UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA

Teoria de *Gauge* para Partículas com *Spin* Contínuo

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Uma dissertação apresentada ao Instituto de Física da Universidade de São Paulo como parte dos requisitos necessários para a obtenção do título de Mestre em Ciências.

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São Paulo

2016

FICHA CATALOGRÁFICA

Preparada pelo Serviço de Biblioteca e Informação do Instituto de Física da Universidade de São Paulo

Avellar, Leonardo Werneck de

Teoria de Gauge para partículas com spin contínuo. São Paulo, 2016.

Dissertação (Mestrado) – Universidade de São Paulo. Instituto de Física. Depto. Física Matemática.

Orientador: Prof. Dr. Victor de Oliveira Rivelles

Área de Concentração: Física de Partículas e Campos.

Unitermos: 1. Teoria de campos; 2. Física matemática; 3. Partícula de spin contínuo; 4. Partícula de spin alto.

USP/IF/SBI-031/2016

UNIVERSITY OF SÃO PAULO

INSTITUTE OF PHYSICS

A Gauge Theory for Continuous Spin Particles

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A dissertation presented to the Institute of Physics of the University of São Paulo in partial fulfillment of the requirements for the degree of Master of Science.

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São Paulo

2016

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To the memory of my late grandmother, Amelia, with gratitude.

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Two things are infinite: the Universe and human stupidity; and I am not yet completely sure about the Universe.

Albert Einstein

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Resumo

Nesta dissertação exploramos as características da formulação de uma Teoria de *Gauge* para partículas de *spin* contínuo (CSP). Para tornar a nossa discussão o mais auto-contida possível, começamos por introduzir todas as informações básicas de Teoria de Grupos − assim como de Teoria de Representações − que são necessárias para enteder de onde surgem as CSPs. A partir daí aplicamos o que foi apresentado sobre Teoria de Grupos para o estudo dos grupos de Lorentz e de Poincaré, até o ponto em que conseguimos construir a representação CSP. Finalmente, após de uma rápida revisão do formalismo de *spin* altos (*Higher Spins*), através do estudo das ações de Schwinger-Fronsdal, damos início ao estudo de uma Teoria de Campos para CSPs. Estudamos e exploramos todas as simetrias locais da ação que descreve uma CSP livre, assim como todas as sutilezas que surgem a partir da introdução de uma nova coordenada, que resulta em um *espaço-tempo estendido* no qual a ação é definida. Terminamos nossa discussão mostrando que todo o conteúdo físico decorrente da ação para uma CSP livre coincide com o que vimos em nossa discussão de Teoria de Grupos.

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Abstract

In this dissertation we explore the features of a Gauge Field Theory formulation for continuous spin particles (CSP). To make our discussion as self-contained as possible, we begin by introducing all the basics of Group Theory − and representation theory − which are necessary to understand where the CSP come from. We then apply what we learn from Group Theory to the study of the Lorentz and Poincaré groups, to the point where we are able to construct the CSP representation. Finally, after a brief review of the Higher-Spin formalism, through the Schwinger-Fronsdal actions, we enter the realm of CSP Field Theory. We study and explore all the local symmetries of the CSP action, as well as all of the nuances associated with the introduction of an enlarged spacetime, which is used to formulate the CSP action. We end our discussion by showing that the physical contents of the CSP action are precisely what we expected them to be, in comparison to our Group Theoretical approach.

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Contents

Acknowledgements

There are many people I would like to thank for their help in many different ways in writing this dissertation. This is definitely a collective work and one I would probably never have completed on my own. Even if indirectly, everyone cited here helped me write this dissertation in some way, and for that I will be forever grateful.

I would like to thank Na, my girlfriend, for everything we experienced together during my years as a graduate student. I know how difficult it is to stay apart while we pursue our dreams in life, but your patience, love, and support are the things that keep me going, and I cannot thank you enough for the privilege of having you by my side.

I would also like to thank my family, who gave me the means and support to pursue a career in Physics, even if I was the first one in the family to do so. Education was always valued by everyone in my family, even if most of them never had the opportunity to seek higher education, and I will never forget everything they have done for me.

I thank my advisor, Victor, for introducing me to the interesting topics of Higher Spin Particles and Continuous Spin Particles. It is also thanks to him that I was able to pursue a Master of Science degree in Theoretical Physics and I will always remember him with great respect and gratitude. Hopefully we will be able to work together again soon.

Among my colleagues, I would like to thank Cedrick Miranda Mello, for the many hours he took from his personal time to work with me through some of the longest calculations in this dissertation. I would like to thank prof. Renata Z. Funchal and prof. Gustavo Burdman for their excellent courses in Quantum Mechanics and Quantum Field Theory, as well as for the discussions we had about studying abroad. I would also like to thank everyone from the String Theory Journal Club, for the many hours of fruitful discussions. There are, perhaps, many friends and colleagues I am not including here, but they know who they are and that I am very thankful for their help.

I would also like to thank everyone who works at the "*Comissão de Pós-Graduação*" of this institute, as they have helped me with everything I ever needed through my

graduate years.

Last, but not least, I would like to thank the "*Coordenação de Aperfeiçoamento de Pessoal de Nível Superior*" (CAPES) for the funding which made this dissertation, and my graduate studies, viable.

A Gauge Theory for Continuous Spin Particles

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Chapter 1

Introduction

The Poincaré group is of fundamental importance in Theoretical Physics. When one attempts to extend the validity of Quantum Mechanics to include particles that move relativistically, one ends up with Quantum Field Theory, the 'marriage' of Quantum Mechanics and Special Relativity. When we take into account the effects of Special Relativity, we find that our systems are invariant under a different, larger symmetry group. In the case of Quantum Mechanics, for instance, our systems are invariant under Galilean transformations, while in Quantum Field Theory, our systems are invariant under Poincaré transformations.

In fact, Poincaré symmetry is so powerful that one needs to go to the extreme scenarios of Nature for it to lose its validity. When one attempts to study large masses of the cosmic scale, that is, one invokes the concepts of the General Theory of Relativity, then one must make use of a even more suitable and powerful group, the group of general coordinate transformations. Although this is an equally interesting group, we will not attempt to study it in this dissertation. Instead, we will explore some aspects of the Poincaré, analysing some not well known aspects.

Among Lie groups, we have what is called a Lie algebra and a *representation* of a Lie group's Lie algebra [3]. In these representations, we can build all the states described by a Quantum Theory. The study of all possible irreducible representations of the Poincaré group shows that they describe states corresponding to massive and massless particles of integer and half-integer spin, and also the *Continuous Spin Particle* (CSP), which was first studied by Wigner [1].

Out of all these particles, only the CSP has not yet been detected. The main reason for this is because we still do not have a Quantum Field Theory that can describe free nor interacting CSPs. In fact, it was only very recently that local, covariant actions describing a single CSP degree of freedom were proposed. In 2014, Schuster & Toro

proposed an action describing a bosonic CSP [12], followed by further analysis by Rivelles [13, 15]. Later, in 2015, an action describing a fermionic CSP was proposed by Bekaert, Najafizadeh, and Setare [16]. As already mentioned, CSP-matter interactions (or even CSP-CSP interactions) are unknown[1], but the actions proposed in [12, 13, 16] are big motivations for us to seek the form of such interactions.

Another motivation for the study of CSPs is their analogues in $2+1$ dimensions, as a form of massless generalization of anyons. Although we will not analyse this case in this dissertation, we refer the reader to [12, 17] and their references for further reading.

In this dissertation, we will study the realm of bosonic CSPs[2], studying them first from a Group Theoretical approach and then from a Field Theoretical approach. In chapter 2, we will introduce everything we will need of Group Theory and Representation Theory in order to understand where CSPs come from [2, 3]. In chapter 3, we study the Poincaré group and its main features [2, 3]. In chapter 4 we study the theory of *higher spin* particles, through the Schwinger-Fronsdal formalism [11]. Finally, in chapter 5, we approach the problem of CSPs from a Field Theoretical point of view, through the analysis of the recently proposed action that describes a single, free, bosonic CSP by Schuster & Toro [12] and the further analysis of this proposal made by Rivelles [13, 15]. In chapter 5 we also show the connection between the formalism proposed in [12, 13, 15] with that of higher spin particles [11] and what was available previously in the CSP literature [1, 14], as well as check the validity of the theory in comparison to our Group Theoretical approach to the problem [2]. We also refer the reader to Appendix A for a better understanding of the notations we will use throughout the dissertation. Appendices B and C contain identities and calculations that correspond to some of the results presented in chapter 5, but were too long to be kept in the main text.

[1] If you look in [12, 16], you can see that CSPs can be coupled to currents, but nothing is known about the currents symmetries.

[2] Studying the fermionic case should be straightforward after the reader becomes familiar with the content of this dissertation.

Chapter 2

Group Theory

In this chapter we would like to introduce all the basic concepts of Group Theory and Group Representations that we will need throughout this dissertation.

Section 2.1 contains an introduction to Group Theory and all the necessary definitions we will need. In section 2.2 the same is done for Group Representations. We then end the chapter applying the presented definitions to a concrete example, namely the $SO(3)$ group.

All of the material presented in this chapter is based on the great texts available on the references [2, 3]. Still, we point out some references to aid the reader to better understand some of the connections between the Group Theory and Quantum Mechanics, which would digress us from the main subject of this dissertation.

2.1 Basic Definitions

Group theory is the natural way to formulate symmetry principles and understand their applications to both Mathematics and Physics. As every physicist learns, symmetries are extremely important when analysing a problem, as they can often lead to drastic simplifications. With this in mind, we provide here the basics of Group Theory which we will need to study the problem of continuous spin particles (CSP).

A non-null set $\{G : a, b, c \ldots\}$ is said to form a *group* if there is an operation, called *group multiplication*, which associates any given pair of elements a, $b \in G$ with a well-defined product $a \cdot b \in G$, such that:

- The operation is associative, i.e. $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad \forall\, a, b, c \in G$;
- Among the elements of G, there is an element E, called the *identity*, which has the property $a \cdot E = E \cdot a = a$.
- For each $a \in G$, there is an element $a^{-1} \in G$, called the *inverse* of a, which has the property $a \cdot a^{-1} = a^{-1} \cdot a = E$.

The group multiplication operation is, in general, dependent on the ordering of the elements of the group involved in said operation. A particular category of groups is that for which the group multiplication is commutative, i.e. $a \cdot b = b \cdot a, \ \forall\, a, b \in G$. In this case, the group is said to be an *Abelian group*. Otherwise, when the group multiplication is not commutative, the group is said to be *non-Abelian*.

Another definition which will be important when we study the Lorentz and the Poincaré groups in the next chapters will be that of a *homomorphism*. A homomorphism from a group G to another group G' is a mapping (not necessarily one-to-one) which preserves group multiplication. In other words, if $g_i \in G \to g'_i \in G'$ and $g_1 g_2 = g_3$, then $g'_1 g'_2 = g'_3$ (from this point forward we will omit the $\cdot$ and leave the group multiplication operation implicit). A special case of a homomorphism is when the mapping *is* one-to-one. This is called an *isomorphism* and the groups are said to be *isomorphic*.

A *subgroup* of G is defined as a subset H of a group G which forms a group under the same multiplication rules as G. Then, let H_1 and H_2 be subgroups of a group G. If every element of H_1 commutes with any element of H_2, i.e. $h_1 h_2 = h_2 h_1$ for all $h_1 \in H_1$ and $h_2 \in H_2$, then G is said to be the *direct product* of H_1 and H_2; symbolically we write $G = H_1 \otimes H_2$.

2.2 Group Representations

If there is a homomorphism from a group G to a group of operators $U(G)$ on a linear vector space V, we say that $U(G)$ forms a *representation* of the group G. The *dimension of the representation* is the dimension of the vector space V.

The group representation is said to be *unitary* if the group representation space is an inner product space and if the operators $U(G)$ are unitary for all $g \in G$. A representation $U(G)$ on V is *irreducible* if there is no non-trivial invariant subspace in V with respect to $U(G)$. Otherwise, the representation is *reducible*. In the latter case, if the orthogonal complement of the invariant subspace is also invariant with respect to $U(G)$, then the representation is said to be *fully reducible*.

For a given finite group G, the *group algebra* $\mathfrak{g}$ consists of all formal linear combinations of g_i, $r = g_i r^i$, where $g_i \in G$ and $\{r^i\}$ are complex numbers. In addition, multiplication of one element of the algebra (q) by another (r) is given by $rq = g_i g_j r^i q^j = g_k(\Delta^k_{ij} r^i q^j)$, where Δ^k_{ij} are determined by the group multiplication rule as indicated. An element C of the group which commutes with all other elements, i.e. $Cr = rC$, $\forall r \in G$, is said to be a *Casimir element*, or *Casimir operator* of the group.

The groups we will be working with are *matrix Lie groups*, which we will define after considering the following. The *General Linear Group*, $GL(N;\mathbb{R})$, is the group of all $N \times N$ invertible matrices with real entries. The General Linear Group over the complex numbers, denoted by $GL(N;\mathbb{C})$, is the group of all $N \times N$ invertible matrices with complex entries. A matrix Lie group is any subgroup G of $GL(N;\mathbb{C})$ with the following property: if A_m is any sequence of matrices in G, and A_m converges to some matrix A, then either $A \in G$ or A is not invertible.

This allows us to cast yet another definition for the group algebra: if G is a matrix Lie group, then the Lie algebra of G, denoted by $\mathfrak{g}$, is the set of all matrices X such that $e^{-itX} \in G$ for all real numbers t. This means we have defined an *exponential mapping*, which takes elements of the algebra to elements of the group. X is said to be the *generator* of the group G.

An example: The $SO(3)$ group

Classical Mechanics and the $SO(3)$ group

We begin our discussion with a simple review of rotations in three dimensional Classical Mechanics. We can label the space coordinates (x, y, z) in the more compact form x_i,

with $i = 1, 2, 3$ so that $(x_1, x_2, x_3) \equiv (x, y, z)$. Following this notation, rotations in three dimensional Euclidean space are transformations that take x_i to x'_i such that

$$x_i \to x'_i = R_{ij} x_j, \tag{2.1}$$

where R is a $\underline{3} \times 3$ matrix that represents a transformation that preserves the length of a vector $|\vec{x}|^2 \equiv x_i x_i$. When we choose these matrices to have determinant equal to 1 (called *special* matrices) and to respect orthogonality ($R^T R = R R^T = \mathbb{1}_{3\times 3}$), we obtain the desired group. One can write this transformation, in an infinitesimal form, as

$$x'_i = x_i - i\delta\theta J_i, \tag{2.2}$$

where $\delta\theta$ is the infinitesimal parameter of the transformation and J_i are the generators of $SO(3)$. These generators can be expressed as

$$(J_i)^j{}_k = -i\epsilon_{ijk}, \tag{2.3}$$

or in their explicit matrix form

$$(J_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad (J_2) = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad (J_3) = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

To see that (2.4) indeed generate rotations in three dimensions, we can use the exponential mapping to relate the elements of the algebra, J_i, with the elements of the group, R, by writing

$$R(\theta) = \exp[-i\theta^i J_i]. \tag{2.5}$$

Say we would like to perform a rotation about the $z-$axis, that is, rotations in the $(x, y)-$plane. This is the equivalent of saying that $\theta^1 = \theta^2 = 0$ in (2.5). By noting that $(J_3)^3 = J_3$, we can write R in the familiar form[1]

$$R_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.6}$$

[1]Here the lower index 3 is used to specify that this is a rotation about the z-axis. The same is done for what follows in the text with the indices 1 and 2 representing rotations about the x-axis and y-axis, respectively.

by doing a simple series expansion of equation (2.5). The same can be done to show that R_1 and R_2 also have their usual forms.

Irreducible Representations of the $SO(3)$ Lie Algebra - $\mathfrak{so}(3)$

We now focus on the role played by the $SO(3)$ group in Quantum Mechanics. The generators (2.4) obey the Lie algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{2.7}$$

From the generators of the $SO(3)$ group, we can construct an operator that commutes with *all* the generators of the group. This is the Casimir operator of $SO(3)$ and is given by

$$J^2 \equiv (J_1)^2 + (J_2)^2 + (J_3)^2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.8}$$

which obviously commutes with all three generators J_i.

To formulate a representation of $\mathfrak{so}(3)$, we choose basis vectors that are eigenvectors of J^2 and one of the generators, which by convention we choose to be J_3 (this choice is completely arbitrary). A representation of $\mathfrak{so}(3)$[2] is characterized by two labels, j and m, where $j(j+1)$ is the eigenvalue of J^2 and m is the eigenvalue of J_3, when these act on one of the eigenvectors of the basis we have chosen. The values of j and m are either integers or half-integers[3]. Here j is called the particle's *spin* and m is the particle's *helicity*. Symbolically, we have

$$\begin{aligned} J^2|j,m\rangle &= j(j+1)|j,m\rangle \\ J_3|j,m\rangle &= m|j,m\rangle \end{aligned}. \tag{2.9}$$

We can also construct two operators from the remaining generators, given by

$$J_\pm \equiv J_1 \pm iJ_2, \tag{2.10}$$

which have the interesting properties

[2]For a complete and detailed discussion of these results, see [2, 4, 5, 6].
[3]We will elaborate further on this later in the text.

$$[J_3, J_\pm] = \pm J_\pm, \tag{2.11}$$

$$[J_+, J_-] = 2J_3, \tag{2.12}$$

$$J_\pm^\dagger = J_\mp, \tag{2.13}$$

$$J_3 J_\pm |m\rangle = [J_3, J_\pm]|m\rangle + J_\pm J_3 |m\rangle = (m \pm 1) J_\pm |m\rangle, \tag{2.14}$$

thus allowing us to conclude that $J_\pm|m\rangle$ are also eigenstates of J_3 with eigenvalues $(m \pm 1)$. This means that $J_\pm$ act on states as *raising* and *lowering* operators, changing the values of m by ± 1 each time they act on a state. We then require the following conditions

$$J_-|m = -j\rangle = 0, \tag{2.15}$$

$$J_+|m = j\rangle = 0, \tag{2.16}$$

so that our representation has dimension $(2j + 1)$, since the possible values of m are $m = -j, -j + 1, \cdots, j - 1, j$, and the possible values of j are $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ (a rigorous way to see this would be to compute $\langle l|J_+ J_-|l\rangle = 0$, where $|l\rangle$ is the last non-vanishing vector and check the result for consistency). We then conclude that the basis vectors have the properties

$$J^2|j, m\rangle = j(j + 1)|j, m\rangle \tag{2.17}$$

$$J_3|j, m\rangle = m|j, m\rangle \tag{2.18}$$

$$J_\pm|j, m\rangle = \sqrt{j(j + 1) - m(m \pm 1)}|j, m \pm 1\rangle \tag{2.19}$$

<u>*Example*</u>: $j = \frac{1}{2}$

This representation has dimension $d = 2$, which means that the operators will be 2×2 matrices. Equations (2.17)-(2.19) tell us that

$$J_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.20}$$

It is easy to see that $J_i = \frac{1}{2}\sigma_i$, where σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.21}$$

and that the possible states are

$$\begin{aligned} &|1/2, 1/2\rangle, \\ J_-|1/2, 1/2\rangle &= |1/2, -1/2\rangle, \\ J_-|1/2, -1/2\rangle &= 0, \\ J_+|1/2, -1/2\rangle &= |1/2, 1/2\rangle, \\ J_+|1/2, 1/2\rangle &= 0 \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} J^2|1/2, 1/2\rangle &= \tfrac{3}{4}|1/2, 1/2\rangle, \\ J^2|1/2, -1/2\rangle &= \tfrac{3}{4}|1/2, -1/2\rangle, \\ J_3|1/2, 1/2\rangle &= \tfrac{1}{2}|1/2, 1/2\rangle, \\ J_3|1/2, -1/2\rangle &= -\tfrac{1}{2}|1/2, -1/2\rangle. \end{aligned} \tag{2.23}$$

This should suffice as a simple, though far from complete, introduction to the rich fields of Group Theory and Group Representation. More complete discussions can be found on the many great books available on the subject, some of which can be found on the references for this dissertation.

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Chapter 3

The Lorentz and Poincaré Groups

Special relativity is the generalization of the homogeneity and isotropy of three-dimensional space to include the time dimension as well. We drop the concept of absolute time and allow for it to *transform* similarly to spatial coordinates, generalizing the concepts of space and time into the new concept of *spacetime*. These generalizations had to be introduced as a consequence of Albert Einstein's proposal that the speed of light is a constant of Nature. The (proper) Lorentz group and the Poincaré group are the symmetry groups of four-dimensional spacetime.

The Lorentz group generalizes the concept of rotations to what is known as *Lorentz transformations*. This is done with the introduction of new transformations, called *boosts*. A boost is a form of "rotation" that mixes time and spatial coordinates, in constrast with regular rotations which only mix spatial coordinates. When we also allow for translations, we then deal with the Poincaré group, which generalizes the so called *Euclidean groups* (these are groups that include rotations *and* translations in Euclidean space).

In section 3.1 we introduce the Lorentz group and the transformations associated with the group. We then move on to find the group's Lie algebra in section 3.1.1. In section 3.1.2 it is shown how the group's Lie algebra can be decomposed into the product of two $\mathfrak{su}(2)$ Lie algebras, thus allowing us to build the representations of the Lorentz group's Lie algebra. The Irreducible, finite-dimensional, non-unitary representations of the Lorentz group are then presented in section 3.1.3.

In section 3.2 we introduce the Poincaré group's multiplication rule, followed by the generators of the Poincaré group and its Lie algebra in section 3.2.1. The Casimir operators of the Poincaré group are introduced in section 3.3. Within section 3.4, sections 3.4.1 - 3.4.3 list all the possible representations of the Poincaré group's Lie algebra.

The starting points of each section are based on the references [2, 3], but we work out every calculation to avoid the necessity for repeated citations. The construction of the representations of the Lorentz and Poincaré groups are based on the reference [2], but we also work them out explicitly. When we cite examples of Quantum Field Theory, further references are offered for the reader, as we cannot cover all these topics in this dissertation.

3.1 The Lorentz Group - $SO(1,3)$

The Lorentz group[1] (also called the $SO(1,3)$ group) is composed by a set of transformations, called *Lorentz transformations*, that leave the length of four-vectors invariant. Here, Lorentz transformations are considered as "rotations" in four-dimensional spacetime, with the transformations that act only on the spatial coordinates called *rotations* and the transformations that mix time and spatial coordinates called *boosts*. Thus, we will refer to Lorentz transformations as the collection of both rotations and boosts that can act on four-dimensional spacetime. An element of the Lorentz group, Λ, acts on a *four-vector* x^μ ($\mu = 0,1,2,3$) as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{3.1}$$

such that the product $x^\mu x_\mu$ remains invariant. This means that

$$x'^\mu x'_\mu = \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} = \eta_{\mu\nu}\Lambda^\mu{}_\sigma x^\sigma \Lambda^\nu{}_\rho x^\rho = \eta_{\mu\nu}\Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho x^\sigma x^\rho, \tag{3.2}$$

where we have introduced the *Minkowski metric* $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Equation (3.2) then implies the following condition on the elements of the Lorentz group

$$\eta_{\mu\nu}\Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho = \eta_{\sigma\rho}. \tag{3.3}$$

Another way of saying this is that Λ is an element of the Lorentz group if, and only if

$$\eta_{\mu\nu}\Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho = \Lambda_\sigma{}^\mu \eta_{\mu\nu}\Lambda^\nu{}_\rho = \eta_{\sigma\rho} \Rightarrow \Lambda^T \eta \Lambda = \eta, \tag{3.4}$$

where in the last equality we have omitted the indices for a more transparent result. The superscript T in (3.4) indicates the matrix transposed. Also, note that $\eta^T = \eta$.

We can interpret Lorentz transformations in analogously to rotations, as we did when we studied the $SO(3)$ group. Indeed, Lorentz transformations can be split in two categories: *rotations* (which act upon the spatial coordinates x^i, $i = 1,2,3$) and *boosts* (which are "rotations" that mix time and spatial coordinates). It is easy to notice that there are six of said transformations - three possible ways to perform rotations and three possible ways to perform boosts. This means we will need six generators for the

[1] Even though we are abusing notation, it is important to know we are dealing here with the *proper* Lorentz group. This is the group where we choose $\Lambda^0{}_0 = 1$. We will, however, drop the word *proper* since we will make no mention to *time-reversal transformations*, which are allowed when $\Lambda^0{}_0 = -1$.

