UNIVERSIDADE DE SÃO PAULO
ESCOLA POLITÉCNICA

FABIO BARBIERI

A mean-field approach for the optimal control of discrete-time linear systems with multiplicative noises

São Paulo
2020

FABIO BARBIERI

A mean-field approach for the optimal control of discrete-time linear systems with multiplicative noises

Thesis submitted for the degree of Doctor in Science to the Escola Politécnica of Universidade de São Paulo.

São Paulo
2020

FABIO BARBIERI

A mean-field approach for the optimal control of discrete-time linear systems with multiplicative noises

Revised Version

Thesis submitted for the degree of Doctor in Science to the Escola Politécnica of Universidade de São Paulo.

Concentration field:
Systems Engineering

Advisor:
Prof. Dr. Oswaldo Luiz do Valle Costa

São Paulo
2020

Este exemplar foi revisado e corrigido em relação à versão original, sob responsabilidade única do autor e com a anuência de seu orientador.

São Paulo, ______ de ____________________ de __________

Assinatura do autor: ________________________

Assinatura do orientador: ________________________

Catalogação-na-publicação

Barbieri, Fabio
A mean-field approach for the optimal control of discrete-time linear systems with multiplicative noises / F. Barbieri -- versão corr. -- São Paulo, 2020.
126 p.

Tese (Doutorado) - Escola Politécnica da Universidade de São Paulo. Departamento de Engenharia de Telecomunicações e Controle.

1.Controle ótimo estocástico 2.Sistemas lineares 3.Restrições intertemporais 4.Solução estabilizadora 5.Otimização de carteira de investimentos I.Universidade de São Paulo. Escola Politécnica. Departamento de Engenharia de Telecomunicações e Controle II.t.

To my family

ACKNOWLEDGEMENT

I want to thank my thesis advisor and my committee for generously offering their time, support, guidance and goodwill throughout the preparation and review of this document.

I also offer my thanks to CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) for the financial support.

ABSTRACT

In this work, we study the stochastic multi-period optimal control for discrete-time linear systems subject to multiplicative noises. Initially, we consider a multi-period mean-variance trade-off performance criterion for the finite-horizon case with and without constraints, and then, its infinite-horizon case with the long-run as well as the discount factor criteria. We adopt the mean-field approach to tackle the problems and get their solutions in terms of a set of two generalised coupled algebraic Riccati equations (GCARE for short). For the finite-horizon case, we derive the optimal control law for a general multi-period mean-variance problem and obtain the optimal control strategy for the constrained problems using the Lagrangian multipliers approach. From the general unrestricted result, we obtain a sufficient condition for a closed-form solution for one of the constrained problems considered in this work. For the infinite-horizon case, we establish sufficient conditions for the existence of the maximal solution, necessary and sufficient conditions for the existence of the mean-square stabilising solution to the GCARE, and derive the optimal control laws for the discounted and long-run problems. When particularised to the portfolio selection problem, we show that our results match some of the results available in the literature. A numerical example illustrates the obtained optimal controls for the multi-period portfolio selection problem in which is desired to optimise the sum of the mean-variance trade-off costs of a portfolio against a benchmark along the time.

Keywords: Stochastic control. Linear systems. Optimal control. Stabilising solution. Portfolio optimisation. Intertemporal restrictions.

RESUMO

Neste trabalho, estudamos o controle ótimo estocástico multi-período de sistemas lineares em tempo discreto sujeitos a ruidos multiplicativos. Inicialmente, consideramos como critério de desempenho a combinação multi-período entre média e variância para o caso de horizonte finito com e sem restrições, e posteriormente consideramos o caso de horizonte infinito com a abordagem de campos de médias para resolvermos os problemas e obtemos suas soluções em termos de um conjunto de duas equações generalizadas de Riccati (GCARE). Para o caso de horizonte finito, derivamos o controle ótimo para um problema geral de média variância multi-período e obtemos as estratégias de controle ótimo para os problemas com restrições através de multiplicadores de Lagrange. Do resultado geral sem restrições, obtemos condições suficientes para uma solução fechada para um dos problemas com restrições considerado neste trabalho. Para o caso de horizonte infinito, são fornecidas condições suficientes para a existência da solução máxima, condições necessárias e suficientes para a existência da solução estabilizadora de média quadrática da GCARE e derivamos as leis de controle ótimo para os problemas com critério de longo prazo e com fator de desconto. Quando particularizado para o problema de alocação de carteiras de investimento, mostramos que nossos resultados são equivalentes há alguns resultados disponíveis na literatura. Concluímos esta tese ilustrando os resultados obtidos com um problema multi-período de alocação de carteira de investimentos no qual é desejado otimizar a soma de médias e variâncias do valor da carteira versus um ativo de referência.

Palavras-chave: Controle estocástico. Sistemas lineares. Controle ótimo. Solução estabilizadora. Otimização de carteiras de investimento. Restrições intertemporais.

LIST OF FIGURES

LIST OF TABLES

CONTENTS

1 INTRODUCTION

In this thesis, we study discrete-time linear systems subject to multiplicative noises. This class of models incorporates a stochastic behaviour in the dynamics of the system and has found applications in many fields of science such as signal processing, biological motor systems, aerospace, finance, among others.

We want to apply linear systems with multiplicative noises to multi-period mean-variance problems and obtain the optimal solution to a variety of situations such as (i) the more general case without constraints, (ii) imposing restrictions on either the output's mean or its variance, and (iii) considering the system stabilisation with infinite horizon, expanding the current results in the literature.

Our methodology consists of applying dynamic programming for the optimisation and Lagrangian multipliers to deal with the imposed restrictions. The formulation follows the same approach as in the mean-field theory, which consists of solving the problems in terms of the state's mean. This approach allows us to overcome some issues that arise from the quadratic term of the variance, in particular to multi-period mean-variance problems with restrictions.

We apply our results in the financial context, specifically in the optimisation of a portfolio of financial assets. The management of financial portfolios requires the possibility of imposing many sorts of situations, for instance, achieving a return higher than inflation, limiting the variance over time, and considering an infinite-time horizon, which suits the application of our findings.

The results compiled here can also be found in the papers (BARBIERI; COSTA, 2020a) and (BARBIERI; COSTA, 2020b).

1.1 Document structure

In Chapter 2, we present a brief literature review with a focus on the main theoretical achievements regarding our system's model. In Chapters 3 and 4, we define the formulations of our problems and detail some previous results, respectively. In Chapter 5, we present the solutions to our problems and, in Chapter 6, we summarise the necessary steps on how to compute the optimal control strategies for the finite and infinite-horizon cases. In Chapter 7, we show how to model a portfolio of risky assets against a benchmark using our system's notation. In Chapter 8, we compare some of our findings to known results from the literature. In Chapter 9, we give an example about how to estimate the model parameters and run numerical simulations to each of our results. Finally, in Chapter 10, we present our final considerations.

2 LITERATURE REVIEW AND CONTRIBUTIONS OF THIS WORK

In this chapter, we present the main literature regarding linear systems with multiplicative noises, mean-field theory, and some aspects of mean-variance control problems that evidence the mathematical difficulties we face when imposing restrictions to such problems.

2.1 Linear systems with multiplicative noises

A relevant goal in engineering is to describe the evolution of observable processes through mathematical equations that can be either continuous or discrete in time, linear or non-linear, stochastic or not, and so on.

We are interested in a class of models regarding discrete-time linear systems with multiplicative noises due to their fundamental variety of applications. For example, they are particularly suited to describe nuclear fission, heat transfer, population immunology, portfolio optimisation, among others. We can refer the book (DRAGAN; MOROZAN; STOICA, 2013) and references therein for an overview for this class of models. Consider the following general linear system with multiplicative noises:

$$\begin{aligned} x(k+1) &= \left(\bar{A}(k) + \tilde{A}(k)w^x(k)\right)x(k) + \left(\bar{B}(k) + \tilde{B}(k)w^u(k)\right)u(k), \\ x(0) &= x_0, \ k = 0, 1, \dots, T-1, \end{aligned}$$

where $x(k)$ denotes the system state at instant k with initial condition x_0, $u(k)$ represents the control input, the matrices $\bar{A}(k)$, $\tilde{A}(k)$, $\bar{B}(k)$, and $\tilde{B}(k)$ represent the environment dynamics, and $w^x(k)$ and $w^u(k)$ are both zero-mean random variables with unitary variance that operate directly on the dynamics and control of the system, respectively.

The optimal control of such systems are obtained through the optimisation of some functional cost. For instance, the linear-quadratic (LQ) cost is a common choice due to the possibility of having a global minimum or maximum. For illustration purposes, consider a general LQ cost problem for the discrete-time as:

$$\inf_u E\left\{\sum_{k=0}^{T-1}[x(k)'Q(k)x(k) + u(k)'R(k)u(k)] + x(T)'Q(T)x(T)\right\}, \tag{2.1}$$

and assume the optimal solution to be of the form

$$u(k) = -K(k)x(k), \tag{2.2}$$

where $K(k)$ is the feedback gain we want to calculate.

Several results related to the control of these class of systems have already been achieved. In (BEGHI; D'ALESSANDRO, 1998), the authors presented the optimal control law by studying a Riccati type equation obtained by applying (2.2) into (2.1) and solving it for $K(k)$.

In (CHEN; LI; ZHOU, 1998), the authors proved that some linear quadratic problems are non-trivial when the control weighting matrix is indefinite. In (BOUKAS; LIU, 2004), the authors combined the linear system with Markov jumps and established sufficient conditions for the solvability of the Riccati equation through linear matrices inequalities (LMI). Linear-systems with Markov jumps and multiplicative noises were studied in more detail in (COSTA; PAULO, 2007; COSTA; PAULO, 2008; COSTA; FRAGOSO; MARQUES, 2005; DOMBROVSKII; LYASHENKO, 2003; ELLIOTT; DUFOUR; MALCOLM, 2005). In (DRAGAN; MOROZAN, 2006a; DRAGAN; MOROZAN, 2006b), the authors studied the stability, observability, and detectability of discrete-time stochastic linear system.

Examples of optimal controls and necessary and sufficient conditions for the solvability of indefinite stochastic control problems can be found in (LI; ZHOU; RAMI, 2003; LIM; ZHOU, 1999; MOORE; ZHOU, 1999; RAMI; CHEN; ZHOU, 2002; RAMI; ZHOU, 2000; WU; ZHOU, 2002; ZHU, 2005).

For the continuous-time control problems, the reader is referred to (ZHOU; LI, 2000; ZHOU; YIN, 2003; BOUKAS; LIU, 2004) for further details.

2.2 Mean-field control problems

The mean-field approach considers the state as well as its expected value in either the system dynamics or the objective functions, or both. Lately, there has been a great deal of attention to the mean-field formulation followed by an increasing number of successful applications in various fields of science, engineering, financial management and economics.

The mean-field games and the mean-field (type) control problems have emerged based on the idea of the mean-field approach. Mean-field theory can be traced back to Kac (KAC, 1956), who presented the McKean-Vlasov stochastic differential equation motivated by a stochastic toy model for the Vlasov kinetic equation of plasma. Mean-field games is a new area of research developed in the engineering community by (HUANG; MALHAMÈ; CAINES, 2006; HUANG; CAINES; MALHAMÈ, 2007), and independently and about the same time by (LASRY; LIONS, 2006a; LASRY; LIONS, 2006b; LASRY; LIONS, 2007).

There are several differences between these two theories. However, in general terms, we can state that the mean-field games can be reduced to a standard control problem, while the search for Nash equilibria in mean-field games is more of a fixed point problem than an optimisation problem. See (BENSOUSSAN; FREHSE; YAM, 2013; HUANG; LI, 2018; GOMES; SAUDE, 2014; CARMONA; DELARUE, 2018; MOON, 2019) for more details on mean-field games.

There are several results related to mean-field linear-quadratic unconstrained problems applied to linear systems with multiplicative noises. See (YONG, 2013; HUANG; LI; YONG, 2015; LI; SUN; XIONG, 2019; MOON; KIM, 2019) and references therein for examples of unconstrained optimal control laws in the continuous-time with finite and infinite horizon.

Regarding the discrete-time mean-field finite-horizon problem, the authors in (ELLIOTT; NI, 2013; NI; ELLIOTT; LI, 2015; NI; LI; ZHANG, 2016a; ZHANG; QI; FU, 2019) investigated the unconstrained multi-period control problem of systems with multiplicative noises defined as:

$$
\begin{aligned}
x(k+1) &= \Big(Ax(k) + \bar{A}\mathbb{E}(x(k)) + Bu(k) + \bar{B}\mathbb{E}(u(k))\Big) \\
&\quad + \Big(Cx(k) + \bar{C}\mathbb{E}(x(k)) + Du(k) + \bar{D}\mathbb{E}(u(k))\Big)\,\omega(k), \\
x(0) &= x_0,
\end{aligned} \tag{2.3}
$$

where, A, $\bar{A}$, B, $\bar{B}$, C, $\bar{C}$, D, and $\bar{D}$ are deterministic matrices of proper dimensions, $x(k)$ and $u(k)$ are the state and control as defined earlier, and $\omega(k)$ is a white noise.

Consider a more general LQ problem associated with (2.3) as:

$$
\begin{aligned}
&\inf_u \sum_{k=0}^{T-1} \mathbb{E}\Big(x(k)'Q(k)x(k) + \mathbb{E}(x(k))'\bar{Q}(k)\mathbb{E}(x(k)) + 2x(k)'L(k)u(k) + 2\mathbb{E}(x(k))'\bar{L}(k)\mathbb{E}(u(k)) \\
&+ u(k)'R(k)u(k) + \mathbb{E}(u(k))'\bar{R}(k)\mathbb{E}(u(k)) + \mathbb{E}(x(T)'G(T)x(T)) + \mathbb{E}(x(T))'\bar{G}(T)\mathbb{E}(x(T))\Big),
\end{aligned} \tag{2.4}
$$

where $Q(k)$, $\bar{Q}(k)$, $L(k)$, $\bar{L}(k)$, $R(k)$, $\bar{R}(k)$, $G(k)$, and $\bar{G}(k)$ are deterministic symmetric matrices of appropriate dimensions.

ELLIOTT; NI (2013) proved that

$$
\begin{aligned}
&Q(k), Q(k) + \bar{Q}(k) \geq 0, \\
&R(k), R(k) + \bar{R}(k) > 0, \\
&G(k), G(k) + \bar{G}(k) \geq 0, k = 0, \dots, T-1,
\end{aligned}
$$

are necessary and sufficient solvability conditions to Problem (2.4), with $L(k)$=$\bar{L}(k) = 0$, and that the unique optimal control is given in terms of some difference Riccati type equations.

NI; ELLIOTT; LI (2015) studied the infinite case of Problem (2.4), with $L(k)$=$\bar{L}(k)$=$G(k)$=$\bar{G}(k) = 0$. They presented a variety of results, including the equivalence of several notions of stability for linear mean-field stochastic difference equations. They also showed that the optimal control of a mean-field linear-quadratic optimal control with an infinite-time horizon uniquely exists, and the optimal control can be expressed as a linear state feedback involving the state and its mean via the minimal non-negative definite solution of two coupled algebraic Riccati equations.

NI; LI; ZHANG (2016a) provided results for the finite as well as the infinite-time horizon case of Problem (2.4). The finite-horizon optimal control is presented based on the assumption that

$$\begin{bmatrix} Q(k) & L(k) \\ L(k)' & R(k) \end{bmatrix} \geq 0, \quad \begin{bmatrix} Q(k)+\bar{Q}(k) & L(k)+\bar{L}(k) \\ L(k)'+\bar{L}(k)' & R(k)+\bar{R}(k) \end{bmatrix} > 0, \quad G(T) \geq 0, \quad \text{and} \quad G(T)+\bar{G}(k) \geq 0.$$

In the same way, the optimal and maximal stabilising solution to the infinite horizon case is also based on the assumption of some matrices and sum of matrices being positive (positive semi-definite).

In (ZHANG; QI; FU, 2019), the authors considered the infinite horizon of Problem (2.4), with $L(k)$=$\bar{L}(k)$=$G(k)$=$\bar{G}(k) = 0$ and under the assumption that $Q \geq 0, Q+\bar{Q} \geq 0$ and $R \geq 0, R+\bar{R} \geq 0$ holds. They obtained that if $(A, \bar{A}, C, \bar{C}, Q^{1/2})$ is exact detectable, then system (2.3) is stabilisable in the mean-square sense if and only if there exists a unique positive semi-definite solution to a coupled algebraic Riccati equation (ARE). Another result states that if $(A, \bar{A}, C, \bar{C}, Q^{1/2})$ is exact observable, then system (2.3) is stabilisable in the mean-square sense if and only if there exists a unique positive definite solution to a coupled ARE. In both cases, the optimal control has the form of $u(k) = \mathcal{K}x(k) + \bar{\mathcal{K}}\mathbb{E}(x(k))$, where $\mathcal{K}$ and $\bar{\mathcal{K}}$ are gains associate to the solution of the coupled ARE calculated.

Finally, the authors in (NI; LI; ZHANG, 2016b) combined System (2.3) with Markov jumps, which means they allowed each of the matrices A, $\bar{A}$, B, $\bar{B}$, C, $\bar{C}$, D, and $\bar{D}$ to change at every step k according to a Markov process (a random process in which the future is independent of the past, given the present). They also considered Problem (2.4), with $L(k)$=$\bar{L}(k) = 0$, but this time the weighting matrices also varies in time according to a Markov process and are re-defined as $Q_i(k)$, $\bar{Q}_i(k)$, $R_i(k)$, $\bar{R}_i(k)$, $G_i(k)$, and $\bar{G}_i(k)$, where i represents the Markov state at time k. They proved that under the assumption that $Q_i(k)$, $\bar{Q}_i(k)$, $G_i(k)$, $\bar{G}_i(k) \geq 0$ and $R_i(k)$, $\bar{R}_i(k) > 0$, then there exists a unique optimal control, which can be explicitly given via solutions of two generalised difference Riccati equations.

The literature on mean-field control is extensive, and for more information, the reader is referred to (BUCKDAHN; LI; PENG, 2009; BUCKDAHN; DJEHICHE; LI, 2009; BUCKDAHN; DJEHICHE; LI, 2011; ANDERSSON; DJEHICH, 2011; MEYER-BRANDIS; ØKSENDAL; ZHOU, 2012) for maximum principles for stochastic differential equations (SDEs) among other results regarding SDEs, (DAWSON, 1983) for mean-field model of cooperative behaviour, and (NI; ZHANG; LI, 2015; ZHANG; QI, 2016) for more results on stabilisation and indefinite LQ optimal control.

As exemplified by the papers above, there are many relevant results in the literature regarding the unconstrained multi-period optimal control strategy using the mean-field formulation. However, to the best of our knowledge, mean-variance and constrained multi-period optimal control problems for systems with multiplicative noises lacks further investigation and poses new challenges in this field.

Also, previous papers regarding the stabilisation using the mean-field formulation considered neither the discounted cost problem nor the long-run problem with linear terms on the performance criterion. They had the stabilisation based on the assumptions of some matrices and sum of matrices being positive definite (semi-definite) and related to the exact observability and detectability of their systems as shown above.

Compared to previous works in the mean-field, we solve classical constrained multi-period problems in a vector space and generalise the stabilisation conditions to just some positive semi-definite matrices and kernels restrictions on some matrices and also considered both the discounted cost problem and the long-run problem with linear terms on the performance criterion.

2.3 Mean-variance optimisation problems

Let us consider the following static form for the multi-period mean-variance (MV) problem:

$$\begin{aligned} MV(\omega): \quad & \max_u \quad \mathbb{E}(x(T)) - \omega Var(x(T)) = \\ & \max_u \quad \mathbb{E}(x(T)) - \omega\mathbb{E}(x(T)^2) + \omega\mathbb{E}(\boldsymbol{x(T)})^{\mathbf{2}}, \end{aligned} \tag{2.5}$$

where $x(T)$ is the state of a system at the final instant T, u is the control applied to the system, and ω is an input parameter. Note that problem $MV(\omega)$ is non-separable in the sense of dynamic programming due to the quadratic term that arises from the variance. In other words, we cannot decompose it by a stage-wise backward recursion and thus, it does not satisfy the principle of optimality. Therefore, all the traditional dynamic programming-based optimal stochastic control solution methods no longer apply in such non-separable situations.

Notwithstanding, there are some approaches in the literature to overcome the difficulty resulted from the non-separability. In (LI; NG, 2000) and (ZHOU; LI, 2000), the authors adopted an embedding scheme and considered the following family of auxiliary problem, $A(\omega,\lambda)$, parameterised in λ,

$$A(\omega,\lambda): \quad \min_{u} \ \mathbb{E}(\omega x(T)^2 - \lambda x(T)). \tag{2.6}$$

Note that problem $A(\omega,\lambda)$ is a separable linear-quadratic stochastic control formulation and can thus be solved analytically. In the same papers, the authors derived the optimal policy to the non-separable problem $MV(\omega)$ via identifying the optimal parameter λ^* under which the optimal policy to $A(\omega,\lambda^*)$ also solves $MV(\omega)$. In (CERNÝ; KALLSEN, 2009; LI; ZHOU; LIM, 2002; SCHWEIZER, 1996) and (XIA; YAN, 2006), we can see other methods to overcome the non-separability issue in the MV problem.

The MV problems mentioned above consider only the final value of the state. However, we can also find in the literature the optimisation problem that considers the intermediate steps, as shown in the more general problem (2.7).

$$MV(\nu,\xi): \quad \min_{u} \ \sum_{t=1}^{T}\Big(\nu(t)Var(y^u(t)) - \xi(t)\mathbb{E}(y^u(t))\Big), \tag{2.7}$$

where $y^u(t)$ is the system's output subjected to the control policy u. The parameters $\xi(t)$ and $\nu(t)$ are inter-temporal weights associated with the expected value of the output and its variance, respectively.

To get an optimal policy for the $MV(\nu,\xi)$ problem, one needs to follow a similar approach as in (LI; NG, 2000) and solve an auxiliary problem that also solves the non-separable mean-variance problem as shown in (COSTA; OLIVEIRA, 2012). However, in the same paper, the authors were only able to develop an optimal policy with restrictions on the minimum expected output over time, leaving the problem with restrictions on the variance without a solution.

In (BARBIERI; COSTA, 2018), the authors managed a workaround to the multi-period mean-variance control problem with restrictions on the variance as defined in (2.8). They obtained an optimal policy by considering an upper-bound value to the total weighted sum of the system's output variance, defined as $\alpha > 0$.

$$MV(\nu,\beta,\alpha): \max_{u\in\mathbb{U}} \sum_{t=1}^{T}\Big[\beta(t)E[y^u(t)]\Big],$$

$$\text{subject to:} \quad \sum_{t=1}^{T} \nu(t)Var[y^u(t)] \leqslant \alpha, \tag{2.8}$$

where $\beta = [\beta(1)\ldots\beta(T)]'$, $\beta(t) > 0$, is an input parameter associated to the expected system's output and $\nu = [\nu(1)\ldots\nu(T)]'$, $\nu(t) \geqslant 0$, is as defined above.

There, they proved that the solution to Problem (2.8) can be obtained through the solution of the unconstrained Problem (2.7) by establishing a linear relationship between ξ and β.

Despite the constraint being on to the total variance, one could manipulate β to (de)increase the variance in particular periods; however, without being able to specify the constrain neither directly nor precisely. Nonetheless, the ingenuity of this workaround exemplifies the difficulties that one faces when dealing with multi-period problems and helps to highlight the benefits and contributions of methodologies that avoid the auxiliary problems to solve them as the mean-field approach explained in the previous section.

2.4 Mean-variance problems applied to finance

From the portfolio optimisation point of view, one of the main applications of linear systems with multiplicative noises is related to the classical portfolio's mean-variance problem.

Mean-variance portfolio optimisation is a classical financial problem introduced by Markowitz (MARKOWITZ, 1959) which paved the foundation for the modern portfolio theory. The main goal is to maximise the expected return for a given level of risk, minimise the expected risk for a given level of expected return, or minimise a trade-off between the variance of the portfolio and its expected return.

There has been nowadays a vast literature about this subject with some extensions for the uni-period case as can be seen, for instance, in (ELTON; GRUBER, 1995; HOWE; RUSTEM; SELBY, 1996; HOWE; RUSTEM, 1997; RUSTEM; BECKER;

MARTY, 1995; STEINBACH, 2001), among others.

The multi-period version of this problem has recently been analysed in continuous as well as in discrete time. The continuous-time multi-period version of the Markowitz's problem was studied in (ZHOU; LI, 2000), using a stochastic linear quadratic theory developed in (CHEN; LI; ZHOU, 1998) with closed-form optimal policies derived, along with an explicit expression of the efficient frontier.

The discrete-time version of the MV allocation problem has dramatically evolved since one of its earlier studies in (LI; NG, 2000), with its generalisation for the risk control over bankruptcy (ZHU; LI; WANG, 2004), the addition of intermediate restrictions (COSTA; NABHOLZ, 2007), and the consideration of liabilities in the portfolio (LEIPPOLD; TROJANI; VANINI, 2004).

Following a different approach to tackle the non-separability issue, the authors in (CUI; LI; LI, 2014) obtained an optimal policies to the multi-period mean-variance problem using the mean-field formulation for the scalar case.They considered the portfolio's wealth dynamic described by

$$x(k+1) = \sum_{j=1}^{n} R^j(k)u^j(k) + \left(x(k) - \sum_{j=1}^{n} u^j(k) \right) s(k), \; k = 0, \ldots, T-1,$$

where $x(k)$ represents the portfolio's wealth, $s(k) > 1$ is the deterministic return of the riskless asset at period k, $R(k) = [R^1(k), \ldots, R^n(k)]'$ is the vector of random returns of the n risky assets at period k, and $u(k) = [u^1(k), \ldots, u^n(k)]'$ represents the asset allocation strategy. Then, they solved the unconstrained Problems (2.7) and (2.5), and also Problem (2.5) subject to a restriction on the variance given by

$$Var(x(k)) \leq a(k) \left(\mathbb{E}(x(k)) - b(k) \right)^2, k = 1, \ldots, T-1,$$

where $a(t)$ represents a maximum probability of the level of wealth $b(t)$ to occur. This kind of problem is also known as control over bankruptcy.

In this work, we generalise the scalar unified framework in (CUI; LI; LI, 2014) for linear systems with multiplicative noises and use the mean-field approach to solve the stabilisation problem and some classical multi-period MV problems with control over bankruptcy and with restrictions on a minimum level of return or maximum level of variance.

2.5 Methodology

Our methodology consists of expanding the state space and study the dynamics and control of a system, $x(k)$, by analysing its expected value, given by $\bar{x}(k)$, together with $z(k) = x(k) - \bar{x}(k)$. Thus, the optimal control regarding the dynamics of $x(k)$ is given now by a set of two related optimal controls associated with the dynamics of $\bar{x}(k)$ and $z(k)$. This expanded state space allows us to tackle directly the mean-variance control problem instead of resorting to embedding schemes, and their family of auxiliary problems (LI; NG, 2000; ZHOU; LI, 2000), to overcome the non-separability issue that arises from the quadratic term in the variance formula.

Regarding the finite-horizon case, we derive the control law for a general multi-period MV problem using dynamic programming and, based on this solution, we obtain the optimal control strategy for the unconstrained problem. In order to solve the constrained problems, we adopt the Lagrangian multipliers approach to re-write the problems with restrictions as unconstrained ones, and in one of these problems a closed-form solution is derived.

The mean-field approach, however, also brings some challenges, especially regarding the stabilisation problem because we have now a set of two GCAREs related to $\bar{x}(k)$ and $z(k)$. In this case, we managed to treat both GCAREs as one and associate the system stabilisation to the spectral radius of some operator following a similar approach as developed in (COSTA; FRAGOSO; MARQUES, 2005; COSTA; PAULO, 2008).

2.6 Contributions

The main contributions of this thesis are summarised below and the results can also be found in the papers (BARBIERI; COSTA, 2020a) and (BARBIERI; COSTA, 2020b).

1) We generalise the scalar unified framework in (CUI; LI; LI, 2014) for discrete-time linear systems with multiplicative noises and obtain the multi-period optimal control law for a general MV problem. The optimal control strategy is derived

from a set of two generalised Riccati difference equations and some parameters obtained from some recursive equations.

2) Based on the solution to this general problem, we consider four other finite-horizon problems. The first one minimises an unconstrained trade-off between the variance and expectation of the output of the system. The second one minimises the variance while keeping the expected output of the system constrained by a minimum value. The third performance criterion maximises the expected output of the system while keeping its variance constrained by a maximum value, and the fourth performance criterion maximises the expected output of the system while restricting its minimum value to a given probability of occurrence. For the last three constrained problems, we adopt the Lagrangian multipliers approach to re-write the problems as unconstrained ones.

3) We derive a sufficient condition for a closed-solution for the problem of minimising the variance while keeping the expected output constrained by a minimum value.

4) We show that when particularised to the portfolio optimisation problem, we retrieve the results obtained in (CUI; LI; LI, 2014) using the mean-field formulation.

5) We derive sufficient conditions for the existence of the maximal solution and necessary and sufficient conditions for the existence of the mean square stabilising solution for a set of two generalised coupled algebraic Riccati equations (GCARE for short) following a similar approach as developed in (COSTA; FRAGOSO; MARQUES, 2005) and applied in (COSTA; PAULO, 2008) for multi-period discrete-time linear systems with Markov jumps and multiplicative noise.

6) A solution to the related infinite-horizon discounted and long-run average cost problems are derived from the mean square stabilising solution to the GCARE. Furthermore, we show that the stabilisation analysis using the mean-field formulation can be greatly simplified when considering the stabilisation of $f(k) = [\bar{x}(k) \;\; z(k)]'$ instead of $x(k)$ directly.

7) We present some numerical examples for the multi-period portfolio selection problem, where we wish to minimise the sum of the mean-variance trade-off costs of a portfolio against a benchmark along the time.

3 PROBLEMS DEFINITIONS AND MEAN-FIELD FORMULATION

In this chapter, we present the notations and problems definitions. Section 3.1 introduces the notation used along the thesis. Section 3.2 presents our systems and the unconstrained, constrained, and infinite-horizon problems. In Section 3.4, we present the same systems and problems that will be solved in this work using the mean-field formulation, and in Section 3.5, we summarise in a table the equivalence from the original equations to the mean-field formulation.

3.1 Notation

In this section, we define our spaces and some matrices notation used throughout the thesis.

For $\mathbb{X}$ and $\mathbb{Y}$ Banach spaces, we set $\mathbb{B}(\mathbb{X}, \mathbb{Y})$ the Banach space of all linear operators of $\mathbb{X}$ into $\mathbb{Y}$, with the uniform induced norm represented by $\|.\|$. For simplicity, we shall set $\mathbb{B}(\mathbb{X}) := \mathbb{B}(\mathbb{X},\mathbb{X})$.

The spectral radius of an operator $\mathcal{T} \in \mathbb{B}(\mathbb{X})$ will be denoted by $r_\sigma(\mathcal{T})$. If $\mathbb{X}$ is a Hilbert space then the inner product will be denoted by $\langle .;. \rangle$ and $\mathcal{T}^*$ will denote the adjoint operator of $\mathcal{T}$ for $\mathcal{T} \in \mathbb{B}(\mathbb{X})$.

Throughout the paper, the n-dimensional real Euclidean space will be denoted by $\mathbb{R}^n$ and the normed linear space of all $n \times m$ real matrices will be expressed by $\mathbb{H}^{n,m}$, with $\mathbb{H}^n = \mathbb{H}^{n,n}$.

For $A \in \mathbb{H}^n$, we use the standard notation $tr(A)$ to represent the trace of a matrix $A \in \mathbb{H}^n$, $A \geq 0$ ($A > 0$ respectively) to denote that the matrix A is positive semi-definite (positive definite) and write $\mathbb{H}^{n+}$ for the set of positive semi-definite matrices.

For $A \in \mathbb{H}^{n,m}$, A' will represent the transpose of A, and the range and null spaces of A will be denoted respectively by $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$. We recall that $\mathrm{Im}(A) = \mathrm{Ker}(A')^\perp$, where $\mathcal{X}^\perp$ represents the orthogonal complement of a linear subspace $\mathcal{X}$.

For a matrix $A \in \mathbb{H}^{n,m}$, the generalized inverse of A (or Moore-Penrose inverse of A) is defined to be the unique matrix $A^\dagger \in \mathbb{H}^{m,n}$ such that i) $AA^\dagger A = A$, ii) $A^\dagger A A^\dagger = A^\dagger$, iii)

$(AA^\dagger)' = AA^\dagger$, and iv) $(A^\dagger A)' = A^\dagger A$, see (SABERI; SANNUTI, 1995), page 12-13.

For $A, B \in \mathbb{H}^{n,m}$, we consider the following norms and inner product in $\mathbb{H}^{n,m}$: $||A||_1 := tr(A)$, $||A||_2 := tr(A'A)^{1/2}$, and $\langle A; B\rangle = tr(A'B)$.

We denote by $\mathbb{T}(\mathbb{H}^{m,n})$ the linear space made up of all matrices of type $V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$ with $V_1, V_2 \in \mathbb{H}^{m,n}$.

In a probabilistic space $(\Omega, \mathbf{P}, \mathcal{F})$, the operator expected value will be represented by $\mathbb{E}(\cdot)$ and the variance will be represented by $Var(\cdot)$.

3.2 Finite-horizon problem formulation

We consider the following linear system with multiplicative noises on a probabilistic space $(\Omega, \mathbf{P}, \mathcal{F})$, with $\mathcal{F}$ being a Borel σ-field, running up to a final time T:

$$\begin{aligned} x(k+1) =& \Big(\bar{A}(k) + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s(k) w_s^x(k)\Big) x(k) + \Big(\bar{B}(k) + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s(k) w_s^u(k)\Big) u(k), \\ x(0) =& x_0, \; k = 0, \ldots, T-1. \end{aligned} \tag{3.1}$$

We consider the following scalar output of system (3.1):

$$y(k) = L(k)x(k), \tag{3.2}$$

where $L(k) \in \mathbb{H}^{1,n}$.

We have for each $k = 0, 1, \ldots, T-1$, $\bar{A}(k) \in \mathbb{H}^n$, $\widetilde{A}_s(k) \in \mathbb{H}^n$, $s = 1, \ldots, \varepsilon^x$, $\bar{B}(k) \in \mathbb{H}^{m,n}$, and $\widetilde{B}_s(k) \in \mathbb{H}^{m,n}$, $s = 1, \ldots, \varepsilon^u$.

The multiplicative noises $\{w_s^x(k); s = 1, \ldots \varepsilon^x, k = 0, 1, \ldots\}$ and $\{w_s^u(k); s = 1, \ldots \varepsilon^u, k = 0, 1, \ldots\}$ are both collections of independent and stationary zero-mean random variables with variance equal to 1 and $\mathbb{E}(w_i^x(k) w_j^x(k)) = 0$, $\mathbb{E}(w_i^u(k) w_j^u(k)) = 0$, for all k and $i \neq j$. We assume without loss of generality that $\varepsilon = \varepsilon^x = \varepsilon^u$ and that $w_s^x(k)$, $w_s^u(k)$, $w_{s'}^x(k')$ and $w_{s'}^u(k')$ are independent for $k \neq k'$ and $s, s' = 1, \ldots, \varepsilon$. The mutual correlation between $w_{s_1}^x(k)$ and $w_{s_2}^u(k)$ is denoted by $\mathbb{E}(w_{s_1}^x(k) w_{s_2}^u(k)) = \rho_{s_1,s_2}(k)$. The initial condition x_0 is assumed to be a random vector in $\mathbb{R}^n$ with finite second moment and independent of $\{w_s^x(k)\}$ and $\{w_s^u(k)\}$.

We define $\mathcal{F}_\tau$ as the σ-field generated by $\{w_s^x(k), w_s^u(k); s = 1, \ldots, \epsilon, k = 0, \ldots, \tau - 1\}$ for $\tau = 1, \ldots, T$, and $\mathcal{F}_0$ the trivial σ-field over Ω, so that the expected value $\mathbb{E}(\cdot|\mathcal{F}_0)$ is just the unconditional expected value $\mathbb{E}(\cdot)$. We write $\mathbb{Q}(k) = \{u(k);\ u(k)$ is an m-dimensional random vector with finite second moments and $\mathcal{F}_k$-measurable$\}$ and $\mathbb{U}(\tau) = \{u_\tau = (u(\tau), \ldots, u(T-1)); u(k) \in \mathbb{Q}(k)$ for each $k = \tau, \ldots, T-1\}$. For simplicity we write $\mathbb{U} = \mathbb{U}(0)$.

3.2.1 General finite-horizon problem, *PG*

The mean-variance general problem, denoted by $PG\,(\nu, \xi, l_M, l_V, L_D)$, will be used as a base problem to solve all our finite-horizon problems and is defined as:

$$PG\,(\nu, \xi, l_M, L_V, l_D) :\ \min_{u\in\mathbb{U}} \sum_{t=0}^{T} \Big(\nu(t) Var\,(y^u\,(t)) - (\xi(t) - l_V(t))\mathbb{E}\,(y^u\,(t)) - l_M(t)\mathbb{E}\,(y^u\,(t))^2 + l_D(t)\Big), \tag{3.3}$$

where y^u is the system's output when the control u is applied, $\nu' = [\nu(1), \ldots, \nu(T)]$, $\nu(t) \geq 0$ and $\xi' = [\xi(1), \ldots, \xi(T)]$, $\xi(t) \geq 0$ are the input parameters and can be seen as risk aversion coefficients, giving a trade-off preference between the expected output and the associated risk (variance) level at time t. We also have the input parameters $l'_V = [l_V(1), \ldots, l_V(T)]$, $l'_M = [l_M(1), \ldots, l_M(T)]$, and $l'_D = [l_D(1), \ldots, l_D(T)]$ introduced just to help the notation of the constrained problems to be defined later. Note that, since l_D does not depend on the control variable, it could be removed from the optimisation problem. The parameters $l_V(t)$, $l_M(t)$, and $l_D(t)$ will be appropriately specified in the sequel.

3.2.2 Finite-horizon unconstrained problem, *PU*

Remark 3.1: In what follows it will be convenient to set $\nu(0) = 0$, $\xi(0) = 0$, $l_V(0) = 0$, $l_M(0) = 0$, and $l_D(0) = 0$.

The mean-variance unconstrained problem is defined as:

$$PU\,(\nu, \xi) :\ \min_{u\in\mathbb{U}} \sum_{t=0}^{T} \Big(\nu(t) Var\,(y^u\,(t)) - \xi(t)\mathbb{E}\,(y^u\,(t))\Big) \tag{3.4}$$

and in this case, we wish an optimal control with no restriction in neither the expected output nor its variance. Notice that Problem $PU\,(\nu,\xi)$ in Equation (3.4) can be re-written as in Equation (3.3) by taking $l_V(t) = 0$, $l_M(t) = 0$, and $l_D(t) = 0$, $t = 1,\ldots,T$.

3.2.3 Constrained finite-horizon problems, $PC1$, $PC2$, and $PC3$

Providing an analytical solution to the optimal control law that takes into consideration a restriction on either the minimum expected output or the maximum variance over time would be relevant to extend the applicability of our formulation. Portfolio managers of pension funds, for instance, would be interested in achieving a return above inflation or even defining a portfolio that has limited risk over specific periods. These two constrained problems are defined as:

$$\begin{aligned} PC1\,(\nu,\epsilon):\ \min_{u\in\mathbb{U}}\ & \sum_{t=0}^{T}\Big(\nu(t)Var\,(y^u(t))\Big) \\ s.t.:\ & \mathbb{E}\,(y^u(t)) \geqslant \epsilon(t) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} PC2\,(\xi,\varphi):\ \min_{u\in\mathbb{U}}\ & -\sum_{t=0}^{T}\Big(\xi(t)\mathbb{E}\,(y^u(t))\Big) \\ s.t.:\ & Var\,(y^u(t)) \leqslant \varphi(t), \end{aligned} \tag{3.6}$$

for $t = 1,\ldots,T$. In problem $PC1$, we wish the control strategy that minimises the weighted sum of the variance while restricting the expected return to a minimum value, $\epsilon(t)$. In problem $PC2$, we wish the control strategy that maximises the weighted sum of the expected output while restricting its variance to a maximum value, $\varphi(t)$. As before, ν and ξ are input parameters that represent a trade-off between risk and return over time.

Another relevant problem for investors, for instance, would involve the risk control that maintains the portfolio value above a minimum value with a given probability. This dynamic mean-variance problem with risk control over a minimum expected output subjected to a maximum probability of occurrence is formulated as the following problem,

$$\min_{u \in \mathbb{U}} \; -\sum_{t=0}^{T} \left(\xi(t) \mathbb{E}\left(y^u(t)\right) \right)$$
$$s.t.: P\left(y(t) \leqslant b(t)\right) \leqslant a(t), \tag{3.7}$$

for $t = 1, \cdots, T$, where $b(t)$ is the disaster level of the output and $a(t)$ is its acceptable maximum probability of occurrence. In a portfolio management perspective, for instance, $b(t)$ can be considered as the minimum level of capital and $a(t)$ as the maximum acceptable probability of achieving $b(t)$. As the problem defined in Equation (3.7) is hard to be directly solved, we replace $P\left(y(t) \leqslant b(t)\right)$ by its upper bound $Var\left(y^u(t)\right) / \left[\mathbb{E}\left(y^u(t)\right) - b(t)\right]^2$ using Tchebycheff inequality as proposed in (ZHU; LI; WANG, 2004), resulting in the following generalised mean-variance model,

$$PC3\left(\xi, a, b\right): \; \min_{u \in \mathbb{U}} \; -\sum_{t=0}^{T} \left(\xi(t) \mathbb{E}\left(y^u(t)\right) \right)$$
$$s.t.: Var\left(y^u(t)\right) \leqslant a(t) \left[\mathbb{E}\left(y^u(t)\right) - b(t)\right]^2, \tag{3.8}$$

for $t = 1, \ldots, T$. The optimal solution to Problem $PC3\left(\xi, a, b\right)$ is feasible for Problem (3.7), thus serving as an approximate solution to this problem.

3.2.4 Lagrangian optimisation problems for $PC1$, $PC2$, and $PC3$

In order to solve Problems (3.5), (3.6), (3.8), we adopt as in (ZHU; LI; WANG, 2004) a primal-dual method by attaching the constraints to the objective function through the Lagrangian multipliers $\omega' = [\omega(1), \ldots, \omega(T)]$, $\omega(t) \geq 0$, $t = 1, \ldots, T$.

The new problems $PC1$, $PC2$, and $PC3$ take the following unconstrained forms:

$$PL1(\omega): \min_{u \in \mathbb{U}} \sum_{t=0}^{T} \left(\nu(t) Var\left(y^u(t)\right) + \omega(t) \left(\epsilon(t) - \mathbb{E}\left(y^u(t)\right)\right) \right), \tag{3.9}$$

$$PL2(\omega): \; \min_{u \in \mathbb{U}} \sum_{t=0}^{T} \left(\omega(t) \left(Var\left(y^u(t)\right) - \varphi(t)\right) - \xi(t) \mathbb{E}\left(y^u(t)\right) \right), \text{ and} \tag{3.10}$$

$$PL3(\omega): \; \min_{u \in \mathbb{U}} \sum_{t=0}^{T} \left(\omega(t) \left(Var\left(y^u(t)\right) - a(t) \left[\mathbb{E}\left(y^u(t)\right) - b(t)\right]^2 \right) - \xi(t) \mathbb{E}\left(y^u(t)\right) \right). \tag{3.11}$$

In order to solve them, we need to solve the Lagrangian dual problem

$$PCi = \max_{\omega \geq 0} \mathcal{H}(\omega), \quad \text{where } \mathcal{H}(\omega) = PLi(\omega), \; i = 1, \; 2, \; 3$$

(see also (BAZARAA; SHERALI; SHETTY, 2013)). Notice that Equations (3.4), (3.9), (3.10), and (3.11) can be re-written as in Equation (3.3) by choosing the parameters $\nu(k)$, $\xi(k)$, $l_V(k)$, $l_M(k)$, and $l_D(k)$ as in Table 1.

Table 1: Input parameters.

Parameter	PU	PL1	PL2	PL3
$\nu(k)$	$\nu(k)$	$\nu(k)$	$\omega(k)$	$\omega(k)$
$\xi(k)$	$\xi(k)$	$\omega(k)$	$\xi(k)$	$\xi(k)$
$l_V(k)$	0	0	0	$2\omega(k)a(k)$
$l_M(k)$	0	0	0	$\omega(k)a(k)$
$l_D(k)$	0	$\omega(k)\epsilon(k)$	$-\omega(k)\xi(k)$	$-\omega(k)a(k)b(k)^2$

Source: Author.

