YEISON ANDRES ZABALA

Constrained Quadratic Control of Discrete-Time Hidden Markovian Jump Linear Systems:

The State Feedback and Static Output Scenarios

São Paulo
2021

YEISON ANDRES ZABALA

Constrained Quadratic Control of Discrete-Time Hidden Markovian Jump Linear Systems:
The State Feedback and Static Output Scenarios

Corrected Version

Ph. D. Thesis presented to the Escola Politécnica of the Universidade de São Paulo to obtain the degree of Doctor of Science.

Area of Concentration:
Systems Engineering

Advisor: Full Professor
Oswaldo Luiz do Valle Costa

São Paulo
2021

Este exemplar foi revisado e corrigido em relação à versão original, sob responsabilidade única do autor e com a anuência de seu orientador.

São Paulo, ______ de ____________________ de __________

Assinatura do autor: ________________________

Assinatura do orientador: ________________________

Catalogação-na-publicação

Zabala, Yeison Andres
Constrained Quadratic Control of Discrete-Time Hidden Markovian Jump Linear Systems: The State Feedback and Static Output Scenarios / Y. A. Zabala -- versão corr. -- São Paulo, 2021.
103 p.

Tese (Doutorado) - Escola Politécnica da Universidade de São Paulo. Departamento de Engenharia de Telecomunicações e Controle.

1.Stochastic Control 2.Linear Matrix Innequalities 3.Hidden Markov Models 4.Jump Linear Systems 5.Optimal Control I.Universidade de São Paulo. Escola Politécnica. Departamento de Engenharia de Telecomunicações e Controle II.t.

ACKNOWLEDGMENTS

To my advisor, Professor Oswaldo Luiz do Valle Costa, for the opportunity to be his student and inspire me in the research area involved in this work. I am also very grateful for your important collaboration, total availability and patience in the development of this thesis.

My wife, Lina M. Osório, for her constant support, her care and love, since she has been the daily energy for my soul and a source of encouragement in difficult times.

To Professor Carlos Humberto Mora Bejarano, for encouraging and guiding me in the postgraduate selection process, without his management, I probably would not have studied at USP.

My family, who is my engine for never giving up on my goals, even though I am always immersed in the great longing resulting from the distance that separates us, at the same time, greater happiness for me, product of the good memories that envision an emotional encounter.

My laboratory colleagues: Rafael, André and Leonardo; for friendship and appropriate academic and personal advices.

To the LAC teachers, for sharing their knowledge with competence and for their good disposition in case of any doubts.

My friends in São Paulo, Brazilians and Colombians, for making me feel at home in the moments that I enjoy with them.

Finally, I would like to thank the Coordination for the Improvement of Higher Education Personnel (CAPES) under Grant PROEX-0487/1595069, for the financial support for the development of this thesis.

“La escuela nos enseña la ubicación geográfica de los ríos, pero jamás nos explica la importancia del agua. Somos un baúl repleto de contenidos, pero vacío de contexto. De ahí nuestra dificultad para aplicar el conocimiento en la realidad”

Rodolfo Llinás.

ABSTRACT

ZABALA, Y.A. **Constrained Quadratic Control of Discrete-Time Hidden Markovian Jump Linear Systems:** The State Feedback and Static Output Scenarios. Thesis (Ph.D) - Escola Politécnica of the University de São Paulo, São Paulo, 2021.

In this thesis 2 global scenarios are considered for the constrained optimal control for hidden MJLS, where it is assumed that the controller only has access to a detector which emits signals $\widehat{\theta}(k)$ providing information on the Markov parameter $\theta(k)$. The State Feedback Quadratic Control (first scenario) and Static Output Feedback Control (second scenario) problems are studied. Hence, they are obtained, via LMIs approach, feedback linear optimal controls so that the respective closed loop systems are stochastically stabilized, an upper-bound for the quadratic cost is minimized, and the constraints on the norm of the state and control variables are satisfied. The Finite Horizon and the Infinite Horizon cases as well as the maximization of the estimate of the domain of an invariant set for a fixed upper-bound of the cost function are also addressed. Finally, some numerical examples are presented for the purpose of illustrating the obtained results.

Keywords: Optimal Control. Stochastic Control. Hidden Markov Models. Linear Matrix Inequalities. Jump Linear Systems.

RESUMO

ZABALA, Y.A. **Controle Quadrático Restrito de Sistemas Lineares com Saltos Markovianos Ocultos em Tempo Discreto:** Os cenários por Realimentação de Estado e Realimetação Estática de Saída. Tese (Doutorado) - Escola Politecnica da Universidade de São Paulo, São Paulo, 2021.

Nesta tese são considerados 2 cenários globais para o controle ótimo restrito para sistemas lineares com salto markoviano com observação parcial, onde se assume que o controlador só tem acesso a um detector que emite sinais $\widehat{\theta}(k)$, o qual fornece informações sobre o parâmetro de Markov $\theta(k)$. Os problemas de Controle Quadrático por Realimentação de Estado (primeiro cenário) e Controle por Realimentação Estática de Saída (segundo cenário) são estudados. Assim, obtém-se, via LMIs, controladores ótimos lineares realimentados tal que os respectivos sistemas de malha fechada sejam estabilizados estocásticamente, o limitante superior para o custo quadrático seja minimizado, e as restrições na norma das variáveis de estado e de controle sejam satisfeitas. Os casos de Horizonte Finito e Horizonte Infinito, bem como a maximização do domínio estimado de um conjunto invariante para um limitante superior fixo da função custo, são abordados. Por fim, são apresentados alguns exemplos numéricos com o objetivo de ilustrar os resultados obtidos.

Palavras Chave: Controle Ótimo. Controle Estocástico. Modelos Ocultos de Markov. Desigualdades Matriciais Lineares. Sistemas Lineares com Salto.

List of Figures

List of Tables

Acronyms

AFTCS	Active Fault-Tolerant Control Systems
CARE	Coupled Algebraic Ricatti Equations
CSF	Constrained State Feedback
CSOF	Constrained Static Output Feedback
FG	Feedback Gains
HMM	Hidden Markov Models
LMIs	Linear Matrix Inequalities
MJ	Markovian Jump
MJLS	Markov Jump Linear Systems
MPC	Model Predictive Control
MSD	Mean Square Detectable
MSS	Mean Square Stable
SOFC	Static Output Feedback Control
SS	Stochastically Stabilizable
TPM	Transition Probability Matrix
USF	Unconstrained State Feedback
USOF	Unconstrained Static Output Feedback

List of Symbols

Ω	Sample space.
$\mathcal{F}$, $\mathcal{F}_k$, $\widehat{\mathcal{F}}_k$	σ-Fields.
k	Discrete-time.
$\theta(k)$	Markov parameter in time.
$\hat{\theta}(k)$	Markov parameter of the detector.
$\mathbb{N}$	Set for the Markov chain $\theta(k)$.
$\mathbb{M}$	Set for the Markov chain $\hat{\theta}(k)$.
p_{ij}	Transition probability from state i to state j.
$\mathbf{P}, [p_{ij}]$	Transition Probability Matrix.
$\alpha_{i\ell}$	Probability of the associated detector, $P(\widehat{\theta}(k) = \ell \mid \theta(k) = i)$.
$[\alpha_{i\ell}]$	Transition probability matrix of the detector.
δ	Upper-bound for the cost function.
$J(K)$	Cost function.
$E(\cdot)$	Expected value.
$\mathbf{1}_A$	Indicator function of the event A.
$\mathbb{X}, \mathbb{Y}$	Banach spaces.
$\mathbb{M}(\mathbb{X}, \mathbb{Y})$	Banach space of all bounded linear operator from $\mathbb{X}$ into $\mathbb{Y}$.
$\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$	Normed linear space of all m by n real matrices.
$\mathcal{H}^{m,n}$	Linear space made up of all N-sequence of matrices.

ℓ_2^n	Hilbert space formed by the sequence of second order random variables.
$\mathbb{R}^n$	n-dimensional real space.
$\text{diag}\{\cdot\}$	Block diagonal matrix.
x_e	Equilibrium point.
$x(k)$	State vector.
$u(k)$	Control variable.
$z(k)$	Controlled output variable.
$y(k)$	Observable output variable.
A, B, C, D	System matrices.
$K(k)$	Feedback gains.
ρ_ι	Value for symmetric constraint.
$Q_i, R_{i\ell}, U_\ell, Y_\ell$	Auxiliary variables.
$L_P(\gamma)$	γ-invariant set.
μ_i	Initial probability for mode i.
θ_0	Initial mode.
x_0	Initial condition for the state.
$\mathcal{D}_\upsilon$	Inner ball with radius $\sqrt{1/\upsilon}$.
$C(k)$	Consumption.
$Y(k)$	National income.
$I(k)$	Governmental expenditure.

Contents

Chapter 1

Introduction

Most behaviors of the real world and phenomena of nature (laws) are classified and mathematically modeled in a deterministic (without uncertainty in the prediction of a result) or stochastic way (with uncertainty in the prediction of a result for the same event) following the same purposes: to understand its operation, to infer about its future and to try to control it. In the stochastic field, there is a wide variety of random processes which model such uncertainties, we have for example: Gaussian processes, Bernoulli processes, autoregressive models, Markov processes and others.

Some specific properties of these processes allow their study and analysis. In this work we are especially interested in the Markov property, which says that if for a value or state of a process in a given time, we will not obtain information on its future dynamics if we add more knowledge of its past, in other words, the conditional distribution probability of any future state of the process depends only on the present state and not on all past states. The Markov property implies that:

$$P(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = P(X_{n+1} = x_{n+1} \mid X_n = x_n),$$

and random process with the Markov property is called the Markov process. Below we will describe an example where the Markov property applies, adapted from [Costa et al., 2005]. Imagine that we have a system with a parameter that can take 2 possible values depending on certain conditions (for instance: climate change, failure of an electronic component in a circuit, change in economic policy in a country, etc.) and with a given probability, like this:

$$x(k+1) = a_{\theta(k)}x(k), \quad \theta(k) = \{1, 2\}, \tag{1.1}$$

where $a_1 = 0.8$, $a_2 = 1.2$ and $\theta(k)$ is a random variable. This change in the dynamics of the system is called in the literature as a jump, and if the Markov property applies and the subsystems are linear, it is the so-called Markovian Jump Linear Systems (MJLS). Suppose there is a 20% probability of the system being in mode 1 and remaining in it, so there is a 80% probability of jumping to mode 2; and the probability of the system to jump from mode 2 to mode 1 is 30% (see the diagram in Figure 1). The matrix containing these state change probabilities, the Transition Probability Matrix (TPM), is given as follows:

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{bmatrix}, \tag{1.2}$$

so we have that the probability of transition from state i to state j is given by the element (i, j) of the matrix $\mathbf{P}$.

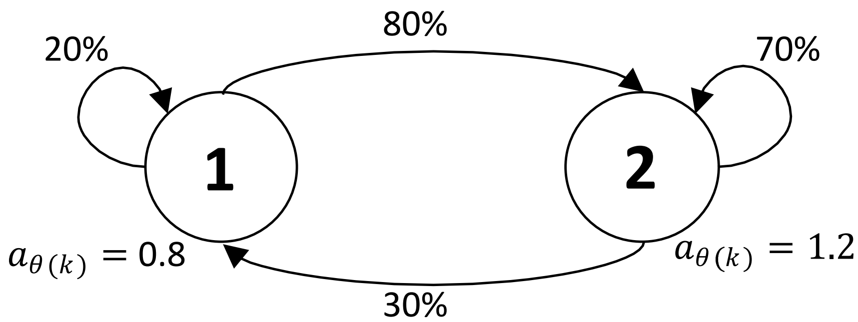


Figure 1: 2-state Markov chain.
Source: author.

If we set the time k varying from 0 to 20, so that $k \in \{0, 1, \dots, 20\}$, and the initial mode $\theta(0)$ at 1, we would have a universe of 2^{20} possible paths. Consider that we pick a path randomly for $\theta(k)$, for example:

$$\boldsymbol{\Theta} = \{1, 1, 2, 2, 2, 1, 2, 2, 1, 1, 1, 2, 1, 2, 1, 1, 1, 1, 2, 2, 1\}$$

and $x(0) = 1$. Thus, the system in time is shown in Figure 2.

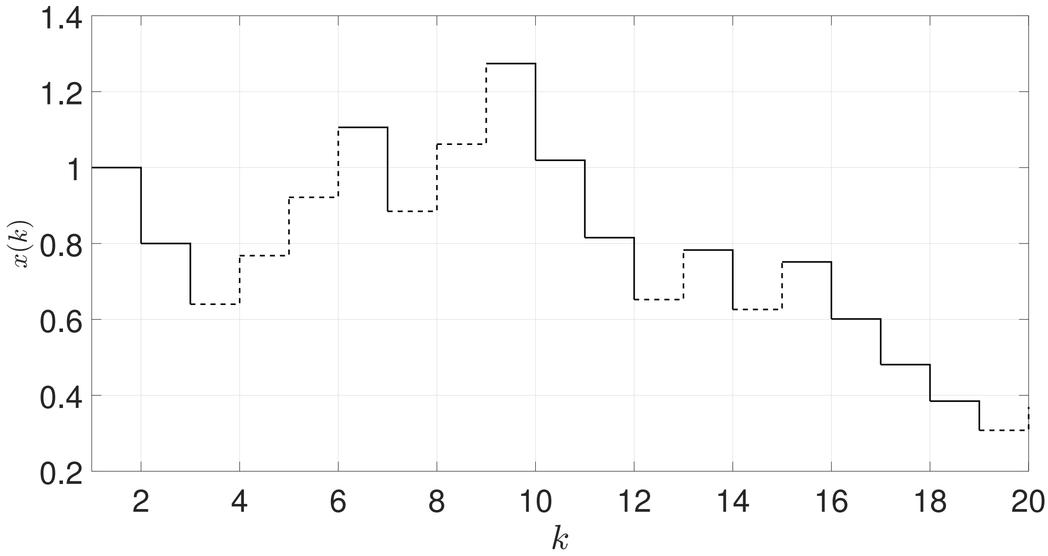


Figure 2: Random path to the system given by Equation (1.1). Mode 1 ($\theta(k) = 1$): solid line; and mode 2 ($\theta(k) = 2$): dashed line.
Source: author.

Figure 3 shows the extreme paths of $x(k)$ (solid lines); its expected steady state value $E(x(k))$ (dashed line) and a limiting range (solid-point line). In the shaded region we have all the possible paths of $x(k)$ that appear as $\theta(k)$ randomly varies in time. We can see in the figure that $E(x(k))$ is increasing and does not converge to any fixed value. Moreover, there exists a region of values that exceeds the hypothetical established limit. In many applications the processes with Markovian Jump (MJ) are naturally limited or are designed with restrictions, in addition to having to ensure that the expected value of the process converges to zero when there is a variable that we can manipulate (control variable) with the purpose of changing the behavior of the process. Such facts are considered in the development and solution of the problems of this thesis.

MJLS have been receiving lately a great deal of attention since that this class of systems can represent models that are subject to sudden changes in their dynamic behavior. As an example of these changes we can mention abrupt environmental disturbances, component failures or repairs, changes in subsystems interconnections, abrupt variations in the operation point for a non-linear plant, etc. This large number of applications lead to a great interest on this field and several results, regarding applications, stability conditions and optimal control problems, can be found in the current literature (see, for instance, [Blair and Sworder, 1975a] for economic systems, [Bar-Shalom and Li, 1993,Gray

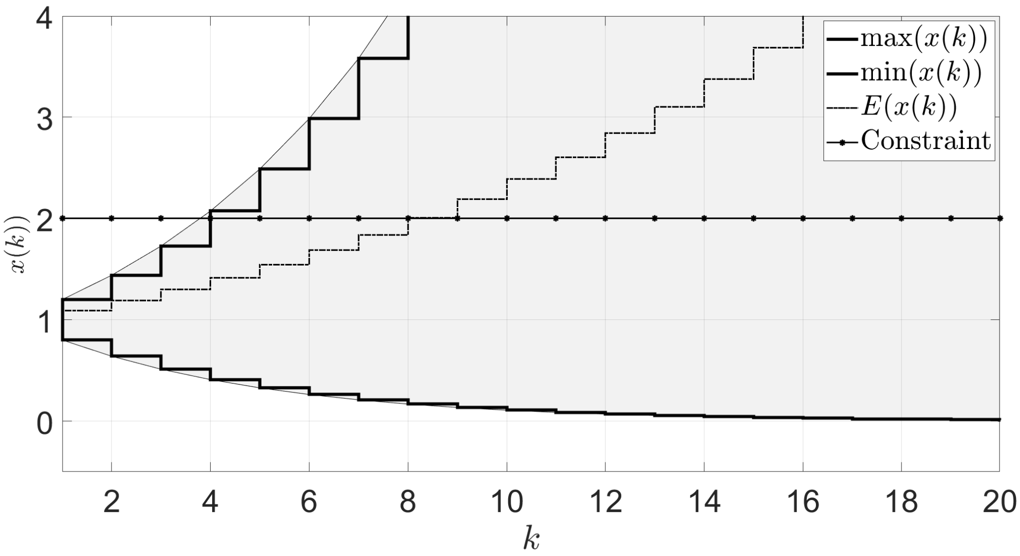


Figure 3: Region of possible paths to the system given by Equation (1.1).
Source: author.

et al., 2000] for aircraft control systems, [Sworder and Rogers, 1983] for control of solar thermal central receivers, [Siqueira and Terra, 2004] for robotic manipulator systems, and [Mariton, 1989, Srichander and Walker, 1993] for active fault-tolerant control systems (AFTCS). We can mention the books [Boukas, 2006, Costa et al., 2005, Costa et al., 2013, Dragan et al., 2010, Mariton, 1990, Mahmoud et al., 2003] and references therein for a sample of works on the subject and applications in AFTCS.

In the previous example (Equation (1.1)) we assume that the Markov parameter, $\theta(k)$, is known all the time (full information), but there are several problems where this does not happen, and we only get an uncertain estimate of its real value (partial information). We also see that in this example that there is no control variable of the process $x(k)$, a manipulated variable $u(k)$ related to $x(k)$, allowing any value of the process (no restrictions) and without guaranteeing convergence. Bearing this in mind and broadly speaking, the following work aims to design an optimal control for processes with MJ and partial information on $\theta(k)$ with state and control constraints, considering the State Feedback and Static Output Feedback scenarios. These problems are structured and solved through Linear Matrix Inequalities (LMIs) instead of the conventional algebraic way. This is the only practical way to solve this class of problems, which allows to include structures that add robustness (polytopes, bounded norm, etc.) at the cost of increasing possible conservatism in the solution (unless the problem is originally convex). In the following chapters we will explain these points in more details and specify the main focus of this thesis.

The thesis is organized in the following way. The literature review of related works and the contribution of this thesis are presented in Chapter 2. Chapter 3 introduces the notation, the framework that will be used throughout the work and deals with the appropriate notions of stability and stabilizability for MJLS, as well as some auxiliary results. In Chapter 4 we show that the problem of constrained state feedback quadratic control for MJLS with partial information on the jump parameter can be stated in terms of an LMI optimization problem, so that convex programming can be used for obtaining an approximation of the optimal solution, in addition to introducing other related problems under different assumptions. **The results obtained in this chapter have been published in the journal IEEE Transactions on Automatic Control** (see [Zabala and Costa, 2019]). The main results regarding the Finite and Infinite cases (Problem 5.4 and Problem 5.6) for constrained static output feedback are shown in Chapter 5, where we suppose that the initial condition for state x_0 and Markov parameter θ_0 are unknown. This chapter also deals with two alternative problems, in which in the first one we consider that the initial state and Markov parameter are known (x_0, θ_0), and in the second problem it is obtained the largest internal ball in a critical region for a fixed cost upper-bound. **These results have been published in the journal IEEE Access** (see [Zabala and Costa, 2020]). Chapter 6 presents some numerical simulations to illustrate the developed results with 2 academic applications. For the Constrained State Feedback scenario, it is considered a simple economic system based on the Samuelson's multiplier-accelerator model; and for the Constrained Static Output Feedback scenario, a linearized model of a small unmanned aerial vehicle in steady flight. We conclude this work with a summary of the contributions of this thesis and some perspectives for future research in Chapter 7. In the Appendices A, B, C we recall some basic facts to help the reading of the thesis.

Chapter 2

Literature review and contributions

In this chapter we contextualize our work in the related literature and point out how this thesis contributes to the theory of control systems.

2.1 Literature review

In this section we will mention some basic works separated into 4 specific fields which were the theoretical focus for the development of the problems presented in this thesis.

2.1.1 Applications

As aforementioned, in recent years systems subject to sudden changes in their dynamics have been the focus of many researches in engineering and related fields. Faced with this situation, MJLS appear as an useful mathematical tool capable of modeling and analyzing these systems, covering several areas of application such as: Systems subject to component failures and repairs [Richter et al., 2008], [Carvalho et al., 2020]; AFTCS [Zhang et al., 2019], [Kang et al., 2012], [Mahmoud et al., 2003], [Faraji-Niri et al., 2016]; economics [Svensson and Williams, 2008]; finance [Yin and Zhou, 2004], [Costa and Araujo, 2008]; energy planning [Song et al., 2000], [Lamond and Boukhtouta, 1996]; reservoir operation [Piantadosi et al., 2010], [Zhou et al., 2017]; etc.

In the last decades, interest in research on this specific topic MJLS has increased exponentially, resulting in hundreds of articles published annually in scientific journals with a high impact factor (see Figure 4), and, consequently, there is by now an extensive literature in which assumptions, extensions, generalizations and different structures are considered (see, for instance, [Shi and Li, 2015]). As a sample of works in this area we can mention [Rodrigues et al., 2017], [Stadtmann and Costa, 2016], [Oliveira et al., 2014] for the continuous-time domain, and [Tugnait., 1982], [Baczynski et al., 2001], [Patrinos et al., 2014] for the discrete-time case. Also, we can cite the books [Costa et al., 2005, Costa et al., 2013], which give a solid basis on stability, filtering, and optimal control of MJLS in the discrete-time and continuous-time cases.

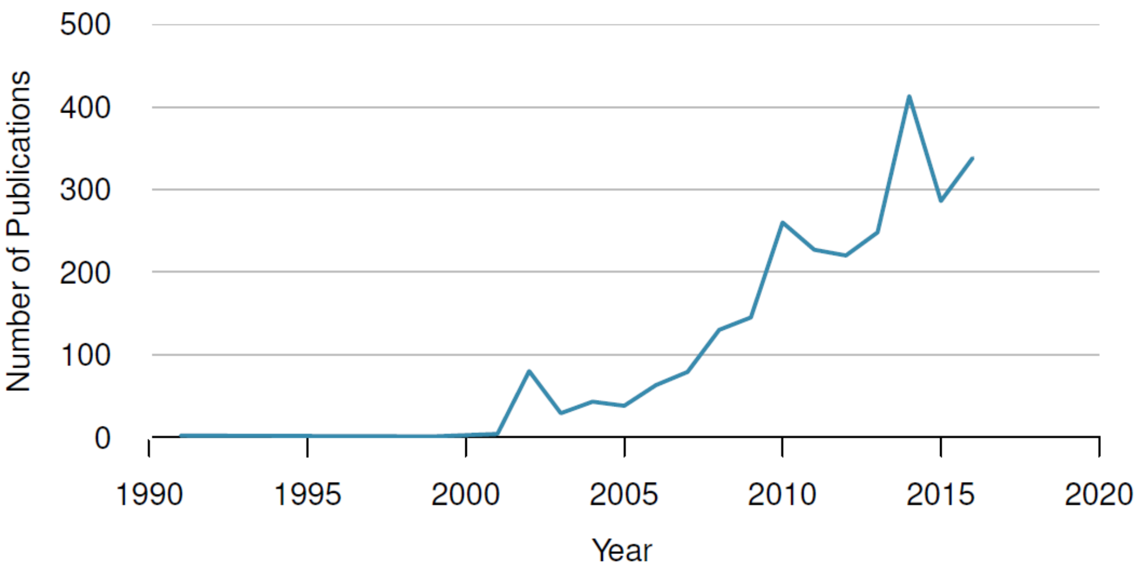


Figure 4: Development of the number of publications tagged with MJLS.
Source: [Stadtmann, 2019].

