ULISSES ALVES MACIEL NETO

Control of an ensemble of half-spin systems replacing Rabi pulses by adiabatic following

Thesis submitted for the degree of Doctor in Science to the Escola Politécnica of Universidade de São Paulo.

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To my mom, Mirian Luciano Maciel: the person who helped me build an honest character that remains even during the storm.

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RESUMO

Considera-se o problema de controle de um conjunto de equações de Bloch (de spin 1/2 sem interação) em um campo magnético estático B_0. O estado $M(t,\cdot)$ pertence ao espaço de Sobolev $H^1((\omega_*,\omega^*),S^2)$, onde o parâmetro $\omega \in (\omega_*,\omega^*)$ é a frequência de Larmor. Trabalhos anteriores propuseram uma lei de controle baseada em uma função de Lyapunov conveniente (na norma H^1) que garante a convergência local L^∞ do estado inicial $M_0(omega)$ para $-e_3$, resolvendo localmente o problema de conduzir o perfil inicial M_0 perto o suficiente de $-e_3$ para uma condiçâo final $-e_3$. No entanto, as leis de controle contém um combo de pulsos de Rabi π-periódicos (impulsos de Dirac), o que representa um controle não limitado. O presente trabalho mostra a existência de uma lei de controle limitada para este problema de convergência aproximada local, onde os pulsos de Rabi são substituídos por pulsos adiabáticos (técnica do rastreamento adiabático). Além disso, simulações têm mostrado que esta nova estratégia produz convergência mais rápida, mesmo para condições iniciais "relativamente distantes" do estado alvo.

Palavras-chave: Sistemas Não Lineares. Sistemas Quânticos. Equações de Bloch. Controle de Equações Diferenciais Parciais. Estabilização de Lyapunov.

ABSTRACT

One considers the control problem of an ensemble of Bloch equations (non-interacting half-spins) in a static magnetic field B_0. The state $M(t,\cdot)$ belongs to the Sobolev space $H^1((\omega_*,\omega^*),S^2)$ where the parameter $\omega \in (\omega_*,\omega^*)$ is the Larmor frequency. Previous works have constructed a Lyapunov based stabilizing feedback in a convenient H^1-norm that assures local L^∞-convergence of the initial state $M_0(\omega)$ to $-e_3$, solving locally the approximate steering problem from M_0 close enough to $-e_3$ to a final condition $-e_3$. However, its control law contains a comb of periodic π-Rabi pulses (Dirac impulses), which represents an unbounded control. The present work shows the existence of a bounded control law for this local steering problem, where the Rabi pulses are replaced by adiabatic pulses (adiabatic following technique). Furthermore, simulations have shown that this new strategy produces faster convergence, even for initial conditions “relatively far" from the target state.

Keywords: Nonlinear Systems. Quantum Systems. Bloch Equations. Control of Partial Differential Equations. Lyapunov Stabilization.

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1 INTRODUCTION

1.1 Presentation

In the process of nuclear magnetic resonance, one of the steps is the inversion of the magnetic moment vector of the protons hydrogen atoms. Magnetic momentum, in the case of protons (which have a positive charge), has the same direction and orientation as spin. Initially, the nuclear magnetic moments point to random directions, so that there is no macroscopic magnetization. However, when subjected to a strong uniform magnetic field, the protons behave like a small compass, tending to align themselves parallel (lower energy state) or antiparallel to this (higher energy state).[1] Assuming as

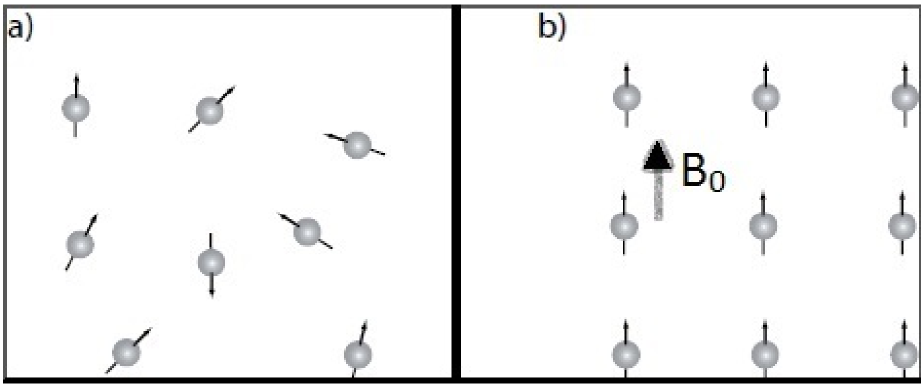

Figure 1 - The spins of the ensemble point in random directions (a), but after the insertion of a uniform magnetic field B_0 (b), they align in its direction.

the initial configuration of the system the magnetic moment vectors already aligned by a magnetic field B_0, we can act on B_0 so that the state vectors of the protons who are in the lower energy state go into the higher energy state.

Fixing a coordinate system in which the direction of B_0 is the axis z and its direction points to $-e_3$, we want to drive the state vectors of the magnetic moments of state $-e_3$ to a final state “close" to $+e_3$ at a time $T > 0$[2]. During this process, the magnetic moment vector performs a precession movement around an axis inside the so-called

[1] In fact this alignment is not perfect and the state vectors form a small angle with $-e_3$ or $+e_3$.

[2] For reasons inherent to Quantum Mechanics, which is not deterministic, it is not possible to specify that the state vector went from one steady state to another without taking a measurement and therefore a consequent collapse of the wave function.

Bloch sphere.

The Bloch sphere is used to represent a two-level quantum system. For the reader less familiar with Quantum Mechanics, we will give a brief explanation here of what this means. Many quantities in the microscopic world are quantized, that is, when measured, they cannot assume any values, but some predetermined values within a discrete set. In the case of a two-level quantum system, when performing a measurement on the variable of interest, it can assume only two possible values. However, before the measurement, the variable may present a state of superposition of these two states and this influences the probability of obtaining one result or the other when we perform a measurement on it. One way to represent this geometrically is using a sphere centered on 0 and radius 1, a vector with center at the origin whose end varies within the sphere representing the state (in this case, M) and, for each measurement performed, we adopt two opposite vectors (in this case, $+e_3$ and $-e_3$) as representatives of the result. Therefore, we have a typical control problem, that is, there is a physical

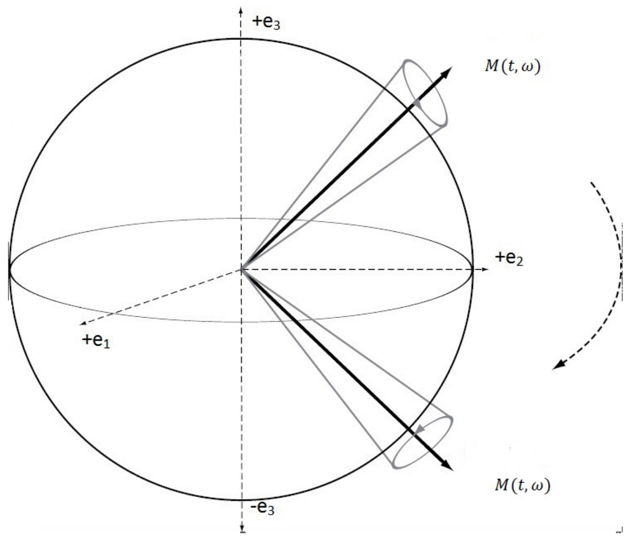

Figure 2 - Precession motion of the magnetic moment vector $M(t, \omega)$ inside the Bloch sphere

quantity which we want to lead, in finite time, from an initial state to a well-defined final state. However, instead of controlling the total magnetization vector M, we will adopt a semiclassical approach.

We will consider each magnetic moment vector as a member of a family (ensemble) parameterized by ω. Since we have a lot of elements, we can consider the Bloch model applied to each one as a good description of the average behavior of the ensemble. Thus, our control problem will consist of driving the entire ensemble from an initial profile $M_0(\omega)$ to a final profile $M_f(\omega)$ using the same control vector (which does not depend on ω). This notion is known in the literature as **ensemble controllability**.

1.2 Literature Review

As stated in the previous section, the goal of ensemble controllability is to simultaneously steer a continuum of systems between two states of interest with the same control inputs. This concept has already been studied by two different ways. In (LI; KHANEJA, 2006) and (LI; KHANEJA, 2009) we have its characterization by the use of Lie algebra tools in the context of quantum systems that are described by Bloch equations depending continuously on a finite number of scalar parameters, and with a finite number of control inputs. In (BEAUCHARD; CORON; ROUCHON, 2010), these aspects are studied under a functional analysis setting, developed for infinite dimensional systems governed by partial differential equations. In particular, this last paper shows that a priori L^2-bounded controls are not sufficient to achieve the exact controllability, but unbounded controls (containing a sum of Dirac masses) are able to recover it. In (BEAUCHARD; SILVA; ROUCHON, 2012) and (BEAUCHARD; SILVA; ROUCHON, 2013) it is shown that the ensemble of Bloch equations is approximately controllable to the south pole of the Bloch sphere (in the Sobolev space H^1) in finite time with unbounded controls. In practice, it is impossible to reproduce exactly the unbounded controls. Therefore, we would like to investigate whether the same effect can be achieved by using bounded controls.

Ensemble controllability with bounded controls is considered in the literature under different approaches. In (BOSCAIN U.; RABITZ, 2014) we have a comprehensive study as well as the time-optimal solution for the transfer population problem for the Bloch equations without dispersion, using geometric methods. In the presence of dispersion, we have in (CHITTARO; GAUTHIER, 2018) a solution for the asymptotic stabilization problem when ω is in a finite or at least countable set by using topological methods and in (AUGIER N., 2018) results for ensemble controllability between eigenstates of generic Hamiltonians using adiabatic approximation techniques where the dispersion parameter lives in a continuum.

1.3 Contribution

The solution presented in (BEAUCHARD; SILVA; ROUCHON, 2012) and (BEAUCHARD; SILVA; ROUCHON, 2013) to the local approximate steering problem for an ensemble of Bloch equations in the continuum case, although mathematically correct, cannot be fully reproduced in practice. In fact, the unit pulse is a mathematical idealization and, when we replace the Dirac delta by pulses of (large) finite amplitude and short duration, there is no asymptotic convergence of the associated error.

In this work we propose an alternative solution with smooth and bounded control inputs of the above mentioned problem. This solution is proved to steer an ensemble of initial conditions close enough to the south pole to an arbitrary neighbourhood of the south pole (vector $-e_3$ here below). As far as we know, this is the first constructive and mathematical result solving locally motion planing towards the south pole with smooth and uniformly bounded control inputs for such ensemble of Bloch systems. This solution combines adiabatic techniques with Lyapunov stabilizing methods to construct open-loop bounded control inputs.

Simulations reported here indicate that the domain of application of the proposed open-loop control algorithm includes a quite large set of initial-value profiles with a significant support in the north hemisphere of the Bloch sphere.

1.4 Organization

In Chapter 2, we briefly review the main concepts and results of functional analysis in Hilbert spaces. The knowledge of the theorems present in this chapter are essential to understand the techniques used in the original proofs present in this work. However, the reader who is already familiar with the numerous properties of the one-dimensional Sobolev space $H^1(a,b)$ can skip reading directly to the next chapter.

Chapter 3 contains the main part of our work. First we present the precise mathematical formulation of our problem and the solution obtained in (BEAUCHARD; SILVA; ROUCHON, 2013). Afterwards, the heuristic of the deduction of our control law and the algorithm for obtaining it will be shown in detail.

In chapter 4 we show the numerical simulations carried out and compare the convergence of the solutions obtained from the control law proposed by (BEAUCHARD; SILVA; ROUCHON, 2013) and the new control law proposed in this work.

Finally, in chapter 5 we present our conclusion, including potential applications and future prospects.

2 MATHEMATICAL PRELIMINARY

2.1 Banach Spaces

We assume that the reader is familiar with the notions of measurable and integrable functions. For a precise definition of these concepts, we recomend the reading of the chapter 11 of (RUDIN, 1976). Every proofs of this chapter can be found in (BREZIS, 2005).

Definition 1. *A set E is called a (real)* ***normed space*** *if E is a (real) vector space, and there exists a function $||\cdot|| : E \times E \longrightarrow \mathbb{R}$ (which plays a role of measure of distances) such that:*
(i) The zero vector 0 has zero length and every other vector has a positive length:
$||x|| \geq 0$ *, and* $||x|| = 0 \iff x = 0$
(ii) $||\alpha x|| = |\alpha| \; ||x||$ *for every scalar* $\alpha \in \mathbb{R}$ *and vector* $x \in E$
(iii) The triangular inequality holds:
$||x + y|| \leq ||x|| + ||y||$ *for any vectors* x *and* y *in* E

Definition 2. *A sequence $(x_n)_{n\in\mathbb{N}}$ in a normed space $(E, ||\cdot||)$ converges to a limit L belonging to E if*

$$\forall \varepsilon > 0, \;\; \exists n_0(\varepsilon) > 0 \text{ such that: } n > n_0(\varepsilon) \Rightarrow ||x_n - L|| < \varepsilon$$

Definition 3. *A sequence $(x_n)_{n\in\mathbb{N}}$ is called a* ***Cauchy sequence*** *if, for each $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that, for all $n, m > N(\varepsilon)$ we have $||x_n - x_m|| < \varepsilon$.*

As we can realize, the importance of Cauchy sequences is its definition do not depend on the knowledge of the limit L. We now present the main properties of Cauchy sequences in Banach spaces.