3.3 Infinite-horizon problem formulation

Consider the multiplicative noises, σ-field, $\mathcal{F}_\tau$, $\mathbb{Q}$, and $\mathbb{U}$ as defined before with $T \to \infty$. We have, for $k = 0, 1, \ldots,$ $s = 1, \ldots, \varepsilon$, $\bar{A} \in \mathbb{H}^n$, $\widetilde{A}_s \in \mathbb{H}^n$, $\bar{B} \in \mathbb{H}^{n,m}$, $\widetilde{B}_s \in \mathbb{H}^{n,m}$, the following linear system with multiplicative noises on a probabilistic space $(\Omega, \mathbf{P}, \mathcal{F})$:

$$\begin{aligned} x(k+1) =& \Big(\bar{A} + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s w_s^x(k)\Big) x(k) + \Big(\bar{B} + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s w_s^u(k)\Big) u(k), \\ x(0) =& \; x_0, \;\; k = 0, 1, \ldots. \end{aligned} \tag{3.12}$$

With the following scalar output of system (3.12):

$$y(k) = Lx(k), \tag{3.13}$$

where $L \in \mathbb{H}^{1,n}$.

3.3.1 Long-run and discounted average cost problems, PL and PD

The long-run and discounted infinite-horizon optimal control problems associated to System (3.12) are defined respectively as:

$$PL(\nu,\xi):\ \inf_{u\in\mathbb{U}_{av}}\ \liminf_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\Big(\nu Var\left(y^{u}\left(t\right)\right)-\xi\mathbb{E}\left(y^{u}\left(t\right)\right)\Big)\ \text{and} \tag{3.14}$$

$$PD(\nu,\xi):\ \inf_{u\in\mathbb{U}_{\alpha}}\ \liminf_{T\to\infty}\sum_{t=0}^{T-1}\alpha^{t}\Big(\nu Var\left(y^{u}\left(t\right)\right)-\xi\mathbb{E}\left(y^{u}\left(t\right)\right)\Big), \tag{3.15}$$

where $\alpha\in(0,1)$ is the discount factor and $\nu\in\mathbb{R}^{+}$ and $\xi\in\mathbb{R}^{+}$ are constant input parameters that represent a trade-off preference between the expected output and the associated risk as before.

We define the set of admissible controllers for the discounted and long-run average cost problems as follows. For the discounted case, with discount factor $\alpha\in(0,1)$, the set of admissible controllers $\mathbb{U}_{\alpha}$ is defined as $\mathbb{U}_{\alpha}$=$\{u\in\mathbb{Q}$ for $x(k)$ as in (3.12), $\lim_{T\to\infty}\alpha^{T}(E(\|x(T)\|^{2})+\|\bar{x}(T)\|)$=$0\}$. For the long-run case, the set of admissible controllers $\mathbb{U}_{av}$ is defined as $\mathbb{U}_{av}=\{u\in\mathbb{Q}$; for $x(k)$ as in (3.12), $\lim_{T\to\infty}\frac{1}{T}(\mathbb{E}(\|x(T)\|^{2})+\|\bar{x}(T)\|)=0\}$.

3.4 Mean-field formulation

In this section, we apply the mean-field approach to our problems and re-write them using the following notation. Notice that the formulation in this section is the one used to solve the original problems presented earlier and the references to their solutions will be interchangeable between them.

Define $\bar{x}(k)=\mathbb{E}(x(k))$, $z(k)=x(k)-\bar{x}(k)$, $\bar{u}(k)=\mathbb{E}(u(k))$, $v(k)=u(k)-\bar{u}(k)$. From Equation (3.1) and the independence hypothesis made on the multiplicative noises, we get that

$$\begin{aligned}\bar{x}(k+1)&=\bar{A}(k)\bar{x}(k)+\bar{B}(k)\bar{u}(k),\\ \bar{x}(0)&=\bar{x}_{0},\ k=0,\dots,T-1,\end{aligned} \tag{3.16}$$

and

$$\begin{aligned} z(k+1) = \Big(\bar{A}(k) + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s(k) w_s^x(k)\Big) z(k) + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s(k) w_s^x(k) \bar{x}(k) + \\ \Big(\bar{B}(k) + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s(k) w_s^u(k)\Big) v(k) + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s(k) w_s^u(k) \bar{u}(k), \\ z(0) = z_0, \ k = 0, \dots, T-1. \end{aligned} \tag{3.17}$$

By the fact that $\mathbb{E}(v(k)) = 0$, we get from Equation (3.17) that $\mathbb{E}(z(k)) = 0$ for all $k = 0, \dots, T$. Indeed, by induction, clearly we have that $\mathbb{E}(z(0)) = 0$ and, considering $\mathbb{E}(z(k)) = 0$, we have from the independence hypothesis made on the multiplicative noises and Equation (3.17) that

$$\begin{aligned} \mathbb{E}(z(k+1)) = &\Big(\bar{A}(k) + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s(k) \mathbb{E}(w_s^x(k))\Big) \mathbb{E}(z(k)) + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s(k) \mathbb{E}(w_s^x(k)) \bar{x}(k) + \\ &\Big(\bar{B}(k) + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s(k) \mathbb{E}(w_s^u(k))\Big) \mathbb{E}(v(k)) + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s(k) \mathbb{E}(w_s^u(k)) \bar{u}(k) = 0. \end{aligned}$$

We define $\mathbb{S}(k)$, $\mathbb{V}(\tau)$, $\mathbb{M}(\tau)$ as follows: we say that $(\bar{u}(k), v(k)) \in \mathbb{S}(k)$ if $\bar{u}(k) \in \mathbb{R}^m$ and $v(k) \in \mathbb{Q}(k)$ satisfying $\mathbb{E}(v(k)) = 0$, that $(\bar{u}_\tau, v_\tau) \in \mathbb{V}(\tau)$ if $(\bar{u}_\tau, v_\tau) = ((\bar{u}(\tau), v(\tau)), \dots, (\bar{u}(T-1), v(T-1)))$ with $(\bar{u}(k), v(k)) \in \mathbb{S}(k)$ for each $k = \tau, \dots, T-1$, and that $(\bar{u}^\tau, v^\tau) = ((\bar{u}(0), v(0)), \dots, (\bar{u}(\tau), v(\tau))) \in \mathbb{M}(\tau)$ if $(\bar{u}(k), v(k)) \in \mathbb{S}(k)$ for each $k = 0, \dots, \tau$. We set $\mathbb{V} = \mathbb{V}(0)$ and write $(\bar{u}, v) = (\bar{u}_0, v_0) \in \mathbb{V}$.

3.4.1 Mean-field formulation for the finite-horizon problems

In this section, we use the mean-field formulation to re-write Problems (3.3), (3.4), (3.9), (3.10), and (3.11), for $t = 1, \cdots, T$, respectively as:

$$\begin{aligned} PG\,(v, \xi, l_M, l_V, l_D) : J_0\,(\bar{x}(0), z(0)) = \min_{(\bar{u}, v) \in \mathbb{V}} \sum_{t=0}^{T} \mathbb{E}\Big(& v(t)(L(t)z(t))^2 - (\xi(t) - l_V(t)) L(t) \bar{x}(t) \\ & - l_M(t)(L(t)\bar{x}(t))^2 + l_D(t)\Big), \end{aligned} \tag{3.18}$$

with $\bar{x}$ and z satisfying Equations (3.16) and (3.17).

$$PU\,(\nu,\xi) := \min_{(\bar{u},v)\in\mathbb{V}} \sum_{t=0}^{T} \mathbb{E}\Big(\nu(t)(L(t)z(t))^2 - \xi(t)L(t)\bar{x}(t)\Big), \tag{3.19}$$

$$PL1(\omega) : J_0^{PL1}\,(\bar{x}(0), z(0)) = \min_{(\bar{u},v)\in\mathbb{V}} \sum_{t=0}^{T} \mathbb{E}\Big(\nu(t)(L(t)z(t))^2 - \omega(t)L(t)\bar{x}(t) + \omega(t)\epsilon(t)\Big), \tag{3.20}$$

$$PL2(\omega) : J_0^{PL2}\,(\bar{x}(0), z(0)) = \min_{(\bar{u},v)\in\mathbb{V}} \sum_{t=0}^{T} \mathbb{E}\Big(\omega(t)(L(t)z(t))^2 - \xi(t)L(t)\bar{x}(t) - \omega(t)\varphi(t)\Big), \text{ and} \tag{3.21}$$

$$\begin{aligned} PL3(\omega) :\ J_0^{PL3}\,(\bar{x}(0), z(0)) = \min_{(\bar{u},v)\in\mathbb{V}} \sum_{t=0}^{T} \mathbb{E}\Big(&\omega(t)\,(L(t)z(t))^2 - \omega(t)a(t)\,(L(t)\bar{x}(t))^2 \\ &- (\xi(t) - 2\omega(t)a(t)b(t))L(t)\bar{x}(t) - \omega(t)a(t)b(t)^2\Big). \end{aligned} \tag{3.22}$$

3.4.2 Mean-field formulation for the infinite-horizon problems

From Equation (3.12) and the independence hypothesis made on the multiplicative noises we get that

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k), \\ \bar{x}(0) &= \bar{x}_0,\ k = 0, 1, \ldots, \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} z(k+1) = &\Big(\bar{A} + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s w_s^x(k)\Big) z(k) + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s w_s^x(k)\bar{x}(k) + \\ &\Big(\bar{B} + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s w_s^u(k)\Big) v(k) + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s w_s^u(k)\bar{u}(k), \\ z(0) = &\ z_0,\ k = 0, 1, \ldots. \end{aligned} \tag{3.24}$$

We define $\mathbb{S}(k)$, $\mathbb{V}(\tau)$, $\mathbb{M}(\tau)$ as before with $T \to \infty$.

For the long-run case, the set of admissible controllers $\mathbb{V}_{av}$ is defined as

$$\mathbb{V}_{av} = \Big\{(\bar{u}, v) \in \mathbb{V};\ \text{for } \bar{x}(k) \text{ and } z(k) \text{ as in } (3.16) \text{ and } (3.17),$$
$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}(\|z(T)\|^2) = 0 \text{ and } \lim_{T\to\infty} \frac{1}{T}\|\bar{x}(T)\|^2 = 0\Big\}.$$

For the discounted case with discount factor α, the set of admissible controllers $\mathbb{V}_\alpha$ is defined as

$$\mathbb{V}_{\alpha} = \Big\{(\bar{u}, v) \in \mathbb{V};\ \text{for } \bar{x}(k) \text{ and } z(k) \text{ as in } (3.16) \text{ and } (3.17),$$
$$\lim_{T\to\infty} \alpha^T\mathbb{E}(\|z(T)\|^2) = 0 \text{ and } \lim_{T\to\infty} \alpha^T\|\bar{x}(T)\|^2 = 0\Big\}.$$

Remark 3.2: Notice that if $\lim_{T\to\infty} \frac{1}{T}\|\bar{x}(T)\|^2 = 0$, then $\lim_{T\to\infty} \frac{1}{T}\|\bar{x}(T)\| = 0$. Similarly, if $\lim_{T\to\infty} \alpha^T\|\bar{x}(T)\|^2 = 0$, then $\lim_{T\to\infty} \alpha^T\|\bar{x}(T)\| = 0$. We also have, from Jensen's inequality, that $\|\bar{x}(T)\|^2 \leq \mathbb{E}(\|x(T)\|^2)$ and, since $z(T) = x(T)-\bar{x}(T)$, we obtain that $\|z(T)\|^2 \leq 2(\|x(T)\|^2 + \|\bar{x}(T)\|^2)$ and $\|x(T)\|^2 \leq 2(\|z(T)\|^2 + \|\bar{x}(T)\|^2)$. Therefore, if $u \in \mathbb{U}_{av}$, then $(\bar{u}, v) \in \mathbb{V}_{av}$ with $\bar{u}(k) = \mathbb{E}(u(k))$, $v(k) = u(k) - \bar{u}(k)$ and, conversely, if $(\bar{u}, v) \in \mathbb{V}_{av}$, then $u \in \mathbb{U}_{av}$ with $u(k) = v(k) + \bar{u}(k)$. The result also holds replacing $\mathbb{U}_{av}$ and $\mathbb{V}_{av}$ by respectively $\mathbb{U}_\alpha$ and $\mathbb{V}_\alpha$.

Problems $PL(\nu,\xi)$ in (3.14) and $PD(\nu,\xi)$ in (3.15) can be re-written now as

$$PL(\nu,\xi) : J_{PL}(\bar{x}(0), z(0)) = \inf_{(\bar{u},v)\in\mathbb{V}_{av}} \liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\Big(\nu(Lz(t))^2 - \xi L\bar{x}(t)\Big) \text{ and} \tag{3.25}$$

$$PD(\nu,\xi) : J_{PD}(\bar{x}(0), z(0)) = \inf_{(\bar{u},v)\in\mathbb{V}_{\alpha}} \liminf_{T\to\infty} \sum_{t=0}^{T-1} \alpha^t\mathbb{E}\Big(\nu(Lz(t))^2 - \xi L\bar{x}(t)\Big), \tag{3.26}$$

with $\bar{x}$ and z satisfying Equations (3.23) and (3.24). Once more, by the fact that $\mathbb{E}(v(k)) = 0$, we get from Equation (3.24) that $\mathbb{E}(z(k)) = 0$ for all $k = 0, 1, \ldots$.

We say that Problems (3.25) or (3.26) is well-posed if $J_{PL}(\bar{x}(0), z(0))$ or $J_{PD}(\bar{x}(0), z(0))$, respectively, is finite for any initial condition $\bar{x}(0)$ and $z(0)$. As we will show in Theorems 5.5 and 5.6, the well-posedness of Problems (3.25) and (3.26) will be derived from the stabilising solution to the GCARE.

3.4.2.1 Formulation adaptation for the discounted problem

In order to solve Problem (3.26), we take a step further and make the following definitions to incorporate the discount factor $\alpha \in (0, 1)$ into a similar notation to the long-run problem. In this way, we will be able to use the same results with few adaptions.

Consider Problem (3.26) with a discount factor $\alpha \in (0, 1)$. Defining $\bar{A}^\alpha = \alpha^{1/2}\bar{A}$, $\bar{B}^\alpha = \alpha^{1/2}\bar{B}$, $\widetilde{A}_s^\alpha = \alpha^{1/2}\widetilde{A}_s$, $\widetilde{B}_s^\alpha = \alpha^{1/2}\widetilde{B}_s$, $\bar{A}^\alpha(k) = \alpha^{1/2}\bar{A}(k)$, $\bar{B}^\alpha(k) = \alpha^{1/2}\bar{B}(k)$, $z^\alpha(k) = \alpha^{k/2}z(k)$, $x^\alpha(k) = \alpha^{k/2}x(k)$, $\bar{x}^\alpha(k) = \alpha^{k/2}\bar{x}(k)$, $\bar{u}^\alpha(k) = \alpha^{k/2}\bar{u}(k)$, and $v^\alpha(k) = \alpha^{k/2}v(k)$, we re-write Equations (3.23), (3.24), and (3.26) as follows:

$$
\begin{aligned}
\bar{x}^\alpha(k+1) &= \bar{A}^\alpha \bar{x}^\alpha(k) + \bar{B}^\alpha \bar{u}^\alpha(k),\\
\bar{x}^\alpha(0) &= \bar{x}_0,\ k = 0, 1, \ldots,
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
z^\alpha(k+1) = \Big(\bar{A}^\alpha + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s^\alpha w_s^x(k)\Big) z^\alpha(k) + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s^\alpha w_s^x(k)\bar{x}^\alpha(k) +& \\
\Big(\bar{B}^\alpha + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s^\alpha w_s^u(k)\Big) v^\alpha(k) + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s^\alpha w_s^u(k)\bar{u}^\alpha(k),& \\
z^\alpha(0) = z_0,\ k = 0, 1, \ldots,&
\end{aligned}
\tag{3.28}
$$

and Problem $PD(\nu, \xi)$ can be re-written as

$$
PD\,(\nu, \xi) : J_{PD}\,(\bar{x}(0), z(0)) = \inf_{(\bar{u}, v) \in \mathbb{V}_\alpha} \ \liminf_{T\to\infty} \Big(\sum_{t=0}^{T-1} \mathbb{E}\Big(\nu (L z^\alpha(t))^2 - \alpha^{t/2} \xi L \bar{x}^\alpha(t)\Big)\Big). \tag{3.29}
$$

3.5 Formulation equivalence guide

Table 2 summarises the equivalence between the equations of our systems and problems.

Table 2: Formulation equivalence.

System / Problem	Original Equation	Lagrangian Equation	Mean-Field Equation	Mean-Field with α
Finite-horizon system	(3.1), (3.2)	-	(3.16),(3.17)	-
PG	(3.3)	-	(3.18)	-
PU	(3.4)	-	(3.19)	-
$PC1$	(3.5)	(3.9)	(3.20)	-
$PC2$	(3.6)	(3.10)	(3.21)	-
$PC3$	(3.8)	(3.11)	(3.22)	-
Infinite-horizon system	(3.12), (3.13)	-	(3.23), (3.24)	(3.27), (3.28)
PL	(3.14)	-	(3.25)	-
PD	(3.15)	-	(3.26)	(3.29)

Source: Author.

Note that for $PC1$, $PC2$, and $PC3$, we need to solve the Lagrangian dual problem

$$PCi = \max_{\omega \geq 0} \mathcal{H}(\omega), \quad \text{where} \quad \mathcal{H}(\omega) = PLi(\omega), i = 1,\ 2,\ 3,$$

with PLi given by the Equations in the column "Mean-Field Equation" in Table 2.

4 MAIN OPERATORS AND AUXILIARY RESULTS

In this chapter, we define the main operators and some auxiliary results that we use to solve our finite and infinite-horizon problems. Section 4.1 refers to the finite-horizon problems while Section 4.2 refers to the infinite-horizon case.

4.1 Main operators and auxiliary results for the finite-horizon problems

We will use the Bellman optimality equation, written in terms of the operators as in Equations (4.1), (4.2), and (4.3) to solve Problem (3.18) through the intermediate problem as in (5.1).

For $k = 0, \dots, T-1$, and $X, Y \in \mathbb{H}^n$, set the following operators $\mathcal{A}(k,.,.) \in \mathbb{H}^n$, $\mathcal{G}(k,.,.) \in \mathbb{H}^n \times \mathbb{H}^n, \mathbb{H}^{n,m}$, $\mathcal{R}(k,.) \in \mathbb{H}^n \times \mathbb{H}^n, \mathbb{H}^m$, and the non-linear operators $\mathcal{K}(k,.,.)$, $\mathcal{M}(k,.,.)$, $\bar{\mathcal{M}}(k,.,.)$, and $\mathcal{P}(k,.)$ as:

$$\mathcal{A}(k,X,Y) = \bar{A}(k)'X\bar{A}(k) + \sum_{s=1}^{\varepsilon} \widetilde{A}_s(k)'Y\widetilde{A}_s(k),$$

$$\mathcal{G}(k,X,Y) = \Big(\bar{A}(k)'X\bar{B}(k) + \sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon} \rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'Y\widetilde{B}_{s_2}(k)\Big)',$$

$$\mathcal{R}(k,X,Y) = \bar{B}(k)'X\bar{B}(k) + \sum_{s=1}^{\varepsilon} \widetilde{B}_s(k)'Y\widetilde{B}_s(k),$$

$$\mathcal{K}(k,X,Y) = \mathcal{R}(k,X,Y)^{\dagger}\mathcal{G}(k,X,Y),$$

$$\mathcal{M}(k,X,Y) = \mathcal{A}(k,X,Y) - \mathcal{G}(k,X,Y)'\mathcal{R}(k,X,Y)^{\dagger}\mathcal{G}(k,X,Y),$$

$$\bar{\mathcal{M}}(k,X,Y) = \mathcal{M}(k,X,Y) - l_M(k)L(k)'L(k), \text{ and}$$
$$\mathcal{P}(k,X) = \mathcal{M}(k,X,X) + \nu(k)L(k)'L(k). \tag{4.1}$$

Define also the non-linear operators $\mathcal{V}(k,.,.)$ and $\mathcal{D}(k,.,.,.)$ as follows. For $X, Y \in \mathbb{H}^n$, $V \in \mathbb{H}^{1,n}$, $\gamma \in \mathbb{R}$,

$$\mathcal{V}(k,X,Y,V) = V\Big(\bar{A}(k) - \bar{B}(k)\mathcal{K}(k,X,Y)\Big) + (\xi(k) - l_V(k))L(k), \text{ and} \tag{4.2}$$

$$\mathcal{D}(k,X,Y,V,\gamma) = \gamma - \frac{1}{4}V\bar{B}(k)\mathcal{R}(k,X,Y)^{\dagger}\bar{B}(k)'V' + l_D(k). \tag{4.3}$$

For $k = T, T-1, \ldots, 0$, set the sequences

$$P(k) = \mathcal{P}(k, P(k+1)), \quad P(T) = \nu(T)L(T)'L(T), \tag{4.4}$$

$$M(k) = \bar{\mathcal{M}}(k, M(k+1), P(k+1)), \quad M(T) = -l_M(T)L(T)'L(T), \tag{4.5}$$

$$V(k) = \mathcal{V}(k, M(k+1), P(k+1), V(k+1)), \quad V(T) = (\xi(T) - L_V(T))L(T), \text{ and} \tag{4.6}$$

$$\gamma(k) = \mathcal{D}(k, M(k+1), P(k+1), V(k+1), \gamma(k+1)), \quad \gamma(T) = l_D(T). \tag{4.7}$$

Remark 4.1: If $\widetilde{A}_s(k) = 0$, $k = 0, \ldots, T-1$, and $l_M(k) = 0$, $k = 1, \ldots, T$, then $M(k) = 0$ for all $k = 0, \ldots, T$. Indeed, by applying induction on $k = T, T-1, \ldots, 0$ in Equation (4.5) we have by definition that $M(T) = 0$. Now, supposing that $M(k+1) = 0$, we get that $\mathcal{A}(k, M(k+1), P(k+1)) = \mathcal{A}(k, 0, P(k+1)) = 0$ and $\mathcal{G}(k, M(k+1), P(k+1)) = \mathcal{G}(k, 0, P(k+1)) = 0$ since $\widetilde{A}_s(k) = 0$, and thus $M(k) = \mathcal{M}(k, M(k+1), P(k+1)) = \mathcal{M}(k, 0, P(k+1)) = 0$ (notice that in this case $\bar{\mathcal{M}} = \mathcal{M}$), completing the induction arguments.

Set also

$$K(k) = \mathcal{R}(k, P(k+1), P(k+1))^{\dagger}\mathcal{G}(k, P(k+1), P(k+1)) \text{ and} \tag{4.8}$$

$$H(k) = \mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\mathcal{G}(k, M(k+1), P(k+1)). \tag{4.9}$$

The following auxiliary propositions will be useful when computing the optimal control law through the generalised inverse of the operator $\mathcal{R}(k, X, Y)$ as defined in (4.1)

Proposition 4.1: Consider $Z \in \mathbb{H}^n$ and $M \in \mathbb{H}^m$ with $Z \geq 0$ and $M \geq 0$. Let A and B be matrices of appropriate dimensions whose entries are random variables. Then $\mathbb{E}(A'ZA) - \mathbb{E}(A'ZB)\Big(\mathbb{E}(B'ZB + M)\Big)^{\dagger}\mathbb{E}(B'ZA) \geq 0$ and $\mathbb{E}(A'ZB) = \mathbb{E}(A'ZB)\Big(\mathbb{E}(B'ZB) + M\Big)^{\dagger}\Big(\mathbb{E}(B'ZB) + M\Big)$.

Proof. See Proposition 3 in (COSTA; PAULO, 2007). □

Proposition 4.2: For $G = G' \in \mathbb{H}^n$ and $H \in \mathbb{H}^{n,m}$, it follows that $H(I - GG^{\dagger}) = 0$ if and only if $\mathrm{Ker}(G) \subseteq \mathrm{Ker}(H)$.

Proof. See Lemma 4.2 in (RAMI; CHEN; ZHOU, 2002). □

Proposition 4.3: For $X, Y \in \mathbb{H}^{n+}$, we have that $\mathcal{P}(k, X) \in \mathbb{H}^{n+}$, $\mathcal{M}(k, X, Y) \in \mathbb{H}^{n+}$ and

$$\mathcal{G}(k, X, Y)' = \mathcal{G}(k, X, Y)'\mathcal{R}(k, X, Y)^{\dagger}\mathcal{R}(k, X, Y). \tag{4.10}$$

Proof. Set in Proposition 4.1 $M = 0$,

$$A = \begin{bmatrix} \bar{A}(k) \\ \sum_{s=1}^{\varepsilon} \widetilde{A}_s(k) w_s^x(k) \end{bmatrix}, \quad B = \begin{bmatrix} \bar{B}(k) \\ \sum_{s=1}^{\varepsilon} \widetilde{B}_s(k) w_s^u(k) \end{bmatrix}, \text{ and } Z = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \geq 0. \tag{4.11}$$

Then, from the hypothesis made for $\{w_s^x(k)\}$ and $\{w_s^u(k)\}$, we have that $\mathbb{E}(A'ZA) = \mathcal{A}(k, X, Y)$, $\mathbb{E}(A'ZB) = \mathcal{G}(k, X, Y)'$, $\mathbb{E}(B'ZB) = \mathcal{R}(k, X, Y)$ and that $\mathcal{M}(k, X, Y) = \mathbb{E}(A'ZA) - \mathbb{E}(A'ZB)\Big(\mathbb{E}(B'ZB)\Big)^{\dagger}\mathbb{E}(B'ZA)$. Then Equation (4.10) follows from Proposition 4.1. □

From Proposition 4.3, we have the following result.

Proposition 4.4: We have that $P(k) \in \mathbb{H}^{n+}$, $M(k) \in \mathbb{H}^{n+}$,

$$\mathcal{G}(k, P(k), P(k))' = \mathcal{G}(k, P(k), P(k))'\mathcal{R}(k, P(k), P(k))^{\dagger}\mathcal{R}(k, P(k), P(k)), \text{ and} \tag{4.12}$$

$$\mathcal{G}(k, M(k), P(k))' = \mathcal{G}(k, M(k), P(k))'\mathcal{R}(k, M(k), P(k))^{\dagger}\mathcal{R}(k, M(k), P(k)). \tag{4.13}$$

Proof. The result follows from Proposition 4.3 after induction on k to show that $P(k) \in \mathbb{H}^{n+}$ and $M(k) \in \mathbb{H}^{n+}$ for all $k = T, \ldots, 0$. □

We make the following assumption:

Assumption 4.1: We assume that for $k = 0, \ldots, T-1$,

$$\bar{B}(k)'V(k+1)' \in \operatorname{Im}(\mathcal{R}(k, M(k+1), P(k+1))) \text{ and} \tag{4.14}$$

$$\mathcal{R}(k, M(k+1), P(k+1)) \geq 0. \tag{4.15}$$

Remark 4.2: Notice that from Proposition 4.3 and Equation (4.4), we have that $P(k) \geq 0$ for all $k = 0, \ldots, T$ since $P(T) \geq 0$. If $l_M(k) = 0$ for all $k = 1, \ldots, T$ (as in Problems PU, PL1 and PL2) then from Equation (4.1) we get that $\bar{\mathcal{M}} = \mathcal{M}$ so that from Proposition 4.3 and Equation (4.5) we have that $M(t) \geq 0$ for all $t = 0, \ldots, T$. In this case, $l_M(k) = 0$, $k = 1, \ldots, T$, and we only require Equation (4.14) in Assumption 4.1 since from the definition of $\mathcal{R}$ in Equation (4.1) and the fact that $M(k) \geq 0$, $P(k) \geq 0$ for all $k = 0, \ldots, T$, we get that Equation (4.15) will always be satisfied.

We have the following result:

Proposition 4.5: We have that for $k = 0, \ldots, T-1$,

$$V(k+1)\bar{B}(k) = V(k+1)\bar{B}(k)\mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\mathcal{R}(k, M(k+1), P(k+1)). \tag{4.16}$$

Proof. Set for simplicity $R = \mathcal{R}(k, M(k+1), P(k+1))$ and $H = \bar{B}(k)'V(k+1)'$. Since $\mathrm{Im}(R) = \mathrm{Im}(R^\dagger)$ and $\mathrm{Im}(R^\dagger) = \mathrm{Ker}(R^\dagger)^\perp$, we have from Equation (4.14) that $H \in \mathrm{Im}(R^\dagger)$, and thus $\mathrm{Ker}(R^\dagger) \subseteq \mathrm{Ker}(H')$. From Proposition 4.2, we get Equation (4.16). □

Conditions (4.14) and (4.15) are equivalent to the following computationally easier to check condition:

$$\begin{bmatrix} \bar{B}(k)V(k+1)\mathcal{R}(k, M(k+1), P(k+1))^\dagger \bar{B}(k)'V(k+1)' & \bar{B}(k)V(k+1) \\ \bar{B}(k)V(k+1) & \mathcal{R}(k, M(k+1), P(k+1)) \end{bmatrix} \geq 0. \quad (4.17)$$

Indeed, from Schur's complement, (4.17) is equivalent to $\mathcal{R}(k, M(k+1), P(k+1)) \geq 0$ and (4.16) (see (SABERI; SANNUTI, 1995), pages 12-13), which is equivalent to (4.14), see Proposition 4.5.

4.1.1 Concavity for discrete functions

Let $f : \mathbb{R}^n \to \mathbb{R}$. The first forward diference of f in the direction of e_i = ith unit vector at a point $\omega \in \mathbb{R}^n$ (provided that $\omega + e_i \in \mathbb{R}^n$) is defined as follows.

$$\Delta_i f(\omega) = f(\omega + e_i) - f(\omega) \quad (4.18)$$

Now, the definition of concavity for discrete functions which states that a discrete function is concanve if its first forward diferences are decreasing (nonincreasing) will be established by the following proposition and corollary.

Proposition 4.6: A discretely concave function of a single variable has its first forward diferences decreasing (nonincreasing). Conversely, if the first forward diferences of a discrete function of a single variable are decreasing (nonincreasing), then it is discretely concave.

Proof. See Theorem 1 in (YÜCEER, 2002). □

Corollary 4.1. *A separable function is discretely concave if and only if it is discretely concave in each component.*

Proof. See Corollary 1 in (YÜCEER, 2002). □

4.2 Main operators and auxiliary results for the infinite-horizon problems

The following operators $\mathcal{A}(.) \in \mathbb{H}^n$, $\mathcal{G}(.) \in \mathbb{H}^n \times \mathbb{H}^n, \mathbb{H}^{m,n}$, $\mathcal{R}(.) \in \mathbb{H}^n \times \mathbb{H}^n, \mathbb{H}^m$, and the non-linear operators $\mathcal{K}(.,.)$, $\mathcal{M}(k,.,.)$, and $\mathcal{P}(.)$ will be useful in the sequel to compute the Riccati equation and optimal control laws. For $Z_1, Z_2 \in \mathbb{H}^n$,

$$\begin{aligned}
\mathcal{A}(Z_1, Z_2) &= \bar{A}' Z_1 \bar{A} + \sum_{s=1}^{\varepsilon} \widetilde{A}'_s Z_2 \widetilde{A}_s,\\
\mathcal{G}(Z_1, Z_2) &= \Big(\bar{A}' Z_1 \bar{B} + \sum_{s_1=1}^{\varepsilon} \sum_{s_2=1}^{\varepsilon} \rho_{s_1,s_2} \widetilde{A}'_{s_1} Z_2 \widetilde{B}_{s_2}\Big)',\\
\mathcal{R}(Z_1, Z_2) &= \bar{B}' Z_1 \bar{B} + \sum_{s=1}^{\varepsilon} \widetilde{B}'_s Z_2 \widetilde{B}_s,\\
\mathcal{K}(Z_1, Z_2) &= -\mathcal{R}(Z_1, Z_2)^{\dagger} \mathcal{G}(Z_1, Z_2),\\
\mathcal{M}(Z_1, Z_2) &= \mathcal{A}(Z_1, Z_2) - \mathcal{G}(Z_1, Z_2)' \mathcal{R}(Z_1, Z_2)^{\dagger} \mathcal{G}(Z_1, Z_2),\\
\mathcal{P}(Z_1) &= \mathcal{M}(Z_1, Z_1) + \nu L' L.
\end{aligned} \tag{4.19}$$

Whenever $Z_1 = Z_2$, the above operators will be displayed with only one input to easy the notation.

The superscript $\breve{}$ applied on a matrix or an operator will represent them in the space $\mathbb{T}$ of appropriate dimension. For instance, $\breve{\mathcal{A}}(Z) = \begin{bmatrix} \mathcal{A}(Z_1, Z_2) & 0 \\ 0 & \mathcal{A}(Z_2) \end{bmatrix}$, and when applied on a constant it will just repeat the constant in a block diagonal such as in $\breve{\bar{A}} = \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{bmatrix} \in \mathbb{T}(\mathbb{H}^n)$. In this way, we can write, for instance, $\breve{\mathcal{A}}(Z) = \begin{bmatrix} \mathcal{A}(Z_1, Z_2) & 0 \\ 0 & \mathcal{A}(Z_2) \end{bmatrix} = \breve{\bar{A}}' \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \breve{\bar{A}} + \sum_{s=1}^{\varepsilon} \breve{\widetilde{A}}'_s \begin{bmatrix} Z_2 & 0 \\ 0 & Z_2 \end{bmatrix} \breve{\widetilde{A}}_s$.

The Riccati operator $\mathcal{T} \in \mathbb{T}(\mathbb{H}^n)$ is defined as follows, for $Z = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathbb{T}(\mathbb{H}^n)$:

$$\mathcal{T}(Z) = \begin{bmatrix} \mathcal{M}(X, Y) & 0 \\ 0 & \mathcal{P}(Y) \end{bmatrix} - Z = -Z + \mathcal{I} + \breve{\mathcal{A}}(Z) - \breve{\mathcal{G}}(Z)' \breve{\mathcal{R}}(Z)^{\dagger} \breve{\mathcal{G}}(Z), \tag{4.20}$$

where $\mathcal{I} = \begin{bmatrix} 0 & 0 \\ 0 & \nu L' L \end{bmatrix}$. We will study the following generalised coupled algebraic Riccati equation (GCARE), with variable in $Z = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathbb{T}(\mathbb{H}^n)$:

$$\mathcal{T}(Z) = 0. \tag{4.21}$$

As we will see later on, in order to assure that the criterion costs will be finite, we need to introduce the following sets:

$$\begin{aligned} &\mathbf{D}(\mathcal{T}) := \{Z \in \mathbb{T}(\mathbb{H}^n); \mathrm{Ker}(\breve{\mathcal{R}}(Z)) \subseteq \mathrm{Ker}(\breve{\mathcal{G}}(Z)')\} \text{ and} \\ &\mathbf{D}_+(\mathcal{T}) := \{Z \in \mathbb{T}(\mathbb{H}^n); \mathrm{Ker}(\breve{\mathcal{R}}(Z)) \subseteq \mathrm{Ker}(\breve{\mathcal{G}}(Z)') \text{ and } \breve{\mathcal{R}}(Z) \geq 0\}. \end{aligned} \tag{4.22}$$

4.2.1 Additional operators and results for the stabilisation problem

We present next some operators, definitions, and results that will be related to the mean square stabilisability of System (3.12).

Defining $f(k) = [\bar{x}(k) \;\; z(k)]'$ and re-arranging Equations (3.23) and (3.24), we obtain that

$$\begin{aligned} f(k+1) &= A(k)f(k) + B(k)c(k), \\ f(0) &= f_0 = [\bar{x}_0 \;\; z_0]', \end{aligned} \tag{4.23}$$

where, $A(k) = \begin{bmatrix} \bar{A} & 0 \\ \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s w_s^x(k) & \bar{A} + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s w_s^x(k) \end{bmatrix}$, $B(k) = \begin{bmatrix} \bar{B} & 0 \\ \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s w_s^u(k) & \bar{B} + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s w_s^u(k) \end{bmatrix}$, and $c(k) = \begin{bmatrix} \bar{u}(k) \\ v(k) \end{bmatrix}$. We define next the stability and stabilisability concepts that we shall consider in the following sections.

Definition 4.1. *We say that $\widetilde{K} \in \mathbb{H}^{m,n}$ stabilises System (3.12) in the mean-square sense if, when we make $u(k) = \widetilde{K}x(k)$ in Equation (3.12), we have that $\mathbb{E}(\|x(k)\|^2) \to 0$ as $k \to \infty$ for any initial condition x_0 such that $\mathbb{E}(\|x_0\|^2) < \infty$. System (3.12) is said to be mean-square stabilisable if for some $\widetilde{K}$, we have that $\widetilde{K}$ stabilises (3.12) in the mean-square sense.*

Definition 4.2. *We say that $K \in \mathbb{T}(\mathbb{H}^{m,n})$ stabilises System (4.23) in the mean-square sense if, when we make $c(k) = Kf(k)$ in Equation (4.23), we have that $\mathbb{E}(\|f(k)\|^2) \to 0$ as $k \to \infty$ for any initial condition $f_0 = [\bar{x}_0 \;\; z_0]'$ with $\mathbb{E}(z_0) = 0$, $\mathbb{E}(\|z_0\|^2) < \infty$ and $\bar{x}_0 \in \mathbb{R}^n$.*

System (4.23) is said to be mean-square stabilisable if for some $K \in \mathbb{T}(\mathbb{H}^{m,n})$, we have that K stabilises (4.23) in the mean-square sense. The set of mean-square stabilising $K \in \mathbb{T}(\mathbb{H}^{m,n})$ will be represented by $\mathbb{K}$.

For $\widetilde{K} \in \mathbb{H}^{m,n}$ define the operator $\mathcal{S}_{\widetilde{K}}$ on $\mathbb{B}(\mathbb{H}^n)$ as follows:

$$\mathcal{S}_{\widetilde{K}}(V) = (\bar{A} + \bar{B}\widetilde{K})V(\bar{A} + \bar{B}\widetilde{K})' + \sum_{s=1}^{\nu} \widetilde{A}_s V \widetilde{A}'_s + \sum_{s_1=1}^{\nu}\sum_{s_2=1}^{\varepsilon} \rho_{s_1,s_2}(\widetilde{A}_{s_1} V \widetilde{K}'\widetilde{B}'_{s_2} + \widetilde{B}_{s_2}\widetilde{K}V\widetilde{A}'_{s_1}) + \sum_{s=1}^{\nu} \widetilde{B}_s \widetilde{K} V K' \widetilde{B}'_s, \tag{4.24}$$

where $V \in \mathbb{H}^n$. Notice that, by defining $U(k) = \mathbb{E}((x(k)x(k)')$ in (3.12) with $u(k) = \widetilde{K}x(k)$, we get that (see (COSTA; PAULO, 2008))

$$U(k+1) = \mathcal{S}_{\widetilde{K}}(U(k)). \tag{4.25}$$

We have the following result.

Proposition 4.7: $r_\sigma(\mathcal{S}_{\widetilde{K}}) < 1$ if and only if $\mathcal{S}^k_{\widetilde{K}}(V) \to 0$ as $k \to \infty$ for any $V \in \mathbb{H}^{n+}$.

Proof. See (COSTA; FRAGOSO; MARQUES, 2005), Proposition 2.5. □

From Proposition 4.7, we have the following equivalence.

Proposition 4.8: System (3.12) is mean-square stabilisable if and only if System (4.23) is mean-square stabilisable.

Proof. Suppose that $\widetilde{K} \in \mathbb{H}^{m,n}$ stabilises System (3.12). Then, we will show that $K = \begin{bmatrix} \widetilde{K} & 0 \\ 0 & \widetilde{K} \end{bmatrix}$ stabilises System (4.23) in the mean-square sense. Indeed, set $x_0 = \bar{x}_0 + z_0$ for any initial condition $f_0 = [\bar{x}_0 \;\; z_0]'$ with $\mathbb{E}(z_0) = 0$, $\mathbb{E}(\|z_0\|^2) < \infty$ and $\bar{x}_0 \in \mathbb{R}^n$. From Definition 4.1, we have that $\mathbb{E}(\|x(k)\|^2) \to 0$. From Jensen's inequality, $\|\bar{x}(k)\|^2 \le \mathbb{E}(\|x(k)\|^2) \to 0$ and, since $z(k) = x(k) - \bar{x}(k)$, we have that $\mathbb{E}(\|z(k)\|^2) \le 2(\mathbb{E}(\|x(k)\|^2) + \|\bar{x}(k)\|^2)) \to 0$, showing that $\mathbb{E}(\|f(k)\|^2) = \mathbb{E}(\|z(k)\|^2) + \|\bar{x}(k)\|^2 \to 0$ as $k \to \infty$, as desired. Suppose now that $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$ stabilises System (4.23) in the mean-square sense. We will show that, by taking $\widetilde{K} = K_2$, we have that $\widetilde{K}$ stabilises (3.12) in the mean-square sense. First notice that for any $V \in \mathbb{H}^{n+}$, we can find z_0 such that $\mathbb{E}(z_0 z'_0) = V$ and $\mathbb{E}(z_0) = 0$ (indeed just take $z_0 = V^{1/2}\chi$ where $\mathbb{E}(\chi) = 0$, $\mathbb{E}(\chi\chi') = I$). By considering $\bar{x}_0 = 0$, we get from (3.12), (4.23), and (4.25) that $\bar{x}(k) = 0$ and $z(k) = x(k)$ for all k, and $tr(U(k)) = \mathbb{E}(\|z(k)\|^2) \to 0$ as $k \to \infty$. From Proposition 4.7, we get that $r_\sigma(\mathcal{S}_{\widetilde{K}}) < 1$ and thus $\widetilde{K}$ stabilises (3.12) in the mean-square sense, completing the proof. □

We define the positive operator $\mathcal{L}_F \in \mathbb{B}(\mathbb{T}(\mathbb{H}^n))$ as follows. For $Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \in \mathbb{T}(\mathbb{H}^n)$ and $F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$, set

$$\mathcal{L}_F(Z) = \begin{bmatrix} \mathcal{L}_{F_1}(Z_1, Z_2) & 0 \\ 0 & \mathcal{L}_{F_2}(Z_2) \end{bmatrix} = \breve{\mathcal{A}}(Z) + F'\breve{\mathcal{R}}(Z)F + F'\breve{\mathcal{G}}(Z) + \breve{\mathcal{G}}(Z)'F, \tag{4.26}$$

where, $\mathcal{L}_{F_1}(Z_1, Z_2)$ and $\mathcal{L}_{F_2}(Z_2) \in \mathbb{H}^n$ are defined as

$$\mathcal{L}_{F_1}(Z_1, Z_2) = \mathcal{A}(Z_1, Z_2) + F_1'\mathcal{R}(Z_1, Z_2)F_1 + F_1'\mathcal{G}(Z_1, Z_2) + \mathcal{G}(Z_1, Z_2)'F_1 \tag{4.27}$$

and

$$\mathcal{L}_{F_2}(Z_2) = \mathcal{A}(Z_2) + F_2'\mathcal{R}(Z_2)F_2 + F_2'\mathcal{G}(Z_2) + \mathcal{G}(Z_2)'F_2. \tag{4.28}$$

The next result shows an important connection between $\mathcal{T}(Z)$ and $\mathcal{L}_F(Z)$, provided that $Z \in \mathbf{D}(\mathcal{T})$.