2.1.2 Constraints

In many practical situations in the control of a dynamic system there are physical constraints on the actuators and state variables, so that these constraints have to be taken into consideration in the design of the controller. Regarding the constrained MJLS with the Markov parameter available to the controller we can mention the papers [Vargas et al., 2007], [Vargas et al., 2013], where constraints on the first and second moments for the state and control variables are imposed; [Patrinos et al., 2014], which deals with a MJLS with subsystems that can be nonlinear; [Lu et al., 2013], which studies a Model Predictive Control (MPC) formulation and adopts uncertainties of the polytopic type in the system matrices as well as in the transition probabilities between modes; [Tonne and Stursberg, 2016], which introduces polytopic constraints on the inputs and states of a robust MPC problem for MJLS; and [Costa et al., 1999], which considers a quadratic state feedback optimization problem for MJLS subject to constraints on the state and control variables.

2.1.3 Partial information, state feedback and filtering

In all the above works it is considered that the Markov parameter is available for the controller. However, in multiple applications the controller does not have direct access to the Markov parameter but, instead, the information on the Markov chain is gleaned from an associated detector. This approach, which closely follows a Hidden Markov Model (HHM) approach, was adopted in [Costa et al., 2015] under the name of the detector approach and, basically, assumes that $(\theta(k), \hat{\theta}(k))$ is a HMM ([Ross, 2010]) in which the mode of operation of the system, represented by the hidden component $\theta(k)$ is not directly observed and only an estimation, represented by the observable component $\hat{\theta}(k)$ (given for instance by some failure detector), is available to the controller. As shown in [Costa et al., 2015], this formulation encompasses several situations regarding the availability of the Markov chain: the complete, cluster, and mode-independent cases. There is by now several works using this framework, such as the H_2 and H_∞ state-feedback control problems studied in [Costa et al., 2015] and [Todorov et al., 2018], the mixed H_2/H_∞ state-feedback control problem in [de Oliveira and Costa, 2017c], the H_2 and H_∞ filtering problems tackled in [de Oliveira and Costa, 2017a] and [de Oliveira and Costa, 2017b], respectively, and for the continuous-time MJLS, the H_2 and H_∞ state-feedback control considered in [Stadtmann and Costa, 2016] and [Rodrigues et al., 2017], and the H_∞ filtering problem, in [Rodrigues et al., 2016].

2.1.4 Partial information and static output feedback

Nowadays, we can find in the literature several works using this approach, also referred to as *asynchronous control* as presented in Song et al [Song et al., 2017, Song et al., 2018], in which the problems of static output feedback control and sliding mode control of MJLS with hidden observations were considered. Alternatively it was considered in Ogura et al [Ogura et al., 2018] an observation process in which the Markov chain is accessed only when some modes of operation of a different Markov process are visited. A new method for the design of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ static output feedback controllers for Hidden MJLS was introduced in [de Oliveira et al., 2020], providing suitable upper-bounds for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms under uncertainty in the transition matrix. Within the discrete-time Finite Horizon setup it was considered in [Song et al., 2017] the stochastic boundedness and $\ell_2 - \ell_\infty$ disturbance attenuation performance with guaranteed upper-bound costs for Hidden MJLS via static output feedback strategy.

2.2 Thesis Contributions

This thesis deals with constrained control for discrete-time MJLS with partial information using LMI optimization problems for minimizing an upper-bound for the quadratic cost function and/or maximizing the estimate of the domain of an invariant set with a fixed upper bound cost, where 2 scenarios are considered: Constrained State Feedback (see Chapter 4) and Constrained Static Output Feedback (see Chapter 5). The main contributions of this work are summarized as follows.

- Differently from [Vargas et al., 2007, Vargas et al., 2013, Patrinos et al., 2014, Lu et al., 2013, Tonne and Stursberg, 2016, Costa et al., 1999], we introduce the constrained quadratic control for discrete-time MJLS considering that the Markov parameter $\theta(k)$ is not available to the controller and, instead, we only have an estimation for this parameter provided by $\hat{\theta}(k)$ with an associated detection probability matrix, following the Hidden MJLS methodology.

- With respect to the works [Costa et al., 2015] and [de Oliveira et al., 2020], which study control problems for the Hidden MJLS without constraints, it is imposed in this work hard symmetrical constraints on the norm of the state and control variables when the Hidden MJLS framework is adopted.

- The Finite Horizon case is tackled in the development of the control law via static output feedback (Chapter 5) in the context of MJLS; which can be considered as a generalization of the work introduced in [Costa et al., 1999], which only addressed the State Feedback scenario within the Infinite Horizon setup.

- Numerical simulations of a simple economic system based on the Samuelson's multiplier accelerator model [Westerhoff, 2006a] (under hard constraints in the state and control variables) and an unmanned aircraft system subject to actuators faults (considering hard constraints on the control variable) are presented as illustrative examples of the derived algorithms.

To our knowledge, there is no other analytical or numerical way of handling this kind of problems with partial information on the jump parameter in the literature.

Chapter 3

Preliminaries

In this chapter we briefly introduce some definitions, concepts and results in the form of theorems for Linear Systems (LS), MJLS and hidden MJLS that will be the basis for the construction of the objective problems of the thesis and its respective solutions.

3.1 Notation

For $\mathbb{X}$ and $\mathbb{Y}$ complex Banach spaces, we set $\mathbb{M}(\mathbb{X}, \mathbb{Y})$ the Banach space of all bounded linear operator of $\mathbb{X}$ into $\mathbb{Y}$. For simplicity we set $\mathbb{M}(\mathbb{X}) := \mathbb{M}(\mathbb{X}, \mathbb{X})$. We denote by $\mathbb{R}^n$ the n-dimensional real space, and set $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ the normed linear space of all m by n real matrices. Whenever $m = n$ we write $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^n) = \mathbb{B}(\mathbb{R}^n)$ for simplicity. The superscript $'$ will indicate transpose. $L \geq 0$ and $L > 0$ will be used if a self-adjoint matrix is positive semi-definite or positive definite respectively and we write $\mathbb{B}(\mathbb{R}^n)^+ = \{L \in \mathbb{B}(\mathbb{R}^n); L = L' \geq 0\}$. We denote by $\|.\|$ either the induced norm in $\mathbb{B}(\mathbb{R}^n)$ or the standard norm in $\mathbb{R}^n$. We set $\text{diag}\{Q_s\}$ as the matrix in $\mathbb{B}(\mathbb{R}^{Sn})$ formed by $Q_1, \ldots, Q_S$ in the diagonal, and zero elsewhere.

We define $\mathcal{H}^{m,n}$ as the linear space made up of all N-sequence of matrices $V = (V_1, \ldots, V_N)$, $V_i \in \mathbb{B}(\mathbb{R}^m, \mathbb{R}^n)$, $i \in \mathbb{N}$. We set $\mathcal{H}^{n,n} = \mathcal{H}^n$ and $\mathcal{H}^{n+} = \{V = (V_1, \ldots, V_N) \in \mathcal{H}^n;\ V_i \in \mathbb{B}(\mathbb{R}^n)^+, i \in \mathbb{N}\}$. For $H = (H_1, \ldots, H_N)$ and $V = (V_1, \ldots, V_N)$ in $\mathcal{H}^{n+}$ the notation $H \leq L$ $(H < L)$ indicates that $H_i \leq L_i$ $(H_i < L_i)$ for each $i \in \mathbb{N}$.

We define ℓ_2^n as the Hilbert space formed by the sequence of second order random variables $z = (z(0), z(1), \ldots)$ with $z(k) \in \mathbb{R}^n$ for each $k = 0, 1, \ldots$ and such that

$$\|z\|_2^2 := \sum_{k=0}^{\infty} \|z(k)\|_2^2 < \infty$$

where $\|z(k)\|_2^2 := E(\|z(k)\|^2)$.

3.2 Linear Matrix Inequality

Several optimization problems with convex objective functions and convex inequalities common in process control applications can be expressed and solved through LMI due to their general form. We can find that linear inequalities convex quadratic inequalities, matrix norm inequalities, and various constraints from control theory such as Lyapunov and Riccati inequalities can all be written as LMI, converting this technique into a powerful tool in the solution of many optimization and control problems. The following is the general definition of LMI [Boyd et al., 1994a].

Definition 3.1 *Any constraint that can be written or converted to*

$$F(x) = F_0 + x_1F_1 + \ldots + x_mF_m < 0, \tag{3.1}$$

*where $x \in \mathbb{R}^m$ and F_i are hermitian matrices $F_i \in \mathbb{B}(\mathbb{R}^n)$ is called a **LMI**.*

Remark 3.2 *Notice that the LMI in Equation (3.1) is a convex constraint on x, thus the set $\{x|F(x) > 0\}$ is convex.*

3.3 Schur Complements

Nonlinear (convex) inequalities can be represented to LMI form using Schur complements. We have the following remarks which will be useful in the sequel.

Remark 3.3

$$W = \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} > 0, \tag{3.2}$$

if and only if

$$R > 0, Q > SR^{-1}S'.$$

For non-strict inequalities this result can be generalized as follows:

$$W = \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0 \tag{3.3}$$

if and only if

$$R \geq 0, Q \geq SR^{\dagger}S', S(I - RR^{\dagger}) = 0,$$

where $R^{\dagger}$ denotes the Moore-Penrose inverse of R (see [Boyd et al., 1994b]).

Remark 3.4 *If $Q > 0$ then*

$$U + U' - Q \leq U'Q^{-1}U. \tag{3.4}$$

3.4 Stability Results for Linear Systems

In this section, some definitions of LS are introduced and more relevant stability results are established, which will be analogous when considering a stochastic environment with MJ. These results are extended and are the basis for the different scenarios described in this thesis.

We consider the following discrete-time homogeneous linear system

$$x(k+1) = Ax(k), \tag{3.5}$$

and the general system given by:

$$x(k+1) = f\left(x(k)\right), \tag{3.6}$$

where $x(k) \in \mathbb{R}^n$, $A \in \mathbb{B}(\mathbb{R}^n)$ and $f : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 3.5 *x_e is called an equilibrium point of the system (3.6) if:*

$$f(x_e) = x_e. \tag{3.7}$$

In this way we have that $x_e = 0$ is an equilibrium point of (3.5).

The definitions below are concepts applied directly or indirectly to all stability results in the rest of the chapter.

Definition 3.6 ***Stability in the sense of Lyapunov.*** *Suppose that f has an equilibrium at x_e, so that $f(x_e) = 0$. An equilibrium point x_e is said to be Lyapunov stable if for each $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $||x(k) - x_e|| \leq \epsilon$ for all $k \geq 0$ whenever $||x(0) - x_e|| \leq \delta_\epsilon$.*

Definition 3.7 ***Asymptotic Stability.*** *The equilibrium point x_e is said to be asymptotically stable if it is Lyapunov stable and there exists $\delta > 0$ such that whenever $||x(0)-x_e|| \leq \delta$ then $x(k) \to x_e$ as $k \to \infty$. Moreover, x_e is called globally asymptotically stable if it is asymptotically stable and $x(k) \to x_e$ for any $x(0)$ in the space state as $k \to \infty$.*

Stability results for the System (3.5) are presented in the next theorem, which is obtained using the Lyapunov's concepts of stability (for more details see [Wiggins, 2003]).

Theorem 3.8 *The system given by Equation (3.5) is globally asymptotically stable, where $x = 0$ is the globally asymptotically stable equilibrium point, if and only if for some $P > 0$ we have that*

$$P - A'PA > 0. \tag{3.8}$$

The assertion given by Theorem 3.8 and (3.8) can be converted to LMI as follows:

$$\begin{bmatrix} -A'P - PA & 0 \\ \bullet & P \end{bmatrix} > 0. \tag{3.9}$$

The finite horizon linear quadratic regulator problem will now be considered, i.e.,

$$x(k+1) = Ax(k) + Bu(k) \tag{3.10}$$

$$z(k) = Cx(k), \tag{3.11}$$

where $B \in \mathbb{B}(\mathbb{R}^m, \mathbb{R}^n), C \in \mathbb{B}(\mathbb{R}^p, \mathbb{R}^n), u(k) \in \mathbb{R}^m$ and $z(k) \in \mathbb{R}^p$. It is desired to minimize the following cost function,

$$J_T^{LQR} = \sum_{k=0}^{T-1} \left(||Cx(k)||^2 + ||Du(k)||^2\right) + E\left(x(T)'\mathcal{V}x(T)\right), \tag{3.12}$$

where $\mathcal{V} \geq 0$ and $Q = D'D > 0$. The solution to this problem is given by equations,

$$u(k) = K(k)x(k), \tag{3.13}$$

$$K(k) = -(B'P_T(k+1)B + D'D)^{-1}B'P_T(k+1)A, \tag{3.14}$$

$$P_T(k) = C'C + A'P_T(k+1)A - A'P_T(k+1)B \times (B'P_T(k+1)B + D'D)^{-1}B'P_T(k+1)A, \tag{3.15}$$

$$P_T(T) = \mathcal{V}. \tag{3.16}$$

Equation (3.15) is the so-called difference Ricatti equation. If the cost is expressed as follows,

$$J_\infty^{LQR} = \sum_{k=0}^{\infty} \left(||Cx(k)||^2 + ||Du(k)||^2\right), \tag{3.17}$$

we have the so-called infinite horizon linear quadratic regulator problem and its solution is:

$$u(k) = K(P)x(k), \tag{3.18}$$

$$K(P) = -(B'PB + D'D)^{-1}B'PA, \tag{3.19}$$

$$P = C'C + A'PA - A'PB(B'PB + D'D)^{-1}B'PA. \tag{3.20}$$

We will introduce the following definitions which are relevant to ensure that the existence and unicity solution of P positive semi-definite of (3.20).

Definition 3.9 ***Stabilizability for LS.*** *The pair (A, B) is stabilizable if there exists $K \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ such that the model (3.10) is stable (Theorem 3.8) with $\bar{A} = A + BK$.*

Definition 3.10 ***Detectability for LS.*** *The pair (C, A) is detectable if there exists $H \in \mathbb{B}(\mathbb{R}^p, \mathbb{R}^n)$ such that the model (3.10) is stable (Theorem 3.8) with $\bar{A} = A + CH$.*

Theorem 3.11 *If the pair (A, B) is stabilizable, $P_T(0)$ converges to a positive semi-definite solution P of (3.20) as $T \to \infty$. Besides that, if the pair (C, A) is detectable, then there exist a unique positive semi-definite stabilizing solution P to (3.20).*

The proof of Theorem 3.11 can be found in [Callier and Desoer, 1999].
These concepts (stabilizability and detectability) for LS are very important to extend when Markov jumps are modeled in the system, which are reduced to Theorem 3.8 when conditions are restricted to the deterministic environment, as it will be seen below in Theorem 3.25.

3.5 Stability Results for MJLS

We consider in this work the following controlled discrete-time linear system with Markov jumps on a probabilistic space $(\Omega, \mathcal{P}, \mathcal{F})$:

$$x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k), \tag{3.21}$$

$$x(0) = \ x_0, \ \ \theta(0) = \theta_0. \tag{3.22}$$

Here the state variable is given by $x(k) \in \mathbb{R}^n$ and the control variable by $u(k) \in \mathbb{R}^m$. We consider that $\theta(k)$ is a Markov chain taking values in the set $\mathbb{N} = \{1, \ldots, N\}$ with TPM $\mathbf{P} = [p_{ij}]$.

For any $V = (V_1, \ldots, V_N) \in \mathcal{H}^n$ define $\mathbb{E}(V) = (\mathbb{E}_1(V), \ldots, \mathbb{E}_N(V))$ as:

$$\mathbb{E}_i(V) := \sum_{j=1}^{N} p_{ij}V_j. \tag{3.23}$$

It will be important to make the following definitions.

Definition 3.12 *System* (3.21) *is mean square stable (MSS) if*

$$E(||x(k)||^2) \to 0 \ as \ k \to \infty, \tag{3.24}$$

for any initial condition (x_0, θ_0).

We introduce below the definitions of mean square stabilizability and mean square detectability.

Definition 3.13 ***Mean square stabilizable.*** *The pair (A,B) is mean square stabilizable, with* $A = (A_1, \ldots, A_N) \in \mathcal{H}^n$ *and* $B = (B_1, \ldots, B_N) \in \mathcal{H}^{n,m}$*, if there exists* $K = (K_1, \ldots, K_N) \in \mathcal{H}^{m,n}$ *such that System (3.21) is MSS with* $\tilde{A}_i = A_i + B_i K_i$*. Thus, we can say that* K *stabilizes* (A, B) *in the mean square sense and set* $\mathcal{K} = \{K \in \mathcal{H}^{m,n} \mid K \text{ stabilizes } (A,B) \text{ in the mean-square sense}\}$.

Definition 3.14 ***Mean square detectable.*** *The pair (C,A) is mean square detectable (MSD), with* $A = (A_1, \ldots, A_N) \in \mathcal{H}^n$ *and* $C = (C_1, \ldots, C_N) \in \mathcal{H}^{p,n}$*, if there exists* $H = (H_1, \ldots, H_N) \in \mathcal{H}^{n,p}$ *such that System (3.21) is MSS with* $\tilde{A}_i = A_i + H_i C_i$*.*

The main result of this section is presented as follows.

Theorem 3.15 *System (3.21) is MSS if and only if there exist* $P = (P_1, \ldots, P_N) > 0$ *such that*

$$P_i - \tilde{A}' \mathbb{E}_i(P) \tilde{A} > 0, \quad \text{for } i = \{1, \ldots, N\}. \tag{3.25}$$

For the proof see [Costa and Fragoso, 1993].

3.6 Constrained Quadratic Control for MJLS

We will introduce the problem of contrained quadratic state feedback control for MJLS tackled in [Costa et al., 1999] and we will present its solution based on an approach using the LMI technique. System (3.21) will be considered together with the following output,

$$z(k) = C_{\theta(k)} x(k) + D_{\theta(k)} u(k), \tag{3.26}$$

where $\theta_0 \in \{1, \ldots, N\}$, x_0 is a second order random variable belonging to $\text{conv}\{x_{0,1}, \ldots, x_{0,s}\}$ with probability 1, $C = (C_1, \ldots, C_N) \in \mathcal{H}^{p,n}$, $D = (D_1, \ldots, D_N) \in \mathcal{H}^{p,m}$.

Notice that $\text{conv}\{W_1, \ldots, W_s\}$ for any set of vectors or matrices of the same dimension $\{W_1, \ldots, W_s\}$, is defined as:

$$\text{conv}\{W_1, \ldots, W_s\} = \{W = \sum_{h=1}^{s} \beta_h W_h, \sum_{h=1}^{s} \beta_h = 1, \beta_h \geq 0\}. \tag{3.27}$$

In this problem, the TPM $\mathbf{P}$ is not exactly known, but it is supposed that:

$$\mathbf{P} \in \mathrm{conv}\{\mathbf{P}_1, \ldots, \mathbf{P}_f\} \tag{3.28}$$

and

$$\mathbf{P}_\ell = [p_{ij,\ell}], \quad \ell = \{1, \ldots, f\}. \tag{3.29}$$

We define the cost function $J(K)$, for $u(k) = K_{\theta(k)}x(k)$ and $K = (K_1, \ldots, K_N) \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, as:

$$J(K) := ||z||_2^2 \tag{3.30}$$

$$= \sum_{k=0}^{\infty} E\left(x(k)'C'_{\theta(k)}C_{\theta(k)}x(k) + u(k)'D'_{\theta(k)}D_{\theta(k)}u(k)\right). \tag{3.31}$$

We want to find K such that it minimizes an upper-bound $\delta > 0$ for $J(K)$,

$$J(K) \leq \delta_{min}, \tag{3.32}$$

subject to constraints,

$$||F_\iota x(k) + G_\iota u(k)|| \leq \rho_\iota, \quad \iota = 1, \ldots, t. \tag{3.33}$$

Set $\Gamma_{i,\ell} = [p_{i1,\ell}^{1/2}\mathbf{I} \ldots p_{iN,\ell}^{1/2}\mathbf{I}] \in \mathbb{B}(\mathbb{R}^{Nn}, \mathbb{R}^n)$ for $i = 1, \ldots, N$ and $\ell = 1, \ldots, f$. Hence, we want to solve the following problem:

Problem 3.16 *Find $\delta > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $Y = (Y_1, \ldots, Y_N)$ such that*

$$\min \delta$$

subject to,

$$\begin{bmatrix} 1 & x'_{0,v} \\ \bullet & Q_i \end{bmatrix} \geq 0, \mathit{for}\ v = 1, \ldots, s, \quad i = 1, \ldots, N, \tag{3.34}$$

$$\begin{bmatrix} Q_i & (Q_iA_i' + Y_i'B_i')\Gamma_{i,\ell} & Q_iC_i' & Y_i'D_i' \\ \bullet & \text{diag}\{Q_\epsilon\} & 0 & 0 \\ \bullet & \bullet & \delta I & 0 \\ \bullet & \bullet & \bullet & \delta I \end{bmatrix} > 0, \tag{3.35}$$

for $\ell = 1, \ldots, f, \quad i = 1, \ldots, N,$

$$\begin{bmatrix} Q_i & Q_iA_i' + Y_i'B_i' \\ \bullet & Q_j \end{bmatrix} > 0, \tag{3.36}$$

for $\ell = 1, \ldots, f, \quad i = 1, \ldots, N,$ j *such that* $p_{ij,\ell} > 0$ *for some* ℓ, *and*

$$\begin{bmatrix} \rho_\iota^2 I - (F_\iota Q_i F_\iota' + F_\iota Y_i' G_\iota' + G_\iota Y_i F_\iota') & G_\iota Y_i \\ \bullet & Q_i \end{bmatrix} > 0, \tag{3.37}$$

for $\iota = 1, \ldots, t$, $i = 1, \ldots, N$,

where $\text{diag}\{Q_\epsilon\}$ is the diagonal matrix in $\mathbb{B}(\mathbb{R}^{Nn})$ formed by $Q_1, \ldots, Q_N$ in the diagonal. The constrained quadratic control state feedback control for MJLS is solved in the following way.

Theorem 3.17 *The constrained quadratic control state feedback control for MJLS when the probability transition matrix is not exactly known is solved by* (δ, Q, Y) *that satisfies the LMIs in Problem 3.16 where,*

$$K_j = Y_j Q_j^{-1}, \quad K \in \mathcal{K}. \tag{3.38}$$

Proof: The proof is presented in [Costa et al., 1999].

3.7 Stability Results for Hidden MJLS

For the results of this section, we will consider System (3.21)-(3.22). We assume that $\theta(k)$ *is not directly observed* but, instead, there is a finite set $\mathbb{M} = \{1, \ldots M\}$ such that a signal $\widehat{\theta}(k) \in \mathbb{M}$ is emitted associated to the Markov chain $\theta(k)$, independently of all previous and present values of the other processes. More precisely, let $\widehat{\mathcal{F}}_0$ be the σ-field generated by $\{x(0), u(0), \theta(0)\}$ and $\widehat{\mathcal{F}}_k$ be the σ-field generated by $\{x(0), u(0), \theta(0), \widehat{\theta}(0), \ldots, x(k), u(k-1), \theta(k)\}$ (therefore excluding $\widehat{\theta}(k)$ at time k). We assume that $\widehat{\theta}(k) \in \{1, \ldots, M\}$ is related to $\theta(k)$ in such a way that

$$P(\widehat{\theta}(k) = \ell \mid \widehat{\mathcal{F}}_k) = P(\widehat{\theta}(k) = \ell \mid \theta(k)) = \alpha_{\theta(k)\ell}, \quad \ell \in \mathbb{M}, \tag{3.39}$$

with $\sum_{\ell=1}^{M} \alpha_{i\ell} = 1$ for each $i \in \mathbb{N}$. Therefore, we have that at each time k we observe the signal $\widehat{\theta}(k)$. We define for each $i \in \mathbb{N}$,

$$\mathcal{I}_i \doteq \{\ell \in \mathbb{M}; \alpha_{i\ell} > 0\} = \{k_1^i, \ldots, k_{\tau^i}^i\}$$

and we assume that $\cup_{i=1}^{N} \mathcal{I}_i = \mathbb{M}$. It will be convenient to define $\tau = \tau^1 + \ldots + \tau^N$. As pointed out in [Costa et al., 2015], we have 2 extreme situations:

a) $M = N$ and $\alpha_{ii} = 1$, for $i \in \mathbb{N}$, which would correspond to the situation in which $\widehat{\theta}(k) = \theta(k)$, that is, $\theta(k)$ is known. In this case $\mathcal{I}_i = \{i\}$ and $\mathbb{M} = \mathbb{N}$.

b) $M = N$ and $\left[\alpha_{i\ell}\right] = \frac{1}{M}$, for all $i \in \mathbb{N}$ and $\ell \in \mathbb{M}$, which corresponds to the situation in which $\widehat{\theta}(k)$ does not provide any information about $\theta(k)$, that is, $\theta(k)$ is totally unknown.

