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Perturbações em torno de Buracos Negros e seus Duais Algébricos

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Dissertação apresentada ao Instituto de Física da Universidade de São Paulo para a obtenção do título de Mestre em Ciências

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Perturbations around Black Holes and their Algebraic Duals

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A thesis submitted to the Physics Institute in partial fullfilment of the requirements for the degree of *Magister Scientiarum in Physics*

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To my beloved parents and grandma,
they mean everything to me.

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Resumo

Nesta tese, nós estabelecemos algumas correspondências entre a dinâmica de campos escalares clássicos em certos espaço-tempos de fundo e duais algébricos apropriados. Os cenários estudados incluem soluções tipo buraco negro com constante cosmológica não nula e os espaços-tempos conhecidos como *geometrias quase-extremas*. Com base em várias propostas na literatura, associamos certos elementos próprios da dinâmica escalar a uma representação apropriada de uma álgebra relacionada com a invariância das equações de movimento escalar sob transformações conformes. Em particular, nós associamos representações de peso maior de dimensão infinita da álgebra $\mathfrak{sl}(2,\mathbb{R})$ com modos e frequências quase-normais de campo escalar em geometrias quase-extremas e geometrias assintóticamente Anti de Sitter.

Abstract

In this thesis, we establish some correspondences between dynamics of classical scalar fields in certain background spacetimes and appropriate algebraic duals. The scenarios studied include black hole solutions with non-zero cosmological constant and the spacetimes known as *near extremal geometries*. Based on several proposals in the literature, we associate certain elements proper to scalar perturbative dynamics to an appropriated representation of an algebra related with the invariance of the scalar equations of motion under conformal transformations. In particular, we associate infinite dimensional highest weight representations of the algebra $\mathfrak{sl}(2,\mathbb{R})$ to quasinormal modes and frequencies of scalar fields in near extremal geometries and asymptotically Anti de Sitter geometries.

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Chapter 1

Introduction

Black holes are one of the more interesting predictions of general relativity and understanding their properties is relevant for both astrophysical observations and the formulation of new physical theories. In the framework of general relativity, black holes are understood as compact objects, concentrating the largest amount of energy in the smallest possible volume; and their defining feature is the existence of an event horizon, a surface past which nothing can leave the black hole. In that sense black holes can be thought of as perfect black bodies since they always absorb but never radiate. In semi-classical approaches considering quantum fields on classical spacetime backgrounds, black holes are found to actually evaporate from the emission of thermal radiation with a characteristic temperature, called Hawking temperature, originating from quantum fluctuations at the event horizon. This led to a surprising connection between black hole physics and thermodynamics [1], in which black holes are the systems with the maximum possible amount of entropy, proportional to the area of the event horizon, but also to some additional conceptual problems regarding the laws of quantum mechanics and the nature of Hawking radiation.

There are still many open problems concerning black holes, both conceptually and observationally. Given their nature, the detection of black holes is difficult, but there has been increasingly indirectly evidence for their existence. It is believed that the milky way has a super massive black named Sagittarius A* with 4.6×10^6 solar masses right at its center [2], and it is thought that the same occurs for a large number of galaxies [3]. In order to gain a better understanding on the nature of black holes and how to detect them, it is important to study how black holes react to perturbations from other forms of matter and energy. Realistic black holes are believed to be formed after the gravitational collapse of massive stars at the final stages of their life, a process that releases large amounts of energy. It is also expected that once a black hole has formed, it continues to grow by absorbing additional matter, such as gas or interstellar dust, which can emit vast amounts of radiation as it falls into the horizon.

From the study of black hole perturbations we can gain insight on what we should look for in astronomical observations. The study of black hole perturbations goes back to the work of Regge and Wheeler [4] in the 50's, where they studied the stability of the Schwarzschild black hole under perturbations. A classical treatment of the subject is given by Chandrasekhar [5], where scalar, electromagnetic and gravitational perturbations on the most familiar black hole scenarios are discussed, including the Schwarzschild, Reissner-Nordstrom and Kerr solutions.

An important characteristic of field perturbations are the so called quasinormal modes. After some transient period, perturbations outside the event horizon of a black hole are followed by oscillations with characteristic frequencies. This oscillations are exponentially damped, such that the associated frequencies, known as quasinormal frequencies, are complex. Quasinormal modes are important because they depend on the black hole properties, but not so much on the details of the initial perturbation. They can be thought of as resonances of the background spacetime and it should be possible to characterize the nature of the black hole from the quasinormal frequencies. However, the analytic determination of quasinormal frequencies is not always possible. In fact, even for well-known geometries such as the Schwarzschild black hole, exact expressions for quasinormal frequencies are not available, and numerical methods or approximations are usually required. Revisions of the subject are found in Nollert [6], and Kokkotas [7], among other references.

The study of black hole perturbations and quasinormal modes has also acquired relevance for other reasons. A recent line of research in theoretical physics has been the pursue for correspondences between otherwise different physical theories, including the gauge/gravity dualities where, in principle, it could be possible to describe gravitational phenomena in terms of some gauge field theory which does not include gravitational interaction [8, 9]. This approach is specially meaningful if one of the theories is found to be difficult to solve but the dual theory is well understood. If these correspondences between gauge theories and gravity are valid, it should be possible to relate black hole perturbations with the correlations functions of a gauge theory.

The main feature of gauge field theories is the invariance of the dynamics under continuous local transformations, so every gauge theory is specified by a continuous *Lie group*. Following this line of thought, in recent works such as Castro *el al.*, [10], Krishnan [11], Chen *el al.* [12, 13], among others, it is suggested that in certain spacetimes and under specific limits, the dynamics of scalar fields is invariant under conformal transformations, establishing a direct relation with the quasinormal modes on those scenarios. These works are the base of this thesis and we aim to find further scenarios where we can apply this reasoning.

The main purpose of this thesis is to establish certain relations between perturbative dynamics of classical fields on certain spacetimes and appropriated algebraic structures. In this work we are particularly interest in certain near extremal geometries. These are spacetimes admitting the existence of two Killing horizons, corresponding to either a event horizon of a black hole or a cosmological horizon, and a certain limit where both Killing horizons become arbitrarily close. The advantage of working with these spacetimes is that the scalar dynamics is greatly simplified, and they can be used to approximate more complicated scenarios. Our goal is to establish a correspondence between perturbative quantities (quasinormal modes and spectrum of quasinormal frequencies) of certain geometries and algebraic elements associated with a representation of a conformal symmetry. In these scenarios, it should be possible to characterize the evolution of classical scalar perturbations with proper algebraic duals related to the invariance of the dynamics under conformal transformations, which in our case will be transformations under the special linear group $SL(2)$ and its algebra $\mathfrak{sl}(2)$.

The outline of this thesis is as follows: In chapter 2 we present a review on the basic topics of general relativity, where we focus on how the symmetries of a spacetime are associated with Killing vectors. We introduce the ideas of staticity, stationarity and spherical symmetry, ultimately defining the concept of what a black hole is, with details of the most well known example: the Schwarzschild solution. In chapter 3 we introduce the relevant geometries for our thesis, which include the generalizations of the Schwarzschild solution to spacetimes with non-zero cosmological constant, namely the Schwarzschild-de Sitter and Schwarzschild-Anti de Sitter spacetimes. We also introduce properly the idea of near extremal geometries explored in the present work.

After having introduced the tools to study general relativity and the geometries of interest, in chapter 4 we proceed to discuss how perturbations of fields are treated on a background spacetime, in particular we develop on the dynamics of scalar fields on static and spherically symmetric spacetimes and we define the notion quasinormal modes and frequencies and their relevance for the evolution of a field perturbation.

In the last two chapters we treat the connection between field dynamics and algebra representations. In chapter 5 we introduce some details on group theory, focusing on Lie groups and representation theory of Lie algebras, and we eventually introduce the group $SL(2)$ and algebra $\mathfrak{sl}(2)$. Finally, in chapter 6 we present the results of our work, where we find a explicit representation of the algebra $\mathfrak{sl}(2)$ in terms of differential operators, from which we are able to obtain the spectrum of quasinormal frequencies for the near extremal geometries. We also present another representation which could be used to model the scalar dynamics in asymptotically Anti de Sitter spacetimes. Final comments and conclusions are presented in chapter 7.

In the development of this thesis, the signature of the metric tensor is $(-,+.+,+)$. With the exception of section 2.2, we take units where $c = 1$, $G = 1$.

Chapter 2

General Relativity

Einstein's general relativity is currently the most widely accepted theory to describe the gravitational interaction and is central to the understanding of a great array of astrophysical phenomena such as black holes, gravitational waves, and the expansion of the universe [14], but remarkably, it is the only known interaction still resisting a consistent quantum mechanical description. Formulated by Albert Einstein in 1915 as an effort to reconcile Newton's gravitational theory with relativistic dynamics, general relativity treats gravity not as a force but as a consequence of a curved spacetime where matter and radiation act as the source of curvature [15, 16]. General relativity has been tested in the solar system matching with great accuracy with experimental observations, ranging from the correct prediction of perihelion precession of Mercury's orbit, the deflection of light by massive bodies and the gravitational redshift of light [17].

General relativity is based upon two principles:

- *Equivalence principle*: Although there is not an unique consensus in the exact formulation of this principle, at its heart, the equivalence principle states the impossibility for an observer to distinguish locally between an acceleration in his own reference frame and the effects of a gravitational field. The base of this principle is the equivalence between inertial and gravitational mass holding for every body, regardless of size or composition, [16].

- *Principle of general covariance*: This principle is based on the requirement for all physics laws to have the same formulation in all reference frames, meaning that there is not such thing as a preferred reference frame. Moreover, special relativity should hold at least locally. The global Lorentz covariance of special relativity becomes a local Lorentz covariance when gravity is introduced [18].

The main feature setting apart gravity from the other known fundamental interactions is its universal character; the gravitational field couples to all forms of matter and energy. Because of that, it is not possible to define a real inertial observer able to properly measure effects of the gravitational field, as it will experience the effects of gravity as well [19, 20].

2.1 Elements of differential geometry

The mathematical formalism used in general relativity is differential geometry, and spacetime, which is the main object of study, is described as a differentiable manifold. An informal idea of a manifold is a space that locally looks like $\mathbb{R}^n$ but globally may posses a nontrivial structure. To provide a formal definition of a manifold some preliminary concepts are introduced :

- Given a set M, a *chart* (also called coordinate system) $\{U,\phi\}$ is a subset U of M along with a one-to-one map $\phi : U \to \mathbb{R}^n$ such that $\phi(U)$ is an open set in $\mathbb{R}^n$, making U an open set in M [21, 22].
- An *atlas* is a collection of charts $\{U_i,\phi_i\}$ such that (i) the union of the subsets U_i covers M and (ii) if two charts overlap, $U_\alpha \cap U_\beta$, the map $(\phi_\alpha \circ {\phi_\beta}^{-1})$ takes points in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ onto an open set $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ [21] .

The last requirement means that if two charts overlap in a certain region of M, there must be a C^∞ continuous coordinate transformation between both charts. With this concepts established a more precise definition of the idea of manifold is the following: A smooth n-dimensional *manifold* is a set M along with a maximal atlas, that is, an atlas that contains every possible compatible chart [15, 20, 22].

In general relativity spacetime is treated as a continuous, connected four dimensional manifold. A point in spacetime is called an event, with three spatial and one temporal coordinate. A coordinate system is by no means unique and physical quantities should be independent from a particular choice of coordinates .

To every point p of a M can be associated the set of tangent vectors of every curve passing through p. These vectors form a vector space V since they can be added together and multiplied by scalars. The vector space V is called *tangent space*. To every smooth function $f : M \to \mathbb{R}$ can be associated a directional derivative with respect to a curve γ passing through p; if such curve is parameterized by a certain $\lambda \in \mathbb{R}$, the directional derivative of f is given by

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\partial_\mu f\,, \tag{2.1}$$

where $\{\partial_\mu\}$ are partial derivatives with respect to some coordinate chart $\{x^\mu\}$. Since f is taken to be an arbitrary function, the directional derivative operation is given by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\partial_\mu \,, \tag{2.2}$$

therefore $\{\partial_\mu\}$ represent a basis for the vector space of directional derivative operators along curves through p, and thus, of the tangent space T_p. This kind of basis is known as coordinate basis; elements of this basis are in general not normalized to unity nor orthogonal to each other.

If $\{\partial_\mu\}$ is a basis for the tangent space at p, any element $V \in T_p$ can be written as $V = V^\mu \partial_\mu$. According to the chain rule of partial derivatives, $\{\partial_\mu\}$ transforms under a change of coordinates $x^\mu \to x^{\mu'}$ as

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu \,, \tag{2.3}$$

thus, for V to remain invariant under this transformation, the components V^μ must transform in the following manner

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}V^\mu . \tag{2.4}$$

If the components of a vector V transform as (2.4), V is said to be a contravariant vector. Given two vector fields X, Y, it is defined the Lie bracket $[X,Y]$ as

$$[X,Y]f \equiv X(Y(f)) - Y(X(f)) \tag{2.5}$$

which in components is

$$[X,Y]^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu \,. \tag{2.6}$$

To every tangent space it can be associated the cotangent space T_p^* as the set of linear maps $\omega : T_p \to \mathbb{R}$. If elements of T_p are identified with directional derivatives of a function f in p, elements on T_p^* can be identified with the gradient $\mathrm{d}f$ of such function. Following the same argument, just as partial derivatives $\{\partial_\mu\}$ with respect to the coordinate functions x^μ constitute a basis for T_p, the gradients $\{\mathrm{d}x^\mu\}$ of the coordinates x^μ provide a basis for the cotangent space T_p^*.

Any element of T_p^* can be expanded as $\omega = \omega_\mu \mathrm{d}x^\mu$. Since under a change of coordinates $x^\mu \to x^{\mu'}$ gradients transform as

$$\mathrm{d}x^{\mu'} = \frac{\partial x_{\mu'}}{\partial x_\mu}\mathrm{d}x^\mu \,, \tag{2.7}$$

the components ω_μ must transform as

$$\omega_{\mu'} = \frac{\partial x_\mu}{\partial x_{\mu'}}\omega_\mu \,, \tag{2.8}$$

for ω to remain unchanged under this transformation. Elements of T_p^* whose components transform as (2.8) are known as covariant vectors or one-forms.

A generalization of the notion of vectors and dual vectors is the idea of tensor. A tensor T of rank (k,l) is a multilinear map from k copies of the cotangent space and l copies of the tangent space to $\mathbb{R}$,

$$T : \underbrace{T_p \otimes \cdots \times T_p}_{k \text{ copies}} \times \underbrace{T_p^* \times \cdots \times T_p^*}_{l \text{ copies}} \to \mathbb{R}\,. \tag{2.9}$$

In components, an arbitrary tensor T can be written as

$$T = T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_k} \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_l}\,. \tag{2.10}$$

Tensors are important in general relativity because a tensorial equation valid in a coordinate system will be valid in every other coordinate systems, as implied the principle of general covariance, suggesting that every equation describing physical quantities should be written in terms of tensors.

A tensor of fundamental importance in general relativity is the metric tensor, a symmetric $(0,2)$ tensor whose components are denoted as $g_{\mu\nu}$. This tensor allows to generalize the notion of Euclidean distance and scalar product of vectors in $\mathbb{R}^n$ to arbitrary curved manifolds. For two vectors V and W the scalar product (V,W) is defined as

$$(V,W) = g_{\mu\nu} V^\mu W^\nu\,, \tag{2.11}$$

where the line element $\mathrm{d}s$ is defined as

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu\,. \tag{2.12}$$

The metric tensor allows to determine of the path length between two points in a manifold, therefore providing a generalization of distance. The metric tensor generalizes the idea of vector norm as well, defined as the scalar product of the vector with itself

In general relativity the signature of the metric is $(-,+,+,+)$ (the signature are the signs of the eigenvalues of the metric). Manifolds with this metric signature are called pseudo-Riemannian or Lorentzian manifolds. In a pseudo-Riemannian metric the norm of a vector is not positive-defined, and vectors are classified according to the value of the norm

$$(V,V) = g_{\mu\nu} V^\mu V^\nu \begin{cases} < 0 & V \text{ is timelike} \\ = 0 & V \text{ is null} \\ > 0 & V \text{ is spacelike} \end{cases}\,. \tag{2.13}$$

This classification also applies to curves and surfaces: A timelike/null/spacelike curve is a curve whose tangent vector is timelike/null/spacelike at every point, and a timelike/null/spacelike surface is a surface whose normal vector is timelike/null/spacelike everywhere. The type of curve that a particle follows through spacetime depends on its mass: massive particles move along timelike curves while massless particles move move along null curves [15, 16].

Since general relativity generalizes Minkowski spacetime to arbitrarily curved Lorentzian spacetimes, the metric tensor also provides a notion of causality. In Minkowski spacetime a light cone, which is the path that a light beam emanating from a single event and traveling in all directions would take, has the same shape at every spacetime point. The same does not hold for an arbitrarily curved spacetime, instead, in general relativity it is said that two events are causally related if they can be connected by a causal curve, that is, a curve that is null or timelike everywhere.

2.1.1 Curvature

In a curved spacetime a generalization of partial derivatives must be introduced since, in general, tangent spaces at different points are not equal and thus, a direct comparison of vectors at different points is not plausible. One such generalization is the covariant derivative ∇, an operator mapping (k,l) tensors to $(k,l+q)$ tensors. As a generalization of partial derivatives, covariant derivatives should satisfy the properties characterizing a differential operator

1. Linearity: $\nabla(X+Y) = \nabla X + \nabla Y$.
2. Leibniz rule: $\nabla(X \otimes Y) = (\nabla X) \otimes Y + X \otimes (\nabla Y)$.

The covariant derivative of a contravariant vector V and a covariant vector ω are respectively (in component notation)

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda \,, \tag{2.14}$$

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu} \omega_\lambda \,, \tag{2.15}$$

where $\Gamma^\nu{}_{\mu\lambda}$ are called connection coefficients. These coefficients allows us to compare vectors between tangent spaces of nearby points. It can be shown that the connection coefficients do not transform as tensor components, however, the covariant derivative does have the transformation properties of a tensor [15],

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu \,. \tag{2.16}$$

To every connection can be associated a tensor knows as the torsion tensor

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} \,. \tag{2.17}$$

The connection coefficients are not unique, as they depend on the procedure employed to compare vectors at different tangent spaces. Nevertheless for every manifold there is a unique connection such that the covariant derivative of the metric with respect to that connection is zero at every point, $\nabla_{\mu} g^{\mu\nu} = 0$, and the associated torsion tensor is zero. This unique, metric compatible connection, is found to be expressed in terms of the metric components and their first order derivatives

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu} \right) \,, \tag{2.18}$$

and it is known as Christoffel connection. In general relativity it is usually assumed a metric compatible connection and a vanishing torsion tensor.

Contrary to the case of partial derivatives, covariant derivatives with respect to different coordinates do not commute; and it is precisely through this non-commutative behavior that the idea of curvature can be quantified. The commutator of covariant derivatives with respect to two different coordinates acting on a vector V is, in component notation,

$$[\nabla_{\mu}, \nabla_{\nu}] V^{\rho} = R^{\rho}{}_{\sigma\mu\nu} V^{\rho} - T^{\lambda}{}_{\mu\nu} \nabla_{\lambda} V^{\rho} \,, \tag{2.19}$$

where $T^{\lambda}{}_{\mu\nu}$ is the torsion tensor (2.17). The fist term defines a tensor of significant importance known as the Riemann tensor, a (1,3) rank tensor whose components are given by

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu} \Gamma^{\rho}{}_{\nu\sigma} - \partial_{\nu} \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda} \Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\sigma} \Gamma^{\lambda}{}_{\mu\sigma} \,. \tag{2.20}$$

If a coordinate system exists such that the components of the metric tensor are coordinate independent, the Riemann tensor will vanish. Another important tensor, known as the Ricci tensor, is defined from the Riemann tensor by contraction of index

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \,. \tag{2.21}$$

The trace of the Ricci tensor is called the Ricci scalar or curvature scalar and it is a quantity that remains invariant under coordinate changes

$$R = R^{\mu}{}_{\mu} \,. \tag{2.22}$$

In the absence of gravity, that is, in flat spacetime, particles move in straight lines. In curved spacetimes a generalization of straight line is the idea of geodesic, as the path of shortest distance between two points. A path $x^\mu(\tau)$, where τ is a parameter of motion, is a geodesic if the tangent vector $\mathrm{d}x^\mu/\mathrm{d}\tau$ satisfies

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\nabla_\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}=0\,, \tag{2.23}$$

which in terms of the covariant derivative (2.14) is known as the geodesic equation

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}+\Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{dx^\beta}{\mathrm{d}\tau}=0\,. \tag{2.24}$$

The world line of a particle free from all external, non-gravitational force, is a particular type of geodesic. In other words, a freely moving or falling particle always moves along a geodesic.

2.1.2 Diffeomorphims

Two manifolds M and N, not necessarily of the same dimension, can be related by some map $\varphi: M\to N$, a rule assigning to each element of M exactly one element of N. If such map exists, a linear map between tangent spaces in both spaces is induced. More precisely, for some $p\in M$, a linear map

$$\varphi^*: T_pM\to T_{\varphi(p)}N\,, \tag{2.25}$$

from the tangent space of M at p to the tangent space of N at $\varphi(p)$. If $v\in T_p(M)$, the vector $\varphi_*v\in T_{\varphi(p)}N$ is called the pushforward of v by φ. Vectors as defined by their action on functions as directional derivatives. If there is a function $f: N\to\mathbb{R}$ the action of φ_*v on f is defined to be equivalent to the action of v on the composition $f\circ\varphi^*: M\to\mathbb{R}$,

$$(\varphi^*v)(f)=v(f\circ\varphi)\,, \tag{2.26}$$

where "$\circ$" indicates composition of maps. Likewise, there is an associated linear map between cotangents spaces, relating dual vectors from $T^*_{\varphi(p)}(N)$ to dual vectors in $T^*_p(M)$

$$\varphi_*: T^*_{\varphi(p)}N\to T^*_p(M)\,, \tag{2.27}$$

if $\omega\in T^*_{\varphi(p)}N$ the pullback one form $\varphi_*\omega_\mu$ is defined requiring that for all $v\in T_p(M)$

$$(\varphi_*\omega_\mu)v^\mu=\omega_\mu(\varphi^*v)^\mu\,. \tag{2.28}$$

If M has coordinates $\{x^\mu\}$ and N has coordinates $\{y^\alpha\}$ the components of a vector $v \in T_pM$ and the pushforward vector $(\phi^* v) \in T_{\varphi(p)}N$ are related by

$$(\phi^* v)^\alpha = v^\mu \frac{\partial y^\alpha}{\partial x^\mu}\,, \tag{2.29}$$

therefore, it is possible to think of a pushforward as a matrix operator of the form $(\varphi^* v)^\alpha = (\varphi_* v)^\alpha{}_\mu v^\mu$ with components given by the Jacobian matrix of the map φ between coordinates

$$(\varphi^*)^\alpha{}_\mu = \frac{\partial y^\alpha}{\partial x^\mu}. \tag{2.30}$$

Since M and N are not necessarily of the same dimension, this pushforward matrix is in general not invertible.

If a map $\varphi : M \to N$ between two manifolds M and N is C^∞ is one-to-one and its inverse $\varphi^{-1} : N \to M$ is C^∞, the map φ is said to be a diffeomorphism. In that case M and N are necessarily of the same dimension, and are said to be diffeomorphic. In particular, the pushforward operation turns out to be invertible.

If M and N are the same manifold, a diffeomorphism induces a change of coordinate system: If $x^\mu : M \to \mathbb{R}$ is a coordinate function defined on M it is possible to define a new coordinate system by $(\varphi^* x)^\mu : M \to \mathbb{R}^n$. This transformation be seen as moving the points of the manifold and evaluate the coordinates at the new points, called active coordinate transformations, contrary to passive coordinate transformations, where new coordinates are introduced as functions of the previous ones.

