FREDERIK STADTMANN

Control of Markov Jump Linear Systems with uncertain detections

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Tese apresentada à Escola Politécnica da Universidade de São Paulo para obtenção do Título de Doutor em ciências.

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Oswaldo L. V. Costa

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RESUMO

Esta monografia aborda problemas de controle e filtragem em sistemas com saltos espontâneos que alteram seu comportamento e cujas mudanças são detectadas e estimadas por um detector imperfeito. Mais precisamente, consideramos sistemas lineares cujos saltos podem ser modelados usando um processo markoviano (Markov Jump Linear Systems) e cujo modo de operação corrente é estimado por um detector. O detector é considerado imperfeito tendo em vista a possibilidade de divergência entre o modo real de operação e o modo de operação detectado. Ademais, as probabilidades das detecções são consideradas conhecidas. Assumimos que o detector possui uma dinâmica própria, o que significa que o modo de operação detectado pode mudar independentemente do modo real de operação. A novidade dessa abordagem está na modelagem das incertezas. Um processo oculto de Markov (HMM) é usado para modelar as incertezas introduzidas pelo detector. Para esses sistemas, os seguintes problemas são abordados: i) estabilidade quadrática ii) controle H_2, iii) controle H_∞ e iv) o problema da filtragem H_∞. Soluções baseadas em Desigualdades de Matriciais Lineares (LMI) são desenvolvidas para cada um desses problemas. No caso do problema de controle H_2, a solução minimiza um limite superior para a norma H_2 do sistema de controle em malha fechada. Para o problema H_∞ -controle é apresentada uma solução que minimiza um limite superior para a norma H_∞ do sistema de controle em malha fechada. No caso da filtragem H_∞, a solução apresentada minimiza a norma H_∞ de um sistema que representa o erro de estimativa. As soluções para os problemas de controle são ilustradas usando um exemplo numérico que modela um processo simples de dois tanques.

Palavras-Chave – controle estocástico, controle ótimo, filtragem, sistemas markovianos.

ABSTRACT

This monograph addresses control and filtering problems for systems with sudden changes in their behavior and whose changes are detected and estimated by an imperfect detector. More precisely it considers continuous-time Markov Jump Linear Systems (MJLS) where the current mode of operation is estimated by a detector. This detector is assumed to be imperfect in the sense that it is possible that the detected mode of operation diverges from the real mode of operation. Furthermore the probabilities for these detections are considered to be known. It is assumed that the detector has its own dynamic, which means that the detected mode of information can change independently from the real mode of operation. The novelty of this approach lies in how uncertainties are modeled. A Hidden Markov Model (HMM) is used to model the uncertainties introduced by the detector. For these systems the following problems are addressed: i) Stochastic Stabilizability in mean-square sense, ii) H_2 control, iii) H_∞ control and iv) the H_∞ filtering problem. Solutions based on Linear Matrix Inequalities (LMI) are developed for each of these problems. In case of the H_2 control problem, the solution minimizes an upper bound for the H_2 norm of the closed-loop control system. For the H_∞ control problem a solution is presented that minimizes an upper bound for the H_∞ norm of the closed-loop control system. In the case of the H_∞ filtering, the solution presented minimizes the H_∞ norm of a system representing the estimation error. The solutions for the control problems are illustrated using a numerical example modeling a simple two-tank process.

Keywords – Stochastic Control, Optimal Control, Filtering, Markov Processes.

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

Symbol	Description
$\boldsymbol{x}$	State vector of a system
$\dot{\boldsymbol{x}}$	Derivative of the state variable
$\boldsymbol{y}$	Output vector
$\boldsymbol{w}$	Disturbance vector
$\boldsymbol{z}$	Vector of the controlled output
$\hat{\boldsymbol{x}}$	Estimated state
$\boldsymbol{B}$	Input gain matrix
$\boldsymbol{C}$	Output gain matrix
$\boldsymbol{D}$	Feedthrough gain matrix for the input
$\boldsymbol{F}$	Feedthrough matrix for the disturbance
$\boldsymbol{A}$	State matrix
$\boldsymbol{H}$	Disturbance matrix
$\boldsymbol{\Pi}$	Transition-rate matrix
$\boldsymbol{K}$	Controller matrix
$\boldsymbol{I}$	Definition of the Identity Matrix
$\boldsymbol{J}$	Input-disturbance Matrix
$\tilde{\boldsymbol{A}}$	State matrix of the closed loop system
$\tilde{\boldsymbol{C}}$	State-Output matrix of the closed-loop system
θ	Mode of operation of the system
$\hat{\theta}$	Estimated mode of operation
K	Controller
Σ	Symbol for a dynamical system
tr	Symbol for the trace operator
Her	$\mathrm{Her}(A) = A + A'$
E	Mathematical expectation
$\mathbb{L}$	Perturbance Operator
$\mathscr{N}$	Set of the hidden Markov states
$\mathscr{M}$	Set of the observed Markov states
$\mathscr{K}$	Set of possible controllers

ACRONYMS

Akronym	Definition
ASS	Almost Sure Stability
CT-HMM	Continuous time hidden Markov model
FDI	Fault Detection and Isolation
FTC	Fault Tolerant Control
HMM	Hidden Markov Model
iMSS	Internal Mean-Square-Stability
LMI	Linear Matrix Inequality
MJLS	Markov Jump Linear System
MSS	Mean-Square Stability
PC	Piecewise continuous

CONTENTS

1 INTRODUCTION

1.1 Motivation

Classical control theory usually considers the dynamic system as time invariant, however, for real systems this is a very restrictive consideration, as the system's properties might change over time. Reasons for these changes can be grouped as follows:

Changes in the environment Many systems depend on external variables which are beyond the influence of the control system. Examples are wind and solar power systems, flotation processes in the mining industry or economic models. All these processes depend on external variables like the availability of sun and wind, the mineral concentration in the ore or the economic situation of some nation. In these cases the setpoint of the controller or even the whole control strategy has to be adjusted according to these external influences [8, 48, 92, 28, 52, 34, 86].

Faults Another reason for changes in systems behavior are faults. A fault is defined as an unpermitted deviation of the expected behavior of a system. Faults can occur in the sensors, the actuators or the system itself. Examples for faults in the system are ruptures in tubulation, tanks etc.

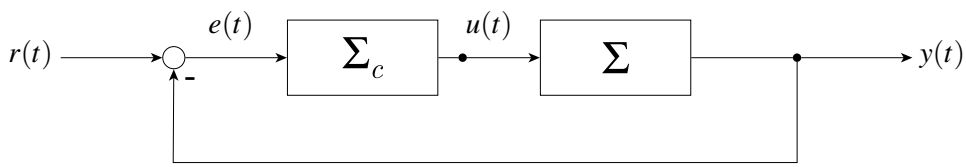
Especially for safety-related systems like industrial plants [20, 56] or aerospace systems [6, 46, 32] it is important to ensure that faults in the systems or altered environmental conditions do not lead to dangers for the environment and people. Other systems are either completely inaccessible like a Mars lander [50] or inaccessible at certain instants like offshore wind turbines in the European winter [41]. In these cases it is desirable that a fault not lead to a complete loss of the system.

Attacks on the system A relatively new reason for a changed behavior of a technical system are attacks on a cyber-physical system where an intruder alters the behavior of a system to cause damage or losses. The most known incident in this category is the stuxnet attack [57].

For many systems it is important that the system maintain an acceptable behavior and meet some performance requirements even in the presence of these changes. Various approaches have been developed to deal with these changes. See for example [9] for an introduction to fault tolerant systems, [58] for switching systems, [17, 16] for Markov Jump Linear Systems, and for robust control see for example [60, 101, 7, 1, 4, 87].

In general the approaches can be divided into two classes: active systems and passive systems. Passive systems are also categorized as robust control. Methods for robust control result in a controller that guarantees an acceptable behavior even under certain variations of operational conditions. As this leads to a trade-off between robustness and performance, the performance in nominal operation is usually suboptimal. Furthermore the range of allowed variations is smaller (see for instance [9]). One advantage of this is that they do not need any additional resources and stability of the overall system can be guaranteed as long as alterations stay within certain bounds. The structure is shown in Figure 1.

Figure 1: Block diagram of the robust feedback system

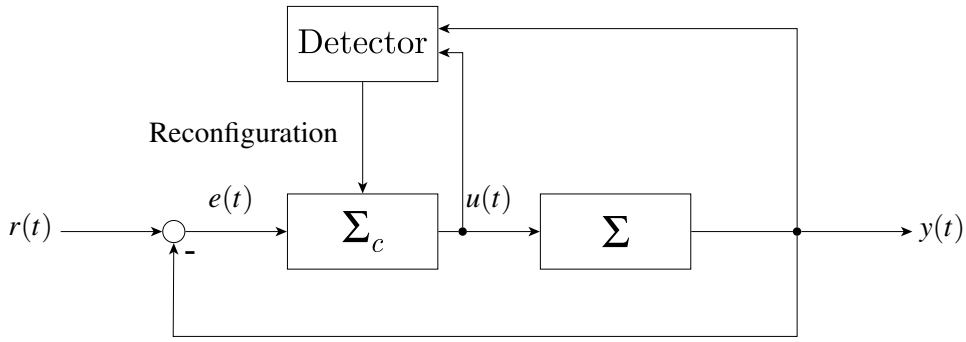


Source: Author

Active approaches address these drawbacks by a reconfiguration of the controller after the detection and identification of a change has taken place. A prerequisite for this is the existence of an additional supervisional layer to detect these changes and prompt a reconfiguration of the controller. The structure is depicted in Figure 2. Hence active approaches come at the cost of additional hardware and energy demands. Furthermore, most of them only guarantee the stability of certain controller/plant combinations, leaving the possibility of delayed detection and possible misdetections aside. However, even the stability of all controller/plant combinations does not necessarily ensure the stability of the complete system [17].

For systems where the occurrence of changes can be modeled by a Markov chain or a Markov process, the theory of Markov Jump Linear Systems (MJLS) combines the advantages of both active and passive approaches by providing an active adjustment of the controller while guaranteeing stability of the overall system. The active adjustment leads to a better performance and a broader range of possible dynamic variations. But until now the theory of MJLS does not provide readily usable tools to include the possibility of misdetections and false alarms into the

Figure 2: Block diagram of the feedback system



Source: Author

systems model. This text deals with these cases where the changes can be modeled using a Markov process.

A formal introduction of the system follows in the next section. For an active reconfiguration, the system depends on the presence of a detector which provides information about the current state of the system. Usually (e.g. [87, 76, 3]) it is assumed that the detector is perfect and thus the information about the nature of the fault always corresponds to the state of the real system. However, studies have shown that this assumption cannot be fulfilled by a real system. For example [68, 73] have shown that there exists a considerable detection delay as well as a high number of false detections in current algorithms concluding that the evaluated techniques need improvements to be useful. Even with more sophisticated methods it is unlikely that the detection delay is zero and that there are no false alarms or detections.

Hence it is necessary to consider the possibility of a false detection of a fault, and to develop methods capable of dealing with this uncertainty.

1.2 Contributions and Structure

This monograph is motivated by the fact that fault tolerant control systems in these days are usually not capable of guaranteeing the stability of the overall system in the presence of false alarms and misdetections. This work approaches the problem by providing a readily implementable solution which guarantees stochastic stability for systems where the dynamic of the fault occurrence can be modeled by a Markov Process (Markov Jump Systems) and its detection can be modeled by a Hidden Markov Process. Chapter 5 provides a condition for mean-square-stability based on linear matrix inequalities (LMI) which can be easily solved using a modern computer and common software. The solution encompasses those special cases which were

discussed in [88, 64, 63, 2]. Section 6.1 extends the LMI-conditions in a way, that the solution guarantees an upper bound on the H_2 cost. The case of H_∞ optimal control is presented in Chapter 7. This Chapter follows the ideas presented in Section 6.1 and [89], extend them for H_∞ control and considers static output feedback rather than state-feedback. This results in a new LMI condition for this control case. The novelty and main difference to [81] lies in the model representing the detector as well as considering the static output feedback case. The probability distribution of the joint process, defined by the detection process and the Markov parameter, is considered to be an exponential hidden Markov process so that the time evolution of the process is well defined and can be easily simulated. Chapter 8 presents results for the H_∞ filtering problem. Again the difference to [78] lies in the model representing the detector. As before the solution of the problem is given as an LMI condition. This monograph is concluded with Chapter 9 which presents a summary of the results obtained, and discusses directions for further research.

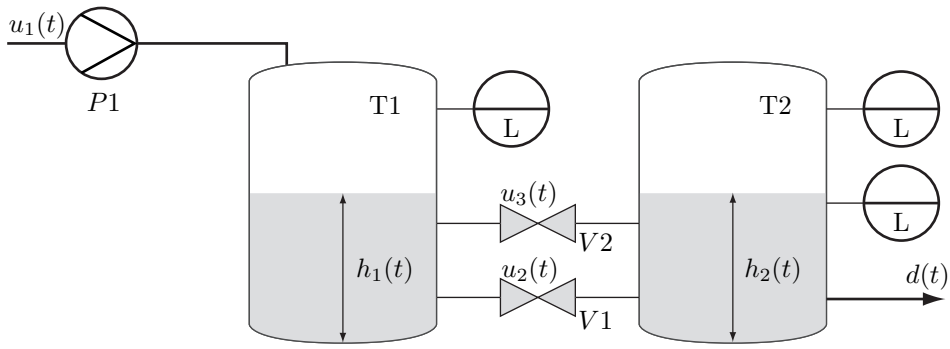
The results presented in Chapter 5 and 6 were partially presented at the 2015 IEEE Conference on Descision and Control [91] and published in the IEEE Transactions on Automatic Control [89]. As for the results in Chapter 7, they were presented at the 2018 IEEE Conference on Decision and Control and published in the IEEE Control Systems Letters [90]. Finally the results presented in Chapter 8 are currently in preparation to be submitted.

1.3 Running Example

This section introduces the example which will be used throughout this monograph for the numerical evaluation of the results.

Example 1. *This example uses a model of a two-tank system and is adapted from [75]. The plant is shown in Figure 3. It consists of two tanks T_1 and T_2 which are connected by two valves V_1 and V_2. The valves are controlled by the signals $u_2(t)$ and $u_3(t)$ respectively. Additionally the tank T_1 can be filled via pump P_1 ($u_1(t)$) and the drain of the tank T_2 is used as a disturbance $d(t)$. It is assumed that all actuators exhibit a linear characteristic and that they can be controlled continuously. Furthermore it is assumed that any dynamic behavior of the actuators can be neglected. Three sensors are included in the model, one continuous level-sensor for tank T_1 (y_1) and two continuous level-sensors for the tank T_2. While the sensor at tank T_1 does not show any dynamic behavior, both sensors installed at tank T_2 are modeled by a first order delay with the states s_1 and s_2. The state vector is given by: $x = (v_1\, v_2\, h_1\, h_2\, s_1\, s_2)'$ where v_1 and v_2 are the flows in valve one and two respectively, h_1 and h_2 refer to the height of the fluid in the tanks while s_1 and s_2 refer to the sensor readings in tank T_2. The input vector consists of the control*

Figure 3: Flow diagram of the two-tanks system



Source: Author

signal for the pump and the control signals for the two valves. The nonlinear model of this plant is discussed in more detail in [74]. The plant is operated around the following operating point:

$$\bar{x} = \begin{pmatrix} 0.74 & 0.2 & 0.4 & 0.06 & 0.06 & 0.06 \end{pmatrix}'$$
$$\bar{u} = \begin{pmatrix} 0.5 & 0.78 & 0.2 \end{pmatrix}'$$

Linearizing the nonlinear model of the system at this point leads to a linear system with the system matrix given by:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3.2 & -3.4 & -7.1 & 3.6 & 0 & 0 \\ 3.2 & 3.4 & 7.1 & -18 & 0 & 0 \\ 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 1/0.3 & 0 & -1/0.3 \end{bmatrix}.$$

The input matrix depends on the mode of operation θ which will be defined later in more detail. The mode-dependent matrix is given by:

$$B_\theta = \begin{bmatrix} 0 & b_{12} & 0 \\ 0 & 0 & 0.2 \\ 8.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.1}$$

It is assumed that the lower valve V_1 of the system is subject to faults which alter the behavior of the valve. Three possible scenarios are considered:

- *The valve is undamaged, resulting in ($b_{21} = 1$).*
- *Opening reduced to 25% of the desired value ($b_{21} = 0.25$).*
- *The valve is stuck in the closed position and cannot be opened ($b_{21} = 0$).*

The occurrence of these three scenarios (modes of operation) is modeled by a continuous-time Markov Process with the transition-rate matrix given by:

$$\boldsymbol{\Pi} = \begin{bmatrix} -0.0894 & 0.0671 & 0.0223 \\ 0.0671 & -0.0671 & 0 \\ 0.0236 & 0 & -0.0236 \end{bmatrix} \tag{1.2}$$

This model is used in the following chapters to evaluate the obtained results numerically.

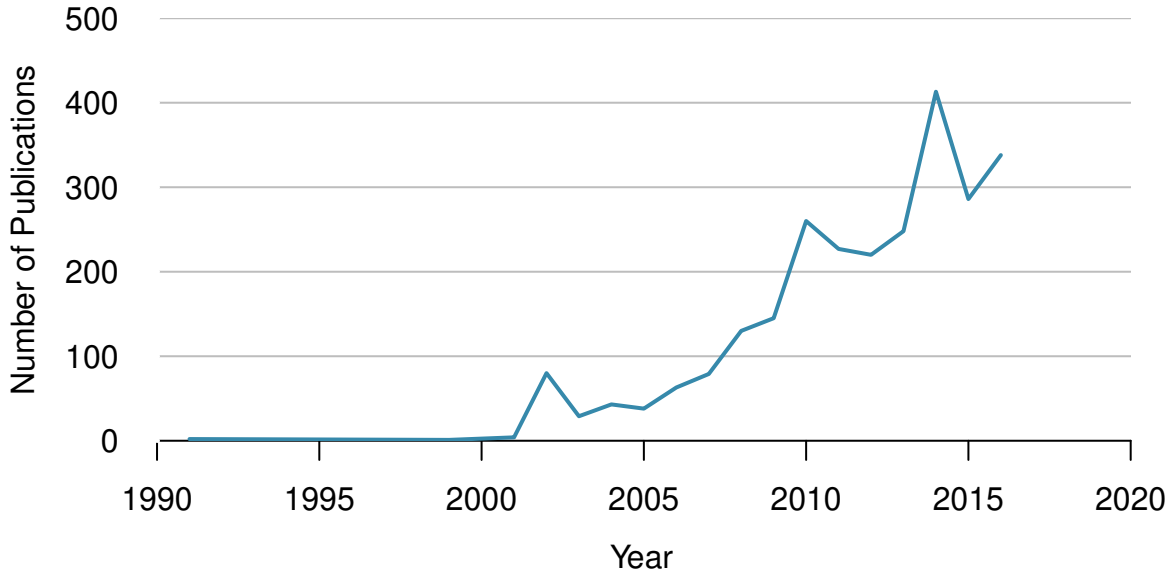
2 LITERATURE SURVEY

2.1 Markov Jump Linear Systems

Markov Jump Linear Systems (MJLS) emerged in the middle of the 20th century as a special case of stochastic differential and difference equations [49, 38]. Research of MJLS gained significant momentum in the early 2000s. Figure 4 shows the development of articles published by the most important publishers which tagged as MJLS (see section A for details). Since then a mature body of system theory was developed, including various problems of stability and optimal control which have been intensively studied. The texts [17, 11, 16, 33, 31, 65, 86] and references therein provide a general overview on MJLS including these topics.

In the early 90s [64] was one of the first considering the possibility of detection delays, for the mode of operation, but the solution was presented in form of a set of coupled Lyapunov equations and hence not readily implementable. Later [88] also took the possibility of false detections into account, but the solution was still given as a Lyapunov function with the known problems. Uncertainty about the state-variable was introduced by [33] who provided a solution in form of coupled Ricatti equations. A more feasible approach was presented in [30], where an LMI condition is presented for discrete-time systems. Moreover the cluster-case was considered in this work, in which the non-observed part of the Markov states is grouped into a number of clusters of observations. A more general model for a discrete-time MJLS was analyzed in [18] where an LMI condition for the stabilization and H_2-optimal control of discrete-time Markov jump linear system was presented. This solution includes the special cases of perfect information, no information and cluster detection. For the perfect information the results recast the usual ones for the H_2 control of discrete-time MJLS as presented in [16]. Finally, [80] presented a bounded real lemma for continuous-time MLJS but under a different hypothesis on the joint process $Z(t) = (\theta(t), \widehat{\theta}(t))$. A similar model using two separate Markov Processes to represent the failure and the FDI process was presented in [63]. Recently similar ideas have been presented in the field of cyber-physical systems to model and detect attacks [84].

Figure 4: Development of the number of publications tagged with MJLS



Source: Author, see appendix A for details.

In order to analyze the imperfect detection case, [89] considered the H_2-control problem, using a hidden Markov model to represent the failure process as well as the detection process. For the H_∞-control problem and also bearing in mind the case of imperfect detections, [2] introduced LMI solutions for robust output feedback control for uncertain systems. The approach uses three separable Markov processes to represent actuator failure, component failure and the FDI process. While this is a very flexible approach, the authors point out that it is difficult to determine the values of the transition rates of the FDI process. Later [81] provided a state-feedback result for the H_∞-control of systems with uncertain detections, considering a continuous-time version of [18] for the detection of the mode of operation. The detector is based on a continuous-time probabilistic Markov type assumption. Note that this assumption in general does not allow to define the distribution of the joint process which is formed by the information coming from the detector and the Markov parameter. This may lead to practical problems concerning the detectability and the ability to implement the method.

The filtering problem was discussed for example in [17] and [93]. Later [78, 77] provided results for continuous time Markov Jump Linear Systems where the mode of operation is subject to uncertainties. As in case of the control problems, the main difference to the monograph in hand lies in the way the uncertainties are modeled.

Remark 2.1. *Some authors [66, 96] emphasize the possibility that the mode of operation and the observed mode are different and call this asynchronous control or asynchronous filtering.*

2.2 Uncertainties

Uncertainties in Markov Jump Linear Systems

As mentioned before there exists a quite comprehensive literature on optimal control of MJLS for the case where the current state of the Markov process (mode of operation) is perfectly known. However, as mentioned before, for real-world applications in fault tolerant control it is necessary to acknowledge that this information is subject to uncertainty. Results for this scenario are more scarce. Table 1 gives an overview on the publications and the uncertainties which were considered. Possible uncertainties in MJLS can be divided into four classes:

Uncertainty about the state variable This refers to the usual case where at least a subset of the state-variables is not measurable.Works which consider this case are for example [33, 10, 19, 22]

Uncertainty about the systems parameters In the case of linear systems this means that the system matrices are subject to uncertainty and may differ from the real parameters. Works which consider this case are for example [3, 2]

Uncertainties in the transition matrix In this case some of the transition probabilities are not exactly known. Examples were given in [17].

Uncertainty about the mode of operation In this case the information about the current mode of operation is unknown or probably wrong. This implies that the chosen controller for the system is probably not the one which was designed for the current mode of operation. An example for the occurrence of this scenario is the case in which the system is subject to a fault which was not (yet) detected by the supervisory system. Considering this uncertainty makes it possible to guarantee stability of the overall system even in case of possible false alarms and wrong detections of the FDI. This makes this case especially interesting for fault-tolerant control. This case was studied in [89, 18, 80, 78, 40, 39]

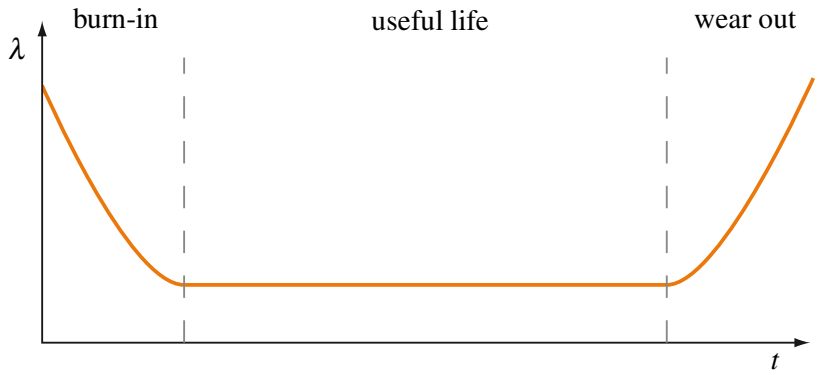
It should be noted that mixed cases are also possible and common and should be taken into consideration.

Ref	Class		Uncertainty				Cases			Control		Filtering			
	Disc.	Cont.	Mode	State	TM	Struct	Perfect	Cluster	No	H_2	H_∞	$l_2 - l_\infty$	H_2	H_∞	Techniques
[2]	x	✓	x	x	x	✓	x	x	x	x	✓	x	x	x	
[18]	✓	x	✓	x	x	x	✓	✓	✓	✓	x	x	x	x	LMI
[33]	x	✓	x	✓	x	x	x	x	x	x	x	x	x	x	Coupled Riccati equations
[64]	x	✓	✓	x	x	x	x	x	x	x	x	x	x	x	Lyapunov Condition
[83]	x	✓	x	x	✓	x	✓	x	✓	✓	x	x	x	x	LMI
[88]	x	✓	✓	x	x	x	✓	✓	x	x	x	x	x	x	Lyapunov condition
[30]	✓	x	✓	x	x	x	✓	✓	x	✓	x	x	x	x	LMI
[81]	x	✓	✓	x	x	x	✓	✓	✓	x	✓	x	x	x	LMI
[100]	✓	x	✓	x	x	x	x	x	x	x	x	x	✓	x	LMI
[96]	✓	x	✓	x	x	x	x	x	x	x	x	x	x	x	LMI
[62]	x	x	x	x	x	x	x	x	x	x	x	x	x	✓	
[98]	✓	x	x	x	x	x	✓	x	x	x	x	x	x	✓	LMI
[27]	x	✓	✓	x	x	x	x	x	x	x	x	x	x	✓	LMI
[69]	x	x	✓	x	x	x	x	x	x	x	✓	x	x	x	LMI
[25]	✓	x	✓	x	x	x	x	x	x	x	x	x	x	✓	LMI
[36]	x	x	x	x	x	x	x	x	x	x	x	x	x	x	
[78]	x	✓	✓	x	x	x	x	x	x	x	✓	x	x	✓	LMI
[26]	x	✓	x	x	x	x	✓	x	x	x	x	x	x	✓	LMI
[77]	x	✓	x	x	x	x	✓	✓	x	x	x	x	x	✓	LMI

Table 1: Uncertainties in MJLS Literature

2.3 The stochastic nature of faults and other changes

Figure 5: The bathtub curve, showing a typical failure rate $\lambda(t)$



Source: Author

As will be later discussed, the stochastic nature of the changes is important for the approach detailed in this monograph. This section discusses justifications for the use of different distributions. In case of faults [51] takes the well known bathtub curve (see Figure 5) as a basis and argues that during the period of useful life the Mean-Time To Failure (MTTF) is constant and hence it is justifiable to use an exponential function to model the reliability.

However, [55] debates whether the bathtub curve is justifiable in the burn-in phase and concludes that most of the references do not provide evidence for the existence of the bathtub curve. He also traces the bathtub curve back to [47] stating that its real origin is unknown [61] also concludes that the bathtub curve is not justified in most of the cases but points out that there are some cases where the bathtub curve can be justified.

Also justifying the exponential function, [63] argues that the occurrence of faults is rare and therefore, according to the law of rare events [71], a Poisson process is the best description for this behavior.

Extensive data is available for faults in hard disks, [70] found that after the first fault the probability of another failure within the next 60 days is over 21 times higher. Hence the Markov-Property is not given here. The authors do not discuss an appropriate distribution. [82] gathered data from more than 100 000 disks, stating that the exponential distribution is not suitable to model the time between failures and suggesting the use of two parameter distributions like the Weibull distribution. Appendix B discusses data available from the storage company Backblaze, the data analyzed gives no clear image, but an exponential function can be ruled out.

For weather models, [95] discusses both history and suitability of different stochastic models, including first and higher level Markov models. The authors conclude that for some applications Markov models are a suitable fit.

As for economic scenarios, Markov models are a common way of modeling, see for example [34] and [53].

Taking all of the aforementioned arguments into account, one can conclude that there are many instances in which an exponential function is adequate to model the changes that were previously discussed. Therefore the exponential function and subsequent MJLS will be studied in this thesis.

3 PRELIMINARIES

3.1 Notation

Spaces

Banach space For the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, the Banach space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$ is written as $\mathbb{B}(\mathbb{X},\mathbb{Y})$. The uniform induced norm is represented by $\|.\|$ and, for simplicity, $\mathbb{B}(\mathbb{X}) := \mathbb{B}(\mathbb{X},\mathbb{X})$. The spectrum of the operator $\mathscr{T} \in \mathbb{B}(\mathbb{X})$ is denoted by $\sigma(\mathscr{T})$ and $\mathrm{Re}(\lambda(\mathscr{T})) := \sup\{\mathrm{Re}(\lambda); \lambda \in \sigma(\mathscr{T})\}$. Where Re() denotes the real part of a complex number.

