}{2\delta}\frac{l_z^2}{\lambda_0}\phi_1\partial_\xi\phi_1 = 0 \\ &\frac{\lambda_0}{\delta}\left(\frac{3}{2}\Theta - \delta\right)\partial_\xi n_2 + 6\partial_\tau\phi_1 + \frac{\lambda_0}{\Omega_{p_i}^2}\left(1 - l_z^2\right)\partial_\xi^3\phi_1 \\ &\qquad + \frac{27\lambda_0}{\delta}\left(\delta - \frac{1}{2}\Theta\right)\phi_1\partial_\xi\phi_1 + \frac{3\lambda_0}{\delta}\partial_\xi\phi_2 = 0 \\ &\frac{\lambda_0}{\delta}\left(3\partial_\xi\phi_2 - \partial_\xi n_2\right) + 6\partial_\tau\phi_1 + \frac{\lambda_0}{\Omega_{p_i}^2}\left(1 - l_z^2\right)\partial_\xi^3\phi_1 \\ &\qquad + \frac{27\lambda_0}{\delta}\left(1 + \Theta\right)\phi_1\partial_\xi\phi_1 = 0, \end{aligned} \tag{4.2.474}$$

and using Eq. (4.2.470), we have

$$6\partial_\tau\phi_1 + \frac{1}{3\lambda_0}\left[\frac{l_z^2}{\delta^2} + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{\left(1 - l_z^2\right)}{l_z^2}\right]\partial_\xi^3\phi_1 + \frac{l_z^2}{\lambda_0\delta}\left(9\Theta + 8\right)\phi_1\partial_\xi\phi_1 = 0, \tag{4.2.475}$$

and finally we get

$$\partial_\tau\phi_1 + \frac{l_z^2}{\lambda_0\delta}\frac{\left(9\Theta + 8\right)}{6}\phi_1\partial_\xi\phi_1 + \frac{1}{18\lambda_0}\left[\frac{l_z^2}{\delta^2} + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{\left(1 - l_z^2\right)}{l_z^2}\right]\partial_\xi^3\phi_1 = 0. \tag{4.2.476}$$

So, we conclude that Eq. (4.2.476) is a corrected KdV-like expression for the magnetized isothermal TF plasma,

$$\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with}$$
$$A = \frac{l_z^2}{\lambda_0 \delta}\frac{(9\Theta + 8)}{6}, \; B = \frac{1}{18\lambda_0}\left[\frac{l_z^2}{\delta^2} + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{\left(1 - l_z^2\right)}{l_z^2}\right], \; \lambda_0^2 = \frac{l_z^2}{3}. \tag{4.2.477}$$

4.2.3.5 *Magnetized Adiabatic Plasma*

To study the adiabatic limit of the magnetized TF plasma, let us consider the dimensionless governing equations

$$\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} + \frac{1}{\beta}\vec{\nabla}\phi + \frac{5}{6}\frac{\Theta}{n^{\frac{1}{3}}}\frac{1}{\beta}\vec{\nabla}n - \Omega_{p_i}\left(\vec{v} \times \hat{z}\right) = 0, \tag{4.2.478}$$
$$\partial_t n + \vec{\nabla} \cdot (n\vec{v}) = 0, \tag{4.2.479}$$
$$\nabla^2 \phi = -\beta\left(n - n_e\right), \tag{4.2.480}$$

where we adopt the transformations stated by Eq. (4.2.34), considering the ion sound speed, c_{SA}, written as Eq. (4.2.429), and β is given by Eq. (4.2.433). If then we introduce the stretched coordinates, ξ and τ, following the Eq. (4.2.90), we obtain from Eqs. (4.2.478) - (4.2.480),

$$-\lambda_0 \epsilon^{\frac{1}{2}}\partial_\xi v_j + \epsilon^{\frac{3}{2}}\partial_\tau v_j + \epsilon^{\frac{1}{2}}\left(\sum_{k=1}^{3} v_k l_k\right)\partial_\xi v_j$$
$$+ \frac{1}{\beta}l_j \epsilon^{\frac{1}{2}}\partial_\xi \phi + \frac{5}{6}\frac{\Theta}{n^{\frac{1}{3}}}\frac{l_j}{\beta}\partial_\xi n - \Omega_{p_i}\left(\vec{v} \times \hat{z}\right)_j = 0, \tag{4.2.481}$$
$$-\lambda_0 \epsilon^{\frac{1}{2}}\partial_\xi n + \epsilon^{\frac{3}{2}}\partial_\tau n + \epsilon^{\frac{1}{2}}\left[\sum_{k=1}^{3} l_k \partial_\xi\left(n v_k\right)\right] = 0, \tag{4.2.482}$$
$$\epsilon \partial_\xi^2 \phi = -\beta\left(n - n_e\right), \tag{4.2.483}$$

and Eq. (4.2.481) returns to each direction

$$\begin{aligned}
&-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_x + \epsilon^{\frac{3}{2}} \partial_\tau v_x + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} v_k l_k \right) \partial_\xi v_x \\
&\qquad = -\frac{1}{\beta} l_x \epsilon^{\frac{1}{2}} \partial_\xi \phi - \frac{5}{6} \frac{\Theta}{n^{\frac{1}{3}}} \frac{l_x}{\beta} \epsilon^{\frac{1}{2}} \partial_\xi n + \Omega_{p_i} v_y,
\end{aligned} \tag{4.2.484}$$

$$\begin{aligned}
&-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_y + \epsilon^{\frac{3}{2}} \partial_\tau v_y + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} v_k l_k \right) \partial_\xi v_y \\
&\qquad = -\frac{1}{\beta} l_y \epsilon^{\frac{1}{2}} \partial_\xi \phi - \frac{5}{6} \frac{\Theta}{n^{\frac{1}{3}}} \frac{l_y}{\beta} \epsilon^{\frac{1}{2}} \partial_\xi n - \Omega_{p_i} v_x,
\end{aligned} \tag{4.2.485}$$

$$\begin{aligned}
&-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_z + \epsilon^{\frac{3}{2}} \partial_\tau v_z + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} v_k l_k \right) \partial_\xi v_z \\
&\qquad = -\frac{1}{\beta} l_z \epsilon^{\frac{1}{2}} \partial_\xi \phi - \frac{5}{6} \frac{\Theta}{n^{\frac{1}{3}}} \frac{l_z}{\beta} \epsilon^{\frac{1}{2}} \partial_\xi n,
\end{aligned} \tag{4.2.486}$$

which after the application of the expansion transformations, Eqs. (4.2.231) - (4.2.234), can be reduced in the lowest perturbation order to

$$v_{1y} = \frac{1}{3} \frac{l_x}{\Omega_{p_i}} \partial_\xi n_1, \tag{4.2.487}$$

$$v_{1x} = -\frac{1}{3} \frac{l_y}{\Omega_{p_i}} \partial_\xi n_1, \tag{4.2.488}$$

$$v_{1z} = \frac{1}{3} \frac{l_z}{\lambda_0} n_1, \tag{4.2.489}$$

while the continuity and Poisson equations, Eqs. (4.2.482) and (4.2.483), return

$$n_1 = \frac{l_z}{\lambda_0} v_{1z}, \tag{4.2.490}$$

$$n_1 = 3\phi_1, \tag{4.2.491}$$

and then we have from Eqs. (4.2.489) - (4.2.491)

$$\frac{\lambda_0}{l_z} = \frac{1}{3} \frac{l_z}{\lambda_0} \quad \Rightarrow \quad \lambda_0^2 = \frac{l_z^2}{3}. \tag{4.2.492}$$

It is also useful to write from Eqs. (4.2.490) and (4.2.491),

$$v_{1z} = 3 \frac{\lambda_0}{l_z} \phi_1. \tag{4.2.493}$$

The next perturbation order of the momentum and continuity equations implies

$$v_{2y} = -\frac{\lambda_0}{\Omega_{p_i}}\partial_\xi v_{1x}, \tag{4.2.494}$$

$$v_{2x} = \frac{\lambda_0}{\Omega_{p_i}}\partial_\xi v_{1y}, \tag{4.2.495}$$

$$v_{1x} = -\frac{l_y}{l_x}v_{1y}, \tag{4.2.496}$$

and the Poisson equation allows us to write

$$\partial_\xi^3\phi_1 - 3\beta\phi_1\partial_\xi\phi_1 = 3\beta\partial_\xi\phi_2 - \beta\partial_\xi n_2. \tag{4.2.497}$$

Analyzing the next higher order of ϵ, we get for the z momentum component

$$\partial_\xi v_{2z} = \frac{3}{l_z}\partial_\tau\phi_1 + 9\frac{\lambda_0}{l_z}\phi_1\partial_\xi\phi_1 + \frac{1}{\beta}\frac{l_z}{\lambda_0}\partial_\xi\phi_2 + \frac{5}{6}\frac{\Theta}{\beta}\frac{l_z}{\lambda_0}\partial_\xi n_2 - \frac{5}{2}\frac{\Theta}{\beta}\frac{l_z}{\lambda_0}\phi_1\partial_\xi\phi_1, \tag{4.2.498}$$

and the continuity equation returns

$$-\lambda_0\partial_\xi n_2 + 3\partial_\tau\phi_1 + l_x\partial_\xi v_{2x} + l_y\partial_\xi v_{2y} + l_z\partial_\xi\left(n_1 v_{1z}\right) + l_z\partial_\xi v_{2z} = 0, \tag{4.2.499}$$

so, associating the previous results, we obtain

$$\begin{aligned}
&-\lambda_0\partial_\xi n_2 + 3\partial_\xi\phi_1 + \frac{l_x^2\lambda_0}{\Omega_{p_i}^2}\partial_\xi^3\phi_1 - \frac{l_y^2\lambda_0}{\Omega_{p_i}^2}\partial_\xi^3\phi_1 \\
&\qquad + 18\lambda_0\phi_1\partial_\xi\phi_1 + l_z\partial_\xi v_{2z} = 0 \\
&\frac{\lambda_0}{\beta}\left(\frac{5}{2}\Theta - \beta\right)\partial_\xi n_2 + 6\partial_\tau\phi_1 + \frac{\lambda_0}{\Omega_{p_i}^2}\left(1 - l_z^2\right)\partial_\xi^3\phi_1 \\
&\qquad + 3\frac{\lambda_0}{\beta}\partial_\xi\phi_2 + \frac{3\lambda_0}{\beta}\left(9\beta - \frac{5}{2}\Theta\right)\phi_1\partial_\xi\phi_1 = 0 \\
&\frac{\lambda_0}{\beta}\left(3\partial_\xi\phi_2 - \partial_\xi n_2\right) + 6\partial_\tau\phi_1 + \frac{\lambda_0}{\Omega_{p_i}^2}\left(1 - l_z^2\right)\partial_\xi^3\phi_1 \\
&\qquad + \frac{3\lambda_0}{\beta}\left(9 + 20\Theta\right)\phi_1\partial_\xi\phi_1 = 0 \\
&\frac{\lambda_0}{\beta}\left(\frac{1}{\beta}\partial_\xi^3\phi_1 - 3\phi_1\partial_\xi\phi_1\right) + 6\partial_\tau\phi_1 + \frac{\lambda_0}{\Omega_{p_i}^2}\left(1 - l_z^2\right)\partial_\xi^3\phi_1 \\
&\qquad + 3\frac{\lambda_0}{\beta}\left(9 + 20\Theta\right)\phi_1\partial_\xi\phi_1 = 0,
\end{aligned}$$

(4.2.500)

which can be rearranged to

$$
\begin{aligned}
6\partial_\tau\phi_1 + \frac{1}{3\lambda_0}\left[\frac{l_z^2}{\beta^2} + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{\left(1-l_z^2\right)}{l_z^2}\right]\partial_\xi^3\phi_1 \\
+ \frac{12}{3}\frac{l_z^2}{\lambda_0\beta}\left(2+5\Theta\right)\phi_1\partial_\xi\phi_1 = 0,
\end{aligned}
\tag{4.2.501}
$$

or simplifying, we get

$$
\begin{aligned}
\partial_\tau\phi_1 + \frac{2}{3}\frac{l_z^2}{\lambda_0\beta}\left(2+5\Theta\right)\phi_1\partial_\xi\phi_1 \\
+ \frac{1}{18\lambda_0}\left[\frac{l_z^2}{\beta^2} + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{\left(1-l_z^2\right)}{l_z^2}\right]\partial_\xi^3\phi_1 = 0.
\end{aligned}
\tag{4.2.502}
$$

Thus, Eq. (4.2.502) is a KdV-like equation considering the corrected ion acoustic speed normalization for a magnetized adiabatic TF fluid, being

$$
\begin{aligned}
&\partial_\tau\phi_1 + A\phi_1\partial_\xi\phi_1 + B\partial_\xi^3\phi_1 = 0 \quad \text{with} \\
&A = \frac{2}{3}\frac{l_z^2}{\lambda_0\beta}\left(2+5\Theta\right), \quad B = \frac{1}{18\lambda_0}\left[\frac{l_z^2}{\beta^2} + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{\left(1-l_z^2\right)}{l_z^2}\right], \quad \lambda_0^2 = \frac{l_z^2}{3}.
\end{aligned}
\tag{4.2.503}
$$

5 RESULTS AND ANALYSIS

Some results can be made clearer from a visual representation. In this chapter, therefore, we adopt a numerical treatment of the system aiming to show the behavior of each profile investigated, allowing a more direct comparison of the original results, whatever the referred normalizations. Finally, we explicitly compute the transition between the different scales, going from the cold-normalized KdV equations to the respective warm-normalized ones, only with the knowledge of λ_0, characterizing it as a control parameter. We use the terms *height* and *breadth* as synonyms of *amplitude* and *width*, respectively, and the color bar is an amplitude indicator at rates of 0.10 (0.05) when the size is greater (smaller) than 0.50.

5.1 RESULTS AND DISCUSSIONS

In Chap. 4, we have presented a complete deduction of ion acoustic solitary waves considering thermal and magnetic effects in a two-component plasma with classical ions and degenerate electrons. Our purpose now is to analyze the previous results. To infer the properties of ion sound waves, we numerically solve the KdV-like expressions, and some consequences are displayed in the next subsections.

From a brief review of the fundamental concepts on plasma waves theory, after applying the *Thomas-Fermi approximation*, we got the parameters that characterize the configuration. Initially, from the consideration of cold, fixed, ions and oscillating electrons, we have determined the *plasma frequency*, ω_p, in which the reflected concepts enable us to identify the *ion plasma frequency*, ω_{p_i} [12]. Likewise, we were able to define the *linear Thomas-Fermi screening length* by studying the linear ion waves limit, and we show general concepts involved in the unmagnetized and magnetized TF fluid regimes, constituting a strong background for the developments made from the analysis of nonlinear ion waves.

[12]: Introduction to Plasma Physics and Controlled Fusion

It is important to note here that the dimensionless variables transformations, represented by Eq. (4.2.34), are a starting point for several results and therefore play a key role in our theory. Unlike what is commonly seen in the literature, all parameters are associated with the system's natural scales, so that the proposed variables changes constitute the only possible set within the validity range. This was possible due to the determination of natural scales from the variational method adopted [40].

[40]: Description limit for soliton waves due to critical scaling of electrostatic potential

5.1.1 *Modified RP Method*

Another important notion introduced in this work is the suggested modification to the traditional *Reductive Perturbation method*, commonly referenced to Washimi and Taniuti [43]. In addition to the introduction of the directional cosines of the wave vector, we allocate a free parameter, λ_0, in our stretched coordinates definition that, later, presented a control character and which will be highlighted shortly.

[43]: Reductive Perturbation Method in Nonlinear Wave Propagation. I

Let us now investigate the numerical results, such as the temporal evolution, amplitude curves, and the solutions shape at a fixed time for certain values of the independent variables, of the diverse plasma configurations contemplated, arising from the combination of the system's magnetization status with thermal effects in the characterization of ions, under normalization weightings.

5.1.2 *Non-Magnetized Cold Ions*

The *unmagnetized cold TF plasma* can be used as the lower limit of all configurations inserted in this project. This is so the motivation to adopt the variables transformations, Eq. (4.2.34), normalized to the cold parameters as a standard implication initially.

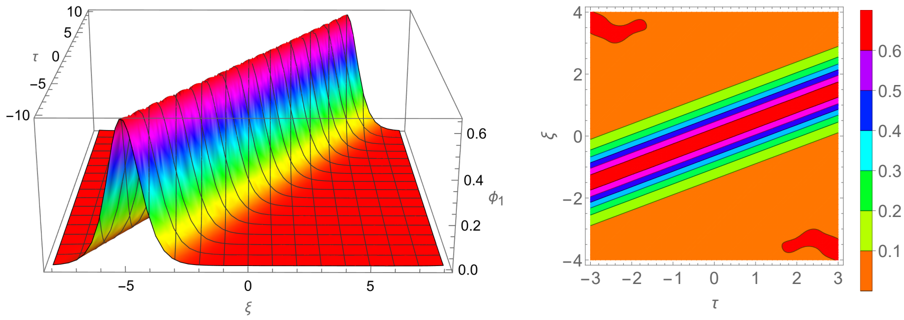
For that limit, we have assumed the moving frame approach to determine the *critical Mach number*, M, in which we can verify the propagation of traveling waves. Admitting the dependence of the variables on a single one, we have as a system property

$$M = \left(\frac{1}{3}\right)^{\frac{1}{2}}. \tag{5.1.1}$$

The investigation of the dynamics starts from the *redefined* stretched variables, Eq. (4.2.90), by the introduction of the *control parameter*, λ_0. We have considered purely the nonlinear regime using relations obtained previously, as well as the dimensionless normalization of the variables and the stretched coordinates, and knowing the scale factors, defining the coordinates of the reference frame, and expanding the parameters up to the second-order, Eqs. (4.2.92) - (4.2.94), it is possible to deduce, from the hydrodynamic equations, the *Korteweg-de Vries* relation assuming the *TF* regime as

$$\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with} \quad A = \frac{4}{3\lambda_0}, \; B = \frac{1}{18\lambda_0}, \; \lambda_0 = \frac{1}{\sqrt{3}}. \tag{5.1.2}$$

Factors A and B, which represent the nonlinear and dispersive terms, respectively, were strategically written in terms of λ_0, despite its known value, to evidence the explicit dependence of the factors on the introduced quantity, which will be useful in further analysis. Solutions to KdV-like equations are well-known, and a simple derivation is presented in Appendix B. We are able then to study the method solutions.