Proposition 1: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a normed space $(E, ||\cdot||)$. Then we have:
(i) If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, it is bounded.
(ii) If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then every subsequence $(x_{\alpha(n)})_{n\in\mathbb{N}}$ is also a Cauchy sequence (for the same norm).
(iii) If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence that has a convergent subsequence, then $(x_n)_{n\in\mathbb{N}}$ is

also convergent.

(iv) If $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, then it is also a Cauchy sequence.

The converse of the last statement (every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges to any $x \in E$) is not always holds. When this happens, we say that the space E is **complete**.

Definition 4. *A complete normed space E is called a **Banach space**.*

A very useful theorem of the Banach spaces is presented below.

Theorem 1. ***(Banach fixed-point theorem)*** *Let E be a Banach space and let $S : E \longrightarrow E$ be a map such that*

$$||S(v_1) - S(v_2)|| \leq k||v_1 - v_2|| \quad \forall v_1, v_2 \in E \quad with \quad k < 1$$

Then S admits a unique fixed point u, i.e., $S(u) = u$.

2.2 Weak convergence

In finite dimensional spaces there are many results about convergence of sequences that require the concept of compactness as a hypothesis. Since those results are not valid for infinite dimensional spaces, we need to define a weaker notion in order to obtain similar results. Before that, we shall introduce some important definitions.

Definition 5. *The set of all **continuous linear functionals** on E is called **topological dual** of E and we shall denote it by E'.*

$$E' = \{f : E \longrightarrow \mathbb{R} \quad | \quad f \quad \text{is linear and continuous} \quad \}$$

Henceforth we shall call the topological dual space as simply **dual space**. The dual of a Banach space is also a Banach space with the norm

$$||f||_{E'} = \sup_{||x||_E \leq 1} |f(x)|$$

In a similar way we define a norm of its **bidual** E'' (dual of the dual) space with the norm

$$||\xi||_{E''} = \sup_{||f||_{E'} \leq 1} ||\xi(f)|| \quad (\xi \in E'')$$

There is a **canonical injection** $J : E \mapsto E''$ given by $J(x) = \xi_x$ where $\xi_x(f) = f(x)$ for all $f \in E'$.

Definition 6. *Let E be a Banach space and let $J : E \mapsto E''$ be the canonical injection from E into E''. The space E is said to be **reflexive** if J is surjective, i.e., $J(E) = E$.*

It is very important to remark that the use of the map J is essential. Now we are going to define an essential concept that will be present throughout our work.

Definition 7. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of E. We say that x_n **weakly converges** to x, and we denote by $x_n \rightharpoonup x$, if $f(x_n) \rightharpoonup f(x)$ for all $f \in E'$.*

Proposition 2: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of E, we have:
(i) If $x_n \to x$ then $x_n \rightharpoonup x$ weakly
(ii) If $x_n \rightharpoonup x$ weakly then $\|x_n\|$ is bounded and $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$
(iii) If $x_n \rightharpoonup x$ weakly and $f_n \to f$ in E' (i.e. $\|f_n - f\| \to 0$) then $f(x_n) \to f(x)$.

Unfortunately, the Heine-Borel theorem (every closed and bounded set is compact) does not apply to Hilbert spaces in general. Although we can define another concept in order to obtain a similar result.

Theorem 2. *Let E a reflexive Banach space and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in E. Then there exists a weakly convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$.*

Definition 8. *We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset E$ is **weakly Cauchy** if, for all $f \in E'$, the sequence $f(x_n)_{n\in\mathbb{N}}$ is Cauchy and the space E is **weakly complete** if every weakly Cauchy sequence in E is weakly convergent.*

2.3 L^p spaces

Throughout this section X shall denote an open set of $\mathbb{R}^N$.

Definition 9. *Let $p \in \mathbb{R}$ with $1 \leq p < \infty$; we set*

$$L^p(X) = \left\{ f : X \mapsto \mathbb{R} \text{ measurable such that } \int_X |f(x)|^p dx < \infty \right\}$$

with the norm

$$\|f\|_p = \left[\int_X |f(x)|^p dx \right]^{1/p}$$

Definition 10. *We set*

$$L^\infty(X) = \{f : X \mapsto \mathbb{R} \textit{ measurable, } \exists C > 0 \textit{ such that } |f(x)| \leq C \textit{ almost everywhere on } X\}$$

with the norm

$$||f||_\infty = \inf\{C > 0, \quad |f(x)| \leq C \textit{ almost everywhere on } X\}$$

Next we shall present some results about the important properties of L^p spaces.

Theorem 3. *L^p is a vector space and $|| \cdot ||_p$ is a norm for any p, $1 \leq p \leq \infty$.*

Theorem 4. *(Fischer-Riesz) L^p is a Banach space for any p, $1 \leq p \leq \infty$.*

Theorem 5. *L^p is reflexive for $1 < p < \infty$.*

Note that L^1 and L^∞ are not reflexive spaces. Give an $1 < p < \infty$ we shall denote by p' (**the conjugate of p**) the real number such that $\frac{1}{p} + \frac{1}{p'} = 1$. When $p = 1$, we set $p' = \infty$ and vice-versa.

Theorem 6. ***(Riesz Representation Theorem)*** *Let $1 < p < \infty$ and let $\phi \in (L^p(X))'$. Then there exits a unique function $u \in L^{p'}$ such that*

$$\phi(f) = \int_X uf \quad \forall f \in L^p$$

In addition, $||u||_{L^{p'}} = ||\phi||_{(L^p)'}$.

Since L^1 and L^∞ are not reflexive spaces, we need a separate formulation.

Theorem 7. *Let $\phi \in \left(L^1(X)\right)'$. Then there exists a unique function $u \in L^\infty$ such that*

$$\phi(f) = \int_X uf \quad \forall f \in L^1.$$

2.4 Hilbert Spaces

Definition 11. *Let E be a vector space. An inner product is defined on E by any map $< \cdot, \cdot >: E \times E \longrightarrow \mathbb{R}$ satisfying the following conditions:*
(i) $< \cdot, \cdot >$ is a bilinear map.

(ii) $< \cdot, \cdot >$ *is symetric:* $\forall (x,y) \in E \times E$, $< x, y >=< y, x >$.

(iii) $< \cdot, \cdot >$ *is positive defined, that is:*

- $< x, x > \geq 0$, *for all* $x \in E$ *(positive)*
- $< x, x >= 0 \Rightarrow x = 0$ *(definite)*

Let $E = \mathbb{R}^N$ and take two arbitrary elements $x = (x_1, x_2, \ldots, x_N)$ and $y = (y_1, y_2, \ldots, y_N)$. We can define the inner product by

$$< x, y >= \sum_{i=1}^{N} x_i y_i$$

Let now X an open bounded set of $\mathbb{R}^N$ and its adherence points set $\overline{X}$. We denote

$$C(\overline{X}) = \{f : \overline{X} \longrightarrow \mathbb{R} \quad | \quad f \quad \text{is continuous}\}$$

For f,$g \in E = C(\overline{X})$, we define

$$< f, g >= \int_X f(x)g(x) \quad dx$$

which is an inner product of E.

Theorem 8. ***(Cauchy-Schwarz inequality)*** *Let* E *be a vector space with an inner product* $< \cdot, \cdot >$. *Then for all* $(u, v) \in E \times E$*:*

$$| < u, v > | \quad \leq \quad < u, u >^{1/2} \quad < v, v >^{1/2}$$

If the equality holds, u *and* v *are colinear.*

Definition 12. *Let* E *be a Banach space. If the application* $u \in E \mapsto < u, u >^{1/2}$ *is a norm, we say that* E *is a* ***Hilbert space****.*

A very important result is that every Hilbert space can be identified with its topological dual.

Theorem 9. ***(Riesz Representation Theorem - general form)*** *Let* E *a Hilbert space and let* $f \in E'$. *There exists an unique element* $u \in E$ *such that, for all* $v \in E$, $< u, v >= f(v)$. *In addition, the application* $L : f \in E' \mapsto u \in E$ *is an isometric isomorphism, i.e.,* $||u||_E = ||f||_{E'}$.

2.5 The Sobolev Space $H^1(a,b)$

In many applied areas, as Engineering or Physics, is very common the use of maps for which the derivative does not exists at all points (for instance, a unit step). Hence, we need of a less restrictive notion, called **weak derivative**.

Definition 13. *Let X an open set of $\mathbb{R}^N$. We define $C_c(X)$ as the space of continuous function with compact support in X, i.e., which vanish outside some compact set $U \subset X$. In addition, we also define $C_c^\infty(X) = C^\infty(X) \cap C_c(X)$.*

Remark 1: When $X = (a,b) \subset \mathbb{R}$, $\varphi \in C_c(a,b) \Longrightarrow \varphi(a) = \varphi(b) = 0$.

Definition 14. *Let $f \in L^1[a,b]$. We say that g is a **weak derivative** of f if*

$$\int_a^b f\varphi' = -\int_a^b g\varphi, \quad \forall \varphi \in C_c(a,b)$$

Proposition 3: The weak derivative is unique and we can denote $g = f'$.

From now on, f' will always denote the weak derivative of f.

Definition 15. *We define the one-dimensional Sobolev space*

$$H^1(a,b) = \left\{ f \in L^2(a,b) \;\middle|\; f' \in L^2(a,b) \right\}$$

equipped with the norm

$$\|f\|_{H^1} = \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2 \right)^{1/2}.$$

Proposition 4: The space H^1 is a separable Hilbert space.

2.6 Vector (wedge) product and map S

Let $c = (c_1\ c_2\ c_3)^\top \in \mathbb{R}^3$ and define the map $S : \mathbb{R}^3 \to SO(3)$ by

$$S(c) = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} \tag{2.1}$$

Note that $S(c)$ is the 3×3 matrix such that $c \wedge v = S(c)v$ for all $c, v \in \mathbb{R}^3$. From the invariance of the dot and the vector products it follows that for all $c, v \in \mathbb{R}^3$ and $A \in SO(3)$

one has:

$$\begin{aligned} \langle c, v \rangle &= \langle Ac, Av \rangle \\ A(c \wedge v) &= (Ac) \wedge (Av) \end{aligned}$$

In particular, $AS(c) = S(Ac)A$ for all $c \in \mathbb{R}^3$ and $A \in SO(3)$.

3 ENSEMBLE ASYMPTOTIC STABILIZATION OF THE BLOCH EQUATIONS USING BOUNDED INPUTS

3.1 The model studied

As described in the Section 1.1, the bulk magnetic moment M, the vector sum of the magnetic moments of individual nuclei ν, arises because a large magnetic field B_0 in the z direction orients the excess of nuclear spins in the low energy state (spin up). The bulk magnetization is proportional to the bulk angular momentum J

$$M = \gamma J \tag{3.1}$$

where the gyromagnetic ratio γ is a characteristic constant for a given nucleus. The resulting M is controlled by an oscillating rf magnetic field $B_{rf}(t) = (B_x(t), B_y(t))$ in the $x - y$ plane, whose magnitude is smaller than B_0 by 4 to 5 orders.

The net field $B = B_x(t)e_1 + B_y(t)e_2 + B_0 e_3$ exerts a torque T on M

$$T = M \wedge B \tag{3.2}$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$.

From Newton's second law:

$$\frac{dJ}{dt} = T, \tag{3.3}$$

and from (3.1) and (3.2),

$$\frac{dM}{dt} = \gamma M \wedge B \tag{3.4}$$

which in vector form is

$$\frac{d}{dt}\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = -\gamma \begin{bmatrix} 0 & -B_0 & B_y(t) \\ B_0 & 0 & -B_x(t) \\ -B_y(t) & B_x(t) & 0 \end{bmatrix} \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} \tag{3.5}$$

Without loss of generality, $M = \left(M_x, M_y, M_z\right)^\top$ can be normalized to 1. We take $\omega_0 = -\gamma B_0$, $u(t) = -\gamma B_y(t)$ and $v(t) = -\gamma B_x(t)$, then the above system becomes

$$\frac{d}{dt}\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = -\begin{bmatrix} 0 & -\omega_0 & u(t) \\ \omega_0 & 0 & -v(t) \\ -u(t) & v(t) & 0 \end{bmatrix}\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} \tag{3.6}$$

The equation (3.6) is called **Bloch equation** and describes the total magnetization of the ensemble (as detailed in the Section 1.1). However, in a semiclassical approach, since we have a large number of hydrogen nuclei, we can extrapolate this model to describe the approximate individual behavior of the magnetization of each nucleus. In this case, we will have a set of systems parametrized by $\omega \in I$ (instead of the ω_0), where I is an open interval of $\mathbb{R}$.

Following the notation introduced in the Section 2.6, we can rewrite the equation (3.6) and consider the **ensemble** $M(t, \omega)$ **of Bloch equations**:

$$\dot{M}(t, \omega) = S\,(u(t)e_1 + v(t)e_2 + \omega e_3)M(t, \omega), \tag{3.7}$$

where $-\infty < \omega_* < \omega^* < +\infty$, $\omega \in (\omega_*, \omega^*)$, $\{e_1, e_2, e_3\}$ is (again) the canonical basis of ^{3} and $S\,(\cdot)$ is the map that defines the wedge product. For simplicity the partial derivative of M with respect to time is denoted by $\dot{M}$, and the partial derivative of M with respect to ω is denoted by M'. For any profile $\omega \in [\omega_*, \omega^*] \mapsto M(\omega) \in \mathbb{R}^3$, its H_1-norm reads

$$||M||_{H^1} = \sqrt{\int_{\omega_*}^{\omega^*} (||M(\omega)||^2 + ||M'(\omega)||^2) d\omega}$$

3.2 Statement of the problem

We will start by stating the local approximate steering problem:

Control Problem: Show the existence of $\delta > 0$ with the following property: for every initial condition $M_0 \in H^1\left((\omega_*, \omega^*), \mathbb{S}^2\right)$ such that $||M_0(\omega) + e_3||_{H^1} < \delta$, and for every $\varepsilon > 0$, it is possible to choose $T_f > 0$ (depending on ε) and to construct bounded controls $u : [0, T_f] \to \mathbb{R}$ and $v : [0, T_f] \to \mathbb{R}$ in a way that $||M(T_f, \cdot) + e_3||_{L^\infty} \leq \varepsilon$.