Lemma 4.1. *For any $Z \in \mathbf{D}(\mathcal{T})$ and $F \in \mathbb{T}(\mathbb{H}^{m,n})$, we have that*

$$\mathcal{T}(Z) + Z - \mathcal{I} = \mathcal{L}_F(Z) - (\breve{\mathcal{K}}(Z) - F)'\breve{\mathcal{R}}(Z)(\breve{\mathcal{K}}(Z) - F). \tag{4.29}$$

Proof. From the properties of the generalized inverse, we have that $\mathcal{R}^\dagger = \mathcal{R}^\dagger\mathcal{R}\mathcal{R}^\dagger$ and $(\mathcal{R}^\dagger)' = \mathcal{R}^\dagger$. By definition, for any $Z \in \mathbf{D}(\mathcal{T})$, $Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}$, we have that $\mathrm{Ker}(\mathcal{R}(Z_1, Z_2)) \subseteq \mathrm{Ker}(\mathcal{G}(Z_1, Z_2)')$ so that $\mathcal{G}(Z_1, Z_2)' = \mathcal{G}(Z_1, Z_2)'\ \mathcal{R}(Z_1, Z_2)^\dagger\ \mathcal{R}(Z_1, Z_2)$, and $\mathrm{Ker}(\mathcal{R}(Z_2)) \subseteq \mathrm{Ker}(\mathcal{G}(Z_2)')$ so that $\mathcal{G}(Z_2)' = \mathcal{G}(Z_2)'\mathcal{R}(Z_2)^\dagger\mathcal{R}(Z_2)$. Thus, using the definition of $\mathcal{K}$ in (4.19), we obtain that

$$\breve{\mathcal{G}}(Z)'\breve{\mathcal{R}}(Z)^\dagger\breve{\mathcal{G}}(Z) = \breve{\mathcal{K}}(Z)'\breve{\mathcal{R}}(Z)\breve{\mathcal{K}}(Z). \tag{4.30}$$

Applying Equation (4.20) into (4.30), we obtain that

$$-\mathcal{T}(Z) - Z + \mathcal{I} + \mathcal{A}(Z) = \breve{\mathcal{K}}(Z)'\breve{\mathcal{R}}(Z)\breve{\mathcal{K}}(Z) \tag{4.31}$$

and substituting $\breve{\mathcal{A}}(Z)$ from Equation (4.26), for some $F \in \mathbb{T}(\mathbb{H}^{m,n})$, we obtain Equation (4.29) after some algebraic manipulation. □

Next, we present the definition of a symmetric solution, maximal solution and stabilising solution to the GCARE.

Definition 4.3. *We say that $X \in \mathbb{T}(\mathbb{H}^n)$ is a symmetric solution to the GCARE if it satisfies Equation (4.21) and $X \in$ **D**$(\mathcal{T})$. We say that X is a maximal solution over **L**, **L** $\subseteq$ **D**$(\mathcal{T})$, if it is a symmetric solution to the GCARE and $X \geq W$ for any $W \in$ **L**. Thus, $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$ is said to be the mean-square stabilising solution if it is a symmetric solution to the GCARE and $\begin{bmatrix} \mathcal{K}(X_1, X_2) & 0 \\ 0 & \mathcal{K}(X_2) \end{bmatrix} \in \mathbb{K}$.*

The next lemma shows that if $W \in$ **D**$_+(\mathcal{T})$ and $Z \in$ **D**$_+(\mathcal{T})$, then $W + Z \in$ **D**$_+(\mathcal{T})$ (**D**$_+(\mathcal{T}) =$ **D**$_+(\mathcal{T}) + \mathbb{T}(\mathbb{H}^n)$).

Lemma 4.2. *If $\widehat{X} \in$ **D**$_+(\mathcal{T})$ and $X > \widehat{X}$, then $X \in$ **D**$_+(\mathcal{T})$.*

Proof. The proof follows a similar approach as in Lemma 7 in (COSTA; PAULO, 2008).

By hypothesis, $\mathrm{Ker}(\breve{\mathcal{R}}(\widehat{X})) \subseteq \mathrm{Ker}(\breve{\mathcal{G}}(\widehat{X})')$ and $\breve{\mathcal{R}}(\widehat{X}) \geq 0$. Since $X \geq \widehat{X}$ we get that $\breve{\mathcal{R}}(X) > \breve{\mathcal{R}}(\widehat{X}) \geq 0$, $\breve{\mathcal{R}}(X) \geq \breve{\mathcal{R}}(X) - \breve{\mathcal{R}}(\widehat{X}) = \breve{\mathcal{R}}(X - \widehat{X})$, and $\mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(\widehat{X})) \subseteq \mathrm{Ker}(\breve{\mathcal{G}}(\widehat{X})')$.

Consider Propositions 4.2 and 4.3 with their operators independent of k to comply with the operators as in (4.19). Thus, from Proposition 4.3, we have that $\breve{\mathcal{G}}(X - \widehat{X})' = \breve{\mathcal{G}}(X - \widehat{X})'\breve{\mathcal{R}}(X - \widehat{X})^{\dagger}\breve{\mathcal{R}}(X - \widehat{X})$ and from Proposition 4.2, we obtain that $\mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X - \widehat{X})) \subseteq \mathrm{Ker}(\breve{\mathcal{G}}(X - \widehat{X})')$.

This means that for any $v \in \mathrm{Ker}(\breve{\mathcal{R}}(X - \widehat{X}))$, $\breve{\mathcal{R}}(X)v = 0$, $\breve{\mathcal{G}}(X - \widehat{X})'v = 0$, and $\breve{\mathcal{G}}(\widehat{X})'v = 0$, so that $\breve{\mathcal{G}}(X)'v = 0$ and hence $v \in \mathrm{Ker}(\breve{\mathcal{G}}(X)')$. Therefore, $\breve{\mathcal{R}}(X) > 0$ and $\mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}(\breve{\mathcal{G}}(X)')$. □

The following Lemmas 4.3 to 4.6 will be required to establish a sufficient condition to obtain the maximal and stabilising solution of the GCARE.

Lemma 4.3. *If $\mathcal{L} \in \mathbb{B}(\mathbb{T}(\mathbb{H}^n))$ is a positive operator, then there exists $Z \in \mathbb{T}(\mathbb{H}^n)$, $Z \neq 0$, such that*

$$\mathcal{L}(Z)^* = r_\sigma(\mathcal{L})Z. \tag{4.32}$$

Proof. The proof follows from the application of the Perron-Frobenius theory for positive operators on general partially-ordered finite-dimensional linear spaces (see Theorem 2.6 in (DAMM; HINRICHSEN, 2003) or Theorem 3.2.3 in (DAMM, 2004) for more general conditions and its proof in (SCHAEFER, 1971), appendix 2.6).

Suppose that for a sufficiently large α, the operator $T = V(\alpha V - \mathcal{L}(V))^{-1}$ is positive. From Theorem 3.2.3 in (DAMM, 2004) (see also appendix 2.6 in (SCHAEFER, 1971)), under more general conditions, we have that there is a vector $v \neq 0$ of proper dimension such that $(\alpha V - \mathcal{L}(V)')^{-1})Vv = r_\sigma^\alpha v$, where $r_\sigma^\alpha = r_\sigma(T)$.

Multiplying this equation from the left by $(\alpha V - \mathcal{L}(V)')$ and after some algebraic manipulation, we obtain $\mathcal{L}(V)' = \lambda_0 V$, where $\lambda_0 = \alpha - 1/r_\sigma^\alpha$.

Moreover, for $\mathcal{L}(V) = \mathcal{L}V$, we have that $T = (\alpha I - \mathcal{L})^{-1}$ and $r_\sigma^\alpha(T) = 1/(\alpha - r_\sigma(\mathcal{L}))$. Therefore, $\lambda_0 = r_\sigma(\mathcal{L})$, completing the proof. □

Lemma 4.4. *Suppose that $M \in \mathbb{T}(\mathbb{H}^n)$ and $F \in \mathbb{T}(\mathbb{H}^{m,n})$ are such that $M \geq 0$ and $\mathrm{Ker}(M) \subseteq \mathrm{Ker}(F')$, then $F'MF \geq \delta F'F$ for some small enough $\delta > 0$.*

Proof. It follows the same arguments as in Lemma 3 in (COSTA; PAULO, 2008) for $i = 1$. Consider the singular decomposition of M (see (HORN; JOHNSON, 1990)), $M = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V'$, where Σ is a non-singular diagonal matrix with the positive eigenvalues of M, V is a matrix such that $VV' = I$ and V can be decomposed as $V = [V_1 \quad V_2]$, where the columns of the matrix V_2 form a basis of $\mathrm{Ker}(M)$.

Since $\mathrm{Ker}(M) \subseteq \mathrm{Ker}(F')$, it follows that $F'V_2 = 0$ and thus $F'V = [F'V_1 \quad 0]$.

Therefore, taking $\delta > 0$ such that $\Sigma \geq \delta I$, we get that $F'MF = (F'V)(V'MV)(V'F) = F'V_1\Sigma V_1'F \geq \delta F'V_1V_1'F = \delta F'VV'F = \delta F'F$. □

Lemma 4.5. *Consider $F, G \in \mathbb{T}(\mathbb{H}^{m,n})$ and $r_\sigma(\mathcal{L}_F) < 1$, with $\mathcal{L}_F$ as defined in Equation*

(4.26). Suppose

$$Y - \mathcal{L}_G(Y) \geq \delta(G - F)'(G - F)$$

for some $Y \in \mathbb{T}(\mathbb{H}^{n+})$ *and* $\delta > 0$, *then* $r_\sigma(\mathcal{L}_G) < 1$.

Proof. See Lemma 9 in (COSTA; PAULO, 2008). □

Lemma 4.6. *If* $K \in \mathbb{K}$, *then for any* $S \in \mathbb{T}(\mathbb{H}^n)$, *there exists a unique solution* $Y \in \mathbb{T}(\mathbb{H}^n)$ *that satisfies*

$$Y - \mathcal{L}_K(Y) = S. \tag{4.33}$$

Moreover, if S *is symmetric* ($\geq 0, > 0$ *respectively), then* Y *is symmetric* ($\geq 0, > 0$). *Conversely, if there is* $Y > 0$ *satisfying Equation (4.33) for some* $S \in \mathbb{T}(\mathbb{H}^n)$, $S > 0$, *then* $r_\sigma(\mathcal{L}_K) < 1$.

Proof. The proof follows the reasoning in Lemma 10 in (COSTA; PAULO, 2008) and recalling that $(I - \mathcal{L}_K)^{-1}(\cdot) = \sum_{j=0}^{\infty} \mathcal{L}_K^j(\cdot)$ (see (WEIDMANN, 1980), page 102). □

Finally, we conclude this section with the following comparisons that will be useful in the next chapter to prove under what conditions there is a maximal stabilising solution to the GCARE.

Lemma 4.7. *Consider that* $X \in \boldsymbol{D}(\mathcal{T})$ *and for some* $\widehat{F} \in \mathbb{T}(\mathbb{H}^{m,n})$, *we have that* $\widehat{X} \in \mathbb{T}(\mathbb{H}^n)$ *satisfies*

$$\widehat{X} - \mathcal{L}_{\widehat{F}}(\widehat{X}) = \mathcal{I}. \tag{4.34}$$

Then,

$$\begin{aligned} &(\widehat{X} - X) - \mathcal{L}_{\widehat{F}}(\widehat{X} - X) = \\ &(\widehat{F} - \breve{\mathcal{K}}(X))'\breve{\mathcal{R}}(X)(\widehat{F} - \breve{\mathcal{K}}(X)) + \mathcal{T}(X). \end{aligned} \tag{4.35}$$

Moreover, if $\widehat{X} \in \boldsymbol{D}(\mathcal{T})$, *then*

$$\begin{aligned} (\widehat{X} - X) - \mathcal{L}_{\breve{\mathcal{K}}(\widehat{X})}(\widehat{X} - X) &= (\breve{\mathcal{K}}(\widehat{X}) - \breve{\mathcal{K}}(X))'\breve{\mathcal{R}}(X)(\breve{\mathcal{K}}(\widehat{X}) \\ &- \breve{\mathcal{K}}(X)) + (\widehat{F} - \breve{\mathcal{K}}(\widehat{X}))'\breve{\mathcal{R}}(\widehat{X})(\widehat{F} - \breve{\mathcal{K}}(\widehat{X})) + \mathcal{T}(X). \end{aligned} \tag{4.36}$$

Furthermore, if $\widehat{Y} \in \mathbb{T}(\mathbb{H}^{n*})$ *and satisfies*

$$\widehat{Y} - \mathcal{L}_{\breve{\mathcal{K}}(\widehat{X})}(\widehat{Y}) = \mathcal{I}, \tag{4.37}$$

then

$$(\widehat{X} - \widehat{Y}) - \mathcal{L}_{\breve{\mathcal{K}}(\widehat{X})}(\widehat{X} - \widehat{Y}) = (\widehat{F} - \breve{\mathcal{K}}(\widehat{X}))' \breve{\mathcal{R}}(\widehat{X})(\widehat{F} - \breve{\mathcal{K}}(\widehat{X})). \tag{4.38}$$

Proof. We have that, taking $Z = X$ and $F = \widehat{F}$ in Equation (4.29) yields

$$X - \mathcal{L}_{\widehat{F}}(X) = -(\breve{\mathcal{K}}(X) - \widehat{F})' \breve{\mathcal{R}}(X)(\breve{\mathcal{K}}(X) - \widehat{F}) - \mathcal{T}(X) + \mathcal{I}, \tag{4.39}$$

and by subtracting Equation (4.39) from Equation (4.34), we get Equation (4.35). From Equation (4.29) with $Z = X$ and $F = \breve{\mathcal{K}}(\widehat{X})$, we have that

$$X - \mathcal{L}_{\breve{\mathcal{K}}(\widehat{X})}(X) = -(\breve{\mathcal{K}}(X) - \breve{\mathcal{K}}(\widehat{X}))' \breve{\mathcal{R}}(X)(\breve{\mathcal{K}}(X) - \breve{\mathcal{K}}(\widehat{X})) - \mathcal{T}(X) + \mathcal{I}. \tag{4.40}$$

By taking $Z = \widehat{X}$ and $F = \widehat{F}$ in Equation (4.29), we have that

$$\widehat{X} - \mathcal{L}_{\widehat{F}}(\widehat{X}) = -(\breve{\mathcal{K}}(\widehat{X}) - \widehat{F})' \breve{\mathcal{R}}(\widehat{X})(\breve{\mathcal{K}}(\widehat{X}) - \widehat{F}) - \mathcal{T}(\widehat{X}) + \mathcal{I}, \tag{4.41}$$

and from Equations (4.34) and (4.41),

$$-\mathcal{T}(\widehat{X}) = (\breve{\mathcal{K}}(\widehat{X}) - \widehat{F})' \breve{\mathcal{R}}(\widehat{X})(\breve{\mathcal{K}}(\widehat{X}) - \widehat{F}). \tag{4.42}$$

Once more, taking $Z = \widehat{X}$ and $F = \breve{\mathcal{K}}(\widehat{X})$ in Equation (4.29), we have from Equation (4.42) that

$$\widehat{X} - \mathcal{L}_{\breve{\mathcal{K}}(\widehat{X})}(\widehat{X}) = (\breve{\mathcal{K}}(\widehat{X}) - \widehat{F})' \breve{\mathcal{R}}(\widehat{X})(\breve{\mathcal{K}}(\widehat{X}) - \widehat{F}) + \mathcal{I}. \tag{4.43}$$

We have that Equation (4.43) minus Equation (4.40) yields Equation (4.36) and that Equation (4.43) minus Equation (4.37) yields Equation (4.38). □

5 MAIN RESULTS

In the following sections, we solve the problems presented in Section 3.4. Section 5.1 shows the optimal control to the unconstrained and constrained problems regarding the finite-time horizon along with some comparison with current results in the literature. In Section 5.2, we present necessary and sufficient conditions for the maximal stabilising solution to the infinite-horizon case and the optimal stabilising control policies to the discounted and long-run problems.

5.1 Constrained and unconstrained finite-horizon control

We start this section with a result that will be useful to define the benefit-to-go function of our problems. At each time $k \in \{1, \ldots, T\}$ and for any $(\bar{u}^{k-1}, v^{k-1}) \in \mathbb{M}(k-1)$, define the following intermediate problem for Problem (3.18):

$$J_k\left(\bar{x}(k), z(k), (\bar{u}^{k-1}, v^{k-1})\right) = \min_{(\bar{u}_k, v_k)\in\mathbb{V}(k)} \sum_{t=k}^{T} \mathbb{E}\Big(v(t)(L(t)z(t))^2 - (\xi(t) - l_V(t))\bar{x}(t) - l_M(t)(L(t)\bar{x}(t))^2 + l_D(t)|\mathcal{F}_k\Big). \tag{5.1}$$

We have the following result.

Lemma 5.1. *Assume that for* $t \in \{1, \ldots, T\}$ *and any* $(\bar{u}^{t-1}, v^{t-1}) \in \mathbb{M}(t-1)$,

$$\mathbb{E}(J_t\left(\bar{x}(t), z(t), (\bar{u}^{t-1}, v^{t-1})\right)|\mathcal{F}_{t-1}) = G^1_{t-1}(\bar{x}(t-1), z(t-1), (\bar{u}^{t-1}, v^{t-1})) + G^2_{t-1}(\bar{x}(t-1), z(t-1), (\bar{u}^{t-1}, v^{t-1})) \tag{5.2}$$

with

$$\mathbb{E}(G^2_{t-1}(\bar{x}(t-1), v(t-1), (\bar{u}^{t-1}, v^{t-1}))) = 0. \tag{5.3}$$

Then, for $t = 0, \ldots, T-1$,

$$(\bar{u}^*(t), v^*(t)) = \textit{arg} \min_{(\bar{u}(t), v(t))\in\mathbb{S}(t)} \Big\{G^1_t(\bar{x}(t), z(t), ((\bar{u}^{t-1}, v^{t-1}), (\bar{u}(t), v(t))) + v(t)(L(t)z(t))^2 - (\xi(t) - l_V(t))L(t)\bar{x}(t) - l_M(t)(L(t)\bar{x}(t))^2 + l_D(t)\Big\}$$

and

$$J_0\left(\bar{x}(0), z(0)\right) = \min_{(\bar{u}^t, v^t)\in\mathbb{M}(t)} \Big\{\mathbb{E}\Big(G_t^1(\bar{x}(t), z(t), (\bar{u}^t, v^t))\Big) + \sum_{j=0}^{t} \mathbb{E}\Big(v(j)(L(j)z(j))^2 - (\xi(t) - l_V(t))L(j)\bar{x}(j) - l_M(j)(L(t)\bar{x}(j))^2 + l_D(j)\Big)\Big\},$$

i.e. $G_t^1(\bar{x}(t), z(t), ((\bar{u}^{t-1}, v^{t-1}), (\bar{u}(t), v(t))) + v(t)(L(t)z(t))^2 - (\xi(t) - l_V(t))L(t)\bar{x}(t) - l_M(t)(L(t)\bar{x}(t))^2 + l_D(t)$ *can be regarded as the benefit-to-go function at time t of problem* PG.

Proof. It is an immediate application of Lemma 3 in (CUI; LI; LI, 2014) □

Define for $k = 0, \ldots, T-1$, and $(\bar{u}^k, v^k) \in \mathbb{M}(k)$

$$\begin{aligned}
&G_k^1(\bar{x}(k), z(k), (\bar{u}^k, v^k)) = z(k)'(\mathcal{A}(k, P(k+1), P(k+1))z(k)+\\
&\bar{x}(k)'(\mathcal{A}(k, M(k+1), P(k+1)))\bar{x}(k)' + v(k)'(\mathcal{R}(k, P(k+1), P(k+1)))v(k)+\\
&\bar{u}(k)'(\mathcal{R}(k, M(k+1), P(k+1)))\bar{u}(k) + 2z(k)'\mathcal{G}(k, P(k+1), P(k+1))'v(k)+\\
&2\bar{x}(k)'\mathcal{G}(k, M(k+1), P(k+1))'\bar{u}(k) - V(k+1)(\bar{A}(k)\bar{x}(k) + \bar{B}(k)\bar{u}(k)) + \gamma(k+1), \text{ and}
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
&G_k^2(\bar{x}(k), z(k), (\bar{u}^k, v^k)) =\\
&2z(k)'\Big(\sum_{s=1}^{\varepsilon} \widetilde{A}_s(k)'P(k)\widetilde{A}_s(k)\bar{x}(k) + \sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon} \rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'P(k)\widetilde{B}_{s_2}(k)\Big)\bar{u}(k)\\
&+ 2\Big(\bar{x}(k)'\sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon} \rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'P(k)\widetilde{B}_{s_2}(k) + \bar{u}(k)'\sum_{s=1}^{\varepsilon} \widetilde{B}_s(k)'P(k)\widetilde{B}_s(k)\Big)v(k).
\end{aligned} \tag{5.5}$$

Note that

$$\begin{aligned}
\mathbb{E}(G_k^2(\bar{x}(k), z(k), (\bar{u}^k, v^k))) = 2\Big(&\mathbb{E}(z(k))'\Big((\sum_{s=1}^{\varepsilon} \widetilde{A}_s(k)'P(k)\widetilde{A}_s(k))\bar{x}(k)\\
&+ (\sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon} \rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'P(k)\widetilde{B}_{s_2}(k))\Big)\bar{u}(k)\\
&+ 2\Big(\bar{x}(k)'(\sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon} \rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'P(k)\widetilde{B}_{s_2}(k))\\
&+ \bar{u}(k)'(\sum_{s=1}^{\varepsilon} \widetilde{B}_s(k)'P(k)\widetilde{B}_s(k))\Big)\mathbb{E}(v(k)) = 0
\end{aligned}$$

since $\mathbb{E}(z(k)) = 0$ and $\mathbb{E}(v(k)) = 0$.

The next theorem presents the optimal control law for PG as in (3.3).

Theorem 5.1. *Suppose that Assumption 4.1 holds. We have that*

$$J_k\left(\bar{x}(k), z(k), (\bar{u}^{k-1}, v^{k-1})\right) = z(k)'P(k)z(k) + \bar{x}(k)'M(k)\bar{x}(k) - V(k)\bar{x}(k) + \gamma(k) \tag{5.6}$$

and Equation (5.2) is satisfied with G_k^1 and G_k^2 as in Equations (5.4) and (5.5), respectively. Moreover, the optimal control strategy for Problem (3.3) is given by $u^*(k) = v^*(k) + \bar{u}^*(k)$, where

$$v^*(k) = -K(k)z(k) \quad \textit{and} \tag{5.7}$$

$$\bar{u}^*(k) = -H(k)\bar{x}(k) + \frac{1}{2}\mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\bar{B}(k)'V(k+1)'. \tag{5.8}$$

Proof. Recall the definitions of our operators in (4.1):

$$\begin{aligned}
\mathcal{A}(k, X, Y) &= \bar{A}(k)'X\bar{A}(k) + \sum_{s=1}^{\varepsilon} \widetilde{A}_s(k)'Y\widetilde{A}_s(k),\\
\mathcal{G}(k, X, Y) &= \left(\bar{A}(k)'X\bar{B}(k) + \sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon} \rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'Y\widetilde{B}_{s_2}(k)\right)',\\
\mathcal{R}(k, X, Y) &= \bar{B}(k)'X\bar{B}(k) + \sum_{s=1}^{\varepsilon} \widetilde{B}_s(k)'Y\widetilde{B}_s(k),\\
\mathcal{K}(k, X, Y) &= \mathcal{R}(k, X, Y)^{\dagger}\mathcal{G}(k, X, Y),\\
\mathcal{M}(k, X, Y) &= \mathcal{A}(k, X, Y) - \mathcal{G}(k, X, Y)'\mathcal{R}(k, X, Y)^{\dagger}\mathcal{G}(k, X, Y),\\
\bar{\mathcal{M}}(k, X, Y) &= \mathcal{M}(k, X, Y) - l_M(k)L(k)'L(k),\\
\mathcal{P}(k, X) &= \mathcal{M}(k, X, X) + \nu(k)L(k)'L(k),\\
\mathcal{V}(k, X, Y, V) &= V\left(\bar{A}(k) - \bar{B}(k)\mathcal{K}(k, X, Y)\right) + (\xi(k) - l_V(k))L(k),\\
\mathcal{D}(k, X, Y, V, \gamma) &= \gamma - \frac{1}{4}V\bar{B}(k)\mathcal{R}(k, X, Y)^{\dagger}\bar{B}(k)'V' + l_D(k),
\end{aligned}$$

where $P(k)$, $M(k)$, $V(k)$, $\gamma(k)$, and their final values are defined as in Equations (4.4), (4.5), (4.6), and (4.7), respectively, for $k = T, T-1, \ldots, 0$:

$$\begin{aligned}
&P(k) = \mathcal{P}(k, P(k+1)), \quad P(T) = \nu(T)L(T)'L(T),\\
&M(k) = \bar{\mathcal{M}}(k, M(k+1), P(k+1)), \quad M(T) = -l_M(T)L(T)'L(T),\\
&V(k) = \mathcal{V}(k, M(k+1), P(k+1), V(k+1)), \quad V(T) = (\xi(T) - L_V(T))L(T), \text{ and}\\
&\gamma(k) = \mathcal{D}(k, M(k+1), P(k+1), V(k+1), \gamma(k+1)), \quad \gamma(T) = l_D(T).
\end{aligned}$$

We apply backward induction on k. For $k = T$ we have that

$$J_T\left(\bar{x}(T), z(T), (\bar{u}^{T-1}, v^{T-1})\right) = \nu(T)z(T)'L(T)'L(T)z(T) - l_M(T)\bar{x}(T)L(T)'L(T)\bar{x}(T)' - (\xi(T) - l_V(T))L(T)\bar{x}(T) + l_D(T)$$

and the results follow with $P(T) = \nu(T)L(T)'L(T)$, $M(T) = -l_M(T)L(T)'L(T)$, $V(T) = (\xi(T) - l_V(T))L(T)$, and $\gamma(T) = l_D(T)$. Suppose that Equation (5.6) holds for $k+1$. In this case we have that

$$\mathbb{E}(J_k\left(\bar{x}(k+1), z(k+1), (\bar{u}^k, v^k)\right)|\mathcal{F}_k) = \mathbb{E}(z(k+1)'P(k+1)z(k+1)+ \bar{x}(k+1)'M(k+1)\bar{x}(k+1) - V(k+1)\bar{x}(k+1)|\mathcal{F}_k) + \gamma(k+1). \tag{5.9}$$

Let us evaluate each term in Equation (5.9). For the first term, we have from Equation (3.17) that

$$\begin{aligned}
&\mathbb{E}(z(k+1)'P(k+1)z(k+1)|\mathcal{F}_k) = z(k)'(\bar{A}(k)'P(k+1)\bar{A}(k)\\
&+ \sum_{s=1}^{\varepsilon}\widetilde{A}_s(k)'P(k+1)\widetilde{A}_s(k))z(k) + \bar{x}(k)'(\sum_{s=1}^{\varepsilon}\widetilde{A}_s(k)'P(k+1)\widetilde{A}_s(k))\bar{x}(k)\\
&+ v(k)'(\bar{B}(k)'P(k+1)\bar{B}(k) + \sum_{s=1}^{\varepsilon}\widetilde{B}_s(k)'P(k+1)\widetilde{B}_s(k))v(k)\\
&+ \bar{u}(k)'(\sum_{s=1}^{\varepsilon}\widetilde{B}_s(k)'P(k+1)\widetilde{B}_s(k))\bar{u}(k) + 2z(k)'(\sum_{s=1}^{\varepsilon}\widetilde{A}_s(k)'P(k+1)\widetilde{A}_s(k))\bar{x}(k)\\
&+ 2z(k)'(\bar{A}(k)'P(k+1)\bar{B}(k) + \sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon}\rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'P(k+1)\widetilde{B}_{s_2}(k))v(k)\\
&+ 2z(k)'(\sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon}\rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'P(k+1)\widetilde{B}_{s_2}(k))\bar{u}(k)\\
&+ 2\bar{x}(k)'(\sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon}\rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'P(k+1)\widetilde{B}_{s_2}(k))v(k)\\
&+ 2\bar{x}(k)'(\sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon}\rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'P(k+1)\widetilde{B}_{s_2}(k))\bar{u}(k)\\
&+ 2\bar{u}(k)'(\sum_{s=1}^{\varepsilon}\widetilde{B}_s(k)'P(k+1)\widetilde{B}_s(k))v(k).
\end{aligned} \tag{5.10}$$

For the second term, we have from Equation (3.16) that

$$\mathbb{E}(\bar{x}(k+1)'M(k+1)\bar{x}(k+1)|\mathcal{F}_k) = \bar{x}(k+1)'M(k+1)\bar{x}(k+1) =$$
$$\bar{x}(k)'\bar{A}(k)'M(k+1)\bar{A}(k)\bar{x}(k) + 2\bar{x}(k)'\bar{A}(k)'M(k+1)\bar{B}(k)\bar{u}(k)$$
$$+ \bar{u}(k)'\bar{B}(k)'M(k+1)\bar{B}(k)\bar{u}(k). \tag{5.11}$$

For the third term, we have again from Equation (3.16) that

$$\mathbb{E}(V(k+1)\bar{x}(k+1)|\mathcal{F}_k) = V(k+1)\bar{x}(k+1) = V(k+1)(\bar{A}(k)\bar{x}(k) + \bar{B}(k)\bar{u}(k)). \tag{5.12}$$

Summing up the terms in Equations (5.10), (5.11), and (5.12), we get from Equation (5.9) that Equations (5.2) and (5.3) are satisfied with G_k^1 and G_k^2 as respectively in Equations (5.4) and (5.5). Notice now that we can write the benefit-to-go function at time k using Lemma 5.1 as

$$G_k^1(\bar{x}(k), z(k), ((\bar{u}^{k-1}, v^{k-1}), (\bar{u}^k, v^k))) + v(k)(L(k)z(k))^2 - l_M(k)(L(k)\bar{x}(k))^2$$
$$- (\xi(k) - l_V(k))L(k)\bar{x}(k) + l_D(k) = z(k)'(\mathcal{A}(k, P(k+1), P(k+1) + v(k)L(k)'L(k))z(k)$$
$$+ \bar{x}(k)'(\mathcal{A}(k, M(k+1), P(k+1)) - l_M(k)L(k)'L(k))\bar{x}(k) - (V(k+1)\bar{A}(k)$$
$$+ (\xi(k) - l_V(k))L(k))\bar{x}(k) + \gamma(k+1) + l_D(k) + F_1(z(k), v(k), k) + F_2(\bar{x}(k), \bar{u}(k), k), \tag{5.13}$$

where

$$F_1(z(k), v(k), k) = v(k)'\mathcal{R}(k, P(k+1), P(k+1))v(k)$$
$$+ 2z(k)'\mathcal{G}(k, P(k+1), P(k+1))'v(k), \text{ and}$$
$$F_2(\bar{x}(k), \bar{u}(k), k) = \bar{u}(k)'\mathcal{R}(k, M(k+1), P(k+1))\bar{u}(k) +$$
$$\left\{2\bar{x}(k)'\mathcal{G}(k, M(k+1), P(k+1))' - V(k+1)\bar{B}(k)\right\}\bar{u}(k).$$

For simplicity, we set next $\mathcal{R}_1 = \mathcal{R}(k, P(k+1), P(k+1))$, $\mathcal{R}_2 = \mathcal{R}(k, M(k+1), P(k+1))$, $\mathcal{G}_1 = \mathcal{G}(k, P(k+1), P(k+1))$, and $\mathcal{G}_2 = \mathcal{R}(k, M(k+1), P(k+1))$.

Recalling the definitions of Equations (4.8) and (4.9):

$$K(k) = \mathcal{R}(k, P(k+1), P(k+1))^{\dagger}\mathcal{G}(k, P(k+1), P(k+1)) \text{ and}$$
$$H(k) = \mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\mathcal{G}(k, M(k+1), P(k+1)),$$

and from Proposition 4.4 and the properties of the generalised inverse, it follows that

$$F_1(z(k), v(k), k) = (v(k) + K(k)z(k))'\mathcal{R}_1(v(k) + K(k)z(k)) - z(k)'\mathcal{G}_1'\mathcal{R}_1^\dagger\mathcal{G}_1 z(k), \tag{5.14}$$

and from Proposition 4.5,

$$\begin{aligned} F_2(\bar{x}(k), \bar{u}(k), k) = {} & (\bar{u}(k) + (H(k)\bar{x}(k) - \frac{1}{2}\mathcal{R}_2^\dagger\bar{B}(k)'V(k+1)'))'\mathcal{R}_2(\bar{u}(k) + (H(k)\bar{x}(k) \\ & - \frac{1}{2}\mathcal{R}_2^\dagger\bar{B}(k)'V(k+1)')) - \bar{x}(k)'\mathcal{G}_2'\mathcal{R}_2^\dagger\mathcal{G}_2\bar{x}(k) + \bar{x}(k)'\mathcal{G}_2'\mathcal{R}_2^\dagger\bar{B}(k)'V(k+1)' \\ & - \frac{1}{4}V(k+1)\bar{B}(k)\mathcal{R}_2^\dagger\bar{B}(k)'V(k+1)'. \end{aligned} \tag{5.15}$$

Replacing Equations (5.14) and (5.15) into Equation (5.13), we get that

$$\begin{aligned} & G_k^1(\bar{x}(k), z(k), ((\bar{u}^{k-1}, v^{k-1}), (\bar{u}^k, v^k))) + \nu(k)(L(k)z(k))^2 - l_M(k)(L(k)\bar{x}(k))^2 - \\ & (\xi(k) - l_V(k))L(k)\bar{x}(k) + l_D(k) = z(k)'(\mathcal{P}(k, P(k+1))z(k) + \bar{x}(k)'(\mathcal{M}(k, M(k+1), P(k+1)) \\ & - l_M(k)L(k)'L(k))\bar{x}(k) - (V(k+1)(\bar{A}(k) - \bar{B}(k)H(k)) + \xi(k) - l_V(k)L(k))\bar{x}(k) \\ & - \frac{1}{4}V(k+1)\bar{B}(k)\mathcal{R}_2^\dagger\bar{B}(k)'V(k+1)' + \gamma(k+1) + l_D(k) + (v(k) + K(k)z(k))'\mathcal{R}_1(v(k) \\ & + K(k)z(k)) + (\bar{u}(k) + (H(k)\bar{x}(k) - \frac{1}{2}\mathcal{R}_2^\dagger\bar{B}(k)'V(k+1)'))'\mathcal{R}_2(\bar{u}(k) + (H(k)\bar{x}(k) \\ & - \frac{1}{2}\mathcal{R}_2^\dagger\bar{B}(k)'V(k+1)')) = z(k)'P(k)z(k) + \bar{x}(k)'M(k)\bar{x}(k) - V(k)\bar{x}(k) + \gamma(k) \\ & + \phi_1(v(k)) + \phi_2(\bar{u}(k)), \end{aligned} \tag{5.16}$$

where

$$\phi_1(v(k)) = (v(k) + K(k)z(k))'\mathcal{R}_1(v(k) + K(k)z(k)) \text{ and} \tag{5.17}$$

$$\begin{aligned} \phi_2(\bar{u}(k)) = {} & (\bar{u}(k) + (H(k)\bar{x}(k) - \frac{1}{2}\mathcal{R}_2^\dagger\bar{B}(k)'V(k+1)'))'\mathcal{R}_2(\bar{u}(k) + (H(k)\bar{x}(k) \\ & - \frac{1}{2}\mathcal{R}_2^\dagger\bar{B}(k)'V(k+1)')). \end{aligned} \tag{5.18}$$

We get that minimising the left hand side of Equation (5.16) in $v(k)$ and $\bar{u}(k)$ is equivalent to minimise $\phi_1(v(k))$ and $\phi_2(\bar{u}(k))$ since the other terms do not depend on $v(k)$ and $\bar{u}(k)$. Since $\mathcal{R}_1 \geq 0$ and $\mathcal{R}_2 \geq 0$, the minimum is $\phi_1(v^*(k)) = 0$ and $\phi_2(\bar{u}^*(k)) = 0$ with $v^*(k)$ and $\bar{u}^*(k)$ given as in Equations (5.7) and (5.8). Note that $\mathbb{E}(v^*(k)) = -K(k)\mathbb{E}(z(k)) = 0$ and thus $(\bar{u}^*(k), v^*(k)) \in \mathbb{S}(k)$. Since

$$J_k\left(\bar{x}(k), z(k), (\bar{u}^{k-1}, v^{k-1})\right) =$$
$$\min_{(\bar{u}(k),v(k))\in\mathbb{S}(k)} \{G_k^1(\bar{x}(k), z(k), ((\bar{u}^{k-1}, v^{k-1}), (\bar{u}(k), v(k)))) + v(k)(L(k)z(k))^2 -$$
$$l_M(k)(L(k)\bar{x}(k))^2 - (\xi(k) - l_V(k))L(k)\bar{x}(k) + l_D(k)\},$$

we get from Equation (5.16) that Equation (5.6) is satisfied, completing the proof. □

We have from Theorem 5.1 that $PU\,(v, \xi) = \mathbb{E}(z(k)'P(k)z(k)) + \bar{x}_0' M(0)\bar{x}_0 - V(0)\bar{x}_0 + \gamma(0)$ and, for $\mathcal{H}(\omega) = PLi(\omega)$, $i = 1, 2$ or 3, with PLi as in Table 2:

$$\mathcal{H}(\omega) = \mathbb{E}(z(k)'P(k)z(k)) + \bar{x}_0' M(0)\bar{x}_0 - V(0)\bar{x}_0 + \gamma(0), \tag{5.19}$$

where the input parameters for problems PU, $PL1$, $PL2$, and $PL3$ as shown in Table 1. If x_0 is known, then $z(0) = 0$ and Equation (5.19) becomes

$$\mathcal{H}(\omega) = \bar{x}_0' M(0)\bar{x}_0 - V(0)\bar{x}_0 + \gamma(0). \tag{5.20}$$

For problems $PC1$, $PC2$, and $PC3$, (see Table 2 for the equations equivalences) we still have to solve the Lagrangian dual problem $\max_{\omega\geq 0}\mathcal{H}$ by applying a search algorithm on ω. As pointed out in (ZHU; LI; WANG, 2004), $\mathcal{H}$ is a concave function so that a primal-dual method based on the gradient method can be applied. Using a different approach, one could also apply Proposition 4.6 and Corollary 4.1 to verify the concavity of $\mathcal{H}$. Notice that some extra care need to be taken since at each iteration it is required to check if Assumption 4.1 is true.

The next proposition shows an explicit formula for the expected output value and its variance in each period k.

For $k = 0, \ldots, T-1$ and $j = 1, \ldots, T$, the following new operators will be useful to present an expression for the output variance $Var\,(y^u(t))$ when the optimal control strategy $u^*(k) = v^*(k) + \bar{u}^*(k)$ is applied to system (3.1). For $Y \in \mathbb{H}^n$, define

$$\begin{aligned}
\bar{\mathsf{P}}(k,Y) &= \mathcal{A}(k,Y,Y) + K(k)'\mathcal{R}(k,Y,Y)K(k) - 2\mathcal{G}(k,Y,Y)'K(k),\\
\bar{\mathsf{Q}}(k,Y) &= \mathcal{A}(k,0,Y) + H(k)'\mathcal{R}(k,0,Y)H(k) - 2\mathcal{G}(k,0,Y)'H(k),\\
\bar{\mathsf{R}}(k,Y) &= \mathcal{G}(k,0,Y)'\mathcal{R}(k,M(k+1),P(k+1))^{\dagger}\bar{B}(k)'V(k+1)'\\
&\quad -H(k)'\mathcal{R}(k,0,Y)\mathcal{R}(k,M(k+1),P(k+1))^{\dagger}\bar{B}(k)'V(k+1)',\\
\bar{\mathsf{S}}(k,Y) &= \frac{1}{4}V(k+1)\bar{B}(k)\mathcal{R}(k,M(k+1),P(k+1))^{\dagger}\mathcal{R}(k,0,Y)\\
&\quad \mathcal{R}(k,M(k+1),P(k+1))^{\dagger}\bar{B}(k)'V(k+1)', \text{ and}\\
\Theta_j^k &= \bar{\mathsf{P}}(j,\ldots\bar{\mathsf{P}}(k-1,\bar{\mathsf{P}}(k,L(k+1)'L(k+1)))\ldots), \text{ with}\\
\Theta_j^k &= L(k+1)'L(k+1) \text{ if } j>k.
\end{aligned} \tag{5.21}$$

We have the following proposition.

Proposition 5.1: Suppose that Assumption 4.1 holds and x_0 is known. If the optimal control strategy $u^*(k) = v^*(k) + \bar{u}^*(k)$ as in (5.7) and (5.8) is applied to system (3.1), then the expected value of the output $\mathbb{E}\left(y^{u^*}(t)\right)$ and the variance output $Var\left(y^{u^*}(t)\right)$ are given respectively by

$$\begin{aligned}
\mathbb{E}\left(y^{u^*}(t)\right) &= L(t)\prod_{j=0}^{t-1}\left(\bar{A}(j)-\bar{B}(j)H(j)\right)x_0 + L(t)\sum_{i=0}^{t-1}\left[\left(\prod_{j=i+1}^{t-1}\left(\bar{A}(j)-\bar{B}(j)H(j)\right)\right)\right.\\
&\quad \left.\left(\frac{1}{2}\bar{B}(i)\mathcal{R}(i,M(i+1),P(i+1))^{\dagger}\bar{B}(i)'V(i+1)'\right)\right] \text{ and}
\end{aligned} \tag{5.22}$$

$$Var\left(y^{u^*}(t)\right) = \sum_{j=0}^{t-1}\left[\bar{x}(j)'\bar{\mathsf{Q}}(j,\Theta_{j+1}^{t-1})\bar{x}(j) + \bar{x}(j)'\bar{\mathsf{R}}(j,\Theta_{j+1}^{t-1}) + \bar{\mathsf{S}}(j,\Theta_{j+1}^{t-1})\right]. \tag{5.23}$$

Proof. To easy the notation, we remove the supercript dependence on u^*. Substituting (5.8) into (3.16) we obtain that

$$\bar{x}(k+1) = (\bar{A}(k)-\bar{B}(k)H(k))\bar{x}(k) + \frac{1}{2}\bar{B}(k)\mathcal{R}(k,M(k+1),P(k+1))^{\dagger}\bar{B}(k)'V(k+1)'. \tag{5.24}$$

Iterating (5.24) for $k = 0,\ldots,T-1$, with $x(0) = x_0$ and using (3.2), we obtain (5.22). To prove Equation (5.23), we use the dynamics in Equation (3.17), the independence hypothesis made on the multiplicative noises, and recall that $Var(y^u(t)) = \mathbb{E}\left((L(t)z(t))^2\right)$.