Remark 3.18 *As pointed out in Chapter 2, there is a close relationship between the detector-based approach and hidden Markov processes, and thus algorithms for estimating the transition p_{ij} and detector $\alpha_{i\ell}$ parameters for HMM could be used in our framework (see, for instance, [Baum et al., 1970]). However the estimation of the parameters p_{ij} and $\alpha_{i\ell}$, although of great interest, is a major problem on its own and falls outside the scope of this work.*

We will consider state-feedback controls using the observed emitted signal $\widehat{\theta}(k)$ instead of the unknown variable $\theta(k)$, that is, $u(k)$ will be of the following form:

$$u(k) = K_{\widehat{\theta}(k)} x(k), \tag{3.40}$$

for $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $\ell \in \mathbb{M}$. Note that for Case b mentioned above, we would have a single feedback gain K for all modes, that is, the controller is blind with respect to the realizations of $\theta(k)$.

Associated to a control as in (3.40) set for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$A_{i\ell} \doteq A_i + B_i K_\ell. \tag{3.41}$$

We define for each $i \in \mathbb{N}$ the following operators $\mathcal{E}$, $\mathcal{L}$ in $\mathbb{M}(\mathcal{H}^n)$. For $V = (V_1, \ldots, V_N) \in \mathcal{H}^n$, and $i, j \in \mathbb{N}$,

$$\mathcal{E}_i(V) = \sum_{j=1}^{N} p_{ij} V_j, \tag{3.42}$$

$$\mathcal{L}_i(V) = \sum_{\ell \in \mathcal{I}_i} \alpha_{i\ell} A'_{i\ell} \mathcal{E}_i(V) A_{i\ell}. \tag{3.43}$$

We recall the following definition of stochastic stabilizability.

Definition 3.19 *We say that System (3.21) is stochastically stabilizable (SS) if there exists $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $\ell \in \mathbb{M}$, such that for $u(k)$ as in (3.40) we have, for every initial condition x_0 with finite second moment and every initial Markov state θ_0, that*

$$\|x\|_2^2 = \sum_{k=0}^{\infty} E(\|x(k)\|^2) < \infty. \tag{3.44}$$

We denote by $\mathcal{K}$ the set of feedback gains (FG) $K = \{K_\ell; \ell \in \mathbb{M}\}$, such that stochastically stabilizes System (3.21).

The following result presents conditions for stochastic stabilizability of System (3.21), the proof can be found in [Costa et al., 2015].

Theorem 3.20 *The following assertions are equivalent:*

i) System (3.21) is SS.

ii) There exists $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $\ell \in \mathbb{M}$ and $P \in \mathcal{H}^n$, $P > 0$, such that for $A_{i\ell}$ as in (3.41),

$$P - \mathcal{L}(P) > 0. \tag{3.45}$$

Chapter 4

Constrained State Feedback for Hidden MJLS

In this chapter, the State Feedback problem in the hidden MJLS context will be expressed and its solution established through a theorem (Theorem 4.2). Two alternative problems related to the first problem will also be treated and their respective solutions will be outlined in corollaries (Corollary 4.5 and Corollary 4.7). This chapter is based on the paper published by the authors in the IEEE Transactions on Automatic Control journal [Zabala and Costa, 2019].

4.1 Preliminaries

Consider again System (3.21) on a probabilistic space $(\Omega, \mathcal{P}, \mathcal{F})$, the σ-field $\widehat{\mathcal{F}}_k$, $k = 0, 1, \ldots$ as defined in Section 3.7, and the output

$$z(k) = C_{\theta(k)}x(k) + D_{\theta(k)}u(k). \tag{4.1}$$

We consider controllers as:

$$u(k) = K_{\widehat{\theta}(k)}(k)x(k). \tag{4.2}$$

For the set of FG $K = \{K_\ell; \ell \in \mathbb{M}\} \in \mathcal{K}$ define:

$$J(K) \doteq \|z\|_2^2 = \sum_{k=0}^{\infty} E(\|z(k)\|^2) \tag{4.3}$$

$$= \sum_{k=0}^{\infty} E(\|C_{\theta(k)}x(k) + D_{\theta(k)}u(k)\|^2), \tag{4.4}$$

with $z = (z(0), \ldots)$ given by (4.1) when $u(k) = K_{\widehat{\theta}(k)}x(k)$. Given $\delta > 0$ we want to find $K \in \mathcal{K}$ and a set $\mathcal{D}_0 \subset \mathbb{R}^n \times \mathbb{N}$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$ we have that the constraints

$$\|F_\iota x(k) + G_\iota u(k)\| \leq \rho_\iota, \text{ for } k = 0, 1, \ldots,\ \iota = 1, \ldots, t. \tag{4.5}$$

are satisfied and $J(K) \leq \delta\|x_0\|^2$.

Notice that we have two main elements that we can play with, the upper-bound value δ that we would like to minimize, and the set $\mathcal{D}_0$, which we would like to make as large as possible. For the case without the constraints (4.5) and with $\mathcal{D}_0 = \mathbb{R}^n \times \mathbb{N}$ the problem of minimizing δ over $K \in \mathcal{K}$ such that $J(K) \leq \delta\|x_0\|^2$ was analyzed in [Costa et al., 2015] via an LMI optimization problem. In the sequel we will present another approach for this problem taking also into account the constraints (4.5) (see also Remark 4.3). The motivation for the restrictions (4.5) is that many systems are subject to constraints on the manipulated and controlled variables. We notice that (4.5) comprises the so-called norm bounds and componentwise peak bounds on the inputs $u(k)$, as well as on a system output $y(k) = Hx(k)$. For instance, the norm bounds and componentwise peak bounds on the inputs $u(k)$ (see, for instance, [Kothare et al., 1996, Boyd et al., 1994b]) are given respectively as $\|u(k)\| \leq u_{\max}$ and $|u_\iota(k)| \leq u_{\iota,\max}$ (where $u_\iota(k)$ represents the ι^{th} element of the vector $u(k)$) for $k = 0, 1, \ldots,$ and $1 \leq \iota \leq m$, and fixed positive upper-bound values $u_{\max}$ and $u_{j,\max}$. By taking $F_\iota = 0$ in (4.5) and $G_\iota = I$, $\rho_\iota = u_{\max}$ we recover the norm bounds constraints while by taking $F_\iota = 0$, $G_\iota = e'_\iota$ (where e_ι is the unitary vector formed by 1 at the ι position, 0 elsewhere), and $\rho_\iota = u_{\iota,\max}$, we obtain the componentwise peak bounds constraints. Similarly, norm bounds and componentwise peak bounds on $y(k)$, given respectively by $\|y(k)\| \leq y_{\max}$ and $|y_\iota(k)| \leq y_{\iota,\max}$ can be, as for the input case, written as in (4.5). In this sense (4.5) is more general since it allows constraints for a linear combination of the input and output. As pointed out in [Kothare et al., 1996], constraints on the input are typically hard constraints, since they represent limitations on process equipment (such as valve saturations in industrial processes), and thus cannot be relaxed. On the other hand, constraints on the output are often associated to performance goals in which it is desired to keep the output $y(k)$ within some upper-bounds norm values $y_{\max}$ and/or peak bounds values $y_{\iota,\max}$. In Subsection 6.1 we present an economic example to illustrate these situations.

We present next an LMI optimization problem that aims at obtaining a $K \in \mathcal{K}$ which minimizes the upper-bound value δ at the same time that obtains an invariant set $\mathcal{D}_0$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$ we have that $(x(k), \theta(k)) \in \mathcal{D}_0$ for all $k = 0, 1, \ldots$ and the constraints (4.5) are satisfied. Other versions of this problem in which it considers the initial condition (x_0, θ_0) fixed or that fixes $\delta > 0$ and it aims at finding the largest inner ball inside an invariant set $\mathcal{D}_0$ will be presented in Corollaries 4.5 and 4.7. To define the LMI optimization problem, set for $i \in \mathbb{N}$, $\mathbf{\Gamma}_i = [p_{i1}^{1/2}\mathbf{I}_1 \ldots p_{iN}^{1/2}\mathbf{I}_N] \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^{\tau n})$, where $\mathbf{I}_i$ is an $n \times \tau^i n$ matrix formed by τ^i identity matrices of dimension n, and

$$\operatorname{diag}\{R_{s\zeta}\} \doteq \operatorname{diag}\{R_{1k_1^1}, \ldots, R_{1k_{\tau^1}^1}, \ldots, R_{Nk_1^N}, \ldots, R_{Nk_{\tau^N}^N}\},$$

a block-diagonal matrix of dimension $n\tau$ and, for fixed $i \in \mathbb{N}$,

$$\operatorname{diag}\{R_{i\zeta}\} \doteq \operatorname{diag}\{R_{ik_1^i}, \ldots, R_{ik_{\tau^i}^i}\},$$

a block-diagonal matrix of dimension $n\tau_i$. Notice that,

$$\operatorname{diag}\{R_{s\zeta}\} = \operatorname{diag}\{\operatorname{diag}\{R_{1\zeta}\}, \ldots, \operatorname{diag}\{R_{N\zeta}\}\}.$$

4.2 The Constrained State Feedback Problem

With the expressions defined above we can establish the Constrained State Feedback problem as follows:

Problem 4.1 *Find $\delta > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, such that*

$$\min \delta$$

subject to,

$$\begin{bmatrix} \delta I & \mathbf{I}_i \\ \bullet & \operatorname{diag}\{R_{i\zeta}\} \end{bmatrix} \geq 0, \textit{for } i \in \mathbb{N}, \tag{4.6}$$

$$\begin{bmatrix} U_\ell' + U_\ell - \alpha_{i\ell} R_{i\ell} & (U_\ell' A_i' \ + Y_\ell' B_i') \mathbf{\Gamma}_i & \begin{bmatrix} U_\ell' C_i' & Y_\ell' D_i' \end{bmatrix} \\ \bullet & \operatorname{diag}\{R_{s\zeta}\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \tag{4.7}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$\begin{bmatrix} U_\ell' + U_\ell - Q_i & (U_\ell' A_i' \ + Y_\ell' B_i') \\ \bullet & Q_j \end{bmatrix} > 0, \tag{4.8}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, j such that $p_{ij} > 0$, and

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota U_\ell + G_\iota Y_\ell \\ \bullet & U_\ell' + U_\ell - Q_i \end{bmatrix} > 0, \tag{4.9}$$

for $\iota = 1, \ldots, t$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$.

Next we define the invariant set that we will consider in this work. For $P = (P_1, \ldots, P_N) > 0$ define the function $P(x, i) = x' P_i x$, $i \in \mathbb{N}$, and, for $\gamma > 0$,

$$L_P(\gamma) := \Big\{ (x, i) \in \mathbb{R}^n \times \mathbb{N}; x' P_i x \leq \frac{1}{\gamma} \Big\}. \tag{4.10}$$

4.3 Solution

The next theorem shows that if there is a solution to the LMI optimization problem Problem 4.1 posed above, then we can get a stochastically stabilizing controller $K = \{K_\ell; \ell \in \mathbb{M}\} \in \mathcal{K}$ such that the quadratic cost $J(K)$ is upper-bounded by δ and there is an invariant set $L_P(1)$, such that whenever the initial conditions $(x_0, \theta_0) \in L_P(1)$ we have that $(x(k), \theta(k)) \in L_P(1)$ for all $k = 0, 1, \ldots,$ and the constraints (4.5) are satisfied.

Theorem 4.2 *Suppose there is a solution $\delta > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, for Problem 4.1. Define $K_\ell = Y_\ell U_\ell^{-1}$, $\ell \in \mathbb{M}$ and $P(x, i) = x' P_i x$, $P_i = Q_i^{-1}$, $i \in \mathbb{N}$. Then the following assertions hold:*

i) $K \in \mathcal{K}$,

ii) $J(K) \leq \delta \|x_0\|^2$.

If $(x_0, \theta_0) \in L_P(1)$ *then*

iii) $(x(k), \theta(k)) \in L_P(1)$ *for all* $k = 0, 1, \ldots,$

iv) the constraints (4.5) are satisfied.

Proof: First of all notice that from Remark 3.4 and (4.7) we get that

$$\begin{bmatrix} U_\ell'(\alpha_{i\ell} R_{i\ell})^{-1} U_\ell & (U_\ell' A_i' + Y_\ell' B_i')\mathbf{\Gamma}_i & \begin{bmatrix} U_\ell' C_i' & Y_\ell' D_i' \end{bmatrix} \\ \bullet & \mathrm{diag}\{R_{s\zeta}\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \tag{4.11}$$

so that by pre and post multiplying (4.11) by $\mathrm{diag}\{(U_\ell')^{-1}, I, I\}$ and its transpose, it yields to:

$$\begin{bmatrix} (\alpha_{i\ell} R_{i\ell})^{-1} & (A_i' + K_\ell' B_i')\mathbf{\Gamma}_i & \begin{bmatrix} C_i' & K_\ell' D_i' \end{bmatrix} \\ \bullet & \mathrm{diag}\{R_{s\zeta}\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \tag{4.12}$$

and from Remark 3.3 we get that (4.12) is equivalent to

$$R_{i\ell}^{-1} > \alpha_{i\ell}\Big\{(A_i + B_iK_\ell)'\left(\sum_{j=1}^{N} p_{ij}\Big(\sum_{\zeta\in\mathcal{I}_j} R_{j\zeta}^{-1}\Big)\right)(A_i + B_iK_\ell) + (C_i + D_iK_\ell)'(C_i + D_iK_\ell)\Big\} \tag{4.13}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$. Set $V_i = \sum_{\zeta\in\mathcal{I}_i} R_{i\zeta}^{-1}$, $i \in \mathbb{N}$, $V = (V_1, \ldots, V_N)$. From (4.13) we have that

$$V_i > \sum_{\zeta\in\mathcal{I}_i} \alpha_{i\zeta}\Big\{(A_i + B_iK_\zeta)'\mathcal{E}_i(V)(A_i + B_iK_\zeta) + (C_i + D_iK_\zeta)'(C_i + D_iK_\zeta)\Big\} \tag{4.14}$$

and thus (4.14) implies that $V - \mathcal{L}(V) > 0$, so that from Theorem 3.20 we get that $K \in \mathcal{K}$, showing i). Let us now show ii). Following the same steps as in the proof of Proposition 4 in [Costa et al., 2015] we get from (4.14) that

$$\begin{aligned}\left\|V_{\theta(k)}^{1/2}x(k)\right\|_2^2 &= E(x(k)'V_{\theta(k)}x(k)) \\ &> E(x(k+1)'V_{\theta(k+1)}x(k+1)) + \|z(k)\|_2^2 \\ &= \left\|V_{\theta(k+1)}^{1/2}x(k+1)\right\|_2^2 + \|z(k)\|_2^2 .\end{aligned} \tag{4.15}$$

Summing up (4.15) from $k = 0$ to ∞, and recalling that $K \in \mathcal{K}$ so that, from the stochastic stability of (3.21), (4.2), $\left\|V_{\theta(k)}^{1/2}x(k)\right\|_2^2 \to 0$ as $k \to \infty$, we obtain that

$$J(K) = \|z\|_2^2 = \sum_{k=0}^{\infty} \|z(k)\|_2^2 \le E(x_0'V_{\theta_0}x_0) \le \delta\|x_0\|_2^2 \tag{4.16}$$

where the last inequality follows from (4.6) since that from Remark 3.3 we derive that (4.6) is equivalent to

$$\delta I \ge \sum_{\zeta\in\mathcal{I}_i} R_{i\zeta}^{-1} = V_i. \tag{4.17}$$

Let us now show iii). From (4.8) and Remark 3.4 we have that

$$\begin{bmatrix} U_\ell' Q_i^{-1} U_\ell & (U_\ell' A_i' \ + Y_\ell' B_i') \\ \bullet & Q_j \end{bmatrix} > 0, \tag{4.18}$$

so that by pre and post multiplying (4.8) by $\mathrm{diag}\{(U_\ell')^{-1}, I\}$ and its transpose, we get that

$$\begin{bmatrix} Q_i^{-1} & (A_i' + K_\ell' B_i') \\ \bullet & Q_j \end{bmatrix} > 0. \tag{4.19}$$

From Remark 3.3 and setting $P_s = Q_s^{-1}$, we get that (4.19) yields to

$$P_i > (A_i + B_i K_\ell)' P_j (A_i + B_i K_\ell), \text{ for } p_{ij} > 0. \tag{4.20}$$

Hence, from (3.21) and (4.2), $x(k+1) = (A_{\theta(k)} + B_{\theta(k)} K_{\widehat{\theta}(k)}) x(k)$, and from (4.20) we get that

$$x(k)' P_{\theta(k)} x(k) > x(k+1)' P_{\theta(k+1)} x(k+1). \tag{4.21}$$

and thus (4.21) yields to

$$x_0' P_{\theta_0} x_0 > x(k)' P_{\theta(k)} x(k) > x(k+1)' P_{\theta(k+1)} x(k+1). \tag{4.22}$$

From (4.22) we have that if $(x_0, \theta_0) \in L_P(1)$ (that is, $x_0' P_{\theta_0} x_0 \leq 1$) then $(x(k), \theta(k)) \in L_P(1)$ for every $k = 0, 1, \ldots$ (since from (4.22) $x(k)' P_{\theta(k)} x(k) \leq 1$), completing the proof of iii). Let us now show iv). From (4.9) and Remark 3.4 again we get that

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota U_\ell + G_\iota Y_\ell \\ \bullet & U_\ell' Q_i^{-1} U_\ell \end{bmatrix} > 0. \tag{4.23}$$

Pre and post multiplying (4.23) by $\mathrm{diag}\{I, (U_\ell')^{-1}\}$ and its transpose, we have that

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota + G_\iota K_\ell \\ \bullet & Q_i^{-1} \end{bmatrix} > 0. \tag{4.24}$$

From (4.24) and Remark 3.3 we derive that

$$\rho_\iota^2 I > (F_\iota + G_\iota K_\ell) Q_i (F_\iota + G_\iota K_\ell)', \tag{4.25}$$

so that, from (4.25), we conclude that $\|(F_\iota + G_\iota K_\ell) Q_i^{1/2}\|^2 \leq \rho_\iota^2$ for all $\ell \in \mathcal{I}_i$, $i \in \mathbb{N}$. Thus we get that

$$\begin{aligned}
&\|F_\iota x(k) + G_\iota u(k)\|^2 = \|(F_\iota + G_\iota K_{\widehat{\theta}(k)}) x(k)\|^2 \\
&= \|(F_\iota + G_\iota K_{\widehat{\theta}(k)}) Q_{\theta(k)}^{1/2} Q_{\theta(k)}^{-1/2} x(k)\|^2 \\
&\leq \|(F_\iota + G_\iota K_{\widehat{\theta}(k)}) Q_{\theta(k)}^{1/2}\|^2 \|Q_{\theta(k)}^{-1/2} x(k)\|^2 \\
&\leq \rho_\iota^2\, x(k)' Q_{\theta(k)}^{-1} x(k) = \rho_\iota^2\, x(k)' P_{\theta(k)} x(k) \leq \rho_\iota^2
\end{aligned} \tag{4.26}$$

since $x(k)' P_{\theta(k)} x(k) \leq 1$, completing the proof. □

Remark 4.3 *From the proof of Theorem 4.2, items i) and ii), we notice that to get the stochastically stabilizing controller* $K = \{K_\ell; \ell \in \mathbb{M}\} \in \mathcal{K}$ *such that the quadratic cost* $J(K)$ *is upper-bounded by* δ *we only need the LMIs (4.6) and (4.7) to be feasible. Therefore for the case without the constraints (4.5) the results of Theorem 4.2 show that the solution of Problem 4.1 without the LMIs (4.8), (4.9) provides a guaranteed upper-bound for the quadratic control problem, based on the observability gramian associated to the operator* $\mathcal{L}$. *A similar result was obtained in Theorem 3 of [Costa et al., 2015], but considering the controllability gramian. In this sense the problem of minimizing* δ *under the LMIs (4.6) and (4.7) (related to the observability gramian) derived in this work can be seen as the dual of the LMI optimization problem considered in Theorem 3 of [Costa et al., 2015] (associated to the controllability gramian). Furthermore, for the full-observation case (and without constraints), that is,* $M = N$ *and* $\alpha_{ii} = 1$, *for* $i \in \mathbb{N}$ *(case a) in Section 3.7 which corresponds to the situation in which* $\widehat{\theta}(k) = \theta(k)$, *that is,* $\theta(k)$ *is known and* $\mathcal{I}_i = \{i\}$, $\mathbb{M} = \mathbb{N}$, *we could take in (4.6)* $U_i = R_{ii}$ *so that from Remark 3.3 the LMI (4.6) would be equivalent to* $P_i > (A_i + B_i K_i)' (\sum_{i \in \mathbb{N}} p_{ij} P_j)(A_i + B_i K_i) + (C_i + D_i K_i)'(C_i + D_i K_i)$, *with* $P_i = R_{ii}^{-1}$, $K_i = Y_i R_{ii}^{-1}$, *which corresponds to the observability gramian of the closed loop system. As seen in Theorem 4.10 in [Costa et al., 2005] in this case the result of Theorem 4.2 items i) and ii) retrieves the* H_2 *optimal control solution associated to the Coupled*

Algebraic Riccati Equations (CARE) associated to the problem (see Chapter 4 in [Costa et al., 2005]).

4.4 Alternative Problems

We conclude this chapter with 2 alternative problems associated to Problem 4.1, referred to as Problem 4.4 and Problem 4.6.

4.4.1 The pair of initial conditions (x_0, θ_0) is fixed

First notice that in Theorem 4.2 the initial condition x_0 and the initial probability for θ_0 are not fixed and it is only assumed that $(x_0, \theta_0) \in L_P(1)$. Suppose now that the initial condition x_0 and the initial probability for θ_0 are fixed, with $\mu_i = \mathcal{P}(\theta_0 = i)$ for $i \in \mathbb{N}$. In this case Problem 4.4 can be defined, taking into account these specific initial conditions, as follows:

Problem 4.4 *Find $\delta > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, such that*

$$\min \delta$$

subject to,

$$\begin{bmatrix} \delta & x_0' \begin{bmatrix} \mu_1^{1/2}\mathbf{I}_1 & \cdots & \mu_N^{1/2}\mathbf{I}_N \end{bmatrix} \\ \bullet & \operatorname{diag}\{R_{s\zeta}\} \end{bmatrix} \geq 0, \tag{4.27}$$

$$\begin{bmatrix} 1 & x_0' \\ \bullet & Q_i \end{bmatrix} \geq 0, \textit{for } i \in \mathbb{N} \textit{ with } \mu_i > 0, \tag{4.28}$$

and the LMIs (4.7), (4.8), (4.9).

We have the following corollary:

Corollary 4.5 *If there is a solution $\delta > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, for Problem 4.4 then, by defining $K_\ell = Y_\ell U_\ell^{-1}$ for $\ell \in \mathbb{M}$ and $P(x, i) = x' P_i x$, $P_i = Q_i^{-1}$, $i \in \mathbb{N}$ we have that i), ii), iii), iv) of Theorem 4.2 are satisfied.*

Proof: Using the same notation as in the proof of Theorem 4.2 we have from (4.27) and Remark 3.3 that

$$\delta \geq x_0'\Big(\sum_{i=1}^{N}\mu_i(\sum_{\zeta\in\mathcal{I}_\rangle} R_{i\zeta}^{-1})\Big)x_0$$
$$= x_0'\Big(\sum_{i=1}^{N}\mu_i V_i\Big)x_0 = E(x_0' V_{\theta_0} x_0) \tag{4.29}$$

so that, from (4.16), we obtain that $J(K) \leq \delta$. From (4.28) it follows that $1 \geq x_0' Q_i^{-1} x_0 = x_0' P_i x_0$ for any $i \in \mathbb{N}$ with $\mu_i > 0$, and thus $(x_0, \theta_0) \in L_P(1)$. From (4.22) and the fact that $(x_0, \theta_0) \in L_P(1)$, we get that $(x(k), \theta(k)) \in L_P(1)$ for every $k = 0, 1, \ldots$, and the remaining of the proof of Theorem 4.2 can be applied. □

4.4.2 The case with fixed upper-bound cost δ

In several applications, it will be required to establish or assume a value for the upper-bound, which could be the electric energy of a system or the risk in the portfolio optimization problem, to mention some examples. In this way, this fact motivates us to investigate this problem.

