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JOSÉ NUNO ALMEIDA DIAS BUENO

Robust Recursive Frameworks for Discrete-Time Linear Systems Subject to Polytopic Uncertainties

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“The day you stop learning is the day you begin decaying.”

Isaac Asimov

RESUMO

BUENO, J. N. A. D. **Métodos Recursivos Robustos para Sistemas Discretos Sujeitos a Incertezas Politópicas**. 2023. Tese de Doutorado (Programa de Doutorado) – Escola de Engenharia de São Carlos, Universidade de São Paulo, São Carlos, 2023.

O problema de regulação quadrática linear para sistemas discretos tem sido assunto de pesquisa desde suas primeiras aparições na literatura nos anos 1960. Desde então, diferentes formulações e aplicações surgiram com objetivo de atender a uma ampla gama de casos teóricos e práticos, como sistemas submetidos aos efeitos de variações paramétricas desconhecidas. Mais especificamente, nesta tese nós investigamos o problema de regulação quadrática para sistemas discretos lineares e com saltos Markovianos sujeitos a incertezas politópicas. Nós definimos os problemas em termos de otimização min-max baseada em mínimos quadrados regularizados incertos e funções de penalidade. Nós consideramos os casos onde incertezas afetam matrizes do modelo e probabilidades de transição, e também sistemas com saltos Markovianos com cadeia não observada. Para cada cenário, nós elaboramos uma função de custo quadrática para acomodar todos os vértices do politopo de uma maneira unificada enquanto mantemos a convexidade dos problemas de otimização. As soluções são recursivas e produzem ganhos de realimentação de estado robustos com esforço computacional relativamente menor que o esforço despendido em abordagens baseadas em desigualdades matriciais lineares. Expandindo as estruturas matriciais das soluções, conseguimos formas reduzidas equivalentes que são mais adequadas para análises de convergência e estabilidade através de equações algébricas de Riccati. Então, considerando que algumas condições de detectabilidade e estabilizabilidade sejam satisfeitas, os ganhos de realimentação garantem a estabilidade dos sistemas em malha fechada associados. O método proposto não exige ajuste adicional de parâmetros durante a operação, o que é desejável em aplicações embarcadas e em sistemas com muitos vértices e modos Markovianos. Ademais, nós providenciamos exemplos numéricos e de aplicações para validarmos nossos resultados e para compará-los com outros controladores disponíveis na literatura de controle robusto.

Palavras-chave: Regulador quadrático linear. Controle robusto. Incertezas politópicas. Sistemas lineares discretos. Sistemas sujeitos a saltos Markovianos. Equações algébricas de Riccati. Otimização.

ABSTRACT

BUENO, J. N. A. D. **Robust Recursive Frameworks for Discrete-Time Linear Systems Subject to Polytopic Uncertainties**. 2023. Doctoral thesis (Doctorate Program) – São Carlos School of Engineering, University of São Paulo, São Carlos, 2023.

The linear quadratic regulation problem for discrete-time systems has been subjected to research since its first appearance in the literature in the 1960s. Thereafter, different formulations and applications came to light to accommodate a wide range of theoretical and practical cases, such as systems undergoing the effects of unknown parametric variations. More specifically, in this thesis, we investigate the quadratic regulation problem for discrete-time linear and Markov jump linear systems subject to polytopic uncertainties. We define the problems regarding min-max optimization based on regularized least squares with uncertain data and penalty functions. We consider the cases where uncertainties affect the model matrices and transition probabilities and Markov jumps systems with unobserved chains. For each scenario, we designed a quadratic cost function to take all polytopic vertices into account in a unified manner while keeping the optimization problems' convexity. The recursive solutions yield robust state feedback gains with a relatively lower computational burden if compared, for instance, with linear matrix inequalities approaches. By expanding the matrix structures of the solutions, we achieved equivalent reduced forms that are more adequate for convergence and stability analyses based on algebraic Riccati equations. Then, provided that some detectability and stabilizability conditions hold, the feedback gains ensure the stability of the associated closed-loop systems. The proposed method requires no further parameter tuning during operation, which is desirable in embedded applications and in systems with many vertices and Markov modes. Furthermore, we provide numerical and application examples to validate our results and to compare them with other approaches available in the literature on robust control.

Keywords: Linear quadratic regulator. Robust control. Polytopic uncertainties. Discrete-time linear systems. Markov jump systems. Algebraic Riccati equations. Optimization.

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LIST OF ABBREVIATIONS AND ACRONYMS

DMJLS	Discrete-time Markov jump linear systems
LMI	Linear matrix inequality
LQR	Linear quadratic regulator
RLQR	Robust recursive linear quadratic regulator
PMRRU	Robust recursive regulator for polytopic DMJLS with unobserved chain

LIST OF SYMBOLS

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	Set of n-dimensional vectors with real elements
$\mathbb{R}^{n\times m}$	Set of $n \times m$ real matrices
A^T	Transpose of matrix A
A^{-1}	Inverse of matrix A
$A^{\dagger}$	Pseudoinverse of matrix A
$A^{1/2}$	Square root of positive semidefinite matrix A
$A^{\circ 1/2}$	Element-wise square root of matrix A
I_n	Identity matrix with dimensions $n \times n$
$\mathbb{E}\{\cdot\}$	Expected value operator
$\otimes$	Kronecker product operator
$P > 0$	P is a positive definite matrix
$P \geq 0$	P is a positive semidefinite matrix
$\|x\|$	Euclidean norm defined as $\left(x^T x\right)^{1/2}$
$\|x\|_P$	Weighted Euclidean norm defined as $\left(x^T P x\right)^{1/2}$ with $P > 0$
$X^T P(\bullet)$	Simplification for $X^T P X$
$\mathbb{1}_n$	Column vector of ones with dimension $n \times 1$
k	Time index
α_k	Coefficients of a polytope with V vertices at instant k, with $\alpha_k \in \mathbf{R}^V$
μ	Penalty parameter
$\leftarrow$	Assignment operator
s	Number of possible modes in a Markov chain
$:=$	Definition operator
θ_k	Markov mode at instant k

Θ	Finite set of Markov modes
Prob$(\cdot)$	Probability operator
p_{ij}	Probability of jumping from mode i to mode j
$r_\sigma(A)$	Spectral radius of a matrix A
$\beta \to \gamma^+$	β approaches γ from the right, where $\beta, \gamma \in \mathbb{R}$
diag$(\cdot)$	Block-diagonal matrix whose diagonal blocks are the operands
$\mathbb{H}^{n,m}$	Linear space composed of all s-sequences of matrices $\mathbf{A} = (A_1, \ldots, A_s)$, $A_i \in \mathbb{R}^{n \times m}$, $i = 1, \ldots, s$
$\mathbb{H}^n_+$	Linear space composed of all s-sequences of matrices $\mathbf{A} = (A_1, \ldots, A_s) \in \mathbb{H}^{n,n}$ such that $A_i = A_i^T > 0$, $i \in \{1, \ldots, s\}$

CONTENTS

1 INTRODUCTION

Since its appearances in the literature in the 1960s and 1970s (KALMAN, 1960a), (KALMAN, 1960b), (DORATO; LEVIS, 1971), the linear quadratic regulation problem (LQR, for short) became a very visited subject regarding control systems. From then on, researchers evolved the LQR formulations to accommodate different practical applications (JAEN et al., 2006), (KO; JATSKEVICH, 2007), (FERRESE et al., 2011), (GATSIS; RIBEIRO; PAPPAS, 2014), (WANG et al., 2019), and theoretical cases (JI; CHIZECK, 1990), (BEMPORAD et al., 2002), (CAO; REN, 2010), (GOMMANS et al., 2014). In this sense, researchers turned their efforts to uncertainty, sensitivity analyses, and disturbance rejection on these class problems, given the need to sidestep the performance degradation generated by unknown parameters and exogenous inputs (SEZER; SILJAK, 1981), (SWORDER, 1977). The robust LQR problem is still a subject of research efforts and relates to different theoretical and practical cases (see, for instance, Petersen and McFarlane (1994), Chizeck et al. (1986), Polyak and Tempo (2001), Tzortzis, Charalambous and Hadjicostis (2020)).

In special, Terra, Cerri and Ishihara (2014) and Cerri and Terra (2017) presented recursive frameworks for the robust recursive linear quadratic regulation problem (RLQR for short) regarding discrete-time linear and Markov jump linear systems with norm-bounded uncertainties. The authors formulated min-max optimization problems regarding regularized least-squares and penalty functions, such that some quadratic cost function is minimized whilst the system undergoes the worst case of uncertainties. They obtained, thereafter, optimal robust solutions defined in terms of symmetric matrix arrangements and Riccati equations, which were later applied with success on different real-world systems (JUTINICO et al., 2017), (NAKAI et al., 2018), (BARBOSA et al., 2019), (BENEVIDES et al., 2019), (BUENO et al., 2019), (MORAIS et al., 2020), (MARCOS et al., 2022).

In this work, on the other hand, we consider the robust linear quadratic regulation of discrete-time dynamic systems whose regions of uncertainties are polytopes. In the following sections, we will introduce the problems under study in this thesis and provide a literature review on robust control for the classes of dynamic systems subject to polytopic uncertainties we are interested in.

1.1 Discrete-Time Linear Systems Subject to Polytopic Uncertainties

In the past few decades, the characteristics of linear systems undergoing the effects of polytopic uncertainties drew the attention of many academics. Undoubtedly, this class of systems has proven to be helpful in a wide range of practical applications, including power systems (CUI et al., 2017), (SADABADI; SHAFIEE; KARIMI, 2018), electronic circuits (ZHAO et al., 2014), robotic manipulation (YU; CHEN; WOO, 2002), (JABALI; KAZEMI, 2017), autonomous navigation of ground vehicles (NGUYEN et al., 2018), (HANG; CHEN;

LUO, 2019), aircraft systems (FENG et al., 2005), (HUANG et al., 2013), among others. Not surprisingly, the success of these implementations is due to advancements in control theory for dynamic systems in polytopic domains. Oliveira, Bernussou and Geromel (1999) presented a highly relevant result on the robust stability of discrete-time polytopic systems. Inserting a matrix variable into the well-known Lyapunov function yielded a less conservative manner of obtaining parameter-dependent Lyapunov functions via linear matrix inequalities (LMI)-based optimization problems. Indeed, this method laid the foundations for further research towards novel conditions for the stability of polytopic systems (DAAFOUZ; BERNUSSOU, 2001), (RAMOS; PERES, 2001), (GEROMEL; OLIVEIRA; BERNUSSOU, 2002), (GERSHON; SHAKED, 2006), (DONG; YANG, 2007), (MORENO-MORA; BECKENBACH; STREIF, 2022), (ZHU; ZHENG, 2020), (CAO; LIU; LU, 2022).

Among the numerous approaches found in the literature on robust control, H_2 and H_∞ feedback synthesis became renowned for treating parametric uncertainties varying inside convex hulls. Caigny et al. (2010), for instance, delineated exponential stability conditions through parameter-dependent LMI to obtain an upper bound for H_2 and H_∞ performance criteria. Also, the authors provided systematic procedures for the computation of gain-scheduled static output feedback controllers and validated their claims in a vibroacoustic system. Geromel, Korogui and Bernussou (2007) achieved less conservative LMI conditions for the guaranteed cost H_2 and H_∞ control problems. In addition, regarding computation speed, the Frank-Wolfe algorithm for quadratic programming (FRANK; WOLFE, 1956) is pointed out to take advantage of the convex structure of the defined problem. Hosoe, Hagiwara and Peaucelle (2018) presented a noteworthy analysis on robust stability and robust stabilization of discrete-time systems expressed by random polytopes. In this study, an auxiliary variable is added with the intention of evaluating the effect of randomness on the system dynamics at the expense of increasing numerical complexity since the new inequality constraints include expectations that are not straightforward to handle. The recent work by Pereira, Oliveira and Kienitz (2021) presented an H_2 control synthesis with reduced conservatism based on a poly-quadratic condition. The authors achieve this goal by extending the results outlined by Pandey and de Oliveira (2017), which, in turn, includes the results presented by Daafouz and Bernussou (2001) as a particular case. For completeness, the reader can also find deeper discussions about LMI-based control techniques and convex optimization in the classic books by Boyd et al. (1994) and Boyd and Vandenberghe (2004).

A common factor of the above references is using LMIs to define stability and stabilizability conditions. However, as the number of inequalities usually depends on the number of vertices describing the uncertainties, the computational effort required for feedback gain realization might become excessive. This aspect motivates us to pursue robust and computationally efficient solutions for the quadratic regulation problem of discrete-time linear systems subject to polytopic uncertainties.

1.2 Discrete-Time Markov Jump Linear Systems Subject to Polytopic Uncertainties

Many systems experience sudden changes in their dynamics due to external disturbances, sensor or actuator failures, and shifting of operation points in linearized plants, among other factors. For minor effects on the system behavior, we can use classical techniques for sensitivity analysis. Otherwise, a stochastic approach is preferable to investigate stability and stabilization (COSTA; FRAGOSO; MARQUES, 2005). In such cases, it is possible to model the dynamics via an ensemble of discrete-time linear systems orchestrated by a Markov chain. The transition from a Markov mode to another, or permanency in the actual mode, obeys a set of transition probabilities inherent to each specific process. Costa, Fragoso and Marques (2005) contributed with valuable discussions on control and filtering problems for discrete-time Markov jump linear systems (DMJLS, abbreviated) and provide a collection of examples to illustrate the relevance of this class of problems.