2.1.3 Lie derivative

Diffeomorphims also provide another alternative to compare vectors at different spacetime points using the operations of pullback and pushforward, and thus, allowing to define another differential operation. It is required a family of diffeomorphism $\{\varphi_t\}$ parameterized by $t \in \mathbb{R}$. The action of $\{\varphi_t\}$ on a point p in M will describe a curve $x^\mu(t)$ parameterized by t. The action of $\{\varphi_t\}$ on every point will generate a set of curves covering M entirely. The set of tangent vectors to each curve at each point defines a vector field $V^\mu(x)$,

$$\frac{\mathrm{d}x^\mu}{dt} = V^\mu\,. \tag{2.31}$$

It is possible to find the variation rate of a tensor T along the vector field V as the difference between the pullback of the tensor from a point q to p and its original value at point p. Since both quantities are well defined tensor at p it is possible to define a operation called Lie derivative of the tensor along a vector field V

$$\mathscr{L}_V T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots} = \lim_{t\to 0} \frac{\varphi_t^*[T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}(\varphi_t(p))] - T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}(p)}{t}\,. \tag{2.32}$$

Lie derivative is a mapping from tensor fields (k,l) to (k,l) manifestly independent of the coordinate system. This operation is linear and satisfies the Leibniz rule:

$$\mathscr{L}_V(aT+bS) = a\mathscr{L}_V T + b\mathscr{L}_V S\,, \tag{2.33}$$

$$\mathscr{L}_V(T\otimes S) = (\mathscr{L}_V T)\otimes S + T\otimes(\mathscr{L}_V S)\,. \tag{2.34}$$

Lie derivative of scalar functions is equivalent to an ordinary directional derivative

$$\mathscr{L}_V f = V(f) = V^\mu\partial_\mu f\,. \tag{2.35}$$

To determine the action of the Lie derivative on tensors it is convenient to chose a coordinate system $x^\mu = (x^1,\ldots x^n)$ such that x^1 is the parameter along the curves. In that case $V = \partial/\partial x^1$ with components $V^\mu = (1,0,...,0)$. A diffeomorphism by t is equivalent to a transformation $x^\mu \to y^\mu = (x^1+t, x^2,...)$ and the pullback matrix is

$$(\varphi_t^*)_\mu{}^\nu = \delta_\mu^\nu\,. \tag{2.36}$$

In this new coordinate system the Lie derivative becomes

$$\mathscr{L}_V T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_k} = \frac{\partial}{\partial x^1} T^{\mu_1\ldots\mu k}{}_{\nu_1\ldots\nu_k}\,. \tag{2.37}$$

For a vector field U^μ

$$\mathscr{L}_V U^\mu = \frac{\partial}{\partial x^1} U^\mu\,. \tag{2.38}$$

This expression is not covariant, but it is equivalent in this coordinate system to the Lie bracket $[V,U]$ between two vector fields V and U

$$[V,U]^\mu = V^\nu\partial_\nu U^\mu - U^\nu\partial_\nu V^\mu\,. \tag{2.39}$$

Since the Lie bracket is a well-defined tensor the Lie derivative of a vector field U along a vector field V is given in any coordinate system by the Lie bracket between both vectors

$$\mathscr{L}_V U^\mu = [V,U]^\mu\,. \tag{2.40}$$

In general, for an arbitrary tensor $T^{\mu 1\ldots\mu k}{}_{\nu 1\ldots\nu k}$ the Lie derivative can be written as [15]

$$\begin{aligned}\mathscr{L}_V T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_k} &= V^{\sigma}(\nabla_{\sigma}T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_k}) \\ &\quad -(\nabla_{\lambda}V^{\mu_1})T^{\lambda\ldots\mu_k}{}_{\nu_1\ldots\nu_k} - \ldots - (\nabla_{\lambda}V^{\mu_k})T^{\mu_1\ldots\lambda}{}_{\nu_1\ldots\nu_k} \\ &\quad +(\nabla_{\nu_1}V^{\lambda})T^{\mu_1\ldots\mu_k}{}_{\lambda\ldots\nu_k} + \ldots + (\nabla_{\nu_l}V^{\lambda})T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\lambda}\,.\end{aligned} \tag{2.41}$$

In particular, the Lie derivative of the metric tensor is

$$\begin{aligned}\mathscr{L}_V g_{\mu\nu} &= V^{\sigma}\nabla_{\sigma}g_{\mu\nu} + (\nabla_{\nu}V^{\lambda})g_{\lambda\nu} + (\nabla_{\nu}V^{\lambda})g_{\mu\lambda} \\ &= \nabla_{\mu}V_{\nu} + \nabla_{\nu}V_{\mu}\,,\end{aligned} \tag{2.42}$$

where it has been used the fact that $\nabla_{\sigma}g_{\mu\nu} = 0$ if ∇_{μ} is the covariant derivative associated to the metric affine connection of $g_{\mu\nu}$.

2.1.4 Symmetries and Killing vectors

A manifold M is said to posses a symmetry if under a certain transformation of the manifold into itself there are quantities remaining invariant. In particular, if the transformation is a diffeomorphism φ, a tensor T is invariant if it remains unchanged under the pullback of φ

$$\varphi^{*}T = T\,. \tag{2.43}$$

If the symmetry is generated by a family of diffeomorphisms φ_t related to a vector field V^{μ} then the Lie derivative of T over the flow of V will be zero

$$\mathscr{L}_V(T) = 0\,. \tag{2.44}$$

This implies that it is always possible to find a coordinate system in which the components of T are independent of one of the coordinates (the coordinates of the integral curves of the vector field).

If the metric tensor $g_{\mu\nu}$ of M is invariant under a diffeomorphism φ, that is, $\varphi^{*}g_{\mu\nu} = g_{\mu\nu}$, then φ is called an isometry. If the isometries are generated by a vector field K^{μ} then

$$\mathscr{L}_K g_{\mu\nu} = 0\,, \tag{2.45}$$

or from equation (2.42)

$$\nabla_{\mu}K_{\nu} + \nabla_{\nu}K_{\mu} = 0\,. \tag{2.46}$$

The vector field K^{μ} is called Killing vector and equation (2.46) is known as Killing equation.

Killing vectors define a conserved current $J^{\mu} = K_{\nu} T^{\mu\nu}$ such that $\nabla_{\mu} J^{\mu} = 0$ [15]. If a spacetime has a Killing vector, it is always possible to find a coordinate system in which the metric is independent of one of the coordinates

2.2 Einstein's field equations

The central idea of general relativity is the association between gravity and spacetime geometry, in particular, the relation between curvature of spacetime and the energy and momentum of any form of matter and radiation present. The content of energy momentum is described by a second rank contravariant tensor $T_{\mu\nu}$ known as the energy-momentum tensor, satisfying the mass-energy conservation condition [16, 19],

$$\nabla^{\mu} T_{\mu\nu} = 0 \,. \tag{2.47}$$

To establish such relation between matter with curvature it is necessary to find a second rank contravariant tensor built from the metric tensor and its derivatives and satisfying the divergenless condition. Even though the Ricci tensor is a second rank tensor containing information of curvature, it is not a good choice since, in general, is not divergenless. However, from both the Ricci tensor (2.21) and the curvature scalar (2.22) a tensor $G_{\mu\nu}$ satisfying the divergenless condition $\nabla^{\mu} G_{\mu\nu} = 0$ is found

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \,, \tag{2.48}$$

and it is called Einstein's tensor. Thus, Einstein's field equations are formulated as a directly proportional relation between $G_{\mu\nu}$ and $T_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{2.49}$$

Einstein's field equation constitute a system of 10 independent non-linear differential equations whose solutions are the components of the metric tensor, representing the gravitational field. This equations reduce to Newton's gravitation law

$$\nabla^2 \Phi = 4\pi G \rho \,, \tag{2.50}$$

in the limit of weak gravitation field and low motion [16].

Einstein's equation can be modified to include an additional term proportional to the metric

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \,. \tag{2.51}$$

This additional term can be thought of as an additional component of the energy-momentum tensor

$$T'_{\mu\nu} = T_{\mu\nu} - \frac{\Lambda c^4}{8\pi G} g_{\mu\nu}\,, \tag{2.52}$$

The constant Λ is called cosmological constant and it commonly interpreted as the energy density of empty space. A useful form of Einstein's equation involve taking trace in (2.49)

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T\, g_{\mu\nu}\right)\,, \tag{2.53}$$

such that vacuum solutions are obtained by solving $R_{\mu\nu} = 0$. Einstein's equation can be derived using a variational method from the Einstein-Hilbert action:

$$S_{EH} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}(R - 2\Lambda) + S_{matter}\,, \tag{2.54}$$

where S_{matter} is the action of whatever matter is present content, whose variation with respect to the metric defines the energy momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g_{\mu\nu}}\,. \tag{2.55}$$

The Einstein-Hilbert action is the only possible action if invariance under change of coordinates is demanded and involving at most second order derivatives of the metric tensor components [15].

2.3 Spherically symmetric and maximally symmetric spacetimes

Given the non-linear character of Einstein's field equations it is not possible to obtain a general solution. Most known solutions suppose a certain number of symmetries and/or simplifications. Among the most relevant symmetries usually considered are

- *Spherical symmetry*: In a spherically symmetric spacetime there is no preferred spatial direction. A coordinate-independent property of spherically symmetric spacetimes is the existence of three spacelike, linearly independent killing vector fields $\{V_i\}_{i=1}^3$ satisfying the algebra of the group $SO(3)$

$$[V_i, V_j] = \varepsilon_{ijk} V_k, \quad i,j,k = 1,2,3\,. \tag{2.56}$$

  where ε_{ijk} is the Levi Civita symbol.

- *Stationarity*: Informally, stationarity means no explicit time dependence. A stationary spacetime is characterized by possessing a vector field which is globally timelike. In a a coordinate system (t, x^1, x^2, x^3) the associated Killing vector is denoted ∂_t with components $\{\partial_t\}^\mu = (1,0,0,0)$.

- *Staticity*: In a non rigurous way, a physical system is static if it does not evolve over time. A spacetime is static if all of the metric components are time independent and invariant under temporal reflection (crossed terms of the form $\mathrm{d}t\mathrm{d}x^i$ or $\mathrm{d}x^i\mathrm{d}t$ are absent). Static spacetimes are characterized in a coordinate independent way by the existence of a timelike killing vector field orthogonal to a family of spatial hypersurfaces parameterized by t constant. A timelike killing vector x^μ is orthogonal to a hypersurface if it satisfies the following equation
$$x_{[\mu}\nabla_\nu x_{\sigma]} = 0\,, \tag{2.57}$$
which is a result following from the well known Frobenius's Theorem (the braces are a notation indicating an antisymmetric, linear combinations of terms with index permutation) for details see for example appendix B of [19].

In a spherically symmetric and static spacetime the metric tensor can be described by a coordinate system (t, r, θ, ϕ), where t is a temporal coordinate associated with the timelike Killing vector defining staticity and (θ, ϕ) are the usual spherical coordinates parameterizing surfaces invariant under rotations (surfaces with area $A = 4\pi r^2$). In this coordinate system the metric tensor can be casted as
$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + \frac{1}{B(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \tag{2.58}$$
where
$$\mathrm{d}\Omega^2 = \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\,, \tag{2.59}$$
is the line element of the 2-sphere and $A(r)$ and $B(r)$ are functions depending only on r and should be positive definite in the case of Lorentzian manifolds. It should be noted that the structure of the metric tensor (2.58) is not obtained from solving Einstein's field equation but rather from considering the most general metric satisfying the conditions of staticity and spherical symmetry [15].

The causal structure of a spacetime is dictated by the behavior of light cones, which can be obtained from the set of radial null curves, that is, curves for which $\mathrm{d}s^2 = 0$ and θ, ϕ are constant. For metrics of the form (2.58) such those curves are given by
$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{1}{\sqrt{A(r)B(r)}}\,, \tag{2.60}$$

which is equivalent to the geodesics of massless particles obtained from solving the geodesic equation (2.24)

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{A(r)B(r)}} \quad \text{and} \quad \frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm 1\,. \tag{2.61}$$

In Minkowski spacetime $\mathrm{d}t/\mathrm{d}r = \pm 1$, that is, light cones form a angle of 45 degrees at every point. However, equation (2.60) indicates that if the metric coefficients depend on r the light cones slope will be different at each point of spacetime. Motivated from this observation, it is convenient to define a new coordinate r_*, called tortoise coordinate, by

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{1}{\sqrt{A(r)B(r)}}\,, \tag{2.62}$$

such that the temporal coordinate t and the new tortoise coordinate are related in the form

$$t = \pm r_* + \text{ constant }, \tag{2.63}$$

implying that for radial null curves $\mathrm{d}t = \pm \mathrm{d}r_*$. In the coordinate system (t, r_*, θ, ϕ) the metric (2.58) becomes

$$\mathrm{d}s^2 = \mathscr{A}(r_*)\left(-\mathrm{d}t^2 + \mathrm{d}{r_*}^2\right) + r^2(r_*)\mathrm{d}\Omega^2\,, \tag{2.64}$$

where $\mathscr{A}(r_*) = A(r(r_*))$. In this particular coordinate system (t, r_*, θ, ϕ) the metric is characterized by the functions $\mathscr{A}(r_*)$ and $r(r_*)$.

Another important coordinate systems are based on the advanced time u and retarded time v, defined as

$$u = t - r_*\,, \tag{2.65}$$

$$v = t + r_*\,. \tag{2.66}$$

Null geodesics with u constant satisfy $\mathrm{d}t = dr_*$ whereas null geodesics with v constant satisfy $\mathrm{d}t = -dr_*$. The coordinate systems (u, r, θ, ϕ) and (v, r, θ, ϕ) are called *ingoing* and *outgoing Eddington-Finkelstein coordinates* respectively [23]. In the coordinate system (v, r, θ, ϕ), the metric (2.58) adopts the following form

$$\mathrm{d}s^2 = -A(r)\mathrm{d}v^2 + \sqrt{\frac{A(r)}{B(r)}}(\mathrm{d}v\mathrm{d}r + \mathrm{d}r\mathrm{d}v)r^2\mathrm{d}\Omega^2\,, \tag{2.67}$$

while in the coordinate system (u, r, θ, ϕ) a similar expression is obtained

$$\mathrm{d}s^2 = -A(r)\mathrm{d}u^2 - \sqrt{\frac{A(r)}{B(r)}}(\mathrm{d}u\mathrm{d}r + \mathrm{d}r\mathrm{d}u) + r^2\mathrm{d}\Omega^2\,, \tag{2.68}$$

It is possible to define another coordinate system using both the retarded and advanced times u, v. From the form of the metric tensor in the coordinates (t, r_*, θ, ϕ) given by (2.64) and the replacements

$$t = \frac{1}{2}(u+v), \quad r_* = \frac{1}{2}(v-u). \tag{2.69}$$

we get the following expression for the metric

$$\mathrm{d}s^2 = -\mathscr{A}(r_*(u,v))\mathrm{d}u\mathrm{d}v + r^2(r_*(u,v))\mathrm{d}\Omega^2, \tag{2.70}$$

with the particularity that there are no quadratic terms in du or dv. The coordinates (u, v) are well suited to describe radial null geodesics, since the condition d$s^2 = 0$ implies that massless particles propagate at either u constant or v constant, which are null curves.

2.3.1 Maximally symmetric spacetimes

An n-dimensional manifold with $\frac{1}{2}n(n+1)$ Killing vectors is said to be a maximally symmetric space, that is, a space with the maximum number of possible isometries. For a maximally symmetric space the curvature is constant everywhere, and the Riemann tensor is [15]

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right). \tag{2.71}$$

Maximally symmetric spaces are characterized locally by the value of the Ricci tensor R, classified according to whether R is positive, negative or zero. For Euclidean spaces $R = 0$ corresponds to $\mathbb{R}^n$, $R > 0$ corresponds to S^n and $R < 0$ corresponds to an n-dimensional hyperboloid.

For Lorentzian manifolds the maximally symmetric spacetime with $R = 0$ is Minkowski space, which in addition to static and spherical symmetries possesses Poincaré invariance under Lorentz boosts and translations. Likewise, the maximally symmetric space with positive curvature is known as de Sitter spacetime (dS) and the negative curvature spacetime is called Anti de Sitter spacetime (AdS). These two geometries are solutions to the Einstein's equation with non-zero cosmological constant (2.51).

de Sitter spacetime is the vacuum solution to Einstein's field equation with positive cosmological constant, $\Lambda > 0$. In four dimensions and it a coordinate system (t, r, θ, ϕ), the metric tensor of de Sitter spacetime is of the form (2.58) with [24]

$$A(r) = B(r) = 1 - \frac{\Lambda r^2}{3}, \tag{2.72}$$

or, equivalently

$$\mathrm{d}s^2 = -\left(1-\frac{r^2}{a^2}\right)\mathrm{d}t^2 + \left(1-\frac{r^2}{a^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \tag{2.73}$$

where $a = \sqrt{3/\Lambda}$ is called the de Sitter radius. As a particular feature, the metric component $g_{rr} = B(r)^{-1}$ becomes singular at $r = a$ and $A(r), B(r) < 0$ for $r > a$. The domain of validity of the radial coordinate is $r \in [0, a)$, in which the functions $A(r)$ and $B(r)$ are positive defined. The surface $r = a$ is said to be a *cosmological horizon*, a surface surrounding any observer and delimiting the space from which the observer can retrieve information. The tortoise coordinate of de Sitter spacetime is given by

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \left(1-\frac{\Lambda r^2}{3}\right)^{-1}\,. \tag{2.74}$$

The solution of this equation is

$$r_*(r) = \sqrt{\frac{3}{\Lambda}}\tanh^{-1}\left(\sqrt{\frac{\Lambda}{3}}r\right)\,. \tag{2.75}$$

Thus, the original domain of the radial coordinate r is extended to $r_* \in [0, \infty)$. The tortoise coordinate $r_*(r)$ can be inverted analytically, giving functions $r(r_*)$ and $\mathscr{A}(r_*)$ as a function of r_*

$$r(r_*) = \sqrt{\frac{3}{\Lambda}}\tanh\left(\sqrt{\frac{\Lambda}{3}}r_*\right)\,, \tag{2.76}$$

$$\mathscr{A}(r_*) = \sqrt{\frac{3}{\Lambda}}\mathrm{sech}\left(\sqrt{\frac{\Lambda}{3}}r_*\right)\,. \tag{2.77}$$

The expression for the metric in (t, r_*, θ, ϕ) coordinates is

$$\mathrm{d}s^2 = \mathrm{sech}^2\left(\frac{r_*}{a}\right)\left(-\mathrm{d}t^2 + \mathrm{d}{r_*}^2\right) + a^2\tanh^2\left(\frac{r_*}{a}\right)\mathrm{d}\Omega^2\,. \tag{2.78}$$

Anti de Sitter spacetime is the maximally symmetric solution of Einstein's equation with negative cosmological constant Λ, corresponding to a negative vacuum energy density and positive pressure. In the coordinate system (t, r, θ, ϕ) the AdS metric is characterized by the functions

$$A(r) = B(r) = 1 + \frac{\Lambda r^2}{3}\,, \tag{2.79}$$

where $\Lambda = -3/R^2 < 0$. In this case the line element is [25]

$$\mathrm{d}s^2 = -\left(1+\frac{r^2}{R^2}\right)\mathrm{d}t^2 + \left(1+\frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega\,. \tag{2.80}$$

The metric tensor is well-defined for $r \in [0,\infty)$. Anti de Sitter resembles Minkowski spacetime for $r \ll R$ but has a different asymptotic behavior near R. The tortoise coordinate is given by

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \left(1+\frac{r^2}{R^2}\right)^{-1}\,, \tag{2.81}$$

with solution

$$r_*(r) = R\arctan\left(\frac{r}{R}\right)\,. \tag{2.82}$$

The tortoise coordinate can be inverted; and the functions $r(r_*)$ and $\mathscr{A}(r_*)$ are given by

$$r(r_*) = R\tan\left(\frac{r_*}{R}\right)\,, \tag{2.83}$$

$$\mathscr{A}(r_*) = \sec^2\left(\frac{r_*}{R}\right)\,. \tag{2.84}$$

The expression for the metric in (t, r_*, θ, ϕ) coordinates is

$$\mathrm{d}s^2 = \sec^2\left(\frac{r_*}{R}\right)\left(-dt^2 + d{r_*}^2\right) + R^2\tan\left(\frac{r_*}{R}\right)^2 d\Omega^2\,. \tag{2.85}$$

The Anti de Sitter spacetime shares very interesting features, and it plays a prominent rule in the AdS/CFT correspondence, first introduced in [26], which has been a very active line of research over the last decade. For example, even though there is a well defined limit $r \to \infty$, AdS has the property that a light beam emitted from any point can reach spatial infinity and bounce back in a finite proper time. In that sense, it is said that Anti de Sitter spacetime has a boundary at $r \to \infty$ (more formally, spatial infinity takes the form of a timelike hypersurface). This property implies that knowledge of equations of motion and of initial data is not enough to determine the time evolution of physical quantities, since information can flow in from infinity, which imposes difficulties for quantizing fields in this spacetime.

2.3.2 Schwarzschild spacetime

A well studied solution of Einstein's equations is the Schwarzschild spacetime [27], which is the unique, spherically symmetric and static vacuum describing spacetime outside a spherical object of mass M. In coordinates (t,r,θ,ϕ) the Schwarzschild metric is is given by

$$A(r) = B(r) = \left(1 - \frac{2M}{r}\right), \tag{2.86}$$

with corresponding line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 . \tag{2.87}$$

In this coordinate system the metric tensor presents two divergences: The component g_{rr} becomes singular at $r = 2M$; it is well known that this divergence is not physical and can be removed by choosing a more appropriated coordinate system [15]. The other divergence corresponds to the component g_{tt} at $r = 0$. This is a physical singularity, since certain coordinate invariant quantities, such as the Kretschmann invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, becomes singular at that point

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} . \tag{2.88}$$

Therefore $r = 0$ should not be considered as part of the manifold, even though it can be reached from other points.

Even though $r = 2M$ is a regular surface, the coordinate system (t,r,θ,ϕ) is only meaningful for the region $r > 2M$. A more appropriated coordinate system is given by the tortoise coordinate r_*, defined by (2.62). For Schwarzschild spacetime the tortoise coordinate is

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \left(1 - \frac{2M}{r}\right)^{-1}, \tag{2.89}$$

which can be integrated to yield

$$r_*(r) = r + 2M\ln\left(\frac{r}{2M} - 1\right) . \tag{2.90}$$

The domain of the radial coordinate $r \in (2M,\infty)$ is extended to $r_* \in (-\infty,\infty)$, with the surface $r = 2M$ pushed to $r_* \to -\infty$. It is to be noted that equation (2.90) cannot be inverted analytically to yield r as a function of r_*. From the tortoise coordinate (2.90) the retarded and advanced times are obtained. In the ingoing Eddington-Finkelstein coordinates (v,r) and (u,r) the metric tensor takes the form

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}v^2 + (\mathrm{d}v\mathrm{d}r + \mathrm{d}r\mathrm{d}v) + r^2\mathrm{d}\Omega^2 , \tag{2.91}$$

while in the outgoing coordinates we have

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - (du dr + dr du) + r^2 d\Omega^2 . \tag{2.92}$$

In this coordinate systems the divergences of the metric components at $r = 2M$ have disappeared, showing that the surface is perfectly regular. Nevertheless new observations are obtained from these change of coordinates. In Schwarzschild spacetime, null geodesics are the worldlines of massless particles either moving directly towards or away from the central mass. Outgoing radial geodesics are characterized by u constant, which in the coordinate system (v, r) translates to

$$\frac{dv}{dr} = 2\left(1 - \frac{2M}{r}\right)^{-1} . \tag{2.93}$$

Equivalently, ingoing null geodesics are characterized by constant v or by the following condition in the coordinate system (u, r)

$$\frac{du}{dr} = -2\left(1 - \frac{2M}{r}\right)^{-1} . \tag{2.94}$$

As t increases we have that for null curves with u constant then r increases while for curves with v constant r decreases, meaning that ingoing particles will reach $r = 0$ at some moment. Outgoing particles at some $r > 2M$ will move away from the surface $r = 2M$, but from (2.93), it is observed that in the region $r < 2M$, all future directed paths of null or timelike outgoing particles are in the direction of decreasing r. Thus particles past this surface cannot reach spatial infinity. This observation will lead us to the introduce the concept of black hole in the next section.

2.4 Black holes

A prediction unique to general relativity is the idea of black hole, a compact region of spacetime where curvature is high enough to prevent the escape of internal observers out of its interior. Schwarzschild black hole, even though the surface $r = 2M$ is regular it separates the spacetime in two regions in only one direction. An hypersurface separating spacetime points that can be connected to infinity by a timelike path from those that cannot is called an *event horizon*. The region bounded by an event horizon and thus casually disconnected from spatial infinity by such surface is what is known as a *black hole* [15, 23].

