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Modelos de transporte e susceptibilidades em AdS/CFT com termo topológico

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AdS/CMT models for transport and susceptibilities with Topological term

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To my dearest parents.

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"[...] la primera condición para cambiar
la realidad consiste en conocerla ."
Eduardo Galeano, "Las venas abiertas de América Latina"

Abstract

The AdS/CFT correspondence, a explicit example of the holographic principle, was started as an application of string theory, in a decoupling limit, as a tool to solve nonperturbative aspects of quantum field theories at the conformal fixed point, since it shows is a special duality between theories that allows one to map operators from one to the other, but inverting the coupling parameter. One important application is to condensed matter theory, dubbed AdS/CMT, and in particular to transport in such systems. Since we are looking at a phenomenological level, we don't need a "top-down model" from string theory.

Transport coefficients has been modeled using holography in ([1–3]). We want to expand these results to include various issues related to S-duality and transport in such models: the effect of a topological term in the gravity theory, needed for S-duality; the presence of both external fields and an explicit breaking of translational invariance due to an axion field; and the equivalence of different models for transport.

We also compare phenomenological models (bottom-up) with those top-down through the calculation of thermodynamical quantities, in particular with the calculation of the susceptibilities.

Key-words: Holography, AdS/CMT, Transport Coefficients, S-duality, Quantum gravity, Nersnt Effect

Resumo

A correspondência AdS/CFT, um exemplo explícito do princípio holográfico, iniciou como uma consequência de teoria das cordas, em um limite de desacoplamento, como uma ferramenta para resolver aspectos não-perturbativos de teoria quântica de campos no ponto fixo conforme, já que esta mostra uma dualidade especial entre teorias que permite mapear operadores de uma para outra (dualidade forte-fraca), mas invertendo o parâmetro de acoplamento, através de um mapa holográfico. Um imporante desenvolvimento é para matéria condensada, chamada AdS/CMT, e em particular para transporte nestes sistemas. Como estamos olhando em um nível fenomenológico, nós não precisamos de um modelo construído a partir de teoria de cordas.

Coeficientes de transporte tem sido estudados usando holografia em ([1–3]). Nós queremos expandir estes resultados para incluir varios efeitos relacionados a S-dualidade e transporte nestes modelos: o efeito de um temo topológico na teoria gravitacional. essencial para a S-dualidade; a existência de ambos campos externos e uma quebra explicita da invariancia translacional através de um campo axion; e a equivalência dos diferentes modelos de transporte.

Também comparamos modelos fenomenológicos (bottom-up) com os construídos a partir de cordas (top-down) através do cálculo de quantidades termodinâmicas gerada por estes, em particular pelo cálculo da susceptibilidade.

Palavras-chave: Holografia, AdS/CMT, Coeficientes de Transporte, S-dualidade, Gravidade Quântica, Efeito Nersnt

Contents

Chapter 1

Introduction

Perturbation theory is quintessential in analysing models in modern physics. The idea is to take intractable problems and approximate to models that we know how to deal with. For example, let's say we have a model that we can get absolute results $\mathcal{H}_0$. We want to study this model under a new interaction $\mathcal{H}_1(\alpha)$, where α is a parameter of this new term. Then we can write the model we are interested in as

$$\mathcal{H}_0 + \mathcal{H}_1(\alpha). \tag{1.1}$$

It often happens that this new model is challenging to calculate (sometimes impossible), then one method that we can use is to expand the interaction term in a series

$$\mathcal{H}_0 + c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 + ... \tag{1.2}$$

where the constants c_i are to be determined.

Of course, this method only makes sense if the power series converges, i.e., $\alpha < 1$. We say that such theories are perturbative, and since we usually have α as a coupling constant of the interaction, we can also call this as a weakly coupled theory.

Not all physical systems are perturbative though, as many materials are studied in condensed matter have strongly coupled electrons. Some of this materials are created and observed in laboratories. An example of great interest to cosmology is the Quark Gluon Plasma (QGP), which strong evidence points it to be a strongly coupled fluid [4].

The holographic principle gives us a technique to deal with strongly coupled theories [5]. The method is to map out the theory to a $d+1$-dimensional theory. The main gain on doing this is that the new theory has a S-duality with the original one, i.e., if the coupling constant of one of the theories is α, the other one is $1/\alpha$. This means that even in the range of nonpertubative parameters ($\alpha > 1$) we can change our focus to a perturbative theory (with $1/\alpha < 1$). Then, since the descriptions of the two theories are the same, we can conjecture that the perturbative results we get from one can be mapped out to the other theory.

Although the principle is a conjecture, we have explicit examples of this realization. The AdS/CFT correspondence [6] matches a quantum field theory with a string theory. When applied to quantum condensed matter theories we use the name AdS/CMT.

The correspondence also works in the other direction, we may use CMT methods to solve problems from quantum gravity. This process would involve coming up with novel effective field theories in order to compute important questions involving the extensions of the standard model to include gravity. This could lead to new experiments for string theory and its competitors.

The scope of this work will be to explore the correspondence in order to calculate transport coefficients of condensed matter using the mathematics of a black hole in AdS space. This was introduced in [7] and [8], with one of the first example of such calculations: a minimal value for the ratio of viscosity and entropy of a fluid, known as the KSS bound [1]. Later work using similar calculations led to the computation of thermoelectric conductivities ([2] and [3]) from a dyonic black hole in the gravity dual, with a more general analysis in [9] (see also, for instance, more recently [10]). We aim to expand the result in these latter references by introducing new terms in the quantum gravity description, and computing what these new terms impact the dual condensed matter theory.

One set of methods was considered, more recently, in [11–17], where one considers an Einstein-Maxwell-dilaton ($g_{\mu\nu}, A_\mu, \Phi$) system in the gravity dual, also coupled to axions (χ_1, χ_2), that allows for a more general analysis of transport, from the horizon of a black hole, based on the application of the membrane paradigm to AdS/CFT in the form started in [18], relating the horizon to the boundary, where standard AdS/CFT quantities are obtained, via a radial evolution equation. In [19–21], an extension of the analysis was considered, by adding a topological $W(\Phi)F_{\mu\nu}\tilde{F}^{\mu\nu}$ term and considering the resulting S-duality properties. One can also use the attractor mechanism and Sen's entropy function [22] to calculate conductivities from black hole horizons [17], and this method was also used in [20].

In this thesis we are interested in considering the effect of a $W(\Phi)$ topological term on the calculation of thermodynamics and tranport in [2,3], as well as the calculations via the attractor mechanism. We will first consider the case of a constant $W(\Phi)$ and then, in order to obtain more nontrivial results, we try to consider varying $W(\Phi)$. However, it turns out that the simplest possible model is to consider directly a fixed $W(\Phi(z)) = W(z)$, with z the radial direction, since otherwise solving the equations is very difficult. This work was summarized in the paper [23].

The AdS/CFT correspondence usually relates strongly coupled field theory to weakly coupled string theory in its classical supergravity limit, with "top-down" models, derived from systems of branes in a decoupling limit. Common applications to condensed matter are usually phenomenological, "bottom-up" constructions. That applies in particular to models of transport in condensed matter systems.

However, there are a few examples of top-down models as well, most notably the ABJM vs. $AdS_4 \times \mathbb{CP}^3$ correspondence [24], which has been used as a sort of a prototype for transport in strongly coupled 2+1 dimensional condensed matter systems. Of course, it is not a top-down model in the sense that there is no *derived* relation of the ABJM model to any condensed matter system (unlike supersymmetric $SU(N)$ gauge theories in 3+1 dimensions, thought of as an extension of the gluon theory for QCD), only a phenomenological one: it gives similar physics. But the holographic map is derived. At nonzero temperature, the dyonic black hole in AdS_4 has been used as a model for 2+1 dimensional transport in the presence of a magnetic field. This is the origin ot the model describe by [2, 3].

In [21], for the transport found the generic case in [20], the Wiedemann-Franz law was obtained by a combination of the two methods. In particular, the matrix of susceptibilities χ_s, calculated as the second order derivatives of the thermodynamic potential in the dyonic black hole background, and was related via the matrix of diffusivities D to the matrix of conductivities (as expected from the general theory of the hydrodynamic limit), for which the results in the perturbative background from [20].

But that implies the assumption that dyonic black hole background of [2, 3] and the perturbative one of [20] give the same thermodynamics, which is not obvious. Therefore in this paper we investigate the possibility of these two results giving the same answer. This has implications beyond the specific case considered here, as it measures the correctness of importing results from a top-down construction to a bottom-up one, or vice versa. This work was summarized in the paper [25].

The following is divided into two parts:

- Part 1 will be a review of how we use AdS/CMT to model transport coefficients of condensed matter theories.
  - Chapter 2 will review the Condensed Matter Theory aspects we will try to reproduce in the Holographic model.

    We show the classical models of transport, from Boltzmann equation and its relativistic version to thermoelectric transport.
  - Chapter 3 will be a lighting review of the AdS/CFT correspondence. We avoid using too much results from string theory in order to keep to the most important aspects of the duality for this work.

    We first define and give examples of the S-duality, an important duality we will look for in our models in Part 2, we also define AdS spaces and conformal fields, to later then introduce the correspondence between them. We take a more historical approach in this review, giving the intuition behind the discoveries.
  - Chapter 4 reviews how we can introduce temperature to a quantum field theory, and apply it to also construct temperature for the dual theory, which is necessary for the calculation of thermoelectric properties of transport.

    The theory of black hole thermodynamics, developed by Beckentein and Hawking, is the starting step for us to develop the appropriate background for our theory in the gravity side of the duality in order to retrieve a dual quantum theory with the wanted properties.
  - Chapter 5 reviews how we can obtain transport coefficients using the holographic dual of a theory. We also give a historical review of the KSS bound.
- Part 2 will discuss the results from this work.
  - In chapter 6 we reproduce the calculation done in [2] in order to obtain thermoelectric transport coefficients from dyonic black holes, and we then introduce

topological terms into the string theory action to expand the condensed matter theories that this method can model. We calculate the thermodynamics, the holographic transport coefficients σ_{ab} and α_{ab} from the Kubo formulas at the boundary, and then the attractor mechanism using Sen's entropy function, to write these in terms of charges and parameters at the horizon.

- In chapter 7 we define a new model with a radially dependent topological term so that we can gain new non-trivial contributions to the transport coefficient. We find the solutions for fluctuations, though the holographic transport coefficients are too complicated to write, as are the results of the entropy function formalism. The effect of S-duality is explained, as is the introducing of anisotropy in the model.
- In chapter 8 we first consider the perturbative model with topological term, but only B, B_1 external fields, and calculating the thermodynamics, the magnetizations and the susceptibilities with this simplified version of the fluctuations.

  We then calculate the transport coefficients for a more general version of the model, with E and $\xi = (\nabla T)/T$ as external fields as well. After that we calculate the susceptibilities for this general case, and compare with the AdS_4 dyonic black hole results.
- In chapter 9 we conclude.

In Appendix A we give a quick review of bosonic string theory as to explain where the particles in our models comes from, and in the Appendices B and C contain some long formulas.

Part I

Review of AdS/CMT models of transport

Chapter 2

CMT

2.1 Classical Transport

We review the transport coefficients in hydrodynamics and electromagnetic theory from classical mechanics. Methods for calculating this properties using the gauge/gravity duality were developed in the last decades as an alternative to methods from quantum field theory that utilizes perturbation theory. The hope is that the duality is going to be able to calculate the coefficients in regimes where perturbation theory is computationally hard or even impossible.

The review of classical Hydrodynamics follows [26]. It's going to be necessary for us to understand the arguments behind the Kubo's formula, the main mechanism behind the AdS/CFT method to calculate the transport coefficients. We then later introduce one of the main first cases of applications of this method, the **KSS bound**.

2.1.1 Boltzmann Equation

We start with the classical Boltzmann equation that describes the fluid dynamics with a thermal gradient:

We start by analysing a fluid with $\mathcal{N}$ particles in some volume. We define $f(\mathbf{r}, \mathbf{v}, t)$ the distribution function of the system, such that:

$$\mathcal{N} = \int f(\mathbf{r}, \mathbf{v}, t) d^3\mathbf{r} d^3\mathbf{v}. \tag{2.1}$$

We also can define the number density of particles:

$$n = \int f(\mathbf{r}, \mathbf{v}, t) d^3 v \tag{2.2}$$

And the particle current density is

$$J_i = \int v_i dn \tag{2.3}$$

The average of any function A can be described as

$$\langle A \rangle = \frac{1}{n} \int A f(\mathbf{r}, \mathbf{v}, t) d^3 v \tag{2.4}$$

Now consider the ideal case of the system flowing in one direction without any type of collision between the particles, then we would have

$$df = 0. \tag{2.5}$$

Expanding the differential form then

$$\frac{\partial f}{\partial t} dt + \nabla_{\mathbf{r}} f \cdot d\mathbf{r} + \nabla_{\mathbf{v}} f \cdot d\mathbf{v} = 0 \tag{2.6}$$

In the existence of collisions, we will have a differential due to it, then

$$\frac{\partial f}{\partial t} dt + \nabla_{\mathbf{r}} f \cdot \mathbf{v} dt + \nabla_{\mathbf{v}} f \cdot \mathbf{a} dt = (\Delta f)_{\text{col}} dt \tag{2.7}$$

where we used the fact that $adt = dv$ and $vdt = dr$.

It's very common to assume the collision part of the above equation to have a dependence of a relaxation time τ_C, such that

$$(\Delta f)_{\text{col}} = \frac{f - f_0}{\tau_C}, \tag{2.8}$$

where f_0 is the distribution at thermal equilibrium. Then the **Boltzmann equation** can be written in its most usual form:

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{r}} f \cdot \mathbf{v} + \nabla_{\mathbf{v}} f \cdot \mathbf{a} = \frac{f - f_0}{\tau_C}. \tag{2.9}$$

Now, let's assume a function $g(v)$ that is conserved in collisions. This means that

$$\int g (\Delta f)_{\text{col}} d^3 \mathbf{v} = 0 \tag{2.10}$$

This comes from the fact that the above average is constant. Then utilizing the Boltzmann equation we have

$$\int g \frac{\partial f}{\partial t} d^3 \mathbf{v} + \int g \nabla_{\mathbf{r}} f \cdot \mathbf{v} d^3 \mathbf{v} + \int g \nabla_{\mathbf{v}} f \cdot \mathbf{a} d^3 \mathbf{v} = 0. \tag{2.11}$$

Note that

$$\nabla_{\mathbf{v}} f \cdot g \mathbf{a} = \nabla_{\mathbf{v}} (g \mathbf{a} v) - f \nabla_{\mathbf{v}} \cdot (g \mathbf{a}) \tag{2.12}$$

So, after some integral manipulations o the Boltzmann equation, we end up with

$$\frac{\partial}{\partial t} (n \langle g \rangle) + \nabla_{\mathbf{r}} \cdot (n \langle g \mathbf{v} \rangle) - n \mathbf{a} \cdot \langle \nabla_{\mathbf{v}} g \rangle = 0, \tag{2.13}$$

where we consider the integral over the first term in the right hand side of 2.12 zero since it is a boundary term. This formula is known as the continuity equation. It's one of the main results of the Boltzmann theory.

2.1.2 Viscous Navier-Stokes Equation

To make it easier, let us start with a fluid without any viscosity. We recall the continuity equation and we take $g = m\mathbf{v}$ as the conserved quantity (the collisions in the fluid conserve momentum). Then we have, where $\rho = mn$ is the mass density,

$$\frac{\partial}{\partial t}(\rho\langle v_i\rangle) + \nabla_{\mathbf{r}} \cdot (\rho\langle v_i \mathbf{v}\rangle) = \rho\mathbf{a} \cdot \langle \cancelto{1}{\nabla_{\mathbf{v}} v_i} \rangle. \tag{2.14}$$

Defining the tensor

$$\Pi_{ij} = \rho\langle v_i v_j\rangle = \rho\langle v_i\rangle\langle v_j\rangle + P_{ij}, \tag{2.15}$$

called momentum flux density tensor, where

$$P_{ij} = \rho\langle(v_i - \langle v_i\rangle)(v_j - \langle v_j\rangle)\rangle, \tag{2.16}$$

and noting that any external forces are given by $F_i = ma_i$ the continuity equation becomes

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = nF_i \tag{2.17}$$

In the case of no external forces and an isotropic fluid we have that $P_{ij} = p\delta_{ij}$, where p is pressure, then we have

$$\begin{aligned} \frac{\partial}{\partial t}(\rho v_i) &= -\frac{\partial}{\partial x_j}\Pi_{ij} \\ \Pi_{ij} &= \rho v_i v_j + p\delta_{ij}. \end{aligned} \tag{2.18}$$

The generalization to viscous fluids comes from an addition of a viscous stress tensor σ_{ij} in Π_{ij}

$$\Pi_{ij} = \rho v_i v_j + p\delta_{ij} + \sigma_{ij}. \tag{2.19}$$

Since we are dealing with a type of friction in fluids, we expect terms that are spatial derivatives in the velocity. Here we assume that we can expand the viscosity term in the number of derivatives, and the hydrodynamic limit is the one which only linear terms of the derivatives is considered. The idea is that in this range we gain an effective theory at large scales. So, separating the trace and traceless part, we get the ansatz

$$\sigma_{ij} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_k v_k\right) + \zeta\delta_{ij}\partial_k v_k, \tag{2.20}$$

where $\eta > 0$ is called shear viscosity and $\zeta > 0$ is called bulk viscosity.

Then the continuity equation becomes the Navier-Stokes equation:

$$\rho\left(\frac{\partial v_i}{\partial t} + (\mathbf{v}\cdot\nabla(v_i)\right) = \frac{\partial}{\partial x_i}p + \frac{\partial}{\partial x_j}\left(\eta\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\delta_{ij}\partial_k v_k\right) + \zeta\partial_k v_k\right). \tag{2.21}$$

2.2 Relativistic Hydrodynamics

The relativity generalization for the Navier-Stokes equation needs to satisfy the conservation of the energy momentum tensor $T^{\mu\nu}$, the generalization of Π_{ij},

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.22}$$

For a viscous fluid, $T^{\mu\nu}$ can be written as, where u_μ is the 4-velocity,

$$T^{\mu\nu} = \rho u^\mu u^\nu + pP^{\mu\nu} + \Pi^{\mu\nu}_{(1)}. \tag{2.23}$$

where

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \tag{2.24}$$

Here the symmetric viscosity term $\Pi^{\mu\nu}_{(1)}$ is going to be in the hydrodynamic limit, i.e., it is expanded up to linear terms of the derivatives ∂u.

We are left with a choice of the frame of reference we choose. Historically it has been studied in the Eckart frame, the one that moves with the particles of the fluid, and the Landau-Lifshitz frame, the one with no energy dissipation fluid. We choose the latter since it has been shown to solve ambiguity problems in the derivation of the equations [27].

In our notation, the Landau frame amounts to

$$\Pi^{\mu\nu}_{(1)} u_\mu = 0. \tag{2.25}$$

Then the general form it can take, considering it is symmetric, composed of linear derivatives and satisfies the equation above,

$$\Pi^{\mu\nu}_{(1)} = -2\eta\sigma^{\mu\nu} - \zeta\theta P^{\mu\nu}, \tag{2.26}$$

where

$$\begin{aligned} \theta &= \nabla_\mu u^\mu, \\ \sigma^{\mu\nu} &= \nabla^{(\mu} u^{\nu)} + u^{(\mu} a^{\nu)} - \frac{1}{d-1}\theta P^{\mu\nu}, \\ a^\mu &= u^\nu \nabla_\nu u^\mu. \end{aligned} \tag{2.27}$$

We see that the coefficients η and ζ play the same role as before, separating a traceless and a trace part.

2.3 Thermo-Electrodynamical transport

The (linear) response of the current function due to an electric field perturbation is the classical Ohm's law

$$\langle \vec{J} \rangle = \sigma \vec{E}. \tag{2.28}$$

There is also a possibility of considering another term coming from a temperature gradient $-\alpha\nabla T$ in the sense of a Nernst thermoelectric effect.

$$\langle \vec{J} \rangle = \sigma \vec{E} - \alpha \vec{\nabla} T \tag{2.29}$$

In the side of the (linear) response of the heat current, we have a response coming from the temperature gradient (as usual)

$$\langle \vec{Q} \rangle = -\bar{\kappa} \vec{\nabla} T \tag{2.30}$$

And a response from the energy generated by the electric field $\alpha T E$ in the same sense as it does the resistance of a material.

$$\langle \vec{Q} \rangle = -\bar{\kappa} \vec{\nabla} T + \alpha T \vec{E} \tag{2.31}$$

The equations above can be summarized in the symmetric matrix below:

$$\begin{pmatrix} \langle J_i \rangle \\ \langle Q_i \rangle \end{pmatrix} = \begin{pmatrix} \sigma & \alpha T \\ \alpha T & \bar{\kappa} \end{pmatrix} \begin{pmatrix} E_i \\ -\nabla_i T / T \end{pmatrix}. \tag{2.32}$$

Note that σ, α and κ are matrices, so we could have, for a flow in the $x-y$ plane, $\sigma = \sigma_{xx} I + \sigma_{xy} \epsilon$. The same for the other matrices.

Chapter 3

AdS/CFT and S-Duality

3.1 S-Duality

Our description of the dynamics of the universe is very sensible to scales our system of interest is in. For example, quarks and gluons under the effects of QCD can experience an asymptotic freedom when at high energy scales, i.e., the forces between them are very weak, so one can use perturbation theory to study this system. But at low energy scale the coupling is strong, so strong that perturbative methods are impossible to be applied.

In this context, the so called **S-duality** (or strong-weak duality) exists. The name duality means that we are going to find a relation between two different theories (usually a quantum field theory or a string theory) so that we can translate a computation from one theory to the other. S-duality is then a duality from one theory with strong coupling to a theory with weak coupling. The usefulness of such relations happens when we have a very hard (or impossible) problem for a theory, then we use the duality to solve this problem in the other theory, and in this new theory the problem may be trivial. We the use the duality again to translate the solutions to the original theory.

There are many examples of S-dualities in the literature. We are going to show the most simple of such examples where this duality happens, the symmetries of the Maxwell's equations.

The Maxwell's equations in the vacuum are written as

$$\begin{aligned} \nabla \cdot E &= 0 \\ \nabla \cdot B &= 0 \\ \nabla \times E &= -\partial_t B \\ \nabla \times B &= \mu_0 \epsilon_0 \partial_t E. \end{aligned} \tag{3.1}$$

Note now that if we make the substitution $E \to B$ and $B \to -\mu_0 \epsilon_0 E$ the equations above are unchanged. This symmetry is called the Maxwell duality, we can relate two different physical systems together since they have the same equations of motion.

The duality breaks when we consider the equations with charges and currents

$$\begin{aligned} \nabla \cdot E &= \frac{\rho_e}{\epsilon_0} \\ \nabla \cdot B &= 0 \\ \nabla \times E &= -\partial_t B \\ \nabla \times B &= \mu_0 J_e + \mu_0 \epsilon_0 \partial_t E. \end{aligned} \tag{3.2}$$

The electric charge is defined as

$$q = \int_V \rho_e dV \tag{3.3}$$

In order to restore the Maxwell duality we consider magnetic charge (h) and current (J_m). Then the equations are

$$\begin{aligned} \nabla \cdot E &= \frac{q}{\epsilon_0} \delta^3(x) \\ \nabla \cdot B &= \mu_0 h \delta^3(x) \\ \nabla \times E &= -\mu_0 J_m - \partial_t B \\ \nabla \times B &= \mu_0 J_e + \mu_0 \epsilon_0 \partial_t E. \end{aligned} \tag{3.4}$$

Then the duality exists with the transformations

$$\begin{aligned} E &\to B \\ B &\to -\mu_0 \epsilon_0 E \\ J_e &\to \mu_0 \epsilon_0 J_m \\ J_m &\to -J_e \\ q &\to \mu_0 \epsilon_0 h \\ h &\to -q \end{aligned} \tag{3.5}$$

This are the equations that describes the Dirac magnetic monopole, developed by Dirac in the 1930's. He found that the quantization condition for this system is

$$qh = 2\pi\hbar N, \quad N \in \mathbb{Z}. \tag{3.6}$$

This condition can be an explanation as to why the electric charge of the electron and the proton are the same. There are no experimental evidence for magnetic monopoles yet, but there are many interesting model that describe it as a feature of the topology of the manifold our physical system lives.

3.1.1 Example of S-Duality: The 2D Ising Model

The Ising Model describes electron's spins that can be either up or down (or any list of variables S_i that can be either $+1$ or -1). Define N as the total number of electrons. The system is in a 2D-lattice space and the lattice has a number of links L where the spins interact with intensity J at temperature T. The Hamiltonian for this system is

$$H = -K \sum_{i,j} S_i S_j \qquad K = J/T. \tag{3.7}$$

This system shows an interesting duality between the model at low temperatures and another model at high temperatures, called Kramers-Wannier duality ([28], [29]). The derivation of the duality in this will follow [30]. For a different and topological way of looking at this and other dualities in the Ising Model see [31].