Since an application of T-periodic adiabatic pulses $(\overline{u}(t), \overline{v}(t))$ can also perform an approximate population inversion, our control strategy relies on considering these adiabatic pulses as a reference control, and to consider an auxiliary transformed system that is obtained by writing (3.7) in the rotating frame of the corresponding adiabatic propagator $A(t,\omega)$, which is the solution of the differential equation

$$\dot{A}(t,\omega) = S(\overline{u}(t)e_1 + \overline{v}(t)e_2 + \omega e_3)A(t,\omega) \tag{3.8}$$

where $A(t,\omega) \in SO(3)$, and the T-periodic adiabatic control $(\overline{u}(t), \overline{v}(t))$ is such that $A(0,\omega) = I$ and $A(kT,\omega) \approx I$, for $k = 1, 2, \ldots, \ell$, for some ℓ big enough. Define the auxiliary state $N(t,\omega)$ by the transformation

$$N(t,\omega) = A(t,\omega)^{\top} M(t,\omega).$$

By time-differentiation of the last equation, it is easy to obtain the auxiliary system

$$\dot{N}(t,\omega) = S\left[A^{\top}(t,\omega)(\widehat{u}(t)e_1 + \widehat{v}(t)e_2)\right] N(t,\omega) \tag{3.9}$$

and to show that an input $(\widehat{u}(t), \widehat{v}(t))$ applied to the auxiliary system (3.9) produces a solution $N(t,\omega)$ if and only if an input $(\overline{u}(t) + \widehat{u}(t), \overline{v}(t) + \widehat{v}(t))$ produces a solution $M(t,\omega) = A(t,\omega)N(t,\omega)$ of (3.7).

In this rotating frame, the drift term of the differential equation is eliminated , and then the idea is to apply the Lyapunov stabilizing techniques of (BEAUCHARD; SILVA; ROUCHON, 2013) to the auxiliary system (3.9). This is not far from what is done in (BEAUCHARD; SILVA; ROUCHON, 2013), and we will return to this aspect later[1].

In this way, the control strategy to be applied would be:

- Compute a T-periodic adiabatic control $(\bar{u}(t), \bar{v}(t))$ with an associate adiabatic propagator $A(t,\omega)$ such that $A(kT,\omega)$ is close enough to the identity matrix, $k = 1, 2, \ldots, \ell$ for some $\ell \in \mathbb{N}$.
- Compute a feedback control $(\widehat{u}(t), \widehat{v}(t))$ for the auxiliary system (3.9) that assures

[1]See equation (3.20) of Section 3.10, that represents the auxiliar dynamics that is considered in (BEAUCHARD; SILVA; ROUCHON, 2013) when $M_f = -e_3$.

that $N(\ell T, \omega)$ is close enough to $-e_3$.;

- Apply the control law $(\overline{u}(t) + \widehat{u}(t), \overline{v}(t) + \widehat{v}(t))$ to system (3.7) in open loop.

As the adiabatic propagator $A(t, \omega)$ is not exactly T-periodic, it will be reinitialized to the identity at $t = kT, k = 1, 2, \ldots$. Otherwise the transformed system (3.9) will not be periodic and the previous techniques of (BEAUCHARD; SILVA; ROUCHON, 2013) cannot be applied. As a consequence the right side of (3.9) will be discontinuous at $t = kT$, for $k = 1, 2, \ldots$, but it will be perfectly T-periodic. In this case the continuous solution $M(t, \omega)$ of the system (3.7) will not be given by $M_1(t) = A(t, \omega)N(t)$ any more (where $N(t, \omega)$ is continuous but $M_1(t, \omega)$ is not). So an error analysis of the term $\|M(t) - M_1(t)\|$ will be needed (see Theorem 11).

3.3 The adiabatic propagator

Consider the adiabatic[2] propagator equation (3.8), where:

- $A(t, \omega) \in SO(3)$, and $A(kT, \omega) = I$, $\forall \omega \in [\omega_*, \omega^*]$, $\forall k \in \mathbb{N}$;
- The pair $(\overline{u}(t), \overline{v}(t))$ is the adiabatic control (3.10) defined as follows.

Consider that one applies the control

$$\overline{u}(t) = B_1(t) \sin \phi(t) \tag{3.10a}$$

$$\overline{v}(t) = B_1(t) \cos \phi(t), \tag{3.10b}$$

where $\phi(t)$ and $B_1(t)$ are defined by:

$$\dot{\phi}(t) = \overline{k}(t)\overline{a}(t), \quad \phi(0) = 0 \tag{3.10c}$$

$$B_1(t) = \overline{k}(t)\overline{b}(t) \tag{3.10d}$$

where $\overline{a}(\cdot)$, $\overline{b}(\cdot)$, and $\overline{k}(\cdot)$ are T-periodic functions defined by $\overline{a}(t) = a(t/T)$, $\overline{b}(t) = b(t/T)$, and $\overline{k}(t) = Kk(t/T)$, where $K > 0$ is a chosen gain and $a(\cdot)$, $b(\cdot)$, and $k(\cdot)$ are

[2] For the adiabatic propagator, for technical reasons concerning the proof of Theorem 10, we consider a compact interval $[\omega_*, \omega^*]$.

1-periodic normalized functions defined as below.

Let $s_0 \in (0, 1/4)$. Define the function $a : [0, 1] \to \mathbb{R}$ by (see Figure 4):

$$a(s) = \begin{cases} -1, \text{if } s \in [0, s_0]; \\ -\cos\left[\dfrac{2\pi(s - s_0)}{1 - 4s_0}\right], \text{if } s \in \left(s_0, \dfrac{1}{2} - s_0\right]; \\ 1, \text{if } s \in \left(\dfrac{1}{2} - s_0, \dfrac{1}{2} + s_0\right]; \\ -\cos\left[\dfrac{2\pi(s - 3s_0)}{1 - 4s_0}\right], \text{if } s \in \left(\dfrac{1}{2} + s_0, 1 - s_0\right]; \\ -1, \text{if; } s \in (1 - s_0, 1]. \end{cases} \tag{3.11a}$$

Define the function $b(\cdot)$ by

$$b(s) = 1 - [a(s)]^2 \tag{3.11b}$$

and $k(\cdot)$ by

$$k(s) = \begin{cases} 1, \text{if } s \in [0, 0.5), \\ -1, \text{if } s \in [0.5, 1] \end{cases} \tag{3.11c}$$

One may extend these functions a, b, k to be 1-periodic functions in a natural way. A computer simulation of the adiabatic propagator $A(t, \omega)$ was done for $T = 10$, $T = 15$ and $T = 20$, with $s_0 = 0.1$ and $K = 10$. The values of $\|A(T^-, \omega) - I\|$ as a function of ω is given in Figure 3. The fast convergence of the maximum value of this norm to zero when $T \to \infty$ is easily seen in that figure.

The Figure 4 show these functions, that are parameterized by s_0, which defines for instance the size of the interval $[0, s_0]$ for which $b(\cdot)$ is null. By (3.10), it is clear that the adiabatic control $(\overline{u}(t), \overline{v}(t))$ is null for $t \in [kT, kT + Ts_0], k \in \mathbb{N}$. This fact is used in the proof of the stabilization result of the auxiliary system in Section 3.10.

We will re-initialize the propagator $A(t, \omega)$ to the identity at $t_{0_k} = kT$, for $k \in \mathbb{N}$.

Definition 16. *Fix $T > 0$. One let $A : \mathbb{R} \times [\omega_*, \omega^*] \to SO(3)$ stands for the T-periodic map such that, in each interval $[t_{0_k}, t_{0_{k+1}}) = [kT, (k + 1)T)$ then $A(t, \omega)$ is the solution of system* (3.8) *with initial condition $A(t_{0_k}, \omega) = I$ for $k \in \mathbb{N}$ and with the T-periodic adiabatic input $(\overline{u}(t), \overline{v}(t))$ that is defined in* (3.10)*.*

Note that $A(t, \omega)$ is not continuous at $t_{0_k} = kT$ for $k \in \mathbb{N}$. One will denote $\lim_{t \to T^-} A(t, \omega)$ by $A(T^-, \omega)$.

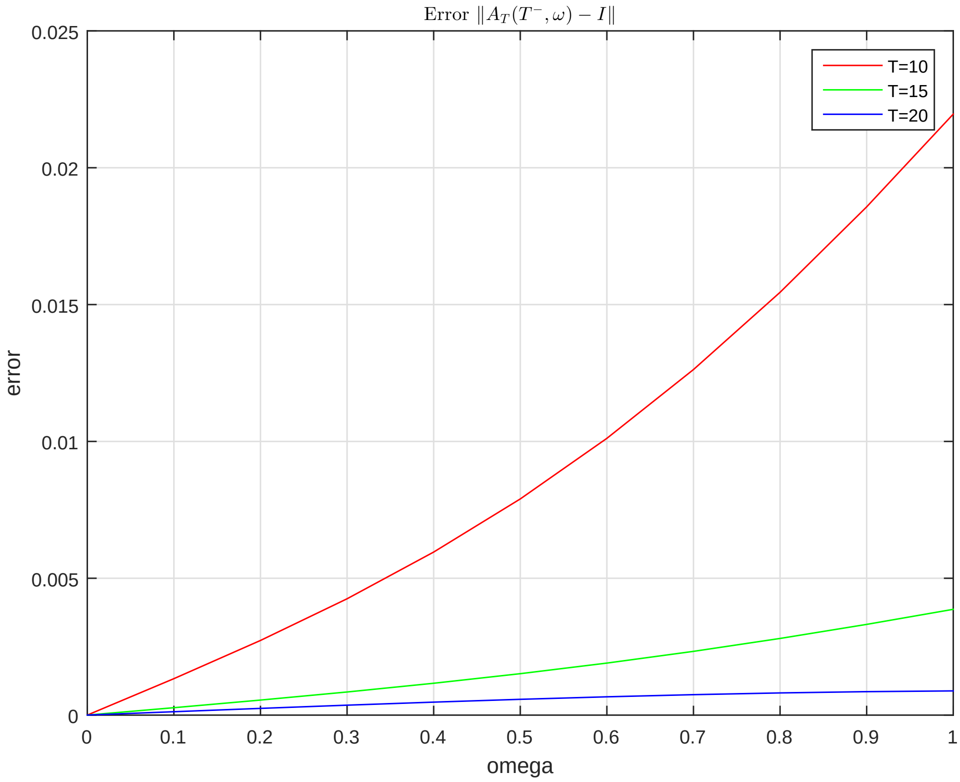

Figure 3 - Plot of the Frobenius norm $\|A(T^-,\omega) - I\|$ as a function of ω with $s_0 = 0.1$ and $K = 10$, for $T = 10$, $T = 15$ and $T = 20$.

Remark 2: The maps $A(\cdot,\omega)$, $\overline{a}(\cdot)$, $\overline{b}(\cdot)$, and $\overline{k}(\cdot)$ depend on the choice of T. This is not explicitly indicated for the sake of simplicity.

The following convergence result is proved[3] in (NETO; SILVA, 2014):

Theorem 10. *Fix* $K > \max\{|\omega_*|, |\omega^*|\}$*. Then one has* $\lim_{T\to\infty} \max_{\omega\in[\omega_*,\omega_*]} \|A(T^-,\omega) - I\| = 0.$

3.4 The auxiliary system and an approximation result

The auxiliary system is the T-periodic auxiliary system given by (3.9). As the (discontinuous) propagator $A(\cdot,\omega)$ depends on T, system (3.9) depends on T, and so it is in fact a family of systems that is parameterized by T. We will study the continuous solutions of the auxiliary system (3.9) and their relationship with the continuous solutions of (3.7) .

Theorem 11. *Fix initial conditions* $N_0 = M_0$ *of systems* (3.9) *and* (3.7)*. Assume that* $N(t,\omega)$ *is the (continuous) solution of* (3.9) *that is obtained by the application of the input*

[3]It may be also proved using the results of (TEUFEL, 2003).