(a) Time Evolution (b) Contour Plot

Figure 5.1.1: Time evolution (a) and amplitude contour (b) of the electrostatic potential in terms of the stretched coordinates, ξ and τ, to cold ions. System $V_0 -$normalized.

First, using the *Numerical Method of Lines* [41], we solve the resulting partial differential equation to get a description of the configuration time evolution, as depicted in Fig. 5.1.1a. After that, from the result, we also graph a *contour plot* of the solution to see in a 2−dimensional format the amplitude lines introduced in the 3−dimensional surface plot, as shown in Fig. 5.1.1b. Finally, to verify the validity of our implementation, we plot the numerical and analytical solutions in the same perspective, setting $\tau = 0$, as we can see in Fig. 5.1.2. For all these results, we choose $\mathcal{U} = 0.5$.

[41]:Mathematica Tutorial: Advanced Numerical Differential Equation Solving

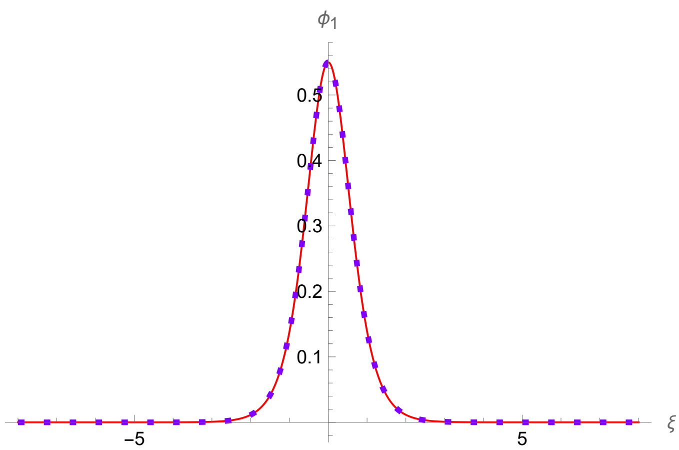


Figure 5.1.2: Numerical (solid, red) and analytical (dashed, blue) solutions for the electrostatic potential in terms of the stretched coordinates, ξ and τ, to cold ions in an unmagnetized system $V_0 -$normalized.

To complete, Fig. 5.1.3 graphs the solutions shape to the system KdV equation considering three distinct values for the frame velocity, $\mathcal{U}$. The highest (smallest) value is related to the higher (smaller) wave amplitude, which, as mentioned before, is in agreement with the literature, since we expected that the taller solitons would also have greater speeds [15].

[15]:Solitons: An Introduction

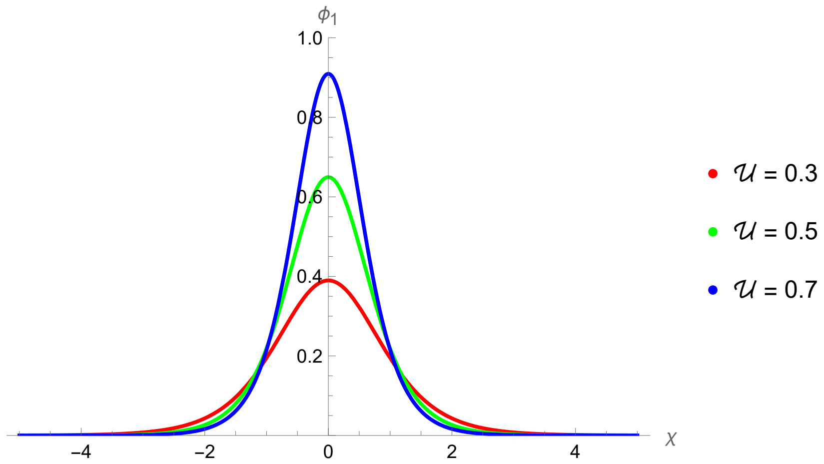


Figure 5.1.3: Electrostatic potential shape in terms of the frame coordinate, χ, to cold ions. Three different values of the frame velocity, $\mathcal{U}$, are considered.

5.1.3 *Non-Magnetized Isothermal Ions*

At first, we have assumed the same dimensionless variables transformation described early, considering the dynamic implications of a cold fluid. Despite this, the investigation of the linear perturbation limit of an unmagnetized isothermal TF plasma returns the correct configuration ion sound speed, in this regime, and the dispersion relation, Eqs. (4.2.138) and (4.2.140).

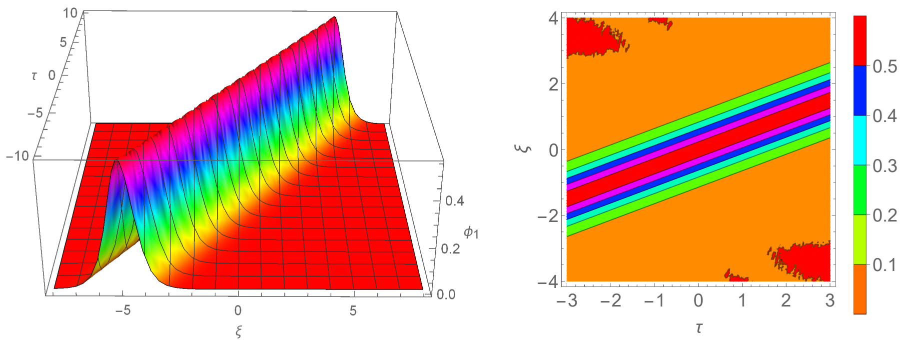
As in the cold case, the nonlinear regime returns from the dimensionless normalization, the transformation in the ξ and τ coordinates, and the equilibrium parameters expansion, a KdV-like equation,

$$\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with}$$
$$A = \frac{(8+9\Theta)}{6\lambda_0}, \quad B = \frac{1}{18\lambda_0}, \quad \lambda_0 = \left(\frac{1}{3} + \frac{\Theta}{2}\right)^{\frac{1}{2}}. \tag{5.1.3}$$

Initially, the previous case indicates that the control parameter takes on the role of the critical Mach number when we investigate the pulse dynamics in our configuration, being

$$M = \left(\frac{1}{3} + \frac{\Theta}{2}\right)^{\frac{1}{2}}, \tag{5.1.4}$$

which corroborates the result obtained from the analysis of cold ions when we apply the limit of $\Theta \to 0$, i.e., we do not have thermal effects.



(a) Time Evolution (b) Contour Plot

Figure 5.1.4: Time evolution (a) and amplitude contour (b) of the electrostatic potential in terms of the stretched coordinates, ξ and τ, to isothermal ions. System V_0 − normalized.

We can then graphically study the behavior of the system under the temperature rate and Mach number dependence, since both parameters appear in determining the shape of the perturbed electrostatic potential term. Following the same steps cited above, for fixed $\mathcal{U} = 0.5$ and $\Theta = 0.5$, we plot the soliton time evolution and the amplitude contour, Figs. 5.1.4a and 5.1.4b, respectively.

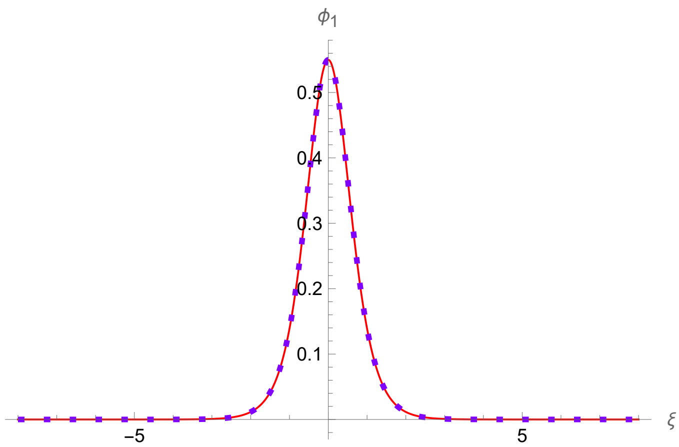


Figure 5.1.5: Numerical (solid, red) and analytical (dashed, blue) solutions for the electrostatic potential in terms of the stretched coordinates, ξ and τ, to isothermal ions in an unmagnetized system V_0 − normalized.

After that, we combine the numerical and analytical solutions assuming the same parameter values at a fixed time, $\tau = 0$, as presented in Fig. 5.1.5.

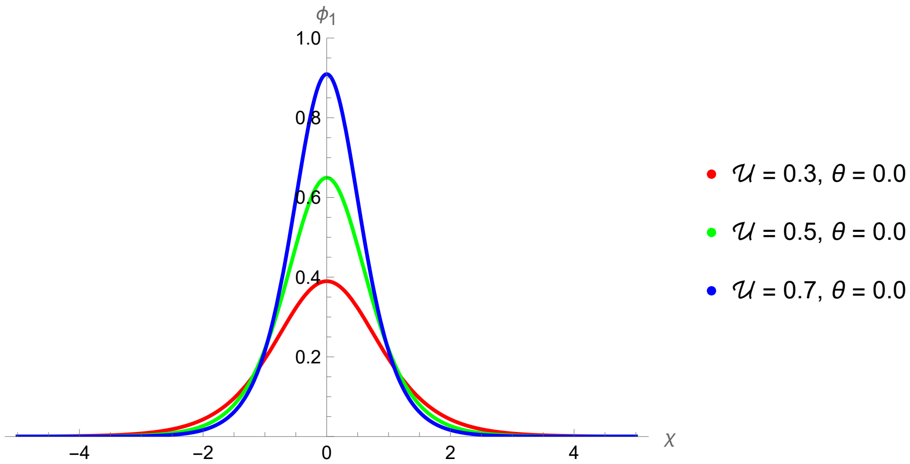
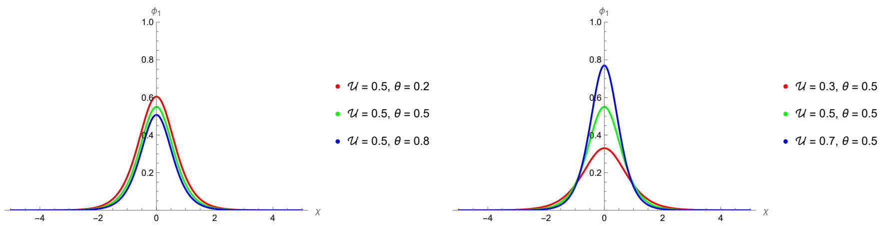


Figure 5.1.6: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermal ions. Three different values of the frame velocity, $\mathcal{U}$, are considered, with $\Theta = 0$.

In Fig. 5.1.6, we can see that as the variables are normalized by the cold ion parameters, under the assumption of $\Theta = 0$ for the same three distinct frame velocities selected earlier, we retrieve the non-thermal threshold, compared to Fig. 5.1.3, as expected. We can also fix the $\mathcal{U}$ (Θ) value and vary Θ ($\mathcal{U}$), Fig 5.1.7.



(a) Θ Variation (b) $\mathcal{U}$ Variation

Figure 5.1.7: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermal ions. Three different values of (a) the temperature ratio, Θ, with a fixed $\mathcal{U} = 0.5$, (b) the frame velocity, $\mathcal{U}$, with fixed $\Theta = 0.5$, are considered.

Fig. 5.1.8 depicts the electrostatic potential shape for $\Theta = (0.2, 0.5, 0.8)$ and $\mathcal{U} = (0.3, 0.5, 0.7)$. We check qualitatively the simultaneous $(\Theta - \mathcal{U})$ response by tracing symmetrical and asymmetrical relations. As usual, in Fig. 5.1.7, we can see the direct relation between velocity and amplitude since the maximum value grows regardless of the temperature rate. Furthermore, we can conclude that the smaller the ratio between T_i and T_F, the greater the wave amplitude, looking at the plotted schemes. Fig. 5.1.8 shows that the asymmetrical assump-

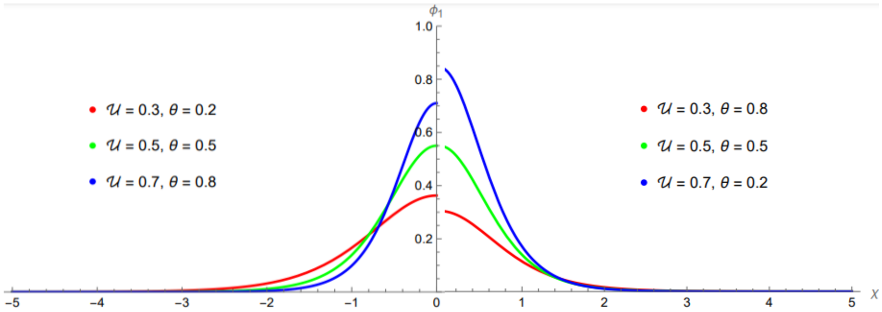


Figure 5.1.8: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to isothermal ions, V_0 − normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of $\mathcal{U}$ and Θ are considered.

tion of $(\Theta - \mathcal{U})$ - right-hand side - implies an escalation in the profile differences while the symmetrical limit - left-hand side - results in a mixed configuration. Both representations return a relation between thermal effects and wave width.

5.1.4 *Non-Magnetized Adiabatic Ions*

Initially, we have revised again the linear regime where it was possible to determine, within this limit, the ion acoustic speed and the dispersion relation for an unmagnetized adiabatic TF plasma, as we can see from Eqs. (4.2.173) and (4.2.175). After that, to analyze the nonlinear regime, we adopted the dimensionless variable transformations, Eq. (4.2.34), as well as the stretched coordinates, ξ and τ, and the expansion of the parameters up to the second order. We got a KdV-like equation that can be written as

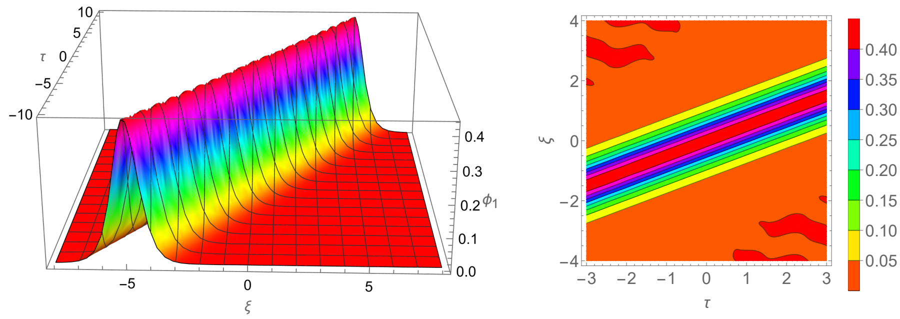
$$\begin{gathered}\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with} \\ A = \frac{2\,(2+5\Theta)}{3\lambda_0},\; B = \frac{1}{18\lambda_0},\; \lambda_0 = \left(\frac{1}{3} + \frac{5\Theta}{6}\right)^{\frac{1}{2}}\end{gathered} \tag{5.1.5}$$

and then we can study the system numerically. First, the critical Mach number character of the control parameter in the dynamic analysis of our system gives us

$$M = \left(\frac{1}{3} + \frac{5\Theta}{6}\right)^{\frac{1}{2}}, \tag{5.1.6}$$

and again we get the cold result by applying the $\Theta \to 0$ limit.

Numerically solving the dynamic governing equation exploring the wave time evolution under the effect of a fixed temperature ratio, $\Theta = 0.5$, and pulse velocity, $\mathcal{U} = 0.5$, we obtain to the behavior and contour plots the results presented in Figs. 5.1.9a and 5.1.9b.



(a) Time Evolution (b) Contour Plot

Figure 5.1.9: Time evolution (a) and amplitude contour (b) of the electrostatic potential in terms of the stretched coordinates, ξ and τ, to adiabatic ions. System V_0 – normalized.

The fit of the analytical and numerical pulse shape curves at $\tau = 0$ is presented in Fig. 5.1.10.

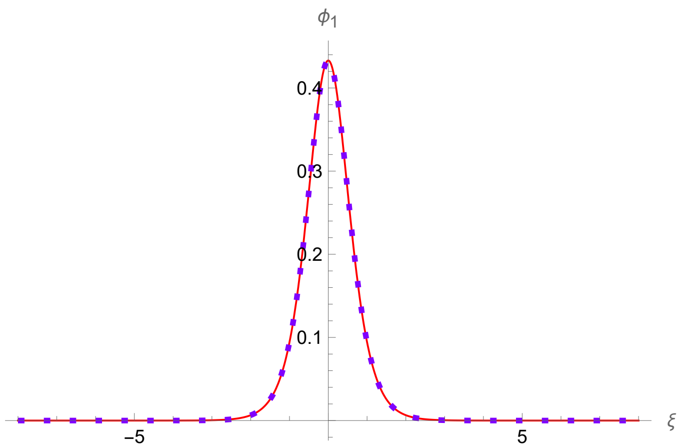


Figure 5.1.10: Numerical (solid, red) and analytical (dashed, blue) solutions for the electrostatic potential in terms of the stretched coordinates, ξ and τ, to adiabatic ions in an unmagnetized system V_0 – normalized.