To easy the notation, we set $Y = L(k+1)'L(k+1)$. Thus, for $t = k+1$, we have that

$$\mathbb{E}(z(k+1)'Yz(k+1)) = \mathbb{E}\{z(k)'\,[\mathcal{A}(k,Y,Y)]\,z(k)\} + \bar{x}(k)'\left[\sum_{s=1}^{\varepsilon}\widetilde{A}_s(k)'Y\widetilde{A}_s(k)\right]\bar{x}(k)+$$
$$v(k)'\,[\mathcal{R}(k,Y,Y)]\,v(k) + u(k)'\left[\sum_{s=1}^{\varepsilon}\widetilde{B}_s(k)'Y\widetilde{B}_s(k)\right]u(k) + 2z(k)'\,[\mathcal{G}(k,Y,Y)']\,v(k)+$$
$$2\bar{x}(k)'\left[\sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon}\rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'Y\widetilde{B}_{s_2}(k)\right]\bar{u}(k) + G(k), \tag{5.25}$$

where

$$G(k) = 2\mathbb{E}(z(k)')\left[\sum_{s=1}^{\varepsilon}\widetilde{A}_s(k)'Y\widetilde{A}_s(k)\right]\bar{x}(k)$$
$$+2\mathbb{E}(z(k)')\left[\sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon}\rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'Y\widetilde{B}_{s_2}(k)\right]\bar{u}(k)+$$
$$2\bar{x}(k)'\left[\sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon}\rho_{s_1,s_2}(k)\widetilde{A}_{s_1}(k)'Y\widetilde{B}_{s_2}(k)\right]\mathbb{E}(v(k))$$
$$+2\mathbb{E}(v(k)')\left[\sum_{s=1}^{\varepsilon}\widetilde{B}_s(k)'Y\widetilde{B}_s(k)\right]\bar{u}(k).$$

Note that $G(k) = 0$ since $\mathbb{E}(z(k)) = 0$ and $\mathbb{E}(v(k)) = 0$. Therefore, applying Equations (5.7) and (5.8) into (5.25), we obtain that

$$\mathbb{E}(z(k+1)'Yz(k+1)) = \mathbb{E}\{z(k)'\,[\mathcal{A}(k,Y,Y)]\,z(k)\} + \bar{x}(k)'\,[\mathcal{A}(k,0,Y)]\,\bar{x}(k)+$$
$$z(k)'\,[K(k)'\mathcal{R}(k,Y,Y)K(k)]\,z(k) + \bar{x}(k)'\,[H(k)'\mathcal{R}(k,0,Y)H(k)]\,\bar{x}(k)+$$
$$\frac{1}{4}V(k+1)\bar{B}(k)\mathcal{R}(k,M(k+1),P(k+1))^{\dagger}\mathcal{R}(k,0,Y)\mathcal{R}(k,M(k+1),P(k+1))^{\dagger}$$
$$\bar{B}(k)'V(k+1)' - \bar{x}(k)'\left[H(k)'\mathcal{R}(k,0,Y)\mathcal{R}(k,M(k+1),P(k+1))^{\dagger}\bar{B}(k)'V(k+1)'\right]$$
$$-2z(k)'\,[\mathcal{G}(k,Y,Y)']\,K(k)z(k) - 2\bar{x}(k)'\,[\mathcal{G}(k,0,Y)'H(k)]\,\bar{x}(k)$$
$$+\bar{x}(k)'\left[\mathcal{G}(k,0,Y)'\mathcal{R}(k,M(k+1),P(k+1))^{\dagger}\bar{B}(k)'V(k+1)'\right]. \tag{5.26}$$

Rearranging the terms in Equation (5.26) and applying the operators (5.21), we obtain that

$$\mathbb{E}(z(k+1)'Yz(k+1)) =$$
$$\mathbb{E}\left\{z(k)'\bar{\mathsf{P}}(k,Y)z(k)\right\} + \bar{x}(k)'\bar{\mathsf{Q}}(k,Y)\bar{x}(k) + \bar{x}(k)'\bar{\mathsf{R}}(k,Y) + \bar{\mathsf{S}}(k,Y). \tag{5.27}$$

Finally, we apply Equation (5.27) recursively on $\mathbb{E}\left\{z(k)'\bar{\mathsf{P}}(k,Y)z(k)\right\}$ and so on to obtain Equation (5.23), completing the proof. □

Next, we present a sufficient condition for a closed form solution to problem $PC1$. From Table 1, notice that in this case $P(k)$ and $M(k)$, as defined in Equations (4.4) and (4.5), will not depend on the parameter ω, so that Assumption 4.1 can be checked independently of ω. Notice also that, as seen in Remark 4.2, $P(k) \geq 0$ and $M(k) \geq 0$ for all k. Define for $k = 0, \ldots, T-1$,

$$\mathbb{A}_H(k) = \bar{A}(k) - \bar{B}(k)H(k), \tag{5.28}$$

$$\tilde{\mathbb{B}}(k,t) = \frac{1}{2}\left(\prod_{j=k+1}^{t-1} \mathbb{A}_H(j)\right)\bar{B}(k)\mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\bar{B}(k)', \tag{5.29}$$

and let the elements in row r and column c of $\mathbb{C} \in \mathbb{H}^T$ and $\mathbb{D} \in \mathbb{H}^{T,1}$ be given as

$$\mathbb{C}_{r,c} = L(r)\sum_{i=0}^{\min(r,c)-1} \tilde{\mathbb{B}}(i,r)\left(\prod_{j=i+1}^{c-1} \mathbb{A}_H(j)\right)' L(c)' \text{ and} \tag{5.30}$$

$$\mathbb{D}_{r,1} = L(r)\prod_{j=0}^{r-1} \mathbb{A}_H(j)x_0. \tag{5.31}$$

The following theorem establishes a sufficient condition for the analytical solution to the Lagrangian dual problem $PC1 = \max_{\omega\geq 0} H(\omega)$, where $\mathcal{H}(\omega) = PL1(\omega)$, $PL1$ as in (3.20).

Theorem 5.2. *Suppose that Assumption 4.1 holds and assume that $det(\mathbb{C}) \neq 0$. Set*

$$\omega^* = \mathbb{C}^{-1}(\epsilon - \mathbb{D}). \tag{5.32}$$

Then, if $\omega^* \geq 0$, we have that $PC1 = \max_{\omega\geq 0}\mathcal{H}(\omega) = \mathcal{H}(\omega^*)$ and an optimal control strategy for problem $PC1$ is given by $u^*(k) = v^*(k) + \bar{u}^*(k)$ as in Equations (5.7) and (5.8), with the parameter $\omega = \omega^*$ as in Table 1.

Proof. Set for any $u \in \mathbb{U}$, $\Psi(u) = \sum_{t=1}^{T}\left(\nu(t)Var\left(y^u(t)\right)\right)$, $\Phi(u) = \sum_{t=1}^{T}\left(\epsilon(t) - \mathbb{E}\left(y^u(t)\right)\right)$, so that $\mathcal{H}(\omega) = \min_{u\in\mathbb{U}}(\Psi(u) + \omega\Phi(u))$. From Theorem 5.1, we have that $\mathcal{H}(\omega) = \mathbb{U}(\Psi(u^\omega) + \omega\Phi(u^\omega))$, with u^ω is as in Equations (5.7) and (5.8), where we have replaced ω^* by ω to indicate the dependence on the parameter ω. The proof consists in developing Equation (5.22) to obtain $\mathbb{E}(y(t))$ explicitly on each $\omega(t)$ and then solving it for ω in order to get $\mathbb{E}(y(t)) = \epsilon(t)$. From Equation (5.22) and using the definitions of $\mathbb{A}_H$ and $\tilde{\mathbb{B}}$ in

Equations (5.28) and (5.29), respectively, we have that

$$\mathbb{E}\left(y^{u^{\omega}}(t)\right) = L(t)\prod_{j=0}^{t-1}\mathbb{A}_H(j)x_0 + L(t)\sum_{i=0}^{t-1}\tilde{\mathbb{B}}(i,t)V(i+1)'. \tag{5.33}$$

Using Equations (4.6) and (5.28), we re-write $V(t)'$ explicitly on each $\omega(t)$, $t = 1, \ldots, T$, as

$$V(k)' = \sum_{c=k}^{T}\left(\prod_{j=k}^{c-1}\mathbb{A}_H(j)\right)' L(c)'\omega(c). \tag{5.34}$$

Applying Equation (5.34) into (5.33), we get that

$$\begin{aligned}\mathbb{E}\left(y^{u^{\omega}}(t)\right) &= L(t)\prod_{j=0}^{t-1}\mathbb{A}_H(j)x_0 + L(t)\sum_{i=0}^{t-1}\tilde{\mathbb{B}}(i,t)\sum_{c=i+1}^{T}\left(\prod_{j=i+1}^{c-1}\mathbb{A}_H(j)\right)' L(c)'\omega(c)\\ &= L(t)\prod_{j=0}^{t-1}\mathbb{A}_H(j)x_0 + L(t)\sum_{c=1}^{T}\sum_{i=0}^{\min(t,c)-1}\tilde{\mathbb{B}}(i,t)\left(\prod_{j=i+1}^{c-1}\mathbb{A}_H(j)\right)' L(c)'\omega(c). \end{aligned}\tag{5.35}$$

Then, we set $\mathbb{E}(y^{u^{\omega}}(t)) = \epsilon(t)$ for $t = 1, \ldots, T$, where $\epsilon(t)$ is a known restriction, and apply Equation (5.35) to obtain a set of T equations on T unknown $\omega(t)$. Finally, using the definitions in Equations (5.30) and (5.31) for $r, c = 1, \ldots, T$, we rearrange this system of equations into a vector form as $\epsilon = \mathbb{C}\omega + \mathbb{D}$, which can be solved for ω as in Equation (5.32) if $det(\mathbb{C}) \neq 0$. Set now for any $u \in \mathbb{U}$, $\Psi(u) = \sum_{t=1}^{T}\left(\nu(t)Var\left(y^u(t)\right)\right)$, $\Phi(u) = \sum_{t=1}^{T}\left(\epsilon(t) - \mathbb{E}\left(y^u(t)\right)\right)$, so that $\mathcal{H}(\omega) = \min_{u\in\mathbb{U}}(\Psi(u) + \omega\Phi(u))$. Consider u^* as in Theorem 5.1 with $\omega = \omega^*$. We have that $\Phi(u^*) = 0$ and for any $\omega \geq 0$, $\mathcal{H}(\omega) \leq \Psi(u^*) + \omega\Phi(u^*) = \Psi(u^*) = \Psi(u^*) + \omega^*\Phi(u^*) = \mathcal{H}(\omega^*)$, so that $\mathcal{H}(\omega^*) \geq \mathcal{H}(\omega)$ and thus $PC1 = \max_{\omega\geq 0}\mathcal{H}(\omega) = \mathcal{H}(\omega^*)$, completing the proof. □

5.2 Infinite-horizon control and stabilisation

The following sections will show the main results regarding the infinite-horizon control and stabilisation. Section 5.2.1 presents sufficient conditions for the maximal and stabilising solutions to the GCARE. Section 5.2.2 shows the optimal stabilising control policies to the discounted and long-run problems along with a numerical approach for obtaining a stabilising solution to the GCARE.

5.2.1 Conditions for maximal and stabilising solutions

In this section, we show some results regarding the sufficient conditions for the existence of the maximal solution to the GCARE and necessary and sufficient conditions for the existence of the mean-square stabilising solution. Those results shall be useful to obtain the stabilising optimal control policies in the following section and a numerical procedure to compute it.

First, we recall the domain definition in (4.22):

$$\mathbf{D}(\mathcal{T}) := \{Z \in \mathbb{T}(\mathbb{H}^n); \mathrm{Ker}(\breve{\mathcal{R}}(Z)) \subseteq \mathrm{Ker}(\breve{\mathcal{G}}(Z)')\} \text{ and}$$
$$\mathbf{D}_+(\mathcal{T}) := \{Z \in \mathbb{T}(\mathbb{H}^n); \mathrm{Ker}(\breve{\mathcal{R}}(Z)) \subseteq \mathrm{Ker}(\breve{\mathcal{G}}(Z)') \text{ and } \breve{\mathcal{R}}(Z) \geq 0\},$$

and make the following definitions. From Lemma 4.6, for $K \in \mathbb{K}$ and $X \in \mathbb{T}(\mathbb{H}^n)$, there is unique solution to

$$X - \mathcal{L}_K(X) = \mathcal{I} \tag{5.36}$$

defined therein as $X(K)$. We also define the following subsets of $\mathbf{D}_+(\mathcal{T})$:

$$\mathbf{M} := \{X \in \mathbf{D}_+(\mathcal{T}); \mathcal{T}(X) \geq 0\},$$
$$\widehat{\mathbf{M}} := \{X \in \mathbf{D}_+(\mathcal{T}); \mathcal{T}(X) = 0\},$$

and, for $K \in \mathbb{K}$,

$$\mathbf{M}(K) := \{X \in \mathbf{M}; \mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}(K') \text{ and } \mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X(K)))\} \text{ and}$$
$$\widehat{\mathbf{M}}(K) := \{X \in \widehat{\mathbf{M}}; \mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}(K') \text{ and } \mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X(K)))\}.$$

The next theorem provides a sufficient condition for the existence of the maximal symmetric solution of Equation (4.21) over $\mathbf{M}(K)$ for $K \in \mathbb{K}$.

Theorem 5.3. *Suppose that $\mathbf{M}(\widetilde{F}) \neq \emptyset$ for some $\widetilde{F} \in \mathbb{K}$, then there exists $X^s \in \mathbf{D}_+(\mathcal{T})$ and $F^s \in \mathbb{T}(\mathbb{H}^{m,n})$, for $s = 0, 1, \ldots$, satisfying the following properties:*

i. $X^0 \geq X^1 \geq \cdots \geq X^s \geq X$, *for an arbitrary $X \in \mathbf{M}(\widetilde{F})$;*

ii. $r_\sigma(\mathcal{L}_{F^s}) < 1$, *with $F^s = \breve{\mathcal{K}}(X^{s-1})$ for $s \geq 1$;*

iii. $X^s = X(F^s)$*;*

iv. $\operatorname{Ker}(\breve{\mathcal{R}}(X^s)) \subseteq \operatorname{Ker}(F^{s\prime})$ *and* $\operatorname{Ker}(\breve{\mathcal{R}}(X^s)) = \operatorname{Ker}(\breve{\mathcal{R}}(X))$.

Moreover, there exists $X^+ \in \widehat{\mathbf{M}}(\widetilde{F})$ *such that* $X^+ \geq X$ *for any* $X \in \mathbf{M}(\widetilde{F})$, $r_\sigma(\mathcal{L}_{\breve{\mathcal{K}}(X^+)}) \leq 1$, *and* $X^s \to X^+$ *as* $s \to \infty$.

Proof. We shall apply induction on s to prove the result following the same reasoning as in Theorem 1 in (COSTA; PAULO, 2008). The auxiliary results used here are in Section 4.2. Consider an arbitrary $X \in \mathbf{M}(\widetilde{F})$ so $\breve{\mathcal{R}}(X) \geq 0$, $\mathcal{T}(X) \geq 0$, and $F = \breve{\mathcal{K}}(X)$. Take $F^0 = \widetilde{F}$ so that $r_\sigma(\mathcal{L}_{F^0}) < 1$. Thus, from Lemma 4.6, there exists a unique $X^0 = X(F^0) \in \mathbb{T}(\mathbb{H}^{n*})$ satisfying Equation (4.33) for $K = F^0$. We have from Equation (4.35) that

$$(X^0 - X) - \mathcal{L}_{F^0}(X^0 - X) = (F^0 - F)'\breve{\mathcal{R}}(X)(F^0 - F) + \mathcal{T}(X)$$

and, since $(F^0 - F)'\breve{\mathcal{R}}(X)(F^0 - F) + \mathcal{T}(X) \geq 0$ and $r_\sigma(\mathcal{L}_{F^0}) < 1$, we have from Lemma 4.6 that $X^0 - X \geq 0$. From Lemma 4.2, we also have that $X^0 \in \mathbf{D}_+(\mathcal{T})$ and $\breve{\mathcal{R}}(X^0) \geq \breve{\mathcal{R}}(X) \geq 0$, which implies that $\operatorname{Ker}(\breve{\mathcal{R}}(X^0)) \subseteq \operatorname{Ker}(\breve{\mathcal{R}}(X))$. But by definition, $X \in \mathbf{M}(F^0)$ implies that $\operatorname{Ker}(\breve{\mathcal{R}}(X)) \subseteq \operatorname{Ker}(F^{0\prime})$ and $\operatorname{Ker}(\breve{\mathcal{R}}(X)) \subseteq \operatorname{Ker}(\breve{\mathcal{R}}(X^0))$ so that $\operatorname{Ker}(\breve{\mathcal{R}}(X)) = \operatorname{Ker}(\breve{\mathcal{R}}(X^0))$ and $\operatorname{Ker}(\breve{\mathcal{R}}(X^0)) \subseteq \operatorname{Ker}(F^{0\prime})$, and the result holds for s=0.

Suppose now that the result holds for $s-1$ and recall that $F^s = \breve{\mathcal{K}}(X^{s-1})$. From Equation (4.36), we get that

$$\begin{aligned}(X^{s-1} - X) - \mathcal{L}_{F^s}(X^{s-1} - X) &= (F^s - F)'\breve{\mathcal{R}}(X)(F^s - F) \\ &\quad + (F^s - F^{s-1})'\breve{\mathcal{R}}(X^{s-1})(F^s - F^{s-1}) + \mathcal{T}(X) \\ &\geq (F^s - F^{s-1})'\breve{\mathcal{R}}(X^{s-1})(F^s - F^{s-1}). \end{aligned} \tag{5.37}$$

We want to use Lemma 4.4 to show that for some small enough $\delta > 0$,

$$(F^s - F^{s-1})'\breve{\mathcal{R}}(X^{s-1})(F^s - F^{s-1}) \geq \delta(F^s - F^{s-1})'(F^s - F^{s-1}). \tag{5.38}$$

Noticing that $F^{s'} = -\breve{\mathcal{G}}(X^{s-1})'\breve{\mathcal{R}}(X^{s-1})^\dagger$ and $\operatorname{Ker}(\breve{\mathcal{R}}(X^{s-1})^\dagger) = \operatorname{Ker}(\breve{\mathcal{R}}(X^{s-1}))$, it follows that $\operatorname{Ker}(\breve{\mathcal{R}}(X^{s-1})) \subseteq \operatorname{Ker}(F^{s'})$. Furthermore, by the induction hypothesis, $\operatorname{Ker}(\breve{\mathcal{R}}(X^{s-1})) \subseteq \operatorname{Ker}(F^{s-1'})$ and combining the results, we get that $\operatorname{Ker}(\breve{\mathcal{R}}(X^{s-1})) \subseteq \operatorname{Ker}((F^s - F^{s-1})')$. Therefore, we obtain Equation (5.38) by applying Lemma 4.4 and we get that $r_\sigma(\mathcal{L}_{F^s}) < 1$ from Equations (5.37), (5.38), and Lemma 4.5. Suppose now that $X^s = X(F^s) \in \mathbb{T}(\mathbb{H}^n)$

so from Lemma 4.7, we have that Equation (4.35) yields

$$(X^s - X) - \mathcal{L}_{F^s}(X^s - X) = (F^s - F)'\breve{\mathcal{R}}(X)(F^s - F) + \mathcal{T}(X)$$

and, since $r_\sigma(\mathcal{L}_{F^s}) < 1$, we get from Lemma 4.6 that $X^s \geq X$ and, from Lemma 4.2, we have that $X^s \in \mathbf{D}_+(\mathcal{T})$. Equation (4.38) yields

$$(X^{s-1} - X^s) - \mathcal{L}_{F^s}(X^{s-1} - X^s) = (F^{s-1} - F^s)'\breve{\mathcal{R}}(X^{s-1})(F^{s-1} - F^s),$$

which shows that $X^{s-1} \geq X^s \geq X$ from the fact that $r_\sigma(\mathcal{L}_{F^s}) < 1$, $(F^{s-1} - F^s)'\breve{\mathcal{R}}(X^{s-1})(F^{s-1} - F^s) \geq 0$, and from Lemma 4.6. We also have that $\breve{\mathcal{R}}(X^{s-1}) \geq \breve{\mathcal{R}}(X^s) \geq \breve{\mathcal{R}}(X) \geq 0$ so that from the induction hypothesis, $\mathrm{Ker}(\breve{\mathcal{R}}(X)) = \mathrm{Ker}(\breve{\mathcal{R}}(X^{s-1})) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X^s)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X))$, showing that $\mathrm{Ker}(\breve{\mathcal{R}}(X^s)) = \mathrm{Ker}\ (\breve{\mathcal{R}}(X)) = \mathrm{Ker}(\breve{\mathcal{R}}(X^{s-1})) \subseteq \mathrm{Ker}(F^{s'})$, completing the induction arguments.

We get that there exists X^+ symmetric such that $X^s \downarrow X^+$ as $s \to \infty$ given that $\{X^s\}_{s=0}^{\infty}$ is a decreasing sequence with $X^s \geq X$ for all $s = 0, 1, \ldots$ (see page 79 in (SONTAG, 1990)). Clearly, $X^+ \geq X$ and, from Lemma 4.2, we get that $X^+ \in \mathbf{D}_+(\mathcal{T})$. Since $X^0 \geq X^+ \geq X$, we get that $\breve{\mathcal{R}}(X^0) \geq \breve{\mathcal{R}}(X^+) \geq \breve{\mathcal{R}}(X) \geq 0$ and thus $\mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X^0)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X^+)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X))$, showing that $\mathrm{Ker}\ (\breve{\mathcal{R}}(X^+)) = \mathrm{Ker}(\breve{\mathcal{R}}(X^0))$ and $\mathrm{Ker}(\breve{\mathcal{R}}(X^+)) = \mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(F^{0'}))$. Moreover, $X^+ \in \widehat{\mathbf{M}}(\widetilde{F})$ since substituting $K = F^s = \breve{\mathcal{K}}(X^{s-1})$ and $X = X^s$ into Equation (5.36) and taking the limit as $s \to \infty$, we get, after rearranging the terms, that $\mathcal{T}(X^+) = 0$. Given that X is arbitrary in $\mathbf{M}(\widetilde{F})$, we have that $X^+ \geq X$ for all $X \in \mathbf{M}(\widetilde{F})$. Finally, note that $r_\sigma(\mathcal{L}_{F^s}) < 1$ implies that $r_\sigma(\mathcal{L}_{F^+}) \leq 1$, where $F^+ = \breve{\mathcal{K}}(X^+)$ (see page 328 in (SONTAG, 1990) regarding the continuity of the eigenvalues on finite-dimensional linear operator entries). □

We show next that there exists at most one mean square stabilising solution to Equation (4.21).

Lemma 5.2. *Suppose that $\mathbf{M}(\widetilde{F}) \neq \emptyset$ for some $\widetilde{F} \in \mathbb{K}$, then there is at most one mean-square stabilising solution to the GCARE (4.21) and it coincides with the maximal solution over $\mathbf{M}(\widetilde{F})$.*

Proof. We follow the same approach as in Lemma 12 in (COSTA; PAULO, 2008) and consider that $\widehat{X}$ is the mean-square stabilising solution to the GCARE (4.21), thus

$\mathcal{T}(\widehat{X}) = 0$. From Lemma 4.1 with $Z = \widehat{X}$ and $F = \breve{\mathcal{K}}(\widehat{X})$, we have that

$$\widehat{X} - \mathcal{L}_{\breve{\mathcal{K}}(\widehat{X})}(\widehat{X}) - \mathcal{I} = 0. \tag{5.39}$$

From Theorem 5.3, there exists the maximal solution $X^+ \in \widehat{\mathbf{M}}(\widetilde{F})$ over $\mathbf{M}(\widetilde{F})$. Again, from Lemma 4.1 with $Z = X^+$, $F = \breve{\mathcal{K}}(\widehat{X})$ and $\mathcal{T}(X^+) = 0$, we have that

$$X^+ - \mathcal{I} - \mathcal{L}_{\breve{\mathcal{K}}(\widehat{X})}(X^+) = -(\breve{\mathcal{K}}(X^+) - \breve{\mathcal{K}}(\widehat{X}))'\breve{\mathcal{R}}(X^+)(\breve{\mathcal{K}}(X^+) - \breve{\mathcal{K}}(\widehat{X})). \tag{5.40}$$

Thus, from Equations (5.39) and (5.40), we obtain that

$$(\widehat{X} - X^+) - \mathcal{L}_{\breve{\mathcal{K}}(\widehat{X})}(\widehat{X} - X^+) = (\breve{\mathcal{K}}(X^+) - \breve{\mathcal{K}}(\widehat{X}))'\breve{\mathcal{R}}(X^+)(\breve{\mathcal{K}}(X^+) - \breve{\mathcal{K}}(\widehat{X})) \geq 0$$

since $\breve{\mathcal{R}}(X^+) \geq 0$. Thereby, from Lemma 4.6, we have that $\widehat{X} - X^+ \geq 0$. However, it also implies that $\breve{\mathcal{R}}(\widehat{X}) \geq \breve{\mathcal{R}}(X^+) \geq 0$. Therefore, $\mathrm{Ker}(\breve{\mathcal{R}}(\widehat{X})) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X^+)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(\widetilde{F}))$ and $\mathrm{Ker}(\breve{\mathcal{R}}(\widehat{X})) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X^+)) \subseteq \mathrm{Ker}(\widetilde{F}')$, showing that $\widehat{X} - X^+ \leq 0$, completing the proof. □

The following theorem provides necessary and sufficient conditions for the existence of the mean square stabilising solution. First, we define for $K \in \mathbb{T}(\mathbb{H}^{m,n})$ and $\Psi \in \mathbb{T}(\mathbb{H}^n)$, the following operator $\mathcal{N}_{\Psi,K} \in \mathbb{T}(\mathbb{H}^n)$: for $Z \in \mathbb{T}(\mathbb{H}^n)$,

$$\mathcal{N}_{\Psi,K}(Z) = \mathcal{L}_K(Z) + (\Psi + \breve{\bar{B}}K)'Z(\Psi + \breve{\bar{B}}K) - (\breve{\bar{A}} + \breve{\bar{B}}K)'Z(\breve{\bar{A}} + \breve{\bar{B}}K). \tag{5.41}$$

Theorem 5.4. *Suppose that $\mathbf{M}(\widetilde{F}) \neq \emptyset$ for some $\widetilde{F} \in \mathbb{K}$, then the following statements are equivalent*

i. There is a $V \in \mathbb{T}(\mathbb{H}^n)$ such that $r_\sigma(\mathcal{N}_{\Gamma(X),\breve{\mathcal{K}}(X)}) < 1$, for some $X \in \mathbf{M}(\widetilde{F})$, where

$$\Gamma(X) = \breve{\bar{A}} + V\mathcal{T}(X)^{1/2}.$$

ii. There exists the mean-square stabilising solution to the GCARE (4.21).

Proof. First, let us prove that i) implies ii). From Theorem 5.3, there exists the maximal solution $X^+ \in \widehat{\mathbf{M}}(\widetilde{F})$ over $\mathbf{M}(\widetilde{F})$. Consider $X \in \mathbf{M}(\widetilde{F})$ and $V \in \mathbb{T}(\mathbb{H}^n)$ satisfying i). Set $F^+ = \breve{\mathcal{K}}(X^+)$ and $F = \breve{\mathcal{K}}(X)$. If $X^+ = X$, then it is easy to verify that $\mathcal{L}_{F^+} = \mathcal{N}_{\Gamma(X),\breve{\mathcal{K}}(X)}$ and the result is proved. Suppose that $X^+ \neq X$. Since from Lemma 4.1, $X^+ - \mathcal{L}_{F^+}(X^+) = \mathcal{I}$, we have from Equation (4.35) that

$$(X^+ - X) - \mathcal{L}_{F^+}(X^+ - X) = (F^+ - F)'\breve{\mathcal{R}}(X)(F^+ - F) + \mathcal{T}(X). \tag{5.42}$$

As seen in Theorem 5.3, $X(\widetilde{F}) \geq X^+ \geq X$ so that $\breve{\mathcal{R}}(X(\widetilde{F})) \geq \breve{\mathcal{R}}(X^+) \geq \breve{\mathcal{R}}(X) \geq 0$ and recalling that $X \in \mathbf{M}(\widetilde{F})$ implies that $\mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X(\widetilde{F})))$, we get that $\mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X(\widetilde{F}))) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X^+)) \subseteq \mathrm{Ker}(\breve{\mathcal{R}}(X))$ and thus $\mathrm{Ker}\ (\breve{\mathcal{R}}(X)) = \mathrm{Ker}(\breve{\mathcal{R}}(X^+))$. Noticing that $F' = -\breve{\mathcal{G}}(X)'\breve{\mathcal{R}}(X)^\dagger$, $F^{+\prime} = -\breve{\mathcal{G}}(X^+)'\breve{\mathcal{R}}(X^+)^\dagger$, and recalling that $\mathrm{Ker}(\breve{\mathcal{R}}(X)^\dagger) = \mathrm{Ker}(\breve{\mathcal{R}}(X))$ and that $\mathrm{Ker}(\breve{\mathcal{R}}(X^+)^\dagger) = \mathrm{Ker}(\breve{\mathcal{R}}(X^+))$, it follows that $\mathrm{Ker}(\breve{\mathcal{R}}(X)) \subseteq \mathrm{Ker}((F^+ - F)')$. From Lemma 4.4, we can find a small enough δ, with $0 < \delta < 1$, such that $(F^+ - F)'\breve{\mathcal{R}}(X)(F^+ - F) \geq \delta(F^+ - F)'(F^+ - F)$. Since $\mathcal{T}(X) \geq \delta\mathcal{T}(X) \geq 0$, we get from Equation (5.42) that

$$(X^+ - X) - \mathcal{L}_{F^+}(X^+ - X) \geq \delta((F^+ - F)'(F^+ - F) + \mathcal{T}(X)). \tag{5.43}$$

Considering $G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}$, $G \in \mathbb{T}(\mathbb{H}^n)$, $F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$, and $F^+ = \begin{bmatrix} F_1^+ & 0 \\ 0 & F_2^+ \end{bmatrix}$, define $\widehat{F}^+ \in \mathbb{H}^{2(m+n),2n}$, $\widehat{F} \in \mathbb{H}^{2(m+n),2n}$, $\widehat{B} \in \mathbb{H}^{2n,2(m+n)}$, and $\widehat{\widetilde{B}} \in \mathbb{H}^{2n,2(m+n)}$ as follows: $\widehat{F}^+ := \begin{bmatrix} 0 & 0 \\ F_1^+ & 0 \\ 0 & 0 \\ 0 & F_2^+ \end{bmatrix}$, $\widehat{F} := \begin{bmatrix} \mathcal{T}_1^{1/2} & 0 \\ F_1 & 0 \\ 0 & \mathcal{T}_2^{1/2} \\ 0 & F_2 \end{bmatrix}$, $\widehat{B} := \begin{bmatrix} G_1 & \bar{B} & 0 & 0 \\ 0 & 0 & G_2 & \bar{B} \end{bmatrix}$, and $\widehat{\widetilde{B}}_s := \begin{bmatrix} 0 & \widetilde{B}_s & 0 & 0 \\ 0 & 0 & 0 & \widetilde{B}_s \end{bmatrix}$. Consider the operator $\widehat{\mathcal{L}}_{\widehat{K}}$ as defined in Equation (4.26) replacing $\breve{B}$, $\breve{\widetilde{B}}_s$, and F by $\widehat{B}$, $\widehat{\widetilde{B}}_s$, and $\widehat{K} \in \mathbb{T}(\mathbb{H}^{2(m+n),2n})$, respectively. Thereby, it is easy to verify that $\widehat{\mathcal{L}}_{\widehat{F}^+} = \mathcal{L}_{F^+}$ and $\widehat{\mathcal{L}}_{\widehat{F}} = \mathcal{N}_{\Gamma(X),F}$. Since Equation (5.43) can be re-written as

$$(X^+ - X) - \widehat{\mathcal{L}}_{\widehat{F}^+}(X^+ - X) \geq \delta(\widehat{F}^+ - \widehat{F})'(\widehat{F}^+ - \widehat{F})$$

and recalling that $(X^+ - X) \geq 0$ and $r_\sigma(\widehat{\mathcal{L}}_{\widehat{F}}) = r_\sigma(\mathcal{N}_{\Gamma(X),F}) < 1$, we can conclude from Lemma 4.5 that $r_\sigma(\widehat{\mathcal{L}}_{\widehat{F}^+}) = r_\sigma(\mathcal{L}_{F^+}) < 1$, showing the first part. Let us prove now that ii) implies i). Consider that $X \in \widehat{\mathbf{M}}(\widetilde{F})$ is the mean-square stabilising solution to Equation (4.21). Then, since $\mathcal{T}(X) = 0$, $\Gamma(X) = \breve{\widetilde{A}}$ implying that $\mathcal{N}_{\breve{\widetilde{A}},\breve{\mathcal{K}}(X)} = \mathcal{L}_{\breve{\mathcal{K}}(X)}$ and the result follows since $r_\sigma(\mathcal{L}_{\breve{\mathcal{K}}(X)}) < 1$. □

5.2.2 Discounted and long-run problems

In this section, we present an optimal control strategy for the long-run and discounted problems as in Equations (3.25) and (3.26). See Table 2 for the equivalence between the different formulations.

5.2.2.1 Preliminaries

In this section, we recall and adapt some results related to the finite-horizon problem, with final time T. At each time $k \in \{1, \ldots, T\}$ and for any $(\bar{u}^{k-1}, v^{k-1}) \in \mathbb{M}(k-1)$, define the following intermediate problem:

$$J_k\left(\bar{x}(k), z(k), (\bar{u}^{k-1}, v^{k-1})\right) = \min_{(\bar{u}_k, v_k)\in\mathbb{V}(k)} \mathbb{E}\Big(\sum_{t=k}^{T-1} \nu(Lz(t))^2 - \xi V(t)\bar{x}(t) \\ + z(T)'P_T z(T) + \bar{x}(T)'M_T\bar{x}(T) - V_T\bar{x}(T)|\mathcal{F}_k\Big). \quad (5.44)$$

Recall the definitions of $\mathcal{A}$, $\mathcal{G}$, $\mathcal{R}$, $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{P}$ as in (4.19):

$$\begin{aligned}
\mathcal{A}(Z_1, Z_2) &= \bar{A}'Z_1\bar{A} + \sum_{s=1}^{\varepsilon} \widetilde{A}_s' Z_2 \widetilde{A}_s, \\
\mathcal{G}(Z_1, Z_2) &= \Big(\bar{A}'Z_1\bar{B} + \sum_{s_1=1}^{\varepsilon}\sum_{s_2=1}^{\varepsilon} \rho_{s_1,s_2}\widetilde{A}_{s_1}' Z_2 \widetilde{B}_{s_2}\Big)', \\
\mathcal{R}(Z_1, Z_2) &= \bar{B}'Z_1\bar{B} + \sum_{s=1}^{\varepsilon} \widetilde{B}_s' Z_2 \widetilde{B}_s, \\
\mathcal{K}(Z_1, Z_2) &= -\mathcal{R}(Z_1, Z_2)^{\dagger}\mathcal{G}(Z_1, Z_2), \\
\mathcal{M}(Z_1, Z_2) &= \mathcal{A}(Z_1, Z_2) - \mathcal{G}(Z_1, Z_2)'\mathcal{R}(Z_1, Z_2)^{\dagger}\mathcal{G}(Z_1, Z_2), \\
\mathcal{P}(Z_1) &= \mathcal{M}(Z_1, Z_1) + \nu L'L.
\end{aligned}$$

We define the following sequences, for $k = 0, 1, \ldots, T-1$:

$$P(k) = \mathcal{P}(P(k+1)), \ P(T) = P_T, \quad (5.45)$$

$$M(k) = \mathcal{M}(k+1, M(k+1), P(k+1)), \ M(T) = M_T, \quad (5.46)$$

$$V(k) = \mathcal{V}(k+1, M(k+1), P(k+1), V(k+1)), \ V(T) = V_T, \quad (5.47)$$

$$\gamma(k) = \mathcal{D}(M(k+1), P(k+1), V(k+1), \gamma(k+1)), \ \gamma(T) = 0, \quad (5.48)$$

where the non-linear operators $\mathcal{V}(.,.)$ and $\mathcal{D}(.,.,.)$ are as follows. For $Z_1, Z_2 \in \mathbb{H}^n$, $V \in$

$\mathbb{H}^{1,n}$, $\gamma \in \mathbb{H}^1$,

$$\mathcal{V}(t, Z_1, Z_2, V) = V\Big(\bar{A} - \bar{B}\mathcal{K}(Z_1, Z_2)\Big) + \xi V(t), \text{ and} \tag{5.49}$$

$$\mathcal{D}(Z_1, Z_2, V, \gamma) = \gamma - \frac{1}{4} V\bar{B}\mathcal{R}(Z_1, Z_2)^\dagger \bar{B}' V'. \tag{5.50}$$

We have the following result.

Proposition 5.2: Suppose that for $k = 0, \dots, T-1$,

$$\bar{B}' V(k+1)' \in \mathrm{Im}(\mathcal{R}(M(k+1), P(k+1))) \text{ and} \tag{5.51}$$

$$\mathcal{R}(M(k+1), P(k+1)) \geq 0. \tag{5.52}$$

Then, we have that

$$J_k\left(\bar{x}(k), z(k), (\bar{u}^{k-1}, v^{k-1})\right) = z(k)' P(k) z(k) + \bar{x}(k)' M(k)\bar{x}(k) - V(k)\bar{x}(k) + \gamma(k). \tag{5.53}$$

Moreover, the optimal control strategy for Problem (5.44) is given by

$$v^*(k) = K(k) z(k) \quad \text{and} \tag{5.54}$$

$$\bar{u}^*(k) = H(k)\bar{x}(k) + \frac{1}{2}\mathcal{R}(M(k+1), P(k+1))^\dagger \bar{B}' V(k+1)', \tag{5.55}$$

where

$$K(k) = \mathcal{R}(P(k+1), P(k+1))^\dagger \mathcal{G}(P(k+1), P(k+1)) \text{ and} \tag{5.56}$$

$$H(k) = \mathcal{R}(M(k+1), P(k+1))^\dagger \mathcal{G}(M(k+1), P(k+1)). \tag{5.57}$$

Proof. See Theorem 5.1 or Theorem 4.1 in (BARBIERI; COSTA, 2020b). □

The following lemma will be useful to prove that the optimal controls are admissible.

Lemma 5.3. *Set $U^{\bar{x}}(k) = \bar{x}(k)\bar{x}(k)'$ in (3.23) and $U^z(k) = \mathbb{E}(z(k)z(k)')$ in (3.24), with $\bar{u}(k) = K_1\bar{x}(k) + \frac{\alpha^{(k+1)/2}}{2}\phi$, $v(k) = K_2 z(k)$, and $\alpha \in (0, 1]$. We have that*

$$U^{\bar{x}}(k+1) = \mathcal{S}^{\bar{x}}_{K_1}(U^{\bar{x}}(k)) + \mathcal{O}^{\bar{x}}(\bar{x}(k)) \text{ and} \tag{5.58}$$

$$U^z(k+1) = \mathcal{S}_{K_2}(U^z(k)) + \mathcal{O}^z(U^{\bar{x}}(k), \bar{x}(k)), \tag{5.59}$$

with $\mathcal{S}_{K_2}(U^z(k))$ *as in Equation (4.24),*

$$
\begin{aligned}
&\mathcal{S}^{\bar{x}}{}_{K_1}(U^{\bar{x}}(k)) = (\bar{A} + \bar{B}K_1)U^{\bar{x}}(k)(\bar{A} + \bar{B}K_1)',\\
&\mathcal{O}^{\bar{x}}(\bar{x}(k)) = \frac{\alpha^{(k+1)/2}}{2}(\bar{B}K_1\bar{x}(k)\phi'\bar{B}' + \bar{B}\phi\bar{x}(k)'K_1'\bar{B}'\\
&\quad + \bar{A}\bar{x}(k)\phi'\bar{B}' + \bar{B}\phi\bar{x}(k)'\bar{A}') + \frac{\alpha^{(k+1)}}{4}\bar{B}\phi'\phi\bar{B}', \quad \textit{and}\\
&\mathcal{O}^{z}(U^{\bar{x}}(k), \bar{x}(k)) =\\
&\quad\sum_{s=1}^{\nu}\Big(\widetilde{A}_s U^{\bar{x}}(k)\widetilde{A}_s' + \widetilde{B}_s K_1 U^{\bar{x}}(k)K_1'\widetilde{B}_s' + \frac{\alpha^{(k+1)/2}}{2}\\
&\quad(\widetilde{B}_s K_1\bar{x}(k)\phi'\widetilde{B}_s' + \widetilde{B}_s\phi\bar{x}(k)'K_1'\widetilde{B}_s' + \frac{\alpha^{(k+1)/2}}{2}\widetilde{B}_s\phi'\phi\widetilde{B}_s')\Big)\\
&\quad+\sum_{s_1=1}^{\nu}\sum_{s_2=1}^{\varepsilon}\rho_{s_1,s_2}\Big(\widetilde{A}_{s_1}U^{\bar{x}}(k)K_1'\widetilde{B}_{s_2}' + \widetilde{B}_{s_2}K_1U^{\bar{x}}(k)\widetilde{A}_{s_1}'\\
&\quad+\frac{\alpha^{(k+1)/2}}{2}(\widetilde{A}_{s_1}\bar{x}(k)\phi'\widetilde{B}_{s_2}' + \widetilde{B}_{s_2}\phi\bar{x}(k)'\widetilde{A}_{s_1}')\Big).
\end{aligned}
$$

Moreover, if $r_\sigma(\bar{A}+\bar{B}K_1) < 1$ *and* $r_\sigma(\mathcal{S}_{K_2}) < 1$*, then* $U^{\bar{x}}(k) \to U^{\bar{x}}$ *and* $U^z(k) \to U^z$ *as* $k \to \infty$*, where* $U^{\bar{x}}$ *and* $U^z \in \mathbb{H}^n$ *are given by:*

i) $U^{\bar{x}} = 0$ *and* $U^z = 0$*, for* $0 < \alpha < 1$*, or*

ii) $U^{\bar{x}} = (I - \mathcal{S}_{K_1}^{\bar{x}})^{-1}\mathcal{O}^{\bar{x}}(\bar{x})$ *and* $U^z = (I - \mathcal{S}_{K_2})^{-1}\mathcal{O}^z(U^{\bar{x}}, \bar{x})$*, where* $\bar{x} = (I - \bar{A} - \bar{B}K_1)^{-1}\bar{B}\phi/2$*, for* $\alpha = 1$*.*

Proof. Consider the control law as defined above. Computing $\bar{x}(k+1)\bar{x}(k+1)'$ using Equation (3.23), we have that the terms with $U^{\bar{x}}(k)$ are captured by $\mathcal{S}_{K_1}^{\bar{x}}(U^{\bar{x}}(k))$ and the remaining terms are captured by $\mathcal{O}^{\bar{x}}(\bar{x}(k))$ so that Equation (5.58) holds. In the same way, calculating $\mathbb{E}(z(k+1)z(k+1)')$ using Equation (3.24) and recalling that $\mathbb{E}(z(k)) = 0$ and that $w_{s_1}^x$ and $w_{s_2}^u$ are white noises with mutual correlation ρ_{s_1,s_2}, we have that the terms with $U^z(k)$ are captured by $\mathcal{S}_{K_2}(U^z(k))$ and the remaining terms different than zero are captured by $\mathcal{O}^z(U^{\bar{x}}(k), \bar{x}(k))$ so that Equation (5.59) holds. Next, we show that $\lim_{k\to\infty} U^{\bar{x}}(k) < \infty$ and $\lim_{k\to\infty} U^z(k) < \infty$ for $\alpha \in (0, 1]$. We first prove the results for $\alpha \in (0, 1)$. From Equation (3.23) and $\bar{u}(k) = K_1\bar{x}(k) + \alpha^{(k+1)/2}\phi/2$, we obtain that $\bar{x}(k+1) = (\bar{A}+\bar{B}K_1)\bar{x}(k)+b(k)$, with $b(k) = \alpha^{(k+1)/2}\phi/2$. Notice that $\sum_{k=0}^{\infty}\sup_{\tau\geq 0}\|b(k+\tau)-b(k)\|$ $= \sum_{k=0}^{\infty}\sup_{\tau\geq 0}|\alpha^{k/2}(\alpha^\tau - 1)|\|\phi\|/2 < \|\phi\|/2(1-\alpha)$ and, thereby, $b(k)$ is a Cauchy summable sequence according to Proposition 2.8 in (COSTA; FRAGOSO; MARQUES, 2005). Given that $r_\sigma(\bar{A} + \bar{B}K_1) < 1$, from Proposition 2.9 in (COSTA; FRAGOSO; MARQUES, 2005), we obtain that $\bar{x}(k)$ is a Cauchy summable sequence and that $\bar{x} = \lim_{k\to\infty} \bar{x}(k) = (I - \bar{A} - \bar{B}K_1)^{-1} \lim_{k\to\infty} \alpha^{(k+1)/2}\bar{B}\phi/2 = 0$. Since $\bar{x}(k)$ is a Cauchy summable sequence, we can easily verify that $O^{\bar{x}}(\bar{x}(k))$ is also a Cauchy summable sequence. Therefore, given that $r_\sigma(\bar{A} + \bar{B}K_1) < 1$ implies that $r_\sigma(\mathcal{S}^{\bar{x}}_{K_1}) < 1$, from Proposition 2.9 in (COSTA; FRAGOSO; MARQUES, 2005) and Equation (5.58), we get that $U^{\bar{x}}(k)$ is a Cauchy summable sequence and that $U^{\bar{x}} = \lim_{k\to\infty} U^{\bar{x}}(k) = (I - \mathcal{S}^{\bar{x}}_{K_1})^{-1} \lim_{k\to\infty} O^{\bar{x}}(\bar{x}(k)) = 0$. Once more, since $\bar{x}(k)$ and $U^{\bar{x}}(k)$ are Cauchy summable sequences and given that $r_\sigma(\mathcal{S}_{K_2}) < 1$, we use the same reasoning as above to obtain that $O^z(U^{\bar{x}}(k), \bar{x}(k))$ is a Cauchy summable sequence and that $U^z = (I - \mathcal{S}_{K_2})^{-1} \lim_{k\to\infty} O^z(U^{\bar{x}}(k)(k), \bar{x}(k)) = 0$, proving (i). Repeating the same steps for $\alpha = 1$ we obtain that (ii) holds. □

5.2.2.2 Long-run average problem $PL(\nu, \xi)$

In this section, we assume that the mean square stabilising solution $Z = \begin{bmatrix} \bar{M} & 0 \\ 0 & \bar{P} \end{bmatrix}$ for Equation (4.21) exists. Set $\breve{\mathcal{K}} = \begin{bmatrix} \mathcal{K}(\bar{M}, \bar{P}) & 0 \\ 0 & \mathcal{K}(\bar{P}) \end{bmatrix}$. Since Z is the mean square stabilising solution, we have that $r_\sigma(\bar{A} + \bar{B}\mathcal{K}(\bar{M}, \bar{P})) < 1$ so that there exist a unique solution $\bar{V}$ satisfying

$$\bar{V} - \bar{V}(\bar{A} + \bar{B}\mathcal{K}(\bar{M}, \bar{P})) = \bar{V}(I - (\bar{A} + \bar{B}\mathcal{K}(\bar{M}, \bar{P})) = \xi L. \tag{5.60}$$

Considering $P_T = \bar{P}$ in (5.45), $M_T = \bar{M}$ in (5.46), $V_T = \bar{V}$ in (5.47) and $V(t) = L$ in (5.49), we get for all k that $P(k) = \bar{P}$, $M(k) = \bar{M}$ and $V(k) = \bar{V}$ from (5.60). We make the following assumption:

Assumption 5.1: We assume that $\bar{B}'\bar{V}' \in \mathrm{Im}(\mathcal{R}(\bar{M}, \bar{P}))$ and $\mathcal{R}(\bar{M}, \bar{P}) \geq 0$.