Lorentz transformations and six parameters associated with these transformations.

A clever way of labelling these is through the use of antisymmetric tensors. If we define $M_{\mu\nu}$ and $\xi^{\mu\nu}$ as the generators of the Lorentz group and the parameters associated with the transformations, respectively, and choose them to be antisymmetric (i.e. $M_{\mu\nu} = -M_{\nu\mu}$ and $\xi^{\mu\nu} = -\xi^{\nu\mu}$), we guarantee that each of them will have only six non-trivial entries.

Then, the exponential mapping that connects the elements of the algebra $\mathfrak{so}(1,3)$ to the elements of the group $SO(1,3)$ is given by

$$\Lambda(\xi) = \exp\left[-\frac{i}{2}\xi^{\mu\nu}M_{\mu\nu}\right]. \tag{3.5}$$

In equation (3.5) we have omitted the matrix indices, but they can be written explicitly as in

$$\Lambda^{\sigma}{}_{\rho} = \left[\exp\left(-\frac{i}{2}\xi^{\mu\nu}M_{\mu\nu}\right)\right]^{\sigma}{}_{\rho}. \tag{3.6}$$

3.1.1 The Lie Algebra of the Lorentz Group - $\mathfrak{so}(1,3)$

Consider, now, an infinitesimal Lorentz transformation, that is, take the transformation parameter $\xi^{\mu\nu}$ to be infinitesimal. This means that a Lorentz transformation can be written as $\Lambda^{\mu}{}_{\nu} = (\mathbb{1}_{4\times4} + \delta\xi)^{\mu}_{\nu}$, where $\delta\xi$ is an infinitesimal transformation parameter. Then, we can write the transformation, using (3.1), as

$$x'^{\sigma} = \Lambda^{\sigma}{}_{\rho}x^{\rho} = (\mathbb{1}_{4\times4} + \delta\xi)^{\sigma}{}_{\rho}\,x^{\rho} = x^{\sigma} + \delta\xi^{\sigma}{}_{\rho}x^{\rho}. \tag{3.7}$$

On the other hand, we can use the exponential mapping (3.6) to see that

$$\begin{aligned} x'^{\sigma} &= \Lambda^{\sigma}{}_{\rho}x^{\rho} = \left[\exp\left(-\frac{i}{2}\delta\xi^{\mu\nu}M_{\mu\nu}\right)\right]^{\sigma}{}_{\rho}x^{\rho} = \left[\mathbb{1}_{4\times4} - \frac{i}{2}\delta\xi^{\mu\nu}M_{\mu\nu}\right]^{\sigma}{}_{\rho}x^{\rho} \\ &= x^{\sigma} - \frac{i}{2}\delta\xi^{\mu\nu}\left(M_{\mu\nu}\right)^{\sigma}{}_{\rho}\,x^{\rho}, \end{aligned} \tag{3.8}$$

These last two results allow us to write

$$\delta\xi^{\sigma}{}_{\rho} = -\frac{i}{2}\delta\xi^{\mu\nu}(M_{\mu\nu})^{\sigma}{}_{\rho}, \tag{3.9}$$

so that $M_{\mu\nu}$ has the matrix form

$$(M_{\mu\nu})^{\sigma}{}_{\rho} = i(\eta_{\mu\rho}\delta^{\sigma}{}_{\nu} - \eta_{\nu\rho}\delta^{\sigma}{}_{\mu}). \tag{3.10}$$

Result (3.10) shows explicitly that $M_{\mu\nu} = -M_{\nu\mu}$. A quick way of checking if (3.10) is correct is by plugging it back in (3.9):

$$\delta\xi^{\sigma}{}_{\rho} = -\frac{i}{2}\delta\xi^{\mu\nu} i(\eta_{\mu\rho}\delta^{\sigma}{}_{\nu} - \eta_{\nu\rho}\delta^{\sigma}{}_{\mu} = \frac{1}{2}(\delta\xi_{\rho}{}^{\nu}\delta^{\sigma}{}_{\nu} - \delta\xi^{\mu}{}_{\rho}\delta^{\sigma}{}_{\mu}) = \delta\xi^{\sigma}{}_{\rho} \checkmark, \tag{3.11}$$

where we used the fact that $\delta\xi$ is antisymmetric under indices exchanges. To find the Lie algebra of the Lorentz group we must then discover what is the commutation relation between two generators of the group. We can do this by noticing that a Lorentz transformations act on $M_{\mu\nu}$ as

$$\Lambda M_{\mu\nu}\Lambda^{-1} = M_{\lambda\sigma}\Lambda^{\lambda}{}_{\mu}\Lambda^{\sigma}{}_{\nu}. \tag{3.12}$$

Then the left-hand side of (3.12) reads

$$\begin{aligned} \Lambda M_{\mu\nu}\Lambda^{-1} &= \left[\mathbb{1}_{4\times4} - \frac{i}{2}\delta\xi^{\lambda\sigma}M_{\lambda\sigma}\right] M_{\mu\nu} \left[\mathbb{1}_{4\times4} + \frac{i}{2}\delta\xi^{\alpha\beta}M_{\alpha\beta}\right] \\ &= M_{\mu\nu} + \frac{i}{2}\delta\xi^{\lambda\sigma}[M_{\mu\nu}, M_{\lambda\sigma}] + \cancelto{0}{\mathcal{O}(\delta\xi^2)}, \end{aligned} \tag{3.13}$$

and the right-hand side

$$\begin{aligned} M_{\lambda\sigma}\Lambda^{\lambda}{}_{\mu}\Lambda^{\sigma}{}_{\nu} &= M_{\lambda\sigma}\left[\mathbb{1}_{4\times4} - \frac{i}{2}\delta\xi^{\alpha\beta}M_{\alpha\beta}\right]^{\lambda}{}_{\mu}\left[\mathbb{1}_{4\times4} - \frac{i}{2}\delta\xi^{\rho\gamma}M_{\rho\gamma}\right]^{\sigma}{}_{\nu} \\ &= M_{\mu\nu} - \frac{1}{2}\delta\xi^{\lambda\sigma}\left(M_{\mu\lambda}\eta_{\nu\sigma} - M_{\mu\sigma}\eta_{\nu\lambda} + M_{\lambda\nu}\eta_{\mu\sigma} - M_{\sigma\nu}\eta_{\mu\lambda}\right) \end{aligned}, \tag{3.14}$$

thus resulting in

$$[M_{\mu\nu}, M_{\lambda\sigma}] = i\left(M_{\mu\lambda}\eta_{\nu\sigma} - M_{\mu\sigma}\eta_{\nu\lambda} + M_{\lambda\nu}\eta_{\mu\sigma} - M_{\sigma\nu}\eta_{\mu\lambda}\right), \tag{3.15}$$

which is the $\mathfrak{so}(1,3)$ Lie algebra.

3.1.2 The Lie Algebra $\mathfrak{so}(1,3)$ as $\mathfrak{su}(2) \times \mathfrak{su}(2)$

To find the representations of the Lie algebra of the $SO(1,3)$ group, it is convenient to separate the generators and parameters in those of boosts and rotations. Let K_i be the boost generators and J_i the rotation generators, with parameters ϕ^i and θ^i, respectively ($i = 1, 2, 3$). We define them as

$$\begin{aligned} M_{0i} &\equiv -K_i \\ M_{ij} &\equiv \epsilon_{ijk} J_k \\ \xi^{0i} &\equiv \phi^i \\ \xi^{ij} &\equiv -\epsilon^{ijk}\theta^k \end{aligned} \quad , \tag{3.16}$$

where ϵ_{ijk} is the totally antisymmetric symbol with $\epsilon_{123} = +1$. The exponential mapping (3.6) then becomes

$$\Lambda(\phi,\theta) = \exp\left[-\frac{i}{2}\Big(-2\phi^i K_i - \underbrace{\epsilon_{ijk}\epsilon^{ij\ell}}_{2\delta_k{}^\ell}\theta^k J_\ell \Big)\right] = \exp\left[i\phi^j K_j + i\theta^j J_j \right]. \tag{3.17}$$

It can be shown, using (3.10) and (3.15), that K_i and J_i satisfy the following commutation relations

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \\ [J_i, K_j] &= i\epsilon_{ijk} K_k \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \end{aligned} \quad . \tag{3.18}$$

The first two commutation relations show that J_i and K_i behave like vectors under ordinary rotations, but the third one shows that K_i do not transform into one another, i.e. the algebra for K_i is not closed. The minus-sign of the third commutator expresses the difference between the non-compact group $SO(1,3)$ and its compact form $SO(4)$ or between $SL(2,\mathbb{C})$ and $SU(2) \times SU(2)$. Because $SL(2,\mathbb{C})$ and $SO(1,3)$ are locally homomorphic, as well as $SO(4)$ and $SU(2) \times SU(2)$, they have homomorphic Lie algebras.

We now perform a basis change by introducing the complex linear combinations

$$\begin{aligned} \mathcal{J}_j &\equiv \frac{1}{2}(J_j + iK_j) \\ \mathcal{K}_j &\equiv \frac{1}{2}(J_j - iK_j) \end{aligned} . \tag{3.19}$$

Using (3.19), we can write (3.17) as

$$\Lambda(\phi, \theta) = \exp\left[i\Big(\theta^j + i\phi^j\Big)\mathcal{J}_j + i\Big(\theta^j - i\phi^j\Big)\mathcal{K}_j\right], \tag{3.20}$$

or, in terms of new transformation parameters, $\alpha^j \equiv \theta^j + i\phi^j$ and $\beta^j \equiv \theta^j - i\phi^j$, as

$$\Lambda(\alpha, \beta) = \exp\left[i\alpha^j\mathcal{J}_j + i\beta^j\mathcal{K}_j\right]. \tag{3.21}$$

We can then use (3.18) to show that these new generators obey the Lie algebra

$$\begin{aligned} [\mathcal{J}_i, \mathcal{J}_j] &= i\epsilon_{ijk}\mathcal{J}_k \\ [\mathcal{K}_i, \mathcal{K}_j] &= i\epsilon_{ijk}\mathcal{K}_k. \\ [\mathcal{J}_i, \mathcal{K}_j] &= 0 \end{aligned} \tag{3.22}$$

This is a very interesting result. We can now see that the generators $\mathcal{J}_i$ and $\mathcal{K}_i$ obey two distinct $\mathfrak{su}(2)$[2] Lie algebras. However, this decomposition holds only for the *complexified*[3] Lie algebra $\mathfrak{so}(1,3)_{\mathbb{C}}$, which contains the *real* Lie algebra $\mathfrak{so}(1,3)$. The Lie algebra $\mathfrak{so}(1,3)_{\mathbb{C}}$ considers the set of real 4×4 matrices $\mathcal{A}$ satisfying

$$\mathcal{A}^T = -\eta\mathcal{A}\eta \tag{3.23}$$

as a *complex vector space*. This allows complex linear combinations of the form $\mathcal{J}_j$ and $\mathcal{K}_j$. Thus, the decomposition

$$\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{su}(2) \times \mathfrak{su}(2), \tag{3.24}$$

is only valid for the complexified Lie algebra of the Lorentz group. However, there is a one-to-one correspondence between representations of a complex Lie algebra and the representations of any of its real forms [3]. This means we can use the irreducible representations of the complex Lie algebra $\mathfrak{so}(1,3)_{\mathbb{C}}$ to find the irreducible representations

[2] We will comment on this in the beginning of the next section.
[3] We have done this when we chose the combinations (3.19).

of the real Lie algebra $\mathfrak{so}(1,3)$.

3.1.3 The Irreducible, Finite-Dimensional, Non-Unitary Representations of $\mathfrak{so}(1,3)$

The $SU(2)$ group, which is the group of special, unitary, 2×2 matrices, is homomorphic to the $SO(3)$ group, which we have seen in chapter 2. The generators of $SU(2)$ obey a Lie algebra that is identical (up to a constant, which can be absorbed by the structure constants) to the $\mathfrak{so}(3)$ Lie algebra.

Since we have seen the irreducible representations of $\mathfrak{so}(3)$, we have, in a sense, also seen the irreducible representations of $\mathfrak{su}(2)$. The Lorentz group's Lie algebra has two Casimir operators, one for each $\mathfrak{su}(2)$. They are, of course, $\mathcal{J}^2$ and $\mathcal{K}^2$, which have eigenvalues $j(j+1)$ and $j'(j'+1)$, respectively, with $j = 0, \frac{1}{2}, 1, \cdots$ and $j' = 0, \frac{1}{2}, 1, \cdots$. We also have that the eigenvalues of the operators $\mathcal{J}_3$ and $\mathcal{K}_3$ are $m = -j, -j+1, \cdots, j-1, j$ and $m' = -j', -j'+1, \cdots, j'-1, j'$. The total spin of the one-particle states is then given by $s = j + j'$. The dimension of the representation[4] is then given by $d_{j,j'} = (2j+1)(2j'+1)$.

It is important to note that these two $\mathfrak{su}(2)$ subalgebras are not independent. A parity transformation acts on J_i and K_i as

$$J_i \to J_i; \quad K_i \to -K_i \tag{3.25}$$

so that $\mathcal{J}_i \leftrightarrow \mathcal{K}_i$. Because we can choose J_i and K_i to be Hermitian, under a Hermitian conjugation we also have the interchange between $\mathcal{J}_i$ and $\mathcal{K}_i$. This means that, for this particular case, a parity transformation is equivalent to Hermitian conjugation.

The representations of $\mathfrak{so}(1,3)$ are then given by the product of two $\mathfrak{su}(2)$ representations. We will now list them, with one remark on notation: we will label the states of the representations of $\mathfrak{so}(1,3)$ keeping both j and j' implicit, that is, $|j, j', m, m'\rangle \to |m, m'\rangle$. The finite-dimensional, non-unitary, irreducible representations of the Lie algebra of the Lorentz group are then listed in the following table.

Table 3.1: **Finite Dimensional, Non-Unitary Representations of $\mathfrak{so}(1,3)$**

[4]The dimension of the representation is the number of allowed states for a particular combination of spin values. This can be seen explicitly in table 3.1.

Spins - (j, j')	Possible States	Total Spin	Dim. of the Rep.
$(0,0)$	$\|0,0\rangle$	$s=0$	$d_{0,0}=1$
$(1/2,0)$	$\|\pm 1/2,0\rangle$	$s=1/2$	$d_{1/2,0}=2$
$(0,1/2)$	$\|0,\pm 1/2\rangle$	$s=1/2$	$d_{0,1/2}=2$
$(1/2,1/2)$	$\|\pm 1/2,1/2\rangle$,$\|\pm 1/2,-1/2\rangle$	$s=1$	$d_{1/2,1/2}=4$
$(1,0)$	$\|0,0\rangle$,$\|\pm 1,0\rangle$	$s=1$	$d_{1,0}=3$
$(0,1)$	$\|0,0\rangle$,$\|0,\pm 1\rangle$	$s=1$	$d_{0,1}=3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
(j,j')	$\|-j,-j'\rangle$,$\|-j+1,-j'\rangle$,$\cdots$,$\|j,j'\rangle$	$s=j+j'$	$d_{j,j'}=(2j+1)(2j'+1)$

3.2 The Poincaré Group - $ISO(1,3)$

We now start our discussion on the Poincaré group, which is our main group of interest. The Poincaré group (also called the $ISO(1,3)$ group) is characterized by transformations on the four-vectors x^μ such that

$$x^\mu \xrightarrow{g} x'^\mu = \Lambda^\mu{}_\nu x^\nu + b^\mu, \tag{3.26}$$

where b^μ is a constant translation of the vector x^μ and $\Lambda^\mu{}_\nu$ is a Lorentz transformation. A group element g is characterized as $g(\Lambda, b)$. When we have two consecutive transformations, we find the group multiplication rule

$$g(\Lambda', b')g(\Lambda, b)x = g(\Lambda', b')\left(\Lambda x + b\right) = \Lambda'\Lambda x + \Lambda' b + b' \Rightarrow$$

$$\Rightarrow g(\Lambda', b')g(\Lambda, b) = g(\Lambda'\Lambda, \Lambda' b + b'), \tag{3.27}$$

where we have, again, suppressed the indices for a more transparent result. We now move on to the ten generators of the Poincaré group (four generators of translations and six generators of Lorentz transformations) and the group's Lie algebra.

3.2.1 Generators of the Poincaré Group and the Lie Algebra $\mathfrak{iso}(1,3)$

We start by considering the infinitesimal versions of $T(b)$ and $\Lambda(\xi)$. Defining P_μ as the generators of translations and $M_{\mu\nu}$ as the generators of Lorentz transformations, we

have

$$T(\delta b) = 1 - \frac{i}{2}\delta b^{\mu} P_{\mu}, \tag{3.28}$$

and

$$\Lambda(\delta\xi) = \mathbb{1}_{4\times 4} - \frac{i}{2}\delta\xi^{\mu\nu} M_{\mu\nu}, \tag{3.29}$$

respectively. Note that we are assuming both δb^{μ} and $\delta\xi^{\mu\nu}$, the transformations parameters, to be infinitesimal. Here, P_{μ} can be identified as the *four-momentum operator*, whose eigenvalue is the four-momentum of the particle $p_{\mu} = (E, \vec{p})$, with E the particle's energy and $\vec{p}$ its momentum. The contravariant generators of translations are defined by $P^{\mu} = \eta^{\mu\nu} P_{\nu}$.

We can then proceed to find the group's Lie algebra. The possible commutation relations between these generators are

$$[P_{\mu}, P_{\nu}]\,, \tag{3.30}$$

$$[P_{\mu}, M_{\lambda\sigma}]\,, \tag{3.31}$$

$$[M_{\mu\nu}, M_{\lambda\sigma}]\,, \tag{3.32}$$

where the last one is known from the last section, and repeated here for the sake of the reader

$$[M_{\mu\nu}, M_{\lambda\sigma}] = i\left(M_{\mu\lambda}\eta_{\nu\sigma} - M_{\mu\sigma}\eta_{\nu\lambda} + M_{\lambda\nu}\eta_{\mu\sigma} - M_{\sigma\nu}\eta_{\mu\lambda}\right). \tag{3.33}$$

We also remind ourselves that the matrix form of $M_{\mu\nu}$ is

$$(M_{\mu\nu})^{\sigma}{}_{\rho} = i(\eta_{\mu\rho}\delta^{\sigma}{}_{\nu} - \eta_{\nu\rho}\delta^{\sigma}{}_{\mu}). \tag{3.34}$$

The first commutation relation we are interested in, namely (3.30) is trivial. This is because P_{μ} are the generators of translations, which commute among themselves. This is known for those who have studied Quantum Mechanics. It is easy enough to demonstrate this by making the identification $P_{\mu} = -i\partial_{\mu}$ and applying the commutator on a trial function $f(x)$

$$[P_\mu, P_\nu]\, f(x) = -\,[\partial_\mu, \partial_\nu]\, f(x) = 0 \Rightarrow [P_\mu, P_\nu] = 0. \tag{3.35}$$

This is merely a consequence of the Abelian nature of the translational subgroup of the Poincaré group. To evaluate (3.31), we remember that P_μ is a vector and thus transforms, under a Lorentz transformation, as

$$\Lambda P_\mu \Lambda^{-1} = P_\nu \Lambda^\nu{}_\mu. \tag{3.36}$$

Using the infinitesimal transformation (3.29) on (3.36) yields, for the left-hand side

$$\begin{aligned} \Lambda P_\mu \Lambda^{-1} &= \left[\mathbb{1}_{4\times4} - \frac{i}{2}\delta\xi^{\lambda\sigma} M_{\lambda\sigma}\right] P_\mu \left[\mathbb{1}_{4\times4} + \frac{i}{2}\delta\xi^{\alpha\beta} M_{\alpha\beta}\right] \\ &= P_\mu + \frac{i}{2}\delta\xi^{\lambda\sigma}[P_\mu, M_{\lambda\sigma}] + \cancelto{0}{\mathcal{O}(\delta\xi^2)}, \end{aligned} \tag{3.37}$$

and, for the right-hand side

$$P_\nu \Lambda^\nu{}_\mu = P_\nu \left[\mathbb{1}_{4\times4} - \frac{i}{2}\delta\xi^{\lambda\sigma} M_{\lambda\sigma}\right]^\nu{}_\mu = P_\mu - \frac{1}{2}\delta\xi^{\lambda\sigma}(P_\lambda \eta_{\mu\sigma} - P_\sigma \eta_{\mu\lambda}). \tag{3.38}$$

Since equation (3.37) must be equal to (3.38), we get

$$[P_\mu, M_{\lambda\sigma}] = i(P_\lambda \eta_{\mu\sigma} - P_\sigma \eta_{\mu\lambda}). \tag{3.39}$$

Results (3.33), (3.35), and (3.39) form the Lie algebra of the Poincaré group, $\mathfrak{iso}(1,3)$.

3.3 The Casimir Operators of the Poincaré Group

The Poincaré group has two Casimir operators. The first one is the four-momentum operator squared, $P^2 = P^\mu P_\mu$, while the second one is the square of the *Pauli-Lubanski pseudo-vector*, which is defined as

$$W_\mu \equiv -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}. \tag{3.40}$$

Since we are claiming that P^2 and $W^2 = W_\mu W^\mu$ are Casimir operators, then we must show that

$$[P^2, P_\mu] = 0, \tag{3.41}$$
$$[P^2, M_{\mu\nu}] = 0, \tag{3.42}$$
$$[W^2, P_\mu] = 0, \tag{3.43}$$
$$[W^2, M_{\mu\nu}] = 0, \tag{3.44}$$
$$[W^2, P^2] = 0. \tag{3.45}$$

Result (3.41) is the simplest to show and is a direct consequence of (3.35). We have

$$[P^2, P_\mu] = P_\nu[P^\nu, P_\mu] + [P_\nu, P_\mu]P^\nu = 0. \tag{3.46}$$

Showing result (3.42) requires us to use (3.35) and (3.39)

$$\begin{aligned}[P^2, M_{\mu\nu}] &= P_\lambda[P^\lambda, M_{\mu\nu}] + [P_\lambda, M_{\mu\nu}]P^\lambda \\ &= iP_\lambda(P_\mu\delta^\lambda{}_\nu - P_\nu\delta^\lambda{}_\mu) + i(P_\mu\eta_{\nu\lambda} - P_\nu\eta_{\mu\lambda})P^\lambda \\ &= i[P_\nu, P_\mu] + i[P_\mu, P_\nu] = 0.\end{aligned} \tag{3.47}$$

For the other commutation relations we will need to use the following result

$$\begin{aligned}[W_\mu, P_\lambda] &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\left(P^\nu[M^{\rho\sigma}, P_\lambda] + \cancelto{0}{[P^\nu, P_\lambda]}M^{\rho\sigma}\right) \\ &= \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu(P^\rho\delta^\sigma{}_\lambda - P^\sigma\delta^\rho{}_\lambda) = 0,\end{aligned} \tag{3.48}$$

where the last equality is achieved upon realizing $\epsilon_{\mu\nu\rho\sigma}$ is antisymmetric under $\nu \leftrightarrow \rho$ and $\nu \leftrightarrow \sigma$, while $P^\nu P^\rho$ and $P^\nu P^\sigma$ are symmetric under the same exchanges. It is also useful to notice that

$$W_\mu P^\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}P^\mu = 0, \tag{3.49}$$

by the same index symmetries mentioned above. This means we have

$$
\begin{aligned}
[W_\mu P^\mu, M_{\alpha\beta}] = 0 = W_\mu[P^\mu, M_{\alpha\beta}] + [W_\mu, M_{\alpha\beta}]P^\mu \\
\Rightarrow [W_\mu, M_{\alpha\beta}]P^\mu = -iW_\mu(P_\alpha \delta^\mu{}_\beta - P_\beta \delta^\mu{}_\alpha) = -i(W_\beta \eta_{\mu\alpha} - W_\alpha \eta_{\mu\beta})P^\mu \\
\Rightarrow [W_\mu, M_{\alpha\beta}] = i(W_\alpha \eta_{\mu\beta} - W_\beta \eta_{\mu\alpha}).
\end{aligned} \tag{3.50}
$$

Hence, using (3.48) and (3.50), we can show that

$$
[W^2, P_\mu] = W_\nu[W^\nu, P_\mu] + [W_\nu, P_\mu]P^\nu = 0, \tag{3.51}
$$

and

$$
\begin{aligned}
[W^2, M_{\alpha\beta}] &= W_\mu[W^\mu, M_{\alpha\beta}] + [W_\mu, M_{\alpha\beta}]W^\mu \\
&= iW_\mu(W_\alpha \delta^\mu{}_\beta - W_\beta \delta^\mu{}_\alpha) + i(W_\alpha \eta_{\mu\beta} - W_\beta \eta_{\mu\alpha})W^\mu \\
&= i[W_\beta, W_\alpha] + i[W_\alpha, W_\beta] = 0.
\end{aligned} \tag{3.52}
$$

The final commutation relation we are interested in is very straightforward to show using (3.51), in the same way we have done in (3.46), and will not be repeated here. Since we have shown that P^2 and W^2 commute with all other generators of the Poincaré group, we conclude that they are indeed Casimir operators of the group.