As seen in Theorem 4.2, by solving Problem 4.1 we minimize the upper-bound δ of the quadratic cost $J(K)$ and derive the invariant set $L_P(1)$ such that the constraints (4.5) will be satisfied whenever the initial conditions (x_0, θ_0) are in $L_P(1)$ (which implies that $(x(k), \theta(k)) \in L_P(1)$ for all $k = 0, 1, \ldots$). Another approach would be, for $\delta > 0$ fixed, consider an objective function that enlarges the invariant set $L_P(1)$ at the same time that ensures that the quadratic cost $J(K)$ is upper-bounded by δ. A possible way to do this would be, for instance, to consider a problem that aims at getting the largest inner ball (with radius $\sqrt{\frac{1}{\upsilon}}$) $\mathcal{D}_\upsilon \doteq \{x_0 \in \mathbb{R}^n; \|x_0\|^2 \leq \frac{1}{\upsilon}\}$ included in the set $\widehat{L}_P(1) := \{x \in \mathbb{R}^n; (x, i) \in L_P(1)$ for some ß $\in \mathbb{N}\}$, in other words, to obtain the minimum $\upsilon > 0$ such that $\mathcal{D}_\upsilon \subseteq \widehat{L}_P(1)$. This problem would be in general too hard to be solved, so that we need to consider a simplified convex version of this problem. Notice that, by restricting our choice for υ as $\upsilon = \max_{i\in\mathbb{N}} \|P_i\|$, where $P_i = Q_i^{-1}$, then we have that $\mathcal{D}_\upsilon \times \mathbb{N} \subset L_P(1)$

since that, if $x_0 \in \mathcal{D}_\upsilon$ then for any $i \in \mathbb{N}$ we have that

$$x_0' P_i x_0 \leq \|x_0\|^2 \|P_i\| \leq \|x_0\|^2 \upsilon \leq 1.$$

Thus, by minimizing $\max_{i\in\mathbb{N}} \|P_i\|$ we get the largest inner ball as defined in $\mathcal{D}_\upsilon$ with $\upsilon = \max_{i\in\mathbb{N}} \|P_i\|$, included in the set $\widehat{L}_P(1)$. Having this in mind, for $\delta > 0$ fixed, we re-write Problem 4.1 as follows:

Problem 4.6 *Find $\upsilon > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, such that*

$$\min \ \upsilon$$

subject to,

$$\begin{bmatrix} \upsilon I & I \\ \bullet & Q_j \end{bmatrix} \geq 0, \tag{4.30}$$

and the LMIs (4.6), (4.7), (4.8), (4.9). We have the following corollary:

Corollary 4.7 *If for $\delta > 0$ fixed there is a solution $\upsilon^* > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, for Problem 4.6 then, by defining $K_\ell = Y_\ell U_\ell^{-1}$ for $\ell \in \mathbb{M}$ and $P(x,i) = x' P_i x$, $P_i = Q_i^{-1}$, $i \in \mathbb{N}$ we have that i), ii), iii), iv) of Theorem 4.2, and $\mathcal{D}_{\upsilon^*} \times \mathbb{N} \subset L_P(1)$ with $\upsilon^* = \max_{i\in\mathbb{N}} \|Q_i^{-1}\|$, are satisfied.*

Proof: Using the same notation as in the proof of Theorem 4.2 we have from (4.30) and Remark 3.3 that $\upsilon I \geq Q_j^{-1}$, so that $\upsilon \geq \max_{j\in\mathbb{N}} \|Q_j^{-1}\|$. Since we want to minimize υ the optimal solution υ^* will be such that $\upsilon^* = \max_{j\in\mathbb{N}} \|Q_j^{-1}\|$. The remaining of the proof follows from the proof of Theorem 4.2. □

Chapter 5

Constrained Static Output Feedback for Hidden MJLS

The problem of Constrained Static Output Feedback for hidden MJLS in the infinite horizon and finite horizon framework are tackled in this chapter (Problems 5.4 and 5.6). The respective solutions will be presented and summarized in the form of theorems (Theorems 5.5 and 5.7). Finally, the chapter will be concluded addressing 2 alternative problems (Problems 5.8 and 5.9). The results of this chapter were published by the authors in IEEE Access journal [Zabala and Costa, 2020].

5.1 Preliminaries

On a probabilistic space $(\Omega, \mathcal{P}, \mathcal{F})$ consider the following controlled discrete-time linear system with Markov jumps :

$$x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k), \tag{5.1}$$

$$y(k) = H_{\theta(k)}x(k), \tag{5.2}$$

$$z(k) = C_{\theta(k)}x(k) + D_{\theta(k)}u(k), \tag{5.3}$$

$$x(0) = x_0, \;\; \theta(0) = \theta_0, \tag{5.4}$$

where $x(k) \in \mathbb{R}^n$ is the state variable, $y(k) \in \mathbb{R}^p$ the observable output variable, $u(k) \in \mathbb{R}^m$ the control variable and $z(k) \in \mathbb{R}^r$ the controlled output variable. The operation mode of the system is determined by a Markov chain $\theta(k)$ taking values in the set $\mathbb{N} = \{1, \dots, N\}$ and with TPM $\mathbf{P} = [p_{ij}]$. We will assume that the controller does not have access to neither $x(k)$ nor $\theta(k)$ but, instead, it can observe the output variable $y(k)$ and a signal $\widehat{\theta}(k) \in \mathbb{M}$. This signal takes values in a finite set $\mathbb{M} = \{1, \dots M\}$, and is related to the Markov chain $\theta(k)$ in the following way. Let $\widehat{\mathcal{F}}_0$ be the σ-field generated by $\{x(0), u(0), \theta(0)\}$ and $\widehat{\mathcal{F}}_k$ be the σ-field generated by $\{x(0), u(0), \theta(0), \widehat{\theta}(0), \dots, x(k), u(k-1), \theta(k)\}$ (therefore excluding $\widehat{\theta}(k)$ at time k). As before, we assume that $\widehat{\theta}(k) \in \{1, \dots, M\}$ is related to $\theta(k)$ in such a way that

$$\begin{aligned} P(\widehat{\theta}(k) = \ell \mid \widehat{\mathcal{F}}_k) &= P(\widehat{\theta}(k) = \ell \mid \theta(k)) \\ &= \alpha_{\theta(k)\ell}, \quad \ell \in \mathbb{M}, \end{aligned} \tag{5.5}$$

with $\sum_{\ell=1}^{M} \alpha_{i\ell} = 1$ for each $i \in \mathbb{N}$. As before, roughly speaking, we have that the values of $\widehat{\theta}(k) \in \mathbb{M}$ depend only on the present value of $\theta(k)$, being thus independent of all previous and present values of the other processes, and $\alpha_{i\ell}$ gives the probability of $\widehat{\theta}(k) = \ell$ whenever $\theta(k) = i$. We recall that for each $i \in \mathbb{N}$,

$$\mathcal{I}_i \doteq \{\ell \in \mathbb{M}; \alpha_{i\ell} > 0\} = \{k_1^i, \dots, k_{\tau^i}^i\}$$

and we assume that $\cup_{i=1}^{N} \mathcal{I}_i = \mathbb{M}$. We recall that $\tau = \tau^1 + \ldots + \tau^N$. As mentioned before and pointed out in [Costa et al., 2015], the model for $\widehat{\theta}(k)$ above encompasses the perfect information case ($M = N$ and $\alpha_{ii} = 1$, for $i \in \mathbb{N}$, which would correspond to the situation in which $\widehat{\theta}(k) = \theta(k)$, that is, $\theta(k)$ is known, and $\mathcal{I}_i = \{i\}$, $\mathbb{M} = \mathbb{N}$), the mode-independent case ($M = 1$ and $\alpha_{i1} = 1$ for all $i \in \mathbb{N}$, which corresponds to the situation in which $\widehat{\theta}(k)$ does not provide any information about $\theta(k)$, that is, $\theta(k)$ is totally unknown), and the cluster case (see [do Val et al., 2002]), which corresponds to the situation such that the state space $\mathbb{N}$ can be decomposed into disjoint sets and it is only known to which of these disjoint sets the Markov chain $\theta(k)$ belongs to.

Remark 5.1 *We are going to study optimization problems for the Finite and Infinite Horizon cases. For the Finite Horizon case all the matrices A_i, B_i, H_i, C_i, D_i, $\mathbf{P}$, and $[\alpha_{i\ell}]$ could be time-dependent but, for notational simplicity, we will consider them time* *invariant.*

We will consider Static Output Feedback Control (SOFC) using the observed emitted signal $\widehat{\theta}(k)$ instead of the unknown variable $\theta(k)$, that is, $u(k)$ will be of the following form:

$$u(k) = K_{\widehat{\theta}(k)}(k)y(k), \tag{5.6}$$

for $K_\ell(k) \in \mathbb{B}(\mathbb{R}^p, \mathbb{R}^m)$, $\ell \in \mathbb{M}$. Associated to a control as in (5.6) set for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$A_{i\ell}(k) \doteq A_i + B_i K_\ell(k) H_i, \tag{5.7}$$

$$C_{i\ell}(k) \doteq C_i + D_i K_\ell(k) H_i. \tag{5.8}$$

For the Infinite Horizon case we will need the concept of stochastic stabilizability (Definition 3.19). In this case we consider time-invariant gains

$$u(k) = K_{\widehat{\theta}(k)} y(k), \tag{5.9}$$

for $K_\ell \in \mathbb{B}(\mathbb{R}^p, \mathbb{R}^m)$, $\ell \in \mathbb{M}$, and set for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$A_{i\ell} \doteq A_i + B_i K_\ell H_i, \;\; C_{i\ell} \doteq C_i + D_i K_\ell H_i. \tag{5.10}$$

For controllers as in (5.6), and the set of FG $K(k) = \{K_\ell(k); \ell \in \mathbb{M}\}$ define the Finite Horizon cost, with final time T_f, as:

$$\begin{aligned} J_{T_f}(K) &\doteq \sum_{k=0}^{T_f-1} E(\|z(k)\|^2) + E(\|C^f_{\theta(T_f)} x(T_f)\|^2) \\ &= \sum_{k=0}^{T_f-1} E(\|C_{\theta(k)\widehat{\theta}(k)}(k) x(k)\|^2) + E(\|C^f_{\theta(T_f)} x(T_f)\|^2), \end{aligned} \tag{5.11}$$

with $z = (z(0), \ldots)$ given by (5.3) when $u(k) = K_{\widehat{\theta}(k)}(k)y(k) = K_{\widehat{\theta}(k)}(k)H_{\theta(k)}x(k)$. For the Infinite Horizon case with $K = \{K_\ell; \ell \in \mathbb{M}\} \in \mathcal{K}$ and control as in (5.9), the cost will be given by

$$J(K) \doteq \|z\|_2^2 = \sum_{k=0}^{\infty} E(\|z(k)\|^2)$$
$$= \sum_{k=0}^{\infty} E(\|C_{\theta(k)\widehat{\theta}(k)}x(k)\|^2). \qquad (5.12)$$

We will also consider the following hard constraints on the state variable $x(k)$ and control varaible $u(k)$. For matrices F_ι of appropriated dimensions, $\iota = 1, \ldots, t$ we introduce the constraints

$$\begin{bmatrix} x(k)' & u(k)' \end{bmatrix} F_\iota' F_\iota \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \leq \rho_\iota, \quad \iota = 1, \ldots, t. \qquad (5.13)$$

Fix $\delta > 0$. We define next the control problems we are interested in;

Definition 5.2 *(The Finite Horizon case) Find $K(k) = \{K_\ell(k); \ell \in \mathbb{M}\}$, $k = 0, \ldots, T_f - 1$, and a set $\mathcal{D}_0 \subset \mathbb{R}^n \times \mathbb{N}$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$ we have that $J_{T_f}(K) \leq \delta\|x_0\|^2$, (5.13) (for $k = 0, \ldots, T_f - 1$) and the final constraint (5.14) (below) are satisfied,*

$$x(T_f)'(G_\iota^f)'G_\iota^f x(T_f) \leq \rho_\iota^f, \quad \iota = 1, \ldots, t. \qquad (5.14)$$

Definition 5.3 *(The Infinite Horizon case) Find $K \in \mathcal{K}$ and a set $\mathcal{D}_0 \subset \mathbb{R}^n \times \mathbb{N}$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$ we have that $J(K) \leq \delta\|x_0\|^2$ and (5.13) are satisfied for all $k = 0, 1, \ldots$.*

We conclude this section with the following condition regarding H_i:

Condition 1 *It is assumed that H_i has full row rank for all $i \in \mathbb{N}$.*

From Condition 1 there exist non-singular matrices S_i such that for each $i \in \mathbb{N}$,

$$H_i S_i = \begin{bmatrix} I & 0 \end{bmatrix}. \qquad (5.15)$$

5.2 The Finite Horizon Case

In this section we analyze the Finite Horizon quadratic control problem as posed in Definition 5.2 through a LMI optimization problem. The goal will be to obtain $K(k) =$

$\{K_\ell(k); \ell \in \mathbb{M}\}$, $k = 0, \ldots, T_f - 1$, which minimizes the upper-bound value δ at the same time that we get a set $\mathcal{D}_0 \subset \mathbb{R}^n \times \mathbb{N}$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$ we have that $J_{T_f}(K) \leq \delta \|x_0\|^2$ and (5.13), (5.14), are satisfied.

In order to define the LMI optimization problem, let us recall the following expressions. For $i \in \mathbb{N}$, $\mathbf{\Gamma}_i = [p_{i1}^{1/2}\mathbf{I}_1 \ldots p_{iN}^{1/2}\mathbf{I}_N] \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^{\tau n})$, where $\mathbf{I}_i$ is an $n \times \tau^i n$ matrix formed by τ^i identity matrices of dimension n, and

$$\operatorname{diag}\{R_{s\zeta}\} \doteq \operatorname{diag}\{R_{1k_1^1}, \ldots, R_{1k_{\tau 1}^1}, \ldots, R_{Nk_1^N}, \ldots, R_{Nk_{\tau N}^N}\},$$

a block-diagonal matrix of dimension $n\tau$ and, for fixed $i \in \mathbb{N}$,

$$\operatorname{diag}\{R_{i\zeta}\} \doteq \operatorname{diag}\{R_{ik_1^i}, \ldots, R_{ik_{\tau^i}^i}\},$$

a block-diagonal matrix of dimension $n\tau_i$. Notice that:

$$\operatorname{diag}\{R_{s\zeta}\} = \operatorname{diag}\{\operatorname{diag}\{R_{1\zeta}\}, \ldots, \operatorname{diag}\{R_{N\zeta}\}\}.$$

We will consider the following problem:

Problem 5.4 *Find $\delta > 0$, $Q(k) = (Q_1(k), \ldots, Q_N(k)) > 0$, $R_{i\zeta}(k) > 0$, $\Phi_{i\zeta}$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, $k = 0, \ldots, T_f$, $Y_\ell(\kappa)$, $U_\ell(\kappa)$, $\ell \in \mathbb{M}$, $\kappa = 0, \ldots, T_f - 1$, such that*

$$\min \delta$$

subject to,

$$\begin{bmatrix} \delta I & \mathbf{I}_i \\ \bullet & \operatorname{diag}\{R_{i\zeta}(0)\} \end{bmatrix} \geq 0, \textit{for } i \in \mathbb{N}, \tag{5.16}$$

$$\begin{bmatrix} U_\ell(k)' + U_\ell(k) - \alpha_{i\ell} S_i^{-1} R_{i\ell}(k) (S_i^{-1})' & (A_i S_i U_\ell(k) + B_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix})' \mathbf{\Gamma}_i & (C_i S_i U_\ell(k) + D_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix})' \\ \bullet & \operatorname{diag}\{R_{s\zeta}(k+1)\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \tag{5.17}$$

$$\begin{bmatrix} \Phi_{i\zeta} + \Phi_{i\zeta}' - R_{i\zeta}(T_f) & \Phi_{i\zeta}'(C_i^f)' \\ \bullet & \tau^i I \end{bmatrix} > 0, \tag{5.18}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, $k = 0, \ldots, T_f - 1$, *where,*

$$U_\ell(k) = \begin{bmatrix} U_{\ell,1}(k) & 0 \\ U_{\ell,2}(k) & U_{\ell,3}(k) \end{bmatrix}, \tag{5.19}$$

and

$$\begin{bmatrix} U_\ell'(k) + U_\ell(k) - S_i^{-1} Q_i(k)(S_i^{-1})' & (A_i S_i U_\ell(k) + B_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix})' \\ \bullet & Q_j(k+1) \end{bmatrix} > 0, \tag{5.20}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, j *such that* $p_{ij} > 0$, $k = 0, \ldots, T_f - 1$, *and*

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota \begin{bmatrix} S_i U_\ell(k) \\ \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix} \end{bmatrix} \\ \bullet & U_\ell(k)' + U_\ell(k) - S_i^{-1} Q_i(k)(S_i^{-1})' \end{bmatrix} > 0, \tag{5.21}$$

for $\iota = 1, \ldots, t$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, $k = 0, \ldots, T_f - 1$, *and*

$$\begin{bmatrix} \rho_\iota^2 I & G_\iota S_i U_\ell(T_f) \\ \bullet & U_\ell(T_f)' + U_\ell(T_f) - S_i^{-1} Q_i(T_f)(S_i^{-1})' \end{bmatrix} > 0, \tag{5.22}$$

for $\iota = 1, \ldots, t$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$.

For $P(k) = (P_1(k), \ldots, P_N(k)) > 0$, as before, we define the function $P(x, i, k) = x' P_i(k) x$, $i \in \mathbb{N}$, and, for $\gamma > 0$,

$$L_P(\gamma, k) := \Big\{ (x, i) \in \mathbb{R}^n \times \mathbb{N}; x' P_i(k) x \leq \frac{1}{\gamma} \Big\}. \tag{5.23}$$

We have the following theorem.

Theorem 5.5 *Suppose there is a solution* $\delta > 0$, $Q(k) = (Q_1(k), \ldots, Q_N(k)) > 0$, $R_{i\zeta}(k) > 0$, $\Phi_{i\zeta}$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, $k = 0, \ldots, T_f$, $Y_\ell(\kappa)$, $U_\ell(\kappa)$, $\ell \in \mathbb{M}$, $\kappa = 0, \ldots, T_f - 1$, *for Problem 5.4. Define* $K_\ell(k) = Y_\ell(k)U_{\ell,1}(k)^{-1}$, $\ell \in \mathbb{M}$ *and* $P(x,i,k) = x'P_i(k)x$, $P_i(k) = Q_i(k)^{-1}$, $i \in \mathbb{N}$. *Then* $J_{T_f}(K) \le \delta\|x_0\|^2$, *and if* $(x_0,\theta_0) \in L_P(1)(0)$ *then* $(x(k),\theta(k)) \in L_P(1,k)$ *for all* $k = 0,1,\ldots,T_f$ *and the constraints (5.13), (5.14), are satisfied.*

Proof: From (5.17) we must have that $U_\ell(k)$ are non-singular. Indeed, if we could find $v \neq 0$ such that $U_\ell(k)v = 0$ then from (5.17) and pre and pos multiplying by the vector $\begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}$ we would end up with $0 < v'U_\ell(k)'v + v'U_\ell(k)v - \alpha_{i\ell}v'S_i^{-1}R_{i\ell}(k)(S_i^{-1})'v = -\alpha_{i\ell}v'S_i^{-1}R_{i\ell}(k)(S_i^{-1})'v < 0$ (since $\alpha_{i\ell} > 0$ and $R_{i\ell}(k) > 0$), which is an absurd. From this and (5.19) it follows that

$$U_\ell(k)^{-1} = \begin{bmatrix} U_{\ell,1}(k)^{-1} & 0 \\ -U_{\ell,3}(k)^{-1}U_{\ell,2}(k)U_{\ell,1}(k)^{-1} & U_{\ell,3}(k)^{-1} \end{bmatrix}, \tag{5.24}$$

From Remark 3.4 and (5.17) we get that

$$\begin{bmatrix} U_\ell(k)'(S_i^{-1}(\alpha_{i\ell}R_{i\ell}(k))(S_i^{-1})')^{-1}U_\ell(k) & (A_iS_iU_\ell(k) + B_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix} U_\ell(k))'\mathbf{\Gamma}_i & (C_iS_iU_\ell(k) + D_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix})' \\ \bullet & \mathrm{diag}\{R_{s\varsigma}(k+1)\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \tag{5.25}$$

so that by pre and post multiplying (5.25) by $\mathrm{diag}\{(U_\ell')(k)^{-1}, I, I\}$ and its transpose, it yields to:

$$\begin{bmatrix} S_i'(\alpha_{i\ell}R_{i\ell}(k))^{-1}S_i & (A_iS_i + B_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix} U_\ell(k)^{-1})'\mathbf{\Gamma}_i & (C_iS_i + D_i \begin{bmatrix} Y_\ell(k)' \\ 0 \end{bmatrix} U_\ell(k)^{-1} \\ \bullet & \mathrm{diag}\{R_{s\varsigma}(k+1)\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \tag{5.26}$$

It follows from (5.24) and (5.15) that

$$\begin{aligned} B_i \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix} U_\ell(k)^{-1} &= B_i \begin{bmatrix} Y_\ell(k) U_{\ell,1}(k)^{-1} & 0 \end{bmatrix} \\ &= B_i \begin{bmatrix} K_\ell(k) & 0 \end{bmatrix} \\ &= B_i K_\ell(k) \begin{bmatrix} I & 0 \end{bmatrix} \\ &= B_i K_\ell(k) H_i S_i, \end{aligned} \tag{5.27}$$

and similarly, $D_i \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix} U_\ell(k)^{-1} = D_i K_\ell(k) H_i S_i$. From (5.27) and pre and post multiplying (5.26) by $\text{diag}\{(S_i')^{-1}, I, I\}$ and its transpose, it yields to

$$\begin{bmatrix} (\alpha_{i\ell} R_{i\ell}(k))^{-1} & (A_i + B_i K_\ell(k) H_i)' \mathbf{\Gamma}_i & (C_i + D_i K_\ell(k) H_i)' \\ \bullet & \text{diag}\{R_{s\zeta}(k+1)\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0. \tag{5.28}$$

From Remark 3.3 we get that (5.28) is equivalent to

$$\begin{aligned} R_{i\ell}(k)^{-1} > & \alpha_{i\ell} \Big\{ (A_i + B_i K_\ell(k) H_i)' \left(\sum_{j=1}^{N} p_{ij} \Big(\sum_{\zeta \in \mathcal{I}_j} R_{j\zeta}(k+1)^{-1} \Big) \right) \\ & \times (A_i + B_i K_\ell(k) H_i) + (C_i + D_i K_\ell(k) H_i)'(C_i + D_i K_\ell(k) H_i) \Big\}, \end{aligned} \tag{5.29}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$. Set $V_i(k) = \sum_{\zeta \in \mathcal{I}_i} R_{i\zeta}(k)^{-1}$, $i \in \mathbb{N}$, $V(k) = (V_1(k), \ldots, V_N(k))$. From (5.29) we have that

$$\begin{aligned} V_i(k) > \sum_{\zeta \in \mathcal{I}_i} \alpha_{i\zeta} \Big\{ & (A_i + B_i K_\zeta(k) H_i)' \mathcal{E}_i(V(k+1))(A_i + B_i K_\zeta(k) H_i) \\ & + (C_i + D_i K_\zeta(k) H_i)'(C_i + D_i K_\zeta(k) H_i) \Big\}. \end{aligned} \tag{5.30}$$

Multiplying (5.30) from the left-hand side by $x(k)'$ and the right-hand by $x(k)$ and taking the expected value we get, from the same arguments as in the proof of Proposition 4 in [Costa et al., 2015], that

$$\begin{aligned}\left\|V_{\theta(k)}(k)^{1/2}x(k)\right\|_2^2 &= E(x(k)'V_{\theta(k)}(k)x(k)) \\ &\geq E(x(k+1)'V_{\theta(k+1)}(k+1)x(k+1)) + \|z(k)\|_2^2 \\ &= \left\|V_{\theta(k+1)}(k+1)^{1/2}x(k+1)\right\|_2^2 + \|z(k)\|_2^2 . \end{aligned} \tag{5.31}$$

Summing up (5.31) from $k = 0$ to $T_f - 1$, we obtain that

$$\begin{aligned}\sum_{k=0}^{T_f-1} \|z(k)\|_2^2 &\leq E(x_0'V_{\theta_0}(0)x_0) - E(x(T_f)'V_{\theta(T_f)}(T_f)x(T_f)) \\ &\leq \delta\|x_0\|_2^2 - E(x(T_f)'V_{\theta(T_f)}(T_f)x(T_f)) \end{aligned} \tag{5.32}$$

where the last inequality follows from (5.16) since that from Remark 3.3 we derive that (5.16) is equivalent to

$$\delta I \geq \sum_{\zeta\in\mathcal{I}_i} R_{i\zeta}^{-1}(0) = V_i(0). \tag{5.33}$$

Notice that, from (5.18) and the same arguments as above, we get that $R_{i\zeta}(T_f)^{-1} > \frac{1}{\tau^i}(C_i^f)'(C_i^f)$, so that $V_i(T_f) = \sum_{\zeta\in\mathcal{I}_i} R_{i\zeta}(T_f)^{-1} > (C_i^f)'C_i^f$ (since $= \sum_{\zeta\in\mathcal{I}_i} 1 = \tau^i$). From this and (5.32) we obtain that $J_{T_f}(K) \leq \delta\|x_0\|_2^2$.