This definition implies that the event horizon is a null hypersurface, where the normal vector to the surface is also a tangent vector. Null hypersurfaces can be seen as a collection of null geodesics $x^\mu(\lambda)$, called the generators of the hypersurface. If a spacetime can be foliated with hypersurfaces Σ defined by $f(x) = constant$ for some function $f(x)$, the event horizon will be located at the region where the hypersurfaces become null [15].

If a Killing vector field X^μ is null along some null hypersurface Σ, it is said that Σ is a Killing horizon of X^μ. In stationary, asymptotically flat spacetimes every event horizon is a Killing horizon for some Killing vector X^μ, but in general, not every Killing horizon is an event horizon. To every Killing horizon there is an associated quantity called surface gravity κ. Since X^μ is normal to the Killing horizon, it obeys the geodesic equation along the Killing horizon

$$X^\mu \nabla_\mu X^\nu = -\kappa X^\nu \,. \tag{2.95}$$

Using Killing's equation (2.46) and Frobenius's' theorem (2.57) a formula that allows to find the value of the surface gravity associated to the Killing horizon can be found [19] (this expression is meant to be evaluated only on the horizon)

$$\kappa^2 = -\frac{1}{2}\left(\nabla_\mu X_\nu\right)\left(\nabla^\mu X^\nu\right) . \tag{2.96}$$

In a static, asymptotically flat spacetime, the surface gravity κ is interpreted as the acceleration (as seen by a static observer at infinity) needed to keep an object at rest at the horizon of events. In the coordinate system (t, r, θ, ϕ) the associated Killing vector is $X^\mu = (1,0,0,0)$, and the corresponding covariant vector is found by lowering index

$$X_\mu = g_{\mu\nu} X^\nu = (-A(r), 0, 0, 0) \,, \tag{2.97}$$

such that the norm of the killing vector is then given by

$$X_\mu X^\mu = -A(r) \,. \tag{2.98}$$

The Killing horizon will correspond to the surface $r = r_+$ for which $A(r_+) = 0$. The surface gravity can be obtained from the covariant derivative of X^μ surface gravity can be obtained

$$\nabla_\mu X_\nu = \partial_\mu X_\nu - \Gamma^\lambda_{\mu\nu} X_\lambda \,. \tag{2.99}$$

The only relevant Christoffel symbol is given by

$$\Gamma^t_{rt} = \frac{1}{2} B(r) \frac{\mathrm{d}}{\mathrm{d}r} A(r) \,, \tag{2.100}$$

and we have that

$$\nabla_\mu X_\nu = \nabla_r X_t = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}r} A(r) \,. \tag{2.101}$$

We raise both index of (2.101) to obtain

$$\nabla^{\mu} X^{\nu} = B(r)\,(\nabla_r X_t)\,A(r)\,, \tag{2.102}$$

and with this the surface gravity reads

$$\kappa^2 = \lim_{r\to r_+} \frac{1}{4}\frac{B(r)}{A(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}A(r)\right)^2 \tag{2.103}$$

We use to following equality

$$\sqrt{\frac{B(r)}{A(r)}}\left(\frac{\mathrm{d}}{\mathrm{d}r}A(r)\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(\sqrt{A(r)B(r)}\right) - A(r)\frac{\mathrm{d}}{\mathrm{d}r}\left(\sqrt{\frac{B(r)}{A(r)}}\right), \tag{2.104}$$

from which we can write

$$\kappa = \lim_{r\to r_+} \frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\sqrt{A(r)B(r)}\right) - A(r)\frac{\mathrm{d}}{\mathrm{d}r}\left(\sqrt{\frac{B(r)}{A(r)}}\right)\right|. \tag{2.105}$$

Taking this into account makes the second term in (2.105) equal to zero, as long as the fraction $B(r)/A(r)$ is differentiable in $r = r_+$

$$\kappa = \lim_{r\to r_+} \frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}r}\sqrt{A(r)B(r)}\right|. \tag{2.106}$$

For the Schwarzschild black hole, where the functions $A(r)$ and $B(r)$ are given by (2.86), the surface gravity is equal to $\kappa = 1/4M$. The surface gravity of a black hole is an important quantity in the analogy between black holes and thermodynamics, where it is defined Hawking temperature as

$$T_H = \frac{\kappa}{2\pi}\,, \tag{2.107}$$

and it is a quantity that remains constant in stationary black holes just as the temperature of a body in thermal equilibrium is constant [1]. This statement is known as the zeroth-law of the thermodynamics of black holes.

Chapter 3

Geometries of Interest

In this chapter we will introduce the geometries under consideration in this thesis. These geometries include the generalizations of the Schwarzschild black hole as a vacuum solution of Einstein's equation when a non-zero cosmological constant is considered, namely Schwarzschild-de Sitter and Schwarzschild-anti de Sitter spacetimes. We also introduce a family of geometries known as near extremal solutions which are solutions admitting two different Killing horizons that become arbitrarily close.

3.1 Schwarzschild-de Sitter spacetime

In the last chapter we introduced Schwarzschild spacetime (2.86), which is a spherically symmetric vacuum solution of Einstein's Field equation. This solution considers a zero cosmological constant and is asymptotically flat, that is, it has the same behavior as the Minkowski spacetime as $r \to \infty$ namely, $A(r) \sim 1$ and $B(r) \sim 1$ in the limit $r \to \infty$.

With the introduction of a non-zero cosmological constant the asymptotically behavior is different. First we introduce Schwarzschild de Sitter spacetime (SdS), describing spacetime outside an object of mass M with a positive cosmological constant Λ. The metric of the SdS spacetime in four dimensions is given by [28]

$$A(r) = B(r) = 1 - \frac{2M}{r} - \frac{r^2}{a^2}\,, \tag{3.1}$$

where a^2 is given in terms of the cosmological constant Λ by $a^2 = 3/\Lambda$. This metric is asymptotically de Sitter for large values of r,

$$A(r) \sim 1 - \frac{r^2}{a^2}\,. \tag{3.2}$$

There is a maximum value for the cosmological constant to admit a black hole, given by

$$\Lambda_{ext} = 1/9M^2\,, \tag{3.3}$$

if $\Lambda > \Lambda_{ext}$ this spacetime will posses a naked singularity. If the cosmological constant is such that $0 < \Lambda < \Lambda_{ext}$, the function $A(r)$ in (3.1) has three real roots, two positive roots r_1 and r_2 with $r_1 < r_2$ and a negative root $r_M = -r_1 - r_2$. In terms of these roots the function $A(r)$ can be expressed as

$$A(r) = \frac{1}{ra^2}(r_2 - r)(r - r_1)(r - r_M)\,. \tag{3.4}$$

The physical parameters M and a^2 can be written in terms of r_1 and r_2 as

$$a^2 = r_1^2 + r_1 r_2 + r_2^2\,, \tag{3.5}$$

$$2Ma^2 = r_2 r_1 (r_1 + r_2)\,. \tag{3.6}$$

The function $A(r)$ is positive definite in the domain $r_1 < r < r_2$, with $r = r_1$ and $r = r_2$ corresponding to Killing horizons related to the Killing vector $\{\partial_t\}^\mu = (1,0,0,0)$. The surface $r = r_1$ corresponds to an event horizon of a black hole while $r = r_2$ corresponds to a cosmological horizon in analogy with pure de Sitter spacetime. For each of the Killing horizons it is defined a surface gravity

$$\kappa_1 = \frac{(r_1 - r_M)(r_2 - r_1)}{2r_1 a^2}\,, \tag{3.7}$$

$$\kappa_2 = \frac{(r_2 - r_1)(r_2 - r_M)}{2r_2 a^2}\,, \tag{3.8}$$

with κ_1 the surface gravity of the event horizon r_1 and κ_2 the surface gravity of the cosmological horizon r_2. To obtain the tortoise coordinate for this spacetime we use partial fractions on the inverse of (3.4),

$$\frac{1}{A(r)} = \frac{1}{2\kappa_1}\left(\frac{1}{r - r_1}\right) + \frac{1}{2\kappa_2}\left(\frac{1}{r_2 - r}\right) + \frac{r_M a^2}{(r_2 - r_M)(r_1 - r_M)}\left(\frac{1}{r - r_M}\right)\,. \tag{3.9}$$

Integrating we obtain

$$r_*(r) = \frac{1}{2\kappa_1}\ln\left(\frac{r}{r_1} - 1\right) - \frac{1}{2\kappa_2}\ln\left(1 - \frac{r}{r_2}\right) + \frac{r_M a^2}{(r_2 - r_M)(r_1 - r_M)}\ln\left(\frac{r}{r_M} - 1\right)\,. \tag{3.10}$$

The event horizon located at $r = r_1$ is mapped to $r_* \to -\infty$ and the cosmological horizon at $r = r_2$ is mapped to $r_* \to \infty$. It is to be noted that (3.10) cannot be inverted analytically, that is, in general we do not have an explicit expression for the inverse function $r(r_*)$.

3.2 Schwarzschild-Anti de Sitter spacetime

Now we introduce Schwarzschild-Anti de Sitter spacetime, which is the extension of Schwarzschild spacetime considering a negative cosmological constant and a static black hole with mass M. The line element is [29]

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + \frac{1}{A(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \tag{3.11}$$

with

$$A(r) = 1 - \frac{2M}{r} + \frac{r^2}{R^2}\,, \tag{3.12}$$

with R^2 given in terms of the cosmological constant Λ by $R^2 = 3/\Lambda$. The function $A(r)$ is positive in the domain $(0,\infty)$ and the metric is asymptotically de Anti de Sitter in the limit $r \to \infty$,

$$A(r) \sim 1 + \frac{r^2}{R^2}\,. \tag{3.13}$$

The function $A(r)$ has a real root $r = r_+$ corresponding to the event horizon of a black hole and two complex roots. We can write (3.12) as

$$A(r) = \frac{(r-r_+)(r^2 + r_+ r + r_+^2 + R^2)}{rR^2}\,. \tag{3.14}$$

In terms of r_+ and R, the mass of the black hole is

$$2M = \frac{(r_+^2 + R^2)r_+}{R^2}\,. \tag{3.15}$$

We can also write the surface gravity κ in terms of r_+ and R

$$\kappa = \frac{1}{2}\left(\frac{2M}{r_+^2} + \frac{2r_+}{R^2}\right)\,, \tag{3.16}$$

or, using (3.15) we have

$$\kappa = \frac{1}{2}\left(\frac{R^2 + 3r_+^2}{R^2 r_+}\right)\,. \tag{3.17}$$

To obtain the tortoise coordinate we use partial fractions on the function $1/A(r)$,

$$\frac{1}{A(r)} = \frac{1}{2\kappa}\left[\frac{1}{r-r_+} - \frac{r}{r^2 + r_+ r + r_+^2 + R^2} + \frac{r_+^2 + R^2}{r_+(r^2 + r_+ r + r_+^2 + R^2)}\right]\,. \tag{3.18}$$

The integral of (3.18) can be solved analytically giving

$$r_*(r) = \frac{1}{2\kappa}\left\{\ln\left(\frac{r}{r_+}-1\right) - \frac{1}{2}\ln\left(\frac{r^2}{r_+^2}+\frac{r}{r_+}+1+\frac{R^2}{r_+^2}\right)\right.$$
$$\left. + \frac{3r_+^2+2R^2}{2r_+}\frac{2}{\sqrt{3r_+^2+4R^2}}\left[\tan^{-1}\left(\frac{2r+r_+}{\sqrt{3r_+^2+4R^2}}\right)-\frac{\pi}{2}\right]\right\}, \tag{3.19}$$

where $\pi/2$ is an integration constant chosen to map $r \to \infty$ to $r_* = 0$. The event horizon located at $r = r_+$ is mapped to $r_* \to -\infty$. Unfortunately, (3.19) cannot be analytically inverted, there is no explicit expression for r as a function of r_*.

3.3 Near extremal geometries

In general, spacetimes admitting a Killing horizon are said to be *extreme geometries* if the Killing horizon has surface gravity equal to zero. Following the same line, *near extremal geometries* are spacetimes admitting a Killing horizon with a surface gravity very close but not exactly zero. We are considering spherically symmetric and static metrics of the form (2.58) where the functions $A(r)$ and $B(r)$ have one of the following properties

- $A(r)$ and $B(r)$ share two simple roots r_1 and r_2.
- $A(r)$ has a single simple root r_1 and $B(r)$ has two simple roots r_1 and r_2.

For this class of geometries the function $r(r_*)$ should satisfy the following conditions

1. The function $r(r_*)$ is such that $r_1 < r < r_2$ for any value of r_*.
2. The function $r(r_*)$ is continuous and C^∞ for $r_1 < r_* < r_2$.

For the metric to be well-defined for any value of r_*, and the radial coordinate r always lying between both Killing horizons.

The near extremal limit will be given when both horizons r_1 and r_2 get arbitrarily close. To characterize this limit it is useful to define a dimensionless parameter δ in terms of r_1 and r_2

$$\delta = \frac{r_2 - r_1}{r_1}, \tag{3.20}$$

such that the near extremal limit is characterized by

$$0 < \delta \ll 1\,. \tag{3.21}$$

3.3.1 Near extremal Schwarzschild de Sitter spacetime

Schwarzschild de Sitter is a common example of a geometry admitting a near extremal limit, and it is in fact the most treated example in the literature [30, 31]. We have seen that *SdS* spacetime admits two Killing horizons r_1 and r_2, and that there is a maximum value the cosmological constant can take for which the geometry admits a black hole. As Λ approaches Λ_{ext}, the two horizons r_1 and r_2 be become arbitrarily closed. Defining a dimensionless parameter δ as

$$\delta = \frac{r_2 - r_1}{r_1}\,, \tag{3.22}$$

we have that $\delta \to 0$ as $\Lambda \to \Lambda_{ext}$. Since r_1 and r_2 are simple roots of (3.4), we can write $A(r)$ as

$$A(r) = R(r)(r_2 - r)(r - r_1)\,, \tag{3.23}$$

We expand the function $A(r)$ in a Taylor series around $r_0 = (r_1 + r_2)/2$, the midpoint of r_1 and r_2,

$$A(r) = A(r_0) + r_1 \left.\frac{\mathrm{d}A(r)}{\mathrm{d}r}\right|_{r=r_0} \left(\frac{r - r_0}{r_1}\right) + \frac{r_1^2}{2} \left.\frac{\mathrm{d}^2 A(r)}{\mathrm{d}^2 r}\right|_{r=r_0} \left(\frac{r - r_0}{r_1}\right)^2 + \mathscr{O}(\delta^3)\,. \tag{3.24}$$

Now we develop each term we have. For the zeroth-order term we have

$$A(r_0) = R(r_0)(r_2 - r_0)(r_0 - r_1) = R(r_0)\left(\frac{r_2 - r_1}{2}\right)^2\,, \tag{3.25}$$

whereas we obtain that the first order term is zero up to higher order in δ^3

$$r_1 \left.\frac{\mathrm{d}A(r)}{\mathrm{d}r}\right|_{r=r_0} \left(\frac{r - r_0}{r_1}\right) = 0 + \mathscr{O}(\delta^3)\,, \tag{3.26}$$

and for the second order term we get

$$\frac{r_1^2}{2} \left.\frac{\mathrm{d}^2 A(r)}{\mathrm{d}^2 r}\right|_{r=r_0} \left(\frac{r - r_0}{r_1}\right)^2 = -R(r_0)(r - r_0)^2 + \mathscr{O}(\delta^3)\,. \tag{3.27}$$

With this we write

$$A(r) = R(r_0)(r_2 - r)(r - r_1) + \mathscr{O}(\delta^3)\,, \tag{3.28}$$

where $R(r_0)$ is a constant given by

$$R(r_0) = \frac{2\kappa_1}{r_2 - r_1}\,. \tag{3.29}$$

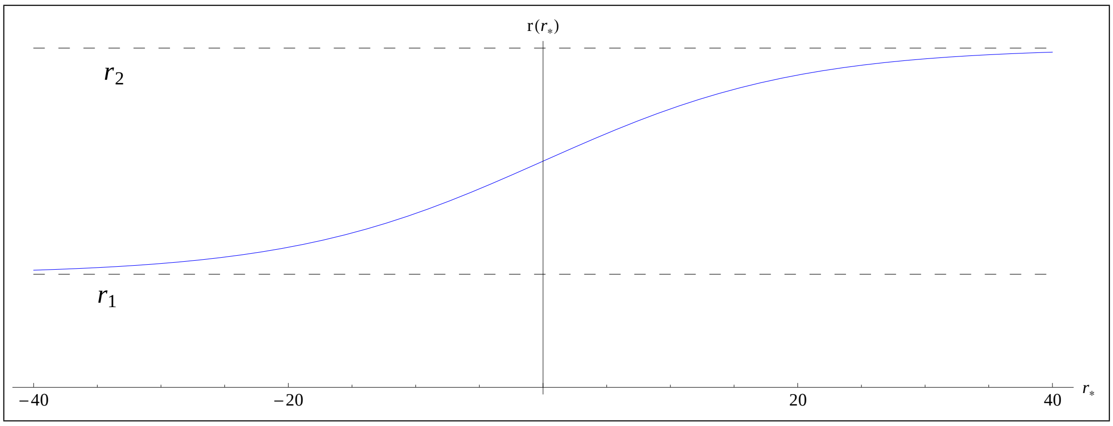

Fig. 3.1 Behavior of $r(r_*)$ in near extremal SdS geometry.

The near extremal surface κ_1 of r_1, obtained from (3.28) is

$$\kappa_1 = \frac{1}{2}R(r_0)(r_2 - r_1) = \frac{1}{2}R(r_0)r_1\delta + \mathcal{O}(\delta^2)\,, \tag{3.30}$$

which effectively approaches zero in the near extremal limit. From the simplified expression of $A(r)$ in (3.28), the tortoise coordinate is found to be

$$r_* = \frac{1}{2\kappa_1}\ln\left(\frac{r - r_1}{r_2 - r}\right) + \mathcal{O}(\delta^3)\,, \tag{3.31}$$

and this allows us to obtain an explicit expression for the function $r(r_*)$

$$r(r_*) = \frac{r_1 \mathrm{e}^{-\kappa_1 r_*} + r_2 \mathrm{e}^{\kappa_1 r_*}}{\mathrm{e}^{-\kappa_1 r_*} + \mathrm{e}^{\kappa_1 r_*}} + \mathcal{O}(\delta^3)\,. \tag{3.32}$$

The form of (3.32) is illustrated in figure 3.1. It is to be noted that (3.32) is monotonic increasing, implying that there are no additional horizons aside from r_1 and r_2. The function $\mathscr{A}(r_*)$, with the result (3.32), is given by

$$\mathscr{A}(r_*) = \frac{2\kappa_1}{r_2 - r_1}\frac{(r_2\mathrm{e}^{-\kappa_1 r_*} - r_1\mathrm{e}^{-\kappa_1 r_*})(r_2\mathrm{e}^{\kappa_1 r_*} - r_1\mathrm{e}^{\kappa_1 r_*})}{(\mathrm{e}^{-\kappa_1 r_*} + \mathrm{e}^{\kappa_1 r_*})^2} + \mathcal{O}(\delta^3)\,, \tag{3.33}$$

which can be simplified to

$$\mathscr{A}(r_*) = \frac{(r_2 - r_1)\kappa_1}{2}\,\mathrm{sech}^2(\kappa r_*) + \mathcal{O}(\delta^3)\,. \tag{3.34}$$

3.3.2 Near extremal wormholes

A second case of interest which admits a near extremal limit are the geometries introduced in [32] as *near extremal wormholes*. Wormholes are compact spacetimes with non trivial topological interiors and topologically simple boundaries, which can be seen as connections between otherwise distant or disconnected parts of the universe [33]. Near extremal wormholes typically appear in spacetimes with a positive cosmological constant, analogous to the near extremal Schwarzschild-de Sitter, and can be interpreted as limits of static and spherically symmetric solutions in world brane scenarios [32].

In the coordinate system (t,r,θ,ϕ) the near extremal limit of this spacetimes is given by

$$A(r) = \tilde{A}_0(r_2 - r)\,, \tag{3.35}$$

$$B(r) = \tilde{B}_0(r_2 - r)(r - r_0)\,, \tag{3.36}$$

where $\tilde{A}_0$ and $\tilde{B}_0$ are positive constants which are defined explicitly in [32]. The coordinate system (t,r,θ,ϕ) is only valid in the region $r_0 < r < r_2$, where r_2 is a Killing horizon assuming the role of a cosmological horizon. We define again a dimensionless parameter δ in terms of r_2 and r_0

$$\delta = \frac{r_2 - r_0}{r_0}\,, \tag{3.37}$$

we have that the surface gravity at $r = r_2$ is given by

$$\kappa = \frac{1}{2}\sqrt{\tilde{A}_0\tilde{B}_0(r_2 - r_0)} = \frac{1}{2}\sqrt{\tilde{A}_0\tilde{B}_0 r_0}\,\delta^{1/2}\,. \tag{3.38}$$

The near-extremal limit is obtained when $r_0 \to r_2$, that is, $0 < \delta \ll 1$, and the surface gravity κ at r_2 approaches zero. Now we extend the coordinate system in the region $r_0 < r < r_2$ by means of the tortoise coordinate (t, r_*, θ, ϕ), solving the following integral

$$r_*(r) = \frac{1}{\sqrt{\tilde{A}_0\tilde{B}_0}} \int \frac{dr}{(r_2 - r)\sqrt{(r - r_0)}}\,. \tag{3.39}$$

The solution of this integral is (based on [34])

$$r_*(r) = \frac{1}{\sqrt{\tilde{A}_0\tilde{B}_0(r_2 - r_0)}} \ln\left(\frac{\sqrt{r_2 - r_0} - \sqrt{r - r_0}}{\sqrt{r_2 - r_0} + \sqrt{r - r_0}}\right)\,, \tag{3.40}$$

which can be reformulated as

$$r_*(r) = \frac{1}{2\kappa} \ln\left[\frac{(r_2 - r_0) - (r - r_0)}{r - r_0 + (r_2 - r_0) + 2\sqrt{r_2 - r_0}\sqrt{r - r_0}}\right]\,, \tag{3.41}$$

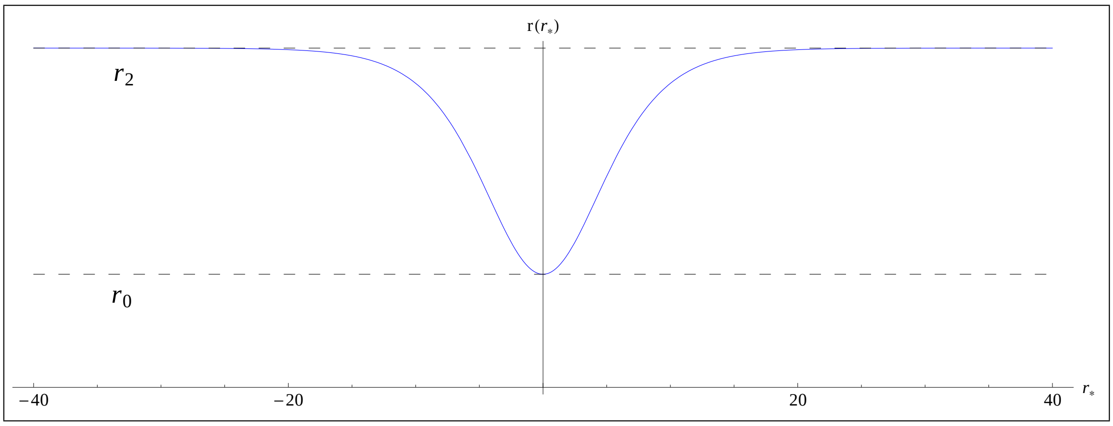

Fig. 3.2 Behavior of $r(r_*)$ in near extremal wormhole geometry.

where the result (3.38) was used. The interval $r_0 < r < r_2$ is mapped to $-\infty < r_* < \infty$, with $r = r_0$ corresponding to $r_* = 0$. Working with the near-extremal limit we have the following approximation

$$\sqrt{r_2 - r_0}\sqrt{r - r_0} \approx r_2 - r_0 \,, \tag{3.42}$$

and the tortoise coordinate takes a simpler form

$$r_*(r) = \frac{1}{2\kappa} \ln\left(\frac{r_2 - r}{r + 3r_2 - 4r_0}\right) . \tag{3.43}$$

Expression (3.43) is invertible, and an analytic expression for $r(r_*)$ is available

$$r(r_*) = \frac{r_2 - (3r_2 - 4r_0)e^{2\kappa r_*}}{1 + e^{2\kappa r_*}} \,, \tag{3.44}$$

which can be written as

$$r(r_*) = \frac{4r_2 \cosh^2(\kappa r_*) - 4(r_2 - r_0)\left(1 + \mathrm{e}^{2\kappa r_*}\right)}{4\cosh^2(\kappa r_*)} \,. \tag{3.45}$$

Further simplification of (3.45) leads to

$$r(r_*) = r_2 - r_0\,\delta\,\mathrm{sech}^2\left(\kappa r_*\right) . \tag{3.46}$$

The form of (3.46) is illustrated in figure 3.2. Reminding that in the near-extremal limit we have $r_0 \to r_2$ we can write

$$r(r_*) = r_0 - r_0\,\delta\,\mathrm{sech}^2\left(\kappa r_*\right) , \tag{3.47}$$

and this allows us to write the function $\mathscr{A}(r_*)$ as

$$\mathscr{A}(r_*) = \tilde{A}_0 r_0 \delta \mathrm{sech}^2(\kappa r_*) \,. \tag{3.48}$$

In this extension of coordinates spacetime limited by two Killing horizons $r_* \to \pm\infty$, both of them corresponding to $r = r_2$. $r_* = 0$ is a local minimum of $r(r_*)$ and the surface $r_* = 0$ ($r = r_0$) is an outer trapping horizon [33], which can be seen as a throat of a wormhole. Spacetimes described by (3.35) and (3.36) are interpreted as a wormhole joining two regions delimited by cosmological horizons.