More specifically, the control problem for DMJLS subject to polytopic uncertainties has been gaining importance and attention in the scientific community. The case with perfectly observed Markov chain was addressed, for instance, in Boukas and Liu (2000) and Alattas et al. (2022) regarding systems with time-delay; Zhang, Song and Cai (2022) for constrained model predictive control; Gabriel, Gonçalves and Geromel (2018) presented a differential LMI approach for optimal H_2 and H_∞ control synthesis; and Vargas et al. (2022) for linear parameter-varying (LPV) systems. Within the class of DMJLS with observed Markov modes, systems with polytopic uncertain probabilities are also widely investigated. In real-world systems, the transition probabilities are hard to obtain and are usually estimated based on experimental data (SHI; LI, 2015). Therefore, these quantities are prone to identification errors which might degenerate the overall performance and even cause instability (XIONG et al., 2005). Clearly, this is a more complex problem with fundamental importance for practical applications. Gonçalves, Fioravanti and Geromel (2012) examined robust and networked control problems under H_∞ performance criteria; Lu, Li and Xi (2013) focused on model predictive control; Lopes et al. (2019) also investigated the model predictive control, but assuming input and state constraints; and Zacchia Lun, D'Innocenzo and Di Benedetto (2019) introduced necessary and sufficient conditions for robust mean-square stability of polytopic time-inhomogeneous DMJLS.

Another problem of interest is the robust control of DMJLS, whose Markov chain is partially or entirely unobserved. In the first, the information about the active mode is intermittent, whereas in the second such information is never available to the controller. Notable results have been reported in the related literature. For example, Costa, Fragoso and Todorov (2015) presented the detector based approach for DMJLS with partial information on the Markov chain and H_2 control, in which the Bernoulli jump case is also handled; Todorov and Fragoso (2016) examined the mixed H_2/H_∞ controller synthesis; de Oliveira, Costa and Daafouz (2020) provided results regarding H_2, H_∞ and mixed H_2/H_∞ control problems aided by the detector approach; and Oliveira, Costa and Gabriel (2022) focused on the H_∞ output feedback control. These robust methods proposed for DMJLS require a certain number of LMIs to be satisfied in the designed optimization problems, whose solutions yield the controller gains. With this in mind, we are interested in finding recursive solutions to sidestep LMI-based synthesis and achieve lower computational times to yield state feedback gains.

1.3 Objectives

We present robust recursive solutions for quadratic regulation problems regarding linear and Markov jump linear systems subject to polytopic uncertainties. Our approach is based on regularized least-squares with uncertain data and penalty functions whilst simultaneously weighting both the actual input signal and future states. We formulate min-max optimization problems whose quadratic cost functions consider the polytopic vertices in a unified manner. The solutions for such problems yield robust state feedback gains. Convergence and stability conditions are well established in terms of algebraic Riccati equations. Thus, the existence of our recursive solutions is not checked via LMIs. Furthermore, we impose no restrictions on how fast the uncertainties vary within the polytopes between two consecutive iterations.

We organized the remaining of this document in the following way:

- In Chapter 2, we outline and discuss some useful preliminary concepts, namely least-squares problems, the penalty function method, and Riccati equations. They will be of fundamental importance throughout the work.

- In Chapter 3, we investigate the quadratic regulation problem for discrete-time linear systems subject to polytopic uncertainties. We provide a robust recursive solution and conditions for convergence and stability. Then, we validate the given solution via numerical examples.

- In Chapter 4, we devote to the quadratic regulation problem for DMJLS subject to polytopic uncertainties. We assume perfect knowledge of transition probabilities to design the quadratic cost function. Conditions for convergence and stability are defined by achieving reduced forms of the solution. We also give numerical examples for validation and comparison purposes.

- In Chapter 5, we address the regulation problem of DMJLS, whose transition probabilities are also subject to polytopic uncertainties. We design a cost function that accounts for the uncertainties in the probabilities whilst keeping its quadratic structure. We then propose the associated recursive solution and validate it with numerical examples.

- In Chapter 6, we focus on the regulation problem of polytopic DMJLS with unobserved (hidden) modes. We yield an augmented system where the knowledge about the actual active mode is interpreted as uncertainty. It is possible to recover the original variables from the augmented ones. We close the chapter with numerical examples.

- We finish the text in Chapter 7 with concluding remarks. Moreover, we discuss some open problems related to polytopic discrete-time systems as promising subjects for future research efforts.

- For completeness, in Appendix A, we give some auxiliary results regarding matrix analysis applied along this research, and we outline the procedure for identification of powertrain model for heavy-duty ground vehicles in Appendix B.

1.4 Published papers

The following journal and conference papers regard the results and studies carried out throughout this research work.

1. J. N. A. D. Bueno, K. D. T. Rocha and M. H. Terra, *Robust Recursive Regulator for Systems Subject to Polytopic Uncertainties*. IEEE Access, 2021.

2. J. N. A. D. Bueno, L. B. Marcos, K. D. T. Rocha and M. H. Terra, *Regulation of Markov Jump Linear Systems Subject to Polytopic Uncertainties*. IEEE Transactions on Automatic Control, 2022.

3. J. N. A. D. Bueno, L. B. Marcos, K. D. T. Rocha and M. H. Terra, *Regulation of Uncertain Markov Jump Linear Systems With Application on Automotive Powertrain Control*. IEEE Transactions on Systems, Man, and Cybernetics: Systems, 2023.

4. J. N. A. D. Bueno, K. D. T. Rocha and M. H. Terra, *Gain-Scheduled Robust Recursive Lateral Control for Autonomous Ground Vehicles Subject to Polytopic Uncertainties*. Latin Amer. Robotics Symp. (LARS), 2020.

5. J. N. A. D. Bueno, L. B. Marcos, K. D. T. Rocha and M. H. Terra, *Longitudinal Control of an Autonomous Truck With Unobserved Gears*. IEEE URUCON, 2021.

6. J. N. A. D. Bueno, K. D. T. Rocha, L. B. Marcos and M. H. Terra, *Mode-Independent Regulator for Polytopic Markov Jump Linear Systems*. 30th Mediterranean Conf. on Control and Automation (MED), 2022.

7. J. N. A. D. Bueno, L. B. Marcos, K. D. T. Rocha and M. H. Terra, *Robust Regulation of Markov Jump Linear Systems with Uncertain Polytopic Transition Probabilities*. European Control Conference (ECC), 2022.

The author also contributed to the following journal and conference papers during the doctorate program.

1. G. A. P. de Morais, L. B. Marcos, J. N. A.D. Bueno, N. F. de Resende, M. H. Terra, V. Grassi Jr, *Vision-based robust control framework based on deep reinforcement learning applied to autonomous ground vehicles*. Control Engineering Practice, 2020.
2. K. D. T. Rocha, J. N. A. D. Bueno, L. B. Marcos and M. H. Terra, *Robust Kalman Filtering for Systems Subject to Polytopic Uncertainties*. 30th Mediterranean Conference on Control and Automation (MED), 2022.
3. L. B. Marcos, J. N. A. D. Bueno, K. D. T. Rocha, and M. H. Terra, *Longitudinal control of self-driving heavy-duty vehicles: a robust Markovian approach*. IEEE 61st Conference on Decision and Control (CDC), 2022.
4. K. D. T. Rocha, J. N. A. D. Bueno, L. B. Marcos, M. H. Terra, *Polytopic Robust Distributed Kalman Consensus Filter for Sensor Networks*. 8th IFAC Symposium on System Structure and Control (SSSC), 2022.

Finally, the author contributed to the following book chapter during the doctorate program.

1. G. A. P. de Morais, L. B. Marcos, J. N. A. D. Bueno, M. H. Terra, and V. Grassi Jr. *Deep learning for autonomous vehicles: a vision-based approach to self adapted robust control*. In: Fei Hu; Iftikhar Rasheed. (Org.). Deep Learning and Its Applications for Vehicle Networks. 1st ed., 2023, v. 1, p. 215-233.

2 THEORETICAL BACKGROUND

In this chapter, we revisit the well-known least-squares problem when data and measures are subject to uncertainties (KAILATH; SAYED; HASSIBI, 2000), (CERRI, 2009), Sayed and Nascimento (1999), (TERRA; CERRI; ISHIHARA, 2014), Sayed, Nascimento and Cipparrone (2002), and Sayed and Chen (2002). Based on optimization problems, we present, from a polytopic point of view, convex solutions that encompass all vertices of the uncertainties. In the following, we present a classical way to deal with constrained optimization problems through penalty functions. They are useful for placing the constraints into the cost function with a penalty parameter. If any constraints are violated, it imposes a high cost in the optimization process, resulting in an equivalent unconstrained optimization problem. More importantly, given that some conditions are satisfied, the solution of this equivalent problem converges to the original constrained problem. We outline some critical results on the penalty functions method, which can be found in greater detail in the specialized literature, such as in Luenberger and Ye (2010), Albert (1972) and Bazaraa, Sherali and Shetty (2006). Fundamental concepts on controllability, observability and mean square stabilizability of o discrete-time linear systems with and without subject to Markovian jumps are provided (LANCASTER; RODMAN, 1995), (BERTSEKAS, 2005), and (COSTA; FRAGOSO; MARQUES, 2005). We also present Riccati algebraic equations for both classes of systems, which define the central framework we will adopt to develop the control approaches of this work.

2.1 Least-Squares Problems

2.1.1 Weighted least-squares

Consider the quadratic optimization problem

$$\min_{x} \; J(x), \tag{1}$$

whose quadratic cost function $J:\ \mathbb{R}^r \to \mathbb{R}$ is defined as

$$J(x) = \|Ax - b\|_W^2 = (Ax - b)^T W (Ax - b), \tag{2}$$

where W is a known symmetric positive definite weighting matrix, $A \in \mathbb{R}^{a\times r}$ and $b \in \mathbb{R}^a$ are known and $x \in \mathbb{R}^r$ is an unknown vector.

Lemma 2.1. *(KAILATH; SAYED; HASSIBI, 2000) A vector $\hat{x} \in \mathbb{R}^r$ is a minimizer of the cost function* (2) *if, and only if, it satisfies the normal equation*

$$A^T W A\hat{x} = A^T W b, \tag{3}$$

and the corresponding minimal value of (2) *is given by*

$$J(\hat{x}) = \|A\hat{x} - b\|_W^2 = b^T W b - b^T W A\hat{x}. \tag{4}$$

If A has full column rank, the solution $\hat{x}$ is unique and given by

$$\hat{x} = (A^T W A)^{-1} A^T W b. \tag{5}$$

And in this case, the minimal cost is

$$J(\hat{x}) = \|A\hat{x} - b\|_W^2 = b^T (W - WA(A^T W A)^{-1} A^T W) b. \tag{6}$$

The following lemma remodels the solution for problem (1)–(2) through a matrix arrangement.

Lemma 2.2. *(CERRI, 2009) Consider the problem* (1)–(2) *with $W = W^T > 0$. The following expressions are equivalent:*

(i)

$$\hat{x} = \arg\min_x (Ax - b)^T W (Ax - b). \tag{7}$$

(ii) $x = \hat{x}$ is a solution to $A^T W A \hat{x} = A^T W b$.

(iii) $(\gamma, x) = (\hat{\gamma}, \hat{x})$ is a solution to

$$\begin{bmatrix} W^{-1} & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \gamma \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}. \tag{8}$$

If A has full column rank, then the solution $\hat{x}$ is unique and given by

$$\hat{x} = (A^T W A)^{-1} A^T W b, \tag{9}$$

and, in this case, the respective minimal cost is

$$J(\hat{x}) = \begin{bmatrix} b^T & 0 \end{bmatrix} \begin{bmatrix} W^{-1} & A \\ A^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix}. \tag{10}$$

The nonsingularity of the central matrix block is assured by Lemma A.3 in Appendix A.

2.1.2 Regularized least-squares

Consider the quadratic optimization problem

$$\min_x \; J(x), \tag{11}$$

whose quadratic cost function $J : \mathbb{R}^r \to \mathbb{R}$ is defined as

$$J(x) = \|x\|_Q^2 + \|Ax - b\|_W^2 = x^T Q x + (Ax - b)^T W (Ax - b), \tag{12}$$

where Q and W are known symmetric positive definite weighting matrices, $A \in \mathbb{R}^{a \times r}$ and $b \in \mathbb{R}^a$ are known and $x \in \mathbb{R}^r$ is an unknown vector. We see that from problem (11)–(12) we recover the least-squares problems introduced in Section 2.1.1 by selecting $Q = 0$.

Lemma 2.3. *(SAYED, 2001) The solution $\hat{x}$ for the optimization problem* (11)–(12) *is given by*

$$\hat{x} = (Q + A^T W A)^{-1} A^T W b, \tag{13}$$

and the minimal value of the cost function (12) *is then*

$$J(\hat{x}) = b^T (W^{-1} + A Q^{-1} A^T)^{-1} b. \tag{14}$$

2.1.3 Regularized least-squares subject to uncertainties

Let us now address the case in which matrices A and b are subject to structured parametric uncertainties, i.e., $A = A_0 + \delta A$, and $b = b_0 + \delta b$. Thus, consider the optimization problem

$$\min_{x} \max_{\delta A,\ \delta b} J(x, \delta A, \delta b). \tag{15}$$

The cost function $J(x, \delta A, \delta B)$ is defined as

$$\begin{aligned} J(x, \delta A, \delta b) &= \|x\|_Q^2 + \|(A_0 + \delta A)x - (b_0 + \delta b)\|_W^2 \\ &= x^T Q x + ((A_0 + \delta A)x - (b_0 + \delta b))^T W ((A_0 + \delta A)x - (b_0 + \delta b)), \end{aligned} \tag{16}$$

where Q and W are known symmetric positive definite weighting matrices, $A_0 \in \mathbb{R}^{a \times r}$ and $b_0 \in \mathbb{R}^a$ are known and $x \in \mathbb{R}^r$ is an unknown vector. The uncertainties $\{\delta A, \delta b\}$ are modeled as:

$$\begin{bmatrix} \delta A & \delta b \end{bmatrix} = M \Gamma \begin{bmatrix} \hat{A} & \hat{b} \end{bmatrix}, \tag{17}$$

with known $M \in \mathbb{R}^{a \times pV}$, $\Gamma = \textbf{diag}\{\alpha_1, \ldots, \alpha_V\} \otimes I_p$, known vertices $A^{(l)} \in \mathbb{R}^{p \times r}$, $b^{(l)} \in \mathbb{R}^p$, $l = 1, \ldots, V$, such that $\hat{A} = \begin{bmatrix} (A^{(1)})^T \cdots (A^{(V)})^T \end{bmatrix}^T$, $\hat{b} = \begin{bmatrix} (b^{(1)})^T \cdots (b^{(V)})^T \end{bmatrix}^T$, and $\alpha = \begin{bmatrix} \alpha_1 \ldots \alpha_V \end{bmatrix}^T$ belongs to the unit simplex Λ_V defined by

$$\Lambda_V := \left\{ \alpha \in \mathbb{R}^V \ : \ \alpha_l \geq 0, \ \sum_{l=1}^{V} \alpha_l = 1 \right\}. \tag{18}$$

Clearly, the uncertain case accommodates the regularized least-squares problem without uncertainties presented in Section 2.1.2 by choosing $A^{(l)} = 0$ and $b^{(l)} = 0$, $l = 1, \ldots, V$.