Usual statistical mechanics methods shows that the model has two different phases depending on the temperature:

- At low T the spins are going to point at the same direction to minimize the energy. This phase is know as ordered phase,
- At high T thermal fluctuations are going to dominate and the spins will point to random direction. This phase is known as disordered phase.

To show the duality, lets compute the partition function:

$$Z(K) = \sum e^{-\beta E(K)} \tag{3.8}$$

At high temperatures (let's use K_h to denote K in this range), one can expand Z to find that

$$Z_{high}(K_h) = (\cosh K_h)^L 2^N (1 + N(\tanh K_h)^2 + ...). \tag{3.9}$$

At low temperatures (let's use K_l to denote K in this range) we have another expansion

$$Z_{low}(K_l) = 2e^{K_l L}(1 + N(e^{-2K_l})^2 + ...). \tag{3.10}$$

If we make the substitution

$$e^{-2K_l} = \tanh K_h \tag{3.11}$$

the two partition functions are going to be the same up to a multiplicative term in front of the series.

The free energy per site is

$$F = -J\frac{1}{N}\ln Z \tag{3.12}$$

Then we have that the free energy for low and high temperatures are going to be related by

$$F_l = F_H + \frac{1}{2}K_h T \ln\left(\sinh\left(2K_h\right)\right). \tag{3.13}$$

The critical value we must have $K_h = K_L = K_C$, then

$$K_C = \frac{2J}{\ln\left(1+\sqrt{2}\right)} \tag{3.14}$$

Note that this duality between a theory with temperature $T > T_C$ and a theory temperature $T < T_C$ is a strong-weak duality between two theories with the same degrees of motion and the same number of dimensions, since in fact is a duality of a theory with itself (usually called a self-duality).

The AdS/CFT duality that is going to be the main topic of this paper, although it is also a strong-weak duality, differs from the Kramers-Wannier duality in the sense that it is a duality between two different theories that have different dimensions and different degrees of motion. This comes from the fact that the principle behind the AdS/CFT is the Holographic principle, to be described in a later section.

3.2 AdS

We start with the Einstein Equation in general relativity:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{3.15}$$

Note that we have a cosmological constant Λ. This equation can also be understood as the Euler-Lagrange equation of the Einstein-Hilbert action:

$$S_{EH} = \frac{1}{2\kappa^2}\int d^dx\sqrt{-g}(R - 2\Lambda + \mathcal{L}_M), \tag{3.16}$$

where $\mathcal{L}_M$ is the Lagrangian of the matter content that generates the tensor $T_{\mu\nu}$. The Einstein equation then tells us that the matter content shapes the geometry of space defined by the Einstein tensor $G_{\mu\nu}$.

Euclidean Geometry

Let us focus on empty space $T_{\mu\nu} = 0$. We want to find solutions for 3.15 such that the spaces have maximal symmetry. For Riemannian manifolds (Locally Euclidean) the full solution to this problem is (using the notation in [32]):

$$ds^2 = \frac{d\chi^2}{1 - k\chi^2} + \chi^2 d\Omega^2_{d-1}, \tag{3.17}$$

where $k = \{0, \pm 1\}$,

- $k = 0$ corresponds to flat Euclidean space,
- $k = 1$ corresponds to a sphere,
- $k = -1$ corresponds to a hyperboloid.

Lorentzian Geometry

If we try to look for Lorentzian manifolds (locally Minkowski), we also have 3 solutions that differ depending on the sign of the cosmological constant Λ[1]:

- For $\Lambda = R = 0$ we have flat Minkowski space,
- For $\Lambda > 0$ we have the de Sitter space,
- For $\Lambda < 0$ we have the Anti-de Sitter space.

The $d+1$ dimensional AdS space can be written as an embedding in a $(2, d)$ Minkowski space (for $i = \{1, ..., d\}$):

$$ds^2 = -(dx^0)^2 + (dx^i)^2 - (dx^{d+1})^2, \tag{3.18}$$

with the constraint

$$-(x^0)^2 + \sum_i (x^i)^2 - (x^{d+1})^2 = -\alpha^2, \tag{3.19}$$

where α is known as the radius of the AdS space.

Thus we say that the AdS space is quasi-sphere[2] of radius α in a $(2, d)$-dimensional space given by the metric (3.18).

This manifold has a boundary at

$$\partial(AdS_{d+1}) = \left\{ \mathbf{x} \middle| -(x^0)^2 + \sum_i (x^i)^2 - (x^{d+1})^2 = 0 \right\}. \tag{3.20}$$

The Einstein equation is satisfied for the AdS space with

$$\Lambda = -\frac{d(d-1)}{2\alpha^2}. \tag{3.21}$$

This result is used to define the relation between the Cosmological constant and the radius of the space.

The Ricci curvature is given by

$$R_{\mu\nu} = \frac{-(d-1)}{\alpha^2} g_{\mu\nu} \tag{3.22}$$

and so the Ricci scalar is

$$R = \frac{-d(d-1)}{\alpha^2}. \tag{3.23}$$

[1] And consequently on the sign of the Ricci scalar R as well.

[2] A generalization to the hyper-sphere and hyper-plane in Euclidean spaces to Lorentzian spaces is called a quasi-sphere.

Global coordinates

We can also define, from the following coordinate changes (**X** $\rightarrow (\tau, \rho, \theta, \phi_1, ..., \phi_{d-3})$:

$$\begin{aligned} X_1 &= \alpha \cosh\rho \cos\tau \\ X_2 &= \alpha \cosh\rho \sin\tau \\ X_i &= \alpha \sinh\rho \hat{x}_i \ . \end{aligned} \tag{3.24}$$

Where $\hat{x}_i$ parametrize S^{d-2} and $\tau \in [0, 2\pi)$. Then the AdS metric becomes

$$ds^2 = \alpha^2(d\rho^2 + \cosh^2\rho\ d\tau^2 + \sinh^2\rho\ d(dx^i)^2) \tag{3.25}$$

Now taking $r = \alpha \sinh\rho$ and $t = \alpha\tau$ we can write the metric of the AdS space as

$$ds^2 = -\left(1 + \frac{r^2}{\alpha^2}\right) dt^2 + \frac{1}{1 + \frac{r^2}{\alpha^2}} dr^2 + r^2(dx^i)^2, \tag{3.26}$$

known as the global coordinates.

Poincaré metric

We can also define, from the following coordinate changes (**X** $\rightarrow (t, r, \vec{x})$:

$$\begin{aligned} X_1 &= \frac{\alpha^2}{2r} \cosh\rho \cos\tau \\ X_2 &= \frac{r}{\alpha} t \\ X_i &= \frac{r}{\alpha} x_i \\ X_{n+i} &= \frac{\alpha^2}{2r}\left(1 - \frac{r^2}{\alpha^4}\left(\alpha^2 - \vec{x}^2 + t^2\right)\right) \ , \end{aligned} \tag{3.27}$$

the metric below

$$ds^2 = \frac{r^2}{\alpha^2}(-dt^2 + (dx^i)^2) + \frac{\alpha^2}{r^2} dr^2 \tag{3.28}$$

known as the Poincaré metric, where the boundary of the AdS-space is located at $r \rightarrow \infty$.

This coordinates only cover half of the space, known as the Poincare patch. Another parameterization commonly used in physics defines $z = \alpha^2/r$ and we get

$$ds^2 = \frac{\alpha^2}{z^2}(-dt^2 + (dx^i)^2) + \frac{\alpha^2}{z^2} dz^2. \tag{3.29}$$

The boundary of the AdS space is then at $z = 0$.

In Figure (3.1) we can view this metric as a Minkowski space being scaled by the coordinate z.

This coordinate system is the one usually used in AdS/CFT correspondence applications.

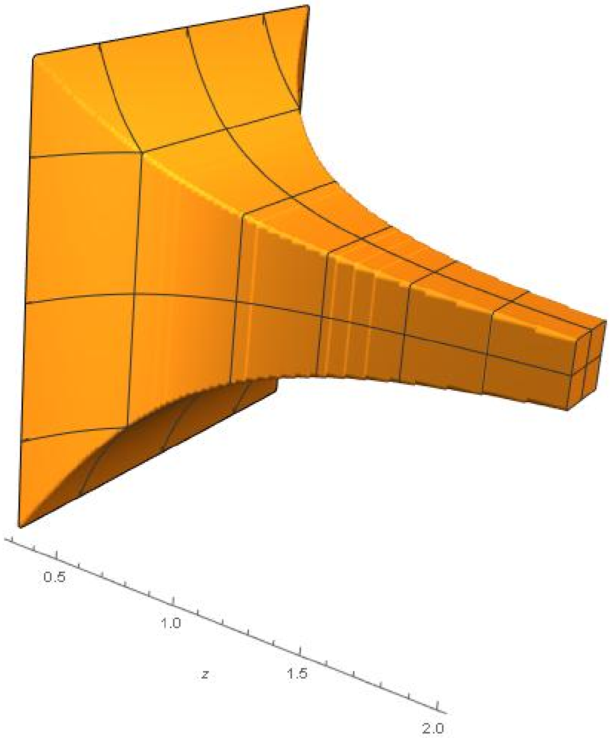

Figure 3.1: The AdS space in the Poincare coordinates (3.29) can be viewed as a Minkowski space $\eta_{\mu\nu}$ being scaled by the coordinate z with $\frac{\alpha^2}{z^2}\eta_{\mu\nu}$. Note that the AdS boundary is at $z \to 0$.

3.3 CFT

Quantum field theories (QFT) with conformal symmetry are called conformal field theories (CFT). They have an immense number of applications, from statistical mechanics, condensed matter and string theory. From one side, they are fixed points of the renormalization group flow, i.e., quantum field theories tend to become conformal at high or low energy scales (outside of some more exotic behaviours like limit cycles [33]), therefore CFT's describe a large number of theories of interest at a critical point. Also, a large number of interesting effective theories in condensed matter have conformal symmetry.

The extra symmetries gives CFTs unique properties, for example at 2 dimensions the algebra is infinite dimensional and sometimes one can solve the theory exactly. Higher dimensional conformal theories became more relevant since the conjecture of the AdS/CFT correspondence (to be explained later). Traditional and relevant references for CFT are [34], [35].

For any metric $g_{\mu\nu}$ of a space with coordinates x_μ, the set o transformations $\delta(x_\mu) = x'_\mu$ that leave the metric invariant up to a scale factor

$$g_{\mu\nu}dx^\mu dx^\nu = \Omega^2(x) g_{\mu\nu}dx'^\mu dx'^\nu \tag{3.30}$$

is called the conformal group (or angle-preserving group). It consist of the Poincaré Group (translations and rotations) generated by P_μ and $M_{\mu\nu}$ as any relativistic quantum

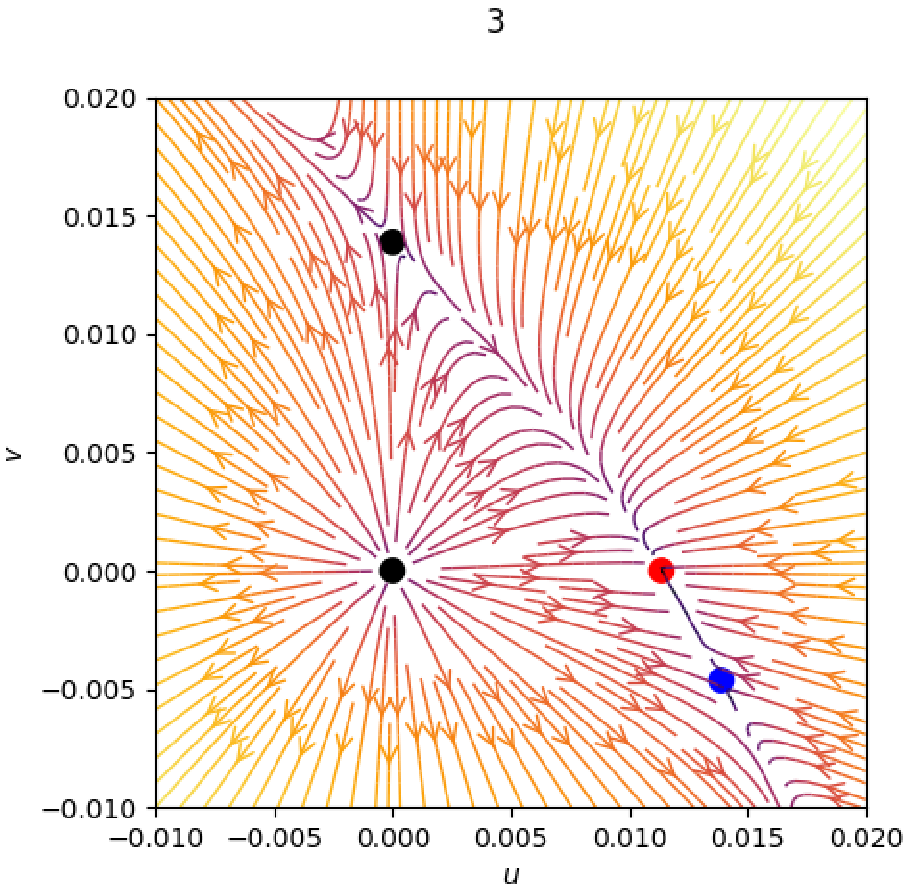

Figure 3.2: CFTs are fixed points in the renormalization group equations. That means that when the energy scale μ changes, the couplings that define a quantum theory also change, unless the theory is conformally invariant. The picture shows the RG flow for the $O3$ QFT, and the points corresponds to the set of parameters where the thoery becomes conformal. Usually QFTs goes either to infinite coupling or to a fixed point at the limits $\mu \to 0$ and $\mu \to \infty$. Image taken from [33].

field theory,

$$x'^{\mu} = a^{\mu} + m^{\mu\nu} x_{\nu}, \tag{3.31}$$

scale transformations generated by D

$$x'^{\mu} = \lambda x^{\mu} \tag{3.32}$$

and the special conformal transformation generated by K_{μ}

$$x'^{\mu} = x^2 b^{\mu} - 2x^{\alpha} b_{\alpha} x^{\mu}, \tag{3.33}$$

where we can write the generators as

$$\begin{aligned} P_{\mu} &= -i\partial_{\mu}, & D &= -x^{\mu}\partial_{\mu}, \\ M_{\mu\nu} &= -i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}), & K &= 2ix^{\mu}x^{\nu}\partial_{\nu} - ix^2\partial_{\mu}. \end{aligned} \tag{3.34}$$

One can now easily check the commutation relations of 3.34 to define the conformal algebra.

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i(\delta_{\mu\rho}P_\nu - \delta_{\nu\rho}P_\mu), \\
[M_{\mu\nu}, M_{\rho,\sigma}] &= i(\delta_{\mu\rho}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\mu\rho} - \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\sigma}M_{\nu\rho}), \\
[M_{\mu\nu}, K_\rho] &= i(\delta_{\mu\rho}K_\nu - \delta_{\nu\rho}K_\mu), \\
[K_\mu, P_\nu] &= 2\delta_{\mu\nu}D - 2iM_{\mu\nu}, \\
[D, K_\mu] &= -K_\mu, \qquad\qquad [D, P_\mu] = P_\mu.
\end{aligned} \tag{3.35}$$

Its usual to write these generators in a single form J_{MN}, $M, N = \{0, 1, ..., d+2\}$ as the $d+2 \times d+2$ dimensional matrix:

$$J_{MN} = \begin{pmatrix} M_{\mu\nu} & \frac{K_\mu - P_\mu}{2} & \frac{K_\mu + P_\mu}{2} \\ -\frac{K_\mu - P_\mu}{2} & 0 & D \\ -\frac{K_\mu + P_\mu}{2} & -D & 0 \end{pmatrix}. \tag{3.36}$$

3.4 Correspondence

In [30] the author gives a good comparison between the terms:

- AdS/CFT: A duality between a Conformal field Theory and a gravity theory formulated in string theory in a Anti de Sitter space. First conjectured by Maldacena [6], more specifically for the CFT $\mathcal{N} = 4$SYM. The methods behind this duality has been expanded to include more than just CFTs and AdS spaces. An important early work on the development of the correspondence was done by Witten [36].

- Holographic Principle: First proposed by Gerard 't Hooft [37], Charles Thorn [38] and Leonard Susskind [39], it conjectures that a theory on volume of a space can described by a theory on its boundary. Therefore we have a description of a theory by a theory with a smaller number of dimensions. The mapping of this two functions is called *Holographic mapping*.

  The idea came from black hole thermodynamics, where the maximum entropy of a region, given by
  $$S = \frac{A}{4}, \tag{3.37}$$
  scales with the surface area A instead of the volume, as one would expect.

  The idea for this came from the idea that if a gas with some entropy falls into a black hole event horizon, that entropy cannot be destroyed since this would violate the second law of thermodynamics. For this reason it was conjectured by Bekenstein [40] that black holes have entropy.

  And since the nature of the event horizon of a black hole says that an infalling matter would take an infinite time to cross it for a frame of reference of a remote observer, the entropy must be proportional to the area of the horizon, and not the volume of the entire black hole. Hawking derivation that the area of a black hole always increase with time [41] was a stronger evidence for it to be a form of entropy.

It is relevant to say that there are solutions to the Einstein equations that violate Bekenstein's rule for the maximum entropy, and collectively they are called *Wheeler's bag of gold*, published first in the rare book [42], which are a class of solutions that shows that a surface area can enclose an arbitrary amount of volume. These conflicts the holographic principle, since it relies on the result of black hole thermodynamics that the entropy is dependent on the area, although recent developments suggest that this is only an apparent paradox in AdS spaces [43].

- Gauge/Gravity duality: It's an umbrella term for any dualities between gauge theories and gravity theories. Can be though of a generalization of the AdS/CFT term since it can includes theories that are not conformal (in the gauge side of the duality) and theories not in AdS (in the gravity side), like the dS/CFT correspondence [44].

The AdS/CFT correspondence comes directly after the discovery by Polchinski of a equality between *D-branes* and extremal *p-branes* [45]. Both object came from String Theory, and a short explanation of them are:

- D-branes: Suppose that we have extended one-dimensional object in space. They are said to be open if their endpoints (their boundary) are not located in the same point in space. A D-brane is a region of space where the endpoints of strings live. For the case of $D = 3$ the theory that lives on them is the conformal field theory $\mathcal{N} = 4$SYM plus corrections.
- p-brane: A p-brane is a p-dimensional object that generalizes the 0-dimensional black-hole. Strings are p-branes with $p = 1$. It can carry electric charge Q, and extremal means $Q = M$. Closed strings on a space curved by A p-brane gives a supergravity theory plus corrections.

The equality described above means that we can write the boundary of a space curved by a D3-brane with the description of $\mathcal{N} = 4$SYM on a D3-brane. Properly writing the metric of this gravitational theory gives us the metric of $AdS_5 \times S^5$ space. Then we have a duality between

- $\mathcal{N} = 4$SYM with gauge group $SU(N)$,
- Gravity theory in the boundary of $AdS_5 \times S^5$.

This is the duality called AdS/CFT correspondence.

The duality is valid for large N and $g_{YM} \to 0$. It is believed that it can be extended for all values of N and g_{YM}. We will see later that this limit is known as the 't Hooft limit.

More details on this argument can be seen in the reviews [46] and [47].

The AdS/CFT duality is formulated in term of string theory. So a proper review of the subject would require some previous knowledge on string and supersymmetric theories. A traditional textbook for it is [48] and [49]. Also relevant for this author is [50]. For a full introduction on AdS/CFT, we recommend the textbooks [32] and [47].

There is a large number of works that try to define the duality, either in more formal description in the hopes that the limits of the correspondence becomes more clear, either by giving an alternative motivation for the existence of this paradigm. We list some references regarding the so called Bottom-Up approach to the correspondence, which is the one we are going to use going forward, [51], [52], [53], [54], [55] and [56].

Also some references for the holographic principle seen by the renormalization group equations are [57] and [58].

3.5 Correspondence at classical gravity level

In the interest of time, we are going to review this topic using hand-waving arguments when it comes to string theory. The already cited reference [30] names as an ”string-less” introduction to AdS/CFT, the method that shows some motivations for the duality that can be understood without much knowledge on string-theory.

The validity of a string-less duality happens when we consider that the gravity side of the duality is in the regime of classical gravity. In this limit, the string theory effects can be neglected and the duality gains a weaker version, named **Bottom-Up**. A good review is in the lecture notes [51].

Essentially, we want the length scales were our system lives L to be much bigger that the string scale l_s, so in this scale there is no extended objects

$$\left(\frac{L}{l_s}\right) \gg 1. \tag{3.38}$$

Also the limit of classical gravity requires that the scales are also much bigger then the plank scale l_p, where quantum effects would show up:

$$\left(\frac{L}{l_p}\right) \gg 1. \tag{3.39}$$

Turns out that this limit is interesting for a bunch of applications. Given a Yang-Mills theory with gauge group $SU(N)$ and coupling g_{YM} on the gauge side of the duality, the Lagrangian is described as:

$$\mathcal{L} = Tr(F_{\mu\nu}F^{\mu\nu} + \textit{fields}) \tag{3.40}$$

with

$$F_{\mu\nu} = \partial\mu A_\nu - \partial\nu A_\mu + ig_{YM}[A_\mu, A_\nu]. \tag{3.41}$$

In 1974 ’t Hooft showed that in the limit of $N \to \infty$ the system is actually easier to solve [59], [60]. The limit can be properly written as

$$\begin{aligned} N &\to \infty, \\ g_{YM} &\to 0, \\ \lambda &= g_{YM}^2 N \text{ fixed.} \end{aligned} \tag{3.42}$$

At this limit, it is possible to expand the Lagrangian in terms of λ. At the limit of large N, only the planar diagrams survives and we can solve this system using perturbation theory. The limit where perturbation theory is valid is for $\lambda \ll 1$, i.e.,

$$\lambda \geq 1 \qquad \text{nonperturbative regime.} \tag{3.43}$$

The limit of large N is especially interesting for the study of *quantum chromodynamics* (QCD), that is a gauge theory that describes the dynamics of quarks and gluons, and the gauge group is $SU(3)$. The discussion if $N = 3$ is large enough for the 't Hooft expansion to be valid is a complicated one and we are not going to present here, but some phenomenological effects have been studied. Also, there has been some efforts to consider higher orders of the expansion. Yet, some interesting effects of QCD only happen at the non-perturbative regime, such as quark confinement and aspects of the theory of quark-gluon plasma. So it would seem that this expansion is not useful for these phenomena.

The mapping described earlier between string theory gravity on AdS and a CFT at large N, the AdS/CFT correspondence, maps the λ and g_{YM} from the CFT side to the string coupling g_s and α', a quantity related to the string tension T

$$T = \frac{1}{2\pi\alpha'}, \tag{3.44}$$

from the gravity side as:

$$\alpha'/R^2 \propto (\lambda)^{-1/2}, \tag{3.45}$$

$$g_s \propto g_{YM}^2, \tag{3.46}$$

where R is given by

$$R = \sqrt{\alpha'}(4\pi g_s N)^{1/4} \tag{3.47}$$

The perturbation theory in String theory is an expansion of order α'/R^2, since we are looking at classical gravity, we want a low value of this quantity. That corresponds to high values of λ in the $\mathcal{N} = 4$SYM theory, the nonperturbative regime of this theory.

Therefore, the limit at which Gauge theories are intractable using traditional methods corresponds the limit at which the string theory approaches gravity solution. This is one of the main motivations for the study of the AdS/CFT correspondence in the last decades. But during this time, a variety of applications of the duality has been found.

3.6 Dictionary

The bridge between the two sides of the duality can be displayed as a list of quantities that are mapped under the holographic mapping. This list is usually known as the *Dictionary*. In this section we aim to give a practical introduction of this concept in the same spirit as it was done in [52].

First let us say that the (d+1)-dimensional gravitational theory on the bulk has an action of the form

$$\mathcal{S}_{bulk}(g_{\mu\nu}, A_\mu, \phi) \tag{3.48}$$

where the fields are respectively the graviton, the dilaton and a connection. The first two appear in the spectrum of string theory (in the vacuum of superstrings)[3], while the last one gives rise to the Maxwell part of the action[4].

We could have more fields corresponding to other interactions. This fields are (d+1)-dimensional as well, so we have $\Phi(r, x)$, where Φ is a list of all the bulk fields.

On the d-dimensional CFT side we have an action

$$\mathcal{S}_{CFT} = \sum \int c_i \mathcal{O}_i, \tag{3.49}$$

where the fields are operators $\mathcal{O}_i$. We can add to the CFT Lagrangian a deformation from an operator $\mathcal{O}$ with source ϕ_0 as

$$\int \phi_0 \mathcal{O} d^d x, \tag{3.50}$$

Standard QFT procedure says that we can define $W(\phi_0)$ from the equation

$$e^{W(\phi_0)} = \langle e^{\int \phi_0 \mathcal{O}} \rangle. \tag{3.51}$$

The dictionary comes from an equation called the GPKW master equation [36], [61] that relates functionals from both sides of the correspondence. It is written as:

$$W(\phi_0) = -\mathcal{S}_{bulk} \left[\lim_{r \to 0} \Phi(r, x) = \phi_0 \right], \tag{3.52}$$

i.e., the CFT is going to equal the gravity theory at the boundary ($r \to 0$). From this equation, we can find the relations from fields by setting the pair (ϕ_0, $\mathcal{O}$) appropriately. In [61] the authors gives a good example on how to do it with full details.