3.3.3 Near extremal black holes in compact universes

As third case of interest of geometries admitting a near extremal limit are solutions of Einstein's equation satisfying isotropic constraints of the energy-momentum tensor, for example the geometries introduced in [35]. If the energy-momentum tensor is of the form

$$T_\nu{}^\mu = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{pmatrix}, \tag{3.49}$$

where ρ is mass density, p_r corresponds to radial pressure and p_t to tangential pressure, the isotropic constraint is given by

$$p_t = p_r\,, \tag{3.50}$$

that is, the radial pressure and the tangential pressure are equal.

Geometries with $\Lambda = 0$ sharing this constraint are given by [35]

$$A(r) = 1 - \frac{2M}{r}\,, \tag{3.51}$$

$$B(r) = \left(1 - \frac{2M}{r}\right)\left[1 + C\left(\frac{r}{M} - 1\right)^2\right], \tag{3.52}$$

where C is a constant taking values between $-1 < C < 0$. Both functions have a simple root $r_+ = 2M$, which is a Killing horizon with associated surface gravity

$$\kappa = \frac{\sqrt{1+C}}{4M}\,, \tag{3.53}$$

and corresponds to the event horizon of a black hole. Additionally, the function $B(r)$ has another simple root r_0 given by

$$r_0 = M\left(1 + \frac{1}{\sqrt{|C|}}\right). \tag{3.54}$$

Given the domain of the parameter C, we have $r_1 < r_0$, and thus, the static region where $A(r) > 0$

and $B(r) > 0$ is given by $r_1 < r < r_0$. From the surface gravity (3.53) it is seen that the near extremal limit corresponds to $C \to -1$. In this limit r_1 gets close to r_0 and $\kappa \to 0$. We define again a dimensionless parameter

$$\delta = \frac{r_0 - r_1}{r_0}. \tag{3.55}$$

Such that the near extreme limit is characterized by $0 < \delta \ll 1$. In the near extremal limit the functions $A(r)$ and $B(r)$ are approximated by a linear and a quadratic polynomial respectively

$$A(r) = \tilde{A}_0(r - r_1)\,, \tag{3.56}$$

$$B(r) = \tilde{B}_0(r - r_1)(r_0 - r)\,, \tag{3.57}$$

and the surface gravity (3.53) is approximated by

$$\kappa = \frac{1}{2}\sqrt{\tilde{A}_0\tilde{B}_0(r_0 - r_1)} = \frac{1}{2}\sqrt{\tilde{A}_0\tilde{B}_0 r_0}\,\delta^{1/2}\,. \tag{3.58}$$

Now we proceed to extend the coordinate system though the tortoise coordinate (t, r_*, θ, ϕ) from the following integral

$$r_*(r) = \frac{1}{\sqrt{\tilde{A}_0\tilde{B}_0}} \int \frac{dr}{(r - r_1)\sqrt{(r_0 - r)}}. \tag{3.59}$$

the solution of this integral is (provided that $r_1 < r_0$)

$$r_*(r) = \frac{2}{\sqrt{\tilde{A}_0\tilde{B}_0(r_0 - r_1)}} \tanh^{-1}\left(-\frac{\sqrt{r_0 - r}}{\sqrt{r_0 - r_1}}\right), \tag{3.60}$$

which can be reformulated in terms of a logarithm function as

$$r_*(r) = \frac{1}{\sqrt{\tilde{A}_0\tilde{B}_0(r_0 - r_1)}} \ln\left(\frac{\sqrt{r_0 - r_1} - \sqrt{r_0 - r}}{\sqrt{r_0 - r_1} + \sqrt{r_0 - r}}\right), \tag{3.61}$$

and thus very similar to (3.40). We can simplify further to obtain

$$r_*(r) = \frac{1}{2\kappa} \ln\left[\frac{(r_0 - r_1) - (r_0 - r)}{r_0 - r + (r_0 - r_1) + 2\sqrt{r_0 - r_1}\sqrt{r_0 - r}}\right], \tag{3.62}$$

The interval $r_0 < r < r_1$ is mapped to $-\infty < r_* < \infty$, with $r = r_0$ mapped to $r_* = 0$ Taking the near-extremal limit

$$\sqrt{r_0 - r_1}\sqrt{r_0 - r} \approx r_0 - r_1\,, \tag{3.63}$$

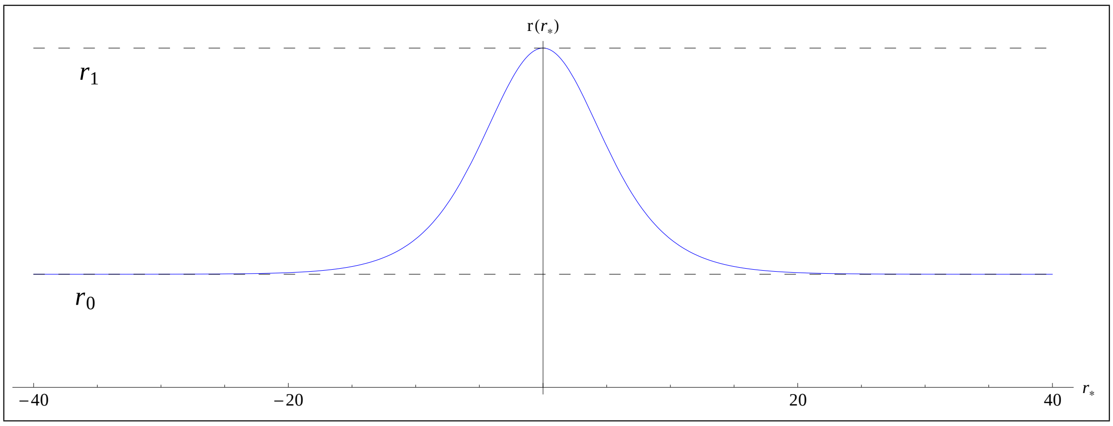

Fig. 3.3 Behavior of $r(r_*)$ of a black hole in compact geometry.

the tortoise coordinate takes a simpler form

$$r_*(r) = \frac{1}{2\kappa}\ln\left(\frac{r-r_1}{-r-3r_1+4r_0}\right). \tag{3.64}$$

Inverting $r_*(r)$ allows us to obtain the function $r(r_*)$

$$r(r_*) = \frac{r_1+(4r_0-3r_1)e^{2\kappa r_*}}{1+e^{2\kappa r_*}}. \tag{3.65}$$

This can also be written in terms of hyperbolic trigonometric functions

$$r(r_*) = \frac{4r_1\cosh^2(\kappa r_*)+4(r_0-r_1)(1+\mathrm{e}^{2\kappa r_*})}{4\cosh^2(\kappa r_*)}. \tag{3.66}$$

or

$$r(r_*) = r_1 + r_0\,\delta\,\mathrm{sech}^2\,(\kappa r_*)\,. \tag{3.67}$$

The form of (3.67) is illustrated in figure 3.3. In the near extremal limit we have

$$r(r_*) = r_1 + r_1\,\delta\,\mathrm{sech}^2\,(\kappa r_*)\,. \tag{3.68}$$

An the function $\mathscr{A}(r_*)$ is found to be

$$\mathscr{A}(r_*) = \tilde{A}_0 r_0\,\delta\,\mathrm{sech}^2(\kappa r_*). \tag{3.69}$$

The surface $r = r_0$ is a Killing horizon, but contrary to the wormhole geometries introduced in the previous section the point $r_* = 0$ is a local maximum of r. The surface $r = r_0$ is an inner trapping horizon [33]. Geometries given by (3.56) and (3.57) are interpreted as black holes in a compact universe.

Chapter 4

Perturbations and Quasinormal Modes

After introducing the basic formalism of general relativity and the spacetimes of interest for this thesis, now we turn to study some dynamical aspects involving propagation of matter content or fields in a spacetime. Solving Einstein's equation means finding the metric tensor $g_{\mu\nu}$ associated to a certain matter-radiation distribution given by the energy-momentum tensor $T_{\mu\nu}$. Once the background metric is established, a natural question is to determine its response to variations of the matter content. This is a highly non-linear problem, as variations in $T_{\mu\nu}$ imply an alteration of spacetime geometry, which at the same time involves a matter redistribution. Nevertheless, if one assumes small variations of the background metric one can treat the problem perturbatively, and in the lowest order, the background reaction can be ignored, implying that only the matter fields are treated dynamically in this approach [36].

In this thesis we will be dealing with classical fields propagating around a fixed spacetime background. The dynamics of such fields is introduced by means of a relativistic motion equation that depends on the spacetime metric.

4.1 Scalar perturbative dynamics

The analysis of the scalar dynamics in a curved spacetime is interesting due to its simplicity and the fact that more complex perturbation have in many times a similar behavior. To study the dynamics of a scalar field in a background spacetime, it is considered initially the simpler case of a massless scalar field Φ satisfying the (massless) Klein-Gordon equation [37]

$$\Box\Phi = 0\,, \tag{4.1}$$

where $\Box$ is the d'Alambertian operator

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu\,. \tag{4.2}$$

As equation (4.1) is written in a covariant form, it is valid in any coordinate system. Now we write (4.1) in terms of the background metric $g_{\mu\nu}$. Since Φ is a scalar function we have that the covariant derivative is equal to an ordinary partial derivative

$$\nabla^\mu\Phi = \partial^\mu\Phi = g^{\mu\nu}\partial_\nu\Phi\,. \tag{4.3}$$

Now, the action of the d'Alambertian operator on Φ corresponds to the divergence of the vector $\nabla^\mu\Phi$,

$$\nabla_\mu\left(\nabla^\mu\Phi\right) = \partial_\mu\nabla^\mu\Phi + \Gamma^\mu_{\mu\sigma}\nabla^\sigma\Phi\,. \tag{4.4}$$

We can write (4.4) in terms of the determinant g of the metric tensor using the following property of the Christoffel symbols [15]

$$\Gamma^\mu{}_{\mu\sigma} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\sigma}\sqrt{-g}\,. \tag{4.5}$$

Thus, we get the following expression

$$\nabla_\mu\nabla^\mu\Phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^{\mu\nu}\partial_\nu\Phi\right)\,, \tag{4.6}$$

and the Klein-Gordon equation (4.1) becomes

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = 0\,. \tag{4.7}$$

Since we will be dealing mostly with spherical symmetric and static spacetimes, we will take the background spacetime to be of the form (2.58) and we will write (4.7) in the coordinate system (t,r,θ,ϕ). In this coordinate system the determinant of the metric is

$$g = -\frac{A(r)}{B(r)}r^4\sin^2\theta\,, \tag{4.8}$$

and

$$\sqrt{-g} = r^2\sin\theta\sqrt{\frac{A(r)}{B(r)}}\,. \tag{4.9}$$

We split the operator $\Box$, as expressed in (4.7), in two parts

$$\Box = \Box_2 + \frac{1}{r^2}\nabla_2\,, \tag{4.10}$$

where we define

$$\Box_2 = \frac{1}{\sqrt{-g}}\left[\partial_t\left(\sqrt{-g}g^{tt}\partial_t\right)+\partial_r\left(\sqrt{-g}g^{rr}\partial_r\right)\right], \tag{4.11}$$

$$\nabla_2 = \frac{r^2}{\sqrt{-g}}\left[\partial_\theta\left(\sqrt{-g}g^{\theta\theta}\partial_\theta\right)+\partial_\phi\left(\sqrt{-g}g^{\phi\phi}\partial_\phi\right)\right]. \tag{4.12}$$

Using (4.9) we have

$$\Box_2 = -\frac{1}{A(r)}\frac{\partial^2}{\partial t^2}+\frac{1}{r^2}\sqrt{\frac{B(r)}{A(r)}}\frac{\partial}{\partial r}\left(r^2\sqrt{A(r)B(r)}\frac{\partial}{\partial r}\right), \tag{4.13}$$

$$\nabla_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{4.14}$$

The operator $\Box_2$ only acts on the variables t and r, while the operator ∇_2 only acts on the variables θ and ϕ, and it is identified with the Laplacian operator in two dimensions, written in spherical coordinates. This suggests that the field Φ should be given by a multipole expansion of the form

$$\Phi(t,r,\theta,\phi) = \sum_{\ell,m}\frac{1}{r}\Psi_\ell(t,r)Y_{l.m}(\theta,\phi), \tag{4.15}$$

where $Y_{\ell.m}$ are the spherical harmonics, eigenfunctions of the Laplacian operator [38]

$$\nabla_2 Y_{\ell.m}(\theta,\phi) = -\ell(\ell+1)Y_{\ell,m}(\theta,\phi). \tag{4.16}$$

The spherical harmonics are a complete set labeled by two integer numbers ℓ, m, with $\ell = 0,1,2,...$ and $m = -\ell,...,0,...,\ell$; and are given explicitly by

$$Y_{\ell m}(\theta,\phi) = P_\ell(\cos\theta)\mathrm{e}^{\pm im\phi}, \tag{4.17}$$

where $P_l(\cos\theta)$ are the Legendre polynomials, which can be obtained from the Rodrigues formula [39]:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell (x^2-1)^\ell. \tag{4.18}$$

Replacing (4.15) into (4.7) gives

$$\sum_{\ell,m}\left[\Box_2\left(\frac{\Psi_\ell(t,r)}{r}\right)Y_{\ell,m}(\theta,\phi)+\left(\frac{\Psi_\ell(t,r)}{r}\right)\frac{1}{r^2}\nabla_2 Y_{\ell,m}(\theta,\phi)\right]=0, \tag{4.19}$$

or

$$\sum_{\ell,m}Y_{\ell,m}(\theta,\phi)\left[\Box_2\left(\frac{\Psi_\ell(t,r)}{r}\right)-\frac{\ell(\ell+1)}{r^2}\left(\frac{\Psi_\ell(t,r)}{r}\right)\right]=0. \tag{4.20}$$

We obtain a differential equation for each ℓ:

$$\left[-\frac{1}{A(r)}\frac{\partial^2}{\partial t^2}+\frac{1}{r^2}\sqrt{\frac{B(r)}{A(r)}}\frac{\partial}{\partial r}\left(r^2\sqrt{A(r)B(r)}\frac{\partial}{\partial r}\right)-\left[\frac{\ell(\ell+1)}{r^2}\right]\right]\left(\frac{\Psi_\ell(t,r)}{r}\right)=0\,. \tag{4.21}$$

Reorganizing terms, we get

$$-\frac{\partial^2}{\partial t^2}\Psi_\ell+\frac{\sqrt{A(r)B(r)}}{r}\frac{\partial}{\partial r}\left(r^2\sqrt{A(r)B(r)}\frac{\partial}{\partial r}\left(\frac{\Psi_\ell}{r}\right)\right)=\frac{\ell(\ell+1)}{r^2}A(r)\Psi_\ell\,. \tag{4.22}$$

After expanding derivatives and canceling some terms the left hand side is equal to

$$-\frac{\partial^2}{\partial t^2}\Psi_\ell-\frac{1}{2r}(A'B+AB')\Psi_\ell+\sqrt{A(r)B(r)}\frac{\partial}{\partial r}\left(\sqrt{A(r)B(r)}\frac{\partial}{\partial r}\Psi_\ell\right), \tag{4.23}$$

where we are denoting the derivatives of the functions $A(r)$ and $B(r)$ by A' and B' respectively. Leaving on the left hand side terms with derivatives we can write the differential equation for the function $\Psi(t,r)$ as

$$-\frac{\partial^2}{\partial t^2}\Psi_\ell(t,r)+\sqrt{A(r)B(r)}\frac{\partial}{\partial r}\left(\sqrt{A(r)B(r)}\frac{\partial}{\partial r}\Psi_\ell(t,r)\right)=V_{sc}(r)\Psi_\ell(t,r), \tag{4.24}$$

where we have defined the function $V_{sc}(r)$ as

$$V_{sc}(r)=\frac{\ell(\ell+1)}{r^2}A(r)+\frac{1}{2r}\left[A'(r)B(r)+A(r)B'(r)\right]\,. \tag{4.25}$$

Equation (4.24) can be expressed in a simpler way in the coordinate system $(t,r_*,\theta;\phi)$, where r_* is the tortoise coordinate defined in (2.62). With the change of variables

$$\frac{\partial}{\partial r}=\frac{\mathrm{d}r_*}{\mathrm{d}r}\frac{\partial}{\partial r_*}=\frac{1}{\sqrt{A(r)B(r)}}\frac{\partial}{\partial r_*}\,, \tag{4.26}$$

we have that (4.24) reduces to

$$-\frac{\partial^2}{\partial t^2}\Psi_\ell(t,r_*)+\frac{\partial^2}{\partial {r_*}^2}\Psi_\ell(t,r_*)=\tilde{V}_{sc}(r_*)\Psi_\ell(t,r_*), \tag{4.27}$$

where we are using the notation

$$\Psi_\ell(t,r_*)=\Psi_\ell(t,r(r_*)), \tag{4.28}$$

$$\tilde{V}_{sc}(r_*)=V_{sc}(r(r_*)). \tag{4.29}$$

We will say that the functions $V_{sc}(r)$ in (4.25) and $\tilde{V}_{sc}(r_*)$ in (4.27) are *effective potentials* for the scalar perturbations, which depends on the details of the geometry from metric coefficients $A(r)$ and $B(r)$. Solutions of equation (4.27) are labeled by the integer ℓ, and depend on the explicit form of the effective potential $\tilde{V}_{sc}(r)$.

4.2 Quasinormal modes

In spherically symmetric and static spacetimes the perturbative dynamics of scalar, electromagnetic and gravitational fields are found as solutions of equations of the form [6, 7]

$$\frac{\partial^2}{\partial t^2}\Psi(t,r_*) + \left(-\frac{\partial^2}{\partial^2 r_*} + V(r_*)\right)\Psi(t,r_*) = 0\,, \tag{4.30}$$

with $V(r_*)$ the effective potential, depending on the details of the (fixed) background spacetime. If one is interested in the evolution of a certain initial perturbation, one can write the problem as a Cauchy initial value problem of the form

$$\Psi(t=0,r_*), \qquad \left.\frac{\partial\Psi(t,r_*)}{\partial t}\right|_{t=0}. \tag{4.31}$$

To solve equation (4.30) with initial conditions of the form (4.31) it is possible to use the technique of Laplace transform [6, 7]. The Laplace transform $\hat{f}(\omega,r_*)$ of a function $f(t,r_*)$ defined for $0 \le t \le \infty$ is

$$\mathscr{L}[f(t,r_*)] = \hat{f}(\omega,r_*) = \int_0^\infty f(t,r_*)\mathrm{e}^{-i\omega t}dt\,, \tag{4.32}$$

where it is required that $f(t,r_*)$ is exponentially bounded for (4.32) to be well defined, that is,

$$\lim_{t\to\infty}|f(t,r_*)\mathrm{e}^{-\alpha t}| = 0\,, \tag{4.33}$$

or equivalently

$$|f(t,r_*)| \le M\mathrm{e}^{\alpha t}\,. \tag{4.34}$$

The boundedness of $f(t,r_*)$ implies that $\hat{f}(\omega,r_*)$ is analytic for $i\omega$ positive and real, therefore it can have an analytic continuation where the variable $i\omega$ can be complex with $Re(i\omega) > 0$. The Laplace transform shares the usual properties of linearity

$$\mathscr{L}[f(t,r_*)+g(t,r_*)] = \mathscr{L}[f(t,r_*)] + \mathscr{L}[g(t,r_*)]\,, \tag{4.35}$$

$$\mathscr{L}[cf(t,r_*)] = c\mathscr{L}[f(t,r_*)]\,. \tag{4.36}$$

An important property that we will use is the Laplace transform for the second derivative of a function $f(t,r_*)$ in the variable t

$$\mathscr{L}(f''(t,r_*)) = -\omega^2 \mathscr{L} f(t,r_*) - i\omega f(t,r_*)|_{t=0} - f'(t,r_*)|_{t=0}\,. \tag{4.37}$$

Using this property the differential equation (4.30) can be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}{r_*}^2}\hat{\Psi}(\omega,r_*) + \omega^2\hat{\Psi}(\omega,r_*) - i\omega\Psi(t=0,r_*) - \Psi'(0,r_*)|_{t=0} - V(r_*)\hat{\Psi}(\omega,r_*) = 0\,. \tag{4.38}$$

Reorganizing terms we have

$$\frac{\mathrm{d}^2}{\mathrm{d}{r_*}^2}\hat{\Psi}(\omega,r_*) + [\omega^2 - V(r_*)]\hat{\Psi}(\omega,r_*) = J(\omega,r_*)\,, \tag{4.39}$$

where

$$\frac{\mathrm{d}^2}{\mathrm{d}{r_*}^2}\hat{\Psi}(\omega,r_*) + [\omega^2 - V(r_*)]\hat{\Psi}(\omega,r_*) = 0\,, \tag{4.40}$$

is the homogeneous differential equation, and $J(\omega,r_*)$ is the inhomogeneous component

$$J(\omega,r_*) = i\omega\Psi(t=0,r_*) + \Psi'(t,r_*)|_{t=0}\,. \tag{4.41}$$

The function $J(\omega,r_*)$ has information of the initial values of the perturbation. The solution $\hat{\Psi}(\omega,r_*)$ of (4.40) is unique and it is given in terms of the Green function of the homogeneous equation

$$\hat{\Psi}(\omega,r_*) = \int_{-\infty}^{\infty} G(\omega,r_*,r'_*)J(\omega,r'_*)dr'_*\,, \tag{4.42}$$

where $G(\omega,r_*,r'_*)$ satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}{r_*}^2}G(\omega,r_*,r'_*) + \left(\omega^2 - V(x)\right)G(\omega,r_*,r'_*) = \delta(r_* - r'_*)G(\omega,r_*,r'_*)\,. \tag{4.43}$$

The Green's function can be obtained from two linearly independent solutions $f_-(\omega,r_*)$ and $f_+(\omega,r_*)$ of the homogeneous equation

$$G(\omega,r_*,r'_*) = \frac{1}{W(\omega)}\begin{cases} f_-(\omega,r'_*)f_+(\omega,r_*) & r'_* < r_* \\ f_-(\omega,r_*)f_+(\omega,r'_*) & r_* < r'_* \end{cases}\,, \tag{4.44}$$

where

$$W(\omega) = \left(\frac{\partial}{\partial r_*}f_-(\omega,r_*)\right)f_+(\omega,r_*) - f_-(\omega,r_*)\frac{\partial}{\partial r_*}f_+(\omega,r_*)\,, \tag{4.45}$$

is the Wronskian of the solutions to the homogeneous differential equation (4.40), with the property that it is different from zero if the solutions are linearly independent and equal to zero otherwise [40].

We will be interested in studying perturbations in spacetimes containing black holes. We will see that the effective potential vanishes at the event horizon, and in many scenarios, it also vanishes at spatial infinity. Therefore, it is not appropriated to impose as boundary condition that solutions of (4.30) vanish in these regions ; as their behavior will resemble spherical plane waves instead [6, 7].