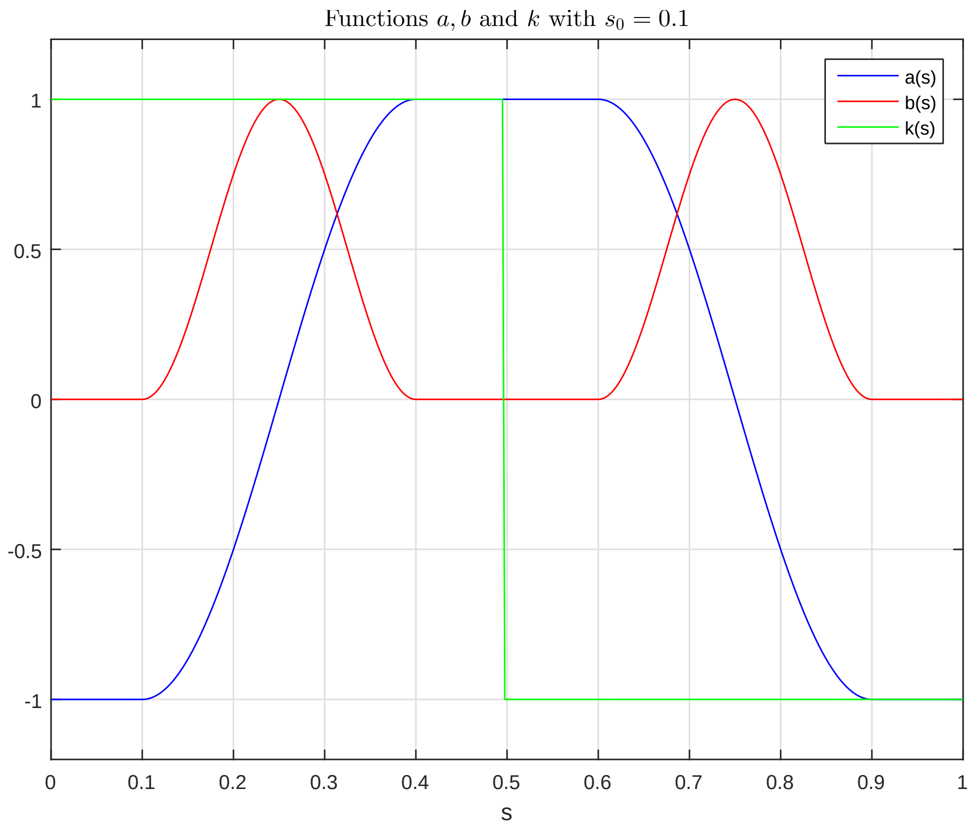

Figure 4 - Functions a, b, and k with $s_0 = 0.1$.

$(\widehat{u}(t), \widehat{v}(t))$. *Assume that* $M(t, \omega)$ *is the (continuous) solution of* (3.7) *that is obtained by the application of the input* $(u(t), v(t))$*, where* $u(t) = \widehat{u}(t) + \overline{u}(t)$*, and* $v(t) = \widehat{v}(t) + \overline{v}(t)$*. Then* $\|M(kT, \cdot) + e_3\|_{L^\infty} \leq k\|A(T^-, \cdot) - I\|_{L^\infty} + \|N(kT, \cdot) + e_3\|_{L^\infty}$

Proof. See Section 3.8. □

The last result clearly indicates that, if it is possible to stabilize the auxiliary system (3.9) uniformly with respect to the choice of T, then we will be close to a solution of the proposed problem.

3.5 Heuristics of the H^1 control law of the auxiliary system

Consider the Lyapunov functional

$$\mathcal{L} = \frac{1}{2}\|N + e_3\|_{H^1}^2 = \int_{\omega_*}^{\omega^*} \left[\frac{1}{2}\langle N', N'\rangle + 1 + \langle N, e_3\rangle\right] d\omega \tag{3.12}$$

In order to compute $\dot{\mathcal{L}}$ note that $\xi = \widehat{u}(t)e_1 + \widehat{v}(t)e_2$ does not depend on ω. One has

$$\dot{N}' = S(A^\top \xi)N' + S((A')^\top \xi)N \tag{3.13}$$

Hence

$$\begin{aligned} \dot{\mathcal{L}} &= \int_{\omega_*}^{\omega^*} \langle N', [(A^\top)'\xi \wedge N]\rangle + \langle e_3, [(A^\top)\xi \wedge N]\rangle d\omega \\ &= H_1\widehat{u} + H_2\widehat{v} \end{aligned} \tag{3.14}$$

where

$$H_i(t) = \int_{\omega_*}^{\omega^*} \langle N', [(A^\top)'e_i \wedge N]\rangle + \langle e_3, [(A^\top)e_i \wedge N]\rangle d\omega, \tag{3.15a}$$

for $i = 1, 2$.

One may construct the control law[4]

$$\begin{aligned} \widehat{u}(t) &= -H_1(A(t,\cdot), N(t,\cdot)), \\ \widehat{v}(t) &= -H_2(A(t,\cdot), N(t,\cdot)), \end{aligned} \tag{3.15b}$$

obtaining

$$\dot{\mathcal{L}} = -(H_1^2 + H_2^2) \leq 0 \tag{3.16}$$

Theorem 12. *For every initial condition* $N_0 \in H^1((\omega_*, \omega^*), \mathbb{S}^2))$*, the closed loop system* (3.9)-(3.15a)-(3.15b) *has a unique solution* $N \in C^1\left([0,\infty), H^1((\omega_*, \omega^*), \mathbb{S}^2)\right)$ *such that* $N(0) = N_0$.

Proof. See Section 3.9. □

3.6 Heuristics of the control strategy

The control strategy is summarized as follows:

- Fix $\ell > 0$. Choose $T > 0$ and $s_0 \in (0, 1/4)$ and construct the T-periodic adiabatic pulses $(\overline{u}(t), \overline{v}(t))$ of (3.10). Compute the adiabatic propagator $A(t,\omega)$ of (3.8) in $[0, \ell T)$ solving numerically (3.8) with initial condition $A(0,\omega) = I$. Extend $A(t,\omega)$ to $[0, \ell T]$ in a way that $A(t,\omega)$ is T-periodic.

[4] Although A and N depend on ω, u_i does not.

- Compute the (continuous) solution $N(t,\omega)$ of the closed loop system (3.9)-(3.15a)-(3.15b), and save the corresponding control law $(\widehat{u}(t),\widehat{v}(t))$ given by (3.15b) in the interval $[0,\ell T]$.
- Apply the open loop control law $(u(t),v(t)) = (\overline{u}(t)+\widehat{u}(t),\overline{v}(t)+\widehat{v}(t))$ to system (3.7) in the interval $[0,\ell T]$.

Roughly speaking, the main result of the paper says that, for ℓ and T big enough, this control strategy provides a solution of the proposed control problem. More precisely, the proof of Theorem 14 shows that, given $\varepsilon > 0$, it is possible to choose positive constants ε_1 and ε_2 such that $\varepsilon = \varepsilon_1+\varepsilon_2$. Then it is possible to fix some $T_0 > 0$ and to find ℓ big enough such that all the members of the family of closed loop auxiliary systems (3.9) are such that $\|N(\ell T,\omega)+e_3\|_{L^\infty} \leq \varepsilon_1$ for all $T \geq T_0$. Furthermore, for T big enough, Theorem 10 implies that the adiabatic control (3.10) assures $\ell\|A(T^-,\cdot) - I\|_{L^\infty} \leq \varepsilon_2$. Then Theorem 11 implies that this control strategy provides a solution of the proposed problem at $T_f = \ell T$, indeed.

3.7 Main results

The following result is the heart of the proof of our stabilization result of this section (Corollary 1). It implies that, if the initial condition is at least ε far from $-e_3$ in the L^∞ norm, then the Lyapunov function $\mathcal{L}(t)$ of the auxiliary system will decrease at least of a quantity c for each period, at least while $\|N+e_3\|_{L^\infty}$ is bigger than ε. The value of s_0 that appears in Theorem 13 is related to definition of adiabatic controls (3.10)-(3.11).

Theorem 13. *Fix $t_0 = kT$ for some $k \in \mathbb{N}$. Let[5] $T_0 > 0$ and $\tau_0 = s_0 T_0 < T/4$ with $s_0 \in (0,1/4)$. It is possible to construct $\delta > 0$ with the following property: for all $\varepsilon > 0$, there exists $c > 0$ (depending on ε) such that, for every $T \geq T_0$, and for every initial condition $N_0 = N(t_0)$ such that $\|N_0+e_3\|_{H^1} \leq \delta$, and $\|N_0+e_3\|_{L^\infty} \geq \varepsilon$, then, one will have $\mathcal{L}(N(t_0+\tau_0)) \leq \mathcal{L}(N(t_0)) - c$ for system* (3.9) *in closed loop with the control law* (3.15a)-(3.15b).

Proof. See Section 3.10. □

[5]By construction, for $T \geq T_0$ the T-periodic adiabatic control $(\overline{u},\overline{v})$ is null for $t \in [0,\tau_0]$).

The control law (3.15a)-(3.15b) stabilizes the auxiliary system uniformly with respect to the choice of T, as shown in the following result.

Corollary 1. *Consider the auxiliary system* (3.9) *in closed-loop of the control law defined in* (3.15a)-(3.15b). *Fix an initial condition* N_0 *such that* $\|N_0 + e_3\|_{H^1} < \delta$ *(where* δ *is defined in the statement of Theorem 13). Fix* $\varepsilon > 0$ *and* $T_0 > 0$. *There exists* $\ell > 0$ *, such that, for all* $T \geq T_0$*, the corresponding closed-loop system is such that* $\|N(\ell T, \cdot) + e_3\|_{L^\infty} < \varepsilon$.

Proof. Let $c > 0$ (that depends on ε) be the constant defined by Theorem 13. Let $p \in \mathbb{N}$ such that $\mathcal{L}(N_0) - pc < 0$. By contradiction, assume that $\|N(\ell T, \cdot) + e_3\|_{L^\infty} \geq \varepsilon$ for all $\ell \in \{0, 1, \ldots, p\}$. Since the Lyapunov functional $\mathcal{L}(t)$ is nonincreasing, the repetitive application of Theorem 13 at the instants $t = kT$ for $k = 0, 1, \ldots, p$ would give $\mathcal{L}(N(pT)) \leq \mathcal{L}(N_0) - pc < 0$. This is not possible since the Lyapunov functional is always non negative. So there must exist some $\ell \in \{0, 1, \ldots, p\}$ with the claimed property. □

From Theorem 11, Corollary 1 and Theorem 10, one may establish the following strategy for solving our control problem:

1. Fix $\varepsilon > 0$. Choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that[6] $\varepsilon = \varepsilon_1 + \varepsilon_2$.

2. From Corollary 1 of section 3.5, it is possible to find $\ell \in \mathbb{N}$, $T_0 > 0$ and a control law $\Omega_T : [0, T_f] \to \mathbb{R}^2$ (depending on T), with $T_f = \ell T$, in a way that the application of $(u_1(t), u_2(t)) = \Omega_T(t)$ to system (3.9) furnishes

$$\|N(\ell T, \cdot) + e_3\|_{L^\infty} \leq \varepsilon_1$$

for all $T > T_0$ for a convenient $T_0 > 0$.

3. Find $T^* > T_0$ big enough (depending on ℓ) such that $\ell\|A(T^-, \cdot) - I\|_{L^\infty} \leq \varepsilon_2$ for all $T \geq T^*$. (application of Theorem 10).

4. Apply the open loop control $(u(t), v(t)) = \Omega_{T^*}(t) + (\overline{u}(t), \overline{v}(t)))$ to system (3.7), obtaining (consequence of Theorem 11):

$$\|M(\ell T, \cdot) + e_3\|_{L^\infty} \leq \varepsilon_1 + \varepsilon_2 = \varepsilon.$$

[6]The numerical experiments have shown that the convergence of $\|A(T^-, \omega) - I\|$ to zero (when $T \to \infty$) is much faster than the convergence of $\|N(\ell T, \cdot) + e_3\|_{L^\infty}$ to zero (when $\ell \to \infty$). Hence it is reasonable to choose ε_1 much larger than ε_2.

One may state the main result of this paper

Theorem 14. *The strategy of the previous steps (1), (2), (3) and (4) always works for solving the control problem for ℓ and T big enough. In particular, there exists ℓ big enough and $T^*(\ell) > 0$ such that the proposed control law furnishes a solution of this problem for all $T \geq T^*(\ell)$ at $T_f = \ell T$.*

Proof. Easy consequence of Corollary 1 of section 3.5, Theorem 11 and Theorem 10. □

3.8 Proof of Theorem 11

The proof of Theorem 11 is a consequence of the following results:

Proposition 5: Fix $\omega \in I$ and let $J = [\tau_0, \tau_1) \subset \mathbb{R}$. Assume that a continuous input $(u(t), v(t))$ defined in J is applied to system

$$\dot{M}(t) = S(u(t)e_1 + v(t)e_2 + \omega e_3)M(t)$$

Let $M_a(t)$ (respectively $M_b(t)$) be the solution of this system defined on J with initial condition $M_a(\tau_0)$ (resp. $M_b(\tau_0)$). Then $\|M_a(t) - M_b(t)\| = \|M_a(\tau_0) - M_b(\tau_0)\|, \forall t \in J$.

Proof. By time-differentiation, it is easy to show that the scalar product $M_a(t)^\top M_b(t)$ is constant in J. Since $M_a(t)$ and $M_b(t)$ are unitary vectors for all $t \in J$, it follows that the angle between them is constant, then $\|M_a(t) - M_b(t)\|$ is constant. □

Proposition 6: Assume that $N(t, \omega)$ and $M(t, \omega)$ are defined as in the statement of Theorem 11. Let $M_1(t, \omega) = A(t, \omega)N(t, \omega)$. Since $A(kT, \omega) = I$, note that, in each interval $J_k = [kT, (k + 1)T)$, $M_1(t, \omega)$ is a solution of (3.7) with initial condition $N(kT, \omega)$. Assume that $L_k = \lim_{t \to kT^-} \|M(t, \omega) - M_1(t, \omega)\|$. Then $L_k = \|M((k - 1)T, \omega) - M_1((k - 1)T, \omega)\|$ and $L_{k+1} \leq L_k + \|A(kT^-, \cdot) - I\|_{L^\infty}$.