3.4 Unitary Irreducible Representations of the Poincaré Group

We will now construct the representations of the Poincaré group based on the concept of *little group*. The little group of the Poincaré group is defined as the set of transformations that leave the particle's four-momentum invariant. Result (3.48) shows us that W_μ commutes with the four-momentum operator and, therefore, leaves the particles four-momentum invariant. This means that the components of the Pauli-Lubanski pseudo-vector will be the generators of the little group of the Poincaré group. This will become clearer once we start constructing the representations explicitly.

3.4.1 Massive Particles

$p^2 > 0$

We start by considering the usual massive particles with timelike four-momentum, which are particles with positive mass squared. Examples of these particles are *electrons* and *quarks*, responsible for most of the matter content we know. We can always boost a massive particle's four-momentum to their rest frame, that is, $p^\mu = (M, 0, 0, 0)$, where M is the particle's mass. In this case, it is simple to see that any transformation that acts upon the spacial components of p^μ will leave it invariant. It seems reasonable, at least as an initial guess, that the little group for this case is $SO(3)$. To confirm this, we construct the generators explicitly

$$W_0 = -\frac{1}{2}\epsilon_{0ijk}P^i M^{jk} = 0, \tag{3.53}$$

$$W_i = -\frac{1}{2}\epsilon_{i0jk}P^0 M^{jk} = \frac{M}{2}\epsilon_{ijk}\epsilon^{ij\ell}J_\ell = MJ_i, \tag{3.54}$$

where in (3.54) we have used (3.16). The Casimir operators in this case are $P^2 = M^2$ and $W^2 = -M^2J^2$. This clearly shows that the symmetry group is $SO(3)$, since all generators are identical (up to a normalization constant) to those of $SO(3)$. The representations are exactly the same as those of $SO(3)$, except they now carry mass and momentum labels[5]

$$\begin{aligned} P^2|M,\mathbf{0};s,\lambda\rangle = M^2|M,\mathbf{0};s,\lambda\rangle,\ \ P_\mu|M,\mathbf{0};s,\lambda\rangle = p^\mu|M,\mathbf{0};s,\lambda\rangle \\ W^2|M,\mathbf{0};s,\lambda\rangle = -M^2 s(s+1)|M,\mathbf{0};s,\lambda\rangle, \\ J^2|M,\mathbf{0};s,\lambda\rangle = s(s+1)|M,\mathbf{0};s,\lambda\rangle, \\ J_3|M,\mathbf{0};s,\lambda\rangle = \lambda|M,\mathbf{0};s,\lambda\rangle, \\ J_\pm|M,\mathbf{0};s,\lambda\rangle = \mathcal{N}_\pm|M,\mathbf{0};s,\lambda\pm 1\rangle, \end{aligned} \tag{3.55}$$

where $\mathcal{N}_\pm$ are normalizations such that $J_\pm|M,\mathbf{0};s,\pm s\rangle = 0$.

[5]When dealing with the Poincaré group we will change notation slightly. The particle's spin will be represented by the letter s and the particle's helicity by λ. A particle's helicity indicates whether its spin is aligned or anti-aligned with its momentum.

$p^2 < 0$

In this case we are dealing with a particle with spacelike four-momentum, or negative mass squared (or imaginary mass). These particles, which move faster than the speed of light, are known as *tachyons*. The appearance of tachyons in a theory is usually an indication of instabilities. One example we can comment on is that of the Higgs boson. In its uncondensed state, the Higgs field is a tachyonic field, which would give rise to a particle of negative mass squared. Through spontaneous symmetry breaking, however, the Higgs field's instability disappears[6].

A standard four-momentum to deal with this case is $p^\mu = (0, 0, 0, Q)$, so that $p^2 = -Q^2$, and thus the desired condition is satisfied (this choice is not unique, as we could have chosen, for example, $p^\mu = (iQ, 0, 0, 0)$, yielding the same result). The generators of the little group for this case are

$$W_0 = -QJ_3, \tag{3.56}$$

and

$$W_i = \epsilon_{ijk} P^j K_k = \begin{cases} W_1 = -QK_2, \\ W_2 = QK_1, \\ W_3 = 0. \end{cases} \tag{3.57}$$

The Lie algebra satisfied by these generators is given by

$$\begin{aligned} [K_1, J_3] &= -iK_2, \\ [K_2, J_3] &= iK_1, \\ [K_1, K_2] &= -iJ_3. \end{aligned} \tag{3.58}$$

This is *almost* the algebra $\mathfrak{so}(3)$, but the last commutator should have a plus sign for this to be true. We have, once again, found a non-compact group. Just as the Lorentz group $SO(1,3)$ was the non-compact version of $SO(4)$, this group is the non-compact version of $SO(3)$, namely $SO(1,2)$. By the same arguments used before, we notice the homomorphism $\mathfrak{so}(1,2)_{\mathbb{C}} \cong \mathfrak{su}(2) \cong \mathfrak{so}(3)$, so that the representations of $\mathfrak{so}(1,2)$ are those of $\mathfrak{so}(3)$, which we have already constructed. However, there is one subtlety regarding the eigenvalues of W^2. Note that

[6]Standard Quantum Field Theory textbooks, such as [7, 8, 9, 10], should cover the details of this wonderful phenomenon, which would take us far away from the scope of this dissertation.

$$W^2 = W_\mu W^\mu = Q^2 \left[(J_3)^2 - (K_1)^2 - (K_2)^2 \right]. \tag{3.59}$$

If we assign to W^2 an eigenvalue ω, then when $\omega > 0$, this automatically implies $(K_1)^2 = (K_2)^2 = 0$, because these do not have finite range. Therefore the representations are exactly the same as the representations of the $\mathfrak{so}(3)$ algebra, with $\omega = s(s+1)$ and $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$ (where we are allowing double-valued representations). This would mean that the states described by the theory would obey

$$\begin{aligned} P^2|p^\mu; s, \lambda\rangle = -Q^2|p^\mu; s, \lambda\rangle,\ P^\mu|p^\mu; s, \lambda\rangle = p^\mu|p^\mu; s, \lambda\rangle, \\ (J_3)^2|p^\mu; s, \lambda\rangle = s(s+1)|p^\mu; s, \lambda\rangle, \\ J_3|p^\mu; s, \lambda\rangle = \lambda|p^\mu; s, \lambda\rangle. \end{aligned} \tag{3.60}$$

However, when we allow ω to be negative, we cannot control its range. This means $-\infty < \omega \leq 0$. This representations cannot be specified by the particle's spin nor its helicity. It is labelled, instead, by a continuous, negative parameter ω

$$\begin{aligned} P^2|p^\mu; \omega, \lambda\rangle = -Q^2|p^\mu; \omega, \lambda\rangle,\ P^\mu|p^\mu; \omega, \lambda\rangle = p^\mu|p^\mu; \omega, \lambda\rangle, \\ W^2|p^\mu; \omega, \lambda\rangle = \omega|p^\mu; \omega, \lambda\rangle \\ J_3|p^\mu; \omega, \lambda\rangle = \lambda|p^\mu; \omega, \lambda\rangle. \end{aligned} \tag{3.61}$$

It is important to notice that we can still build our raising and lowering operators through $K_\pm = K_1 \pm iK_2$, which raise and lower the value of λ by unit. In the case where $\omega > 0$, we have that the representation has dimension $d_+ = 2s + 1$. However, in the case where $\omega < 0$, we have no limit to how many times we can raise or lower the helicity of the particle. This means we have a *infinite dimensional representation.*

3.4.2 The Null Vector Representation

This is a special case of the Poincaré group, where we consider a particle with four-momentum $p^\mu = (0, 0, 0, 0)$. In this case, p^μ is invariant under *all Lorentz transformations.* This means that the little group of this representation is $SO(1, 3)$, i.e. the Lorentz group. The representation is then given by that of section 3.1.3. The only difference would be that the states described by the theory now carry an extra lable $p^\mu = 0$, however there is no Lorentz transformation that can change this quantity.

3.4.3 Massless Particles

When we deal with massless particles, we have $p^2 = 0$. We can consider a standard four-momentum of the form $p^\mu = (E, 0, 0, E)$, where E is the energy of the particle. In this case, the Pauli-Lubanski pseudo-vector takes the form

$$\begin{aligned} W_0 &= -\frac{1}{2}\epsilon_{ijk}P^i M^{jk} = -P^i J_i = -EJ_3, \\ W_i &= -\frac{1}{2}\epsilon_{i0jk}P^0 M^{jk} - \epsilon_{ij0k}P^j M^{0k} = EJ_i - \epsilon_{ijk}P^j K^k, \\ W_1 &= E(J_1 + K_2), \\ W_2 &= E(J_2 - K_1), \\ W_3 &= EJ_3. \end{aligned} \tag{3.62}$$

This means

$$W_\mu = E\Big(- J_3, J_1 + K_2, J_2 - K_1, J_3\Big), \tag{3.63}$$

and

$$W^\mu = E\Big(- J_3, -J_1 - K_2, -J_2 + K_1, -J_3\Big), \tag{3.64}$$

so that

$$W^2 = W_\mu W^\mu = -(W_1)^2 - (W_2)^2. \tag{3.65}$$

This means that the eigenvalues of W^2 are either zero or negative. We call them $-\rho^2 \leq 0$, where $\rho \in \mathbb{R}$. We now have two possibilities, which we will discuss separately.

Usual Massless Particles - $\rho = 0$

When $\rho = 0$, we can look at (3.65) and see that the eigenvalue of W^2 must vanish. This is the equivalent of saying that this operator annihilate physical states. The representation is then labelled by the particle's four-momentum and its helicity, p^μ and λ, which are respectively the eigenvalues of P^μ and J_3. The states described by the theory must obey

$$
\begin{aligned}
P^2|p^\mu;\lambda\rangle = 0, P^\mu|p^\mu;\lambda\rangle = p^\mu|p^\mu;\lambda\rangle,\\
J_3|p^\mu;\lambda\rangle = \lambda|p^\mu;\lambda\rangle,\\
W^2|p^\mu;\lambda\rangle = W_1|p^\mu;\lambda\rangle = W_2|p^\mu;\lambda\rangle = 0.
\end{aligned}
\tag{3.66}
$$

These are the states that describe the usual massless particles we encounter, for example the *gauge bosons*[7]. These elementary particles are the force carriers of all the known interactions (e.g. the photon is the force carrier of the electromagnet field, while the gluons are the force carriers of the strong force). These particles are extremely important to the understanding of the fundamental interactions of nature and thus it is more than relevant to include this discussion in this dissertation.

The Continuous Spin Particles - $\rho \neq 0$

Finally, we enter the realm of the last possible representation of the Poincaré group's Lie algebra: that of the *continuous spin particles* (CSP) [1]. These are massless particles that have $\rho \neq 0$, which make them differ greatly from the usual massless particles. In this case, we have that

$$
\begin{aligned}
[W_1, W_2] = 0,\\
[W_2, J_3] = iW_1,\\
[W_1, J_3] = -iW_2,
\end{aligned}
\tag{3.67}
$$

which follow from (3.62). This is the Lie algebra of the Euclidean group in two dimensions (rotations and translations in two dimensional Euclidean space), denoted by E_2 or $ISO(2)$. This is then the little group of the CSP representation. In this case, the operators

$$
W_\pm \equiv -(W_1 \pm iW_2) \tag{3.68}
$$

act on states as raising and lowering operators of the helicity λ by unit. The states described by the theory must, then, obey

[7] We use the term *gauge* here to differ these bosons from the Higgs boson, which we mentioned earlier. The Higgs boson is a *scalar boson*. This means that the class of bosons we are describing here arise from gauge fields, while the Higgs boson arises from a scalar field. Again, more information can be found in the references [7, 8, 9, 10]

$$
\begin{aligned}
P^2|p^\mu;\rho;\lambda\rangle = 0,\ \ P^\mu|p^\mu;\rho;\lambda\rangle &= p^\mu|p^\mu;\rho;\lambda\rangle,\\
J_3|p^\mu;\rho;\lambda\rangle &= \lambda|p^\mu;\rho;\lambda\rangle,\\
W^2|p^\mu;\rho;\lambda\rangle &= -\rho^2|p^\mu;\rho;\lambda\rangle,\\
W_+|p^\mu;\rho;\lambda\rangle &= i\rho|p^\mu;\rho;\lambda+1\rangle,\\
W_-|p^\mu;\rho;\lambda\rangle &= -i\rho|p^\mu;\rho;\lambda-1\rangle,
\end{aligned}
\tag{3.69}
$$

and the representation is also infinite dimensional, since in this case we do not have a limit on how many times we can act with our operators $W_\pm$. In other words, *all* values of λ are needed to furnish the representation. It is also important to note that we have two classes of CSPs. When λ is integer, then the states are $|p^\mu;\rho;0,\pm1,\pm2,\cdots\rangle$, and we say this is a *bosonic CSP* representation. On the other hand, if λ is half-integer, then the representation is $|p^\mu;\rho;0,\pm\frac{1}{2},\pm\frac{3}{2},\cdots\rangle$, and we say this describes a *fermionic CSP*. The field theoretical analysis we will do for CSPs will deal with the case of a bosonic CSP.

The last two relations in (3.69) fix[8] the eigenvalues of $W_\pm$ upon realizing that $W^2 = -W_\pm W_\mp$ and that, therefore

$$
-W_\pm W_\mp|p^\mu;\rho;\lambda\rangle = W^2|p^\mu;\rho;\lambda\rangle = -\rho^2|p^\mu;\rho;\lambda\rangle. \tag{3.70}
$$

Although they are predicted by theory, CSPs are not observed in Nature. The main reason for this is because we do not have a Quantum Field Theory that describes CSPs, meaning we cannot predict how these particles would interact with the other known particles. The lack of a local, covariant actions that can describe bosonic and fermionic CSPs restricted our capability of studying such particles. However, recently, action were proposed that can describe bosonic CSPs [12, 13, 15] and fermionic CSPs [16], at least at the classical level. These are major progresses towards a Quantum Field Theory describing CSPs.

This concludes the group theoretical description of our CSPs. We will now begin a field theoretical approach to the subject, following the references [12, 13, 15], and study a more useful approach for dealing with bosonic CSPs.

[8] We could have also chosen to set the eigenvalues of $W_\pm$ to $-i\rho$, but we chose the positive sign since this is an arbitrary choice.

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Chapter 4

The Schwinger-Fronsdal Formalism

In this chapter we present the Schwinger-Fronsdal formalism for massless Bosons. We take the general spin-s action, given by (4.1), and study explicitly, as examples, the cases where $s \leq 4$, finding the equations of motion for each case and checking the actions' gauge invariance. Through the study of these examples, we are able to infer the form of the equations of motion in the general case, which is the main result of this chapter.

In section 4.1 we introduce the general Schwinger-Fronsdal formalism that will be used throughout this chapter. From sections 4.1.1 to 4.1.5 we study the cases of $s \leq 4$, generalizing our results to a particle of spin-s in section 4.1.6.

4.1 Massless Bosons

Here we present the general action for a *spin*$-s$ Boson (integer spin) as proposed by Fronsdal [11] and written in terms of spacetime derivatives instead of the particle's momenta [12]

$$\begin{aligned} S_s = (-1)^s \int d^dx \Big[& \tfrac{1}{2}(\partial_\alpha \phi)^2 - \tfrac{s}{2}(\partial \cdot \phi)^2 - \tfrac{s(s-1)}{2}\phi' \cdot (\partial \cdot \partial \cdot \phi) - \tfrac{s(s-1)}{4}(\partial_\alpha \phi')^2 \\ & - \tfrac{s(s-1)(s-2)}{8}(\partial \cdot \phi')^2 - \phi \cdot J^{(h)} \Big], \end{aligned} \tag{4.1}$$

where ϕ is a rank-s completely symmetric tensor field, restricted to be double traceless (i.e. $\phi'' = 0$), s is the particle's spin, J is a rank-s tensor source, and the factor $(-1)^s$ ensures a canonical kinetic term with our mostly-negative metric. The notation $\partial \cdot \phi^{(s)}$ indicates a contraction between the derivative and the *first* index in ϕ, i.e.

$\partial \cdot \phi^{(s)} \equiv \partial^{\mu_1}\phi_{\mu_1\mu_2\cdots\mu_s}$. Although we will explore the gauge invariance of this action, it is clear from our discussion of the Lorentz and Poincaré groups in the previous chapter that (4.1) is invariant under Poincaré transformations[1].

The action (4.1) is also invariant (when $J = 0$) under the gauge transformations [11, 12]

$$\delta\phi^{(s)} = \partial \circ \varepsilon^{(s-1)}, \tag{4.2}$$

where ε is a traceless rank-$(s-1)$ tensor ($\varepsilon' = 0$). Both our field $\phi^{(s)}$ and our gauge parameter $\varepsilon^{(s-1)}$ are *symmetric* under indices exchanges.

It is not obvious *a priori* to understand the implications of the conditions on the field $\phi^{(s)}$ and the gauge parameter $\varepsilon^{(s-1)}$. Thus, we will study a few examples explicitly, which will allow us to find the necessity for these conditions, as well as allow us to infer the form of the equations of motion associated with the action (4.1).

Starting with the cases of spin zero, one, and two, we will notice that the actions we obtain from (4.1) are equivalent to the Klein-Gordon, Maxwell, and linearized Einstein actions, respectively. We will then move on to study the spin three and four cases, where the necessity of a traceless condition upon the gauge parameter and a double-traceless condition upon our tensor field will become explicit. In what follows, all actions will be studied for the free theory case.

4.1.1 The Spin-0 Action

By setting $s = 0$ in (4.1) we obtain the familiar action for a massless scalar field

$$S_0 = \int d^dx \Big[\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)\Big] = S_{\text{KG}}, \tag{4.3}$$

also known as the *Klein-Gordon action*. We cannot construct a gauge transformation such as (4.2) for this case.

A variation in our field ϕ by an infinitesimal amount $\phi \to \phi + \delta\phi$ yields a variation in the action given by

$$\delta S_{KG} = \int d^dx \Big[(\partial_\mu\phi)(\partial^\mu\delta\phi)\Big] = 0, \tag{4.4}$$

[1]The action is clearly Lorentz invariant because all indices are fully contracted. It is also invariant under spacetime translations by a constant amount $\phi^{(h)}(x) \to \phi^{(h)}(x-a)$, where a is a constant vector in spacetime.

which upon an integration by parts[2], and the principle of least action[3], allow us to obtain the equations of motion

$$\partial^2\phi \equiv \Box\phi \equiv \mathcal{F} = 0, \tag{4.5}$$

known as the *Klein-Gordon equation*. The notation $\mathcal{F}$, which is the symbol we will use to denote the equations of motion, is here introduced for the first time, and although it might not accomplish much in this case, it will be useful when we attempt to generalize our results to the case of a particle with spin-s.

4.1.2 The Spin-1 Action

To make the connection with electromagnetism simpler, we will rename our rank-1 tensor field (or *vector field*) ϕ_μ - obtained by setting $s = 1$ in (4.1) - to A_μ. We then have

$$\begin{aligned} S_{\pm 1} &= -\int d^dx\Big[\tfrac{1}{2}(\partial_\mu A_\nu)^2 - \tfrac{1}{2}(\partial_\mu A^\mu)^2\Big] = -\frac{1}{2}\int d^dx\Big[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A^\mu)(\partial_\nu A^\nu)\Big] \\ &= -\frac{1}{2}\int d^dx\Big[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A^\nu)(\partial_\nu A^\mu)\Big], \end{aligned} \tag{4.6}$$

where in the last equality we have integrated the last term by parts twice. If we remind ourselves of the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \Rightarrow$$

$$\Rightarrow F_{\mu\nu}F^{\mu\nu} = 2\Big[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A^\nu)(\partial_\nu A^\mu)\Big], \tag{4.7}$$

we can rewrite (4.6) using (4.7) as

$$S_{\pm 1} = \int d^dx\Big[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big] = S_{\text{Maxwell}}, \tag{4.8}$$

which is the *Maxwell action*. Varying the field A_μ in the action (4.8) yields the equations

[2]We will not keep track of the surface terms that arise when we perform integrations by parts, assuming that all of them vanish.

[3]Which we already used on (4.4) when we set the variation of the action to zero.

of motion

$$\mathcal{F}_\mu \equiv \Box A_\mu - \partial_\mu \partial \cdot A = 0, \tag{4.9}$$

where we again have used the notation $\mathcal{F}_\mu$ to denote our equation of motion.