To obtain the solutions $f_-(\omega, r_*)$ and $f_+(\omega, r_*)$ we use the condition of the potential $V(r_*)$ vanishing for $|r_*| > b$, then, solutions of (4.40) for $|r_*| > b$ are of the form

$$f_+(\omega, r_*) = \mathrm{e}^{-i\omega r_*}, \qquad \text{for } r_* > b \tag{4.46}$$

$$f_-(\omega, r_*) = \mathrm{e}^{+i\omega r_*}, \qquad \text{for } r_* < -b\,. \tag{4.47}$$

For $|r_*| < b$, solutions of $f_-(\omega, r_*)$ and $f_+(\omega, r_*)$ will be in general a linear combination of the form $a(\omega)e^{-i\omega r_*} + b(\omega)e^{i\omega r_*}$.

The property that the inner region of a black hole is casually disconnected from spatial infinity leads to impose the condition that near the event horizon perturbations should behave as purely ingoing waves (only entering into the black hole). Additionally, if perturbations coming from spatial infinity are disregarded, thus considering only localized perturbations, solutions at spatial infinity behave as purely outgoing waves. If the event horizon is located at $r_* \to -\infty$ and spatial infinity at $r_* \to \infty$ the boundary conditions for perturbations are

$$\Psi(r_*) \sim \begin{cases} \mathrm{e}^{-i\omega r_*} & \text{as } r_* \to -\infty \\ \mathrm{e}^{i\omega r_*} & \text{as } r_* \to \infty \end{cases}, \tag{4.48}$$

meaning that they satisfy both conditions (4.46) and (4.47) simultaneously. Perturbations satisfying these conditions are called *quasinormal modes*. The complex numbers $\{\omega_n\}$, for which both functions $f_-(\omega, r_*)$ and $f_+(\omega, r_*)$ become linearly dependent

$$f_+(\omega_n, r_*) = c(\omega_n) f_-(\omega_n, r_*)\,, \tag{4.49}$$

are called *quasinormal frequencies*. In this case the Wronskian (4.45) vanishes and the Green function (4.43) is singular. If a solution with boundary condition (4.48) does exist, then an initial perturbation outside the event horizon of the black hole will be followed by exponentially damped oscillations, given the complex character of the frequencies [6, 7]. Quasinormal modes are important because they depend on the black hole parameters, but not much on the details of the initial perturbation, thus they can be though of as resonances of the spacetime in response to the mentioned perturbation.

4.2.1 Completeness of quasinormal modes

With the Laplace transform, the problem of solving (4.30) reduces to obtain the Green function of (4.40). Once (4.42) is solved the time dependent solution $\Psi(t,r_*)$ can be determined by means of Laplace inverse transformation

$$\Psi(t,r_*) = \frac{1}{2\pi}\int_\gamma e^{i\omega t}\hat{\Psi}(\omega,r_*)d\omega\,, \tag{4.50}$$

The integration path $\gamma = i\omega$ is chosen to lie at the right of every singularity of $\hat{\Psi}(\omega,r_*)$. The curve γ is then turned into a closed curve by taking a semicircle C of radius R and the limit $R\to\infty$.

Since the quasinormal frequencies $\{\omega_n\}$ are singularities of the Green function, and therefore of $\hat{\Psi}(\omega,r_*)$, we can use the theorem of residues to solve (4.50)

$$\oint e^{i\omega t}\hat{\Psi}(\omega,x)d\omega = 2\pi i\sum_{\omega_i} Res(\omega_i)\,, \tag{4.51}$$

where $Res(\omega_i)$ is the residue associated with the pole ω_i. On the other hand, as we shall see later, the quasinormal frequencies $\{\omega_n\}$ might not be the only source of singularities for (4.50), and other contributions must be taken into account. Quasinormal modes form a complete set only if the Green's function can be expressed as a sum over quasinormal modes [6, 7].

In [7], it was shown that by deforming the path of the complex integration, the late time behavior of the function $\Psi(t,r_*)$ can be approximated by a finite sum of the form

$$\Psi(t,r_*) \sim \sum_{n=1}^{N} a_n e^{(\alpha_n+i\beta_n)t} f_+(\omega_n,r_*)\,, \tag{4.52}$$

where $i\omega_n = \alpha_n + i\beta_n$. This approximation implies the existence of a constant C such that for $t > t_0$

$$\left|\Psi(t,r_*) - \sum_{n=1}^{N} a_n e^{(\alpha_n+i\beta_n)t} f_+(\omega_n,r_*)\right| \le Ce^{(-|\alpha_{n+1}+i\beta_{n+1}+\varepsilon|)t}\,, \tag{4.53}$$

implying that solutions decay exponentially in time over spatially bounded regions.

In general, quasinormal modes do not constitute a complete set, and it is not possible to express a perturbation in terms of quasinormal modes only. For any given perturbation, and depending on the nature of the effective potential, other possible contributions may appear:

1. *Tail contributions*: These contributions appear for some potentials that decay faster than exponentials, for example, with a power-tail law. In [41] it was found that for effective

potentials of the form

$$V(r) = \frac{\ell(\ell+1)}{r^2} + \frac{1}{r^\alpha}\ln(r)^\beta, \quad \beta = 0, 1, \tag{4.54}$$

the Green function possesses singularities on the -Im ω axis, and therefore a branch cut. The predominant late time behavior are power-law tails of the form

$$t^{-(2\ell+1)}\ln(t)^\beta. \tag{4.55}$$

The Schwarzschild black hole is a particular case of (4.54) with $\beta = 0$, and normally, after an initial perturbation there is a quasinormal mode oscillation followed by a power-tail law decaying behavior .

2. *Prompt-contributions*: This contribution corresponds from the integration over the semi-circle $|\omega| = R$ with $R \to \infty$. These are large frequency $|\omega|$, or equivalent, short time contributions which vanishes after a certain time and does not affect the late time behavior of the field evolution [6].

4.3 Effective potential of the SdS spacetime

Here we will give some details of perturbative dynamics for scalar fields in the geometries introduced in chapter 3, starting with scalar fields on the Schwarzschild-de Sitter spacetime. The effective potential associated with a massless scalar field perturbation is of the form

$$V(r) = A(r)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} - \frac{2}{R^2}\right). \tag{4.56}$$

where $A(r)$, given by (3.1), has two roots r_1 and r_2, with r_1 corresponds to a black hole horizon and r_2 to a cosmological horizon. Potential (4.56) is zero at both r_1 and r_2 and it is positive defined as long as $\ell > 0$.

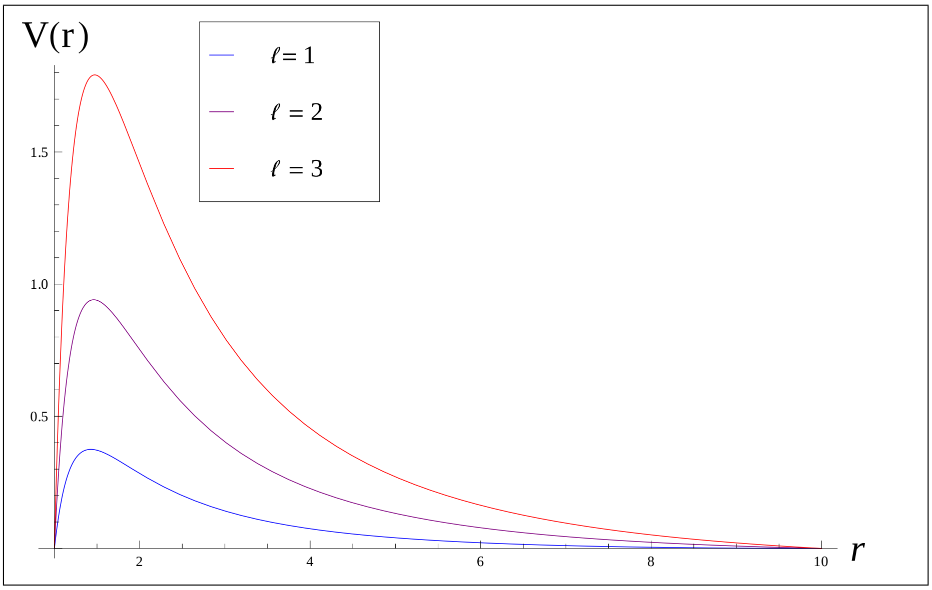

Fig. 4.1 Effective potential for Schwarzschild-de Sitter spacetime as a function of the radial coordinate. Parameters $r_1 = 1$, $r_2 = 10$.

Normally, it is not possible to obtain an explicit expression for the function of $r(r_*)$ by finding the inverse of (3.10), but we have seen that in the near extremal we can get analytical results. We write (4.56)

$$V(r) = A(r)\Omega(r), \tag{4.57}$$

where $\Omega(r)$ is given by

$$\Omega(r) = \left(\frac{\ell(\ell+1)}{r^2} + \frac{1}{r}\frac{\mathrm{d}A(r)}{\mathrm{d}r}\right). \tag{4.58}$$

We will expand the function $\Omega(r)$ around r_0 at the lowest order

$$\Omega(r) = \Omega(r_0) + \mathscr{O}(\delta). \tag{4.59}$$

From (3.26) we have that the first derivative of $A(r)$ at r_0 is zero at the leading order

$$\left.\frac{\mathrm{d}A(r)}{\mathrm{d}r}\right|_{r=r_0} = 0 + \mathscr{O}(\delta^2)\,, \tag{4.60}$$

and the function $\Omega(r)$ reduces to

$$\Omega(r) = \frac{\ell(\ell+1)}{r_0^2} + \mathscr{O}(\delta)\,. \tag{4.61}$$

Now, in the near extremal limit we have the following relation between r_0 and r_1

$$r_0 = r_1 + \mathscr{O}(\delta)\,, \tag{4.62}$$

and then, at the lowest order in δ, and taking into account (3.34), the effective potential in the near extremal limit becomes

$$V(r) = \left[\frac{\ell(\ell+1)}{r_1^2}\right]\frac{2\kappa_1(r_2-r)(r-r_1)}{r_2-r_1} + \mathscr{O}(\delta)\,, \tag{4.63}$$

which we note is only meaningful for $\ell > 0$. In the near extremal approximation, we have a function $r(r_*)$ given by (3.32), and we get

$$(r_2 - r) = \frac{(r_2-r_1)\mathrm{e}^{-\kappa r_*}}{\mathrm{e}^{-\kappa r_*} + e^{\kappa r_*}} \qquad \text{and} \qquad (r-r_1) = \frac{(r_2-r_1)\mathrm{e}^{\kappa r_*}}{\mathrm{e}^{-\kappa r_*} + e^{\kappa r_*}}\,. \tag{4.64}$$

With this, the effective potential for scalar perturbations reduces to

$$V(r_*) = \left[\frac{\ell(\ell+1)}{r_1^2}\right]\frac{2\kappa_1(r_2-r_1)}{(\mathrm{e}^{-\kappa r_*} + e^{\kappa r_*})^2} + \mathscr{O}(\lambda)\,. \tag{4.65}$$

We can write (4.65) in terms of an hyperbolic cosine function as,

$$V(r_*) = \frac{V_0}{\cosh^2(\kappa r_*)}\,, \tag{4.66}$$

where the constant V_0 is the peak of the potential, given by

$$V_0 = \left[\frac{\ell(\ell+1)}{r_1^2}\right]\frac{\kappa_1(r_2-r_1)}{2}\,. \tag{4.67}$$

The effective potential (4.66) is known as the *Pöschl-Teller* potential [42]. It is positive defined and decays to zero when $r_\star \to \pm\infty$. The quasinormal modes for (4.66) can be solved analytically by turning the scalar equation into an hypergeometric equation [43], and the quasi-

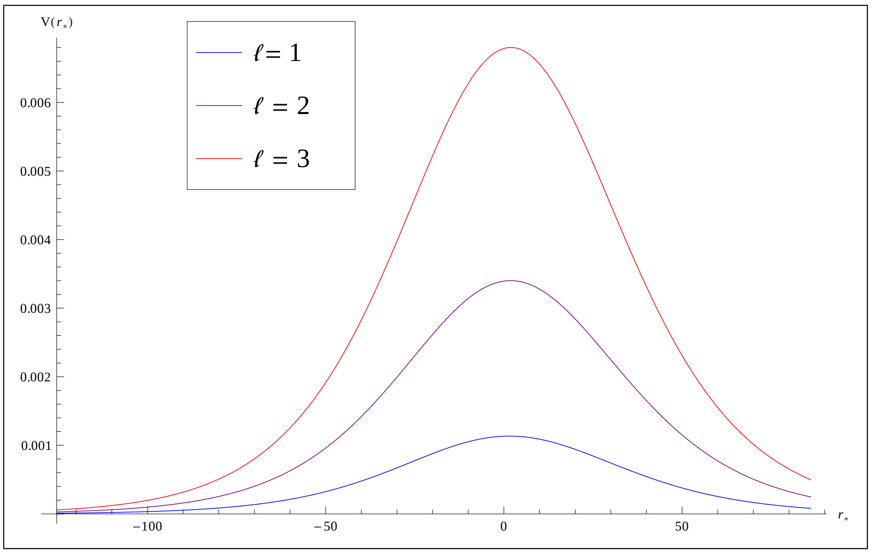

Fig. 4.2 Effective potential for near extremal Schwarzschild-de Sitter as a function of r_*. Parameters: $r_1 = 1$, $r_2 = 1.05$, $\ell = 1$.

normal frequencies are well-known for this case

$$\omega = \kappa \left[\sqrt{\frac{V_0}{\kappa^2} - \frac{1}{4}} - i \left(n + \frac{1}{2} \right) \right] , \tag{4.68}$$

where $n \in \{0, 1, \dots\}$ is an integer number labeling each quasinormal mode. In Beyer [44] it is discussed the completeness of quasinormal modes of the Pöschl-Teller potential, where they characterize entirely the scalar dynamics.

As it was mentioned before, the Schwarzschild-de Sitter potential is positive defined for $\ell > 0$. For the spherically symmetric modes $\ell = 0$ it is suggested in [28] that the evolution of the scalar field is stable, and the field tends to a non-zero constant for every fixed value of r.

We close this section showing that the Pöschl-Teller potential can also be used to approximate the dynamics of scalar fields on the near extremal geometries introduced in subsections 3.3.2 and 3.3.3. In the case of near extremal wormholes in asymptotically de Sitter spacetimes, we had that the metric function was approximated by the functions (3.35) and (3.36). Since $A(r)$ and $B(r)$ are different we must take expression (4.25) into account. However, we also use the following approximation

$$\left. \frac{\mathrm{d}B(r)}{\mathrm{d}r} \right|_{r=\frac{r_2+r_0}{2}} = 0 + \mathscr{O}(\delta) . \tag{4.69}$$

With that, the scalar effective potential is given by

$$V(r) = \frac{\ell(\ell+1)}{r_0^2}\tilde{A}_0(r_2 - r) - \frac{1}{2r_0}\tilde{A}_0\tilde{B}_0(r_2 - r)(r - r_0)\,. \tag{4.70}$$

We write potential (4.70) above in terms of the tortoise coordinate (3.46), from where we obtain the following

$$V(r) = \tilde{A}_0 r_0 \delta\, \mathrm{sech}^2(\kappa r_*)\left[\frac{\ell(\ell+1)}{r_0^2} - \frac{\tilde{B}_0}{2}\delta \tanh^2(\kappa r_*)\right]. \tag{4.71}$$

Taking into account the near extremal limit $\delta \to 0$ we can neglect the second term, since it would be proportional to δ^2, and write

$$V(r) = \tilde{A}_0 \delta \frac{\ell(\ell+1)}{r_0}\frac{1}{\cosh^2(\kappa r_*)} + \mathscr{O}(\delta^2)\,. \tag{4.72}$$

The scenario of black holes in compact universes is not very different. From the functions (3.56) and (3.57) and also taking the following approximation

$$\left.\frac{\mathrm{d}B(r)}{\mathrm{d}r}\right|_{r=\frac{r_1+r_0}{2}} = 0 + \mathscr{O}(\delta)\,, \tag{4.73}$$

the effective potential is

$$V(r) = \frac{\ell(\ell+1)}{r_0^2}\tilde{A}_0(r - r_1) + \frac{1}{2r_0}\tilde{A}_0\tilde{B}_0(r - r_1)(r_0 - r)\,. \tag{4.74}$$

Using the tortoise coordinate given in (3.67) we get

$$V(r) = \tilde{A}_0 r_0 \delta\, \mathrm{sech}^2(\kappa r_*)\left[\frac{\ell(\ell+1)}{r_0^2} + \frac{\tilde{B}_0}{2}\delta \tanh^2(\kappa r_*)\right]. \tag{4.75}$$

which at first order in δ yields the same result as (4.72).

4.4 Effective potential of the SAdS spacetime

Now we turn to study scalar dynamics in the Schwarzschild-Anti de Sitter scenario. The effective potential associated with a massless scalar field perturbation in is of the form

$$V(r) = A(r)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} + \frac{2}{R^2}\right), \tag{4.76}$$

The effective potential (4.76) is zero at r_+ and it is positive defined for for $r > r_+$ since both $A(r)$ and the factor

$$\Omega(r) = \frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} + \frac{2}{R^2}, \tag{4.77}$$

are positive for $r > r_+$. Unlike the *SdS* potential, the *SAdS* potential is divergent in the limit $r \to \infty$. We also have that the shape of the potential depends on the relation between r_+ and R, as seen in figure 4.3

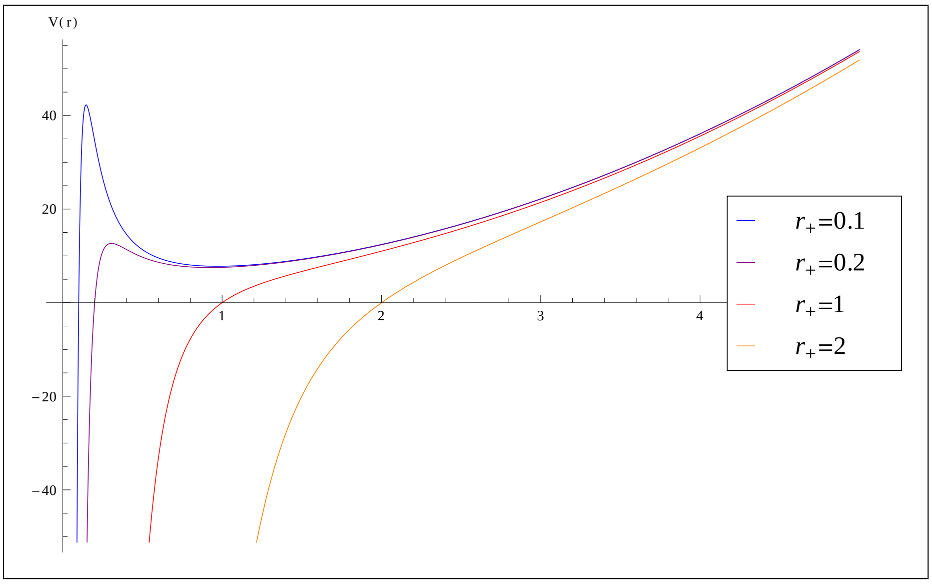

Fig. 4.3 Effective potential for Schwarzschild-Anti de Sitter spacetime as a function of the radial coordinate for different event horizons in relation with $R = 1$.

There is no analytic expression for (4.76) in terms of the tortoise coordinate r_*, neither a near extremal approximation since there is only one horizon. Nevertheless, we can study the behavior of the potential near the horizon and in the limit $r \to \infty$. At the event horizon the effective potential (4.76) vanishes. In the near horizon limit the tortoise coordinate is

$$r_*(r) = \frac{1}{2\kappa}\ln\left(\frac{r}{r_+} - 1\right) + r_*^+ + \mathcal{O}\left(\frac{r - r_+}{r_+}\right)^2, \tag{4.78}$$

where

$$r_*^+ = -\frac{1}{4\kappa}\ln\left(3+\frac{R^2}{r_+^2}\right)+\frac{R^2}{\sqrt{3r_+^2+4R^2}}\frac{3r_+^2+2R^2}{3r_+^2+R^2}\left[\tan^{-1}\left(\frac{3r_+}{\sqrt{3r_+^2+4R^2}}\right)-\frac{\pi}{2}\right]. \tag{4.79}$$

Expression (4.78) can be inverted giving

$$r(r_*) = r_+\left(1+\mathrm{e}^{2\kappa r_*}\mathrm{e}^{-2\kappa r_*^+}\right)+\mathcal{O}(\mathrm{e}^{2\kappa r_*})^2. \tag{4.80}$$

Now, we approximate the expressions of the function $A(r)$ and the effective potential $V(r)$ near r_+. With $A(r_+)=0$ we have the following first order Taylor expansion

$$A(r) = \left.\frac{\mathrm{d}A(r)}{\mathrm{d}r}\right|_{r_+}(r-r_+)+\mathcal{O}\left(r-r_+\right)^2. \tag{4.81}$$

Using the definition for the surface gravity (3.17) and the expression for the tortoise coordinate (4.78) we get

$$A(r_*) = 2\kappa r_+\mathrm{e}^{2\kappa r_*}\mathrm{e}^{-2\kappa r_*^+}+\mathcal{O}\left(r_+\mathrm{e}^{2\kappa r_*}\right). \tag{4.82}$$

Near the horizon, the first-order Taylor expansion of the scalar effective potential is

$$V(r) = V(r_+)+\left.\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right|_{r_+}(r-r_+)+\mathcal{O}\left(r-r_+\right)^2 \tag{4.83}$$

which in terms of $A(r)$ and $\Omega(r)$, reminding that $A(r)$ is zero at r_+, reduces to

$$V(r) = \left(\Omega(r_+)\left.\frac{\mathrm{d}A(r)}{\mathrm{d}r}\right|_{r_+}\right)(r-r_+)+\mathcal{O}\left(r-r_+\right)^2, \tag{4.84}$$

or, equivalently

$$V(r) = 2\kappa\Omega(r_+)(r-r_+)+\mathcal{O}\left(r-r_+\right)^2. \tag{4.85}$$

Using this result, we have that near $r_+\to-\infty$ the effective potential, as a function of r_*, has the following form

$$V(r_*) = 2\kappa\left[\frac{\ell(\ell+1)}{r_+^2}+\frac{2M}{r_+^3}+\frac{2}{R^2}\right]r_+\mathrm{e}^{2\kappa r_*}\mathrm{e}^{-2\kappa r_*^+}+\mathcal{O}\left(r_+\mathrm{e}^{2\kappa r_*}\right)^2. \tag{4.86}$$

Using the expressions (3.15) for the black hole mass and (3.17) of the surface gravity we can write this as

$$V(r_*) = 2\kappa\left[\frac{\ell(\ell+1)}{r_+}+2\kappa\right]\mathrm{e}^{2\kappa r_*}\mathrm{e}^{-2\kappa r_*^+}+\mathcal{O}\left(r_+\mathrm{e}^{2\kappa r_*}\right)^2. \tag{4.87}$$

We have that the effective potential vanishes exponentially near the event horizon. Now we analyze the limit $r \to \infty$. In this limit the tortoise coordinate is approximately

$$r_*(r) = -\frac{R^2}{r} + \mathscr{O}\left(\frac{r_+}{r}\right)^2 . \tag{4.88}$$

Inverting this function we get

$$r(r_*) = \frac{-R^2}{r_*} + \mathscr{O}\left(r_*\right)^2 , \tag{4.89}$$

In the effective potential the dominant factor is

$$V(r) = \frac{2}{R^4}r^2 + \frac{2+\ell(\ell+1)}{R^2} + \mathscr{O}\left(\frac{r_+}{r}\right)^2 , \tag{4.90}$$

Implying that the potential diverges quadratically as $r \to \infty$. In terms of the tortoise coordinate we have

$$V(r_*) = \frac{2}{r_*^2} + \frac{2+\ell(\ell+1)}{R^2} + \mathscr{O}\left(r_*\right)^2 , \tag{4.91}$$

Thus, the effective potential as a function of r_* diverges at $r_* = 0$. The divergent behavior of the scalar potential at spatial infinite imposes a modification on boundary conditions of quasinormal modes. In asymptotically Anti de Sitter spacetimes, the boundary conditions (4.48) are not appropriated since information can flow from infinity. To prevent that, instead of outgoing solutions at infinity it is adapted a Dirichlet boundary condition and ingoing solutions at the black hole horizon horizon

$$\Psi(r_*) \sim \begin{cases} \mathrm{e}^{-i\omega r_*} & \text{as } r_* \to -\infty \\ 0 & \text{as } r_* \to \infty \end{cases} . \tag{4.92}$$

Most treatments of quasinormal modes of Schwarzschild-Anti de Sitter spacetime are numeric. As a matter of fact, except for a limited number of geometries, quasinormal modes cannot be found analytically, and one has to resort to either a direct numerical integration of (4.30) or approximations of the effective potential. Reviews on the topic of quasinormal modes with results for a number of geometries can be found in references such as [45, 46]

In this thesis we will use an algebraic approach to obtain quasinormal modes for the geometries introduced in chapter 3 in certain specific limits. In the following chapter we give a review on some topics of group theory, where we will focus on representations of Lie groups and algebras, and later, we will associate the solutions of scalar dynamics with a particular algebraic representation.