The next result is based on Cerri (2009) and Sayed and Nascimento (1999) and provides the unique solution for the min-max optimization problem (15)–(16).

Lemma 2.4. *Consider the optimization problem* (15)–(16)*. The following sentences are equivalent:*

(i) For $Q > 0$, there is a unique $\hat{x} := x(\hat{\lambda})$ such that

$$\begin{aligned} \hat{x} = \arg \ \min_{x} \max_{\delta A, \delta b} J(x, \delta A, \delta b), \\ \begin{bmatrix} \delta A & \delta b \end{bmatrix} = M \Gamma \begin{bmatrix} \hat{A} & \hat{b} \end{bmatrix}. \end{aligned}$$

(ii) For $Q > 0$, there is a unique $\hat{x}$ such that

$$\hat{x} = \arg \min_{x} \left\{ \left(\begin{bmatrix} I \\ A_0 \\ \hat{A} \end{bmatrix} x - \begin{bmatrix} 0 \\ b_0 \\ \hat{b} \end{bmatrix} \right)^T \begin{bmatrix} Q & 0 & 0 \\ 0 & W(\hat{\lambda}) & 0 \\ 0 & 0 & \hat{\lambda} I \end{bmatrix} \left(\bullet \right) \right\},$$

where the Lagrange multiplier $\hat{\lambda}$ is given by

$$\hat{\lambda} := \arg \min_{\lambda > \|M^T W M\|} \{ f(\lambda) \}, \tag{19}$$

with

$$f(\lambda) := \|x(\lambda)\|_Q^2 + \lambda \|\hat{A} x(\lambda) - \hat{b}\|^2 + \|A_0 x(\lambda) - b_0\|_{W(\lambda)}^2, \tag{20}$$

$$x(\lambda) := [Q(\lambda) + A_0^T W(\lambda) A_0]^{-1} [A_0^T W(\lambda) b_0 + \lambda \hat{A}^T \hat{b}], \tag{21}$$

$$Q(\lambda) := Q + \lambda \hat{A}^T \hat{A}, \tag{22}$$

$$W(\lambda) := W + WM \left(\lambda I - M^T W M \right)^{\dagger} M^T W. \tag{23}$$

(iii) $(\alpha, \zeta, \gamma, x) = (\hat{\alpha}, \hat{\zeta}, \hat{\gamma}, \hat{x})$ is a solution to

$$\begin{bmatrix} Q^{-1} & 0 & 0 & I \\ 0 & W(\hat{\lambda})^{-1} & 0 & A_0 \\ 0 & 0 & \hat{\lambda}^{-1} I & \hat{A} \\ I & A_0^T & \hat{A}^T & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \zeta \\ \gamma \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ b_0 \\ \hat{b} \\ 0 \end{bmatrix}.$$

In addition, the unique solution $\hat{x}$ and the corresponding cost $J(\hat{x})$ are obtained by

$$\begin{bmatrix} \hat{x} \\ J(\hat{x}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & b_0 \\ 0 & \hat{b} \\ I & 0 \end{bmatrix}^T \underbrace{\begin{bmatrix} Q^{-1} & 0 & 0 & I \\ 0 & W(\hat{\lambda})^{-1} & 0 & A_0 \\ 0 & 0 & \hat{\lambda}^{-1} I & \hat{A} \\ I & A_0^T & \hat{A}^T & 0 \end{bmatrix}}_{\mathscr{W}}^{-1} \begin{bmatrix} 0 \\ b_0 \\ \hat{b} \\ 0 \end{bmatrix}, \tag{24}$$

where $\hat{A} = \begin{bmatrix} (A^{(1)})^T & \dots & (A^{(V)})^T \end{bmatrix}^T$, and $\hat{b} = \begin{bmatrix} (b^{(1)})^T & \dots & (b^{(V)})^T \end{bmatrix}^T$.

Proof. We cast the original regularized least-squares problem with uncertain data discussed in (TERRA; CERRI; ISHIHARA, 2014) from a polytopic perspective by choosing $\{\delta A, \delta b\}$ as in (17). Then, the steps follow directly with the procedures thoroughly described in Cerri (2009). Moreover, since both weight matrices Q and W are positive definite, we have $\mathscr{W}$ nonsingular according to Lemma A.3 (see Appendix A). □

Remark 2.1. *Authors in Sayed, Nascimento and Cipparrone (2002), Sayed and Nascimento (1999), and Sayed and Chen (2002) meticulously investigated problem* (15)–(16) *with respect to its convexity. Observe that* (15)–(17) *can be expressed by*

$$\min_{x} \left\{ \|x\|_Q^2 + \max_{\delta A, \delta b} \{ \mathcal{T}(x, \delta A, \delta b) \} \right\},$$

where $\mathcal{T}(x, \delta A, \delta b) = \|(A_0 + \delta A)x - (b_0 + \delta b)\|_W^2$*. The residual function* $\mathcal{T}(x, \delta A, \delta b)$ *is convex in* x *for any pair* $\delta A, \delta b$*. As such, the maximum*

$$\mathcal{L}(x) = \max_{\delta A, \delta b} \{\mathcal{T}(x, \delta A, \delta b)\}$$

is also convex in x*. In addition, given that* $Q > 0$*, both* $\|x\|_Q^2$ *and* $\|x\|_Q^2 + \mathcal{L}(x)$ *are strictly convex in* x*. Therefore, the uniqueness of the minimizing solution* $\hat{x}$ *provided in Lemma 2.4 for problem* (15)–(16) *is guaranteed.*

Remark 2.2. *As mentioned by Sayed and Chen (2002), for any* W *positive definite and* $\lambda > \|M^T W M\|$ *we have* $\left(\hat{\lambda} I - M^T W M\right)$ *positive definite and, as such, nonsingular. Therefore, the inverse operation can replace the pseudoinverse in* (23) *and we have* $W(\lambda) = W + WM\left(\lambda I - M^T W M\right)^{-1} M^T W$*.*

Note that the solution provided in Lemma 2.4 depends on the multiplier $\hat{\lambda}$, which is the minimizer in (19). From a practical point of view, this is a drawback since an additional optimization problem over λ restricted to the open interval $(\|M^T W M\|,\ +\infty)$ must be solved to obtain $\hat{x}$ ultimately. According to Sayed (2001), $W(\lambda)$ in (23) is positive definite if $\lambda > \|M^T W M\|$, which implies $f(\lambda)$ in (20) being also positive definite with a unique global minimum. In fact, as stated in Sayed (2001) and confirmed by results shown in Sayed and Chen (2002), $f(\lambda)$ reaches amplitudes close to its minimum value for arguments λ that are not far from the lower bound $\|M^T W M\|$. Thus, we make the approximation $\hat{\lambda} \approx \beta \|M^T W M\|$, for some $\beta > 1$, to sidestep the need of finding $\hat{\lambda}$ through (19) in online applications.

2.2 Penalty Functions

Consider the following constrained minimization problem.

$$\min_x \{f(x)\}, \tag{25}$$
$$\text{subject to } g(x) = 0,$$

and assume it is replaced by

$$\min_{x \in \mathbb{R}^n} \{q(x, \mu) := f(x) + g(x)^T \mu g(x)\}, \tag{26}$$

whose solution is given by $\hat{x}(\mu)$. The term $g(x)^T \mu g(x)$, $\mu > 0$, is called penalty function and satisfies

(i) $g(x)^T g(x)$ is continuous;

(ii) $g(x)^T g(x) \geq 0$ for all $x \in \mathbb{R}^r$.

The steps to solve (25) via penalty functions are defined as follows (BAZARAA; SHERALI; SHETTY, 2006):

Step 1: Let $\{\mu_k\}$, $k = 1,\ 2,\ ...,\ \infty$, be a sequence such that, for each k, $\mu_k \geq 0$ and $\mu_{k+1} > \mu_k$. Define the problem

$$\min_{x \in \mathbb{R}^n} \{q(x, \mu_k) := f(x) + g(x)^T \mu_k g(x)\}. \tag{27}$$

Step 2: Define a real $\epsilon > 0$ as the termination criteria. Select $\vartheta > 1$ and initial conditions x_1 and $\mu_1 > 0$.

Step 3: Set $k = 1$, then:

Step 3.1: Starting with x_k, solve the follwing optimization problem:

$$\min_{x_k \in \mathbb{R}^n} \{q(x_k, \mu_k)\},$$

and obtain the optimal solution x_{k+1}. Go *Step 3.2*.

Step 3.2: If $g(x_{k+1})^T \mu_k g(x_{k+1}) < \epsilon$, then stop. Else, set $\mu_{k+1} = \vartheta \mu_k$ and go back to *Step 3.1*.

Ideally, if x^* solves (25) and $\hat{x}(\mu)$ minimizes $q(x, \mu)$ in (26), then we have $\lim_{\mu \to \infty} \hat{x}(\mu) = x^*$ and $\hat{x}(\mu)$ is optimal as stated by the next lemmas. Otherwise, if $\mu \to \infty$ is not allowed for some reason, then $\hat{x}(\mu)$ is a sub-optimal solution for the constrained problem.

Lemma 2.5. *(LUENBERGER; YE, 2010) Let $\{\mu_k\}$, $k = 1,\ 2,\ ...,\ \infty$ be a sequence such that, for each k, $\mu_k \geq 0$, $\mu_{k+1} > \mu_k$, and $q(x, \mu) = f(x) + g(x)^T \mu g(x)$. Then,*

(i) $q(x_k, \mu_k) \leq q(x_{k+1}, \mu_{k+1})$;

(ii) $g(x_k)^T g(x_k) \geq g(x_{k+1})^T g(x_{k+1})$;

(iii) $f(x_k) \leq f(x_{k+1})$.

Lemma 2.6. *(LUENBERGER; YE, 2010) Let x^* be a solution to* (25)*. Then, for each $k = 1,\ 2,\ ...,\ \infty$ we have*

$$f(x^*) \geq q(x_k, \mu_k) \geq f(x_k).$$

Theorem 2.1. *(LUENBERGER; YE, 2010) Let $\{x_k\}$, $k = 1,\ 2,\ ...,\ \infty$, be a sequence generated by the penalty functions method. Then, any limit point of this sequence is a solution to* (25)*.*

Observe that we incorporate, via penalty functions, the problem constraints into the cost function. As we will see in the next chapters, this allows us to encompass all polytope vertices at once. Thenceforth we must ultimately solve a single matrix equation with minimal parameter tuning in order to obtain the state feedback gains.

Remark 2.3. *Although the literature on optimal control introduces μ_k as a variable, we will tune it to a constant value μ. As such, this parameter can be interpreted as an additional weight in the robust regularized least-squares problem.*

2.3 Bellman's Principle of Optimality

Let us consider the following finite horizon optimization problem:

$$\min_{x_{k+1},u_k} \left\{ \|x_N\|^2_{P_N} + \sum_{t=0}^{N-1} (\|x_t\|^2_{Q_t} + \|u_t\|^2_{R_t}) \right\}, \quad (28)$$
$$\text{subject to } x_{k+1} = F_k x_k + G_k u_k,$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $P_N > 0$, $Q_k > 0$ and $R_k > 0$, $k = 0, \ldots, N$.

The optimal solution for (28) satisfies Bellman's Principle of Optimality, which can be stated as follows (BERTSEKAS, 2005).

> ***Bellman's Principle of Optimality.*** *Let $\hat{X} = \{\hat{X}_0, \hat{X}_1, \ldots, \hat{X}_{N-1}\}$ be the optimal policy that solves* (28)*, where $\hat{X}_k = \{\hat{x}_{k+1}, \hat{u}_k\}$. Consider now the sub-problem*
>
> $$\min_{x_{k+1},u_k} \left\{ \|x_N\|^2_{P_N} + \sum_{t=t_0}^{N-1} (\|x_t\|^2_{Q_t} + \|u_t\|^2_{R_t}) \right\},$$
> $$\text{subject to } x_{k+1} = F_k x_k + G_k u_k.$$
>
> *Then, the truncated policy $\{\hat{X}_{t_0}, \hat{X}_{t_0+1}, \ldots, \hat{X}_{N-1}\}$ is optimal for this sub-problem.*

Two fundamental aspects arise when we consider Bellman's Principle of Optimality to solve a finite horizon optimization problem. First, we solve the problem in a backward fashion, which means that we use the information from instant $t = N$ as the boundary to solve the step $t = N - 1$, then we use $t = N - 1$ as the boundary to the step $t = N - 2$ and so forth until $t = t_0$. Second, the solution is, therefore, recursive. Cerri (2009) explored these features to demonstrate in detail how to split the problem (28) into one-step problems and, furthermore, how to treat the robust case in a similar manner.