Proof. In the interval J_k, both curves $M(t, \omega)$ and $M_1(t, \omega)$ are solutions with the same applied input for $k \in \mathbb{N}$. By Proposition 5, the distance $\|M(t, \omega) - M_1(t, \omega)\|$ is constant on $J_k, k \in \mathbb{N}$. By Proposition 5 it follows that $L_k = \|M(kT^-, \omega) - M_1(kT^-, \omega)\| = \|M((k - 1)T, \omega) - M_1((k - 1)T, \omega)\|$.

As $M(t,\omega)$ and $N(t,\omega)$ are continuous in time, then $M(kT^-,\omega) = M(kT,\omega)$ and $N(kT^-,\omega) = N(kT,\omega)$. Now note that $\|M_1(kT^-,\omega) - M_1(kT,\omega)\| = \|A(kT^-,\omega)N(kT,\omega) - A(kT,\omega)N(kT,\omega)\| = \|(A(kT^-,\omega)-I)N(kT,\omega)\| \leq \|A(kT^-,\omega)-I\|_{L^\infty}\|N(kT,\omega)\| = \|A(kT^-,\omega)-I\|_{L^\infty}$. In particular, $\|M(kT,\omega) - M_1(kT,\omega)\| = \|M(kT,\omega) - M_1(kT^-,\omega) + M_1(kT^-,\omega) - M_1(kT,\omega)\| \leq \|M(kT,\omega) - M_1(kT^-,\omega)\| + \|M_1(kT^-,\omega) - M_1(kT,\omega)\|$. Now, from the continuity of $M(t,\omega)$ in t and by Proposition 5 applied in J_{k-1}, it follows that $\|M(kT,\omega) - M_1(kT^-,\omega)\| = \|M(kT^-,\omega) - M_1(kT^-,\omega)\| = \|M((k-1)T,\omega) - M_1((k-1)T,\omega)\| = L_{k-1}$. This concludes the proof. □

Now, to prove Theorem 11, note that $N(0,\omega) = M(0,\omega) = M_0$ and so Proposition 5 implies that $L_1 = 0$. Then, by induction, it follows from the last Proposition that $\|M(kT,\omega) - M_1(kT,\omega)\| = \|M(k,T,\omega) - N(kT,\omega)\| = L_k \leq k\|A(kT^-,\cdot) - I\|_{L^\infty}$. Hence, $\|M(k,T,\omega) + e_3\| \leq \|M(k,T,\omega) - N(kT,\omega) + N(kT,\omega) + e_3\| \leq \|M(k,T,\omega) - N(kT,\omega)\| + \|N(kT,\omega) + e_3\| \leq k\|A(kT^-,\cdot) - I\|_{L^\infty} + \|N(kT,\omega) + e_3\|$, showing Theorem 11.

3.9 Proof of Theorem 12

Proposition 7: Let a, b, c, d and e non-negative constants such that $e \geq \max\{a,c\}$. Then $(ab+cd)^2 \leq e^2(c+d)^2$.

Proof. Note that
$e^2(b+d)^2 - (ab+cd)^2 = (eb+ed)^2 - (ab+cd)^2 = (eb+ed+ab+cd)(eb+ed-ab-d)$
$= (eb+ed+ab+cd)[(e-a)b+(e-c)d]$
which is non-negative because $e \geq \max\{a,c\}$. □

Proposition 8: Let I be an open interval of $\mathbb{R}$ and fix $f \in C^1(\bar{I},\mathbb{R})$. For all $g \in H^1(I,\mathbb{R})$ there exists a constant $C > 0$ such that $\|fg\|_{H^1} \leq C\|g\|_{H^1}$.

Proof. Indeed, by definition one has

$$\|fg\|_{H^1}^2 = \int_I \left(|f(\omega)g(\omega)|^2 + \left| \frac{d}{d\omega}[f(\omega)g(\omega)] \right|^2 \right) d\omega$$

$$= \int_I |f(\omega)g(\omega)|^2 d\omega + \int_I |f'(\omega)g(\omega) + f(\omega)g'(\omega)|^2\, d\omega$$

Then, using the proposition 7,

$$\|fg\|_{H^1}^2 \le \|f\|_{C^1}^2 \int_I |g(\omega)|^2 d\omega + \|f\|_{C^1}^2 \int_I |g(\omega) + g'(\omega)|^2\, d\omega = \|f\|_{C^1}^2 \|g\|_{L^2}^2 + \|f\|_{C^1}^2 \|g + g'\|_{L^2}^2$$

$$\le \|f\|_{C^1}^2 \left[\|g\|_{L^2}^2 + \left(\|g\|_{L^2}^2 + \|g'\|_{L^2}^2 \right)^2 \right] \le \|f\|_{C^1}^2 \left[\|g\|_{H^1}^2 + 4\|g\|_{H^1}^2 \right] = 5 \cdot \|f\|_{C^1}^2 \|g\|_{H^1}^2$$

$$\Longrightarrow \|fg\|_{H^1} \le \sqrt{5}\|f\|_{C^1} \cdot \|g\|_{H^1}$$

□

Proposition 9: Let $M \in L^\infty\left([0,T], C^1(\bar{I}, M_3(\mathbb{R}))\right)$ (that is, $M_{ij}(t) \in C^1(\bar{I}, \mathbb{R})$ for all $i, j \in \{1,2,3\}$) and let $v : I \longrightarrow \mathbb{R}^3$ such that $v_j \in H^1(I, \mathbb{R})$ for all $j \in \{1,2,3\}$. Then there exists a constant $C > 0$ such that $\|M(t)v\|_{H^1} \le C\|v\|_{H^1}$, $\forall t \in [0,T]$.

Proof. First of all, we fix $t \in [0,T]$. For now on, we omit the dependency of t for reasons of simplifying the notation. Define $B_i = \sum_j M_{ij} v_j$, then we have $Mv = [B_1\ B_2\ B_3]^\top$ and $\|Mv\|_{H^1}^2 = \sum_i \|B_i\|_{H^1}^2$.
Let $K_i = \max_j \|M_{ij}\|_{C^1}$, for each $i \in \{1,2,3\}$ one obtains

$$\|B_i\|_{H^1}^2 = \int_I \left(\sum_j M_{ij} v_j \right)^2 + \left(\sum_j \frac{d}{d\omega}(M_{ij} v_j) \right)^2 d\omega = \int_I \left(\sum_j M_{ij} v_j \right)^2 + \left(\sum_j M'_{ij} v_j + M_{ij} v'_j \right)^2 d\omega$$

$$\le K_i^2 \left\{ \int_I \left(\sum_j v_j \right)^2 + \left(\sum_j v_j + v'_j \right)^2 d\omega \right\} \le 5 \cdot K_i^2 \max\{\|v_j\|_{L^2}^2, \|v'_j\|_{L^2}^2\} \le 5 \cdot K_i^2 \|v_i\|_{H^1}^2$$

Now we take $K = \max_i \{K_i\}$ and one has

$$\|Mv\|_{H^1}^2 \le 3 \cdot 5K\|v_i\|_{H^1}^2$$

It is enough to take $C(t) = \sqrt{15}K$ for this specified t. Therefore, since $M \in L^\infty\left([0,T], C^1\right)$, we can take

$$C = \sup_{t \in [0,T]} \{C(t)\}$$

and one obtains the expected result. □

Proposition 10: Let I be an open interval of $\mathbb{R}$ and fix $f \in C^1(\bar{I}, \mathbb{R}^3)$. For all $g \in B_R\left[H^1(I, \mathbb{R}^3)\right]$ there exists a constant $C > 0$ such that $\|f \wedge g\|_{H^1} \le C\|f\|_{H^1}$.

Proof. Consider $f = (f_1, f_2, f_3)^\top$ and $g = (g_1, g_2, g_3)^\top$. One has

$$f \wedge g = (f_2 g_3 - f_3 g_2)e_1 + (f_3 g_1 - f_1 g_3)e_2 + (f_1 g_2 - f_2 g_1)e_3$$

Therefore, $\|f \wedge g\|_{H^1}^2 = \|f_2 g_3 - f_3 g_2\|_{H^1}^2 + \|f_3 g_1 - f_1 g_3\|_{H^1}^2 + \|f_1 g_2 - f_2 g_1\|_{H^1}^2$.
For the third component, we have

$$\|f_1 g_2 - f_2 g_1\|_{H^1}^2 =$$

$$\int_I |f_1(\omega)g_2(\omega) - f_2(\omega)g_1(\omega)|^2 \, d\omega + \int_I \left| f_1'(\omega)g_2(\omega) + f_1(\omega)g_2'(\omega) - f_2'(\omega)g_1(\omega) - f_2(\omega)g_1'(\omega) \right|^2 d\omega$$

For the first integral

$$\int_I |f_1(\omega)g_2(\omega) - f_2(\omega)g_1(\omega)|^2 \, d\omega \leq \int_I (|f_1(\omega)g_2(\omega)| + |f_2(\omega)g_1(\omega)|)^2 \, d\omega$$

$$\leq \|g\|_{L^\infty}^2 \int_I (|f_1(\omega)| + |f_2(\omega)|)^2 \, d\omega$$

Using the same arguments of the proposition 8, we can conclude that

$$\|g\|_{L^\infty}^2 \int_I (|f_1(\omega)| + |f_2(\omega)|)^2 \, d\omega \leq 4 \cdot \|g\|_{L^\infty}^2 \|f\|_{H^1}^2$$

Then, we can take $C_1 = 2R > 2\|g\|_\infty$.
For the second one

$$\int_I \left| f_1'(\omega)g_2(\omega) + f_1(\omega)g_2'(\omega) - f_2'(\omega)g_1(\omega) - f_2(\omega)g_1'(\omega) \right|^2 d\omega$$

$$\leq \int_I (|f_1'(\omega)g_2(\omega)| + |f_2'(\omega)g_1(\omega)| + |f_1(\omega)g_2'(\omega)| + |f_2(\omega)g_1'(\omega)|)^2 \, d\omega$$

Using the relation

$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)$$

we must analyze ten terms separately. However, for sake of simplicity, we will show only the analysis of the first one. The rest is completely analogous.
Remembering that, in one dimension, $f \in C^1 \Rightarrow f \in H^1$, we have

$$\int_I |f_1'(\omega)g_2(\omega)|^2 d\omega = \int_I |f_1'(\omega)| \left| f_1'(\omega)g_2(\omega)^2 \right| \leq \|f_1'\|_{L^2} \|f_1' g_2^2\|_{L^2}$$

$$\leq \|f_1'\|_{L^2} \left(\int_I f_1'(\omega) g_2^2(\omega) d\omega \right)^{1/2} \leq \|f_1'\|_{L^2} \|f_1'\|_{L^\infty}^{1/2} \|g_2\|_{L^2}^2$$

Since f is fixed, we take $C_2 > 0$ such that $\|f_1'\|_{L^\infty}^{1/2} < C_2$. Moreover, we always have

$\|g_2\|_{L^2}^2 < R^2$ and $\|f_1'\|_{L^2} < \|f\|_{H^1}$. Hence,

$$\int_I |f_1'(\omega)g_2(\omega)|^2 d\omega \leq C_2 R^2 \|f\|_{H^1}$$

□

Proposition 11: There exists a constant $C > 0$ such that $\|\Omega(t, N_1) - \Omega(t, N_2)\|_{H^1} \leq C\|N_1 - N_2\|_{H^1}$ for all $N_1, N_2 \in B_R[H^1(I, \mathbb{R}^3)]$, $\forall t \in [0, T]$.

Proof. By definition,

$$\Omega(t, N) = \sum_{i=1}^{2} \left[\int_I \left(\left\langle \frac{\partial N}{\partial \omega}, \frac{\partial A^\top}{\partial \omega}(t) e_i \wedge N \right\rangle + \langle e_i, A^\top(t) e_i \wedge N \rangle \right) d\omega \right] e_i$$

For a fixed $t \in [0, T]$ (but omitting the dependence of t for simplicity), we have

$$|\Omega(t, N_1) - \Omega(t, N_2)|$$

$$\leq \sum_{i=1}^{2} \left| \int_I \left(\left\langle \frac{\partial N_1}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_1 \right\rangle + \langle e_i, A^\top e_i \wedge N_1 \rangle - \left\langle \frac{\partial N_2}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_2 \right\rangle - \langle e_i, A^\top e_i \wedge N_2 \rangle \right) d\omega \right|$$

$$\leq \sum_{i=1}^{2} \left\{ \left| \int_I \left(\left\langle \frac{\partial N_1}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_1 \right\rangle - \left\langle \frac{\partial N_2}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_2 \right\rangle \right) d\omega \right| + \left| \int_I (\langle e_i, A^\top e_i \wedge N_1 \rangle - \langle e_i, A^\top e_i \wedge N_2 \rangle) \, d\omega \right| \right\}$$

$$\leq \sum_{i=1}^{2} \left\{ \left| \int_I \left(\left\langle \frac{\partial N_1}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_1 \right\rangle - \left\langle \frac{\partial N_2}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_2 \right\rangle \right) d\omega \right| + \left| \int_I \langle N_1 - N_2, e_i \wedge A^\top e_i \rangle \, d\omega \right| \right\}$$

We will analyse the two terms of this sum separately. We start by the second term:

$$\left| \int_I \langle N_1 - N_2, e_i \wedge A^\top e_i \rangle \, d\omega \right| \leq \int_I \left| \langle N_1 - N_2, e_i \wedge A^\top e_i \rangle \right| d\omega \leq \|e_i \wedge A^\top e_i\|_{L^2} \, \|N_1 - N_2\|_{L^2}$$

where the last inequality is the Hölder inequality.