Let us show now that if $(x_0, \theta_0) \in L_P(1)(0)$ then $(x(k), \theta(k)) \in L_P(1,k)$ for all $k = 0, 1, \ldots, T_f$. From (5.20) and Remark 3.4, we have that

$$\begin{bmatrix} U_\ell'S_i'Q_i(k)^{-1}S_iU_\ell & (A_iS_iU_\ell(k) + B_i\begin{bmatrix}Y_\ell(k) & 0\end{bmatrix})' \\ \bullet & Q_j \end{bmatrix} > 0, \tag{5.34}$$

so that by pre and post multiplying (5.20) by $\text{diag}\{(U_\ell')(k)^{-1}, I\}$ and its transpose, and repeating the same reasoning as in (5.27) we get that

$$\begin{bmatrix} S_i'Q_i(k)^{-1}S_i & (A_iS_i + B_i \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix} U_\ell(k)^{-1})') \\ \bullet & Q_j(k+1) \end{bmatrix}$$
$$= \begin{bmatrix} S_i'Q_i(k)^{-1}S_i & ((A_i + B_iK_\ell(k)H_i)S_i)') \\ \bullet & Q_j(k+1) \end{bmatrix} > 0. \tag{5.35}$$

Pre and post multiplying (5.35) by $\mathrm{diag}\{(S_i')^{-1}, I\}$ and its transpose and applying Remark 3.3 we get, after setting $P_s(k) = Q_s(k)^{-1}$, that for $p_{ij} > 0$

$$P_i(k) > (A_i + B_iK_\ell(k)H_i)'P_j(k+1)(A_i + B_iK_\ell(k)H_i). \tag{5.36}$$

Hence, from (5.1) and (5.6), $x(k+1) = (A_{\theta(k)} + B_{\theta(k)}K_{\widehat{\theta}(k)}(k)H_{\theta(k)})x(k)$, and from (5.36) we get that

$$x(k)'P_{\theta(k)}(k)x(k) \geq x(k+1)'P_{\theta(k+1)}(k+1)x(k+1). \tag{5.37}$$

and thus (5.37) yields to

$$\begin{aligned} x_0'P_{\theta_0}(0)x_0 &\geq x(k)'P_{\theta(k)}(k)x(k) \\ &> x(k+1)'P_{\theta(k+1)}(k+1)x(k+1). \end{aligned} \tag{5.38}$$

From (5.38) we have that if $(x_0, \theta_0) \in L_P(1)(0)$ (that is, $x_0'P_{\theta_0}(0)x_0 \leq 1$) then $(x(k), \theta(k)) \in L_P(1, k)$ for every $k = 0, 1, \ldots$ (since from (5.38) $x(k)'P_{\theta(k)}(k)x(k) \leq 1$). Finally let us show that the constraints (5.13), (5.14), are satisfied. From (5.21) and Remark 3.4 again we get that

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota \begin{bmatrix} S_iU_\ell(k) \\ \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix} \end{bmatrix} \\ \bullet & U_\ell(k)'(S_i^{-1}Q_i(k)(S_i^{-1})')^{-1}U_\ell(k) \end{bmatrix} > 0. \tag{5.39}$$

Pre and post multiplying (5.39) by $\mathrm{diag}\{I, (U_\ell')(k)^{-1}\}$ and its transpose, and repeating the same reasoning as in (5.27) we have that

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota \begin{bmatrix} S_i U_\ell(k) \\ \begin{bmatrix} Y_\ell(k) & 0 \end{bmatrix} \end{bmatrix} U_\ell^{-1} \\ \bullet & S_i' Q_i(k)^{-1} S_i \end{bmatrix} = \begin{bmatrix} \rho_\iota^2 I & F_\iota \begin{bmatrix} I \\ K_\ell(k) H_i \end{bmatrix} S_i \\ \bullet & S_i' Q_i^{-1}(k) S_i \end{bmatrix} > 0. \tag{5.40}$$

Pre and post multiplying (5.40) by $\operatorname{diag}\{I, (S_i')^{-1}\}$ and its transpose, and Remark 3.3, we derive that

$$\rho_\iota^2 I > F_\iota \begin{bmatrix} I \\ K_\ell(k) H_i \end{bmatrix} Q_i(k) (F_\iota \begin{bmatrix} I \\ K_\ell(k) H_i \end{bmatrix})', \tag{5.41}$$

so that, from (5.41), we conclude that

$$\|F_\iota \begin{bmatrix} I \\ K_\ell(k) H_i \end{bmatrix} Q_i(k)^{1/2}\|^2 \leq \rho_\iota^2 \text{for all} \ell \in \mathcal{I}_i, i \in \mathbb{N}.$$

Thus we get that

$$\begin{aligned} \|F_\iota \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}\|^2 &= \|F_\iota \begin{bmatrix} I \\ K_{\widehat{\theta}(k)} H_{\theta(k)} \end{bmatrix} x(k)\|^2 \\ &= \|F_\iota \begin{bmatrix} I \\ K_{\widehat{\theta}(k)} H_{\theta(k)} \end{bmatrix} Q_{\theta(k)}(k)^{1/2} Q_{\theta(k)}(k)^{-1/2} x(k)\|^2 \\ &\leq \|F_\iota \begin{bmatrix} I \\ K_{\widehat{\theta}(k)} H_{\theta(k)} \end{bmatrix} Q_{\theta(k)}(k)^{1/2}\|^2 \|Q_{\theta(k)}(k)^{-1/2} x(k)\|^2 \\ &\leq \rho_\iota^2 \ x(k)' Q_{\theta(k)}^{-1} x(k) \\ &= \rho_\iota^2 \ x(k)' P_{\theta(k)} x(k) \leq \rho_\iota^2 \end{aligned} \tag{5.42}$$

since $x(k)' P_{\theta(k)} x(k) \leq 1$, showing the result for $k = 0, \ldots, T_f - 1$. From (5.22) and repeating the same arguments as above we get that $\|G_\iota x(T_f)\|^2 \leq \rho_\iota^2$, completing the proof. $\square$

5.3 The Infinite Horizon Case

The next LMI optimization problem aims at obtaining a $K \in \mathcal{K}$ which minimizes the upper-bound value δ at the same time that obtains an invariant set $\mathcal{D}_0$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$ we have that $(x(k), \theta(k)) \in \mathcal{D}_0$ for all $k = 0, 1, \ldots$ and the constraints (5.13) are satisfied. In Section 5.4 we will present other versions of this problem, either by fixing the initial condition (x_0, θ_0) or by fixing $\delta > 0$ and aiming to find the largest inner ball inside an invariant set $\mathcal{D}_0$.

Problem 5.6 *Find $\delta > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, such that*

$$\min \delta$$

subject to,

$$\begin{bmatrix} \delta I & \mathbf{I}_i \\ \bullet & \mathrm{diag}\{R_{i\zeta}\} \end{bmatrix} \geq 0, \textit{for } i \in \mathbb{N}, \tag{5.43}$$

$$\begin{bmatrix} U_\ell' + U_\ell - \alpha_{i\ell} S_i^{-1} R_{i\ell} (S_i^{-1})' & (A_i S_i U_\ell + B_i \begin{bmatrix} Y_\ell' \\ 0 \end{bmatrix})' \mathbf{\Gamma}_i & (C_i S_i U_\ell + D_i \begin{bmatrix} Y_\ell' \\ 0 \end{bmatrix})' \\ \bullet & \mathrm{diag}\{R_{s\zeta}\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \tag{5.44}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$U_\ell = \begin{bmatrix} U_{\ell,1} & 0 \\ U_{\ell,2} & U_{\ell,3} \end{bmatrix}, \tag{5.45}$$

$$\begin{bmatrix} U_\ell' + U_\ell - S_i^{-1} Q_i (S_i^{-1})' & (A_i S_i U_\ell + B_i \begin{bmatrix} Y_\ell & 0 \end{bmatrix})' \\ \bullet & Q_j \end{bmatrix} > 0, \tag{5.46}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, j *such that* $p_{ij} > 0$, *and*

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota \begin{bmatrix} S_i U_\ell \\ \begin{bmatrix} Y_\ell & 0 \end{bmatrix} \end{bmatrix} \\ \bullet & U_\ell' + U_\ell - S_i^{-1} Q_i (S_i^{-1})' \end{bmatrix} > 0, \tag{5.47}$$

for $\iota = 1, \dots, t$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$.

For $P = (P_1, \dots, P_N) > 0$, set the function $P(x,i) = x' P_i x$, $i \in \mathbb{N}$, and, for $\gamma > 0$,

$$L_P(\gamma) := \Big\{ (x,i) \in \mathbb{R}^n \times \mathbb{N}; x' P_i x \leq \frac{1}{\gamma} \Big\}. \tag{5.48}$$

We have the following result.

Theorem 5.7 *Suppose there is a solution* $\delta > 0$, $Q = (Q_1, \dots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, *for Problem 5.6. Define* $K_\ell = Y_\ell U_{\ell,1}^{-1}$, $\ell \in \mathbb{M}$ *and* $P(x,i) = x' P_i x$, $P_i = Q_i^{-1}$, $i \in \mathbb{N}$. *Then: i)* $K \in \mathcal{K}$*; ii)* $J(K) \leq \delta \|x_0\|^2$. *If* $(x_0, \theta_0) \in L_P(1)$ *then: iii)* $(x(k), \theta(k)) \in L_P(1)$ *for all* $k = 0, 1, \dots$*; iv) the constraints (5.13) are satisfied.*

Proof: Following the same steps as in the proof of Theorem 5.5 we obtain from (5.44) that

$$\begin{bmatrix} (\alpha_{i\ell} R_{i\ell})^{-1} & (A_i + B_i K_\ell H_i)' \mathbf{\Gamma}_i & (C_i + D_i K_\ell H_i)' \\ \bullet & \operatorname{diag}\{R_{s\zeta}\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \tag{5.49}$$

and thus, from Remark 3.3, we get that (5.49) is equivalent to

$$\begin{aligned} R_{i\ell}^{-1} > & \alpha_{i\ell} \Big\{ (A_i + B_i K_\ell H_i)' \left(\sum_{j=1}^{N} p_{ij} \Big(\sum_{\zeta \in \mathcal{I}_j} R_{j\zeta}^{-1} \Big) \right) \\ & \times (A_i + B_i K_\ell H_i) + (C_i + D_i K_\ell H_i)' (C_i + D_i K_\ell H_i) \Big\} \end{aligned} \tag{5.50}$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$. Set $V_i = \sum_{\zeta \in \mathcal{I}_i} R_{i\zeta}^{-1}$, $i \in \mathbb{N}$, $V = (V_1, \dots, V_N)$. From (5.50) we have that

$$\begin{aligned} V_i > \sum_{\zeta \in \mathcal{I}_i} \alpha_{i\zeta} \Big\{ & (A_i + B_i K_\zeta H_i)' \mathcal{E}_i(V) (A_i + B_i K_\zeta H_i) \\ & + (C_i + D_i K_\zeta H_i)' (C_i + D_i K_\zeta H_i) \Big\} \end{aligned} \tag{5.51}$$

and thus (5.51) implies that $V - \mathcal{L}(V) > 0$, so that, from Theorem 3.20, $K \in \mathcal{K}$, showing i). Let us now show ii). Following the same steps as in the proof of Proposition 4 in [Costa et al., 2015] we get from (5.51) that

$$\begin{aligned}\left\|V_{\theta(k)}^{1/2}x(k)\right\|_2^2 &= E(x(k)'V_{\theta(k)}x(k)) \\ &\geq E(x(k+1)'V_{\theta(k+1)}x(k+1)) + \|z(k)\|_2^2 \\ &= \left\|V_{\theta(k+1)}^{1/2}x(k+1)\right\|_2^2 + \|z(k)\|_2^2 .\end{aligned} \tag{5.52}$$

Since $K \in \mathcal{K}$ we have, from the stochastic stability of (5.1), (5.6), that $\left\|V_{\theta(k)}^{1/2}x(k)\right\|_2^2 \to 0$ as $k \to \infty$. From Remark 3.3 we derive that (5.43) is equivalent to

$$\delta I \geq \sum_{\zeta \in \mathcal{I}_i} R_{i\zeta}^{-1} = V_i. \tag{5.53}$$

Taking the sum in (5.52) for $k = 0$ to ∞, and using (5.53) we obtain that

$$J(K) = \|z\|_2^2 = \sum_{k=0}^{\infty} \|z(k)\|_2^2 \leq E(x_0'V_{\theta_0}x_0) \leq \delta\|x_0\|_2^2. \tag{5.54}$$

Let us now show iii). From (5.46) and repeating the same reasoning as in the proof of Theorem 5.5 we get, after setting $P_s = Q_s^{-1}$, that

$$P_i > (A_i + B_iK_\ell H_i)'P_j(A_i + B_iK_\ell H_i), \text{ for } p_{ij} > 0, \tag{5.55}$$

and recalling that $x(k+1) = (A_{\theta(k)} + B_{\theta(k)}K_{\widehat{\theta}(k)}H_{\theta(k)})x(k)$, we get from (5.55) that

$$x(k)'P_{\theta(k)}x(k) \geq x(k+1)'P_{\theta(k+1)}x(k+1). \tag{5.56}$$

so that (5.56) yields to

$$x_0'P_{\theta_0}x_0 \geq x(k)'P_{\theta(k)}x(k) \geq x(k+1)'P_{\theta(k+1)}x(k+1). \tag{5.57}$$

Therefore (5.57) implies that if $(x_0, \theta_0) \in L_P(1)$ (that is, $x_0'P_{\theta_0}x_0 \leq 1$) then $(x(k), \theta(k)) \in L_P(1)$ for every $k = 0, 1, \ldots$ (since from (5.38) $x(k)'P_{\theta(k)}x(k) \leq 1$), completing the proof of iii). Let us now show iv). Repeating the same reasoning as in the proof of Theorem 5.5, we get from (5.47) that

$$\rho_\iota^2 I > F_\iota \begin{bmatrix} I \\ K_\ell H_i \end{bmatrix} Q_i (F_\iota \begin{bmatrix} I \\ K_\ell H_i \end{bmatrix})', \tag{5.58}$$

and from (5.58) we conclude that $\|F_\iota \begin{bmatrix} I \\ K_\ell H_i \end{bmatrix} Q_i^{1/2}\|^2 \leq \rho_\iota^2$ for all $\ell \in \mathcal{I}_i$, $i \in \mathbb{N}$. As in the proof of Theorem 5.5, this implies that

$$\begin{aligned} \|F_\iota \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \|^2 &\leq \|F_\iota \begin{bmatrix} I \\ K_\ell H_i \end{bmatrix} Q_i^{1/2}\|^2 \|Q_{\theta(k)}^{-1/2} x(k)\|^2 \\ &\leq \rho_\iota^2 \; x(k)' Q_{\theta(k)}^{-1} x(k) = \rho_\iota^2 \; x(k)' P_{\theta(k)} x(k) \\ &\leq \rho_\iota^2 \end{aligned}$$

since $x(k)' P_{\theta(k)} x(k) \leq 1$, completing the proof. □

5.4 Alternative Problems

We have 2 alternative problems associated to Problems 5.4 (Finite Horizon case) and 5.6 (Infinite Horizon case). For simplicity we will present the results only for the Infinite Horizon case (Problem 5.6). These 2 new problems will be referred to as Problem 5.8 and Problem 5.9.

5.4.1 The pair of initial conditions (x_0, θ_0) is fixed

If the initial condition x_0 and the initial probability for θ_0 are fixed, with $\mu_i = \mathcal{P}(\theta_0 = i)$ for $i \in \mathbb{N}$, we can re-write Problem 5.6, taking into account these specific initial conditions, as Problem 5.8 in the following way:

Problem 5.8 *Find $\delta > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\varsigma} > 0$, $i \in \mathbb{N}$, $\varsigma \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, such that*

$$\min \delta$$

subject to,

$$\begin{bmatrix} \delta & x_0' \begin{bmatrix} \mu_1^{1/2}\mathbf{I}_1 & \cdots & \mu_N^{1/2}\mathbf{I}_N \end{bmatrix} \\ \bullet & \operatorname{diag}\{R_{s\zeta}\} \end{bmatrix} \geq 0, \tag{5.59}$$

$$\begin{bmatrix} 1 & x_0' \\ \bullet & Q_i \end{bmatrix} \geq 0, \textit{for } i \in \mathbb{N} \textit{ with } \mu_i > 0, \tag{5.60}$$

and the LMIs (5.44)-(5.47).

Proof: As shown in Corollary 4.5, we have that (5.59) and Remark 3.3 implies that

$$\begin{aligned} \delta &\geq x_0' \Big(\sum_{i=1}^{N} \mu_i (\sum_{\zeta \in \mathcal{I}_\rangle} R_{i\zeta}^{-1}) \Big) x_0 \\ &= x_0' \Big(\sum_{i=1}^{N} \mu_i V_i \Big) x_0 = E(x_0' V_{\theta_0} x_0) \end{aligned} \tag{5.61}$$

so that, from (5.54), we obtain that $J(K) \leq \delta$. From (5.60) it follows that $1 \geq x_0' Q_i^{-1} x_0 = x_0' P_i x_0$ for any $i \in \mathbb{N}$ with $\mu_i > 0$, and thus $(x_0, \theta_0) \in L_P(1)$. From (5.57) and the fact that $(x_0, \theta_0) \in L_P(1)$ we get that $(x(k), \theta(k)) \in L_P(1)$ for every $k = 0, 1, \ldots$, and the reaming of the proof of Theorem 5.7 can be applied.

5.4.2 The case with fixed upper-bound cost δ

The second alternative problem would be, for a fixed $\delta > 0$, to get an approximation for the largest inner ball (with radius $\sqrt{\frac{1}{\upsilon}}$) $\mathcal{D}_\upsilon \doteq \{x_0 \in \mathbb{R}^n; \|x_0\|^2 \leq \frac{1}{\upsilon}\}$ included in the set $\widehat{L}_P(1) := \{x \in \mathbb{R}^n; (x, i) \in L_P(1) \text{ for some } i \in \mathbb{N}\}$, in other words, to obtain the minimum $\upsilon > 0$ such that $\mathcal{D}_\upsilon \subseteq \widehat{L}_P(1)$. This problem would be too hard to be solved, so that a simplified convex version of this problem would be in the following way. Notice that, with $\upsilon = \max_{i \in \mathbb{N}} \|P_i\|$, we have that $\mathcal{D}_\upsilon \times \mathbb{N} \subset L_P(1)$ since that, if $x_0 \in \mathcal{D}_\upsilon$ then for any $i \in \mathbb{N}$ we have that $x_0' P_i x_0 \leq \|x_0\|^2 \|P_i\| \leq \|x_0\|^2 \upsilon \leq 1$. Thus, by minimizing $\max_{i \in \mathbb{N}} \|P_i\|$ we get the largest inner ball as defined in $\mathcal{D}_\upsilon$ with $\upsilon = \max_{i \in \mathbb{N}} \|P_i\|$, included in the set $\widehat{L}_P(1)$. Having this in mind, for $\delta > 0$ fixed, we re-write Problem 5.6 as follows:

Problem 5.9 *Find* $\upsilon > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, *such that*

$$\min \ \upsilon$$

subject to,

$$\begin{bmatrix} \upsilon I & I \\ \bullet & Q_j \end{bmatrix} \geq 0, \tag{5.62}$$

and the LMIs (5.43)-(5.47).

As shown in Corollary 4.7, we have from (5.62) and Remark 3.3 that $\upsilon I \geq Q_j^{-1}$, so that $\upsilon \geq \max_{j \in \mathbb{N}} \|Q_j^{-1}\|$. Since we want to minimize υ the optimal solution υ^* will be such that $\upsilon^* = \max_{j \in \mathbb{N}} \|Q_j^{-1}\| = \max_{j \in \mathbb{N}} \|P_j\|$ since, by definition, $P_j = Q_j^{-1}$.

Chapter 6

Simulation

In this chapter, we present the numerical implementation of the theoretical results as illustrative examples. In Section 6.1, the solutions to the problems tackled in Chapter 4 will be applied considering an economic example. For the results of Chapter 5, an unmanned aircraft model with failures on the actuators will be addressed in Section 6.2.

In the sequel the numerical results will be presented, solving the previous LMI optimization problems, using the YALMIP [Lofberg, 2004] and SeDuMi [Sturm, 1999] numerical tool packages.

6.1 Application for the Constrained State Feedback scenario

For the simulations it is considered a simple economic system based on the Samuelson's multiplier–accelerator model (see [Samuelson, 1939]). According to this model, the national income at time k, denoted by $Y(k)$, may be written as the sum of three components: consumption, $C(k)$, induced private investment, $I(k)$, and governmental expenditure, $G(k)$, so that $Y(k) = C(k) + I(k) + G(k)$. Moreover the terms $C(k)$ and $I(k)$ are related to the national income $Y(k)$ through the equations $C(k) = (1 - s)Y(k - 1)$, and $I(k) = w(Y(k - 1) - Y(k - 2))$, where s represents the marginal propensity to save and w the accelerator coefficient (see [Samuelson, 1939, Blair and Sworder, 1975b, Westerhoff, 2006b]). We end up with the following difference equation:

$$Y(k) = (1-s)Y(k-1) + w(Y(k-1) - Y(k-2)) + G(k). \tag{6.1}$$

Considering a target governmental expenditure $\bar{G}$ we have that the fixed point of (6.1) is given by $\bar{Y} = \frac{\bar{G}}{s}$. Setting $\widetilde{Y}(k) = Y(k) - \bar{Y}$, $\widetilde{G}(k) = G(k) - \bar{G}$, we have that (6.1) can be re-written as

$$\widetilde{Y}(k) = (1-s)\widetilde{Y}(k-1) + w(\widetilde{Y}(k-1) - \widetilde{Y}(k-2)) + \widetilde{G}(k). \tag{6.2}$$

A state-space version of (6.2) is

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -w & 1-s+w \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{6.3}$$

where $x_2(k) = \widetilde{Y}(k)$, $x_1(k) = \widetilde{Y}(k-1)$, and $u(k) = \widetilde{G}(k+1)$, and it is desirable that the expected value of $E(x_2(k))$ converges to zero as k goes to infinity.

In [Blair and Sworder, 1975b] it was presented a MJLS version of (6.3) by considering a Markov chain $\theta(k) \in \mathbb{N} = \{1, 2, 3\}$ and three possible values for the parameters s_i and w_i representing the possible states of the economy (see [Ackley, 1969] for more details); $i = 1$ for the "Norm" state, $i = 2$ for the "Boom" state, and $i = 3$ for the "Slump" state. Here we adopted the following values for these parameters: $w_1 = 1.5$, $s_1 = 0.5$; $w_2 = 2.5$, $s_2 = 0.4$; and $w_3 = 1$, $s_3 = 0.6$ (see Table 1). Since the classification of these states rely on some economical indexes that may take some time to be evaluated, it would be possible to have some mismatch between the estimated value (represented by $\widehat{\theta}(k)$) and the real state of the economy (represented by $\theta(k)$).