The most relevant results for this procedure gives the first lines of the dictionary:

CFT	gravity
energy-momentum tensor $T^{\mu\nu}$	metric $g^{\mu\nu}$
current J^μ	connection A^μ
scalar operator $\mathcal{O}$	scalar field ϕ
...	...

We can increase the dictionary with any fields we can input in the master equation. For now we only show some of the most common fields in these theories.

[3]Which we show how in the appendix A in a quick review of bosonic strings

[4]Or technically speaking Yang-Mills term if the connection forms a non-abelian group

Chapter 4

Thermodynamics

4.1 Finite Temperature

A common method to include temperature at quantum field theories at equilibrium is using the imaginary time formalism. It is possible to describe non-equilibrium process [62], but that's beyond what we need for this thesis.

A better insight from this formalism come from the relationship between QFTs and statistical field theory. We review the explanation o chapter V.2 of [63]. An understandable review of thermal QFT can be found in [64][1].

Intuitively, we can interpret a mapping from a real time t to an imaginary time τ, $t \to -i\tau$, as a mapping from a Lorentzian spacetime

$$ds^2 = -dt^2 + (d\vec{x})^2 \tag{4.1}$$

to an Euclidean spacetime

$$ds^2 = -(d(-i\tau))^2 + (d\vec{x})^2 = d\tau^2 + (d\vec{x})^2 \ . \tag{4.2}$$

We are going to see how this mapping also connects quantum mechanics to statistical physics.

The transition amplitude between two states $|n\rangle$ and $|m\rangle$ in quantum mechanics is given by

$$\langle m| e^{-itH} |n\rangle \ , \tag{4.3}$$

where H is the Hamiltonian of the system, and t is the time difference between the two states.

Now consider that the end state is the same as the beginning state $|n\rangle = |m\rangle$, which we are going to say is a part of a periodic process, and we make the change $t \to -i\tau$, a imaginary time, we get that the amplitude becomes

$$\langle n| e^{-\tau H} |n\rangle = Tr\left[e^{-\tau H}\right] \ . \tag{4.4}$$

[1] A more complete reference by the same author is [65].

Now note that the partition function in statistical mechanics for a system with Hamiltonian H and temperature $T = 1/\beta$ is given by

$$Z = Tr\left[e^{-\beta H}\right] \ . \tag{4.5}$$

This lead us to assert that if τ is made to be equal to β we connect quantum mechanics and statistical mechanics.

To generalize this to field theory, let's consider the fact that the transition amplitude can also be written in terms of the path integral formalism as

$$\langle m| e^{-\tau H} |n\rangle = \int \mathcal{D}q(t) e^{iS[q(t)]} \ , \tag{4.6}$$

where $q(t)$ is the position of the state.

And on statistical field theory we have that the partition function for a state $\phi(\vec{q}, t)$ over periodic trajectories with period β is given by in Euclidean time

$$Z = Tr\left[e^{-\beta H}\right] = \int \mathcal{D}\phi e^{-S_E[\phi]} \tag{4.7}$$

Then we understand that the if we take the Wick rotation $t \to -i\beta$ and $S \to iS_E$ for going from Lorentzian time to Euclidean time we connect a $(D+1)$-dimensional QFT to a D-dimensional quantum statistical field theory with temperature T.

4.2 Black Hole Thermodynamics

We are not going to review the work by Stephen Hawking on deriving the thermodynamics of a black hole and the fact that they emit radiation [66], but we are going to indicate the results of it. A good review applied to AdS/CFT discussions can be found at [67] and at [26].

We first start with the Schwarzschild metric $(G = 1)$ for the black hole

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} + r^2 d\Omega^2 \ . \tag{4.8}$$

Following the procedure of the last section, we Wick rotate it to Euclidean time

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} + r^2 d\Omega^2 \ . \tag{4.9}$$

Note that the horizon $r_h = 2M$ is special for this metric, since crossing would change the signature of the space. That does not happens in Lorentzian time, since time and space can change roles when crossing the horizon. So it seems that r_h is a singularity that should not exist.

Now one can show that for a periodic time $\tau/4M$, it the period is 2π then the metric is exactly flat space and there are no singularities at the horizon.

We discussed that this imaginary time should be periodic with period β. Therefore we have that $\beta = 8M\pi$, or equivalently we have the temperature of the black hole

$$T_{BH} = \frac{1}{8\pi M} \ , \tag{4.10}$$

where BH stands for *Bekenstein* and *Hawking*. This method of finding the temperature of black holes is the one used throughout this thesis for many different metrics.

Most times this quantity is given in terms of the surface gravity κ, which for the black hole described above is $\kappa = 1/4M$, so

$$T_{BH} = \frac{\kappa}{2\pi} \ . \tag{4.11}$$

The entropy of a black hole can also be defined and we discussed this briefly in section 3.4. Let's take the result seen there that the entropy is related to the area of a black hole;

$$S = \frac{A}{4} \ . \tag{4.12}$$

When summarizing the discoveries of the field of black hole thermodynamics, the authors[2] of [68] defined four laws that can be compared to the four laws of classical thermodynamics:

- Zero-th law: The surface gravity κ is constant for a stationary solution.

  This gives the understanding that κ is the black hole version of temperature.

- First law: The variation of mass is given by $dM = \frac{\kappa}{8\pi} dA$.

- Second law: $dA \geq 0$.

  The first and second laws confirms the entropy nature of the area of a black hole, while the mass is related to the energy.

- Third law: It is impossible for a process of finite steps to lead κ to zero.

4.3 Witten Metric

In this section we are going to show a intuitive path to deform pure AdS/CFT correspondence by introducing finite temperature.

We have seen the correspondence from the AdS gravity with a scale invariant field theory. The gravity theory has a dilatation symmetry (giving rise to the dilaton field) and that corresponds to the scale invariance in the dual theory. We can think of breaking this symmetry by adding relevant operators and removing the scale invariance or by adding finite temperature to the ensemble, and due to the duality this corresponds to the breaking of the dilatation symmetry in the gravity theory [5].

[2] James M. Bardeen, B. Carter and S.W. Hawking

If we start with the metric of the AdS space:

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + (dx^i)^2 + dz^2)), \tag{4.13}$$

we can break the scale invariance by deforming this metric with functions $f(z)$, $g(z)$ and $h(z)$ such that

$$ds^2 = \frac{L^2}{z^2}(-f(z)dt^2 + g(z)(dx^i)^2 + h(z)dz^2)). \tag{4.14}$$

Note that this metric is still rotationally and translationally symmetric.

Let's take, for simplicity, $g(z) = 1$ and $h(z) = 1/f(z)$. Then, imposing the Einstein's equation for this metric

$$R_{\mu\nu} + \frac{3}{L^2}g_{\mu\nu} = 0, \tag{4.15}$$

we need for $f(z)$ to satisfy

$$3 - 3f(z) + zf'(z) = 0, \tag{4.16}$$

which leads our final metric to

$$ds^2 = \frac{L^2}{z^2}(-f(z)dt^2 + (dx^i)^2 + \frac{1}{f(z)}dz^2)) \tag{4.17}$$

with

$$f(z) = 1 - (z/z_h)^3. \tag{4.18}$$

Here, z_h is a constant of integration coming from the solution of (4.16), and for now we set $z_h = 1$. Since $f(z)$ goes to 1 as z goes to 0, we have an asymptotically AdS space.

The metric we arrived is the same as the one from a black hole. This suggests that the dual to a CFT at finite temperature is a gravity theory with a black hole in the space.

In short, say we have an (Euclidean) action of a quantum gravity theory

$$S = -\frac{2}{\kappa_4^2}\int d\tau dx^2 dz\sqrt{-g}\left(-\frac{1}{4}R - \frac{3}{2}\frac{1}{L^2}\right) + I_{ct}, \tag{4.19}$$

with euclidean time τ where I_{ct} is the counterterms for this actions (see [69], [70] and [71]):

$$I_{ct} = \lim_{z\to 0}\frac{1}{2\kappa_4^2}\int d\tau dx^2\sqrt{\gamma}\left(-K + \frac{2}{L}\right), \tag{4.20}$$

where γ is the induced metric and K is the trace of the extrinsic curvature.

With the metric given by (4.17), this now tells us that there is a constraint in this Euclidean space time, since for $z = 1$ we have $f(1) = 0$.

The necessity for the space-time to be regular can be achieved if we demand periodicity in τ

$$\tau \to \tau + \frac{4\pi}{|f'(z)|} = \tau + \frac{4\pi}{3} \tag{4.21}$$

The prescription to include temperature in AdS/CFT was found by Witten , and thus the AdS black hole metric we used is known as the Witten metric[3]. In Euclidean time τ the metric is an altered version of the global coordinates

$$ds^2 = \left(\frac{\rho^2}{L^2} - \frac{L^{d-2}}{\rho^{d-2}}\right) d\tau^2 + \frac{1}{\frac{\rho^2}{L^2} - \frac{L^{d-2}}{\rho^{d-2}}} d\rho^2 + \rho^2 (dx^i)^2 \tag{4.22}$$

Then following Hawking's derivation of the temperature of a black hole, for us to avoid singularities we need a periodic τ with period (ρ_h being the location of the horizon)

$$\beta = \frac{1}{T} = \frac{\frac{\rho_h^2}{L^2} - \frac{L^{d-2}}{\rho_h^{d-2}}}{4\pi} = \frac{d}{4\pi}. \tag{4.23}$$

Going to the more familiar coordinates we have (in Minkowiski time t)

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + (dx^i)^2 + \frac{dz^2}{f(z)}\right), \tag{4.24}$$

where $f(z) = 1 - z^3/z_0^3$, z_0 being the location of the horizon of the black hole, usually taken to be $z_0 = 1$. The previous procedure gives the temperature to be, in $d = 3$,

$$T = \frac{3}{4\pi}. \tag{4.25}$$

4.4 Dyonic Black Hole

Finite temperature is added to quantum field theories considering a wick rotation to a periodic euclidean time of period $1/T$. We saw in the previous section that for gravity the procedure to consider temperatures comes from the thermodynamics of black hole. Thus its important for us to study black holes in AdS spaces if we want to compare them to QFTs.

The description given in the previous section gave us a black hole with no charges. To include electric and magnetic charges we start by imposing a gauge term in the action giving us the Einstein-Maxwell action:

$$S = -\frac{2}{\kappa_4^2}\int d\tau dx^2 dz \sqrt{-g}\left(-\frac{1}{4}R - \frac{3}{2}\frac{1}{L^2} + \frac{L^2}{4}F^{\mu\nu}_{\mu\nu}\right) + I_{ct}, \tag{4.26}$$

where the curvature tensor is determined by $F = dA$, where the connection A is given by

$$A = B(z)xdy + E(z)dt. \tag{4.27}$$

Here, the magnetic (B) and electric (E) fields can depend on the scale dimension z. Let's take $B = h$ a magnetic charge and $E = q$ and electric charge.

[3] You can read a review in the lecture notes [72] or in the books [26] and [47].

Then the solution to the Einstein equations of motion

$$R_{\mu\nu} = 2L^2 F_{\mu\sigma} F^{\sigma}_{\nu} - \frac{L^2}{2} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} - \frac{3}{L^2} g_{\mu\nu}, \tag{4.28}$$

$$\nabla_{\mu} F^{\mu\nu} = 0. \tag{4.29}$$

Gives us the Reissner-Nordstrom black hole[4].

$$ds^2 = \frac{L^2}{z^2}(-f(z)dt^2 + (dx^i)^2 + \frac{1}{f(z)}dz^2)) \tag{4.30}$$

with

$$f(z) = 1 - (1 + h^2 + q^2)z^3 + (h^2 + q^2)z^4. \tag{4.31}$$

Therefore, we only need to rewrite the $f(z)$ term from the description of the black holes without charges.

The temperature of the black hole then becomes, using the previous procedure,

$$T = \frac{(3 - h^2 - q^2)}{4\pi}. \tag{4.32}$$

[4]The Reissner-Nordstrom black hole is a solution to gravity and Maxwelll equations where a black hole has electrical charge, temperature and a mass M ([73], [74], [75], [76]). A generalization would be if the black hole is rotating, which is called a **Kerr–Newman** metric.

Chapter 5

Transport Coefficients

5.1 Kubo's Formula

The Ads/CFT correspondence can be applied to compute transport coefficients of strongly correlated CFT's, that using traditional quantum field theories would be a intractable problem, through the use of equations called **Kubo formulas**.

Named after Ryogo Kubo, who derived these equations in 1957 ([77]- [78]), defines the linear response of an observable when its under a time-dependant perturbation. More specifically, we can write the linear response in terms of integrals of correlation functions. In this form, they receive the name of **Green-Kubo relations**.

We review in this section how correlators in AdS/CFT correspondence is defined and how we can apply them using the Kubo formulas. The reference [26] has a good review of this subject. A quick introduction can also be found in [79]. The original references that were also used in this text are [7] and [8].

One of the most import results of this method is the so called KSS bound, where the shear viscosity of a fluid is found to obey the relation:

$$\frac{\eta}{s} \geq \frac{1}{4\pi}. \tag{5.1}$$

We will show this result in the next section and, at the writing of this, experimental results seems to hold truth. The original reference is [1].

5.1.1 Retarded Green Function

Given two observables $\mathcal{O}_A$ and $\mathcal{O}_B$, the retarded propagator is defined as, in the metric of a Minkowski space in d-dimensions:

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(k) = -i\int d^dx e^{-ik\cdot x}\theta(t)\langle[\mathcal{O}_A(x),\mathcal{O}_B(0)]\rangle. \tag{5.2}$$

Here we defined $x = (t, \vec{X})$ and $k = (\omega, \vec{k})$.

Note that the use of the Heaviside function $\theta(t)$ ensures that this Green function is retarded. The advanced propagator, in its turn, is defined as

$$G^{A}_{\mathcal{O}_A\mathcal{O}_B}(k) = i\int d^d x e^{-ik\cdot x}\theta(-t)\langle[\mathcal{O}_A(x), \mathcal{O}_B(0)]\rangle. \tag{5.3}$$

The other relevant propagator is the symmetrized Wightman function:

$$G_{\mathcal{O}_A\mathcal{O}_B}(k) = \frac{1}{2}\int d^d x e^{-ik\cdot x}\langle(\mathcal{O}_A(x)\mathcal{O}_B(0) + \mathcal{O}_B(0)\mathcal{O}_A(x))\rangle. \tag{5.4}$$

Every other propagator can be written in terms of the propagators $G^{R}_{\mathcal{O}_A\mathcal{O}_B}$, $G^{A}_{\mathcal{O}_A\mathcal{O}_B}$ and $G_{\mathcal{O}_A\mathcal{O}_B}$.

We are dealing with Minkowski space propagators. But the original prescription of AdS/CFT allowed the calculations of correlators in the boundary CFT using Euclidean space.

In Euclidean space we define $x_E = (t_E, \vec{X})$ and $k_E = (\omega_E, \vec{k})$, where t_E is the Euclidean time and ω_E are called Matsubara frequencies. The most common propagator in Euclidean time is the Matsubara propagator:

$$G^{E}_{\mathcal{O}_A\mathcal{O}_B} = \int d^d x_E e^{-ik_E\cdot x_E}\langle T_E\mathcal{O}_A(x_E)\mathcal{O}_B(0)\rangle, \tag{5.5}$$

where T_E is the Euclidean time-ordering operator

$$T_E\mathcal{O}_A(x)\mathcal{O}_B(0) = \begin{cases} \mathcal{O}_A(x_E)\mathcal{O}_B(0) & \text{if } t_E < 0 \\ \mathcal{O}_B(0)\mathcal{O}_A(x_E) & \text{if } t_E > 0. \end{cases} \tag{5.6}$$

If $\mathcal{O}_B$ (0) and $\mathcal{O}_A$ are bosonic operators, the Matsubara frequencies are multiples of $2\pi T$.

The relation between the Matsubara correlator and the previously defined Minkowski correlators are [7]:

$$G^{R}_{\mathcal{O}_A\mathcal{O}_B}(2\pi i T n, \vec{k}) = -G^{E}_{\mathcal{O}_A\mathcal{O}_B}(2\pi T n, \vec{K}), \tag{5.7}$$

for the retarded Green function and

$$G^{A}_{\mathcal{O}_A\mathcal{O}_B}(-2\pi i T n, \vec{k}) = -G^{E}_{\mathcal{O}_A\mathcal{O}_B}(-2\pi T n, \vec{K}), \tag{5.8}$$

for the advanced Green function.

5.1.2 Response of a perturbation

The propagators defined last section calculate the response of the observable $\mathcal{O}_A$ due to a time dependent perturbation coupled to $\mathcal{O}_B$.

To study this, let's consider the Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \delta H(t) \tag{5.9}$$

where δH is the perturbation

$$\delta H = \int d^{d-1}\vec{x}\delta\phi_{B(0)}(x)\mathcal{O}_B(x). \tag{5.10}$$

Here $\delta\phi_{B(0)}$ is the source of the perturbation of $\mathcal{O}_B$.

The (unnormalized) density matrix $\rho(t)$

$$\rho(t) = e^{-\beta\mathcal{H}} \tag{5.11}$$

satisfies the Liouville–von Neumann equation

$$i\frac{\partial\rho}{\partial t} = [\mathcal{H}, \rho]. \tag{5.12}$$

We want to compute the VEV of the operator $\mathcal{O}_A$ under this perturbation. The VEV can be written as

$$\langle\mathcal{O}_A\rangle = \mathrm{Tr}(\rho\mathcal{O}_A). \tag{5.13}$$

In the interaction picture, the time evolution of the perturbation $\mathcal{H}_0$ is absorbed by the operators. Then we obtain the equation

$$\langle\mathcal{O}_A\rangle(x) = \mathrm{Tr}(\rho_0 U^{-1}(t)\mathcal{O}_A(\vec{x})U^1)). \tag{5.14}$$

In here, ρ_0 is the density matrix of the original Hamiltonian

$$\rho_0 = e^{-\beta\mathcal{H}_0}, \tag{5.15}$$

and the operator $U(t)$ is the time evolution operator

$$U(t) = \frac{1}{\beta}e^{-i\int_0^t \delta H(t)dt'}. \tag{5.16}$$

Now taking the variation of the equation (5.14) we obtain

$$\begin{aligned}\delta\langle\mathcal{O}_A\rangle(x) &= \mathrm{Tr}\left[\rho_0 U^{-1}\mathcal{O}_A(\vec{x})U^1\left(-i\int_0^t \delta H(t)dt'\right)\right] \\ &= -i\mathrm{Tr}\left[\rho_0\int_0^t \mathcal{O}_A(x)\delta H(t)dt'\right].\end{aligned} \tag{5.17}$$

Now inputting the explicit definition of δH

$$\begin{aligned}\delta\langle\mathcal{O}_A\rangle(x) &= -i\mathrm{Tr}\left[\rho_0\int_0^t \mathcal{O}_A(x)\left(\int d^{d-1}\vec{x}\phi_{B(0)}(x)\mathcal{O}_B(x)\right)dt'\right] \\ &= -i\mathrm{Tr}\left[\rho_0\int d^{d-1}\vec{x}\int_0^t dt'\mathcal{O}_A(x)\mathcal{O}_B(x)\phi_{B(0)}(x)\right] \\ &= -i\int d^{d-1}\vec{x}\int_0^t dt'\mathrm{Tr}\left[\rho_0\mathcal{O}_A(x)\mathcal{O}_B(x)\right]\phi_{B(0)}(x).\end{aligned} \tag{5.18}$$

Now using the definition of the VEV

$$\delta\langle\mathcal{O}_A\rangle(x) = -i\int d^d x\theta(t)\langle[\mathcal{O}_A(x)\mathcal{O}_B(x)]\rangle\phi_{B(0)}(x) \tag{5.19}$$

This is pretty much what we needed to relate the Green function with the response of the perturbation, except that we defined the Green functions in momentum space $(\omega, \vec{k})$. So we only need to do a Fourier transformation on the previous equation for us to get

$$\begin{aligned}\delta\langle\mathcal{O}_A\rangle(k) &= -i\int d^d x e^{-ik\cdot x}\theta(t)\langle[\mathcal{O}_A(k)\mathcal{O}_B(k)]\rangle\phi_{B(0)}(k) \\ &= G^R_{\mathcal{O}_A\mathcal{O}_B}(k)\phi_{B(0)}(k).\end{aligned} \tag{5.20}$$

Therefore we have the final result

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(k) = \frac{\delta\langle\mathcal{O}_A\rangle(k)}{\phi_{B(0)}(k)}. \tag{5.21}$$

Then we have the proof that the Green function measures the linear response of the operator $\mathcal{O}_A$ under a perturbation sourced by $\phi_{B(0)}(k)$.

5.1.3 Thermal Susceptibility

In order to calculate the Green functions we found in the last subsections, we have to find mathematical tools to help us. An important equation of functions of the complex plane to the complex plane are the Kramers-Kronig relations, stated as:

Kramers-Kronig Relations

Given a complex function

$$\chi : \mathcal{C} \to \mathcal{C}.$$

Let us write $\chi(\omega)$ of complex ω such that

$$\chi(\omega) = \chi_1(\omega) + i\chi_2(\omega)$$

where χ_1 and χ_2 are real functions.

Assume that χ satisfies:

1. χ is analytic in the closed upper-half plane (Im$(\omega) \geq 0$).
2. χ goes to zero as $|\omega| \to \infty$ faster that $1/|\omega|$.

Then we have that

$$\chi_1(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\chi_2(\omega')}{\omega' - \omega}d\omega'$$

$$\chi_2(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\chi_1(\omega')}{\omega' - \omega}d\omega'$$

where $\mathcal{P}$ denotes the Cauchy principal value.

There are a number of ways to derive these equations. In order to see them, as well as many applications on physics, we invite the reader to check chapter 19 of [80].

The Cauchy principal value before the integral indicates that when we are close to the singularity at $\omega' = \omega$ we consider the limit, since without it the integral would become ill-defined.

Now let us take the retarded Green function $G^R_{\mathcal{O}_A\mathcal{O}_B}(k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, \vec{k})$ and assume ω can take complex values, in order for us to take the analytic continuation thatg gives us the function $G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega)$.

Let us also assume that this functions satisfies property 2. of the box above. The the Kramers-Kronig relations can be applied and we write for the first equation:

$$\mathrm{Re}G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\mathrm{Im}G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega')}{\omega' - \omega} d\omega' \tag{5.22}$$

Now we define the quantity

$$\chi_{AB} = \lim_{\omega\to 0} \mathrm{Re}G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega) \tag{5.23}$$

We call this as the static thermodynamic susceptibility, since it satisfies

$$\chi_{AB} = \frac{\partial\langle\mathcal{O}_A\rangle}{\partial\phi_{B(0)}}, \tag{5.24}$$

as an allusion to the way we define electric susceptibility

$$\chi_E = \frac{\partial D}{\partial E}. \tag{5.25}$$

5.1.4 Electric Conductivity

A very simple application of the study of linear response is given as the calculation of the transport coefficient of the electric field.

Given a perturbation of the electric for some Hamiltonian given by a gauge perturbation δA. We assume the gauge $\delta A_0 = 0$. Then we have a perturbed electric field given by

$$E_i = \partial_t \delta A_i. \tag{5.26}$$

In Fourier space this becomes

$$E_i = -i\omega\delta A_i. \tag{5.27}$$

At a linear level, the response is calculated by the Ohm's law

$$\langle J_i\rangle = \sigma E_i, \tag{5.28}$$

where σ is the electric conductivity. But according to what we calculated in subsection 5.1.2, the linear response is given by

$$\langle J_i \rangle = G^R_{J_i J_i}(\omega, \vec{k}) \delta A_i. \tag{5.29}$$

Therefore we have the equation

$$\sigma(\omega, \vec{k}) = i \frac{G^R_{J_i J_i}(\omega, \vec{k})}{\omega} \tag{5.30}$$

This is known as the Kubo's formula for the electric conductivity.