Notice that from Assumption 5.1, we have that the conditions of Proposition 5.2 are satisfied (see Equations (5.51) and (5.52)). We have the following theorem.

Theorem 5.5. *Suppose that the mean square stabilising solution* $Z = \begin{bmatrix} \bar{M} & 0 \\ 0 & \bar{P} \end{bmatrix}$ *for Equation (4.21) exists, and that Assumption 5.1 holds. Set* $\breve{\mathcal{K}} = \begin{bmatrix} \mathcal{K}(\bar{M}, \bar{P}) & 0 \\ 0 & \mathcal{K}(\bar{P}) \end{bmatrix}$ *and* $\bar{V}$ *as (5.60). We have that an optimal control strategy for Problem (3.25) is given by*

$\hat{u}(k) = \hat{v}(k) + \hat{\bar{u}}(k)$, *where*

$$\hat{v}(k) = \mathcal{K}(\bar{P})z(k) \quad \textit{and} \tag{5.61}$$

$$\hat{\bar{u}}(k) = \mathcal{K}(\bar{M}, \bar{P})\bar{x}(k) + \frac{1}{2}\mathcal{R}(\bar{M}, \bar{P})^{\dagger}\bar{B}'\bar{V}'. \tag{5.62}$$

Moreover, Problem (3.25) is well-posed and the optimal cost is

$$J_{PL}(\bar{x}(0), z(0)) = -\frac{1}{4}\bar{V}\bar{B}\mathcal{R}(\bar{M}, \bar{P})^{\dagger}\bar{B}'\bar{V}'.$$

Proof. Set for simplicity $J_k = J_k\left(\bar{x}(k), z(k), (\bar{u}^{k-1}, v^{k-1})\right)$ in (5.44). From Proposition 5.2 with $V(t) = L$ in (5.44) and (5.49), we have that

$$J_k\left(\bar{x}(k), z(k), (\bar{u}^{k-1}, v^{k-1})\right) = z(k)'\bar{P}z(k) + \bar{x}(k)'\bar{M}(k)\bar{x}(k) - \bar{V}\bar{x}(k) + \gamma(k), \tag{5.63}$$

with the optimal control strategy as in (5.54) and (5.55), which coincides with (5.61) and (5.62). From (5.48) we get that $\gamma(k) = \frac{T-k}{4}\bar{V}\bar{B}\mathcal{R}(\bar{M}, \bar{P})^{\dagger}\bar{B}'\bar{V}'$ so that, from (5.63), we obtain that

$$J_k = z(k)'\bar{P}z(k) + \bar{x}(k)'\bar{M}\bar{x}(k) - \bar{V}\bar{x}(k) - \frac{T-k}{4}\bar{V}\bar{B}\mathcal{R}(\bar{M}, \bar{P})^{\dagger}\bar{B}'\bar{V}'. \tag{5.64}$$

Then, considering $k = 0$ in (5.64) and from (5.44), we get that for any $(\bar{u}, v) \in \mathbb{V}_{av}$,

$$\begin{aligned} &\mathbb{E}(z_0'\bar{P}z_0) + \bar{x}_0'\bar{M}\bar{x}_0 - \bar{V}\bar{x}_0 - \frac{T}{4}\bar{V}\bar{B}\mathcal{R}(\bar{M}, \bar{P})^{\dagger}\bar{B}'\bar{V}' \leq \\ &\mathbb{E}\Big(\sum_{t=0}^{T-1} \nu(Lz(t))^2 - \xi L\bar{x}(t) + z(T)'\bar{P}z(T) + \bar{x}(T)'\bar{M}\bar{x}(T) - \bar{V}\bar{x}(T)\Big), \end{aligned} \tag{5.65}$$

with equality for the control strategy given by (5.61) and (5.62).

We can show now that $(\hat{\bar{u}}, \hat{v}) \in \mathbb{V}_{av}$. Set $K_1 = \mathcal{K}(\bar{M}, \bar{P})$, $K_2 = \mathcal{K}(\bar{P})$, and $\phi = \mathcal{R}(\bar{M}, \bar{P})^{\dagger}\bar{B}'\bar{V}'$, then Equations (5.58) and (5.59) hold for $v(k)$ and $\bar{u}(k)$ as in (5.61) and (5.62), respectively. Since Z is the mean-square stabilising solution, we have that $r_\sigma(\bar{A} + \bar{B}K_1) < 1$, $r_\sigma(\mathcal{S}_{K_1}^{\bar{x}}) < 1$, $r_\sigma(\mathcal{S}_{K_2}) < 1$, and for $\alpha = 1$, Lemma 5.3 item (ii) gives the convergence of $U^{\bar{x}}(k)$ and $U^z(k)$ to stationary matrices $U^{\bar{x}}$ and U^z as $k \to \infty$, respectively. Therefore, $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}(\|\bar{x}(T)\|^2) = \lim_{T\to\infty} \frac{1}{T}tr[U^{\bar{x}}(T)] = 0$ and $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}(\|z(T)\|^2) = \lim_{T\to\infty} \frac{1}{T}tr[U^z(T)] = 0$, showing that $(\hat{\bar{u}}, \hat{v}) \in \mathbb{V}_{av}$.

Since $(\bar{u}, v) \in \mathbb{V}_{av}$, we have that

$$\lim_{T\to\infty} \frac{1}{T} |\mathbb{E}(z(T)'\bar{P}z(T)) + \bar{x}(T)'\bar{M}\bar{x}(T) - \bar{V}\bar{x}(T)|$$
$$\leq \|\bar{P}\| \lim_{T\to\infty} \frac{1}{T}\mathbb{E}(\|z(T)\|^2) + \|\bar{M}\| \lim_{T\to\infty} \frac{1}{T}\|\bar{x}(T)\|^2 + \|\bar{V}\| \lim_{T\to\infty} \frac{1}{T}\|\bar{x}(T)\| = 0. \tag{5.66}$$

Thus, dividing Equation (5.65) by T, we get from (5.66), after taking the $\liminf$ as $T \to \infty$, that

$$-\frac{1}{4}\bar{V}\bar{B}\mathcal{R}(\bar{M}, \bar{P})^{\dagger}\bar{B}'\bar{V}' \leq \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\Big(\sum_{t=0}^{T-1} \nu(Lz(t))^2 - \xi L\bar{x}(t)\Big),$$

with equality for the strategy $(\hat{\bar{u}}, \hat{v}) \in \mathbb{V}_{av}$, completing the proof. □

5.2.2.3 Discounted problem $PD(\nu, \xi)$

In this section, we consider Problem (3.26) with a discount factor $\alpha \in (0, 1)$ as defined in Equation (3.29):

$$PD\,(\nu, \xi) : J_{PD}\,(\bar{x}(0), z(0)) = \inf_{(\bar{u},v)\in\mathbb{V}_\alpha} \liminf_{T\to\infty} \Big(\sum_{t=0}^{T-1} \mathbb{E}\Big(\nu(Lz^\alpha(t))^2 - \alpha^{t/2}\xi L\bar{x}^\alpha(t)\Big)\Big).$$

Recall from Section 3.4.2.1 that $\bar{A}^\alpha = \alpha^{1/2}\bar{A}$, $\bar{B}^\alpha = \alpha^{1/2}\bar{B}$, $\widetilde{A}^\alpha_s = \alpha^{1/2}\widetilde{A}_s$, $\widetilde{B}^\alpha_s = \alpha^{1/2}\widetilde{B}_s$, $\bar{A}^\alpha(k) = \alpha^{1/2}\bar{A}(k)$, $\bar{B}^\alpha(k) = \alpha^{1/2}\bar{B}(k)$, $z^\alpha(k) = \alpha^{k/2}z(k)$, $x^\alpha(k) = \alpha^{k/2}x(k)$, $\bar{x}^\alpha(k) = \alpha^{k/2}\bar{x}(k)$, $\bar{u}^\alpha(k) = \alpha^{k/2}\bar{u}(k)$, $v^\alpha(k) = \alpha^{k/2}v(k)$, and Equations (3.23), (3.24), and (3.26) respectively as:

$$\begin{aligned}
\bar{x}^\alpha(k+1) &= \bar{A}^\alpha\bar{x}^\alpha(k) + \bar{B}^\alpha\bar{u}^\alpha(k),\\
\bar{x}^\alpha(0) &= \bar{x}_0,\ k = 0, 1, \ldots,\\
z^\alpha(k+1) &= \Big(\bar{A}^\alpha + \sum_{s=1}^{\varepsilon^x} \widetilde{A}^\alpha_s w^x_s(k)\Big)z^\alpha(k) + \sum_{s=1}^{\varepsilon^x} \widetilde{A}^\alpha_s w^x_s(k)\bar{x}^\alpha(k) +\\
&\qquad \Big(\bar{B}^\alpha + \sum_{s=1}^{\varepsilon^u} \widetilde{B}^\alpha_s w^u_s(k)\Big)v^\alpha(k) + \sum_{s=1}^{\varepsilon^u} \widetilde{B}^\alpha_s w^u_s(k)\bar{u}^\alpha(k),\\
z^\alpha(0) &= z_0,\ k = 0, 1, \ldots.
\end{aligned}$$

Next, we adapt the operators notation to incorporate the discount factor in order to use the same reasoning and similar formulation as in the previous section to solve our problem PD. Define the operators $\mathcal{A}^\alpha$, $\mathcal{G}^\alpha$, $\mathcal{R}^\alpha$, $\mathcal{K}^\alpha$, $\mathcal{M}^\alpha$, $\mathcal{P}^\alpha$, and $\mathcal{T}^\alpha$ as in Equations

(4.19) and (4.29), replacing $\bar{A}$, $\bar{B}$, $\widetilde{A}_s$, and $\widetilde{B}_s$ by $\bar{A}^\alpha$, $\bar{B}^\alpha$, $\widetilde{A}^\alpha_s$, and $\widetilde{B}^\alpha_s$, respectively. We suppose in this subsection that the mean square stabilising solution $Z^\alpha = \begin{bmatrix} \bar{M}^\alpha & 0 \\ 0 & \bar{P}^\alpha \end{bmatrix}$ to the GCARE $\mathcal{T}^\alpha(Z^\alpha) = 0$ exists and we set $\breve{\mathcal{K}}^\alpha = \begin{bmatrix} \mathcal{K}^\alpha(\bar{M}^\alpha, \bar{P}^\alpha) & 0 \\ 0 & \mathcal{K}^\alpha(\bar{P}^\alpha) \end{bmatrix}$. Since Z^α is the mean square stabilising solution, we have that $r_\sigma(\bar{A}^\alpha + \bar{B}^\alpha \mathcal{K}^\alpha(\bar{M}^\alpha, \bar{P}^\alpha)) < 1$ so that there exist a unique solution $\bar{V}^\alpha$ satisfying

$$\bar{V}^\alpha - \alpha^{1/2}\bar{V}^\alpha(\bar{A}^\alpha + \bar{B}^\alpha \mathcal{K}^\alpha(\bar{M}^\alpha, \bar{P}^\alpha)) = \xi L. \tag{5.67}$$

By setting $P^\alpha_T = \bar{P}^\alpha$ and $M^\alpha_T = \bar{M}^\alpha$ in (5.45) and (5.46), respectively, we get that $P^\alpha(k) = \bar{P}^\alpha$ and $M^\alpha(k) = \bar{M}^\alpha$ for all k. Consider $V(t) = \alpha^{t/2}L$ in (5.49), $V^\alpha_T = \alpha^{T/2}\bar{V}^\alpha$, and set $V^\alpha(k)$ as in (5.49). Then from (5.67), we have that $V^\alpha(k) = \alpha^{k/2}\bar{V}^\alpha$. We make the following assumption.

Assumption 5.2: We assume that $\bar{B}^{\alpha\prime}\bar{V}^{\alpha\prime} \in \mathrm{Im}(\mathcal{R}(\bar{M}^\alpha, \bar{P}^\alpha))$ and $\mathcal{R}(\bar{M}^\alpha, \bar{P}^\alpha)) \geq 0$.

Notice that from Assumption 5.2, we have that the conditions of Proposition 5.2 (see (5.51), (5.52)) are satisfied. We have the following theorem.

Theorem 5.6. *Suppose that the mean square stabilising solution* $Z^\alpha = \begin{bmatrix} \bar{M}^\alpha & 0 \\ 0 & \bar{P}^\alpha \end{bmatrix}$ *to the GCARE* $\mathcal{T}^\alpha(Z^\alpha) = 0$ *exists and that Assumption 5.2 is satisfied. Then, an optimal control strategy for Problem (3.26) is given by* $\hat{u}(k) = \hat{v}(k) + \hat{\bar{u}}(k)$, *where*

$$\hat{v}(k) = \mathcal{K}^\alpha(\bar{P}^\alpha)z(k) \quad \text{and} \tag{5.68}$$

$$\hat{\bar{u}}(k) = \mathcal{K}^\alpha(\bar{M}^\alpha, \bar{P}^\alpha)\bar{x}(k) + \frac{\alpha^{1/2}}{2}\mathcal{R}^\alpha(\bar{M}^\alpha, \bar{P}^\alpha)^\dagger \bar{B}^{\alpha\prime}\bar{V}^{\alpha\prime}. \tag{5.69}$$

Furthermore, Problem (3.26) is well-posed and the optimal cost is

$$J_{PD}(\bar{x}(0), z(0)) = \mathbb{E}(z_0'\bar{P}^\alpha z_0) + \bar{x}_0'\bar{M}^\alpha\bar{x}_0 - \bar{V}^\alpha\bar{x}_0 - \frac{\alpha}{4(1-\alpha)}\bar{V}^\alpha\bar{B}^\alpha\mathcal{R}^\alpha(\bar{M}^\alpha, \bar{P}^\alpha)^\dagger\bar{B}^{\alpha\prime}\bar{V}^{\alpha\prime}. \tag{5.70}$$

Proof. Following the same reasoning as in Theorem 5.5, we have from Proposition 5.2

with $V(t) = \alpha^{t/2}L$ in (5.44) and (5.49), that

$$\begin{aligned}
J_k &= z^\alpha(k)'\bar{P}^\alpha z^\alpha(k) + \bar{x}^\alpha(k)'\bar{M}^\alpha\bar{x}^\alpha(k) - V^\alpha(k)\bar{x}^\alpha(k) + \gamma^\alpha(k)\\
&= z^\alpha(k)'\bar{P}^\alpha z^\alpha(k) + \bar{x}^\alpha(k)'\bar{M}^\alpha\bar{x}^\alpha(k) - \alpha^{k/2}\bar{V}^\alpha\bar{x}^\alpha(k) - \frac{1}{4}\bar{V}^\alpha\bar{B}^\alpha\mathcal{R}^\alpha(\bar{M}^\alpha,\bar{P}^\alpha)^\dagger\bar{B}^{\alpha\prime}\bar{V}^{\alpha\prime}\sum_{t=k}^{T}\alpha^t\\
&\le \mathbb{E}\Big(\sum_{t=k}^{T-1}\Big(\nu(Lz^\alpha(t))^2 - \xi\alpha^{t/2}L\bar{x}^\alpha(t)\Big) + z^\alpha(T)'\bar{P}^\alpha z^\alpha(T) + \bar{x}^\alpha(T)'\bar{M}^\alpha\bar{x}(T) - \alpha^{T/2}\bar{V}^\alpha\bar{x}^\alpha(T)\Big),
\end{aligned} \tag{5.71}$$

with equality in (5.71) when we take the optimal control given by Equations (5.68) and (5.69).

We can show now that $(\hat{\bar{u}},\hat{v}) \in \mathbb{V}_\alpha$. Set $K_1 = \mathcal{K}^\alpha(\bar{M}^\alpha,\bar{P}^\alpha)$, $K_2 = \mathcal{K}^\alpha(\bar{P}^\alpha)$, and $\phi = \mathcal{R}^\alpha(\bar{M}^\alpha,\bar{P}^\alpha)^\dagger\bar{B}^{\alpha\prime}\bar{V}^{\alpha\prime}$, then Lemma 5.3 holds substituting $\bar{A}$, $\bar{B}$, $\widetilde{A}$, $\widetilde{B}$ for $\bar{A}^\alpha$, $\bar{B}^\alpha$, $\widetilde{A}^\alpha$, $\widetilde{B}^\alpha$ and $\bar{x}(k)$, $z(k)$, $v(k)$, and $\bar{u}(k)$ as in (3.27), (3.28), (5.68), and (5.69), respectively. Since Z^α is the mean-square stabilising solution, we have that $r_\sigma(\bar{A}^\alpha + \bar{B}^\alpha K_1) < 1$, $r_\sigma(\mathcal{S}^{\bar{x}}_{K_1}) < 1$, $r_\sigma(\mathcal{S}_{K_2}) < 1$, and for $\alpha \in (0,1)$, Lemma 5.3 item (i) gives the convergence of $U^{\bar{x}}(k)$ and $U^z(k)$ to zero as $k \to \infty$. Therefore, $\lim_{T\to\infty}\frac{1}{T}\mathbb{E}(\|\bar{x}^\alpha(T)\|^2) = \lim_{T\to\infty}\frac{1}{T}tr[U^{\bar{x}}(T)] = 0$, $\lim_{T\to\infty}\frac{1}{T}\mathbb{E}(\|\bar{x}^\alpha(T)\|) = 0$, and $\lim_{T\to\infty}\frac{1}{T}\mathbb{E}(\|z^\alpha(T)\|^2) = \lim_{T\to\infty}\frac{1}{T}tr[U^z(T)] = 0$, showing that $(\hat{\bar{u}}^\alpha,\hat{v}^\alpha) \in \mathbb{V}_\alpha$.

Since $(\hat{\bar{u}},\hat{v}) \in \mathbb{V}_\alpha$ and recalling that $z^\alpha(t) = \alpha^{t/2}z(t)$ and $\bar{x}^\alpha(t) = \alpha^{t/2}\bar{x}(t)$, we have

$$\begin{aligned}
&\lim_{T\to\infty}|\mathbb{E}(z^\alpha(T)'\bar{P}^\alpha z^\alpha(T))| \le \|\bar{P}^\alpha\|\lim_{T\to\infty}\alpha^T\mathbb{E}(\|z(T)\|^2) = 0,\\
&\lim_{T\to\infty}\|\bar{x}^\alpha(T)'\bar{M}^\alpha\bar{x}(T)\| \le \|\bar{M}^\alpha\|\lim_{T\to\infty}\alpha^T\|\bar{x}(T)\|^2 = 0,\\
&\lim_{T\to\infty}\alpha^{t/2}\|L\bar{x}^\alpha(t)\| \le \|L\|\lim_{T\to\infty}\alpha^T\|\bar{x}(T)\| = 0,
\end{aligned} \tag{5.72}$$

and taking the $\liminf$ as T goes to infinity in (5.71) with $k = 0$, we get that, from (5.72), for any $(\bar{u},v) \in \mathbb{V}_\alpha$,

$$\begin{aligned}
&\mathbb{E}(z(0)'\bar{P}^\alpha z(0)) + \bar{x}(0)'\bar{M}^\alpha\bar{x}(0) - \bar{V}^\alpha\bar{x}(0) - \frac{1}{4(1-\alpha)}\bar{V}^\alpha\bar{B}^\alpha\mathcal{R}^\alpha(\bar{M}^\alpha,\bar{P}^\alpha)^\dagger\bar{B}^{\alpha\prime}\bar{V}^{\alpha\prime}\\
&\le \liminf_{T\to\infty}\mathbb{E}\Big(\sum_{t=0}^{T-1}\Big(\nu(Lz^\alpha(t))^2 - \xi\alpha^{t/2}L\bar{x}^\alpha(t)\Big)\Big) = \liminf_{T\to\infty}\mathbb{E}\Big(\sum_{t=0}^{T-1}\alpha^t\Big(\nu(Lz(t))^2 - \xi L\bar{x}(t)\Big)\Big),
\end{aligned} \tag{5.73}$$

with equality for the control $(\hat{\bar{u}},\hat{v}) \in \mathbb{V}_\alpha$, completing the proof. □

Remark 5.1: Lemma 5.3 allows us to analyse the behaviour of $x(k)$ as $k \to \infty$. As shown in Theorems 5.5 and 5.6, the optimal control law will assume the form as presented in Lemma 5.3 and the spectral radius conditions will be satisfied leading to $U^{\bar{x}}(k) \to U^{\bar{x}}$ and $U^{z}(k) \to U^{z}$ as $k \to \infty$. Therefore, since $\mathbb{E}\left(x(k)x(k)'\right) = U^{\bar{x}}(k) + U^{z}(k)$, when we apply the optimal control strategy $\bar{u}(k) = K_1\bar{x}(k) + \frac{\alpha^{(k+1)/2}}{2}\phi$ and $v(k) = K_2 z(k)$, as $k \to \infty$ we obtain that:

i) $\alpha^k \mathbb{E}\left(x(k)x(k)'\right) \to 0$ for $0 < \alpha < 1$ and

ii) $\mathbb{E}\left(x(k)x(k)'\right) \to (I - \mathcal{S}^{\bar{x}}_{K_1})^{-1}\mathcal{O}^{\bar{x}}(\bar{x}) + (I - \mathcal{S}_{K_2})^{-1}\mathcal{O}^{z}(U^{\bar{x}}, \bar{x})$ for $\alpha = 1$,

where $\bar{x} = (I - \bar{A} - \bar{B}K_1)^{-1}\bar{B}\phi/2$. It shows that the obtained optimal control strategy for Problem PL leads to the convergence of $\mathbb{E}\left(x(k)x(k)'\right)$ to a stationary matrix, while for Problem PD, it leads to the convergence of $\mathbb{E}(x^{\alpha}(k)x^{\alpha}(k)')$ to zero without $\mathbb{E}\left(x(k)x(k)'\right)$ necessarily converging. In fact, as we will show in the numerical example in Chapter 9, $\bar{x}(k)$ diverges to infinity while $\bar{x}^{\alpha}(k)$ converges to zero with probability 1, and $\mathbb{E}(\|x(k)\|^2) \to \infty$ while $\mathbb{E}(\|x^{\alpha}(k)\|^2) \to 0$. If the optimal control did not have a constant term, then $\mathbb{E}\left(x(k)x(k)'\right)$ would converge to zero for $\alpha = 1$ as can be easily verified from Lemma 9 with $\phi = 0$.

Remark 5.2: We could consider the vector output $y(t) \in \mathbb{R}^N$ and replace the cost $\nu Var(y(t)) - \xi E(y(t))$ by the cost

$$\sum_{i=1}^{N} \left(\nu Var(y_i(t)) - \xi \mathbb{E}(y_i(t))\right),$$

$$y(t) = \mathbb{L}x(t), \quad \mathbb{L} = [\mathbb{L}_1' \ \cdots \ \mathbb{L}_N']' \in \mathbb{H}^{N,n}$$

and applying the mean-field formulation, we would obtain that

$$\begin{aligned}
&\sum_{i=1}^{N} \mathbb{E}\left(\nu(\mathbb{L}_i z(t))^2 - \xi \mathbb{L}_i \bar{x}(t)\right) = \\
&\mathbb{E}(\nu z(t)'(\mathbb{L}_1'\mathbb{L}_1 + \cdots + \mathbb{L}_N'\mathbb{L}_N)z(t) - \xi(\mathbb{L}_1 + \cdots + \mathbb{L}_N)\bar{x}(t)) = \\
&\mathbb{E}\left(\nu z(t)'\mathbb{L}'\mathbb{L}z(t) - \xi\,[1 \ \cdots \ 1]\,\mathbb{L}\bar{x}(t)\right).
\end{aligned}$$

Therefore, we could apply our solution to this new cost by replacing $\nu L'L$ by $\nu\mathbb{L}'\mathbb{L}$ in Equations (4.19) and (4.20) and L by $(\mathbb{L}_1 + \cdots + \mathbb{L}_N)$ in Equations (5.60) and (5.67).

5.2.2.4 A numerical approach for the stabilising solution to the GCARE

In this subsection, we establish a link between a LMI (linear matrix inequality) optimisation problem and the maximal solution $X^+ \in$ **M**. This link provides a numerical way of obtaining stabilising solution to the GCARE and the optimal control laws as proposed in Theorems 5.5 and 5.6. In particular, using the Python library CVXPY and its class "Problem" and method "Solve", one could easily set the LMI as the constraint and maximise the unknown variable X. CVXPY relies on the open source solvers ECOS, OSQP, and SCS. For background on convex optimisation, see the book (DIAMOND; BOYD, 2016). Suppose that all matrices below are real and that $\mathbf{D}_+(\mathcal{T})$ is as defined previously. Consider the following convex optimisation problem:

$$\max \ tr(X) \ \text{ subject to } \begin{bmatrix} -X + \mathcal{I} + \breve{\mathcal{A}}(X) & \breve{\mathcal{G}}(X)' \\ \breve{\mathcal{G}}(X) & \breve{\mathcal{R}}(X) \end{bmatrix} \geq 0,$$

$$\breve{\mathcal{R}}(X) > 0. \tag{5.74}$$

Lemma 5.4. *Suppose that Equation (4.23) is mean square stabilisable. Then, there exist $X^+ \in \widehat{\mathbf{M}}$ such that $X^+ \geq X$ for all $X \in$ **M** if and only if there exists a solution $\widehat{X}$ for the above convex programming problem (5.74). Moreover, $\widehat{X} = X^+$.*

Proof. Note that, from Schur's complement, $X \in \mathbb{T}(\mathbb{H}^n)$ satisfies the restriction (5.74) if and only if $-X + \mathcal{I} + \breve{\mathcal{A}}(X) - \breve{\mathcal{G}}(X)'\breve{\mathcal{R}}(X)^\dagger\breve{\mathcal{G}}(X) \geq 0$ and $\breve{\mathcal{R}}(X) > 0$, that is, if and only if $X \in$ **M**. Thus, if $X^+ \in$ **M** is such that $X^+ > X$ for all $X \in$ **M** clearly $tr(X^+) \geq tr(X)$ for all $X \in$ **M** and, since $X^+ \in \widehat{\mathbf{M}} \subseteq \mathbf{M}$, it follows that X^+ is the solution of the convex programming problem (5.74). On the other hand, suppose that $\hat{X}$ is the solution of the convex programming problem (5.74). Thus $\hat{X} \in \mathbf{M} \neq \emptyset$ and from Theorem 5.4, there exists $X^+ \in$ **M** such that $X^+ \geq \hat{X}$. But from the optimality of X and the fact that $\widehat{\mathbf{M}} \subseteq \mathbf{M}$, we have that $tr(X^+ - X) \leq 0$. Since $X^+ - X \geq 0$, we have that $X^+ = X$. □

6 IMPLEMENTATION PROCEDURES FOR THE FINITE AND INFINITE-HORIZON CASES

In this chapter, we consolidate the results obtained above into chronological steps in order to compute the optimal control for each of our problems using the mean-field formulation. The reader is referred to Table 2 for the formulation equivalence of our problems. Section 6.1 shows the steps to obtain a stabilising optimal control law for the finite-horizon problems and Section 6.2 shows the steps for the infinite horizon cases. In the following sections, we are going to repeat some equations to easy the reading.

6.1 Implementation procedure for the finite-horizon case

In Section 6.1.1, we show the procedures to solve the general problem $PG\,(\nu, \xi, l, D)$ and then, in Section 6.1.2, we particularise the adaptations and extra steps to solve the unconstrained Problem PU and the constrained Problems $PL1$, $PL2$, and $PL3$.

6.1.1 General problem

To solve the general problem PG as in Equation (3.18):

$$\begin{aligned} PG\,(\nu, \xi, l, D) : J_0\,(\bar{x}(0), z(0)) = \min_{(\bar{u},\nu)\in\mathbb{V}} \sum_{t=0}^{T} \mathbb{E}\Big(&\nu(t)(L(t)z(t))^2 - (\xi(t) - l_V(t))L(t)\bar{x}(t) \\ &- l_M(t)(L(t)\bar{x}(t))^2 + l_D(t)\Big), \end{aligned}$$

we start by computing backwards the operators $\bar{\mathcal{M}}(k, P(k+1))$, $\mathcal{M}(k, M(k+1), P(k+1))$, $\mathcal{V}(k, M(k+1), P(k+1), V(k+1))$, and $\mathcal{D}(k, M(k+1), P(k+1), V(k+1), \gamma(k+1))$ using the definitions in (4.1), (4.2), and (4.3) for some $X, Y \in \mathbb{H}^n$:

$$\begin{aligned}
\mathcal{A}(k, X, Y) &= \bar{A}(k)' X \bar{A}(k) + \sum_{s=1}^{\varepsilon} \widetilde{A}_s(k)' Y \widetilde{A}_s(k), \\
\mathcal{G}(k, X, Y) &= \Big(\bar{A}(k)' X \bar{B}(k) + \sum_{s_1=1}^{\varepsilon} \sum_{s_2=1}^{\varepsilon} \rho_{s_1,s_2}(k) \widetilde{A}_{s_1}(k)' Y \widetilde{B}_{s_2}(k)\Big)', \\
\mathcal{R}(k, X, Y) &= \bar{B}(k)' X \bar{B}(k) + \sum_{s=1}^{\varepsilon} \widetilde{B}_s(k)' Y \widetilde{B}_s(k), \\
\mathcal{K}(k, X, Y) &= \mathcal{R}(k, X, Y)^{\dagger} \mathcal{G}(k, X, Y), \\
\mathcal{M}(k, X, Y) &= \mathcal{A}(k, X, Y) - \mathcal{G}(k, X, Y)' \mathcal{R}(k, X, Y)^{\dagger} \mathcal{G}(k, X, Y), \\
\bar{\mathcal{M}}(k, X, Y) &= \mathcal{M}(k, X, Y) - l_M(k) L(k)' L(k), \\
\mathcal{P}(k, X) &= \mathcal{M}(k, X, X) + \nu(k) L(k)' L(k), \\
\mathcal{V}(k, X, Y, V) &= V\Big(\bar{A}(k) - \bar{B}(k)\mathcal{K}(k, X, Y)\Big) + (\xi(k) - l_V(k)) L(k), \\
\mathcal{D}(k, X, Y, V, \gamma) &= \gamma - \frac{1}{4} V \bar{B}(k) \mathcal{R}(k, X, Y)^{\dagger} \bar{B}(k)' V' + l_D(k),
\end{aligned}$$

where $P(k)$, $M(k)$, $V(k)$, $\gamma(k)$, and their final values are defined as in Equations (4.4), (4.5), (4.6), and (4.7), respectively, for $k = T, T-1, \ldots, 0$:

$$\begin{aligned}
&P(k) = \mathcal{P}(k, P(k+1)), \quad P(T) = \nu(T) L(T)' L(T), \\
&M(k) = \bar{\mathcal{M}}(k, M(k+1), P(k+1)), \quad M(T) = -l_M(T) L(T)' L(T), \\
&V(k) = \mathcal{V}(k, M(k+1), P(k+1), V(k+1)), \quad V(T) = (\xi(T) - L_V(T)) L(T), \text{ and} \\
&\gamma(k) = \mathcal{D}(k, M(k+1), P(k+1), V(k+1), \gamma(k+1)), \quad \gamma(T) = l_D(T).
\end{aligned}$$

Then, using Equations (4.8) and (4.9) given by

$$\begin{aligned}
&K(k) = \mathcal{R}(k, P(k+1), P(k+1))^{\dagger} \mathcal{G}(k, P(k+1), P(k+1)) \text{ and} \\
&H(k) = \mathcal{R}(k, M(k+1), P(k+1))^{\dagger} \mathcal{G}(k, M(k+1), P(k+1)),
\end{aligned}$$

we can compute $v^*(k)$ and $\bar{u}^*(k)$, $k = 0, \ldots, T-1$, applying Equations (5.7) and (5.8):

$$\begin{aligned}
&v^*(k) = -K(k) z(k) \quad \text{and} \\
&\bar{u}^*(k) = -H(k) \bar{x}(k) + \frac{1}{2} \mathcal{R}(k, M(k+1), P(k+1))^{\dagger} \bar{B}(k)' V(k+1)'.
\end{aligned}$$

Finally, from Theorem 5.1, we obtain the optimal control of System (3.1) using the fact that $u^*(k) = v^*(k) + \bar{u}^*(k)$ and computing $\bar{x}(k)$ and $z(k)$ applying Equations 3.16 and

(3.17) as in

$$\bar{x}(k+1) = \bar{A}(k)\bar{x}(k) + \bar{B}(k)\bar{u}(k),$$
$$\bar{x}(0) = \bar{x}_0,\ \ k = 0, \dots, T-1,$$

and

$$\begin{aligned} z(k+1) = &\Big(\bar{A}(k) + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s(k) w_s^x(k)\Big) z(k) + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s(k) w_s^x(k) \bar{x}(k) + \\ &\Big(\bar{B}(k) + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s(k) w_s^u(k)\Big) v(k) + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s(k) w_s^u(k) \bar{u}(k), \\ z(0) = z_0,\ \ &k = 0, \dots, T-1. \end{aligned}$$

To check if conditions (4.14) and (4.15) are true in each step of the interaction we use Equation (4.17):

$$\begin{bmatrix} \bar{B}(k)V(k+1)\mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\bar{B}(k)'V(k+1)' & \bar{B}(k)V(k+1) \\ \bar{B}(k)V(k+1) & \mathcal{R}(k, M(k+1), P(k+1)) \end{bmatrix} \geq 0.$$

The expected output and its variance can be calculate using either Proposition 5.1 or a simulation for verification.

6.1.2 Unconstrained and constrained problems

To solve problems PU, $PL1$, $PL2$, and $PL3$:

$$PU\,(\nu, \xi) := \min_{(\bar{u}, v) \in \mathbb{V}} \sum_{t=0}^{T} \mathbb{E}\Big(\nu(t)(L(t)z(t))^2 - \xi(t)L(t)\bar{x}(t)\Big),$$

$$PL1(\omega) : J_0^{PL1}\,(\bar{x}(0), z(0)) = \min_{(\bar{u}, v) \in \mathbb{V}} \sum_{t=0}^{T} \mathbb{E}\Big(\nu(t)(L(t)z(t))^2 - \omega(t)L(t)\bar{x}(t) + \omega(t)\epsilon(t)\Big),$$

$$PL2(\omega) : J_0^{PL2}\,(\bar{x}(0), z(0)) = \min_{(\bar{u}, v) \in \mathbb{V}} \sum_{t=0}^{T} \mathbb{E}\Big(\omega(t)(L(t)z(t))^2 - \xi(t)L(t)\bar{x}(t) - \omega(t)\varphi(t)\Big),$$ and

$$PL3(\omega): \; J_0^{PL3}(\bar{x}(0), z(0)) = \min_{(\bar{u},v)\in\mathbb{V}} \sum_{t=0}^{T} \mathbb{E}\Big(\omega(t)\left(L(t)z(t)\right)^2 - \omega(t)a(t)\left(L(t)\bar{x}(t)\right)^2$$
$$- (\xi(t) - 2\omega(t)a(t)b(t))L(t)\bar{x}(t) - \omega(t)a(t)b(t)^2\Big),$$

we follow the same steps as in Section 6.1.1 to solve PG and apply the input parameters as shown in Table 1.

Parameter	PU	PL1	PL2	PL3
$v(k)$	$v(k)$	$v(k)$	$\omega(k)$	$\omega(k)$
$\xi(k)$	$\xi(k)$	$\omega(k)$	$\xi(k)$	$\xi(k)$
$l_V(k)$	0	0	0	$2\omega(k)a(k)$
$l_M(k)$	0	0	0	$\omega(k)a(k)$
$l_D(k)$	0	$\omega(k)\epsilon(k)$	$-\omega(k)\xi(k)$	$-\omega(k)a(k)b(k)^2$

Source: Author.

Note that for problems $PC1$, $PC2$, and $PC3$, we have to solve the Lagrangian dual problem $\max_{\omega\geq 0}\mathcal{H}$, where $\mathcal{H}(\omega) = PLi(\omega)$, $i = 1, 2$ or 3 by applying a search algorithm on ω using Equation (5.20) according to each problem PLi,

$$\mathcal{H}(\omega) = \bar{x}_0' M(0)\bar{x}_0 - V(0)\bar{x}_0 + \gamma(0),$$

for $x(0) = x_0$ and $z(0) = 0$. In this thesis, we adopt the Nelder-Mead simplex method to solve the Lagrangian problems, which is an available option of the Python optimisation function "scipy.optimize.sco.fmin".

In particular for Problem $PL1$, we can also obtain ω^* using Theorem 5.2 with

$$\omega^* = \mathbb{C}^{-1}(\epsilon - \mathbb{D}),$$

where, the elements in row r and column c of $\mathbb{C} \in \mathbb{H}^T$ and $\mathbb{D} \in \mathbb{H}^{T,1}$ are given by Equations (5.30) and (5.31):

$$\mathbb{C}_{r,c} = L(r) \sum_{i=0}^{\min(r,c)-1} \tilde{\mathbb{B}}(i, r) \left(\prod_{j=i+1}^{c-1} \mathbb{A}_H(j) \right)' L(c)' \text{ and}$$
$$\mathbb{D}_{r,1} = L(r) \prod_{j=0}^{r-1} \mathbb{A}_H(j) x_0,$$

with

$$\mathbb{A}_H(k) = \bar{A}(k) - \bar{B}(k)H(k),$$
$$\tilde{\mathbb{B}}(k,t) = \frac{1}{2}\left(\prod_{j=k+1}^{t-1} \mathbb{A}_H(j)\right)\bar{B}(k)\mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\bar{B}(k)',$$

for $k = 0, \ldots, T-1$, and $t = 1, \ldots, T$.

6.2 Implementation procedure for the infinite-horizon case

In Section 6.2.1, we show how to use our results to obtain a stabilising optimal control law for the long-run problem $PL(\nu, \xi)$ and, in Section 6.2.2, we adapt the same procedures to solve the discounted problem $PD(\nu, \xi)$.

Recall that the superscript $\breve{}$ applied on a matrix or an operator will represent them in the space $\mathbb{T}$ of appropriate dimension. For instance, $\breve{\mathcal{A}}(Z) = \begin{bmatrix} \mathcal{A}(X,Y) & 0 \\ 0 & \mathcal{A}(Y) \end{bmatrix}$, and when applied on a constant it will just repeat the constant in a block diagonal such as in $\breve{\bar{A}} = \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{bmatrix} \in \mathbb{T}(\mathbb{H}^n)$. In this way, we can write, for instance, $\breve{\mathcal{A}}(Z) = \begin{bmatrix} \mathcal{A}(X,Y) & 0 \\ 0 & \mathcal{A}(Y) \end{bmatrix} = \breve{\bar{A}}' \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \breve{\bar{A}} + \sum_{s=1}^{\varepsilon} \breve{\tilde{A}}'_s \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix} \breve{\tilde{A}}_s$.

6.2.1 Long-run problem

We first consider the long-run problem as in Equation (3.25):

$$PL(\nu, \xi) : J_{PL}(\bar{x}(0), z(0)) = \inf_{(\bar{u}, \nu) \in \mathbb{V}_{av}} \liminf_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\Big(\nu(Lz(t))^2 - \xi L\bar{x}(t)\Big).$$

To solve the stabilisation problem, we need to obtain a mean square stabilising solution to the GCARE $\mathcal{T}(Z) = 0$ defined in Equation (4.21) for $Z = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathbb{T}(\mathbb{H}^n)$:

$$\mathcal{T}(Z) = \begin{bmatrix} \mathcal{M}(X,Y) & 0 \\ 0 & \mathcal{P}(Y) \end{bmatrix} - Z = -Z + \mathcal{I} + \breve{\mathcal{A}}(Z) - \breve{\mathcal{G}}(Z)'\breve{\mathcal{R}}(Z)^{\dagger}\breve{\mathcal{G}}(Z) = 0,$$

where the operators in (4.19) are defined as

$$\begin{aligned}
\mathcal{A}(X, Y) &= \bar{A}'X\bar{A} + \sum_{s=1}^{\varepsilon} \widetilde{A}_s' Y \widetilde{A}_s, \\
\mathcal{G}(X, Y) &= \Big(\bar{A}'X\bar{B} + \sum_{s_1=1}^{\varepsilon} \sum_{s_2=1}^{\varepsilon} \rho_{s_1,s_2} \widetilde{A}_{s_1}' Y \widetilde{B}_{s_2}\Big)', \\
\mathcal{R}(X, Y) &= \bar{B}'X\bar{B} + \sum_{s=1}^{\varepsilon} \widetilde{B}_s' Y \widetilde{B}_s, \\
\mathcal{M}(X, Y) &= \mathcal{A}(X, Y) - \mathcal{G}(X, Y)'\mathcal{R}(X, Y)^{\dagger}\mathcal{G}(X, Y), \\
\mathcal{P}(X) &= \mathcal{M}(X, X) + \nu L'L.
\end{aligned}$$

We obtain the stabilising solution to the GCARE by solving the LMI Equation (5.74) numerically:

$$\max \ tr(Z) \ \text{ subject to } \begin{bmatrix} -Z + \mathcal{I} + \breve{\mathcal{A}}(Z) & \breve{\mathcal{G}}(Z)' \\ \breve{\mathcal{G}}(Z) & \breve{\mathcal{R}}(Z) \end{bmatrix} \geq 0,$$

$$\breve{\mathcal{R}}(Z) > 0.$$

Then, we can compute the gain $\breve{\mathcal{K}}(Z)$ as

$$\breve{\mathcal{K}}(Z) = \begin{bmatrix} \mathcal{K}(X, Y) & 0 \\ 0 & \mathcal{K}(Y) \end{bmatrix} = \begin{bmatrix} -\mathcal{R}(X, Y)^{\dagger}\mathcal{G}(X, Y) & 0 \\ 0 & -\mathcal{R}(Y)^{\dagger}\mathcal{G}(Y) \end{bmatrix}$$

.

In the next steps, we wish to compute $\bar{V}$. From Theorem 5.4 and $\mathcal{T}(Z) = 0$, we obtain that $\Gamma(X) = \breve{\bar{A}}$. It follows from the definitions of $\mathcal{N}_{\breve{\bar{A}},\breve{\mathcal{K}}}(Z)$ in (5.41) and $\mathcal{L}_{\breve{\mathcal{K}}}(Z)$ in (4.26) that $\mathcal{N}_{\breve{\bar{A}},\breve{\mathcal{K}}}(Z) = \mathcal{L}_{\breve{\mathcal{K}}}(Z)$ for $\Gamma(X) = \breve{\bar{A}}$, with

$$\mathcal{L}_{\breve{\mathcal{K}}}(Z) = \begin{bmatrix} \mathcal{L}_{\mathcal{K}}(X, Y) & 0 \\ 0 & \mathcal{L}_{\mathcal{K}}(Y) \end{bmatrix} = \breve{\mathcal{A}}(Z) + \breve{\mathcal{K}}(Z)'\breve{\mathcal{R}}(Z)\breve{\mathcal{K}}(Z) + \breve{\mathcal{K}}(Z)'\breve{\mathcal{G}}(Z) + \breve{\mathcal{G}}(Z)'\breve{\mathcal{K}}(Z).$$

Now, we are in a position to check whether $r_{\sigma}(\mathcal{L}_{\breve{\mathcal{K}}}) = r_{\sigma}(\mathcal{L}_{\breve{\mathcal{K}}}) < 1$, and if so there is a unique solution $\bar{V}$ satisfying (5.60):

$$\bar{V}(I - (\bar{A} + \bar{B}\mathcal{K}(X, Y)) = \xi L.$$

Finally, from Theorem 5.5, we can compute an stabilising optimal strategy for Problem PD in Equation (3.25) using Equations (5.61) and (5.62):

$$\hat{v}(k) = \mathcal{K}(Y)z(k) \quad \text{and}$$
$$\hat{\bar{u}}(k) = \mathcal{K}(X,Y)\bar{x}(k) + \frac{1}{2}\mathcal{R}(X,Y)^{\dagger}\bar{B}'\bar{V}'.$$

Similarly as in the finite case, we obtain the optimal control for System (3.12) using the fact that $\hat{u}(k) = \hat{v}(k) + \hat{\bar{u}}(k)$ and computing $\bar{x}(k)$ and $z(k)$ applying Equations (3.23) and (3.24) as in

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k),$$
$$\bar{x}(0) = \bar{x}_0, \ k = 0, 1, \ldots,$$

and

$$\begin{aligned} z(k+1) = {} & \Big(\bar{A} + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s w_s^x(k)\Big) z(k) + \sum_{s=1}^{\varepsilon^x} \widetilde{A}_s w_s^x(k)\bar{x}(k) + \\ & \Big(\bar{B} + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s w_s^u(k)\Big) v(k) + \sum_{s=1}^{\varepsilon^u} \widetilde{B}_s w_s^u(k)\bar{u}(k), \\ z(0) = {} & z_0, \ k = 0, 1, \ldots. \end{aligned}$$

To verify if Assumption 5.1 is true in each step of the interaction, we check if the adapted Equation (4.17) to our system's dynamics holds:

$$\begin{bmatrix} \bar{B}\bar{V}\mathcal{R}(X,Y)^{\dagger}\bar{B}'\bar{V}' & \bar{B}\bar{V} \\ \bar{B}\bar{V} & \mathcal{R}(X,Y) \end{bmatrix} \geq 0.$$

The expected output and its variance can be calculate using either Proposition 5.1 or a simulation for verification.