That said, in the next chapters we address a series of quadratic optimization problems with finite horizon, and separate them into one-step quadratic problems by means of the aforementioned concepts. We refer the reader to (CERRI, 2009), where the complete splitting procedure is outlined, and to (BERTSEKAS, 2005) for deeper details about Bellman's Principle of Optimality and dynamic programming.

2.4 Algebraic Riccati equations

The following concepts regarding algebraic Riccati equations are fundamental for the analyses of convergence and stability of the recursive solutions achieved throughout this thesis. The proofs are omitted, but the reader can easily find them in the classical literature about control systems and algebraic Riccati equations.

2.4.1 Discrete-time linear systems

Theorem 2.2. *(LANCASTER; RODMAN, 1995) Consider the system*

$$x_{k+1} = Ax_k + Bu_k, \tag{29}$$

$$y_k = Cx_k, \quad k = 1, ..., N-1$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. The system is said be controllable if and only if the matrix pair $\{A, B\}$ is controllable, i.e.,

$$\text{rank}\left(\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}\right) = n.$$

Theorem 2.3. *(LANCASTER; RODMAN, 1995) The system* (29) *is observable if and only if the matrix pair $\{C, A\}$ is observable, i.e.,*

$$\text{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n.$$

Definition 2.1. *(BERTSEKAS, 2005) The system $x_{k+1} = Lx_k$, where $L = A + BK$, is stable if all eigenvalues of L are located inside the open unit disc. If this is the case, $\|x_k\| \to 0$ as $k \to \infty$.*

Definition 2.2. *(LANCASTER; RODMAN, 1995) The matrix pair $\{A, B\}$ is stabilizable if there exists a feedback matrix $K \in \mathbb{R}^{n\times n}$ such that $L = A + BK$ is stable.*

Theorem 2.4. *(LANCASTER; RODMAN, 1995) If the matrix pair $\{A, B\}$ is controllable, the it is stabilizable.*

Theorem 2.5. *(BERTSEKAS, 2005) Assume the pair $\{A, B\}$ is controllable and $\{A, C\}$ is observable. Then, there exists a unique $P > 0$ such that the discrete-time algebraic Riccati equation*

$$P_k = Q + A^T P_{k+1} A - A^T P_{k+1} B \left(R + B^T P_{k+1} B\right)^{-1} B^T P_{k+1} A, \tag{30}$$

where $Q = CC^T$, converges to P as $k \to \infty$. Moreover, the eigenvalues of the corresponding closed-loop matrix $L = A + BK$, with matrix $K = -(R + B^T PB)^{-1} B^T PA$, are located inside the open unit disc.

2.4.2 Discrete-time Markov jump linear systems

Consider $\boldsymbol{A} = (A_1, \dots, A_s) \in \mathbb{H}^{n,n}$, $\boldsymbol{B} = (B_1, \dots, B_s) \in \mathbb{H}^{n,m}$, $\boldsymbol{C} = (C_1, \dots, C_s) \in \mathbb{H}^{p,n}$, $\Theta = \{1, \dots, s\}$, transition probability matrix $\mathbb{P} \in \mathbb{R}^{s\times s}$, and recall the following results borrowed from the literature on the control of discrete-time Markov jump linear systems.

Theorem 2.6. *(COSTA; FRAGOSO; MARQUES, 2005) Consider the DMJLS*

$$x_{k+1} = A_{\theta_k} x_k + B_{\theta_k} u_k, \tag{31}$$
$$y_k = C_{\theta_k} x_k,$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$, $A_{\theta_k} \in \mathbb{R}^{n\times n}$, and $B_{\theta_k} \in \mathbb{R}^{n\times m}$, for $k = 0, ..., N-1$ and $\theta_k \in \Theta$. The system with $u_k = 0$ is mean square stable (MSS) if for any initial condition $x_0 \in \mathbb{R}^n$, $\theta_0 \in \Theta$, we have $\|\mathbb{E}\{x_k\}\| \to 0$ as $k \to \infty$.

Definition 2.3. *(COSTA; FRAGOSO; MARQUES, 2005) The pair $\{\boldsymbol{A}, \boldsymbol{B}\}$ is mean square stabilizable if there exists $\boldsymbol{K} = (K_1, ..., K_s) \in \mathbb{H}^{m,n}$ such that when $u_k = K_{\theta_k} x_k$, $\theta_k \in \Theta$,* (31) *is MSS. In this case, $\boldsymbol{K}$ is said to stabilize the pair $\{\boldsymbol{A}, \boldsymbol{B}\}$.*

Definition 2.4. *(COSTA; FRAGOSO; MARQUES, 2005) The pair $\{\boldsymbol{C}, \boldsymbol{A}\}$ is mean square detectable if there exists $\boldsymbol{H} = (H_1, ..., H_s) \in \mathbb{R}^{n\times p}$ such that $r_\sigma(\mathscr{D}) < 1$, where $\mathscr{D} = (\mathbb{P}^T \otimes I_{n^2})$ **diag** $\big((A_{\theta_k} + H_{\theta_k} C_{\theta_k}) \otimes (A_{\theta_k} + H_{\theta_k} C_{\theta_k})\big)$, $\theta_k \in \Theta$.*

Theorem 2.7. *(COSTA; FRAGOSO; MARQUES, 2005) The following statements are equivalent:*

(i) System (31) *is MSS.*

(ii) $r_\sigma(\mathcal{C}\mathcal{N}) < 1$, where $\mathcal{C} = \left(\mathbb{P}^T \otimes I_{n^2}\right)$ and $\mathcal{N} = diag\{(A_{\theta_k} + H_{\theta_k} C_{\theta_k}) \otimes (A_{\theta_k} + H_{\theta_k} C_{\theta_k})\}$, for $\theta_k \in \Theta$.

(iii) For all $x_0 \in \mathbb{R}^n$ and $\theta_0 \in \Theta$, we have $\sum_{k=0}^{\infty} \mathbb{E}\left\{\|x_k\|^2\right\} < \infty$.

Definition 2.5. *(COSTA; FRAGOSO; MARQUES, 2005) $\boldsymbol{P} = (P_1, ..., P_s) \in \mathbb{H}^n_+$ is a stabilizing solution for the coupled Riccati equations*

$$P_{\theta_k k} = Q_{\theta_k} + A^T_{\theta_k} \mathcal{E}_{k+1} A_{\theta_k} - A^T_{\theta_k} \mathcal{E}_{k+1} B_{\theta_k} (R_{\theta_k} + B^T_{\theta_k} \mathcal{E}_{k+1} B_{\theta_k})^{-1} B^T_{\theta_k} \mathcal{E}_{k+1} A_{\theta_k}, \tag{32}$$

where $Q_{\theta_k} = C_{\theta_k} C^T_{\theta_k}$, if $\boldsymbol{P}$ satisfies (32) *for all $\theta_k \in \Theta$ and $\boldsymbol{K} = (K_1, ..., K_s) \in \mathbb{H}^{m,n}$ stabilizes $(\boldsymbol{A}, \boldsymbol{B})$ in the mean square sense when $u_k = K_{\theta_k} x_k$, where $\mathcal{E}_{k+1} = \sum_{j=1}^{s} p_{ij} P_{j,k+1}$ and K_{θ_k} is given by*

$$K_{\theta_k} = -(R_{\theta_k} + B^T_{\theta_k} \mathcal{E} B_{\theta_k})^{-1} B^T_{\theta_k} \mathcal{E} A_{\theta_k}, \tag{33}$$

with $\mathcal{E} = \sum_{j=1}^{s} p_{ij} P_j$.

Corollary 2.1. *(COSTA; FRAGOSO; MARQUES, 2005) If the pair $\{\boldsymbol{A}, \boldsymbol{B}\}$ is mean square stabilizable and $\{\boldsymbol{C}, \boldsymbol{A}\}$ is mean square detectable, then the stabilizing solution for the coupled algebraic Riccati equations* (32) *exists.*

Theorem 2.8. *(COSTA; FRAGOSO; MARQUES, 2005) Assume that the pair* $\{\boldsymbol{A}, \boldsymbol{B}\}$ *is mean square stabilizable. Then, for any initial condition* $\boldsymbol{P}_0 = (P_{1,0}, ..., P_{s,0}) \in \mathbb{H}_+^n$*, the sequence* $\{\boldsymbol{P}_k\} = \{(P_{1,k}, ..., P_{s,k})\}$ *converges to a solution* $\boldsymbol{P} = (P_1, ..., P_s) \in \mathbb{H}_+^n$ *for* (32) *when* $k \to \infty$*. Additionally, if the pair* $\{\boldsymbol{C}, \boldsymbol{A}\}$ *is mean square detectable, then* $\boldsymbol{P}$ *is the unique positive semidefinite stabilizing solution for* (32)*.*

The concepts outlined in Sections 2.1.3 and 2.2 are fundamental to solving the regulation problems we formulate in the next chapters for systems subject to polytopic uncertainties. As such, we derive solutions structured as matrix arrangements and, in sequence, we achieve equivalent forms suitable for convergence and stability analysis through the Riccati equations presented in Section 2.4.

3 ROBUST REGULATOR FOR SYSTEMS SUBJECT TO POLYTOPIC UNCERTAINTIES

In this chapter, we present the solution for the linear quadratic optimization problem for systems subject to uncertainties varying within a convex hull. The formulation is recursive and exhibits different characteristics when compared to well-known solutions found in the literature on polytopic systems, such as Boyd et al. (1994), Colaneri, Geromel and Locatelli (1997), Oliveira, Geromel and Bernussou (2010), Duan and Yu (2013), among others.

We write the vertices of uncertainties related to state and input matrices under the form of subsystems to ensemble a constraint set for the min-max optimization problem. Then, it is possible to rearrange this set as a single equality constraint and, by means of a penalty function, we place the vertices into the original cost. As consequence, we obtain an equivalent unconstrained minimization problem that takes into account the whole set of polytope vertices at once. As the new penalized cost function is quadratic with respect to the minimization variables, we derive the robust recursive solution in the form of a symmetric matrix arrangement based on the preliminary concepts described in Chapter 2.

In the next sections, we will formulate the control problem, outline the proposed solution and demonstrate the convergence and stability of the presented method. Finally, we provide numerical and real-world examples for validation purposes.

3.1 Problem Formulation

Consider the uncertain discrete-time linear system

$$x_{k+1} = (F_k + \delta F_k)\, x_k + (G_k + \delta G_k)\, u_k, \tag{34}$$

for $k = 0, ..., N-1$, where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input, $F_k \in \mathbb{R}^{n\times n}$ and $G_k \in \mathbb{R}^{n\times m}$ are known nominal system matrices, and $\{\delta F_k, \delta G_k\}$ are uncertainty matrices described by

$$\begin{bmatrix}\delta F_k & \delta G_k\end{bmatrix} = \sum_{l=1}^{V} \alpha_{l,k} \begin{bmatrix}F_k^{(l)} & G_k^{(l)}\end{bmatrix}, \tag{35}$$

with $F_k^{(l)} \in \mathbb{R}^{n\times n}$ and $G_k^{(l)} \in \mathbb{R}^{n\times m}$ known and coefficients $\alpha_k = \begin{bmatrix}\alpha_{1,k} \dots \alpha_{V,k}\end{bmatrix}^T$ belonging to the unit simplex with V vertices

$$\Lambda_V = \left\{ \alpha_k \in \mathbb{R}^V \;:\; \alpha_{l,k} \geq 0,\; \sum_{l=1}^{V} \alpha_{l,k} = 1 \right\}. \tag{36}$$

Assume known initial condition x_0 and states x_k observed at each instant k. Our task is to find a sequence of input signals $\{u_0^*, ..., u_{N-1}^*\}$ which makes the states x_k converge to zero despite the maximum influence of uncertainties. To this end, we define the following min-max optimization problem based on the uncertain model (34)–(35):

$$\min_{x_{k+1},u_k} \max_{\delta F_k,\delta G_k} \left\{ \|x_N\|^2_{P_N} + \sum_{t=0}^{N-1} \left(\|x_t\|^2_{Q_t} + \|u_t\|^2_{R_t} \right) \right\} \tag{37}$$

$$\text{subject to} \quad \begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} x_{k+1} = \begin{bmatrix} F_k + V\delta F_k^{(1)} \\ \vdots \\ F_k + V\delta F_k^{(V)} \end{bmatrix} x_k + \begin{bmatrix} G_k + V\delta G_k^{(1)} \\ \vdots \\ G_k + V\delta G_k^{(V)} \end{bmatrix} u_k.$$

Based on Bellman's Principle of Optimality and on principles of dynamic programming (BERTSEKAS, 2005), we approach problem (37) by splitting it into N one-step optimization problems of the form

$$\min_{x_{k+1},u_k} \max_{\delta F_k,\delta G_k} \left\{ \|x_{k+1}\|^2_{P_{k+1}} + \|x_k\|^2_{Q_k} + \|u_k\|^2_{R_k} \right\}, \tag{38}$$

$$\text{subject to:} \quad \begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} x_{k+1} = \begin{bmatrix} F_k + V\delta F_k^{(1)} \\ \vdots \\ F_k + V\delta F_k^{(V)} \end{bmatrix} x_k + \begin{bmatrix} G_k + V\delta G_k^{(1)} \\ \vdots \\ G_k + V\delta G_k^{(V)} \end{bmatrix} u_k, \tag{39}$$

for $k = N-1, \dots, 0$, where $\delta F_k^{(l)} := \alpha_{l,k} F_k^{(l)}$, $\delta G_k^{(l)} := \alpha_{l,k} G_k^{(l)}$, $l = 1, \dots, V$, and weighting matrices $P_{k+1} > 0$, $Q_k > 0$, and $R_k > 0$.