In the spin-1 case, the action is invariant under the gauge transformation

$$\delta A_\mu = \partial_\mu \varepsilon. \tag{4.10}$$

To check whether what we are claiming is true or not, we could set

$$A_\mu \to A_\mu + \delta A_\mu, \tag{4.11}$$

in (4.6) and keep only the linear terms in δA_μ (which is the same procedure used to obtain the equations of motion). But, because $[\partial_\mu, \partial_\nu] = 0$, it is much simpler to notice that

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = [\partial_\mu, \partial_\nu]\varepsilon = 0, \tag{4.12}$$

which implies that (4.8) is invariant under the transformation (4.10). The equations of motion are also clearly invariant under (4.10), since

$$\delta \mathcal{F}_\mu = \Box \delta A_\mu - \partial_\mu \partial_\nu \delta A^\nu = \Box \partial_\mu \varepsilon - \partial_\mu \Box \varepsilon = 0. \tag{4.13}$$

4.1.3 The Spin-2 Action

For $s = 2$, the action (4.1) becomes

$$S_{\pm 2} = \int d^d x \left[\tfrac{1}{2}(\partial_\alpha \phi_{\mu\nu})^2 - (\partial \cdot \phi_\nu)^2 - \phi'(\partial_\mu \partial_\nu \phi^{\mu\nu}) - \tfrac{1}{2}(\partial_\alpha \phi')^2 \right]. \tag{4.14}$$

The gauge transformation that leaves (4.14) invariant is given by

$$\delta \phi_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu, \tag{4.15}$$

where our gauge transformation parameter is now a vector field. To see that (4.15) indeed leaves (4.14) invariant, we vary our field $\phi_{\mu\nu}$ in (4.14) and maintain only the terms linear in the variation

$$\begin{aligned}\delta S_{\pm 2} &= \int d^dx \Big[\partial_\alpha \phi_{\mu\nu}\partial^\alpha \delta\phi^{\mu\nu} - 2\partial\cdot\phi_\nu \partial\cdot\delta\phi^\nu - \phi'\partial_\mu\partial_\nu\delta\phi^{\mu\nu} - \delta\phi'\partial_\mu\partial_\nu\phi^{\mu\nu} - \partial_\alpha\phi'\partial^\alpha\delta\phi'\Big] \\ &= \int d^dx \Big[\underbrace{\partial_\alpha\phi_{\mu\nu}(\partial^\alpha\partial^\mu\varepsilon^\nu + \partial^\alpha\partial^\nu\varepsilon^\mu)}_{2(\partial\cdot\phi_\nu)\Box\varepsilon^\nu} - 2(\partial\cdot\phi_\nu)\Box\varepsilon^\nu \underbrace{-2(\partial\cdot\phi_\nu)\partial^\nu(\partial\cdot\varepsilon)}_{+2(\partial\cdot\partial\cdot\phi)(\partial\cdot\varepsilon)} - 2\phi'\Box(\partial\cdot\varepsilon) \\ &\quad - 2(\partial\cdot\varepsilon)(\partial\cdot\partial\cdot\phi)\underbrace{-2\partial_\mu\phi'\partial^\mu(\partial\cdot\varepsilon)}_{+2\phi'\Box\varepsilon}\Big] \Rightarrow \delta S_{\pm 2} = 0,\end{aligned} \tag{4.16}$$

where the brackets below the terms in the expression above indicate the results obtained after we used integration by parts. Thus, (4.14) is invariant under transformations of the form (4.15). The equations of motion obtained by varying $\phi_{\mu\nu}$ in (4.14) can be read (after some integrations by parts) directly from the first line of equation (4.16)

$$\Box\phi_{\mu\nu} - \partial_\mu(\partial\cdot\phi_\nu) - \partial_\nu(\partial\cdot\phi_\mu) + \partial_\mu\partial_\nu\phi' + g_{\mu\nu}(\partial\cdot\partial\cdot\phi) - g_{\mu\nu}\Box\phi' = 0. \tag{4.17}$$

We can then write equation (4.17) in a more compact form by defining

$$\mathcal{F}_{\mu\nu} \equiv \Box\phi_{\mu\nu} - \partial_\mu(\partial\cdot\phi_\nu) - \partial_\nu(\partial\cdot\phi_\mu) + \partial_\mu\partial_\nu\phi', \tag{4.18}$$

and noticing that

$$\mathcal{F}' \equiv g_{\sigma\lambda}\mathcal{F}^{\sigma\lambda} = \Box\phi' - 2(\partial\cdot\partial\cdot\phi) + \Box\phi' \Rightarrow -\tfrac{1}{2}g_{\mu\nu}\mathcal{F}' = g_{\mu\nu}(\partial\cdot\partial\cdot\phi) - g_{\mu\nu}\Box\phi'. \tag{4.19}$$

We can thus rewrite (4.17) using (4.18) and (4.19) as

$$\mathcal{F}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{F}' = 0. \tag{4.20}$$

This equation of motion can be split into two equations of motion upon realizing that the trace of (4.20) gives

$$\mathcal{F}' - \tfrac{d}{2}\mathcal{F}' = 0 \Rightarrow \mathcal{F}' = 0 \Rightarrow \mathcal{F}_{\mu\nu} = 0, \tag{4.21}$$

so that

$$\mathcal{F}_{\mu\nu} = 0. \tag{4.22}$$

Finally, we can also check that (4.17) is gauge invariant by noticing that a variation of the form (4.15) yields

$$\begin{aligned}\Box(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu) - \partial_\mu\Big[\Box\varepsilon_\nu + \partial_\nu(\partial\cdot\varepsilon)\Big] - \partial_\nu\Big[\Box\varepsilon_\mu + \partial_\mu(\partial\cdot\varepsilon)\Big] \\ + 2\partial_\mu\partial_\nu(\partial\cdot\varepsilon) + 2g_{\mu\nu}\Box(\partial\cdot\varepsilon) - 2g_{\mu\nu}\Box(\partial\cdot\varepsilon) = 0.\end{aligned} \tag{4.23}$$

The Linearized Einstein's Equations

The equations of motion (4.22) resemble the linearized Einstein's equations of motion. To notice that they are indeed those equations, we will now derive them from a different approach.

Consider a curved spacetime metric $G_{\mu\nu}$, which differs from the flat spacetime metric $g_{\mu\nu}$ by a small deviation $\phi_{\mu\nu}$

$$G_{\mu\nu} = g_{\mu\nu} + \phi_{\mu\nu}. \tag{4.24}$$

Now, because $\phi_{\mu\nu}$ is a small deviation, we will disregard all terms that are "quadratic" in ϕ and its derivatives, such as $\phi\phi$, $\phi\partial\phi$, $\partial\phi\partial\phi$, etc. We use $g_{\mu\nu}$ to raise and lower $\phi_{\mu\nu}$ indices and, for completeness, we give the inverse metric

$$G^{\mu\nu} = g^{\mu\nu} - \phi^{\mu\nu}, \tag{4.25}$$

so that, as usual, $G_{\mu\nu}G^{\nu\rho} = \delta^\rho_\mu$. With these considerations, the Christoffel symbol associated with $G_{\mu\nu}$ becomes

$$\Gamma_{\mu\nu\rho} = \tfrac{1}{2}(\partial_\mu\phi_{\nu\rho} + \partial_\nu\phi_{\mu\rho} - \partial_\rho\phi_{\mu\nu}), \tag{4.26}$$

so that the Riemann curvature tensor can be constructed as follows

$$R_{\rho\sigma\mu\nu} = \partial_\mu\Gamma_{\rho\nu\sigma} - \partial_\nu\Gamma_{\rho\mu\sigma} = \tfrac{1}{2}(\partial_\mu\partial_\sigma\phi_{\rho\nu} - \partial_\mu\partial_\rho\phi_{\nu\sigma} - \partial_\nu\partial_\sigma\phi_{\rho\mu} + \partial_\nu\partial_\rho\phi_{\mu\sigma}). \tag{4.27}$$

Result (4.27) does not contain terms such as $\Gamma_{\rho\mu\lambda}\Gamma^\lambda_{\nu\sigma}$ because these would contain

terms of order $\mathcal{O}(\partial\phi\partial\phi)$. From (4.27) we can construct the Ricci tensor

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu} = -\tfrac{1}{2}\Big[\Box\phi_{\mu\nu} - \partial_\mu(\partial\cdot\phi_\nu) - \partial_\nu(\partial\cdot\phi_\mu) + \partial_\mu\partial_\nu\phi'\Big], \tag{4.28}$$

and, finally, the Ricci scalar

$$R \equiv R^{\mu}{}_{\mu} = -\Big[\Box\phi' - (\partial\cdot\partial\cdot\phi)\Big]. \tag{4.29}$$

If we now compare results (4.28) and (4.29) with (4.18) and (4.19), respectively, we can make the identifications

$$\mathcal{F}_{\mu\nu} = -\tfrac{1}{2}R_{\mu\nu}, \tag{4.30}$$
$$\mathcal{F}' = -\tfrac{1}{2}R, \tag{4.31}$$

so that we obtain

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 0, \tag{4.32}$$
$$R_{\mu\nu} = 0, \tag{4.33}$$
$$R = 0. \tag{4.34}$$

Thus the Schwinger-Fronsdal formalism for a massless spin 2 Boson (the *graviton*) is completely equivalent to that of the linearized Einstein's equations (up to an overall normalization factor that does not affect the equations of motion).

4.1.4 The Spin-3 Action

We now start studying the action (4.1) when $s = 3$. This will lead us to find the necessity for the condition $\varepsilon' = 0$. The $s = 3$ action is

$$S_{\pm 3} = \int d^dx\Big[\tfrac{1}{2}(\partial_\alpha\phi_{\mu\nu\sigma})^2 - \tfrac{3}{2}(\partial\cdot\phi_{\mu\nu})^2 - 3\phi'\cdot(\partial\cdot\partial\cdot\phi) - \tfrac{3}{2}(\partial_\alpha\phi'_\mu)^2 - \tfrac{3}{4}(\partial\cdot\phi')^2\Big]. \tag{4.35}$$

This action is invariant under gauge transformations of the form

$$\delta\phi_{\mu\nu\sigma} = \partial_\mu\varepsilon_{\nu\sigma} + \partial_\nu\varepsilon_{\mu\sigma} + \partial_\sigma\varepsilon_{\mu\nu}. \tag{4.36}$$

Varying our field ϕ in (4.35) and keeping only the linear terms in $\delta\phi$ we get

$$\begin{aligned}
\delta S_{\pm 3} = \int d^dx \Big[& (\partial_\alpha\phi)\cdot(\partial^\alpha\delta\phi) - 3(\partial\cdot\phi)\cdot(\partial\cdot\delta\phi) - 3\phi'\cdot(\partial\cdot\partial\cdot\delta\phi) - 3\delta\phi'\cdot(\partial\cdot\partial\cdot\phi) \\
& - 3(\partial_\alpha\phi')\cdot(\partial^\alpha\delta\phi') - \tfrac{3}{2}(\partial\cdot\phi')(\partial\cdot\delta\phi')\Big] \\
= \int d^dx \Big[& \underbrace{\partial_\alpha\phi_{\mu\nu\sigma}\partial^\alpha(\partial^\mu\varepsilon^{\nu\sigma} + \partial^\nu\varepsilon^{\mu\sigma} + \partial^\sigma\varepsilon^{\mu\nu})}_{3(\partial\cdot\phi)\cdot(\Box\varepsilon)} \underbrace{-3(\partial\cdot\phi_{\mu\nu})(\partial^\mu\partial\cdot\varepsilon^\nu + \partial^\nu\partial\cdot\varepsilon^\mu + \Box\varepsilon^{\mu\nu})}_{6(\partial\cdot\partial\cdot\phi)\cdot(\partial\cdot\varepsilon) - 3(\partial\cdot\phi)\cdot(\Box\varepsilon)} \\
& - 6(\Box\partial\cdot\varepsilon)\cdot\phi' \underbrace{-3\phi'^\sigma\partial_\sigma(\partial\cdot\partial\cdot\varepsilon)}_{+3(\partial\cdot\phi)(\partial\cdot\partial\cdot\varepsilon)} - 6(\partial\cdot\varepsilon)\cdot(\partial\cdot\partial\cdot\phi) - \cancelto{0,\,\varepsilon \text{ is traceless}}{3(\partial\cdot\partial\cdot\phi)\cdot(\partial\varepsilon')} \\
& \underbrace{-6\partial_\alpha\phi'\cdot(\partial^\alpha\partial\cdot\varepsilon)}_{+6\phi'\cdot(\Box\partial\cdot\varepsilon)} - 3(\partial\cdot\phi')(\partial\cdot\partial\cdot\varepsilon)\Big] \Rightarrow \delta S_{\pm 3} = 0,
\end{aligned} \tag{4.37}$$

where we must use the traceless condition of our gauge parameter ε to obtain the desired result. Here, for the first time, we that it is key that ε is traceless in order for the action (4.35) to be gauge invariant. This condition will also be necessary when we study higher spins. Again, on equation (4.37), every term with a bracket underneath them indicates that we have to perform an integration by parts, and the results of said integrations are indicated below those brackets.

The equations of motion for the action (4.35) can be found after a simple manipulation of the first equality in (4.37), yielding

$$\begin{aligned}
\Box\phi_{\mu\nu\sigma} - \Big[\partial_\mu(\partial\cdot\phi_{\nu\sigma}) + \text{perm.}'\Big] + \Big[\partial_\mu\partial_\nu\phi'_\sigma + \text{perm.}'\Big] + \Big[g_{\mu\nu}(\partial\cdot\partial\cdot\phi_\sigma) + \text{perm.}'\Big] \\
- \Big[g_{\mu\nu}\Box\phi'_\sigma + \text{perm.}'\Big] + \tfrac{1}{2}\Big[g_{\mu\nu}\partial_\sigma(\partial\cdot\phi') + \text{perm.}'\Big] = 0,
\end{aligned} \tag{4.38}$$

where "perm.$'$" stands for inequivalent permutations of the involved indices with no symmetry factor, e.g.

$$\Big[\partial_\mu(\partial\cdot\phi_{\nu\sigma}) + \text{perm.}'\Big] = \partial_\mu(\partial\cdot\phi_{\nu\sigma}) + \partial_\nu(\partial\cdot\phi_{\mu\sigma}) + \partial_\sigma(\partial\cdot\phi_{\mu\nu}). \tag{4.39}$$

Note that on (4.39) we get only *half* the terms of the "full" permutation because the field $\phi_{\mu\nu\sigma}$ is symmetric under indices exchanges. Since terms like $\partial_\mu(\partial\cdot\phi_{\nu\sigma})$ and $\partial_\mu(\partial\cdot\phi_{\sigma\nu})$ are equivalent to each other, we drop one of them.

For the purpose of generalization, we also try to write down the equations of motion (4.38) in terms of an $\mathcal{F}$ tensor. First we define

$$\mathcal{F}_{\mu\nu\sigma} \equiv \Box\phi_{\mu\nu\sigma} - \Big[\partial_\mu(\partial\cdot\phi_{\nu\sigma}) + \text{perm.}'\Big] + \Big[\partial_\mu\partial_\nu\phi'_\sigma + \text{perm.}'\Big]. \tag{4.40}$$

Then, taking the trace of (4.40) we get

$$\mathcal{F}'_\sigma = 2\Box\phi'_\sigma - 2(\partial\cdot\partial\cdot\phi_\sigma) + \partial_\sigma(\partial\cdot\phi'). \tag{4.41}$$

If we compare results (4.40) and (4.41) with (4.38) we see we can write it in the very elegant form

$$\mathcal{F}_{\mu\nu\sigma} - \tfrac{1}{2}(g\circ\mathcal{F}')_{\mu\nu\sigma} = 0, \tag{4.42}$$

which resembles the linearized Einstein's equations. We can also trace (symmetrically) equation (4.42), so that (4.42) can be written as

$$\mathcal{F}_{\mu\nu\sigma} = 0. \tag{4.43}$$

The equations of motion (4.43) are gauge invariant under transformation (4.46), as long as $\varepsilon' = 0$. We can see this explicitly by doing

$$\begin{aligned}\delta\mathcal{F}_{\mu\nu\sigma} &= \Box\Big[\partial_\mu\varepsilon_{\nu\sigma} + \text{perm.}'\Big] - \Big[\partial_\mu(\partial\cdot\varepsilon_{\nu\sigma}) + \text{perm.}'\Big] + \Big[\partial_\mu\partial_\nu g^{\alpha\beta}\partial_\alpha\varepsilon_{\beta\sigma} + \text{perm}'\Big]\\ &= 3\partial_\mu\partial_\nu\partial_\sigma\varepsilon' = 0.\end{aligned} \tag{4.44}$$

4.1.5 The Spin-4 Action

The action for a spin-4 particle can be obtained by setting $s = 4$ in (4.1)

$$S_{\pm 4} = \int d^dx\Big[\tfrac{1}{2}(\partial_\alpha\phi_{\mu\nu\rho\sigma})^2 - 2(\partial\cdot\phi_{\mu\nu\rho})^2 - 6\phi'_{\mu\nu}(\partial\cdot\partial\cdot\phi^{\mu\nu}) - 3(\partial_\alpha\phi'_{\mu\nu})^2 - 3(\partial\cdot\phi'_\mu)^2\Big]. \tag{4.45}$$

This action is invariant under the gauge transformation

$$\delta\phi_{\mu\nu\rho\sigma} = 3\left[\partial_\mu\varepsilon_{\nu\rho\sigma} + \partial_\nu\varepsilon_{\mu\rho\sigma} + \partial_\rho\varepsilon_{\mu\nu\sigma} + \partial_\sigma\varepsilon_{\mu\nu\rho}\right]. \tag{4.46}$$

To check that the action is gauge invariant, we will abuse notation and gather all permutations of the terms as a single term multiplied by the total amount of permutations. This will make the notation less clustered, but could be confusing[4]. We will return to the proper notation once we write down the equations of motion. We then have

$$\begin{aligned}
\delta S_{\pm 4} &= \int d^dx\Big[\partial_\alpha\phi_{\mu\nu\rho\sigma}\partial^\alpha\delta\phi^{\mu\nu\rho\sigma} - 4\partial^\alpha\phi_{\alpha\nu\rho\sigma}\partial_\mu\delta\phi^{\mu\nu\rho\sigma} - 6\phi'_{\mu\nu}(\partial\cdot\partial\cdot\delta\phi^{\mu\nu}) \\
&\quad - 6\delta\phi'_{\mu\nu}(\partial\cdot\partial\cdot\phi^{\mu\nu}) - 6\partial_\alpha\phi'_{\mu\nu}\partial^\alpha\delta\phi'^{\mu\nu} - 6(\partial\cdot\phi'_\mu)(\partial\cdot\delta\phi'^\mu)\Big] \\
&= 3\int d^dx\Big[\underbrace{4\partial_\alpha\phi_{\mu\nu\rho\sigma}\partial^\alpha\partial^\mu\varepsilon^{\nu\rho\sigma}}_{4(\partial\cdot\phi_{\mu\nu\rho})\Box\varepsilon^{\mu\nu\rho}}\underbrace{-4(\partial\cdot\phi_{\mu\nu\rho})\Big(\Box\varepsilon^{\mu\nu\rho} + 3\partial^\mu(\partial\cdot\varepsilon^{\nu\rho})\Big)}_{-4(\partial\cdot\phi_{\mu\nu\rho})\Box\varepsilon^{\mu\nu\rho}+12(\partial\cdot\partial\cdot\phi_{\mu\nu})(\partial\cdot\varepsilon^{\mu\nu})} \\
&\quad \underbrace{-6\phi'_{\mu\nu}\Big(2\Box(\partial\cdot\varepsilon^\mu) + 2\partial^\mu(\partial\cdot\partial\cdot\varepsilon^\nu)\Big)}_{-12\phi'_{\mu\nu}\Box(\partial\cdot\varepsilon^{\mu\nu})+12(\partial\cdot\phi'_{\mu\nu})(\partial\cdot\partial\cdot\varepsilon^{\mu\nu})} - 12(\partial\cdot\partial\cdot\phi_{\mu\nu})(\partial\cdot\varepsilon^{\mu\nu}) \\
&\quad \underbrace{-12\partial_\alpha\phi'_{\mu\nu}\partial^\alpha(\partial\cdot\varepsilon^{\mu\nu})}_{12\phi'_{\mu\nu}\Box(\partial\cdot\varepsilon^{\mu\nu})} - 12(\partial\cdot\phi'_\mu)(\partial\cdot\partial\cdot\varepsilon^\mu)\Big] \Rightarrow \delta S_{\pm 4} = 0.
\end{aligned} \tag{4.47}$$

The equations of motion obtained by varying ϕ in (4.45) are obtained by looking at (4.47). We get

$$\begin{aligned}
&\Box\phi_{\mu\nu\rho\sigma} - \left[\partial_\mu(\partial\cdot\phi_{\nu\rho\sigma}) + \text{perm.}'\right] + \Big[\partial_\mu\partial_\nu\phi'_{\rho\sigma} + \text{perm.}'\Big] + \left[g_{\mu\nu}(\partial\cdot\partial\cdot\phi_{\rho\sigma}) + \text{perm.}'\right] \\
&\quad - \frac{1}{2}\left[g_{\mu\nu}\partial_\rho(\partial\cdot\phi'_\sigma) + \text{perm.}'\right] - \Big[g_{\mu\nu}\Box\phi'_{\rho\sigma} + \text{perm.}'\Big] = 0
\end{aligned} \tag{4.48}$$

To write them in a way that resembles Einstein's equations, we define

$$\mathcal{F}_{\mu\nu\rho\sigma} = \Box\phi_{\mu\nu\rho\sigma} - \left[\partial_\mu(\partial\cdot\phi_{\nu\rho\sigma}) + \text{perm.}'\right] + \Big[\partial_\mu\partial_\nu\phi'_{\rho\sigma} + \text{perm.}'\Big], \tag{4.49}$$

which has trace

[4]The main example is the third term in the action (4.45). Once you write down $\phi'_{\sigma\rho} = g^{\mu\nu}\phi_{\mu\nu\sigma\rho}$, you can see that there are twelve possible inequivalent permutations of the indices, but the term has an overall factor of $\frac{1}{2}$, which hides this fact.

$$\mathcal{F}'_{\mu\nu} = -\Big[2(\partial\cdot\partial\cdot\phi_{\mu\nu}) - \partial_\mu(\partial\cdot\phi'_\nu) - \partial_\nu(\partial\cdot\phi'_\mu) - 2\Box\phi'_{\mu\nu}\Big]. \tag{4.50}$$

To arrive at result (4.50), we must use the property $\phi'' = 0$. Still, this is just a way of rewriting the equations of motion and should not necessarily have any physical importance. We will discuss the double-traceless condition of our field in the next section. Thus, we can write down equation (4.48) as

$$\mathcal{F}_{\mu\nu\rho\sigma} - \tfrac{1}{2}(g\circ\mathcal{F}')_{\mu\nu\rho\sigma} = 0, \tag{4.51}$$

where

$$\begin{aligned}\tfrac{1}{2}(g\circ\mathcal{F}')_{\mu\nu\rho\sigma} = &-\Big[g_{\mu\nu}(\partial\cdot\partial\cdot\phi_{\rho\sigma}) + \text{perm.}'\Big] + \frac{1}{2}\Big[g_{\mu\nu}\partial_\rho(\partial\cdot\phi'_\sigma) + \text{perm.}'\Big] \\ &+ \Big[g_{\mu\nu}\Box\phi'_{\rho\sigma} + \text{perm.}'\Big].\end{aligned} \tag{4.52}$$

As before, we can trace out equation (4.51) symmetrically, thus reaching

$$\mathcal{F}_{\mu\nu\rho\sigma} = 0. \tag{4.53}$$

$$\tag{4.54}$$

The Double-Traceless Condition

Up to spin-3, the double-traceless condition upon our tensorial field ϕ was not used. In fact, we also did not need the condition to find the equations of motion (4.48) for our spin-4 action. Why do we require a double-traceless condition?

Remember the linearized Einstein's equations, (4.32). That equation can be written as the Einstein tensor, $E_{\mu\nu}$, as

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{4.55}$$

The Einstein tensor obeys the following property

$$\partial\cdot E_\nu = 0, \tag{4.56}$$

known as the *Bianchi identity*. Since we are attempting to write every equation of

motion in a form that resembles the linearized Einstein's equation, we could imagine a *generalized Einstein tensor*, which would obey a *generalized Bianchi identity*. In fact, the generalized Einstein tensor for the spin-3 particle, given by (4.42), obeys a generalized Bianchi identity

$$\partial \cdot \mathcal{E}_{\mu\nu} = \partial^{\mu} \left[\mathcal{F}_{\mu\nu\sigma} - \tfrac{1}{2}(g \circ \mathcal{F}')_{\mu\nu\sigma} \right] = 0. \tag{4.57}$$

However, consider equation (4.50) without the use of the double-traceless condition. We would have

$$\widetilde{\mathcal{F}'}_{\mu\nu} = -2\Big[(\partial \cdot \partial \cdot \phi_{\mu\nu}) - \partial_{\mu}(\partial \cdot \phi'_{\nu}) - \Box\phi'_{\mu\nu}\Big] + \partial_{\mu}\partial_{\nu}\phi'' = \mathcal{F}'_{\mu\nu} - \partial_{\mu}\partial_{\nu}\phi''. \tag{4.58}$$

If we then built the equations of motion similarly to (4.51), but with $\widetilde{\mathcal{F}}$ instead of $\mathcal{F}$, we would get

$$\widetilde{\mathcal{E}}_{\mu\nu\rho\sigma} \equiv \mathcal{F}_{\mu\nu\rho\sigma} - \tfrac{1}{2}(g \circ \mathcal{F}')_{\mu\nu\rho\sigma} - \tfrac{1}{4}\left[g_{\mu\nu}\partial_{\rho}\partial_{\sigma}\phi'' + \text{perm.}'\right] = 0. \tag{4.59}$$

This "modified" Einstein tensor does not satisfy a generalized form of the Bianchi identity, since we would have

$$\partial \cdot \widetilde{\mathcal{E}}^{(4)} \sim \partial\partial\partial\phi'', \tag{4.60}$$

where the (4) indicates the tensor's rank. This identity is satisfied when $\phi'' = 0$. If we did not choose it to be this way, the generalized Einstein tensors of rank-4 and above would always contain terms with two or more traces of our field ϕ, which would not vanish. Thus the double-traceless condition is required in order for the generalized versions of the Einstein tensors of rank-4 and above to satisfy the generalized versions of the Bianchi identity. A physical interpretation to this is that we are eliminating the propagation of undesired degrees of freedom[5].

Thus, for the case of spin-4, if we define the correct generalized Einstein tensor

$$\mathcal{E}_{\mu\nu\rho\sigma} \equiv \mathcal{F}_{\mu\nu\rho\sigma} - \tfrac{1}{2}(g \circ \mathcal{F}')_{\mu\nu\rho\sigma}, \tag{4.61}$$

it satisfies the generalized Bianchi identity

[5]For example, the double trace of our spin-4 field would be $\phi'' = g^{\mu\nu}g^{\rho\sigma}\phi_{\mu\nu\rho\sigma}$, which satisfies the Klein-Gordon equation (4.5) and is, therefore, a spin-0 field.

$$\partial \cdot \mathcal{E}_{\nu\rho\sigma} = 0. \tag{4.62}$$

4.1.6 The General Case – Spin-s

Now that we have studied five specific cases explicitly, we are ready to study the general case for a massless Bosonic particle of spin-s. The results obtained in this section would have definitely simplified the discussions of the previous ones if we chose to put this section first, but the explicit analysis of the other cases can be very fruitful for the reader unfamiliar with the material.