Chapter 5

Elements of Representations of Lie Groups and Algebras

In the last chapter we introduced some ideas of perturbative dynamics of classical scalar fields on spherically symmetric and static spacetimes by means of a relativistic field equation. In some cases, it happens that these field equations possess additional symmetries which are not directly related to the isometries of the background metric. One of the central ideas of this thesis is to take advantage of those symmetries to obtain quasinormal modes solutions using group theoretical methods.

As the mathematical language used to formalize the idea of symmetries is group theory, this chapter is dedicated to introduce some elements of this subject, specially on representation theory of Lie groups and Lie algebras, and ultimately on the group $SL(2,\mathbb{R})$ and its algebra $\mathfrak{sl}(2,\mathbb{R})$, the group we will study as a symmetry of the equations of motions.

5.1 Basic concepts

A *group* $(G,\cdot)$ is a nonempty set G, together with a binary operation $\cdot : G \times G \to G$, called the group product of G, associating to any ordered pair of elements (a,b) another element in G, denoted $a \cdot b$ or ab, such that both the set and the group product satisfy the following group axioms [22]:

- *Associativity*: The defined product is associative, i.e., for all a,b,c in G, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

- *Existence of an identity*: There is an (unique) identity element $\mathbf{e}$ in G such that $\mathbf{e} \cdot g = g \cdot \mathbf{e} = g$ for every element g in G.

- *Existence of an inverse*: For each element g of G, the set contains an (unique) element g^{-1} such that $g \cdot g^{-1} = g^{-1} \cdot g = \mathbf{e}$.

In general the group product is not associative. If $a \cdot b = b \cdot a$ for all $a, b \in G$ the group G is called Abelian. A group can have either a finite or infinite number or elements, and the topology of the group can be discrete (its elements are countable) or continuous (a set with infinitely, non-countable elements).

A *subgroup* $(H, \cdot)$ of $(G, \cdot)$ is a subset H of G also forming a group under the group operation of G. More precisely, H is a subgroup of G if the restriction of $\cdot$ to $H \times H$ is a group operation on H.

A *group homomorphism* between two groups $(G, \cdot)$ and $(H, *)$ is a map $f : G \to H$ preserving the group structure

$$f(g_1 \cdot g_2) = f(g_1) * f(g_2)\,, \tag{5.1}$$

for all elements g_1, g_2 in G. This requirement ensures that

$$f(e_G) = e_H, \quad \text{and} \quad f(g)^{-1} = f(g^{-1})\,, \tag{5.2}$$

for all g in G, where e_G is the identity in G and e_H is the identity in H. Even though a group homomorphism preserves the group structure, in general $f(G)$ is not equal to H. A group homomorphism of G into itself is called *endomorphism*

A *group isomorphism* is a group homomorphism that is both invertible and one-to-one. If there exists a group isomorphism between G and H, the groups are called isomorphic $G \cong H$. A group isomorphism of G into itself is called *automorphism*.

5.2 Group representations

A group is an abstract construction and sometimes not useful on its own. Instead of dealing directly with groups, physicist often work with group representations, which are realizations of groups in terms of operators, usually finite or infinite dimensional matrices, acting on a particular vector space.

First we define the notion of group action. If X is a set and G is a group, the *group action* of G on X is a map $g \in G : X \to X$ that takes an element $x \in X$ to another element $gx \in X$, such that [47]

$$g_2(g_1 x) = (g_2 g_1)x, \quad and \quad ex = x\,, \tag{5.3}$$

for every $g_1, g_2 \in G$.

A *group representation* of a group G on a vector space V is a group homomorphism from G

to the group $GL(V)$ of non-singular linear transformations of V. That is, a representation of G is a map $D : G \to GL(V)$ such that

$$D(g_1)D(g_2) = D(g_1 \cdot g_2)\,, \tag{5.4}$$

for every $g_1, g_2 \in G$ [48]. In particular if e is the identity element of G, $D(e)$ is the identity matrix acting on the vector space V,

$$D(eg) = D(ge) = D(g)D(e) = D(e)D(g) = D(g)\,, \tag{5.5}$$

for all g in G. A group representation is a set of operators acting on a vector space V. The vector space V is called the representation space and the dimension of V is called the dimension of the representation. For every vector space there is the representation $D(g) = \mathbb{1}$ for all g. If $\dim V = 1$ such representation is called the trivial representation.

If the vector space V is equipped with a positive definite scalar product, a representation D is said to be unitary if the scalar product is preserved

$$(u, v) = (D(g)u, D(g)v)\,. \tag{5.6}$$

Unitarity implies $D(g^{-1}) = D(g)^\dagger$, with $D(g)^\dagger$ being the adjoint of $D(g)$.

Most groups admit different representations, even on different vector spaces. Given two vector spaces V and W, two representations $D^{(1)} : G \to GL(V)$ and $D^{(2)} : G \to GL(W)$ are said to be equivalent or isomorphic if they are related by a vector space isomorphism $S : V \to W$, sometimes called similarity transformation, satisfying

$$S \circ D^{(1)}(g) \circ S^{-1} = D^{(2)}(g) \text{ for all } g \text{ in } G\,. \tag{5.7}$$

Given two representations D_1 and D_2 of G, one can obtain new representations as either a direct sum $D^{(1)}(a) \oplus D^{(2)}$, which is a block diagonal matrix of the form

$$\begin{pmatrix} D^{(1)}(g) & 0 \\ 0 & D^{(2)}(g) \end{pmatrix}, \tag{5.8}$$

or as a direct product $D^{(1)}(a) \otimes D^{(2)}$, which is the action of the group on the tensor product of the representation spaces [47].

It might happen that operators of a group representation leave a proper subspace $U \subset V$ invariant. A representation D is said to be reducible if the vector space V possesses an non-trivial invariant subspace under the action of the representation, that is, if there is a linear subspace $W \subset V$ with $D(g)w \in W$ for all $g \in G$ and all $w \in W$. In a reducible representation it can always be found a basis by means of a similarity transformation, such that all representation elements $D(g)$ are simultaneously of a block-diagonal form

$$D(g) = \begin{pmatrix} D^{(1)}(g) & 0 & \cdots & 0 \\ 0 & D^{(2)}(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^{(k)}(g) \end{pmatrix}, \tag{5.9}$$

with each of the blocks being a representation of the group independent of other representations. In that case, the representation $D(g)$ can be decomposed into a direct sum of k representations

$$D(g) = D^{(1)}(g) \oplus D^{(2)}(g) \oplus \cdots \oplus D^{(k)}(g)\,, \tag{5.10}$$

and the dimension of $D(g)$ is the sum of the dimensions of the blocks:

$$\dim[D(g)] = \sum_{j=1}^{k} \dim[D^{(k)}(g)]\,. \tag{5.11}$$

If for a representation D this decomposition is not possible, the representation D is called irreducible, with the only invariant subspaces being the the whole vector space V, and $\{0\}$.

A very important result in group representation theory is Schur's Lemma [49, 50]: If $D(g)$ is a irreducible representation of a group G acting on a vector space V and if there is an operator $A : V \to V$ that commutes with all the elements of this representation, then A must be proportional to the identity of V. As a corollary, if there is a representation with an operator commuting with all elements of such representation but is not proportional to the identity then the representation is reducible.

5.3 Lie groups

A (real) *Lie group* is a group G sharing as well the structure of a finite dimensional (real) differentiable manifold, with the group operations of multiplication and inversion being differentiable functions with respect to the manifold structure. Both this conditions resume in that the map

$$\mu : G \times G \to G, \qquad \mu(x,y) \mapsto x^{-1}y, \tag{5.12}$$

be differentiable [51]. It follows from the manifold structure that Lie groups have an infinite, non-countable number of elements, and each group element can be parametrized by a set of real numbers $\{x_i\}$ as $g(x_1,\ldots,x_n)$. In this sense, the parameters $\{x_i\}$ act as a set of local coordinates of the Lie group. It is customary to identify the group identity e with $g(0,\ldots,0)$.

The differentiable structure of a Lie group establishes a sense of continuity between group elements. This allows, in particular, to define continuous curves. If $x_i(t)$ is a continuous function of t for all i in a certain domain of t-values, then the elements with parameter values $x_i(t)$ are said to define a continuous curve in G. Two elements of G are said to be (path-wise) connected if there is a continuous curve in G on which both elements are contained [21]. How different elements of a Lie group are connected defines the following global properties of Lie groups:

- A group G is *connected* if any two group elements in G can be connected by a curve lying wholly within G. A *connected component* of G is the maximal subset of G that can be obtained by continuous variation of the parameters of one element of G. The connected component of a linear Lie group that contains the identity e is an invariant (or normal) Lie-subgroup of G.
- A connected Lie group is said to be *simply connected*, if any closed curve can be shrunk to a point [21] (in the case of Lie groups, shrunk to a single group element) continuously.
- A Lie group of dimension n with a finite number of connected components is said to be compact if the parameters range over a compact set, i.e. a closed and bounded set [21].

In group theory, a *simple Lie group* is a connected non-Abelian Lie group G which does not have nontrivial connected normal subgroups. A direct sum of simple Lie algebras is called *semisimple* [50].

5.3.1 Lie algebras

Since Lie groups have the structure of manifolds, to every Lie group can be associated a tangent space at the identity element, with the Lie bracket between tangent vectors defining an algebra. A *Lie algebra* $\mathfrak{g}$ is a vector space endowed with a bilinear operation

$$[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{5.13}$$

which is neither commutative nor associative [47, 49], and whose defining properties are anti-symmetry

$$[x,y] = -[y,x], \qquad \text{for all } x,y \in \mathfrak{g}\,, \tag{5.14}$$

and the Jacoby identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \qquad \text{for all } x,y,z \in \mathfrak{g}\,. \tag{5.15}$$

If the dimension of $\mathfrak{g}$ is as a vector space is finite or countably infinite, it can always be chosen a basis for the algebra

$$\mathscr{B} = \{T_a | a = 1,2,\ldots d\}\,. \tag{5.16}$$

The elements T_a of $\mathscr{B}$ are called generators of the group, and every element $x \in \mathfrak{g}$ can be written as $x = x^a T_a$ [49]. Given a set of generators $\{T_a\}$, the Lie bracket is fixed by its action on the basis elements

$$[T_a, T_b] = \sum_{c=1}^{d} C_{ab}{}^{c} T_c\,, \tag{5.17}$$

where the numbers $C_{ab}{}^{c} \in \mathbb{F}$ are called structure constants of the Lie algebra $\mathfrak{g}$.

Just as with the case of groups, it is possible to maps between different Lie algebras. A homomorphism between two algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a map ϕ that preserves Lie brackets

$$[x,y] \to \varphi([x,y]) = [\varphi(x), \varphi(y)]\,. \tag{5.18}$$

A map φ which is both injective and surjective is called an isomorphism from $\mathfrak{g}$ to $\mathfrak{h}$.

Every Lie group possesses a unique Lie algebra, but the opposite does not generally hold, since there are different Lie groups with isomorphic Lie algebras. Nevertheless, there is an important relation between Lie algebras and simply connected Lie groups: For every finite-dimensional Lie algebra $\mathfrak{g}$ over F, there is a simply connected Lie group G with $\mathfrak{g}$ as Lie algebra, unique up to isomorphism [48]. As a corollary, for each non-simply connected Lie group G, there is a unique, simply connected group with the same algebra, called the universal covering group of G, and denoted $\tilde{G}$.

The connection between Lie groups and Lie algebras is given by the exponential map: to any element $x \in \mathfrak{g}$ it can be associated a curve $\phi_x(t)$, with parameter $t \in \mathbb{R}$, given by

$$\left.\frac{\mathrm{d}\phi_x(t)}{\mathrm{d}t}\right|_{t=0} = x \tag{5.19}$$

with initial condition $\phi_x(0) = e_G$. This defines a group homomorphism between the real numbers and G, which is called a *one-parameter* subgroup of G

$$\phi_x(t_1)\phi_x(t_2) = \phi_x(t_1 + t_2), \quad t_1, t_2 \in G\,. \tag{5.20}$$

The exponential map is defined as a map

$$\exp : \mathfrak{g} \to G, \quad x \mapsto \exp(x)\,, \tag{5.21}$$

such that $\exp(x) = \phi_x(1)$, where $\phi_x : \mathbb{R} \to G$ is the unique one-parameter subgroup of G whose tangent vector at the identity is equal to x [22]. Therefore

$$\exp(tx) = \phi_x(t)\,. \tag{5.22}$$

In the case of finite dimensional Lie groups it can be shown in a neighborhood near the identity element, any element $g \in G$ can be written as $g = \exp(x)$ for some $x \in \mathfrak{g}$. If $\mathfrak{g}$ is spanned by a set of generators $\{T_a\}$, this relation reads

$$g = \exp\left(\sum_{a=1}^{d} \xi^a T_a\right)\,, \tag{5.23}$$

for a suitable choice of real coefficient $\{\xi^a\}$. The exponential map of a compact connected Lie group is surjective globally and any element of the Lie group can be written as the exponential of some element of the Lie algebra.

Most common examples of Lie groups in physics are matrix groups, that is, groups whose elements are matrices. If $G \subset GL(n,\mathbb{R})$ is a matrix Lie group and $\mathfrak{gl}(n,\mathbb{R})$ its Lie algebra, the exponential map corresponds to the usual exponential function for $n \times n$ matrices

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}\,. \tag{5.24}$$

5.3.2 Representations of Lie algebras

For Lie groups, representations are defined as in section 5.2 with the additional condition that the group homomorphism must be smooth in its arguments. Representations can be defined for Lie algebras as well, in terms of operators acting on a vector space. A representation of a Lie algebra $\mathfrak{g}$ on a vector space is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}_n(V) = End(V)\,, \tag{5.25}$$

from $\mathfrak{g}$ to the set of endomorphisms of V, the set of linear maps from V to itself, not necessarily invertible [48, 49]. It is a linear map preserving the bracket relation

$$\rho([x,y]) = [\rho(x),y] - [x,\rho(y)], \tag{5.26}$$

for all x,y in $\mathfrak{g}$. The vector space V, together with the representation ρ, is called a $\mathfrak{g}$-module.

Given the relation between Lie groups and Lie algebras, a natural question is if it is possible to derive representations of a Lie algebra from a representations of the corresponding Lie group. Let D_G be an analytic $d-$dimensional representation of G. Then, there exists a $d-$dimensional representation $D_{\mathfrak{g}}$ of $\mathfrak{g}$ given by [52]

$$D_{\mathfrak{g}}(x) = \frac{\mathrm{d}}{\mathrm{d}t} D_G(\exp tx)\bigg|_{t=0}\,, \tag{5.27}$$

and for all $x \in \mathfrak{g}$ and $t \in \mathbb{R}$

$$\exp(tD_{\mathfrak{g}}(x)) = D_G(\exp(tx))\,. \tag{5.28}$$

The exponential map relates representations of a Lie group with the representations of the Lie algebra. The finite-dimensional representations of a compact Lie algebra $\mathfrak{g}$ and of its universal covering group are one-to-one correspondence and act on the same representation spaces.

If two representations of G are equivalent the corresponding representations of $\mathfrak{g}$ are equivalent, also a representation D_G of G is irreducible if $D_{\mathfrak{g}}$ is irreducible. The converse arguments also hold if G is connected. With this, it is sufficient to classify the irreducible representations of a Lie algebra, which is usually much easier. However Lie algebras might have more irreducible representations than Lie groups, since different Lie groups may have the same Lie algebra, and not all representations of a Lie algebra become representation of a Lie group upon exponentiation.

Given an analytic function f on G, for any element x of the Lie algebra $\mathfrak{g}$ of G, the Lie derivative of f in the direction x is given by [49, 53],

$$\mathscr{L}_x f(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(tx))\bigg|_{t=0} . \tag{5.29}$$

The Lie derivatives are a first order linear differential operator. The set of all Lie derivatives forms a Lie algebra under commutation, and it is homomorphic to the Lie algebra $\mathfrak{g}$ of G [54, 55].

$$\mathscr{L}_{[x,y]} f(g) = [\mathscr{L}_x, \mathscr{L}_y] f(g) . \tag{5.30}$$

This is important for us since it will allow us to obtain representations from differential operators that we can relate with a differential equation which is invariant under a certain group G.

5.3.3 Adjoint representation

Besides the trivial representation, for every Lie algebra there exits at least another representation, called the adjoint representation. Since a Lie algebra is a vector space itself, for every Lie algebra $\mathfrak{g}$ there is an automorphism $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined via the Lie bracket, any element x of $\mathfrak{g}$ gives a linear transformation given by [48]

$$\mathrm{ad}_x(y) = [x, y] . \tag{5.31}$$

By virtue of the Jacobi identity, it is verified that (5.31) is a Lie algebra homomorphism since it preserves the Lie bracket,

$$\begin{aligned} [\mathrm{ad}_{x_1}, \mathrm{ad}_{x_2}](y) &= [x_1, [x_2, y]] - [x_2, [x_1, y]] \\ &= [[x_1, x_2], y] \\ &= \mathrm{ad}_{[x_1, x_2]}(y) . \end{aligned} \tag{5.32}$$

The dimension of the adjoint representation is equal to the dimension of the algebra. In terms of the generators of the algebra, the matrix elements of the adjoint representation are given by the structure constants,

$$\mathrm{ad}_{T_a}(T^b) = [T_a \cdot T_b] = \sum_c C_{ab}{}^c T_c . \tag{5.33}$$

In the adjoint representation of a finite dimensional Lie algebra, it is possible to define an inner product $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by means of the *Killing form*, a bilinear and symmetric form defined as [52]

$$\kappa(x, y) = \mathrm{Tr}\left(\mathrm{ad}_x \circ \mathrm{ad}_y\right) . \tag{5.34}$$

If $\mathfrak{g}$ is spanned by a basis of generators $\{T_a\}$ we can associate a matrix κ_{ab} to the Killing form

$$\kappa_{ab} = \mathrm{Tr}(\mathrm{ad}_{T_a} \circ \mathrm{ad}_{T_b}) = \sum_{c,e}^{d} C_{ae}{}^{c} C_{bc}{}^{e} \,. \tag{5.35}$$

In terms of the matrix κ_{ab}, the inner product between two elements $x = x^a T_a$ and $y = y^b T_b$ in $\mathfrak{g}$ reads

$$(x,y) = \kappa_{ab} x^a y^b \,. \tag{5.36}$$

The Killing form allows us to raise and lower index in the components of a vector $x \in \mathfrak{g}$, for example $x_b = \kappa_{ab} x^a$. It can also be used to raise and lower index of structure constants, for example $C_{abc} = \kappa_{cd} C_{ab}{}^{d}$.

5.3.4 Casimir invariant

Given a representation of a Lie algebra, an important issue is to determine if the representation is either irreducible or reducible. According to Schur's Lemma, a representation is irreducible if there is an operator commuting with every element of the algebra and it is proportional to the identity. A notable operator satisfying this property is the *Casimir operator*, which we define as follows: Let $\{T_a\}_{a=1}^{d}$ be a basis of a Lie algebra $\mathfrak{g}$ on a certain representation $D_{\mathfrak{g}}$. If in the representation it can be defined an invariant bilinear form $B(x,y)$, for example the Killing form in the adjoint representation, one can define a dual basis $\{T^a\}_{a=1}^{d}$, satisfying

$$B(T^a, T_b) = \delta^a{}_b \,, \tag{5.37}$$

then the Casimir invariant Ω is defined by

$$\Omega = \sum_{a=1}^{d} T_a T^a \,. \tag{5.38}$$

Since the Casimir operator is proportional to the identity, the constant of proportionality can be used to classify the representations of the Lie algebra [47, 49].

Using the antisymmetry of the structure constants it can be proved that (5.38) commutes with every generator of the algebra. In the adjoint representation the Casimir Ω is given in terms of the Killing form as

$$\Omega = \kappa_{ab} T^a T^b \,, \tag{5.39}$$

such that

$$[\Omega, T^e] = \kappa_{ab} [T^a T^b, T^e]. \tag{5.40}$$

Expanding commutators gives us

$$[\Omega, T^e] = \kappa_{ab} C^{be}{}_d T^a T^d + \kappa_{ab} C^{ae}{}_d T^d T^b \,. \tag{5.41}$$

$$= \kappa_{ab} \kappa_{dk} C^{bek} T^a T^d + \kappa_{ab} \kappa_{kd} C^{aek} T^d T^b \,. \tag{5.42}$$

We can write this as

$$[\Omega, T^e] = \kappa_{ab} \kappa_{dk} C^{bek} \{T^a, T^d\} \,, \tag{5.43}$$

where $\{T^a, T^d\} = T^a T^d \; + \; T^d T^a$. We have as a result that $[\Omega, T^e] = 0$, since the Killing form is symmetric but the structure constants are antisymmetric.

5.3.5 Weight representations

Finding the irreducible representations of a Lie group (or algebra) is equivalent to knowing the invariant subspaces under the action of a certain representation. Therefore, it is important to know which subspaces can be diagonalized simultaneously under the same similarity transformation. Since a Lie algebra is in general non commutative, not all elements of the Lie algebra elements can be diagonalized simultaneously under the same basis. [48, 52].

Let $\mathfrak{g}$ be a Lie algebra, the maximal set $\mathfrak{h}$ of commutative elements is called Cartan subalgebra, that is, a basis $\{h_a\} \in \mathfrak{g}$ such that

$$[h_a, h_b] = 0 \,. \tag{5.44}$$

The maximal number of commuting generators is called the rank of the algebra. The Cartan subalgebra allows us to diagonalize the elements of the algebra to obtain the irreducible representations of the algebra. For a semisimple algebra of rank r there will be r independent Casimirs invariants.

The remaining elements of the algebra do not commute with the Cartan subalgebra, but it can be chosen a basis $\{E_\alpha\}$ of such elements satisfying

$$[h_i, E_\alpha] = \alpha_i E_\alpha \,. \tag{5.45}$$

The real constants α_i are called roots of the algebra, and they constitute an r-component vector called root vector. In particular, if α_i is a root, then $-\alpha_i$ is a root as well.

With this construction, the algebra is decomposed into the Cartan subalgebra (5.44) and the algebra of the root generators [50],

$$[E_\alpha, E_\beta] = \begin{cases} \alpha_i h_i & \text{if } \alpha = -\beta \\ e_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ 0 & \text{if } \alpha+\beta \text{ is not a root} \end{cases} . \tag{5.46}$$

From the Cartan subalgebra, one can define the weight representation. Let V be a representation of $\mathfrak{g}$ (not necessarily finite dimensional). An element $v \in V$ is said to be of weight λ if for all $h \in \mathfrak{h}$, we have [49]

$$hv = \lambda(h)v. \tag{5.47}$$

The set of all elements of weight λ is a vector space V_λ called a weight space of V, with dimension equal to the rank of the algebra. If V_λ is non zero then λ is called a weight of the representation V. Moreover, if V is the adjoint representation of $\mathfrak{g}$, the weights are just the roots of the Lie algebra.

If V is the direct sum of its weight spaces,

$$V = \bigoplus_{\lambda \in \mathfrak{g}^*} V_\lambda, \tag{5.48}$$

then it is called a weight module; this corresponds to having an a basis of eigenvectors, i.e., a diagonalizable matrix.

If in the weight space it is defined an ordering of elements such that a nonnegative linear combination of positive vectors with at least one nonzero coefficient is another positive vector, then a representation is said to have highest weight λ if λ is a weight and no other weight of V is larger than λ. Similarly, it is said to have lowest weight λ if λ is a weight and all its other weights are greater than it.

5.4 Group $SL(2,\mathbb{R})$ and algebra $\mathfrak{sl}(2,\mathbb{R})$

In this section we introduce the Lie group $SL(2,\mathbb{R})$, a subgroup of $GL(2,\mathbb{C})$, the set of all 2×2 matrices with complex entries and non-zero determinant. This is an important group with both physical and mathematical applications. Following some proposals in the quasinormal mode literature [10–13], we will associate quasinormal modes with representations of the algebra of this group.