Since we are usually interested on the real part of the conductivity, we also could have called the following relation as the Kubo formula:

$$\sigma(\omega, \vec{k}) = -\frac{\mathrm{Im} G^R_{J_i J_i}(\omega, \vec{k})}{\omega} \tag{5.31}$$

And the DC conductivity is obtained taking ω to zero

$$\sigma(0, \vec{k}) = -\lim_{\omega \to 0} \frac{\mathrm{Im} G^R_{J_i J_i}(\omega, \vec{k})}{\omega} \tag{5.32}$$

5.1.5 Thermoelectric transport

We have seen how we can compute the linear response due to an electric perturbation. Let's add to that a thermal response due to a mix between electrical current and thermal current (heat).

$$\begin{pmatrix} \langle J_i \rangle \\ \langle Q_i \rangle \end{pmatrix} = \begin{pmatrix} \sigma & \alpha T \\ \alpha T & \bar{\kappa} \end{pmatrix} \begin{pmatrix} E_i \\ -\nabla_i T / T \end{pmatrix}. \tag{5.33}$$

We have shown the Kubo's formula for σ. Now using the same logic as the previous subsection, we write the Kubo's formula for:

$$\alpha(\omega, \vec{k}) = i \frac{G^R_{Q_i J_i}(\omega, \vec{k})}{\omega} \tag{5.34}$$

and

$$\bar{\kappa}(\omega, \vec{k}) = i \frac{G^R_{Q_i Q_i}(\omega, \vec{k})}{\omega}. \tag{5.35}$$

As before, we could have defined these formulas only for the real part of α and $\bar{\kappa}$:

$$\alpha(\omega, \vec{k}) = -\frac{\mathrm{Im} G^R_{Q_i J_i}(\omega, \vec{k})}{\omega} \tag{5.36}$$

and

$$\bar{\kappa}(\omega, \vec{k}) = -\frac{\mathrm{Im} G^R_{Q_i Q_i}(\omega, \vec{k})}{\omega}. \tag{5.37}$$

The static quantities are given by, as before, taking ω going to zero:

$$\alpha(0,\vec{k}) = -\lim_{\omega\to 0}\frac{\mathrm{Im}G^R_{Q_iJ_i}(\omega,\vec{k})}{\omega} \tag{5.38}$$

and

$$\bar{\kappa}(0,\vec{k}) = -\lim_{\omega\to 0}\frac{\mathrm{Im}G^R_{Q_iQ_i}(\omega,\vec{k})}{\omega}. \tag{5.39}$$

5.2 KSS Bound

Of course, there are many other interesting Kubo's formulas to be described for a number of different perturbations. The prescription described in the previous section can be applied for many other perturbations, and from the Green functions we are going to have interesting results for different Hamiltonians. In this section we want to cover one of the main results of the theory, the KSS-bound, named after the authors of [1].

This result sets a bound for the ratio of the shear viscosity with the entropy. The calculation of the value of the viscosity comes from a Kubo's formula, and then applied to a model that inputs the Green function to the formula.

We then first show the derivation of this equation.

5.2.1 Kubo's Formula for Shear Viscosity

We have seen in section 2.1 that the relativistic hydrodynamics equation sets the energy-momentum tensor to satisfy

$$\nabla_\mu T^{\mu\nu} = 0 \tag{5.40}$$

Here, the energy-momentum tensor is of the form

$$T^{\mu\nu} = \rho u^\mu u^\nu + pP^{\mu\nu} - 2\eta\sigma^{\mu\nu} - \zeta\theta P^{\mu\nu}, \tag{5.41}$$

where η is the shear viscosity and ζ is the bulk viscosity and the formulas for $P^{\mu\nu}$ and $\sigma^{\mu\nu}$ were given in section 2.1:

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \tag{5.42}$$

and

$$\sigma^{\mu\nu} = \nabla^{(\mu}u^{\nu)} + u^{(\mu}u^\alpha\nabla_\alpha u^{\nu)} - \frac{1}{d-1}\nabla_\alpha u^\alpha P^{\mu\nu}. \tag{5.43}$$

Let us take the case of the fluid at rest $u^\mu = (1,0,0,0)$ and to it we will add a perturbation that will give the viscosity term. The fluid at rest is given by

$$T^{\mu\nu} = \rho\delta^\mu_t\delta^\nu_t + p(g^{\mu\nu} + \delta^\mu_t\delta^\nu_t). \tag{5.44}$$

Where we see that the viscosity terms disappear.

Now we consider the perturbation h_{xy} to the metric $g_{\mu\nu}$. Since we want to focus on the shear viscosity term, we assume that the bulk viscosity term can be neglected for this perturbation. Later calculations may include this term.

At linear level, the energy-momentum tensor with the perturbation is changed at the indices $(\mu, \nu) = (x, y)$, where this term becomes

$$T_{xy} = ph_{xy} + \eta\partial_t h_{xy} + \mathcal{O}(h_{xy}^2) + \mathcal{O}(\partial^2 h_{xy}) \tag{5.45}$$

Linearization of a Perturbation

Here we show a algorithmic way of linearizing equations with perturbation. Of course, the equation above is quite easy to linearize, but later we are going to do this for much more complicated equations, and we are going to use the power of the algorithms to help us.

To start, we take the bare metric without the perturbation

$$g^0_{\mu\nu}.$$

Then we add to this the perturbation

$$g_{\mu\nu} = g^0_{\mu\nu} + \epsilon h_{\mu\nu}$$

Here, $\epsilon > 0$ plays the role of counting the order of perturbations. Then in order to get the quantity

$$T^{\mu\nu}$$

in linear term of the perturbation we just need to take the expansion of $T^{\mu\nu}$ in powers of ϵ, then the term of order ϵ^1 is the result we are looking for.

Since we want the equation in terms of ω we transform to Fourier space

$$T_{xy}(\omega, \vec{k}) = -i\omega\eta(\omega, \vec{k})h_{xy}. \tag{5.46}$$

The prescription of linear response developed in the last section tell us that

$$\langle T_{xy}(\omega, \vec{k})\rangle = G^R_{T_{xy}T_{xy}}(\omega, \vec{k})h_{xy}. \tag{5.47}$$

Then we have that

$$\eta(\omega, \vec{k}) = i\frac{G^R_{T_{xy}T_{xy}}(\omega, \vec{k})}{\omega}. \tag{5.48}$$

This is the Kubo's formula for the shear viscosity. As before, usually we only care about the real part of the viscosity, so we can write as the formula the following:

Then we have that

$$\eta(\omega, \vec{k}) = \frac{\text{Im}G^R_{T_{xy}T_{xy}}}{\omega}. \tag{5.49}$$

And for static viscosity we take ω to zero

$$\eta(0, \vec{k}) = \lim_{\omega\to 0}\frac{\text{Im}G^R_{T_{xy}T_{xy}}(\omega, \vec{k})}{\omega}. \tag{5.50}$$

5.2.2 The original calculation of the bound

Now we get to the main result calculated by [1]. The authors start with the thermal field theory dual to the gravity theory of a black brane (a D-dimensional generealization of a black hole) with some metric.

$$ds^2 = f(\xi)(dx^2 + dy^2) + g_{\mu\nu}(\xi)d\xi^\mu d\xi^\nu. \tag{5.51}$$

As an example of such space, the authors gives an example of the near-extremal D3-brane in type IIB supergravity, the space that is dual to the $\mathcal{N} = 4$SYM

$$ds^2 = \frac{r^2}{R^2}\left[-\left(1-\frac{r_0^4}{r^4}\right)dt^2 + dx^2 + dy^2 + dz^2\right] + \frac{R^2}{r^2(1-r_0^4/r^4)}dr^2. \tag{5.52}$$

The results found here works for the general case of equation 5.51.

The temperature in the field theory is going to be the Hawking temperature of the black brane. At the same time, the entropy of the field is also going to be found calculating the entropy of the brane, which is proportional to the area of the horizon

$$S = \frac{A}{4G}. \tag{5.53}$$

Now the authors aim to solve the (static) Kubo's formula for the shear viscosity

$$\eta = \lim_{\omega\to 0}\frac{\mathrm{Im}G^R_{T_{xy}T_{xy}}(\omega,\vec{k})}{\omega} = \lim_{\omega\to 0}\frac{1}{2\omega}\int dtdxe^{i\omega t}\langle[T_{xy}(t,x),T_{xy}(0,0)]\rangle. \tag{5.54}$$

Now we consider a perturbation on the xy component of the metric h_{xy}. The Einstein equation, that can be written in the form

$$R_{MN} = T_{MN} - \frac{T}{D-2}g_{MN}, \tag{5.55}$$

can be linearized to the order of the perturbation, like we said in the previous subsection.

In order for these equations to be respected, we get (after a calculation described in some detail in [1]) the constraint on h_{xy}

$$\Box h_{xy} = 0. \tag{5.56}$$

Then the authors cite that this situation is the same as massless scalar, which has a theorem, described in [81] and [82], that tells us that the cross section defined by

$$\sigma(\omega) = -2\frac{\kappa^2}{\omega}\mathrm{Im}G^R_{T_{xy}T_{xy}}(\omega,\vec{k}), \tag{5.57}$$

is given by the fraction of the area over the volume

$$\lim_{\omega\to 0}\sigma(\omega) = A/V. \tag{5.58}$$

Therefore, we get that

$$\eta = \frac{\sigma(0)}{2\kappa^2} = \frac{1}{2\kappa^2}\frac{A}{V}. \tag{5.59}$$

Recalling that the entropy is proportional to the area, and taking s as the entropy density

$$s = \frac{S}{V} \tag{5.60}$$

we get

$$\eta = \frac{4G}{2\kappa^2}s. \tag{5.61}$$

The equation above can be rewritten in its most familiar form since $\kappa^2 = 8\pi G$

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{5.62}$$

This value is conjectured to set a minimal bound for more general scenarios

$$\frac{\eta}{s} \geq \frac{1}{4\pi}. \tag{5.63}$$

An argument for this is the calculations of the ratio η/s in more orders of perturbation, and those corrections are always positive. We can see that experimentally the bound is respected for common fluids in figure 5.1.

It is of importance to say that there are theoretical calculations that can violate the bound, specifically in a case of a system with a Bjorken-flow [93] or in the case of Gauss-Bonnet gravity [94,95], which is an extension of Einstein's equations that includes quadratic terms. That is to say, if we find experimental cases where the bound is violated, this could be an indication that we need extra terms in our calculations of general relativity, although this is still up to debate.

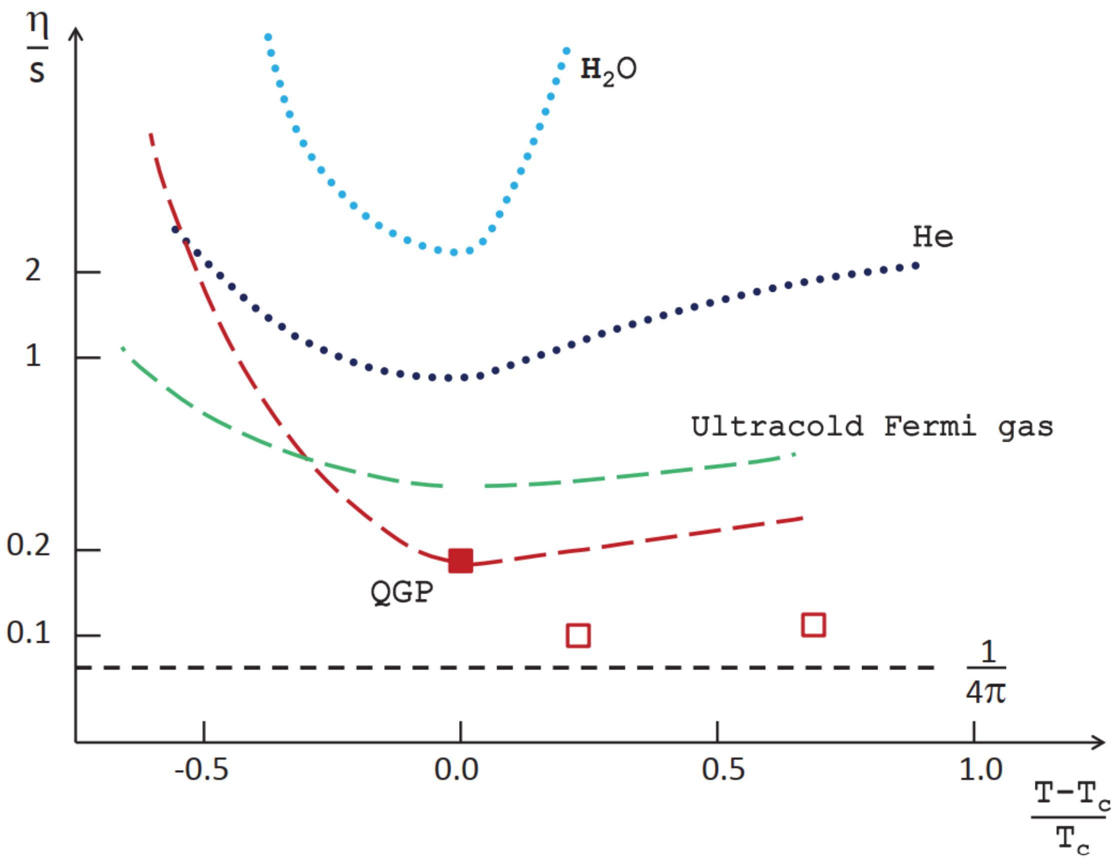

Figure 5.1: Experimental data of the shear viscosity-entropy bound for some known fluids. Figure taken from [83]. Earlier versions of this can be found at [84] and [85]. The experimental data comes from [86], [87], [88], [89], [90], [91] and [92]. The figure shows that the KSS bound is respected for these fluids.

Part II

Results and Future work

Chapter 6

Thermoelectric conductivities from Kubo formulas, with topological term

6.1 Model with a constant Topological Term

We want to expand the results found in [2, 3], where the author calculated the transport coefficients from fluctuations around a dyonic black hole without a topological term, and the results in [9], where it was calculated for magnetic field $B = 0$. We are interested in the case $B \neq 0$ and with topological term, considered in [20, 21].

The gravitational action considered in [2] includes a Maxwell term and a cosmological constant $-\frac{1}{L^2}$. We modify this action to include a topological term proportional to a constant W.

$$I = \frac{2}{\kappa_4^2} \int d^4x \sqrt{-g} \left(-\frac{1}{4}R + \frac{L^2}{4} F_{\mu\nu}F^{\mu\nu} - \frac{3}{2}\frac{1}{L^2} + W\frac{L^2}{4} F_{\mu\nu}\tilde{F}^{\mu\nu} \right). \tag{6.1}$$

Right now we could set $W = 1$, but we make a choice to keep from doing it since it will help us for future calculations and it's going to be easier to see the results of including this new term to this action.

In the action we have the curvature from a connection A_μ:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{6.2}$$

and its dual

$$\tilde{F}_{\mu\nu} = \frac{1}{2} E_{\mu\nu\rho\sigma} F^{\rho\sigma}, \tag{6.3}$$

where $E^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor, which can be written in terms of the Levi-Civita symbol ϵ as

$$E_{\mu\nu\rho\sigma} = \sqrt{-g}\epsilon_{\mu\nu\rho\sigma}. \tag{6.4}$$

In terms of differential forms, the dual utilizes the Hodge star operator $\star$

$$\tilde{F} = \star F. \tag{6.5}$$

Some good references for electromagnetic fields in curved spacetimes are [96–98].

Also, g is the determinant of $g_{\mu\nu}$, R the Ricci scalar and κ_4^2 a constant for 4d-gravitational theories that normalizes to

$$\frac{2L^2}{\kappa_4^2} = \frac{\sqrt{2}N^{3/2}}{6\pi}. \tag{6.6}$$

The equations of motion for the gravity and for the connection are:

$$R_{\mu\nu} = 2L^2 F_{\mu\sigma}F^{\sigma}_{\nu} - \frac{L^2}{2}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} - \frac{3}{L^2}g_{\mu\nu}, \tag{6.7}$$

$$\nabla_\mu \left(F^{\mu\nu} + W\tilde{F}^{\mu\nu}\right) = 0. \tag{6.8}$$

Since since F is antisymmetric rank 2 tensor, we can write

$$\begin{aligned}\nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\mu\rho}F^{\rho\nu} + \Gamma^{\nu}_{\mu\rho}F^{\mu\rho} \\ &= \partial_\mu F^{\mu\nu} + \left(\frac{1}{\sqrt{-g}}\partial_\rho\sqrt{-g}\right)F^{\rho\nu} \\ &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g})F^{\mu\nu}.\end{aligned} \tag{6.9}$$

Then we can write the equations of motion in the same form as in [20]:

$$\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\left(F^{\mu\nu} + W\tilde{F}^{\mu\nu}\right) = 0. \tag{6.10}$$

Note that the topological term does not affect the gravity equation. Then we can use the solution in [2]:

$$\frac{1}{L^2}ds^2 = -\frac{\alpha^2}{z^2}f(z)dt^2 + \frac{\alpha^2}{z^2}(dx^2+dy^2) + \frac{1}{z^2}\frac{dz^2}{f(z)}, \tag{6.11}$$

where f(z) carries the magnetic and electric charge,

$$f(z) = 1 + (h^2+q^2)z^4 - (1+h^2+q^2)z^3. \tag{6.12}$$

The horizon of the black hole is at $z=1$. The space boundary of AdS is in the asymptotic $z \to 0$. We can now see that

$$\sqrt{-g} = \frac{L^4\alpha^3}{z^4}. \tag{6.13}$$

The solution for the connection in [2] still works for 6.8:

$$A = h\alpha^2 x dy + q\alpha(z-1)dt, \tag{6.14}$$

giving the curvature equal to (using coordinates $\{t,x,y,z\}$):

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -q\alpha \\ 0 & 0 & h\alpha^2 & 0 \\ 0 & -h\alpha^2 & 0 & 0 \\ q\alpha & 0 & 0 & 0 \end{pmatrix}, \tag{6.15}$$

and the dual becomes:

$$\tilde{F}^{\mu\nu} = \frac{z^4}{L^4\alpha^3}\begin{pmatrix} 0 & 0 & 0 & h\alpha^2 \\ 0 & 0 & -q\alpha & 0 \\ 0 & q\alpha & 0 & 0 \\ -h\alpha^2 & 0 & 0 & 0 \end{pmatrix}. \tag{6.16}$$

We can compute the temperature of the black hole by

$$T = \frac{\alpha f'(1/z)|_{z=1}}{4\pi} = \frac{\alpha(3-h^2-q^2)}{4\pi}. \tag{6.17}$$

In order to compute the thermodynamics of the black hole we use the definition for the Grand Canonical Ensemble $\Omega = TI$. But a the solutions found the action becomes infinite. The reason is that we actually need to renormalize the action.

$$I_{ren} = I + \text{counterterms} \tag{6.18}$$

We can use the counterterms that comes from the boundary of the AdS found in the references [69–71]:

$$I_{\text{ct}} = \frac{1}{\kappa_4^2}\int d^3x\sqrt{-\gamma}\theta - \frac{2}{\kappa_4^2}\frac{1}{L}\int d^3x\sqrt{-\gamma}. \tag{6.19}$$

We will show what the terms above mean. First we define a unit vector that is normal to the boundary $n^\mu \propto (0,0,0,1)$. We find the normalization from

$$n^\mu n_\mu = 1. \tag{6.20}$$

In our case

$$n^\mu = z\frac{\sqrt{f(z)}}{L}. \tag{6.21}$$

The boundary metric γ_{MN} is the 3d-metric of the boundary induced by $g_{\mu\nu}$. Note that the metric goes to infinity at the boundary $z \to 0$. To avoid this, we will consider our calculations at some point near the boundary at $z = \epsilon$ for some small $\epsilon > 0$. Then the induced metric becomes

$$\gamma_{MN} = -\frac{\alpha^2}{\epsilon^2}f(\epsilon)dt^2 + \frac{\alpha^2}{\epsilon^2}(dx^2+dy^2). \tag{6.22}$$

We could had considered the induced metric in 4d by defining $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. One can check that this gives the same metric.

The other term to be considered is the extrinsic curvature defined by

$$\theta_{MN} = -\frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu) = \begin{pmatrix} \frac{L\sqrt{f(\epsilon)}(\epsilon f'(\epsilon)-2f(\epsilon))}{2\epsilon^2} & 0 & 0 \\ 0 & -\frac{L\sqrt{f(\epsilon)}\alpha^2}{\epsilon^2} & 0 \\ 0 & 0 & -\frac{L\sqrt{f(\epsilon)}\alpha^2}{\epsilon^2} \end{pmatrix}, \tag{6.23}$$

and its trace is

$$\theta = \gamma^{MN}\theta_{MN} = \frac{\epsilon f'(\epsilon) - 6f(\epsilon)}{2L\sqrt{f(\epsilon)}}. \tag{6.24}$$

Then plugging this in 6.19 and realising that the integral in t becomes a finite temperature β we get, in a power series of ϵ

$$I_{\text{ct}} = \beta V \frac{\sqrt{2}N^{3/2}}{6\pi}\alpha^3 \left(\frac{-1}{\epsilon^3} + \frac{1+h^2+q^2}{2} + \mathcal{O}(\epsilon)\right), \tag{6.25}$$

where $V = \int dxdy$.

With this we can now see our solutions in the unrenormalized action and we are going to have (integrating in z form ϵ to 1):

$$I = \beta V \frac{\sqrt{2}N^{3/2}}{6\pi}\alpha^3 \left(\frac{1}{\epsilon} - 1 + h^2 - q^2 - 2Whq + \mathcal{O}(\epsilon)\right). \tag{6.26}$$

Then the full action becomes,

$$I_{ren} = \lim_{\epsilon\to 0}(I + I_{ct}) = \beta V \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{\alpha^3}{4}\left(-1 - q^2 + 3h^2 - 4Whq\right). \tag{6.27}$$

The Grand Canonical Ensemble is defined as the action times the temperature

$$\Omega = T I_{ren}, \tag{6.28}$$

therefore

$$\Omega = \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{\alpha^3 V}{4}\left(-1 - q^2 + 3h^2 - 4Whq\right). \tag{6.29}$$

The magnetic field and the chemical potential are found from the $z \to 0$ limit of A in (6.14) and its field strength F in (6.15),

$$B = \alpha^2 h\ , \qquad \mu = -\alpha q. \tag{6.30}$$

The grand canonical ensemble in terms of this parameters is:

$$\Omega = \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{\alpha^3 V}{4}\left(-1 - \frac{\mu^2}{\alpha^2} + 3\frac{B^2}{\alpha^4} + 4W\frac{\mu B}{\alpha^3}\right). \tag{6.31}$$

And the temperature is

$$T = \frac{\alpha}{4\pi}\left(3 - \frac{\mu^2}{\alpha^2} - \frac{B^2}{\alpha^4}\right). \tag{6.32}$$

While this potential depends on W, the entropy S and energy E derived from it must not, since they depend only on the geometry of the black hole. Indeed, for the entropy

$$S = -\left.\frac{\partial\Omega}{\partial T}\right|_{B,\mu} = -\left.\left(\frac{\partial\Omega}{\partial\alpha}\right)\right|_{B,\mu} \Big/ \left.\left(\frac{\partial T}{\partial\alpha}\right)\right|_{B,\mu}, \tag{6.33}$$

we first find from the relations for Ω (6.31) in and the T in (6.32) that

$$dT|_{B,\mu} = \frac{d\alpha}{4\pi}\left(3 + 3\frac{B^2}{\alpha^4} + \frac{\mu^2}{\alpha^2}\right) \ , \tag{6.34}$$

$$d\Omega|_{B,\mu} = -\frac{\sqrt{2}N^{3/2}}{6\pi}\frac{\alpha^2 d\alpha V}{4}\left(3 + 3\frac{B^2}{\alpha^4} + \frac{\mu^2}{\alpha^2}\right) \ , \tag{6.35}$$

so that

$$S = -\left.\frac{\partial\Omega}{\partial T}\right|_{B,\mu} = \frac{\sqrt{2}N^{3/2}}{6}\alpha^2 V, \tag{6.36}$$

As for the energy, we first need to compute the electric charge Q, so, from deriving equation (6.32) with T, B fixed, we get $\mu = \mu(\alpha)$ as

$$d\alpha\left(3 + \frac{\mu^2}{\alpha^2} + 3\frac{B^2}{\alpha^4}\right) = 2\mu\frac{d\mu}{\alpha} \ , \tag{6.37}$$

and then

$$\begin{aligned} d\Omega|_{T,B} &= \left.\frac{\partial\Omega}{\partial\alpha}\right|_{\mu,T,B} d\alpha + \left.\frac{\partial\Omega}{\partial\mu}\right|_{\alpha,T,B} d\mu \\ &= \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{V}{4}\left\{d\alpha\left[-3\alpha^2 - \mu^2 - 3\frac{B^2}{\alpha^2}\right] + d\mu\left[-2\alpha\mu + 4BW\right]\right\} \\ &= \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{V}{4}\left\{-2\mu\frac{d\mu}{\alpha}\alpha^2 + d\mu\left[-2\alpha\mu + 4BW\right]\right\} \ , \end{aligned} \tag{6.38}$$

so that

$$Q = -\left.\frac{\partial\Omega}{\partial\mu}\right|_{T,B} = \frac{\sqrt{2}N^{3/2}}{6\pi}V(\alpha\mu - BW). \tag{6.39}$$

Then the energy is

$$E = \Omega + TS + \mu Q = \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{\alpha^3 V}{2}\left(1 + q^2 + h^2\right) \ . \tag{6.40}$$

Again from (6.32), when T, μ are fixed, we get $\alpha = \alpha(B)$,

$$d\alpha\left(3 + \frac{\mu^2}{\alpha^2} + 3\frac{B^2}{\alpha^4}\right) = 2B\frac{dB}{\alpha^3} \ , \tag{6.41}$$

and then

$$\begin{aligned} d\Omega|_{T,\mu} &= \left.\frac{\partial\Omega}{\partial\alpha}\right|_{B,T,\mu} d\alpha + \left.\frac{\partial\Omega}{\partial\mu}\right|_{\alpha,T,\mu} d\mu \\ &= \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{V}{4}\left\{d\alpha\left[-3\alpha^2 - \mu^2 - 3\frac{B^2}{\alpha^2}\right] + 6\frac{BdB}{\alpha} + 4W\mu dB\right\} \\ &= \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{V}{4}\left\{-2B\frac{dB}{\alpha^3}\alpha^2 + 6\frac{BdB}{\alpha} + 4W\mu dB\right\} \ . \end{aligned} \tag{6.42}$$

so that

$$M = \left.\frac{d\Omega}{dB}\right|_{T,\mu} = \frac{\sqrt{2}N^{3/2}}{6\pi}\frac{V}{\alpha}\left(B + W\alpha\mu\right). \tag{6.43}$$

So the charge and magnetization do get a W term, while the entropy and energy don't.