Hilbert space In the Hilbert space $\mathbb{X}$ $\langle .;. \rangle$ stands for the inner product, and for $\mathscr{T} \in \mathbb{B}(\mathbb{X})$, $\mathscr{T}^*$ indicates the adjoint operator of $\mathscr{T}$. For operators $\mathscr{T} \in \mathbb{B}(\mathbb{X})$ and symmetric matrices $\boldsymbol{B} = \boldsymbol{B}'$ the expression $\mathscr{T} \succ 0$ $(\mathscr{T} \succeq 0)$ defines positive (semi) definiteness.

Euclidean space The n-dimensional real Euclidean space is denoted by $\mathbb{R}^n$, the interval $[0,\infty)$ by $\mathbb{R}^+$ and the real part of a complex number z by $\mathrm{Re}(z)$. The normed bounded linear space of all $m \times n$ real matrices is denoted by $\mathbb{B}(\mathbb{R}^n,\mathbb{R}^m)$, with $\mathbb{B}(\mathbb{R}^n) := \mathbb{B}(\mathbb{R}^n,\mathbb{R}^n)$, and $\mathbb{B}(\mathbb{R}^n)^+ := \{L \in \mathbb{B}(\mathbb{R}^n); L = L' \geq 0\})$.

Consider N and M positive integers. Define $\mathbb{H}^{n,m}$ as the linear space made up of all sequence of NM matrices $\mathbf{V} = \{V_{ik}; i = 1,\ldots,N, k = 1,\ldots,M\}$ with $\boldsymbol{V}_{ik} \in \mathbb{B}(\mathbb{R}^n,\mathbb{R}^m)$. For simplicity, set $\mathbb{H}^n := \mathbb{H}^{n,n}$ and $\mathbb{H}^{n+} := \{\mathbf{V} = \{\boldsymbol{V}_{ik}\} \in \mathbb{H}^n; \boldsymbol{V}_{ik} \geq 0, i = 1,\ldots,N, k = 1,\ldots,M\}$ and write, for $\mathbf{V} \in \mathbb{H}^n$ and $\mathbf{S} \in \mathbb{H}^n$, that $\mathbf{V} \geq \mathbf{S}$ if $\mathbf{V} - \mathbf{S} = \{\boldsymbol{V}_{ik} - \boldsymbol{S}_{ik}\} \in \mathbb{H}^{n+}$, and that $\mathbf{V} > \mathbf{S}$ if $\boldsymbol{V}_{ik} - \boldsymbol{S}_{ik} > 0$ for each $i = 1,\ldots,N$, $k = 1,\ldots,M$. For $\mathbf{V},\mathbf{S} \in \mathbb{H}^{n,m}$, we consider the following inner product in $\mathbb{H}^{n,m}$:

$$\langle \mathbf{V}; \mathbf{S} \rangle = \sum_{i=1}^{N} \sum_{k=1}^{M} \mathrm{tr}\left(\boldsymbol{V}_{ik}' \boldsymbol{S}_{ik}\right), \tag{3.1}$$

Matrices

Throughout this monograph the following notation is applied: matrices are denoted with bold upper-case letters $\boldsymbol{A}$, vectors are denoted with lower-case bold letters: $\boldsymbol{v}$. The transpose of a matrix $\boldsymbol{A}$ is represented by $\boldsymbol{A}'$. A Hermetian matrix is a matrix where $\boldsymbol{A} = \boldsymbol{A}^*$ holds, with $\boldsymbol{A}^*$ being the conjugate transpose. For a Hermetian matrix $\boldsymbol{A}$ the following notations are defined: $\boldsymbol{A} \prec 0$ ($\boldsymbol{A} \preceq 0$) stands for a negative (semi-) definite matrix. This means that all eigenvalues of this matrix are strictly negative (or null). The symbol $\succ$ ($\succeq$) stands for positive (semi-) definiteness, which means that the eigenvalues are strictly positive (or null). For square matrices $\boldsymbol{A}$ the following operators are defined:

Definition 3.1 (trace)**.**

$$\mathrm{tr}(\boldsymbol{A}) = \sum_{v=1}^{n} a_{vv}$$

with n being the dimension of the matrix and a_{ij} being the element of the matrix in the i-th row and j-th column.

Definition 3.2 (her)**.**

$$\mathrm{Her}(\boldsymbol{A}) = \boldsymbol{A} + \boldsymbol{A}'.$$

Symmetric block matrices are abbreviated using $\bigstar$:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}' & \boldsymbol{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \bigstar & \boldsymbol{C} \end{bmatrix}$$

Many problems in control theory can be expressed as linear matrix inequalities, which then can be easily solved using computational methods and packages [94, 13]. A linear matrix inequality (LMI) is defined as follows:

Definition 3.3 (LMI [13])**.** *With $\boldsymbol{F}_i = \boldsymbol{F}_i' \in \mathbb{R}^n, i = 0,\ldots,m$ and $\boldsymbol{x} = (x_1,\ldots,x_m)' \in \mathbb{R}^m$ as a decision variable, the inequality*

$$\sum_{i=1}^{m} x_i \boldsymbol{F_i} \succ \boldsymbol{0}$$

is called linear matrix inequality.

For linear matrix inequalities of block matrices the following assertions which are also known as Schur Complement, are equivalent [13]:

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \bigstar & \boldsymbol{R} \end{bmatrix} \prec 0$$

$$\boldsymbol{Q} \prec 0 \text{ and } \boldsymbol{R} - \boldsymbol{S}'\boldsymbol{Q}^{-1}\boldsymbol{S} \prec 0$$

$$\boldsymbol{R} \prec 0 \text{ and } \boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}' \prec 0$$

This also holds for the case where $\prec$ is substituted by $\succ$.

For square matrices $\boldsymbol{P} > \boldsymbol{0}$ and $\boldsymbol{G}$ of compatible dimension the following holds:

$$\boldsymbol{G}'\boldsymbol{P}^{-1}\boldsymbol{G} \succeq \mathrm{Her}(\boldsymbol{G}) - \boldsymbol{P}. \tag{3.2}$$

Lemma 3.1. *The following statements are equivalent:*

a) $\tilde{\boldsymbol{B}}'\boldsymbol{A}\tilde{\boldsymbol{B}} > 0$ *where* $\boldsymbol{B}\tilde{\boldsymbol{B}} = 0$.

b) $\boldsymbol{A} + \boldsymbol{X}\boldsymbol{B} + \boldsymbol{B}'\boldsymbol{X}' > 0$ *for some matrix* $\boldsymbol{X}$.

Probability and stochastic processes The following probability-space is defined: $(\Omega, \mathscr{F}_t, P)$ where $\mathscr{F}_t$ represents a measurable right-continuous filtration. In this space $(\Omega, \mathscr{F}_t, P)$, the mathematical expectation with respect to P is denoted by $\mathrm{E}(.)$. The Dirac measure over a set $A \in \mathscr{F}$ is defined by $1_A(.)$, meaning that

$$1_A(\omega) = \begin{cases} 1, \text{ if } \omega \in A \\ 0, \text{ otherwise.} \end{cases} \tag{3.3}$$

The notation L_2^n denotes the space of square integrable stochastic processes $\boldsymbol{x} = \{\boldsymbol{x}(t) \in \mathbb{R}^n, t \in \mathbb{R}^+\}$ with $\boldsymbol{x}(t)$ $\mathscr{F}_t$-measurable for each $t \in \mathbb{R}^+$

3.2 Markov Jump Linear System

As discussed in Section 2.3 the occurrence of faults for many cases can be modelled by an exponential distribution. A common framework to model a chain of events where the time between the events is exponentially distributed is a continuous-time Markov chain. When combined with a linear system where the parameters depend on the state of this Markov chain, the model is called continuous-time Markov Jump Linear System (CT-MJLS). This Section intro-

duces the two distinctive parts of a CT-MJLS: first the Markov process which governs the values of the systems matrices is introduced. Afterwards, the corresponding linear system is defined in Section 3.2.2. Finally, Section 3.3 discusses the most important properties of MJLS.

3.2.1 Markov processes

For the Markov chain, the following set is defined: $\mathscr{N} = \{1 \dots N\}$ where N is a positive integer. This set of numbers defines the possible modes of operation of the linear system. This mode of operation is denoted by $\theta(t)$ with $\theta(t) \in \mathscr{N}\ \forall t \in \mathbb{R}^+$. Now define a probability space $(\Omega, \mathscr{F}_t, P)$ where the following assumptions hold:

Assumption 3.1. *[17] The filtration $\mathscr{F}_t; t \in \mathbb{R}^+$ is a right-continuous filtration augmented by all null sets in the P-completion of $\mathscr{F}$.*

The probabilities of the continuous time Markov chain are defined as follows:

$$P(\theta(t+h) = j | \theta(t) = i) = \begin{cases} \lambda_{ij} h + o(h), & (i) \neq (j) \\ 1 + \lambda_{ii} h + o(h), & (i) = (j). \end{cases} \tag{3.4}$$

with the following properties

$$\begin{aligned} &0 \leq \lambda_{ij}, && i \neq j \\ &0 \leq \lambda_i := -\lambda_{ii} = \sum_{j:j \neq i} \lambda_{ij} && \forall i \in \mathscr{N} \end{aligned}$$

The values λ_{ij} are called transition rates and form the transition rate matrix Π. This matrix, together with the initial condition θ_0, define the Markov process which governs the dynamic system introduced in the next section.

Remark 3.1. *In the remaining parts of this monograph, the time dependence of $\theta(t)$ will be omitted to improve readability. Hence: $\theta = \theta(t)$*

3.2.2 Dynamic system

The dynamic part of a Markov Jump Linear System in general is defined by:

$$\begin{aligned} \Sigma_\theta : &\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_\theta \boldsymbol{x}(t) + \boldsymbol{B}_\theta \boldsymbol{u}(t) + \boldsymbol{H}_\theta \boldsymbol{w}(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}_\theta \boldsymbol{x}(t) \end{cases} \\ &\boldsymbol{x}(0) = \boldsymbol{x}_0 \\ &\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0 \end{aligned} \tag{3.5}$$

where the development of the state $\boldsymbol{x}(t)$ depends on the state himself, the input $\boldsymbol{u}(t)$ and the disturbance $\boldsymbol{w}(t)$ which will be defined in the following chapters. Furthermore, there are the gain-matrices $\boldsymbol{A}_\theta$, $\boldsymbol{B}_\theta$ and $\boldsymbol{H}_\theta$. The output $\boldsymbol{y}(t)$ depends on the state, input, and disturbance with the corresponding gain matrices $\boldsymbol{C}_\theta$, $\boldsymbol{D}_\theta$ and $\boldsymbol{F}_\theta$ respectively. All matrices depend on the mode of operation $\boldsymbol{\theta}$ which was defined in the previous section. It is assumed that all matrices and vectors are of compatible dimension.

Furthermore, the following decomposition of the state $\boldsymbol{x}(t)$ is defined:

$$\boldsymbol{x}(t) = \boldsymbol{x}_{zs}(t) + \boldsymbol{x}_{zi}(t)$$

where $\boldsymbol{x}_{zs}(t)$ is the unique solution to

$$\dot{\boldsymbol{x}}_{zs}(t) = \boldsymbol{A}_\theta \boldsymbol{x}_{zs}(t) + \boldsymbol{H}_\theta \boldsymbol{w}(t)$$

with $\boldsymbol{x}_{zs}(0) = 0$, and is called the zero-state response. The second part $\boldsymbol{x}_{zi}$, called the zero-input response, is defined as the unique solution to:

$$\dot{\boldsymbol{x}}_{zi}(t) = \boldsymbol{A}_\theta \boldsymbol{x}_{zi}(t)$$

with $\boldsymbol{x}_{zi}(0) = x_0$.

3.2.3 The general control problem

The general control problem is stated as follows: Find a controller that controls the system (3.5) via input $\boldsymbol{u}(t)$ using the controller

$$\begin{aligned} \Sigma_C : &\begin{cases} \boldsymbol{x}_c(t) = h(\boldsymbol{x}_c(t), \boldsymbol{y}(t)) \\ \boldsymbol{u}(t) = f(\boldsymbol{x}_c(t), \boldsymbol{y}(t)) \end{cases} \\ &\boldsymbol{x}_c(0) = \boldsymbol{x}_{c0} \end{aligned} \tag{3.6}$$

in a way that certain stability and performance criteria are met. These criteria are discussed in more depth in the following sections. As the properties of the dynamic system depend on the parameter θ, the goal is to find a set of N controllers:

$$\Sigma_{K(\theta)} : \begin{cases} \boldsymbol{x}_c(t) = h_\theta\left(\boldsymbol{x}_c(t), \boldsymbol{y}(t)\right) \\ \boldsymbol{u}(t) = f_\theta\left(\boldsymbol{x}_c(t), \boldsymbol{y}(t)\right) \end{cases} \\ \boldsymbol{x}_c(0) = \boldsymbol{x}_{c0} \tag{3.7}$$

This monograph studies two particular cases of the control problem: linear output feedback control and linear state-feedback control as a special case of the latter.

Linear output feedback control describes a memoryless controller which applies a gain matrix to the output-signal $\boldsymbol{y}(t)$ of the system (3.5) and feeds the resulting signal to the input $\boldsymbol{u}(t)$ of the same system. The controller matrix $\boldsymbol{K}$ depends on θ. In this case, the following control law will be applied.

$$\boldsymbol{u}(t) = \boldsymbol{K}_\theta \boldsymbol{y}(t).$$

Resulting in the closed-loop system:

$$\Sigma_{clo} : \begin{cases} \dot{\boldsymbol{x}}(t) = \left(\boldsymbol{A}_\theta + \boldsymbol{B}_\theta \boldsymbol{K}_\theta \boldsymbol{C}_\theta\right)\boldsymbol{x}(t) + \boldsymbol{H}_\theta \boldsymbol{w}(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}_\theta \boldsymbol{x}(t) \end{cases} \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \theta(0) = \theta_0 \tag{3.8}$$

For convenience the following feedback system matrix is defined:

$$\tilde{\boldsymbol{A}}_\theta = \left(\boldsymbol{A}_\theta + \boldsymbol{B}_\theta \boldsymbol{K}_\theta \boldsymbol{C}_\theta\right)$$
$$\tilde{\boldsymbol{C}}_\theta = \boldsymbol{C}_\theta$$

State feedback control State feedback control can be considered as a special case of (3.8) where $\boldsymbol{C} = \boldsymbol{I}$ hence the control law reduces to:

$$\boldsymbol{u}(t) = \boldsymbol{K}_\theta \boldsymbol{x}(t) \tag{3.9}$$

and the closed-loop system:

$$\Sigma_{cls}\begin{cases} \dot{\boldsymbol{x}}(t) = (\boldsymbol{A}_\theta + \boldsymbol{B}_\theta \boldsymbol{K}_\theta)\,\boldsymbol{x}(t) + \boldsymbol{H}_\theta \boldsymbol{w}(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}_\theta \boldsymbol{x}(t) \end{cases} \tag{3.10}$$

Again $\tilde{\boldsymbol{A}}_\theta$ and $\tilde{\boldsymbol{C}}_\theta$ are defined as:

$$\tilde{\boldsymbol{A}}_\theta = (\boldsymbol{A}_\theta + \boldsymbol{B}_\theta \boldsymbol{K}_\theta)$$
$$\tilde{\boldsymbol{C}}_\theta = \boldsymbol{C}_\theta$$

3.3 Results

3.3.1 Stability

For this section the system (3.5) is reduced to the following system:

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_\theta \boldsymbol{x}(t) + \boldsymbol{B}_\theta \boldsymbol{u}(t) \\ &\boldsymbol{x}(0) = \boldsymbol{x}_0 \\ &\theta(0) = \theta_0 \end{aligned} \tag{3.11}$$

Definition 3.4 (Stochastic Stability [17]). *The system* (3.8) *is called mean square stabilizable if there exists a set of controllers* $\mathscr{K} = (K_1 \dots K_n)$ *that for arbitrary initial conditions* $\boldsymbol{x}_0,\ \theta_0$

$$\int_0^\infty E\left(\|\boldsymbol{x}(t)\|^2\right) dt < \infty \tag{3.12}$$

holds.

Definition 3.5 (internal Mean-Square-Stability (iMSS) [17]). *System* (3.11) *is called internally mean-square-stable if for arbitrary initial conditions* $\boldsymbol{x}_0,\ \theta_0$,

$$\lim_{t\to\infty} \mathrm{E}\left(\|\boldsymbol{x}_{zi}(t)\|^2\right) = 0$$

holds.

In order to obtain differential equations for the second moments of $x(t)$ in (3.8) the following linear operators are defined:

Definition 3.6. *[17][Operators]* $\mathscr{L} \in \mathbb{B}(\mathbb{H}^n)$ *and* $\mathscr{T} \in \mathbb{B}(\mathbb{H}^n)$*: for* $\mathbf{P} := \{P_i\} \in \mathbb{H}^n$*, set* $\mathscr{L}(\mathbf{P}) := \{\mathscr{L}_i(\mathbf{P}); (i) \in \mathscr{N}\} \in \mathbb{H}^n$ *and* $\mathscr{T}(\mathbf{P}) := \{\mathscr{T}_i(\mathbf{P}); (i) \in \mathscr{N}\} \in \mathbb{H}^n$ *as*

$$\mathscr{L}_i(\mathbf{P}) := \tilde{\boldsymbol{A}}_i \boldsymbol{P}_i + \boldsymbol{P}_i \tilde{\boldsymbol{A}}_i' + \sum_{j \in \mathscr{N}} \lambda_{ij} \boldsymbol{P}_j, \tag{3.13}$$

$$\mathscr{T}_i(\mathbf{P}) := \tilde{\boldsymbol{A}}_i' \boldsymbol{P}_i + \boldsymbol{P}_i \tilde{\boldsymbol{A}}_i + \sum_{j \in \mathscr{N}} \lambda_{ji} P_j. \tag{3.14}$$

hold.

Equivalent criteria for stability

Theorem 3.1. *The following assertions are equivalent:*

i) there exists $\mathbf{K} = \{K_i; i \in \mathscr{N}\} \in \mathscr{K}$ *such that it stabilizes system* (3.10) *as in Definition 3.4.*

ii) there exists $\mathbf{K} = \{K_i; i \in \mathscr{N}\}$ *such that* $Re(\lambda(\mathscr{L})) < 0$.

iii) there exist $\mathbf{K} = \{K_i; i \in \mathscr{N}\}$ *and* $\mathbf{P} = \{P_i\} \in \mathbb{H}^{n+}$, $P_i > 0$, $(i) \in \mathscr{N}$ *such that* $\mathscr{L}(\mathbf{P}) < 0$.

iv) there exists $\mathbf{K} = \{K_i; i \in \mathscr{N}\}$ *such that* $Re(\lambda(\mathscr{T})) < 0$.

v) there exist $\mathbf{K} = \{K_i; i \in \mathscr{N}\}$ *and* $\mathbf{P} = \{P_i\} \in \mathbb{H}^{n+}$, $P_i > 0$, $(i) \in \mathscr{N}$, *such that* $\mathscr{T}(\mathbf{P}) < 0$.

Proof. See Lemma 3.37 in [17]. □

3.3.2 Optimal Control

Optimal control aims at controlling a system while minimizing some performance measure [54]. These measures are also called costs. In this monograph two performance measures are considered: H_2 norm and H_∞ norm. For these cases the following system is considered:

$$\Sigma_\theta : \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_\theta \boldsymbol{x}(t) + \boldsymbol{B}_\theta \boldsymbol{u}(t) + \boldsymbol{H}_\theta \boldsymbol{w}(t) \\ \boldsymbol{z}(t) = \boldsymbol{C}_\theta \boldsymbol{x}(t) + \boldsymbol{D}_\theta \boldsymbol{u}(t) + \boldsymbol{F}_\theta \boldsymbol{w}(t) \\ \boldsymbol{y}(t) = \boldsymbol{E}_\theta \boldsymbol{x}(t) \end{cases} \tag{3.15}$$

$$\boldsymbol{x}(0) = \boldsymbol{x}_0$$

$$\theta(0) = \theta_0$$

H_2-control

The H_2-norm measures the impact of the disturbance $\boldsymbol{w}(t)$ via the system (3.15) onto the output $\boldsymbol{z}(t)$. The matrix $\boldsymbol{F}$ is considered to be $\mathbf{0}$ in the following. The system Σ_K refers to the system (3.15) controlled by a state feedback controller:

$$\boldsymbol{u}(t) = \boldsymbol{K}_\theta \boldsymbol{x}(t)$$

Definition 3.7. *[17] The H_2-norm of the system Σ_K is defined as*

$$\|\Sigma_{\mathbf{K}}\|_2^2 := \sum_{s=1}^{r} \sum_{i \in \mathscr{N}} \eta_i \, \|z_{s,i}\|_2^2$$

where $\|\cdot\|$ represents the Euclidean norm and $z_{s,i}$ represents the output $\{\boldsymbol{z}(t); t \in \mathbb{R}^+\}$ given by (3.15) *when:*

(i) the input is given by $w = \{w(t); t \geq 0\}$, $w(t) = e_s \delta(t)$, $\delta(t)$ the unitary impulse, and e_s the r-dimensional unitary vector formed by 1 at the sth position and zero elsewhere, and

(ii) $(\theta_0) = (i) \in \mathscr{N}$ with probability η_i.

The H_2-norm as defined above can be calculated via the solution of the continuous-time coupled observability and controllability Gramians, a result that mirrors its deterministic counterpart, see the last chapter for details.

H_∞ control

Definition 3.8 (H_∞-Cost [17])**.** *For the system* (3.5) *with $\boldsymbol{u} = 0$ the following operator is defined:*

$$\mathbb{L}\boldsymbol{w}(t) = \boldsymbol{C}_\theta \boldsymbol{x}_{zs}(t) + \boldsymbol{L}_\theta \boldsymbol{w}(t) \qquad t \in \mathbb{R}^+$$

where $\mathbb{L} : L_2^r(\Omega, \mathscr{F}_t, P) \to L_2^p(\Omega, \mathscr{F}_t, P)$ is a well-defined bounded operator if iMSS holds (see [17]). This operator describes the impact of disturbances $\boldsymbol{w}(t)$ on the output of the system (3.15)*. The H_∞-norm is defined as:*

$$\|\mathbb{L}\| = \sup\left\{ \frac{\|\mathbb{L}\boldsymbol{w}\|_2}{\|\boldsymbol{w}\|_2} : \boldsymbol{w} \in L_2^r(\Omega, \mathscr{F}_t, P), \|w\|_2 \neq 0 \right\} = \|\Sigma_\theta\|_\infty$$

The norm defined above represents a measure for the worst-case effect of finite-energy disturbances on the output.

The bounded-real lemma established an LMI condition for the stability and optimality of a dynamic system:

Lemma 3.2 (Bounded Real Lemma [17])**.** *A System* (3.5) *is internally mean-square-stable, with an H_∞ cost smaller than γ if the following condition hold:*

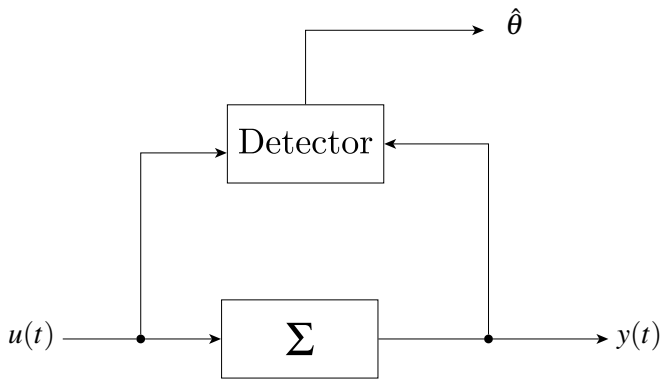
$$\begin{bmatrix} \boldsymbol{R}_i\boldsymbol{A}_i + \boldsymbol{A}_i'\boldsymbol{R}_i + \sum_{j\in\mathcal{N}} \lambda_{ij}\boldsymbol{R}_j & \boldsymbol{R}_i\boldsymbol{H}_i & \boldsymbol{C}_i' \\ \boldsymbol{H}_i'\boldsymbol{R}_i & -\gamma\boldsymbol{I} & \boldsymbol{F}_i' \\ \boldsymbol{C}_i & \boldsymbol{F}_i & -\gamma\boldsymbol{I} \end{bmatrix} < 0 \tag{3.16}$$

with $\boldsymbol{R}_i > 0$, for all $i \in \mathcal{N}$.

4 THE HIDDEN MARKOV PROCESS

As discussed in Section 2.2, the detection of the mode of operation should not be considered as perfect; it is subject to uncertainties. Therefore, additional to the mode of operation a *detected mode of operation* has to be considered. This detected mode of operation corresponds to the information that is emitted by a detector which observes the actual plant and tries to estimate the mode of operation. This estimated mode of operation will be denoted by $\hat{\theta}(t)$.

Figure 6: Estimation of the mode of operation using a detector



Source: Author

While the practical realization of such a detector is beyond the scope of this thesis, a mathematical model of the detection process and the combined process is introduced in this chapter. Section 4.1 introduces the underlying idea, followed by Section 4.1.1, in which the mathematical model is introduced. Section 4.2 extends the results from Section 3.3 to the new model and Section 4.3 closes the chapter with some example cases.

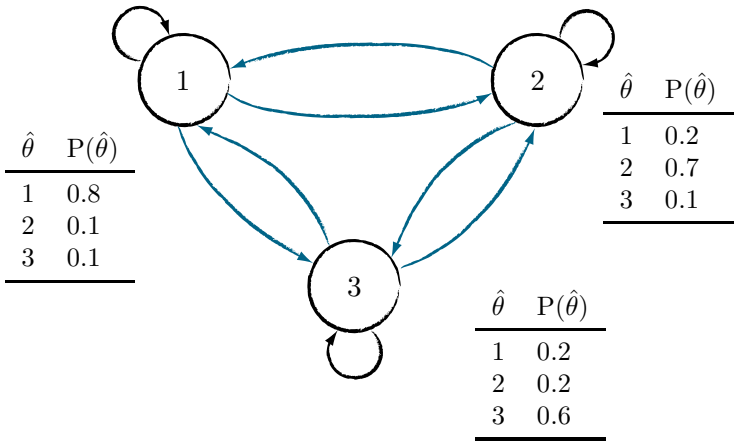
Choice of the detector In the literature the choice of a detector (if given) can be divided into two different concepts. Some [18, 80, 77] define the detector using a σ-field, see for example [18] for details. To others [91, 89, 90, 23, 24], the evolution of the real mode and the observed mode of operation can be modelled as a hidden-Markov process where the real mode of operation is modelled as the hidden state and the observed state models the signal

which is emitted by the detector. The second approach has the advantage that the distribution of the detection process is exactly known and can easily be modelled In this work the second approach, more specifically a Hidden Markov-process consisting of two elements as described in section 4.1.1, is considered. This leads to the fact that the joint process by design fulfils the Markov-property while the detector alone does not necessarily fulfil this property as the transition rate at a time t depends on $\theta(t)$, see also Remark 4.1.

4.1 The underlying concept

To model the uncertainties which were discussed before the following is considered: to every mode of operation all possible detections are assigned with their corresponding probabilities (see Figure 7 for an example). This idea is known as a Hidden Markov Model. For the combined process this opens a space describing the possible combinations of the two processes. In this monograph both the underlying Markov process and the detection process are modelled as one Markov process where every state of the process is formed by a combination of θ and $\hat{\theta}$.

Figure 7: Markov Chain with 3 modes of operation



Source: Author

4.1.1 Hidden Markov Process

In a probability space $(\Omega, \mathscr{F}_t, P)$ a continuous-time hidden Markov model (CT-HMM) $Z(t) = (\theta(t), \widehat{\theta}(t)), t \in \mathbb{R}^+$, is formed by two components, the hidden state $\theta(t)$ taking values in the set $\mathscr{N} := \{1, \ldots, N\}$, and the observation state $\widehat{\theta}(t)$ taking values in the set $\mathscr{M} := \{1, \ldots, M\}$. Both sets form the invariant set $\mathscr{V}$ as follows: $\mathscr{V} \subseteq \mathscr{N} \times \mathscr{M}$.