The group $SL(2,\mathbb{R})$. is the set of all 2×2 matrices with real entries and unitary determinant [56],

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \text{ and } ad-bc=1 \right\}. \tag{5.49}$$

The group structure of $SL(2,\mathbb{R})$ can be verified from the following properties:

- Since for two matrices A,B the determinant of AB is $\det(AB)=\det(A)\det(B)$ the product of two matrices with unitary determinant is a matrix with unitary determinant as well.
- The identity 2×2 matrix $I_{2\times 2} = diag(1,1)$ belongs to this group and acts as the identity element.
- Any matrix $g\in(2,\mathbb{R})$ has an inverse matrix $g^{-1}\in SL(2,\mathbb{R})$ since $\det(g)\neq 0$ for all $g\in SL(2,\mathbb{R})$.

The group $SL(2,\mathbb{R})$ is non-compact, as its parameters range over a non-compact set. It is connected, but not simply connected. In contrast, $SL(2,\mathbb{C})$ is simply connected.

In general, the elements of the group $GL(2,\mathbb{C})$ can be associated with the linear fractional transformations of the complex plane of the form

$$f(z) = \frac{az+b}{cz+d}, \quad z\in\mathbb{C}, \tag{5.50}$$

where a,b,c,d are any complex numbers satisfying $ad-bc\neq 0$. If $c\neq 0$, it is defined

$$f(-d/c)=\infty \text{ and } f(\infty)=a/c\,. \tag{5.51}$$

Since $SL(2,\mathbb{R})$ is a subgroup of $GL(2,\mathbb{C})$, there is a subset of the linear fractional transformations corresponding to the matrices of $SL(2,R)$. These transformations are given by a,b,c,d real and satisfying $ad-bc=1$; they map the upper and the lower half-planes into themselves and are isomorphic with the orientation-preserving isometries of the upper half-plane, that is, they preserve angles on the upper half-plane [57, 58].

The Lie algebra of $SL(2,\mathbb{R})$, called $\mathfrak{sl}(2,\mathbb{R})$, is the algebra of all real traceless 2×2 matrices. To see this we note that $X \in sl(2,\mathbb{R})$ if and only if

$$\det \mathrm{e}^{tX} = 1\,, \tag{5.52}$$

for all $t \in \mathbb{R}$. Since for any matrix A

$$\det \mathrm{e}^{A} = \mathrm{e}^{\mathrm{tr}(A)}\,, \tag{5.53}$$

then X must be traceless. The lie algebra $\mathfrak{sl}(2,\mathbb{R})$ has dimension three. The matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{5.54}$$

span a basis of $\mathfrak{sl}(2,\mathbb{R})$ as a vector space and satisfy the following algebra

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H. \tag{5.55}$$

In terms of the algebra generators (5.54), any element γ of $SL(2,\mathbb{R})$ of the form

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.56}$$

can be written in a small neighborhood of the identity uniquely as [49]

$$g = \exp\left(\frac{b}{d}L_+\right)\exp\left(cdL_-\right)\exp\left(-\ln(d)L_0\right)\,. \tag{5.57}$$

5.4.1 Representations of $\mathfrak{sl}(2,\mathbb{R})$

The group $SL(2,\mathbb{R})$ admits both finite and infinite dimensional representations, but not every representation will be unitary. Furthermore, $SL(2,\mathbb{R})$ is not simply connected. Finite dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$ are in a one-to-one correspondence with representations of $SL(2,\mathbb{R})$ but the same does not hold for infinite dimensional representations. The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ has the same complexification as the algebra $\mathfrak{su}(2)$, therefore, the finite-dimensional representation theory of $SL(2,\mathbb{R})$ is equivalent to the representation theory of $SU(2)$. From the generators (5.54) it is conventional to select H as a basis for the compact Cartan subalgebra. The weight space is the set of vectors $\{v_n\}$ satisfying

$$Hv_n = nv_n \tag{5.58}$$

for some integer n. The matrix E expands the root space and acts as a raising operator since

$$\begin{aligned} H(E\nu_n) &= [H,E]\nu_n + E(H\nu_n) \\ &= (n+2)E\nu_n\,, \end{aligned} \tag{5.59}$$

that is, $E\nu_n$ is proportional to vectors of the set $\{\nu_{n+2}\}$. Similarly, F acts as a lowering operator

$$\begin{aligned} H(F\nu_n) &= [H,F]\nu_n + F(H\nu_n) \\ &= (n-2)F\nu_n\,. \end{aligned} \tag{5.60}$$

In a highest weight representation V there will be a largest eigenvalue of H with eigenvector v, which is also annihilated by E

$$H\nu = n\nu, \quad E\nu = 0\,, \tag{5.61}$$

and the set $\{F^i\nu\}$, with $i \in \mathbb{Z}$, spans V as an invariant subspace. In general

$$H(F^i\nu) = (n-2i)F^i\nu\,, \tag{5.62}$$

$$E(F^i\nu) = (ni - i(i-1))F^{i-1}\nu\,. \tag{5.63}$$

In a finite dimensional representation, there is a limited number of times for F to act on v, that is, there is some integer k such that $F^k v = 0$, but $F^{k-1}v \neq 0$. We have that

$$0 = E(F^k v) = [nk - k(k-1)]F^{k-1}v\,. \tag{5.64}$$

Since $F^{k-1}v$ is different from zero, it must be that $nk - k(k-1) = 0$. Since k is not zero, then $n-(k-1) = 0$, or $k = n+1$. For each nonnegative integer n, the algebra $\mathfrak{sl}(2,\mathbb{R})$ has a finite dimensional irreducible representation of dimension $n+1$, unique up to isomorphisms.

The Casimir invariant allows us to classify the different irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$. Since $\mathfrak{sl}(2,\mathbb{R})$ has rank 1, there is an unique Casimir operator Ω. We can obtain Ω from the Killing form of the adjoint representation. From the structure constants the inner product defined by the Killing form give

$$(H,H) = 2, \quad (E,F) = 1\,, \tag{5.65}$$

and $(E,E) = (F,F) = (F,H) = (E,H) = 0$. For the basis H,E,F we have dual basis $H^* = H/2$, $E^* = F$, and $F^* = E$. With this, the Casimir operator of $\mathfrak{sl}(2,\mathbb{R})$ is

$$\Omega = HH^* + XX^* + YY^* = \frac{1}{2}H^2 + EF + FE. \tag{5.66}$$

If V is a highest weight representation, the action of the Casimir invariant to the highest weigh vector v gives

$$\Omega v = \left(\frac{1}{2}H^2 + EF + FE\right) v = \left(\frac{n^2}{2} + n\right) v\,. \tag{5.67}$$

That is, Ω has eigenvalue $n^2/2 + n$ on the irreducible representation of dimension $n+1$.

Since $SL(2,\mathbb{R})$ is noncompact the only finite dimensional unitary representation is the trivial representation. These representation are in correspondence with finite dimensional representations of $SL(2,\mathbb{R})$. A known finite dimensional representation of the group $SL(2,R)$ is given by its action on the space of homogeneous polynomials of degree n in two variables, $\{f(x,y)\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x,y) = f(ax+by, cx+dy)\,. \tag{5.68}$$

This is a $n+1$ dimensional space with basis $x^i y^{n-i}$, and the case $n=0$ corresponds to the trivial representation. The following operators

$$H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad E = x\frac{\partial}{\partial y}, \quad F = y\frac{\partial}{\partial x}, \tag{5.69}$$

satisfy an $\mathfrak{sl}(2,\mathbb{R})$ algebra, and define an action of $SL(2,\mathbb{R})$ on the space of polynomials with degree $n+1$, preserving the total degree of polynomials in two variables

$$Hx^i y^{n-i} = (2i-n)x^i y^{n-i}, \tag{5.70}$$

$$Ex^i y^{n-i} = (n-i)x^{i+1} y^{n-i-1}, \tag{5.71}$$

$$Fx^i y^{n-i} = nx^{i-1} y^{n-i+1}\,. \tag{5.72}$$

Regarding infinite dimensional representations, the main difference with the finite dimensional case is that there is no reason for a highest weight to exist. In general, there will be a basis $\{\dots v_{n-2}, v_n, v_{n+2}, \dots\}$ with

$$Hv_n = nv_n\,, \tag{5.73}$$

$$Ev_n = \text{ multiple of } v_{n+2}\,, \tag{5.74}$$

$$Fv_n = \text{ multiple of } v_{n-2}\,, \tag{5.75}$$

and three possibilities for the structure of a infinite dimensional weight representation:

- no highest weight or lower weight, and therefore, infinite in both directions;
- a lowest weight and infinite in the other direction;
- a highest weight and infinite in the other direction.

In an infinite dimensional highest weight representation there will be a vector v satisfying (5.61) and an infinite number of vectors $\{F^i v\}$, with $i \in \mathbb{Z}$, spanning the infinite dimensional space. From (5.62) it is obtained that the highest weight n must be either negative or positive-non integral for the series to never terminate.

As a final remark, since $SL(2,\mathbb{R})$ is not simply connected, in general the infinite dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$ have no correspondence with infinite dimensional representations of $SL(2,\mathbb{R})$.

In the next chapter, we present an application of this highest weight representation method, in which we obtain first-order differential operators satisfying the $\mathfrak{sl}(2,\mathbb{R})$ algebra, from which we get a Casimir operator that has the form of a second-order differential operator in two variables that we will relate with the equations of motion of a scalar field.

Chapter 6

Quasinormal Modes through Group Theoretical Methods

In chapter 4 we introduced the concept of quasinormal modes and frequencies, which characterize the response of a spacetime to localized perturbations, with the benefit that they do not depend on the initial details of the perturbation. However, analytical determination of quasinormal modes is usually not possible [6, 7]. In this chapter we present a computation of quasinormal modes and frequencies associated with the geometries introduced in chapter 3, in which the equations of motion possess a $\mathfrak{sl}(2,\mathbb{R})$ symmetry that will allow us to obtain exact solutions through group theoretical methods.

6.1 Differential representations of the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra

The dynamics of a relativistic field on a fixed spacetime is dictated by a certain equation of motion, which might possess further symmetries in addition to the isometries of the background spacetime. We are interested in dynamics of scalar fields on static and spherically symmetric spacetimes, in which the isometries are invariance under time translation and rotations (global $SO(3)$ symmetry). In this chapter we explore some cases in where there is an additional invariance under the algebra $\mathfrak{sl}(2,\mathbb{R})$, which was introduced in chapter 5.

In order to establish a connection with $\mathfrak{sl}(2,\mathbb{R})$, what we should do is to relate the equations of motion with a particular representation of the algebra. We have seen that the scalar equations of motion assume the form of a second-order differential operator acting on the field at each spacetime point, and given the spherical symmetry, the field can be factorized into spherical harmonics and a function $\Psi(t,x)$ independent of the angular coordinates θ and ϕ (here x stands for a radial variable, either r or r_*), following to an equation of the form (4.30). What we need is a representation of $\mathfrak{sl}(2,\mathbb{R})$ in terms of differential operators in the variables t and x.

Now, the operators $\{\partial_t, \partial_x\}$ are elements of the coordinate basis of the tangent space of all spacetime points covered by the chart (t, x, θ, ϕ). Since the field is to be evaluated at each point, we will have a local representation in which the representation space will be associated with the tangent space at each point. The problem of obtaining a representation of $\mathfrak{sl}(2,\mathbb{R})$ is reduced to finding a representation of the basis $\{H, E, F\}$ given in (5.54). Thus our representation will be a map ρ from the matrices $\{H,E,F\}$ to a set vectors $\{\hat{L}_0, \hat{L}_{+1}, \hat{L}_{-1}\}$ in terms of the basis $\{\partial_t, \partial_x\}$,

$$\rho(H) = \hat{L}_0, \quad \rho(E) = \hat{L}_1, \quad \rho(F) = \hat{L}_{-1}, \tag{6.1}$$

with components depending of the coordinates of the spacetime point where they are evaluated, and preserving the Lie bracket structure (5.55)

$$[\hat{L}_0, \hat{L}_1] = 2\hat{L}_1, \quad [\hat{L}_0, \hat{L}_{-1}] = -2\hat{L}_{-1}, \quad [\hat{L}_1, \hat{L}_{-1}] = \hat{L}_0. \tag{6.2}$$

We will obtain a weight representation, in which the first step is to introduce a diagonalizable operator corresponding to a representation of the Cartan generator H of $\mathfrak{sl}(2,\mathbb{R})$. Since quasinormal modes can be expressed as

$$\Psi(t,x) = \frac{1}{2\pi}\int \hat{\Psi}(\omega,x)\mathrm{e}^{i\omega t}\mathrm{d}\omega\,, \tag{6.3}$$

with $\omega \in \mathbb{C}$, an appropriate choice of diagonalizable operator should be of the form

$$\hat{L}_0 = \lambda\partial_t\,, \tag{6.4}$$

where λ is a proportionality constant, such that the action of $\hat{L}_0$ on $\Psi(t,x)$ gives us an eigenvalue equation

$$\hat{L}_0\Psi(t,x) = i\lambda\omega\Psi(t,x)\,. \tag{6.5}$$

After defining the Cartan operator the following step is to introduce the differential operators acting as the root generators E and F. We assume form linear in both $\{\partial_t\}$ and $\{\partial_x\}$, with coefficients depending on both t and x

$$\hat{L}_+ = J_+(t,x)\partial_t + K_+(t,x)\partial_x\,, \tag{6.6}$$

$$\hat{L}_- = J_-(t,x)\partial_t + K_-(t,x)\partial_x\,. \tag{6.7}$$

Since operators (6.4), (6.6) and (6.7) must preserve the Lie bracket relations given in (6.2), we expect to find restrictions on the form of coefficients. First we obtain the Lie bracket between $\hat{L}_\pm$ and $\hat{L}_0$

$$[L_0, L_\pm] = \lambda\left[\dot{J}_\pm(t,x))\partial_t + \dot{K}_\pm(t,x)\partial_x\right]\,, \tag{6.8}$$

where we will use "$\cdot$" as a shorthand notation to indicate derivative with respect to t. For this result to reproduce the Lie bracket $[L_0, L_\pm] = \pm 2L_0$, the following conditions must be satisfied

$$\lambda \dot{J}_\pm(t,x) = \pm 2J_\pm(t,x)\,, \tag{6.9}$$

$$\lambda \dot{K}_\pm(t,x) = \pm 2K_\pm(t,x)\,, \tag{6.10}$$

which have as solution

$$J_\pm(t,x) = \exp(\pm 2t/\lambda) J_\pm(x)\,, \tag{6.11}$$

$$K_\pm(t,x) = \exp(\pm 2t/\lambda) K_\pm(x)\,. \tag{6.12}$$

With this, equations (6.6) and (6.7) become

$$\hat{L}_\pm = \mathrm{e}^{\pm\frac{2}{\lambda}t}\left[J_\pm(x)\partial_t + K_\pm(x)\partial_x\right]. \tag{6.13}$$

Now we compute the Lie bracket between the operators $\hat{L}_\pm$

$$\begin{aligned}\left[\hat{L}_+, \hat{L}_-\right] = &\left(-\frac{4}{\lambda}J_+J_- + J'_-K_+ - J'_+K_-\right)\partial_t \\ &+ \left(K_+K'_- - K'_+K_- - \frac{2}{\lambda}(J_+K_- + J_-K_+)\right)\partial_x\,,\end{aligned} \tag{6.14}$$

where we use "$'$" as shorthand notation for the derivative with respect to the spatial variable x. Comparing with the Lie bracket (6.2), we obtain additional constraints

$$-\frac{4}{\lambda}J_+J_- + J'_-K_+ - J'_+K_- = \lambda\,, \tag{6.15}$$

$$K_+K'_- - K'_+K_- - \frac{2}{\lambda}(J_+K_- + J_-K_+) = 0\,. \tag{6.16}$$

To solve these constraints, we turn to establish a direct connection between the Casimir obtained from $\hat{L}_0$ and $\hat{L}_\pm$, and the scalar equations of motion. From the expressions (6.4), (6.6) and (6.7) and the general form of the $\mathfrak{sl}(2,\mathbb{R})$ Casimir, given by (5.66), we get

$$\begin{aligned}\hat{L}^2 = &\left(\frac{\lambda^2}{2} + 2J_+J_-\right)\partial_t^2 + 2K_+K_-\partial_x^2 + \left(K_+J'_- + K_-J'_+\right)\partial_t \\ &+ 2\left(K'_+K_- - \frac{2}{\lambda}J_+K_-\right)\partial_x + 2\left(J_+K_- + J_-K_+\right)\partial_{tx}\,,\end{aligned} \tag{6.17}$$

where in the term linear in ∂_x, the constraint (6.16) was used. Since in static spacetimes there should be invariance under time reversal ($t \to -t$), the equations of motion should not possess terms proportional to ∂_t or ∂_{tx}, giving us further restriction on the coefficient functions

$$K_+ J'_- + K_- J'_+ = 0\,, \tag{6.18}$$

$$J_+ K_- + J_- K_+ = 0\,. \tag{6.19}$$

By taking constraint (6.19) into (6.16) we get

$$\frac{1}{K_+}\frac{\mathrm{d}}{dx}K_+ = \frac{1}{K_-}\frac{\mathrm{d}}{\mathrm{d}x}K_-\,, \tag{6.20}$$

implying that K_- is proportional to K_+. By choosing $K_- = -K_+$ we get from (6.19) that $J_+ = J_-$. Now we define $J(x) = J_+ = J_-$ and $K(x) = K_+ = -K_-$, allowing us to reduce (6.17) to a simpler form

$$\begin{aligned}\hat{L}^2 = &\left(\frac{\lambda^2}{2} + 2[J(x)]^2\right)\partial_t^2 - 2[K(x)]^2\partial_x^2 \\ &- 2\left(\frac{2}{\lambda}J(x) - K'(x)\right)K(x)\partial_x\,.\end{aligned} \tag{6.21}$$

If the radial coordinate x corresponds to the tortoise coordinate r_*, the scalar field equation, as seen in (4.27) will only contain second order derivatives in t and r_*, meaning that the term proportional to ∂_x in (6.21) should be absent. This requirement will result in a direct relation between the coefficients $J(x)$ and $K(x)$

$$J(x) = \frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}x}K(x)\,, \tag{6.22}$$

and with this we get an even simpler expression for the representations of the Casimir and the root operators

$$\hat{L}_\pm = \mathrm{e}^{\pm\frac{2}{\lambda}t}\left[\frac{\lambda}{2}K'(x)\partial_t \pm K(x)\partial_x\right]\,, \tag{6.23}$$

$$\hat{L}^2 = \frac{\lambda^2}{2}\left(1 + [K'(x)]^2\right)\partial_t^2 - 2[K(x)]^2\partial_x^2\,. \tag{6.24}$$

If we want to write (4.27) in terms of the Casimir operator (6.24), then the coefficients of the terms ∂_t^2 and ∂_x^2 should differ only in a minus sign. For this to occur the following equality must be solved

$$\frac{\lambda^2}{2}\left(1 + [K'(x)]^2\right) = 2[K(x)]^2\,. \tag{6.25}$$

which can also be expressed as

$$\frac{\mathrm{d}K(x)}{\mathrm{d}x} = \pm\sqrt{\left(\frac{2}{\lambda}K(x)\right)^2 - 1}\,. \tag{6.26}$$

The solution of (6.26) is

$$K(x) = \frac{1}{2}\left(\frac{\lambda^2 B}{4}\mathrm{e}^{\pm\frac{2}{\lambda}x} + \frac{1}{B}\mathrm{e}^{\mp\frac{2}{\lambda}x}\right), \tag{6.27}$$

where B results from an integration constant. We note that if $B = 2/\lambda$ we get

$$K(x) = \frac{\lambda}{2}\cosh\left(\frac{2}{\lambda}x\right), \tag{6.28}$$

which is a result that will be closely related when we find the quasinormal modes of the geometries whose effective potential can be approximated by the Pöschl-Teller potential.

If instead of $K_-(x) = -K_+(x)$ we would had chosen $K_-(x) = K_+(x)$, then we would get $J_+(x) = -J_-(x)$. In this case it is defined $J(x) = J_+(x) = -J_-(x)$ and $K(x) = K_+(x) = K_-(x)$ to get the following Casimir

$$\hat{L}^2 = \left(\frac{\lambda^2}{2} - 2[J(x)]^2\right)\partial_t^2 + 2[K(x)]^2\partial_x^2 \tag{6.29}$$

$$+2\left(K'(x) - \frac{2}{\lambda}J(x)\right)K(x)\partial_x\,. \tag{6.30}$$

If (6.22) is imposed again to discard the term proportional to ∂_x we get

$$\hat{L}_\pm = \mathrm{e}^{\pm\frac{2}{\lambda}t}\left[K(x)\partial_x \pm \frac{\lambda}{2}K'(x)\partial_t\right], \tag{6.31}$$

and

$$\hat{L}^2 = \frac{\lambda^2}{2}\left(1 - [K'(x)]^2\right)\partial_t^2 + 2[K(x)]^2\partial_x^2\,. \tag{6.32}$$

Now, the condition for the coefficients in ∂_t^2 and ∂_x^2 to only differ in a sign is

$$-\frac{\lambda^2}{2}\left(1 - [K'(x)]^2\right) = 2K^2(x)\,, \tag{6.33}$$

or equivalently

$$\frac{\mathrm{d}K(x)}{\mathrm{d}x} = \pm\sqrt{1 + \left(\frac{2}{\lambda}K(x)\right)^2}\,. \tag{6.34}$$

Integration by separation of variables gives us

$$\frac{\lambda}{2}\sinh^{-1}\left(\frac{2}{\lambda}K\right) = \pm x + c\,, \tag{6.35}$$

with c being an integration constant. Solving for $K(x)$ we get

$$K(x) = \frac{\lambda}{2}\sinh\left(\pm\frac{2}{\lambda}x + c'\right)\,. \tag{6.36}$$

6.2 Quasinormal modes of near extremal geometries

In chapter 3 we saw that certain geometries, such as the Schwarzschild de Sitter black hole, admit a near extremal limit in where the effective potential of a scalar perturbation can be approximated by the Pöschl-Teller potential (4.66). Thus, we aim to solve the following equation

$$\frac{\partial^2}{\partial t^2}\Psi(t,r_*) + \left(-\frac{\partial^2}{\partial^2 r_*} + \frac{V_0}{\cosh^2(\kappa r_*)}\right)\Psi(t,r_*) = 0\,, \tag{6.37}$$

where the quasinormal modes will be solutions with boundary conditions given by (4.48). In the particular case of perturbations in SdS the peak of the potential is $V_0 = \ell(\ell+1)\kappa^2$ [31]. As an illustration, in figure 6.1 we show the plot of the Schwarzschild-de Sitter scalar potential and its approximation by (4.66).

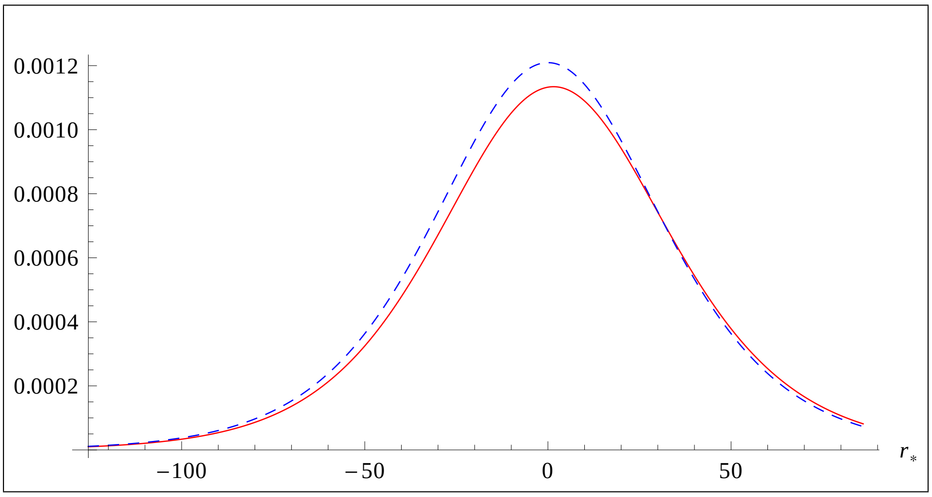

Fig. 6.1 Scalar effective potential for near extremal Schwarzschild-de Sitter (red) and approximation by Pöschl-Teller potential (dashed blue). The parameters used were $r_1 = 1$, $r_2 = 1.05$, $\ell = 1$.