6.1.1 Susceptibilities

We also wish to compute the susceptibilities of this theory so that we may compare with other models in the following chapters.

Let's first aim to calculate

$$\chi_{E,E} = -\frac{1}{V}\left.\frac{\partial^2\Omega}{\partial\mu^2}\right|_{T,B} , \tag{6.44}$$

which can be also written as

$$\chi_{E,E} = -\frac{1}{V}\left.\frac{\partial Q}{\partial\mu}\right|_{T,B} . \tag{6.45}$$

We can write

$$\begin{aligned} dQ|_{T,B} &= \left.\frac{\partial Q}{\partial\alpha}\right|_{B,T,\mu} d\alpha + \left.\frac{\partial Q}{\partial\mu}\right|_{\alpha,T,\mu} d\mu \\ &= \frac{\sqrt{2}N^{3/2}}{6\pi}V(\mu d\alpha + \alpha d\mu) \\ &= \frac{\sqrt{2}N^{3/2}}{6\pi}V\left(\alpha + \frac{2\alpha^3\mu^2}{3B^2+3\alpha^4+\alpha^2\mu^2}\right)d\mu \; . \end{aligned} \tag{6.46}$$

So then

$$\begin{aligned} \chi_{E,E} &= -\frac{\sqrt{2}N^{3/2}}{6\pi}\alpha\left(\frac{3\alpha^4+3B^2+3\alpha^2\mu^2}{3\alpha^4+3B^2+\alpha^2\mu^2}\right) \\ &= -\frac{\sqrt{2}N^{3/2}}{6\pi}3\alpha\left(\frac{1+h^2+q^2}{3+3h^2+q^2}\right) \; . \end{aligned} \tag{6.47}$$

We can do the same for

$$\chi_{T,T} = -\frac{1}{V}\left.\frac{\partial^2\Omega}{\partial T^2}\right|_{B,\mu} = -\frac{1}{V}\left.\frac{\partial S}{\partial T}\right|_{B,\mu} . \tag{6.48}$$

Note that

$$\begin{aligned} \left.\frac{\partial S}{\partial T}\right|_{B,\mu} &= \left.\frac{\partial S}{\partial\alpha}\right|_{B,\mu} \Big/ \left.\frac{\partial T}{\partial\alpha}\right|_{B,\mu} \\ &= \frac{\sqrt{2}N^{3/2}}{6}V\frac{8\alpha^5}{3\alpha^4+3B^2+\alpha^2\mu^2} \; . \end{aligned} \tag{6.49}$$

So we have

$$\chi_{T,T} = -\frac{1}{V}\left.\frac{\partial^2\Omega}{\partial T^2}\right|_{B,\mu} = -\frac{\sqrt{2}N^{3/2}}{6}\frac{8\alpha}{3+3h^2+q^2} \; . \tag{6.50}$$

We can also calculate the cross term

$$\chi_{T,E} = -\frac{1}{V}\left.\frac{\partial}{\partial T}\right|_{B,\mu}\left.\frac{\partial}{\partial \mu}\right|_{T,B}\Omega \tag{6.51}$$

by taking either a derivative on Q or in S

$$\chi_{T,E} = -\frac{1}{V}\left.\frac{\partial Q}{\partial \alpha}\right|_{B,\mu}\Bigg/\left.\frac{\partial T}{\partial \alpha}\right|_{B,\mu} = -\frac{1}{V}\left.\frac{\partial S}{\partial \mu}\right|_{T,B} . \tag{6.52}$$

Regardless of the choice, the result is

$$\begin{aligned}\chi_{T,E} &= -\frac{\sqrt{2}N^{3/2}}{6\pi}\frac{4\alpha^4\mu}{3\alpha^4+3B^2+\alpha^2\mu^2}\\ &= -\frac{\sqrt{2}N^{3/2}}{6\pi}\frac{4\mu}{3+3h^2+q^2} .\end{aligned} \tag{6.53}$$

This reproduces the results in [21]n although the authors use a different notation. But we can move forwad and also compute

$$\chi_{B,B} = -\frac{1}{V}\left.\frac{\partial^2\Omega}{\partial^2 B}\right|_{T,\mu} \tag{6.54}$$

$$\chi_{T,B} = -\frac{1}{V}\left.\frac{\partial}{\partial T}\right|_{B,\mu}\left.\frac{\partial}{\partial B}\right|_{T,\mu}\Omega \tag{6.55}$$

$$\chi_{B,E} = -\frac{1}{V}\left.\frac{\partial}{\partial B}\right|_{T,\mu}\left.\frac{\partial}{\partial \mu}\right|_{T,B}\Omega . \tag{6.56}$$

We can compute these in a similarly fashion than the others before, and we can get the following matrix

$$\chi = \begin{pmatrix}\chi_{E,E} & \chi_{T,E} & \chi_{T,B}\\ \chi_{T,E} & \chi_{T,T} & \chi_{B,E}\\ \chi_{T,B} & \chi_{B,E} & \chi_{B,B}\end{pmatrix} \tag{6.57}$$

to be equal to

$$\chi = -\frac{\sqrt{2}N^{3/2}}{6}\begin{pmatrix}\frac{\alpha(1+h^2+q^2)}{3+3h^2+q^2} & \frac{4\mu}{3+3h^2+q^2} & \frac{4\pi B}{\alpha^2(3+3h^2+q^2)}\\ \frac{4\mu}{3+3h^2+q^2} & \frac{8\pi\alpha}{3+3h^2+q^2} & \frac{2qh}{3+3h^2+q^2}+W\\ \frac{4\pi B}{\alpha^2(3+3h^2+q^2)} & \frac{2qh}{3+3h^2+q^2}+W & \frac{3+h^2+q^2}{\alpha(3+3h^2+q^2)}\end{pmatrix} . \tag{6.58}$$

Note that only one of the entries depend on the topological term as can be seen by the presence of the constant W.

6.2 Magnetization term

In order to get the transport coefficients from the Kubo Formula we need to consider green functions of fluctuations of the AdS background. The general fluctuations was calculated in [2] for the theory without the topological term and with translational symmetry, and in [3] the authors considered a specific ansatz for the fluctuations. We aim to use this ansatz in our analysis.

The Kubo formula for systems with $B \neq 0$ must subtract the magnetic currents [99]

$$J_i^{\text{mag}} = \epsilon_{ij}\partial_j M, \tag{6.59}$$

$$T_{ti}^{\text{mag}} = \epsilon_{ij}\partial_j M^E, \tag{6.60}$$

where M and M^E are respectively the magnetization and the energy magnetization densities.

We consider background fields δA_μ^0 that is coupled to a source current J_μ and a $\delta g_{t\nu}^0$ that sources $T_{t\nu}$. We consider fluctuations that at the boundary gives us this fields:

$$\lim_{z\to 0} \delta A_y = xB, \tag{6.61}$$

$$\lim_{z\to 0} \delta G_y = xB^E, \tag{6.62}$$

where $G_y = g_{ty}z^2/\alpha$. The ansatz we take from [3] For these fluctuations is

$$\delta A_y = x(B - qB^E z), \tag{6.63}$$

$$\delta G_y = xf(z)B^E. \tag{6.64}$$

Also, we consider that we are going to need only $\delta A_t(z)$ to compensate δA_y and δG_y in the equations of motion at the linear level of perturbation.

We then proceed to consider the on-shell action 6.1 at the linear level of these fluctuations. The process is similar to the linearization of gravity described in chapter 40 of [100] and in chapter 18 of [101]. Here we are going to have a more "brute force" approach to this method. We introduce a parameter ϵ so that our new metric and connection becomes

$$g_{\mu\nu} = g_{\mu\nu}^{\text{background}} + \epsilon\delta g_{\mu\nu}, \tag{6.65}$$

$$A_\mu = A_\mu^{\text{background}} + \epsilon\delta A_\mu. \tag{6.66}$$

Then we compute the equations of motion 6.8 and 6.7 using the definitions above, and we expand the result in a Taylor series in $epsilon$. We expect the 0-th term to be zero since they are just the equations of motion of the background. The result for the gravity equation is:

$$0 = 2L^2 F_{\mu\sigma}F_\nu^\sigma - \frac{L^2}{2}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} - \frac{3}{L^2}g_{\mu\nu} - R_{\mu\nu}. \tag{6.67}$$

For example, for $(\mu,\nu) = (x,x)$ the right hand side of the above expression gives

$$2z^2(Bh - B^E qhz + q\alpha\delta A_t'(z))\epsilon + \mathcal{O}(\epsilon^2). \tag{6.68}$$

Prime denotes partial derivation of the coordinate z. So in order for it to respect the equations of motion at linear level of perturbation we need

$$\delta A_t'(z) = -\frac{Bh - B^E qhz}{q\alpha} \tag{6.69}$$

The full calculation shows that the same expression above cancels every linear term for every pair $(\mu, \nu) = \{t, x, y, z\}$.

For the gauge equation of motion (first without the W term) we do the same procedure

$$\nabla_\mu F^{\mu t} = \frac{z^4}{\alpha^3}(B^E h - \alpha \delta A_t''(z))\epsilon + \mathcal{O}(\epsilon^2). \tag{6.70}$$

Then

$$\delta A_t''(z) = \frac{B^E h}{\alpha}. \tag{6.71}$$

The solution for the differential equations 6.70 and 6.71 is

$$\delta A_t(z) = \frac{hB^E}{2\alpha}(z^2 - 1) - \frac{hB}{q\alpha}(z - 1), \tag{6.72}$$

where the integration constants were chosen such that δA_t goes to 0 at the horizon. This result matches [3].

We then add the equations of motion for the topological term. As expected since the topological term is a constant, we get

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0, \tag{6.73}$$

therefore in order for the fluctuations to impact on the equations of motion we will have to consider the coefficient W as a function.

6.3 Computing the Green functions

6.3.1 Recipe

In other to compute the Green function of our models, we use the prescription from [8]:

1. From the classical action for a field ϕ, recognize the function $A(z)$ in front of the kinetic term $(\partial_\mu \phi)^2$

$$S_{cl} = \frac{1}{2}\int dz dx^2 A(z)(\partial_\mu \phi)^2 \tag{6.74}$$

2. Solve the linearized field equations for phi in terms of its value at the boundary ϕ_0:

$$\phi(z) = f\omega(z)\phi_0, \tag{6.75}$$

therefore we require that $f\omega(z) = 1$ at the boundary. Also, we need to impose incoming-wave boundary conditions

3. The retarded Green function is given by

$$G^R = \lim_{z\to 0} A(z) f_{-\omega}(z)\partial_z f_\omega. \tag{6.76}$$

We are going to show how [2] implemented this prescription for their model[1]. First we assume fluctuations of the form $\delta A_i = A_i(z)e^{i\omega t}$ and $\delta G_i = G_i(z)e^{-i\omega t}$ for $i \in \{x, y\}$ and $\alpha G_i = g_{ti}z^2$, and then we compute the linearized Maxwell's equations of motion for theses fluctuations:

$$\begin{aligned} f(fA_x')' + \bar\omega A_x + i\bar\omega h G_y + qfG_x' = 0 \\ f(fA_y')' + \bar\omega A_y + i\bar\omega h G_x + qfG_y' = 0 \end{aligned} \tag{6.77}$$

Here, prime denotes derivation by z. One can check that the gravity equations of motion leads to the same equations. We have defined, in the notation of [2], the dimensionless frequency $\bar\omega = \omega/\alpha$.

Now we impose incoming wave solutions

$$\begin{aligned} A_i(z) =& f(z)^\nu a_i(z) \\ G_i(z) =& f(z)^{(1+\nu)} g_i(z), \end{aligned} \tag{6.78}$$

where $\nu = i\bar\omega/(h^2 + q^2 - 3)$. We require the functions a_i and g_i to be regular at the horizon $z = 1$.

We also have a constant solution

$$\begin{aligned} G_y =& \frac{i\bar\omega}{h} A_x \\ G_x =& -\frac{i\bar\omega}{h} A_y. \end{aligned} ;. \tag{6.79}$$

6.3.2 Fluctuations in the Hydrodynamic Limit

We want to solve the solutions at the hydrodynamic limit $\bar\omega \to 0$. for this, we expand the fluctuations around the frequency while maintaining h and q fixed:

$$\begin{aligned} a_i(z) =& a^{(0)} + \bar\omega a_i^{(1)}(z) + ..., \\ g_i(z) =& g^{(0)} + \bar\omega g_i^{(1)}(z) + ... \end{aligned} \tag{6.80}$$

The solution for this can be found in detail in [2]. In first order we get

$$G_x(z) = -\frac{i\bar\omega}{h}\delta_y + f(z)^{1+\nu}\left(G_x^0 + \frac{i\bar\omega}{h}\delta_y - i\bar\omega G_x^0 \int_0^z \frac{du}{\psi^2(u)} P_5(u)\right), \tag{6.81}$$

and

$$\begin{aligned} A_x(z) = \delta_x + f(z)^\nu \Bigg(A_x^0 - \delta_x - (G_x^0 + \frac{i\bar\omega}{h})qz + i\bar\omega q G_x^0 \int_0^z \frac{du(z-u)}{\psi^2(u)} P_5(u) \\ -i\bar\omega \int_0^z \frac{du}{f(u)} (G_x^0 Q_4(u) + G_y^0 Q_3(u)) \Bigg). \end{aligned} \tag{6.82}$$

[1]The model topological term with a constant W doesn't affect linear perturbations, therefore the results showed here are going to work for the model that includes it.

Where $\psi(z) = f(z)/z$, the functions $P_5(u)$, $Q_3(u)$ and $Q_4(u)$ are polynomial found at Appendix 1 of [2] and δ_i are constants that depends on the boundary values of the fields

$$\delta_x = A_x^0 + \frac{G_y^0 h(h^2+q^2-3) - 3G_x^0 q(1+h^2+q^2)}{4(h^2+q^2)}. \tag{6.83}$$

And similarly for δ_y.

6.3.3 Quadratic Action

We need the classical quadratic action for the description of the Green functions. We do this by writing 6.1 up to linear order in perturbation an then expanding in on shell.

It may be easier to see how this is done term by term. First, let us neglect the metric perturbations and only consider those coming from the Maxwell fields, then we have

$$\begin{aligned}\frac{2}{\kappa_4^2}\int d^4x\sqrt{-g}\left(\frac{L^2}{4}F_{\mu\nu}F^{\mu\nu} + W\frac{L^2}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \rightarrow \\ \frac{\alpha L^2}{\kappa_4^2}\int d^4x\left(\frac{\bar{\omega}}{f(z)}(A_x^2+A_y^2) + f(z)((A_x')^2+(A_y')^2) + 2iW\bar{\omega}(A_x'A_y - A_xA_y')\right)\end{aligned} \tag{6.84}$$

Integrating by parts and neglecting terms calculated at the horizon $(z=1)$ and at lowest order in ω gives us

$$\begin{aligned}\frac{2}{\kappa_4^2}\int d^4x\sqrt{-g}\left(\frac{L^2}{4}F_{\mu\nu}F^{\mu\nu} + W\frac{L^2}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \rightarrow \\ \frac{\alpha L^2}{\kappa_4^2}\int d^3x \lim_{z\to 0} f(z)\left[(A_x'A_x + A_y'A_y) + 2i\bar{\omega}W\epsilon^{ab}A_aA_b\right]\end{aligned} \tag{6.85}$$

Bringing back the perturbations from the metric gives us the result

$$\begin{aligned}I \rightarrow \frac{\alpha L^2}{\kappa_4^2}\int d^3x \lim_{z\to 0}\bigg(&\frac{f^{1/2}-1}{2z^3 f^{1/2}}(G_xG_x + G_yG_y) + q(A_xG_x + A_yG_y) + \\ &-\frac{1}{8z^2}(G_xG_x' + G_yG_y') + \frac{f}{2}(A_xA_x' + A_yA_y') + 2i\bar{\omega}W\epsilon^{ab}A_aA_b\bigg).\end{aligned} \tag{6.86}$$

which, except for the new extra term with W matches the result in [2].

We can now expand this result in Fourier modes for the fields

$$A_x^0(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}A_x^0(\omega)e^{-i\omega t} \tag{6.87}$$

Then inputting the on-shell linearized fluctuations 6.82 we have:

$$
\begin{aligned}
I_{\text{quadratic}} = &\frac{i}{2}\int \frac{d\omega}{2\pi} dx^2 \omega (A_x^0(\omega)A_y^0(-\omega) - A_y^0(\omega)A_x^0(-\omega))\left(\frac{q}{h} - W\right) + \\
&\frac{-3i(1+h^2+q^2)}{4h}\int \frac{d\omega}{2\pi} dx^2 \omega (A_x^0(\omega)G_y^0(-\omega) - A_y^0(\omega)G_x^0(-\omega)) + \\
&\frac{9iq(1+h^2+q^2)^2}{32(h^2+q^2)}\int \frac{d\omega}{2\pi} dx^2 \omega (G_x^0(\omega)G_y^0(-\omega) - G_y^0(\omega)G_x^0(-\omega)) + \\
&\frac{i(-3+h^2+q^2)^2}{32(h^2+q^2)}\int \frac{d\omega}{2\pi} dx^2 \omega (G_x^0(\omega)G_x^0(-\omega) - G_y^0(\omega)G_y^0(-\omega)).
\end{aligned}
\tag{6.88}
$$

which is now in the form where we can use the recipe for calculating holographic retarded Green's functions. Note that the leading terms of order $\mathcal{O}(\bar{\omega}^0)$ have cancelled, and the only contributions are: from the mixing of terms alluded before (G_x containing A_y^0, etc.) for the Maxwell and gravity terms, leading to an extra $\bar{\omega}$, but from the leading term in the topological one, and seeing as we already had a $\bar{\omega}$, this is of the same order.

We can now go back to the recipe to read off the Green functions. Recalling that the magnetic field is$B = h\alpha^2$, The charge density is $\rho = q\alpha^2$ and the energy density is $\epsilon = (1 + h^2 + q^2)$, we have

$$
G^R_{J_a,J_b} = -i\omega\epsilon_{a,b}\left(\frac{q}{h} - W\right) = -i\omega\epsilon_{a,b}\left(\frac{\rho}{B} - W\right) , \tag{6.89}
$$

$$
G^R_{J_a,T_{tb}} = -\, i\omega\epsilon_{a,b}\frac{3(1+h^2+q^2)}{2h} = -i\omega\epsilon_{a,b}\frac{\epsilon}{2B} , \tag{6.90}
$$

and finally the correlator between energy-momentum tensors:

$$
G^R_{T_{ta},T_{tb}} = -i\omega\epsilon_{ab}\frac{9q(1+h^2+q^2)^2}{32(h^2+q^2)} + i\omega\delta_{ab}\frac{(-3+h^2+q^2)^2}{32(h^2+q^2)} . \tag{6.91}
$$

Plugging (6.89) to the Kubo formula gives us

$$
\sigma_{ab} = -\lim_{\omega\to 0}\frac{\mathrm{Im}G^R_{J_aJ_b}}{\omega} = \epsilon_{ab}\left(\frac{\rho}{B} - W\right) . \tag{6.92}
$$

The formulas for α and $\bar{\kappa}$ follow the same path. As noted in section 6.2, we need to consider the magnetization term from [99] in the Kubo formulas:

$$
\alpha_{ab} = -\lim_{\omega\to 0}\frac{\mathrm{Im}G^R_{T_{ta}J_b}}{\omega} + \frac{M}{T}\epsilon_{ab} = \frac{s}{B}\epsilon_{ab} , \tag{6.93}
$$

and

$$
\bar{\kappa}_{ab} = -\lim_{\omega\to 0}\frac{\mathrm{Im}G^R_{T_{ta}T_{tb}}}{\omega} + \frac{2(M^E - \mu M)}{T}\epsilon_{ab} . \tag{6.94}
$$

6.3.4 Anisotropy

In order to break isotropy, we consider two parameters (k_x, k_y) multiplying the boundary space coordinates, giving a metric

$$
\frac{1}{L^2}ds^2 = -\frac{\alpha^2}{z^2}f(z)dt^2 + \frac{\alpha^2}{z^2}(k_x dx^2 + k_y dy^2) + \frac{1}{z^2}\frac{dz^2}{f(z)}. \tag{6.95}
$$

Note that then we obtain $\sqrt{-g} = \sqrt{k_x k_y} L^4 \alpha^3 / z^4$.

From the equations of motion for gravity, we obtain the condition

$$k_x = 1/k_y. \tag{6.96}$$

Nothing new is obtained from this moment on, and the analysis proceeds as before, with no new physics. We have shown this, however, since in the next section, when we have varying $W(z)$, there will be new results.

6.3.5 Conductivity from entropy function via attractor mechanism and membrane paradigm

The conductivity can also be obtained using the attractor mechanism, for calculations at the horizon, via Sen's entropy function [22]. The application of Sen's entropy function formalism to holography was considered in [102] (see also [103]), while the application to the calculation of conductivity was done in [17], and was also used in [20] for the case of metrics with charge density and magnetic field, defined only near the horizon. The fact that we can calculate the conductivity at the horizon, as well as at its natural AdS/CFT location, the boundary, is related to the application of the membrane paradigm to the AdS black holes, as argued initially by Iqbal and Liu [18].

The near-horizon geometry of an *extremal* 4-dimensional planar black hole (such as obtained in the attractor mechanism) is $AdS_2 \times \mathbb{R}^2$, written as

$$ds^2 = -vr^2\, dt^2 + w(dx^2 + dy^2) + \frac{v}{r^2} dr^2 \ , \tag{6.97}$$

where $r = 1/z$. Note that this is for a planar black hole, but the formalism works as well as for the spherical (S^2 instead of $\mathbb{R}^2$) black hole case [22]. Also, sometimes v and w are denoted by v_1 and v_2, in order to emphasize their similarities.

The values of the magnetic and electric fields at the horizon are $F^A_{xy} = B^A$ (soemetimes denoted p^A) and $F^A_{zt} = e_A$, and for the scalars we have the horizon values $\phi_s = u_s$.