For the general case, we take our starting point to be the action (4.1), repeated here for the sake of the reader

$$\begin{aligned} S_s = (-1)^s \int d^dx \Big[& \tfrac{1}{2}(\partial_\alpha\phi)^2 - \tfrac{s}{2}(\partial\cdot\phi)^2 - \tfrac{s(s-1)}{2}\phi'\cdot(\partial\cdot\partial\cdot\phi) - \tfrac{s(s-1)}{4}(\partial_\alpha\phi')^2 \\ & - \tfrac{s(s-1)(s-2)}{8}(\partial\cdot\phi')^2 - \phi\cdot J^{(s)}\Big]. \end{aligned} \tag{4.63}$$

Our discussion from the specific cases already allows us to impose the double-traceless condition upon our field. Furthermore, we want to obtain equations of motion that can lead to the results (4.5), (4.9), (4.20), (4.42), and (4.51), for particles of spin 0, 1, 2, 3, and 4, respectively, once we specify one of these spins for our particle.

A variation of our action with respect to ϕ gives[6]

$$\begin{aligned} 0 = \delta S_s = (-1)^s \int d^dx \Big[& -(\Box\phi) + \partial\circ\partial\cdot\phi - \partial\circ\partial\circ\phi' - g\circ(\partial\cdot\partial\cdot\phi) \\ & + g\circ(\Box\phi') + \frac{1}{2}g\circ(\partial\circ\partial\cdot\phi')\Big]\cdot\delta\phi, \end{aligned} \tag{4.64}$$

so that the equations of motion can be written as

$$(\Box\phi) - \partial\circ\partial\cdot\phi + \partial\circ\partial\circ\phi' + g\circ(\partial\cdot\partial\cdot\phi) - g\circ(\Box\phi') - \frac{1}{2}g\circ(\partial\circ\partial\cdot\phi') = 0. \tag{4.65}$$

[6] We will write one term explicitly for the sake of the reader. The variation of the second term in (4.63) gives (omitting the integral) $-s(\partial\cdot\phi)(\partial\cdot\delta\phi)$. Upon integration by parts, we arrive at (dropping the surface term) $s(\partial_{\mu_1}\partial\cdot\phi_{\mu_2\cdots\mu_s})\delta\phi^{\mu_1\cdots\mu_s}$. Because the only permutations that are inequivalent are those where we exchange the index of the derivative with the $(s-1)$ indices of $\partial\cdot\phi$, we see that there are precisely s inequivalent permutations for this term. Thus we can cast the result in the form $(\partial\circ\partial\cdot\phi)\cdot\delta\phi$, which is the second term in the second equality of (4.64).

Now we wish to construct a tensor $\mathcal{F}^{(s)}$, in analogy with what we have done in the previous sections. Since we know the structure of the equations of motions from the previous section, we know that a good choice for $\mathcal{F}^{(s)}$ would be

$$\mathcal{F}^{(s)} \stackrel{?}{=} (\Box\phi) - \partial \circ \partial \cdot \phi + \partial \circ \partial \circ \phi', \tag{4.66}$$

but we have to check if this is a consistent choice. Taking the trace of (4.66) gives

$$\mathcal{F}'^{\ (s-2)} \equiv g \cdot \mathcal{F}^{(s)} = -2\left(\partial \cdot \partial \cdot \phi - \Box\phi'\right) + \partial \circ \partial \cdot \phi', \tag{4.67}$$

so that our equations of motion (4.65) can be written precisely in the form we wanted

$$\mathcal{F}^{(s)} - \frac{1}{2} g \circ \mathcal{F}'^{\ (s-2)} = 0, \tag{4.68}$$

thus allowing us to conclude that (4.66) is indeed the correct form of $\mathcal{F}^{(s)}$[7]. To compare our result with the literature, we can take the trace of equation (4.68) to find that

$$\mathcal{F}'^{\ (s-2)} = 0 \Rightarrow \mathcal{F}^{(s)} = (\Box\phi) - \partial \circ \partial \cdot \phi + \partial \circ \partial \circ \phi' = 0, \tag{4.69}$$

which agrees with result (2.4) of [12]. Now we have to check the gauge invariance of our action under the general transformation

$$\delta\phi^{(s)} = \partial \circ \varepsilon^{(s-1)}, \tag{4.70}$$

where the gauge parameter $\varepsilon^{(s-1)}$ is a traceless rank-$(s-1)$ tensor. This is a rather straightforward calculation but one must take care not to get confused with notation. For example, remember the following

$$[\partial \circ \partial \cdot \phi]_{\mu_1 \cdots \mu_s} \equiv \underbrace{[\partial_{\mu_1} \partial \cdot \phi_{\mu_2 \cdots \mu_s} + \text{perm.}']}_{s \text{ ineq. perm.}} \stackrel{!}{=} s \partial_{\mu_1} \partial \cdot \phi_{\mu_2 \cdots \mu_s}, \tag{4.71}$$

where we have used the notation $\stackrel{!}{=}$ because this is not strictly true, since the terms do not have identical properties. However, not using this trick can make the computation extremely long. We can then write (4.64)[8] as

[7] At this point, one can check explicitly that the equations of motion obtained in this section reduces to the equations of motion in the previous sections.

[8] Of course, since we are now specifying the form of our variation, we do not intend to force $\delta S_s = 0$. Instead we want to find this equality in order to show that the action is gauge invariant.

$$
\begin{aligned}
\delta S_s &= (-1)^s \int d^dx \Big[-(\Box\phi) + \partial\circ\partial\cdot\phi - \partial\circ\partial\circ\phi' - g\circ(\partial\cdot\partial\cdot\phi) \\
&\quad + g\circ(\Box\phi') + \frac{1}{2} g\circ(\partial\circ\partial\cdot\phi')\Big]\cdot(\partial\circ\varepsilon) \\
&= (-1)^s \int d^dx \Big\{ \Big[s\Box\partial\cdot\phi_{\mu_2\cdots\mu_s} - s\Box\partial\cdot\phi_{\mu_2\cdots\mu_s} - s(s-1)\partial_{\mu_2}\partial\cdot\partial\cdot\phi_{\mu_3\cdots\mu_s} \\
&\quad + 2\frac{s(s-1)}{2}\partial_{\mu_2}\partial\cdot\partial\cdot\phi_{\mu_3\cdots\mu_s} + \frac{s(s-1)(s-2)}{2}\partial_{\mu_2}\partial_{\mu_3}\partial\cdot\phi'_{\mu_4\cdots\mu_s}\Big]\varepsilon^{\mu_2\cdots\mu_s} \\
&\quad + \Big[-s(s-1)\partial\cdot\partial\cdot\phi_{\mu_3\cdots\mu_s} + s(s-1)\Box\phi'_{\mu_3\cdots\mu_s} + \frac{s(s-1)(s-2)}{2}\partial_{\mu_3}\partial\cdot\phi'_{\mu_4\cdots\mu_s}\Big]\partial\cdot\varepsilon^{\mu_3\cdots\mu_s}\Big]\Big\} \\
&= 0.
\end{aligned}
\tag{4.72}
$$

For the second equality to be achieved, one must realize that

$$
s g_{\mu_1\mu_2}\partial^{\mu_1}\varepsilon^{\mu_2\cdots\mu_s} = g_{\mu_1\mu_2}\left[\partial^{\mu_1}\varepsilon^{\mu_2\cdots\mu_s} + \text{perm.}'\right] = 2\partial\cdot\varepsilon^{\mu_3\cdots\mu_s} + \cancelto{0}{(s-2)\partial^{\mu_3}\varepsilon'^{\mu_4\cdots\mu_s}}, \tag{4.73}
$$

where we used the traceless property of ε. Similar manipulations are also required for the integrations by parts, for example (omitting the integral)

$$
\begin{aligned}
-\Big[\frac{s(s-1)}{2}\partial_{\mu_1}\partial_{\mu_2}\phi'_{\mu_3\cdots\mu_s}\Big] s\partial^{\mu_1}\varepsilon^{\mu_2\cdots\mu_s} &\to 2\frac{s(s-1)}{2}\partial_{\mu_2}\Box\phi'_{\mu_3\cdots\mu_s}\epsilon^{\mu_2\cdots\mu_s} \\
&\quad + (s-2)\frac{s(s-1)}{2}\partial_{\mu_2}\partial_{\mu_3}\partial\cdot\phi'_{\mu_4\cdots\mu_s}\epsilon^{\mu_2\cdots\mu_s},
\end{aligned}
\tag{4.74}
$$

where the arrow in (4.74) corresponds to an integration by parts. Thus, result (4.72) shows that our action (4.63) is invariant under the gauge transformations (4.70).

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Chapter 5

A Continuous Spin Particle Gauge Field Theory

5.1 The Action for a Single CSP

In 2014, Schuster and Toro proposed an action that describes a single free CSP particle [12]. The particle is described through the usual spacetime coordinates x^μ *and* an additional four-vector coordinate η^μ. The CSP field, $\Psi(x,\eta)$, is a scalar field, analytic in η. The action is given by [12]

$$S = \frac{1}{2}\int d^4x\, d^4\eta \left\{\delta'(\eta^2+1)\left[\partial_x\Psi(\eta,x)\right]^2 + \frac{1}{2}\delta(\eta^2+1)\left[\Delta\Psi(\eta,x)\right]^2\right\}, \tag{5.1}$$

where δ' indicates the derivative of the Dirac's delta function with respect to its argument, $\Delta \equiv \partial_\eta \cdot \partial_x + \rho$, and ρ is the particle's *continuous spin*. The action is invariant under the gauge transformation [12]

$$\delta\Psi(\eta,x) = \left[\eta\cdot\partial_x - \frac{1}{2}(\eta^2+1)\Delta\right]\epsilon(\eta,x) + \frac{1}{4}(\eta^2+1)^2\chi(\eta,x) \equiv \delta_\epsilon\Psi(\eta,x) + \delta_\chi\Psi(\eta,x), \tag{5.2}$$

for arbitrary gauge parameter $\epsilon(\eta,x)$ and $\chi(\eta,x)$. To check this gauge invariance explicitly, we will first consider a χ-transformation and then a ϵ-transformation, verifying that the action (5.1) is invariant under each of these transformations. We start by performing an infinitesimal variation of our field Ψ of the form $\Psi \to \Psi + \delta\Psi$ in our action (5.1), which leads to a variation of the action given by

$$\delta S = \int d^4x\ d^4\eta \left\{ \delta'(\eta^2+1)\,(\partial_x\Psi)\cdot(\partial_x\delta\Psi) + \frac{1}{2}\delta(\eta^2+1)\Delta\Psi\Delta\delta\Psi \right\}. \tag{5.3}$$

Then, when we choose a χ-transformation, we are setting $\delta\Psi = \delta_\chi\Psi$ in (5.3), giving

$$\delta_\chi S = \frac{1}{4}\int d^4x\ d^4\eta \left\{ \delta'(\eta^2+1)\,(\partial_x\Psi)\cdot\left[(\eta^2+1)^2\partial_x\chi\right] + \frac{1}{2}\delta(\eta^2+1)\Delta\Psi\Delta\left[(\eta^2+1)^2\chi\right] \right\}. \tag{5.4}$$

Now we make use of the Dirac's delta function identities $x\delta(x) = 0$ and $x^2\delta'(x) = 0$ so that

$$\delta_\chi S = \frac{1}{4}\int d^4x\ d^4\eta \left\{ \frac{1}{2}\delta(\eta^2+1)\Delta\Psi\left[2(\eta^2+1)\eta\cdot\partial_x\chi + (\eta^2+1)^2\Delta\chi\right] \right\} = 0, \tag{5.5}$$

and thus the action (5.1) is invariant under χ-transformations. In the same way, performing a ϵ-transformation means setting $\delta\Psi = \delta_\epsilon\Psi$ in (5.3), giving

$$\begin{aligned} \delta_\epsilon S = \int d^4x\ d^4\eta \Bigg\{ & \delta'(\eta^2+1)\,(\partial_x\Psi)\cdot\partial_x\left[\eta\cdot\partial_x - \frac{1}{2}(\eta^2+1)\Delta\right]\epsilon \\ & + \frac{1}{2}\delta(\eta^2+1)\Delta\Psi\Delta\left[\eta\cdot\partial_x - \frac{1}{2}(\eta^2+1)\Delta\right]\epsilon \Bigg\} \end{aligned}. \tag{5.6}$$

To continue, we must make use of the identity $x\delta'(x) = -\delta(x)$ and perform a few integrations by parts[1], so that

$$\begin{aligned} \delta_\epsilon S = & -\int d^4x\ d^4\eta\ \delta'(\eta^2+1)\Psi\left[\eta\cdot\partial_x - \frac{1}{2}(\eta^2+1)\Delta\right]\Box_x\epsilon \\ & + \frac{1}{2}\int d^4x\ d^4\eta\ \Psi\Delta\left[\delta(\eta^2+1)\Box_x\epsilon\right] \\ = & \int d^4x\ d^4\eta\ \Psi\Big\{ -\delta'(\eta^2+1)\eta\cdot\partial_x\Box_x\epsilon - \frac{1}{2}\delta(\eta^2+1)\Delta\Box_x\epsilon + \delta'(\eta^2+1)\eta\cdot\partial_x\Box_x\epsilon \\ & + \frac{1}{2}\delta(\eta^2+1)\Delta\Box_x\epsilon \Big\} = 0, \end{aligned} \tag{5.7}$$

where $\Box_x \equiv \partial_x\cdot\partial_x$. Thus, we have $\delta S = \delta_\epsilon S + \delta_\chi S = 0$ and the action (5.1) is invariant

[1] In this chapter we will also consider that all the surface terms that arise from these manipulations vanish.

under the gauge transformation (5.2).

The equations of motion for our field Ψ can easily be obtained through (5.3) after integrating both terms by parts, and are given by

$$\delta'(\eta^2+1)\Box_x\Psi - \frac{1}{2}\Delta\left[\delta(\eta^2+1)\Delta\Psi\right] = 0, \tag{5.8}$$

in agreement with result (4.4) of [12]. The equation of motion (5.8) are trivially invariant under χ-transformations upon usage of the Dirac's delta function identities we have presented in this discussion. They are also invariant under ϵ-transformations, as can be seen by doing a variation $\Psi \to \Psi + \delta_\epsilon\Psi$ in (5.8)

$$\delta'(\eta^2+1)\left[\eta\cdot\partial_x - \frac{1}{2}(\eta^2+1)\Delta\right]\Box_x\epsilon - \frac{1}{2}\Delta\left\{\delta(\eta^2+1)\Delta\left[\left(\eta\cdot\partial_x - \frac{1}{2}(\eta^2+1)\Delta\right)\epsilon\right]\right\}. \tag{5.9}$$

It is easy to notice that (5.9) vanishes upon realizing that the second term can be written, using $\delta(x) = -x\delta'(x)$, as

$$-\frac{1}{2}\left\{\delta'(\eta^2+1)2\eta\cdot\partial_x\Box_x\epsilon + \delta(\eta^2+1)\Delta\Box_x\epsilon\right\}, \tag{5.10}$$

which cancels the first term in (5.9).

We could continue our discussion using action (5.4) and check that it indeed describes a single CSP degree of freedom. However, in 2015, Rivelles [13] proposed an expansion of our field Ψ which uses two scalar fields, ψ_0 and ψ_1, to describe our CSP instead of one scalar field. The action obtained through this expansion contains a simpler local symmetry. We explore these features in the next section and move on to the physical contents of our theory through these new approach.

5.2 Reducibility of the Local Transformations

When we look at action (5.1), we can notice that the role played by the Dirac's delta function is that of making the equation of motion (5.8) non-trivial in the hyperboloid $\eta^2+1=0$. We can then perform the following expansion for our field Ψ [13]

$$\Psi(\eta,x) = \sum_{n=0}^{\infty}\frac{1}{n!}(\eta^2+1)^n\psi_n(\eta,x), \tag{5.11}$$

where ψ_n are also scalar fields that can be written as [13]

$$\psi_n(\eta, x) = \sum_{s=0}^{\infty} \frac{1}{s!} \eta^{\mu_1} \dots \eta^{\mu_s} \psi^{(n,s)}_{\mu_1 \dots \mu_s}(x). \tag{5.12}$$

In (5.12), $\psi^{(n,s)}_{\mu_1 \dots \mu_s}(x)$ are arbitrary completely symmetric tensor fields that depend only on our spacetime coordinates x^μ [13]. Although the choices (5.11) and (5.12) are not unique, they are quite natural given the structure of our action, and they also lead to a new transformation, a Ξ-transformation, defined as

$$\delta\psi_n(\eta, x) = \sum_{p=1}^{\infty} \frac{n!}{(n+p)!} (\eta^2 + 1)^p \Xi_{n,n+p}(\eta, x) - \sum_{p=0}^{n-1} \Xi_{n,p}(\eta, x), \tag{5.13}$$

which leaves Ψ, as written in (5.11), invariant. This is *not* a gauge transformation, as we are not able to remove any degrees of freedom from it. To check that Ψ is left invariant, we can explicitly perform the substitution (5.13) in (5.11), giving

$$\begin{aligned}
\delta\Psi &= \sum_{n=0}^{\infty} \frac{1}{n!} (\eta^2 + 1)^n \delta\psi_n \\
&= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\eta^2 + 1)^{n+p}}{(n+p)!} \Xi_{n,n+p} - \sum_{p=0}^{n-1} \sum_{n=1}^{\infty} \frac{(\eta^2 + 1)^n}{n!} \Xi_{n,p} \\
&= \left[(\eta^2 + 1)\Xi_{0,1} + \frac{(\eta^2 + 1)^2}{2!} (\Xi_{0,2} + \Xi_{1,2}) + \frac{(\eta^2 + 1)^3}{3!} (\Xi_{0,3} + \Xi_{1,3} + \Xi_{2,3}) + \cdots \right] \\
&- \left[(\eta^2 + 1)\Xi_{1,0} + \frac{(\eta^2 + 1)^2}{2!} (\Xi_{2,0} + \Xi_{2,1}) + \frac{(\eta^2 + 1)^3}{3!} (\Xi_{3,0} + \Xi_{3,1} + \Xi_{3,2}) + \cdots \right],
\end{aligned} \tag{5.14}$$

where the first and second terms in square brackets of the third equality are the explicit forms of the first and second double sums present in the second equality. This means that, for $\delta\Psi = 0$, we must have $\Xi_{a,b} = \Xi_{b,a}$, which is precisely the condition presented in [13].

Now, notice that our gauge transformations in (5.2) are themselves invariant under the local transformations [13]

$$\delta\epsilon = \frac{1}{2}(\eta^2 + 1)\Lambda(\eta, x), \tag{5.15}$$

$$\delta\chi = \Delta\Lambda(\eta, x), \tag{5.16}$$

where $\Lambda(\eta, x)$ is a new local transformation parameter. This is precisely the definition

of *reducibility*, so that we can say that the gauge transformations (5.2) are *reducible*. To verify this gauge invariance, we perform the gauge transformations (5.15) and (5.16) in (5.2), getting

$$\begin{aligned}\left[\eta\cdot\partial_x-\frac{1}{2}(\eta^2+1)\Delta\right]\delta\epsilon+\frac{1}{4}(\eta^2+1)^2\delta\chi &= \frac{1}{2}\left[\eta\cdot\partial_x-\frac{1}{2}(\eta^2+1)\Delta\right](\eta^2+1)\Lambda+\frac{1}{4}(\eta^2+1)^2\Lambda\\ &= \frac{1}{2}\left[(\eta^2+1)-(\eta^2+1)\right]\eta\cdot\partial_x\Lambda\\ &\quad-\frac{1}{4}(\eta^2+1)^2\Lambda+\frac{1}{4}(\eta^2+1)^2\Lambda\\ &= 0.\end{aligned} \tag{5.17}$$

If we now expand $\epsilon(\eta,x)$ and $\chi(\eta,x)$ in analogy to the way we have expanded our field Ψ in (5.11), we find that the gauge transformations (5.2) can be written as

$$\begin{aligned}\delta\Psi &= \sum_{n=0}^{\infty}\frac{(\eta^2+1)^n}{n!}\eta\cdot\partial_x\epsilon_n-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(\eta^2+1)}{n!}\left[n(\eta^2+1)^{n-1}2\eta\cdot\partial_x\epsilon_n+(\eta^2+1)^n\Delta\epsilon_n\right]\\ &+\frac{1}{4}\sum_{n=0}^{\infty}\frac{(\eta^2+1)^{n+2}}{n!}\chi_n\\ &= \sum_{n=0}^{\infty}\frac{(\eta^2+1)^n}{n!}\left[(1-n)\eta\cdot\partial_x\epsilon_n-\frac{1}{2}n\Delta\epsilon_{n-1}+\frac{1}{4}n(n-1)\chi_{n-2}\right].\end{aligned} \tag{5.18}$$

Comparing (5.18) with (5.11) allows us to conclude that

$$\delta\psi_n=(1-n)\eta\cdot\partial_x\epsilon_n-\frac{1}{2}n\Delta\epsilon_{n-1}+\frac{1}{4}n(n-1)\chi_{n-2}, \tag{5.19}$$

which is in agreement with result (8) from [13]. We can use the same expansions in (5.15) and (5.16), which, in an analogous manner, allow us to conclude that

$$\delta\epsilon_n=\frac{1}{2}n\Lambda_{n-1}, \tag{5.20}$$

$$\delta\chi_n=2\eta\cdot\partial_x\Lambda_{n+1}+\Delta\Lambda_n. \tag{5.21}$$

First, notice that $\delta\epsilon_0=0$, using (5.20) with an arbitrary Λ. Now we can use our Λ gauge symmetry to choose $\Lambda_{n-1}=-(2/n)\epsilon_n$, for $n\neq 0$ [13], which upon substitution

in (5.20) yields

$$\delta\epsilon_n = \frac{n}{2}\left(-\frac{2}{n}\epsilon_n\right) = -\epsilon_n,\ n \neq 0. \tag{5.22}$$

This means we can gauge away every ϵ_n with $n \neq 0$. Then we can use our χ gauge symmetry to choose $\chi_{n-2} = -\frac{4}{n(n-1)}\psi_n$, $n \geq 2$, which gauges away every $\delta\psi_n$ in (5.19), except for $\delta\psi_0$ and $\delta\psi_1$ [13]. This means that all our transformations are reduced to

$$\delta\psi_0 = \eta \cdot \partial_x \epsilon_0, \tag{5.23}$$

$$\delta\psi_1 = -\frac{1}{2}\Delta\epsilon_0, \tag{5.24}$$

$$\delta\epsilon_0 = 0. \tag{5.25}$$

Since the gauge parameter ϵ_0 does not transform, as can be seen from (5.25), we now say that the gauge transformations are *irreducible*.

5.3 The CSP Action in Terms of ψ_0 and ψ_1

Now that we have studied the consequences to our gauge transformations when making the expansion (5.11), we will study the consequences of this expansion in our action (5.1). Remember the identities $x\delta(x) = 0$, $x\delta'(x) = -\delta(x)$ and $x^2\delta'(x) = 0$, as they are all used when attempting to write the action in terms of the fields ψ_0 and ψ_1. Our analysis of the reducibility of the gauge transformations allowed us to notice that these are the relevant fields for our study of the CSP action.

When we substitute (5.11) in (5.1), the Dirac's delta function in the first term of the action will eliminate all terms of the expansion except for those that contain ψ_0 and ψ_1, while the Dirac's delta function of the second term of the action will eliminate all terms of the expansion except for those that contain ψ_0. We are then left with

$$S = \frac{1}{2}\int d^4x\, d^4\eta \left\{\delta'(\eta^2+1)(\partial_x\psi_0)^2 + \frac{1}{2}\delta(\eta^2+1)\left[(\Delta\psi_0 + 2\eta\cdot\partial_x\psi_1)^2 - 4(\partial_x\psi_0)\cdot(\partial_x\psi_1)\right]\right\}. \tag{5.26}$$

which is the action (14), proposed by Rivelles in [13]. Even though we have fixed the Λ and χ symmetries in the previous section, the action (5.26) does not allow any terms ψ_n with $n \geq 2$, as already explained.