After, if we plot the boundary condition of non-thermal effects, $\Theta \to 0$, we obtain the result shown in Fig. 5.1.11, and it is straightforward that the cold regime is recovered, as could be predicted.
Then, from Fig. 5.1.12a, we can see how the shape behaves when we consider fixed frame velocities and vary the temperature parameter. For a selected value of $\mathcal{U}$, the amplitude of the ion wave scales with the decrease in $(T_i$ - $T_F)$ ratio, the same as we observed for the isothermal regime, while the growth of the normalized speed variable, $\mathcal{U}$, increases extension, as theorized. This result is shown by Fig. 5.1.12b. In Fig. 5.1.13, the distinction between the direct relation, on the left-half, and the inverse relation, on the right-half, of $\Theta - \mathrm{U}$ is depicted.

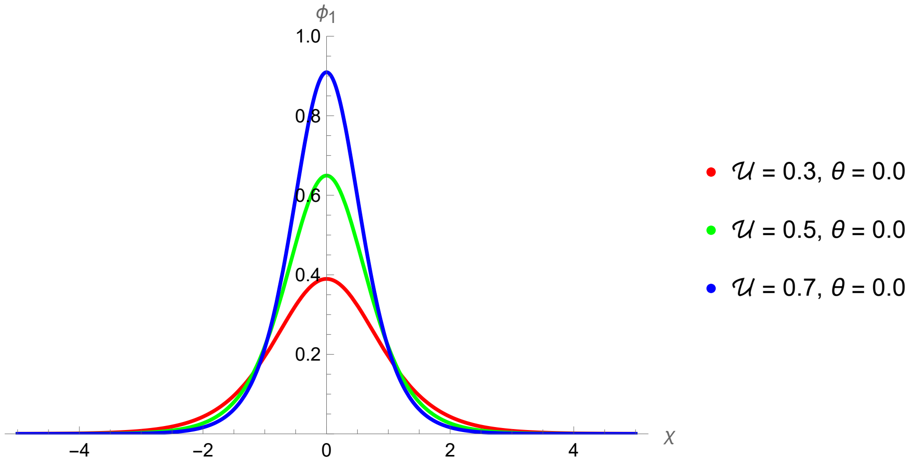
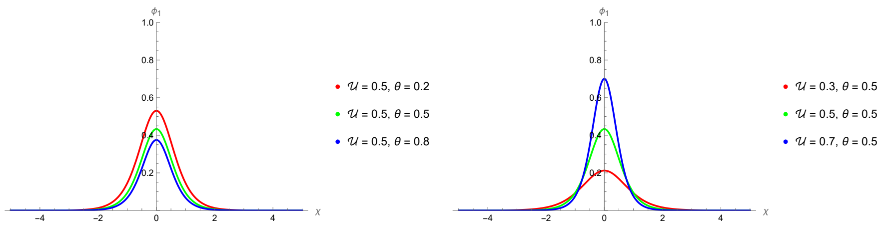


Figure 5.1.11: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions. Three different values of the frame velocity, $\mathcal{U}$, are considered, with $\Theta = 0$.



(a) Θ Variation (b) $\mathcal{U}$ Variation

Figure 5.1.12: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions. Three different values of (a) the temperature ratio, Θ, with a fixed $\mathcal{U} = 0.5$, (b) the frame velocity, $\mathcal{U}$, with fixed $\Theta = 0.5$, are considered.

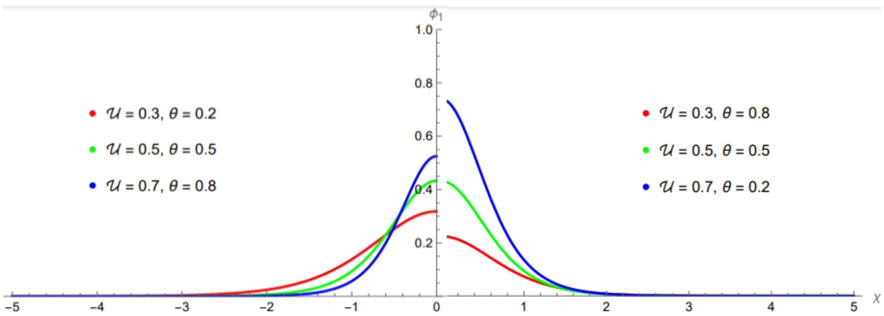


Figure 5.1.13: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to adiabatic ions, V_0-normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of $\mathcal{U}$ and Θ are considered.

5.1.5 *Magnetized Cold Ions*

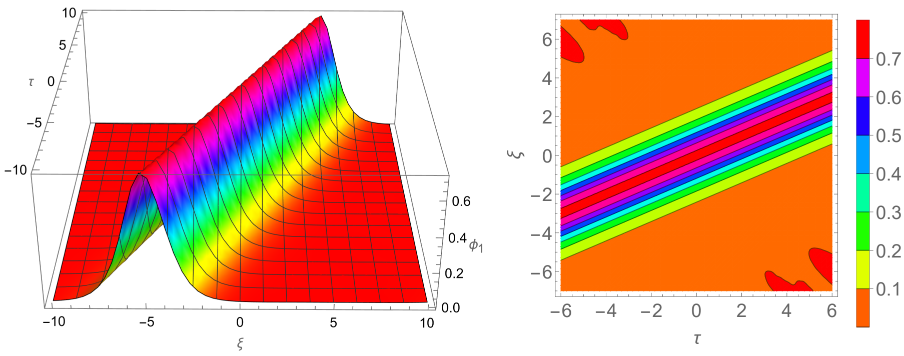
By introducing a $z-$direction constant external magnetic field into the problem, $\vec{B} = B_0\hat{z}$, we are inserting new free parameters into the theory that appears in waveform determination. Therefore, as we have been dealing from the beginning with extensions of unmagnetized cold TF fluids, in a certain reduction we have to recover the initial shape, which also occurs in this regime.

To start our study, we have linearized the plasma to obtain the dispersion relation and the ion acoustic speed, Eqs. (4.2.218) and (4.2.220), respectively, where the velocity was found by assuming *pure* ion sound waves, which exist only for $\vec{k} \parallel \vec{B}$, confirming the deduction carried out in Sec. 3.2. After that, associating the transformation of the variables, the stretched coordinates definition, and the modified expansion, as stated in Eqs. (4.2.231) - (4.2.234), we obtain the KdV-like equation for the electrostatic potential,

$$\begin{gathered}\partial_\tau\phi_1 + A\phi_1\partial_\xi\phi_1 + B\partial_\xi^3\phi_1 = 0 \quad \text{with} \\ A = 4\lambda_0, \; B = \frac{\lambda_0}{6}\left[1 + \frac{(1-l_z^2)}{\Omega_{p_i}^2}\right], \; \lambda_0 = \frac{l_z}{\sqrt{3}},\end{gathered} \tag{5.1.7}$$

and the control parameter indicates as critical Mach number

$$M = l_z\left(\frac{1}{3}\right)^{\frac{1}{2}}. \tag{5.1.8}$$



(a) Time Evolution

(b) Contour Plot

Figure 5.1.14: Time evolution (a) and amplitude contour (b) of the electrostatic potential in terms of the stretched coordinates, ξ and τ, to cold ions in a magnetized plasma. System V_0 −normalized.

Following the procedure, the soliton temporal behavior, the amplitude map, and the adjustment of analytic-numerical solutions in a fixed time are shown in Figs. 5.1.14a, 5.1.14b, and 5.1.15.
In Fig. 5.1.16, we can see that the unmagnetized case is obtained by considering the director cosine in the z−direction as one, i. e., the coordinate asymmetry induced by the magnetic field can be compensated by the propagation obliqueness, and $l_z = 1$ is exactly the *pure* case.

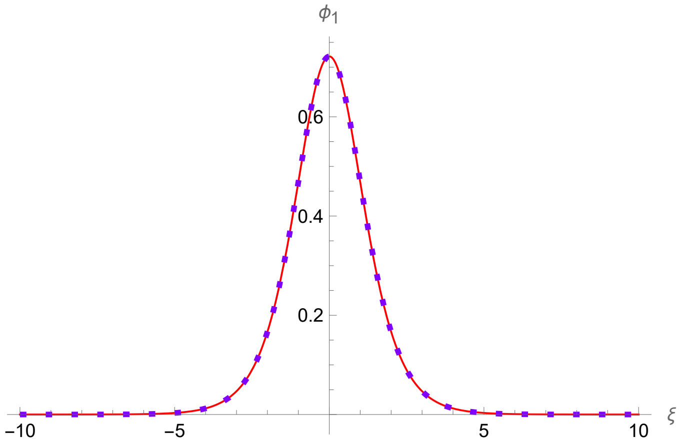


Figure 5.1.15: Numerical (solid, red) and analytical (dashed, blue) solutions for the electrostatic potential in terms of the stretched coordinates, ξ and τ, to cold ions in a magnetized system V_0 − normalized.

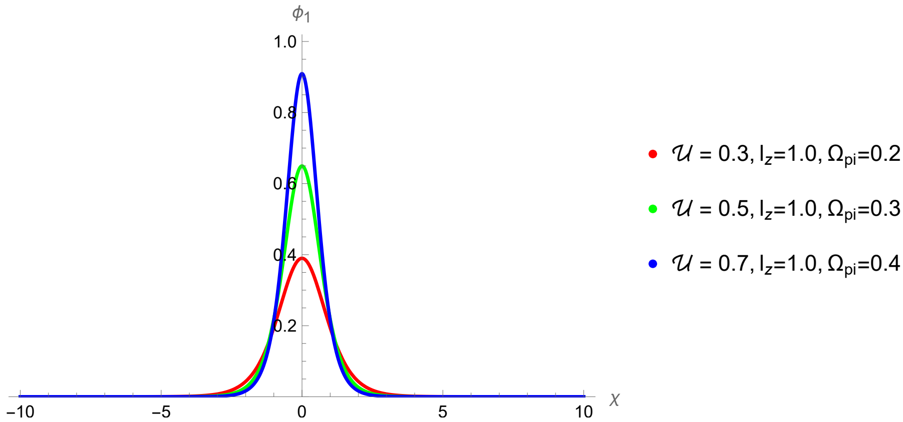
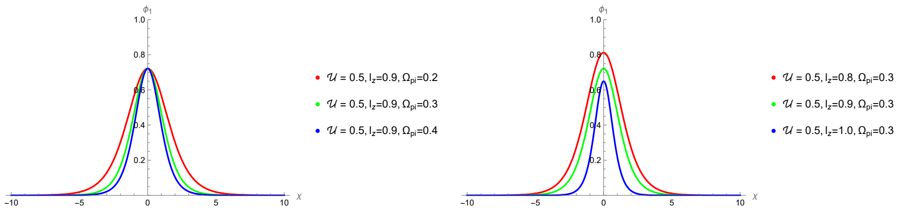


Figure 5.1.16: Electrostatic potential shape in terms of the frame coordinate, χ, to cold ions in a magnetized system for $l_z = 1$.

If we now fix the directional parameter and apply magnetic fields of distinct magnitudes, we conclude from Fig. 5.1.17a that, for a given frame velocity, Ω_{p_i} is connected with the wave width fluctuation. Ultimately, by setting the value of the magnetic quantity and shifting l_z, we find that the increase in obliquity causes a decrease in the amplitude of the potential nonlinear steady structure, as shown in Fig. 5.1.17b.

5.1.6 *Magnetized Isothermal Ions*

For the magnetized isothermal TF plasma, we have studied the linear and nonlinear regimes under inspection of the influence of the free parameters, $\mathcal{U}$, Θ, l_z, and Ω_{p_i}, on the wave profile. First, the linear fluid dynamic equations return the system dispersion relation



(a) Ω_{p_i} Variation (b) l_z Variation

Figure 5.1.17: Electrostatic potential shape in terms of the frame coordinate, χ, to cold ions. Three different values of (a) the magnetic field magnitude, Ω_{p_i}, with fixed $l_z = 0.9$ and $\mathcal{U} = 0.5$, (b) the z-directional cosine, l_z, with fixed $\Omega_{p_i} = 0.3$ and $\mathcal{U} = 0.5$, are considered.

and ion acoustic speed, respectively, as expressed in Eqs. (4.2.280) and (4.2.282), where the second was obtained by adopting $\vec{k} = k_z\hat{z}$, as before.

The investigation of the nonlinear approach enables us to write from the governing equations and appropriate assumptions that

$$
\begin{aligned}
&\partial_\tau\phi_1 + A\phi_1\partial_\xi\phi_1 + B\partial_\xi^3\phi_1 = 0 \quad \text{with} \\
&A = \frac{l_z^2}{6\lambda_0}\left(9\Theta + 8\right), \;\; B = \frac{1}{18\lambda_0}\left(l_z^2 + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\right), \\
&\lambda_0^2 = l_z^2\left(\frac{1}{3} + \frac{1}{2}\Theta\right),
\end{aligned}
\tag{5.1.9}
$$

which implies as the critical Mach number, from the control parameter,

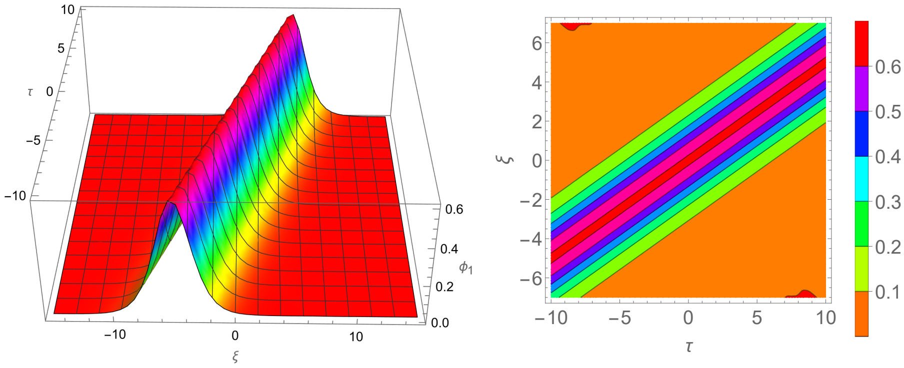
$$
M = l_z\left(\frac{1}{3} + \frac{1}{2}\Theta\right)^{\frac{1}{2}}, \tag{5.1.10}
$$

and numerically plotting the solutions, in Fig. 5.1.18a we have the pulse evolution, while Figs. 5.1.18b and 5.1.19 depict the amplitude contour plot and the numerical and analytical fit results.

The cold and isothermal unmagnetized cases are recovered by choosing $l_z = 1$ for $\Theta = 0$ and $\Theta = 0.5$, respectively, for example, as we can see in Figs. 5.1.20a and 5.1.20b, for the different frame velocities.

Commonly done here, we can confirm the role of parameters $\mathcal{U}$ and Ω_{p_i} in the theory. Fig. 5.1.21 shows that the magnetic pressure is again related to the wave width, while the velocity is connected to the amplitude, as known from the solitary waves background.

It is also interesting to see how the temperature acts when we arbitrarily set the other free quantities. Fig. 5.1.22 depicts exactly this



(a) Time Evolution (b) Contour Plot

Figure 5.1.18: Time evolution (a) and amplitude contour (b) of the electrostatic potential in terms of the stretched coordinates, ξ and τ, to isothermic ions in a magnetized plasma. System V_0 — normalized.

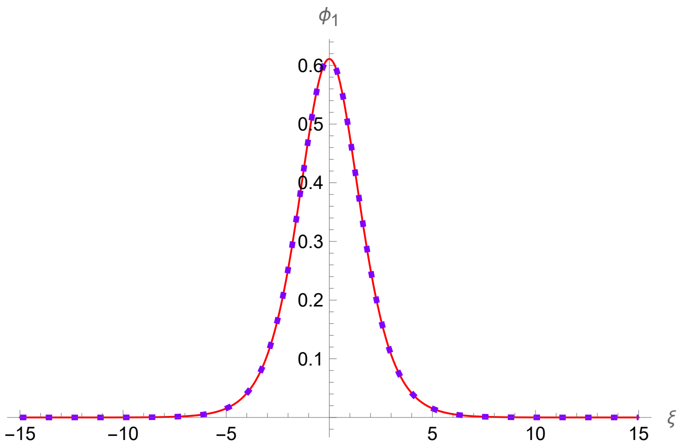
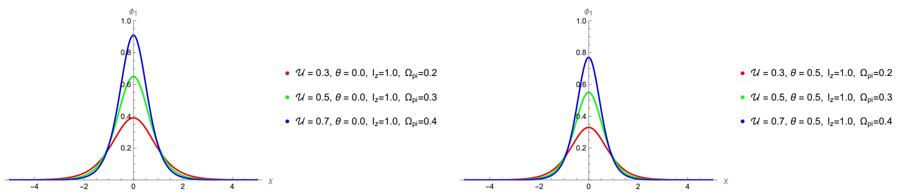
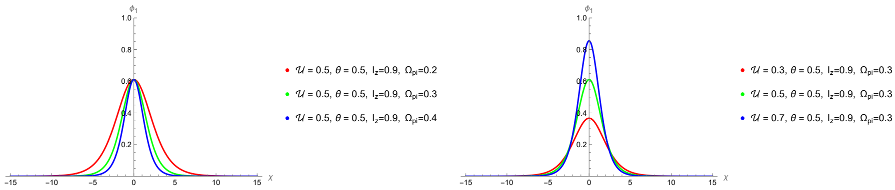


Figure 5.1.19: Numerical (solid, red) and analytical (dashed, blue) solutions for the electrostatic potential in terms of the stretched coordinates, ξ and τ, to isothermic ions in a magnetized system V_0 — normalized.



(a) Unmagnetized Cold Case (b) Unmagnetized Isothermal Case

Figure 5.1.20: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermic ions. Three different values of the magnetic field magnitude, Ω_{p_i}, and frame velocity, $\mathcal{U}$, with fixed $l_z = 1.0$, are considered, for (a) $\Theta = 0$ (b) $\Theta = 0.5$.