The following assumptions are made:

Assumption 4.1. *$Z(t)$ is a homogeneous Markov process taking values in $\mathscr{N} \times \mathscr{M}$ with the transition rates $\nu_{(i,k)(j,l)}$, fulfilling the following properties:*

$$\nu_{(i,k)(j,l)} \geq 0 \, for \, (j,l) \neq (i,k)$$
$$\nu_{(i,k)(i,k)} = -\sum_{(j,l)\neq(i,k)} \nu_{(i,k)(j,l)}.$$

Therefore:

$$P(Z(t+h) = (j,l)|Z(t) = (i,k)) = \begin{cases} \nu_{(i,k),(j,l)}h + o(h), & (j,l) \neq (i,k) \\ 1 + \nu_{(i,k),(i,k)}h + o(h), & (j,l) = (i,k). \end{cases} \tag{4.1}$$

Assumption 4.2. *The transition rates $\nu_{(i,k)(j,l)}$, for (i,k), (j,l) in $\mathscr{N} \times \mathscr{M}$ satisfy*

$$\nu_{(i,k)(j,l)} = \begin{cases} \alpha^k_{jl}\lambda_{ij}, & j \neq i, \, l \in \mathscr{M} \\ q^i_{kl}, & l \neq k, j = i, \, i \in \mathscr{N} \\ \lambda_{ii} + q^i_{kk}, & j = i, l = k \end{cases} \tag{4.2}$$

where

$$\begin{aligned} &\sum_{l=1}^{M} \alpha^k_{jl} = 1, \\ &\lambda_{ij} \geq 0 \, \forall j \neq i, \\ &q^i_{kl} \geq 0 \, \forall l \neq k \\ &\lambda_{ii} = -\sum_{j\neq i} \lambda_{ij} \\ &q^i_{kk} = -\sum_{l\neq k} q^i_{kl}. \end{aligned} \tag{4.3}$$

Remark 4.1. *It should be noted, that $\theta(t)$ alone is a Markov process with the transition rates given by λ_{ij}, but the observed mode of operation $\hat{\theta}(t)$ alone may not be a Markov process since in general its transition rate at time t will depend on $\theta(t)$.*

Considering an invariant set $\mathscr{V} \subseteq \mathscr{N} \times \mathscr{M}$ for $Z(t)$ (that is, $P(Z(t) \in \mathscr{V}) = 1$ whenever $Z(0) \in \mathscr{V}$), the following distinct situations can be modelled:

Perfect Information In case of $\mathscr{M} = \mathscr{N}$, $q^i_{kl} = 0$, $\alpha^k_{jj} = 1$, $\alpha^k_{jl} = 0$ for $l \neq j$, and invariant set $\mathscr{V} = \{(i,i); i \in \mathscr{N}\}$. Therefore the mode of operation θ and the observed mode of operation $\hat{\theta}$ will always jump at the same time to the same state ($\hat{\theta}(t) = \theta(t) \, \forall t$). In this case the transition-rate matrix formed by $\nu_{(i,k)(j,l)}$ can be reduced to a square-matrix of the dimension *NxN* just containing the jumps of θ.

No Information If $\mathscr{M} = \{1\}$, $q^i_{kl} = 0$ and $\alpha^1_{j1} = 1$, the detector will always provide the same information about the mode of operation and thus $\hat{\theta}$ will be constant. Therefore no information about the actual mode of operation exists and the control problem reduces to robust control.

No Mutual Jumps Supposing $\alpha^k_{jk} = 1$, $\alpha^k_{jl} = 0$ for $l \neq k$. In this case, with probability one, one jump at a time will occur for $\theta(t)$ (with rate λ_{ij}) and $\widehat{\theta}(t)$ (with rate q^i_{kl}, conditioned on $\theta(t) = i$). This kind of situation was considered, for instance, in [88, 64, 63, 2].

The Cluster Case Considering that the Markov-chain can be modelled as a union of M disjoint sets (or clusters) $\mathscr{N}_i$, such that $\mathscr{N} = \cup_{j=1}^{M} \mathscr{N}_j$, using $\mathscr{M} = \{1, \ldots, M\}$, with $M \leq N$. A function $g : \mathscr{N} \to \mathscr{M}$ can be defined such that $g(i) = j$ for all $i \in \mathscr{N}_j$, with $g(i)$ representing to which cluster the state i belongs to. This concept was discussed in [30, 45, 37]. It is assumed that the detector emits information about the cluster to which θ belongs to while the mode itself is unknown.

Delayed Detections Another possible case is the one where the observed mode of operation $\hat{\theta}$ follows the trajectory of the mode of operation θ, but with a time delay (see Figure 8). This case can be modelled by using

$$\alpha^k_{jl} \begin{cases} \neq 0 \text{ for } l = k \\ = 0 \text{ for } l \neq k \end{cases}$$

and

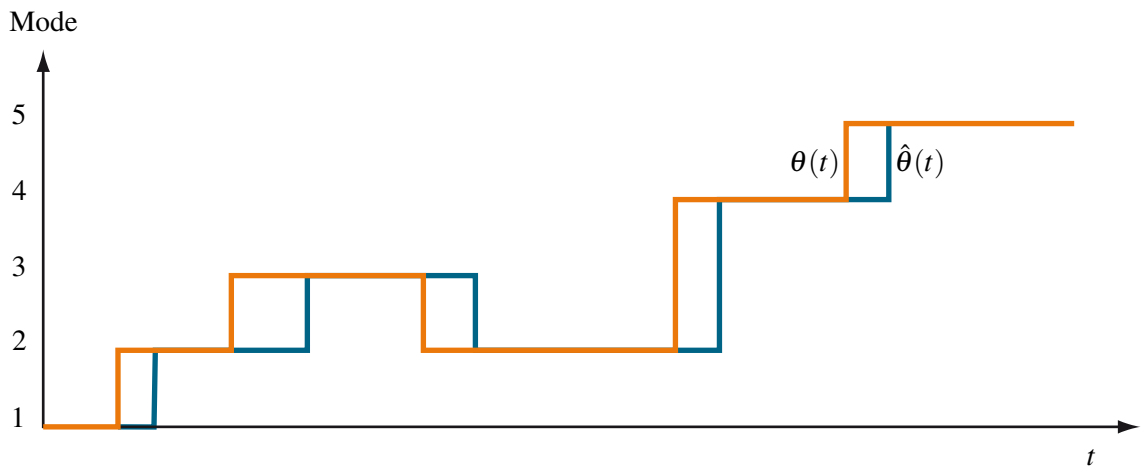
$$q^i_{kl} \begin{cases} \neq 0 \text{ for } i = l \\ = 0 \text{ otherwise.} \end{cases}$$

These situations and combinations of them allow to model a large share of the cases discussed in the literature, for example those discussed in [88, 64, 63, 2, 30, 18], see also Table 1 for more examples.

4.1.2 Modified control problem

Considering the model introduced in the previous section, the modified control problem is stated as follows: find a set of controllers $\mathscr{K}$ that control the system (3.5) via input $\boldsymbol{u}(t)$ using

Figure 8: Trajectory of a system with delayed detections, $\theta(t)$ in orange and $\hat{\theta}(t)$ in blue.



Source: Author

a controller

$$\Sigma_{K(\hat{\theta})} : \begin{cases} \boldsymbol{x}_c(t) = h_{\hat{\theta}}\left(\boldsymbol{x}_c(t), \boldsymbol{y}(t)\right) \\ \boldsymbol{u}(t) = f_{\hat{\theta}}\left(\boldsymbol{x}_c(t), \boldsymbol{y}(t)\right) \end{cases} \tag{4.4}$$
$$\boldsymbol{x}_c(0) = \boldsymbol{x}_{c0}$$

in a way that certain stability and performance criteria are met, while the controller has only access to an estimate of the mode of operation. As the properties of the dynamic system depend on the parameter θ, there are N different systems which should be controlled. As the controller has access to an estimated value of θ, there is a set of M controllers. As before two particular cases of the control problem are considered: linear output feedback control and linear state-feedback control as a special case of the latter. The difference to the cases in the previous chapter lies in the fact that the controller now depends on $\hat{\theta}$.

Linear output feedback control The control law now turns into:

$$\boldsymbol{u}(t) = \boldsymbol{K}_{\hat{\theta}}\boldsymbol{y}(t).$$

Resulting in the closed-loop system:

$$\Sigma_{clo}: \begin{cases} \dot{\boldsymbol{x}}(t) = \left(\boldsymbol{A}_\theta + \boldsymbol{B}_\theta \boldsymbol{K}_{\hat{\theta}} \boldsymbol{C}_\theta\right) \boldsymbol{x}(t) + \boldsymbol{J}_\theta \boldsymbol{w}(t) \\ \boldsymbol{z}(t) = \boldsymbol{C}_\theta \boldsymbol{x}(t) + \boldsymbol{L}_\theta \boldsymbol{w}(t) \end{cases}$$
$$\boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{4.5}$$
$$\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0$$
$$\hat{\boldsymbol{\theta}}(0) = \hat{\boldsymbol{\theta}}_0$$

For convenience the following feedback system matrix is defined:

$$\tilde{\boldsymbol{A}}_{\theta,\hat{\theta}} = \left(\boldsymbol{A}_\theta + \boldsymbol{B}_\theta \boldsymbol{K}_{\hat{\theta}} \boldsymbol{C}_\theta\right) \tag{4.6}$$

State feedback control In this case the control law turns into:

$$\boldsymbol{u}(t) = \boldsymbol{K}_{\hat{\theta}} \boldsymbol{x}(t) \tag{4.7}$$

and the closed-loop system:

$$\Sigma_{cls} \begin{cases} \dot{\boldsymbol{x}}(t) = \left(\boldsymbol{A}_\theta + \boldsymbol{B}_\theta \boldsymbol{K}_{\hat{\theta}}\right) \boldsymbol{x}(t) + \boldsymbol{J}_\theta \boldsymbol{w}(t) \\ \boldsymbol{z}(t) = \boldsymbol{C}_\theta \boldsymbol{x}(t) + \boldsymbol{L}_\theta \boldsymbol{w}(t) \end{cases} \tag{4.8}$$

4.2 Results

In this section the results shown in Section 3.3 are extended to the case where the controller has only access to the estimated mode of operation.

4.2.1 Stability

Definition 4.1 (Stochastic Stabilizability)**.** *The System* (3.5) *is stochastically stabilizable if there exists* $K_l \in \mathbb{B}(\mathbb{R}^{n,m})$, $l \in \mathscr{M}$, *such that for arbitrary initial conditions* $(\theta_0, \widehat{\theta}_0) \in \mathscr{V}$ *and* x_0, *we have that* $\int_0^\infty E\left(\|x(t)\|^2\right) dt < \infty$ *where* $x(t)$ *is given by (3.10) with* $t \in \mathbb{R}^+$. *In this case we say that* K_l *stabilizes (4.5) and write* $\mathbf{K} := \{K_l; l \in \mathscr{M}\}$. *We denote the set of* $\mathbf{K}$ *that stabilizes (4.5) by* $\mathscr{K}$.

Definition 4.2 (Operators)**.** *Since* $\mathscr{V}$ *is an invariant set for* $Z(t)$, *only* A_{ik} *with the states* $(i,k) \in \mathscr{V}$ *will matter for the stochastic stability of* (4.5) *whenever* $(\theta_0, \widehat{\theta}_0) \in \mathscr{V}$. *In order to obtain differential equations for the second moments of* $x(t)$ *in* (4.5) *we define the following linear*

operators $\mathscr{L} \in \mathbb{B}(\mathbb{H}^n)$ *and* $\mathscr{T} \in \mathbb{B}(\mathbb{H}^n)$*: for* $\mathbf{P} := \{P_{ik}\} \in \mathbb{H}^n$*, set* $\mathscr{L}(\mathbf{P}) := \{\mathscr{L}_{jl}(\mathbf{P}); (j,l) \in \mathscr{V}\} \in \mathbb{H}^n$ *and* $\mathscr{T}(\mathbf{P}) := \{\mathscr{T}_{ik}(\mathbf{P}); (i,k) \in \mathscr{V}\} \in \mathbb{H}^n$ *as*

$$\mathscr{L}_{jl}(\mathbf{P}) := \tilde{A}_{jl}P_{jl} + P_{jl}\tilde{A}'_{jl} + \sum_{(i,k)\in\mathscr{V}} \nu_{(i,k)(j,l)}P_{ik}, \tag{4.9}$$

$$\mathscr{T}_{ik}(\mathbf{P}) := \tilde{A}'_{ik}P_{ik} + P_{ik}\tilde{A}_{ik} + \sum_{(j,l)\in\mathscr{V}} \nu_{(i,k)(j,l)}P_{jl}. \tag{4.10}$$

Lemma 4.1. *This result establishes a link between* $\mathscr{T}$ *and* $\mathscr{L}$*. With the inner product as defined in Section 3.1, we have that* $\mathscr{T}^* = \mathscr{L}$*, i.e.,* $\mathscr{T}$ *is the adjoint operator of* $\mathscr{L}$*. Moreover* $Re(\lambda(\mathscr{T})) < 0$ *if and only if* $Re(\lambda(\mathscr{L})) < 0$*.*

Proof. It follows the same reasoning as in Lemma 3.5 and Proposition 3.11 in [17]. □

4.2.2 Cost measures

H_2-control

The H_2-norm measures the impact of the disturbance $\boldsymbol{w}(t)$ via the system (3.15) onto the output $\boldsymbol{z}(t)$. The matrix $\boldsymbol{F}$ is considered to be $\mathbf{0}$ in the following. The system Σ_K refers to the system (3.15) controlled by a state feedback controller:

$$\boldsymbol{u}(t) = \boldsymbol{K}_{\hat{\theta}}\boldsymbol{x}(t)$$

If $\boldsymbol{K} = \{K_l; l \in \mathscr{M}\} \in \mathscr{K}$, K_l stabilizes (4.5) (see Definition 4.1) then from Lemma 4.1 and Theorem 3.1 it follows that $\mathrm{Re}(\lambda(\mathscr{L})) < 0$ and $\mathrm{Re}(\lambda(\mathscr{T})) < 0$. Considering $w = \{w(t); t \in \mathbb{R}^+\}$ an impulse input (that is, $w(t) = v\delta(t)$ where v is an r-dimensional vector and $\delta(t)$ the unitary impulse) then, as a consequence of Theorem 3.15 *(v)* in [17], there exists $b > 0$ and $a > 0$, such that for each $t \in \mathbb{R}^+$,

$$E(\|z(t)\|^2) \leq a\, e^{-bt} E(\|x_0\|^2) \tag{4.11}$$

and $\int_0^\infty E(\|z(t)\|^2)dt < \infty$.

The H_2-norm as defined above can be calculated via the solution of the continuous-time coupled observability and controllability Gramians, a result that mirrors its deterministic counterpart.

For

$$\mathbf{K} = \{K_l; l \in \mathscr{M}\} \in \mathscr{K}$$

define

$$\tilde{C}_{il} := C_i + D_i K_l$$

as well as $\mathscr{T}$, $\mathscr{L}$ as in (4.9), (4.10), and

$$\begin{aligned}
&\mathbf{M} := \{\widetilde{C}_i'\widetilde{C}_i\} \in \mathbb{H}^{n+} \\
&\eta\mathbf{N} := \{\eta_i H_i H_i'\} \in \mathbb{H}^{n+} \\
&\mathbf{S} := \{S_i\} \in \mathbb{H}^{n+} \\
&\mathbf{P} := \{P_i\} \in \mathbb{H}^{n+}
\end{aligned}$$

the unique solution of the equations (see Theorem 3.25 in [17]): $\mathscr{T}(\mathbf{S}) + \mathbf{M} = 0$ (observability Gramian) and $\mathscr{L}(\mathbf{P}) + \eta\mathbf{N} = 0$ (controllability Gramian).

The following result establishes a connection between the H_2-norm with the observability and controllability Gramians.

Theorem 4.1.

$$\|\Sigma_{\mathbf{K}}\|_2^2 = \sum_{(j,l)\in\mathscr{V}} \eta_{jl}\,\mathrm{tr}\big(H_{jl}' S_{jl} H_{jl}\big) = \sum_{(j,l)\in\mathscr{V}} \mathrm{tr}\big(\widetilde{C}_{jl} P_{jl} \widetilde{C}_{jl}'\big).$$

Proof. The proof follows the same reasoning as the proof of Theorem 5.4 in [17]. For the first equality with $i \in \mathscr{N}$

$$\mathscr{T}_i(\mathbf{S}) + \widetilde{C}_{jl}^* \widetilde{C}_{jl} = \widetilde{A}_i^* S_i + S_i \widetilde{A}_i + \sum_{j\in\mathscr{S}} \lambda_{ij} S_j + \widetilde{C}_{jl}^* \widetilde{C}_{jl} = 0.$$

Consider $z = \{z(t); t \geq 0\}$ an impulse response of (4.5). Then

$$\begin{aligned}
E\big(z(t)^* z(t)\big) &= E\Big(x(t)^* \widetilde{C}_{\theta(t)}^* \widetilde{C}_{\theta(t)} x(t)\Big) \\
&= -E\Big(x(t)^* \mathscr{T}_{\theta(t)}(\mathbf{S}) x(t)\Big) \\
&= -\sum_{i\in\mathscr{S}} E\Big(x(t)^* \mathscr{T}_i(\mathbf{S}) x(t) 1_{\{\theta(t)=i\}}\Big) \\
&= -\sum_{i\in\mathscr{S}} \mathrm{tr}\Big(E\Big(x(t)x(t)^* 1_{\{\theta(t)=i\}}\Big)\mathscr{T}_i(\mathbf{S})\Big) \\
&= -\sum_{i\in\mathscr{S}} \mathrm{tr}\Big(Q_i(t)\mathscr{T}_i(\mathbf{S})\Big) \\
&= -\big\langle \mathbf{Q}(t); \mathscr{T}(\mathbf{S})\big\rangle \\
&= -\big\langle \mathscr{L}(\mathbf{Q}(t)); \mathbf{S}\big\rangle \\
&= -\big\langle \dot{\mathbf{Q}}(t); \mathbf{S}\big\rangle.
\end{aligned}$$

Taking the integral over t from 0 to ∞, and recalling that $\mathbf{Q}(t) \to 0$ as $t \to \infty$ (since the system is MSS) and that $\theta(0) = j$, $x(0) = H_j e_s$, results in

$$\begin{aligned}
\left\|z_{s,j}\right\|_2^2 &= \int_0^\infty E(\|z(t)\|^2)dt \\
&= -\int_0^\infty \left\langle \dot{\mathbf{Q}}(t);\mathbf{S}\right\rangle dt \\
&= -\left\langle \mathbf{Q}(t);\mathbf{S}\right\rangle\Big]_0^\infty \\
&= \left\langle \mathbf{Q}(0);\mathbf{S}\right\rangle \\
&= \sum_{i\in\mathscr{S}} \operatorname{tr}\big(Q_i(0)S_i\big) \\
&= e_s^* H_j^* S_j H_j e_s.
\end{aligned}$$

Therefore,

$$\|\Sigma_K\|_2^2 = \sum_{s=1}^{r}\sum_{j\in\mathscr{S}} \nu_j \left\|z_{s,j}\right\|_2^2 = \sum_{s=1}^{r}\sum_{j\in\mathscr{S}} \nu_j e_s' H_j' S_j H_j e_s = \sum_{j\in\mathscr{S}} \nu_j \operatorname{tr}(H_j^* S_j H_j)$$

holds, proving the first equality.

The second equality is proven in the following:

$$\begin{aligned}
\|\Sigma_K\|_2^2 &= \sum_{j=1}^{N} \nu_j \operatorname{tr}\big(H_j^* S_j H_j\big) \\
&= \sum_{j=1}^{N} \operatorname{tr}\Big(\nu_j H_j^* S_j H_j + P_j\big(\widetilde{C}_j^*\widetilde{C}_j + \mathscr{T}_j(\mathbf{S})\big)\Big) \\
&= \sum_{j=1}^{N} \operatorname{tr}\Big(\nu_j H_j H_j^* S_j + P_j\widetilde{C}_j^*\widetilde{C}_j\Big) + \left\langle \mathbf{P};\mathscr{T}(\mathbf{S})\right\rangle \\
&= \left\langle \mathbf{P};\mathbf{M}\right\rangle + \left\langle \nu\mathbf{N};\mathbf{S}\right\rangle + \left\langle \mathscr{L}(\mathbf{P});\mathbf{S}\right\rangle \\
&= \left\langle \mathbf{P};\mathbf{M}\right\rangle + \left\langle (\mathscr{L}(\mathbf{P}) + \nu\mathbf{N});\mathbf{P}\right\rangle \\
&= \left\langle \mathbf{P};\mathbf{M}\right\rangle = \sum_{j=1}^{N} \operatorname{tr}\big(\widetilde{C}_j P_j \widetilde{C}_j^*\big)
\end{aligned}$$

completing the proof of the theorem. $\square$

H_∞ control

Now the following system is considered:

$$\Sigma_C : \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_\theta \boldsymbol{x}(t) + \boldsymbol{B}_\theta \boldsymbol{u}(t) + \boldsymbol{J}_\theta \boldsymbol{w}(t) \\ \boldsymbol{z}(t) = \boldsymbol{C}_\theta \boldsymbol{x}(t) + \boldsymbol{D}_\theta \boldsymbol{u}(t) + \boldsymbol{L}_\theta \boldsymbol{w}(t) \\ \boldsymbol{y}(t) = \boldsymbol{E}_\theta \boldsymbol{x}(t) \end{cases} \tag{4.12}$$

where $\boldsymbol{u}(t) \in \mathbb{R}^p$ denotes the vector of control and $\boldsymbol{y} \in \mathbb{R}^s$ the measured output. Again all matrices are considered to be of compatible dimensions. The following assumption regarding $\boldsymbol{E}_i$ is made:

Assumption 4.3. *It is assumed that $\boldsymbol{E}_i$ has full row rank for all $i \in \mathscr{N}$.*

4.2.3 Bounded Real Lemma

As pointed out before, the controller considered does not depend on θ, but on $\hat{\theta}$. Therefore the matrix $\boldsymbol{A}_i$ in (3.16) changes to $\tilde{\boldsymbol{A}}_{ik} = \boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{K}_k$ and matrix $\tilde{\boldsymbol{C}}_i$ to $\tilde{\boldsymbol{C}}_{ik} = \boldsymbol{C}_i + \boldsymbol{D}_i \boldsymbol{K}_k$. Using the same reasoning as in [89] with the hidden Markov process as explained before, Lemma 3.2 turns into the following one:

Lemma 4.2 (Extended Bounded Real Lemma)**.** *The system* (4.5) *is iMSS with a H_∞ cost smaller than γ if the following conditions hold: for $(j,l) \in \mathscr{V}$,*

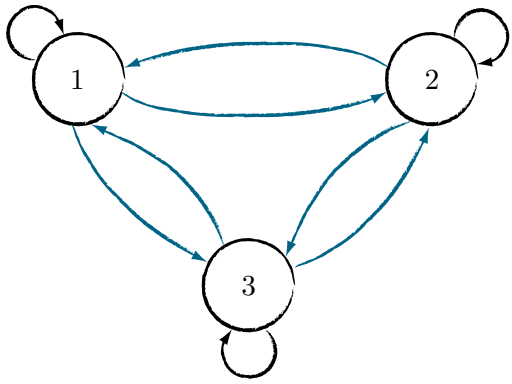
$$\begin{bmatrix} \boldsymbol{R}_{jl}\tilde{\boldsymbol{A}}_{jl} + \tilde{\boldsymbol{A}}'_{jl}\boldsymbol{R}_{jl} + \sum_{(i,k)\in\mathscr{V}} \nu_{(i,k)(j,l)}\boldsymbol{R}_{ik} & \boldsymbol{R}_{jl}\boldsymbol{H}_j & \tilde{\boldsymbol{C}}'_{jl} \\ \boldsymbol{H}'_j\boldsymbol{R}_{jl} & -\gamma\boldsymbol{I} & \boldsymbol{F}'_j \\ \tilde{\boldsymbol{C}}_{jl} & \boldsymbol{F}_j & -\gamma\boldsymbol{I} \end{bmatrix} < 0 \tag{4.13}$$

with $\boldsymbol{R}_{jl} > 0$ for all $(j,l) \in \mathscr{V}$.

4.3 Examples

This section gives some examples to illustrate the findings of this chapter. For this purpose a Markov Process with 3 modes of operation as depicted in Figure 9 is considered.

Figure 9: Markov Chain with 3 modes of operation



Source: Author

This means that $\mathscr{N} = [1,2,3]$. To each mode θ probabilities are assigned. In the most general case $\mathscr{M} = \mathscr{N}$ holds, so the joint Markov process takes in the space $\mathscr{V} = \mathscr{N} \times \mathscr{M}$. For the depicted chain this opens a space of 9 possible combinations of θ and $\hat{\theta}$. Using the relations (4.2) and (4.3) the general form of the transition matrix is shown below:

$$
\Pi = \begin{array}{c} \theta,\hat{\theta} \\ 1,1 \\ 1,2 \\ 1,3 \\ 2,1 \\ 2,2 \\ 2,3 \\ 3,1 \\ 3,2 \\ 3,3 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1,1 & 1,2 & 1,3 & 2,1 & 2,2 & 2,3 & 3,1 & 3,2 & 3,3 \end{array} \\
\begin{bmatrix}
\lambda_{11}+q^1_{11} & q^1_{12} & q^1_{13} & \alpha^1_{21}\lambda_{12} & \alpha^1_{22}\lambda_{12} & \alpha^1_{23}\lambda_{12} & \alpha^1_{13}\lambda_{13} & \alpha^1_{13}\lambda_{13} & \alpha^1_{13}\lambda_{13} \\
q^1_{21} & \lambda_{11}+q^1_{22} & q^1_{23} & \alpha^2_{21}\lambda_{12} & \alpha^2_{22}\lambda_{12} & \alpha^2_{23}\lambda_{12} & \alpha^1_{13}\lambda_{13} & \alpha^1_{13}\lambda_{13} & \alpha^1_{13}\lambda_{13} \\
q^1_{31} & q^1_{32} & \lambda_{11}+q^i_{kk} & \alpha^3_{21}\lambda_{12} & \alpha^3_{22}\lambda_{12} & \alpha^3_{23}\lambda_{12} & \alpha^1_{13}\lambda_{13} & \alpha^1_{13}\lambda_{13} & \alpha^1_{13}\lambda_{13} \\
\alpha^1_{11}\lambda_{21} & \alpha^1_{12}\lambda_{21} & \alpha^1_{13}\lambda_{21} & \lambda_{22}+q^2_{11} & q^2_{12} & q^2_{13} & \alpha^1_{11}\lambda_{23} & \alpha^1_{12}\lambda_{23} & \alpha^1_{13}\lambda_{23} \\
\alpha^2_{11}\lambda_{21} & \alpha^2_{12}\lambda_{21} & \alpha^2_{13}\lambda_{21} & q^2_{21} & \lambda_{22}+q^i_{22} & q^2_{23} & \alpha^1_{11}\lambda_{23} & \alpha^1_{12}\lambda_{23} & \alpha^1_{13}\lambda_{23} \\
\alpha^3_{11}\lambda_{21} & \alpha^3_{12}\lambda_{21} & \alpha^3_{13}\lambda_{21} & q^2_{31} & q^2_{32} & \lambda_{22}+q^2_{33} & \alpha^1_{11}\lambda_{23} & \alpha^1_{12}\lambda_{23} & \alpha^1_{13}\lambda_{23} \\
\alpha^1_{11}\lambda_{31} & \alpha^1_{12}\lambda_{31} & \alpha^1_{13}\lambda_{31} & \alpha^1_{21}\lambda_{32} & \alpha^1_{22}\lambda_{32} & \alpha^1_{23}\lambda_{32} & \lambda_{33}+q^3_{11} & q^3_{12} & q^3_{13} \\
\alpha^2_{11}\lambda_{31} & \alpha^2_{12}\lambda_{31} & \alpha^2_{13}\lambda_{31} & \alpha^2_{21}\lambda_{32} & \alpha^2_{22}\lambda_{32} & \alpha^2_{23}\lambda_{32} & q^3_{21} & \lambda_{33}+q^3_{22} & q^3_{23} \\
\alpha^3_{11}\lambda_{31} & \alpha^3_{12}\lambda_{31} & \alpha^3_{13}\lambda_{31} & \alpha^3_{21}\lambda_{32} & \alpha^3_{22}\lambda_{32} & \alpha^3_{23}\lambda_{32} & q^3_{31} & q^3_{23} & \lambda_{33}+q^3_{33}
\end{bmatrix}
\end{array}
\tag{4.14}
$$

The transition matrix (4.14) shows a block diagonal structure. The blocks on the main diagonal model these situations where the real mode of operation does not change. The values for q^i_{kl} are used to model spontaneous jumps, which means that the estimated mode of operation $\hat{\theta}$ changes, even though the underlying mode of operation θ of the plant does not change. The values α^k_{jl} are used to model the detection probabilities when both the mode of operation θ and the estimated mode of operation jump at the same time. As for the remaining values, the values

for λ_{ij} are taken from the transition matrix of the original plant and the others are calculated using the relations (4.3).