Analysing the first one:

$$\left| \int_I \left(\left\langle \frac{\partial N_1}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_1 \right\rangle - \left\langle \frac{\partial N_2}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_2 \right\rangle \right) d\omega \right| =$$

$$\left| \int_I \left(\left\langle \frac{\partial N_1}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_1 \right\rangle - \left\langle \frac{\partial N_2}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_1 \right\rangle + \left\langle \frac{\partial N_1}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_1 \right\rangle \right.\right.$$

$$\left.\left. \left\langle \frac{\partial N_2}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_2 \right\rangle \right) d\omega \right| \leq \int_I \left| \left\langle \frac{\partial N_1}{\partial \omega} - \frac{\partial N_2}{\partial \omega}, \frac{\partial A^\top}{\partial \omega} e_i \wedge N_1 \right\rangle \right| d\omega + \int_I \left| \left\langle N_1 - N_2, \frac{\partial N_2}{\partial \omega} \wedge \frac{\partial A^\top}{\partial \omega} e_i \right\rangle \right| d\omega$$

$$\leq \left\| \frac{\partial N_1}{\partial \omega} - \frac{\partial N_2}{\partial \omega} \right\|_{L^2} \left\| \frac{\partial A^\top}{\partial \omega} e_i \wedge N_1 \right\|_{L^2} + \|N_1 - N_2\|_{L^2} \left\| \frac{\partial N_2}{\partial \omega} \wedge \frac{\partial A^\top}{\partial \omega} e_i \right\|_{L^2}$$

Using the same ideas of the proposition 10, it is very easy to show that the same statement holds replacing $\left(H^1, \|\cdot\|_{H^1}\right)$ by $\left(L^2, \|\cdot\|_{L^2}\right)$. Therefore, there exist $C_1(t), C_2(t) > 0$ such that

$$\left\|\frac{\partial A^\top}{\partial \omega} e_i \wedge N_1\right\|_{L^2} \leq C_1(t)\left\|\frac{\partial A^\top}{\partial \omega} e_i\right\|_{L^2} \quad \text{and} \quad \left\|\frac{\partial N_2}{\partial \omega} \wedge \frac{\partial A^\top}{\partial \omega} e_i\right\|_{L^2} \leq C_2(t)\left\|\frac{\partial A^\top}{\partial \omega} e_i\right\|_{L^2}$$

Let $C_3(t) = \max\left\{\|e_i \wedge A^\top(t)e_i\|_{L^2},\ C_1(t)\left\|\frac{\partial A^\top}{\partial \omega}(t)e_i\right\|_{L^2},\ C_2(t)\left\|\frac{\partial A^\top}{\partial \omega}(t)e_i\right\|_{L^2}\right\}$.
Since $\left\|\frac{\partial N_1}{\partial \omega} - \frac{\partial N_2}{\partial \omega}\right\|_{L^2}, \|N_1 - N_2\|_{L^2} \leq \|N_1 - N_2\|_{H^1}$ and taking $C = \sup_{t\in[0,T]}\{C_3(t)\}$, we have the statement. □

Proposition 12: There exists $C > 0$ such that $\|F(t, N_1) - F(t, N_2)\|_{H^1} \leq C\|N_1 - N_2\|_{H^1}$ for all $N_1, N_2 \in B_R\left[H^1(I, \mathbb{R}^3)\right]$, for all $t \in [0, T]$.

Proof. By definition, $F(t, N) = S\left[A^\top(t)\Omega(t, N)\right] N = A^\top(t)\Omega(t, N) \wedge N$.
Therefore,

$$\|F(t, N_1) - F(t, N_2)\|_{H^1} = \|A^\top(t)\Omega(t, N_1) \wedge N_1 - A^\top(t)\Omega(t, N_2) \wedge N_2\|_{H^1}$$

$$= \|A^\top(t)\Omega(t, N_1) \wedge N_1 - A^\top(t)\Omega(t, N_2) \wedge N_1 + A^\top(t)\Omega(t, N_2) \wedge N_1 - A^\top(t)\Omega(t, N_2) \wedge N_2\|_{H^1}$$

$$\leq \|A(t)^\top\left[\Omega(t, N_1) - \Omega(t, N_2)\right] \wedge N_2\|_{H^1} + \|A^\top(t)\Omega(t, N_2) \wedge (N_1 - N_2)\|_{H^1}$$

Since $N_2 \in B_R\left[H^1\left(I, \mathbb{R}^3\right)\right]$ and $A^\top\left[\Omega(t, N_1) - \Omega(t, N_2)\right] \in L^\infty\left(C^1(\bar{I}, \mathbb{R}^3)\right)$, for each $t \in [0, T]$, by Proposition 10 there exists a constant $C_1(t) > 0$ such that

$$\|A(t)^\top\left[\Omega(t, N_1) - \Omega(t, N_2)\right] \wedge N_2\|_{H^1} \leq C_1\|A^\top(t)\left[\Omega(t, N_1) - \Omega(t, N_2)\right]\|_{H^1}$$

Since $A^\top_{ij}(t) \in C^1(\bar{I}, \mathbb{R})$ for all $i, j \in \{1, 2, 3\}$ and $\forall t \in [0, T]$, by Proposition 9 there exists a constant $C_2 > 0$ such that

$$\|A^\top(t)\left[\Omega(t, N_1) - \Omega(t, N_2)\right]\|_{H^1} \leq C_2\|\Omega(t, N_1) - \Omega(t, N_2)\|_{H^1}$$

And, by Proposition 11, there exists a constant $C_3 > 0$ such that

$$\|\Omega(t, N_1) - \Omega(t, N_2)\|_{H^1} \leq C_3\|N_1 - N_2\|_{H^1}$$

Putting $K_1 = C_1 C_2 C_3$, one has

$$\|A(t)^\top\left[\Omega(t, N_1) - \Omega(t, N_2)\right] \wedge N_2\|_{H^1} \leq K_1\|N_1 - N_2\|_{H^1}$$

On the other hand, by Proposition 9, there exists a constant $K_2 > 0$ such that

$$\|A^\top(t)\Omega(t,N_2) \wedge (N_1 - N_2)\|_{H^1} = \|S\left[A^\top(t)\Omega(t,N_2)\right](N_1 - N_2)\|_{H^1} \le K_2\|N_1 - N_2\|_{H^1}$$

Then, it is enough to take $C = \max\{K_1, K_2\}$. □

Proof of Theorem 12

Proof. Let $N^0 \in H^1(I,\mathbb{S}^2)$ and $R > 0$ be such that $R > \max\left\{\|N^0\|_{H^1},\ \sqrt{2\mathcal{L}(0)} + \sqrt{\omega^\star - \omega_\star}\right\}$. By Proposition 12, there exists a constant $K > 0$ such that

$$\|F(t,N_1) - F(t,N_2)\|_{H^1} \le K\|N_1 - N_2\|_{H^1},\quad \forall N_1,N_2 \in B_R\left[H^1(I,\mathbb{S}^2)\right], \forall t \in [0,T]$$

In addition, proceeding in the same way as was done in the Proposition 12, we can easily show (by using the same arguments) that there exists a constant $C > 0$ such that

$$\|F(t,N(t))\|_{H^1} \le C\|N(t)\|_{H^1},\quad \forall N \in B_R\left[H^1(I,\mathbb{S}^2)\right], \forall t \in [0,T] \tag{3.17}$$

Let $T^\star = T^\star(R) > 0$ be small enough so that

$$\|N^0\|_{H^1} + T^\star CR < R \text{ and } T^\star K < 1 \tag{3.18}$$

Now, define $E = B_R\left[C^0\left([0,T^\star], H^1(I,\mathbb{R}^3)\right)\right]$ and consider the map $\Theta : E \longrightarrow C^0\left([0,T^\star], H^1(I,\mathbb{R}^3)\right)$ defined by

$$\Theta(N)(t,\omega) = N^0(\omega) + \int_0^t F(s,N(s,\omega))\ ds$$

Statement 1: Θ takes values in E

Let $N \in E$. For a fixed $t \in [0,T^\star]$, we have:

$$\|\Theta(N)(t)\|_{H^1} \le \|N^0\|_{H^1} + \int_0^t \|F(s,N(s))\|_{H^1}\ ds \le \|N^0\|_{H^1} + \int_0^t C\|N(s)\|_{H^1}\ ds$$

$$\le \|N^0\|_{H^1} + \int_0^t CR\ ds = \|N^0\|_{H^1} + tCR$$

By inequality (3.18) , we have

$$\|\Theta(N)\|_{L^\infty\left((0,T^\star),H^1\right)} \le \|N_0\|_{H^1} + T^\star CR < R$$

Therefore, $\Theta(N) \in E$.

Statement 2: Θ is a contraction

For $N_1, N_2 \in E$ and $t \in [0, T^\star]$, one has

$$\|\Theta(N_1)(t) - \Theta(N_2)(t)\|_{H^1} \leq \int_0^t \|F(s, N_1(s)) - F(s, N_2(s))\|_{H^1} \ ds \leq \int_0^t K \ \|N_1(s) - N_2(s)\|_{H^1} \ ds$$

Therefore,

$$\|\Theta(N_1) - \Theta(N_2)\|_{L^\infty((0,T^\star),H^1)} \leq KT^\star \|N_1 - N_2\|_{L^\infty((0,T^\star),H^1)} < \|N_1 - N_2\|_{L^\infty((0,T^\star),H^1)}$$

by inequality (3.18).

Statement 3: Existence and uniqueness of strong solution

Thanks to two previous steps, using the Theorem 1 (Banach Fixed-Point Theorem), we can conclude that Θ has a unique fixed point. Therefore, for every $R > 0$, there exists a unique weak solution $N \in C^0\left([0, T^\star], H^1(I, \mathbb{R}^3)\right)$ and the variation of constants formula

$$N(t) = N^0 + \int_0^t F(s, N(s)) \quad ds \tag{3.19}$$

holds in $H^1(I, \mathbb{R}^3)$, for all $t \in [0, T^\star]$. Thus, from 3.19, we conclude that

$$N \in C^1_{pw}\left([0, T^\star], H^1(I, \mathbb{R}^3)\right)$$

(the continuity of the derivative of N is inherited from the continuity of F) and $\frac{dN}{dt}(t) = F(t, N(t))$ holds in $H^1(I, \mathbb{R}^3)$ for all $t \in [0, T^\star] - \mathbb{N}T$.

However, H^1 can be compactly embedded in C^0, which implies

$$\frac{dN}{dt}(t, \omega) = F(t, N(t, \omega)), \quad \forall t \in [0, T^\star], \quad \forall \omega \in I$$

(here we are looking at N as a function of two variables).

This has two consequences:

(i) $N(t, \cdot)$ takes values in $\mathbb{S}^2$, $\forall t \in [0, T^\star]$:

Indeed,

$$\frac{d}{dt}\|N(t, \omega)\|^2_{\mathbb{R}^3} = \frac{d}{dt}\langle N(t, \omega), N(t, \omega)\rangle = 2\left\langle N(t, \omega), \dot{N}(t, \omega)\right\rangle = 2\langle N(t, \omega), F(t, N(t, \omega))\rangle$$

$$= 2\left\langle N(t,\omega), A^{\top}(t,\omega)\Omega(t) \wedge N(t,\omega)\right\rangle = 2\left\langle A^{\top}(t,\omega)\Omega(t), N(t,\omega) \wedge N(t,\omega)\right\rangle$$

$$= 2 < A^{\top}(t,\omega)\Omega(t), 0 >= 0$$

Therefore, $\|N(t,\omega)\|_{\mathbb{R}^3}$ is constant. Since $\|N(0,\omega)\|_{\mathbb{R}^3} = 1, \forall\omega \in I$, it follows that $\|N(t,\omega)\|_{\mathbb{R}^3} = 1, \forall(t,\omega) \in [0,\infty) \times I \Longrightarrow N(t,\omega) \in \mathbb{S}^2, \forall(t,\omega) \in [0,\infty) \times I$.

(ii) As the solution N actually exists, we are allowed to calculate the expression 3.14 and, furthermore, $\mathcal{L}$ is not increasing.

Therefore, we have[7]

$$\|N(T^\star)\|_{H^1} = \|N(T^\star) + e_3 - e_3\|_{H^1} \leq \|N(T^\star) + e_3\|_{H^1} + \|-e_3\|_{H^1}$$

$$= \sqrt{2\mathcal{L}(T^\star)} + \sqrt{\omega^\star - \omega_\star} \leq \sqrt{2\mathcal{L}(0)} + \sqrt{\omega^\star - \omega_\star} < R$$

The previous inequality shows that $N(T^\star)$ remains in $B_R\left[H^1(I,\mathbb{R}^3)\right]$. Hence, we can apply the previous steps replacing N^0 by $N(T^\star)$, which yields a solution on $[0, 2T^\star]$. By repeating this process indefinitely, we get a solution defined on $[0, +\infty)$. □

3.10 Proof of Theorem 13

The Proof of Theorem 13 relies on Lemmas 1 an 2 stated in the sequel.