We will obtain the quasinormal modes and frequencies of (6.37) using the representations of the algebra $\mathfrak{sl}(2,\mathbb{R})$ obtained last section. We write (6.37) as

$$\cosh^2(\kappa r_*)\left[\frac{\partial^2}{\partial^2 r_*}-\frac{\partial^2}{\partial t^2}\right]\Psi(t,r_*)=V_0\Psi(t,r_*)\,. \tag{6.38}$$

Comparing with (6.24), we observe that an appropriated choice for the function $K(r_*)$ (in this case, $x=r_*$) is the following

$$K(r_*)=\frac{1}{\kappa}\cosh(\kappa r_*)\,. \tag{6.39}$$

With this, the operators $\hat{L}_0$ and $\hat{L}_\pm$ take the form

$$\hat{L}_\pm=\frac{1}{\kappa}\mathrm{e}^{\mp\kappa t}\left[-\sinh(\kappa r_*)\partial_t\pm\cosh(\kappa r_*)\partial_{r_*}\right], \tag{6.40}$$

$$\hat{L}_0=-\frac{2}{\kappa}\partial_t, \tag{6.41}$$

where $\lambda=-2/\kappa$, and the Casimir operator is found to be

$$\begin{aligned}\hat{L}^2&=\frac{2}{\kappa^2}\left(1+\sinh^2(\kappa r_*)\right)\partial_t^2-2\left(\frac{\cosh(\kappa r_*)}{\kappa}\right)^2\partial_{r_*}^2\\&=\frac{2}{\kappa^2}\cosh^2(\kappa r_*)\left(\partial_t^2-\partial_{r_*}^2\right)\,.\end{aligned} \tag{6.42}$$

We can write the equation of motion (6.38) in terms of the Casimir (6.42) as

$$\hat{L}^2\Psi(t,r_*)=-2\frac{V_0}{\kappa^2}\Psi(t,r_*)\,. \tag{6.43}$$

We get that the value of the Casimir, which labels every different irreducible representation, will be related with the peak of the effective potential V_0.

Now we use the highest weight procedure to find the quasinormal modes. We choose a function $\Psi^{(0)}(t,r)$ acting as the highest weight, satisfying

$$\hat{L}_0\Psi^{(0)}(t,r_*)=h\Psi^{(0)}(t,r_*)\,, \tag{6.44}$$

$$\hat{L}_+\Psi^{(0)}(t,r_*)=0\,. \tag{6.45}$$

where h is the highest weight. The action of $\hat{L}_0$ on $\Psi^{(0)}(t,r_*)$ is

$$\hat{L}_0\Psi^{(0)}(t,r_*)=-i2\frac{\omega_0}{\kappa}\Psi^{(0)}(t,r_*)\,, \tag{6.46}$$

relating the quasinormal frequency ω_0 with the highest weight h

$$\omega_0 = i\frac{1}{2}\kappa h\,. \tag{6.47}$$

In terms of h the action of the Casimir $\hat{L}^2$ on $\Psi^{(0)}(t,r)$ is

$$\hat{L}^2\Psi^{(0)}(t,r_*) = \left(\frac{h^2}{2}+h\right)\Psi^{(0)}(t,r_*)\,. \tag{6.48}$$

Direct comparison of (6.43) and (6.48) allows to solve h in terms of V_0 and κ

$$h = -1 \pm 2i\sqrt{\frac{V_0}{\kappa^2}-\frac{1}{4}}\,, \tag{6.49}$$

and the term inside the square root is positive since $\kappa = 0+\mathscr{O}(\delta)$ in the near extremal limit. The quasinormal frequency for $\Psi^{(0)}(t,r)$ is found using (6.47)

$$\omega_0 = \kappa\left(-i\frac{1}{2}\pm\sqrt{\frac{V_0}{\kappa^2}-\frac{1}{4}}\right)\,, \tag{6.50}$$

which matches to the quasinormal frequencies (4.68) for $n=0$ and a positive sign of the real part of (6.50). To find the corresponding quasinormal mode we use the highest weight condition (6.45) which translates to the following differential equation

$$\left[-\sinh(\kappa r_*)\frac{\partial}{\partial t}+\cosh(\kappa r_*)\frac{\partial}{\partial r_*}\right]\Psi^{(0)}(t,r_*) = 0\,, \tag{6.51}$$

or

$$\left[-i\omega_0\sinh(\kappa r_*)+\cosh(\kappa r_*)\frac{\partial}{\partial r_*}\right]\Psi^{(0)}(\omega,r_*) = 0\,. \tag{6.52}$$

Equation (6.52), can be solved exactly, giving us an expression for the first quasinormal mode

$$\Psi^{(0)}(\omega,r_*) = A\cosh(\kappa r_*)^{i\frac{\omega_0}{\kappa}}\,, \tag{6.53}$$

or

$$\Psi^{(0)}(t,r_*) = \frac{1}{2\pi}\int A\cosh(\kappa r_*)^{i\frac{\omega_0}{\kappa}}\mathrm{e}^{i\omega_0 t}\mathrm{d}\omega\,, \tag{6.54}$$

where A is an integration constant. This function has the correct behavior of a quasinormal mode solution, namely, the asymptotic behavior of (6.53) matches (4.48),

$$\Psi^{(0)}(\omega,r_*) \sim \begin{cases} \mathrm{e}^{i\omega r_*} & \text{as } r_*\to\infty \\ \mathrm{e}^{-i\omega r_*} & \text{as } r_*\to-\infty \end{cases}\,. \tag{6.55}$$

The remaining quasinormal modes can be obtained by successively applying lowering operators to the $\Psi^{(0)}(\omega, r_*)$, and there will be an infinite number of them, giving us a infinite dimensional representation. Thus, the n-th quasinormal mode obtained by applying n times the operators $\hat{L}_{-1}$ on $\Psi^{(0)}$

$$\Psi^{(n)}(t, r_*) = (\hat{L}_-)^n \Psi^{(0)}(t, r_*) . \tag{6.56}$$

The associated quasinormal frequencies can be found. For that it is useful to prove by induction the following property:

$$[\hat{L}_0, \hat{L}_\pm^n] = \pm(2n)\hat{L}_\pm^n . \tag{6.57}$$

We have that $n = 1$, expression (6.57) corresponds to the Lie bracket of the algebra given in (6.2) If (6.57) is valid for n, for $n+1$ we have

$$\begin{aligned}\left[\hat{L}_0, \hat{L}_\pm^{n+1}\right] &= \hat{L}_\pm[\hat{L}_0, \hat{L}_\pm^n] + [\hat{L}_0, \hat{L}_\pm]\hat{L}_\pm^n \\ &= \pm(2n)\hat{L}_\pm\hat{L}_\pm^n + 2\hat{L}_\pm\hat{L}_\pm^n \\ &= \pm 2(n+1)\hat{L}_\pm^{n+1} .\end{aligned} \tag{6.58}$$

and thus (6.57) is proved. The quasinormal frequency associated to the quasinormal mode $\Psi^{(n)}(t, r_*)$ is given by the action of the operator $\hat{L}_0$, for which we obtain by use of (6.57) and (6.56),

$$\begin{aligned}\hat{L}_0 \Psi^{(n)}(t, r_*) &= \left([\hat{L}_0, \hat{L}_-^n] + \hat{L}_- \hat{L}_0\right) \Psi^{(0)}(t, r_*) \\ &= (h - 2n)\,\Psi^{(n)}(t, r_*) ,\end{aligned} \tag{6.59}$$

meaning that the quasinormal frequencies will be equally spaced

$$\omega_n = i\frac{\kappa}{2}(h - 2n) = \kappa\left(-i\left(n + \frac{1}{2}\right) \pm \sqrt{\frac{V_0}{\kappa^2} - \frac{1}{4}}\right) . \tag{6.60}$$

In general, the n-th quasinormal mode is of the form

$$\Psi^{(n)}(t, r_*) = \frac{1}{2\pi}\int \Psi^{(n)}(\omega, r_*) e^{i(\omega_0 - in\kappa)t} \mathrm{d}\omega . \tag{6.61}$$

Applying the operator $\hat{L}_0$ we calculated the first three quasinormal modes and frequencies explicitely; we display them in table 6.1.

n	Quasinormal frequency	Mode $\Psi^{(n)}(\omega, r_*)$
0	$\kappa\left(-i\frac{1}{2} \pm \sqrt{\frac{V_0}{\kappa^2} - \frac{1}{4}}\right)$	$A\cosh(\kappa r_*)^{i\frac{\omega_0}{\kappa}}$
1	$\kappa\left(-i\frac{3}{2} \pm \sqrt{\frac{V_0}{\kappa^2} - \frac{1}{4}}\right)$	$2\frac{i\omega_0}{\kappa}A\sinh(\kappa r_*)\cosh(\kappa r_*)^{i\frac{\omega_0}{\kappa}}$
2	$\kappa\left(-i\frac{5}{2} \pm \sqrt{\frac{V_0}{\kappa^2} - \frac{1}{4}}\right)$	$2\frac{i\omega_0}{\kappa^2}A\left(2i\omega_0\sinh^2(\kappa r_*) + \kappa\cosh^2(\kappa r_*)\right)\cosh(\kappa r_*)^{i\frac{\omega_0}{\kappa}}$

Table 6.1 Lowest quasinormal modes and frequencies of the Pöschl-Teller potential.

6.3 Quasinormal modes of asymptotically Anti-de Sitter geometries

In the first section of this chapter we obtained two possible representations for the $\mathfrak{sl}(2,\mathbb{R})$ algebra, characterized by two functions (6.28), and (6.36). From (6.28) we introduced the representation given by the differential operators (6.40) and (6.41), which turned to be applicable for solving the quasinormal modes and frequencies of the Pöschl-Teller potential. In this section we turn to study the representation given by (6.36) and the possible geometries where it could approximate the scalar dynamics.

In (6.36), we take $K(r_*) = \frac{1}{\kappa}\sinh(\kappa r_*)$, where we are choosing $\lambda = -2/\kappa$. The operators $\hat{L}_0$ and $\hat{L}_\pm$ assume the form

$$\hat{L}_\pm = \frac{1}{\kappa}e^{\mp\kappa t}\left[-\sinh(\kappa r_*)\partial_{r_*} \pm \cosh(\kappa r_*)\partial_t\right], \tag{6.62}$$

$$\hat{L}_0 = -\frac{2}{\kappa}\partial_t, \tag{6.63}$$

and the Casimir operator of this representation is the following

$$\begin{aligned}\hat{L}^2 &= -\frac{2}{\kappa^2}\left(\cosh^2(\kappa r_*) - 1\right)\partial_t^2 + 2\left(\frac{\sinh(\kappa r_*)}{\kappa}\right)^2\partial_{r_*}^2 \\ &= \frac{2}{\kappa^2}\sinh^2(\kappa r_*)\left(-\partial_t^2 + \partial_{r_*}^2\right).\end{aligned} \tag{6.64}$$

An effective potential with the following form

$$V(r_*) = \frac{V_0}{\sinh^2(\kappa r_*)}, \tag{6.65}$$

can be related with the Casimir (6.64) by their action on a function $\Psi(t,r_*)$

$$\hat{L}^2\Psi(t,r_*) = \frac{2V_0}{\kappa^2}\Psi(t,r_*)\,, \tag{6.66}$$

where it should be noted the difference of sign with (6.43), since we are taking potential (6.65) to be positive defined, making the constant V_0 a positive number. We will restrict the domain of (6.65) to $r_* \in (-\infty, 0)$, where it has the following behavior

$$V(r) \sim \begin{cases} \infty & \text{as } r_* \to 0 \\ \mathrm{e}^{2\kappa r_*} & \text{as } r_* \to -\infty \end{cases}\,, \tag{6.67}$$

near $r_* = 0$ we have the following Taylor series for the hyperbolic cosecant function

$$\sinh^{-1}(r_*) = \operatorname{csch}(r_*) = \frac{1}{r_*} - \frac{r_*}{6} + \frac{7r_*^3}{360} + \ldots \quad 0 < |r_*| < \pi\,, \tag{6.68}$$

then, the Taylor expansion for (6.65) is given by

$$V_{app}(r_*) = \frac{V_0}{(\kappa r_*)^2} - \frac{V_0}{3} + \mathscr{O}(\kappa r_*)^2\,, \tag{6.69}$$

which diverges at $r_* = 0$. Now, as $r_* \to -\infty$, we get the following limit

$$\lim_{r_* \to -\infty} V_{app}(r_*) = \lim_{r_* \to -\infty} \frac{V_0}{(\mathrm{e}^{\kappa r_*} - \mathrm{e}^{-\kappa r_*})^2} = V_0 e^{2\kappa r_*}\,. \tag{6.70}$$

We also note the following: the function (6.65) is a monotonically increasing function in the negative r_*-axis. Its derivative is given by

$$\frac{\mathrm{d}V_{app}(r_*)}{\mathrm{d}r_*} = -2\kappa V_0 \frac{\coth(\kappa r_*)}{\sinh^2(\kappa r_*)}\,, \tag{6.71}$$

which is positive with $r_* \in (-\infty, 0)$ since the function $\coth(\kappa r_*)$ is negative there. With this, we show that unlike the Pöschl-Teller potential, the potential (6.65) has no local maximum, and neither does it have a local minimum.

We proceed to solve (6.66) by a highest weight representation. Following with the procedure, we introduce a function $\Psi^{(0)}(t,r)$ taking the role of a highest weight vector with weight h, satisfying (6.44) and (6.45). The action of the Casimir on this function is given by (6.48), but now the relation between h and V_0 is slightly different,

$$\frac{h^2}{2} + h = \frac{2V_0}{\kappa^2}\,, \tag{6.72}$$

which has the following real solutions as long as V_0 is a positive number

$$h = -1 \pm \sqrt{1 + \frac{4V_0}{\kappa^2}}. \tag{6.73}$$

Following the relation between h and ω given in (6.47) we obtain the following frequencies

$$\omega_0 = i\kappa \left(-\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{V_0}{\kappa^2}} \right). \tag{6.74}$$

Unlike the case of the Pöschl-Teller potential, we are obtaining purely imaginary frequencies. Now we proceed to find the associated quasinormal modes. From the highest weight condition (6.45) we get the following differential equation for $\Psi^{(0)}(\omega, r_*)$

$$\left[i\omega_0 \cosh(\kappa r_*) - \sinh(\kappa r_*) \frac{\mathrm{d}}{\mathrm{d}r_*} \right] \Psi^{(0)}(\omega, r_*) = 0, \tag{6.75}$$

whose solution is of the form

$$\Psi^{(0)}(\omega, r_*) = A \sinh(\kappa r_*)^{i\frac{\omega_0}{\kappa}}, \tag{6.76}$$

where once again A is an integration constant. The remaining quasinormal modes can be obtained by successively applying lowering operators to the $\Psi^{(0)}(\omega, r_*)$, and the quasinormal frequencies are found to be equally spaced

$$\omega_n = i\frac{\kappa}{2}(h - 2n) = i\kappa \left(-\left(n + \frac{1}{2}\right) \pm \sqrt{\frac{1}{4} + \frac{V_0}{\kappa^2}} \right). \tag{6.77}$$

Thus, we obtain an infinite number of quasinormal modes related to a infinite dimensional highest weight representation of $\mathfrak{sl}(2,\mathbb{R})$ as long as h is negative or positive non-integer. The first three solutions are shown in table 6.2

n	Quasinormal frequency	Mode $\Psi^{(n)}(\omega, r_*)$
0	$i\kappa\left(-\frac{1}{2} \pm \sqrt{\frac{V_0}{\kappa^2} - \frac{1}{4}}\right)$	$A\sinh(\kappa r_*)^{i\frac{\omega_0}{\kappa}}$
1	$i\kappa\left(-\frac{3}{2} \pm \sqrt{\frac{V_0}{\kappa^2} - \frac{1}{4}}\right)$	$2\frac{i\omega_0}{\kappa} A \cosh(\kappa r_*) \sinh(\kappa r_*)^{i\frac{\omega_0}{\kappa}}$
2	$i\kappa\left(-\frac{5}{2} \pm \sqrt{\frac{V_0}{\kappa^2} - \frac{1}{4}}\right)$	$2\frac{i\omega_0}{\kappa^2} A \left(2i\omega_0 \cosh^2(\kappa r_*) + \kappa \sinh^2(\kappa r_*)\right) \sinh(\kappa r_*)^{i\frac{\omega_0}{\kappa}}$

Table 6.2 Lowest quasinormal modes and frequencies of the potential (6.65).

Now we discuss the characteristics of the geometries where (6.65) could be applied to model the scalar dynamics. First of all, we argue that they should be geometries possessing a Killing horizon r_+, where we can approximate the function $A(r)$ by

$$A(r) = 2\kappa(r - r_+) + \mathscr{O}(r - r_+)^2 \,. \tag{6.78}$$

where κ corresponds to the gravity surface on r_+. Assuming that $B(r)$ has the same approximation near r_+, the tortoise coordinate obtained from (6.78) is

$$r_*(r) = \frac{1}{2\kappa}\ln\left(\frac{r}{r_+} - 1\right) + \mathscr{O}\left(\frac{r - r_+}{r_+}\right)^2 , \tag{6.79}$$

mapping $r = r_+$ to $r_* \to -\infty$. Equation (6.79) can be inverted to yield a function $r(r_*)$

$$r(r_*) = r_+\left(1 + \mathrm{e}^{2\kappa r_*}\right) + \mathscr{O}(\mathrm{e}^{2\kappa r_*})^2 \,. \tag{6.80}$$

The scalar potential will be a function of the form $V(r) = A(r)\Omega(r)$, and taking into account (6.78) we get

$$V(r) = 2\kappa\Omega(r_+)(r - r_+) + \mathscr{O}(r - r_+)^2 \,, \tag{6.81}$$

or, in terms of the tortoise coordinate

$$V(r) = 2\kappa\Omega(r_+)r_+\mathrm{e}^{2\kappa r_*} + \mathscr{O}(r - r_+)^2 \,. \tag{6.82}$$

which we note has the same behavior of the potential (6.65) as $r_* \to -\infty$, as shown in (6.70). Second, we argue that the geometries should be asymptotically Anti de Sitter. For that, we take the function $A(r)$ of the AdS metric in equation (2.79). The tortoise coordinate is given by equation (2.82). In the limit $r \to \infty$, or equivalently $R/r \to 0$, the coordinate r_* is approximately given by

$$r_*(r) = -\frac{R^2}{r} + \mathscr{O}\left(\frac{R}{r}\right)^2 , \tag{6.83}$$

where $r \to \infty$ is mapped to $r_* = 0$. From (2.79) the effective potential for scalar perturbations is

$$V(r) = \left(1 + \frac{r^2}{R^2}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2}{R^2}\right] , \tag{6.84}$$

which is divergent as $r \to \infty$, where we have the following behavior

$$V(r) = \frac{2r^2}{R^4} + \frac{\ell(\ell+1)}{R^2} + \mathscr{O}\left(\frac{R}{r}\right)^2 . \tag{6.85}$$

In terms of the tortoise coordinate, the potential is of the following form

$$V(r_*) = \frac{2}{r_*^2} + \frac{\ell(\ell+1)}{R^2} + \mathscr{O}(r_*)^2 \,, \tag{6.86}$$

which diverges at $r_* = 0$ in a similar way to (6.69).

Finally, we discuss about the behavior of the solutions. Taking as a starting point the $n = 0$ quasinormal mode obtained in (6.76), we note the behavior of the hyperbolic sine function

$$\lim_{x\to 0} \sinh(x) = 0, \quad \lim_{x\to -\infty} \sinh(x) = \lim_{x\to -\infty} -e^{-x} \to -\infty, \tag{6.87}$$

In order to recover the correct the asymptotic behavior at $r_* = 0$, it is selected the quasinormal frequency with negative imaginary part in (6.74), that is,

$$\omega_0 = -i\kappa \left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{V_0}{\kappa^2}} \right) \,, \tag{6.88}$$

since otherwise, the quantity $i\omega_0/\kappa$ would be negative and the solution would be divergent (a division by zero). Taking the negative imaginary frequency we get that $\Psi^{(0)}$ behaves as

$$\Psi^{(0)}(\omega, r_*) \sim \begin{cases} 0 & \text{as } r_* \to 0 \\ \mathrm{e}^{-i\omega r_*} & \text{as } r_* \to -\infty \end{cases} \,, \tag{6.89}$$

which is the same behavior expected from quasinormal modes in asymptotically AdS spacetimes, as indicated in (4.92).

We note that there are many similarities with the scalar potential of the Schwarzschild-AdS spacetime studied in section 4.4; however, it seems that the representation we obtained is not appropriated to obtain quasinormal modes for that geometry since, based on numerical results presented in [59], the scalar quasinormal modes in the $SAdS$ black hole present frequencies with both real and imaginary components, which is not the case here. Now, there is a larger set of asymptotically AdS geometries that could be associated with the potential (6.65), based on their asymptotic behavior. These geometries are characterized by functions $A(r)$ and $B(r)$ such that the effective potential (4.25) reduces to the form presented in (6.65) and solutions to the scalar equations satisfying the behavior indicated in (6.89).

Chapter 7

Conclusions

In this thesis there were derived some connections between perturbative aspects of scalar dynamics in certain geometries and the conformal algebra $\mathfrak{sl}(2)$. The initial step was to identify a relation between the equations of motions of a scalar field and a representation of the algebra, in particular, with the Casimir operator which is an invariant of any irreducible representation. Given that connection, we managed to associate quasinormal modes of a scalar fields with an infinite dimensional highest weight representation of the algebra, in which the associated quasinormal frequencies correspond to the weights of the representation.

The algebraic approach is well-suited for near extremal geometries whose dynamics can be modeled by the Pöschl-Teller potential (4.66). The quasinormal modes and frequencies are obtained from the representation of the algebra $\mathfrak{sl}(2,\mathbb{R})$ given by the differential operators (6.40) and (6.41). We calculated the quasinormal modes and frequencies for the Pöschl-Teller potential using a highest weight representation, where the highest weight element corresponds to the fundamental mode $n = 0$. The frequency obtained for the $n = 0$ mode, given by (6.50), is in good correspondence with the results from the literature and the solution (6.54) has the expected asymptotic behavior of quasinormal modes in de Sitter spacetimes. The remaining modes can be obtained by applying successively the lowering operator to the fundamental mode, and with that the spectrum of frequencies is found to be equally spaced.

We also obtained a second representation of the algebra $\mathfrak{sl}(2,\mathbb{R})$ given by the operators (6.62) and (6.62), and related with the potential (6.65). We presented arguments that given the asymptotic behavior of the potential and the solutions obtained, the representation could be applicable for some asymptotically Anti de Sitter spacetimes admitting a Killing horizon. The quasinormal frequencies obtained where purely imaginary numbers, making this potential not entirely suitable for modeling the scalar dynamics on the Schwarzschild-Anti de Sitter.

The black holes considered in this work only had as a physical parameter the black hole mass and the cosmological constant, but one could also consider other parameters such as a black hole charge or angular momentum. It is possible that the potential (6.65) approximates another, more complicated, Anti de Sitter black hole in a specific limit. It would be a very interesting scenario if one could obtain the correct quasinormal modes and frequencies of a certain black holes in Anti de Sitter spacetime from a different representation of the algebra, and even more, if it could be possible related with the results obtained for the near extremal asymptotically de Sitter geometries.

We should also remark that the analysis in this thesis was concerned only with scalar fields. It would be interest to see if this relation between quasinormal modes and algebra representations can be generalized to vectorial or gravitational perturbations, which are expected to model more common forms of matter, since in many scenarios the equations of motion assume a form similar to the scalar field case and the dynamics turns out to be very similar. Another possible line is to try generalize the analysis done in this work to consider dynamics of fields in background with a different number of dimensions.

As a closing remark, and motivated by the so-called gauge/gravity dualities, which have been an extensive area of research in the last two decades, a possible next step for this work would be to study if given a relation between quasinormal modes and a representation of a conformal algebra it is possible to establish a correspondence between perturbations of black holes and the dynamics of some gauge field theory with group symmetry $SL(2)$ (or another group with an isomorphic algebra). For example, in the AdS/CFT conjecture it is stated that a black hole inside the AdS spacetime corresponds to a thermal state of a conformal field theory living in the boundary, and the decay of the test field in the black hole spacetime corresponds to the decay of the perturbed state in the conformal field theory. Since the results we obtained seem to be more appropriated for black holes in asymptotically de Sitter spacetimes, and particularly, for near extremal black holes, it would be interest to see if a similar relation between gravitational phenomena and gauge fields can be established, where one could relate properties of a black hole inside the de Sitter background with a conformal field theory defined at the boundary.

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