In the following section the parametrization for the special cases introduced at the end of Section 4.1.1 are discussed in more detail. For the initial transition matrix the following numerical values for the transition matrix are used:

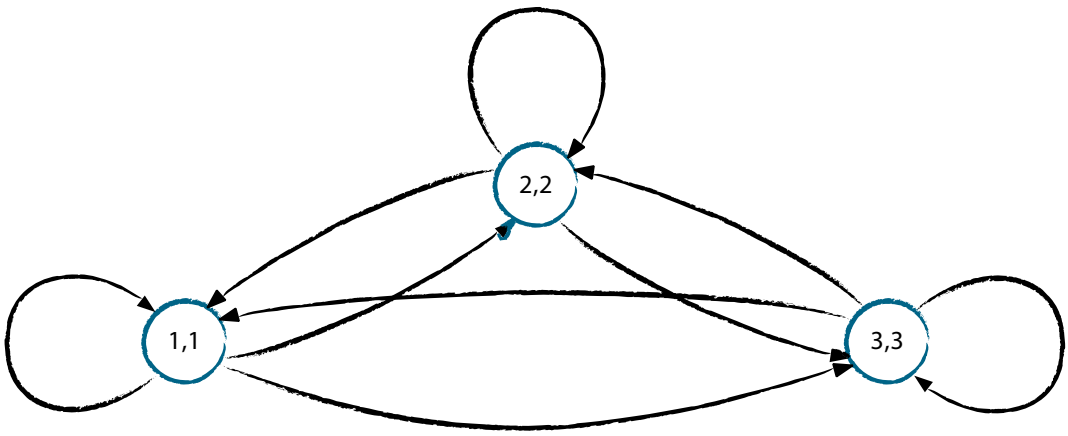
$$\Pi = \begin{bmatrix} -0{,}5 & 0{,}4 & 0{,}1 \\ 0{,}2 & -0{,}5 & 0{,}3 \\ 0{,}4 & 0{,}2 & -0{,}6 \end{bmatrix} \tag{4.15}$$

For all cases a graphical representation is shown. The numbers in the nodes represent the actual values for the mode of operation and the estimated mode of operation, where the first number corresponds to θ and the second one to $\hat{\theta}$

Remark 4.2. *In the following section elements of matrices which are* 0 *will be left blank to enhance readability.*

Remark 4.3. *To illustrate the affects on the matrix, in all cases a full $N \times N$ matrix is shown. If a combination of $\theta, \hat{\theta}$ does not exist the corresponding node is considered as inaccessible but existing.*

Figure 10: Graph representation of the reachable nodes in the case of perfect information



Source: Author

Example 2 (Perfect Information)**.** *This means that at every instance of time $\hat{\theta} = \theta$ holds and therefore the detector always provides a perfect information about the mode of operation. A graphical representation is shown in Figure 10 To achieve this, the following detection probabilities are considered:*

	$\theta = 1$	$\theta = 2$	$\theta = 3$
$P(\hat{\theta} = 1)$	1	0	0
$P(\hat{\theta} = 2)$	0	1	0
$P(\hat{\theta} = 3)$	0	0	1

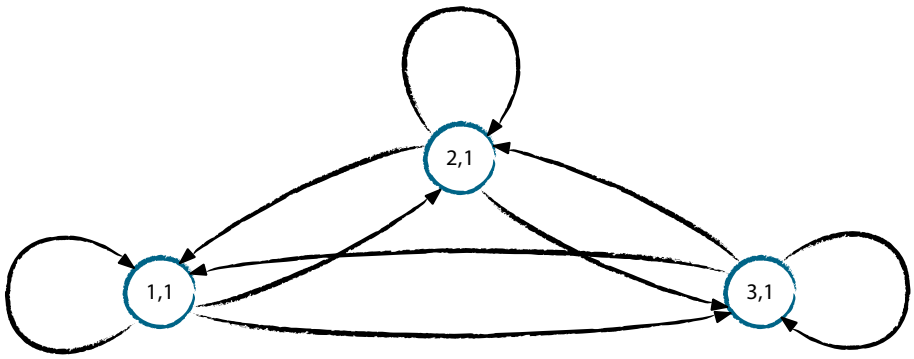
Using these values, the transition matrix turns into:

$$\Pi = \begin{array}{c} \theta,\hat{\theta} \\ 1{,}1 \\ 1{,}2 \\ 1{,}3 \\ 2{,}1 \\ 2{,}2 \\ 2{,}3 \\ 3{,}1 \\ 3{,}2 \\ 3{,}3 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1{,}1 & 1{,}2 & 1{,}3 & 2{,}1 & 2{,}2 & 2{,}3 & 3{,}1 & 3{,}2 & 3{,}3 \end{array} \\ \left[\begin{array}{ccc|ccc|ccc} -0{,}5 & & & & 0{,}4 & & & & 0{,}1 \\ & & & & & & & & \\ & & & & & & & & \\ \hline & & & & & & & & \\ 0{,}2 & & & & -0{,}5 & & & & 0{,}3 \\ & & & & & & & & \\ \hline & & & & & & & & \\ & & & & & & & & \\ 0{,}4 & & & & 0{,}2 & & & & -0{,}6 \end{array}\right] \end{array}$$

This assumes that $\theta_0 = \hat{\theta}_0$ holds. If this is not the case it is necessary to contemplate the possibility that the first jump occurs from a joint mode of observation $(\theta, \hat{\theta})$ where $\theta_0 \neq \hat{\theta}_0$, to $(\theta, \hat{\theta})$ where $\theta_0 = \hat{\theta}_0$, thus introducing more elements to the matrix.

$$\Pi = \begin{array}{c} \theta,\hat{\theta} \\ 1{,}1 \\ 1{,}2 \\ 1{,}3 \\ 2{,}1 \\ 2{,}2 \\ 2{,}3 \\ 3{,}1 \\ 3{,}2 \\ 3{,}3 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1{,}1 & 1{,}2 & 1{,}3 & 2{,}1 & 2{,}2 & 2{,}3 & 3{,}1 & 3{,}2 & 3{,}3 \end{array} \\ \left[\begin{array}{ccc|ccc|ccc} -0{,}5 & & & & 0{,}4 & & & & 0{,}1 \\ & -0{,}5 & & & 0{,}4 & & & & 0{,}1 \\ & & -0{,}5 & & 0{,}4 & & & & 0{,}1 \\ \hline 0{,}2 & & & -0{,}5 & & & & & 0{,}3 \\ 0{,}2 & & & & -0{,}5 & & & & 0{,}3 \\ 0{,}2 & & & & & -0{,}5 & & & 0{,}3 \\ \hline 0{,}4 & & & & 0{,}2 & & -0{,}6 & & \\ 0{,}4 & & & & 0{,}2 & & & -0{,}6 & \\ 0{,}4 & & & & 0{,}2 & & & & -0{,}6 \end{array}\right] \end{array}$$

Figure 11: Graphical representation of the reachable nodes in the no information case



Source: Author

Example 3 (No information)**.** *This case considers $\hat{\theta} = const.$ For the example $\hat{\theta} = 1 \; \forall \; t$ and $\hat{\theta}_0 = 1$ is chosen, the graphical representation is shown in Figure 11. As for the numerical values, the following detection probabilities are used:*

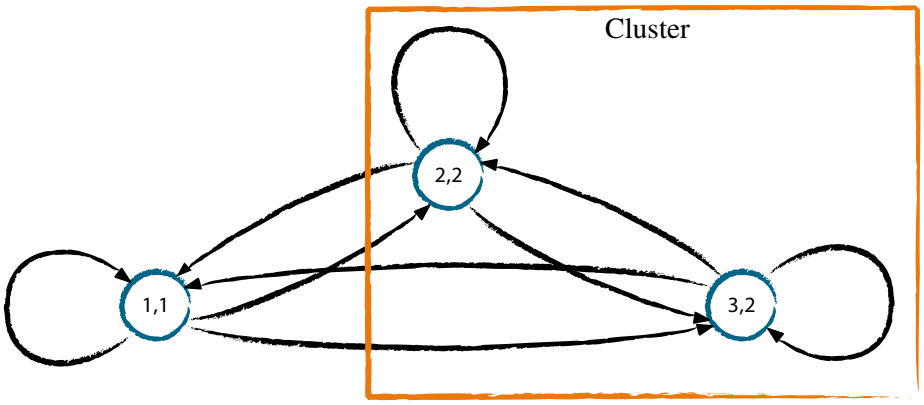
	$\theta = 1$	$\theta = 2$	$\theta = 3$
$P(\hat{\theta} = 1)$	1	1	1
$P(\hat{\theta} = 2)$	0	0	0
$P(\hat{\theta} = 3)$	0	0	0

This turns the transition matrix into:

$$\Pi = \begin{array}{c} \theta, \hat{\theta} \\ 1,1 \\ 1,2 \\ 1,3 \\ 2,1 \\ 2,2 \\ 2,3 \\ 3,1 \\ 3,2 \\ 3,3 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1,1 & 1,2 & 1,3 & 2,1 & 2,2 & 2,3 & 3,1 & 3,2 & 3,3 \end{array} \\ \left[\begin{array}{ccc|ccc|ccc} -0{,}5 & & & 0{,}4 & & & 0{,}1 & & \\ & & & & & & & & \\ & & & & & & & & \\ \hline 0{,}2 & & & -0{,}5 & & & 0{,}3 & & \\ & & & & & & & & \\ & & & & & & & & \\ \hline 0{,}4 & & & 0{,}2 & & & -0{,}6 & & \\ & & & & & & & & \\ & & & & & & & & \end{array}\right] \end{array}$$

It can be seen that, as in the perfect information case, the matrix could be reduced to the original one, the only difference in this case is the information emitted by the detector.

Figure 12: Graphical representation of the cluster case



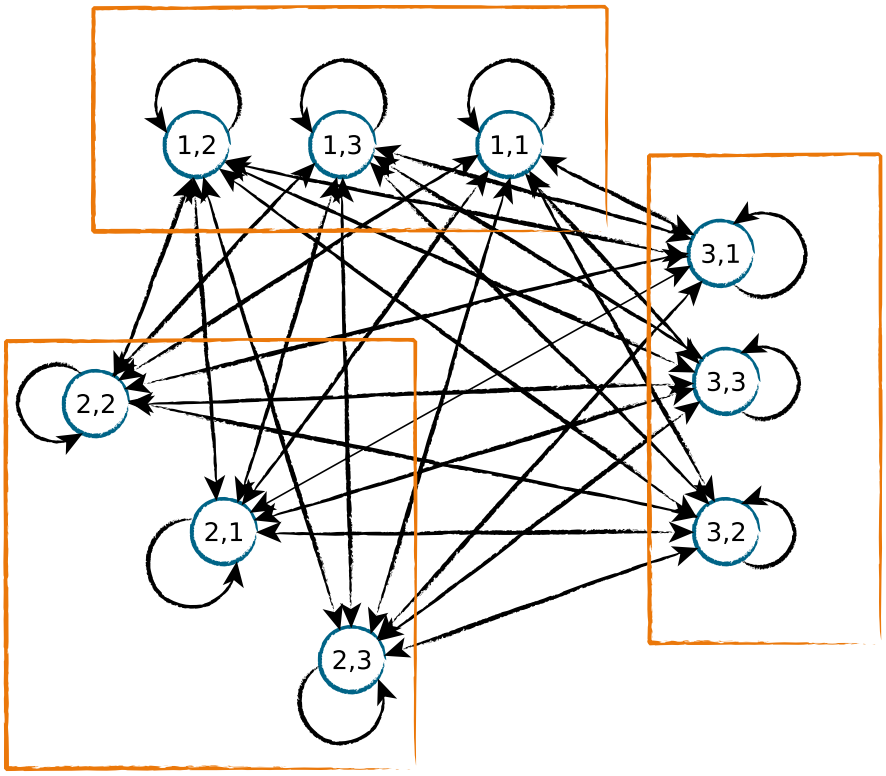
Source: Author

Example 4 (The Cluster Case)**.** *This is a combination of the cases presented before, for a subset $\mathcal{N}_D \subset \mathcal{N}$ perfect information is assumed, while for the remaining $\mathcal{N}_{ND} = \mathcal{N} - \mathcal{N}_D$ the no information case is considered, that means that, while $\theta \in \mathcal{N}_{ND}$ $\hat{\theta} = const$ is considered. In general a multitude of and $\mathcal{N}_{ND}$ can be considered, meaning that there are various clusters (which possibly emit different $\hat{\theta}$). For this example it is considered that $\mathcal{N}$ is divided in $\mathcal{N}_D = 1$ and $\mathcal{N}_{ND} = [2,3]$. The corresponding graph is shown in Figure 12. Using the detection probabilities it is modelled by:*

	$\theta = 1$	$\theta = 2$	$\theta = 3$
$P(\hat{\theta} = 1)$	1	0	0
$P(\hat{\theta} = 2)$	0	1	1
$P(\hat{\theta} = 3)$	0	0	0

$$\Pi = \begin{array}{c} \theta, \hat{\theta} \\ 1,1 \\ 1,2 \\ 1,3 \\ 2,1 \\ 2,2 \\ 2,3 \\ 3,1 \\ 3,2 \\ 3,3 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1,1 & 1,2 & 1,3 & 2,1 & 2,2 & 2,3 & 3,1 & 3,2 & 3,3 \end{array} \\ \left[\begin{array}{ccc|ccc|ccc} -0{,}5 & & & & 0{,}4 & & & 0{,}1 & \\ & & & & & & & & \\ & & & & & & & & \\ \hline & & & & & & & & \\ 0{,}2 & & & & -0{,}5 & & & 0{,}3 & \\ & & & & & & & & \\ \hline & & & & & & & & \\ 0{,}4 & & & & 0{,}2 & & & -0{,}6 & \\ & & & & & & & & \end{array}\right] \end{array}$$

Figure 13: Graphical representation of the case where there are no mutual jumps allowed



Source: Author

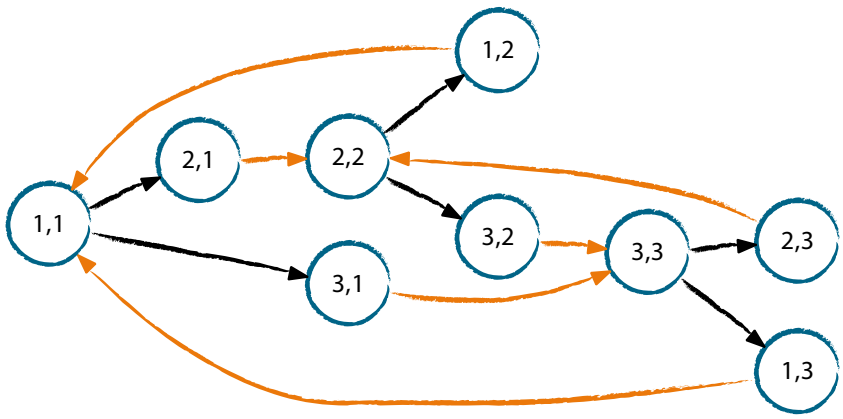
Example 5 (No mutual jumps)**.** *This case models that $\hat{\theta}$ only jumps when θ also jumps, hence the detector does not show an independent dynamic. This affects the blocks on the main diagonal which themselves exhibit a diagonal structure thus eliminating the internal dynamic of the detector. The corresponding graph is shown in Figure 13. The nodes which belong to the same mode of operation θ are enclosed by orange boxes. While it is difficult to trace every connection between the nodes, it can be seen that there is no connection between the nodes where θ is constant. For this example the following probabilities are considered:*

	$\theta = 1$	$\theta = 2$	$\theta = 3$
$P(\hat{\theta} = 1)$	0,6	0,1	0,1
$P(\hat{\theta} = 2)$	0,3	0,7	0,1
$P(\hat{\theta} = 3)$	0,1	0,2	0,8

Furthermore it is considered that these probabilities are independent from the previous mode of operation. The corresponding transition matrix turns into:

$$
\Pi = \begin{array}{c} \theta, \hat{\theta} \\ 1,1 \\ 1,2 \\ 1,3 \\ 2,1 \\ 2,2 \\ 2,3 \\ 3,1 \\ 3,2 \\ 3,3 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1,1 & 1,2 & 1,3 & 2,1 & 2,2 & 2,3 & 3,1 & 3,2 & 3,3 \end{array} \\
\left[\begin{array}{ccc|ccc|ccc}
-0,5 & & & 0,04 & 0,28 & 0,08 & 0,01 & 0,01 & 0,08 \\
 & -0,5 & & 0,04 & 0,28 & 0,08 & 0,01 & 0,01 & 0,08 \\
 & & -0,5 & 0,04 & 0,28 & 0,08 & 0,01 & 0,01 & 0,08 \\
\hline
0,12 & 0,06 & 0,02 & -0,5 & & & 0,03 & 0,03 & 0,24 \\
0,12 & 0,06 & 0,02 & & -0,5 & & 0,03 & 0,03 & 0,24 \\
0,12 & 0,06 & 0,02 & & & -0,5 & 0,03 & 0,03 & 0,24 \\
\hline
0,24 & 0,12 & 0,04 & 0,02 & 0,14 & 0,04 & -0,6 & & \\
0,24 & 0,12 & 0,04 & 0,02 & 0,14 & 0,04 & & -0,6 & \\
0,24 & 0,12 & 0,04 & 0,02 & 0,14 & 0,04 & & & -0,6
\end{array}\right]
\end{array}
$$

Figure 14: Graphical representation for the case of delayed detections



Source: Author

Example 6 (Delayed detections)**.** *This case can occur when there is a perfect detector which is able to detect any change of θ, but with a delay (see Figure 8 for an example trajectory). A graphical representation is shown in Figure 14, where the transitions of the detector are shown in orange and the transitions of the mode of operation in black. It is assumed that the dynamic of the detector is much faster than the dynamic of the Markov Process (here factor 10*

is considered) which governs the evolution of θ.

$$\Pi = \begin{array}{c} \theta,\hat{\theta} \\ 1{,}1 \\ 1{,}2 \\ 1{,}3 \\ 2{,}1 \\ 2{,}2 \\ 2{,}3 \\ 3{,}1 \\ 3{,}2 \\ 3{,}3 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1{,}1 & 1{,}2 & 1{,}3 & 2{,}1 & 2{,}2 & 2{,}3 & 3{,}1 & 3{,}2 & 3{,}3 \end{array} \\ \left[\begin{array}{ccc|ccc|ccc} -0{,}5 & & & 0{,}4 & & & 0{,}1 & & \\ 5 & -5 & & & & & & & \\ 5 & & -5 & & & & & & \\ \hline & & & -5 & 5 & & & & \\ & 0{,}2 & & & -0{,}5 & & & 0{,}3 & \\ & & & & 5 & -5 & & & \\ \hline & & & & & & -6 & & 6 \\ & & & & & & & -6 & 6 \\ & & 0{,}4 & & & 0{,}2 & & & -0{,}6 \end{array}\right] \end{array}$$

These cases represent the extreme cases where just one specific characteristic is modelled When modelling real systems a mixture of the cases presented here would be the most common case. The influence of some of the parameters on the cost function is explored in the corresponding chapters.

5 STABILITY

The goal of this section is to present LMI sufficient conditions to obtain $\mathbf{K} = \{K_l; l \in \mathscr{M}\} \in \mathscr{K}$, in a way that stability in the mean-square sense is guaranteed for the system in a closed loop. The solution is based on the Lyapunov-like equation $\mathscr{L}(\mathbf{P}) < 0$ and its dual $\mathscr{T}(\mathbf{P}) < 0$.

The problem is stated as follows: find a set of controllers $\mathscr{K}$ with $\boldsymbol{K}_l \in \mathbb{B}(\mathbb{R}^{n,m})$, $l \in \mathscr{M}$, which stabilize the closed-loop system (4.5) in a way that it is mean square stable (see Definition 4.1) for arbitrary initial conditions in $\mathscr{V}$.

In this chapter $A_{ik} := A_i + B_i K_k$ holds for $i \in \mathscr{N}$, $k \in \mathscr{M}$. Therefore the feedback-loop can be rewritten as

$$\dot{\boldsymbol{x}}(t) = A_{\theta,\hat{\theta}}\boldsymbol{x}(t) = \boldsymbol{A}_{Z(t)}\boldsymbol{x}(t). \tag{5.1}$$

5.0.1 Primal case

The first result is obtained using the Lyapunov-like equation $\mathscr{L}(\mathbf{P}) < 0$. For $\mathbf{P} = \{P_{ik}\} \in \mathbb{H}^{n+}$, $\mathbf{J} := \{J_{ik}\} \in \mathbb{H}^{n+}$, L_l, $l \in \mathscr{M}$, $j \in \mathscr{I}$, set for $(j,l) \in \mathscr{V}$,

$$\Psi_{jl} := A_j P_{jl} + P_{jl} A_j' + \sum_{(i,k)\in\mathscr{V}} \nu_{(i,k)(j,l)} P_{ik} + B_j L_l + L_l' B_j' + J_{jl}.$$

For fixed parameters $\zeta_l > 0$, the following result provides an LMI sufficient condition for the stochastic stabilizability of the system (4.5):

Theorem 5.1. *If it is possible to find* $\mathbf{P} = \{P_{ik}\} \in \mathbb{H}^{n+}$, L_l, U_l, $l \in \mathscr{M}(j)$, $j \in \mathscr{I}$, $\mathbf{J} = \{J_{ik}\} \in \mathbb{H}^{n+}$ *and* ζ_l, *such that for all* $(j,l) \in \mathscr{V}$.

$$\Phi_{jl} := \begin{bmatrix} \Psi_{jl} & B_j L_l & 0 \\ \bigstar & -\mathrm{Her}(U_l) & \zeta_l P_{jl} - U_l \\ \bigstar & \bigstar & -J_{jl} \end{bmatrix} < 0 \tag{5.2}$$

holds, then $K_l = \zeta_l L_l U_l^{-1}$ *stochastically stabilizes system (6.2).*

Proof. This proof is inspired by the proof of Theorem 4 in [43]. For simplicity, set $\bar{P}_{jl} = \zeta_l P_{jl}$.

From (5.2) it follows that $\mathrm{Her}(U_l) = U_l + U_l' > 0$ and thus U_l is non-singular, $J_{jl} > 0$ and, without loss of generality, it can be assumed that $\bar{P}_{jl} - U_l$ is non-singular (after perturbing slightly (5.2) if necessary).

From (3.2) it follows that

$$U_l'\big((\bar{P}_{jl} - U_l)J_{jl}^{-1}(\bar{P}_{jl} - U_l)'\big)^{-1}U_l \geq \mathrm{Her}(U_l) - (\bar{P}_{jl} - U_l)J_{jl}^{-1}(\bar{P}_{jl} - U_l)'. \tag{5.3}$$

Define

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & -I & -(\bar{P}_{jl} - U_l)J_{jl}^{-1} \end{bmatrix}.$$

Obviously T has full rank, and hence is invertible. Pre and pos multiplying (5.2) by T and T' results in

$$T\Phi_{jl}T' = \begin{bmatrix} \Psi_{jl} & -B_jL_l \\ \bigstar & -\mathrm{Her}(U_l) + (\bar{P}_{jl} - U_l)J_{jl}^{-1}(\bar{P}_{jl} - U_l)' \end{bmatrix} < 0. \tag{5.4}$$

Combining (5.3) and (5.4) results in

$$\begin{bmatrix} \Psi_{jl} & -B_jL_l \\ \bigstar & -U_l'\big((\bar{P}_{jl} - U_l)J_{jl}^{-1}(\bar{P}_{jl} - U_l)'\big)^{-1}U_l \end{bmatrix} < 0. \tag{5.5}$$

for $(j,l) \in \mathscr{V}$. Now applying the Schur complement yields to:

$$\Psi_{jl} + B_jL_lU_l^{-1}\big((\bar{P}_{jl} - U_l)J_{jl}^{-1}(\bar{P}_{jl} - U_l)'\big)(U_l^{-1})'(B_jL_l)' < 0. \tag{5.6}$$

Recalling that $K_l = \zeta_l L_l U_l^{-1}$ it is possible to obtain from (3.2):

$$\begin{aligned} &B_jL_lU_l^{-1}\left((\bar{P}_{jl} - U_l)J_{jl}^{-1}(\bar{P}_{jl} - U_l)'\right)\left(U_l^{-1}\right)'\left(B_jL_l\right)' \\ &= (B_jK_lP_{jl} - B_jL_l)J_{jl}^{-1}(B_jK_lP_{jl} - B_jL_l)' \\ &\geq (B_jK_lP_{jl} - B_jL_l) + (B_jK_lP_{jl} - B_jL_l)' - J_{jl}. \end{aligned} \tag{5.7}$$

Combining (5.6) and (5.7) results in

$$\begin{aligned} 0 &> \Psi_{jl} + B_jL_lU_l^{-1}\big((\bar{P}_{jl} - U_l)J_{jl}^{-1}(\bar{P}_{jl} - U_l)'\big)(B_jL_lU_l^{-1})' \\ &\geq \Psi_{jl} + (B_jK_lP_{jl} - B_jL_l) + (B_jK_lP_{jl} - B_jL_l)' - J_{jl} \\ &= (A_j + B_jK_l)P_{jl} + P_{jl}(A_j + B_jK_l)' + \sum_{(i,k)\in\mathscr{V}} \nu_{(i,k)(j,l)}P_{ik}. \end{aligned} \tag{5.8}$$

From (5.8) it follows that $\mathscr{L}(\mathbf{P}) < 0$ holds, completing the proof. □

Remark 5.1. *For the perfect information case ($\widehat{\theta}(t) = \theta(t)$) with $\mathscr{V} = \{(i,i); i \in \mathscr{N}\}$ from Theorem 3.1 iii) it follows that if the system (3.5) is stochastically stabilizable then there exists K_i and $P_{ii} > 0$, $i \in \mathscr{N}$ such that for all $j \in \mathscr{N}$,*

$$(A_j + B_j K_j)P_{jj} + P_{jj}(A_j + B_j K_j)' + \sum_{i \in \mathscr{N}} \nu_{(i,i)(j,j)} P_{ii} < 0. \tag{5.9}$$

Considering $\zeta_j = \zeta$ for all j, $L_j = K_j P_{jj}^{-1}$, $U_j = \zeta P_{jj}$, $J_{jj} = \varepsilon I$, it is easy to see that (5.9) implies that $\mathrm{Her}(A_j P_{jj} + B_j L_j) + \sum_{i \in \mathscr{N}} \nu_{(i,i)(j,j)} P_{ii} + \varepsilon I + \frac{1}{2\zeta} B_j L_j P_{jj}^{-1} L_j' B_j' < 0$ *for ε sufficiently small and ζ sufficiently large, and thus from Schur complement we have that (5.2) will hold.*

5.0.2 Dual formulation

In the following the dual Lyapunov-like equation $\mathscr{T}(\mathbf{P}) < 0$ is used to obtain $\mathbf{K} = \{K_l; l \in \mathscr{M}\} \in \mathscr{K}$ using an LMI sufficient condition.

The following definitions are used: for $\mathbf{X} = \{X_{ik}\} \in \mathbb{H}^{n+}$, $\mathbf{J} = \{J_{ik}\} \in \mathbb{H}^{n+}$, L_l, $l \in \mathscr{M}(j)$, $j \in \mathscr{I}$, T_{ik}, $\mathbf{Q}_{ik} = \{Q_{(ik),(jl)}\} \in \mathbb{H}^{n+}$ for $(i,k) \in \mathscr{V}$, set

$$\begin{aligned}
\Upsilon_{ik} &:= X_{ik} A_i' + A_i X_{ik} + B_i L_k + L_k' B_i' + \nu_{(i,k)(i,k)} X_{ik} + J_{ik} \\
\Delta_{ik} &:= -\mathrm{Her}(T_{ik}) + \sum_{(j,l) \in \mathscr{I}(i,k)} \nu_{(i,k)(j,l)} Q_{(ik),(jl)}.
\end{aligned}$$

Theorem 5.2. *If it is possible to find $\mathbf{X} = \{X_{ik}\} \in \mathbb{H}^{n+}$, L_k, U_k, T_{ik}, $\mathbf{Q}_{ik} = \{Q_{(ik),(jl)}\} \in \mathbb{H}^{n+}$, $k \in \mathscr{M}(i)$, $i \in \mathscr{I}$, $\mathbf{J} = \{J_{ik}\} \in \mathbb{H}^{n+}$, for fixed scalars $\zeta_l > 0$, for all $(i,k) \in \mathscr{V}$, that*

$$\begin{bmatrix} \Upsilon_{ik} & X_{ik} & B_i L_k & 0 \\ \star & \Delta_{ik} & 0 & 0 \\ \star & \star & -\mathrm{Her}(U_k) & \zeta_k X_{ik} - U_k \\ \star & \star & \star & -J_{ik} \end{bmatrix} < 0 \tag{5.10}$$

$$\begin{bmatrix} Q_{(ik),(jl)} & T_{ik}' \\ \star & X_{jl} \end{bmatrix} > 0, \ (j,l) \in \mathscr{V} \tag{5.11}$$

holds, then $K_k = \zeta_k L_k U_k^{-1}$ stochastically stabilizes system (4.5).