(a) Ω_{p_i} Variation (b) $\mathcal{U}$ Variation

Figure 5.1.21: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermic ions. Three different values of (a) the magnetic field magnitude, Ω_{p_i}, with fixed $l_z = 0.9$, $\Theta = 0.5$, and $\mathcal{U} = 0.5$, (b) the frame velocity, $\mathcal{U}$, with fixed $\Omega_{p_i} = 0.3$, $l_z = 0.9$, and $\Theta = 0.5$, are considered.

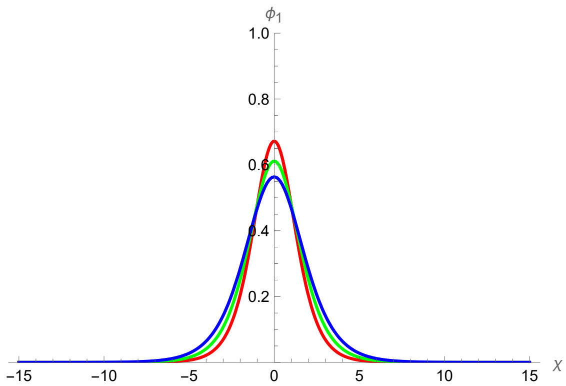


Figure 5.1.22: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermic ions in a magnetized system, varying Θ.

situation and we can conclude that, as well as the previous behaviors, Θ is connected to variations in both shape height and breadth.
Even more, looking at Fig. 5.1.23 we have the implication of the symmetrical - left-hand side - and asymmetrical - right-hand side - fluctuations of Θ and Ω_{pi}, and, as we conclude before, since the magnetic field is related to the width, maintaining the temperature unchanged for each case, we verify only a broadening or compression of the waveform.
In Fig. 5.1.24, we fixed only the directional propagation variable, being clear the role of the frame velocity in controlling the amplitude. The left and right side results differ in the symmetry of the quantities fluctuation.

To complete, we choose l_z to vary individually. Then, as observed in the magnetized non-thermal limit, the decrease in the wave height can be seen in several cases as a consequence of the increase in the obliqueness, comparing structures. This result is presented in Fig. 5.1.25.

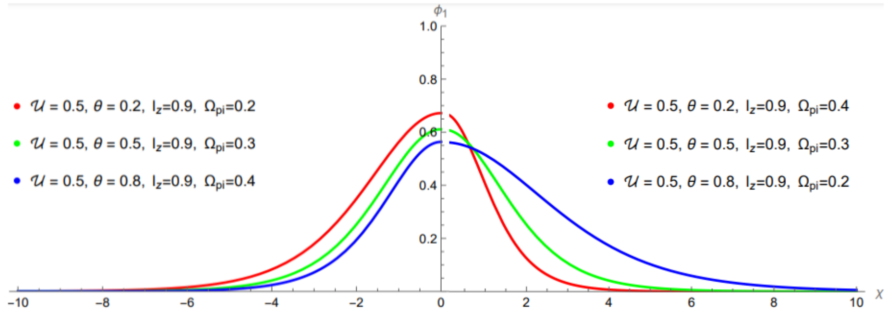


Figure 5.1.23: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to isothermic ions in a magnetized plasma, V_0 – normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of Ω_{p_i} and Θ are considered.

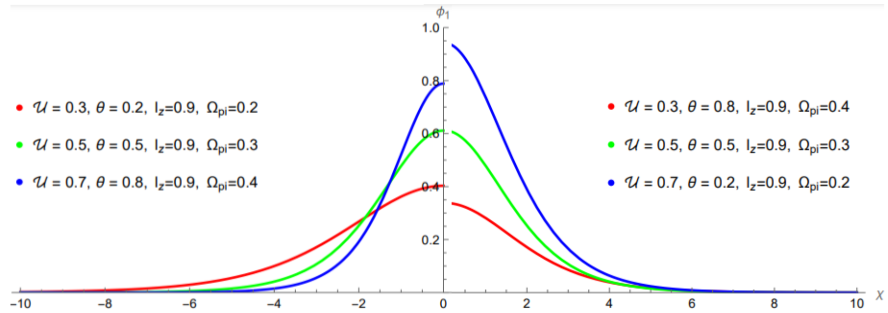


Figure 5.1.24: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to isothermic ions in a magnetized plasma, V_0 – normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of $\mathcal{U}$, Ω_{p_i} and Θ are considered.

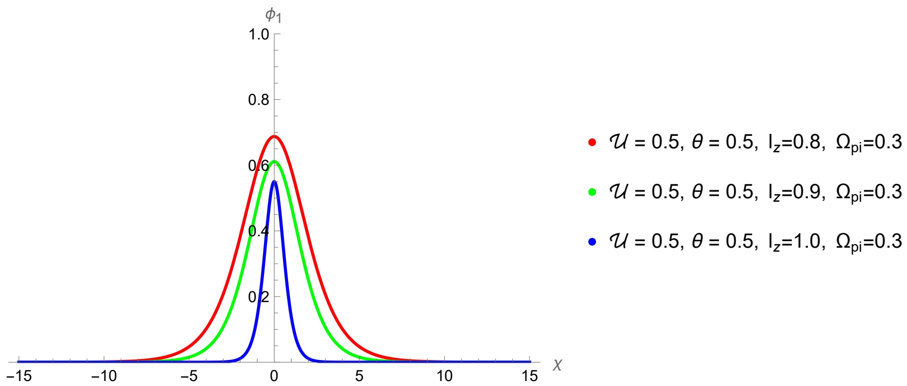


Figure 5.1.25: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermal ions in a magnetized system, V_0 – normalized, setting as fixed parameters Θ, $\mathcal{U}$ and Ω_{p_i}, and varying l_z. Three different values of l_z are considered.

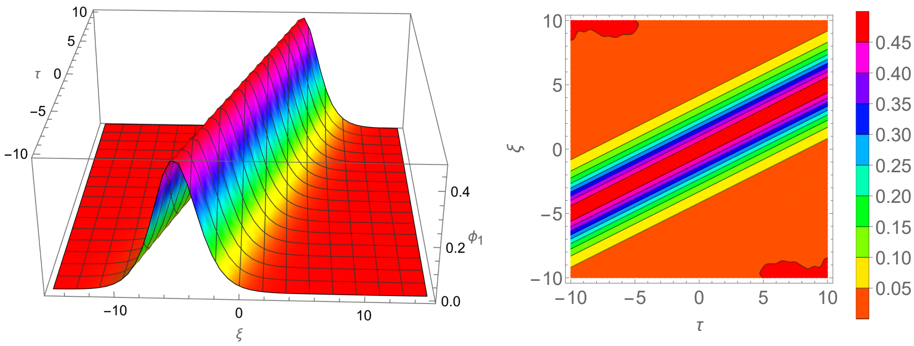
5.1.7 *Magnetized Adiabatic Ions*

We can make a similar analysis for the adiabatic ions when we consider the application of an external magnetic field, since the free parameters are the same as in the previous case, but here the temperature responds differently when compared. Thus, starting from the linearization, we got the dispersion relation and ion sound speed as stated in Eqs. (4.2.335) and (4.2.337), and then we have adopted the traditional procedure for the variables transformation, stretched coordinates refit, and magnetized equilibrium expansion to obtain the KdV-like equation for the electrostatic potential,

$$\begin{aligned} &\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with} \\ &A = 4\lambda_0, \;\; B = \frac{1}{18\lambda_0}\left(l_z^2 + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\right), \;\; \lambda_0^2 = l_z^2\left(\frac{1}{3}+\frac{5}{6}\Theta\right), \end{aligned} \tag{5.1.11}$$

and it is straightforward that

$$M = l_z\left(\frac{1}{3}+\frac{5}{6}\Theta\right)^{\frac{1}{2}}. \tag{5.1.12}$$



(a) Time Evolution (b) Contour Plot

Figure 5.1.26: Time evolution (a) and amplitude contour (b) of the electrostatic potential in terms of the stretched coordinates, ξ and τ, to adiabatic ions in a magnetized plasma. System V_0 − normalized.

As traditionally, the first investigation lies in the numerical study of the configuration. From Figs. 5.1.26a, 5.1.26b and 5.1.27 we can see the time evolution, the amplitude profile and the solutions correspondence, respectively.

The reduction to the cold and adiabatic non-magnetized cases occurs by setting $\Theta = 0$ and $\Theta = 0.5$, for instance, for $l_z = 1.0$, as could be predicted from other warm regimes results. The results are graphed in Figs. 5.1.28a and 5.1.28b.

Next, to see the direct relation of the velocity $\mathcal{U}$ and the wave amplitude, in addition to the influence of the magnetic pressure in the

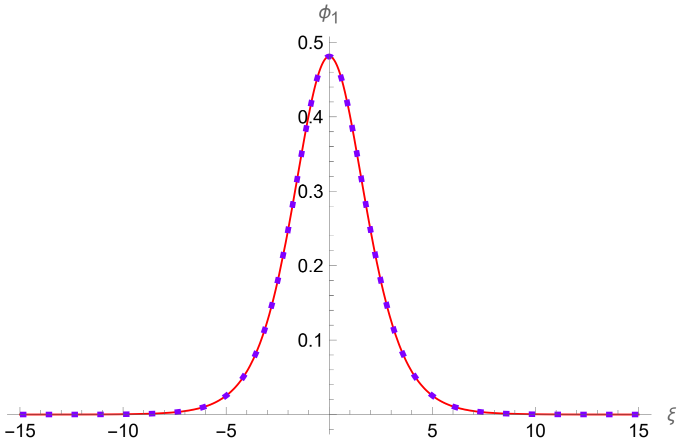
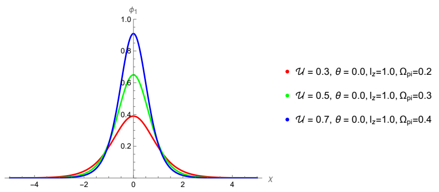
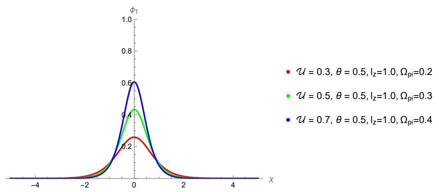


Figure 5.1.27: Numerical (solid, red) and analytical (dashed, blue) solutions for the electrostatic potential in terms of the stretched coordinates, ξ and τ, to adiabatic ions in a magnetized system V_0 – normalized.



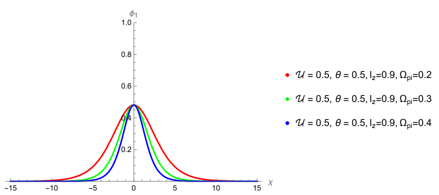
(a) Unmagnetized Cold Case



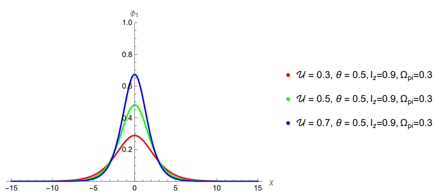
(b) Unmagnetized Adiabatic Case

Figure 5.1.28: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions. Three different values of the magnetic field magnitude, Ω_{p_i}, and frame velocity, $\mathcal{U}$, with fixed l_z = 1.0, are considered, for (a) $\Theta = 0$ (b) $\Theta = 0.5$.

determination of the shape width, we plot, considering Θ and l_z constants, the fluctuation of Ω_{p_i} for a selected frame speed value, and the consequences are presented in the set of Figs. 5.1.29, as well as the effect of $\mathcal{U}$ in the amplitude.



(a) Ω_{p_i} Variation



(b) $\mathcal{U}$ Variation

Figure 5.1.29: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions. Three different values of (a) the magnetic field magnitude, Ω_{p_i}, with fixed $l_z = 0.9$, $\Theta = 0.5$, and $\mathcal{U} = 0.5$, (b) the frame velocity, $\mathcal{U}$, with fixed $\Omega_{p_i} = 0.3$, $l_z = 0.9$, and $\Theta = 0.5$, are considered.

Temperature evidently plays a key role in the theory, and we have seen that in a thermal fluid it can be associated at the same time with both pulse height and breadth. We confirm this by choosing appropriate fixed values to $\mathcal{U}$, Ω_{p_i}, and l_z, as illustrated in Fig. 5.1.30, for the three different temperature ratios indicated.

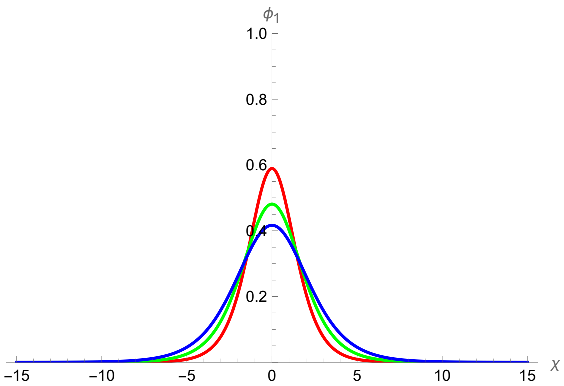


Figure 5.1.30: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions in a magnetized system, varying Θ.

As temperature and magnetic field are independent parameters, it is interesting to investigate the behavior of the system to the simultaneous variation of both. As we concluded, Θ is related to all shape characteristics, while Ω_{p_i} does not interfere in the amplitude scale, therefore, as shown in Fig. 5.1.31, the only difference occurs in the width, since we kept constant the temperature ratio and varied the magnetic term, symmetrically - left-hand side - and asymmetrically - right-hand side, in each curve.

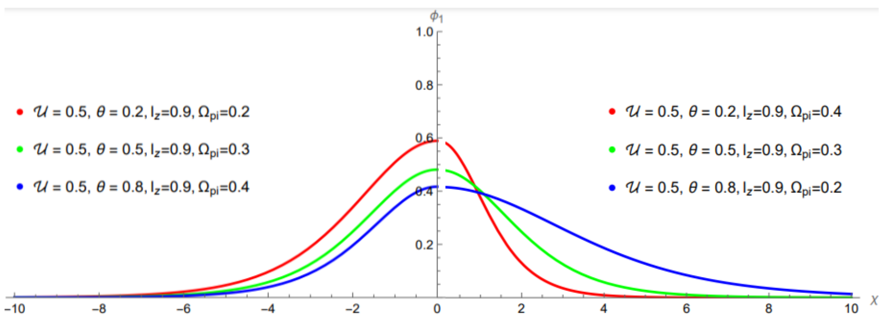


Figure 5.1.31: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to adiabatic ions in a magnetized plasma, V_0 − normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of Ω_{p_i} and Θ are considered.

The results in Fig. 5.1.32 include the fluctuations in the velocity parameter, which is directly linked to the profile height, and we can see a potential problem in determining the plasma configuration if we properly combine the variations.

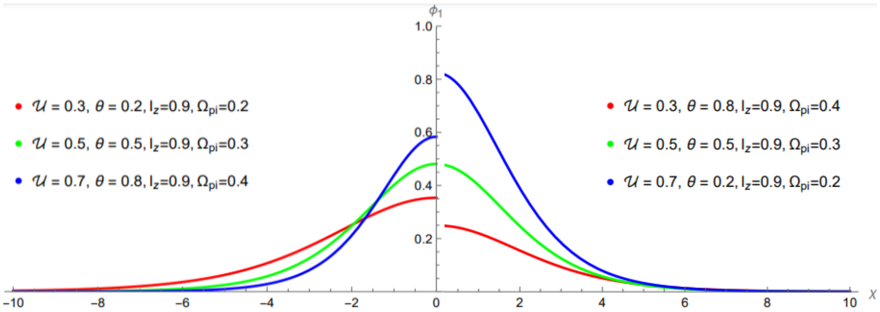


Figure 5.1.32: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to adiabatic ions in a magnetized plasma, V_0 − normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of $\mathcal{U}$, Ω_{p_i} and Θ are considered.

Finally, the directional cosine in the z−direction, which returns the *pure* regime when considering the limit of $l_z \to 1$, may also imply changes in the waveform. Fig. 5.1.33 shows its relation with the shape structure and confirms previous conclusions about the parameter.

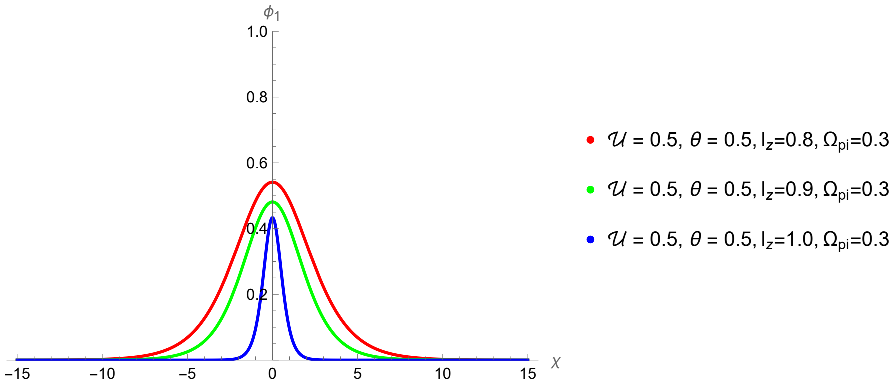


Figure 5.1.33: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions in a magnetized system, V_0 − normalized, setting as fixed parameters Θ, $\mathcal{U}$ and Ω_{p_i}, and varying l_z. Three different values of l_z are considered.