	Norm	Boom	Slump
i	1	2	3
w	1.5	2.5	1
s	0.5	0.4	0.6

Table 1: Parameters for the Samuelson's multiplier–accelerator model.
Source: author.

Thus from (6.3) considering the MJLS, the matrices for our example are:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1.5 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -2.5 & 3.1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 1.4 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1.5477 & -1.0976 \\ -1.0976 & 1.9145 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1.7025 & -1.2074 \\ -1.2074 & 2.1060 \\ 0 & 0 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1.3929 & -0.9878 \\ -0.9878 & 1.7231 \\ 0 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0 \\ 0 \\ 1.6125 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 0 \\ 1.7738 \end{bmatrix}, D_3 = \begin{bmatrix} 0 \\ 0 \\ 1.4513 \end{bmatrix}.$$

The transition matrix that relates the system operation modes (Markov diagram is shown in Figure 5) is given by (as used in [Blair and Sworder, 1975b]):

$$\mathbf{P} = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.30 & 0.47 & 0.23 \\ 0.26 & 0.10 & 0.64 \end{bmatrix},$$

where we have adopted the same cost weighting matrices C_i and D_i as in [Costa et al., 2005]. The initial condition x_0 and the initial state θ_0 are assumed to be known (Corollary 4.5) with the following values:

$$x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}', \quad \theta_0 = 1.$$

To limit great variations on the amount invested in the economy, it is considered a constraint related to the private investment (that depends on $x_2(k) - x_1(k)$) and the

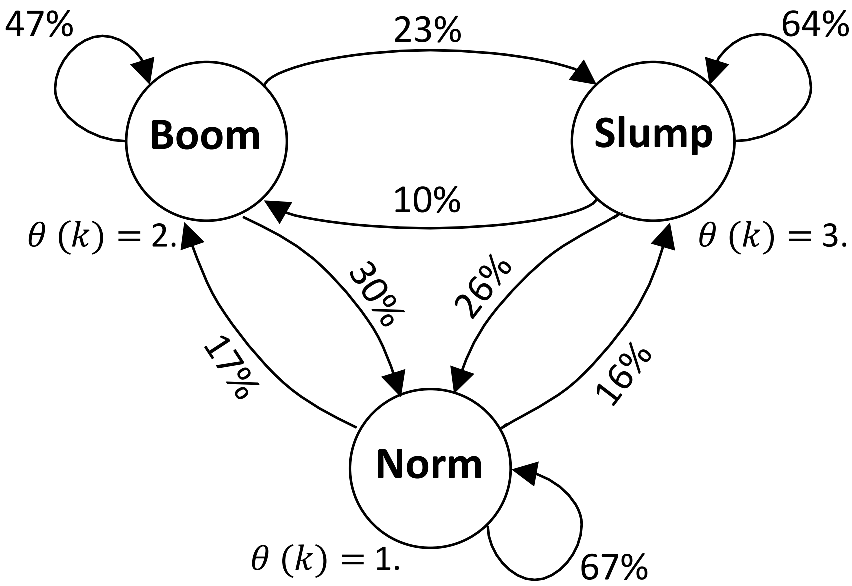


Figure 5: Markov diagram for economic system.
Source: author.

government expenditure $u(k)$ defined by the following matrices:

$$F_1 = c_I \begin{bmatrix} -1 & 1 \end{bmatrix}, c_I = 2, G_1 = 1 \text{ and } \rho_1 = 0.2,$$

thus,

$$\|2(x_2(k) - x_1(k)) + u(k)\| \leq 0.2.$$

We consider three cases as below:

(a) Algorithm developed in [Costa et al., 1999], with full information on $\theta(k)$.

(b) Ideal detector, i.e., the detector provides full information (similar to Case (a) above, and also discussed in Section 3.7).

(c) Detector provides information according to the probability matrix

$$\begin{bmatrix} \alpha_{i\ell} \end{bmatrix} = \begin{bmatrix} 0.8 & 0 & 0.2 \\ 0.15 & 0.85 & 0 \\ 0 & 0.25 & 0.75 \end{bmatrix}.$$

Table 3 presents the obtained results for the three cases (upper-bounds, total cost and controllers). Notice that Case (a) attained a larger values for the upper-bound δ and cost $J(K)$ than the equivalent Case (b). Hence, for this example, the algorithm developed in

Case	δ	$J(K)$	K_ℓ
(a)	5.9547	5.0408	$K_1 = \begin{bmatrix}1.4929 & -1.4089\end{bmatrix}$ $K_2 = \begin{bmatrix}1.5958 & -1.5137\end{bmatrix}$ $K_3 = \begin{bmatrix}1.4442 & -1.3845\end{bmatrix}$
(b)	4.6133	4.5526	$K_1 = \begin{bmatrix}1.6537 & -1.5257\end{bmatrix}$ $K_2 = \begin{bmatrix}2.1303 & -1.9797\end{bmatrix}$ $K_3 = \begin{bmatrix}1.5670 & -1.4422\end{bmatrix}$
(c)	5.6768	5.0996	$K_1 = \begin{bmatrix}1.8253 & -1.6355\end{bmatrix}$ $K_2 = \begin{bmatrix}1.7771 & -1.5891\end{bmatrix}$ $K_3 = \begin{bmatrix}1.6274 & -1.4742\end{bmatrix}$

Table 2: Performance parameters for different cases (State Feedback scenario).
Source: author.

the present work is an improvement with respect to the one introduced in [Costa et al., 1999]. We can also conclude that, comparing Case (b) and Case (c), a more reliable information (regarding the parameters $\alpha_{i\ell}$) yields to a lower value for δ and total cost $J(K)$.

The mean value of the state (second component), obtained out of 1000 Monte Carlo simulations, is shown in Figure 6. It can be seen that Case (a) obtained the best performance, but total cost $J(k)$ is larger than Case (b) due to its expected control signal; and Case (c), as expected, attained the slowest stabilization behavior. Figure 7 presents the extreme values of all the realizations of $F_\iota x(k) + G_\iota u(k)$ for unconstrained (dashed line) and constrained (solid line) algorithm. We can observe that the extreme realizations for constrained algorithm are bounded for the pre-fixed limit $\rho_\iota = 0.2$ (constant star line) in all the cases (in comparison with unconstrained case) but these values are distant from that limit, from which we can infer that the constraint is conservative.

The problem for maximizing the invariant set derived from Corollary 4.7 was also implemented using the parameters of Case (c) (set $\rho = 0.4$) and fixing δ attained by computing the algorithm from Theorem 4.2 ($\delta = 104.4839$). For this problem, it was obtained the optimal value $\upsilon^* = 1.6032$ and the same solution $Q = (Q_1, \dots, Q_N) > 0$ and

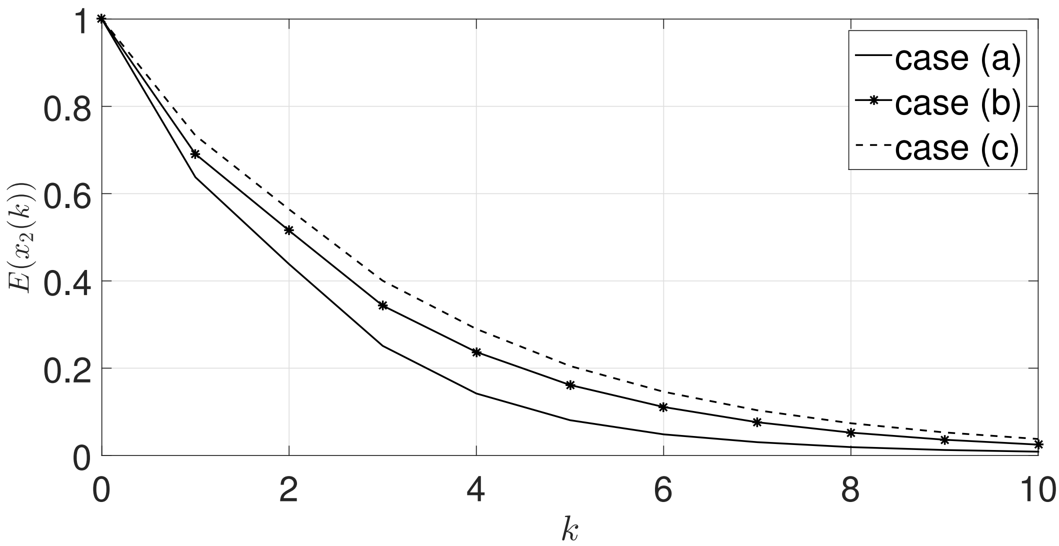
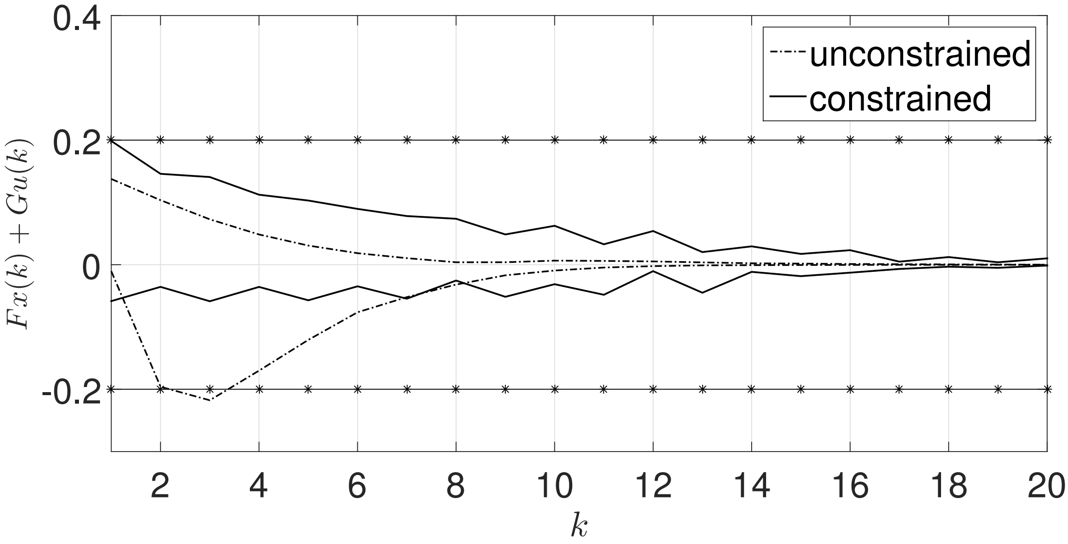


Figure 6: Mean value of the state for different cases.
Source: author.

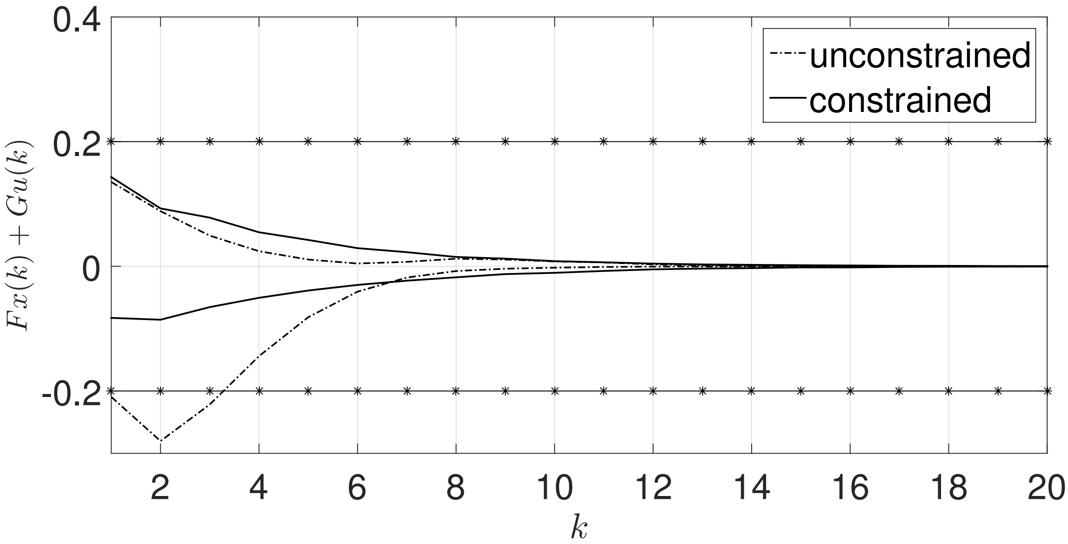
K_ℓ as achieved from Theorem 4.2

$$K_1 = \begin{bmatrix} 1.8832 & -1.7640 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 1.7349 & -1.6055 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} 1.7146 & -1.6076 \end{bmatrix}.$$

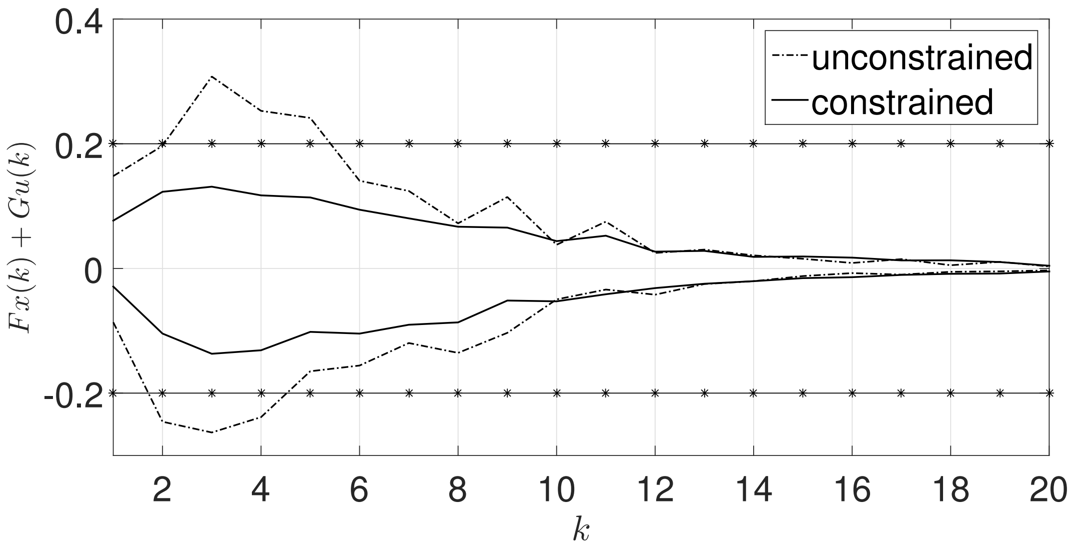
Thus in this case there is no gain in increasing the invaraint set $L_P(1)$. But if we increase δ by 10%, that is, by fixing $\delta = 114.9323$, it is obtained a lower value for υ^* ($\upsilon^* = 1.2624$) and, as expected, we get a bigger invariant region $\widehat{L}_P(1)$ for this case (recall that the radius of the inner ball inside $\widehat{L}_P(1)$ is given by $\sqrt{\frac{1}{\upsilon^*}}$).


unconstrained
constrained
0.4
0.2
0
-0.2
2
4
6
8
10
12
14
16
18
20
k
$Fx(k) + Gu(k)$

(a)


unconstrained
constrained
0.4
0.2
0
-0.2
2
4
6
8
10
12
14
16
18
20
k
$Fx(k) + Gu(k)$

(b)



(c)

Figure 7: Extreme values of realizations of $F_\iota x(k) + G_\iota u(k)$ for the Cases (a), (b) and (c). Source: author.

6.2 Application for the Constrained Static Output Feedback scenario

In this section, the theoretical results of Chapter 5 are numerically implemented, and for this purpose it is considered the linearized model of a small unmanned aerial vehicle in steady flight with some modifications (see [de Oliveira et al., 2020]). The state variable $x(k)$ is represented by small perturbations on the roll rate, yaw rate, sideslip, and roll angles, and the control variable $u(k)$ corresponds to the aileron and rudder commands. We assume that the aircraft's motion has two operation modes ($N = 2$), with the nominal operation model assigned by $\theta(k) = 1$ and the faulty operation mode by $\theta(k) = 2$. The system parameters for the nominal operation mode are:

$$A_1 = \begin{bmatrix} 0.5637 & 0.1133 & -0.6607 & -0.0062 \\ 0.0198 & 0.8368 & 1.0512 & 0.0089 \\ 0.0033 & -0.0450 & 0.9481 & 0.0159 \\ 0.0381 & 0.0073 & -0.0164 & 0.9999 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 2.9735 & -0.0618 \\ -0.1175 & 0.6414 \\ 0.0112 & -0.0165 \\ 0.0812 & -0.0006 \end{bmatrix}.$$

For the faulty operation mode it is assumed that the aileron command is ineffective, i.e.,

$$A_2 = A_1, \quad B_2 = B_1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrices H_i and S_i are chosen as follows ($i \in \{1, 2\}$):

$$H_i = \begin{bmatrix} 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad S_i = \begin{bmatrix} 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \\ 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \end{bmatrix}.$$

Note that states $x_1(k)$ and $x_2(k)$ are not included in the output $y(k)$ (Equation (5.3)). The remaining parameters are set in the canonical form

$$C_i = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix}, \quad D_i = \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix}, \quad i \in \{1, 2\}.$$

The transition matrix that relates the system operation modes $\mathbf{P}$ (see Markov diagram in Figure 8), detection probability matrix $\left[\alpha_{i\ell}\right]$ and initial probability of Markov parameter

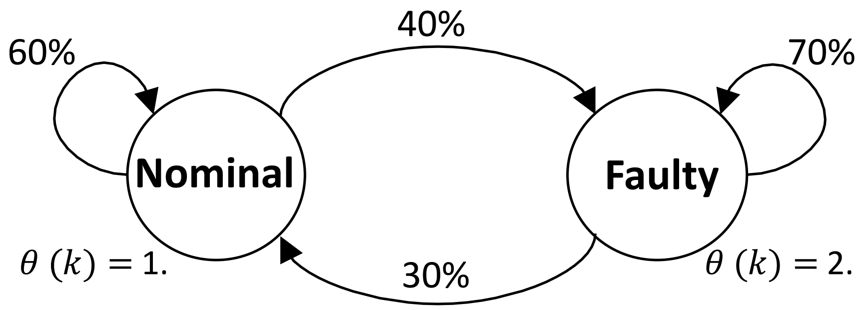


Figure 8: Markov diagram for unmanned aerial vehicle.
Source: author.

θ_0 are given by:

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}, \begin{bmatrix} \alpha_{i\ell} \end{bmatrix} = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix}, \mu_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}.$$

For the constrained cases it is considered the following hard restriction,

$$|u_2(k)| \leq 0.1,$$

and therefore we have that,

$$F_\iota = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \rho_1 = 0.1.$$

Figure 9 shows the convergence of the upper-bound δ for the Finite Horizon case (Theorem 5.5) as the final time T_f increases, in comparison with the upper-bound obtained for the Infinite Horizon case (Theorem 5.7). As expected, the limit value for the Finite Horizon case is lower than for the Infinite Horizon case, since the Finite Horizon case allows time varying gains, being thus less restrictive.

The solution of Problem 5.8 is implemented and presented in Figure 10. We can observe that for both cases in the figure (the State Feedback and the Static Output Feedback cases), the upper-bound δ is symmetric with respect to the value $\alpha = 0.5$, and it varies with the degree of information (entropy), attaining its maximum value in the equiprobable scenario ($\alpha = 0.5$). We can also corroborate that the upper-bound is lower when we

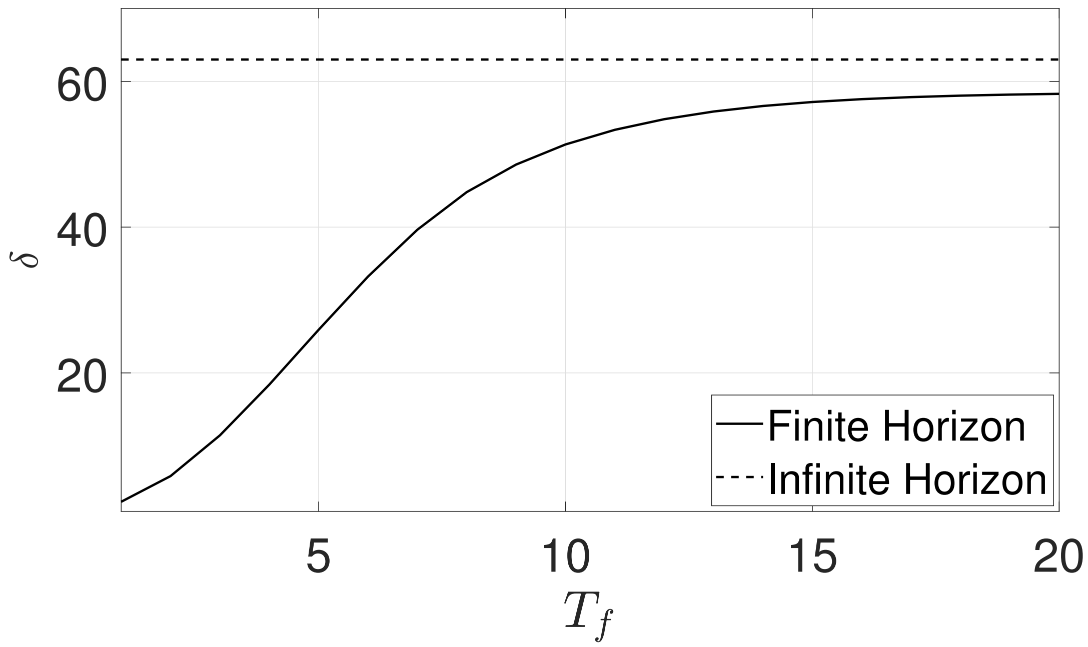


Figure 9: The upper-bound δ for the Finite Horizon Case as a function of the final time T_f ($\alpha = 0.75$), and the upper-bound value for the Infinite Horizon Case (dashed line).
Source: author.

have complete observation of the state vector, which results in a better performance of the control system.

For the elaboration of Figure 11 and Figure 12, Montecarlo simulations were performed with 1000 experiments. The mean value of the state (second component, $x_2(k)$) is shown in Figure 11 for the unconstrained and constrained cases. We can observe that even without considering $x_2(k)$ in the output vector (Static Output Feedback case), it is possible to stabilize the closed loop system. Notice that the unconstrained case performs better than the constrained case in terms of oscillations and stabilization time, as expected.

Figure 12 presents the maximum values among all the realizations of $u(k)$ (the second component $u_2(k)$ of $u(k)$ for both cases). The maximum realization values for the constrained case are bounded by the pre-fixed bound $\rho_1 = 0.1$ (constant lines) as expected in the design for the constrained algorithm. On the other hand, the values for the unconstrained case are much more distant from the bound (hard constraint), showing the importance of including these constraints in the optimization problem.

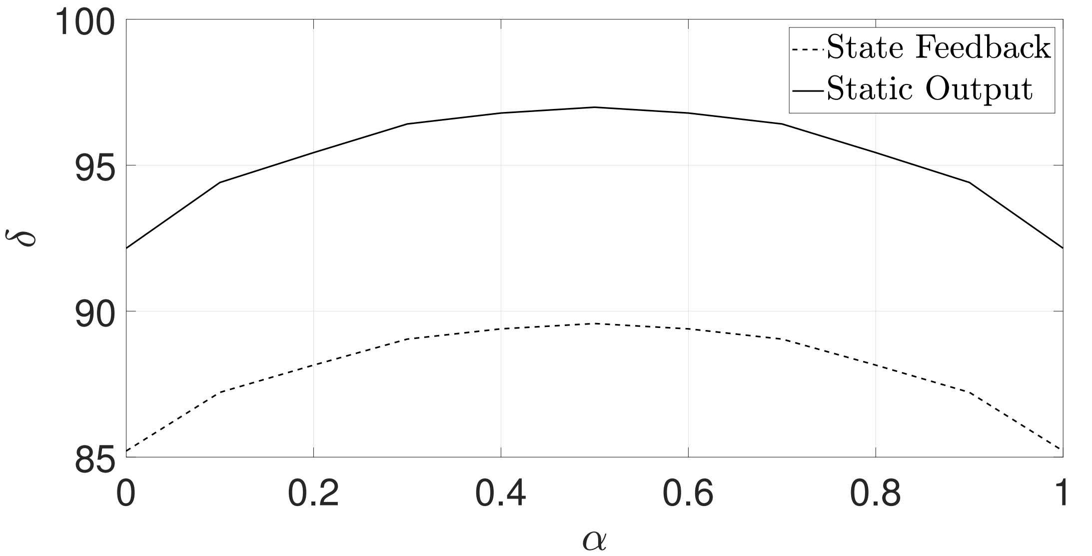


Figure 10: The upper-bound δ for State Feedback and Static Output Feedback cases as a function of the detection probability α.
Source: author.