Proof. As shown in [14], we have from (5.11) and (3.2) that $\Delta_{ik} \geq -\widetilde{X}_{ik}$, where

$$\widetilde{X}_{ik} := \Big(\sum_{(j,l) \in \mathscr{I}(i,k)} \nu_{(i,k)(j,l)} X_{jl}^{-1} \Big)^{-1}.$$

From this, (5.10), and applying a transformation as in (5.4) we get from (3.2) that

$$\begin{bmatrix} \Upsilon_{ik} & X_{ik} & -B_iL_k \\ \bigstar & -\widetilde{X}_{ik} & 0 \\ \bigstar & \bigstar & -U_k'\big((\bar{X}_{ik}-U_k)J_{ik}^{-1}(\bar{X}_{ik}-U_k)'\big)^{-1}U_k \end{bmatrix} < 0 \tag{5.12}$$

where $\bar{X}_{ik} = \zeta_k X_{ik}$. From (5.12) and the Schur complement we get that $\Upsilon_{ik} + X_{ik}\widetilde{X}_{ik}^{-1}X_{ik} + B_iL_kU_k^{-1}\big((\bar{X}_{ik}-U_k)J_{ik}^{-1}(\bar{X}_{ik}-U_k)'\big)(U_k^{-1})'(B_iL_k)' < 0$. Recalling that $K_k = \zeta_k L_k U_k^{-1}$ we obtain from the last inequality and a similar reasoning as in the proof of Theorem 5.1 that $\mathscr{T}(\mathbf{P}) < 0$ is satisfied with $P_{ik} = X_{ik}^{-1}$, completing the proof. □

Remark 5.2. *For the perfect information case ($\widehat{\theta}(t) = \theta(t)$) with $\mathscr{V} = \{(i,i); i \in \mathscr{N}\}$ from Theorem 3.1 v) it follows that, if system* (3.5) *is stochastically stabilizable, then there exists K_i and $P_{ii} > 0$, $i \in \mathscr{N}$ such that for all $j \in \mathscr{N}$, $P_{ii}(A_i+B_iK_i)+(A_i+B_iK_i)'P_{ii}+\sum_{j\in\mathscr{N}} \nu_{(i,i)(j,j)}P_{jj} <$* 0 *and thus*

$$\mathrm{Her}(A_iX_{ii}+B_iL_i)+X_{ii}\Big(\sum_{j\in\mathscr{N}} \nu_{(i,i)(j,j)}X_{jj}^{-1}\Big)X_{ii} < 0 \tag{5.13}$$

where $X_{ii} = P_{ii}^{-1}$, $L_i = K_iP_{ii}^{-1}$.
Now considering $\zeta_i = \zeta$ for all i, $U_i = \zeta P_{ii}^{-1}$, $J_{ii} = \varepsilon I$, $T_{ii} = \Big(\sum_{j\neq i} \nu_{(i,i)(j,j)}X_{jj}^{-1}\Big)^{-1}$ and $Q_{(ii),(jj)} = T_{ii}P_{jj}T_{ii}+\varepsilon I$, it is easy to see that (5.11) is satisfied and that $\Delta_{ii} = -T_{ii} - \nu_{(i,i)(i,i)}\varepsilon I$. From (5.13) it follows that, if it is possible to find $\varepsilon_0 > 0$ sufficiently small and ζ sufficiently large such that $\mathrm{Her}(A_iX_{ii}+B_iL_i)+\nu_{(i,i)(i,i)}X_{ii}-X_{ii}\Delta_{ii}^{-1}X_{ii}+\varepsilon_0 I+\frac{1}{2\zeta}B_iL_iX_{ii}L_i'B_i' < 0$ *, and thus using the Schur complement (5.10) holds.*

Remark 5.3. *Remarks 5.1 and 5.2 show that for the perfect information case the LMIs in Theorems 5.1 and 5.2 will provide for ε sufficiently small and ζ sufficiently large a necessary and sufficient condition for the existence of a stochastic stabilizing controller for (4.5). In general the conditions are just sufficient and it is not possible to determine if one is stronger than the other. Thus it is highly recommended to test both methods and choose the one which leads to the best result for the given problem.*

6 H_2-CONTROL

This chapter discusses the H_2-control problem for MJLS, in particular the state-feedback case. The problem is stated as follows: Consider the system

$$\Sigma_{H2}: \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_\theta \boldsymbol{x}(t) + \boldsymbol{B}_\theta \boldsymbol{u}(t) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0,\ \theta(0) = \theta_0 \end{cases} \tag{6.1}$$

where the matrices $\boldsymbol{A}_\theta$ and $\boldsymbol{B}_\theta$ depend on the mode of operation as described in the previous chapter. The goal is to find controllers $\boldsymbol{K} = \{\boldsymbol{K}_l\} \in \mathscr{K}$ that stabilize the closed loop system

$$\Sigma_{H2cl}: \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_\theta \boldsymbol{x}(t) + \boldsymbol{B}_\theta \boldsymbol{K}_{\hat{\theta}} \boldsymbol{x}(t) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0,\ \theta(0) = \theta_0 \end{cases} \tag{6.2}$$

in the sense of mean square stability and has guaranteed H_2 cost.

The chapter is divided in 3 sections: the first section discusses the problem of stabilizing the system in the mean square sense. In the following section the solution is extended to the H_2 control problem. The chapter finishes with numerical results in the last section.

6.1 The H_2-control problem

This section extends the previous result for mean-square-stability by including a measure of optimality. Specifically the notion of H_2-Optimality is used.

The system

$$\Sigma_\theta: \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_\theta \boldsymbol{x}(t) + \boldsymbol{B}_\theta \boldsymbol{u}(t) + \boldsymbol{H}_\theta \boldsymbol{w}(t) \\ \boldsymbol{z}(t) = \boldsymbol{C}_\theta \boldsymbol{x}(t) + \boldsymbol{D}_\theta \boldsymbol{u}(t) \\ \boldsymbol{x}(0) = 0,\ \theta(0) = \theta_0 \end{cases} \tag{6.3}$$

with $\boldsymbol{H}_i \in \mathbb{B}(\mathbb{R}^{n,r})$ for each $i \in \mathscr{N}$ should be stabilized by controllers $\boldsymbol{K} = \{\boldsymbol{K}_l\} \in \mathscr{K}$ using state-feedback, such that the closed loop system (6.2) is mean-square-stable (see Definition 4.1)

and has a guaranteed H_2 cost (see Definition 3.7). As before, it is assumed that the controller depends on the estimated mode of operation $\hat{\theta}$ and not on θ.

6.1.1 Primal Case

The following result provides an LMI optimization problem in order to get K_l that stabilizes (6.2) with a guaranteed H_2 cost, based on the controllability Gramian and Theorem 5.1.

Theorem 6.1. *For fixed scalars $\zeta_l > 0$ if we can find $\mathbf{P} = \{P_{ik}\} \in \mathbb{H}^{n+}$, $\mathbf{W} = \{W_{ik}\} \in \mathbb{H}^{n+}$, L_l, U_l, $l \in \mathscr{M}(j)$, $j \in \mathscr{I}$, $\mathbf{J} = \{J_{ik}\} \in \mathbb{H}^{n+}$, solution of the following LMI optimization problem:*

$$\min \sum_{(i,k)\in\mathscr{V}} \operatorname{tr}(W_{ik})$$

subject, for all $(j,l) \in \mathscr{V}$, to

$$\begin{bmatrix} W_{jl} & \zeta_l^{-1/2} C_j U_l + \zeta_l^{1/2} D_j L_l \\ \bigstar & \operatorname{Her}(U_l) - \zeta_l P_{jl} \end{bmatrix} > 0 \tag{6.4}$$

$$\begin{bmatrix} \Psi_{jl} + \eta_{jl} H_j H_j' & B_j L_l & 0 \\ \bigstar & -\operatorname{Her}(U_l) & \zeta_l P_{jl} - U_l \\ \bigstar & \bigstar & -J_{jl} \end{bmatrix} < 0 \tag{6.5}$$

then $K_l = \zeta_l L_l U_l^{-1}$ stochastically stabilizes system (6.3) and

$$\inf_{\bar{\mathbf{K}}\in\mathscr{K}} \|\Sigma_{\bar{\mathbf{K}}}\|_2^2 \le \sum_{(i,k)\in\mathscr{V}} \operatorname{tr}(W_{ik}). \tag{6.6}$$

Proof. From Theorem 5.1 and (6.5) it follows that $K_l = L_l U_l^{-1}$ stabilizes system (6.2) and moreover $\mathscr{L}(\mathbf{P}) + \eta\mathbf{N} < 0$. Therefore it is possible to find $\mathbf{R} = \{R_{jl}\} \in \mathbb{H}^{n+}$ such that $\mathscr{L}(\mathbf{P}) + \eta\mathbf{N} + \mathbf{R} = 0$. From Theorem 4.1 it follows that $\|\Sigma_{\mathbf{K}}\|_2^2 = \sum_{(j,l)\in\mathscr{V}} \operatorname{tr}\big((C_j + D_j K_l)\widehat{P}_{jl}(C_j + D_j K_l)'\big)$, where $\mathbf{K} = \{K_l\}$ and $\widehat{\mathbf{P}} = \{\widehat{P}_{jl}\} \in \mathbb{H}^{n+}$ is the unique solution of the controllability Gramian $\mathscr{L}(\widehat{\mathbf{P}}) + \eta\mathbf{N} = 0$. Thus $\mathscr{L}(\mathbf{P} - \widehat{\mathbf{P}}) + \mathbf{R} = 0$ and from Theorem 3.25 in [17] it follows that $\mathbf{P} - \widehat{\mathbf{P}} \ge 0$. If we show that $W_{jl} \ge (C_j + D_j K_l) P_{jl} (C_j + D_j K_l)'$ then clearly we have that

$$\inf_{\bar{\mathbf{K}}\in\mathscr{K}} \|\Sigma_{\bar{\mathbf{K}}}\|_2^2 \le \|\Sigma_{\mathbf{K}}\|_2^2 = \sum_{(j,l)\in\mathscr{V}} \operatorname{tr}\big((C_j + D_j K_l)\widehat{P}_{jl}(C_j + D_j K_l)'\big)$$
$$\le \sum_{(j,l)\in\mathscr{V}} \operatorname{tr}\big((C_j + D_j K_l) P_{jl} (C_j + D_j K_l)'\big) \le \sum_{(j,l)\in\mathscr{V}} \operatorname{tr}\big(W_{jl}\big)$$

showing (6.6). From (3.2) and (6.4) it follows that

$$\begin{bmatrix} W_{jl} & \zeta_l^{-1/2} C_j U_l + \zeta_l^{1/2} D_j L_l \\ \bigstar & \frac{1}{\zeta_l} U_l' P_{jl}^{-1} U_l \end{bmatrix} > 0. \tag{6.7}$$

Using the Schur complement and (6.7):

$$\begin{aligned} W_{jl} &> (C_jU_l+\zeta_lD_jL_l)U_l^{-1}P_{jl}(U_l^{-1})'(C_jU_l+\zeta_lD_jL_l)' \\ &= (C_j+D_jK_l)P_{jl}(C_j+D_jK_l)' \end{aligned}$$

completing the proof. □

Remark 6.1. *Using the same reasoning as in Remark 5.1 for the perfect information case and considering* $W_{ii}=(C_i+D_iK_i)P_{ii}(C_i+D_iK_i)'+\varepsilon I$ *in (6.4), follows that for* ε *sufficiently small and* ζ *sufficiently large the LMI optimization problem in Theorem 6.1 will provide an* ε'*-optimal solution for the* H_2 *control problem.*

6.1.2 Dual Case

The following LMI optimization problem is based on the observability Gramian and Theorem 5.2. It provides a solution to calculate K_l that stabilizes (6.2) with a guaranteed H_2 cost.

Theorem 6.2. *For fixed scalars* $\zeta_l>0$ *and* $\varepsilon>0$ *if it is possible to find* $\mathbf{X}=\{X_{ik}\}\in\mathbb{H}^{n+}$, L_k, U_k, T_{ik}, $\mathbf{Q}_{ik}=\{Q_{(ik),(jl)}\}\in\mathbb{H}^{n+}$, $k\in\mathscr{M}(i)$, $i\in\mathscr{I}$, $\mathbf{J}=\{J_{ik}\}\in\mathbb{H}^{n+}$, $\mathbf{Z}=\{Z_{ik}\}\in\mathbb{H}^{n+}$, $\Xi=\{\Xi_{ik}\}\in\mathbb{H}^{n+}$, $\mathbf{Y}=\{Y_{ik}\}\in\mathbb{H}^{n+}$, *solution of the following LMI optimization problem:*

$$\min \sum_{(i,k)\in\mathscr{V}} \eta_{ik}\mathrm{tr}(\Xi_{ik})$$

subject, for all $(i,k)\in\mathscr{V}$, *to* (5.11) *and*

$$\begin{bmatrix} \Upsilon_{ik}+Z_{ik} & X_{ik} & B_iL_k & 0 \\ \bigstar & \Delta_{ik} & 0 & 0 \\ \bigstar & \bigstar & -\mathrm{Her}(U_k) & \zeta_kX_{ik}-U_k \\ \bigstar & \bigstar & \bigstar & -J_{ik} \end{bmatrix} < 0 \tag{6.8}$$

$$\begin{bmatrix} \varepsilon Z_{ik} & X_{ik} \\ \bigstar & \mathrm{Her}(U_k)-\varepsilon Y_{ik} \end{bmatrix} > 0, \quad \begin{bmatrix} \Xi_{ik} & H_i' \\ \bigstar & X_{ik} \end{bmatrix} > 0 \tag{6.9}$$

$$\begin{bmatrix} Y_{ik} & (C_iU_k+\zeta_kD_iL_k)' \\ \bigstar & I \end{bmatrix} > 0 \tag{6.10}$$

then $K_k=\zeta_kL_kU_k^{-1}$ *stochastically stabilizes system (6.3) and*

$$\inf_{\bar{\mathbf{K}}\in\mathscr{K}} \|\Sigma_{\bar{\mathbf{K}}}\|_2^2 \le \sum_{(i,k)\in\mathscr{V}} \eta_{ik}\mathrm{tr}(\Xi_{ik}). \tag{6.11}$$

Proof. From Lemma 1 in [44] and the first inequality in (6.9) it follows that

$$Z_{ik} > X_{ik}(U_k^{-1})' Y_{ik} U_k^{-1} X_{ik} \tag{6.12}$$

and by using (6.10):

$$Y_{ik} > (C_i U_k + \zeta_k D_i L_k)'(C_i U_k + \zeta_k D_i L_k).$$

Combining these results leads to

$$Z_{ik} > X_{ik}(C_i + D_i K_k)'(C_i + D_i K_k) X_{ik} \tag{6.13}$$

Following the same reasoning as in Theorem 5.2, from (5.11) and (6.8) it follows that

$$\Upsilon_{ik} + Z_{ik} + X_{ik}\widetilde{X}_{ik}^{-1} X_{ik} + B_i L_k U_k^{-1}\big((\bar{X}_{ik} - U_k) J_{ik}^{-1} (\bar{X}_{ik} - U_k)'\big)(U_k^{-1})'(B_i L_k)' < 0 \tag{6.14}$$

Using the results presented before and recalling that $K_k = \zeta_k L_k U_k^{-1}$ it follows that:

$$\begin{aligned}&\Upsilon_{ik} + X_{ik}(C_i + D_i K_k)'(C_i + D_i K_k) X_{ik} + X_{ik}\widetilde{X}_{ik}^{-1} X_{ik} \\ &\quad + B_i L_k U_k^{-1}\big((\bar{X}_{ik} - U_k) J_{ik}^{-1} (\bar{X}_{ik} - U_k)'\big)(U_k^{-1})'(B_i L_k)' < 0\end{aligned}$$

From the last inequality and a similar reasoning as in the proof of Theorem 5.2 it follows that $\mathscr{T}(\mathbf{P}) + \mathbf{M} < 0$ with $P_{ik} = X_{ik}^{-1}$. Take $\widehat{\mathbf{P}} = \{\widehat{P}_{jl}\} \in \mathbb{H}^{n+}$ the unique solution of the observability Gramian $\mathscr{T}(\widehat{\mathbf{P}}) + \mathbf{M} = 0$. Thus $\mathscr{T}(\mathbf{P} - \widehat{\mathbf{P}}) + \mathbf{R} = 0$ where $\mathbf{R} = \{R_{jl}\} \in \mathbb{H}^{n+}$ is such that $\mathscr{T}(\mathbf{P}) + \mathbf{M} + \mathbf{R} = 0$. From Theorem 3.25 in [17] it follows that $\mathbf{P} - \widehat{\mathbf{P}} \geq 0$. Using the second inequality in (6.9) results in $\Xi_{ik} > H_{ik}' P_{ik} H_{ik} \geq H_{ik}' \widehat{P}_{ik} H_{ik}$, and thus $\inf_{\bar{\mathbf{K}} \in \mathscr{K}} \|\Sigma_{\bar{\mathbf{K}}}\|_2^2 \leq \|\Sigma_{\mathbf{K}}\|_2^2 = \sum_{(i,k)\in\mathscr{V}} \eta_{ik} \mathrm{tr}\big(H_{ik}' \widehat{P}_{ik} H_{ik}\big) \leq \sum_{(i,k)\in\mathscr{V}} \eta_{ik} \mathrm{tr}\big(\Xi_{ik}\big)$ showing (6.11). □

Remark 6.2. *For the primal case (Theorem 6.1) it follows that the maximum number of matrix variables is $n_V = 3NM + 2M$ and the the maximum number of LMIs is $n_L = 2NM$. The dual case (Theorem 6.2) results in $n_V = 6NM + (NM)^2 + 2M$ and $n_L = 4NM$, being thus more computationally demanding.*

As pointed out in Remark 5.3 these conditions are only sufficient and it is not possible to judge if one is stronger than the other.

6.2 Numerical Example

The following section presents a numerical evaluation of the results obtained before. The plant shown in Example 1 is used to evaluate the effects of the two parameters α_{jl}^k and q_{kl}^i on the H_2 upper-bound-cost.

The number of modes of operation θ is considered as 3. This represents the worst-case scenario where in every mode of operation the detector could detect any mode of operation.

For the evaluation MATLAB is used in combination with the YALMIP-Toolbox [59] and the solver MOSEK.

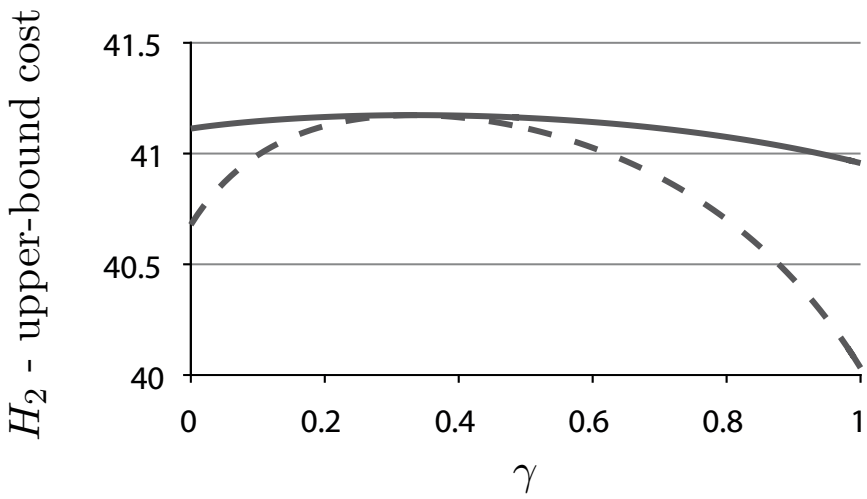
Recall from (4.2) that λ_{ij} represents the transition rate for $\theta(t)$, q_{kl}^{i} the transition rate for $\widehat{\theta}(t)$ conditioned on $\theta(t) = i$. The factor α_{jl}^{k} describes the probability of a jump of the observation state $\widehat{\theta}(t)$ from state k to state l whenever there is a jump of the hidden Markov parameter $\theta(t)$ from a state i to a state j. Hence, by varying this parameter it is possible to describe the probability of the right or wrong detection of a jump. For simplicity it is assumed that all α_{jl}^{k} are modified equally as follows (recall that $N = 3$):

$$\alpha_{jl}^{k} = \begin{cases} \gamma & \text{for } j = l \\ (1-\gamma)/(N-1) & \text{for } j \neq l \end{cases}$$

and also that q_{kl}^{i} will be equal for all i, k, and l. The dashed line in Figure 15 shows the H_2 upper bound cost in function of γ with $q_{kl}^{i} = 0$ while the solid line shows the cost with $q_{kl}^{i} = 1$ for all i,k,l. As expected, the upper-bound H_2 cost increases for a greater value of q_{kl}^{i} since this parameter increases the uncertainty of the detector. Both curves reach their minimum at $\gamma = 1$. The highest cost is located at $\gamma = 1/3$ where the detection probabilities are equally likely to $1/3$. In this case the feedback controller matrices are equal for all modes of operation, and hence the information from the detector is useless (this case is equivalent to the robust control case). According to Remark 6.2 the number of variables in this case for the primal problem is equal to 33 and the total number of LMIs is 18. On a standard system (i5 / Windows 7 64 Bit, 8GB Ram, Matlab 2015b, YALMIP-Toolbox [59] and the solver MOSEK) this LMI-condition is solved in less than 0.4 seconds. For the dual case there are 141 variables and 36 LMIs, and the computation time was 2.7 seconds.

Figure 16 shows the behaviour of the state variable $x_3 = h_1$ corresponding to the switches given by $\theta(t)$ and $\widehat{\theta}(t)$. It is assumed that at $t = 2.3s$ the lower valve is affected by a failure which reduces the flow to 25% of its original value. Thus at this moment the mode of operation (θ) jumps from 1 to 2. As we allow random jumps of the detector ($q_{kl}^{i} = 1$) at $t = 0.1s$ we have that $\widehat{\theta}$ jumps from 1 to 2, which represents a false alarm. Since the system is in a steady-state condition at this moment, this false alarm does not have an impact on the state x_3, as shown in Figure 16. At time $t = 3s$ the reference value for the water level in tank T_1 is changed to $h_1 = 0.1$. It can be seen that the system tracks this change, even though θ and $\widehat{\theta}$ do not

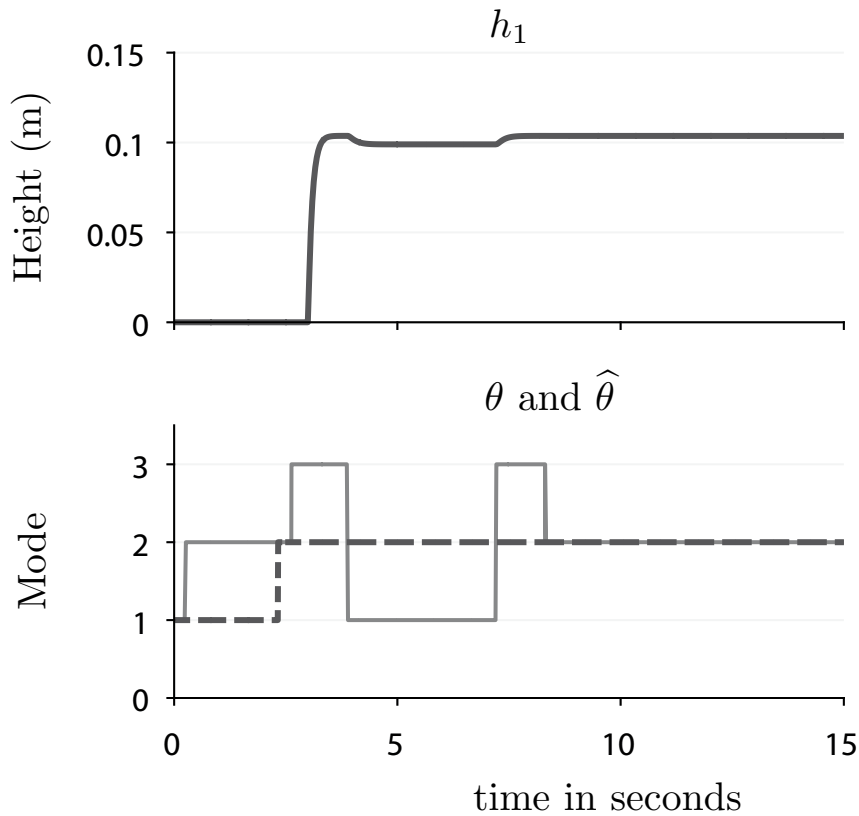
Figure 15: H_2 upper-bound cost for the two-tanks example in function of the parameters γ and $q_{kl}^i = 0$ (dashed line), $q_{kl}^i = 1$ (continuous line), and $\zeta = 40$



Source: Author

correspond to each other at all times and the controller in some moments is reconfigured for another mode of operation.

Figure 16: Simulation of the Systems behaviour, $x_3(t) = h_1(t)$ (in meters), $\theta(t)$ (dashed) and $\widehat{\theta}(t)$ (solid) for the two-tanks system



Source: Author

7 H_∞-CONTROL

This chapter considers the H_∞ control problem. The goal is to control the system (4.12) with an output feedback law using only the observed mode of operation $\hat{\theta}$, that is, in the form

$$\boldsymbol{u}(t) = \boldsymbol{K}_{\hat{\theta}}\boldsymbol{y}(t) \tag{7.1}$$

such that the closed loop system

$$\Sigma_{cl} : \begin{cases} \dot{\boldsymbol{x}}(t) = \left(\boldsymbol{A}_\theta + \boldsymbol{B}_\theta \boldsymbol{K}_{\hat{\theta}} \boldsymbol{E}_\theta\right) \boldsymbol{x}(t) + \boldsymbol{H}_\theta \boldsymbol{w}(t) \\ z(t) = \left(\boldsymbol{C}_\theta + \boldsymbol{D}_\theta \boldsymbol{K}_{\hat{\theta}} \boldsymbol{E}_\theta\right) \boldsymbol{x}(t) + \boldsymbol{F}_\theta \boldsymbol{w}(t) \end{cases} \tag{7.2}$$

is internally mean square stable and has a guaranteed H_∞ cost (see Definitions 3.5 and 3.8).

7.1 Main results

First the following result is recalled (see [99]). From Assumption 4.3 there exist non-singular matrices $\boldsymbol{T}_j$ such that for each $j \in \mathscr{N}$,

$$\boldsymbol{E}_j \boldsymbol{T}_j = \begin{bmatrix} \boldsymbol{I} & 0 \end{bmatrix}. \tag{7.3}$$

In the remainder the sum in element (1,1) in the LMI of Lemma 4.2 is represented in the following way:

$$\begin{aligned} \sum_{(j,l)\in\mathscr{V}} \nu_{(i,k)(j,l)} \boldsymbol{R}_{ik} &= \sum_{(i,k)\in\mathscr{V}_{(j,l)}} \nu_{(i,k)(j,l)} \boldsymbol{X}_{ik}^{-1} + \nu_{(i,k)(i,k)} \boldsymbol{X}_{jl}^{-1} \\ &= \boldsymbol{\Pi}_{jl} \mathscr{D}_{jl}^{-1} \boldsymbol{\Pi}_{jl} + \nu_{(j,l)(j,l)} \boldsymbol{X}_{jl}^{-1} \end{aligned} \tag{7.4}$$

where

$$\begin{aligned}
\mathscr{V}_{(j,l)} &= \{(i,l) \in \mathscr{V}; (i,l) \neq (j,k) \text{ and } \nu_{(i,k)(j,l)} \neq 0\} \\
&= \{r_{(j,l)}(1), \ldots, r_{(j,l)}(\tau_{(j,l)}); r_{(j,l)}(\iota) \in \mathscr{V}, \iota = 1, \ldots, \tau_{(j,l)}\} \\
\Pi_{jl} &= \left[\sqrt{\nu_{(j,l)r_{(j,l)}(1)}}\boldsymbol{I} \quad \cdots \quad \sqrt{\nu_{(j,l)r_{(j,l)}(\tau_{(j,l)})}}\boldsymbol{I}\right] \\
\mathscr{D}_{jl} &= diag(\boldsymbol{X}_{r_{(j,l)}(1)}, \ldots, \boldsymbol{X}_{r_{(j,l)}(\tau_{(j,l)})})
\end{aligned}$$

and $\boldsymbol{X}_{jl} = \boldsymbol{R}_{jl}^{-1}$.

The next Theorem presents, through an LMI constraint, a way to design a controller so that system (2) is internally mean-square-stable with an H_∞ cost smaller than γ.