Lemma 1. *Fix $t_0 = kT$. It is possible to construct $\delta > 0$ such that, if an initial condition $N_0 = N(t_0,\cdot)$ of the auxiliary system* (3.9) *is such that $\|N_0 + e_3\|_{H^1} < \delta$ and the control law $(u_1(t), u_2(t))$ defined by* (3.15a)-(3.15b) *is null in $[t_0, t_0 + \tau_0]$. Then $N_0 = -e_3$.*

Proof. Since the auxiliary system (3.9) is T-periodic, it suffices to show the result for $t_0 = 0$; The idea is to show that, in the interval $[0, \tau_0]$, Lemma 1 is a particular case of (BEAUCHARD; SILVA; ROUCHON, 2013, Prop. 3). For this, note that the dynamics that is considered in that paper when $M_f = e_3$ (that implies that $R(\omega) = I$) is

$$\dot{N}(t,\omega) = S\left[F(t,\omega)(u_1e_1 + u_2e_2)\right]N(t,\omega) \tag{3.20}$$

where $F(t,\omega) = \exp(\sigma(t)S(\omega e_3))$, and $\dot{\sigma}(t) = (-1)^{E(t/T)}$, $\sigma(0) = 0$ and $E(x)$ denotes the integer part of x. In particular, for $t \in [0,\tau_0)$, one has $\sigma(t) = t$ for $\tau_0 < T/2$. As the Proof of (BEAUCHARD; SILVA; ROUCHON, 2013, Prop. 3) refers only to small neighborhood

[7] Here we interpret the symbol $-e_3$ as the function $f : I \longrightarrow \mathbb{R}^3$ defined as $f(\omega) = (0,0,-1)^{\top}, \forall\omega \in I$.

of t_0, it suffices to note that, for null control $\overline{u}(t) = \overline{v}(t) = 0$, the solution of (3.8) is $A(t,\omega) = \exp(tS(\omega e_3))$, hence the dynamics of the auxiliary system (3.9) is analogous to the dynamics of (3.20), but with $F(t,\omega) = \exp(tS(\omega e_3))$ replaced by $F(-t,\omega)$. Hence similar arguments of the proof of (BEAUCHARD; SILVA; ROUCHON, 2013, Prop. 3) may be applied to (3.9). □

Since the auxiliary system is T-periodic, we will state the next result only for $t_0 = 0$.

Lemma 2. *Consider a sequence of initial conditions* $N_0^n \in H^1((\omega_*,\omega^*),\mathbb{S}^2)$ *such that* $N_0^n \rightharpoonup N_0^\infty$ *in* H^1 *weakly. Then the solution* $N^n(t,\cdot) \rightharpoonup N^\infty(t,\cdot)$ *weakly in* H^1 *and the control* $\Omega^n(t) = (u_1^n(t), u_2^n(t)) \to \Omega^\infty(t)$ *for* $t \in [0,\tau_0]$, *where* $\Omega_\infty(t)$ *is the control* (3.15a)-(3.15b) *that is obtained with the initial condition* N_0^∞.

Proof. Using the same arguments of the Proof of Lemma 1, it suffices to apply the same arguments of the proof of (BEAUCHARD; SILVA; ROUCHON, 2013, Prop. 4) in the interval $[0,\tau_0]$ in the particular case where $\tau_n = 0, \forall n \in \mathbb{N}$. □

Proof. (of Theorem 13) Since the auxiliary system is T-periodic, there is no loss of generality in considering $t_0 = 0$. The proof of this theorem is based on Lemmas 1 and 2. By contradiction, if the result does not hold, one may construct a sequence $N_0^n, n \in \mathbb{N}$ of initial conditions of the auxiliary system with the following properties[8]:

(i) $\|N_0^n + e_3\|_{L^\infty} \geq \varepsilon, \forall n \in \mathbb{N}$;

(ii) $\|N_0^n + e_3\|_{H^1} \leq \delta, \forall n \in \mathbb{N}$;

(iii) $\int_0^{\tau_0}[(u_1^n)^2(t) + (u_2^n)^2(t)]dt \leq 1/n, \forall n \in \mathbb{N}, n > 0$.

By (ii), passing to a convenient subsequence if necessary, one may assume $N_0^n \rightharpoonup N_0^\infty$ weakly in H^1. In particular, $N_0^n \to N_0^\infty$ strongly in the L^∞ norm. By (i), this strong convergence gives $\|N_0^\infty + e_3\|_{L^\infty} \geq \varepsilon$. Hence, it is clear that $\|N_0^\infty + e_3\|_{H^1} \geq \epsilon(\omega^* - \omega_*) > 0$.

Now, due to weak convergence, $\|N_0^\infty + e_3\|_{H^1} \leq \lim_{n\to\infty} \|N_0^n + e_3\|_{H^1} \leq \delta$ (BREZIS, 2005, Prop. 3.5). Then, we will show that the initial condition N_0^∞ produces null controls for $t \in [0,\tau_0]$. Then Lemma 1 will imply that $\|N_0^\infty + e_3\|_{H^1} = 0$, which is a contradiction. By Lemma 2, one has that $\Omega_n(t) = (u_1^n(t), u_2^n(t)) \to \Omega_\infty(t)$ where $\Omega_\infty(t)$ is the control

[8] Note that (iii) follows from (3.15b) and (3.16).

that is obtained with the initial condition N_0^∞. An extra work (that is analogous to the first step of the proof of (BEAUCHARD; SILVA; ROUCHON, 2012, Theorem 1)) shows that the controls are of class C^1, and they are uniformly bounded, as well as their time-derivatives. In particular, the sequence of controls $\Omega_n(t)$ are uniformly bounded and equicontinuous, and so by Ascoli-Arzela Theorem, passing to a subsequence if necessary, Ω_n converges to Ω_∞ in C^0 with the $\sup$ norm. Assuming that Ω_∞ is not identically null, this gives a contradiction with the fact that the L^2 norm of Ω_n tends to zero (assured by (iii)). □

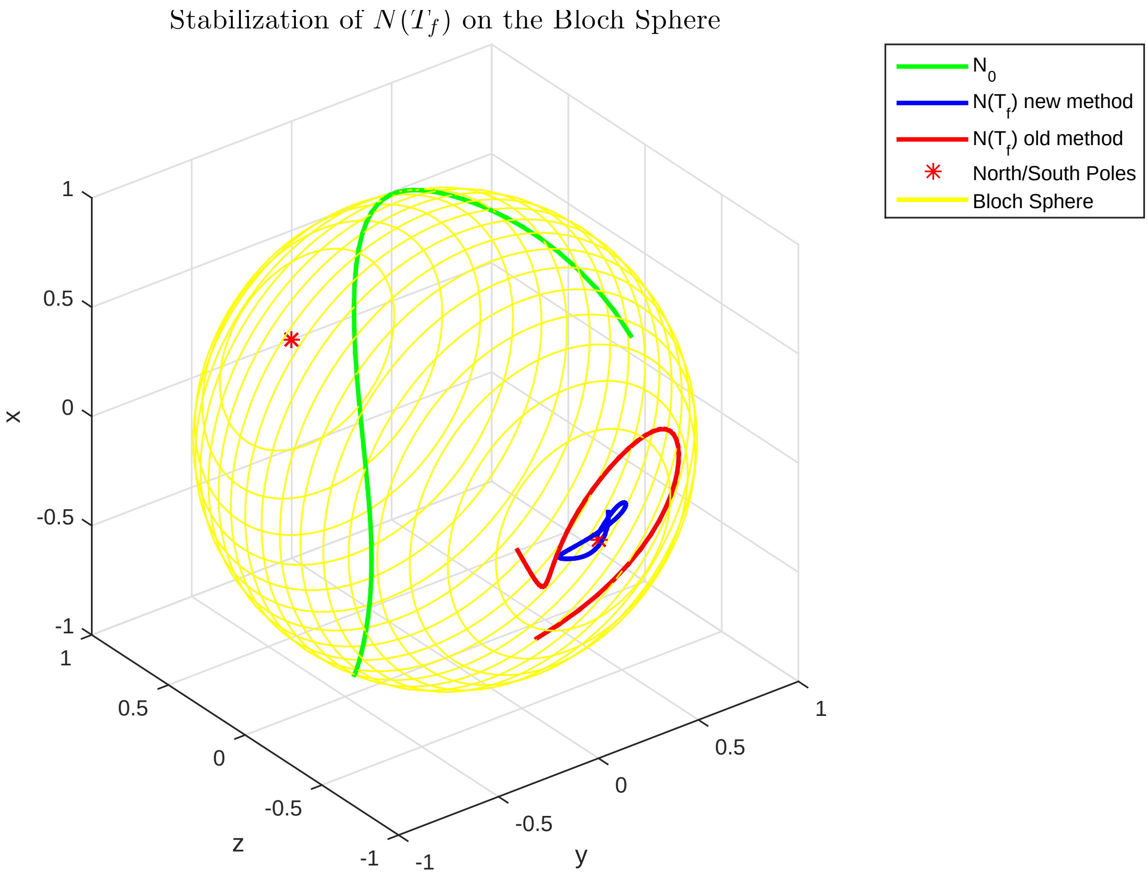

Figure 5 - Results on the Bloch sphere for both the new and the old method.

4 NUMERICAL EXPERIMENTS

For simplicity we shall refer to the method of (BEAUCHARD; SILVA; ROUCHON, 2013) as the “old method” and the method described in this paper will be referred as “the new method”. One has chosen, $\epsilon = 0.2$, $T = 20$, $s_0 = 0.1$, $K = 10$, $\ell = 16$ and $T_f = \ell T$ for the new method. As defined in (3.15a)-(3.15b), we have chosen unitary gains of the feedback law (we mean, there is no gain multiplying H_i of (3.15b)). For the old method, we have chosen $T = 1$ and unitary gains as well. We have verified that greater values of T than 1 for the old method are worse, but smaller values of T will not improve the result. Figure 5 shows the simulation results in the Bloch sphere for these choices.

The obtained error of the adiabatic propagator (see Theorem 10) is $\|A(T^-, \cdot) - I\|_{L^\infty} \leq 0.0009$ (see Figure 3). So $\ell\|A(T^-, \cdot) - I\|_{L^\infty} \leq 0.015$. It is very small in this case. In the simulations we have found that $\|N(\ell T) + e_3\|_{L^\infty}$ is more than ten times greater than

$\ell \|A(T^-, \cdot) - I\|_{L^\infty}$. Hence one will show only the behaviour of the auxiliary state $N(t, \omega)$.

In figure 5 one may see the initial condition $N_0 = M_0$ and the final condition $N(\ell T)$. One has obtained $\|N(T_f) + e_3\|_{L^\infty} = 0.185$ with our new method, and $\|N(T_f) + e_3\|_{L^\infty} = 0.58$ with our old method, using the same unitary gains multiplying H_i. The expressions of the feedback of the old method is analogous to (3.15a)-(3.15b), with the difference that $A(t, \omega)$ is replaced by the matrix $\exp(\sigma(t) S(\omega e_3))$, where $\dot{\sigma}(t) = (-1)^{E(t/T)}$, $E(s)$ is the integer part of s, and $\sigma(0) = 0$ (see the proof of Lemma 1). Our new method have produced a result that is more than 3 times better than the old method with respect to the final L^∞ norm. Note that, for the new method, $\ell \|A(T^-, \cdot) - I\|_{L^\infty} + \|N(T_f) + e_3\|_{L^\infty} \leq 0.015 + 0.185 = 0.2 = \varepsilon$. So Theorem 13 assures that the problem is solved for the given ε.

Figure 6 regards only the new method. It shows the evolution of the Lyapunov functional $\mathcal{L}(t) = \frac{1}{2}\|N(t) + e_3\|^2_{H^1}$. In that figure one shows also the evolution of $\|N(t) + e_3\|_{L^\infty}$. The controls $u_1(t)$ and $u_2(t)$ are also depicted in that figure. The Figure 7 is a "zoom" of the last one. This allows to see the “microstructure” of the control of the new strategy.

The Figure 8 presents a comparison of the input norms of the old and the new method. The Figure 9 shows the plot of $\log(\|N(t) + e_3\|^2_{H^1})$ versus time. The slope of the curves of $\log(\|N(t) + e_3\|^2_{H^1})$ would give a measure of the exponential rate of decaying of $\|N(t) + e_3\|^2_{H^1} = 2\mathcal{L}(t)$. The slope is much bigger for the first method in the beginning, and this inclination decreases faster for the old method with respect to the new one. This indicates that the new method seems to be more effective than the old one.