5.1.8 *Corrected Non-Magnetized Isothermal Ions*

As mentioned before, the approach so far assumes the transformation of variables to dimensionless form under normalization by the parameters associated with the cold TF plasma ion sound speed, Eq. (4.2.105). Despite being a commonly observed procedure in the literature, a correction must be made if we want to study the system more faithfully, adopting the ion acoustic velocity of the respective configuration, following the results obtained previously.

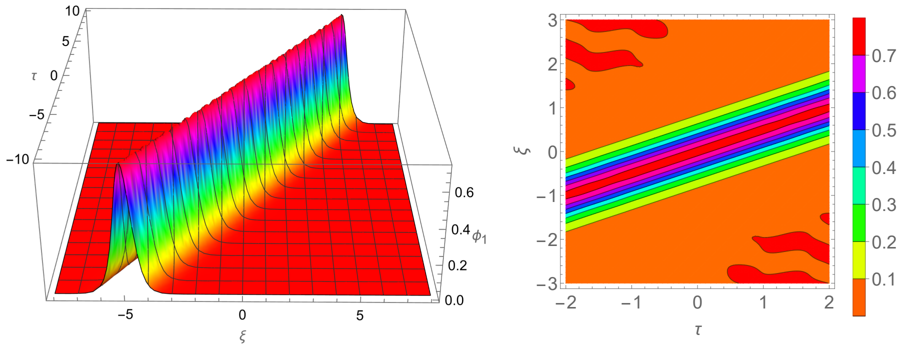
Then changing v_0 to ${}^{Iso}v_0$, as specified in Eq. (4.2.408), and taking the same steps as listed above, the nonlinear regime returns as the adjusted KdV-like equation for the unmagnetized isothermal TF fluid,

$$\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with}$$
$$A = \frac{1}{3\lambda_0}\frac{(9\Theta+8)}{(3\Theta+2)}, \quad B = \frac{2}{9\lambda_0}\frac{1}{(3\Theta+2)^2}, \quad \lambda_0 = \frac{1}{\sqrt{3}} \tag{5.1.13}$$

where we have now the λ_0 as

$$\lambda_0 = M = \frac{1}{\sqrt{3}}, \tag{5.1.14}$$

just as in the cold limit, and corroborating the relation between the Mach number and the control parameter in our theory.



(a) Time Evolution

(b) Contour Plot

Figure 5.1.34: Time evolution (a) and amplitude contour (b) of the electrostatic potential in terms of the stretched coordinates, ξ and τ, to isothermal ions. System ${}^{Iso}V_0$ – normalized.

As done previously, we restudy the dynamics of the system assuming the changes and plot the temporal behavior, the amplitude profile, and the numerical and analytical solutions for a fixed time instant, as shown in Figs. 5.1.34a, 5.1.34b and 5.1.35.
To confirm our expectation, in Fig. 5.1.36, we have plotted the null Θ limit for distinct U to recover the unmagnetized cold regime.
Furthermore, from the new formulation, we can conclude that the temperature continues to affect the wave amplitude, but in a different way if we compare with Fig. 5.1.7, since the set of graphics in Fig. 5.1.37 shows that the curves representing the Θ variations assume closer values when are crossing the ϕ_1 axes, besides the influence on the width. The $\mathcal{U}$ fluctuations, however, keep modifying the amplitude.

Ultimately, the response of the system to the symmetrical and asymmetrical $(\Theta - \mathcal{U})$ fluctuations is shown in Fig. 5.1.38, whose small-amplitude variation and significant change in width corroborate the conclusions taken before for the implications of the temperature on the system.

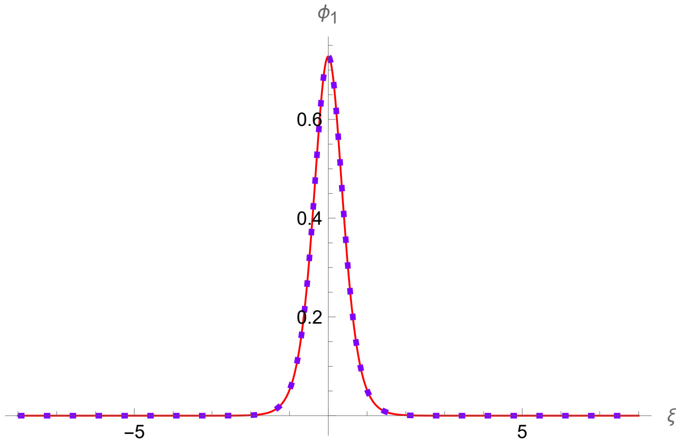


Figure 5.1.35: Numerical (solid, red) and analytical (dashed, blue) solutions for the electrostatic potential in terms of the stretched coordinates, ξ and τ, to isothermal ions in an unmagnetized system ${}^{\text{Iso}}V_0$ − normalized.

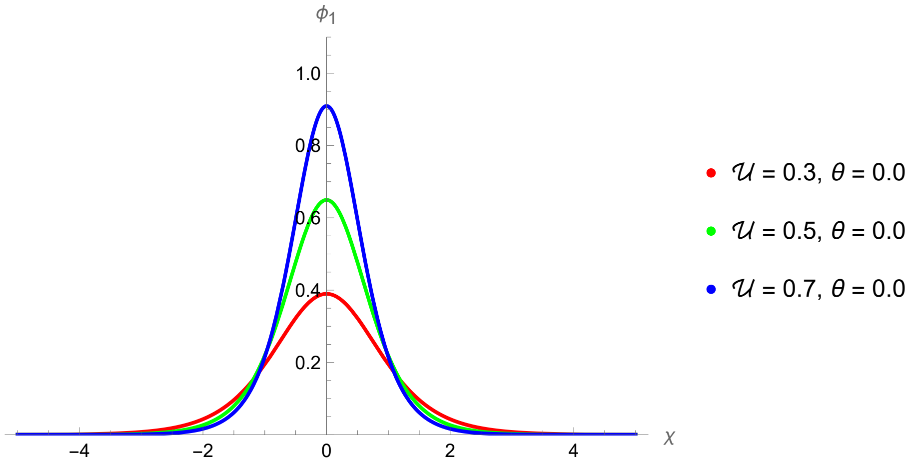
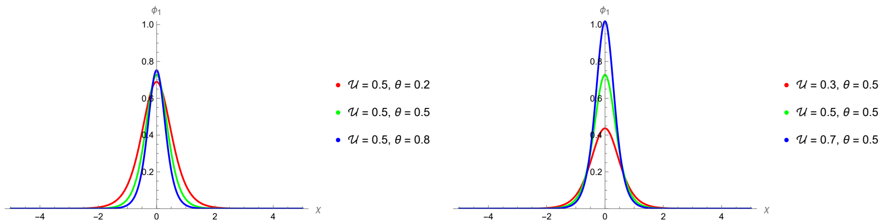


Figure 5.1.36: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermal ions. Three different values of the frame velocity, $\mathcal{U}$, are considered, with $\Theta = 0$. System ${}^{\text{Iso}}V_0$ − normalized.

5.1.9 *Corrected Non-Magnetized Adiabatic Ions*

To the adiabatic case, we apply the modifications as specified by Eq. (4.2.408), but replacing ${}^{\text{Iso}}\nu_0$ to ${}^{\text{Adb}}\nu_0$, associated with the ion sound velocity that characterizes this TF fluid configuration, and then we can follow the nonlinear regime analysis, as traditionally done, to obtain

$$\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with}$$
$$A = \frac{4}{3\lambda_0}, \quad B = \frac{2}{9\lambda_0}\frac{1}{(2+5\Theta)^2}, \quad \lambda_0 = \frac{1}{\sqrt{3}}, \tag{5.1.15}$$



(a) Θ Variation (b) $\mathcal{U}$ Variation

Figure 5.1.37: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermal ions. Three different values of (a) the temperature ratio, Θ, with a fixed $\mathcal{U} = 0.5$, (b) the frame velocity, $\mathcal{U}$, with fixed $\Theta = 0.5$, are considered in a system ${}^{Iso}V_0-$normalized.

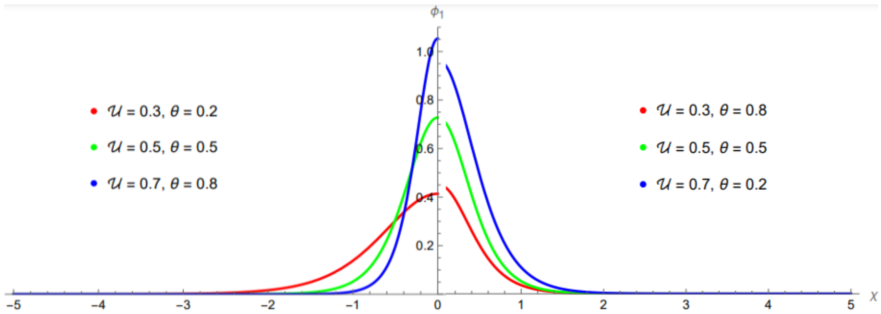


Figure 5.1.38: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to isothermal ions, ${}^{Iso}V_0-$normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of $\mathcal{U}$ and Θ are considered.

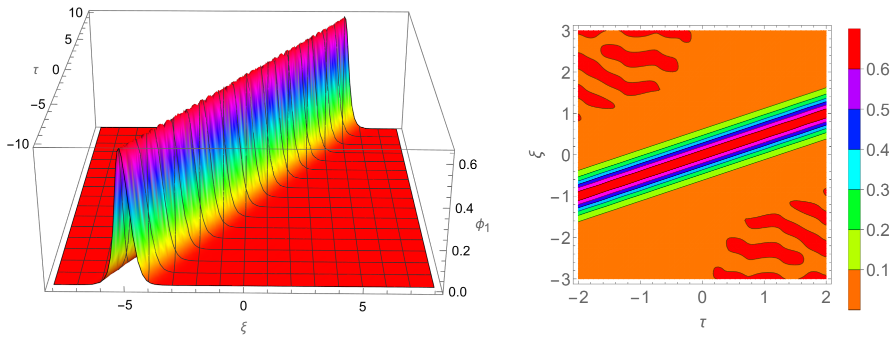
verifying the $\lambda_0 = M$ result as in the cold and corrected isothermal cases, with

$$M = \frac{1}{\sqrt{3}}. \tag{5.1.16}$$

The temporal evolution of the pulse is depicted in Fig. 5.1.39a, as before, as well as the amplitude contour plot and the solutions fit for $\tau = 0$, Figs. 5.1.39b and 5.1.40, respectively.
We can see in Fig. 5.1.41 that the cold limit for a TF plasma is obtained for $\Theta = 0$, varying the frame velocity values, as expected.

Under the changes applied to the theory, Fig. 5.1.42 allows us to affirm that the temperature does not affect the wave amplitude, contrary to what we had concluded in Fig. 5.1.12a, being the frame velocity here the only one that can induce modifications in the profile height, since Fig. 5.1.42a shows that the curves representing the Θ variation cross the ϕ_1 axis at the same value for a fixed $\mathcal{U}$.

Graphically representing the symmetrical and asymmetrical $(\Theta - U)$ changes on the left and right hand sides of Fig. 5.1.43, respectively, and comparing with Fig. 5.1.13, we can see a rectification



(a) Time Evolution (b) Contour Plot

Figure 5.1.39: Time evolution (a) and amplitude contour (b) of the electrostatic potential in terms of the stretched coordinates, ξ and τ, to adiabatic ions. System ${}^{Adb}V_0$ – normalized.

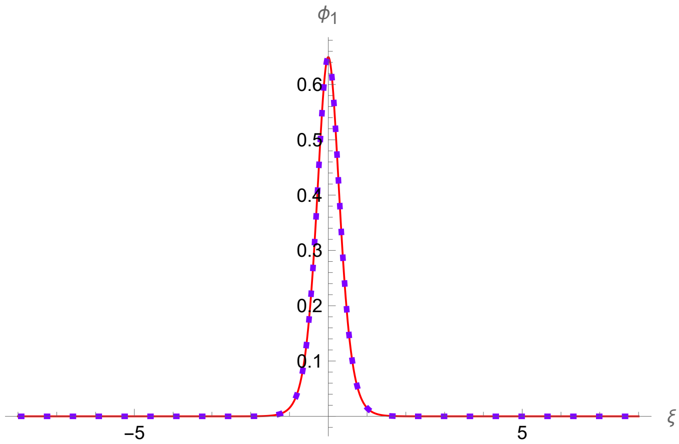


Figure 5.1.40: Numerical (solid, red) and analytical (dashed, blue) solutions for the electrostatic potential in terms of the stretched coordinates, ξ and τ, to adiabatic ions in an unmagnetized system ${}^{Adb}V_0$ – normalized.

in the profile, with U being the one responsible for determining the height, while the ratio between T_i and T_F adjusts the breadth.

5.1.10 *Corrected Magnetized Isothermal Ions*

Since *pure* ion acoustic waves in unmagnetized and magnetized TF plasmas have the same dispersion relation, we are considering normalization by ${}^{Iso}v_0$ for the fits, as mentioned before. Thus, investigating the nonlinear regime, the hydrodynamic equations return as the corrected KdV-like expression

$$\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with}$$
$$A = \frac{l_z^2}{\lambda_0 \delta}\frac{(9\Theta + 8)}{6}, \quad B = \frac{1}{18\lambda_0}\left[\frac{l_z^2}{\delta^2} + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\right], \quad \lambda_0^2 = \frac{l_z^2}{3},$$

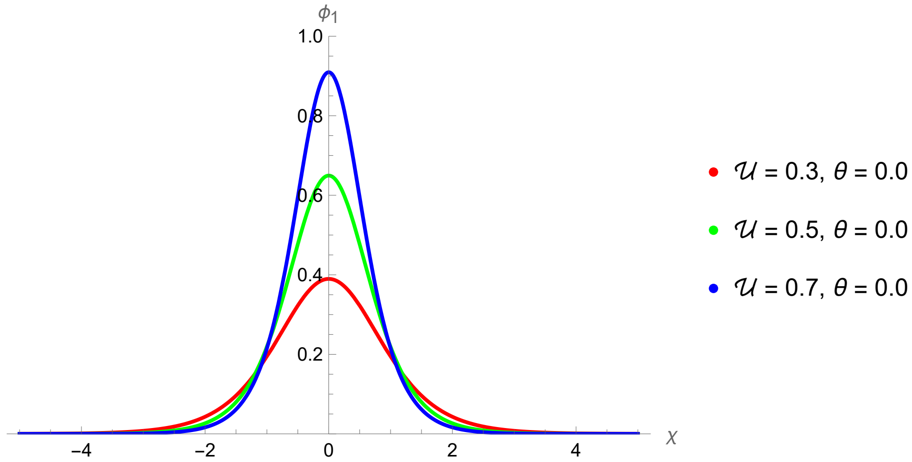
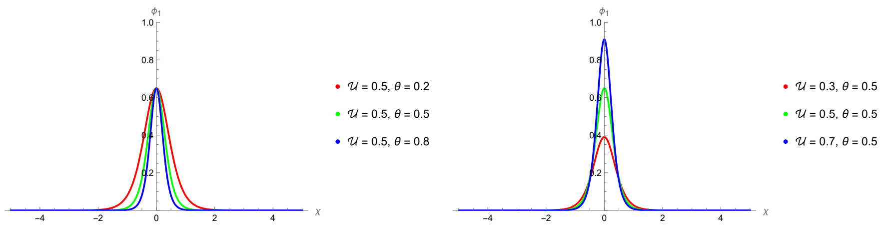


Figure 5.1.41: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions. Three different values of the frame velocity, $\mathcal{U}$, are considered, with $\Theta = 0$. System $^{Adb}V_0$ – normalized.



(a) Θ Variation (b) $\mathcal{U}$ Variation

Figure 5.1.42: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions. Three different values of (a) the temperature ratio, Θ, with a fixed $\mathcal{U} = 0.5$, (b) the frame velocity, $\mathcal{U}$, with fixed $\Theta = 0.5$, are considered in a system $^{Adb}V_0$ – normalized.

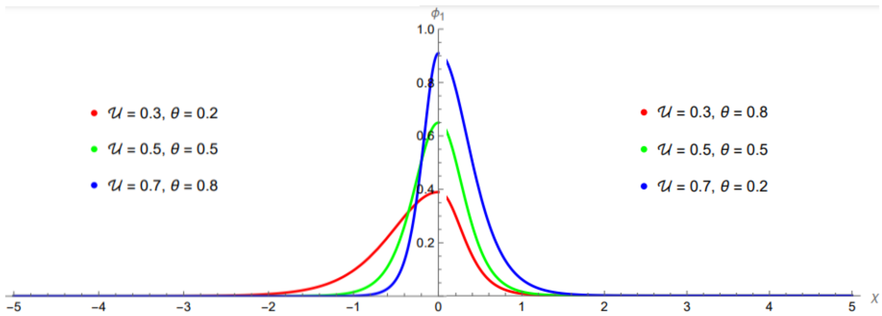
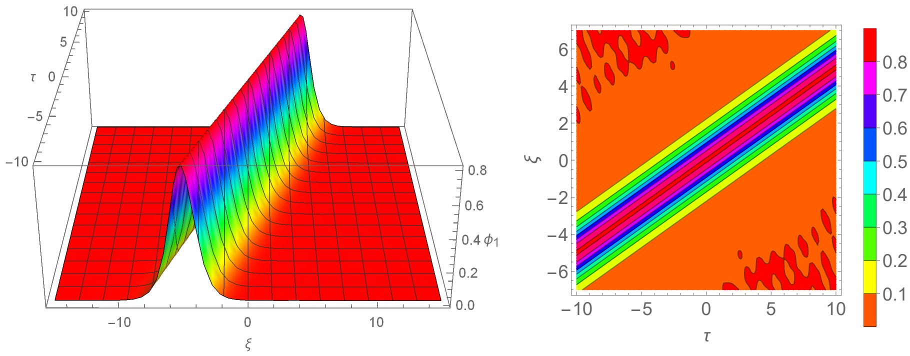


Figure 5.1.43: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to adiabatic ions, $^{Adb}V_0$ – normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of $\mathcal{U}$ and Θ are considered.

$$(5.1.17)$$

where we have written the parameters in terms of δ, declared in Eq. (4.2.451), and the factor λ_0 is related to l_z in the same way as in the Mach number for the cold magnetized case, i. e.,

$$\lambda_0 = M = \frac{l_z}{\sqrt{3}}. \tag{5.1.18}$$



(a) Time Evolution (b) Contour Plot

Figure 5.1.44: Time evolution (a) and amplitude contour (b) of the electrostatic potential in terms of the stretched coordinates, ξ and τ, to isothermic ions in a magnetized plasma. System ${}^{\mathrm{Iso}}V_0$ – normalized.