Remark 3.1. *We explicitly express the vertices $\{F_k^{(l)}, G_k^{(l)}\}$, $l = 1, \dots, V$, in the constraint* (39) *instead of directly considering only their convex combination. Nonetheless, we recover the original system* (34) *by premultiplying both sides of* (39) *by $\mathbb{1}_V^T \otimes I_n$, yielding*

$$V x_{k+1} = \left(V F_k + V \sum_{l=1}^{V} \alpha_{l,k} F_k^{(l)} \right) x_k + \left(V G_k + V \sum_{l=1}^{V} \alpha_{l,k} G_k^{(l)} \right) u_k,$$

$$x_{k+1} = (F_k + \delta F_k) x_k + (G_k + \delta G_k) u_k.$$

All constraints in (39) can be incorporated into the quadratic cost function by means of a penalty parameter. In this case, all vertices of the polytopic model will be weighted in the cost function in a unified way and the coefficients $\alpha_{l,k}$ will be interpreted as a contraction. Notice that (39) can be redefined as

$$\begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} x_{k+1} = \left(\begin{bmatrix} F_k \\ \vdots \\ F_k \end{bmatrix} + \begin{bmatrix} V\delta F_k^{(1)} \\ \vdots \\ V\delta F_k^{(V)} \end{bmatrix} \right) x_k + \left(\begin{bmatrix} G_k \\ \vdots \\ G_k \end{bmatrix} + \begin{bmatrix} V\delta G_k^{(1)} \\ \vdots \\ V\delta G_k^{(V)} \end{bmatrix} \right) u_k, \tag{40}$$

from which we define $g(x_{k+1}, u_k)$ as

$$g(x_{k+1}, u_k) = \left(\begin{bmatrix} I_n & -G_k \\ \vdots & \vdots \\ I_n & -G_k \end{bmatrix} + \begin{bmatrix} 0 & -V\delta G_k^{(1)} \\ \vdots & \vdots \\ 0 & -V\delta G_k^{(V)} \end{bmatrix} \right) \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} - \left(\begin{bmatrix} F_k \\ \vdots \\ F_k \end{bmatrix} + \begin{bmatrix} V\delta F_k^{(1)} \\ \vdots \\ V\delta F_k^{(V)} \end{bmatrix} \right) x_k, \tag{41}$$

and $C(x_{k+1}, u_k) = g(x_{k+1}, u_k)^T \mu g(x_{k+1}, u_k)$, with the penalty parameter $\mu > 0$.

Therefore, the constraints are placed into the cost function via $C(x_{k+1}, u_k)$ so that, after some algebraic manipulation, problem (38)–(39) becomes

$$\min_{x_{k+1}, u_k} \max_{\delta F_k, \delta G_k} J_k(x_{k+1}, u_k, \delta F_k, \delta G_k), \tag{42}$$

for $k = N-1, \dots, 0$, with one-step cost function given by

$$J_k(x_{k+1}, u_k, \delta F_k, \delta G_k) = \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}^T \begin{bmatrix} P_{k+1} & 0 \\ 0 & R_k \end{bmatrix} \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}$$
$$+ \left\{ \left(\begin{bmatrix} 0 & 0 \\ I_n & -G_k \\ \vdots & \vdots \\ I_n & -G_k \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -V\delta G_k^{(1)} \\ \vdots & \vdots \\ 0 & -V\delta G_k^{(V)} \end{bmatrix} \right) \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} - \left(\begin{bmatrix} -I_n \\ F_k \\ \vdots \\ F_k \end{bmatrix} + \begin{bmatrix} 0 \\ V\delta F_k^{(1)} \\ \vdots \\ V\delta F_k^{(V)} \end{bmatrix} \right) x_k \right\}^T \begin{bmatrix} Q_k & 0 \\ 0 & \mu I_{nV} \end{bmatrix} \{\bullet\}. \tag{43}$$

The subsequent section presents the process through which we come to a solution for (42)–(43) and, as consequence, how we obtain the robust recursive linear quadratic regulator (RLQR, for short) for system (34) subject to polytopic uncertainties (35).

3.2 RLQR for Discrete-Time Linear Systems Subject to Polytopic Uncertainties

In order to solve (42)–(43), the formulation presented in Section 3.1 intended to fit the optimization problem into the quadratic framework given in Chapter 2. In this sense, the following identifications are necessary between (43) and (16):

$$J \leftarrow J_k(x_{k+1}, u_k, \delta F_k, \delta G_k), \quad x \leftarrow \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}, \quad Q \leftarrow \begin{bmatrix} P_{k+1} & 0 \\ 0 & R_k \end{bmatrix}, \quad W \leftarrow \begin{bmatrix} Q_k & 0 \\ 0 & \mu I_{nV} \end{bmatrix},$$
$$A_0 \leftarrow \begin{bmatrix} 0 & 0 \\ I_n & -G_k \\ \vdots & \vdots \\ I_n & -G_k \end{bmatrix}, \quad \delta A \leftarrow \begin{bmatrix} 0 & 0 \\ 0 & -V\delta G_k^{(1)} \\ \vdots & \vdots \\ 0 & -V\delta G_k^{(V)} \end{bmatrix}, \quad b_0 \leftarrow \begin{bmatrix} -I_n \\ F_k \\ \vdots \\ F_k \end{bmatrix} x_k, \quad \delta b \leftarrow \begin{bmatrix} 0 \\ V\delta F_k^{(1)} \\ \vdots \\ V\delta F_k^{(V)} \end{bmatrix} x_k. \tag{44}$$

Also, by comparing (43) and (17) we map

$$M \leftarrow \begin{bmatrix} 0 \\ I_{nV} \end{bmatrix}, \quad A^{(l)} \leftarrow \begin{bmatrix} 0 & -VG_k^{(l)} \end{bmatrix}, \quad b^{(l)} \leftarrow VF_k^{(l)} x_k, \quad l = 1, \dots, V. \tag{45}$$

Given that W in our case is definite positive, we rewrite (23) as

$$W(\lambda) = \left(W^{-1} - \lambda^{-1} M M^T \right)^{-1},$$

which implies, with W in (44) and M in (45),

$$W(\lambda) \leftarrow \begin{bmatrix} Q_k & 0 \\ 0 & \Phi^{-1} \end{bmatrix}, \tag{46}$$

where $\Phi = (\mu^{-1} - \lambda^{-1})I_{nV}$. In addition, since $\lambda \in \big(\|M^T W M\|, \infty\big)$ and $\hat{\lambda} \approx \beta\|M^T W M\|$ for some $\beta > 1$, as discussed in Section 2.1.3, the associations (44) and (45) yield

$$\hat{\lambda} \approx \beta \left\| \begin{bmatrix} 0 & I_{nV} \end{bmatrix} \begin{bmatrix} Q_k & 0 \\ 0 & \mu I_{nV} \end{bmatrix} \begin{bmatrix} 0 \\ I_{nV} \end{bmatrix} \right\| = \beta\mu. \tag{47}$$

We are now in a position to state the main result of this chapter:

Theorem 3.1. *Consider the optimization problem* (42)–(43)*, with known* $P_N > 0$*,* $Q_k > 0$*, and* $R_k > 0$*. For a fixed* $\mu > 0$*, the solution and corresponding cost are given by*

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{u}_k \\ J_k(\hat{x}_{k+1}, \hat{u}_k) \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & x_k^T \end{bmatrix} \begin{bmatrix} L_k \\ K_k \\ P_k \end{bmatrix} x_k, \qquad k = 0, \dots, N-1, \tag{48}$$

with $\{L_k, K_k, P_k\}$ *recursively given by*

$$\begin{bmatrix} L_k \\ K_k \\ P_k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & -I_n & \hat{F}_k^T & \hat{F}_{V,k}^T & 0 & 0 \end{bmatrix} \times$$
$$\begin{bmatrix} (P_{k+1})^{-1} & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & R_k^{-1} & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & Q_k^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & \hat{I} & -\hat{G}_k \\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{G}_{V,k} \\ I_n & 0 & 0 & \hat{I}^T & 0 & 0 & 0 \\ 0 & I_m & 0 & -\hat{G}_k^T & -\hat{G}_{V,k}^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I_n \\ \hat{F}_k \\ \hat{F}_{V,k} \\ 0 \\ 0 \end{bmatrix}, \qquad k = N-1, \dots, 0, \tag{49}$$

where $\Phi := \mu^{-1}(1 - \beta^{-1})I_{nV}$*,* $\Sigma := (\beta\mu)^{-1} I_{nV}$*, with* $\beta > 1$*,*

$$\hat{F}_k := \begin{bmatrix} F_k \\ \vdots \\ F_k \end{bmatrix} \in \mathbb{R}^{nV \times n}, \quad \hat{G}_k := \begin{bmatrix} G_k \\ \vdots \\ G_k \end{bmatrix} \in \mathbb{R}^{nV \times m},$$

$$\hat{F}_{V,k} := \begin{bmatrix} F_k^{(1)} \\ \vdots \\ F_k^{(V)} \end{bmatrix} \in \mathbb{R}^{nV \times n}, \quad \hat{G}_{V,k} := \begin{bmatrix} G_k^{(1)} \\ \vdots \\ G_k^{(V)} \end{bmatrix} \in \mathbb{R}^{nV \times m}, \quad \hat{I} := \begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} \in \mathbb{R}^{nV \times n}.$$

Proof. Recall that $\delta F_k^{(l)} = \alpha_{l,k} F_k^{(l)}$ and $\delta G_k^{(l)} = \alpha_{l,k} G_k^{(l)}$, where $F_k^{(l)}$ and $G_k^{(l)}$ are vertices of a polytope with coefficients $\alpha_k \in \Lambda_V$. Since (42)–(43) is a robust regularized least-squares problem, we make use of the identifications (44)–(45) to fit the problem under study into the framework provided by Lemma 2.4. With this association and based on Lemma 2.4, we yield the recursive solution for problem (42)–(43) as a symmetric matrix arrange of the form

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{u}_k \\ J_k(\hat{x}_{k+1}, \hat{u}_k) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & -x_k^T I_n & x_k^T \hat{F}_k^T & x_k^T \hat{F}_{V,k}^T & 0 & 0 \end{bmatrix} \times$$
$$\begin{bmatrix} (P_{k+1})^{-1} & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & R_k^{-1} & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & Q_k^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & \hat{I} & -\hat{G}_k \\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{G}_{V,k} \\ I_n & 0 & 0 & \hat{I}^T & 0 & 0 & 0 \\ 0 & I_m & 0 & -\hat{G}_k^T & -\hat{G}_{V,k}^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I_n x_k \\ \hat{F}_k x_k \\ \hat{F}_{V,k} x_k \\ 0 \\ 0 \end{bmatrix}, \quad (50)$$

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{u}_k \\ J_k(\hat{x}_{k+1}, \hat{u}_k) \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & x_k^T \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & -I_n & \hat{F}_k^T & \hat{F}_{V,k}^T & 0 & 0 \end{bmatrix} \times$$
$$\underbrace{\begin{bmatrix} (P_{k+1})^{-1} & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & R_k^{-1} & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & Q_k^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & \hat{I} & -\hat{G}_k \\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{G}_{V,k} \\ I_n & 0 & 0 & \hat{I}^T & 0 & 0 & 0 \\ 0 & I_m & 0 & -\hat{G}_k^T & -\hat{G}_{V,k}^T & 0 & 0 \end{bmatrix}^{-1}}_{\mathscr{W}} \begin{bmatrix} 0 \\ 0 \\ -I_n \\ \hat{F}_k \\ \hat{F}_{V,k} \\ 0 \\ 0 \end{bmatrix} x_k, \quad (51)$$

which yields (48) and (49). We approximate $\hat{\lambda}$ by $\hat{\lambda} \approx \beta\mu$, for some $\beta > 1$, to sidestep additional computational effort as discussed in Section 2.1.3. As such, we attain $\Sigma := (\beta\mu)^{-1} I_{nV}$, and $\Phi = \mu^{-1}(1 - \beta^{-1}) I_{nV}$. In general, by choosing $\beta \in (1, 2]$ leads to adequate results. The mappings presented in (44) imply $Q > 0$, since both P_{k+1} and R_k are positive definite. The convexity of problem (42) is, therefore, ensured by Remark 2.1 and the solution (48)–(49) is in fact unique. Finally, the block matrix $\mathscr{W}$ is nonsingular according to Lemma A.3 (see Appendix A). □

Theorem 3.1 brings up to light some interesting aspects. At each instant, a single matrix equation is solved considering all vertices of the polytope in a unified manner. The method accommodates time-varying uncertainties without further online parameter tuning. Instead, approaches such as in Boyd et al. (1994) and Hosoe, Hagiwara and Peaucelle (2018) require the solution of optimization problems subject to coupled LMI constraints and, for a higher number of vertices, the region of feasible solutions becomes more restrained to satisfy all inequalities. The RLQR for systems subject to polytopic uncertainties is, therefore, also suitable for online applications, which is its main advantage. Finally, (48) computes the future state as $x_{k+1} = L_k x_k$ and, in turn, L_k is equivalent to the closed-loop matrix of system (34) when $u_k = K_k x_k$.

In the following statement, we extend the result from Theorem 3.1 and provide a manner to compute matrices $\{L_k, K_k, P_k\}$ in an equivalent reduced form. As we will see in the subsequent section, the reduced forms will be useful for convergence and stability analysis.