Defining the function f as the integral over the horizon of the Lagrangian density,

$$f(u, v, w, B^A, e_A) = \int dxdy \sqrt{-g}\mathcal{L} \ , \tag{6.98}$$

as shown by Sen, the Einstein equations imply that the u_s, v and w are extrema of f, while the charges Q_A can be defined as its variations with respect to the electric fields, so

$$\frac{\partial f}{\partial u_s} = 0 \ , \quad \frac{\partial f}{\partial v} = 0 \ , \quad \frac{\partial f}{\partial w} = 0 \ , \quad \frac{\partial f}{\partial e_A} = Q^A. \tag{6.99}$$

Then Sen's entropy function is

$$\mathcal{E}(u_s, v, w, e_A, B^A; Q^A) = 2\pi(e_A Q^A - f(u_s, v, w, e_A, B^A)) \ , \tag{6.100}$$

and is value *at its extremum*, defined by the *attractor equations*

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0 \ , \quad \frac{\partial \mathcal{E}}{\partial v} = 0 \ , \quad \frac{\partial \mathcal{E}}{\partial w} = 0 \ , \quad \frac{\partial \mathcal{E}}{\partial e_A} = 0. \tag{6.101}$$

In our case we don't have a scalar, and we have a single electromagnetic field, with electric field e, so we only have 3 attractor equations. The function f in our case is then (with $4\pi G_N = 1$)

$$\begin{aligned} -f &= \int dx\,dy\,\sqrt{-g}\left(-\frac{1}{4}R+\frac{L^2}{4}F_{\mu\nu}F^{\mu\nu}-\frac{3}{2}\frac{1}{L^2}+W\frac{L^2}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \\ &= \frac{1}{2}\int dxdy\left(\frac{B^2L^2v}{w}+w\left(1-\frac{3v}{L^2}-\frac{e^2L^2}{v}\right)+2BeL^2W\right)\ , \end{aligned} \tag{6.102}$$

and the entropy function is

$$\mathcal{E} = 2\pi e(\tilde{Q}-BL^2W)+\pi\frac{B^2L^2v}{w}+\pi w\left(1-\frac{3v}{L^2}-\frac{e^2L^2}{v}\right)\ , \tag{6.103}$$

where $\tilde{Q} = Q/\text{Vol}$. The attractor equations are then

$$\begin{aligned} w^2\left(3-\frac{e^2L^4}{v^2}\right)-B^2L^4 &= 0 \\ L^2\left(\frac{B^2v}{w^2}+\frac{e^2}{v}\right)+\frac{3v}{L^2}-1 &= 0 \\ \frac{L^2\left(ew-BvW\right)}{v}-\tilde{Q} &= 0. \end{aligned} \tag{6.104}$$

We use the last equation to solve for e in terms of $\tilde{Q}$ (inverting it), and the other two to solve for v and w. Then the parameters v, w, e of the ansatz are written in terms of the charge density $\tilde{Q}$ and the magnetic field B, as well as the parameters L and W in the action, as

$$\begin{aligned} v &= \frac{L^2}{6}\ , \\ w &= \frac{\sqrt{B^2L^4\left(W^2+1\right)-2BL^2\tilde{Q}W+\tilde{Q}^2}}{\sqrt{3}}\ , \\ e &= \frac{BL^2W-\tilde{Q}}{2\sqrt{3}z^2\sqrt{B^2L^4\left(W^2+1\right)-2BL^2\tilde{Q}W+\tilde{Q}^2}}. \end{aligned} \tag{6.105}$$

Since the charge density of the field theory, ρ, is dual (couples) to A_t, we have

$$\rho = \frac{\delta S}{\delta A_t}\ , \tag{6.106}$$

which gives, on the solution, the same result as the charge density parameter of the black hole,

$$\rho = \sqrt{-g}(F^{tz}+W\tilde{F}^{tz}) = \frac{\tilde{Q}}{L^2}. \tag{6.107}$$

For the entropy density, the extremum of the entropy function gives the same as the Hawking formula,

$$s = \frac{\sqrt{B^2L^4\left(W^2+1\right)-2BL^2\tilde{Q}W+\tilde{Q}^2}}{4\sqrt{3}G_N} = \frac{w}{4G_N}\ , \tag{6.108}$$

and this is identified with the entropy density in the field theory.

We can now use these formulas for the entropy density s and the charge density ρ of the field theory in the formulas already derived, from Kubo formulas, for the electric and thermoelectric conductivities, to find these as functions of the charges of the black hole. We find

$$\begin{aligned} \sigma_{ab} &= \epsilon_{ab}\left(\frac{\rho}{B} - W\right) = \epsilon_{ab}\left(\frac{\tilde{Q}}{BL^2} - W\right) , \\ \alpha_{ab} &= \epsilon_{ab}\frac{\sqrt{B^2L^4\left(W^2+1\right) - 2BL^2\tilde{Q}W + \tilde{Q}^2}}{4\sqrt{3}BG_N} . \end{aligned} \tag{6.109}$$

We note that now, with the holographic transport coefficients written in terms of $\tilde{Q}$, B and W via the attractor mechanism, the explicit W dependence becomes more complicated.

6.3.6 S-duality

As observed in [20], and made more precise in [21], the general formulas for transport coefficients obtained from fluctuations around a solution near the horizon of a black hole have an action of S-duality ($Sl(2;\mathbb{Z})$) on them, coming from the S-duality invariance of the gravitational action.

If we consider the action

$$I = -\frac{4}{2\kappa_4^2}\int d^4x\sqrt{-g}\left(-\frac{1}{4}R + Z\frac{L^2}{4}F_{\mu\nu}F^{\mu\nu} + W\frac{L^2}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right) - \frac{3}{2}\frac{1}{L^2} , \tag{6.110}$$

Which is the same as before but with the addition of Z in front of the Maxwell action, the invariance is under

$$\begin{aligned} F_{\mu\nu} &\to Z\tilde{F}_{\mu\nu} - WF_{\mu\nu} \\ Z &\to -\frac{Z}{Z^2+W^2} \\ W &\to \frac{W}{Z^2+W^2} . \end{aligned} \tag{6.111}$$

This led to the duality relation on the complex conductivity $\sigma \equiv \sigma_{xy} + i\sigma_{xx}$,

$$\sigma' = -\frac{1}{\sigma} . \tag{6.112}$$

In our case, this is less obvious, since we have effectively fixed Z to 1[2], but the duality is there.

In the case of the dyonic black hole of [2, 3], which gives a subset of the formulas in [20], the action of S-duality was hard to understand, as the only relevant limit is the

[2]We worked with $Z = 1$ so far to approximate to the results in [2] with just the W term added. Going forward we are not going to set $Z = 1$ anymore.

one that takes $\rho \to 0$, followed by $s \to 0$ in the transport coefficients, and previously there was nothing left.

With the introduction of the W term in this dyonic black hole calculation, and the associated entropy function result, it becomes relevant to take the limit $\rho \to 0$, and obtain

$$\sigma_{xx} = 0 \;, \quad \sigma_{xy} = -W \;, \quad \alpha_{xx} = 0 \;, \quad \alpha_{xy} = \frac{s}{B}. \tag{6.113}$$

S-duality then acts nontrivially, as

$$W \to \frac{1}{W} \Rightarrow \sigma_{xy} \to -\frac{1}{\sigma_{xy}}. \tag{6.114}$$

The limit $\rho \to 0$ is understood, from the point of view of the dyonic black hole, as the limit when the electric charge goes to zero, keeping the magnetic charge finite. Considering also the entropy function calculation leading to (6.108), we obtain that in this limit,

$$\alpha_{xy} = \frac{s}{B} = \frac{c}{\sqrt{3}}\sqrt{W^2+1} \;, \tag{6.115}$$

where

$$c = \frac{L^2}{4G_N} = \frac{\pi Z}{g_4^2} \tag{6.116}$$

is the central charge of the dual field theory, with the second form being due to our fixing Z/g_4^2.

But, moreover, if we would keep Z free, we would obtain $\sqrt{W^2+Z^2}$ in (6.115), which is an S-duality invariant, see (6.111).

Chapter 7

Thermoelectric conductivities from Kubo formulas, with radially varing terms

7.1 Model with a dilaton field

In this section we consider a more general model, with a field dependent topological term, so an Einstein-Maxwell-dilaton model with a nontrivial dilaton, as considered for instance in [20, 21],

$$
\begin{aligned}
I = \frac{2}{\kappa_4^2}\int d^4x\sqrt{-g}\left(-\frac{1}{4}R - \frac{1}{2}\left[(\partial\phi)^2 + \Phi(\phi)((\partial\chi_1)^2 + (\partial\chi_2)^2)\right] - V(\phi) + \right. \\
\left. + Z(\phi)\frac{L^2}{4}F_{\mu\nu}F^{\mu\nu} + W(\phi)\frac{L^2}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right) .
\end{aligned}
\tag{7.1}
$$

This model without the topological term has been studied by [11] and in [20] the authors added the topological term to it.

The reason for introducing $W(\phi)$ is to have a nontrivial contribution to the Einstein equation (and the Maxwell equation); otherwise, as we saw in the previous section, this is a topological term.

Here the axions χ_i are introduced to have a breaking of translational invariance of the theory, via a linear axion ansatz

$$
\begin{aligned}
\chi_1 &= k_1 x \\
\chi_2 &= k_2 y .
\end{aligned}
\tag{7.2}
$$

We will go back to the isotropic case by considering $k_1 = k_2 = k$.

In order for the ansatz above to be consistent with the axion equations of motion,

$$
\Phi(\phi)\partial_\mu\partial^\mu\chi_{(1,2)} + \Phi'(\phi)\partial_\mu\chi_{(1,2)}\partial^\mu\phi = 0 ,
\tag{7.3}
$$

we will assume that the dilaton is static and only depends on the radial direction, so $\phi = \phi(z)$.

The equation of motion for the dilaton is

$$\partial_\mu\partial^\mu\phi - V'(\phi) - \frac{1}{2}\Phi'(\phi)\left[(\partial\chi_1)^2 + (\partial\chi_2)^2\right] + \frac{L^2}{4}(Z'(\phi)F_{\mu\nu} + W'(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu}) = 0. \quad (7.4)$$

To guarantee that we have a solution, we must impose that

$$\begin{aligned} V(0) &= \frac{-6}{L^2}\,, \\ V'(0) &= 0. \end{aligned} \quad (7.5)$$

7.2 Radially varying topological term

However, the ansatz considered so far is still too complicated to solve, so instead of the arbitrary functions of the dilaton $W(\phi), \Phi(\phi), V(\phi)$ and $Z(\phi)$, on top of the radially varying dilaton $\phi(z)$, we will simplify further and directly consider independent functions of the radial coordinate z, so $W(z), \Phi(z), V(z)$ and $Z(z)$. The resulting reduced Einstein-Maxwell-dilaton model is

$$\begin{aligned} I = \frac{2}{\kappa_4^2}\int d^4x\sqrt{-g}\bigg(-\frac{1}{4}R - \frac{1}{2}\left[(\partial\phi)^2 + \Phi(z)((\partial\chi_1)^2 + (\partial\chi_2)^2)\right] - V(z) + \\ + Z(z)\frac{L^2}{4}F_{\mu\nu}F^{\mu\nu} + W(z)\frac{L^2}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\bigg)\,. \end{aligned} \quad (7.6)$$

Now the gauge field equation of motion becomes

$$\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\left(Z(z)F^{\mu\nu} + W(z)\tilde{F}^{\mu\nu}\right) = 0. \quad (7.7)$$

We take the same ansatz for the metric as in the previous section,

$$\frac{1}{L^2}ds^2 = -\frac{\alpha^2}{z^2}f(z)dt^2 + \frac{\alpha^2}{z^2}(dx^2 + dy^2) + \frac{1}{z^2}\frac{dz^2}{f(z)}\,, \quad (7.8)$$

where $f(z)$ is the same function,

$$f(z) = 1 + (h^2 + q^2)z^4 - (1 + h^2 + q^2)z^3\,, \quad (7.9)$$

and the ansatz for the field strength is also unchanged,

$$F = h\alpha^2 dx \wedge dy + q\alpha dz \wedge dt \Rightarrow F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -q\alpha \\ 0 & 0 & h\alpha^2 & 0 \\ 0 & -h\alpha^2 & 0 & 0 \\ q\alpha & 0 & 0 & 0 \end{pmatrix}. \quad (7.10)$$

Then the Maxwell equation of motion gives

$$hW'(z) - qZ'(z) = 0\,, \quad (7.11)$$

solved by

$$Z(z) = \frac{hW(z)}{q} + Z_0 \ , \tag{7.12}$$

where Z_0 is a constant.

The Einstein equations,

$$R_{\mu\nu} = K_{\mu\nu} + \frac{1}{2} g_{\mu\nu} V(z) + Z(z) \left(2L^2 F_{\mu\sigma} F^{\sigma}_{\nu} - \frac{L^2}{2} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} \right) \ , \tag{7.13}$$

give 3 independent equations, which can be taken to be the xx, zz and xy components, for instance, and can be used to fix $K_{zz}(z)$, $K_{xx}(z) = K_{yy}(z)$ and $V(z)$, where

$$K_{zz} = \frac{1}{2}(\partial_z \phi)^2 \ , \quad K_{xx} = \frac{1}{2}\Phi(z) k_1^2 \ , \quad K_{yy} = \frac{1}{2}\Phi(z) k_2^2. \tag{7.14}$$

The solution of the equations of motion is then

$$\begin{aligned} K_{zz}(z) &= 0 \ , \\ K_{xx}(z) &= -\frac{2\alpha^2 z^2 \left(h^2+q^2\right) \left(hW(z) + q(Z_0 - 1)\right)}{q} \ , \\ V(z) &= \frac{2hz^4 \left(h^2+q^2\right) W(z) + 2q \left(z^4 (Z_0 - 1) \left(h^2+q^2\right) - 3\right)}{L^2 q}. \end{aligned} \tag{7.15}$$

Note that, strictly speaking, the above solution means that ϕ is constant, so Φ, V, Z, W should have been constant as well. Except, of course, in a correct solution we should have varied $\Phi(\phi), V(\phi), Z(\phi), W(\phi)$ with the chain rule to obtain the correct dilaton equation of motion, which was not done here.

So the above must be thought of as a simple toy model for the correct case. We have fixed $Z(z), \Phi(z), V(z)$ in terms of the independent $W(z)$ from the equations of motion, considered as the only variable, set by hand.

7.2.1 Fluctuations

To calculate the quadratic action for fluctuations around the background solution, and find the holographic Green's functions and the transport coefficients through Kubo formulas, we proceed as in the previous section.

We add fluctuations to the off-diagonal metric and gauge field in the spatial boundary directions,

$$\begin{aligned} g_{tx} &= \frac{\alpha\epsilon G_x(z) e^{-it\omega}}{z^2} \ , \\ g_{ty} &= \frac{\alpha\epsilon G_y(z) e^{-it\omega}}{z^2} \ , \\ dA_x &= \epsilon A_x(z) e^{-it\omega} \ , \\ dA_y &= \epsilon A_y(z) e^{-it\omega} \ , \end{aligned} \tag{7.16}$$

and consider the same ansatz with infalling boundary conditions at the horizon,

$$A_x(z) = f(z)^{\nu} a_x(z) \ ,$$

$$G_x(z) = f(z)^{(1+\nu)} g_x(z). \tag{7.17}$$

At linear level, we now obtain the equations of motion

$$\begin{aligned} i\bar{\omega}G'_x(z) &= -\frac{4z^2(hW(z)+qZ_0)}{q}\left[-iq\bar{\omega}A_x(z)\right. \\ &\left. +h(z-1)A'_y(z)\left\{z\left(z\left[z\left(h^2+q^2\right)-1\right]-1\right)-1\right\}+hqG_y(z)\right] , \\ & q(z-1)\left\{z\left(z\left[z\left(h^2+q^2\right)-1\right]-1\right)-1\right\}\times \\ & \times\left[4z^3A'_x(z)(hW(z)+qZ_0)-2G'_x(z)+zG''_x(z)\right] \\ &= 4ih\bar{\omega}z^3A_y(z)(hW(z)+qZ_0)+4qz^3G_x(z)\left(h^2-hqW(z)-q^2(Z_0-1)\right). \end{aligned} \tag{7.18}$$

We expand again the solutions in orders of $\bar{\omega}$:

$$\begin{aligned} a_x(z) &= a_x^0(z)+\bar{\omega}a_x^1(z)+... \\ g_x(z) &= g_x^0(z)+\bar{\omega}g_x^1(z)+... \end{aligned} \tag{7.19}$$

We solve the equations of motion order by order in ϵ and in $\bar{\omega}$, as before, focusing on the linearized perturbations (first order in ϵ) At zeroth order in $\bar{\omega}$, we obtain

$$\begin{aligned} a_x^{0\prime}(z) &= -qg_x^0(z) , \\ g_x^{0\prime\prime}(z) &= -\frac{2g_x^{0\prime}(z)\left(3h^2z^4-2h^2z^3+3q^2z^4-2q^2z^3-2z^3-1\right)}{(z-1)z\left(h^2z^3+q^2z^3-z^2-z-1\right)} \\ &= -2\frac{\psi'(z)}{\psi(z)}g_x^{0\prime}(z) , \end{aligned} \tag{7.20}$$

which matches what we got in the previous section for constant W, as expected. So the same solution as before is also valid now,

$$a_x^0(z) = \alpha_x - qz\gamma_x g_x^0(z) = \gamma_x \tag{7.21}$$

where γ_x and α_x are constants.

At first order in $\bar{\omega}$, we obtain the equation for a_x

$$a_x^{1\prime}(z) = \frac{\theta(z)+\gamma(z)+\psi(z)}{4h(z-1)z^2\left(z\left(z\left(z\left(h^2+q^2\right)-1\right)-1\right)-1\right)\left(hW(z)+qZ_0\right)} , \tag{7.22}$$

where we have defined the functions

$$\begin{aligned} \theta(z) &= -4qh(z-1)z^2g_x^1(z)\left(z\left(z\left(z\left(h^2+q^2\right)-1\right)-1\right)-1\right)\left(hW(z)+qZ_0\right) , \\ \gamma(z) &= q\frac{4iz^2(hW(z)+qZ_0)}{q\left(h^2+q^2-3\right)}\left(h^3z^2(4z-3)(\gamma_x qz-\alpha_x)+h^2q(\gamma_y qz-\alpha_y)+\right. \\ &\left. +hz^2\left(q^2(4z-3)-3\right)(\gamma_x qz-\alpha_x)+q\left(q^2-3\right)(\gamma_y qz-\alpha_y)\right) , \\ \psi(z) &= -qi\gamma_y z^2\left(4z\left(h^2+q^2\right)-3\left(h^2+q^2+1\right)\right). \end{aligned} \tag{7.23}$$

There is also a similarly complicated equation for g_x^1. But it turns out that we can write both these equations (for a_x and g_x^1) in the same form as we did in the previous section,

$$a_x^{1\prime}(z)+qg_x^1(z) = \mathcal{A}_x^0(z) ,$$

$$g_x^{1\prime\prime}(z) + 2\frac{\psi'(z)}{\psi(z)}g_x^{1\prime}(z) \quad = \quad \mathcal{G}_x^0(z) \; , \tag{7.24}$$

except with different $\mathcal{G}_x^0$ and $\mathcal{A}_x^0(z)$. Solving the equations in the same way (see [2]), we obtain

$$g_x^{1\prime}(z) = \frac{c_2}{\psi(z)^2} + \frac{1}{\psi(z)^2}\int_0^z \mathcal{G}_x^0(u)\psi(u)^2 du. \tag{7.25}$$

This again blows up at $z = 1$, so we must impose regularity by putting $c_2 = 0$, and

$$\int_0^1 \mathcal{G}_x^0(u)\psi(u)^2 du = 0 \; , \tag{7.26}$$

which relates γ_x to the other integration constants,

$$\begin{aligned}\gamma_x = &-\frac{(h^2+q^2-3)\left(\gamma_y q\left(2Z_0\left(h^2+q^2\right)+h^2+q^2+3\right)-4\alpha_y Z_0\left(h^2+q^2\right)\right)}{3h\left(h^2+q^2+1\right)}+\\ &+\frac{i\left(h^2+q^2-3\right)}{3\left(h^2+q^2+1\right)}\int_0^1 -\frac{4i\left(h^2+q^2\right)W(z)(\gamma_y q z-\alpha_y)}{q}\,dz\end{aligned} \tag{7.27}$$

and similarly for γ_y. Therefore we have the solution

$$\begin{aligned} a_x^1(z) &= \alpha_x - q\int_0^z g_x^1(u)du - i\int_0^z \mathcal{A}_x^0(u)du \; , \\ g_x^1(z) &= \gamma_x - i\int_0^z \frac{1}{\psi(u)^2}\gamma_x \mathcal{W}(u)du \; , \end{aligned} \tag{7.28}$$

where $\mathcal{W}(u)$ is a function depending on the parameters of the solution, which we write, together with $\mathcal{A}_x^0(u)$, in the Appendix B.

Next one would need to calculate the quadratic action as we did in (6.3.3) and (6.88), and extract the transport coefficients, but it is now too involved (one could do numerics for it, but we leave that for further work).

Note that, as before, besides the solutions with ansatz (7.17), with $\pm\nu$, we also have a constant solution,

$$\begin{aligned} A_x &= \delta_x \; , \quad G_x = -\frac{i\bar{\omega}}{h}\delta_x \; , \\ A_y &= \delta_y \; , \quad G_y = \frac{i\bar{\omega}}{h}\delta_y . \end{aligned} \tag{7.29}$$

The solutions become near the boundary at $z = 0$

$$\begin{aligned} A_x^0 &= \delta_x + \alpha_x(\gamma_x, \gamma_y) \; , \\ G_x^0 &= -\frac{i\bar{\omega}}{h}\delta_x + \gamma_x \; , \\ A_y^0 &= \delta_y + \alpha_y(\gamma_x, \gamma_y) \; , \\ G_y^0 &= \frac{i\bar{\omega}}{h}\delta_y + \gamma_y . \end{aligned} \tag{7.30}$$

7.3 Anisotropy

We can introduce anisotropy as in the previous chapter, via a metric ansatz with $k_x \neq k_y$,

$$\frac{1}{L^2}ds^2 = -\frac{\alpha^2}{z^2}f(z)dt^2 + \frac{\alpha^2}{z^2}(k_x dx^2 + k_y dy^2) + \frac{1}{z^2}\frac{dz^2}{f(z)} \ , \tag{7.31}$$

and now it is not trivial anymore. The Einstein equations give now different values for K_{xx} and K_{yy}, and we find

$$\begin{aligned} V(z) &= \frac{2\left(hz^4W(z)\left(h^2+k_xk_yq^2\right)+q\sqrt{k_xk_y}\left(h^2z^4Z_0-k_xk_y\left(z^4\left(h^2-q^2Z_0+q^2\right)+3\right)\right)\right)}{L^2q(k_xk_y)^{3/2}} \ , \\ K_{xx}(z) &= -\frac{2\alpha^2k_xz^2\left(W(z)\left(h^3+hk_xk_yq^2\right)+q\sqrt{k_xk_y}\left(h^2(Z_0-k_xk_y)+k_xk_yq^2(Z_0-1)\right)\right)}{q(k_xk_y)^{3/2}} \ , \\ K_{yy} &= -\frac{2\alpha^2k_yz^2\left(W(z)\left(h^3+hk_xk_yq^2\right)+q\sqrt{k_xk_y}\left(h^2(Z_0-k_xk_y)+k_xk_yq^2(Z_0-1)\right)\right)}{q(k_xk_y)^{3/2}} . \end{aligned} \tag{7.32}$$

Repeating the procedure from the previous subsection, we obtain the γ_x and γ_y given in the Appendix C.

7.4 Conductivity from entropy function

Considering the same $AdS_2 \times \mathbb{R}^2$ ansatz for the near-horizon metric of the planar extremal black hole in 4 dimensions,

$$ds^2 = -\frac{v}{z^2}(dt^2 - dz^2) + w(dx^2 + dy^2) \ , \tag{7.33}$$

we compute as before the boundary spatial integral of the Lagrangian density,

$$f = \int dxdy \int d^4x \sqrt{-g}\mathcal{L} \ , \tag{7.34}$$

and finally Sen's entropy function, which becomes

$$\mathcal{E} = \frac{\pi}{\alpha^2 vwz^2}\Gamma + \frac{\pi}{\alpha^2 e_A vw^2z^2}\Theta \ , \tag{7.35}$$

where

$$\begin{aligned} \Gamma =& 4\alpha^4h^2vwz^2\left(w^2-L^2vZ_0\right)-4e^2w^2z^4\left(v-L^2Z_0\right)+ \\ &+\alpha^2(L^2Z_0\left(-B^2v^2+e^2w^2z^2\left(z^2-4w\right)+4h^2v^2z^4\right)+ \\ &+vw\left(2ez^2\left(2ew^2+Z_0\right)-4h^2wz^4-13w\right)) \end{aligned} \tag{7.36}$$

and

$$\begin{aligned} \Theta =& \alpha L^2vW(z)(e^2w^2z^2\left(-2\alpha B+\alpha^2h\left(z^2-4w\right)+4hz^2\right)+ \\ &-\alpha^2hv^2\left(B^2-4h^2z^2\left(z^2-\alpha^2w\right)\right)). \end{aligned} \tag{7.37}$$

The attractor equations are, as before

$$\frac{\partial \mathcal{E}}{\partial v} = 0 \; , \quad \frac{\partial \mathcal{E}}{\partial w} = 0 \; , \quad \frac{\partial \mathcal{E}}{\partial e_A} = 0. \tag{7.38}$$

The full solutions for v, w and e are too big to be shown here, though it should be possible to find them numerically.

7.5 S-duality

One important reason to consider the more general action (7.1) is S-duality. As explained in the previous chapter, such an action is manifestly invariant under S-duality acting on $Z(\phi)$ and $W(\phi)$.

However, we considered the "toy model" with only z dependence for $W(z), Z(z)$, $V(z)$ and $\Phi(z)$, so we need to check S-duality on the solutions. Of course, we see that the Maxwell equation (7.11) is indeed S-duality invariant, but in order to have the solutions be as well, we need that $Z_0 = 0$ in (7.12).

Since $W(z)$ and $Z(z)$ are functions of the radial coordinate z, now we have to ask: at what position z is the action of S-duality relevant to transport coefficients to be considered? On the one hand, by virtue of calculating the holographic Green's functions and using the Kubo formulas at the boundary, that is where it seems we should consider them. But on the other hand, the conductivity calculated from Sen's entropy function in the attractor mechanism is obtained at the horizon, so that is where it seems to be needed in this case.