As usual, we briefly study the corrected dynamics. In Fig. 5.1.44a we can visualize the system from its temporal evolution, and the respective amplitude map is graphed in Fig. 5.1.44b. Next, we present the fit of the numerical and analytical solutions, Fig. 5.1.45.

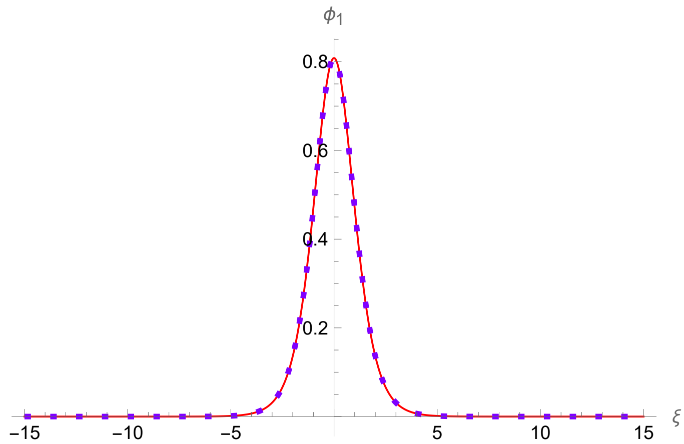
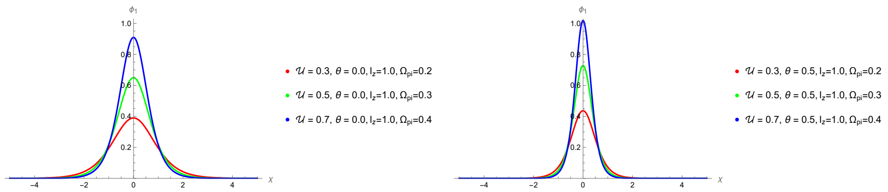


Figure 5.1.45: Numerical (solid, red) and analytical (dashed, blue) solutions for the electrostatic potential in terms of the stretched coordinates, ξ and τ, to isothermic ions in a magnetized system ${}^{\mathrm{Iso}}V_0$ – normalized.

Confirming the complete reduction to the non-magnetized cold regime and the respective unmagnetized warm limit, we again set

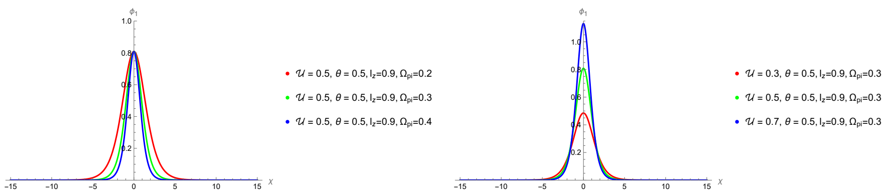
$\Theta = 0$ and $\Theta = 0.5$, for example, to $l_z = 1$, resulting in the configuration profiles depicted in Figs. 5.1.46a and 5.1.46b, which are exactly the desired results.



(a) Unmagnetized Cold Case (b) Unmagnetized Isothermal Case

Figure 5.1.46: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermic ions. Three different values of the magnetic field magnitude, Ω_{p_i}, and frame velocity, $\mathcal{U}$, with fixed $l_z = 1.0$, are considered, for (a) $\Theta = 0$ (b) $\Theta = 0.5$. System ${}^{Iso}V_0 -$ normalized.

Furthermore, if we look at the magnetic pressure effect on the system, we have that it is related to the determination of the shape breadth, just as the magnetized isothermal fluid returned earlier, and $\mathcal{U}$ is closely connected with the profile height, as can be concluded from Fig. 5.1.47, demonstrating the consistency of our formulation.



(a) Ω_{p_i} Variation (b) $\mathcal{U}$ Variation

Figure 5.1.47: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermic ions. Three different values of (a) the magnetic field magnitude, Ω_{p_i}, with fixed $l_z = 0.9$, $\Theta = 0.5$, and $\mathcal{U} = 0.5$, (b) the frame velocity, $\mathcal{U}$, with fixed $\Omega_{p_i} = 0.3$, $l_z = 0.9$, and $\Theta = 0.5$, are considered in a system ${}^{Iso}V_0 -$ normalized.

Next, we turn to the temperature consequences in theory. Setting the other free quantities as constants and plotting three different values to quantify the response of the system to fluctuations in Θ, we get the results presented in Fig. 5.1.48.

Confronting the curves shown in Fig. 5.1.22 and those obtained after correction, we can observe a difference in both amplitude and width scales of the generated pulses, with Θ here contributing synergistically with $\mathcal{U}$ to the pulse size.

Depicting the simultaneous variation of the temperature ratio and the magnetic term, we obtain the profiles shown in Fig. 5.1.49. As standardized, on the left we present a symmetrical variation while on the right the asymmetrical fluctuation, and then we can confirm that

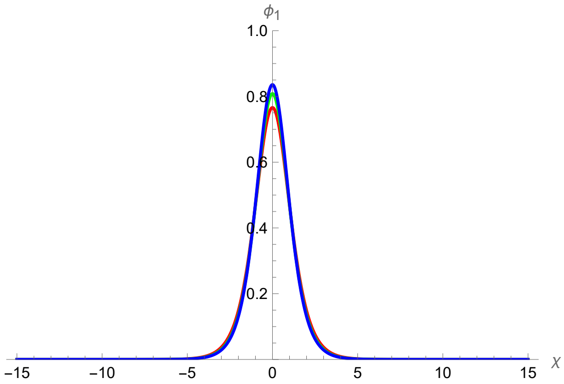


Figure 5.1.48: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermic ions in a magnetized system ${}^{Iso}V_0$ – normalized, varying Θ.

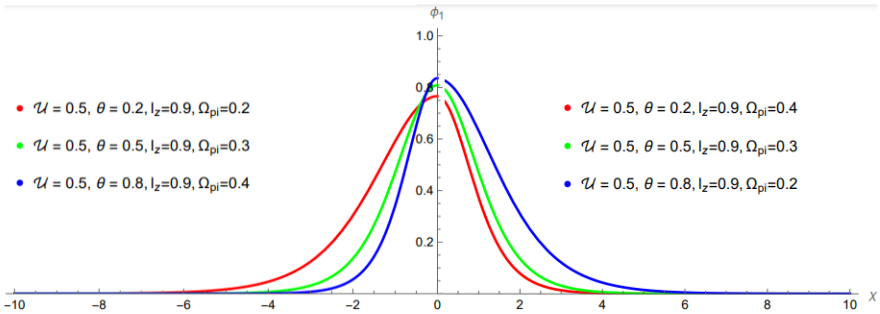


Figure 5.1.49: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to isothermic ions in a magnetized plasma, ${}^{Iso}V_0$ – normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of Ω_{p_i} and Θ are considered.

Ω_{p_i} does not interfere with the shape height. The temperature ratio, on the other hand, is the one responsible for the amplitude and also has a subtle importance in the wave width.

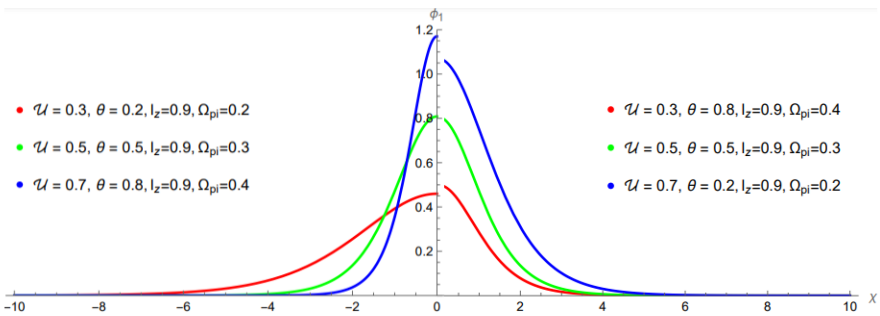


Figure 5.1.50: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to isothermic ions in a magnetized plasma, ${}^{Iso}V_0$ – normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of $\mathcal{U}$, Ω_{p_i} and Θ are considered.

The introduction of the changes in the frame velocity returns the waveform as established in Fig. 5.1.50, and then we have a significant change in amplitudes.

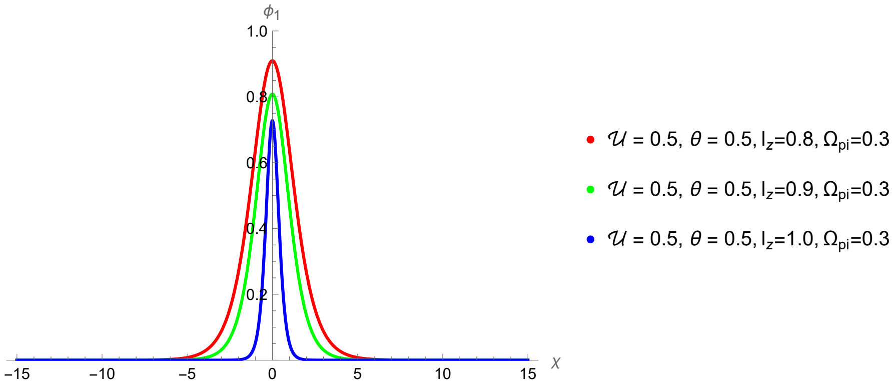


Figure 5.1.51: Electrostatic potential shape in terms of the frame coordinate, χ, to isothermal ions in a magnetized system, ${}^{Iso}V_0$ – normalized, setting as fixed parameters Θ, $\mathcal{U}$ and Ω_{p_i}, and varying l_z. Three different values of l_z are considered.

Concluding, in Fig. 5.1.51, we have a representation of the fluctuations in the directional cosine parameter for certain values of the other independent quantities, and we can see that, in comparison with Fig. 5.1.25, this variable has a strong importance in the pulse shape, with the decrease of l_z implying a taller and wider profile.

5.1.11 *Corrected Magnetized Adiabatic Ions*

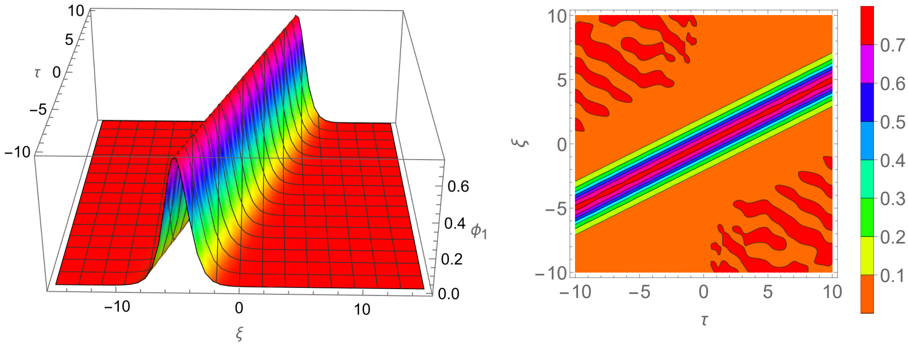
To deduce the modified KdV-like equation for the adiabatic ion configuration from the governing equations, we have considered the dimensionless variables transformation normalized by ${}^{Adb}v_0$ instead of the initial v_0 assumption. Then, the investigation of the nonlinear regime returns

$$\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0 \quad \text{with}$$
$$A = \frac{2}{3}\frac{l_z^2}{\lambda_0 \beta}(2+5\Theta)\,, \quad B = \frac{1}{18\lambda_0}\left[\frac{l_z^2}{\beta^2} + 9\frac{\lambda_0^4}{\Omega_{p_i}^2}\frac{(1-l_z^2)}{l_z^2}\right], \quad \lambda_0^2 = \frac{l_z^2}{3}, \tag{5.1.19}$$

where $\lambda_0 = M$ corroborates the magnetized cases of cold and corrected isothermal TF plasma, being

$$M = \frac{l_z}{\sqrt{3}}, \tag{5.1.20}$$

and factor β is expressed by Eq. (4.2.433).



(a) Time Evolution (b) Contour Plot

Figure 5.1.52: Time evolution (a) and amplitude contour (b) of the electrostatic potential in terms of the stretched coordinates, ξ and τ, to adiabatic ions in a magnetized plasma. System ${}^{Adb}V_0$ − normalized.

The corrected soliton temporal behavior and the amplitude contour plot are depicted in Figs. 5.1.52a and 5.1.52b. To validate the results, as previously done, we have in Fig. 5.1.53 the comparison between the analytical and numerical results.

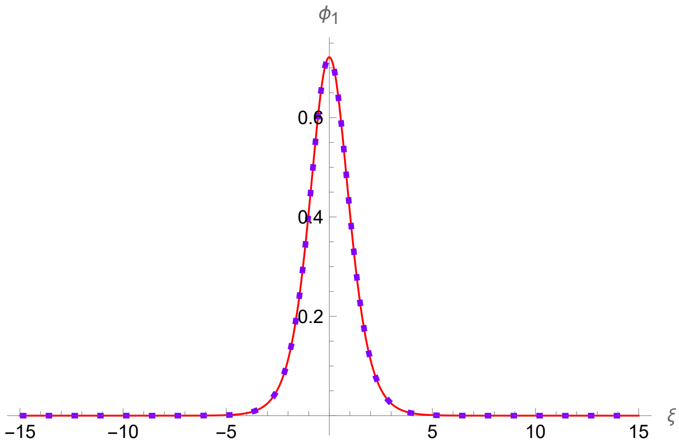
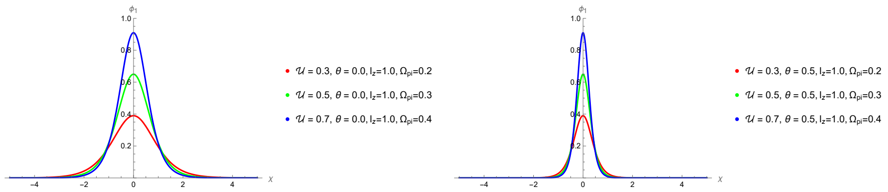


Figure 5.1.53: Numerical (solid, red) and analytical (dashed, blue) solutions for the electrostatic potential in terms of the stretched coordinates, ξ and τ, to adiabatic ions in a magnetized system ${}^{Adb}V_0$ − normalized.

As we know, the cold regime must be obtained for purely directed non-thermal ions. Therefore, by setting $l_z = 1$ and $\Theta = 0$, we get the results shown in Fig. 5.1.54a, which are the expected pulse representations for the three frame velocities indicated. The same can be done to the unmagnetized adiabatic configuration, choosing here $\Theta = 0.5$, as presented in Fig 5.1.54b.

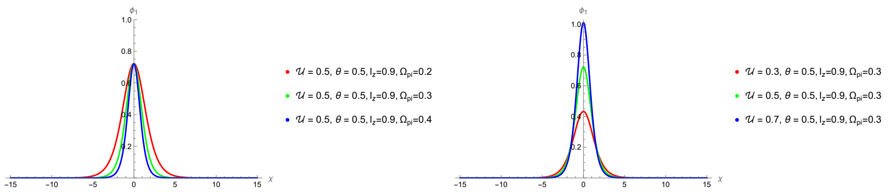
For constant temperature ratios and directional cosines, we investigate how the Ω_{p_i} term influences wave profiles. As we can see from the set of results compiled in Fig. 5.1.55, the magnitude of the mag-



(a) Unmagnetized Cold Case (b) Unmagnetized Adiabatic Case

Figure 5.1.54: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions. Three different values of the magnetic field magnitude, Ω_{p_i}, and frame velocity, $\mathcal{U}$, with fixed $l_z = 1.0$, are considered, for (a) $\Theta = 0$, (b) $\Theta = 0.5$, in a system ${}^{Adb}V_0-$normalized.

netic field is directly related to the shape width, considering a fixed $\mathcal{U}$, while the latter has consequences on the amplitude.



(a) Ω_{p_i} Variation (b) $\mathcal{U}$ Variation

Figure 5.1.55: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions. Three different values of (a) the magnetic field magnitude, Ω_{p_i}, with fixed $l_z = 0.9$, $\Theta = 0.5$, and $\mathcal{U} = 0.5$, (b) the frame velocity, $\mathcal{U}$, with fixed $\Omega_{p_i} = 0.3$, $l_z = 0.9$, and $\Theta = 0.5$, are considered, in a system ${}^{Adb}V_0-$normalized.

For the thermal implications, since the same value of $\mathcal{U}$ was considered in Fig. 5.1.56 for all curves presented, we can conclude that the parameter Θ is not related to variations in wave height and implies subtle changes in the breadth.

As Θ and Ω_{p_i} have been shown to have common implications, we can expect that the variation of both will result in fluctuations in pulse width. Fig. 5.1.57 depicts the corrected behavior of the shape under symmetrical and asymmetrical modifications, compared to Fig. 5.1.31, without changes in the amplitude.