Theorem 3.2. *Consider the optimization problem* (42)–(43)*. For a fixed $\mu > 0$, the solution given by* (48)–(49) *is equivalent to*

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{u}_k \\ J_k(\hat{x}_{k+1}, \hat{u}_k) \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & x_k^T \end{bmatrix} \begin{bmatrix} L_k \\ K_k \\ P_k \end{bmatrix} x_k, \qquad k = 0, \dots, N-1, \tag{52}$$

with

$$L_k = P_{k+1}^{-1} \mathscr{P}_{k+1} \bar{\mathcal{F}}_k - P_{k+1}^{-1} \mathscr{P}_{k+1} \bar{G}_k \left(I_m + \bar{G}_k^T \mathscr{P}_{k+1} \bar{G}_k \right)^{-1} \bar{G}_k^T \mathscr{P}_{k+1} \bar{\mathcal{F}}_k, \tag{53}$$

$$K_k = -\bar{R}_k V G_k^T (\bar{P}_{k+1} + V G_k \bar{R}_k G_k^T)^{-1} \bar{\mathcal{F}}_k + (\Sigma + \hat{G}_{V,k} R_k^{-1} \hat{G}_{V,k}^T)^{-1} \hat{F}_{V,k}, \tag{54}$$

$$P_k = \bar{Q}_k + \bar{\mathcal{F}}_k^T \mathscr{P}_{k+1} \bar{\mathcal{F}}_k - \bar{\mathcal{F}}_k^T \mathscr{P}_{k+1} \bar{G}_k \big(I_m + \bar{G}_k^T \mathscr{P}_{k+1} \bar{G}_k \big)^{-1} \bar{G}_k^T \mathscr{P}_{k+1} \bar{\mathcal{F}}_k, \tag{55}$$

for $k = N-1, \dots, 0$, where

$$\bar{\mathcal{F}}_k = F_k - G_k R_k^{-1} \hat{G}_{V,k}^T \left(\Sigma + \hat{G}_{V,k} R_k^{-1} \hat{G}_{V,k}^T \right)^{-1} \hat{F}_{V,k},$$

$$\bar{R}_k = R_k^{-1} \left(I_m - \hat{G}_{V,k}^T \left(\Sigma + \hat{G}_{V,k} R_k^{-1} \hat{G}_{V,k}^T \right)^{-1} \hat{G}_{V,k} R_k^{-1} \right),$$

$$\bar{Q}_k = Q_k + \hat{F}_{V,k}^T \left(\Sigma + \hat{G}_{V,k} R_k^{-1} \hat{G}_{V,k}^T \right)^{-1} \hat{F}_{V,k},$$

$$\Sigma = (\beta\mu)^{-1} I_{nV}, \quad \bar{G}_k = G_k \bar{R}^{1/2},$$

$$\mathscr{P}_{k+1} = V \bar{P}_{k+1}^{-1}, \quad \bar{P}_{k+1} = \varphi + V P_{k+1}^{-1},$$

$$\varphi = \mu^{-1}(1 - \beta^{-1}) I_n, \quad \beta > 1.$$

Proof. From Theorem 3.1, we see that (49) holds since the system of equations

$$\begin{bmatrix} (P_{k+1})^{-1} & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & R_k^{-1} & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & Q_k^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & \hat{I} & -\hat{G}_k \\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{G}_{V,k} \\ I_n & 0 & 0 & \hat{I}^T & 0 & 0 & 0 \\ 0 & I_m & 0 & -\hat{G}_k^T & -\hat{G}_{V,k}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ \bar{d} \\ e \\ L_k \\ K_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -I_n \\ \hat{F}_k \\ \hat{F}_{V,k} \\ 0 \\ 0 \end{bmatrix} \tag{56}$$

has a unique solution, where $\bar{d} = \mathbb{1}_V \otimes d$, $d \in \mathbb{R}^{n \times n}$. Note that

$$\Phi = \mu^{-1}(1 - \beta^{-1}) I_{nV} = \begin{bmatrix} \varphi & 0 & \dots & 0 \\ 0 & \varphi & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \varphi \end{bmatrix}, \tag{57}$$

with $\varphi = \mu^{-1}(1 - \beta^{-1}) I_n$. Let us expand (56) to get the following set of equations:

$$\begin{cases} P_{k+1}^{-1} a + L_k = 0, & \text{(58a)} \\ R_k^{-1} b + K_k = 0, & \text{(58b)} \\ Q_k^{-1} c = -I_n, & \text{(58c)} \\ \Phi \bar{d} + \hat{I} L_k - \hat{G}_k K_k = \hat{F}_k, & \text{(58d)} \\ \Sigma e - \hat{G}_{V,k} K_k = \hat{F}_{V,k}, & \text{(58e)} \\ I_n a + \hat{I}^T \bar{d} = 0, & \text{(58f)} \\ I_m b - \hat{G}_k^T \bar{d} - \hat{G}_{V,k}^T e = 0. & \text{(58g)} \end{cases}$$

An additional equation results from the combination of (56) and (49):

$$P_k = -c + \hat{F}_k^T \bar{d} + \hat{F}_{V,k}^T e,$$

$$P_k = -c + \begin{bmatrix} F_k^T & \dots & F_k^T \end{bmatrix} \begin{bmatrix} d \\ \vdots \\ d \end{bmatrix} + \hat{F}_{V,k}^T e,$$

$$P_k = -c + V F_k^T d + \hat{F}_{V,k}^T e. \tag{59}$$

By manipulating (58a) and (58c), respectively, we have:

$$a = -P_{k+1} L_k, \tag{60}$$

$$c = -Q_k. \tag{61}$$

Meanwhile, the development of (58d) results in

$$\begin{bmatrix} \varphi & 0 & \dots & 0 \\ 0 & \varphi & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \varphi \end{bmatrix} \begin{bmatrix} d \\ d \\ \vdots \\ d \end{bmatrix} + \begin{bmatrix} I_n \\ I_n \\ \vdots \\ I_n \end{bmatrix} L_k - \begin{bmatrix} G_k \\ G_k \\ \vdots \\ G_k \end{bmatrix} K_k = \begin{bmatrix} F_k \\ F_k \\ \vdots \\ F_k \end{bmatrix},$$

$$\varphi d + L_k - G_k K_k = F_k. \tag{62}$$

From (58f),

$$I_n a + \begin{bmatrix} I_n & \dots & I_n \end{bmatrix} \begin{bmatrix} d \\ \vdots \\ d \end{bmatrix} = 0,$$

$$V d = -I_n a, \tag{63}$$

and substituting a from (60) yields

$$d = V^{-1} P_{k+1} L_k. \tag{64}$$

From (58b),

$$b = -R_k K_k, \tag{65}$$

and by placing into (58g) results

$$I_m(-R_k K_k) - \begin{bmatrix} G_k^T & \dots & G_k^T \end{bmatrix} \begin{bmatrix} d \\ \vdots \\ d \end{bmatrix} - \hat{G}_{V,k}^T e = 0,$$

$$-R_k K_k - V G_k^T d - \hat{G}_{V,k}^T e = 0,$$

$$K_k = -R_k^{-1} V G_k^T d - R_k^{-1} \hat{G}_{V,k}^T e. \tag{66}$$

By joining (66) and (58e) produces

$$\Sigma e + \hat{G}_{V,k} R_k^{-1} \left(V G_k^T d + \hat{G}_{V,k}^T e \right) = \hat{F}_{V,k},$$

$$\Sigma e + \hat{G}_{V,k} R_k^{-1} V G_k^T d + \hat{G}_{V,k} R_k^{-1} \hat{G}_{V,k}^T e = \hat{F}_{V,k},$$

$$\hat{G}_{V,k} R_k^{-1} V G_k^T d + \left(\Sigma + \hat{G}_{V,k} R_k^{-1} \hat{G}_{V,k}^T \right) e = \hat{F}_{V,k}. \tag{67}$$

By substituting K_k from (66) into (62) and isolating matrix L_k:

$$\varphi d + L_k + G_k \left(R_k^{-1} V G_k^T d + R_k^{-1} \hat{G}_{V,k}^T e \right) = F_k,$$

$$\varphi d + L_k + G_k R_k^{-1} V G_k^T d + G_k R_k^{-1} \hat{G}_{V,k}^T e = F_k,$$

$$L_k = F_k - \left(\varphi + G_k R_k^{-1} V G_k^T \right) d - G_k R_k^{-1} \hat{G}_{V,k}^T e, \tag{68}$$

and by placing into (64) yields

$$d = V^{-1}P_{k+1}\left[F_k - \left(\varphi + G_k R_k^{-1} V G_k^T\right) d - G_k R_k^{-1}\hat{G}_{V,k}^T e\right],$$

$$d = V^{-1}P_{k+1}F_k - V^{-1}P_{k+1}\left(\varphi + G_k R_k^{-1} V G_k^T\right) d - V^{-1}P_{k+1}G_k R_k^{-1}\hat{G}_{V,k}^T e,$$

$$\left[I_n + V^{-1}P_{k+1}\left(\varphi + G_k R_k^{-1} V G_k^T\right)\right] d + V^{-1}P_{k+1}G_k R_k^{-1}\hat{G}_{V,k}^T e = V^{-1}P_{k+1}F_k. \tag{69}$$

By isolating e in (67) results

$$e = \left(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T\right)^{-1}\left(\hat{F}_{V,k} - \hat{G}_{V,k}R_k^{-1}VG_k^T d\right), \tag{70}$$

and placing it into (69) yields

$$\begin{aligned} V^{-1}P_{k+1}F_k = &\left[I_n + V^{-1}P_{k+1}\left(\varphi + G_k R_k^{-1} V G_k^T\right)\right] d \\ &+ V^{-1}P_{k+1}G_k R_k^{-1}\hat{G}_{V,k}^T\left(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T\right)^{-1}\left(\hat{F}_{V,k} - \hat{G}_{V,k}R_k^{-1}VG_k^T d\right), \end{aligned}$$

$$\begin{aligned} V^{-1}P_{k+1}F_k = &\left[I_n + V^{-1}P_{k+1}\left(\varphi + G_k R_k^{-1} V G_k^T\right)\right] d \\ &+ V^{-1}P_{k+1}G_k R_k^{-1}\hat{G}_{V,k}^T\left(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T\right)^{-1}\hat{F}_{V,k} \\ &- V^{-1}P_{k+1}G_k R_k^{-1}\hat{G}_{V,k}^T\left(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T\right)^{-1}\hat{G}_{V,k}R_k^{-1}VG_k^T d, \end{aligned}$$

$$\begin{aligned} &\Big[I_n + V^{-1}P_{k+1}\left(\varphi + G_k R_k^{-1} V G_k^T\right) \\ &\qquad - P_{k+1}G_k R_k^{-1}\hat{G}_{V,k}^T\left(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T\right)^{-1}\hat{G}_{V,k}R_k^{-1}G_k^T\Big] d = \\ &\qquad\qquad V^{-1}P_{k+1}F_k - V^{-1}P_{k+1}G_k R_k^{-1}\hat{G}_{V,k}^T\left(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T\right)^{-1}\hat{F}_{V,k}, \end{aligned}$$

$$\begin{aligned} &\left\{I_n + V^{-1}P_{k+1}\varphi + P_{k+1}G_k R_k^{-1}\left[I_m - \hat{G}_{V,k}^T\left(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T\right)^{-1}\hat{G}_{V,k}R_k^{-1}\right] G_k^T\right\} d = \\ &\qquad V^{-1}P_{k+1}\left[F_k - G_k R_k^{-1}\hat{G}_{V,k}^T\left(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T\right)^{-1}\hat{F}_{V,k}\right]. \end{aligned} \tag{71}$$

We define

$$\bar{\mathcal{F}}_k = F_k - G_k R_k^{-1}\hat{G}_{V,k}^T\left(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T\right)^{-1}\hat{F}_{V,k}, \tag{72}$$

$$\bar{R}_k = R_k^{-1}\left[I_m - \hat{G}_{V,k}^T\left(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T\right)^{-1}\hat{G}_{V,k}R_k^{-1}\right]. \tag{73}$$

Therefore,

$$\left(I_n + V^{-1}P_{k+1}\varphi + P_{k+1}G_k\bar{R}_k G_k^T\right) d = V^{-1}P_{k+1}\bar{\mathcal{F}}_k,$$

$$d = \left(I_n + P_{k+1}\left(V^{-1}\varphi + G_k\bar{R}_k G_k^T\right)\right)^{-1} V^{-1}P_{k+1}\bar{\mathcal{F}}_k,$$

$$d = \left(V^{-1}P_{k+1}(VP_{k+1}^{-1} + \varphi + VG_k\bar{R}_k G_k^T)\right)^{-1} V^{-1}P_{k+1}\bar{\mathcal{F}}_k,$$

$$d = (VP_{k+1}^{-1} + \varphi + VG_k\bar{R}_k G_k^T)^{-1}\bar{\mathcal{F}}_k.$$

Considering $\bar{P}_{k+1} = \varphi + VP_{k+1}^{-1}$, then,

$$d = (\bar{P}_{k+1} + VG_k\bar{R}_kG_k^T)^{-1}\bar{\mathcal{F}}_k. \tag{74}$$

By placing (74) in (64), we obtain

$$L_k = VP_{k+1}^{-1}d,$$

$$L_k = VP_{k+1}^{-1}(\bar{P}_{k+1} + VG_k\bar{R}_kG_k^T)^{-1}\bar{\mathcal{F}}_k,$$

$$L_k = VP_{k+1}^{-1}\left((I_n + VG_k\bar{R}_kG_k^T\bar{P}_{k+1}^{-1})^{-1}\bar{P}_{k+1}\right)^{-1}\bar{\mathcal{F}}_k,$$

$$L_k = VP_{k+1}^{-1}\bar{P}_{k+1}^{-1}(I_n + VG_k\bar{R}_kG_k^T\bar{P}_{k+1}^{-1})^{-1}\bar{\mathcal{F}}_k,$$

with $\bar{G}_k = G_k\bar{R}_k^{1/2}$, then

$$L_k = VP_{k+1}^{-1}\bar{P}_{k+1}^{-1}(I_n + V\bar{G}_kI_m\bar{G}_k^T\bar{P}_{k+1}^{-1})^{-1}\bar{\mathcal{F}}_k,$$