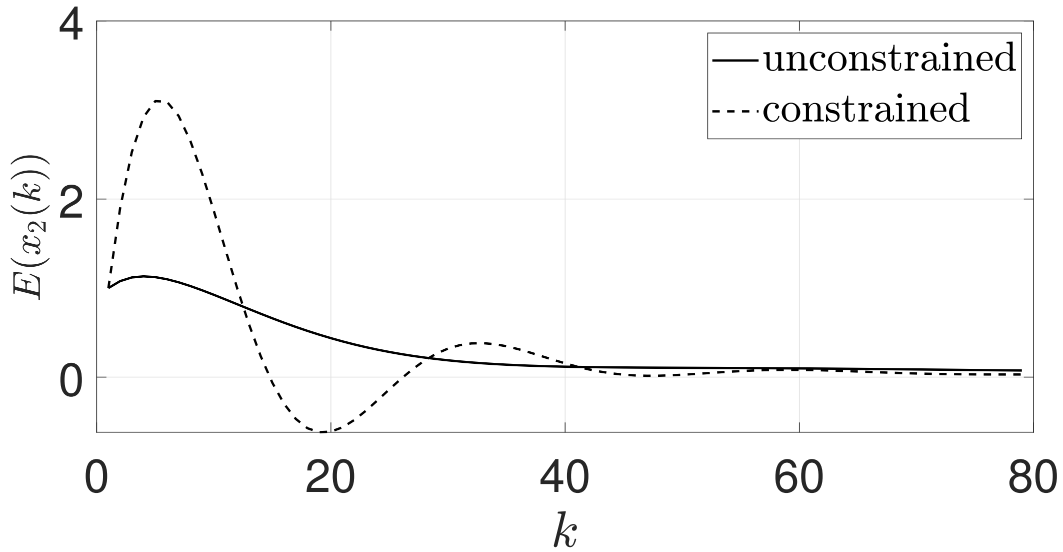


Figure 11: Mean value of the state (second component) $E\left(x_2(k)\right)$ for the Unconstrained and Constrained cases ($\alpha = 0.85$).
Source: author.

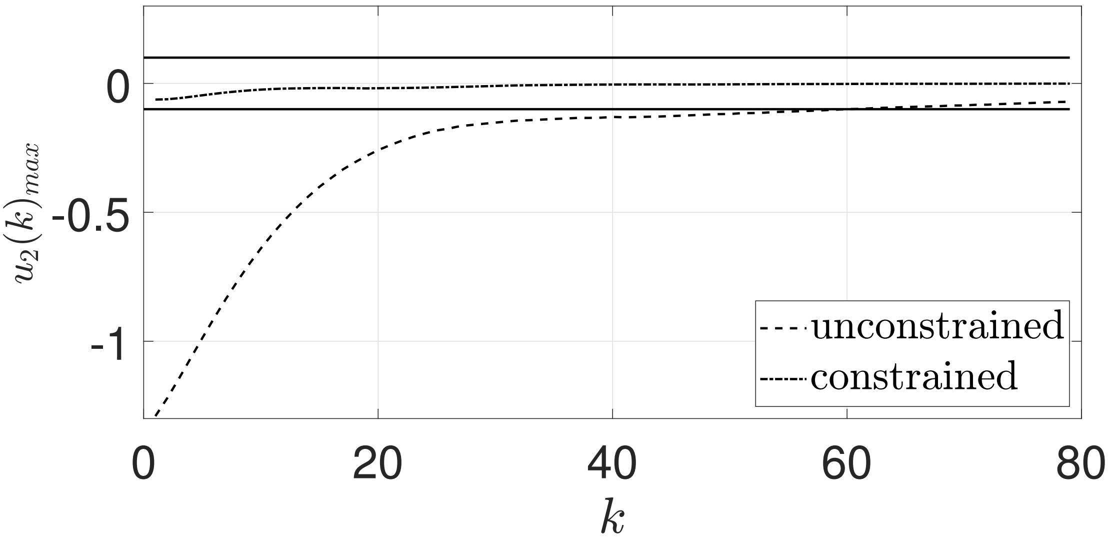


Figure 12: Maximum realization of the control variable (second component) $u_2(k)$ for the Unconstrained (dashed line) and Constrained cases ($\alpha = 0.85$).
Source: author.

Some performance parameters are shown in Table 3. We can see that in all cases the real cost $J(K)$ is less than its corresponding upper-bound δ. We can also see that the State Feedback case has better indices (δ and $J(K)$) than the Static Output Feedback case, as expected. The respective controllers for each case are shown in the fourth column.

Case	δ	$J(K)$	K_ℓ
(USF)	36.1063	35.2050	$K_1 = \begin{bmatrix} -0.1699 & -0.0713 & 0.2494 & -0.1401 \\ -0.0783 & -0.7363 & -0.1282 & -0.0680 \end{bmatrix}$ $K_2 = \begin{bmatrix} -0.0632 & 0.0549 & -0.0100 & -0.0654 \\ -0.0833 & -0.7063 & -0.1671 & -0.0743 \end{bmatrix}$
(CSF)	89.2089	81.9124	$K_1 = \begin{bmatrix} -0.1683 & -0.0320 & 0.1795 & -0.2850 \\ -0.0034 & -0.0080 & -0.0177 & -0.0382 \end{bmatrix}$ $K_2 = \begin{bmatrix} -0.0477 & -0.0102 & 0.0695 & -0.1421 \\ -0.0035 & -0.0082 & -0.0168 & -0.0382 \end{bmatrix}$
(USO)	37.9658	36.1680	$K_1 = \begin{bmatrix} 1.2627 & -0.6897 \\ -12.6984 & -0.2926 \end{bmatrix}$ $K_2 = \begin{bmatrix} 0.7927 & -0.4362 \\ -12.5349 & -0.3704 \end{bmatrix}$
(CSOF)	96.6686	85.8033	$K_1 = \begin{bmatrix} 0.4318 & -2.1775 \\ -0.2322 & -0.4054 \end{bmatrix}$ $K_2 = \begin{bmatrix} 0.2617 & -1.3386 \\ -0.2208 & -0.4059 \end{bmatrix}$

Table 3: Performance parameters for different cases (Static Output Feedback scenario). Source: author.

Chapter 7

Conclusion and Future Work

Finally, we close this thesis with some conclusions and summarize the main contributions to the literature in Section 7.1. We also mention some open problems for future works on which this thesis can serve as a basis (see Section 7.2).

7.1 Conclusions

In this thesis were considered 2 global scenarios where different cases were explored in each one. The main contributions of this thesis are divided into 2 parts (for each scenario) and summarized as follows.

1. In the first scenario (Chapter 4), it was considered a state feedback quadratic control problem of discrete time MJLS with constraints on the norm of the state and control variables, presented as Problem 4.1. A feedback linear control is derived using the information provided by the detector, so that the closed loop system is stochastically stabilized, an upper-bound δ for the quadratic cost is minimized, and the constraints are satisfied provided that the initial conditions are inside an invariant set, Theorem 4.2. We also show that 2 other problems can be formulated by LMI optimization problems under the proposed framework; one in which δ is fixed and it is desired to maximize the estimate of the domain of an invariant set, Problem 4.4; and the other to minimize the guaranteed quadratic cost for fixed initial conditions, Problem 4.6. The solutions for these 2 problems were outlined in the Corollary 4.5 and Corollary 4.7, respectively. This part is illustrated in Chapter 6 with the nu-

merical simulation of a simple economic system based on the Samuelson's multiplier accelerator model in which it was observed that the algorithm developed in this work, for the considered example, presented better results than the one introduced in [Costa et al., 1999]. Moreover, as expected, it was noticed that a more reliable detector yields to a lower the value of δ and the respective total cost $J(K)$, Table 3. The constraint imposed on the norm of the state and control variables for this example has been satisfied, Figure 7. These results were published in the IEEE Transactions on Automatic Control journal (see paper [Zabala and Costa, 2019]).

2. In the second scenario (Chapter 5), it was studied the constrained static output feedback control problem for discrete-time MJLS considering the Finite Horizon as well as the Infinite Horizon cases, where also the Markov parameter $\theta(k)$ is not directly observed. It is assumed that the only information available to the controller with respect to $\theta(k)$ comes from a detector which provides a signal $\hat{\theta}(k)$ (estimation), where it is considered that $\big(\theta(k), \hat{\theta}(k)\big)$ follows a HMM. For the Infinite Horizon case the obtained results in this part can be seen as a generalization of the State Feedback case introduced in [Zabala and Costa, 2019], by setting the output matrix as the identity matrix. Theorems 5.5 and 5.7 show that, by obtaining a solution for the LMI optimization Problems 5.4 and 5.6, we can find a static output feedback controller as in (5.6) for the Finite Horizon case, stabilizing static output feedback controller as in (5.9) for the Infinite Horizon case, such that the hard constraints (5.13) (and (5.14) for the Finite Horizon case) are satisfied and the quadratic costs (5.11) and (5.12) are lower than the upper-bound $\delta\|x_0\|^2$. Alternative problems (as in the previous scenario) in which the initial conditions are fixed and in which it is desired to maximize an estimate of the domain of an invariant set are also analyzed (Problems 5.8 and 5.9). Lastly, a numerical simulation of an unmanned aircraft system subject to actuators faults were implemented in Chapter 6 showing, as expected, that the limit value for the Finite Horizon case is lower than for the Infinite Horizon case, since the Finite Horizon case allows time varying gains, being thus less restrictive, Figure 9. It is also corroborated that the scenario with State Feedback has better indices (δ and $J(K)$) than the Static Output Feedback scenario,

and that a more reliable detector yields to a lower value for the upper-bound (cost function), Figure 10. The constrain on the norm of the control variable specified in this example was obeyed, Figure 12. The results of this part were published in the IEEE Access journal (see paper [Zabala and Costa, 2020]).

7.2 Perspectives

In this work it was considered quadratic optimal control for MJLS where, broadly speaking, the following topics were tackled:

1. a hidden MJLS framework;
2. state feedback and static output feedback control;
3. deterministic constraints on the norm and control variables;
4. the finite and infinite horizon cases;
5. LMIs technique as solution method.

With this in mind, possible extensions or some directions for future research could be:

- considering probabilistic constraints as, for instance, analyzed in [Cinquemani et al., 2011], for MJLS under the detector-based approach;
- analyzing the case with second moment constraints as introduced in [Vargas et al., 2013] instead of hard constraints, see Equation (4.5), which should yield to less conservative controllers;
- introducing the condition where the TPM $\mathbf{P}$ is not exactly known as [Costa et al., 1999] (see Equation (3.29)), for hidden MJLS context addressing the finite and infinite horizon cases;
- investigating and extending the portfolio selection problems addressed in [Costa and Araujo, 2008], assuming the existence of a detector for the market parameters, posing these problems as Problem 4.1 and Problem 4.6, and representing the upper-bound of the cost δ as some financial index of interest (for instead, the multi-period portfolio risk);

- studying the so-called Positive Markov Jump Linear Systems (PMJLS) (see for instance [Lian et al., 2015]) in order to apply the derived algorithms to the fields of reservoir control and energy planning (for instead [Bertolucci and Costa, 2011]).

Bibliography

[Ackley, 1969] Ackley, G. (1969). *Macroeconomic Theory.* Macmillan Library Reference.

[Baczynski et al., 2001] Baczynski, J., Fragoso, M. D., and Lopes, E. P. (2001). On a discrete time linear jump stochastic dynamic game. *International Journal of Systems Science*, 32:979–988.

[Bar-Shalom and Li, 1993] Bar-Shalom, Y. and Li, X. R. (1993). *Estimation and Tracking. Principles, Techniques, and Software.* Artech House.

[Baum et al., 1970] Baum, L. E., Petrie, T., Soules, G., and Weiss, N. (1970). A maximization technique occurring in the statistical analysis of probabilistic functions of markov chains. *Ann. Math. Statist.*, 41(1):164–171.

[Bertolucci and Costa, 2011] Bertolucci, L. H. B. and Costa, E. F. (2011). Reference tracking controller applied to the operation of hydrothermal systems. *IFAC Proceedings Volumes*, 44(1):12279–12281.

[Bertsekas, 2000] Bertsekas, D. P. (2000). *Dynamic Programming and Optimal Control.* Athena Scientific, 2nd edition.

[Blair and Sworder, 1975a] Blair, W. P. and Sworder, D. D. (1975a). Continuous-time regulation of a class of econometric models. *IEEE Transactions on Systems, Man and Cybernetics*, SMC-5:341–346.

[Blair and Sworder, 1975b] Blair, W. P. and Sworder, D. D. (1975b). Feedback control of a class of linear discrete systems with jump parameters and quadratic cost criteria. *International Journal of Control*, 21:833–844.

[Boukas, 2006] Boukas, E. K. (2006). *Stochastic Switching Systems: Analysis and Design.* Birkhäuser.

[Boyd et al., 1994a] Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994a). *Linear Matrix Inequalities in System and Control Theory.* SIAM Studies in Applied Mathematics.

[Boyd et al., 1994b] Boyd, S., Ghaoui, L. E., Feron, E., and Balakrishnan, V. (1994b). *Linear Matrix Inequalities and Control Theory.* SIAM, Philadelphia.

[Callier and Desoer, 1999] Callier, F. M. and Desoer, C. A. (1999). *Linear System Theory.* Springer-Verlag.

[Carvalho et al., 2020] Carvalho, L. D. P., Oliveira, A. M. D., and Costa, O. L. D. V. (2020). $\mathcal{H}_2$ /$\mathcal{H}_\infty$ simultaneous fault detection and control for Markov jump linear systems with partial observation. *IEEE Access*, 8:11979–11990.

[Cinquemani et al., 2011] Cinquemani, E., Agarwal, M., Chatterjee, D., and Lygeros, J. (2011). Convexity and convex approximations of discrete-time stochastic control problems with constraints. *Automatica*, 47(9):2082 – 2087.

[Costa and Fragoso, 1993] Costa, O. and Fragoso, M. (1993). Stability results for discrete-time linear systems with Markovian jumping parameters. *Journal of Mathematical Analysis and Applications*, 179:154–178.

[Costa and Araujo, 2008] Costa, O. L. and Araujo, M. V. (2008). A generalized multi-period meanâ€"variance portfolio optimization with Markov switching parameters. *Automatica*, 44(10):2487 – 2497.

[Costa et al., 1999] Costa, O. L. V., Assumpção, E. O., Boukas, E. K., and Marques, R. P. (1999). Constrained quadratic state feedback control of discrete-time Markovian jump linear systems. *Automatica*, 35:617–626.

[Costa et al., 2005] Costa, O. L. V., Fragoso, M. D., and Marques, R. P. (2005). *Discrete-Time Markov Jump Linear Systems*. Springer.

[Costa et al., 2013] Costa, O. L. V., Fragoso, M. D., and Todorov, M. G. (2013). *Continuous-Time Markov Jump Linear Systems*. Springer, New York, NY, USA.

[Costa et al., 2015] Costa, O. L. V., Fragoso, M. D., and Todorov, M. G. (2015). A detector-based approach for the H_2 control of Markov jump linear systems with partial information. *IEEE Transactions on Automatic Control*, 60(5):1219–1234.

[de Oliveira and Costa, 2017a] de Oliveira, A. M. and Costa, O. L. V. (2017a). H_2-filtering for discrete-time hidden Markov jump systems. *International Journal of Control*, 90(3):599–615.

[de Oliveira and Costa, 2017b] de Oliveira, A. M. and Costa, O. L. V. (2017b). $\mathcal{H}_\infty$-filtering for Markov jump linear systems with partial information on the jump parameter. *IFAC Journal of Systems and Control*, 1:13 – 23.

[de Oliveira and Costa, 2017c] de Oliveira, A. M. and Costa, O. L. V. (2017c). Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ of hidden Markov jump systems. *International Journal of Robust and Nonlinear Control*, 28:1261–1280.

[de Oliveira et al., 2020] de Oliveira, A. M., do Valle Costa, O. L., and Daafouz, J. (2020). Suboptimal $\mathcal{H}_2$- and $\mathcal{H}_\infty$- static output feedback control of hidden Markov jump linear systems. *European Journal of Control*, 51:10 – 18.

[do Val et al., 2002] do Val, J. B. R., Geromel, J. C., and Gonçalves, A. P. C. (2002). The H_2-control for jump linear systems: Cluster observations of the Markov state. *Automatica*, 38:343–349.

[Dragan et al., 2010] Dragan, V., Morozan, T., and Stoica, A. M. (2010). *Mathematical Methods in Robust Control of Linear Stochastic Systems (Mathematical Concepts and Methods in Science and Engineering)*. Springer-Verlag.

[Faraji-Niri et al., 2016] Faraji-Niri, M., Jahed-Motlagh, M. R., and Barkhordari-Yazdi, M. (2016). Stabilization of active fault-tolerant control systems by uncertain nonhomogeneous Markovian jump models. *Complexity*, 21(S1):318–329.

[Gray et al., 2000] Gray, W. S., González, O. R., and Doğan, M. (2000). Stability analysis of digital linear flight controllers subject to electromagnetic disturbances. *IEEE Transactions on Aerospace and Electronic Systems*, 36:1204–1218.

[Kang et al., 2012] Kang, Y., Li, Z., Dong, Y., and Xi, H. (2012.). Markovian-based fault-tolerant control for wheeled mobile manipulators. *IEEE Transactions on Control Systems Technology*, 20(1):266–276.

[Kothare et al., 1996] Kothare, M. V., Balakrishnan, V., and Morari, M. (1996). Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32:1361–1379.

[Lamond and Boukhtouta, 1996] Lamond, B. F. and Boukhtouta, A. (1996). Optimizing long-term hydro-power production using Markov decision processes. *International Transactions in Operational Research*, 3(3):223 – 241.

[Lian et al., 2015] Lian, J., Liu, J., and Zhuang, Y. (2015). Mean stability of positive Markov jump linear systems with homogeneous and switching transition probabilities. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 62(8):801–805.

[Lofberg, 2004] Lofberg, J. (2004). Yalmip : a toolbox for modeling and optimization in matlab. In *2004 IEEE International Conference on Robotics and Automation (IEEE Cat. No.04CH37508)*, pages 284–289.

[Lu et al., 2013] Lu, J., Li, D., and Xi, Y. (2013). Constrained model predictive control synthesis for uncertain discrete-time Markovian jump linear systems. *IET Control Theory and Applications*, 7(5):707–719.

[Mahmoud et al., 2003] Mahmoud, M., Jiang, J., and Zhang, Y. (2003). *Active Fault Tolerant Control Systems - Stochastic Analysis and Synthesis*. Springer, Germany.

[Mariton, 1989] Mariton, M. (1989). Detection delays, false alarm rates and the reconfiguration of control systems. *International Jounal of Control*, 49:981–992.

[Mariton, 1990] Mariton, M. (1990). *Jump Linear Systems in Automatic Control*. CRC Press, New York, NY, USA.

[Naidu et al., 2002] Naidu, D. S., Naidu, S., and Dorf, R. C. (2002). *Optimal Control Systems*. CRC Press, Inc., USA.

[Ogura et al., 2018] Ogura, M., Cetinkaya, A., Hayakawa, T., and Preciado, V. M. (2018). State-feedback control of Markov jump linear systems with hidden-Markov mode observation. *Automatica*, 89:65 – 72.

[Oliveira et al., 2014] Oliveira, R. C. L. F., Vargas, A. N., d. Val, J. B. R., and Peres, P. L. D. (2014). Mode-independent $\langle_2$ -control of a dc motor modeled as a Markov jump linear system. *IEEE Transactions on Control Systems Technology*, 22(5):1915–1919.

[Patrinos et al., 2014] Patrinos, P., Sopasakis, P., Sarimveis, H., and Bemporad, A. (2014). Stochastic model predictive control for constrained discrete-time markovian switching systems. *Automatica*, 50(10):2504 – 2514.

[Piantadosi et al., 2010] Piantadosi, J., Howlett, P. G., Bean, N. G., and Beecham, S. (2010). Modelling systems of reservoirs using structured Markov chains. *Proceedings of the Institution of Civil Engineers - Water Management*, 163(8):407–416.

[Richter et al., 2008] Richter, J., Lunze, J., and Schlage, T. (2008). Control reconfiguration after actuator failures by Markov parameter matching. *International Journal of Control StaticIOReconf International Journal of Control DD Month*, 9:15–1.

[Rodrigues et al., 2016] Rodrigues, C. C. G., Todorov, M. G., and Fragoso, M. D. (2016). H_∞ filtering for Markovian jump linear systems with mode partial information. In *2016 IEEE 55th Conference on Decision and Control (CDC)*, pages 640–645.

[Rodrigues et al., 2017] Rodrigues, C. C. G., Todorov, M. G., and Fragoso, M. D. (2017). H_∞ control of continuous-time Markov jump linear systems with detector-based mode information. *International Journal of Control*, 90(10):2178–2196.

[Ross, 2010] Ross, S. M. (2010). *Introduction to Probability Models*. Academic Press, Inc., Orlando, FL, USA, tenth edition.

[Samuelson, 1939] Samuelson, P. A. (1939). Interactions between the multiplier analysis and the principle of accelleration. *Review of Economic Statistics*, 21:75–85.

[Shi and Li, 2015] Shi, P. and Li, F. (2015). A survey on Markovian jump systems: Modeling and design. *International Journal of Control, Automation and Systems*, 13(1):1–16.

[Siqueira and Terra, 2004] Siqueira, A. A. G. and Terra, M. H. (2004). Nonlinear and Markovian $\mathcal{H}_\infty$ controls of underactuated manipulators. *IEEE Transactions on Control Systems Technology*, 12:635–648.

[Song et al., 2000] Song, H., Liu, C. ., Lawarree, J., and Dahlgren, R. W. (2000). Optimal electricity supply bidding by Markov decision process. *IEEE Transactions on Power Systems*, 15(2):618–624.

[Song et al., 2017] Song, J., Niu, Y., Zhao, H., and Cao, Z. (2017). Finite-time $\ell_2 - \ell_\infty$-control of Markovian jump linear systems with partly accessible hidden information via asynchronous output feedback. In *2017 11th Asian Control Conference (ASCC)*, pages 2447–2452.

[Song et al., 2017] Song, J., Niu, Y., and Zou, Y. (2017). Asynchronous output feedback control of time-varying Markovian jump systems within a finite-time interval. *Journal of the Franklin Institute*, 354(15):6747 – 6765.

[Song et al., 2018] Song, J., Niu, Y., and Zou, Y. (2018). Asynchronous sliding mode control of Markovian jump systems with time-varying delays and partly accessible mode detection probabilities. *Automatica*, 93:33 – 41.

[Srichander and Walker, 1993] Srichander, R. and Walker, B. K. (1993). Stochastic stability analysis for continuous-time fault tolerant control systems. *International Journal of Control*, 57(2):433–452.

[Stadtmann, 2019] Stadtmann, F. (2019). *Control of Markov Jump Linear Systems with uncertain detections*. PhD thesis, Universidade de Sao Paulo.

[Stadtmann and Costa, 2016] Stadtmann, F. and Costa, O. L. V. (2016). H_2-control of continuous-time hidden Markov jump linear systems. *IEEE Transactions on Automatic Control*, 62:4031–4037.

[Sturm, 1999] Sturm, J. F. (1999). Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1-4):625–653.

[Svensson and Williams, 2008] Svensson, L. E. and Williams, N. (2008). Optimal monetary policy under uncertainty in dsge models: A Markov jump-linear-quadratic approach. Working Paper 13892, National Bureau of Economic Research.

[Sworder and Rogers, 1983] Sworder, D. D. and Rogers, R. O. (1983). An LQG solution to a control problem with solar thermal receiver. *IEEE Transactions on Automatic Control*, 28:971–978.

[Todorov et al., 2018] Todorov, M. G., Fragoso, M. D., and Costa, O. L. V. (2018). Detector-based H_∞ results for discrete-time Markov jump linear systems with partial observations. *Automatica*, 91:159 – 172.