In addition, shifting U, we can see in Fig. 5.1.58 the composed consequences on the shape by the symmetric and asymmetric considerations of U with the variations of $(\Theta - \Omega_{p_i})$. Note that the obtained shapes are strongly different from the curves in Fig. 5.1.32.

Finally, we plotted the directional cosine influence on the waveform, and we can see that it affects both the profile height and breadth at the same time, as represented in Fig. 5.1.59, but now in a distinguished way.

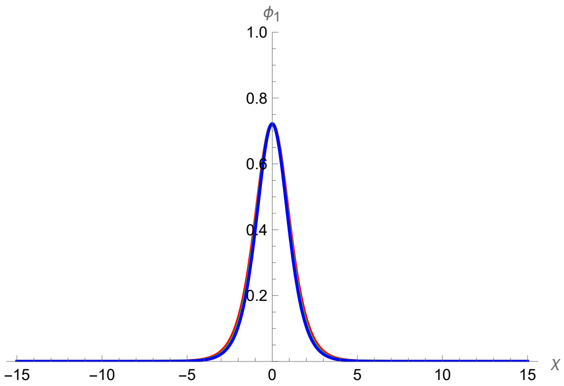


Figure 5.1.56: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions in a magnetized system ${}^{Adb}V_0$ − normalized, varying Θ.

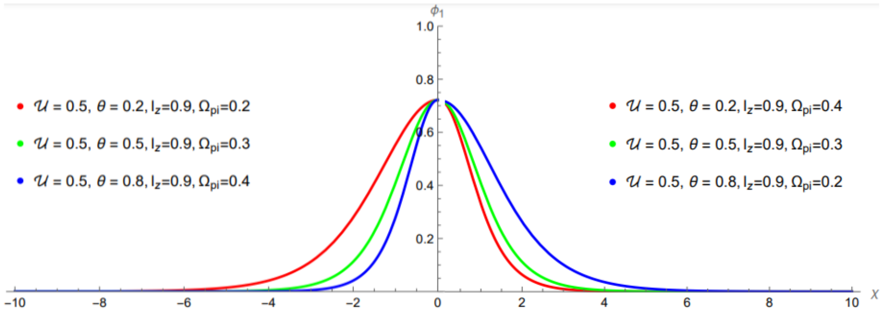


Figure 5.1.57: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to adiabatic ions in a magnetized plasma, ${}^{Adb}V_0$ − normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of Ω_{p_i} and Θ are considered.

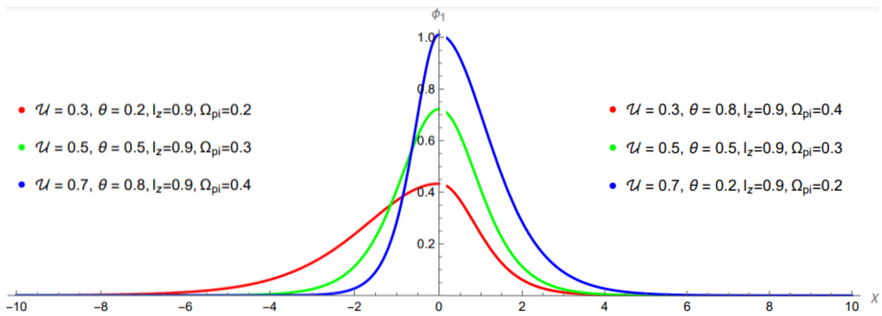


Figure 5.1.58: Electrostatic potential half-shapes in terms of the frame coordinate, χ, to adiabatic ions in a magnetized plasma, ${}^{Adb}V_0$ − normalized. Simultaneous symmetric (left-hand side) and asymmetric (right-hand side) variations of $\mathcal{U}$, Ω_{p_i} and Θ are considered.

5.1.12 *The Control Parameter*

Investigating the existence and shape of SW in the TF plasma under the analysis of the nonlinear regime, we have defined the stretched

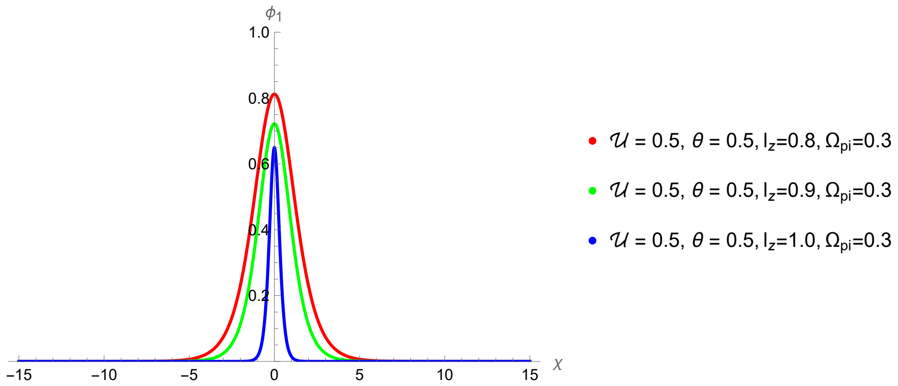


Figure 5.1.59: Electrostatic potential shape in terms of the frame coordinate, χ, to adiabatic ions in a magnetized system, ${}^{Adb}V_0$ − normalized, setting as fixed parameters Θ, U and Ω_{p_i}, and varying l_z. Three different values of l_z are considered.

coordinates, Eq. (4.2.90), with the introduction of a *dimensionless parameter*, λ_0, which helps to deduce the solution and characterize the *modified RP method*. So, by studying the cold and warm configurations of our fluid, we obtain results that explicitly depend on the control variable.

First, as we initially set the normalization by V_0, Eq. (4.2.367), which characterizes the ion acoustic velocity of the system on the analysis scale, it is easy to mention the KdV equations resulting from the cold regime, for both the unmagnetized and magnetized limits, without needing to consider any modification. From Eqs. (4.2.127) and (4.2.256), we got, respectively, $\lambda_0^2 = \frac{1}{3}$ and $\lambda_0^2 = \frac{l_z^2}{3}$, where the latter in the *pure* limit reduces to the former since $l_z = 1$. Therefore, we are dealing with a frame that moves with $M \equiv \lambda_0 = \frac{1}{\sqrt{3}}$.

Second, turning our attention to the thermal effects in the system, considering the absence and presence of an external magnetic field, we got KdV-like expressions whose temperature dependence also appears in the determination of the parameter λ_0. For isothermal cases, we have the results presented in Eqs. (4.2.161) and (4.2.310), respectively, and the adiabatic regime returns Eqs. (4.2.198) and (4.2.366), in that order, with the quantities differing only by a product of the z-directional cosine term.

The explicit dependence of the dimensionless moving frame parameter on the temperature is questionable, which leads to believe in a possible correction based on the selected normalization. So, as the linear regime analysis allowed us to deduce the configuration ion sound speed, and after associating this to the corrected acoustic velocities, changing V_0 to ${}^{Iso}V_0$ in isothermal cases and ${}^{Adb}V_0$ in adiabatic systems, we obtain the relations expressed in Eqs. (4.2.428) and (4.2.477), when $\gamma = 1$, and Eqs. (4.2.449) and (4.2.503), when $\gamma = \frac{5}{3}$, where γ is the polytropic index.

The non-magnetized equalities obtained for the control parameter set $\lambda_0^2 = \frac{1}{3}$, while $\lambda_0^2 = \frac{l_z^2}{3}$ is stated for the magnetized ones, as we had deduced for cold fluids. Thus, we can conclude that *(pure)* SW in TF plasmas are subsonic, and the analysis can be done considering a frame moving with $M = \frac{1}{\sqrt{3}}$. To complete, note that the resulting term in the control parameters ratio, considering the expressions reached under normalization by V_0 and ${}^{Iso}V_0$, or ${}^{Adb}V_0$, is exactly the difference between these systems ion sound speed, since

$$\text{Non-magnetized Isothermal:} \quad \frac{\lambda_0}{\lambda_0^{Iso}} = \left(1 + \frac{3\Theta}{2}\right)^{\frac{1}{2}} \quad \text{and} \quad {}^{Iso}V_0 = V_0\left(1 + \frac{3\Theta}{2}\right)^{\frac{1}{2}},$$

$$\text{Magnetized Isothermal:} \quad \frac{\lambda_0}{\lambda_0^{Iso}} = \frac{l_z}{l_z}\left(1 + \frac{3\Theta}{2}\right)^{\frac{1}{2}} \quad \text{and} \quad {}^{Iso}V_0 = V_0\left(1 + \frac{3\Theta}{2}\right)^{\frac{1}{2}},$$

$$\text{Non-magnetized Adiabatic:} \quad \frac{\lambda_0}{\lambda_0^{Adb}} = \left(1 + \frac{5\Theta}{2}\right)^{\frac{1}{2}} \quad \text{and} \quad {}^{Adb}V_0 = V_0\left(1 + \frac{5\Theta}{2}\right)^{\frac{1}{2}},$$

$$\text{Magnetized Adiabatic:} \quad \frac{\lambda_0}{\lambda_0^{Adb}} = \frac{l_z}{l_z}\left(1 + \frac{5\Theta}{2}\right)^{\frac{1}{2}} \quad \text{and} \quad {}^{Adb}V_0 = V_0\left(1 + \frac{5\Theta}{2}\right)^{\frac{1}{2}},$$

implying that ${}^{J}V_0\lambda_0^{J} = V_0\lambda_0$, with $J = Iso, Adb$, being then λ_0 a normalization *control* parameter.

5.1.13 *Association of Normalized Results*

Unmagnetized Isothermal Plasma

Finally, knowing the implications of our previous analysis, we can take the route from the V_0-normalized KdV-like equations to the ${}^{J}V_0$-normalized ones. Let us first consider Eq. (4.2.161), which can be explicitly written as

$$\partial_\tau\phi_1 + \frac{\sqrt{6}\,(8 + 9\Theta)}{6\sqrt{2 + 3\Theta}}\phi_1\partial_\xi\phi_1 + \frac{\sqrt{6}}{18\sqrt{2 + 3\Theta}}\partial_\xi^3\phi_1 = 0, \qquad (5.1.21)$$

and the velocity-normalized entity involved in the expression, according to the transformations stated in Eq. (4.2.34) and the resulting derivatives, is only ∂_ξ, since this is related to the spatial variable by Eqs. (4.2.36) and (4.2.91). So, looking for the isothermal ion sound speed expressed in terms of the cold ion acoustic velocity within the regime of interest, presented in Eq. (4.2.406) and rewritten as

$$^{\mathrm{Iso}}V_0 = V_0 \frac{\lambda_0}{\lambda_0^{\mathrm{Iso}}} \quad \text{with} \quad \frac{\lambda_0}{\lambda_0^{\mathrm{Iso}}} = \sqrt{1+\nu}, \tag{5.1.22}$$

where ν can be seen as $\nu(\Theta)$, and adjusting the normalization, we get from Eqs. (5.1.21) and (5.1.22)

$$\partial_\tau\phi_1 + \frac{\sqrt{6}\,(8+9\Theta)}{6\sqrt{2+3\Theta}}\frac{1}{\sqrt{1+\nu}}\phi_1\partial_\xi\phi_1 + \frac{\sqrt{6}}{18\sqrt{2+3\Theta}}\frac{1}{\left(\sqrt{1+\nu}\right)^3}\partial_\xi^3\phi_1 = 0. \tag{5.1.23}$$

Then, we can manipulate Eq. (5.1.23), identifying $\nu = \frac{3\Theta}{2}$,

$$\begin{aligned}
&\partial_\tau\phi_1 + \frac{\sqrt{6}\,(8+9\Theta)}{6\sqrt{2+3\Theta}}\sqrt{\frac{2}{2+3\Theta}}\phi_1\partial_\xi\phi_1 \\
&\qquad\qquad + \frac{\sqrt{6}}{18\sqrt{2+3\Theta}}\left(\sqrt{\frac{2}{2+3\Theta}}\right)^3\partial_\xi^3\phi_1 = 0 \\
&\partial_\tau\phi_1 + \frac{2\sqrt{3}}{6}\frac{8+9\Theta}{2+3\Theta}\phi_1\partial_\xi\phi_1 + \frac{2\sqrt{3}}{18}\frac{1}{\sqrt{2+3\Theta}}\frac{2}{\left(\sqrt{2+3\Theta}\right)^3}\partial_\xi^3\phi_1 = 0 \\
&\partial_\tau\phi_1 + \frac{\sqrt{3}}{3}\frac{8+9\Theta}{2+3\Theta}\phi_1\partial_\xi\phi_1 + \frac{2\sqrt{3}}{9}\frac{1}{(2+3\Theta)^2}\partial_\xi^3\phi_1 = 0,
\end{aligned} \tag{5.1.24}$$

and we can write the KdV-like equation as

$$\begin{aligned}
&\partial_\tau\phi_1 + A\phi_1\partial_\xi\phi_1 + B\partial_\xi^3\phi_1 = 0 \quad \text{with} \\
&\qquad A = \frac{1}{\sqrt{3}}\frac{(9\Theta+8)}{(3\Theta+2)}, \quad B = \frac{2}{3\sqrt{3}}\frac{1}{(3\Theta+2)^2},
\end{aligned} \tag{5.1.25}$$

which is exactly the corrected expression obtained earlier, Eq. (4.2.428), since $\lambda_0 = \frac{1}{\sqrt{3}}$ in this case.

Unmagnetized Adiabatic Plasma

For the adiabatic regime, from Eq. (4.2.198), we have

$$\partial_\tau\phi_1 + \frac{2\,(2+5\Theta)}{3}\sqrt{\frac{6}{2+5\Theta}}\phi_1\partial_\xi\phi_1 + \frac{1}{18}\sqrt{\frac{6}{2+5\Theta}}\partial_\xi^3\phi_1 = 0, \tag{5.1.26}$$

and proceeding similarly to the previous situation, adopting

$$ {}^{Adb}V_0 = V_0 \frac{\lambda_0}{\lambda_0^{Adb}} \quad \text{with} \quad \frac{\lambda_0}{\lambda_0^{Adb}} = \sqrt{1+\nu}, \tag{5.1.27} $$

we get, by combining Eqs. (5.1.26) and (5.1.27),

$$ \begin{aligned} \partial_\tau \phi_1 + \frac{2}{3} \frac{(2+5\Theta)}{\sqrt{1+\nu}} \sqrt{\frac{6}{2+5\Theta}} \phi_1 \partial_\xi \phi_1 \\ + \frac{1}{18\left(\sqrt{1+\nu}\right)^3} \sqrt{\frac{6}{2+5\Theta}} \partial_\xi^3 \phi_1 = 0. \end{aligned} \tag{5.1.28} $$

So, since $\nu = \frac{5\Theta}{2}$, it is possible to manipulate Eq. (5.1.28), getting

$$ \begin{aligned} \partial_\tau \phi_1 + \frac{2\sqrt{6}\,(2+5\Theta)}{3\sqrt{2+5\Theta}} \sqrt{\frac{2}{2+5\Theta}} \phi_1 \partial_\xi \phi_1 \\ + \frac{1}{18} \sqrt{\frac{6}{2+5\Theta}} \left(\sqrt{\frac{2}{2+5\Theta}} \right)^3 \partial_\xi^3 \phi_1 = 0 \\ \partial_\tau \phi_1 + \frac{4\sqrt{3}}{3} \frac{(2+5\Theta)}{\left(\sqrt{2+5\Theta}\right)^2} \phi_1 \partial_\xi \phi_1 + \frac{4\sqrt{3}}{18} \frac{1}{(2+5\Theta)^2} \partial_\xi^3 \phi_1 = 0 \\ \partial_\tau \phi_1 + \frac{4\sqrt{3}}{3} \phi_1 \partial_\xi \phi_1 + \frac{2\sqrt{3}}{9} \frac{1}{(2+5\Theta)^2} \partial_\xi^3 \phi_1 = 0, \end{aligned} \tag{5.1.29} $$

being Eq. (5.1.29) a KdV-like expression,

$$ \begin{aligned} \partial_\tau \phi_1 + A \phi_1 \partial_\xi \phi_1 + B \partial_\xi^3 \phi_1 = 0 \quad \text{with} \\ A = \frac{4}{\sqrt{3}}, \quad B = \frac{2}{3\sqrt{3}} \frac{1}{(2+5\Theta)^2}. \end{aligned} \tag{5.1.30} $$

Thus, we recognize Eq. (4.2.449), substituting the respective λ_0 value.