Now we will explore *all* the local symmetries of our action (5.26). At first we see that our action is invariant under the transformations [13]

$$\delta\psi_0 = \eta \cdot \partial_x \epsilon_0 + (\eta^2+1)^2 \chi_0 + (\eta^2+1)\Xi, \tag{5.27}$$

$$\delta\psi_1 = -\frac{1}{2}\Delta\epsilon_0 + (\eta^2+1)\chi_1 - \Xi, \tag{5.28}$$

where we have defined $\Xi \equiv \Xi_{0,1} = \Xi_{1,0}$, as found in (5.13). These transformations are also invariant under the gauge transformations [13]

$$\delta\Xi = (\eta^2+1)\theta(\eta, x), \tag{5.29}$$

$$\delta\chi_0 = -\theta(\eta, x), \tag{5.30}$$

$$\delta\chi_1 = \theta(\eta, x), \tag{5.31}$$

for a new local parameter $\theta(\eta, x)$. This indicates that transformations (5.27) and (5.28) are reducible. Checking the invariance of (5.27) and (5.28) under the transformations (5.29), (5.30), and (5.31) is extremely simple. We have

$$(\eta^2+1)^2(-\theta) + (\eta^2+1)^2\theta = 0, \tag{5.32}$$

and

$$(\eta^2+1)\theta - (\eta^2+1)\theta = 0, \tag{5.33}$$

respectively.

Since these transformations act only upon χ_0, χ_1, and Ξ, it means that (at least) one of these parameters can be removed. To understand the implications of the reducibility of these transformations, we first find the equations of motion obtained by varying ψ_0 and ψ_1 in (5.26). Varying the action with respect to ψ_0 gives

$$\begin{aligned}\delta S = \int d^4x\, d^4\eta \Big\{ &\delta'(\eta^2+1)(\partial_x\psi_0)\cdot(\partial_x\delta\psi_0) + \frac{1}{2}\delta(\eta^2+1)\Big[\Delta\psi_0\Delta\delta\psi_0 \\ &+ 2\eta\cdot\partial_x\psi_1\Delta\delta\psi_0 - (\partial_x\psi_1)\cdot(\partial_x\delta\psi_0)\Big]\Big\}.\end{aligned} \tag{5.34}$$

Making use of the Dirac's delta function identities, integration by parts, and the prin-

ciple of least action, we get

$$\delta'(\eta^2+1)\left[\Box_x\psi_0 - \eta\cdot\partial_x\Delta\psi_0 - 2(\eta\cdot\partial_x)^2\psi_1\right] - 2\delta(\eta^2+1)\left[\Box_x\psi_1 + \frac{1}{2}\eta\cdot\partial_x\Delta\psi_1 + \frac{1}{4}\Delta^2\psi_0\right] = 0, \tag{5.35}$$

which is result (20) of [13]. Doing the same thing for a variation of ψ_1

$$\delta S = \int d^4x\ d^4\eta\ \delta(\eta^2+1)\Big[-(\partial_x\psi_0)\cdot(\partial_x\delta\psi_1) + \Delta\psi_0(\eta\cdot\partial_x\delta\psi_1) + 2(\eta\cdot\partial_x\psi_1)(\eta\cdot\partial_x\delta\psi_1)\Big], \tag{5.36}$$

resulting in the equation of motion

$$\delta(\eta^2+1)\left[\Box_x\psi_0 - \eta\cdot\partial_x\Delta\psi_0 - 2(\eta\cdot\partial_x)^2\psi_1\right] = 0, \tag{5.37}$$

which is result (21) of [13]. These equations of motion are not independent, since multiplying (5.35) by η^2+1 eliminates the second term of that result and the first term becomes (5.37) upon using a Dirac's delta function identity.

We are now ready to explore the reducibility of the transformations (5.27) and (5.28). First we note, as we mentioned before, that the equations of motion (5.35) and (5.37) are defined around the hyperboloid $\eta^2+1=0$. This will make our analysis very difficult because this is not a trivial constraint. Our plan is then to extend the validity of the equations of motion to all η-space and continue our analysis from there.

5.4 Reducibility Revisited

We start by calling the first term in square brackets in (5.35) of $A(\eta,x)$ and the second term in square brackets of $B(\eta,x)$. This means our equations of motion now become

$$\delta'(\eta^2+1)A(\eta,x) - 2\delta(\eta^2+1)B(\eta,x) = 0, \tag{5.38}$$

$$\delta(\eta^2+1)A(\eta,x) = 0. \tag{5.39}$$

We can see that A and B are not independent by considering

$$\begin{aligned}\Delta A &= \Delta\Box_x\Psi_0 - \Delta\Box_x\psi_0 - \eta\cdot\partial_x\Delta^2\psi_0 - 4(\eta\cdot\partial_x)\Box_x\psi_1 - 2(\eta\cdot\partial_x)^2\Delta\psi_1 \\ &= -\eta\cdot\partial_x\Delta^2\psi_0 - 4(\eta\cdot\partial_x)\Box_x\psi_1 - 2(\eta\cdot\partial_x)^2\Delta\psi_1,\end{aligned} \tag{5.40}$$

and

$$\eta\cdot\partial_x B = (\eta\cdot\partial_x)\Box_x\psi_1 + \frac{1}{2}(\eta\cdot\partial_x)^2\Delta\psi_1 + \frac{1}{4}\eta\cdot\partial_x\Delta^2\psi_0, \tag{5.41}$$

so that [13]

$$\Delta A(\eta, x) = -4\eta\cdot\partial_x B(\eta, x). \tag{5.42}$$

We now can notice that A and B are invariant under ϵ_0 transformations

$$\begin{aligned}\delta A &= \Box_x\eta\cdot\partial_x\epsilon_0 - \eta\cdot\partial_x\Delta\left(\eta\cdot\partial_x\epsilon_0\right) + (\eta\cdot\partial_x)^2\Delta\epsilon_0 \\ &= \eta\cdot\partial_x\Box_x\epsilon_0 - (\eta\cdot\partial_x)^2\Delta\epsilon_0 - \eta\cdot\partial_x\Box_x\epsilon_0 + (\eta\cdot\partial_x)^2\Delta\epsilon_0 \\ &= 0,\end{aligned} \tag{5.43}$$

and

$$\begin{aligned}\delta B &= -\frac{1}{2}\Delta\Box_x\epsilon_0 - \frac{1}{4}\eta\cdot\partial_x\Delta^2\epsilon_0 + \frac{1}{4}\Delta^2(\eta\cdot\partial_x\epsilon_0) \\ &= -\frac{1}{2}\Delta\Box_x\epsilon_0 - \frac{1}{4}\eta\cdot\partial_x\Delta^2\epsilon_0 + \frac{1}{2}\Delta\Box_x\epsilon_0 + \frac{1}{4}\eta\cdot\partial_x\Delta^2\epsilon_0 \\ &= 0,\end{aligned} \tag{5.44}$$

respectively. However, they are not invariant under χ_0, χ_1, and Ξ transformations. Under χ_0 and χ_1 transformations, A transforms as

$$\begin{aligned}\delta A &= (\eta^2+1)^2\Box_x\chi_0 - \eta\cdot\partial_x\Delta\left[(\eta^2+1)^2\chi_0\right] - 2(\eta^2+1)(\eta\cdot\partial_x)^2\chi_1 \\ &= (\eta^2+1)^2\left[\Box_x - \eta\cdot\partial_x\Delta\right]\chi_0 - 2(\eta^2+1)(\eta\cdot\partial_x)^2\left[2\chi_0+\chi_1\right],\end{aligned} \tag{5.45}$$

while under Ξ transformations it transforms as

$$\begin{aligned}\delta A &= (\eta^2+1)\Box_x\Xi - \eta\cdot\partial_x\Delta\left[(\eta^2+1)\Xi\right] + 2(\eta\cdot\partial_x)^2\Xi \\ &= (\eta^2+1)\left[\Box_x - \eta\cdot\partial_x\Delta\right]\Xi.\end{aligned} \tag{5.46}$$

This means that under the transformations (5.27) and (5.28), we get

$$\delta A = (\eta^2+1)\left[(\Box_x - \eta\cdot\partial_x\Delta)\,\Xi - 2(\eta\cdot\partial_x)^2\,(2\chi_0+\chi_1) + (\eta^2+1)\,(\Box_x - \eta\cdot\partial_x\Delta)\,\chi_0\right]. \tag{5.47}$$

The same can be done, analogously, for B, giving

$$\begin{aligned}\delta B = -\frac{1}{2}\,(\Box_x - \eta\cdot\Delta)\,\Xi + (\eta\cdot\partial_x)^2\,(2\chi_0+\chi_1) + (\eta^2+1)\Bigg[&(\Box_x + 2\eta\cdot\partial_x\Delta)\,\chi_0 \\ + \left(\Box_x + \frac{1}{2}\eta\cdot\partial_x\Delta\right)\chi_1 + \frac{1}{4}\Delta^2\Xi + \frac{1}{4}(\eta^2+1)\Delta^2\chi_0&\Bigg].\end{aligned} \tag{5.48}$$

These last two results are, however, invariant under θ transformations of the form (5.29)-(5.31), which we can check explicitly for (5.47)

$$\begin{aligned}(\eta^2+1)\Big[&(\eta^2+1)\Box_x\theta - (\eta^2+1)\eta\cdot\partial_x\Delta\theta - 2(\eta\cdot\partial_x)^2\theta + 4(\eta\cdot\partial_x)^2\theta - 2(\eta\cdot\partial_x)^2\theta \\ &- (\eta^2+1)\Box_x\theta + (\eta^2+1)\eta\cdot\partial_x\Delta\theta\Big] = 0,\end{aligned} \tag{5.49}$$

and for (5.48)

$$\begin{aligned}&(\eta^2+1)\left[\frac{1}{4}(\eta^2+1)\Delta^2\theta + \eta\cdot\partial_x\Delta\theta + \frac{1}{2}\Box_x\theta - \Box_x\theta - 2\eta\cdot\partial_x\Delta\theta + \Box_x\theta + \frac{1}{2}\eta\cdot\partial_x\Delta\theta - \frac{1}{4}(\eta^2+1)\Delta^2\theta\right] \\ &- \frac{1}{2}(\eta^2+1)\Box_x\theta + \frac{1}{2}\eta\cdot\partial_x\Delta\theta + (\eta\cdot\partial_x)^2\theta - (\eta\cdot\partial_x)^2\theta = 0.\end{aligned} \tag{5.50}$$

Now we will use our χ_0 and χ_1 symmetries to extend the validity of (5.39) to all of η-space. This means that even without the Dirac's delta function, we will have $A(\eta, x) = 0$ [13]. This imposes the condition

$$(\eta^2+1)^2\left[\Box_x-\eta\cdot\partial_x\Delta\right]\chi_0-2(\eta^2+1)(\eta\cdot\partial_x)^2\left[2\chi_0+\chi_1\right]=0, \tag{5.51}$$

which comes from (5.38) after setting $A(\eta,x)=0$. Using the θ symmetry, we will set $\chi_0=0$, which leads to $(\eta\cdot\partial_x)^2\chi_1=0$, using (5.51). Now, if we Fourier transform this last result to momentum space, we read it as $(\partial_\omega\cdot p)^2\tilde{\chi}_1=0$, where ω is the Fourier conjugate of our coordinate η. Since we do not want to constraint the momentum of our particle, we conclude that $\tilde{\chi}_1=0$. Now, if we remember result (5.42), setting $A=0$ leads to $\eta\cdot\partial_x B(\eta,x)=0$ which, by the same arguments given for χ_1, sets $B(\eta,x)=0$ [13].

In this way, we have explored some of the local symmetries of our Lagrangian in order to extend the equations of motion outside the hyperboloid $\eta^2+1=0$. We have that the equations of motion, extended to all of η-space, are

$$\Box_x\psi_0-\eta\cdot\partial_x\Delta\psi_0-2(\eta\cdot\partial_x)^2\psi_1=0, \tag{5.52}$$

$$\Box_x\psi_1+\frac{1}{2}\eta\cdot\partial_x\Delta\psi_1+\frac{1}{4}\Delta^2\psi_0=0, \tag{5.53}$$

and they are invariant under the transformations

$$\delta\psi_0=\eta\cdot\partial_x\epsilon_0+(\eta^2+1)\Xi, \tag{5.54}$$

$$\delta\psi_1=-\frac{1}{2}\Delta\epsilon_0-\Xi, \tag{5.55}$$

which are now irreducible. These last four results are, respectively, in complete agreement to what we see in equations (30), (31), (32), and (33) of [13].

5.5 Connections to the CSP Literature

We will now show that the extended equations of motion (5.52) and (5.53) are in agreement to the results obtained in [14]. The notation used in most of the CSP literature is that of momentum space, thus, to make the connection clearer, we will Fourier transform our results to momentum space. As mentioned before, we give the name ω to the Fourier conjugate of our coordinate η.

We perform the usual Fourier transformation (on both x^μ and η^μ) on our fields ψ_i as

$$\psi_i(\eta, x) = \int d^4\omega \; d^4p \; e^{i(\eta\cdot\omega + p\cdot x)}\tilde{\psi}_i(\omega, p), \tag{5.56}$$

and similarly for ϵ_0. This implies that equation (5.52)

$$\begin{aligned} &-p^2\tilde{\psi}_0 + p\cdot\partial_\omega(-p\cdot\omega+\rho)\tilde{\psi}_0 - 2(p\cdot\partial_\omega)^2\tilde{\psi}_1 = 0 \\ \Rightarrow &-p^2\tilde{\psi}_0 - p^2\tilde{\psi}_0 - (p\cdot\omega-\rho)p\cdot\partial_\omega\tilde{\psi}_0 - 2(p\cdot\partial_\omega)^2\tilde{\psi}_1 = 0 \\ \Rightarrow &\; p^2\tilde{\psi}_0 + \tfrac{1}{2}(p\cdot\omega-\rho)p\cdot\partial_\omega\tilde{\psi}_0 + (p\cdot\partial_\omega)^2\tilde{\psi}_0 = 0, \end{aligned} \tag{5.57}$$

and, analogously, (5.53) can be written as

$$p^2\tilde{\psi}_1 - (p\cdot\omega-\rho)p\cdot\partial_\omega\tilde{\psi}_1 - \tfrac{1}{2}(p\cdot\omega-\rho)^2\tilde{\psi}_0 = 0. \tag{5.58}$$

We then use our Ξ symmetry to choose the gauge [13]

$$\tilde{\psi}_0 + (\Box_\omega - 1)\tilde{\psi}_1 = 0, \tag{5.59}$$

so that our transformations (5.54) and (5.55) are reduced to

$$\delta\tilde{\psi}_0 = -p\cdot\partial_\omega\tilde{\epsilon}_0, \tag{5.60}$$

$$\delta\tilde{\psi}_1 = \tfrac{1}{2}(p\cdot\omega-\rho)\tilde{\epsilon}_0. \tag{5.61}$$

Once we apply our gauge choice (5.59) to (5.58), favouring the field ψ_1, we find

$$p^2\tilde{\psi}_1 - (p\cdot\omega-\rho)p\cdot\partial_\omega\tilde{\psi}_1 + \tfrac{1}{2}(p\cdot\omega-\rho)^2(\Box_\omega-1)\tilde{\psi}_1 = 0, \tag{5.62}$$

which is equation (5.2) of [14]. The field constraint (5.3) of [14], which is the analogous version of the double-traceless conditions of our fields ϕ in the Schwinger-Fronsdal formalism, can also be found by using our gauge choice in (5.57)

$$\begin{aligned} 0 &= p^2(\Box_\omega-1)\tilde{\psi}_1 + \tfrac{1}{2}(p\cdot\omega-\rho)p\cdot\partial_\omega(\Box_\omega-1)\tilde{\psi}_1 + (p\cdot\partial_\omega)^2\tilde{\psi}_0 \\ &= -\tfrac{1}{4}(p\cdot\omega-\rho)^2(\Box_\omega-1)^2\tilde{\psi}_1 - \tfrac{1}{2}(p\cdot\omega-\rho)p\cdot\partial_\omega(\Box_\omega-1)\tilde{\psi}_1 + (p\cdot\partial_\omega)^2\tilde{\psi}_1 \\ &\quad + \tfrac{1}{2}(p\cdot\omega-\rho)p\cdot\partial_\omega(\Box_\omega-1)\tilde{\psi}_1 - (p\cdot\partial_\omega)^2\tilde{\psi}_1 \\ &= (p\cdot\omega-\rho)^2(\Box_\omega-1)^2\tilde{\psi}_1, \end{aligned} \tag{5.63}$$

where to go from the first to the second equality in (5.63), we must multiply equation (5.62) by $(\Box_\omega - 1)$ and substitute the result in favor of the term of the last equality. We can then proceed in analogy with what we argued when we found $B(\eta, x) = 0$ in the

previous section. Because we do not want to constrain the momentum of our particle, we end up with the result

$$(\Box_\omega - 1)^2 \tilde{\psi}_1 = 0, \tag{5.64}$$

which reproduces result (5.3) of [14][2]. By using (5.59) in (5.60), we can also reproduce the following condition on our gauge parameter ϵ_0,

$$\begin{aligned}
\delta\tilde{\psi}_0 = -p\cdot\partial_\omega\tilde{\epsilon}_0 = -(\Box_\omega - 1)\delta\tilde{\psi}_1 &= \frac{1}{2}(\Box_\omega - 1)(p\cdot\omega - \rho)\tilde{\epsilon}_0 \\
\Rightarrow -p\cdot\partial_\omega\tilde{\epsilon}_0 &= \frac{1}{2}(p\cdot\omega-\rho)\tilde{\epsilon}_0 - \frac{1}{2}\partial_\omega^{\ \mu}\left[p_\mu\tilde{\epsilon}_0 + (p\cdot\omega-\rho)\partial_{\omega\mu}\tilde{\epsilon}_0\right] \\
\Rightarrow \frac{1}{2}(p\cdot\omega-\rho)\tilde{\epsilon}_0 &- \frac{1}{2}\left[2p\cdot\partial_\omega + (p\cdot\omega-\rho)\Box_\omega\right]\tilde{\epsilon}_0 \\
\Rightarrow (p\cdot\omega-\rho)(\Box_\omega - 1)\tilde{\epsilon}_0 &= 0 \\
\Rightarrow (\Box_\omega - 1)\tilde{\epsilon}_0 &= 0,
\end{aligned} \tag{5.65}$$

where in the last line we used the fact that we do not want to constrain our particle's momentum. (5.65) is result (5.6) from [14]. Even though we did not mention previously, our gauge transformation (5.61) is equivalent to the gauge transformation (5.5) of [14].

5.6 Connections to the Higher-Spin Literature

This section is devoted to analysing the considerations proposed in [13] that shows the equivalente between the action (5.26) (in the limit $\rho \to 0$) to a sum of Schwinger-Fronsdal actions [11]. We start by considering the action (5.26) with $\rho = 0$

$$S = \frac{1}{2}\int d^4x\, d^4\eta \left\{\delta'(\eta^2+1)(\partial_x\psi_0)^2 + \frac{1}{2}\delta(\eta^2+1)\left[(\partial_\eta\cdot\partial_x\psi_0 + 2\eta\cdot\partial_x\psi_1)^2 - 4(\partial_x\psi_0)\cdot(\partial_x\psi_1)\right]\right\}. \tag{5.66}$$

The equations of motion for ψ_0 and ψ_1 can be obtained by explicit variations of these fields in (5.66). However, since we have already found the equations of motion for $\rho \neq 0$, namely (5.35) and (5.37), we will just set $\rho = 0$ in those results, finding

[2]There is a sign difference between our result and the one found in [14]. This is due to the fact that we have chosen $\Delta \equiv \partial_\eta\cdot\partial_x + \rho$, while the choice made in [14] is $\Delta \equiv \partial_\eta\cdot\partial_x - \rho$. There is also a difference in the way the eigenvalue of W^2 are set, since theirs is $+\rho^2$ (in their notation, μ^2), while in our discussion of chapter 3 we found the eigenvalue $-\rho^2$. These differences arise from the fact that the Minkowski metric we are using in this dissertation is mostly negative, while the one used in [14] is mostly positive.

$$\begin{aligned}\delta'(\eta^2+1)\Big[\Box_x\psi_0 - \eta\cdot\partial_x\partial_\eta\cdot\partial_x\psi_0 - 2(\eta\cdot\partial_x)^2\psi_1\Big] \\ - 2\delta(\eta^2+1)\Big[\Box_x\psi_1 + \frac{1}{2}\eta\cdot\partial_x\partial_\eta\cdot\partial_x\psi_1 + \frac{1}{4}(\partial_\eta\cdot\partial_x)^2\psi_0\Big] = 0,\end{aligned} \tag{5.67}$$

and

$$\delta(\eta^2+1)\left[\Box_x\psi_0 - \eta\cdot\partial_x\partial_\eta\cdot\partial_x\psi_0 - 2(\eta\cdot\partial_x)^2\psi_1\right] = 0, \tag{5.68}$$

respectively. Now we can proceed in a completely analogous manner as we have done in section 5.4 and extend the validity of (5.67) and (5.68) to all of η-space. We get [13]

$$\Box_x\psi_0 - \eta\cdot\partial_x\partial_\eta\cdot\partial_x\psi_0 - 2(\eta\cdot\partial_x)^2\psi_1 = 0, \tag{5.69}$$

$$\Box_x\psi_1 + \frac{1}{2}\eta\cdot\partial_x\partial_\eta\cdot\partial_x\psi_1 + \frac{1}{4}(\partial_\eta\cdot\partial_x)^2\psi_0 = 0, \tag{5.70}$$

where (5.69) would be the $\rho = 0$ analogous of $A(\eta, x) = 0$ and (5.70) would be the $\rho = 0$ analogous of $B(\eta, x) = 0$. These results are invariant under the transformations

$$\delta\psi_0 = \eta\cdot\partial_x\epsilon_0 + (\eta^2+1)\Xi, \tag{5.71}$$

$$\delta\psi_1 = -\frac{1}{2}\partial_\eta\cdot\partial_x\epsilon_0 - \Xi, \tag{5.72}$$

which are the $\rho = 0$ analogous of (5.54) and (5.55), respectively. Now we can use our Ξ symmetry to choose the gauge [13]

$$\psi_1 + \frac{1}{4}\Box_\eta\psi_0 = 0, \tag{5.73}$$

which implies the relation

$$(1+\eta\cdot\partial_\eta)\Xi + \frac{1}{4}(\eta^2+1)\Box_\eta\Xi = 0, \tag{5.74}$$

when we use (5.71) and (5.72). We can then use (5.73) back in our equations of motion to find

$$\Box_x\psi_0 - \eta\cdot\partial_x\partial_\eta\cdot\partial_x\psi_0 + \frac{1}{2}(\eta\cdot\partial_x)^2\Box_\eta\psi_0 = 0, \tag{5.75}$$

$$\Box_x\Box_\eta\psi_0 + \frac{1}{2}\eta\cdot\partial_x\partial_\eta\cdot\partial_x\Box_\eta\psi_0 - (\partial_\eta\cdot\partial_x)^2\psi_0 = 0. \tag{5.76}$$

Now applying $\Box_\eta$ in (5.75) we get

$$\Box_x\Box_\eta\psi_0 - (\partial_\eta\cdot\partial_x)^2\psi_0 + \frac{1}{2}\eta\cdot\partial_x\partial_\eta\cdot\partial_x\Box_\eta\psi_0 + \frac{1}{4}(\eta\cdot\partial_x)^2\Box_\eta^2\psi_0 = 0, \tag{5.77}$$

which, when compared to (5.76), implies

$$(\eta\cdot\partial_x)^2\Box_\eta^2\psi_0 = 0. \tag{5.78}$$

Because we do not want to constrain the particle's momentum, equation (5.78) reduces to $\Box_\eta^2\psi_0 = 0$. This is a double-traceless condition[3], in analogy to what we had in the Schwinger-Fronsdal formalism [11, 13]. Making a Ξ transformation in (5.78) implies that $\Box_\eta^2\delta_\Xi\psi_0 = 4\Box_\eta\Xi = 0$, which when used back in (5.74) implies $\Xi = 0$. If we apply an ϵ_0 transformation to (5.78), we are left with $\Box_\eta\epsilon_0 = 0$, which is the traceless condition of our gauge parameter [11, 13]. Now that we have fixed the Ξ symmetry, we found all the conditions of our field and gauge parameter to be exactly the same as the ones we had in chapter 4.