The two calculations are related by the application to AdS/CFT of the membrane paradigm, as done by Iqbal and Liu, and as shown for instance in [20], but in the case of our toy model, that is guaranteed by the fact that $Z(z)$, as well as $V(z)$ and $\Phi(z)$, are related to $W(z)$, and $Z(z)$ is related to it via a duality-invariant proportionality relation,

$$Z(z) = \frac{h}{q} W(z) \; , \tag{7.39}$$

which is one reason why the toy model set-up is a sensible one.

Chapter 8

Susceptibilities for the model with dilaton field

So far, we have seen that, at nonzero temperature, the dyonic black hole in AdS_4 has been used as a model for 2+1 dimensional transport in the presence of a magnetic field [2, 3] and to the presence of a topological term in the action in [23]. We have calculated the thermodynamic quantities, and transport from fluctuations around the dyonic background.

The generic transport is necessarily obtained from a background obtained by adding perturbations at infinity (and perhaps the horizon of the black hole), so that the full background solution is not known, following the method in [11–17]. One rather generic case was considered in [20]. In [21], the Wiedemann-Franz law was obtained by a combination of the two methods. In particular, the matrix of susceptibilities χ_s, calculated as the second order derivatives of the thermodynamic potential in the dyonic black hole background, and was related via the matrix of diffusivities D to the matrix of conductivities (as expected from the general theory of the hydrodynamic limit), for which the results in the perturbative background from [20]. In this thesis we reproduced the calculation of χ_s giving enough details.

But that implies the assumption that dyonic black hole background of [2, 3] and the perturbative one of [20] give the same thermodynamics, which is not obvious. Therefore we aim to investigate the possibility of these two results giving the same answer. This has implications beyond the specific case considered here, as it measures the correctness of importing results from a top-down construction to a bottom-up one, or vice versa.

In the next section we consider the perturbative model with topological term, but only B, B_1 external fields, and calculating the thermodynamics, the magnetizations and the susceptibilities with this simplified version of the fluctuations, and for the other sections we construct the susceptibilities for the general case.

8.1 AdS/CMT perturbative model and boundary conditions at the black hole horizon

The action in [20] is given by:

$$I = \int dx^4\sqrt{-g}\left[\frac{1}{16\pi G_N}\left(R - V(\phi) - \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}((\partial\chi_1)^2 + (\partial\chi_2)^2)\Phi(\phi)\right) - \frac{F_{\mu\nu}F^{\mu\nu}Z(\phi)}{4g_4^2} - F_{\mu\nu}\tilde{F}^{\mu\nu}W(\phi)\right], \tag{8.1}$$

Thus is the same action we have studied before, but now we are not going to make the assumption that $W(\phi) = W(z)$.

We want to study electrical and thermal transport in the presence of a magnetic field. We consider the *Euclidean* action in the bulk, in the absence of axion perturbations, as

$$S_E = \int d^4x\sqrt{g}\left(\frac{1}{16\pi G_N}\left(R + \frac{1}{2}(\partial\phi)^2 + V(\phi)\right) + \frac{Z(\phi)}{4g_4^2}F_{\mu\nu}F^{\mu\nu} - W(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu}\right). \tag{8.2}$$

In this case, the response of the euclidean action with the change in the magnetic field gives the magnetization density,

$$M = -\frac{1}{\text{Vol}}\frac{\partial S_E}{\partial B}. \tag{8.3}$$

We also need to consider the response of the action with respect to a fluctuations in the metric of the type $\delta g_{tx} = -B_1 y$, which gives the energy magnetization density,

$$M_E = -\lim_{B_1\to 0}\frac{1}{\text{Vol}}\frac{\partial S_E}{\partial B_1}. \tag{8.4}$$

These two affect the background solutions by adding a term to A_x and a non-diagonal term to the metric,

$$A = a(r)dt + (-B_1 + (a(r) - \mu)B_1 y)dx, \tag{8.5}$$

$$ds^2 = -U(r)(dt + B_1 y dx)^2 + \frac{dr^2}{U(r)} + e^{2V(r)}(dx^2 + dy^2). \tag{8.6}$$

Then on-shell, the gauge terms equal

$$F_{\mu\nu}F^{\mu\nu} = 2E^{-4V(r)}(B + B_1\mu - B_1 a(r))^2 - 2a'(r)^2 \tag{8.7}$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = 4e^{-2V(r)}a'(r)(-B_1 a(r) + B + B_1\mu), \tag{8.8}$$

and the Ricci scalar is given by

$$R = U(r)\left(\frac{1}{2}B_1^2 e^{-4V(r)} - 6V'(r)^2 - 4V''(r)\right) - 4U'(r)V'(r) - U''(r). \tag{8.9}$$

Taking the derivatives

$$\frac{-1}{\text{Vol}}\frac{\partial S_E}{\partial B} = \int_{r_h}^{\Lambda} dr\left(\frac{B + B_1\mu - B_1 a(r)}{g_4^2}e^{-2V(r)}Z(\phi) - 4W(\phi)a'(r)\right) \tag{8.10}$$

$$\frac{-1}{\text{Vol}}\frac{\partial S_E}{\partial B_1} = \int_{r_h}^{\Lambda} dr \left[\frac{B_1 e^{-2V(r)}U(r)}{16\pi G_N} + \left(\frac{B + B_1\mu - B_1 a(r)}{g_4^2} e^{-2V(r)} Z(\phi) - 4W(\phi)a'(r) \right) (\mu - a(r)) \right] \quad (8.11)$$

and then the limit of B_1 going to zero, we get the magnetization densities,

$$M = \int_{r_h}^{\Lambda} dr \left(\frac{Be^{-2V(r)}Z(\phi)}{g_4^2} - 4W(\phi)a'(r) \right) \quad (8.12)$$

$$M_E = \int_{r_h}^{\Lambda} dr \left(\frac{Be^{-2V(r)}Z(\phi)}{g_4^2} - 4W(\phi)a'(r) \right) (\mu - a(r)). \quad (8.13)$$

We can also define the heat magnetization density as $M_Q = M_E - \mu M$, giving

$$M_Q = -\int_{r_h}^{\Lambda} dr \left(\frac{Be^{-2V(r)}Z(\phi)}{g_4^2} - 4W(\phi)a'(r) \right) a(r). \quad (8.14)$$

We can also define

$$Q = -\frac{\partial S_E}{\partial \mu} \quad (8.15)$$

which give us

$$Q = -\text{Vol} \int_{r_h}^{\Lambda} dr \left(\frac{B_1 e^{-2V(r)} Z(\phi)}{g_4^2} (B + B_1(\mu - a(r))) - 4B_1 W(\phi)a'(r) \right). \quad (8.16)$$

Note that this goes to zero when $B_1 \to 0$.

8.1.1 Susceptibilities

We define the susceptibilities, as usual, as the second derivatives of the thermodynamical potential, which in holography equals the Euclidean action, which is *a priori* a function of B, B_1, μ and T, and in the most general case to be considered in the next section, also of E and ξ, $S_E(B, B_1, \mu, T; E, \xi)$. In this case, the magnetization susceptibility is the derivative of the magnetization with respect to B,

$$\chi_{BB} = -\frac{\partial}{\partial B}\left(-\frac{1}{\text{Vol}}\frac{\partial S_E}{\partial B} \right)\bigg|_{B_1,\mu,T}, \quad (8.17)$$

and more generally, we can also define other susceptibilities if we replace any B with a B_1, so we can write

$$\chi_{B_iB_j} = -\frac{\partial}{\partial B_i}\left(-\frac{1}{\text{Vol}}\frac{\partial S_E}{\partial B_j} \right)\bigg|_{B_k,\mu,T}. \quad (8.18)$$

Then, by taking the derivative of equations (8.10-8.11), we get

$$\begin{aligned}
\chi_{BB} &= \int_{r_h}^{\Lambda} \frac{e^{-2V(r)}Z(\phi)}{g_4^2} dr, \\
\chi_{BB_1} &= -\int_{r_h}^{\Lambda} \frac{e^{-2V(r)}Z(\phi)}{g_4^2}(\mu - a(r))dr, \\
\chi_{B_1B_1} &= \int_{r_h}^{\Lambda} \frac{e^{-2V(r)}U(r)}{16\pi G_n} + \frac{e^{-2V(r)}Z(\phi)}{g_4^2}(\mu - a(r))^2 dr.
\end{aligned} \tag{8.19}$$

For completeness, we can calculate also the derivatives of the heat magnetizations M_Q,

$$\left.\frac{\partial M_Q}{\partial B}\right|_{\mu,T} = -\int_{r_h}^{\Lambda} \frac{e^{-2V(r)}Z(\phi)}{g_4^2}(a(r))dr \tag{8.20}$$

and

$$\left.\frac{\partial M_Q}{\partial B_1}\right|_{\mu,T} = -\int_{r_h}^{\Lambda} \frac{e^{-2V(r)}U(r)}{16\pi G_n} + \frac{e^{-2V(r)}Z(\phi)}{g_4^2}(-\mu a(r) + a(r)^2)dr. \tag{8.21}$$

Here we see that no terms proportional to $W(\phi)$ appear in the susceptibilities $\chi_{B_iB_j}$.

We can also compute the double derivatives coming from Q:

$$\begin{aligned}
\chi_{EE} &= -\frac{\partial}{\partial \mu}\left(-\frac{1}{\mathrm{Vol}}\frac{\partial S_E}{\partial \mu}\right)\bigg|_{B_1,B,T}, \\
\chi_{EB} &= -\frac{\partial}{\partial \mu}\left(-\frac{1}{\mathrm{Vol}}\frac{\partial S_E}{\partial \mu}\bigg|_{B_1,\mu,T}\right|_{B_1,B,T}, \\
\chi_{EB_1} &= -\frac{\partial}{\partial \mu}\left(-\frac{1}{\mathrm{Vol}}\frac{\partial S_E}{\partial \mu}\bigg|_{\mu,B,T}\right|_{B_1,B,T}.
\end{aligned} \tag{8.22}$$

Taking the derivatives we get

$$\begin{aligned}
\chi_{EE} &= -\int_{r_h}^{\Lambda} \frac{B_1^2 e^{-2V(r)}Z(\phi)}{g_4^2} dr\ , \\
\chi_{EB} &= -\int_{r_h}^{\Lambda} \frac{B_1 e^{-2V(r)}Z(\phi)}{g_4^2} dr\ , \\
\chi_{EB_1} &= -\int_{r_h}^{\Lambda} \left(\frac{(B + 2B_1(\mu - a(r))e^{-2V(r)}Z(\phi)}{g_4^2} - 4W(\phi)a'(r)\right) dr\ ,
\end{aligned} \tag{8.23}$$

Note that as we had in the more simple model in 6.1.1 only one of the double derivatives depends on the topological terms a seen by the presence of the W term.

Also, note that we haven't calculated the derivatives related to the temperature. This because we are leaving the results in terms of functions to be discovered, and the temperature would appear as we calculate the limits of the integrals, since it comes from the horizon term

$$U(r_h) = 4\pi T\ , \tag{8.24}$$

following the 0-th law of black hole thermodynamics.

To complete these calculations we may look for an ansatz that solves the equations of motion.

8.2 Ansatz and Transport Coefficients

We are interested in calculating the susceptibilities for the model with general perturbations, using the ansatz from [20]. In this section we review the calculation of the transport coefficients in [20], since the results are going to be used later.

The source of fluctuations are the same as in the previous section at $B_1 = 0$, a magnetic field $A_x^{(0)} = -By$, but now we also consider a nonzero electric field $E_x = E$ and thermal gradient $\frac{1}{T}\nabla_x T = \xi$. We also add general fluctuations for all fields depending on the sources from the Einstein equations of motions, $\delta h_{\mu\nu}$ for the metric, δA_μ for the gauge field and $\delta\chi_i$ for the axion fields.

The resulting fields with all their fluctuations are: -the metric field,

$$g_{\mu\nu} = \begin{pmatrix} -U(r) & 0 & \epsilon(\delta h_{tx}e^{2V(r)} - \xi t U(r)) & \epsilon\delta h_{ty}e^{2V(r)} \\ 0 & \frac{1}{U(r)} & \epsilon\delta h_{rx}e^{2V(r)} & \epsilon\delta h_{ry}e^{2V(r)} \\ \epsilon(\delta h_{tx}e^{2V(r)} - \xi t U(r)) & \epsilon\delta h_{rx}e^{2V(r)} & e^{2V(r)} & 0 \\ \epsilon\delta h_{ty}e^{2V(r)} & \epsilon\delta h_{ry}e^{2V(r)} & 0 & e^{2V(r)} \end{pmatrix} \tag{8.25}$$

-the gauge field,

$$\begin{aligned} A_t =& a(r) \\ A_x =& -By + t\epsilon(\xi a(r) - E) + \epsilon\delta A_x \\ A_y =& \epsilon\delta A_y \end{aligned} \tag{8.26}$$

-and the axion fields

$$\chi_1(r) = k_1 x + \epsilon\delta\chi_1 \tag{8.27}$$

$$\chi_2(r) = k_2 y + \epsilon\delta\chi_2. \tag{8.28}$$

Here, ϵ is added as a mathematical tool in order to account for the order in the fluctuations, since we are considering B, E and ξ small.

8.2.1 Maxwell's equations of motion

The gauge equations of motion are given by

$$\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\left(F^{\mu\nu} + W\tilde{F}^{\mu\nu}\right) = 0. \tag{8.29}$$

In the equation for $\nu = x$ only the $\mu = r$ term survives

$$\partial_r\left[\sqrt{-g}\left(F^{rx} + W\tilde{F}^{rx}\right)\right] = 0. \tag{8.30}$$

and in the equation for $\nu = y$ we have an extra term that also survives

$$\partial_r \left[\sqrt{-g}\left(F^{ry} + W\tilde{F}^{ry}\right)\right] = -4\xi a'(r)W(\phi)\epsilon - \frac{B\xi Z(\phi)}{g_4^2}\sqrt{e^{-2V(r)}}\epsilon. \tag{8.31}$$

Note that the second term in the equation above is the same as the integrand times ξ of the magnetization found in (8.12).

As explained in [20], in order to do a calculation without having the full solution, we can take advantage of the fact that there are generalized currents that are r-independent, $\mathcal{J}_i$, following the general idea of the membrane paradigm in the form of Iqbal and Liu [18]. We can define these currents for the model modified by the magnetization term as

$$\begin{aligned}\mathcal{J}^x =& \sqrt{-g}\left(F^{rx} + W\tilde{F}^{rx}\right)\\ \mathcal{J}^y =& \sqrt{-g}\left(F^{ry} + W\tilde{F}^{ry}\right) - \epsilon\xi M(r).\end{aligned} \tag{8.32}$$

Note that we do in fact have $\partial_r \mathcal{J}^i = 0$. On-shell, we have

$$\begin{aligned}\mathcal{J}^x =& \frac{\epsilon e^{4V(r)}Z(\phi)\left(\delta h_{tx}a'(r) + U(r)e^{-2V(r)}\left(B\delta h_{ry} + \delta A_x'\right)\right)}{g_4^2}\\ \mathcal{J}^y =& \frac{\epsilon e^{4V(r)}Z(\phi)\left(\delta h_{ty}a'(r) + U(r)e^{-2V(r)}\left(\delta A_y' - B\delta h_{rx}\right)\right)}{g_4^2} + \\ & + 4\epsilon e^{4V(r)}W(\phi)(\xi a(r) - E) - \epsilon\xi M(r).\end{aligned} \tag{8.33}$$

The currents are going to be useful for us because they don't depend on the coordinate r, and thus we can relate the fields for any r to values at the horizon or the boundary:

$$\mathcal{J}^i(r) = \mathcal{J}^i(r_h) = \lim_{r\to\infty} \mathcal{J}^i(r). \tag{8.34}$$

At the horizon we have that the magnetization vanishes, $M(r_h) = 0$, directly from the result in (8.12). Also, we can impose regularity conditions near the horizon [11],

$$\delta A_x = -\frac{E\ln(r - r_h)}{4\pi T} + \mathcal{O}(r - r_h) \tag{8.35}$$

$$\delta A_y = \mathcal{O}(r - r_h) \tag{8.36}$$

$$\delta h_{rx} = \frac{\delta h_{tx}}{U(r)} + \frac{\xi e^{-2V(r)}\log(r - r_h)}{4\pi T} + \mathcal{O}(r - r_h) \tag{8.37}$$

$$\delta h_{ry} = \frac{\delta h_{ty}}{U(r)} + \mathcal{O}(r - r_h) \tag{8.38}$$

$$\delta \chi_i = \mathcal{O}(r - r_h). \tag{8.39}$$

We then obtain that at the horizon the usual currents J^i equal the generalized currents

$\mathcal{J}^i$, and equal

$$J^x = \mathcal{J}_x(r_h) = \lim_{r\to r_h} \frac{e^{2V(r)}Z(\phi)\left(\delta h_{tx}e^{2V(r)}a'(r) + B\delta h_{ty} - E\right)}{g_4^2} \tag{8.40}$$

$$J^y = \mathcal{J}_y(r_h) = \lim_{r\to r_h} \frac{e^{2V(r)}Z(\phi)\left(\delta h_{ty}e^{2V(r)}a'(r) - B\delta h_{tx}\right)}{g_4^2} + 4W(\phi)(\xi a(r) - E). \tag{8.41}$$

(8.42)

We still need to deal with δh_{ti}, appearing in the above formulas, and for that we must use the gravity equations of motion.

8.2.2 Einstein's equations of motion

The equations of motion for gravity are

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}V(\phi) + \frac{16\pi G_N}{4g_4^2}\left(2F_{\mu\sigma}F^\sigma_\nu - \frac{1}{2}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}\right). \tag{8.43}$$

We calculate them on-shell at linear level in ϵ. The derivation of the formulas can get involved, so, since we are interested in the result near horizon, we can expand the background field near this region,

$$\begin{aligned} a(r) &= a_h(r - r_h) + ... \\ V(r) &= V_h + ... \\ \phi &= \phi_h + ... \end{aligned} \tag{8.44}$$

Using this, for $\mu\nu = ty$, we end up with the Einstein equation

$$\begin{aligned} &\frac{1}{2}U(r)e^{2V(r)}\left(\delta h''_{ty} + 4\delta h'_{ty}V'(r)\right) - B^2\delta h_{ty}e^{-2V(r)}Z(\phi) + \\ &\quad + \frac{1}{4}\left(k_1^2 + k_2^2\right)\delta h_{ty}e^{2V(r)}\Phi(\phi) - 2Bh_{rx}U(r)a'(r)Z(\phi) \\ &= -2U(r)a'(r)\delta A'_y Z(\phi) + 2Be^{-2V(r)}Z(\phi)(\xi a(r) - E). \end{aligned} \tag{8.45}$$

We can rewrite the above expression so we get a more familiar result [11]

$$\begin{aligned} U(e^{4V}\delta h'_{ty})' - \left(\frac{\kappa}{g_4^2}B^2 Z + \frac{1}{2}(k_1^2 + k_2^2)e^{4V}\Phi\right)\delta h_{ty} - \frac{2\kappa}{g_4^2}UBZe^{2V}a'h_{rx} = \\ = -\frac{2\kappa}{g_4^2}UZe^{2V}a'\delta A'_y + \frac{2\kappa}{g_4^2}BZ(\xi a - E). \end{aligned} \tag{8.46}$$

A similar expression can be found for $\mu\nu = tx$,

$$\begin{aligned} U(e^{4V}\delta h'_{tx})' - \left(\frac{\kappa}{g_4^2}B^2 Z + \frac{1}{2}(k_1^2 + k_2^2)e^{4V}\Phi\right)\delta h_{tx} - \frac{2\kappa}{g_4^2}UBZe^{2V}a'h_{ry} = \\ = -\frac{2\kappa}{g_4^2}UZe^{2V}a'\delta A'_x. \end{aligned} \tag{8.47}$$

We need impose the regularity conditions (8.35 - 8.37), and another expansion near the horizon for the function $U(r)$,

$$U(r) = (r - r_h)U'(r_h) + ..., \tag{8.48}$$

where the coefficient in the expansion is given, as usual, by the temperature

$$U'(r_h) = 4\pi T. \tag{8.49}$$

Note that

$$\delta A'_x = -\frac{E}{4\pi T}\frac{1}{r - r_h} = -\frac{E}{U}. \tag{8.50}$$

Then we have

$$\begin{aligned}
\left(\frac{\kappa}{g_4^2}ZB^2 + \frac{1}{2}e^{2V}(k_1^2 + k_2^2)\Phi\right)\delta h_{tx} - \frac{2\kappa}{g_4^2}ZBe^{2V}a_h\delta h_{ty} &= -\frac{2\kappa}{g_4^2}Ze^{2V}a_hE + e^{2V}4\pi T\xi \\
\left(\frac{\kappa}{g_4^2}ZB^2 + \frac{1}{2}e^{2V}(k_1^2 + k_2^2)\Phi\right)\delta h_{ty} - \frac{2\kappa}{g_4^2}ZBe^{2V}a_h\delta h_{tx} &= -\frac{2\kappa}{g_4^2}ZBE\ ,
\end{aligned} \tag{8.51}$$

and we can solve for δh_{tx} and δh_{ty} in terms of ξ, E, B.

With this result, we can rewrite the currents (8.40-8.41) and then equate with the general formula for transport

$$J_i = \sigma_{xi}E - \alpha_{xi}T\xi\ , \tag{8.52}$$

and thus we can identify the thermoeletric transport coefficients, obtaining (when comparing with [20] note that here we have considered the more general case with $k_1 \neq k_2$)

$$\sigma_{xx} = \frac{1}{2}\frac{e^{2V}(k_1^2 + k_2^2)\Phi(2\kappa_4^2 g_4^4\rho^2 + 2\kappa_4^2 B^2 Z^2 + g_4^2 Ze^{2V}(k_1^2 + k_2^2)\Phi/2)}{4\kappa_4^4 g_4^4 B^2\rho^2 + (2\kappa_4^2 B^2 Z + g_4^2 e^{2V}(k_1^2 + k_2^2)\Phi/2)^2}\Bigg|_{r_h} \tag{8.53}$$

$$\sigma_{xy} = 4\kappa_4^2 B\rho\frac{\kappa_4^2 g_4^4\rho^2 + \kappa_4^2 B^2 Z^2 + g_4^2 Ze^{2V}(k_1^2 + k_2^2)\Phi/2}{4\kappa_4^4 g_4^4 B^2\rho^2 + (2\kappa_4^2 B^2 Z + g_4^2 e^{2V}(k_1^2 + k_2^2)\Phi/2)^2} - 4W\Bigg|_{r_h} \tag{8.54}$$

$$\alpha_{xx} = \frac{2\kappa_4^2 g_4^4 s\rho e^{2V}(k_1^2 + k_2^2)\Phi/2}{4\kappa_4^4 g_4^4 B^2\rho^2 + (2\kappa_4^2 B^2 Z + g_4^2 e^{2V}(k_1^2 + k_2^2)\Phi/2)^2}\Bigg|_{r_h} \tag{8.55}$$

$$\alpha_{xy} = 2\kappa_4^2 sB\frac{2\kappa_4^2 g_4^4\rho^2 + 2\kappa_4^2 B^2 Z^2 + g_4^2 Ze^{2V}(k_1^2 + k_2^2)\Phi/2}{4\kappa_4^4 g_4^4 B^2\rho^2 + (2\kappa_4^2 B^2 Z + g_4^2 e^{2V}(k_1^2 + k_2^2)\Phi/2)^2}\Bigg|_{r_h}. \tag{8.56}$$

Here

$$\rho = -Ze^{2V}a_h \tag{8.57}$$

is the charge density and

$$s = 4\pi e^{2V_h} \tag{8.58}$$

is the entropy density.

One important observation for the following is that there are is *explicit* dependence on T in the above formulas (the only explicit dependence on T in $\delta h_{tx}, \delta h_{ty}$ was through the factor $T\xi$, which was factored out in order to obtain the coefficients α_{xi}, σ_{xi}).

8.3 Susceptibilities of the general model with perturbations

The susceptibilities of the model are the double derivatives of the thermodynamic potential,

$$\chi_{ab} = \frac{1}{\mathrm{Vol}}\,\frac{\partial^2\Omega}{\partial a\partial b}\bigg|_{\text{other vars.}}\,, \tag{8.59}$$

where a and b stand for the thermodynamic variables.

The potential is given by the on-shell Euclidean action times the temperature

$$\Omega = TS_E\,, \tag{8.60}$$

so we need to compute the Euclidean action

$$\begin{aligned} S_E \;&=\; \int d^4\mathbf{x}\left(-\frac{F_{\mu\nu}F^{\mu\nu}Z(\phi)}{4g_4^2} - F_{\mu\nu}\tilde{F}^{\mu\nu}W(\phi)\right.\\ &\qquad \left. R - V(\phi) + \frac{-\frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}((\partial\chi_1)^2 + (\partial\chi_2)^2)\Phi(\phi)}{16\pi G_N}\right) \end{aligned} \tag{8.61}$$

on the ansatz (8.25 8.28), this time up to quadratic terms in ϵ.