Magnetized Isothermal Plasma

The results of the magnetized approach to the isothermal system is shown in Eq. (4.2.310), which can be manipulated to

$$ \begin{aligned} \partial_\tau \phi_1 + \frac{l_z\,(9\Theta+8)}{6} \sqrt{\frac{6}{2+3\Theta}} \phi_1 \partial_\xi \phi_1 \\ + \frac{1}{18 l_z} \sqrt{\frac{6}{2+3\Theta}} \left(l_z^2 + 9 \frac{l_z^2\,(1-l_z^2)}{\Omega_{p_i}^2} \left(\frac{2+3\Theta}{6} \right)^2 \right) \partial_\xi^3 \phi_1 = 0, \end{aligned} \tag{5.1.31} $$

and assuming the usual process, as

$$ {}^{\mathrm{Iso}}V_0 = V_0 \frac{\lambda_0}{\lambda_0^{\mathrm{Iso}}} \quad \text{with} \quad \frac{\lambda_0}{\lambda_0^{\mathrm{Iso}}} = \sqrt{1+\nu}, \tag{5.1.32} $$

we can write from Eq. (5.1.31)

$$ \begin{aligned} &\partial_\tau \phi_1 + \frac{l_z\,(9\Theta+8)}{6\sqrt{1+\nu}} \sqrt{\frac{6}{2+3\Theta}} \phi_1 \partial_\xi \phi_1 \\ &+ \frac{1}{18 l_z \left(\sqrt{1+\nu}\right)^3} \sqrt{\frac{6}{2+3\Theta}} \left(l_z^2 + 9 \frac{l_z^2\left(1-l_z^2\right)}{\Omega_{p_i}^2} \left(\frac{2+3\Theta}{6}\right)^2 \right) \partial_\xi^3 \phi_1 = 0. \end{aligned} \tag{5.1.33} $$

Since we have $\nu = \frac{3\Theta}{2}$, it is possible to write

$$ \begin{aligned} &\partial_\tau \phi_1 + \frac{l_z\sqrt{3}\,(9\Theta+8)}{6} \left(\sqrt{\frac{2}{2+3\Theta}}\right)^2 \phi_1 \partial_\xi \phi_1 \\ &\quad + \frac{\sqrt{3}}{18 l_z} \left(\sqrt{\frac{2}{2+3\Theta}}\right)^4 \left(l_z^2 + 9 \frac{l_z^2\left(1-l_z^2\right)}{\Omega_{p_i}^2} \left(\frac{2+3\Theta}{6}\right)^2 \right) \partial_\xi^3 \phi_1 = 0 \\ &\partial_\tau \phi_1 + \frac{2\sqrt{3}}{6} \frac{l_z\,(9\Theta+8)}{(2+3\Theta)} \phi_1 \partial_\xi \phi_1 \\ &\quad + \frac{4\sqrt{3}}{18 l_z} \frac{1}{(2+3\Theta)^2} \left(l_z^2 + 9 \frac{l_z^2\left(1-l_z^2\right)}{\Omega_{p_i}^2} \left(\frac{2+3\Theta}{6}\right)^2 \right) \partial_\xi^3 \phi_1 = 0 \\ &\partial_\tau \phi_1 + \frac{\sqrt{3}}{3} \frac{l_z\,(9\Theta+8)}{(2+3\Theta)} \phi_1 \partial_\xi \phi_1 \\ &\quad + \left[\frac{2\sqrt{3}}{9} \frac{l_z}{(2+3\Theta)^2} + \frac{\sqrt{3}}{18} \frac{l_z\left(1-l_z^2\right)}{\Omega_{p_i}^2} \right] \partial_\xi^3 \phi_1 = 0, \end{aligned} \tag{5.1.34} $$

and we obtain the KdV-like equation,

$$ \begin{aligned} &\partial_\tau \phi_1 + A \phi_1 \partial_\xi \phi_1 + B \partial_\xi^3 \phi_1 = 0 \quad \text{with} \\ &\quad A = \frac{l_z}{\sqrt{3}} \frac{(9\Theta+8)}{(2+3\Theta)}, \;\; B = \frac{1}{3\sqrt{3}} \left[\frac{2 l_z}{(2+3\Theta)^2} + \frac{l_z\left(1-l_z^2\right)}{2\Omega_{p_i}^2} \right]. \end{aligned} \tag{5.1.35} $$

Note that if we rewrite Eq. (5.1.35) to be expressed in terms of $\delta = 1 + \frac{3}{2}\Theta$, we get

$$ \partial_\tau \phi_1 + \frac{l_z}{\delta} \frac{(9\Theta+8)}{2\sqrt{3}} \phi_1 \partial_\xi \phi_1 + \frac{1}{6\sqrt{3}} \left[\frac{l_z}{\delta^2} + \frac{l_z\left(1-l_z^2\right)}{\Omega_{p_i}^2} \right] \partial_\xi^3 \phi_1 = 0 \tag{5.1.36} $$

allowing us to identify Eq. (4.2.477), since $\lambda_0^2 = \frac{l_z^2}{3}$.

Magnetized Adiabatic Plasma

Finally, we will start the analysis of the magnetized adiabatic configuration from Eq. (4.2.366), which can be expressed as

$$\begin{aligned}\partial_\tau\phi_1 &+ 4l_z\sqrt{\frac{2+5\Theta}{6}}\phi_1\partial_\xi\phi_1\\ &+\frac{1}{18l_z}\sqrt{\frac{6}{2+5\Theta}}\left(l_z^2+9l_z^2\frac{(2+5\Theta)^2}{36}\frac{\left(1-l_z^2\right)}{\Omega_{p_i}^2}\right)\partial_\xi^3\phi_1=0\end{aligned} \tag{5.1.37}$$

and using the relation

$$^{Adb}V_0 = V_0\frac{\lambda_0}{\lambda_0^{Adb}} \quad \text{with} \quad \frac{\lambda_0}{\lambda_0^{Adb}}=\sqrt{1+\nu}, \tag{5.1.38}$$

we can manipulate Eq. (5.1.37) to write

$$\begin{aligned}\partial_\tau\phi_1 &+ \frac{4l_z}{\sqrt{1+\nu}}\sqrt{\frac{2+5\Theta}{6}}\phi_1\partial_\xi\phi_1\\ &+\frac{1}{18l_z}\sqrt{\frac{6}{2+5\Theta}}\frac{1}{\left(\sqrt{1+\nu}\right)^3}\left(l_z^2+l_z^2\frac{(2+5\Theta)^2}{4}\frac{\left(1-l_z^2\right)}{\Omega_{p_i}^2}\right)\partial_\xi^3\phi_1=0.\end{aligned} \tag{5.1.39}$$

Thus, we have

$$\begin{aligned}&\partial_\tau\phi_1 + 4l_z\sqrt{\frac{2}{2+5\Theta}}\sqrt{\frac{2+5\Theta}{6}}\phi_1\partial_\xi\phi_1\\ &\qquad+\frac{\sqrt{3}}{18l_z}\left(\sqrt{\frac{2}{2+5\Theta}}\right)^4\left(l_z^2+l_z^2\frac{(2+5\Theta)^2}{4}\frac{\left(1-l_z^2\right)}{\Omega_{p_i}^2}\right)\partial_\xi^3\phi_1=0\\ &\partial_\tau\phi_1+\frac{4l_z}{\sqrt{3}}\phi_1\partial_\xi\phi_1\\ &\qquad+\frac{4\sqrt{3}}{18}\frac{1}{l_z\left(2+5\Theta\right)^2}\left(l_z^2+l_z^2\frac{(2+5\Theta)^2}{4}\frac{\left(1-l_z^2\right)}{\Omega_{p_i}^2}\right)\partial_\xi^3\phi_1=0\\ &\partial_\tau\phi_1+\frac{4l_z}{\sqrt{3}}\phi_1\partial_\xi\phi_1+\left[\frac{2}{3\sqrt{3}}\frac{l_z}{(2+5\Theta)^2}+\frac{l_z}{6\sqrt{3}}\frac{\left(1-l_z^2\right)}{\Omega_{p_i}^2}\right]\partial_\xi^3\phi_1=0,\end{aligned} \tag{5.1.40}$$

identifying from Eq. (5.1.40) the KdV-like expression,

$$\begin{aligned}&\partial_\tau\phi_1+A\phi_1\partial_\xi\phi_1+B\partial_\xi^3\phi_1=0 \quad \text{with}\\ &\quad A=\frac{4l_z}{\sqrt{3}},\ B=\frac{1}{3\sqrt{3}}\left[\frac{2l_z}{(2+5\Theta)^2}+\frac{l_z\left(1-l_z^2\right)}{2\Omega_{p_i}^2}\right].\end{aligned} \tag{5.1.41}$$

Rewriting Eq. (5.1.41) as a function of β, with $\beta = 1 + \frac{5}{2}\Theta$,

$$\partial_\tau \phi_1 + \frac{4 l_z}{\sqrt{3}} \phi_1 \partial_\xi \phi_1 + \frac{1}{6\sqrt{3}} \left[\frac{l_z}{\beta^2} + \frac{l_z \left(1 - l_z^2\right)}{\Omega_{p_i}^2} \right] \partial_\xi^3 \phi_1 = 0, \quad (5.1.42)$$

which is Eq. (4.2.503) presented above, conveniently substituting the relation $\lambda_0^2 = \frac{l_z^2}{3}$.

5.2 CONCLUSION

In this work, we have investigated the nonlinear ion acoustic oscillations described by the Korteweg-de Vries equation in a Thomas-Fermi gas. Initially, we proposed a non-thermal configuration composed of cold ions and degenerate electrons, in which the density is given by the TF distribution instead of the Boltzmann one, focusing on verifying the KdV description and its solutions.

Moving away from the standard process, which consists of investigating the analytical properties from the pseudo-potential, commonly referred to as the Sagdeev potential, we made an analysis from the variational method to determine the existence of steady solutions only for a well-defined Mach number value, where $M = \frac{1}{\sqrt{3}}$ for our system. We then use the M excess to define the expansion parameter, ϵ, which is associated with the stretched coordinates stated by the modified reductive perturbation method. By modified RP we are calling the introduction of the λ_0 parameter in the theory, and this modification gives us the possibility to control our description by ${}^{J}V_0\lambda_0^{J} = V_0\lambda_0$, whatever the configuration under analysis.

Furthermore, the derivation of the KdV equation allows us to conclude that in the cold regime the nonlinear effects are stronger than those of a Boltzmann gas, while the dispersive ones are weaker. To complete the non-thermal study, we set a magnetic field in a specific direction in order to see the influences of plasma magnetization on the results, and then we see an explicit dependence of the nonlinear term on the magnetic field direction and also a composition between the directional cosine and the field magnitude to determine the consequences of dispersion.

Investigations of warm regimes, from the consideration of isothermal and adiabatic ions embedded in a background of non-thermal electrons, assuming corrected system normalization, were made for both non-magnetized and magnetized cases. Isothermally, the two considerations return a dependence of the nonlinear and dispersive terms on the fluid temperature and, where appropriate, on the magnetic field direction. On the other hand, the nonlinear parameters in adiabatic fluids are independent of temperature, while the dispersion terms are weighted by it, and both are changed depending on the directional cosine. Magnetized dispersive sets carry the consequences of magnetic intensity.

Finally, besides the graphical examination, we have used the constant control parameter relation to adjust the distinguished thermal TF plasma approaches, ranging from the cold normalization to the warm expressions, verifying the validity and the usability of the result.

Part IV

APPENDIX

A
PARAMETER EXPANSION

The main purpose of this appendix is to obtain the parameters expansion about the equilibrium to study ion acoustic waves in a magnetized plasma, which proved useful in Sec. 4.2. The deduction follows the reasoning presented in [21].

[21]:Nonlinear Waves, Solitons and Chaos

A.1 MAGNETIZED PLASMA APPROACH

To simplify the analysis, let us consider the magnetized cold TF plasma system with $\vec{B} = B_0\hat{z}$. The fluid dynamics is described in a dimensionless form by

$$\partial_t\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}\phi + \frac{\Omega}{\omega_{p_i}}\left(\vec{v}\times\hat{z}\right), \tag{A.1.1}$$

$$\partial_t n + \vec{\nabla}\cdot(n\vec{v}) = 0, \tag{A.1.2}$$

$$\nabla^2\phi = -\left(n - n_e\right), \tag{A.1.3}$$

where we are considering the transformations presented in Eq. (4.2.34), and then we can use the stretched coordinates defined as

$$\xi = \epsilon^{\frac{1}{2}}\left(l_x x + l_y y + l_z z - \lambda_0 t\right) \qquad \tau = \epsilon^{\frac{3}{2}} t, \tag{A.1.4}$$

and the derivative implications,

$$\partial_{x_j} = \epsilon^{\frac{1}{2}} l_j \partial_\xi \qquad \partial_t = \epsilon^{\frac{3}{2}}\partial_\tau - \epsilon^{\frac{1}{2}}\lambda_0\partial_\xi \qquad \partial^2_{x_j} = \epsilon l_j^2 \partial_\xi^2, \tag{A.1.5}$$

to get the governing equations,

$$\begin{aligned}-\lambda_0\epsilon^{\frac{1}{2}}\partial_\xi v_j + \epsilon^{\frac{3}{2}}\partial_\tau v_j + \epsilon^{\frac{1}{2}}\left(\sum_{k=1}^{3} l_k v_k\right)\partial_\xi v_j \\ = -l_j\epsilon^{\frac{1}{2}}\partial_\xi\phi + \Omega_{p_i}\left(\vec{v}\times\hat{z}\right)_j,\end{aligned} \tag{A.1.6}$$

$$-\lambda_0\epsilon^{\frac{1}{2}}\partial_\xi n + \epsilon^{\frac{3}{2}}\partial_\tau n + \epsilon^{\frac{1}{2}}\left[\sum_{k=1}^{3} l_k\partial_\xi\left(n v_k\right)\right] = 0, \tag{A.1.7}$$

$$\epsilon\partial_\xi^2\phi = -\left(n - n_e\right), \tag{A.1.8}$$

with Eq. (A.1.6) returning to each Cartesian direction, as we know,

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_x + \epsilon^{\frac{3}{2}} \partial_\tau v_x + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} l_k v_k \right) \partial_\xi v_x = -l_x \epsilon^{\frac{1}{2}} \partial_\xi \phi + \Omega_{p_i} v_y, \tag{A.1.9}$$

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_y + \epsilon^{\frac{3}{2}} \partial_\tau v_y + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} l_k v_k \right) \partial_\xi v_y = -l_y \epsilon^{\frac{1}{2}} \partial_\xi \phi - \Omega_{p_i} v_x, \tag{A.1.10}$$

$$-\lambda_0 \epsilon^{\frac{1}{2}} \partial_\xi v_z + \epsilon^{\frac{3}{2}} \partial_\tau v_z + \epsilon^{\frac{1}{2}} \left(\sum_{k=1}^{3} l_k v_k \right) \partial_\xi v_z = -l_z \epsilon^{\frac{1}{2}} \partial_\xi \phi. \tag{A.1.11}$$

We expect that the dependent variables take the form

$$n = 1 + \epsilon^{r_n} n_1 + \cdots, \tag{A.1.12}$$

$$\phi = \epsilon^{r_\phi} \phi_1 + \cdots, \tag{A.1.13}$$

$$v_{x,y} = \epsilon^{r_{(x,y)}} v_{1(x,y)} + \cdots, \tag{A.1.14}$$

$$v_z = \epsilon^{r_z} v_{1z} + \cdots, \tag{A.1.15}$$

and to lowest order, meaningful equations can only be obtained if

$$r_n = r_\phi = r_z, \tag{A.1.16}$$

resulting in the following relations

$$n_1 = \frac{l_z}{\lambda_0} v_{1z}, \tag{A.1.17}$$

$$v_{1z} = \frac{l_z}{\lambda_0} \phi_1, \tag{A.1.18}$$

$$n_1 = 3\phi_1, \tag{A.1.19}$$

where we expanded the electron TF distribution, n_e, considering the dimensionless expression presented in Eq. (4.2.70). Thus, from Eq. (A.1.16), we have that $r_\parallel \equiv r_z$ will yield all other powers from the lower order equations, with Eqs. (A.1.9) and (A.1.10) then returning

$$r_{(x,y)} = r_\parallel + \frac{1}{2}. \tag{A.1.20}$$

To generate all other r's we introduce a nonlinear argument stating that for the next order we have a balance of nonlinearity, $v_\parallel \partial_\xi v_\parallel$, and dispersion, $\partial_\xi^3 v_\parallel$, i. e., we assume a competition between nonlinear and dispersive effects. So, we obtain from the exponents of the expansion

$$2r_{\parallel} + \frac{1}{2} = r_{\parallel} + \frac{3}{2}, \tag{A.1.21}$$

with Eq. (A.1.21) giving

$$r_{\parallel} = 1, \tag{A.1.22}$$

and we have, therefore, obtained the entire expansion by invoking one nonlinear argument.

B KORTEWEG-DE VRIES EQUATION SOLUTION

This appendix contains a simple deduction of the KdV equation one-soliton solution, which we used in the approach adopted in Sec. 5.1. The same steps can be taken from a reading of [14].

[14]:*Physics of Solitons*

B.1 SOLITON SOLUTION

Writing the Korteweg-de Vries equation in a general way, with A and B being the nonlinear and dispersive factors, respectively,

$$\partial_\tau \phi_1 + A\phi_1 \partial_\xi \phi_1 + B\partial_\xi^3 \phi_1 = 0, \tag{B.1.1}$$

we can briefly deduce the well-known solutions. To get steady profiles, we can introduce a new variable χ,

$$\chi = \xi - \mathcal{U}\tau, \tag{B.1.2}$$

where $\mathcal{U}$ is the V_0-normalized velocity term, and the transformed derivatives

$$\partial_\xi = d_\chi \qquad \partial_\tau = -\mathcal{U} d_\chi, \tag{B.1.3}$$

which imply

$$-\mathcal{U} d_\chi \phi_1 + A\phi_1 d_\chi \phi_1 + B d_\chi^3 \phi_1 = 0. \tag{B.1.4}$$

Admitting as appropriate boundary conditions

$$\left(\phi_1, d_\chi \phi_1, d_\chi^2 \phi_1\right) \to 0 \quad \text{as} \quad \chi \to \pm\infty, \tag{B.1.5}$$

and integrating Eq. (B.1.4), it is possible to write

$$-\mathcal{U}\phi_1 + \frac{A}{2}\phi_1^2 + B d_\chi^2 \phi_1 = 0. \tag{B.1.6}$$

Now, multiplying Eq. (B.1.6) by $d_\chi \phi_1$ and then integrating, we obtain

$$d_\chi \phi_1 = \phi_1 \sqrt{\frac{\mathcal{U}}{B} - \frac{A}{3B}\phi_1}, \tag{B.1.7}$$

whose solution returns

$$\phi_1 = \frac{3\mathcal{U}}{A} \operatorname{sech}^2\left(\sqrt{\frac{\mathcal{U}}{4B}}\chi\right). \tag{B.1.8}$$

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