We now rewrite the expansion (5.12) as

$$\psi_0(\eta,x) = \sum_{n=0}^{\infty}\frac{1}{n!}\eta^{\mu_1}\cdots\eta^{\mu_n}\psi^{(0,n)}_{\mu_1\cdots\mu_n}(x) \equiv \sum_{n=0}^{\infty}\frac{1}{n!}\eta^{\mu_1}\cdots\eta^{\mu_n}\psi_{\mu_1\cdots\mu_n}(x), \tag{5.79}$$

where we drop the superscript $(0,n)$ to keep the notation less cluttered. Using (5.79) in (5.75) yields

$$\sum_{n=0}^{\infty}\frac{1}{n!}\eta^{\mu_1}\cdots\eta^{\mu_n}\left[\Box_x\psi_{\mu_1\cdots\mu_n}(x) - n\partial_{\mu_1}\partial\cdot\psi_{\mu_2\cdots\mu_n}(x) + \frac{1}{2}n(n-1)\partial_{\mu_1}\partial_{\mu_2}\psi'_{\mu_3\cdots\mu_n}(x)\right] = 0, \tag{5.80}$$

[3]To understand why this means a double-traceless condition, try looking at ψ_0 as in (5.12). When we apply $\Box_\eta^2$ the left-hand side of (5.12), we are effectively taking two traces of $\psi^{(0,s)}_{\mu_1\cdots\mu_s}(x)$, on the right-hand side.

where every derivative that appears in (5.80) is taken with respect to x^μ. Also, as usual, we have $\partial \cdot \psi_{\mu_2 \cdots \mu_n}(x) \equiv \partial^{\mu_1} \psi_{\mu_1 \cdots \mu_n}$ and $\psi'_{\mu_1 \cdots \mu_n} \equiv \psi^\rho{}_{\rho \mu_1 \cdots \mu_n} = g^{\rho\sigma} \psi_{\rho\sigma\mu_1 \cdots \mu_n}$. Comparing (5.80) with (4.65), allows us to notice that (5.80) is a sum of Schwinger-Fronsdal equations for all integer helicities[4] [11, 13]. This means that the action proposed by Rivelles in [13] reduces to a sum of Schwinger-Fronsdal actions, as discussed in chapter 4, in the limit $\rho \to 0$.

5.7 The Eigenvalues of P^2 and W^2

After completing the analysis of our action when $\rho \to 0$, we go back to the case where $\rho \neq 0$ and study its physical content. As we have seen in chapter 3, a CSP representation is characterized by the eigenvalues of two operators (called the Casimir operators of the Poincaré group), P^2 and W^2, acting on the eigenstates of the theory with eigenvalues 0 and $(-\rho^2)$, respectively. We then wish to write down these operators using our η-space notation and compute their eigenvalues when they act on our fields ψ_0 and ψ_1.

First we construct the Pauli-Lubanski pseudo-vector in η-space. In chapter three we have cast a definition for this vector, which we present here again

$$W^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}. \tag{5.81}$$

When we take into consideration the effects of our η-space, we see that we can write [13] $P_\mu = -i\partial_\mu$ and $M_{\mu\nu} = -i\left(x_{[\mu}\partial_{x\ \nu]} + \eta_{[\mu}\partial_{\eta\ \nu]}\right)$. This means that (5.81) becomes

$$\begin{aligned} W^\mu &= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left[\underbrace{\partial_{x\nu} x_{[\mu}\partial_{x\ \nu]}}_{0,\text{ using the }\epsilon} + \partial_{x\nu}\left(\eta_\rho \partial_{\eta\sigma} - \eta_\sigma \partial_{\eta\rho}\right)\right] \\ &= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_{x\nu}\eta_\rho\partial_{\eta\sigma} + \frac{1}{2}\underbrace{\epsilon^{\mu\nu\sigma\rho}\partial_{x\nu}\eta_\sigma\partial_{\eta\rho}}_{\sigma\leftrightarrow\rho} \\ &= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_{x\nu}\eta_\rho\partial_{\eta\sigma} - \frac{1}{2}\underbrace{\epsilon^{\mu\nu\rho\sigma}\partial_{x\nu}\eta_\rho\partial_{\eta\sigma}}_{\epsilon^{\mu\nu\sigma\rho}=-\epsilon^{\mu\nu\rho\sigma}} \\ &= -\epsilon^{\mu\nu\rho\sigma}\partial_{x\nu}\eta_\rho\partial_{\eta\sigma} \end{aligned} \tag{5.82}$$

Now we must compute W^2. To do this, we will need two identities involving the product

[4] The factors of n that appear in (5.80) are equivalent to the number of symmetric, inequivalent permutations of the terms in the Schwinger-Fronsdal equations.

between two *Levi-Civita symbols*, $\epsilon^{\mu\nu\rho\sigma}$, which are

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\alpha\beta\gamma} = -\Big[&\delta^{\nu}{}_{\alpha}\delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\gamma} - \delta^{\nu}{}_{\alpha}\delta^{\rho}{}_{\gamma}\delta^{\sigma}{}_{\beta} + \delta^{\nu}{}_{\beta}\delta^{\rho}{}_{\gamma}\delta^{\sigma}{}_{\alpha} \\ &- \delta^{\nu}{}_{\beta}\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\gamma} + \delta^{\nu}{}_{\gamma}\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta} - \delta^{\nu}{}_{\gamma}\delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}\Big],\end{aligned} \tag{5.83}$$

and

$$\epsilon^{\mu\sigma\nu\rho}\epsilon_{\mu\sigma\alpha\gamma} = -2\left(\delta^{\nu}{}_{\alpha}\delta^{\rho}{}_{\gamma} - \delta^{\nu}{}_{\gamma}\delta^{\rho}{}_{\alpha}\right). \tag{5.84}$$

This is a very lengthy, though simple, computation which we will leave to Appendix B. The result we get is

$$\begin{aligned}W^2\psi_i = [&(\eta\cdot\partial_\eta)(1+\eta\cdot\partial_\eta)\Box_x - \eta^2\Box_\eta\Box_x - 2(\eta\cdot\partial_x)(\partial_\eta\cdot\partial_x)(\eta\cdot\partial_\eta) \\ &+ (\eta\cdot\partial_x)^2\Box_\eta + \eta^2(\partial_\eta\cdot\partial_x)^2]\psi_i,\ i = 0 \text{ or } 1.\end{aligned} \tag{5.85}$$

Now we must work on the right-hand side of equation (5.85) for both ψ_0 and ψ_1, subject to their equations of motion in all of η-space, given by (5.52) and (5.53), respectively. After another pair of very long computations(which we leave to Appendix C) we can show that

$$W^2\psi_0 = -\rho^2\psi_0 + \delta_\epsilon\psi_0 + \delta_\Xi\psi_0, \tag{5.86}$$

and

$$W^2\psi_1 = \eta^2\rho^2\psi_1 + \delta_\epsilon\psi_1 + \delta_\Xi\psi_1, \tag{5.87}$$

where the transformation parameters ϵ_0 and Ξ are given by

$$\begin{aligned}\epsilon_0 =&\eta\cdot\partial_\eta(1+\eta\cdot\partial_\eta)\partial_\eta\cdot\partial_x\psi_0 + \rho\left[2+3\eta\cdot\partial_\eta+(\eta\cdot\partial_\eta)^2\right]\psi_0 - (\eta^2\Delta - \eta\cdot\partial_x)\Box_\eta\psi_0 \\ &+ 2\left[2+3\eta\cdot\partial_\eta+(\eta\cdot\partial_\eta)^2\right]\eta\cdot\partial_x\psi_1 - 2\eta^2\left(\eta\cdot\partial_x\Box_\eta + 3\partial_\eta\cdot\partial_x - \rho\right)\psi_1,\end{aligned} \tag{5.88}$$

and

$$\Xi = \rho^2 \psi_0. \tag{5.89}$$

These results agree with results (78)-(81) of [13]. It is important to notice that the equations of motion we have chosen to find results (5.86) and (5.87) where the equations of motion extended to *all* of η-space. However, result (5.87) makes it clear that we will only get the correct eigenvalues to both ψ_0 and ψ_1 when we are on the hyperboloid $\eta^2 + 1 = 0$, that is, when we are in the presence of the Dirac's delta functions of the previous sections. In other words, the correct eigenvalues for W^2 are obtained when

$$\delta(\eta^2 + 1) W^2 \psi_0 = \delta(\eta^2 + 1) \left[-\rho^2 \psi_0 + \delta_\epsilon \psi_0 + \delta_\Xi \psi_0 \right], \tag{5.90}$$

$$\delta(\eta^2 + 1) W^2 \psi_1 = \delta(\eta^2 + 1) \left[-\rho^2 \psi_1 + \delta_\epsilon \psi_1 + \delta_\Xi \psi_1 \right], \tag{5.91}$$

These results indicate that CSP degrees of freedom live only on the hyperboloid and not on all of η-space [13].

The other eigenvalue we are interested in is that of the operator P^2. However, if we look at the equations of motion (5.52) and (5.53) a little bit more carefully, we see they can be written in the form

$$\Box_x \psi_0 = 0 + \delta_{\epsilon_0} \psi_0, \tag{5.92}$$

$$\Box_x \psi_1 = 0 + \delta_{\epsilon_0} \psi_1, \tag{5.93}$$

where

$$\epsilon_0 = \Delta \psi_0 + 2\eta \cdot \partial_x \psi_1, \tag{5.94}$$

and the ϵ_0-transformations are given by the first terms of equations (5.54) and (5.55). Since we can write $P^2 = -\Box_x$, then we have $P^2 \psi_i = 0$ and $W^2 \psi_i = -\rho^2 \psi_i$, which are the correct eigenvalues for our two Casimir operators. Of course, if you look directly at results (5.86), (5.87), (5.92), and (5.93), you will see the the eigenvalues of P^2 and W^2 hold *up to a gauge transformation* (or up to a *pure gauge* term, if you will). This happens because we are studying a gauge theory, thus we can only demand these kinds of relations up to pure gauge terms [12, 13].

5.8 Physical Contents

To find our physical degrees of freedom, we will work with the same gauge choice we made in section 5.6, namely (5.73), but now favouring ψ_0 instead of ψ_1. We start by reminding ourselves that our expansion (5.11) actually only have two terms (that are relevant)

$$\Psi(\eta, x) = \psi_0(\eta, x) + (\eta^2 + 1)\psi_1(\eta, x). \tag{5.95}$$

We also remember that when we do a similar expansion for our gauge parameter $\epsilon(\eta, x)$, we can use the symmetry in Λ to remove all terms of the expansion except one, so that $\epsilon(\eta, x) = \epsilon_0(\eta, x)$ [13, 15]. If we choose only these considerations as our starting point, then we have no condition on the traces of ψ_i and ϵ_0. This means that we still have our Ξ symmetry, so that our local transformations are given by (5.54) and (5.55). Also, the equations of motion we are working with are the ones valid for all of η-space, given by (5.52) and (5.53).

Applying our gauge choice to (5.52) gives

$$\left[\Box_x - \eta \cdot \partial_x \Delta + \frac{1}{2}\left(\eta \cdot \partial_x\right)^2 \Box_\eta\right] \psi_0 = 0, \tag{5.96}$$

and, for (5.53),

$$\begin{aligned} 0 &= \left[-\frac{1}{4}\Box_x \Box_\eta - \frac{1}{2}\eta \cdot \partial_x \Delta \Box_\eta + \frac{1}{4}\Delta^2 \right]\psi_0 \\ &= \left[(\eta \cdot \partial_x)^2 \Box_\eta^2 + 4\rho(\Delta - \eta \cdot \partial_x \Box_\eta)\right]\psi_0, \end{aligned} \tag{5.97}$$

where in the last equality we used (5.96). If we take a gauge transformation of (5.73), we get

$$\begin{aligned} \delta\psi_1 + \frac{1}{4}\Box_\eta \delta\psi_0 &= -\frac{1}{2}\Delta\epsilon_0 + \frac{1}{4}\Box_\eta \eta \cdot \partial_x \epsilon_0 = -2\Delta\epsilon_0 + \Box_\eta\Big[\eta \cdot \partial_x \epsilon_0\Big] \\ &= -2\Delta\epsilon_0 + 2\partial_\eta \cdot \partial_x \epsilon_0 + \eta \cdot \partial_x \Box_\eta \epsilon_0 \\ &= \Big(\eta \cdot \partial_x \Box_\eta - 2\rho\Big)\epsilon_0 = 0. \end{aligned} \tag{5.98}$$

We can notice that (5.96) reduces to the Schwinger-Fronsdal equations of motion for ψ_0 and (5.98) reduces to the traceless condition for the gauge parameter, both when

$\rho = 0$ [15]. Now we have partially fixed our gauge, but we still have a residual gauge symmetry for a parameter ϵ_0 satisfying (5.98). To continue, we will choose a harmonic gauge

$$\Big(\eta \cdot \partial_x \Box_\eta - 2\Delta\Big)\psi_0 = 0, \tag{5.99}$$

so that the equations of motion (5.96) and (5.97) are reduced to

$$\Box_x \psi_0 = 0, \tag{5.100}$$

$$\Big(\eta \cdot \partial_x \Box_\eta - 2\rho\Big)\Box_\eta \psi_0 = 0, \tag{5.101}$$

respectively. Also, taking a gauge transformation of (5.100) and keeping the particle's momentum arbitrary means that our gauge parameter must satisfy the condition

$$\Box_x \epsilon_0 = 0. \tag{5.102}$$

These are the new conditions our field ψ_0 and our gauge parameter ϵ_0 must satisfy *after* gauge fixing. Now we want to use our gauge symmetry in ϵ_0 to gauge away some components of ψ_0 in the expansion (5.12). We will work on a Lorentz frame where the light-cone components of the momentum are $(p_+, p_-, p_1, p_2) = (p_+, 0, 0, 0)$. We will also be a bit more explicit with our series expansions in order to avoid confusion with the notation. Expanding ϵ_0, as in (5.12), in (5.98) gives

$$\sum_0^\infty \frac{1}{n!}\Big(\eta \cdot \partial_x \Box_\eta - 2\rho\Big)\left[\eta^{\mu_1} \cdots \eta^{\mu_n} \epsilon_{0\ \mu_1 \cdots \mu_n}(x)\right] = 0. \tag{5.103}$$

The first term in (5.103) can be read as

$$\begin{aligned}\sum_0^\infty \frac{1}{n!}\eta \cdot \partial_x \Box_\eta \left[\eta^{\mu_1} \cdots \eta^{\mu_n} \epsilon_0(x)\right] &= \eta^{\mu_1} \partial_{x\mu_1} \epsilon_0'(x) + \frac{1}{2!}\eta^{\mu_1}\eta^{\mu_2} \partial_{x(\mu_1} \epsilon_{0\mu_2)}'(x) \\ &+ \frac{1}{3!}\left[\frac{1}{2!}\eta^{\mu_1}\eta^{\mu_2}\eta^{\mu_3} \partial_{x(\mu_1} \epsilon_{0\mu_2\mu_3)}'(x)\right] \\ &+ \frac{1}{4!}\left[\frac{1}{3!}\eta^{\mu_1}\eta^{\mu_2}\eta^{\mu_3}\eta^{\mu_4} \partial_{x(\mu_1} \epsilon_{0\mu_2\mu_3\mu_4)}'(x)\right] + \cdots ,\end{aligned} \tag{5.104}$$

while the second term can be read as

$$-2\rho\sum_{0}^{\infty}\frac{1}{n!}\left[\eta^{\mu_1}\cdots\eta^{\mu_n}\epsilon_0(x)\right] = -2\rho\Big[\epsilon_0(x)+\eta^{\mu_1}\epsilon_{0\mu_1}(x)+\frac{1}{2!}\eta^{\mu_1}\eta^{\mu_2}\epsilon_{0\mu_1\mu_2}(x) + \frac{1}{3!}\eta^{\mu_1}\eta^{\mu_2}\eta^{\mu_3}\epsilon_{0\mu_1\mu_2\mu_3}(x)+\frac{1}{4!}\eta^{\mu_1}\eta^{\mu_2}\eta^{\mu_3}\eta^{\mu_4}\epsilon_{0\mu_1\mu_2\mu_3\mu_4}(x)+\cdots\Big], \tag{5.105}$$

so that, comparing (5.104) and (5.105), we can find (after going to momentum space) [15]

$$\frac{1}{(n-1)!}ip_{(\mu_1}\tilde{\epsilon}'_{\mu_2\cdots\mu_n)}(p) - 2\rho\tilde{\epsilon}_{\mu_1\cdots\mu_n}(p) = 0, \tag{5.106}$$

where $\tilde{\epsilon}$ is the Fourier transform of ϵ_0 and $\tilde{\epsilon}'$ is the trace of $\tilde{\epsilon}$. Because of our choice of Lorentz frame, we can simplify result (5.106) to [15]

$$imp_+\tilde{\epsilon}'_{\underbrace{+\cdots+}_{m-1\ \text{times}}A_1\cdots A_n} - 2\rho\tilde{\epsilon}_{\underbrace{+\cdots+}_{m\ \text{times}}A_1\cdots A_n} = 0, \tag{5.107}$$

where $A = (-, i)$. This can be achieved because of the symmetric permutation notation we are using in (5.106), out of all the terms that arise from those permutations, the only ones that survive are the ones that are listed in (5.107). We can note that setting $\rho =$ in (5.107) will give us the usual traceless condition on the gauge parameter ϵ_0. However, in the CSP case, we no longer have this condition! For $m = 0$, we learn from (5.107) that

$$\tilde{\epsilon}_{A_1\cdots A_n} = 0 \Rightarrow \epsilon_{i_1\cdots i_n} = 0,\ m \geq 1. \tag{5.108}$$

Then, the gauge part of our transformation (5.54) can be written as

$$\delta\tilde{\psi}_{0\mu_1\cdots\mu_n} = \frac{1}{(n-1)!}ip_{(\mu_1}\tilde{\epsilon}_{\mu_2\cdots\mu_n)}, \tag{5.109}$$

which because of our choice of Lorentz frame can be written as

$$\delta\tilde{\psi}_{\underbrace{+\cdots+}_{m\ \text{times}}A_1\cdots A_n} = imp_+\tilde{\epsilon}_{\underbrace{+\cdots+}_{m-1\ \text{times}}A_1\cdots A_n},\ m \geq 0. \tag{5.110}$$

When $m = 0$, $\psi_{i_1\cdots i_n}$ is trivially gauge invariant because of (5.108). For $m \geq 1$, we can gauge away the components of $\tilde{\psi}$ with $+$ indices by making use of our $\tilde{\epsilon}$ symmetry. In other words, we set [15]

$$\tilde{\psi}^{\text{new}}_{\underbrace{+\cdots+}_{m\ \text{times}}A_1\cdots A_n} = \tilde{\psi}_{\underbrace{+\cdots+}_{m\ \text{times}}A_1\cdots A_n} + imp_+\tilde{\epsilon}_{\underbrace{+\cdots+}_{m-1\ \text{times}}A_1\cdots A_n} = 0,\ p \geq 1. \tag{5.111}$$

For this condition to be reached, we must solve (5.111) for $\tilde{\epsilon}$ in terms of $\tilde{\psi}$, substitute the result in (5.106) and find an analogous condition for $\tilde{\psi}$[5]. The condition $\tilde{\psi}$ must satisfy in order for (5.111) to be reached is

$$imp_+\tilde{\psi}'_{\underbrace{+\cdots+}_{m-1\ \text{times}}A_1\cdots A_n} - 2ip_+\tilde{\psi}_{-\underbrace{+\cdots+}_{m\ \text{times}}A_1\cdots A_n} - 2\rho\tilde{\psi}_{\underbrace{+\cdots+}_{m\ \text{times}}A_1\cdots A_n} = 0,\ p \geq 0. \tag{5.113}$$

This means we can reach the gauge $\tilde{\psi}^{\text{new}}_{+\cdots+A_1\cdots A_n} = 0$ [15]. We can rewrite the harmonic gauge condition (5.99) as

$$\frac{i}{(n-1)!}p_{(\mu_1}\tilde{\psi}'_{\mu_2\cdots\mu_n)} - 2ip_+\tilde{\psi}_{-\mu_1\cdots\mu_n} - 2\rho\tilde{\psi}_{\mu_1\cdots\mu_n} = 0, \tag{5.114}$$

which is exactly the same as (5.113) [15]. Now that our gauge is completely fixed, we can find, using (5.113) with $m = 0$, that

$$\tilde{\psi}_{\underbrace{-\cdots-}_{\ell\ \text{times}}i_1\cdots i_n} = \left(-\frac{\rho}{ip_+}\right)^{\ell}\tilde{\psi}_{i_1\cdots i_n}. \tag{5.115}$$

Result (5.115) can be obtained iteratively. After setting $m = 0$, we get result (5.115) with $\ell = 1$, that is

$$\tilde{\psi}_{-i_1\cdots i_n} = -\frac{\rho}{ip_+}\tilde{\psi}_{i_1\cdots i_n}. \tag{5.116}$$

Then, setting one of the i indices to "minus" on both sides of (5.116) gives

$$\tilde{\psi}_{--i_1\cdots i_n} = -\frac{\rho}{ip_+}\tilde{\psi}_{-i_1\cdots i_n} = \left(-\frac{\rho}{ip_+}\right)^2\tilde{\psi}_{i_1\cdots i_n}. \tag{5.117}$$

[5]Remember that, for a symmetric tensor, $A' = g^{\mu\nu}A_{\mu\nu} = 2A_{-+} - A_{ii}$, where we are working with light-cone coordinates with metric

$$(g^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5.112}$$

This can be done ℓ times, for instance, thus reaching result (5.115). When $\rho = 0$, then every component of ψ with a $-$ index vanishes. With result (5.115), we also find that our condition (5.101) is satisfied.

Setting $m = 1$ in (5.113) we can conclude that (using, again, the trace of $\tilde{\psi}$ in our light-cone coordinates) $\tilde{\psi}_{jji_1\cdots i_n} = 0$ [15]. This is a traceless condition upon our field $\tilde{\psi}$.

Now we are ready to interpret our results. We found that the $\tilde{\psi}_{i_1\cdots i_n}$ contain *all* the degrees of freedom of our theory, since they are the only independent components of our field $\tilde{\psi}$. We also found that the field is subject to the traceless condition $\tilde{\psi}_{jji_1\cdots i_n} = 0$. This means we have, as expected from our discussion in chapter 3, that the CSP field carries *all integer helicities*, from $-\infty$ to ∞, each one appearing only once. In the $\rho = 0$ limit, we also have that our fields ψ_0 decouple into a sum of Schwinger-Fronsdal fields for all integer helicities [15].

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Appendices

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Appendix A

Notation and Conventions

Throughout this dissertation we will be using the mostly minus Minkowski metric[1] signature, that is

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A.1}$$

with $\mu, \nu = 0, 1, 2, 3$. We use the metric to raise and lower spacetime indices, i.e.

$$x^\mu = \eta^{\mu\nu} x_\nu, \tag{A.2}$$

and

$$x_\nu = \eta_{\mu\nu} x^\nu. \tag{A.3}$$

When we talk about groups in chapter 2 and 3, the notation $SO(N, M)$ indicates the special orthogonal group with N time coordinates and M spatial coordinates (N positive metric eigenvalues and M negative metric eigenvalues, if you wish).

Sometimes, we may simplify some antisymmetrization operations with the shorthand notation

$$A_{[\mu}{}^{[\nu} B_{\sigma]}{}^{\rho]} \equiv A_\mu{}^\nu B_\sigma{}^\rho - A_\mu{}^\rho B_\sigma{}^\nu - A_\sigma{}^\nu B_\mu{}^\rho + A_\sigma{}^\rho B_\mu{}^\nu. \tag{A.4}$$

[1] In chapters 2-4 we will denote the Minkowski metric by $\eta_{\mu\nu}$, but in chapter 5 we switch notation to $g_{\mu\nu}$ in order to avoid confusion with the extra coordinate η of the enlarged spacetime.