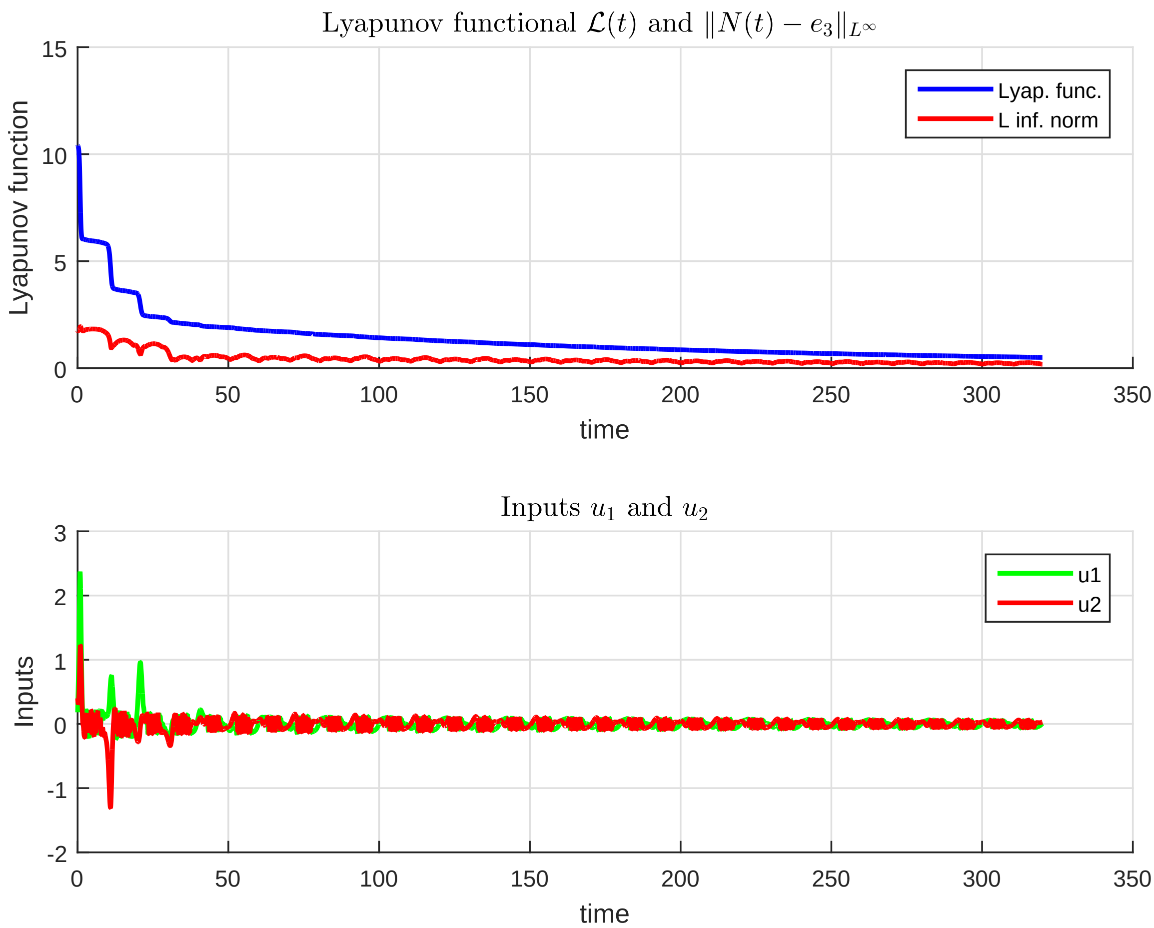

Figure 6 - Lyapunov functional and inputs for the new method.

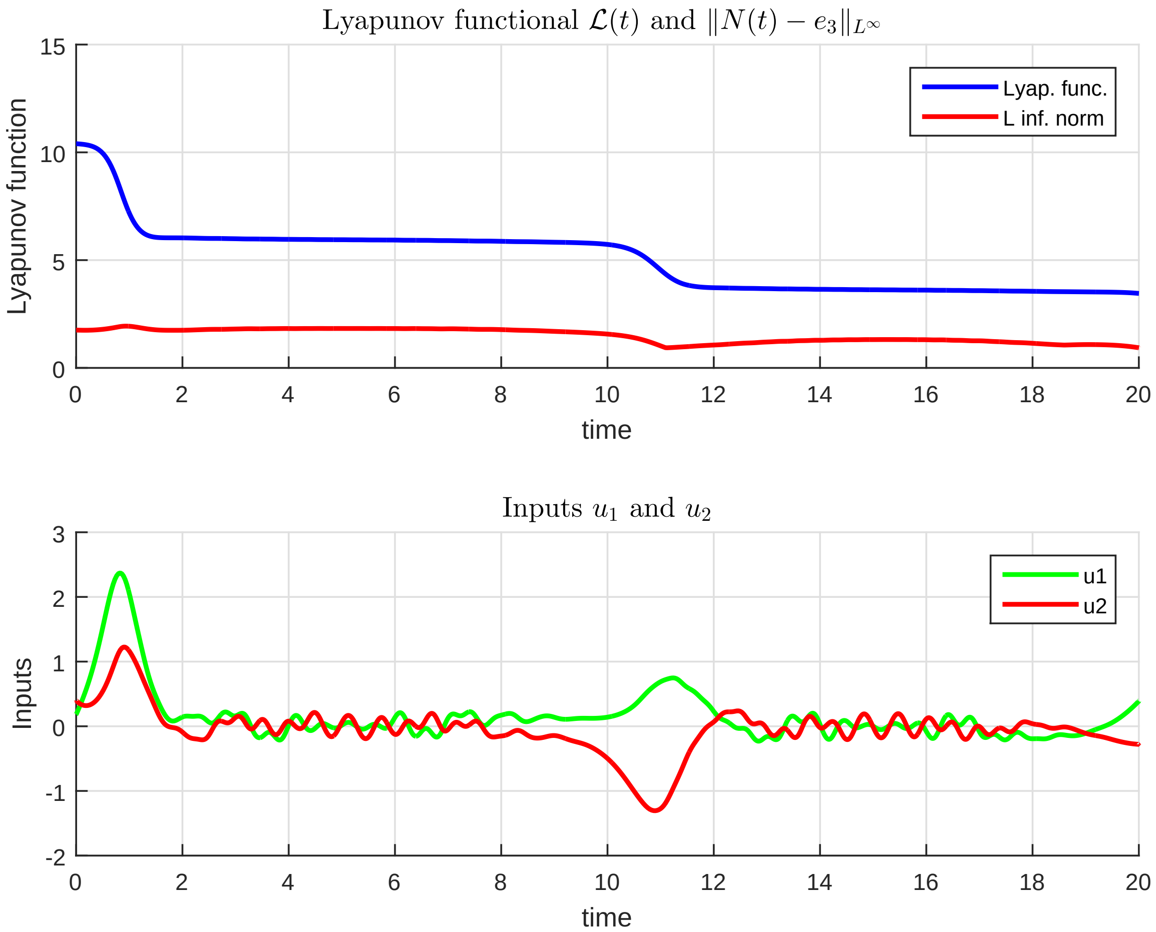

Figure 7 - Lyapunov functional and inputs for the new method (zoom).

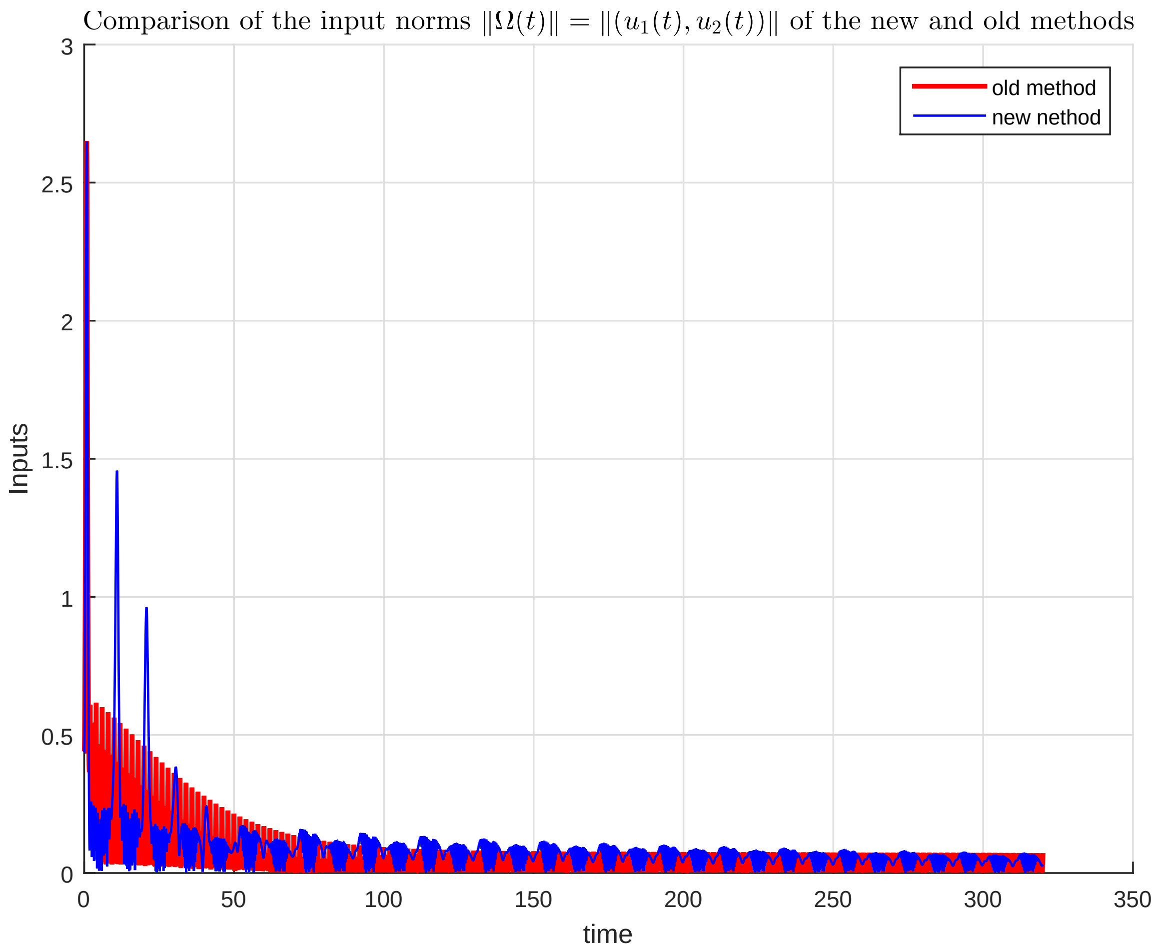

Figure 8 - Plot of input norms $\|u_1(t) + u_2(t)\|$.

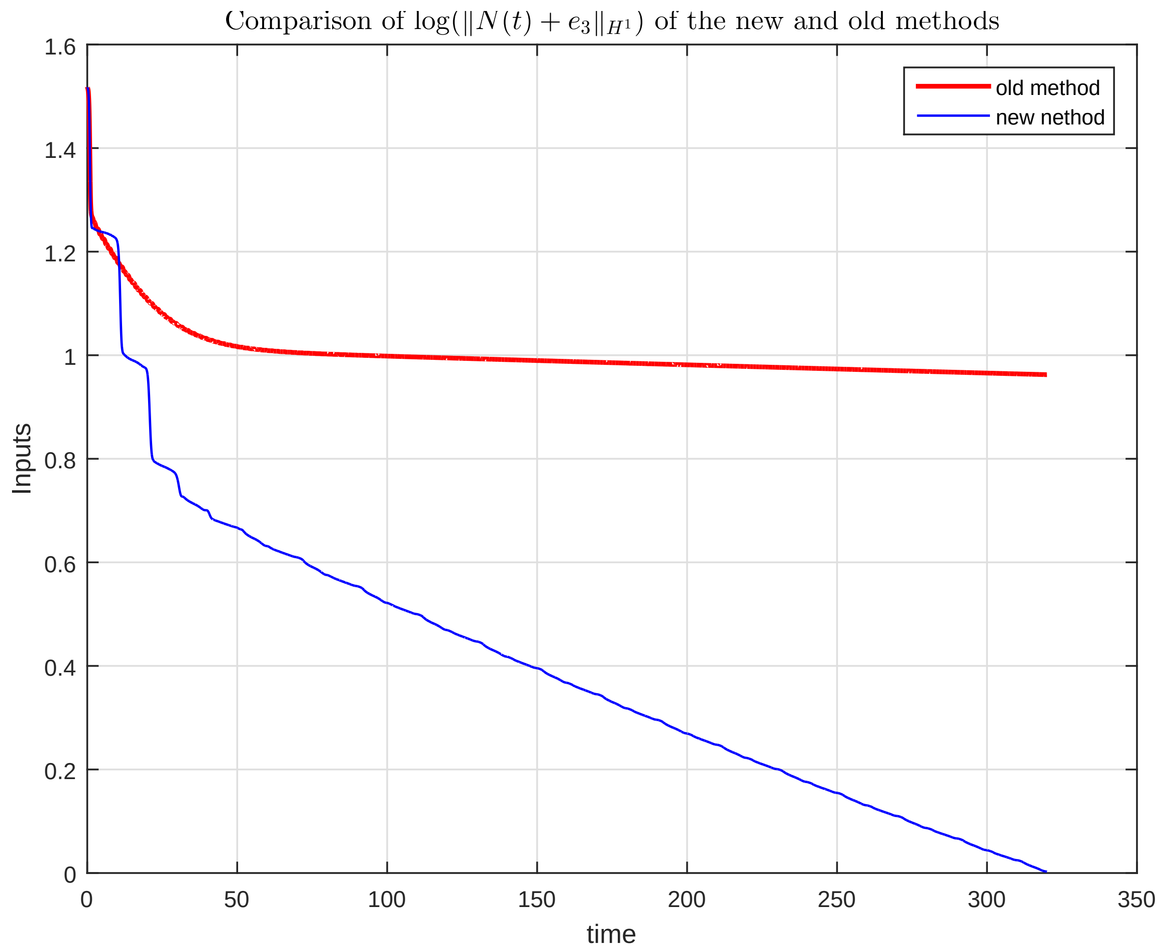

Figure 9 - Plot of the natural logarithm of the square of the H^1 norm of $(N(t) + e_3)$.

5 CONCLUSION

The main result of this work indicates that the Rabi pulses that are commonly encountered in *Nuclear Magnetic Ressonance (NMR)* techniques (for instance spin-echo pulses) are not a mandatory ingredient for an efficient open loop control law.

One might ask if this could imply that one may develop NMR methods with pulses with less intensity than the ones that are found in the present state of the art. This could be an interesting topic of future research, which may lead to produce less “agressive” NMR techniques for medical (and other possible) applications.

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