Theorem 7.1. *The system* (4.5) *is iMSS with an H_∞ - cost smaller than γ if there are matrices $\boldsymbol{X}_{jl} > 0$, $\boldsymbol{G}_l$ and $\boldsymbol{V}_l$, and scalars $\varepsilon_j > 0$ with $j \in \mathscr{N}$ and $l \in \mathscr{M}$ such that the following set of LMI is satisfied:*

$$\begin{bmatrix} \nu_{(j,l)(j,l)}\boldsymbol{X}_{jl} & \boldsymbol{H}_j & \boldsymbol{0} & \boldsymbol{X}_{jl} & \boldsymbol{X}_{jl}\Pi_{jl} \\ \boldsymbol{H}_j' & -\gamma\boldsymbol{I} & \boldsymbol{F}_j' & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{F}_j & -\gamma\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{X}_{jl}' & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\Pi}_{jl}'\boldsymbol{X}_{jl}' & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\mathscr{D}_{jl} \end{bmatrix} + \\ + \mathrm{Her}\left(\begin{bmatrix} \boldsymbol{A}_j\boldsymbol{T}_j\boldsymbol{G}_l + \boldsymbol{B}_j\begin{bmatrix}\boldsymbol{V}_l & \boldsymbol{0}\end{bmatrix} \\ \boldsymbol{0} \\ \boldsymbol{C}_j\boldsymbol{T}_j\boldsymbol{G}_l + \boldsymbol{D}_j\begin{bmatrix}\boldsymbol{V}_l & \boldsymbol{0}\end{bmatrix} \\ -\boldsymbol{T}_j\boldsymbol{G}_l \\ \boldsymbol{0} \end{bmatrix}\begin{bmatrix}\varepsilon_j\boldsymbol{I} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{I} \\ \boldsymbol{0}\end{bmatrix}'\right) < 0 \tag{7.5}$$

for all $(j,l) \in \mathscr{V}$, with $\boldsymbol{G}_l$ in the following form:

$$\boldsymbol{G}_l = \begin{bmatrix} \boldsymbol{G}_{l1} & 0 \\ \boldsymbol{G}_{l2} & \boldsymbol{G}_{l3} \end{bmatrix}. \tag{7.6}$$

Moreover the feedback controller matrices are given by:

$$\boldsymbol{K}_l = \boldsymbol{V}_l\boldsymbol{G}_{l1}^{-1}, \; k \in \mathscr{M}. \tag{7.7}$$

Proof. The proof is inspired by the proofs in [81] and [99]. Suppose that there are matrices $\boldsymbol{X}_{jl} > 0$, $\boldsymbol{G}_l$ and $\boldsymbol{V}_l$, and scalars $\varepsilon_j > 0$ with $j \in \mathscr{N}$ and $k \in \mathscr{M}$ such that (7.5) and (7.6) are satisfied. From (7.5) we get that $\boldsymbol{T}_j\boldsymbol{G}_l + \boldsymbol{G}_l'\boldsymbol{T}_j' > 0$ and since $\boldsymbol{T}_j$ is non-singular, we get that $\boldsymbol{G}_l + \boldsymbol{G}_l' > 0$, so that $\boldsymbol{G}_l$ is non-singular. From this and (7.6) we get that $\boldsymbol{G}_{l1}$ is non-singular, so

that $\boldsymbol{K}_l$ in (7.7) is well defined. Noticing that $\boldsymbol{V}_l = \boldsymbol{K}_l \boldsymbol{G}_{l1}$ we get that

$$\begin{bmatrix} \boldsymbol{K}_l & 0 \end{bmatrix} \boldsymbol{G}_l = \begin{bmatrix} \boldsymbol{K}_l & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{G}_{l1} & \boldsymbol{0} \\ \boldsymbol{G}_{l2} & \boldsymbol{G}_{l3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{V}_l & 0 \end{bmatrix}. \tag{7.8}$$

Thus from (7.3),

$$\begin{aligned} \boldsymbol{A}_j \boldsymbol{T}_j \boldsymbol{G}_l + \boldsymbol{B}_j \begin{bmatrix} \boldsymbol{V}_l & \boldsymbol{0} \end{bmatrix} &= \boldsymbol{A}_j \boldsymbol{T}_j \boldsymbol{G}_l + \boldsymbol{B}_j \begin{bmatrix} \boldsymbol{K}_l & \boldsymbol{0} \end{bmatrix} \boldsymbol{G}_l \\ &= \boldsymbol{A}_j \boldsymbol{T}_j \boldsymbol{G}_l + \boldsymbol{B}_j \boldsymbol{K}_l \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \boldsymbol{G}_l \\ &= \left(\boldsymbol{A}_j + \boldsymbol{B}_j \boldsymbol{K}_l \boldsymbol{E}_j \right) \boldsymbol{T}_j \boldsymbol{G}_l. \end{aligned}$$

Using the same reasoning we get that

$$\boldsymbol{C}_j \boldsymbol{T}_j \boldsymbol{G}_l + \boldsymbol{D}_j \begin{bmatrix} \boldsymbol{V}_l & 0 \end{bmatrix} = \left(\boldsymbol{C}_j + \boldsymbol{D}_j \boldsymbol{K}_l \boldsymbol{E}_j \right) \boldsymbol{T}_j \boldsymbol{G}_l.$$

Thus we conclude that (7.5) is equivalent to

$$\begin{aligned} &\begin{bmatrix} \nu_{(j,l)(j,l)} \boldsymbol{X}_{jl} & \boldsymbol{H}_j & \boldsymbol{0} & \boldsymbol{X}_{jl} & \boldsymbol{X}_{jl} \boldsymbol{\Pi}_{jl} \\ \boldsymbol{H}_j' & -\gamma \boldsymbol{I} & \boldsymbol{F}_j' & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{F}_j & -\gamma \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{X}_{jl}' & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\Pi}_{jl}' \boldsymbol{X}_{jl}' & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\mathscr{D}_{jl} \end{bmatrix} + \\ &+ \mathrm{Her} \left(\begin{bmatrix} \boldsymbol{A}_j + \boldsymbol{B}_j \boldsymbol{K}_l \boldsymbol{E}_j \\ \boldsymbol{0} \\ \boldsymbol{C}_j + \boldsymbol{D}_j \boldsymbol{K}_l \boldsymbol{E}_j \\ -\boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{T}_j \boldsymbol{G}_l \begin{bmatrix} \varepsilon_j \boldsymbol{I} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix}' \right) < 0. \end{aligned} \tag{7.9}$$

Set $\tilde{\boldsymbol{A}}_{jl} = (\boldsymbol{A}_j + \boldsymbol{B}_j \boldsymbol{K}_l \boldsymbol{E}_j)$, $\tilde{\boldsymbol{C}}_{jl} = (\boldsymbol{C}_j + \boldsymbol{D}_j \boldsymbol{K}_l \boldsymbol{E}_j)$ and notice that

$$\begin{bmatrix} \tilde{\boldsymbol{A}}_{jl}' & \boldsymbol{0} & \tilde{\boldsymbol{C}}_{jl}' & -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \boldsymbol{W} = 0$$

where $\boldsymbol{W}$ is defined as

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \tilde{\boldsymbol{A}}_{jl}' & \boldsymbol{0} & \tilde{\boldsymbol{C}}_{jl}' & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}.$$

We have from the Finsler's Lemma (see Lemma 3.1) that (7.9) is equivalent to

$$\boldsymbol{W}'\begin{bmatrix} \nu_{(j,l)(j,l)}\boldsymbol{X}_{jl} & \boldsymbol{H}_j & \boldsymbol{0} & \boldsymbol{X}_{jl} & \boldsymbol{X}_{jl}\Pi_{jl} \\ \boldsymbol{H}_j' & -\gamma\boldsymbol{I} & \boldsymbol{F}_j' & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{F}_j & -\gamma\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{X}_{jl}' & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\Pi}_{jl}'\boldsymbol{X}_{jl}' & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\mathscr{D}_{jl} \end{bmatrix}\boldsymbol{W}<0$$

resulting in

$$\begin{bmatrix} \nu_{(j,l)(j,l)}\boldsymbol{X}_{jl}+\boldsymbol{X}_{jl}\tilde{\boldsymbol{A}}_{jl}+\tilde{\boldsymbol{A}}_{jl}'\boldsymbol{X}_{jl}' & \boldsymbol{H}_j & \boldsymbol{X}_{jl}\tilde{\boldsymbol{C}}_{jl}' & \boldsymbol{X}_{jl}\Pi_{jl} \\ \boldsymbol{H}_j' & -\gamma\boldsymbol{I} & \boldsymbol{F}_j' & \boldsymbol{0} \\ \tilde{\boldsymbol{C}}_{jl}\boldsymbol{X}_{jl}' & \boldsymbol{F}_j & -\gamma\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{\Pi}_{jl}'\boldsymbol{X}_{jl}' & \boldsymbol{0} & \boldsymbol{0} & -\mathscr{D}_{jl} \end{bmatrix}<0.$$

From Schur's Lemma it follows that

$$\begin{bmatrix} \Gamma_{jl} & \boldsymbol{H}_j & \boldsymbol{X}_{jl}\tilde{\boldsymbol{C}}_{jl}' \\ \boldsymbol{H}_j' & -\gamma\boldsymbol{I} & \boldsymbol{F}_j' \\ \tilde{\boldsymbol{C}}_{jl}\boldsymbol{X}_{jl}' & \boldsymbol{F}_j & -\gamma\boldsymbol{I} \end{bmatrix}<0 \tag{7.10}$$

with $\Gamma_{jl}=\nu_{(j,l)(j,l)}\boldsymbol{X}_{jl}+\boldsymbol{X}_{jl}\tilde{\boldsymbol{A}}_{jl}+\tilde{\boldsymbol{A}}_{jl}'\boldsymbol{X}_{jl}'+\boldsymbol{X}_{jl}\Pi_{jl}\mathscr{D}_{jl}^{-1}\boldsymbol{\Pi}_{jl}'\boldsymbol{X}_{jl}'$. Now by pre and pos multiplying (7.10) by $\mathrm{diag}(\boldsymbol{X}_{jl}^{-1},\boldsymbol{I},\boldsymbol{I})$ and substituting $\boldsymbol{X}_{jl}^{-1}$ by $\boldsymbol{R}_{jl}$ we get that

$$\begin{bmatrix} \Xi_{jl} & \boldsymbol{R}_{jl}\boldsymbol{H}_j & \tilde{\boldsymbol{C}}_{jl}' \\ \boldsymbol{H}_j'\boldsymbol{R}_{jl} & -\gamma\boldsymbol{I} & \boldsymbol{F}_j' \\ \tilde{\boldsymbol{C}}_{jl} & \boldsymbol{F}_j & -\gamma\boldsymbol{I} \end{bmatrix}<0 \tag{7.11}$$

where

$$\Xi_{jl}=\nu_{(j,l)(j,l)}\boldsymbol{X}_{jl}^{-1}+\tilde{\boldsymbol{A}}_{jl}\boldsymbol{X}_{jl}^{-1}+\boldsymbol{X}_{jl}^{-1}\tilde{\boldsymbol{A}}_{jl}'+\Pi_{jl}\mathscr{D}_{jl}^{-1}\boldsymbol{\Pi}_{jl}'$$

so that, from (7.4), $\Xi_{jl}=\sum_{(j,l)\in\mathscr{V}}\nu_{(i,k)(j,l)}\boldsymbol{R}_{ik}$. From (7.11) and Lemma 4.2 we get the desired result. □

Remark 7.1 (On the size of the problem)**.** *Recalling that m denotes the number of outputs, n the number of state variables, r the number of inputs, while N stands for the number of Markov states (modes of operation) and M for the number of observed modes, it is worth to mention that the dimension of the LMI matrix given in Theorem 7.1 is, in the worst case (with $\mathscr{V}$ with NM elements, and $\mathscr{V}_{ik}$ with $NM-1$ elements) equal to $(NM+1)n+m+r$ and it is necessary to consider NM LMIs, and $(N+2)M$ matrix variables. In comparison, the method given by [81] yields to N LMI matrices with dimension $(N+1)n+r+mM$, and $NM+N+M$*

matrix variables. The reason for this difference lies in the modelling of the uncertainties. The authors of [81] incorporate the uncertainties in the systems matrices while in our approach it is necessary to create a LMI for every possible combination of θ and $\hat{\theta}$

Remark 7.2 (LMI optimization problem)**.** *To minimize the upper bound for the H_∞ cost of the control problem, the parameter γ in* (7.5) *needs to be minimized, since $\|G_w\|_\infty < \gamma$*

Remark 7.3 (Perfect Information)**.** *For the case in which $\hat{\theta}(t) = \theta(t)$ for all $t \in \mathbb{R}^+$ the LMI condition* (7.5) *reduces to the same as in Corollary 4.1 in [81].*

Remark 7.4 (On the influence of ε_i)**.** *There exists no obvious rule on how to choose the parameters ε_i in* (7.5)*. Therefore it is necessary to apply an appropriate algorithm to find the set of ε_i which will result in the lowest parameter γ in* (7.5)*. Depending on the size of the problem this might be computationally expensive. Sequence convex optimization is a possible approach to this problem.*

Remark 7.5. *As in Remark of [99] the static output feedback design can also be used to design a dynamic output control. In this scenario it is desired to obtain a control in the following form:*

$$\Sigma_F : \begin{cases} \dot{\hat{\boldsymbol{x}}}(t) = \hat{\boldsymbol{A}}_{\hat{\theta}}\hat{\boldsymbol{x}}(t) + \hat{\boldsymbol{B}}_{\hat{\theta}}\boldsymbol{y}(t) \\ \boldsymbol{u}(t) = \hat{\boldsymbol{C}}_{\hat{\theta}}\hat{\boldsymbol{x}}(t) + \hat{\boldsymbol{D}}_{\hat{\theta}}\boldsymbol{y}(t) \end{cases} \tag{7.12}$$

with, as before, $\boldsymbol{y}(t) = \boldsymbol{E}_\theta \boldsymbol{x}(t)$. From (7.12) *and* (4.5) *we get the following closed-loop system:*

$$\Sigma_{CLF} : \begin{cases} \dot{\boldsymbol{x}}(t) = \left(\boldsymbol{A}_\theta + \boldsymbol{B}_\theta \hat{\boldsymbol{D}}_{\hat{\theta}} \boldsymbol{E}_\theta\right)\boldsymbol{x}(t) + \boldsymbol{B}_\theta \hat{\boldsymbol{C}}_{\hat{\theta}}\hat{\boldsymbol{x}}(t) + \boldsymbol{H}_\theta \boldsymbol{w}(t) \\ \dot{\hat{\boldsymbol{x}}}(t) = \hat{\boldsymbol{A}}_{\hat{\theta}}\hat{\boldsymbol{x}}(t) + \hat{\boldsymbol{B}}_{\hat{\theta}}\boldsymbol{E}_\theta \boldsymbol{x}(t) \\ \boldsymbol{z}(t) = \left(\boldsymbol{C}_\theta + \boldsymbol{D}_\theta \hat{\boldsymbol{D}}_{\hat{\theta}} \boldsymbol{E}_\theta\right)\boldsymbol{x}(t) + \boldsymbol{D}_\theta \hat{\boldsymbol{C}}_{\hat{\theta}}\hat{\boldsymbol{x}}(t) + \boldsymbol{F}_\theta \boldsymbol{w}(t) \end{cases} \tag{7.13}$$

Now defining for $i \in \mathscr{N}$,

$$\begin{aligned} \bar{\boldsymbol{x}}(t) &= \begin{bmatrix} \boldsymbol{x}(t) \\ \hat{\boldsymbol{x}}(t) \end{bmatrix} \\ \bar{\boldsymbol{E}}_i = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{E}_i & \boldsymbol{I} \end{bmatrix}, \quad \bar{\boldsymbol{A}}_i &= \begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \\ \bar{\boldsymbol{B}}_i = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{B}_i \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \quad \bar{\boldsymbol{H}} &= \begin{bmatrix} \boldsymbol{H}_i \\ \boldsymbol{0} \end{bmatrix} \\ \bar{\boldsymbol{C}}_i = \begin{bmatrix} \boldsymbol{C}_i & \boldsymbol{0} \end{bmatrix}, \quad \bar{\boldsymbol{D}}_i &= \begin{bmatrix} \boldsymbol{0} & \boldsymbol{D}_i \end{bmatrix} \end{aligned} \tag{7.14}$$

we get from (7.13) and (7.14) that

$$\dot{\bar{\boldsymbol{x}}}(t) = \left(\bar{\boldsymbol{A}}_\theta + \bar{\boldsymbol{B}}_\theta \hat{\boldsymbol{K}}_{\hat{\theta}} \bar{\boldsymbol{E}}_\theta\right) \bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{H}}_\theta \boldsymbol{w}(t)$$
$$\boldsymbol{z}(t) = \left(\bar{\boldsymbol{C}}_\theta + \bar{\boldsymbol{D}}_\theta \hat{\boldsymbol{K}}_{\hat{\theta}} \bar{\boldsymbol{E}}\right) \bar{\boldsymbol{x}}(t) + \boldsymbol{F}_\theta \boldsymbol{w}(t)$$

where for $l \in \mathscr{M}$,

$$\hat{\boldsymbol{K}}_l = \begin{bmatrix} \hat{\boldsymbol{A}}_l & \hat{\boldsymbol{B}}_l \\ \hat{\boldsymbol{C}}_l & \hat{\boldsymbol{D}}_l \end{bmatrix} \tag{7.15}$$

so that the static output feedback design method can be applied with the matrices as in (7.14).

7.2 Numerical evaluation

This section presents a numerical evaluation of the results presented in the previous section. The following model is used:

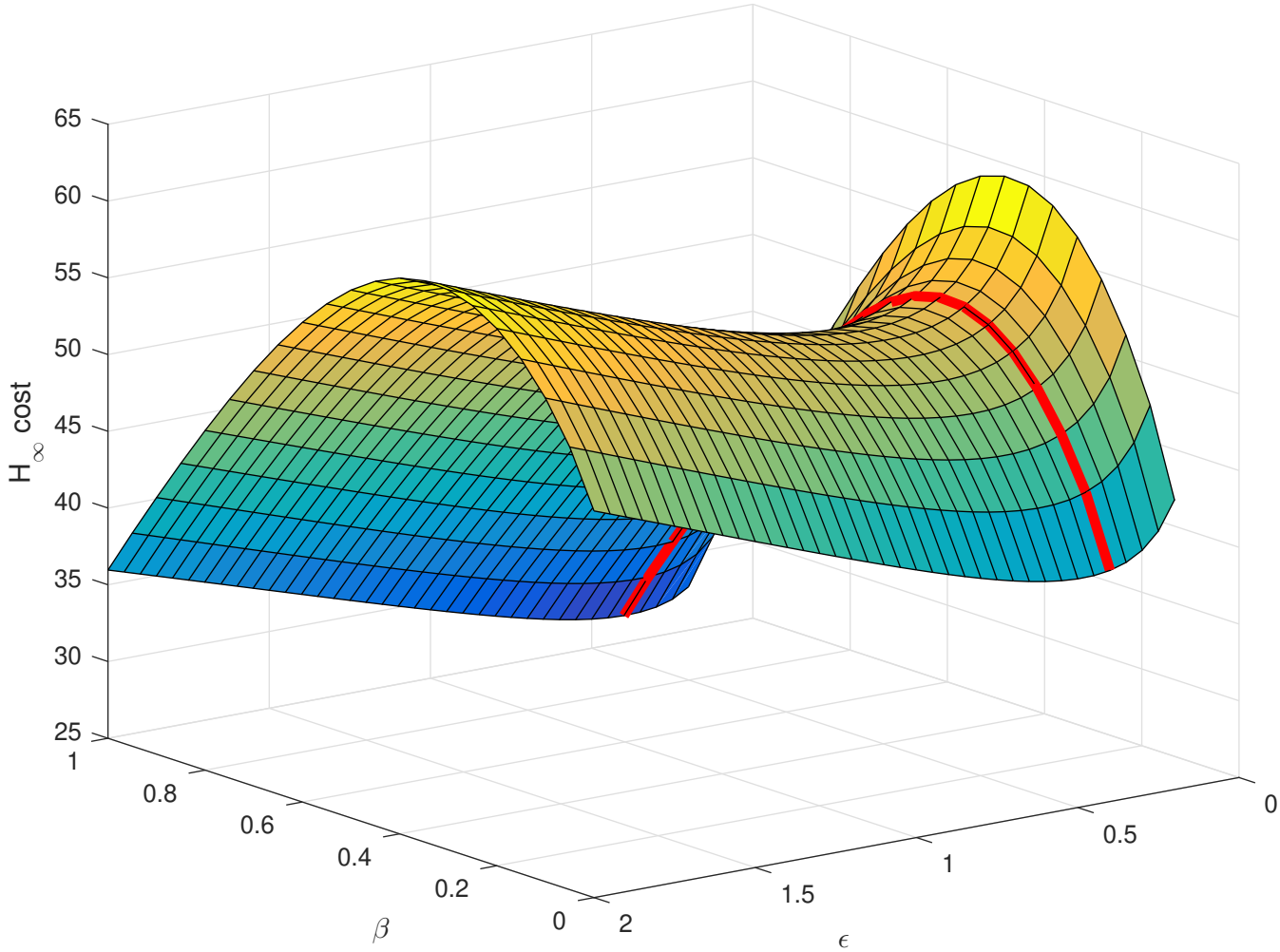
Example 7. *The example is adapted from [75] and consists of two coupled tanks T_1 and T_2 as shown in Figure 3.*

We assume that a detector $\widehat{\theta}(t)$, also taking values in $\mathscr{M} = \mathscr{N} = \{1,2,3\}$, will provide an estimate of the real mode of operation (valve condition) according to the model described in (4.1), (4.2).

This model is used in the following to evaluate the influence of the parameter ε (see equation 7.5) and the detection probability as well as discussing the behaviour of the resulting controller. The first graph (Figure 17) shows the influence of the parameters ε_i and β. To facilitate the analysis we use a constant value for all ε_i. As in [89] the detection probability is changed in the following way: All α_{jl}^k are modified as follows: $\alpha_{jl}^k = \beta$ for $j = l$, $\alpha_{jl}^k = (1-\beta)/2$ for $j \neq l$, and also that q_{kl}^i will be equal for all i, k, and l. The parameter ε_i is continuously changed from 0.5 to 2 while β is changed from 0 to 1. Looking at the influence of ε_i, it can be seen that there is a minimum at about $\varepsilon_i = 0.4$. Looking at the influence of the uncertainty, which is represented in β, it can be seen that the curve follows a similar trajectory as in [89].

For the minimal epsilon, the trajectory is shown in Figure 18. The highest cost is located at $\beta = 1/3$, which is highlighted with a red line, as this is the point where the uncertainties are at their highest. At this point the result is equal to robust control as there exists no information on the mode of operation at all. Hence the calculated controller-gains are the same for all modes of operation. The lowest costs are located at $\beta = 1$, which is expected as this is the

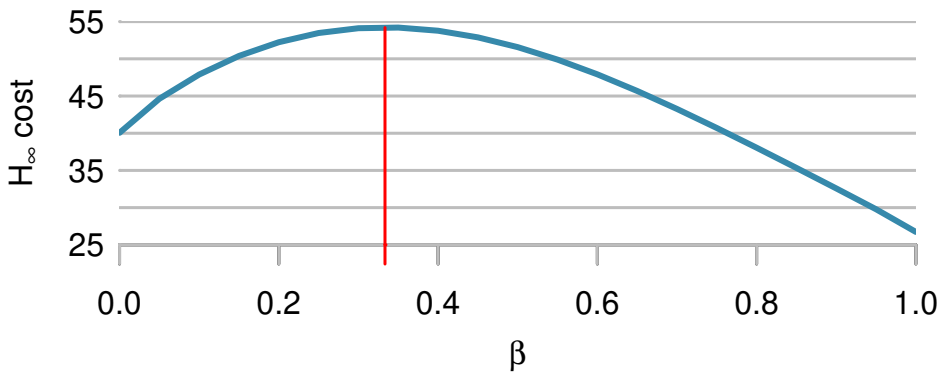
Figure 17: Cost-Surface of the two tank example



Source: Author

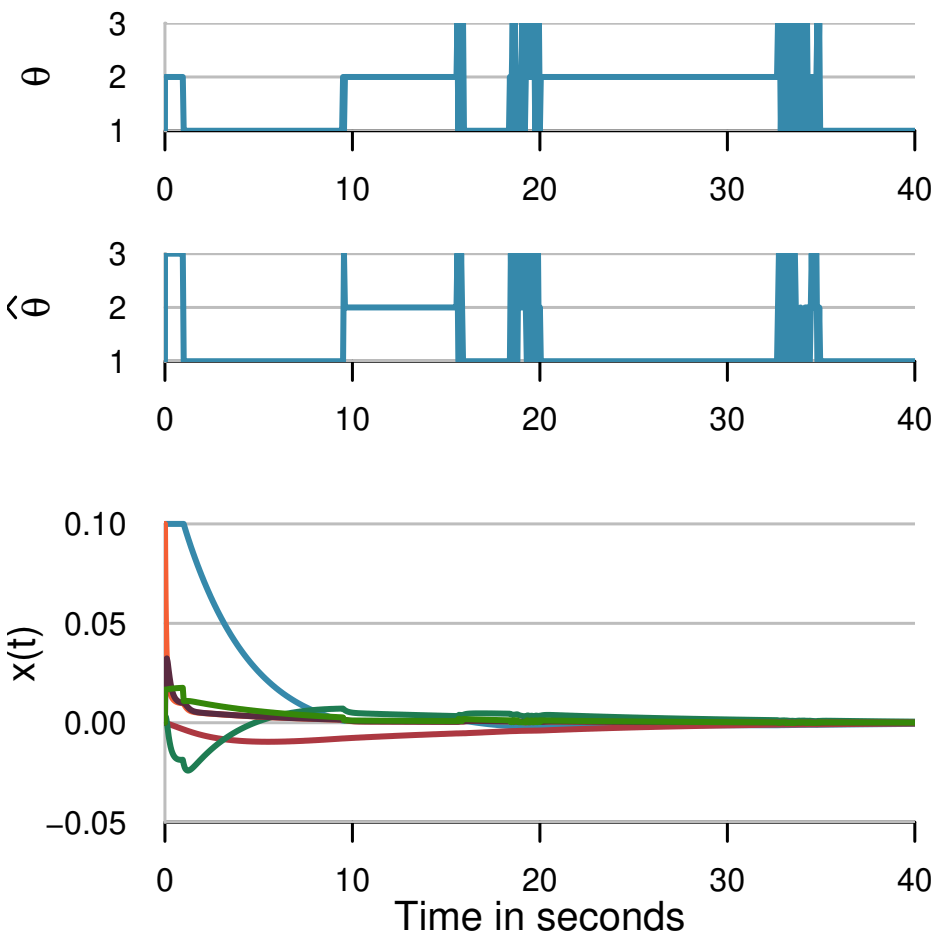
point of the perfect information case. A simulation of the system with the initial state $\boldsymbol{x} = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0.1 & 0 & 0 \end{bmatrix}'$ is shown in Figure 19. The first graph shows the mode of operation, the second shows the observed mode of operation and the third shows the development of the states. It can be seen that, despite the fact that θ and $\hat{\theta}$ sometimes show diverging values during the simulation, all states converge to the operating point at the end of the simulation, showing that the method presented in this monograph is capable to stabilize the simulated system even in the presence of uncertainties concerning the mode of operation.

Figure 18: Cost for the two tank example at optimal epsilon



Source: Author

Figure 19: Response to initial conditions with $\varepsilon = 0.4$ and $\beta = 0.8$



Source: Author

8 FILTERING

The goal is to design a filter that estimates the output of a linear system while guaranteeing that the estimation error is subject to certain bounds. This section introduces the system under consideration and defines the bounds more precisely. The following Markov Jump Linear System (MJLS) is considered:

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_\theta \boldsymbol{x}(t) + \boldsymbol{B}_\theta \boldsymbol{w}(t) \\ \boldsymbol{z}(t) = \boldsymbol{C}_\theta \boldsymbol{x}(t) + \boldsymbol{D}_\theta \boldsymbol{w}(t) \\ \boldsymbol{y}(t) = \boldsymbol{L}_\theta \boldsymbol{x}(t) \end{cases} \tag{8.1}$$

with $\boldsymbol{x}(t)$ as the state of the system, the disturbance input $\boldsymbol{w}(t)$, the outputs $\boldsymbol{y}(t)$ and $\boldsymbol{z}(t)$. All matrices are of compatible dimensions and their values depend on the development of a continuous-time Markov process as before.