$$L_k = VP_{k+1}^{-1}\bar{P}_{k+1}^{-1}(I_n - V\bar{G}_k(I_m + \bar{G}_k^TV\bar{P}_{k+1}^{-1}\bar{G}_k)^{-1}\bar{G}_k^T\bar{P}_{k+1}^{-1})\bar{\mathcal{F}}_k,$$

$$L_k = VP_{k+1}^{-1}\bar{P}_{k+1}^{-1}\bar{\mathcal{F}}_k - VP_{k+1}^{-1}\bar{P}_{k+1}^{-1}V\bar{G}_k(I_m + \bar{G}_k^TV\bar{P}_{k+1}^{-1}\bar{G}_k)^{-1}\bar{G}_k^T\bar{P}_{k+1}^{-1}\bar{\mathcal{F}}_k,$$

and, by defining $\mathscr{P}_{k+1} = V\bar{P}_{k+1}^{-1}$, we have

$$L_k = P_{k+1}^{-1}\mathscr{P}_{k+1}\bar{\mathcal{F}}_k - P_{k+1}^{-1}\mathscr{P}_{k+1}\bar{G}_k\left(I_m + \bar{G}_k^T\mathscr{P}_{k+1}\bar{G}_k\right)^{-1}\bar{G}_k^T\mathscr{P}_{k+1}\bar{\mathcal{F}}_k,$$

which corresponds to (53). Now, substitute (74) and (70) into (66), so that

$$K_k = -R_k^{-1}VG_k^Td - R_k^{-1}\hat{G}_{V,k}^T\Big(-(\Sigma + G_{V,k}R_k^{-1}G_{V,k}^T)^{-1}V\hat{G}_{V,k}R_k^{-1}G_k^Td + (\Sigma + G_{V,k}R_k^{-1}G_{V,k}^T)^{-1}\hat{F}_{V,k}\Big),$$

$$K_k = -R_k^{-1}\left(I_m - \hat{G}_{V,k}^T(\Sigma + G_{V,k}R_k^{-1}G_{V,k}^T)^{-1}\hat{G}_{V,k}R_k^{-1}\right)VG_k^Td + (\Sigma + G_{V,k}R_k^{-1}G_{V,k}^T)^{-1}\hat{F}_{V,k},$$

$$K_k = -\bar{R}_kVG_k^T + (\Sigma + G_{V,k}R_k^{-1}G_{V,k}^T)^{-1}\hat{F}_{V,k},$$

$$K_k = -\bar{R}_kVG_k^T(\bar{P}_{k+1} + VG_k\bar{R}_kG_k^T)^{-1}\bar{\mathcal{F}}_k + (\Sigma + G_{V,k}R_k^{-1}G_{V,k}^T)^{-1}\hat{F}_{V,k},$$

which is the same as (54). Next, we substitute (61), (74) and (70) into (59), thus we have:

$$P_k = Q_k + VF_k^Td + \hat{F}_{V,k}^T\left(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T\right)^{-1}\left(\hat{F}_{V,k} - \hat{G}_{V,k}R_k^{-1}VG_k^Td\right),$$

$$P_k = Q_k + \left(VF_k^T - \hat{F}_{V,k}^T(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T)^{-1}V\hat{G}_{V,k}R_k^{-1}G_k^T\right)d + \hat{F}_{V,k}^T(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T)^{-1}\hat{F}_{V,k},$$

$$P_k = Q_k + \bar{\mathcal{F}}_k^TVd + \hat{F}_{V,k}^T(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T)^{-1}\hat{F}_{V,k}.$$

We define

$$\bar{Q}_k = Q_k + \hat{F}_{V,k}^T(\Sigma + \hat{G}_{V,k}R_k^{-1}\hat{G}_{V,k}^T)^{-1}\hat{F}_{V,k}, \tag{75}$$

then,

$$P_k = \bar{Q}_k + \bar{\mathcal{F}}_k^T V d,$$

$$P_k = \bar{Q}_k + \bar{\mathcal{F}}_k^T V(\bar{P}_{k+1} + VG_k\bar{R}_kG_k^T)^{-1}\bar{\mathcal{F}}_k,$$

$$P_k = \bar{Q}_k + \bar{\mathcal{F}}_k^T V\left((I_n + V\bar{G}_kI_m\bar{G}_k^T\bar{P}_{k+1}^{-1})\bar{P}_{k+1}\right)^{-1}\bar{\mathcal{F}}_k,$$

$$P_k = \bar{Q}_k + \bar{\mathcal{F}}_k^T V\bar{P}_{k+1}^{-1}(I_n + V\bar{G}_kI_m\bar{G}_k^T\bar{P}_{k+1}^{-1})^{-1}\bar{\mathcal{F}}_k,$$

$$P_k = \bar{Q}_k + \bar{\mathcal{F}}_k^T V\bar{P}_{k+1}^{-1}\left(I_n - V\bar{G}_k(I_m + \bar{G}_k^TV\bar{P}_{k+1}^{-1}\bar{G}_k)^{-1}\bar{G}_k^T\bar{P}_{k+1}^{-1}\right)\bar{\mathcal{F}}_k,$$

$$P_k = \bar{Q}_k + \bar{\mathcal{F}}_k^T V\bar{P}_{k+1}^{-1}\bar{\mathcal{F}}_k - \bar{\mathcal{F}}_k^T V\bar{P}_{k+1}^{-1}\bar{G}_k(I_m + \bar{G}_k^TV\bar{P}_{k+1}^{-1}\bar{G}_k)^{-1}\bar{G}_k^TV\bar{P}_{k+1}^{-1}\bar{\mathcal{F}}_k,$$

$$P_k = \bar{Q}_k + \bar{\mathcal{F}}_k^T\mathscr{P}_{k+1}\bar{\mathcal{F}}_k - \bar{\mathcal{F}}_k^T\mathscr{P}_{k+1}\bar{G}_k\big(I_m + \bar{G}_k^T\mathscr{P}_{k+1}\bar{G}_k\big)^{-1}\bar{G}_k^T\mathscr{P}_{k+1}\bar{\mathcal{F}}_k,$$

which corresponds to (55).

□

Remark 3.2. *Bearing in mind the set of equations in* (56)*, we are able to clarify why the optimization problem* (42)–(43) *is solved over both variables* $\{u_k,\, x_{k+1}\}$*. This selection of variables allows us to provide, in a unified manner, both stability and robustness to the control system by solving the following equations:*

$$\hat{I}L_k = (\hat{F}_k + \hat{G}_kK_k) - \Phi\bar{d},$$
$$\Sigma e = (\hat{F}_{V,k} + \hat{G}_{V,k}K_k),$$

which involve all polytope vertices of (34)*. Meanwhile, if* $\mu \to \infty$*, we have* $\Phi \to 0$ *and* $\Sigma \to 0$*, hence* $(\hat{F}_{V,k} + \hat{G}_{V,k}K_k) \to 0$ *and* $(\hat{F}_k + \hat{G}_kK_k) \to \hat{I}L_k$*. As such, we achieve the optimal RLQR. If it is not possible to tune* $\mu \to \infty$*, we adjust* $\mu^{-1} \to \epsilon$ *in order to obtain a sub-optimal robust recursive regulator.*

Remark 3.3. *For any penalty parameter* $\mu > 0$*, if* $\beta \to 1^+$*, then* $\varphi \to 0$ *and* $\mathscr{P}_{k+1} \to P_{k+1}$*. As such,* (55) *becomes a standard Riccati equation given by*

$$P_k = \bar{Q}_k + \bar{\mathcal{F}}_k^TP_{k+1}\bar{\mathcal{F}}_k - \bar{\mathcal{F}}_k^TP_{k+1}\bar{G}_k\big(I_m + \bar{G}_k^TP_{k+1}\bar{G}_k\big)^{-1}\bar{G}_k^TP_{k+1}\bar{\mathcal{F}}_k. \tag{76}$$

The penalty μ can be interpreted as a weighting parameter and is closely related to the optimality of the solution. In fact, (48) (as well as (52)) converges to the optimal solution of the original constrained problem as $\mu \to \infty$. Nonetheless, although a finite positive penalty parameter yields a sub-optimal solution, the resulting feedback gain still stabilizes system (34) when $u_k = K_kx_k$. In the following section, we elaborate on this aspect based on the reduced forms introduced in Theorem 3.2.

3.3 Convergence and Stability

To perform the analysis, let us consider invariant system parameters while allowing coefficients α_k to be time-varying, and $\beta \to 1^+$. As such, we have the following discrete-time system realization:

$$x_{k+1} = (F + \sum_{l=1}^{V} \alpha_{l,k} F^{(l)}) x_k + (G + \sum_{l=1}^{V} \alpha_{l,k} G^{(l)}) u_k, \quad \alpha_k \in \Lambda_V.$$

We are now in a position to establish conditions for convergence and stability of the RLQR for systems subject to polytopic uncertainties.

Theorem 3.3. *Assume that $\{\bar{\mathcal{F}}, \bar{G}\}$ is controllable, $\{\bar{Q}^{1/2}, \bar{\mathcal{F}}\}$ is observable and consider* (76) *with initial condition $P_N > 0$. Then, there exists $P > 0$ symmetric such that $\lim_{k\to\infty} P_k = P$. Moreover, P is the unique stabilizing solution for* (55) *and the closed-loop system matrix*

$$L = \bar{\mathcal{F}} - \bar{G}\left(I_m + \bar{G}^T P \bar{G}\right)^{-1} \bar{G}^T P \bar{\mathcal{F}},$$

such that $x_{k+1} = L x_k$, is stable.

Proof. Notice that (55) conforms to the standard Riccati recursive equation (30), namely

$$P_k = Q + A^T P_{k+1} A - A^T P_{k+1} B \left(R + B^T P_{k+1} B\right)^{-1} B^T P_{k+1} A,$$

through the identifications $A \leftarrow \bar{\mathcal{F}}$, $Q \leftarrow \bar{Q}$, $R \leftarrow I_m$, and $B \leftarrow \bar{G}$. Therefore, as thoroughly discussed in (BERTSEKAS, 2005, Chapter 4) and (LANCASTER; RODMAN, 1995, Chapter 12) and given the above equivalences, it follows that $\lim_{k\to\infty} P_k = P$, where $P > 0$ is the unique solution for (55). In addition, the feedback gain K, such that $u_k = K x_k$, makes the eigenvalues of L lie within the open unit disc. □

3.4 Illustrative Examples

We present two illustrative examples to validate the proposed robust recursive regulator. For comparison purposes, we adopt the robust controller presented in Oliveira, Bernussou and Geromel (1999) computed with the YALMIP Toolbox (LÖFBERG, 2004). The first example focuses on the computational efficiency and behavior of closed-loop eigenvalues, while the second example is an application of the RLQR on a commercial quadrotor model.

Example 3.1. *Consider the discrete-time system, based on Oliveira, Bernussou and Geromel (1999), with state-space matrices and initial conditions given by*

$$F = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.03 \\ 0 & 0 & 1.0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1.0 \\ 3.0 \\ -0.5 \\ -1.0 \end{bmatrix},$$

subject to polytopic uncertainties with parameterized vertices $(\rho F^{(i)}, G^{(i)})$*,* $i = 1, 2$*,* $\rho \in \mathbb{R}$*, where*

$$F^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.8 & -0.5 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F^{(2)} = -F^{(1)}, \quad G^{(1)} = \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}, \quad G^{(2)} = -G^{(1)}, \quad \rho > 0.$$

We search for the maximum values $\{\bar{\rho}_{\textit{RLQR}},\ \bar{\rho}_{\textit{ref}}\}$ *for which the closed-loop system is stable with the robust recursive regulator for any* $\rho \leq \bar{\rho}_{\textit{RLQR}}$*, and with the controller from Oliveira, Bernussou and Geromel (1999, Theorem 3) for any* $\rho \leq \bar{\rho}_{\textit{ref}}$*. To this end, we set up a horizon* $N = 100$*,* $\mu = 10^{15}$*,* $\beta = 1.2$ *and weighting matrices* $P_N = I_4$*,* $Q_k = I_4$*, and* $R_k = 1$ *for the quadratic cost function* (43)*. The feedback gain and the Riccati solution obtained with Lemma 3.1 converged to*

$$K = \begin{bmatrix} -0.6901 & 0.4589 & -0.2739 & -0.9830 \end{bmatrix},$$

$$P = 10^{15} \begin{bmatrix} 3.5861 & -2.1681 & -0.8085 & 4.2750 \\ -2.1681 & 1.3437 & 0.1812 & -2.7604 \\ -0.8085 & 0.1812 & 3.1651 & 0.6119 \\ 4.2750 & -2.7604 & 0.6119 & 6.1168 \end{bmatrix}.$$

The robust controller adopted for comparison purposes yielded feedback gain

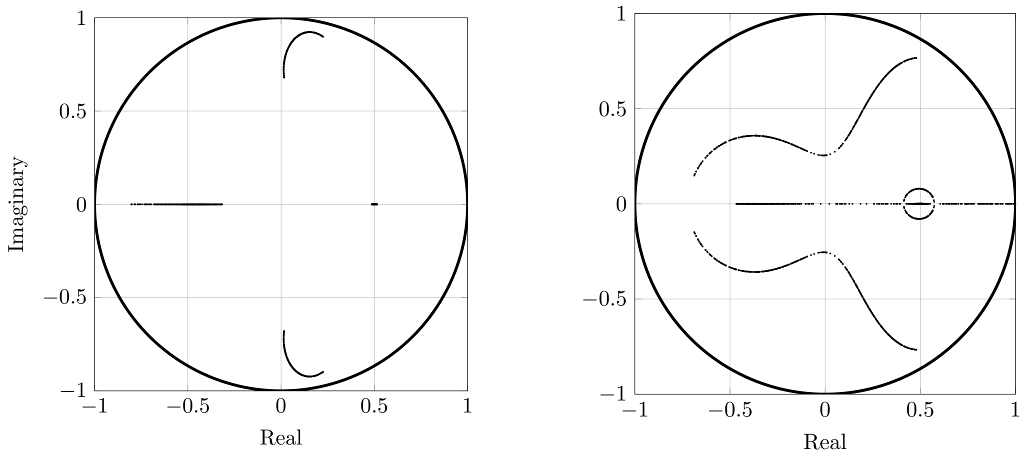
$$K_{ref} = \begin{bmatrix} 0.0233 & 0.0668 & -0.8731 & -0.2706 \end{bmatrix}.$$