[Tonne and Stursberg, 2016] Tonne, J. and Stursberg, O. (2016). Constraint Robust Model Predictive Control for Jump Markov Linear Systems with Additive Disturbances. In *2016 European Control Conference (ECC)*, pages 1315–1321.

[Tugnait., 1982] Tugnait., J. K. (1982). Adaptive estimation and identification for discrete systems with Markov jump parameters. *IEEE Transactions on Automatic Control*, 27:1054–1064.

[Vargas et al., 2007] Vargas, A. N., Furloni, W., and do Val, J. B. (2007). Control of Markov jump linear systems with state and input constraints: A necessary optimality condition. *IFAC Proceedings Volumes*, 40(20):250 – 255. 3rd IFAC Symposium on System Structure and Control.

[Vargas et al., 2013] Vargas, A. N., Furloni, W., and do Val, J. B. R. (2013). Second moment constraints and the control problem of Markov jump linear systems. *Numerical Linear Algebra with Applications*, 20(2, SI):357–368.

[Westerhoff, 2006a] Westerhoff, F. H. (2006a). Samuelson's multiplier–accelerator model revisited. *Applied Economics Letters*, 13:89–92.

[Westerhoff, 2006b] Westerhoff, F. H. (2006b). Samuelson's multiplier–accelerator model revisited. *Applied Economics Letters*, 13(2):89–92.

[Wiggins, 2003] Wiggins, S. (2003). *Introduction to applied nonlinear dynamical systems and chaos*, volume 2. Springer Science & Business Media.

[Yin and Zhou, 2004] Yin, G. and Zhou, X. Y. (2004). Markowitz's mean-variance portfolio selection with regime switching: from discrete-time models to their continuous-time limits. *IEEE Transactions on Automatic Control*, 49(3):349–360.

[Zabala and Costa, 2019] Zabala, Y. A. and Costa, O. L. V. (2019). A detector-based approach for the constrained quadratic control of discrete-time Markovian jump linear systems. *IEEE Transactions on Automatic Control*, 65:1211 – 1217.

[Zabala and Costa, 2020] Zabala, Y. A. and Costa, O. L. V. (2020). Static output constrained control for discrete-time hidden markov jump linear systems. *IEEE Access*, 8:62969–62979.

[Zhang et al., 2019] Zhang, T., Deng, F., and Zhang, W. (2019). H_{Index} for linear time-varying Markov jump stochastic systems and its application to fault detection. *IEEE Access*, 7:23698–23712.

[Zhou et al., 2017] Zhou, J., Xie, M., He, Z., Qin, H., and Yuan, L. (2017). Medium-term hydro generation scheduling (mthgs) with chance constrained model (ccm) and dynamic control model (dcm). *Water Resources Management*, 31(11):3543–3555.

Appendix A

Some Important Matrix Properties

Below are exposed some of the main matrix properties that were used in this work (for more details, see [Naidu et al., 2002]).

A.1 Matrix

An $n \times m$ matrix $\mathbf{A}$ is an arrangement of nm elements a_{ij} $(i =, 1, 2, ..., n; j = 1, 2, .., m)$ into n rows and m columns

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \tag{A.1}$$

The $n \times m$ of the matrix $\mathbf{A}$ is also referred to as the *size* or *dimension* of the matrix.

A.2 Symmetric Matrix

A *symmetric* matrix is one whose row elements are the same as the corresponding column elements. Thus, $a_{ij} = a_{ji}$. In other words, if $\mathbf{A} = \mathbf{A}'$, then the matrix $\mathbf{A}$ is symmetric. Some basic properties of symmetric matrices are:

a) The sum and difference of two symmetric matrices is again symmetric.

b) This is not always true for the product: given symmetric matrices $\mathbf{A}$ and $\mathbf{B}$, then $\mathbf{AB}$ is symmetric if and only if $\mathbf{A}$ and $\mathbf{B}$ commute, i.e., if $\mathbf{AB} = \mathbf{BA}$.

c) For integer n, $\mathbf{A}^n$ is symmetric if $\mathbf{A}$ is symmetric.

d) If $\mathbf{A}^{-1}$ exist, it is symmetric if and only if $\mathbf{A}$ is symmetric.

A.3 Norm of a Matrix

For matrices, the various norms are defined as

1. $||\mathbf{Ax}|| \leq ||\mathbf{A}||.||\mathbf{x}||$,
2. $||\mathbf{A} + \mathbf{B}|| \leq ||\mathbf{A}|| + ||\mathbf{B}||$, called the Schwartz inequality,
3. $||\mathbf{AB}|| \leq ||\mathbf{A}||.||\mathbf{B}||$.

A.4 Matrix Inversion Formulas

The inverse of sum of matrices is given as

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{DA}^{-1}B + \mathbf{C}^{-1})^{-1}\mathbf{DA}^{-1} \tag{A.2}$$

Where, $\mathbf{A}$ and $\mathbf{C}$ are nonsingular matrices, the matrix $(\mathbf{A} + \mathbf{BCD})$ can be formed and is nonsingular and the matrix $(\mathbf{DA}^{-1}B + \mathbf{C}^{-1})$ is nonsingular. As special case

$$(\mathbf{I} - \mathbf{F}(\mathbf{sI} - \mathbf{A})^{-1}\mathbf{B})^{-1} = \mathbf{I} + \mathbf{F}(\mathbf{sI} - \mathbf{A} - \mathbf{BF})^{-1}\mathbf{B} \tag{A.3}$$

If a matrix $\mathbf{A}$ consists of submatrices as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{A.4}$$

then

$$|\mathbf{A}| = |\mathbf{A}_{11}|.|\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}|$$
$$= |\mathbf{A}_{22}|.|\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}|$$
$$\mathbf{A}^{-1} = \begin{pmatrix} E_1^{-1} & -E_1^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}E_1^{-1} & E_2^{-1} \end{pmatrix}$$

where, the inverse of $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ exist and

$$\mathbf{E}_1 = (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}), \qquad \mathbf{E}_2 = (\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}) \tag{A.5}$$

A.5 Quadratic Forms and Definiteness

Quadratic Forms

Consider the inner product of a real symmetric matrix $\mathbf{P}$ and a vector $\mathbf{x}$ or the norm vector $\mathbf{x}$ w.r.t. the real symmetric matrix $\mathbf{P}$ as

$$< \mathbf{x}, \mathbf{P}\mathbf{x} > = \mathbf{x}'\mathbf{P}\mathbf{x} = ||\mathbf{x}||_{\mathbf{P}}$$
$$= (x_1 x_2 \dots x_n) \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{12} & p_{22} & \dots & p_{2n} \\ \dots & \dots & \dots & \dots \\ p_{1n} & p_{2n} & \dots & p_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$
$$= \sum_{i,j=1}^{n} p_{ij} x_i x_j.$$

The scalar quantity $\mathbf{x}'\mathbf{P}\mathbf{x}$ is called a *quadratic form* since it contains quadratic terms such as $x_1^2 p_{11}, x_1 x_2 p_{12}$,...

Definiteness

Let $\mathbf{P}$ be a real and symmetric matrix and $\mathbf{x}$ be a nonzero real vector, then:

1. $\mathbf{P}$ is *positive definite* if the scalar quantity $\mathbf{x}'\mathbf{P}\mathbf{x} > 0$ or is *positive.*

2. $\mathbf{P}$ is *positive semidefinite* if the scalar quantity $\mathbf{x}'\mathbf{P}\mathbf{x} \gneqq 0$ or is *nonnegative.*

3. $\mathbf{P}$ is *negative definite* if the scalar quantity $\mathbf{x}'\mathbf{P}\mathbf{x} < 0$ or is *nonnegative.*

4. $\mathbf{P}$ is *negative semidefinite* if the scalar quantity $\mathbf{x}'\mathbf{P}\mathbf{x} \leq 0$ or is *nonpositive*

A test for real symmetric matrix $\mathbf{P}$ to be positive definite is that all its *principal* or *leading minors* must be positive, that is,

$$p_{11} > 0, \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} > 0, \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} > 0 \tag{A.6}$$

for a 3×3 matrix $\mathbf{P}$. The $>$ sign is changed accordingly for positive semidefinite ($\geq$), negative definite ($<$), and negative semidefinite (≤ 0) cases. Another simple test for definiteness is using eigenvalues (all eigenvalues positive for positive definiteness, etc.) Also, note that

$$(\mathbf{x}'\mathbf{P}\mathbf{x})' = \mathbf{x}'\mathbf{P}'\mathbf{x} = \mathbf{x}'\mathbf{P}\mathbf{x}$$
$$\mathbf{P} = \sqrt{\mathbf{P}}\sqrt{\mathbf{P}'} = \sqrt{\mathbf{P}'}\sqrt{\mathbf{P}}$$

A.6 Some Properties

Let $\mathbf{A}$ and $\mathbf{B}$ be matrices and c be a scalar

1. $(\mathbf{A}')' = \mathbf{A}$.

The operation of taking the transpose is an involution (self-inverse).

2. $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$.

The transpose respects addition

3. $(\mathbf{A}\mathbf{B})' = \mathbf{B}'\mathbf{A}'$.

The order of the factors reverses. From this one can deduce that a square matrix $\mathbf{A}$ is invertible if and only if $\mathbf{A}'$ is invertible, and in this case we have $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$. By induction, this result extends to the general case of multiple matrices, where it is found that $(\mathbf{A}_1\mathbf{A}_2...\mathbf{A}_{k-1}\mathbf{A}_k)' = \mathbf{A}_k'\mathbf{A}_{k-1}'...\mathbf{A}_2'\mathbf{A}_1'$.

4. $(c\mathbf{A}') = c\mathbf{A}'$

The transpose of a scalar is the same scalar. Together with (2), this states that the transpose is a linear map from the space of $m \times n$ matrices.

The determinant of a square matrix is the same as the determinant of its transpose.

5. If $\mathbf{A}$ has only real entries, then $\mathbf{A}'\mathbf{A}$ is a positive-semidefinite matrix.

6. $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$.

7. If $\mathbf{A}$ is a square matrix, then its eigenvalues are equal to the eigenvalues of its transpose, since they share the same characteristic polynomial.

Appendix B

Probability Concepts

In this appendix, we introduce some basic concepts of probability, which are explained in [Bertsekas, 2000].

B.1 Probability

Probability is the study of random or non-deterministic experiments. A probability space consists of:

1) A set Ω.

2) A collection $\mathcal{F}$ of subsets of Ω, called *events*, which includes Ω and has the following properties:

    a) If A is an event, then the complement $\overline{\mathrm{A}} = \{\omega \in \Omega \mid \omega \notin A\}$ is also an event (The complement of Ω is the empty set and is considered to be an event.)

    b) If $A_1, A_2, ..., A_k, ...$ are events, then $\cup_{k=1}^{\infty} A_k$ is also an event.

    c) If $A_1, A_2, ..., A_k, ...$ are events, then $\cap_{k=1}^{\infty} A_k$ is also an event.

3) A function $P(\cdot)$ assigning to each event A a real number $P(A)$, called the *probability of the event* A, and satisfying:

    a) $P(A) \geq 0$ for every event A.

    b) $P(\Omega) = 1$

    c) $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ for every pair of disjoint events A_1, A_2

d) $P(\cup_{k=1}^{\infty} A_k) = (\Sigma_{k=1}^{\infty} P(A_k)$ for every sequence of mutually disjoint events $A_1, \ldots, A_k, ...$

The function P is referred to as a *probability measure.*

Convention for Finite and Countable Probability Spaces

The case of a probability space where the set Ω is a countable set is encountered frequently. When Ω is specified as finite or countable, it is assumed that the associated collection of events is the collection of *all* subsets of Ω (including Ω and the empty set). Then, if Ω is a finite set, $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$, the probability space is specified by the probabilities $p_1, p_2, ..., p_n$, where p_i denotes the probability of the event consisting of just ω_i. Similarly, if $\Omega = \{\omega_1, \omega_2, ..., \omega_k, ...\}$, the probability space is specified by the corresponding probabilities $(p_1, p_2, ..., p_k, ...)$ as a *probability distributions over* Ω.

B.2 Random Variables

A *random variable* on a probability space $(\Omega, \mathcal{F}, P)$ is a function $x : \Omega \to \mathbb{R}$ such that for every scalar λ the set

$$\{\omega \in \Omega \mid x(\omega) \leqslant \lambda\} \tag{B.1}$$

is an event (i.e., belongs to the collection $\mathcal{F}$). An n-dimensional *random vector* $x = (x_1, x_2, ..., x_n)$ is an n-tuple of random variables $x_1, x_2, ..., x_n$, each defined on the same probability space.

The *distribution function* $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ (or *cumulative distribution function*- CDF) of a random variable x is defined by

$$F(z) = P\left(\{\omega \in \Omega \mid x(\omega) \leqslant z\}\right); \tag{B.2}$$

i.e., $F(z)$ is the probability that the random variable takes a value less than or equal to z. The distribution function $F : \mathbb{R}^n \to \mathbb{R}$ of a random vector $x = (x_1, x_2, ..., x_n)$ is defined by

$$F(z_1, z_2, ..., z_n) = P\left(\{\omega \in \Omega \mid x_1(\omega) \leqslant z_1, x_2(\omega) \leqslant z_2, ..., x_n(\omega) \leqslant z_n\}\right). \tag{B.3}$$

Given the distribution function of a random vector $x = (x_1, ..., x_n)$, the (marginal) distribution function of each random variable x_i is obtained from

$$F_i(z_i) = \lim_{z_j \to \infty, j \neq i} F(z_1, z_2..., z_n). \tag{B.4}$$

The random variables $x_1, ...x_n$ are said to be *independent* if

$$F(z_1, z_2, ..., z_n) = F_1(z_1)F_2(z_2)...F_n(z_n), \tag{B.5}$$

B.3 Expected Value

The **mean**, **expected value**, or **expectation** of a random variable is written as $E(X)$ or $\mu(x)$. Let X be a continuous random variable with p.d.f. $f_X(x)$. The expected value of X is

$$E(x) = \int_{-\infty}^{\infty} x(w) f_x(w)\, dw \tag{B.6}$$

The expectation value of a discrete random variable X taking the values $x_1, x_2, ...$ and with probability mass function p is the number

$$E(x) = \sum_i w_i P(x(w) = w_i) = \sum_i w_i p(w_i). \tag{B.7}$$

B.3.1 Expectation of $g(x)$

Let $g(x)$ be a function of x. Considering there is a long-term average of $g(x)$ as there is a long-term average of x. This average can be written as $E(g(x))$. Now, it has many observations of x to give results $w_1, w_2, ..., w_N$. Applying the function g to each of these observations, we have $g(w_1), ..., g(w_N)$. Then the mean of $g(w_1), g(w_2), ..., g(w_N)$ approaches $E(g(x))$ as the of observations N tends to infinity.

Let x be a continuous random variable, and let g be a function.The expected value of $g(x)$ is

$$E(g(x)) = \int_{-\infty}^{\infty} g(w) f_X(w)\, dw \tag{B.8}$$

Let x be a discrete random variable, and let g be a function. The expected value of $g(x)$ is

$$E(g(x)) = \sum_{w} g(w) f_x(w) = \sum_{w} g(w) P(x = w). \tag{B.9}$$

B.3.2 Properties of Expectation

i) Let g and h be functions, and let a ad b be constants. For any random variable x (discrete or continuous),

$$E\{ag(x) + bh(x)\} = aE\{g(x)\} + bE\{h(x)\}. \tag{B.10}$$

In particular,

$$E(ax + b) = aE(x) + b. \tag{B.11}$$

ii) Let x and y be any random variables. Then

$$E(x + y) = E(x) + E(y). \tag{B.12}$$

More generally, for any random variable $x_1, ...x_n$

$$E(x_1 + \cdots + x_n) = E(x_1) + \cdots + E(x_n). \tag{B.13}$$

iii) Let x and y be independent random variables, and g, h be functions. Then

$$E(xy) = E(x)E(y). \tag{B.14}$$

$$E(g(x)h(y)) = E(g(x))E(h(y)). \tag{B.15}$$

B.3.3 Probability of the Indicator Function

Let A be any event. We can write $\mathbb{P}(A)$ as an expectation, as follows.

Define the *indicator function*:

$$I_A = \begin{cases} 1, & \text{if event } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases} \tag{B.16}$$

Then I_A is a variable random,

$$\begin{aligned} E(I_A) = & \sum_{r=0}^{1} r\mathbb{P}(I_A = 1) \\ = & 0 \times \mathbb{P}(I_A = 0) + 1 \times \mathbb{P}(I_A = 1) \\ = & \mathbb{P}(I_A = 1) \\ = & \mathbb{P}(I_A) \end{aligned}$$

Thus,

$$\mathbb{P}(A) = E(I_A) \quad \text{for any event } A. \tag{B.17}$$

B.4 Conditional Probability

Considering the case where the underlying probability space Ω is a countable (possibly finite) set and the set events is the set of all subsets of Ω.

Given two events A and B, the *conditional probability of* B *given* A is defined by

$$P(B \mid A) = \left\{ \begin{matrix} \dfrac{P(A \cap B)}{P(A)}, & \text{if } P(A) > 0 \\ 0, & \text{if } P(A) = 0 \end{matrix} \right\} \tag{B.18}$$

The notation $P\{B \mid A\}$ is also used in place of $P(B \mid A)$. If $B_1, B_2, ...$ are a countable (possibly finite) collection of mutually exclusive and exhaustive events (i.e., the sets B_i

are disjoint and their union is Ω) and A is an event, then

$$P(A) = \sum_i P(A \cap B_i). \tag{B.19}$$

From the relations above, it is obtained the *total probability theorem*

$$P(A) = \sum_i P(B_i)P(A \mid B_i). \tag{B.20}$$

Thus, for every k

$$P(B_k \mid A) = \frac{P(A \cap B_k).}{P(A)} = \frac{P(B_k)P(A \mid B_k)}{\sum_i P(B_i)P(A \mid B_i)}; \tag{B.21}$$

assuming that $P(A) > 0$. This relations is referred to as *Bayes' rule*.

Considering two random vectors x and y taking values in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively (i.e., $x(\omega) \in \mathbb{R}^n$, $y(\omega) \in \mathbb{R}^m$ for all $\omega \in \Omega$).

For a fixed vector $v \in \mathbb{R}^n$, it is defined the *conditional distributions function* of x given v by

$$F(z \mid v) = P(\{\omega \mid x(\omega) \leq z\} \mid \{\omega \mid y(\omega) = v\}), \tag{B.22}$$

and the *conditional expectation* of x given v by

$$E\{x \mid v\} = \int_{\mathbb{R}^n} z \, dF(z \mid v), \tag{B.23}$$

assuming that the integral is well-defined. $E\{x \mid v\}$ is a functions mapping v into $\mathbb{R}^n$.

Now, for random vectors, if $\omega_1, \omega_2, ...$ are the elements of Ω, denote as

$$z_i = x(\omega_i), \quad v_i = y(\omega_i), \quad i = 1, 2... \tag{B.24}$$

Also for any vectors $z \in \mathbb{R}^n, v \in \mathbb{R}^m$, denoting

$$P(z) = P(\{\omega \mid x(\omega) = z\}), P(v) = P(\{\omega \mid y(\omega) = v\}). \tag{B.25}$$

We have $P(z) = 0$ if $z \neq z_i$, $i = 1, 2, ...$, and $P(v) = 0$ if $v \neq v_i$, $i = 1, 2, ..$

Denote also,

$$P(z \mid v) = P(\{\omega \mid x(\omega) = z\} \mid \{\omega \mid y(\omega) = v\}), \tag{B.26}$$

$$P(v \mid z) = P(\{\omega \mid x(\omega) = v\} \mid \{\omega \mid y(\omega) = z\}). \tag{B.27}$$

Then, for all $k = 1, 2, ...,$, Bayes' rule yields

$$P(z_k \mid v) = \begin{cases} \dfrac{P(z_k)P(v \mid z_k)}{\sum_i P(z_i)P(v \mid z_i)}, & \text{if } P(v) > 0 \\ 0, & \text{if } P(v) = 0. \end{cases} \tag{B.28}$$

B.5 Summary of the main properties of the expectation

Let x, y, x be random variables, $a, b \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Assuming all the following expectations exist, we have:

a) $E(a \mid y) = a$.

b) $E(ax + bz \mid y) = aE(x)(y) + bE(z \mid y)$.

c) $E(x)(y) \leq 0$ if $x \leq 0$.

d) $E(x \mid y) = E(x)$ if x and y are independent.

e) $E(E(x \mid y)) = E(x)$.

f) $E(xg(y) \mid y) = g(y)E(x|y)$. In particular, $E(g(y) \mid y) = g(y)$.

g) $E(x \mid y, g(y)) = E(x \mid y)$.

h) $E(E(x \mid y, z) \mid y) = E(x)(y)$.

Appendix C

Controllability and Observability

In this part, 2 fundamental concepts in the control theory are introduced: controllability and observability [Callier and Desoer, 1999].

C.1 Controllability

Considering a non-homogeneous system:

$$x(k+1) = Ax(k) + Bu(k). \tag{C.1}$$

where $B \in \mathbb{B}(\mathbb{R}^m, \mathbb{R}^n)$ and $u(k) \in \mathbb{R}^m$ is a vector of inputs to the system.

The concept of Controllability is related to the following idea: for a certain pair (A, B), is it possible to apply a sequence of $u(k)$ that drive the system from any $x(0)$ to a specified final state x_f in a finite time. Hence, it can be defined controllability as:

Definition C.1 ***(Controllability)*** The pair (A, B) is said to be controllable, if for any $x(0)$ and any given final state x_f, there exists a finite positive integer T and a sequence of inputs $u(0), u(1), ...u(T-1)$ that, applied to system C.1, yields $x(T) = x_f$.

The following theorem can be used to establish if a system is controllable.

Theorem C.2 *The following assertions are equivalent.*

i) The pair (A, B) is controllable.

ii) The following $n \times nm$ matrix (called a controllability matrix) has a rank n:

$$(B\ AB \cdots A^{n-1}B). \tag{C.2}$$

iii) The controllability Grammian $S_c \in \mathbb{B}(\mathbb{R}^n)$ *given by*

$$S_c(k) = \sum_{i=0}^{k} A^i BB'(A')^i \tag{C.3}$$

is nonsingular for some $k < \infty$.

iv) For A and B real, given any monic real polynomial ψ of degree n, there exists $F \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ such that $det(sI - (A + BF)) = \psi(s)$.

The proof of this theorem can be found in [Callier and Desoer, 1999]).

Moreover, if $r_\sigma(A) < 1$ then the pair (A, B) is controllable if and only if the unique solution (S_c) of $(S = ASA' + BB')$ is positive-definite.

Item *iv* of Theorem C.2 involves the *state feedback*, then, supposing that for some $F \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, it can be applied $u(k) = Fx(k)$ in system C.1, yielding

$$x(k+1) = (A + BF)x(k), \tag{C.4}$$

According to Theorem C.2, an adequate choice of F (for A and B and F real) allows to perform pole placement for the closed loop system $(A + BF)$. For instance, it can be use state feedback to stabilize an unstable system.

C.2 Observability

Definition C.3 ***(Observability)*** The pair (L, A) is said to be observable, if exists a finite positive integer T such that knowledge of the outputs $y(0), y(1), ..., y(T-1)$, is sufficient to determine the initial state $x(0)$.

Observability allows to answer if is it possible to infer the internal behavior of a system by observing its outputs.

The following theorem is dual to Theorem C.2

Theorem C.4 *The following assertions are equivalent.*

i) The pair (L, A) is observable.

ii) The following $pn \times n$ matrix (called an observability matrix) has a rank n:

$$\begin{pmatrix} L \\ LA \\ \dots \\ LA^{n-1} \end{pmatrix} \tag{C.5}$$

iii) The observability Grammian $S_o \in \mathbb{B}(\mathbb{R}^n)$ *given by*

$$S_o(k) = \sum_{i=0}^{k} (A')^i L' L A^i \tag{C.6}$$

is nonsingular for some $k < \infty$.

iv) For A and L real, given any monic real polynomial ψ of degree n, there exists $K \in \mathbb{B}(\mathbb{R}^p, \mathbb{R}^n)$ such that $det(sI - (A + KL)) = \psi(s)$.

For the proof, see [Callier and Desoer, 1999].

Moreover, if $r_\sigma(A) < 1$ then the pair (L, A) is observable if and only if the unique solution (S_o) of $(S = A'SA + L'L)$ is positive-definite.