8.1 Problem statement

The problem that should be solved is the following: the value of the output $\boldsymbol{z}(t)$ of system (8.1) should be estimated by using the linear filter

$$\Sigma_F : \begin{cases} \dot{\hat{\boldsymbol{x}}}(t) = \hat{\boldsymbol{A}}_{\hat{\theta}} \hat{\boldsymbol{x}}(t) + \hat{\boldsymbol{B}}_{\hat{\theta}} \boldsymbol{y}(t) \\ \hat{\boldsymbol{z}}(t) = \hat{\boldsymbol{L}}_{\hat{\theta}} \hat{\boldsymbol{x}}(t) + \hat{\boldsymbol{E}}_{\hat{\theta}} \boldsymbol{y}(t) \end{cases} \tag{8.2}$$

while the estimation error is limited. The values of the matrices $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$, $\hat{\boldsymbol{L}}$ and $\hat{\boldsymbol{E}}$ depend on the estimated mode of operation. The estimation error $\boldsymbol{z}_\Delta = \boldsymbol{z}(t) - \hat{\boldsymbol{z}}(t)$ is expressed as follows:

$$\boldsymbol{z}(t) - \hat{\boldsymbol{z}}(t) = \boldsymbol{L}_\theta \boldsymbol{x}(t) - \hat{\boldsymbol{L}}_{\hat{\theta}} \hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{E}}_{\hat{\theta}} \boldsymbol{C}_\theta \boldsymbol{x}(t) - \hat{\boldsymbol{E}}_{\hat{\theta}} \boldsymbol{D}_\theta \boldsymbol{w}(t) \tag{8.3}$$

$$= \left(\boldsymbol{L}_\theta - \hat{\boldsymbol{E}}_{\hat{\theta}} \boldsymbol{C}_\theta\right) \boldsymbol{x}(t) - \hat{\boldsymbol{L}}_{\hat{\theta}} \hat{\boldsymbol{x}}(t) - \hat{\boldsymbol{E}}_{\hat{\theta}} \boldsymbol{D}_\theta \boldsymbol{w}(t) \tag{8.4}$$

and the state $\dot{\hat{\boldsymbol{x}}}(t)$:

$$\dot{\hat{\boldsymbol{x}}}(t) = \hat{\boldsymbol{A}}_{\hat{\theta}}\hat{\boldsymbol{x}}(t) + \hat{\boldsymbol{B}}_{\hat{\theta}}\boldsymbol{C}_{\theta}\boldsymbol{x}(t) + \hat{\boldsymbol{B}}_{\hat{\theta}}\boldsymbol{D}_{\theta}\boldsymbol{w}(t) \tag{8.5}$$

$$\tag{8.6}$$

Both equations can be combined forming an augmented system:

$$\begin{aligned}
\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\hat{\boldsymbol{x}}}(t) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A}_{\theta} & \boldsymbol{0} \\ \hat{\boldsymbol{B}}_{\hat{\theta}}\boldsymbol{C}_{\theta} & \boldsymbol{A}_{\hat{\theta}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \hat{\boldsymbol{x}}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_{\theta} \\ \hat{\boldsymbol{B}}_{\hat{\theta}}\boldsymbol{D}_{\theta} \end{bmatrix} \boldsymbol{w}(t) \\
\boldsymbol{z}_{\Delta}(t) &= \begin{bmatrix} \boldsymbol{L}_{\theta} - \hat{\boldsymbol{E}}_{\hat{\theta}}\boldsymbol{C}_{\theta} & -\hat{\boldsymbol{L}}_{\hat{\theta}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\hat{\boldsymbol{x}}}(t) \end{bmatrix} - \hat{\boldsymbol{E}}_{\hat{\theta}}\boldsymbol{D}_{\theta}\boldsymbol{w}(t)
\end{aligned} \tag{8.7}$$

For the remainder, the following abbreviations are used:

$$\begin{aligned}
\bar{\boldsymbol{A}}_{\theta,\hat{\theta}} &= \begin{bmatrix} \boldsymbol{A}_{\theta} & 0 \\ \hat{\boldsymbol{B}}_{\hat{\theta}}\boldsymbol{C}_{\theta} & \boldsymbol{A}_{\hat{\theta}} \end{bmatrix} \\
\bar{\boldsymbol{B}}_{\theta,\hat{\theta}} &= \begin{bmatrix} \boldsymbol{B}_{\theta} \\ \hat{\boldsymbol{B}}_{\hat{\theta}}\boldsymbol{D}_{\theta} \end{bmatrix} \\
\bar{\boldsymbol{L}}_{\theta,\hat{\theta}} &= \begin{bmatrix} \boldsymbol{L}_{\theta} - \hat{\boldsymbol{E}}_{\hat{\theta}}\boldsymbol{C}_{\theta} & -\hat{\boldsymbol{L}}_{\hat{\theta}} \end{bmatrix} \\
\bar{\boldsymbol{E}}_{\theta,\hat{\theta}} &= -\hat{\boldsymbol{E}}_{\hat{\theta}}\boldsymbol{D}_{\theta}
\end{aligned}$$

and hence:

$$\begin{aligned}
\dot{\bar{\boldsymbol{x}}}(t) &= \bar{\boldsymbol{A}}_{\theta,\hat{\theta}}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{B}}_{\theta,\hat{\theta}}\boldsymbol{w}(t) \\
\boldsymbol{z}_{\Delta}(t) &= \bar{\boldsymbol{L}}_{\theta,\hat{\theta}}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{E}}_{\theta,\hat{\theta}}\boldsymbol{w}(t)
\end{aligned} \tag{8.8}$$

It should be noted that all the matrices in the last equation depend as much on the mode of operation θ as on the estimated mode of operation $\hat{\theta}$. The goal of a bounded error is defined as follows:

- The System (8.8) representing the error should be internally mean-square-stable. A mean-square stable error system guarantees that the estimation-error is bounded, and
- The effect of external disturbance in sense of the H_{∞} norm of the error should be bounded by γ.

The following paragraphs introduce a design-method for the filter (8.2) based on linear matrix inequalities which guarantees the two goals defined above.

8.2 Main result

Using the same reasoning as in [89, 91] the LMI-Condition of the bounded real Lemma (see Chapter 3) turns into:

$$M_i(P) = \begin{bmatrix} A'_{ik}P_{ik} + P_{ik}A_{ik} + \sum_{j,l \in Sx\mathscr{M}} \nu_{(ik),(jl)}P_{jl} + C'_{ik}C_{ik} & P_{ik}J_{ik} + C'_{ik}L_{ik} \\ \bigstar & L'_{ik}L_{ik} - \gamma^2 I \end{bmatrix} \prec 0 \tag{8.9}$$

Applied to the system (8.8) the LMI condition transforms in:

$$M_i(P) = \begin{bmatrix} \bar{\boldsymbol{A}}'_{ik}P_{ik} + P_{ik}\bar{\boldsymbol{A}}_{ik} + \sum_{j,l \in Sx\mathscr{M}} \nu_{(ik),(jl)}P_{jl} + \bar{\boldsymbol{L}}'_{ik}\bar{\boldsymbol{L}}_{ik} & P_{ik}\bar{\boldsymbol{B}}_{ik} + \bar{\boldsymbol{L}}'_{ik}\bar{\boldsymbol{E}}_{ik} \\ \bigstar & \bar{\boldsymbol{E}}'_{ik}\bar{\boldsymbol{E}}_{ik} - \gamma^2 I \end{bmatrix} \prec 0 \tag{8.10}$$

Now the new LMI based method is presented, see [79] for a similar result under different assumptions.

Theorem 8.1. *There exists a filter guaranteeing that the System* (8.3) *representing the error is iMSS and guarantees a H_∞ cost smaller than γ if the following is feasible: there are matrices $\boldsymbol{P}_{ik} \in \mathbb{R}^{2n\times 2n}$, $\boldsymbol{Q}_i \in \mathbb{R}^{(4n+r)\times n}$, $\boldsymbol{U}_{ik} \in \mathbb{R}^{2n\times 2n}$, $\boldsymbol{V}_{ik} \in \mathbb{R}^{r\times 2n}$, $\boldsymbol{W}_{ik} \in \mathbb{R}^{r\times r}$, $\boldsymbol{X}_k \in \mathbb{R}^{n\times n}$, $\boldsymbol{Y}_k \in \mathbb{R}^{n\ timesq}$, $\boldsymbol{S}_k \in \mathbb{R}^{p\times n}$, $\boldsymbol{T}_k \in \mathbb{R}^{p\times q}$ and $\Upsilon_{ik} \in \mathbb{R}^{(4n+r)\times n}$ such that the conditions*

$$P_{ik} \succ 0 \tag{8.11}$$

$$\Gamma_{ik} + \mathrm{Her}\left(\boldsymbol{Q}_i\boldsymbol{\Phi}_i\right) + \mathrm{Her}\left(\Upsilon_{ik}\Psi_{ik}\right) \prec 0 \tag{8.12}$$

$$\Xi_{ik} = \begin{bmatrix} \boldsymbol{U}_{ik} & \boldsymbol{V}'_{ik} & \begin{bmatrix} -\boldsymbol{C}'_i\boldsymbol{T}'_k + \boldsymbol{L}'_i \\ -\boldsymbol{S}'_k \end{bmatrix} \\ \boldsymbol{V}_{ik} & \boldsymbol{W}_{ik} & -\boldsymbol{D}'_i\boldsymbol{T}'_k \\ \begin{bmatrix} -\boldsymbol{T}_k\boldsymbol{C}_i + \boldsymbol{L}_i & -\boldsymbol{S}_k \end{bmatrix} & -\boldsymbol{T}_k\boldsymbol{D}_i & \boldsymbol{I} \end{bmatrix} \succ 0 \tag{8.13}$$

can be fulfilled, with

$$\Gamma_{ik} = \begin{bmatrix} \boldsymbol{U}_{il} + \sum_{j,l \in V} \nu_{(i,k)(j,l)}P_{j,l} & \boldsymbol{P}'_{ik} & \boldsymbol{V}'_{ik} \\ \boldsymbol{P}_{ik} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{V}_{ik} & \boldsymbol{0} & -\gamma^2\boldsymbol{I} + \boldsymbol{W}_{ik} \end{bmatrix} \tag{8.14}$$

$$\Psi_{ik} = \begin{bmatrix} \boldsymbol{Y}_k\boldsymbol{C}_i & \boldsymbol{X}_k & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{Y}_k\boldsymbol{D}_i \end{bmatrix} \tag{8.15}$$

$$\Phi_i = \begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{B}_i \end{bmatrix} \tag{8.16}$$

If the problem stated above is feasible, the matrices of the filter (8.2) *are given by*

$$\hat{\boldsymbol{A}}_k = \boldsymbol{X}_k \tag{8.17}$$

$$\hat{\boldsymbol{B}}_k = \boldsymbol{Y}_k \tag{8.18}$$

$$\hat{\boldsymbol{L}}_k = \boldsymbol{S}_k \tag{8.19}$$

$$\hat{\boldsymbol{E}}_k = \boldsymbol{T}_k \tag{8.20}$$

Proof. Using the relations stated above, it is possible to make the following substitutions in (8.15):

$$\begin{aligned}\Psi_{ik} &= \begin{bmatrix}\boldsymbol{Y}_k\boldsymbol{C}_i & \boldsymbol{X}_k & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{Y}_k\boldsymbol{D}_i\end{bmatrix} \\ &= \begin{bmatrix}\hat{B}_k\boldsymbol{C}_i & \hat{A}_k & \boldsymbol{0} & -\boldsymbol{I} & \hat{B}_k\boldsymbol{D}_i\end{bmatrix}\end{aligned}$$

If (8.12) is feasible, we get:

$$\Gamma_{i,k} + \mathrm{Her}\left(\begin{bmatrix}\boldsymbol{Q}_i & \Upsilon_{ik}\end{bmatrix}\begin{bmatrix}\Phi_i \\ \Psi_{ik}\end{bmatrix}\right) \prec 0 \tag{8.21}$$

Consider (8.7):

$$\begin{bmatrix}\Phi_i \\ \Psi_{ik}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}_i & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{B}_i \\ \hat{B}_k\boldsymbol{C}_i & \hat{A}_k & \boldsymbol{0} & -\boldsymbol{I} & \hat{B}_k\boldsymbol{D}_i\end{bmatrix}$$

which can be written as:

$$\begin{bmatrix}\Phi_i \\ \Psi_{ik}\end{bmatrix} = \begin{bmatrix}\bar{\boldsymbol{A}}_{ik} & -I & \bar{\boldsymbol{B}}_{ik}\end{bmatrix} \tag{8.22}$$

For the projection lemma [42] the following fact is used:

$$\ker\begin{bmatrix}\bar{\boldsymbol{A}}_{ik} & -I & \bar{\boldsymbol{B}}_{ik}\end{bmatrix} = \mathrm{im}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} \\ \bar{A}_{ik} & \bar{B}_{ik} \\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix} \tag{8.23}$$

Using

$$T_{ik} = \begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} \\ \bar{A}_{ik} & \bar{B}_{ik} \\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}$$

as a transformation matrix on Γ_{ik} results in:

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \bar{A}_{ik} & \bar{B}_{ik} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \Gamma_{ik} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \bar{A}_{ik} & \bar{B}_{ik} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}' \tag{8.24}$$

$$= \begin{bmatrix} \boldsymbol{P}'_{ik}\bar{A}_{ik} + \bar{\boldsymbol{A}}'_{ik}\boldsymbol{P}_{ik} + \boldsymbol{U}_{il} + \sum_{j,l\in V} \boldsymbol{\nu}_{(i,k)(j,l)} P_{j,l} & \boldsymbol{P}'_{ik}\bar{B}_{ik} + \boldsymbol{V}'_{ik} \\ \bar{\boldsymbol{B}}'_{ik}\boldsymbol{P}_{ik} + \boldsymbol{V}_{ik} & -\gamma^2\boldsymbol{I} + \boldsymbol{W}_{ik} \end{bmatrix} \tag{8.25}$$

$$= \begin{bmatrix} \boldsymbol{P}'_{ik}\bar{A}_{ik} + \bar{\boldsymbol{A}}'_{ik}\boldsymbol{P}_{ik} + \sum_{jl\in V} \boldsymbol{\nu}_{(i,k)(j,l)} P_{j,l} & \boldsymbol{P}'_{ik}\bar{B}_{ik} \\ \bar{\boldsymbol{B}}'_{ik}\boldsymbol{P}_{ik} & -\gamma^2\boldsymbol{I} \end{bmatrix} + \begin{bmatrix} \boldsymbol{U}_{ik} & \boldsymbol{V}'_{ik} \\ \boldsymbol{V}_{ik} & \boldsymbol{W}_{ik} \end{bmatrix} \prec 0 \tag{8.26}$$

The second matrix of (8.26) is equal to the upper left block ($[1:2][1:2]$) of Ξ (8.13). Applying the Schur complement to Ξ_{ik} leads to:

$$\Xi_{ik} > 0 \tag{8.27}$$

$$\begin{bmatrix} \boldsymbol{U}_{ik} & \boldsymbol{V}'_{ik} \\ \boldsymbol{V}_{ik} & \boldsymbol{W}_{ik} \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} -\boldsymbol{C}'_i\boldsymbol{T}'_k + \boldsymbol{L}'_i \\ -\boldsymbol{S}'_k \end{bmatrix} \\ -\boldsymbol{D}'_i\boldsymbol{T}'_k \end{bmatrix} \boldsymbol{I} \begin{bmatrix} \begin{bmatrix} -\boldsymbol{T}_k\boldsymbol{C}_i + \boldsymbol{L}_i & -\boldsymbol{S}_k \end{bmatrix} & -\boldsymbol{T}_k\boldsymbol{D}_i \end{bmatrix} \succ 0 \tag{8.28}$$

From (8.28) it follows that:

$$\begin{bmatrix} \boldsymbol{U}_{ik} & \boldsymbol{V}'_{ik} \\ \boldsymbol{V}_{ik} & \boldsymbol{W}_{ik} \end{bmatrix} > \begin{bmatrix} \begin{bmatrix} -\boldsymbol{C}'_i\boldsymbol{T}'_k + \boldsymbol{L}'_i \\ -\boldsymbol{S}'_k \end{bmatrix} \\ -\boldsymbol{D}'_i\boldsymbol{T}'_k \end{bmatrix} \begin{bmatrix} \begin{bmatrix} -\boldsymbol{T}_k\boldsymbol{C}_i + \boldsymbol{L}_i & -\boldsymbol{S}_k \end{bmatrix} & -\boldsymbol{T}_k\boldsymbol{D}_i \end{bmatrix}$$

Now recalling relation (8.20) and (8.19):

$$\begin{bmatrix} \boldsymbol{U}_{ik} & \boldsymbol{V}'_{ik} \\ \boldsymbol{V}_{ik} & \boldsymbol{W}_{ik} \end{bmatrix} > \begin{bmatrix} \begin{bmatrix} -\boldsymbol{C}'_i\hat{\boldsymbol{E}}'_k + \boldsymbol{L}'_i \\ -\hat{\boldsymbol{L}}'_k \end{bmatrix} \\ -\boldsymbol{D}'_i\hat{\boldsymbol{E}}'_k \end{bmatrix} \begin{bmatrix} \begin{bmatrix} -\hat{E}_k\boldsymbol{C}_i + \boldsymbol{L}_i & -\hat{L}_k \end{bmatrix} & -\hat{E}_k\boldsymbol{D}_i \end{bmatrix}$$

so:

$$\begin{aligned} 0 &\succ \begin{bmatrix} \boldsymbol{P}'_{ik}\bar{A}_{ik} + \bar{\boldsymbol{A}}'_{ik}\boldsymbol{P}_{ik} + \sum_{jl\in V} \boldsymbol{\nu}_{(i,k)(j,l)} P_{j,l} & \boldsymbol{P}'_{ik}\bar{B}_{ik} \\ \bar{\boldsymbol{B}}'_{ik}\boldsymbol{P}_{ik} & -\gamma^2\boldsymbol{I} \end{bmatrix} + \begin{bmatrix} \boldsymbol{U}_{ik} & \boldsymbol{V}'_{ik} \\ \boldsymbol{V}_{ik} & \boldsymbol{W}_{ik} \end{bmatrix} \\ &\succ \begin{bmatrix} \boldsymbol{P}'_{ik}\bar{A}_{ik} + \bar{\boldsymbol{A}}'_{ik}\boldsymbol{P}_{ik} + \sum_{jl\in V} \boldsymbol{\nu}_{(i,k)(j,l)} P_{j,l} & \boldsymbol{P}'_{ik}\bar{B}_{ik} \\ \bar{\boldsymbol{B}}'_{ik}\boldsymbol{P}_{ik} & -\gamma^2\boldsymbol{I} \end{bmatrix} + \begin{bmatrix} \bar{\boldsymbol{L}}'_{ik} \\ \bar{\boldsymbol{E}}'_{ik} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{L}}_{ik} & \bar{\boldsymbol{E}}_{ik} \end{bmatrix} \end{aligned}$$

which is equivalent to the bounded real lemma (8.9) and thus closing the proof. □

9 CONCLUSIONS

9.1 Summary

In the previous chapters, continuous-time MJLS with partial information on the mode of operation have been discussed and design procedures based on LMI equations for the following problems have been derived:

- Stochastic stabilizability
- H_2-control
- H_∞-control
- H_∞-filtering

Numerical simulations have shown the usefulness and applicability of the procedures for both the H_2 and H_∞ control problems. In both cases it could be shown that both methods lead to a controller able to stabilize the system while achieving a minimal cost for the given detection probabilities. As for the filtering case, due to its numerical complexity no results have been obtained yet.

Different from other existing works, the approach considered in this paper allows the simulation of the system and gives a clear answer on the nature of the detector, since the joint process formed by the Markov parameter and detector parameter is an exponential hidden Markov process. As shown, the formulation encompasses the cases with perfect information, no information, the cases considered in [2, 63, 64, 88], and a continuous-time clustering information case, which was analyzed in ([30]) for discrete-time MJLS. In contrast to the results in [88, 64, 63], which either do not deal with the control design problem ([88, 64]) or present necessary conditions for the existence of a stabilizing feedback control law and a computational algorithm which relies on the limiting behavior of the solutions of some Riccati-like and covariance-like equations ([63], Chapter 8), the design procedures can be readily implemented using available LMI toolboxes. Moreover the controller obtained in our LMI formulation depends only on the

information coming from the detector and thus can be implemented in real applications, unlike the LMI controller design formulation presented in [2], which depends on both the detector information and the jump parameter.

H_2-control For the H_2-control case two formulations have been presented, one based on the controllability gramian and one on observability gramian. The solution based on the controllability gramian is computationally more expensive and has no guarantee of convergence. As expected for the primal case, by an appropriate choice of the parameter ζ it is possible to reach H_2 costs close to the optimal costs for the perfect information case.

H_∞-control In this monograph we have presented a new LMI based method for the design of an H_∞ output feedback controller for continuous time MJLS, with a detector subject to false detections. It is assumed that the joint process formed by the Markov parameter $\theta(k)$ and the detector information $\hat{\theta}(k)$ is an exponential hidden Markov model. Theorem 7.1 provides a sufficient LMI condition for the existence and design of an output feedback controller as in (4.5) with an H_∞ guaranteed upper bound norm, which was numerically evaluated in Section 7.2.

Filtering For the filtering problem this monograph presented a new LMI based method which followed the idea of [79, 77], but as before the assumptions on the detector were different, leading to a model which can be implemented and simulated. However, due to the numerical complexity of this solution no simulational results have been derived for the example presented in this work.

9.2 Open Problems

This monograph focuses on the open problem of stabilizing a MJLS where the information about the mode of operation is subject to uncertainties. Aside from the core problem discussed in this text there are extensions and problems worthy to be discussed in future works. Some important ones are:

- Dynamic Output Feedback Control
- Numerical Evaluation
- Uncertainties and Robustness

- Mixed H_2/H_∞ Control

These points are detailed below:

Dynamic output feedback control In this monograph only static output control has been considered. The next natural step would be to extend the results to the dynamic case. For the perfect information case results have been presented in [22, 17].

Evaluation The numerical evaluation of the design procedures for both the H_2 and the H_∞ case have been evaluated using a model of a simple process. This allows to show the most important features and characteristics of the methods without unnecessary complexity. The natural next step would be evaluating its performance using a model with more complexities or even a small-scale plant.

Uncertainties and Robustness This work lays the basis to incorporate uncertainties about the mode of operation into the model of MJLS. However, this is not the only possible source of uncertainty. The model of the system is usually just an approximation of the real world system. Which is why it is interesting to combine the approach presented with approaches incorporating other uncertainties. Some approaches for other uncertainties are shown in [85, 97, 12] and probably can be used as a starting point.

The mixed H_2 H_∞ case The mixed case is another possible extension, for systems without uncertainties a result is shown in [17], and for cases where the transition matrix is not exactly known [67] presented a result. This case would be a natural extension of the results presented here since it combines optimality and robustness.

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APPENDIX A – ANALYSIS OF THE LITERATURE

This chapter discusses the methods used for the quantitative literature and author analysis which were presented in Chapter 2. For the analysis the publishers listed in Table 2 where considered. To the author's knowledge these incorporate the most important publications in the field. All publishers' websites were queried for the following terms:

- Markov Jump Linear System
- Markov Jump Linear Systems
- Markovian Jump Linear System
- Markovian Jump Linear Systems

All in all 3033 articles were gathered and analysed. All search results were stored in a database for later analysis. After the initial import a manual cleanup of the database has been done to eliminate duplicates both in the used keywords and in authors which were caused by different spellings or the use of initials instead of given names. For these cleaned results the number of publications per year was counted and plotted.

Name	Number
Elsevier	593
IEEE	1530
Institution of Engineering and Technology (IET)	185
Society for Industrial & Applied Mathematics (SIAM)	9
Springer	262
Taylor and Francis	116
Wiley	323
Other	15
Sum	3033

Table 2: Publications per publisher

APPENDIX B – THE BACKBLAZE DATA

The cloud-storage company Backblaze publishes daily data on the status of all the hard disk drives (HDD) that have been used by the company [5] since April 2013. This dataset consists of the data of 118 269 drive with 2 322 729 251 hours of operation. 6942 of these drives have failed during operation.

The Data The dataset consists of daily data in various fields for every operational hard-drive in the data center. For the following analysis only the following fields are of interest:

serialnumber The serial number is unique and used to separate the observations.

model All serial numbers belonging to one model are grouped.

failure This field is set to one if a hard disk fails. There exists no exact a definition of failure, but the result is a replacement of the hard disk in question.

smart_9_raw This field contains the uptime of the HDD in hours, as the statistics are collected once per day this data has an accuracy of about 24 hours.

The data has to be considered as censored data, as most of the disks haven't failed yet, which is also a result of the fact that the number of drives is permanently changing and some disks are retired before they fail. From the total of 118 269 drives registered in the database only 88 388 were in use by the end of September 2017.

Analysis For the analysis the language R [72] is used together with the *fitdrplus* package [29]. The failed drives are divided into 99 different models. For models with more than 100 disks in use, the corresponding numbers are listed in table 3. It can be noted that the failure-ratio varies widely across the different models which is why each model is discussed separately. In the following pages all models with more than 100 failed disks are examined. To

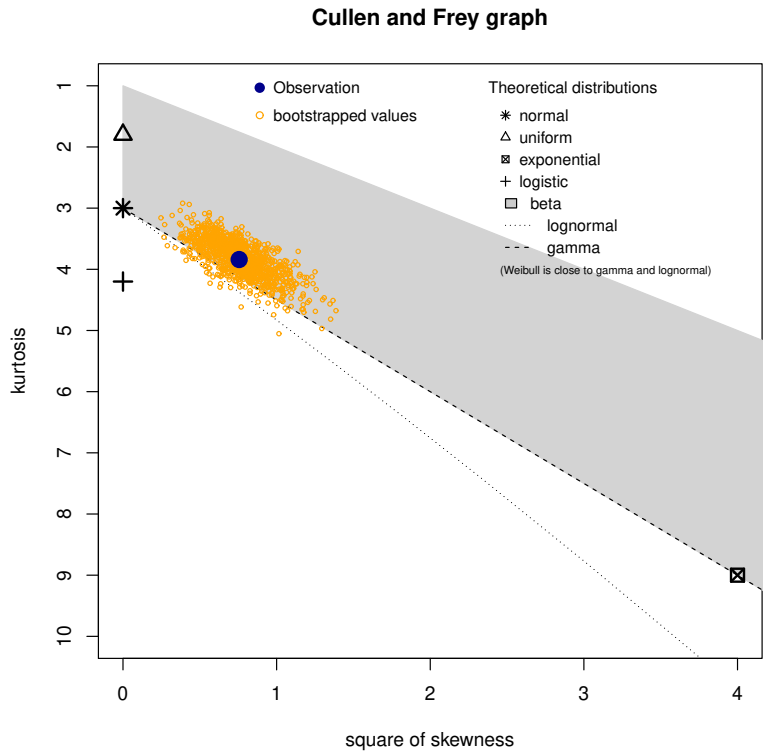
exclude outliers, disks with an uptime of more than 10 years or empty *smart_9_raw* values have been excluded.

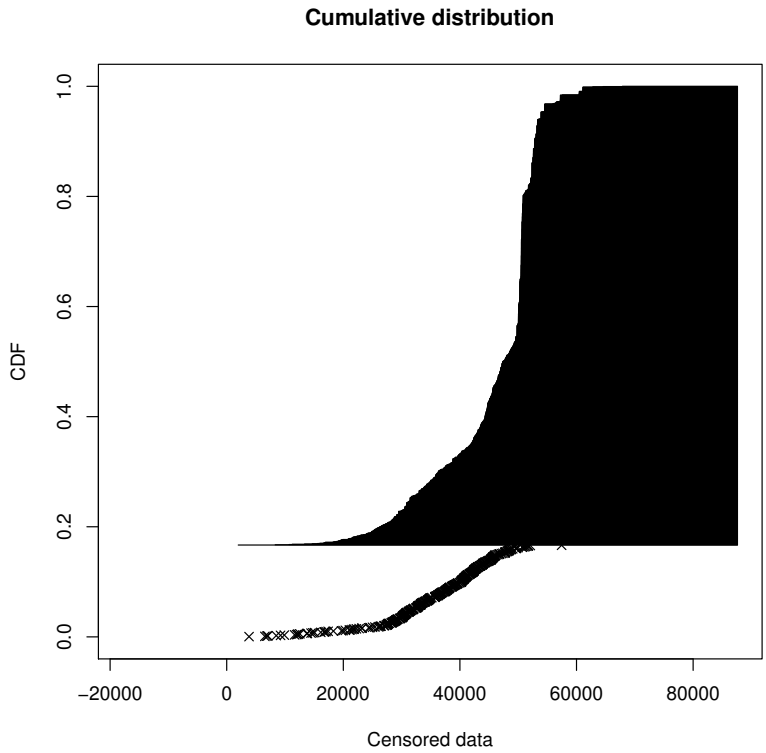
The analysis is carried out in two different steps: First the censored data (the disks which have not failed yet) is dismissed and for the remaining data a Cullen and Frey [21] graph is plotted. To improve the result, bootstrapped values [35, 15] are also included in the graphs. In a second step the packages functions for censored-data are used and all data is analysed and plotted as a distribution. If the disk has not failed yet, the latest uptime will be used as a lower bound. Hence the data will be considered as interval-censored data and plotted as a horizontal line. In all cases it can be visually seen that the exponential distribution is highly unlikely to be a good fit for any model, even though the distributions show much variation between the manufacturers. The Seagate ST drives all show a similar distribution which looks very much like a log-normal function, especially the ST3000DM001 which is the model with the highest percentage of failures. For the other manufacturers the data is not that obvious, but based on this data the exponential distribution can be ruled out and so does the applicability of MJLS for this scenario.