Throughout chapter 4 and 5, contractions between two tensors are defined by $X \cdot Y \equiv X^{\mu\nu\cdots} Y_{\mu\nu\ldots}$, keeping in mind that contractions with derivatives and the metric are always taken to be outer products (i.e. $\partial \cdot X \equiv \partial_\mu X^{\mu\cdots}$ and $g \cdot X \equiv g_{\mu\nu} X^{\mu\nu\cdots}$). Also, we define the *inequivalent symmetric contraction with no symmetry factor* between two tensors, X and Y, by $X \circ Y$. Here, $X \circ Y$ is a $(n+m)$ tensor if the ranks of X and Y are n and m, respectively. This means that for $n = m = 1$, for example, we have $X \circ Y \equiv X_\mu Y_\nu + X_\nu Y_\mu$. The trace of a tensor is written as X' and is defined as $X' \equiv Tr(X) \equiv g \cdot X$.

Appendix B

Computing W^2 in the Field Theory Formalism

Now we will elaborate further on how to get to result (5.85). In this appendix we will use $g_{\mu\nu}$ for the Minkowski metric.

We will start by writing again the identities for contractions between two Levi-Civita symbols that we listed on chapter 5

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\alpha\beta\gamma} &= \det(g)\Big[\delta^{\nu}{}_{\alpha}\delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\gamma} - \delta^{\nu}{}_{\alpha}\delta^{\rho}{}_{\gamma}\delta^{\sigma}{}_{\beta} + \delta^{\nu}{}_{\beta}\delta^{\rho}{}_{\gamma}\delta^{\sigma}{}_{\alpha} \\ &\quad - \delta^{\nu}{}_{\beta}\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\gamma} + \delta^{\nu}{}_{\gamma}\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta} - \delta^{\nu}{}_{\gamma}\delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}\Big] \\ &= -\Big[\delta^{\nu}{}_{\alpha}\delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\gamma} - \delta^{\nu}{}_{\alpha}\delta^{\rho}{}_{\gamma}\delta^{\sigma}{}_{\beta} + \delta^{\nu}{}_{\beta}\delta^{\rho}{}_{\gamma}\delta^{\sigma}{}_{\alpha} \\ &\quad - \delta^{\nu}{}_{\beta}\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\gamma} + \delta^{\nu}{}_{\gamma}\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta} - \delta^{\nu}{}_{\gamma}\delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}\Big],\end{aligned} \tag{B.1}$$

and

$$\begin{aligned}\epsilon^{\mu\sigma\nu\rho}\epsilon_{\mu\sigma\alpha\gamma} &= \det(g)2!\left(\delta^{\nu}{}_{\alpha}\delta^{\rho}{}_{\gamma} - \delta^{\nu}{}_{\gamma}\delta^{\rho}{}_{\alpha}\right) \\ &= -2\left(\delta^{\nu}{}_{\alpha}\delta^{\rho}{}_{\gamma} - \delta^{\nu}{}_{\gamma}\delta^{\rho}{}_{\alpha}\right).\end{aligned} \tag{B.2}$$

Let us start by reviewing our definition of the Pauli-Lubanski pseudo vector

$$W^{\mu} \equiv -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_{\nu}M_{\rho\sigma}. \tag{B.3}$$

Then we write.

$$P_\nu = -i\partial_{x\nu}, \tag{B.4}$$

$$M_{\rho\sigma} = -i\left(x_\rho\partial_{x\sigma} - x_\sigma\partial_{x\rho} + \eta_\rho\partial_{\eta\sigma} - \eta_\sigma\partial_{\eta\rho}\right). \tag{B.5}$$

Using (B.4) and (B.5) in (B.3) we get

$$\begin{aligned} W^\mu &= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_{x\nu}\left(x_\rho\partial_{x\sigma} - x_\sigma\partial_{x\rho} + \eta_\rho\partial_{\eta\sigma} - \eta_\sigma\partial_{\eta\rho}\right) \\ &= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(g_{\nu\rho}\partial_{x\sigma} - g_{\nu\sigma}\partial_{x\rho} + \partial_{x\nu}\eta_\rho\partial_{\eta\sigma}\right) + \frac{1}{2}\underbrace{\epsilon^{\mu\nu\sigma\rho}}_{-\epsilon^{\mu\nu\rho\sigma}}\partial_{x\nu}\eta_\rho\partial_{\eta\sigma}. \\ &= -\epsilon^{\mu\nu\rho\sigma}\partial_{x\nu}\eta_\rho\partial_{\eta\sigma} \end{aligned} \tag{B.6}$$

Now we can compute $W^2\Psi(x,\eta)$ using (B.6)

$$\begin{aligned} W^2\Psi &= \left(\epsilon^{\mu\nu\rho\sigma}\partial_{x\nu}\eta_\rho\partial_{\eta\sigma}\right)\left(\epsilon_{\mu\alpha\beta\gamma}\partial_x^\alpha\eta^\beta\partial_\eta^\gamma\right)\Psi \\ &= \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\alpha\beta\gamma}\left(\eta_\rho\eta^\beta\partial_{\eta\sigma}\partial_\eta^\gamma + \delta^\beta{}_\sigma\eta_\rho\partial_\eta^\gamma\right)\partial_{x\nu}\partial_x^\alpha\Psi. \end{aligned} \tag{B.7}$$

Making use of our identity (B.1), the first term in (B.7) can be written as

$$\begin{aligned} &-(\delta^\nu{}_\alpha\delta^\rho{}_\beta\delta^\sigma{}_\gamma - \delta^\nu{}_\alpha\delta^\rho{}_\gamma\delta^\sigma{}_\beta + \delta^\nu{}_\beta\delta^\rho{}_\gamma\delta^\sigma{}_\alpha - \delta^\nu{}_\beta\delta^\rho{}_\alpha\delta^\sigma{}_\gamma + \delta^\nu{}_\gamma\delta^\rho{}_\alpha\delta^\sigma{}_\beta - \delta^\nu{}_\gamma\delta^\rho{}_\beta\delta^\sigma{}_\alpha)\times \\ &\quad\times\left(\eta_\rho\eta^\beta\partial_{\eta\sigma}\partial_\eta^\gamma\partial_{x\nu}\partial_x^\alpha\right)\Psi \\ &\quad= [-\eta^2\Box_\eta\Box_x + (\eta\cdot\partial_\eta)^2\Box_x - 2(\eta\cdot\partial_x)(\eta\cdot\partial_\eta)(\partial_\eta\cdot\partial_x) + \\ &\quad+ (\eta\cdot\partial_x)^2\Box_\eta + \eta^2(\partial_\eta\cdot\partial_x)^2]\Psi. \end{aligned} \tag{B.8}$$

Now, using (B.2), the second term in (B.7) can be written as

$$\begin{aligned} \underbrace{\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\alpha\sigma\gamma}}_{-\epsilon^{\mu\sigma\nu\rho}\epsilon_{\mu\sigma\alpha\gamma}}\eta_\rho\partial_\eta^\gamma\partial_{x\nu}\partial_x^\alpha\Psi &= 2\left(\delta^\nu{}_\alpha\delta^\rho{}_\gamma - \delta^\nu{}_\gamma\delta^\rho{}_\alpha\right)\eta_\rho\partial_\eta^\gamma\partial_{x\nu}\partial_x^\alpha\Psi \\ &= 2\left[(\eta\cdot\partial_\eta)\Box_x - (\eta\cdot\partial_x)(\partial_\eta\cdot\partial_x)\right]\Psi. \end{aligned} \tag{B.9}$$

Thus, adding results (B.8) and (B.9) we get

$$
\begin{aligned}
W^2\Psi = [&(\eta\cdot\partial_\eta)(1+\eta\cdot\partial_\eta)\Box_x - \eta^2\Box_\eta\Box_x - 2(\eta\cdot\partial_x)(\partial_\eta\cdot\partial_x)(\eta\cdot\partial_\eta) \\
&+ (\eta\cdot\partial_x)^2\Box_\eta + \eta^2(\partial_\eta\cdot\partial_x)^2]\Psi,
\end{aligned}
\tag{B.10}
$$

which is precisely result (5.85) once we change $\Psi \to \psi_i$, $i = 0, 1$.

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Appendix C

Computing the Eigenvalues of W^2

When performing the manipulations of the next few sections, it is necessary to use commutation relations between a few operators of our theory. For example, we can write the term $\eta \cdot \partial_x \Delta f(\eta, x)$, where $f(\eta, x)$ is an arbitrary test function, as

$$\Delta\Big(\eta \cdot \partial_x f\Big) = \eta \cdot \partial_x \Delta f + \Box_x f \Rightarrow \eta \cdot \partial_x \Delta f = \Delta\Big(\eta \cdot \partial_x f\Big) - \Box_x f. \tag{C.1}$$

These manipulations occur frequently on what follows and it might be difficult for the reader to keep track of all of them. Still, we would like to list a few of the most useful ones here

$$\eta \cdot \partial_\eta \Delta f = \Delta\Big(\eta \cdot \partial_\eta f) - \partial_\eta \cdot \partial_x f, \tag{C.2}$$

$$(\eta \cdot \partial_\eta)^2 \Delta f = \Delta\Big[(\eta \cdot \partial_\eta)^2 f\Big] - (2\eta \cdot \partial_\eta + 1)\delta \cdot \partial_x f. \tag{C.3}$$

Notice that most calculations which involve $\partial_\eta \cdot \partial_x$, that is, $(\Delta - \rho)$, are very similar to the ones already listed here, just making the substitution $\Delta \to \partial_\eta \cdot \partial_x$.

C.1 Computing $W^2\psi_0$

First, we write down, for the sake of the reader, the equations of motion for ψ_0 and ψ_1 extended to all of η-space and how these fields transform under gauge transformations. The equations of motion are

$$\Box_x \psi_0 - \eta \cdot \partial_x \Delta \psi_0 - 2(\eta \cdot \partial_x)^2 \psi_1 = 0, \tag{C.4}$$

$$\Box_x \psi_1 + \frac{1}{2}\eta \cdot \partial_x \Delta \psi_1 + \frac{1}{4}\Delta^2 \psi_0 = 0, \tag{C.5}$$

and the transformations are

$$\delta\psi_0 = \eta\cdot\partial_x\epsilon_0 + (\eta^2+1)\Xi, \tag{C.6}$$

$$\delta\psi_1 = -\frac{1}{2}\Delta\epsilon_0 - \Xi. \tag{C.7}$$

Now we want to use our result (B.10) with $\Psi \to \psi_0$, that is

$$\begin{aligned} W^2\psi_0 = {} & [(\eta\cdot\partial_\eta)(1+\eta\cdot\partial_\eta)\Box_x - \eta^2\Box_\eta\Box_x - 2(\eta\cdot\partial_x)(\partial_\eta\cdot\partial_x)(\eta\cdot\partial_\eta) \\ & + (\eta\cdot\partial_x)^2\Box_\eta + \eta^2(\partial_\eta\cdot\partial_x)^2]\psi_0, \\ & = [(\eta\cdot\partial_\eta)\Box_x + (\eta\cdot\partial_\eta)^2\Box_x - \eta^2\Box_\eta\Box_x - 2(\eta\cdot\partial_x)(\partial_\eta\cdot\partial_x)(\eta\cdot\partial_\eta) \\ & + (\eta\cdot\partial_x)^2\Box_\eta + \eta^2(\partial_\eta\cdot\partial_x)^2]\psi_0, \end{aligned} \tag{C.8}$$

where in the second equality we have expanded the first term inside the square brackets of the first equality. Because each term in (C.8) requires lots of manipulations, we will work out each of them separately. The first term in the second equality of (C.8) can be written as

$$\eta\cdot\partial_\eta\Big[\eta\cdot\partial_x\Delta\psi_0 + 2(\eta\cdot\partial_x)^2\psi_1\Big] = \eta\cdot\partial_x\Big[\Delta\psi_0 + \eta\cdot\partial_\eta\Delta\psi_0 + 4\eta\cdot\partial_x\psi_1 + 2\eta\cdot\partial_x\eta\cdot\partial_\eta\psi_1\Big]. \tag{C.9}$$

The second term can be written as

$$\begin{aligned} (\eta\cdot\partial_\eta)^2\Box_x\psi_0 = \eta\cdot\partial_x\Big[& \Delta\psi_0 + 2\eta\cdot\partial_\eta\Delta\psi_0 + (\eta\cdot\partial_\eta)^2\Delta\psi_0 + 8\eta\cdot\partial_x\Psi_1 + 8\eta\cdot\partial_x\eta\cdot\partial_\eta\psi_1 \\ & + 2\eta\cdot\partial_x(\eta\cdot\partial_\eta)^2\psi_1\Big]. \end{aligned} \tag{C.10}$$

The third term can be written as

$$\begin{aligned} -\eta^2\Box_\eta\Box_x\psi_0 &= -\eta^2\Big[2\partial_\eta\cdot\partial_x\Delta\psi_0 + \eta\cdot\partial_x\Box_\eta\Delta\psi_0 + 4\Box_x\psi_1 + 8\eta\cdot\partial_x\partial_\eta\cdot\partial_x\psi_1 + 2(\eta\cdot\partial_x)^2\Box_\eta\psi_1\Big] \\ &= \eta^2\Delta^2\psi_0 + \eta\cdot\partial_x\Big[2\eta^2\Big(\Delta\psi_1 - 2\partial_\eta\cdot\partial_x\Delta\psi_0 - \Box_\eta\Delta\psi_0 - 8\partial_\eta\cdot\partial_x\psi_1 - 2\eta\cdot\partial_x\Box_\eta\psi_1\Big)\Big] \end{aligned} \tag{C.11}$$

The fourth and fifth terms in (C.8) do not require further manipulations since they are

already in pure gauge form. The last term can be written as

$$\eta^2(\partial_\eta \cdot \partial_x)^2\psi_0 = \eta^2(\Delta - \rho)^2\psi_0 = \eta^2\Delta^2\psi_0 + \eta^2\rho^2\psi_0 - 2\eta^2\rho\Delta\psi_0. \tag{C.12}$$

Then, adding (C.9), (C.10), (C.11), (C.12), and the two terms we did not manipulate, gives

$$\begin{aligned} W^2\psi_0 &= \eta^2\psi_0 + \delta_\epsilon\psi_0 = \eta^2\psi_0 + \rho^2\psi_0 - \rho^2\psi_0 + \delta_\epsilon\psi_0 = -\rho^2\psi_0 + \delta_\epsilon\psi_0 + (\eta^2+1)\rho^2\psi_0 \\ &= -\rho^2\psi_0 + \delta_\epsilon\psi_0 + \delta_\Xi\psi_0, \end{aligned} \tag{C.13}$$

which is precisely result (5.86). Note, however, that although our Ξ parameter already has its correct form, $\Xi = \rho^2\psi_0$, our ϵ_0 parameter is still messy when compared to (5.88). We will fix that a couple of sections below.

C.2 Computing $W^2\psi_1$

Now we will perform the same calculation for $W^2\psi_1$. This calculation is more difficult because the ϵ-transformation of ψ_1 contain η derivatives. Still, the principle is the same and no further difficulties should arise. We have

$$\begin{aligned} W^2\psi_1 = [&(\eta\cdot\partial_\eta)\Box_x + (\eta\cdot\partial_\eta)^2\Box_x - \eta^2\Box_\eta\Box_x - 2(\eta\cdot\partial_x)(\partial_\eta\cdot\partial_x)(\eta\cdot\partial_\eta) \\ &+ (\eta\cdot\partial_x)^2\Box_\eta + \eta^2(\partial_\eta\cdot\partial_x)^2]\psi_1. \end{aligned} \tag{C.14}$$

We will start our calculation with the last term of expression (C.14). We have

$$\begin{aligned} \eta^2(\Delta-\rho)^2\psi_1 &= \eta^2\rho^2\psi_1 + \Delta(\eta^2\Delta\psi_1) - 2\eta\cdot\partial_x\Delta\psi_1 - 2\Delta(\eta^2\rho\psi_1) + 4\rho\eta\cdot\partial_x\psi_1 \\ &= \eta^2\rho^2\psi_1 + 4\rho\eta\cdot\partial_x\psi_1 + \Delta(\eta^2\Delta\psi_1 - 2\eta^2\rho\psi_1 - 4\eta\cdot\partial_x\psi_1 - \Delta\psi_0). \end{aligned} \tag{C.15}$$

Now, the third term can be written as

$$
\begin{aligned}
-\eta^2\Box_\eta\Box_x\psi_1 &= \frac{1}{2}\Box_\eta\Delta(2\eta\cdot\partial_x\psi_1+\Delta\psi_0)\\
&= \frac{1}{2}\Delta(2\Box_\eta\eta\cdot\partial_x\psi_1+\Box_\eta\Delta\psi_0).
\end{aligned}
\tag{C.16}
$$

The fourth term can be written as

$$
\begin{aligned}
2\eta\cdot\partial_x(\Delta-\rho)\eta\cdot\partial_\eta\psi_1 &= -2\eta\cdot\partial_x\Delta\eta\cdot\partial_\eta\psi_1+2\rho\eta\cdot\partial_x\eta\cdot\partial_\eta\psi_1\\
&= -2\rho\eta\cdot\partial_x\psi_1+2\rho\eta\cdot\partial_x\eta\cdot\partial_\eta\psi_1-\Delta\Big(2\eta\cdot\partial_\eta\eta\cdot\partial_x\psi_1+\eta\cdot\partial_\eta\Delta\psi_0\\
&\qquad -\partial_\eta\cdot\partial_x\psi_0-2\eta\cdot\partial_x\psi_1+2\eta\cdot\partial_x\eta\cdot\partial_\eta\psi_1\Big).
\end{aligned}
\tag{C.17}
$$

The fifth term needs no manipulation, so all we are left with is the first two terms. The first term gives

$$
\begin{aligned}
\eta\cdot\partial_\eta\Box_x\psi_1 &= -\frac{1}{2}\eta\cdot\partial_\eta\Delta(2\eta\cdot\partial_x\psi_1+\Delta\psi_0)\\
&= -\frac{1}{2}\Delta(2\eta\cdot\partial_\eta\eta\cdot\partial_x\psi_1+\eta\cdot\partial_\eta\Delta\psi_0)+\frac{1}{2}\partial_\eta\cdot\partial_x(2\eta\cdot\partial_x\psi_1+\Delta\psi_0)\\
&= -\rho\eta\cdot\partial_x\psi_1-\frac{1}{2}\Delta(2\eta\cdot\partial_\eta\eta\cdot\partial_x\psi_1+\eta\cdot\partial_\eta\Delta\psi_0-\partial_\eta\cdot\partial_x\psi_0-2\eta\cdot\partial_x\psi_1).
\end{aligned}
\tag{C.18}
$$

Finally, the second term gives

$$
\begin{aligned}
(\eta\cdot\partial_\eta)^2\Box_x\psi_1 &= \frac{1}{2}(\eta\cdot\partial_\eta)^2(2\eta\cdot\partial_x\psi_1+\Delta\psi_0)\\
&= \rho\eta\cdot\partial_x\psi_1+\frac{1}{2}\rho\Delta\psi_0-2\rho\eta\cdot\partial_x\psi_1-2\rho\eta\cdot\partial_x\eta\cdot\partial_\eta\psi_1\\
&\quad -\frac{1}{2}\Delta\Big\{\Big[(\eta\cdot\partial_\eta^2-2\eta\cdot\partial_\eta+1\Big](2\eta\cdot\partial_x\psi_1+\Delta\psi_0)-\rho\psi_0\Big\}\\
&= -\rho\eta\cdot\partial_x\psi_1-2\rho\eta\cdot\partial_x\eta\cdot\psi_1\\
&\quad -\frac{1}{2}\Delta\Big\{\Big[(\eta\cdot\partial_\eta^2-2\eta\cdot\partial_\eta+1\Big](2\eta\cdot\partial_x\psi_1+\Delta\psi_0)-\rho\psi_0\Big\}.
\end{aligned}
\tag{C.19}
$$

Then, adding every term of (C.14) after the manipulations above we get the desired result

$$W^2\psi_1 = \eta^2\rho^2\psi_1 + \delta_\epsilon\psi_1 + \delta_\Xi\psi_1, \tag{C.20}$$

again with the correct form of Ξ but a messy form of ϵ_0. Result (C.20) is precisely result (5.87).

C.3 Working out ϵ_0

Now that we have reproduced results (5.86) and (5.87), we want to manipulate the form of the ϵ_0 parameter in the results above to show that they can be cast in the form (5.88). The calculations are very similar for the ϵ_0 parameter of ψ_0 and ψ_1, so will perform the calculation for ψ_0 only.

Writing down the ϵ_0 of result (C.13) explicitly we get

$$
\begin{aligned}
\epsilon_0 &= 2\Delta\psi_0 + 3\eta\cdot\partial_\eta\Delta\psi_0 + 12\eta\cdot\partial_x\psi_1 + 10\eta\cdot\partial_x\eta\cdot\partial_\eta\psi_1 \\
&+ (\eta\cdot\partial_\eta)^2\Delta\psi_0 + 2\eta\cdot\partial_x(\eta\cdot\partial_\eta)^2\psi_1 - 2(1+\eta\cdot\partial_\eta)\partial_\eta\cdot\partial_x\psi_0 \\
&+ \eta\cdot\partial_x\Box_\eta\psi_0 - \eta^2\Box_\eta\Delta\psi_0 - 8\eta^2\partial_\eta\cdot\partial_x\psi_1 - 2\eta^2\eta\cdot\partial_x\Box_\eta\psi_1 + 2\eta^2\Delta\psi_1 \\
&= (\eta\cdot\partial_x - \eta^2\Delta)\Box_\eta\psi_0 - 2\eta^2(\eta\cdot\partial_x\Box_\eta + 3\partial_\eta\cdot\partial_x - \rho)\psi_1 \\
&+ \cancel{2\partial_\eta\cdot\partial_x\psi_0} + 2\rho\psi_0 + \cancel{3}\eta\cdot\partial_\eta\partial_\eta\cdot\partial_x\psi_0 + 3\rho\eta\cdot\partial_\eta\psi_0 \\
&+ 12\eta\cdot\partial_x\psi_1 + 10\eta\cdot\partial_x\eta\cdot\partial_\eta\psi_1 + (\eta\cdot\partial_\eta)^2\partial_\eta\cdot\partial_x\psi_0 \\
&+ \rho(\eta\cdot\partial_\eta)^2\psi_0 + 2\eta\cdot\partial_x(\eta\cdot\partial_\eta)^2\psi_1 - \cancel{2\partial_\eta\cdot\partial_x\psi_0} - \cancel{2\eta\cdot\partial_\eta\partial_\eta\cdot\partial_x\psi_0} \\
&= \rho(2 + 3\eta\cdot\partial_\eta + (\eta\cdot\partial_\eta)^2)\psi_0 + (\eta\cdot\partial_x - \eta^2\Delta)\Box_\eta\psi_0 \\
&- 2\eta^2(\eta\cdot\partial_x\Box_\eta + 3\partial_\eta\cdot\partial_x - \rho)\psi_1 + \eta\cdot\partial_\eta(1+\eta\cdot\partial_\eta)\partial_\eta\cdot\partial_x\psi_0 \\
&+ 10\eta\cdot\partial_x\eta\cdot\partial_\eta\psi_1 + 12\eta\cdot\partial_x\psi_1 + 2\eta\cdot\partial_x\eta\cdot\partial_\eta\psi_1 \\
&+ 2\eta\cdot\partial_x\eta_\mu\eta\cdot\partial_\eta\partial_\eta^\mu\psi_1 \\
&= \eta\cdot\partial_\eta(1+\eta\cdot\partial_\eta)\partial_\eta\cdot\partial_x\psi_0 + \rho\left[2 + 3\eta\cdot\partial_\eta + (\eta\cdot\partial_\eta)^2\right]\psi_0 - (\eta^2\Delta - \eta\cdot\partial_x)\Box_\eta\psi_0 \\
&+ 2\left[2 + 3\eta\cdot\partial_\eta + (\eta\cdot\partial_\eta)^2\right]\eta\cdot\partial_x\psi_1 - 2\eta^2\left(\eta\cdot\partial_x\Box_\eta + 3\partial_\eta\cdot\partial_x - \rho\right)\psi_1,
\end{aligned}
\tag{C.21}
$$

where the last equality in (C.21) reproduces result (5.88), as desired.

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