6.2.2 Discounted problem

Recall the definitions of $\bar{A}^{\alpha} = \alpha^{1/2}\bar{A}$, $\bar{B}^{\alpha} = \alpha^{1/2}\bar{B}$, $\widetilde{A}_s^{\alpha} = \alpha^{1/2}\widetilde{A}_s$, $\widetilde{B}_s^{\alpha} = \alpha^{1/2}\widetilde{B}_s$, $\bar{A}^{\alpha}(k) = \alpha^{1/2}\bar{A}(k)$, $\bar{B}^{\alpha}(k) = \alpha^{1/2}\bar{B}(k)$, $z^{\alpha}(k) = \alpha^{k/2}z(k)$, $x^{\alpha}(k) = \alpha^{k/2}x(k)$, $\bar{x}^{\alpha}(k) = \alpha^{k/2}\bar{x}(k)$, $\bar{u}^{\alpha}(k) = \alpha^{k/2}\bar{u}(k)$, and $v^{\alpha}(k) = \alpha^{k/2}v(k)$ leading to the new system dynamics as in (3.27) and (3.28).

The steps to obtain an stabilising optimal control law for Problem (3.26),

$$PD\,(\nu,\xi) : J_{PD}\,(\bar{x}(0), z(0)) = \inf_{(\bar{u},\nu)\in\mathbb{V}_\alpha} \liminf_{T\to\infty} \sum_{t=0}^{T-1} \alpha^t \mathbb{E}\Big(\nu(Lz(t))^2 - \xi L\bar{x}(t)\Big),$$

re-written as in (3.29),

$$PD\,(\nu,\xi) : J_{PD}\,(\bar{x}(0), z(0)) = \inf_{(\bar{u},\nu)\in\mathbb{V}_\alpha} \liminf_{T\to\infty} \Big(\sum_{t=0}^{T-1} \mathbb{E}\Big(\nu(Lz^\alpha(t))^2 - \alpha^{t/2}\xi L\bar{x}^\alpha(t)\Big)\Big),$$

follow the same procedures as described for the long-run problem in Section 6.2.1 with the following differences.

Define the operators $\mathcal{A}^\alpha$, $\mathcal{G}^\alpha$, $\mathcal{R}^\alpha$, $\mathcal{K}^\alpha$, $\mathcal{M}^\alpha$, $\mathcal{P}^\alpha$, and $\mathcal{T}^\alpha$ as in Equations (4.19) and (4.29), replacing $\bar{A}$, $\bar{B}$, $\widetilde{A}_s$, and $\widetilde{B}_s$ by $\bar{A}^\alpha$, $\bar{B}^\alpha$, $\widetilde{A}^\alpha_s$, and $\widetilde{B}^\alpha_s$, respectively.

Then, we compute the mean square stabilising solution $Z^\alpha = \begin{bmatrix} X^\alpha & 0 \\ 0 & Y^\alpha \end{bmatrix}$ to the GCARE $\mathcal{T}^\alpha(Z^\alpha) = 0$ by solving the adapted LMI Equation (5.74) numerically:

$$\max\ tr(Z^\alpha) \ \text{ subject to } \begin{bmatrix} -Z^\alpha + \mathcal{I} + \breve{\mathcal{A}}(Z^\alpha) & \breve{\mathcal{G}}^\alpha(Z^\alpha)' \\ \breve{\mathcal{G}}^\alpha(Z^\alpha) & \breve{\mathcal{R}}^\alpha(Z^\alpha) \end{bmatrix} \geq 0,$$

$$\breve{\mathcal{R}}^\alpha(Z^\alpha) > 0$$

and set $\breve{\mathcal{K}}^\alpha = \begin{bmatrix} \mathcal{K}^\alpha(X^\alpha, Y^\alpha) & 0 \\ 0 & \mathcal{K}^\alpha(Y^\alpha) \end{bmatrix}$. As beforfe and since Z^α is the mean square stabilising solution, we have that $r_\sigma(\bar{A}^\alpha + \bar{B}^\alpha\mathcal{K}^\alpha(X^\alpha, Y^\alpha)) < 1$ so that there exist a unique solution $\bar{V}^\alpha$ satisfying Equation (5.67):

$$\bar{V}^\alpha - \alpha^{1/2}\bar{V}^\alpha(\bar{A}^\alpha + \bar{B}^\alpha\mathcal{K}^\alpha(X^\alpha, Y^\alpha)) = \xi L.$$

Finally, from Theorem 5.6, we obtain the optimal control using the fact that $\hat{u}(k) = \hat{v}(k) + \hat{\bar{u}}(k)$ and Equations (5.68) and (5.69):

$$\hat{v}(k) = \mathcal{K}^\alpha(Y^\alpha)z(k) \quad \text{and}$$
$$\hat{\bar{u}}(k) = \mathcal{K}^\alpha(X^\alpha, Y^\alpha)\bar{x}(k) + \frac{\alpha^{1/2}}{2}\mathcal{R}^\alpha(X^\alpha, Y^\alpha)^\dagger \bar{B}^{\alpha\prime}\bar{V}^{\alpha\prime},$$

with $\bar{x}(k)$ and $z(k)$ as in (3.23) and (3.24), respectively.

7 PORTFOLIO MANAGEMENT MODEL

A specific problem of great interest regards the management of a portfolio of assets. This challenge is probably as old as the economy itself, but only with Markowitz, it was framed in proper technical terms and improved in many ways since then.

In this chapter, we show how to model the dynamics of a portfolio of assets using the notation describe in detail in Chapter 3, allowing us to use our results in control theory to find the optimal allocation of its assets.

This chapter is organised as follows: In Section 7.1 we cite some examples on how the complexity of portfolio management models evolved to meet specific needs such as the consideration of changes in expectation, use of benchmarks, and computation of cash flows. Then, in Section 7.2, we develop a portfolio selection formulation that matches the notation of our system.

7.1 Brief historical overview

The seminal works of Markowitz (MARKOWITZ, 1952; MARKOWITZ, 1959) verified the benefits of diversification and framed the asset allocation in a way to maximise the expected portfolio's return while minimising its variance. However, he incorporated expectations about the future in a single period and did not consider the liabilities and leverages.

Naturally, subsequent studies took into consideration more characteristics such as leverage (TOBIN, 1958), liabilities (SHARPE; TINT, 1990), and a multi-period investment horizon (MOSSIN, 1968; SAMUELSON, 1969; HAKANSSON, 1970).

A variety of portfolio planning models have been proposed and investigated besides the mean-variance model of Markowitz. They include the mean absolute variance, the weighted goal programming, the minimax model which use alternative metrics for risk, and the use of genetic algorithms for efficiently selecting a subset of stocks to trade. The reader is referred to (SATCHEL; SCOWCROFT, 2003) for detailed information on the subject.

Other relevant characteristics of portfolio management models include the possibility of considering a benchmark, cash flows within the investment period, and a risk-free security. The relevance of these characteristics becomes evident due to their practical applications, exemplified below.

Exchange-traded funds or pension funds with a mandate to track the return of an index is a classic example of a practical problem that led to the asset allocation formulation with a benchmark. There, the optimisation considers the maximisation of the excess return over the benchmark while minimising its variance.

Another example of model regards the Asset Liability Management (ALM) theory in which we must consider cash inflows and outflows besides the benchmark and risky assets. This type of model would be of great value for pension funds that must provide returns higher than inflation in a long time horizon while creating wealth to honour the actuarial liabilities.

A very relevant model described in detail in the next section regards a portfolio with risky assets and a reference security (potentially risk-free). This type of model is a typical application for portfolio management and, in Chapter 9, it was chosen to exemplify our results.

7.2 Model formulation

In this section, we will examine a portfolio of market securities against a benchmark. We consider m financial assets with random prices represented by the vector $\bar{S}(t) \in \mathbb{R}^m$,

$$\bar{S}(t) = [S_1(t), \ \ldots \ , S_m(t)]' \in \mathbb{R}^m, \tag{7.1}$$

with the first security representing a reference asset and consider a benchmark with random prices $B(t) \in \mathbb{R}$, $t = 0, 1, \ \ldots \ , T$.

Set the random return vector $\bar{R}(t) \in \mathbb{R}^{m+1}$ with relative returns as

$$\bar{R}(t) = \begin{bmatrix} R_1(t) \\ \widehat{R}(t) \\ R_{m+1}(t) \end{bmatrix}, \quad \widehat{R}(t) = \begin{bmatrix} R_2(t) \\ \vdots \\ R_m(t) \end{bmatrix}, \tag{7.2}$$

with $R_i(t) = \frac{S_i(t+1)}{S_i(t)}$, $i = 1, \ldots, m$, and $R_{m+1}(t) = \frac{B(t+1)}{B(t)}$ satisfying the following equation:

$$\bar{R}(t) = (\,\bar{e} + \bar{\mu}(t)\,) + \bar{\sigma}(t)w(t), \tag{7.3}$$

where $\bar{e} = [1, e]'$, $e \in \mathbb{R}^m$, is a vector with 1's in all its components, $\bar{\mu}(t) \in \mathbb{R}^{m+1}$ represents the expected returns of the assets, while $\bar{\sigma}(t)\bar{\sigma}(t)' \in \mathbb{R}^{m+1,m+1}$ is the covariance matrix of the returns.

The vectors $\{w(t)' = [w_1(t), \ldots, w_{m+1}(t)]; \quad t = 0, \ldots, T-1\}$ constitute a sequence of random and independent vectors of $m+1$ dimension with zero mean and covariance equal to the identity matrix.

For convenience, we write

$$\bar{\mu}(t) = \begin{bmatrix} \mu_1(t) \\ \widehat{\mu}(t) \\ \mu_{m+1}(t) \end{bmatrix}, \quad \widehat{\mu}(t) = \begin{bmatrix} \mu_2(t) \\ \vdots \\ \mu_m(t) \end{bmatrix}. \tag{7.4}$$

Repeating the decomposition above to $\bar{\sigma}(t)$, we have that

$$\bar{\sigma}(t) = \begin{bmatrix} \sigma_1(t) \\ \widehat{\sigma}(t) \\ \sigma_{m+1}(t) \end{bmatrix}, \tag{7.5}$$

where

$$\sigma_1(t) = \left[\sigma_{1,1}(t), \ldots, \sigma_{1,m+1}(t)\right], \tag{7.6}$$

$$\widehat{\sigma}(t) = \begin{bmatrix} \sigma_2(t) \\ \vdots \\ \sigma_m(t) \end{bmatrix} = \begin{bmatrix} \sigma_{2,1}(t) & \cdots & \sigma_{2,m+1}(t) \\ \vdots & \ddots & \vdots \\ \sigma_{m,1}(t) & \cdots & \sigma_{m,m+1}(t) \end{bmatrix}, \tag{7.7}$$

and

$$\sigma_{m+1}(t) = \left[\sigma_{m+1,1}(t), \ldots, \sigma_{m+1,m+1}(t)\right]. \tag{7.8}$$

Consider the wealth allocated to the i^{th} asset at time t as $U_i(t)$, $U(t) = [\,U_1(t), \ldots, U_m(t)\,]' = \begin{bmatrix} U_1(t) \\ u(t) \end{bmatrix}$.

Let $X^U(t)$ be the portfolio's value process associated with the investment strategy U at each $t = 0, 1, \ \ldots \ , T$. Suppressing the superscript ^{U} for simplicity and assuming $X(0) > 0$, $B(0) > 0$, then the portfolio's value at time t can be described as

$$X(t) = U_1(t) + u(t)'e', \tag{7.9}$$

and the wealth allocated in the reference asset will be given by

$$U_1(t) = X(t) - u(t)'e'. \tag{7.10}$$

Considering there are neither cash inflows nor cash outflows, the portfolio is self-financed and the wealth process is given by

$$X(t+1) = R_1(t)U_1(t) + \widehat{R}(t)'u(t) \tag{7.11}$$

and the benchmark process is given by

$$B(t+1) = R_{m+1}B(t). \tag{7.12}$$

Define the random vector

$$\eta(t) = \widehat{R}(t) - R_1(t)e' \tag{7.13}$$

and, as shown in Proposition 1.1.3 in (DAVIS; VINTER, 1985), we can write

$$\eta(t) = \mathbb{E}(\eta(t)) + \widehat{\sigma}(t)w(t), \quad \mathbb{E}(\eta(t)) = \widehat{\mu}(t) - \mu_1(t)e', \quad \mathbb{E}(w(t)) = 0, \quad cov(w(t)) = I, \tag{7.14}$$

with $\widehat{\sigma}(t) = cov(\eta(t))^{1/2}$.

Set $\widehat{\sigma}(t) = [\widehat{\sigma}^1(t), \ldots, \widehat{\sigma}^{m+1}(t)]$, that is, $\widehat{\sigma}^j(t)$ is the j^{th} column of $\widehat{\sigma}(t)$, and noticing that

$$\widehat{\sigma}(t)\widehat{\sigma}(t)' = \sum_{j=1}^{m+1} \widehat{\sigma}^j(t)\widehat{\sigma}^j(t)' = \left[\widehat{\sigma}^1(t), \quad \ldots \quad, \widehat{\sigma}^{m+1}(t)\right] \begin{bmatrix} \widehat{\sigma}^1(t)' \\ \vdots \\ \widehat{\sigma}^{m+1}(t)' \end{bmatrix},$$

the evolution of $X(t)$ and $B(t)$ in Equations (7.9), (7.11), and (7.12) can be represented as

$$X(t+1) = \left[(1+\mu_1(t)) + \sum_{s=1}^{m+1} \sigma_1(t)^s w^s(t)\right] X(t)$$
$$+ \left[(\widehat{\mu}(t) - \mu_1(t)e')' + \sum_{s=1}^{m+1} (\widehat{\sigma}^s(t) - \sigma_1^s(t)e')' w^s(t)\right] u(t), \tag{7.15}$$
$$B(t+1) = \left[(1+\mu_{m+1}(t)) + \sum_{s=1}^{m+1} \sigma_{m+1}^s(t) w^s(t)\right] B(t). \tag{7.16}$$

Setting $x(t) = [X(t), B(t)]'$, the first asset as the reference (or risk-free) security, and rearranging Equations (7.15) and (7.16), we recover Equations (3.1) and (3.2) by considering

$$\bar{A}(t) = \begin{bmatrix} 1+\mu_1(t) & 0 \\ 0 & 1+\mu_{m+1}(t) \end{bmatrix}, \quad \widetilde{A}_s(t) = \begin{bmatrix} \sigma_1^s(t) & 0 \\ 0 & \sigma_{m+1}^s(t) \end{bmatrix},$$
$$\bar{B}(t) = \begin{bmatrix} (\widehat{\mu}(t) - \mu_1(t)e')' \\ 0 \end{bmatrix}, \quad \widetilde{B}_s(t) = \begin{bmatrix} (\widehat{\sigma}^s(t) - \sigma_1^s(t)e')' \\ 0 \end{bmatrix},$$
$$\text{and } L = [1, \;\; -1]. \tag{7.17}$$

The system considered for the stabilisation problem is recovered considering the same arguments as above, but with constant expected returns and covariances over time leading to constant matrices in Equation (7.17). Note that the covariances are assumed to be homoscedastic within each step t for the finite horizon and homoscedastic over all steps for the infinite horizon.

8 COMPARISON WITH RESULTS IN THE CURRENT LITERATURE

In this chapter, we apply the results obtained in Section 5.1 to recover some known results analysed in (CUI; LI; LI, 2014) for the scalar portfolio selection problem using the mean-field formulation. In Section 8.1, we write the portfolio selection problem as the linear system with multiplicative noises introduced in Chapter 3, and show that the solution derived from Theorem 5.1 coincide with the one obtained in (CUI; LI; LI, 2014). In Section 8.2, we present the portfolio selection problem considering the risk control over the bankruptcy problem.

We start by recalling the following result, known as the Schur's complement.

Proposition 8.1: (Schur's complement) Suppose that $Q > 0$ and $R > 0$. The following assertions are equivalent.

a) $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0.$

b) $Q \geq SR^{-1}S'$.

c) $R \geq S'Q^{-1}S$.

Recalling that $\eta(k) = \widehat{R}(k) - R_1(k)e'$ as defined in Section 7.2 and defining $\mathcal{B}(k) = \mathbb{E}(\eta(k))'\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))$, we have the following auxiliary result.

Proposition 8.2: For given scalars X and $Y > 0$, if $\mathcal{B}(k)X + (1 - \mathcal{B}(k))Y > 0$ then

$$\mathcal{R}(k, X, Y) = \mathbb{E}(\eta(k)\eta(k)')Y + \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))'(X - Y) > 0. \tag{8.1}$$

Proof. From $\mathcal{B}(k)X + (1 - \mathcal{B}(k))Y > 0$ we have that $X - Y > \frac{-Y}{\mathcal{B}(k)}$ and thus

$$\mathbb{E}(\eta(k)\eta(k)')Y + \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))'(X - Y) \geq \left(\mathbb{E}(\eta(k)\eta(k)') - \frac{\mathbb{E}(\eta(k))\mathbb{E}(\eta(k))'}{\mathcal{B}(k)}\right)Y. \tag{8.2}$$

From the definition of $\mathcal{B}(k)$ and Schur's complement (Proposition 8.1), we have that

$$\begin{bmatrix} \mathbb{E}(\eta(k)\eta(k)') & \mathbb{E}(\eta(k)) \\ \mathbb{E}(\eta(k))' & \mathcal{B}(k) \end{bmatrix} \geq 0 \Leftrightarrow \left(\mathbb{E}(\eta(k)\eta(k)') - \frac{\mathbb{E}(\eta(k))\mathbb{E}(\eta(k))'}{\mathcal{B}(k)}\right) \geq 0. \tag{8.3}$$

Applying Lemma 4 in (CUI; LI; LI, 2014), we have that $\mathcal{R}(k,X,Y)$ has inverse since by assumption $\mathcal{B}(k)X + (1-\mathcal{B}(k))Y > 0$. From this and (8.3) and (8.2), we get (8.1). □

8.1 Portfolio selection considering problem PU

Let us consider an asset allocation model as described in Chapter 7 with m financial assets, one riskless asset ($\sigma_1 = 0$), and no benchmark ($\mu_{m+1} = 0$ and $\sigma_{m+1} = 0$). Define $s(k) = 1 + \mu_1(k)$ as the deterministic return of the riskless asset at period k and, as in Equation (7.13), set the random vector $\eta(k) = (1 + \widehat{R}(k)) - s(k)e'$.

As shown in (CUI; LI; LI, 2014), $cov(\eta(k)) > 0$ and thus $\mathbb{E}(\eta(k)\eta(k)') > 0$. Note that

$$\sum_{j=1}^{m} \sigma^j(k)\sigma^j(k)' = \begin{bmatrix}\sigma^1(k) & \dots & \sigma^m(k)\end{bmatrix}\begin{bmatrix}\sigma^1(k)' \\ \vdots \\ \sigma^m(k)'\end{bmatrix} = \widehat{\sigma}(k)\widehat{\sigma}(k)' = cov(\eta(k)). \tag{8.4}$$

Finally, set $\mathcal{B}(k) = \mathbb{E}(\eta(k))'\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))$. From Lemma 2 in (CUI; LI; LI, 2014), we have that

$$cov(\eta(k))^{-1}\mathbb{E}(\eta(k)) = \frac{\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))}{1-\mathcal{B}(k)}. \tag{8.5}$$

Using the formulation as in Chapter 7, we recover Equations (3.1) and (3.2) considering $\bar{A}(k) = s(k)$, $\widetilde{A}_s(k) = 0$, $\bar{B}(k) = \mathbb{E}(\eta(k))'$, $\widetilde{B}_j(k) = \sigma^j(k)'$, $\varepsilon = m$, and $L(k) = 1$ so that $y(k) = x(k)$.

In what follows, consider $l_M(k) = 0$, $l_V(k) = 0$, and $l_D(k) = 0$ for all $k = 1,\dots,T$. Since $\widetilde{A}_s(k) = 0$, we have from Remark 4.1 that $M(k) = 0$, $\mathcal{A}(k,X,Y) = s(k)^2X$, and $\mathcal{G}(k,X,Y) = s(k)\mathbb{E}(\eta(k))X$. From Equation (8.4),

$$\mathcal{R}(k,X,Y) = \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))'X + \sum_{j=1}^{n}\sigma^j(k)\sigma^j(k)'Y = \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))'X + cov(\eta(k))Y,$$

so that we have

$$\mathcal{R}(k,0,P(k+1)) = cov(\eta(k))P(k+1) > 0 \tag{8.6}$$

provided that $P(k+1) > 0$. Noticing that $\mathbb{E}(\eta(k)\eta(k)') = cov(\eta(k)) + \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))' =$

$cov(\eta(k)) + \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))'$, we have that for $Y > 0$,

$$\begin{aligned}\mathcal{P}(k, Y) &= s(k)^2 Y - s(k)^2 Y \mathbb{E}(\eta(k))'(\mathbb{E}(\eta(k))\mathbb{E}(\eta(k))' + cov(\eta(k)))^{-1}\mathbb{E}(\eta(k)) + \nu(k)\\ &= s(k)^2 Y(1 - \mathbb{E}(\eta(k))'\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))) + \nu(k)\\ &= s(k)^2 Y(1 - \mathcal{B}(k)) + \nu(k). \qquad (8.7)\end{aligned}$$

From this and Equation (4.4), we get that $P(k) > 0$ and $P(k) = s(k)^2(1-\mathcal{B}(k))P(k+1)+\nu(k)$, $k = 0, \ldots, T-1$, $P(T) = \nu(T)$. From Equation (8.6), we have that Assumption 4.1 holds true. Note now that, since $\mathcal{G}(k, M(k+1), P(k+1)) = \mathcal{G}(k, 0, P(k+1)) = 0$, we have from Equation (4.6) that

$$V(k) = \mathcal{V}(k, M(k+1), P(k+1), V(k+1)) = s(k)V(k+1) + \xi(k), \quad V(T) = \xi(T)$$

and since from Equation (8.5),

$$\begin{aligned}\bar{B}(k)\mathcal{R}(k, M(k+1), P(k+1))^{\dagger}\bar{B}(k)' &= \frac{1}{P(k+1)}\mathbb{E}(\eta(k))'cov(\eta(k))^{-1}\mathbb{E}(\eta(k))\\ &= \frac{\mathcal{B}(k)}{P(k+1)(1-\mathcal{B}(k))},\end{aligned}$$

we get from Equations (4.3) and (4.7) that for $k = T-1, \ldots, 0$,

$$\gamma(k) = \gamma(k+1) - \frac{V(k+1)^2}{4P(k+1)}\frac{\mathcal{B}(k)}{(1-\mathcal{B}(k))}, \quad \gamma(T) = 0.$$

Repeating the arguments above, we have from Equation (4.8) that

$$K(k) = s(k)(\mathbb{E}(\eta(k))\mathbb{E}(\eta(k))' + cov(\eta(k)))^{-1}\mathbb{E}(\eta(k)) = s(k)\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))$$

and from Equation (4.9) that $H(k) = 0$. From Equations (5.7), (5.8) and (8.5), we get that

$$\begin{aligned}v^*(k) &= -s(k)\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))(x(k) - \mathbb{E}(x(k))),\\ \bar{u}^*(k) &= \frac{V(k+1)}{2P(k+1)}cov(\eta(k))^{-1}\mathbb{E}(\eta(k)) = \Big(\frac{V(k+1)}{2P(k+1)}\Big)\frac{\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))}{1-\mathcal{B}(k)},\end{aligned}$$

and from Equation (5.6), we obtain that $J_k(\mathbb{E}(x(k)), x(k) - \mathbb{E}(x(k))) = P(k)(x(k)-\mathbb{E}(x(k)))^2 - V(k)\mathbb{E}(x(k)) + \gamma(k)$.

Finally, for problem $PU(\nu, \xi)$, we have that

$$\bar{A}(k) - \bar{B}(k)H(k) = s(k) - \mathbb{E}(\eta(k)')0 = s(k) \qquad (8.8)$$

and

$$\begin{aligned}\frac{1}{2}\bar{B}(k)R^{\dagger}(k)\bar{B}(k)'V(k+1)' &= \frac{\mathbb{E}(\eta(k)')cov(\eta(k))^{-1}\mathbb{E}(\eta(k))V(k+1)}{2P(k+1)}\\ &= \frac{\mathbb{E}(\eta(k)')\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))V(k+1)}{2(1-\mathcal{B}(k))P(k+1)}\\ &= \frac{\mathcal{B}(k)V(k+1)}{2(1-\mathcal{B}(k))P(k+1)}. \end{aligned} \tag{8.9}$$

Applying Equations (8.8) and (8.9) into Equation (5.22), we obtain that

$$\mathbb{E}\left(y^u(t)\right) = x_0\prod_{j=0}^{t-1}s(j) + \sum_{i=0}^{t-1}\left(\prod_{j=i+1}^{t-1}s(j)\right)\frac{\mathcal{B}(i)V(i+1)}{2((1-\mathcal{B}(i))\,P(i+1))}.$$

These results coincide with those obtained in Proposition 1 in (CUI; LI; LI, 2014).

8.2 Portfolio selection considering the risk control over bankruptcy

We now apply the results regarding the mean-variance with risk-control over a minimum expected output obtained in Section 5.1 to recover some known results analysed in (CUI; LI; LI, 2014) using the mean-field formulation. Let us consider a financial market as defined in Section 8.1 and a modification of problem $PC3(\omega)$ similar to the one in (CUI; LI; LI, 2014) and stated as

$$PC3\left(\xi,a,b\right): \max_{u\in\mathbb{U}}\Big(\xi(T)\mathbb{E}\left(y^u(T)\right) - \omega(T)Var\left(y^u(T)\right)\Big) \tag{8.10}$$

$$s.t.: Var\left(y^u(t)\right) \leqslant a(t)\left[\mathbb{E}\left(y^u(t)\right) - b(t)\right]^2. \tag{8.11}$$

Taking $L(t) = 1$, $\xi(t) = 0$, $t = 1,\dots,T-1$, $\xi(T) = 1$, and $a(T) = 0$, we get the problem as defined in Equation (3.11) for the Lagrangian multipliers $\omega(t)$, $t = 1,\dots,T-1$. We also have that $P(T) = \omega(T)$, $M(T) = 0$, $V(T) = 1$, and $\gamma(T) = 0$. Since $\widetilde{A}_s(k) = 0$, we have that $\mathcal{A}(k,X,Y) = s(k)^2X$ and $\mathcal{G}(k,X,Y) = s(k)\mathbb{E}(\eta(k))X$. From Equations (4.4) and (8.7), we have that $P(k) = s(k)^2(1-\mathcal{B}(k))P(k+1)+\omega(k)$, $k = 0,\dots,T-1$, $P(T) = \omega(T)$, and thus $P(k) > 0$. From Equation (8.4), we have that

$$\begin{aligned}\mathcal{R}(k,X,Y) &= \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))'X + \sum_{j=1}^{n}\sigma^j(k)\sigma^j(k)'Y = \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))'X + cov(\eta(k))Y\\ &= \mathbb{E}(\eta(k)\eta(k)')Y - \mathbb{E}(\eta(k))\mathbb{E}(\eta(k))'(Y-X). \end{aligned} \tag{8.12}$$

Applying Lemma 4 in (CUI; LI; LI, 2014), we have that

$$\mathcal{R}(k, M(k+1), P(k+1))^{-1}\mathbb{E}(\eta(k)) = \frac{\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))}{\mathcal{B}(k)M(k+1) + (1-\mathcal{B}(k))\,P(k+1)} \tag{8.13}$$

provided that $\mathcal{B}(k)M(k+1) + (1-\mathcal{B}(k))\,P(k+1) \neq 0.$ Define

$$\delta(k+1) = \frac{(1-\mathcal{B}(k))\,P(k+1)}{\mathcal{B}(k)M(k+1) + (1-\mathcal{B}(k))\,P(k+1)}.$$

Since $\mathcal{B}(k) = \mathbb{E}(\eta(k))'\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))$, we have from Equations (4.5) and (8.13),

$$\begin{aligned}
&M(k) = \mathcal{A}(k, X, Y) - \mathcal{G}(k, X, Y)'\mathcal{R}(k, X, Y)^{-1}\mathcal{G}(k, X, Y) - \omega(k)a(k)L(k)'L(k) \\
&= s(k)^2M(k+1) - \frac{s(k)^2M(k+1)^2\mathbb{E}(\eta(k))'\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))}{\mathcal{B}(k)M(k+1) + (1-\mathcal{B}(k))\,P(k+1)} - \omega(k)a(k) \\
&= \frac{s(k)^2M(k+1)^2\mathcal{B}(k) + s(k)^2M(k+1)\,(1-\mathcal{B}(k))\,P(k+1) - s(k)^2M(k+1)^2\mathcal{B}(k)}{\mathcal{B}(k)M(k+1) + (1-\mathcal{B}(k))\,P(k+1)} \\
&- \omega(k)a(k) = s(k)^2\delta(k+1)M(k+1) - \omega(k)a(k).
\end{aligned} \tag{8.14}$$

From Proposition 8.2, we have that if $\mathcal{B}(k)M(k+1)+(1-\mathcal{B}(k))\,P(k+1) > 0$ then Assumption 4.1 will hold and $M(k)$ is given by (8.14). From Equation (4.6),

$$\begin{aligned}
&V(k) = \\
&V(k+1)\Big(s(k) - \frac{s(k)M(k+1)\mathbb{E}(\eta(k))'\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))}{\mathcal{B}(k)M(k+1) + (1-\mathcal{B}(k))\,P(k+1)}\Big) - \omega(k)a(k)b(k) \\
&= V(k+1)\Big(\frac{s(k)\mathcal{B}(k)M(k+1) + s(k)\,(1-\mathcal{B}(k))\,P(k+1) - s(k)\mathcal{B}(k)M(k+1)}{\mathcal{B}(k)M(k+1) + (1-\mathcal{B}(k))\,P(k+1)}\Big) \\
&- \omega(k)a(k)b(k) = s(k)\delta(k+1)V(k+1) - \omega(k)a(k)b(k)
\end{aligned} \tag{8.15}$$

and, from Equation (4.7),

$$\begin{aligned}
\gamma(k) &= \gamma(k+1) + \frac{V(k+1)^2\mathbb{E}(\eta(k))'\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))}{\mathcal{B}(k)M(k+1) + (1-\mathcal{B}(k))\,P(k+1)} + \omega(k)a(k)b(k)^2 \\
&= \gamma(k+1) + \frac{V(k+1)^2\mathcal{B}(k)}{\mathcal{B}(k)M(k+1) + (1-\mathcal{B}(k))\,P(k+1)} + \omega(k)a(k)b(k)^2.
\end{aligned} \tag{8.16}$$

Note that repeating the arguments above, we have from Equation (4.8) that

$$K(k) = s(k)\mathbb{E}(\eta(k)\eta(k)')^{\dagger}\mathbb{E}(\eta(k)) \tag{8.17}$$

and from Equation (4.9) that

$$H(k) = \frac{s(k)M(k+1)\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))}{\mathcal{B}(k)M(k+1) + (1-\mathcal{B}(k))\,P(k+1)}. \tag{8.18}$$

From Equations (5.7) and (5.8), we get that

$$
\begin{aligned}
v^*(k) =& -s(k)\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))'(k),\\
\bar{u}^*(k) =& -\frac{s(k)M(k+1)\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))}{\mathcal{B}(k)M(k+1)+(1-\mathcal{B}(k))\,P(k+1)}\bar{x}(k)\\
&+\frac{V(k+1)\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))}{2\mathcal{B}(k)M(k+1)+(1-\mathcal{B}(k))\,P(k+1)}\\
=& \frac{0.5V(k+1)-s(k)M(k+1)\bar{x}(k)}{\mathcal{B}(k)M(k+1)+(1-\mathcal{B}(k))\,P(k+1)}\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))
\end{aligned}
$$

and, from Equation (5.20), we obtain that

$$
\begin{aligned}
\mathcal{H}(\omega) =& M(1)\delta(1)s(0)^2x(0)^2 - V(1)\delta(1)s(0)x(0)\\
&-\sum_{j=0}^{T-1}\left[\frac{V(j+1)^2\mathcal{B}(j)}{\mathcal{B}(j)M(j+1)+(1-\mathcal{B}(j))\,P(j+1)}+\omega(j)a(j)b(j)^2\right].
\end{aligned}
$$

Finally, we apply Equations (5.22), (5.23), and the operators in Equation (5.21) to recover the expected output and its variance formulas obtained in (CUI; LI; LI, 2014). For problem $PL3(\omega)$, we have that

$$
\begin{aligned}
\bar{A}(k)-\bar{B}(k)H(k) =& s(k)-\frac{s(k)M(k+1)\mathbb{E}(\eta(k)')\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))}{\mathcal{B}(k)M(k+1)+(1-\mathcal{B}(k))\,P(k+1)}\\
=& \frac{(1-\mathcal{B}(k))\,P(k+1)}{\mathcal{B}(k)M(k+1)+(1-\mathcal{B}(k))\,P(k+1)}s(k) = \delta(k+1)s(k) \qquad (8.19)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}\bar{B}(k)R^\dagger(k)\bar{B}(k)'V(k+1)' =& \frac{\mathbb{E}(\eta(k)')\mathbb{E}(\eta(k)\eta(k)')^{-1}\mathbb{E}(\eta(k))V(k+1)}{2(\mathcal{B}(k)M(k+1)+(1-\mathcal{B}(k))\,P(k+1))}\\
=& \frac{\mathcal{B}(k)V(k+1)}{2(\mathcal{B}(k)M(k+1)+(1-\mathcal{B}(k))\,P(k+1))}. \qquad (8.20)
\end{aligned}
$$

Applying Equations (8.19) and (8.20) into Equation (5.22), we obtain that

$$
\begin{aligned}
\mathbb{E}\left(y^u(t)\right) =& x_0\prod_{j=0}^{t-1}\delta(j+1)s(j)\\
&+\sum_{i=0}^{t-1}\left(\prod_{j=i+1}^{t-1}\delta(j+1)s(j)\right)\frac{\mathcal{B}(i)V(i+1)}{2(\mathcal{B}(i)M(i+1)+(1-\mathcal{B}(i))\,P(i+1))}.
\end{aligned}
$$

From Equations (5.21) and (5.23), we obtain that

$$Var\,(y^u(t)) =$$
$$\sum_{j=0}^{t-1} \Big[\frac{(0.5V(j+1) - s(j)M(j+1)\bar{x}(j))^2 \left(\mathcal{B}(j) - \mathcal{B}(j)^2\right)}{(\mathcal{B}(j)M(k+1) + (1-\mathcal{B}(j))\,P(j+1))^2} \Big] \prod_{l=j+1}^{t-1} s(l)^2\,(1-\mathcal{B}(l))\,.$$

These results coincide with those obtained in Section IV in (CUI; LI; LI, 2014).

9 NUMERICAL EXAMPLES

In this chapter, we illustrate the application of our results in the management of a portfolio of financial assets against a benchmark. Section 9.1 shows examples regarding the unconstrained and constrained problems in finite-horizon while, in Section 9.2, we present a simulation regarding the stabilisation of the discounted problem with infinite horizon.

We consider the Brazilian market with the reference asset represented by the CDI ("Cédula de Crédito Interbancário") and the benchmark represented by the inflation index IPCA ("Índice de Preços ao Consumidor Amplo"). The risk assets are represented by the Ibovespa stock market index (IBOV), the fixed income index (IRF-M), the US dollar versus the Brazilian reais (PTAX), and the gold. Table 3 shows the Bloomberg's ticker applied to retrieve the historical data used in our simulations and the assigned index to each security as in our formulation in Chapter 7.

Table 3: Securities used in our simulations.

Index	Security	Ticker
1	Reference asset - CDI	BZACCETP
2	Equity market index - Ibovespa	IBOV
3	Fixed income index - IRF-M	BZRFIRFM
4	Exchange rate - R$/US$	BZFXPINT
5	Gold	XAU BGN
6	Benchmark - IPCA	BZPIIPCA

Source: Bloomberg.

We obtained the historical prices from February, 3^{rd} 2006 to February, 21^{st} 2020 in a interval of seven days, except for the IPCA, which is a monthly index. It led to 733 weekly returns assuming the IPCA has constant weekly returns within each month.

The weekly expected returns and their covariance matrix are shown below in Equations (9.1) and (9.2) for $k = 0, \ldots, T-1,$.

$$\mu(k) = [0.09 \;\; 0.16 \;\; 0.23 \;\; 0.09 \;\; 0.14 \;\; 0.05]' \; 10^{-2}, \tag{9.1}$$

$$\sigma(k)\sigma(k)' = \begin{bmatrix} 0.003 & -0.006 & 0.003 & -0.0007 & 0.002 & 0.0002 \\ -0.006 & 11.82 & 0.50 & -4.49 & 1.61 & -0.01 \\ 0.003 & 0.50 & 0.16 & -0.34 & 0.11 & -0.002 \\ -0.0007 & -4.49 & -0.34 & 4.29 & -1.18 & 0.001 \\ 0.002 & 1.61 & 0.11 & -1.18 & 5.91 & -0.003 \\ 0.0002 & -0.01 & -0.002 & 0.001 & -0.003 & 0.004 \end{bmatrix} 10^{-4}. \tag{9.2}$$

Finally, from Equations (9.1) and (9.2), we obtain the system dynamics using (7.17):

$$\bar{A}(t) = \begin{bmatrix} 1+\mu_1(t) & 0 \\ 0 & 1+\mu_{m+1}(t) \end{bmatrix}, \quad \widetilde{A}_s(t) = \begin{bmatrix} \sigma_1^s(t) & 0 \\ 0 & \sigma_{m+1}^s(t) \end{bmatrix},$$

$$\bar{B}(t) = \begin{bmatrix} (\widehat{\mu}(t) - \mu_1(t)e')' \\ 0 \end{bmatrix}, \quad \widetilde{B}_s(t) = \begin{bmatrix} (\widehat{\sigma}^s(t) - \sigma_1^s(t)e')' \\ 0 \end{bmatrix},$$

and $L = [1, \ -1]$.

For all problems, we set $x_0 = [1.0 \ \ 1.0]'$ and $\rho_{s_1,s_2}(k) = 1$ for $s_1 = s_2$ and 0 otherwise. The finite-horizon problems will have a time horizon of $T = 5$ weeks.

In the following sections, we solve the finite and infinite-horizon control problems.

9.1 Constrained and unconstrained finite-horizon control

We solve problems PU, $PC1$, $PC2$, and $PC3$ by applying Theorem 5.1 together with Table 1 and the risk coefficients and their respective restrictions as in Table 4 for $t = 1, \ldots, T$.

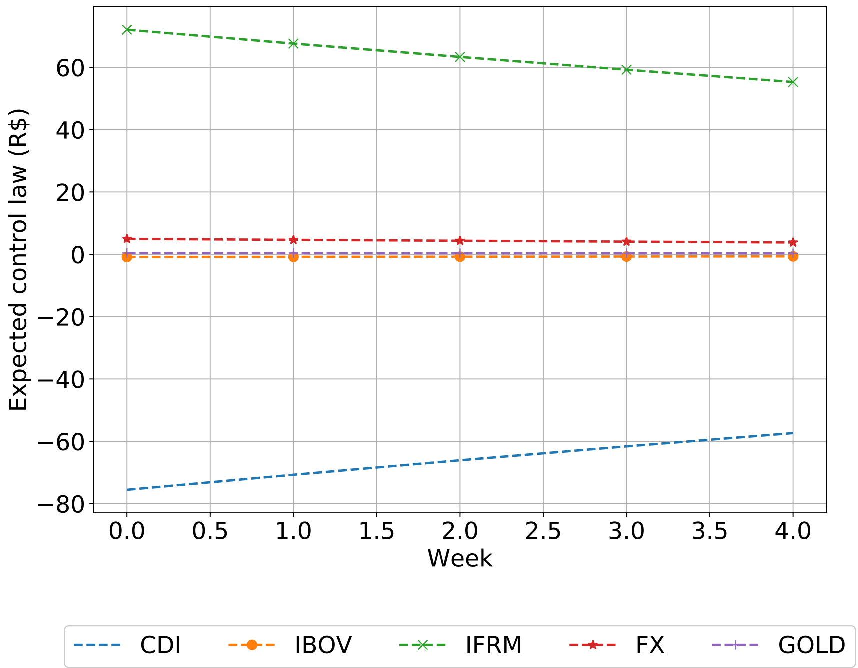
Table 4: Risk and restrictions coefficients.

Problem	$\nu(t)$	$\xi(t)$	Restriction
PU	1	1	-
PL1	1	$\omega(t)$	$\epsilon = [0.12,\ 0.2,\ 0.3,\ 0.4,\ 0.5]'$
PL2	$\omega(t)$	1	$\varphi = [0.005,\ 0.01,\ 0.01,\ 0.015,\ 0.015]'$
PL3	$\omega(t)$	1	$a(t) = 0.05$ and $b(t) = 0.1$

Source: Author.

To solve problem $PU(\nu, \xi)$, we follow the same steps as described in Sections 6.1.1 and 6.1.2 and start by computing backwards the operators in Equations (4.1), (4.2), and (4.3) using the definitions as in Equations (4.4), (4.5), (4.6), and (4.7). Then, using Equations (4.8) and (4.9), we can compute $v^*(k)$ and $\bar{u}^*(k)$, $k = 0, \ldots, T-1$, applying Equations (5.7) and (5.8). Figure 1 shows the expected optimal control for the simulation.

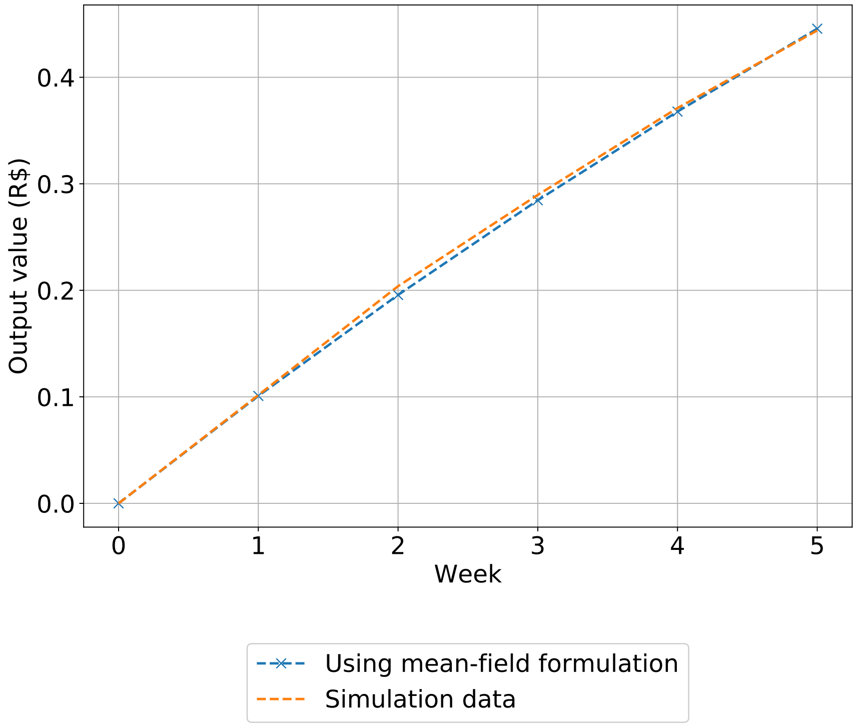
Figure 1: Expected optimal control law for the unconstrained problem PU.