Model	Number of Disks	Number of Failures
ST4000DM000	36944	2635
HGST HMS5C4040BLE640	16309	128
ST8000NM0055	14449	44
ST8000DM002	9995	113
HGST HMS5C4040ALE640	8699	130
Hitachi HDS722020ALA330	4774	229
ST3000DM001	4707	1708
Hitachi HDS5C3030ALA630	4664	147
Hitachi HDS5C4040ALE630	2719	86
ST31500541AS	2188	397
ST6000DX000	1938	58
WDC WD30EFRX	1331	168
ST10000NM0086	1220	0
Hitachi HDS723030ALA640	1048	73
ST500LM012 HN	807	38
ST31500341AS	787	216
WDC WD10EADS	550	64
WDC WD30EZRX	500	22
WDC WD60EFRX	499	58
ST4000DM001	424	24
ST32000542AS	385	33
ST33000651AS	351	31
WDC WD5000LPVX	350	37
TOSHIBA MQ01ABF050	348	7
ST4000DX000	222	77
WDC WD20EFRX	167	15
TOSHIBA MD04ABA400V	150	4
TOSHIBA MQ01ABF050M	140	0
WDC WD1600AAJS	125	19
HGST HDS5C4040ALE630	118	5
ST1500DL003	116	90
WDC WD10EACS	109	8

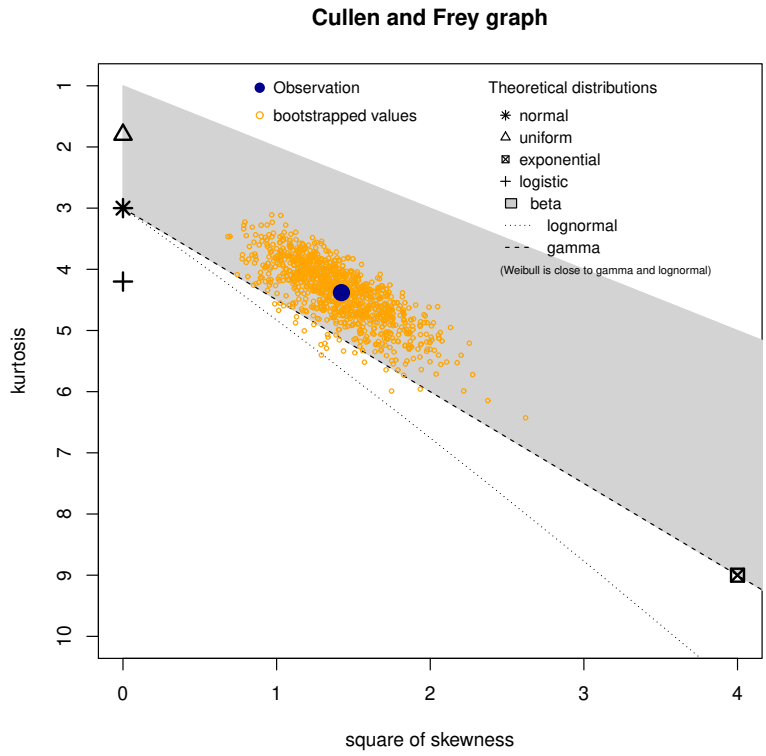
Table 3: Data of Backblaze Dataset

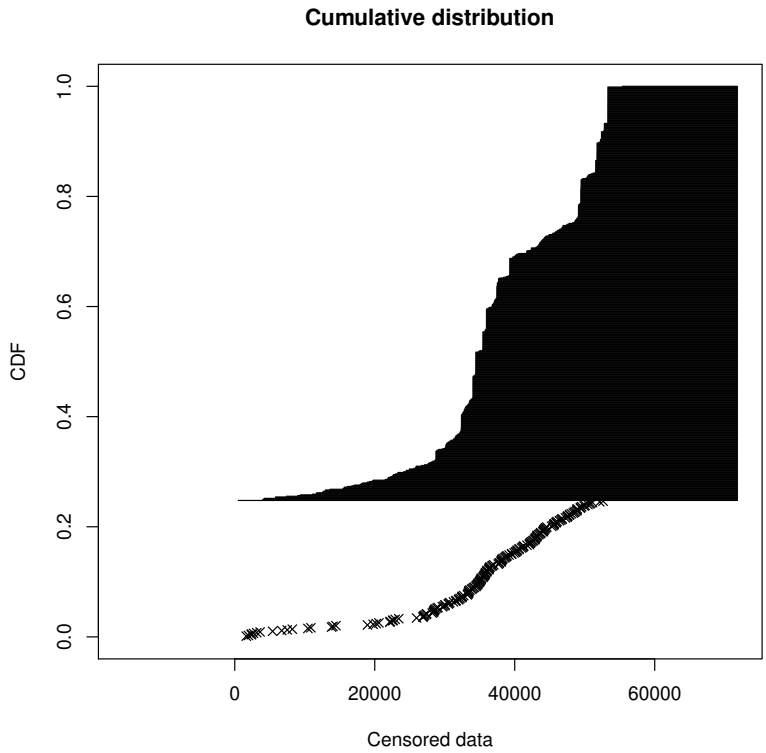
ST31500541AS



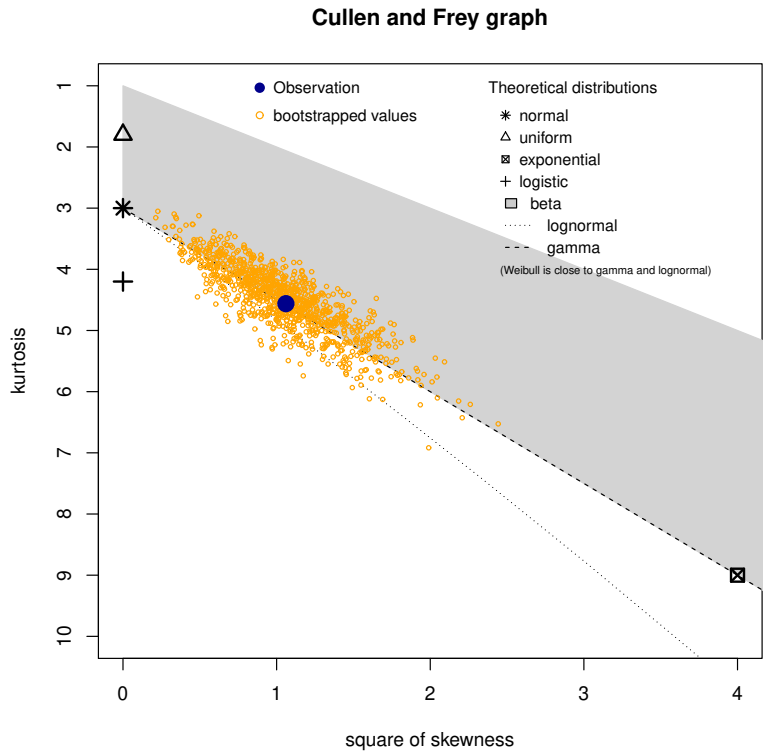


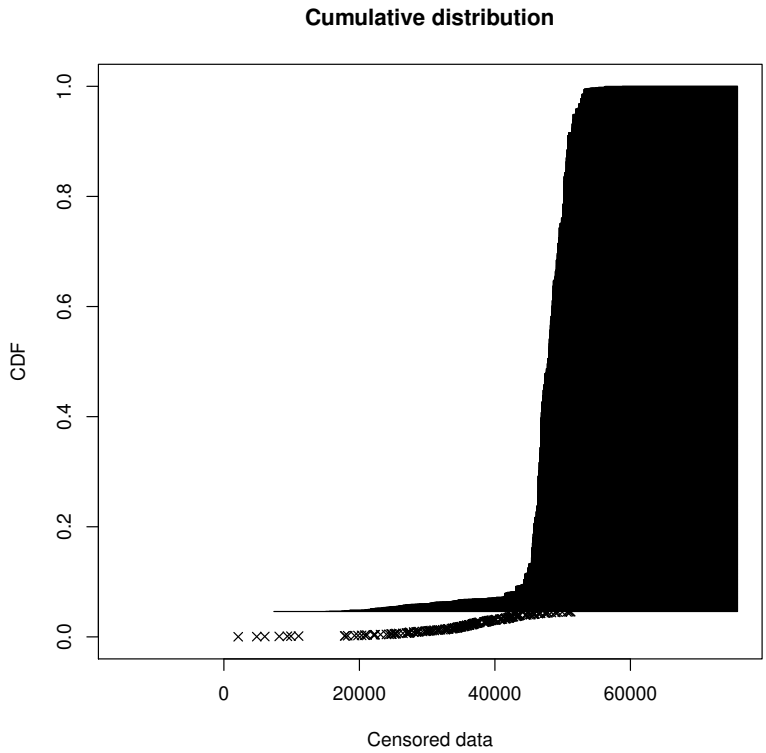
ST31500341AS





Hitachi HDS722020ALA330





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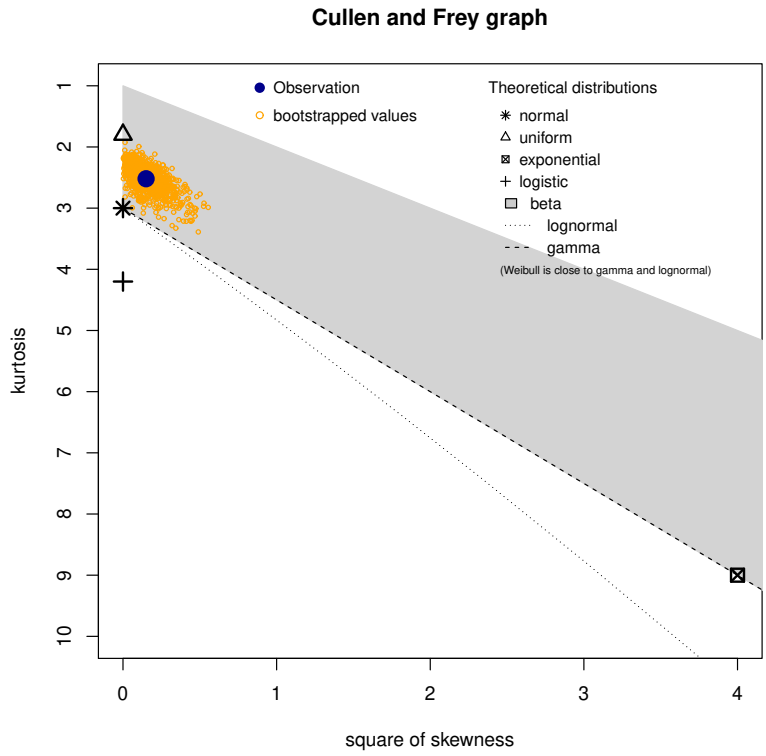

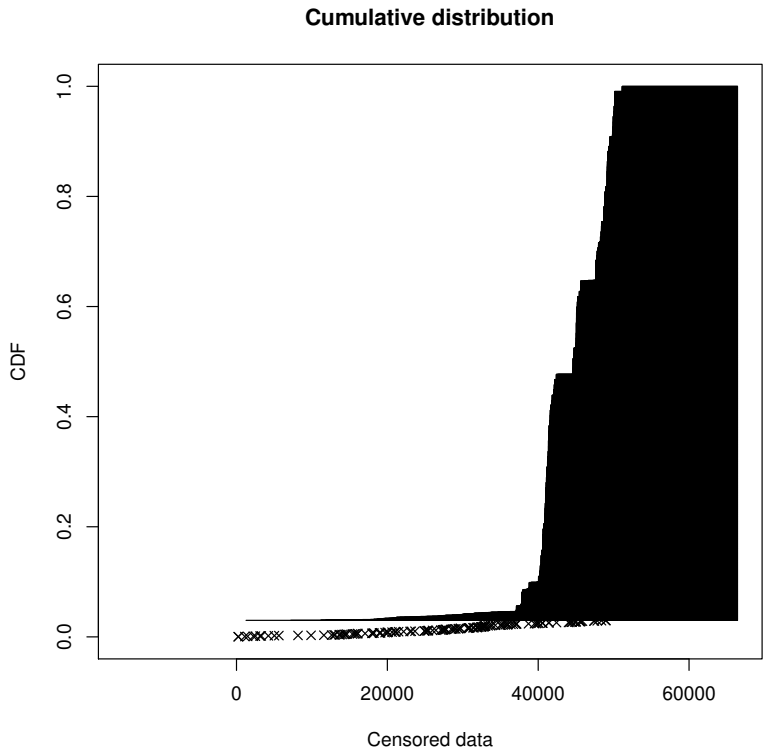

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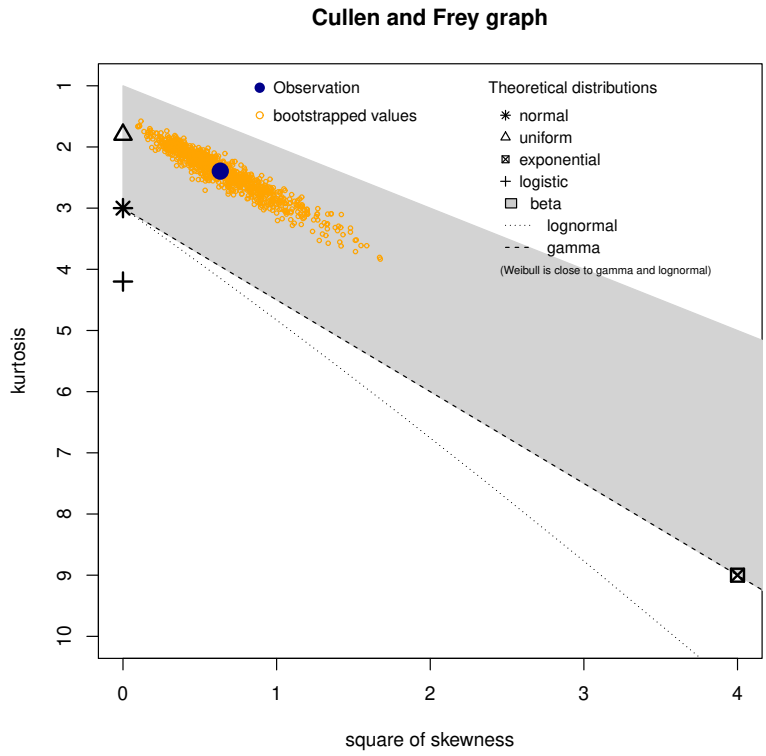

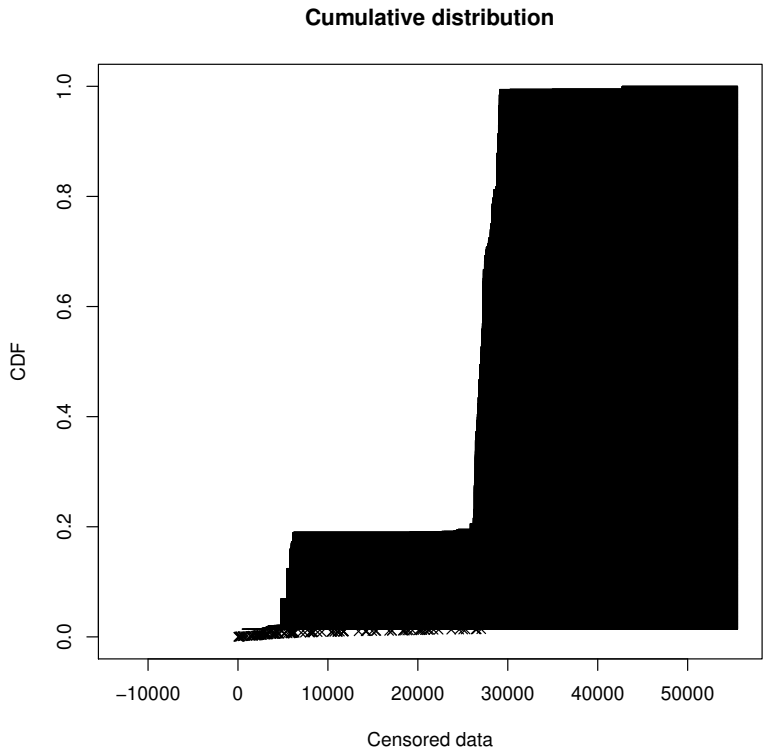

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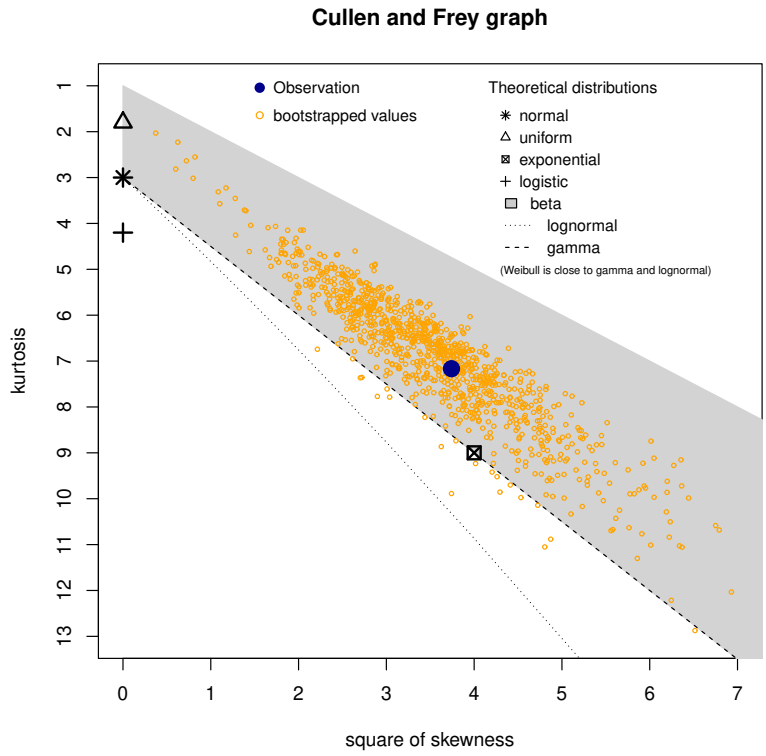

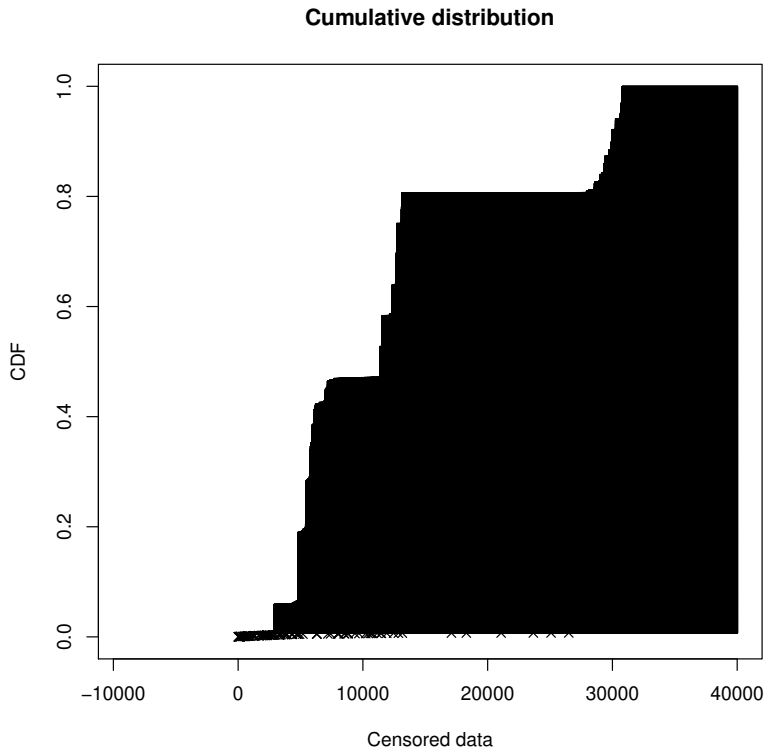

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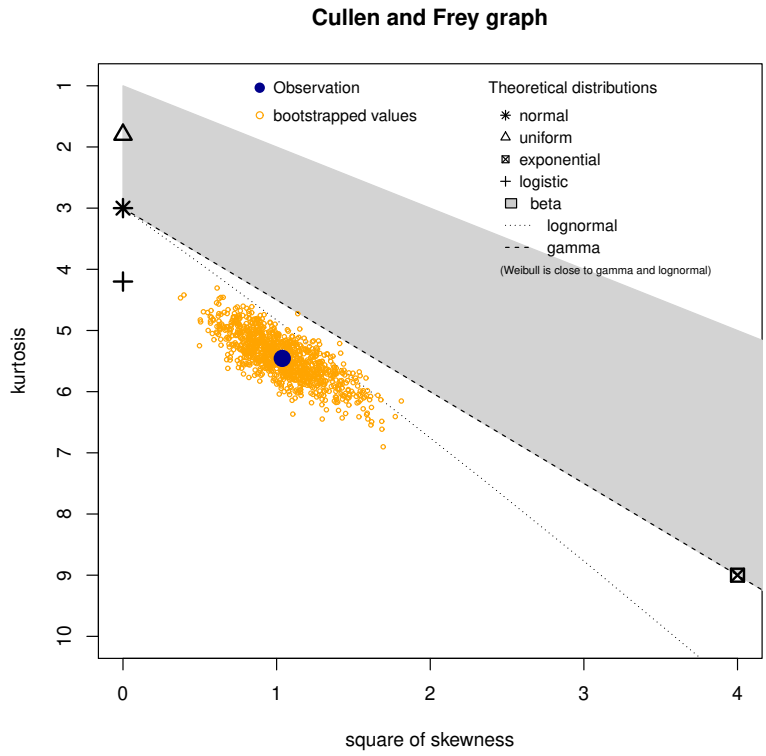

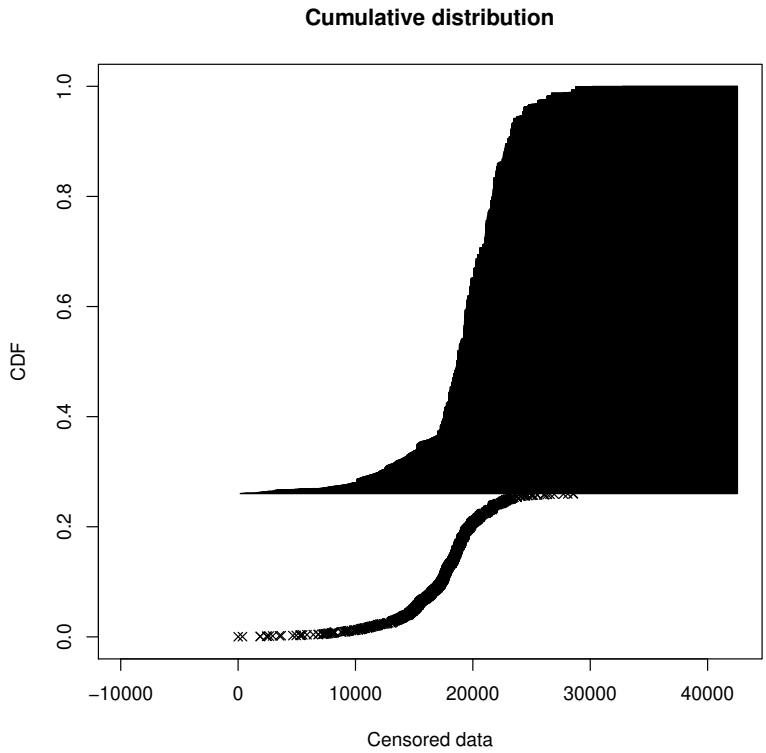

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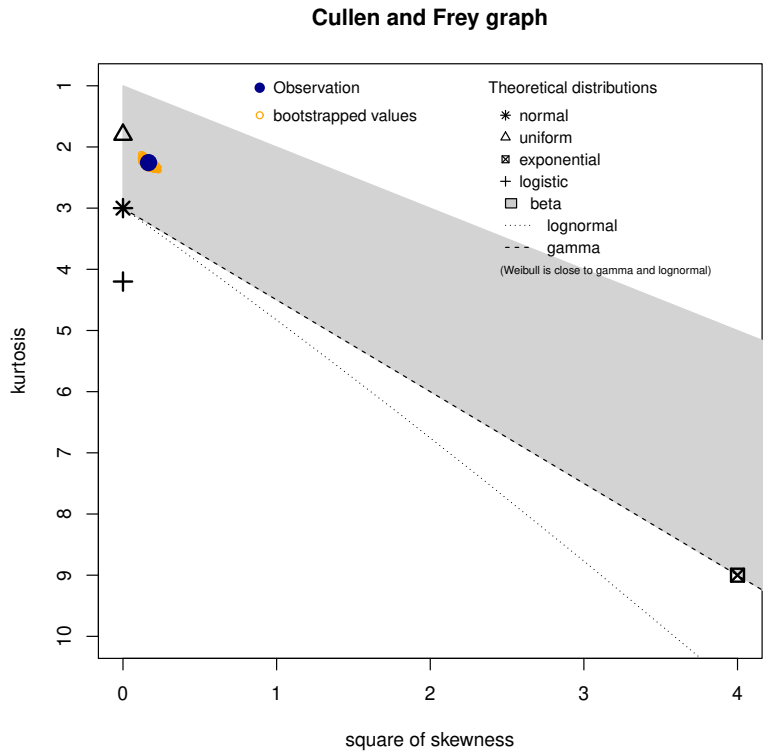

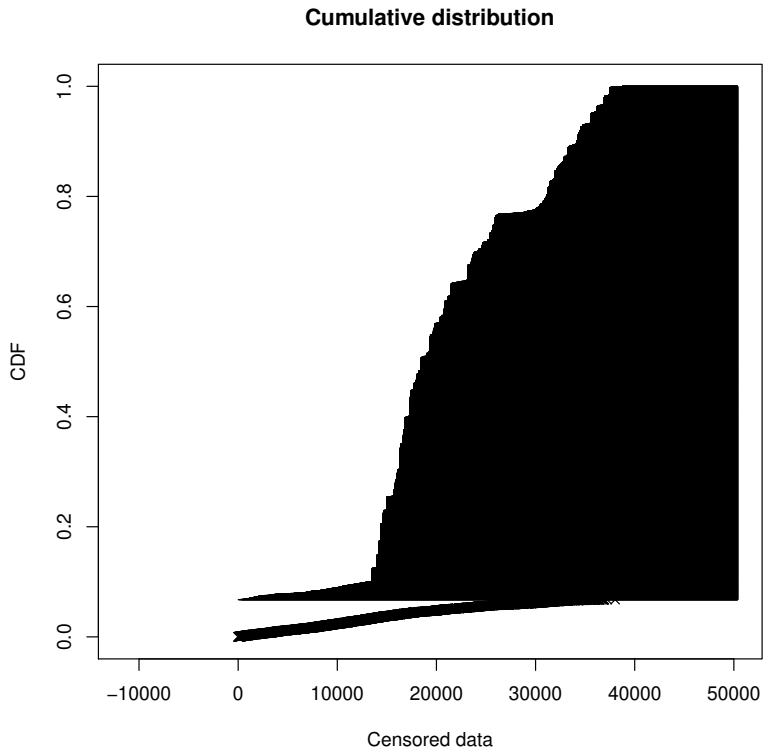

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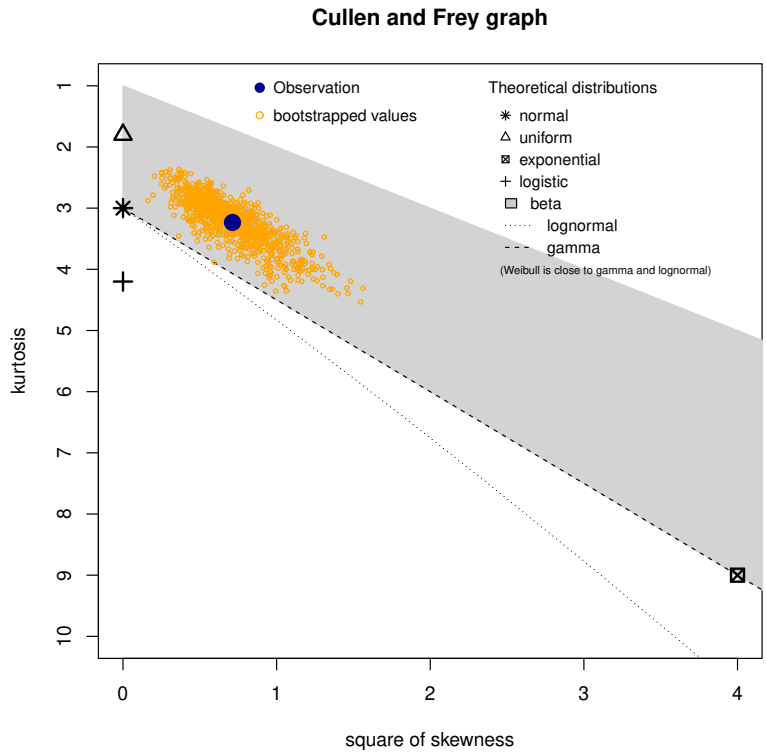

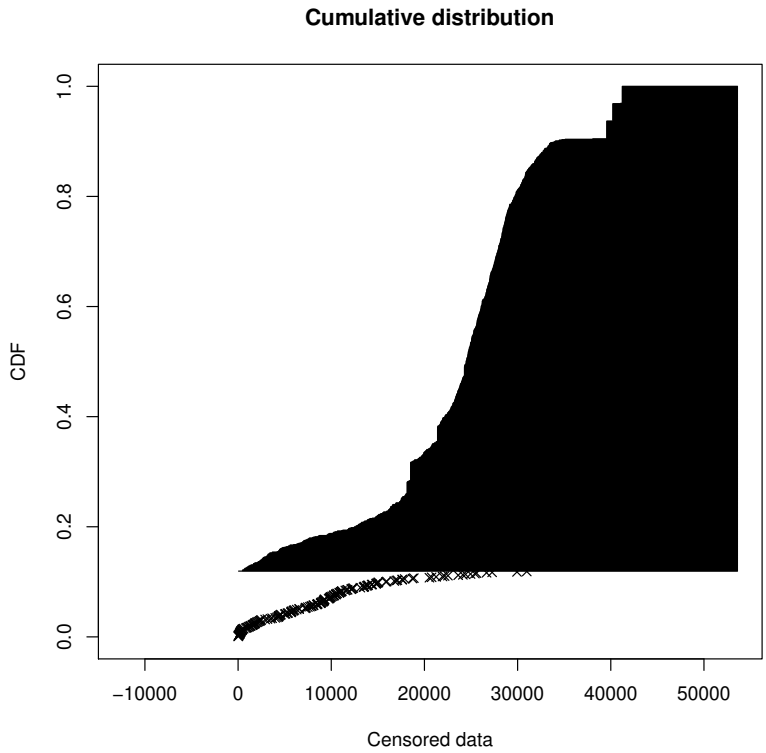

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