The integral over time cancels with the temperature in (8.60), and the integrals over x and y turn into an overall volume Vol $= \int dx \int dy$, so in the end our result will only depend on an integral over r.

8.3.1 Susceptibilities with $(a, b) \in (\xi, E, B)$

The full quadratic Lagrangian is too big for us to show in this paper, but luckily a lot of terms goes to zero when we take the double derivatives. Furthermore, counterterms also do not contribute at this level.

Here we calculate and show these facts for the off-diagonal susceptibilities involving the magnetic B and electric E fields and the thermal gradient ξ:

$$\chi_{E\xi} = \int_{r_h}^{\Lambda} dr\left(-\frac{a(r)Z(\phi)}{g_4^2 U(r)}\right), \tag{8.62}$$

$$\chi_{BE} = \int_{r_h}^{\Lambda} dr\left(-\frac{\delta h_{ty}Z(\phi)}{g_4^2 U(r)}\right), \tag{8.63}$$

$$\chi_{\xi B} = \int_{r_h}^{\Lambda} dr\left(\frac{a(r)\delta h_{ty}Z(\phi)}{g_4^2 U(r)} + \mathcal{O}(t)\right). \tag{8.64}$$

We also obtain formulas for the diagonal susceptibilities involving the same:

$$\chi_{EE} = \int_{r_h}^{\Lambda} dr \left(\frac{Z(\phi)}{g_4^2 U(r)} \right), \tag{8.65}$$

$$\chi_{\xi\xi} = \int_{r_h}^{\Lambda} dr \left(\frac{a(r)^2 Z(\phi)}{g_4^2 U(r)} + \mathcal{O}(t^2) \right), \tag{8.66}$$

$$\chi_{BB} = \int_{r_h}^{\Lambda} dr \left(\frac{Z(\phi)}{g_4^2 U(r)} \left(\frac{\left(\delta h_{tx}^2 + \delta h_{ty}^2 - U(r)^2 \left(\delta h_{rx}^2 + \delta h_{ry}^2\right)\right)}{2} - U(r) e^{-2V(r)} \right) + \mathcal{O}(t) \right). \tag{8.67}$$

8.3.2 Susceptibilities with $(a, b) = (T, ...)$

We wish to also compute the susceptibilities involving the temperature T as (at least) one of the variables (a, b). One way to do this is to solve the integral of r and get a result that depends on the fields at the boundary and at the horizon, while the latter is related to the temperature. This proved to be a hard challenge in this general case, since we obtain functions that are not calculable with the methods we employ.

Instead, the path we explored is to make use of the already computed result for the electrical currents (8.32), and consider only the case that the T dependence comes only from explicit dependence, not from *implicit* T dependence in the conductivities σ_{xx}, α_{xx} (previously computed) and in the metric fluctuations $\delta h_{tx}, \delta h_{ry}$.

First, we use the fact that

$$\mathcal{J}^i(r) = \mathcal{J}^i(r_h) = J^i , \tag{8.68}$$

since $\mathcal{J}_i$ does not depend on r. Thus we can relate the fields at any r through the result for the thermoelectric response (8.52),

$$J^i = \sigma_{xi} E - \alpha_{xi} T \xi , \tag{8.69}$$

where we have computed σ_{xi} and α_{xi} in subsection 8.2, where we noted that they had no *explicit* T dependence.

Then we obtain

$$\sqrt{-g} \left(F^{ri} + W \tilde{F}^{ri} \right) - \epsilon \xi M(r) \delta_{iy} = \sigma_{xi} E - \alpha_{xi} T \xi. \tag{8.70}$$

Solving for ξ the above equation for $i = x$, we have

$$\xi = \frac{Z(\phi) \left(\delta h_{tx} e^{2V(r)} a'(r) + U(r) \left(B \delta h_{ry} + \delta A_x'\right)\right) + E g_4^2 \sigma_{xx}}{\alpha_{xx} g_4^2 T}. \tag{8.71}$$

Then we substitute ξ as a function of T from the above formula in the quadratic Lagrangian, and after taking derivatives (*and assuming $\delta h_{tx}, \delta h_{ry}$ and σ_{xx}, α_{xx} are T-independent, i.e., considering only the explicit dependence in their formulas*) we have, at lowest order in T,

$$\chi_{ET} = \int_{r_h}^{\Lambda} dr \frac{2\sigma_{xx} a(r)^2 Z(\phi)}{\alpha_{xx}^2 g_4^4 T^3 U(r)} \left(Z(\phi) \left(\delta h_{tx} e^{2V(r)} a'(r) \right.\right.$$

$$+U(r)\left(B\delta h_{ry}+\delta A'_x\right))+Eg_4^2\sigma_{xx}\Big)\,. \tag{8.72}$$

Rewriting this, we get the final form,

$$\chi_{ET}=\int_{r_h}^{\Lambda}dr\left(\frac{2\sigma_{xx}a(r)^2Z(\phi)\xi}{\alpha_{xx}g_4^2T^2U(r)}\right). \tag{8.73}$$

We can do the same procedure to find the other susceptibilities involving T, at the lowest order in T,

$$\begin{aligned}\chi_{BT} &= \int_{r_h}^{\Lambda}dr\;2\epsilon^2a(r)^2\delta h_{ry}Z(\phi)^2\left(\frac{Z(\phi)\delta h_{tx}e^{2V(r)}a'(r)}{\alpha_{xx}^2g_4^6T^3}\right.\\ &\quad\left.+\frac{Z(\phi)U(r)\left(B\delta h_{ry}+\delta A'_x(r)\right)+Eg_4^2\sigma_{xx}}{\alpha_{xx}^2g_4^6T^3}\right)\\ \chi_{TT} &= -\frac{T}{\mathrm{Vol}}\left.\frac{\partial\Omega}{\partial T^2}\right|_{B,\mu}=\int_{r_h}^{\Lambda}dr3\epsilon^2a(r)^2Z(\phi)\times\\ &\quad\times\left(\frac{\left(Z(\phi)\left(\delta h_{tx}e^{2V(r)}a'(r)+U(r)\left(B\delta h_{ry}+\delta A'_x\right)\right)+Eg_4^2\sigma_{xx}\right)^2}{\alpha_{xx}^2g_4^6T^4U(r)}\right)\end{aligned} \tag{8.74}$$

Note the sign difference, and the multiplication by T, which are standard for χ_{TT}.

Rewriting these, we get

$$\begin{aligned}\chi_{BT} &= \int_{r_h}^{\Lambda}dr\left(\frac{2\epsilon^2a(r)^2\delta h_{ry}Z(\phi)^2\xi}{\alpha_{xx}g_4^4T^2}\right)\\ \chi_{TT} &= \int_{r_h}^{\Lambda}dr\left(\frac{3\epsilon^2a(r)^2Z(\phi)\xi^2}{\alpha_{xx}g_4^4T^2U(r)}\right).\end{aligned} \tag{8.75}$$

Chapter 9

Conclusion

The duality between $SU(N)$ Yang-Mills theory at the conformal fixed point and string theory on AdS space 1-dimensional higher, called AdS/CFT correspondence, provides us with a new framework to study both quantum gravity and quantum field theories. We used this duality to study strong correlated operators in CFT by creating a holographic map to a gravity dual using the GPKW master rule. This leads us to a theory that is weakly coupled, so perturbative methods are applicable again.

We gave an utilitarian review of the correspondence, giving the historical context of its discovery. The first ideas of a duality between theories of different dimensions, called holographic principle, that would help us understand the quantum nature of gravity was inspired by black hole thermodynamics developed by Beckenstein-Hawking, where the entropy of a black hole depends on its area instead on its volume.

We reviewed the AdS/CMT method for computing low frequency transport coefficients for strongly coupled condensed matter models in the hydrodynamic limit by considering the gravity dual of a dyonic black hole in AdS-space. We utilize the previously mentioned theory by Hawking to understand how we can introduce temperature and other thermodynamics notions to the black hole in the quantum gravity theory.

The method of holographic transport can be applied as an algorithm for different models for the gravity background and the connection to our goal is made from the Kubo's Formula. This allows us to study effects related to transport, such as quantum Hall effect, Nersnt effect and viscosity, in regimes that weren't possible to model before.

We expanded the model studied in [2, 3] to include a topological term $WF_{\mu\nu}\tilde{F}^{\mu\nu}$ in the 3+1 dimensional gravitational action for the Einstein-Maxwell model, and we used it to holographically calculate transport coefficients in strongly coupled 2+1 dimensional materials, using a dyonic black hole background that asymptotes to AdS space.

Considering first a constant W, we have found that the results match the general results in [20] from the calculation in an a priori unknown metric, at the horizon of the black hole. Using also the attractor mechanism and Sen's entropy function, the transport coefficients were written in terms of $\rho = \tilde{Q}$, B and W, and we have found the action of S-duality on them.

In order to gain new terms, we include a dilaton field, a type of field very common in quantum gravity theories. We also include the presence of axions to possibly break spatial symmetries and thus consider even more generic models.

A toy model for this complicated case of a general solution with nontrivial $W(\phi), Z(\phi), V(\phi), \Phi(\phi)$ was the case with functions of the radial coordinate z, $W(z), Z(z), V(z)$ and $\Phi(z)$. We obtained the solutions for fluctuations, which were very complicated, so we did not proceed to find the transport coefficients, though those could be found numerically from our results. We have also shown how to introduce anisotropy in this case, and found the solutions for fluctuations in this case. We have set up the case of the attractor mechanism and Sen's entropy function, again leaving the complete (and very ugly) formulas for later. S-duality arguments sharpened the idea of the toy model, as well as its relevance.

Anisotropy might give us a path to find a model that can interpret an interesting class of condensed matter materials, like the one Strongly Coupled Anisotropic Plasma studied in [104], where a possible future work would be to utilise the methods seen here for this material.

This work led to the paper [23] co-written by this author. We are still interested if its possible to obtain the numerical values of the transport coefficients. This is viewed as a first step towards a case based on a more complete solution, involving not only a dyonic black hole, but nontrivial $\phi(z)$ and $W(\phi), Z(\phi), V(\phi), \Phi(\phi)$.

We also can expand the methods seen here to explore other models. For example, in [105], the authors used this formalism to compute the conductivities in in Gauss-Bonnet gravity with momentum dissipation, and found that they are independent of the Gauss-Bonnet coupling, so their results (even after the Sen's entropy calculations) are the same as ours.

We have also calculated thermodynamic susceptibilities, the second order derivatives of the thermodynamic potential, whose matrix is related to the conductivity matrix by the general theory of the hydrodynamic limit.

We first did it for the simple model in [2] with constant topological term, and we expanded the results in [21] to include the derivatives in the chemical potential μ. We showed that only one of this second derivatives depends on the topological factor.

We then compute the susceptibilities for a general holographic model with external fields B, B_1, where we also get that only of them depends on the topological term.

And then for the general model with E, B, μ, ξ introduced as perturbations at infinity. In the process, we have also found more general formulas for the thermoelectric conductivities in the case that not only translational invariance, but isotropy is also broken, through general linear dilatons $\chi_1 = k_1 x, \chi_2 = k_2 y, k_1 \neq k_2$.

We have then compared the formulas with formulas obtained in the standard analysis using the "top-down" AdS_4 dyonic black hole, and we have found that the results do not match. While there is a possibility that one of the assumptions in our calculation is unwarranted, we think that unlikely. More likely, calculations using different types of assumptions (the fields are nonperturbatively introduced in the dyonic black hole, while perturbatively introduced at infinity in the case considered here) are not expected to match in general, so one should be careful when exporting them from one model to the other.

This work led to the paper [25] co-written by this author.

The holographic principle can also be applied to more broader problems in physics, and can lead to a many other research options with models like the one studied in this work. Quark-gluon plasma is believed to be a strong coupled fluid, so we believe that the duality may give us quantitative results that we can observe in laboratories in the future. Also, we incorporate models for Holographic QCD, and it is hoped that effects that only happens at the strong coupling regime of chromodyamics, such as confinement, can be modeled from a gravity dual. We also hope for quantum gravity models for cosmology that can be studied from a condensed matter dual, and thus creating a bridge string theory and experiments to be tested in laboratories.

Appendix A

Lightining review of Bosonic Strings

Polyakov Action

We start our review with the Polyakov action that define the strings

$$S_P = \frac{-1}{4\pi\alpha'}\int_M d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\ . \tag{A.1}$$

In this, the fields are γ that represents a metric for the spacetime where $X(\tau,\sigma)$ is the embedding that describes a surface (the string).

The equations of motion for γ can be defined as constraints for X,

$$T_{ab} = -4\pi \frac{1}{\sqrt{-\gamma}}\frac{\delta S_P}{\delta\gamma^{ab}}/;\ , \tag{A.2}$$

while the eom for X is

$$\frac{\partial}{\partial\sigma^+}\frac{\partial}{\partial\sigma^-}X^\mu = 0 \tag{A.3}$$

In the following, we will work with $\gamma^{ab} = \eta^{ab}$ (called conformal gauging) and with the coordinates

$$\sigma^\pm = \tau \pm \sigma\ , \tag{A.4}$$

(called light-cone coordinates).

Mode expansion

In the light cone coordinates the equation of motion

$$\frac{\partial}{\partial\sigma^+}\frac{\partial}{\partial\sigma^-}X^\mu = 0 \tag{A.5}$$

ha a solution given by a left-moving and a right-moving parts

$$X^\mu(\sigma,\tau) = X_L(\sigma^+) + X_R(\sigma^-)\ . \tag{A.6}$$

For closed strings that obey the boundary conditions

$$X^\mu(\tau, l) = X^\mu(\tau, 0), \tag{A.7}$$

we have the expansions:

$$\begin{aligned} X_L^m u(\sigma^+) &= \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\sigma^+) + \frac{i\sqrt{2\alpha'}}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+}. \\ X_R^m u(\sigma^-) &= \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\sigma^-) + \frac{i\sqrt{2\alpha'}}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}. \end{aligned} \tag{A.8}$$

where we can identify

$$(\alpha_{-n}^\mu) = (\alpha_n^\mu)^\dagger. \tag{A.9}$$

While for open strings that satisfies Neumann boundary conditions with string length $= \pi$,

$$\frac{\partial}{\partial\sigma}X(\tau, \sigma = 0, \pi) = 0 \tag{A.10}$$

we have

$$X^m u(\sigma, \tau) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma \tag{A.11}$$

We identify the momentum with the zero modes: For open strings,

$$\alpha_0^\mu = \sqrt{2\alpha'}p^\mu \tag{A.12}$$

and for closed strings:

$$\tilde{\alpha}_0^\mu = \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu \tag{A.13}$$

Quantization

The constraints we need to satisfy is

$$T_{ab} = 0, \tag{A.14}$$

where

$$\begin{aligned} T_{00} =& T_{11} = \frac{1}{2\alpha'}(\partial_\tau X^2 + \partial_\sigma X^2) = 0, \\ T_{01} =& T_{10} = \frac{1}{\alpha'}(\partial_\tau X^2 \cdot \partial_\sigma X^2) = 0. \end{aligned} \tag{A.15}$$

We can redefine this on-shell as,

$$\begin{aligned} T_{++} &= = \frac{1}{2}(T_{00} + T_{01}) = \frac{1}{\alpha'}\partial_\tau X_L^2, \\ T_{--} &= = \frac{1}{2}(T_{10} + T_{11}) = \frac{1}{\alpha'}\partial_\tau X_R^2. \end{aligned} \tag{A.16}$$

Then wrapping around the constraints we have

$$\begin{aligned} L_m =& \frac{1}{2\pi}\int d\sigma e^{-im\sigma}T_{--} = \frac{1}{2}\sum_n \alpha^\mu_{m-n}\alpha^\mu_n, \\ \tilde{L}_m =& \frac{1}{2\pi}\int d\sigma e^{-im\sigma}T_{++} = \frac{1}{2}\sum_n \tilde{\alpha}^\mu_{m-n}\tilde{\alpha}^\mu_n. \end{aligned} \tag{A.17}$$

The Hamiltonian of this system, given by

$$H = \frac{1}{2\pi}\int_0^\pi d\sigma T_{00}, \tag{A.18}$$

can be calculated so we get

$$H = \frac{1}{2}\sum_n (\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n). \tag{A.19}$$

So the constraint for L_0 gives $H = 0$, which in turn gives us

$$\alpha_0^2 = -2\sum_{n\geq 1}\alpha^\mu_{-n}\alpha^\mu_n \tag{A.20}$$

The constraints for the light cone then give

$$\alpha^+_n = \alpha^\tau_n + \alpha^\sigma_n = 0 \tag{A.21}$$

and

$$\alpha^-_n = \alpha^\tau_n - \alpha^\sigma_n = \frac{\sqrt{2\alpha'}}{2p^+}\sum_m \alpha^i_{n-m}\alpha^i_m. \tag{A.22}$$

where $i = 1, .., D-2$ is a transverse dimension.

The quantization process (exchanging Poisson brackets for $-i$ times commutators) then gives similar results to the operators in the quantum harmonic oscillator a^μ_n but with a rescaling

$$\alpha^\mu_n = \sqrt{n}a^\mu_n. \tag{A.23}$$

Number of dimensions

At quantum level, the $n = 0$ mode is ambiguous, so we modify the above expression

$$\alpha^-_n = \alpha^\tau_n - \alpha^\sigma_n = \frac{\sqrt{2\alpha'}}{2p^+}\sum_m \alpha^i_{n-m}\alpha^i_m - a\delta_{n,0} \tag{A.24}$$

for some number a to be determined.

This modifies the mass operator as such

$$M^2 = \frac{1}{\alpha'}(N - a) \tag{A.25}$$

where we have defined the number operator

$$N = \sum_{n\geq 1} \alpha^i_{-n}\alpha^i_n \qquad \text{and} \qquad \tilde{N} = \sum_{n\geq 1} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n. \tag{A.26}$$

Now lets propose we have a vacuum state $|0\rangle$. The next excited state, $(\alpha^i_1)^\dagger |0\rangle$ would have mass

$$M^2(\alpha^i_1)^\dagger |0\rangle = \frac{1-a}{\alpha'}. \tag{A.27}$$

But this state would be a transverse vector in D-dimensions, which only makes sense if it is massless, since a massive vector would have $D-1$ physical components, and not $D-2$ as is this case. Therefore we have $a = 1$.

we can also compute the zero point energy of our system comparing with the harmonic oscillator (remeber the rescaling n)

$$H|0\rangle = -a|0\rangle = \sum_n \sum_{i=1}^{D-2} \frac{n}{2}|0\rangle = \frac{D-2}{2}\sum_n n|0\rangle. \tag{A.28}$$

This final sum looks a dead end, but we can interpret as the Riemann zeta function

$$\zeta(s) = \sum_n n^{-s} \tag{A.29}$$

calculated at $s = 1$. There is an unique analytical continuation to the zeta function, and we get

$$\zeta(-1) = -\frac{1}{12}. \tag{A.30}$$

So we have that our zero point energy is

$$a = \frac{D-2}{24}, \tag{A.31}$$

and since $a = 1$ we have that we need $D = 26$. Therefore the theory for bosonic string must be a 26-dimensional theory.

Closed String Spectrum

We have the following equivalent formulas for the mass operator $M^2 = -p^\mu p_\mu$ for closed strings:

$$M^2 = \frac{2}{\alpha'}\sum_{n\geq 1}(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n - 2) = \frac{2}{\alpha'}(N + \tilde{N} - 2), \tag{A.32}$$

So for the vacuum state $|0\rangle$ we have $M^2 = -4/\alpha'$. This tell us that our theory has a tachyon present.

The next state is given by $N = 1$ and $\tilde{N} = 1$, so

$$(\alpha^i_1)^\dagger(\tilde{\alpha}^i_1)^\dagger |0\rangle = |ij\rangle. \tag{A.33}$$

This state is decomposed of three irreducible parts of the $SO(D-2)$ group.

- A symmetric tensor $|(i,j)\rangle$, which can be identified as the graviton g_{ij}.
- An antisymmetric tensor $|[i,j]\rangle$, which can be identified as the Kalb-Rammond field B_{ij}.
- A scalar trace given by $|ii\rangle$, identified as the dilaton ϕ.

It is currently unknown if the instability of the theory caused by the tachyon presence can be fixed or not. Moreover, bosonics strings have an infinite number of states, all interacting with each other, so an instability of one particle means an instability for the entire theory. That is one reason why we use **superstring** theories for realistic applications, since in this the tachyon is not present, and bosonic string theory is regarded as a toy model.

Even so, the bosonic states defined earlier appear in the superstring spectrum, but now as the states of the vacuum. The mass for the states still increases with $1/\sqrt{\alpha'}$, so in the classical limit $\alpha' \to 0$ we are left with only the massless modes as the classical background, plus α' quantum corrections.

The effect on them in the action are: $g_{\mu\nu}$ gives the curvature replacing $\eta_{\mu\nu}$, $B_{\mu\nu}$ couples to an antisymetric tensor and gives a quantity called *Wess-Zumino* term, and the dilaton ϕ couples with a topological invariant term and is identified with extra holes in the worldsheet, so it is identified with the string coupling g_s as $g_s = e^{\phi}$.

Appendix B

Solutions for fluctuations in the radially varing case

$$
\begin{aligned}
\mathcal{W}(u) = &- \frac{i}{\gamma_x} \int_0^u -i \left(\frac{4\left(h^2+q^2\right) W(z)(\gamma_y q z - \alpha_y)}{q} \right) dz \\
&- u \frac{\gamma_x h^3 \left(u^2 \left(2q^2(u-1)(4u-3) - 7u + 6\right) + 4\right)}{\gamma_x h \left(h^2+q^2-3\right)} \\
&- u \frac{2h^2 \left(\gamma_y q^3 (3-2u) + \left(2q^2-3\right) Z_0 (\gamma_y q u - 2\alpha_y) + 3\gamma_y q (u-1)\right)}{\gamma_x h \left(h^2+q^2-3\right)} \\
&- u \frac{h^4 (\gamma_y q (2u(Z_0-1)+3) - 4\alpha_y Z_0) + \gamma_x h^5 (u-1) u^2 (4u-3)}{\gamma_x h \left(h^2+q^2-3\right)} \\
&- u \frac{\gamma_x h \left(q^2 \left(u^2 \left(q^2(u-1)(4u-3) - 7u + 6\right) + 4\right) + 3u^2\right)}{\gamma_x h \left(h^2+q^2-3\right)} \\
&- u \frac{q\left(q^2-3\right)\left(3\gamma_y + \gamma_y q^2 (2u(Z_0-1)+3) - 4\alpha_y q Z_0\right)}{\gamma_x h \left(h^2+q^2-3\right)} .
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
\mathcal{A}_x^0(u) = &\, i \frac{(h^3 z^2 (4z-3)(\gamma_x q z - \alpha_x) + h^2 q (\gamma_y q z - \alpha_y)}{h(z-1)\left(h^2+q^2-3\right)\left(z\left(z\left(z\left(h^2+q^2\right)-1\right)-1\right)-1\right)} \\
&+ i \frac{h z^2 \left(q^2(4z-3)-3\right)(\gamma_x q z - \alpha_x) + q\left(q^2-3\right)(\gamma_y q z - \alpha_y))}{h(z-1)\left(h^2+q^2-3\right)\left(z\left(z\left(z\left(h^2+q^2\right)-1\right)-1\right)-1\right)} \\
&+ \frac{i\gamma_y q \left(h^2(3-4z) + q^2(3-4z) + 3\right)}{4h(z-1)\left(z\left(z\left(z\left(h^2+q^2\right)-1\right)-1\right)-1\right)\left(hW(z) + qZ_0\right)}
\end{aligned}
\tag{B.2}
$$

and similarly for $\mathcal{A}_y^0(u)$.

Appendix C

Solution for fluctuations for the anisotropic model

$$
\begin{aligned}
\gamma_y = \frac{(h^2+q^2-3)}{3hk_x\,(h^2+q^2+1)} \Bigg(& -ihk_x \int_0^1 \frac{4ik_yW(z)\,(h^2+k_xk_yq^2)\,(\gamma_x qz-\alpha_x)}{q(k_xk_y)^{3/2}}\,dz \\
& -4\alpha_x Z_0\left(h^2+k_xk_yq^2\right) \\
& +\gamma_x q\left(k_xk_y\left(h^2+2q^2Z_0+q^2+3\right)+2h^2Z_0\right)\Bigg)
\end{aligned} \tag{C.1}
$$

and

$$
\begin{aligned}
\gamma_x = \frac{(h^2+q^2-3)}{3hk_y\,(h^2+q^2+1)} \Bigg(& ihk_y \int_0^1 \frac{4ik_xW(z)\,(h^2+k_xk_yq^2)\,(\gamma_y qz-\alpha_y)}{q(k_xk_y)^{3/2}}\,dz \\
& -4\alpha_y Z_0\left(h^2+k_xk_yq^2\right) \\
& +\gamma_y q\left(k_xk_y\left(h^2+2q^2Z_0+q^2+3\right)+2h^2Z_0\right)\Bigg)
\end{aligned} \tag{C.2}
$$

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