Source: Author.

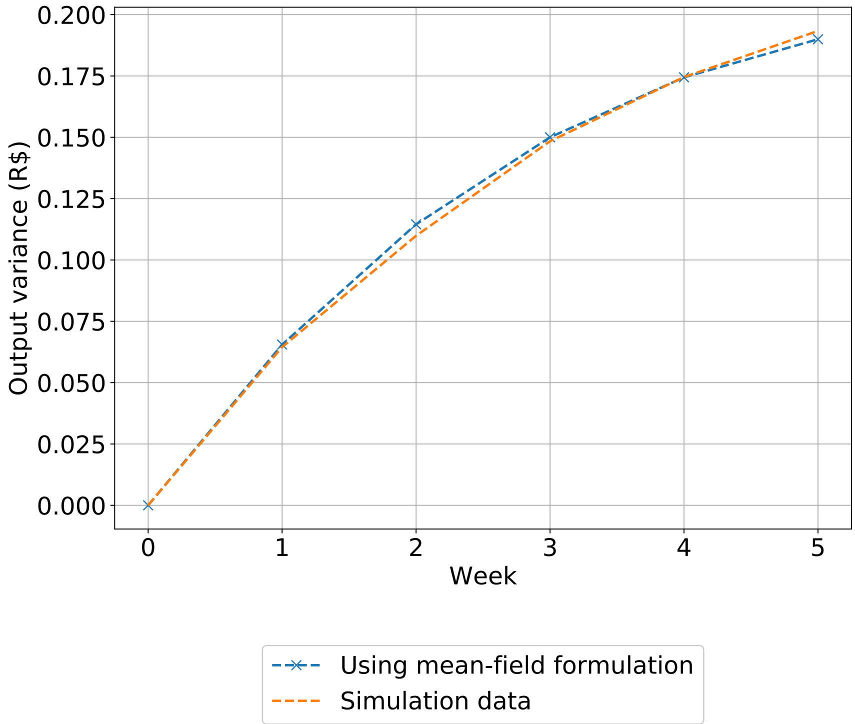
Finally, the expected output and variance is calculated using Proposition 5.1 and simulation data leading to the results in Figures 2 and 3.

Figure 2: Expected output for the unconstrained problem PU versus simulation data.



Source: Author.

Figure 3: Output variance for the unconstrained problem PU versus simulation data.



Source: Author.

In order to compare our results, we also solve PU using an embedding scheme as applied in (COSTA; OLIVEIRA, 2012), where an auxiliary problem parameterised in λ is solved. This technique led to exact the same optimal control law, expected output and variance as before, corroborating our formulation.

In problems $PC1$, $PC2$, and $PC3$, we solved the Lagrangian dual problem $PCi = \max_{\omega \geq 0} \mathcal{H}(\omega)$, where $\mathcal{H}(\omega) = PLi(\omega)$, $i = 1, 2$, or 3, is given by Equation (5.20) with their respective input parameters as in Table 1. In this thesis, we adopt the Nelder-Mead simplex method to solve the Lagrangian problems, which is an available option of the Python optimisation function "scipy.optimize.sco.fmin".

The resulting Lagrangian multipliers for each problem are shown in Table 5.

Table 5: Lagrangian multipliers.

Problem	$\omega^{*\prime}$
PL1	$[2.581,\ 0,\ 0.972,\ 1.118,\ 1.290]$
PL2	$[4.837,\ 0,\ 11.940,\ 1.112,\ 4.413]$
PL3	$[0,\ 33.034,\ 31.923,\ 30.672,\ 29.271]$

Source: Author.

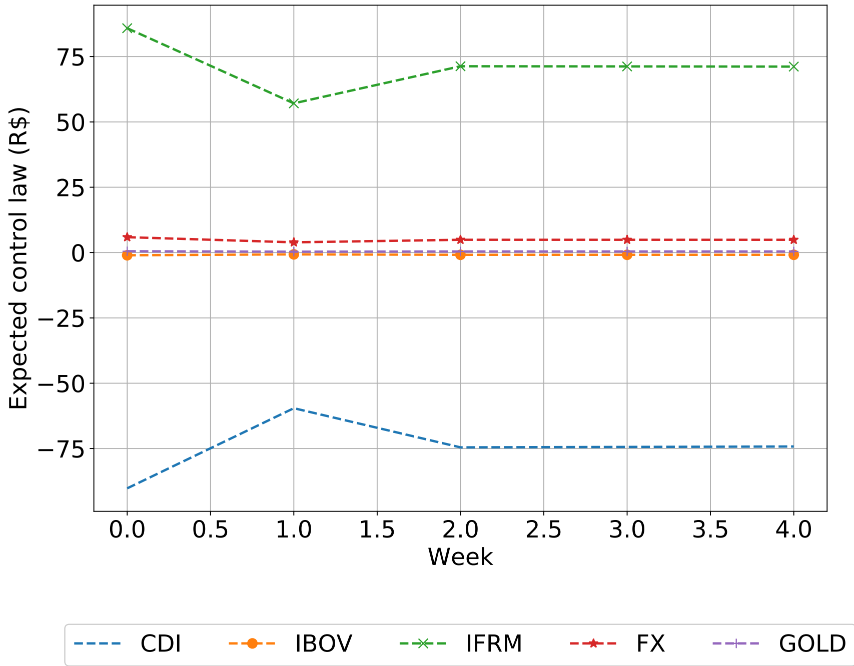
In the case of problem $PC1(\omega)$, we can also obtain ω^* analytically. Thus, applying Theorem 5.2, we obtain that

$$\mathbb{C} = \begin{bmatrix} 2.007 & 2.009 & 2.010 & 2.012 & 2.014 \\ 2.009 & 4.365 & 4.369 & 4.373 & 4.377 \\ 2.010 & 4.369 & 7.315 & 7.321 & 7.328 \\ 2.012 & 4.373 & 7.321 & 11.456 & 11.466 \\ 2.014 & 4.377 & 7.328 & 11.466 & 19.188 \end{bmatrix} 10^{-2},$$

$det(\mathbb{C}) = 4.43 \times 10^{-8}$, and $\mathbb{D} = [0.378,\ 0.757,\ 1.136,\ 1.516,\ 1.897]'\, 10^{-3}$. Finally, applying Equation (5.32), we get the same ω^* as in Table 5 for $PL1$, corroborating our results.

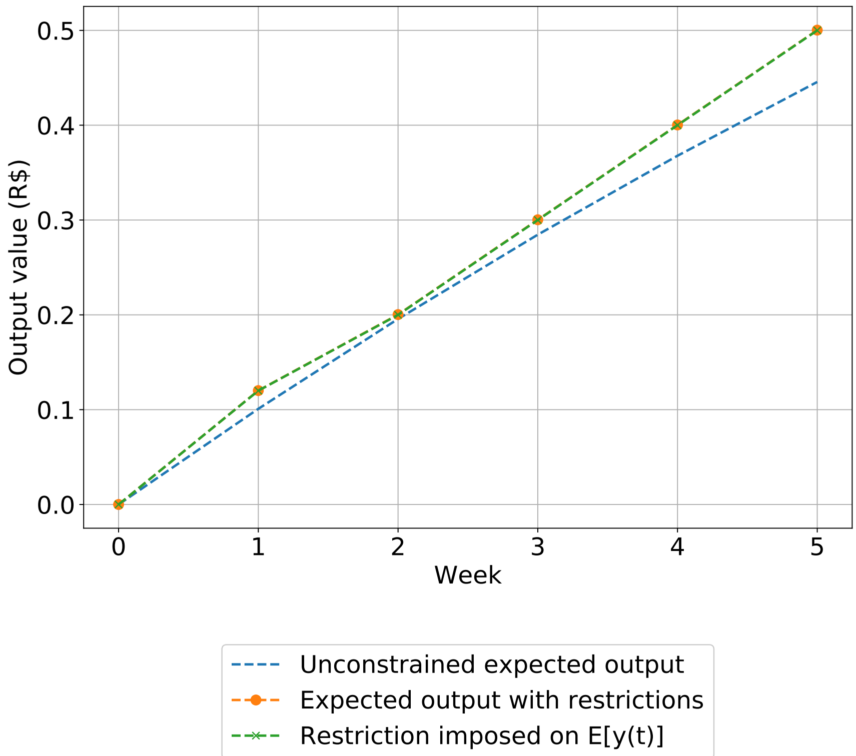
Figures 4 to 12 show the expected optimal control, output and its variance for each problem PL1, PL2, and PL3.

Figure 4: Expected optimal control law for the constrained problem PC1.



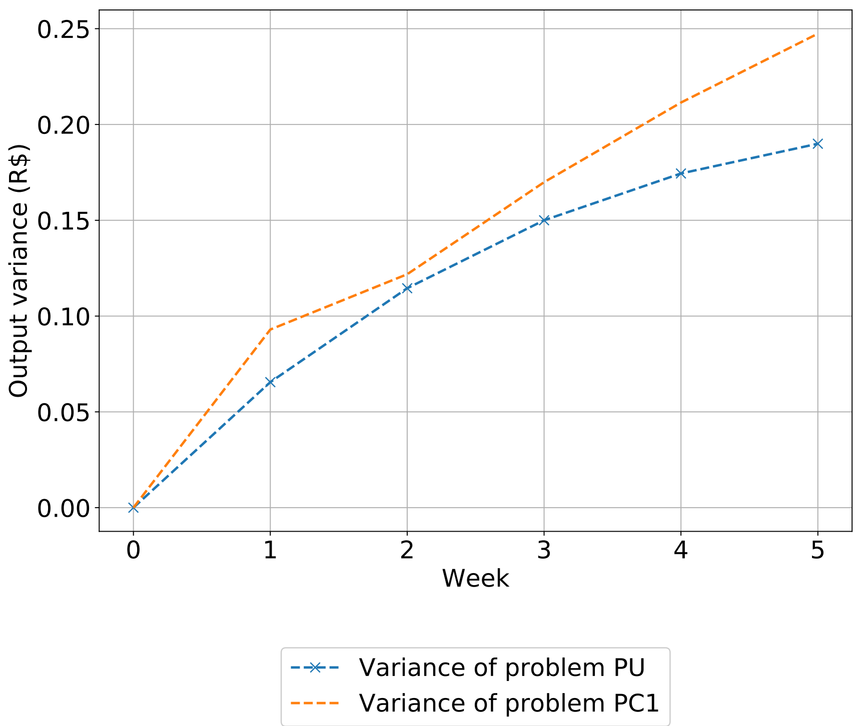
Source: Author.

Figure 5: Expected output for the constrained problem PC1 versus PU.



Source: Author.

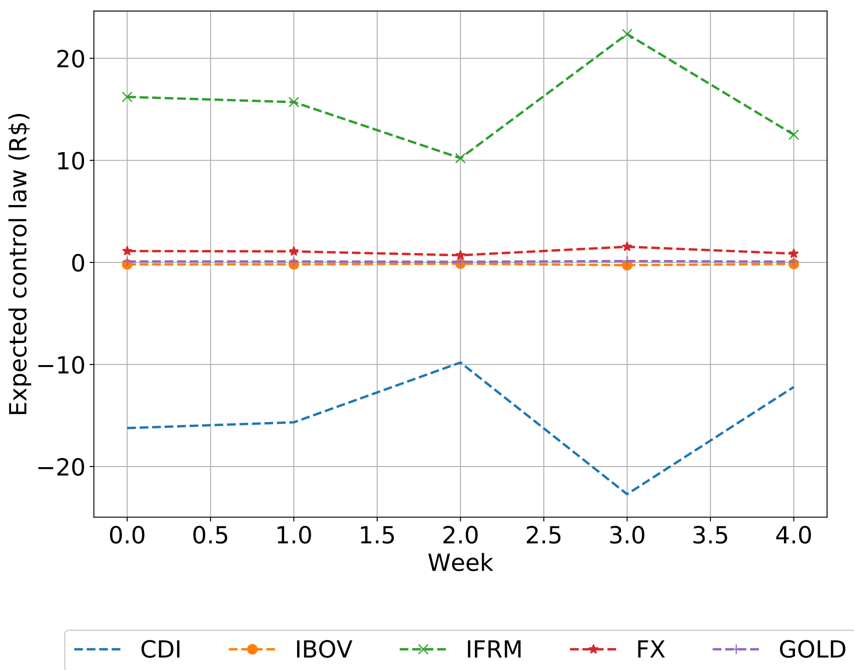
Figure 6: Output variance for the unconstrained problem PC1 versus PU.



Source: Author.

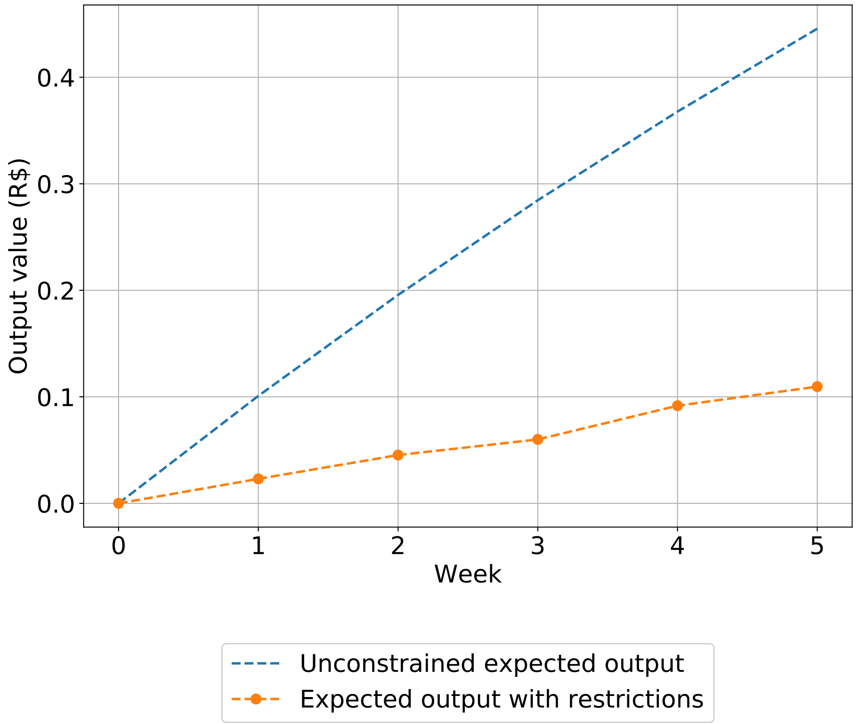
Note the variance for $PC1$ is higher than the one for PU as expected, given that we are imposing a higher expected return for $PC1$ than for PU.

Figure 7: Expected optimal control law for the constrained problem PC2.



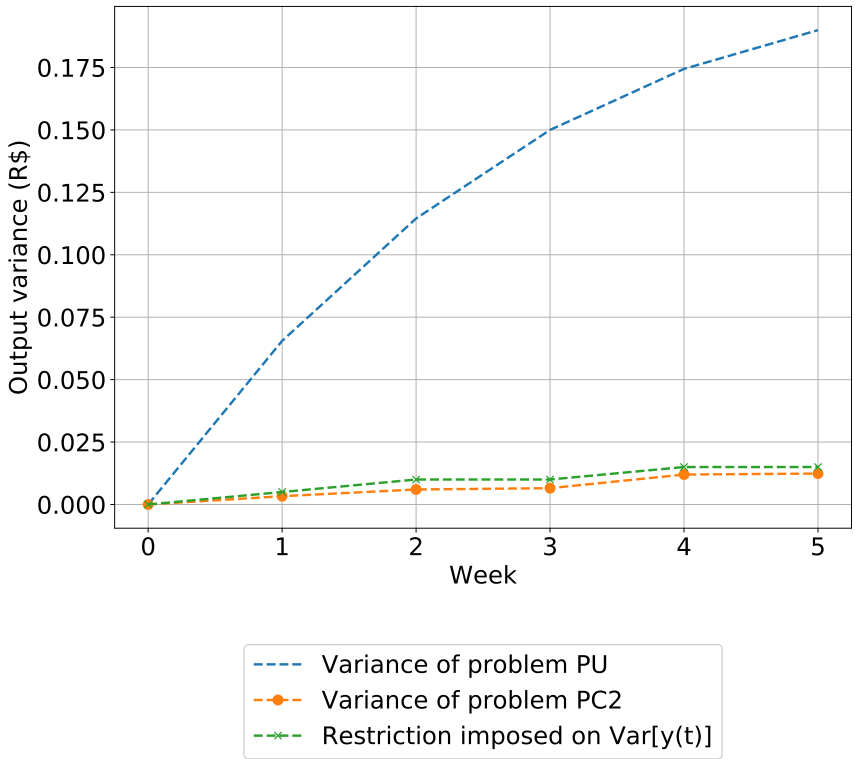
Source: Author.

Figure 8: Expected output for the constrained problem PC2 versus PU.



Source: Author.

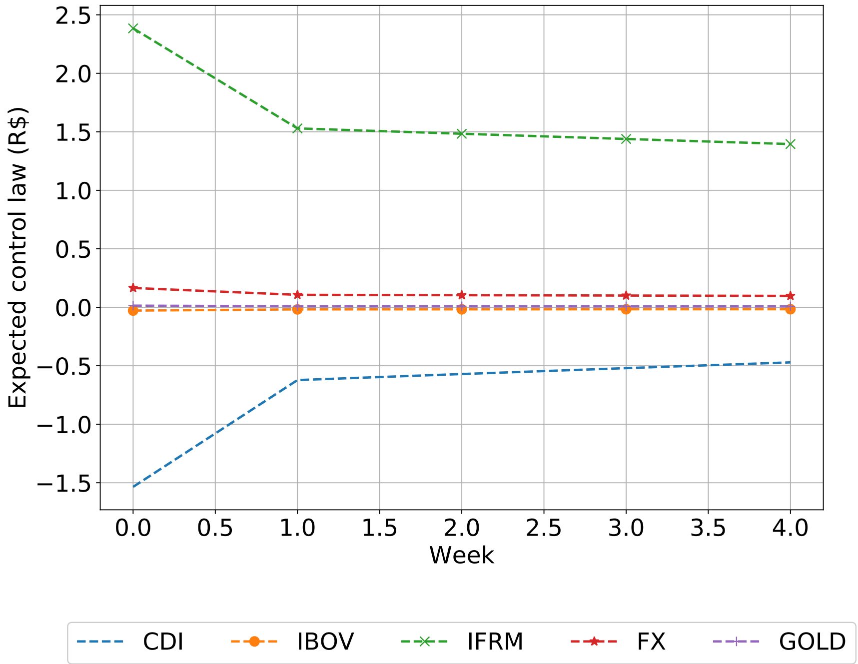
Figure 9: Output variance for the unconstrained problem PC2 versus PU.



Source: Author.

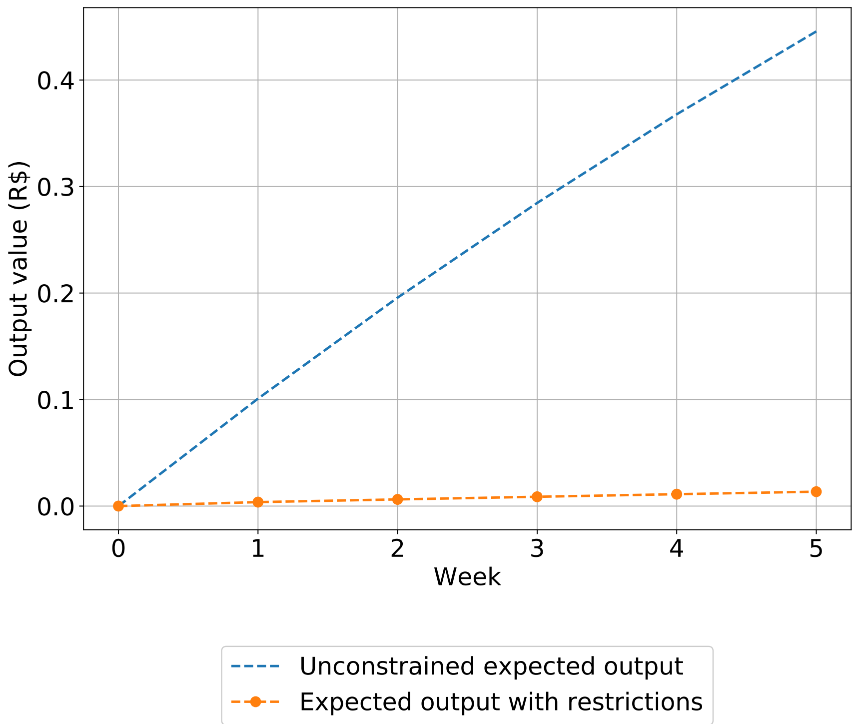
For $PC2$, we are imposing a lower variance than the one for PU, and as a result, we obtain a lower expected return for $PC2$ as expected.

Figure 10: Expected optimal control law for the constrained problem PC3.



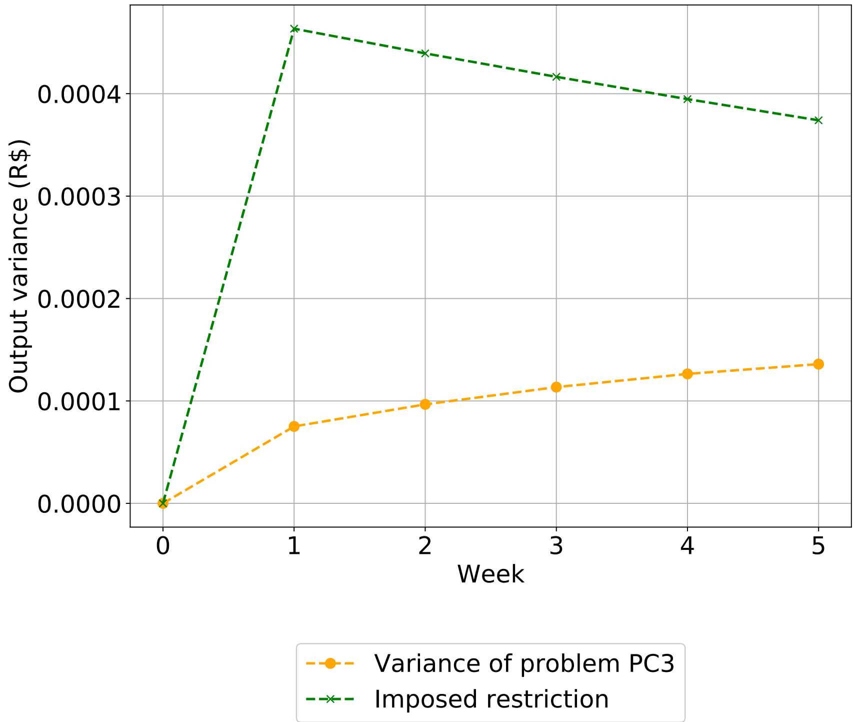
Source: Author.

Figure 11: Expected output for the constrained problem PC3 versus PU.



Source: Author.

Figure 12: Output variance for the unconstrained problem PC3.



Source: Author.

Once more, the imposed restriction was attained with a lower variance for problem $PC3$ and a resulting lower expected output as expected.

9.2 Infinite-horizon control and stabilisation

In the following example, we follow the procedures as described in Section 6.2.2 and solve the discounted problem as in Equation (3.29) using the system notations as stated in Equations (3.27) and (3.28), in order to consider the discount factor, α.

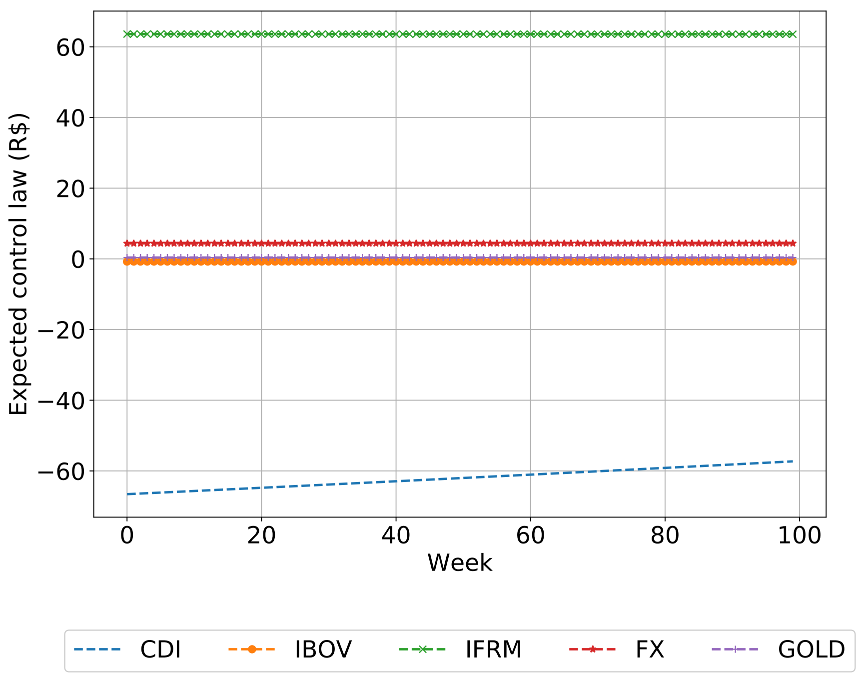
Set $\alpha = 0.7$ and $\xi = \nu = 1$. The problem is to find the optimal portfolio allocation at each time t that minimizes the functional cost (3.29), which can be solved applying the results presented in Theorem 5.6.

To get the mean square stabilising solution to the GCARE, we solved the LMI optimization problem presented in Equation (5.74). For the optimal solution $Z^\alpha \in \mathbb{T}(\mathbb{H}^n)$,

$$Z^\alpha = \begin{bmatrix} 2,9027.9 & -2,9366.5 & 0 & 0 \\ -2,9366.5 & 2,8798.1 & 0 & 0 \\ 0 & 0 & 1.552 & 0.655 \\ 0 & 0 & 0.655 & 1.210 \end{bmatrix} \times 10^{-4},$$

we get that $\mathcal{T}(Z^\alpha) = 0$, $\Gamma(Z^\alpha) = \breve{\bar{A}}$ and $r_\sigma(\mathcal{L}_{\breve{K}^\alpha}) = 0.7013$, so that Z^α is indeed the mean-square stabilising solution to the GCARE and we can apply Theorem 5.6 to obtain $\bar{V}^\alpha$ and the optimal control policies using Equations (5.68) and (5.69). Figure 13 shows the resulting expected control law for PD.

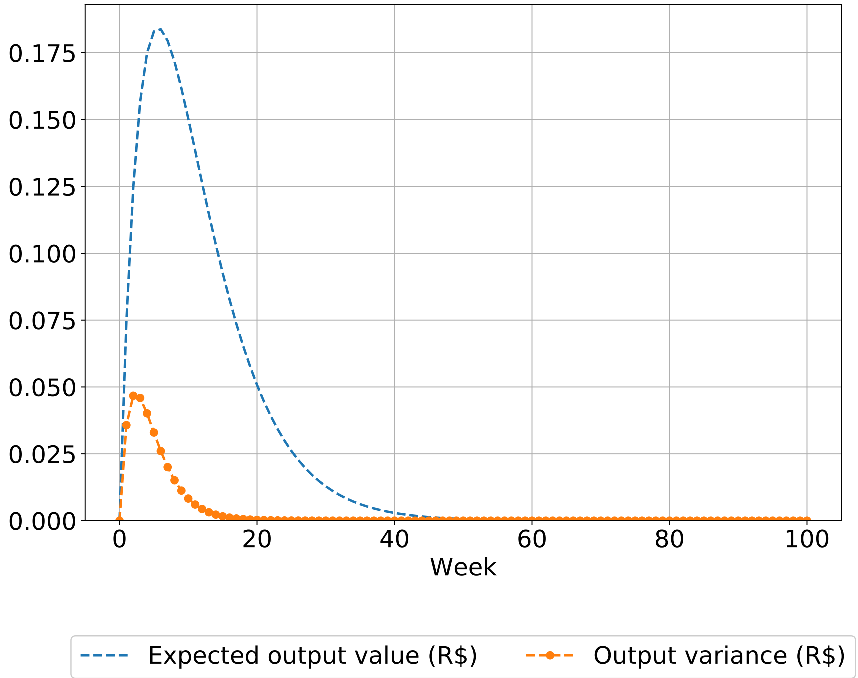
Figure 13: Optimal asset allocation.



Source: Author.

We present in Figure 14 the expected tracking error output of the portfolio value against its benchmark, $y^\alpha(t) = Lx^\alpha(t)$, and its variance, $Var(y^\alpha(t))$, after 10,000 simulations.

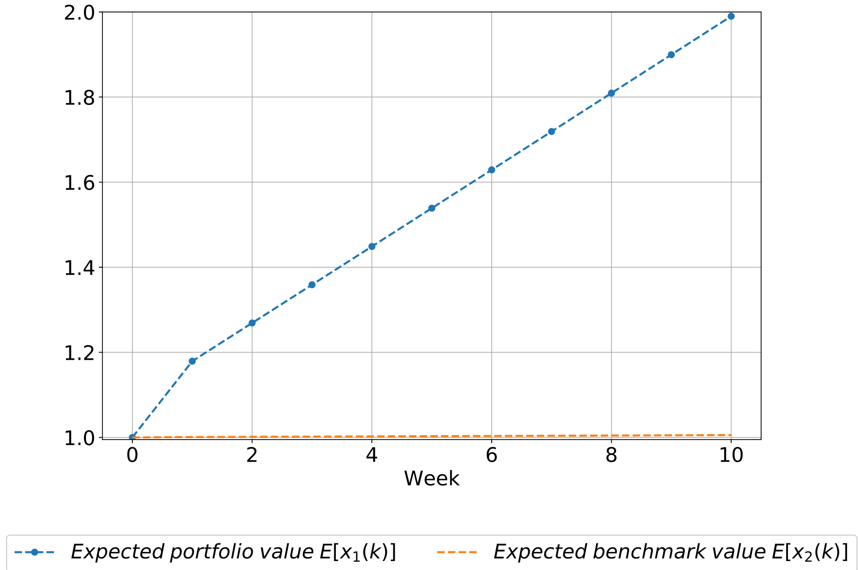
Figure 14: Expected output and its variance with a discount factor α.



Source: Author.

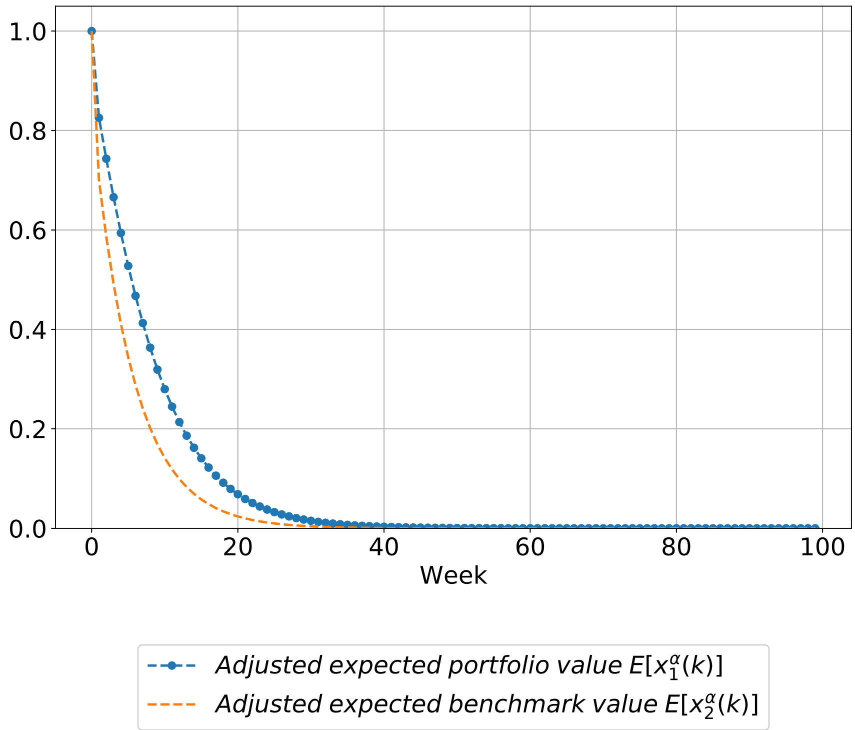
Figure 15 shows the individual evolution of both the expected portfolio value and the benchmark without considering the effects of the discount factor α. Figure 16 shows the behaviour of the state $x^{\alpha}(k)$, confirming its convergence to zero with probability 1 even though the state $x(k)$ does not converge in this particular example.

Figure 15: Expected portfolio value versus the benchmark.



Source: Author.

Figure 16: Expected value of $x^{\alpha}(k)$.



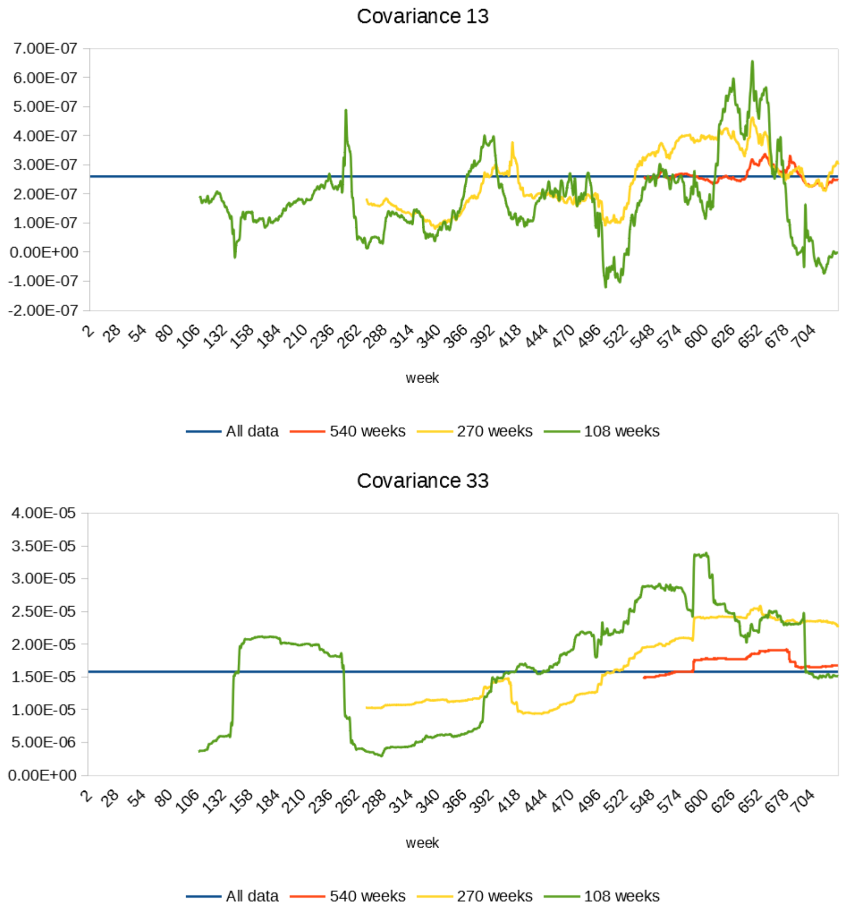
Source: Author.

9.3 A note on the homoscedasticity hypothesis and future works including Markov chains

In statistics, a sequence (or a vector) of random variables is homoscedastic if all its random variables have the same finite variance. The complementary notion is called heteroscedasticity. In the same sense, two or more normal distributions are homoscedastic if they share a common covariance (or correlation) matrix. Therefore, a homoscedastic sequence will display a constant covariance matrix over time.

To illustrate the homoscedastic (or heteroscedastic) of our dataset, we present below how the covariance changed over time for some assets. In Appendix A, we present the covariances among all assets. Figure 17 shows the covariance of CDI against IFR-M (covariance 13) and the variance of gold (covariance 55) considering different windows of measurement of 108, 270, 540 and all weeks in the dataset. Please, refer to Table 3 for the index attributed to each asset.

Figure 17: Covariances of CDI against IFR-M (covariance 13) and the variance of gold (covariance 55).



Source: Author.

In the previous sections, we assumed that the covariances are homoscedastic within each step k for the finite horizon and over all steps for the infinite horizon. However, as illustrated above, the covariances vary over time and a more appropriate way of estimating it is paramount to obtain more significant results. There is an ample range of possible ways of modelling covariances, and the most immediate choice is usually between static and dynamic models.

The most common static models are those in which the covariance matrix is unconditionally estimated based on a sample of asset returns or estimated on a factor model that captures cross-sectional characteristics of asset returns or estimated by shrinking the sample covariance matrix towards alternative targets. See (CHAMBERLAIN, 1983; CHAMBERLAIN; ROTHSCHILD, 1983; STOCK; WATSON, 1989; BAI; NG, 2002; BAI; NG, 2007) for instance.

Dynamic models are based on the idea that next period's covariances depend on the covariances of previous periods, and they are updated according to alternative autoregressive structures such as multivariate GARCH and stochastic volatility models, see (SILVENNOINEN; ASVIRTA, 2005; SILVENNOINEN; ASVIRTA, 2009; BAUWENS; LAURENT, 2005; BAUWENS; LAURENT; ROMBOUTS, 2006). On the other hand, traditional multivariate time series methods, as a rule, are quite helpless in large samples, and alternatives methodologies such as dynamic factor models have been developed. In these models, the market and the stock-specific components are assessed independently in an attempt to improve the covariances estimates (SHIOHAMA et al., 2010).

Another approach considers stochastic volatility models which treat price volatility as a random variable, allowing the price to vary over time and improving the accuracy of calculations and forecasts. In particular, a Markov process or a Markov chain is a stochastic model that seems well suited for financial modelling. A Markov chain describes a sequence of possible events in which the probability of each event depends only on the state attained in the previous event. For example, imagine a market that operates in two states, $i = 1$ ("bearish"), $i = 2$ ("bull"), with their respective expected returns and covariances given by $\mu = [-7\% \quad 5\%]$ and $cov = [0.25\% \quad 0.09\%]$, and assume that the transition probability matrix from state i (row) to j (column) is given by $P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$. Thus, given the current market state, say $i = 1$, it will continue in the current state 70% of the time and show a -7% return or jump to another state in 30% of the cases and present a return of 5%. More generally, we could imagine the expected returns and their covariances jumping into a series os possible states over time.

Systems that incorporate such models have already been the subject of many studies and several results related to the control of these systems have already been derived in the literature, see (COSTA; FRAGOSO; MARQUES, 2005; DRAGAN; MOROZAN, 2006a; DRAGAN; MOROZAN, 2006b; COSTA; PAULO, 2008; ZHANG; WANG, 2015; MA; JIA, 2013) as a sample of works in this area. Therefore, the mean-variance optimal control of linear systems with Markov jumps and multiplicative noises seems an excellent candidate for future works regarding the mean-field approach developed in this thesis.

10 CONCLUSION

In this work, we have considered stochastic multi-period mean-variance optimal control problems and adopted the mean-field formulation to solve them.

Regarding the finite-horizon case, we generalise previous works in the literature by considering discrete-time linear systems with multiplicative noises and tackle the problem by expanding the state space to $(\mathbb{E}(x(t)),\quad x(t)-\mathbb{E}(x(t)))$ and develop the optimal control in terms of $(\mathbb{E}(u(t)),\quad u(t)-\mathbb{E}(u(t)))$.

Thereby, under these new state and solution space, we can eliminate the quadratic term from the variance in our initial problem and solve it to a variety of situations. We first applied this method to a general problem with finite horizon and no constraints. Then, we solved finite-horizon problems with inter-temporal restrictions on either the expected value of the output or its variance and with restrictions on the minimum value of the output associated with a given probability of occurrence.

An explicit sufficient condition for the existence of an optimal control strategy for the general unconstrained problem and the value functions for the dual Lagrangian optimisation problems for the constrained cases were derived. The solution to the general problem was derived from a set of two generalised Riccati difference equations interconnected with a set of linear recursive equations (see the definitions of $P(k)$, $M(k)$, $V(k)$ in Equations (4.4), (4.5), and (4.6)). We also presented a sufficient condition for an explicit solution for the problem that restricts the output to a minimum value while minimising its variance over time.

We then studied the multi-period infinite-horizon stabilisation problem of discrete-time linear systems with multiplicative noises under the mean-field approach. We considered the existence of the maximal and mean square stabilising solutions for a set of two generalised coupled algebraic Riccati equations associated to the infinite-horizon stochastic model, see the definitions of $\mathcal{M}$, $\mathcal{P}$, and $\mathcal{T}$ in Equations (4.19) and (4.21).

Regarding the stabilisation problem, the results include a necessary and sufficient condition under which there exists the mean square stabilising solution and a sufficient condition under which there exists the maximal solution to the GCARE, all in terms of the spectral radius of an operator. Compared to previous works, we generalise the stabilisation conditions to just some positive semi-definite matrices and kernels restrictions on some matrices and also solved both the discounted cost problem and the long-run problem with linear terms on the performance criterion.

When specialised to the optimal asset allocation problem, we showed that our results retrieve some known outcomes in the literature. We also applied our formulation to a numerical case of a multi-period portfolio selection problem with a benchmark, where we find the best asset allocation to optimise the sum of the trade-off between the variance and the excess return of the portfolio against a benchmark.

Regarding future developments, we would consider the following relevant topics:

1. Development of optimal control policies using the mean-field formulation considering systems that follow a Markov process. Systems modelled by Markov jumps are well suited to represent environment dynamics that are subjected to significant changes. There are many examples of situations that would require such complex models, for instance:

   i) Aircraft control systems dealing with abrupt changes in pressure, altitude, and speed.

   ii) Economic models facing the burst of financial crisis or changes of governments and policies.

   iii) Population models with the advent of diseases.

   It would also lead to a more robust methodology to estimate the non-stationary input parameters as described in Section 9.3.

2. Consideration of transactions costs and restriction on the maximum and minimum limits allocated per asset. For a portfolio manager, for instance, imposing such restrictions seems vital to the proper use of our system. Transactions costs are always present as well as investment policies that limit the portfolio exposure to specific assets, countries, currencies, or leverage.

3. Expansion of our results by applying the mean-field approach to filtering, quadratic optimal control with partial information, and H_∞-control.

REFERENCES

ANDERSSON, D.; DJEHICH, B. A maximum principle for SDEs of mean-field type. **Applied Mathematics & Optimization**, v. 63, p. 341–356, 2011.

BAI, J.; NG, S. Determining the number of factors in approximate factor models. **Econometrica**, v. 70, p. 191–223, 2002.

____. Determining the number of primitive shocks in factor models. **J. Bus. Econ. Stat.**, v. 25, p. 52–60, 2007.

BARBIERI, F.; COSTA, O. L. V. Optimal control with constrained total variance for Marvov jump linear systems with multiplicative noises. **International Journal of Systems Science**, Taylor & Francis, 2018. Available at: <https://www.tandfonline.com/doi/abs/10.1080/00207721.2018.1441469>.

____. A mean-field formulation for the infinite-horizon mean-variance control of discrete-time linear systems with multiplicative noises. **IET Control Theory & Applications**, Institution of Engineering and Technology, 2020. Available at: <http://dx.doi.org/10.1049/iet-cta.2020.0442>.

____. A mean-field formulation for the mean-variance control of discrete-time linear systems with multiplicative noises. **International Journal of Systems Science**, Taylor & Francis, 2020. Available at: <http://dx.doi.org/10.1080/00207721.2020.1780340>.

BAUWENS, L.; LAURENT, S. A new class of multivariate skew densities, with application to GARCH models. **J. Bus. Econ. Stat.**, v. 23, p. 346–354, 2005.

BAUWENS, L.; LAURENT, S.; ROMBOUTS, J. V. Multivariate GARCH models: A survey. **Journal of Applied Econometrics**, v. 21, n. 1, p. 79–109, 2006.

BAZARAA, M. S.; SHERALI, H. D.; SHETTY, C. M. **Nonlinear Programming: Theory and Algorithms**. 3rd. ed. [S.l.]: Wiley Publishing, 2013. ISBN 1118857569, 9781118857564.

BEGHI, A.; D'ALESSANDRO, D. Discrete-time optimal control with control-dependent noise and generalized Riccati difference equations. **Automatica**, v. 34, p. 1031–1034, 1998.

BENSOUSSAN, A.; FREHSE, J.; YAM, P. **Mean field games and mean field type control theory**. New York: Springer Briefs in Mathematics: Springer, 2013.

BOUKAS, E.; LIU, Z. Delay-dependent stabilization of singularly perturbed jump linear systems. **International Journal of Control**, v. 77, p. 310–319, 2004.

BUCKDAHN, R.; DJEHICHE, B.; LI, J. Mean-field backward stochastic differential equations: A limit approach. **Annals of Probability**, v. 37, p. 1524–1565, 2009.

____. A general stochastic maximum principle for SDEs of mean-field type. **Applied Mathematics & Optimization**, v. 64, p. 197–216, 2011.

BUCKDAHN, R.; LI, J.; PENG, S. Mean-field backward stochastic differential equations and related partial differential equations. **Stochastic Processes and their Applications**, v. 119, p. 3133–3154, 2009.

CARMONA, R.; DELARUE, F. **Probabilistic theory of mean field games with applications I**. Gewerbestrasse 11, 6330 Cham, Switzerland: Springer International Publishing AG, 2018.

CERNÝ, A.; KALLSEN, J. Hedging by sequential regressions revisited. **Math. Finance**, v. 19, p. 591–617, 2009.

CHAMBERLAIN, G. Funds, factors, and diversification in arbitrage pricing models. **Econometrica**, v. 51, p. 1281–1304, 1983.

CHAMBERLAIN, G.; ROTHSCHILD, M. Arbitrage, factor structure and mean-variance analysis in large asset markets. **Econometrica**, v. 51, p. 1305–1324, 1983.

CHEN, S.; LI, X.; ZHOU, X. Y. Stochastic linear quadratic regulators with indefinite control weight costs. **SIAM Journal of Control and Optimization**, v. 36, p. 1685–1702, 1998.

COSTA, O. L. V.; FRAGOSO, M. D.; MARQUES, R. P. **Discrete-time Markov jump linear systems**. London: Springer-Verlag, 2005.

COSTA, O. L. V.; NABHOLZ, R. B. Multi-period mean-variance optimization with intertemporal restrictions. **Journal of Optimization Theory and Applications**, v. 134, p. 257–274, 2007.

COSTA, O. L. V.; OLIVEIRA, A. Optimal mean-variance control for discrete-time linear systems with Markovian jumps and multiplicative noise systems. **Automatica**, v. 48, p. 304–315, 2012.

COSTA, O. L. V.; PAULO, W. L. de. Indefinite quadratic with linear cost optimal control of Markovian jump with multiplicative noise systems. **Automatica**, v. 43, p. 587–597, 2007.

____. Generalized coupled algebraic Riccati equations for discrete-time Markov jump with multiplicative noise systems. **European Journal of Control**, v. 14, p. 391–408, 2008.

CUI, X.; LI, X.; LI, D. Unified framework of mean-field formulations for optimal multi-period mean-variance portfolio selection. **IEEE Transactions on Automatic Control**, v. 59, n. 7, p. 1833–1844, 2014.

DAMM, T. **Rational matrix equations in stochastic control**. Berlin Heidelberg: Springer-Verlag, 2004. v. 297 of Lecture notes in control and information sciences.

DAMM, T.; HINRICHSEN, D. Newton's method for concave operators with resolvent positive derivatives in ordered banach spaces. **Linear Algebra Applications**, v. 363, p. 43–64, 2003.

DAVIS, M.; VINTER, R. B. **Stochastic modelling and control**. London: Chapman and Hall, 1985.

DAWSON, D. A. Critical dynamics and fluctuations for a mean-field model of cooperative behavior. **Journal of Statistical Physics**, v. 31, p. 29–85, 1983.

DIAMOND, S.; BOYD, S. CVXPY: A Python-embedded modeling language for convex optimization. **Journal of Machine Learning Research**, v. 17, n. 83, p. 1–5, 2016.

DOMBROVSKII, V. V.; LYASHENKO, E. A. A linear quadratic control for discrete systems with random parameters and multiplicative noise and its application to investment portfolio optimization. **Automatic and Remote Control**, v. 64, p. 1558–1570, 2003.

DRAGAN, V.; MOROZAN, T. Mean square exponential stability for some stochastic linear discrete-time systems. **European Journal of Control**, v. 12, p. 373–395, 2006.

____. Observability and detectability of a class of discrete-time stochastic linear systems. **IMA Journal of Mathematical Control and Information**, v. 23, p. 371–394, 2006.

DRAGAN, V.; MOROZAN, T.; STOICA, A.-M. **Mathematical methods in robust control of linear atochastic aystems**. 3rd. ed. [S.l.]: Springer-Verlag New York, New York, NY, USA, 2013.

ELLIOTT, R.; DUFOUR, F.; MALCOLM, W. State and mode estimation for discrete-time jump Markov systems. **SIAM Journal on Control and Optimization**, v. 44, p. 1081–1104, 2005.

ELLIOTT, R. J.; NI, Y. Discrete time mean-field stochastic linear quadatic optimal control problems. **Automatica**, v. 49(11), p. 3222–3233, 2013.

ELTON, E. J.; GRUBER, M. J. **Modern portfolio theory and investment analysis**. New York: Wiley, 1995.

GOMES, D. A.; SAUDE, J. Mean field games models - A brief survey. **Dyn Games Appl**, v. 4, p. 110–154, 2014.

HAKANSSON, N. H. Optimal investment and consumption strategies under risk for a class of utility functions. **Econometrica**, v. 38, p. 587–607, 1970.

HORN, R. A.; JOHNSON, C. R. **Matrix analysis**. New York: Cambridge University Press, 1990.

HOWE, M. A.; RUSTEM, B. A robust hedging algorithm. **Journal of Economic Dynamics and Control**, v. 21, p. 1065–1092, 1997.

HOWE, M. A.; RUSTEM, B.; SELBY, M. J. P. Multi-period minimax hedging strategies. **European Journal of Operational Research**, v. 93, p. 185–204, 1996.

HUANG, J.; LI, N. Linear-quadratic mean-field game for stochastic delayed systems. **IEEE Transactions on Automatic Control**, v. 63, n. 8, p. 2722–2729, 2018.

HUANG, J.; LI, X.; YONG, J. A linear quadratic optimal control problem for mean-field stochastic differential equations in infinite horizon. **Mathematical Control and Related Fields**, v. 5, n. 1, p. 97–139, 2015.

HUANG, M.; CAINES, P. E.; MALHAMÈ, R. P. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ε-Nash equilibria. **IEEE Transactions on Automatic Control**, v. 52, n. 9, p. 1560–1571, 2007.

HUANG, M.; MALHAMÈ, R. P.; CAINES, P. E. Large population stochastic dynamic games: closed-loop mckean-vlasov systems and the Nash certainty equivalence principle. **Communications in Information and Systems**, v. 6, n. 3, p. 221–252, 2006.

KAC, M. Foundations of kinetic theory. In: **Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume 3: Contributions to Astronomy and Physics**. Berkeley, California: University of California Press, 1956. p. 171–197. Available at: <https://projecteuclid.org/euclid.bsmsp/1200502194>.

LASRY, J.-M.; LIONS, P.-L. Jeux à champ moyen. I - le cas stationnaire. **Comptes Rendus de l'Académie des Sciences Paris**, v. 343, p. 619–625, 2006.

____. Jeux à champ moyen. II - horizon fini et contrôle optimal. **Comptes Rendus de l'Académie des Sciences Paris**, v. 343, p. 679–684, 2006.

____. Mean field games. **Japanese Journal of Mathematics**, v. 2, p. 229–260, 2007.

LEIPPOLD, M.; TROJANI, F.; VANINI, P. A geometric approach to multi-period mean variance optimization of assets and liabilities. **Journal of Economic Dynamics and Control**, v. 28, p. 1079–1113, 2004.

LI, D.; NG, W. L. Optimal dynamic portfolio selection: Multi-period mean-variance formulation. **Mathematical Finance**, v. 10, p. 387–406, 2000.

LI, X.; SUN, J.; XIONG, J. Linear quadratic optimal control problems for mean-field backward stochastic differential equations. **Applied Mathematics Optimization**, v. 80, p. 223–250, 2019.

LI, X.; ZHOU, X. Y.; LIM, A. E. B. Dynamic mean-variance portfolio selection with no-shorting constraints. **SIAM Journal of Control Optimization**, v. 40, p. 1540–1555, 2002.

LI, X.; ZHOU, X. Y.; RAMI, M. A. Indefinite stochastic linear quadratic control with Markovian jumps in infinite time horizon. **Journal of Global Optimization**, v. 27, p. 149–175, 2003.

LIM, A.; ZHOU, X. Y. Stochastic optimal LQR control with integral quadratic constraints and indefinite control weights. **IEEE Transactions on Automatic Control**, v. 44, p. 1359–1369, 1999.

MA, H.; JIA, Y. H2 control of discrete-time periodic systems with Markovian jumps and multiplicative noise. **International Journal of Control**, v. 86, Issue 10, p. 1837–1849, 2013.

MARKOWITZ, H. Portfolio selection. **The Journal of Finance**, v. 7, p. 77–91, 1952.

____. **Portfolio selection: efficient diversification of investments**. New York: John Wiley, 1959.

MEYER-BRANDIS, T.; ØKSENDAL, B.; ZHOU, X. Y. A mean-field stochastic maximum principle via malliavin calculus. **Stochastics**, v. 84, p. 643–666, 2012.

MOON, J. A sufficient condition for linear-quadratic stochastic zero-sum differential games for Markov jump systems. **IEEE Transactions on Automatic Control**, v. 64, n. 4, p. 1619–1626, 2019.

MOON, J.; KIM, Y. Linear exponential quadratic control for mean field stochastic systems. **IEEE Transactions on Automatic Control**, v. 64, n. 12, p. 5094–5100, 2019.

MOORE, J. B.; ZHOU, X. Y. Discrete-time LQG controls with control dependent noise. **Systems & Control Letters**, v. 36, p. 199–206, 1999.

MOSSIN, J. Optimal multiperiod portfolio policies. **Journal of Business**, v. 41, p. 215–229, 1968.

NI, Y.-H.; ELLIOTT, R.; LI, X. Discrete-time mean-field stochastic linear-quadratic optimal control problems, II: Infinite horizon case. **Automatica**, v. 57, p. 65–77, 2015.

NI, Y.-H.; LI, X.; ZHANG, J.-F. Indefinite mean-field stochastic linear-quadratic optimal control: From finite horizon to infinite horizon. **IEEE Transactions on Automatic Control**, v. 61, n. 11, p. 3269–3284, 2016.

____. Mean-field stochastic linear quadratic optimal control with Markov jump parameters. **Systems & Control Letters**, v. 93, p. 69–76, 2016.

NI, Y.-H.; ZHANG, J.-F.; LI, X. Indefinite mean-field stochastic linear-quadratic optimal control. **IEEE Transactions on Automatic Control**, v. 60, n. 7, p. 1786–1800, 2015.

RAMI, M. A.; CHEN, X.; ZHOU, X. Y. Discrete-time indefinite LQ control with state and control dependent noises. **Journal of Global Optimization**, v. 23, p. 245–265, 2002.

RAMI, M. A.; ZHOU, X. Y. Linear matrix inequalities, Riccati equations, and indefinite stochastic linear quadratic controls. **IEEE Transactions on Automatic Control**, v. 45, p. 1131–1143, 2000.

RUSTEM, B.; BECKER, R. G.; MARTY, W. Robust min-max portfolio strategies for rival forecast and risk scenarios. **Journal of Economic Dynamics and Control**, v. 24, p. 1591–1621, 1995.

SABERI, A.; SANNUTI, P. **H2-optimal control**. New Jersey: Prentice Hall, 1995.

SAMUELSON, P. Lifetime portfolio selection by dynamic stochastic programming. **Review of Economics and Statistics**, v. 51, p. 239–246, 1969.

SATCHEL, S.; SCOWCROFT, A. **Advances in portfolio construction and implementation**. 1st. ed. Oxford: Butterworth-Heinemann, 2003.

SCHAEFER, H. H. **Topological vector spaces**. Berlin, Heidelberg, New York: Springer-Verlag, 1971.

SCHWEIZER, M. Approximation pricing and the variance-optimal martingale measure. **The Annals of Probability**, v. 24, p. 206–236, 1996.

SHARPE, W. F.; TINT, L. G. Liabilites - a new approach. **Journal of Portfolio Management**, 1990.

SHIOHAMA, T. et al. Dynamic portfolio optimization using generalized dynamic conditional heteroskedastic factor models. **J. Japan Statist. Soc.**, v. 40, n. 1, p. 145–166, 2010.

SILVENNOINEN, A.; ASVIRTA, T. T. Multivariate autoregressive conditional heteroskedasticity with smooth transitions in conditional correlations. **SSE/EFIWorking Paper Series in Economics and Finance**, v. 577, 2005.

____. Out-of-sample predictions of bond excess returns and forward rates: An asset allocation perspective. **Review of Financial Studies**, v. 25, n. 10, p. 3141–3168, 2009.

SONTAG, E. D. **Mathematical control theory**. New York: Springer-Verlag, 1990.

STEINBACH, M. C. Markowitz revisited: mean-variance models in financial portfolio analysis. **SIAM Review**, v. 43, p. 31–85, 2001.

STOCK, J. H.; WATSON, M. W. New indices of coincident and leading indicators in o. j. blanchard and s. fischer. In: ____. [S.l.]: M.I.T. Press, 1989.

TOBIN, J. Liquidity preference as behaviour toward risk. **Review of Economic Studies**, v. 25, p. 65–86, 1958.

WEIDMANN, J. **Linear operators in Hilbert spaces**. New York: Springer-Verlag, 1980.

WU, H.; ZHOU, X. Y. Characterizing all optimal controls for an indefinite stochastic linear quadratic control problem. **IEEE Transactions on Automatic Control**, v. 47, p. 1119–1122, 2002.

XIA, J. M.; YAN, J. A. Markowitz's portfolio optimization in an incomplete market. **Math. Finance**, v. 16, p. 203–216, 2006.

YONG, J. Linear-quadratic optimal control problems for mean-field stochastic differential equations. **SIAM Journal on Control and Optimization**, v. 51, p. 2809–2838, 2013.

YÜCEER, U. Discrete convexity: convexity for functions defined on discrete spaces. **Discrete Applied Mathematics**, v. 119, p. 297–304, 2002.

ZHANG, H.; QI, Q. A complete solution to optimal control and stabilization for mean-field systems: Part I, discrete-time case. **arXiv:1608.06363v1**, p. 1–20, 2016.

ZHANG, H.; QI, Q.; FU, M. Optimal stabilization control for discrete-time mean-field stochastic systems. **IEEE Transactions on Automatic Control**, v. 64, p. 1125–1136, 2019.

ZHANG, Y.; WANG, C. Robust stochastic stability of uncertain discrete-time impulsive Markovian jump delay systems with multiplicative noises. **International Journal of Systems Science**, v. 46, Issue 12, p. 2210–2220, 2015.

ZHOU, X. Y.; LI, D. Continuous-time mean-variance portfolio selection: A stochastic LQ framework. **Applied Mathematics & Optimization**, v. 42, p. 19–33, 2000.

ZHOU, X. Y.; YIN, G. Markowitz's mean-variance portfolio selection with regime switching: a continuous-time model. **SIAM Journal on Control and Optimization**, v. 42, p. 1466–1482, 2003.

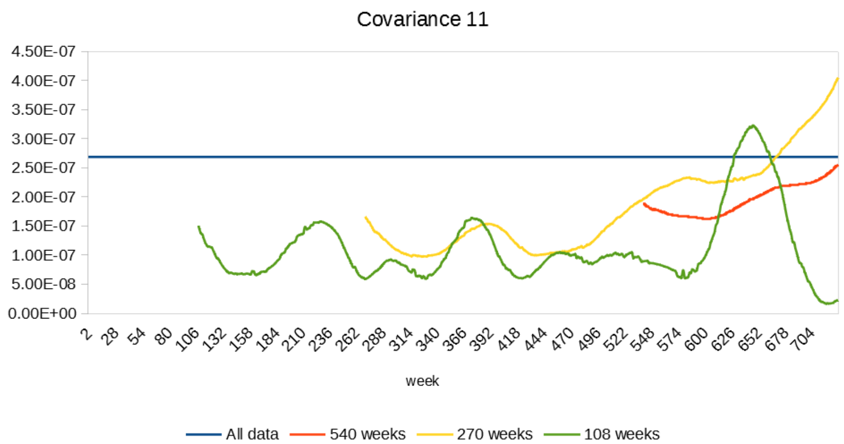
ZHU, J. On stochastic Riccati equations for the stochastic LQR problem. **Systems & Control Letters**, v. 54, p. 119–124, 2005.

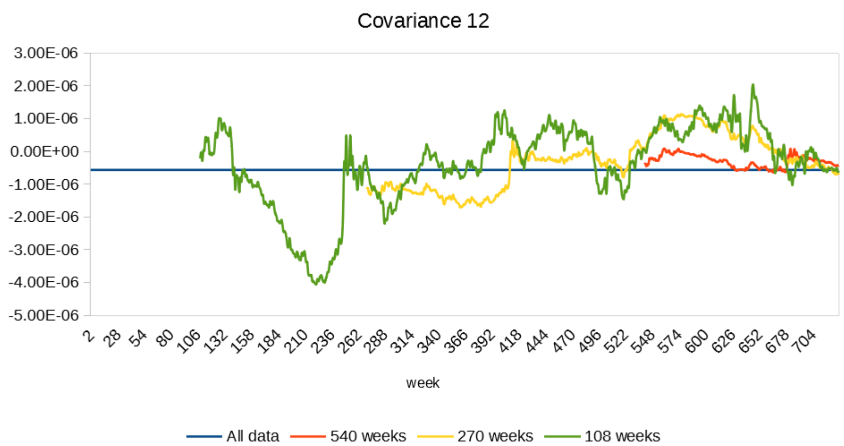
ZHU, S. S.; LI, D.; WANG, S. Y. Risk control over bankruptcy in dynamic portfolio selection: A generalized mean-variance formulation. **IEEE Transactions on Automatic Control**, v. 49, p. 447–457, 2004.

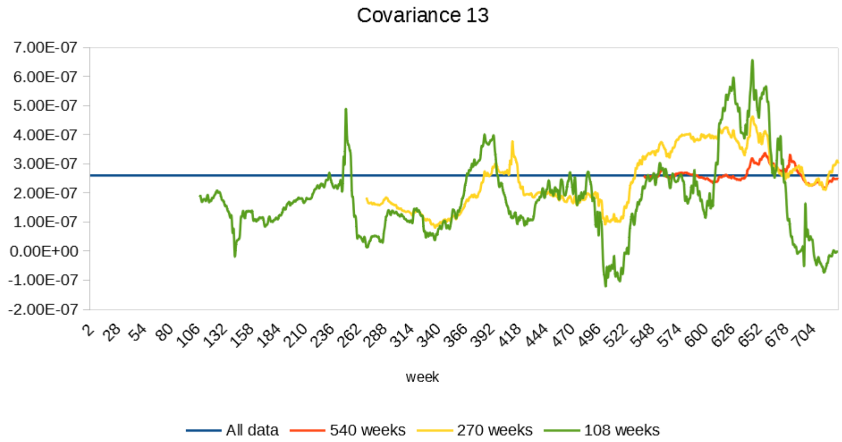
APPENDIX A – COVARIANCE CHARTS

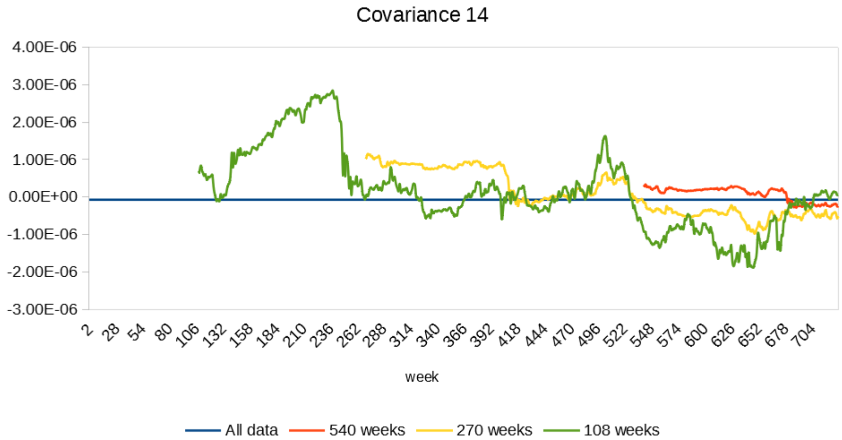
Figure 18 shows the covariance of each security as defined in Chapter 9, Table 3, assuming different windows of measurement of 108, 270, 540 and all weeks in the dataset.

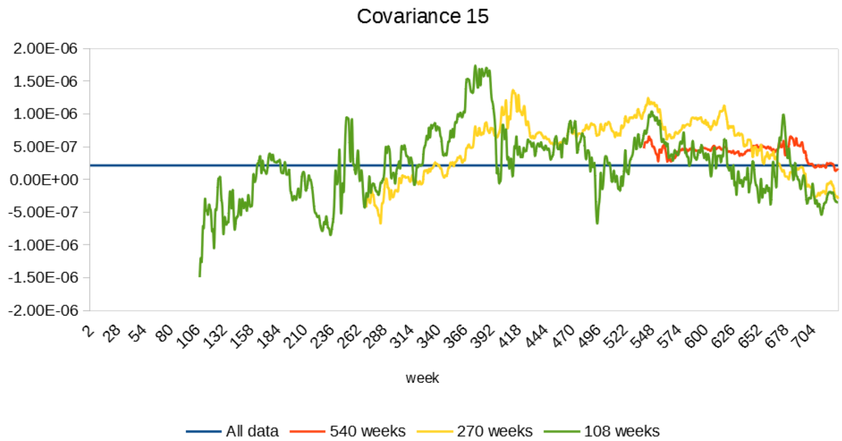
Figure 18: Covariances of assets 1 to 6 (see index in Table 3).

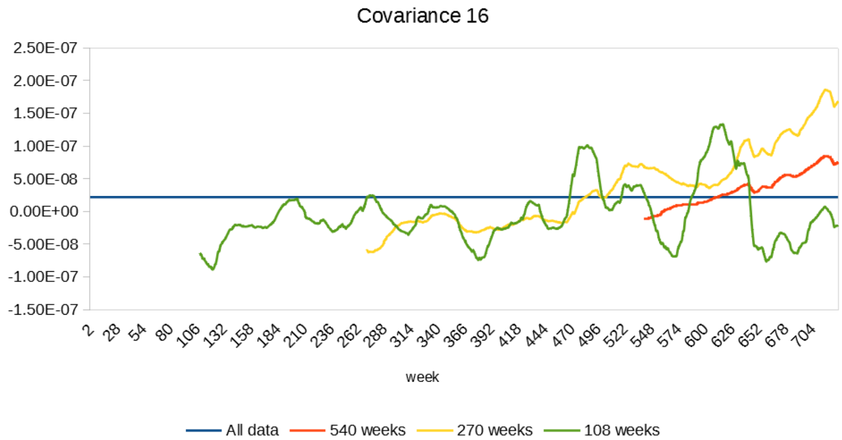


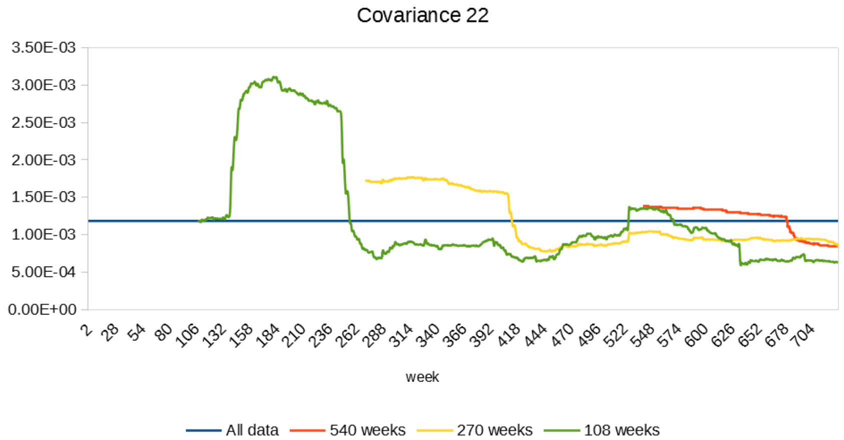


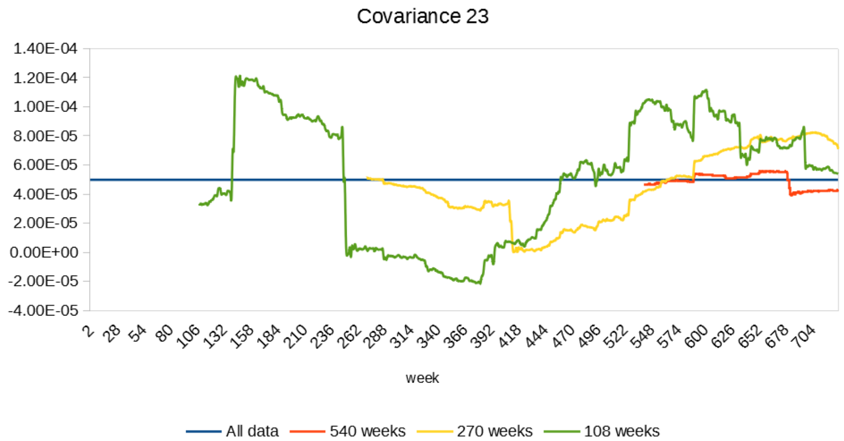

Covariance 13
7.00E-07
6.00E-07
5.00E-07
4.00E-07
3.00E-07
2.00E-07
1.00E-07
0.00E+00
-1.00E-07
-2.00E-07
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks

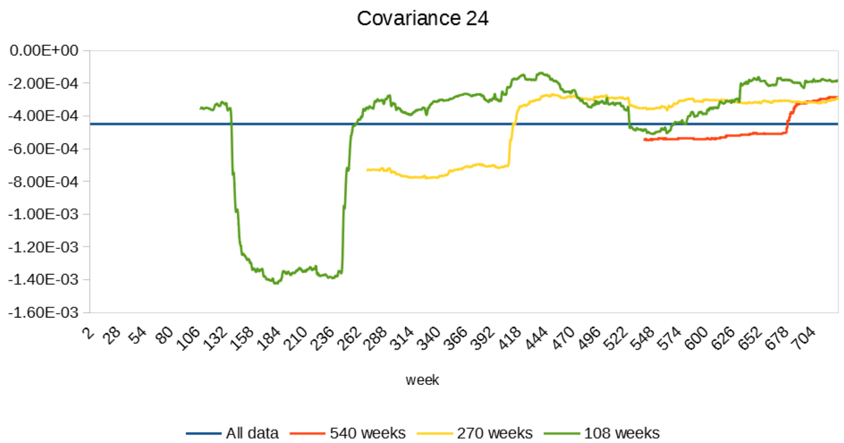

Covariance 14
4.00E-06
3.00E-06
2.00E-06
1.00E-06
0.00E+00
-1.00E-06
-2.00E-06
-3.00E-06
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks

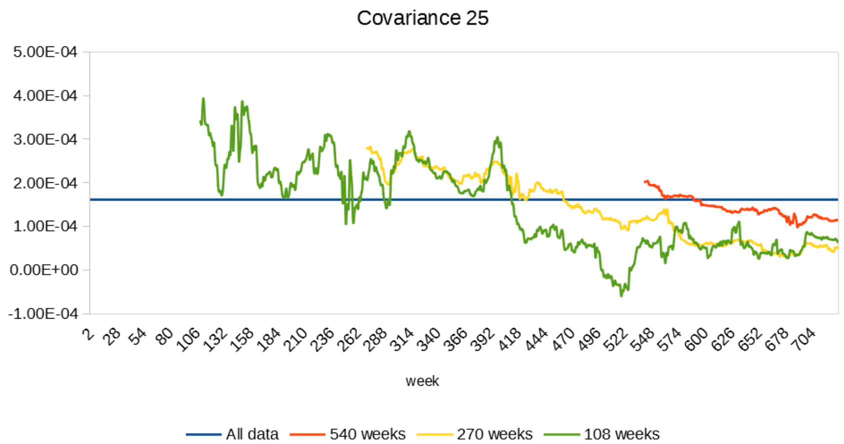

Covariance 15
2.00E-06
1.50E-06
1.00E-06
5.00E-07
0.00E+00
-5.00E-07
-1.00E-06
-1.50E-06
-2.00E-06
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks

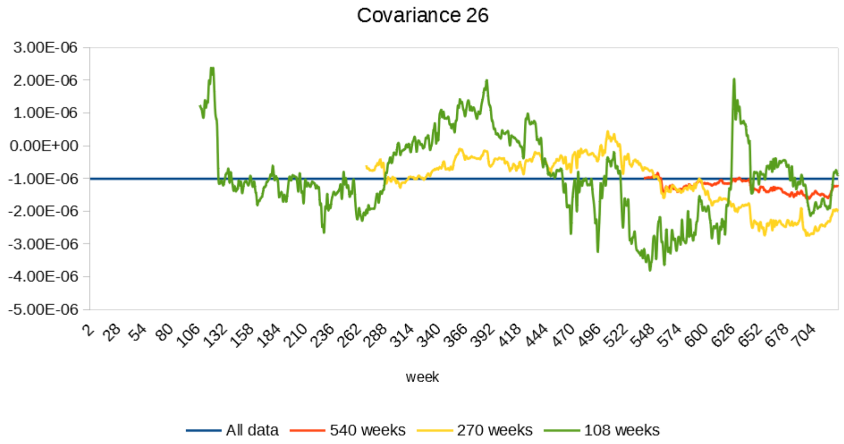

Covariance 16
2.50E-07
2.00E-07
1.50E-07
1.00E-07
5.00E-08
0.00E+00
-5.00E-08
-1.00E-07
-1.50E-07
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks

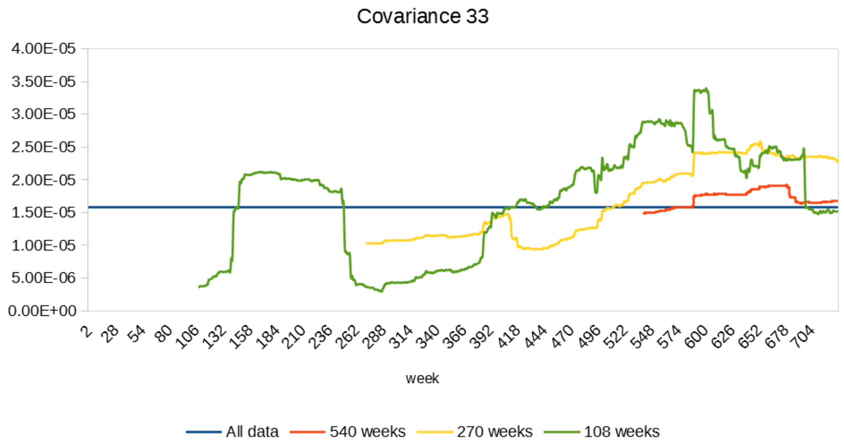

Covariance 22
3.50E-03
3.00E-03
2.50E-03
2.00E-03
1.50E-03
1.00E-03
5.00E-04
0.00E+00
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks

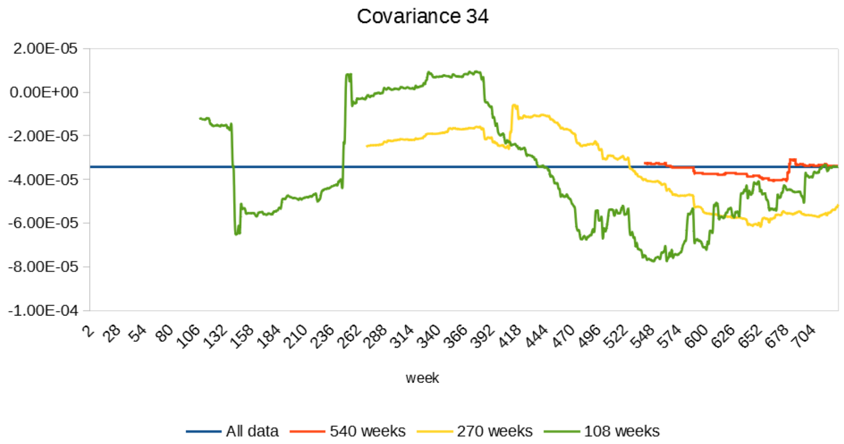

Covariance 23
1.40E-04
1.20E-04
1.00E-04
8.00E-05
6.00E-05
4.00E-05
2.00E-05
0.00E+00
-2.00E-05
-4.00E-05
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks

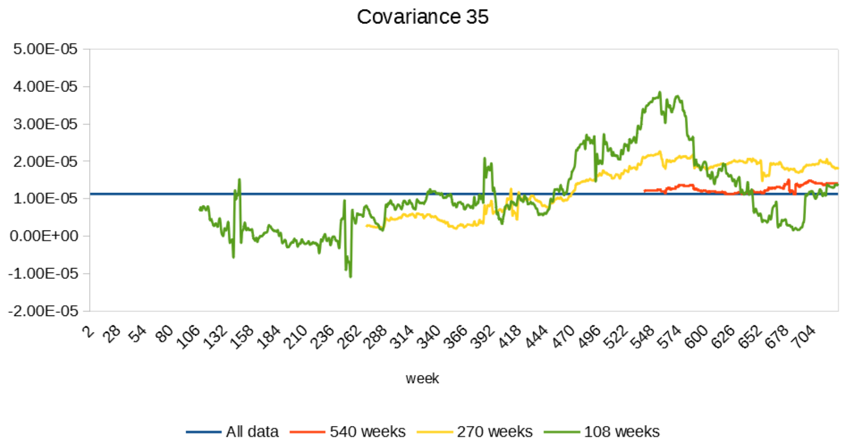

Covariance 24
0.00E+00
-2.00E-04
-4.00E-04
-6.00E-04
-8.00E-04
-1.00E-03
-1.20E-03
-1.40E-03
-1.60E-03
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks

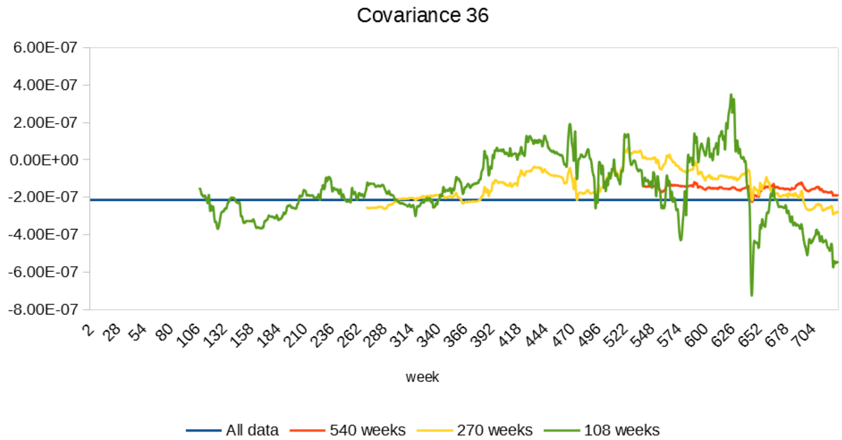

Covariance 25
5.00E-04
4.00E-04
3.00E-04
2.00E-04
1.00E-04
0.00E+00
-1.00E-04
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks


Covariance 26
3.00E-06
2.00E-06
1.00E-06
0.00E+00
-1.00E-06
-2.00E-06
-3.00E-06
-4.00E-06
-5.00E-06
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data 540 weeks 270 weeks 108 weeks

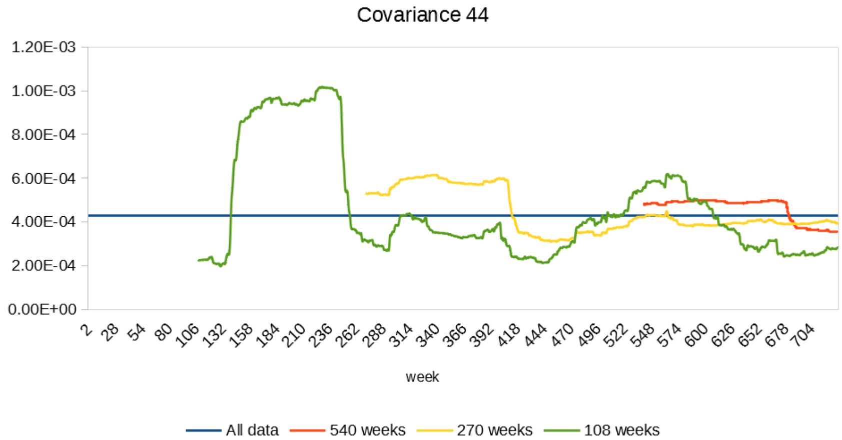

Covariance 33
4.00E-05
3.50E-05
3.00E-05
2.50E-05
2.00E-05
1.50E-05
1.00E-05
5.00E-06
0.00E+00
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data 540 weeks 270 weeks 108 weeks


Covariance 34
2.00E-05
0.00E+00
-2.00E-05
-4.00E-05
-6.00E-05
-8.00E-05
-1.00E-04
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data 540 weeks 270 weeks 108 weeks

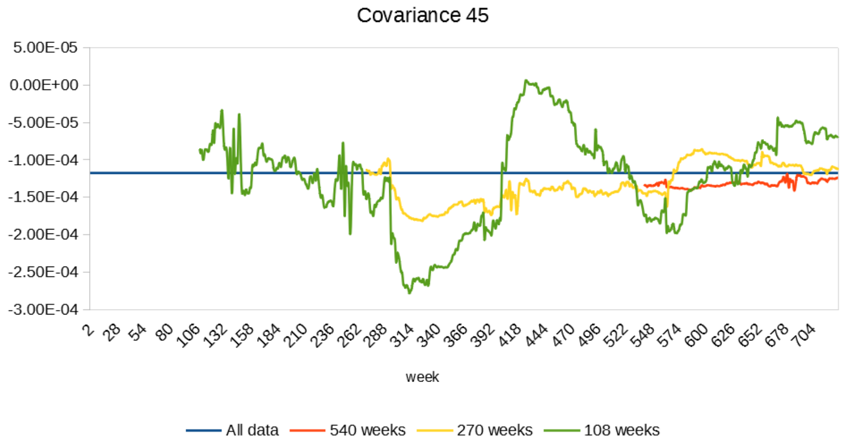

Covariance 35
5.00E-05
4.00E-05
3.00E-05
2.00E-05
1.00E-05
0.00E+00
-1.00E-05
-2.00E-05
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data 540 weeks 270 weeks 108 weeks



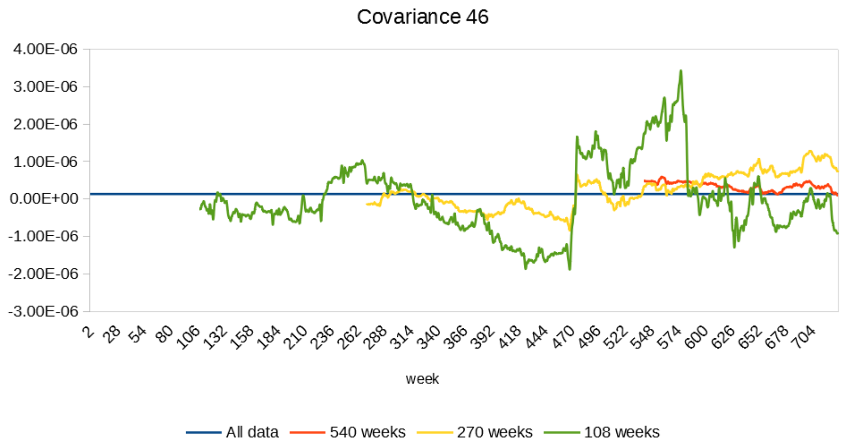
Covariance 36
6.00E-07
4.00E-07
2.00E-07
0.00E+00
-2.00E-07
-4.00E-07
-6.00E-07
-8.00E-07
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks



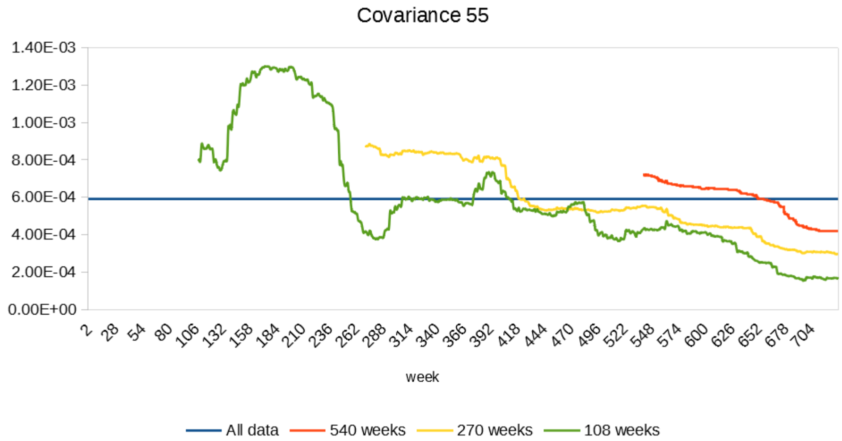
Covariance 44
1.20E-03
1.00E-03
8.00E-04
6.00E-04
4.00E-04
2.00E-04
0.00E+00
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks

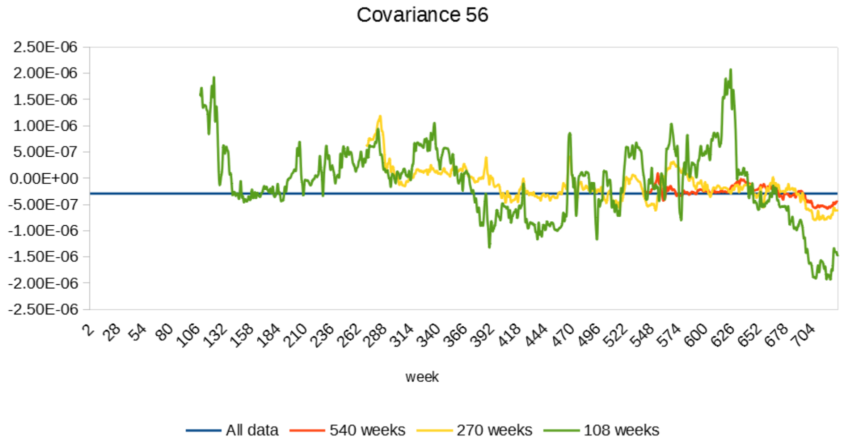


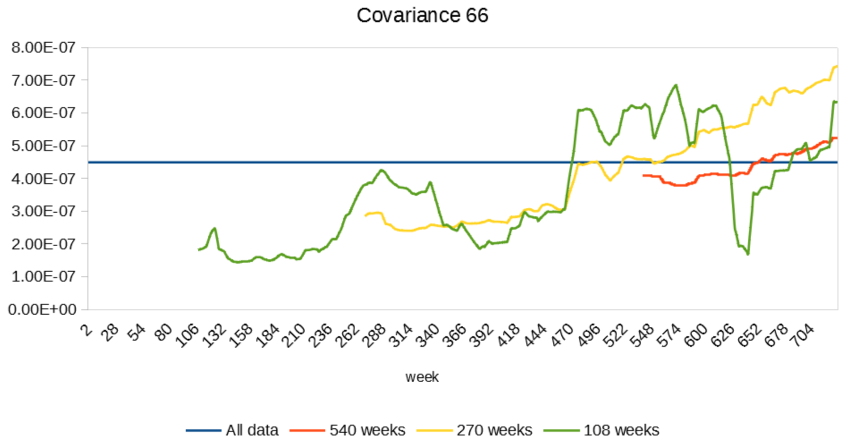
Covariance 45
5.00E-05
0.00E+00
-5.00E-05
-1.00E-04
-1.50E-04
-2.00E-04
-2.50E-04
-3.00E-04
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks



Covariance 46
4.00E-06
3.00E-06
2.00E-06
1.00E-06
0.00E+00
-1.00E-06
-2.00E-06
-3.00E-06
2 28 54 80 106 132 158 184 210 236 262 288 314 340 366 392 418 444 470 496 522 548 574 600 626 652 678 704
week
All data
540 weeks
270 weeks
108 weeks







Source: Bloomberg and author.