We perform an iterative search procedure and attain $\bar{\rho}_{\textit{RLQR}} = 1.9130$ *and* $\bar{\rho}_{\textit{ref}} = 1.0511$*. Such a result indicates that the robust recursive regulator is able to handle a wider range of uncertainties. We show the eigenvalues of the closed-loop system, denoted by* ν*, in Figs. 1 and 2. Observe that, when* $\rho = \bar{\rho}_{\textit{ref}}$*, both approaches are capable of stabilizing the closed-loop system, while only the robust recursive regulator ensures stability when* $\rho = \bar{\rho}_{\textit{RLQR}}$*.*

Now, let us assume different values of μ *and, for each of them, we search for the maximum* $\bar{\rho}_{\textit{RLQR}}$ *for which the closed-loop system is stable with the robust recursive regulator for any* $\rho \leq \bar{\rho}_{\textit{RLQR}}$*. We summarize the results in Table 1. Observe that* $\bar{\rho}_{\textit{RLQR}}$ *converges to* 1.9130 *as* μ *increases. It is noteworthy, moreover, that* $\bar{\rho}_{\textit{RLQR}} > \bar{\rho}_{\textit{ref}}$ *and the closed-loop system remains stable even for small values of* μ*.*

Finally, we examine the computational effort demanded to calculate the feedback gains. The average elapsed time to compute the gain with Lemma 3.1 was 1.7 *ms, whilst the controller from Oliveira, Bernussou and Geromel (1999, Theorem 3) required* 149.7 *ms on average. As such, the diminished computational effort indicates that the proposed robust recursive approach is also adequate for online applications.*

Figure 1 – Eigenvalues of the closed-loop system with $\rho = \bar{\rho}_{\text{ref}}$: max$\{\|\nu\|_{\text{RLQR}}\} = 0.937381$, max$\{\|\nu\|_{\text{ref}}\} = 0.999707$.

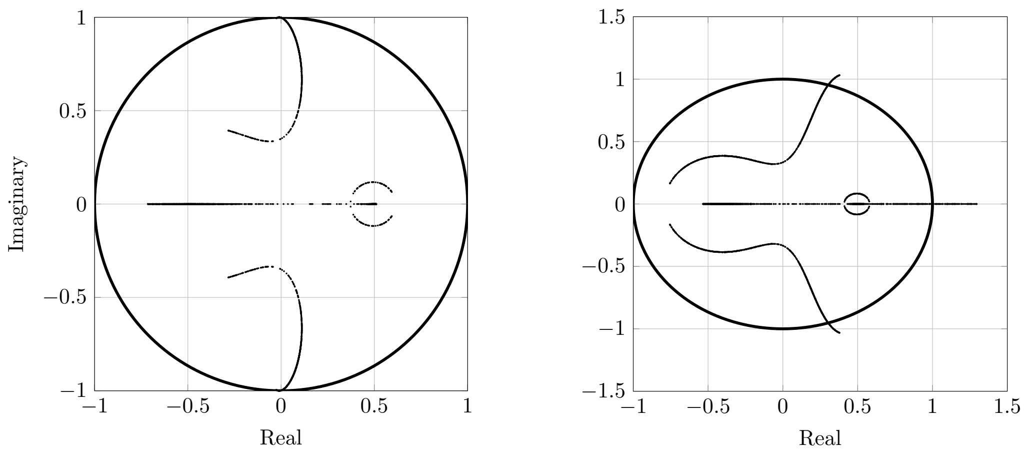


(a) Robust regulator.

(b) Robust controller for comparison.

Source: author.

Figure 2 – Eigenvalues of the closed-loop system with $\rho = \bar{\rho}_{\text{RLQR}}$: max$\{\|\nu\|_{RLQR}\} = 0.999980$, max$\{\|\nu\|_{\text{ref}}\} = 1.296115$.



(a) Robust recursive regulator.

(b) Robust controller for comparison.

Source: author

Example 3.2. *The following 4-DOF system is based on Rosales et al. (2017) and describes a trajectory tracking model of a commercial quadrotor, more specifically, a Parrot AR 2.0:*

$$x_{k+1} = \begin{bmatrix} \Xi & 0 \\ 0.01I_4 & I_4 \end{bmatrix} x_k + \begin{bmatrix} 0.01I_4 \\ 0 \end{bmatrix} u_k, \tag{77}$$

Table 1 – Effects of μ over $\bar{\rho}_{\text{RLQR}}$ and maximum norms of closed-loop eigenvalues with the robust recursive regulator.

μ	$\bar{\rho}_{\text{RLQR}}$	$\|\nu_\mu\|$
1	1.20730	0.999966
10	1.75800	0.999970
10^5	1.91290	0.999973
10^{10}	1.91300	0.999980
10^{12}	1.91300	0.999980

Source: author.

with

$$\Xi = \begin{bmatrix} 0.9985 & 0.0003 & 0 & 0 \\ 0.0003 & 0.9970 & 0 & 0 \\ 0 & 0 & 0.9755 & 0 \\ 0 & 0 & 0 & 0.9893 \end{bmatrix},$$

where $x_k = \begin{bmatrix} e_{\mathsf{v}_k}^T & e_{\mathsf{p}_k}^T \end{bmatrix}^T \in \mathbb{R}^8$ *represents the error state vector, with* $e_{\mathsf{v}_k} = \begin{bmatrix} e_{\mathsf{v}x} & e_{\mathsf{v}y} & e_{\mathsf{v}z} & e_{\mathsf{v}_\psi} \end{bmatrix}^T$ *and* $e_{\mathsf{p}_k} = \begin{bmatrix} e_x & e_y & e_z & e_\psi \end{bmatrix}^T$*, in which* $\{e_{\mathsf{v}x}, e_x\}$*,* $\{e_{\mathsf{v}y}, e_y\}$*, and* $\{e_{\mathsf{v}z}, e_z\}$ *are the velocity and position errors along the global* x*,* y *and* z *axes, in this order, and* $\{e_{\mathsf{v}_\psi}, e_\psi\}$ *are the angular velocity and orientation errors, respectively. The commands are computed via* $v_{drone_k} = u_{ref_k} - u_k$*, where* u_{ref_k} *is the reference control input and* u_k *is calculated with the selected control law. To design the reference signal* u_{ref_k}*, we used the Parrot's official simulator Sphinx to perform flight with the desired trajectory.*

For this example, the main task is to control the quadrotor so as to track an 8-shaped reference trajectory beginning at the origin of the global coordinate frame. Polytopic uncertainties δF_k *and* δG_k *represent variations on elements of* F_k *and* G_k *that could result from unmodeled dynamics, nonlinearities and disturbances. In this manner, we consider two polytope vertices to compose* δF_k *and* δG_k*, such that*

$$F^{(i)} = \begin{bmatrix} 10^{-2}E_{\Xi_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad G^{(i)} = \begin{bmatrix} 10^{-3}\Upsilon_i \\ 0 \end{bmatrix}, \quad i = 1, 2,$$

where

$$E_{\Xi_1} = -\begin{bmatrix} 1.37 & 5.99 & 0 & 0 \\ 4.64 & 6.82 & 0 & 0 \\ 0 & 0 & 1.29 & 0 \\ 0 & 0 & 0 & 3.12 \end{bmatrix}, \qquad \Upsilon_1 = -\begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 1.1 & 0.3 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 2.0 \end{bmatrix},$$

$$E_{\Xi_2} = \begin{bmatrix} 0.80 & 2.66 & 0 & 0 \\ 3.10 & 6.21 & 0 & 0 \\ 0 & 0 & 1.04 & 0 \\ 0 & 0 & 0 & 5.13 \end{bmatrix}, \qquad \Upsilon_2 = \begin{bmatrix} 6.5 & 7.4 & 0 & 0 \\ 0.8 & 4.3 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 1.9 \end{bmatrix}.$$

For the quadratic cost function (43)*, we select* $\mu = 10^{10}$*,* $\beta = 1.00001$*, and weights* $P_N = 10^{12} I_8$*,* $Q_k =$ *diag*$\{0.5 \cdot 10^{10} I_4,\ 10^{10} I_4\}$*, and* $R_k = I_4$*. Therefore, the feedback gain* K *attained with Theorem 3.1 is*

$$K = \begin{bmatrix} -60.7374 & 22.3468 & 0 & 0 & -82.7444 & 31.6448 & 0 & 0 \\ 23.1583 & -48.7937 & 0 & 0 & 33.4338 & -64.1943 & 0 & 0 \\ 0 & 0 & -99.4591 & 0 & 0 & 0 & -140.2319 & 0 \\ 0 & 0 & 0 & -74.5335 & 0 & 0 & 0 & -102.0202 \end{bmatrix},$$

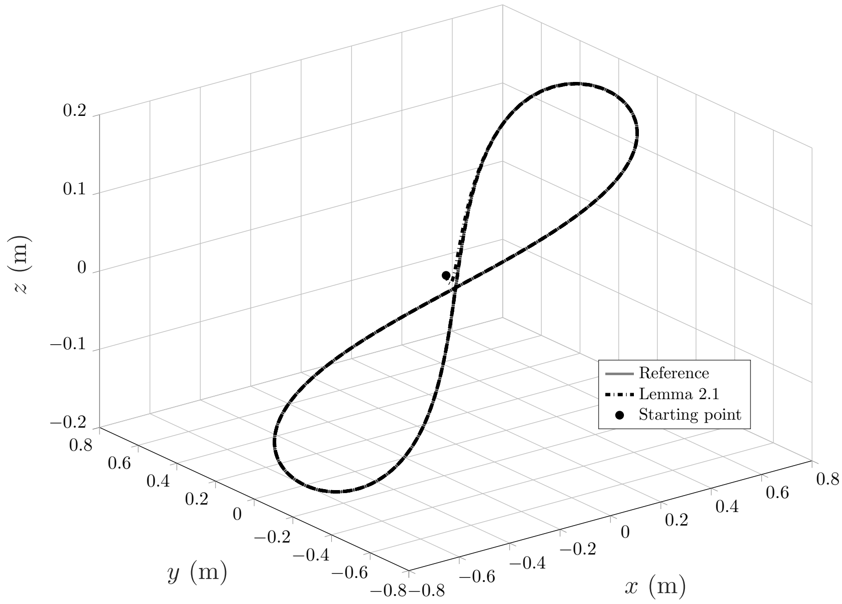
while Oliveira, Bernussou and Geromel (1999, Theorem 3) returned, for comparison,

$$K_{ref} = \begin{bmatrix} -72.3857 & 25.2426 & 0 & 0 & -41.1403 & 13.8486 & 0 & 0 \\ 2.6812 & -78.7751 & 0 & 0 & 1.7914 & -43.3649 & 0 & 0 \\ 0 & 0 & -96.8010 & 0 & 0 & 0 & -56.3373 & 0 \\ 0 & 0 & 0 & -97.2112 & 0 & 0 & 0 & -54.8025 \end{bmatrix}.$$

We carried out 1000 *Monte Carlo experiments, each with time horizon equal to* $N = 3000$*, meaning flights with duration of* 30 *seconds. For both controllers, the initial condition is* $x_0 = \begin{bmatrix} 0.40 & 0.33 & -0.64 & 0.01 & 0 & 0 & 0.01 & 0 \end{bmatrix}^T$*. The resulting motion of the quadrotor and the reference trajectory in the global coordinate frame are presented in Fig. 3, while the norms of errors and input vectors are shown in Figs. 4a and 4b, respectively. Additionally, the norms and standard deviations of velocity and position tracking errors,* $\|e_v\|_{\mathcal{L}_2}$*,* σ_v*,* $\|e_p\|_{\mathcal{L}_2}$*, and* σ_p*, respectively, and of the control input,* $\|u\|_{\mathcal{L}_2}$ *and* σ_u*, in this order, are summarized in Table 2. The results show that the robust recursive regulator was able to successfully minimize the errors while consuming less energy to perform trajectory tracking, as can be seen in Fig. 4b and Table 2.*

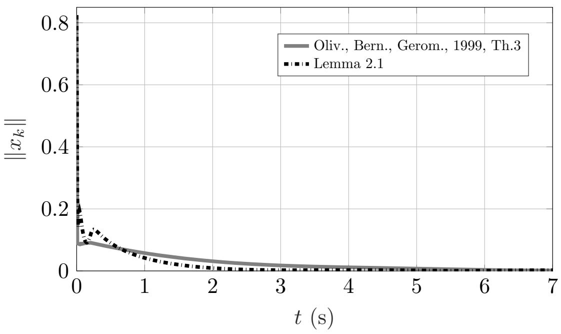
Let us now assume parameterized vertices $(\rho F^{(i)}, G^{(i)})$ *and search for the maxima* $\{\bar{\rho}_{RLQR},\ \bar{\rho}_{ref}\}$ *such that the closed-loop system is stable with the RLQR for* $\rho \leq \bar{\rho}_{RLQR}$ *and with the robust controller adopted for comparison for* $\rho\bar{\rho}_{ref}$*. We find* $\bar{\rho}_{RLQR} = 11.5002$ *and* $\bar{\rho}_{ref} = 11.4967$*, and in this case that both approaches provide similar levels of robustness.*

Figure 3 – Resulting trajectory of the quadrotor in the global coordinate frame.

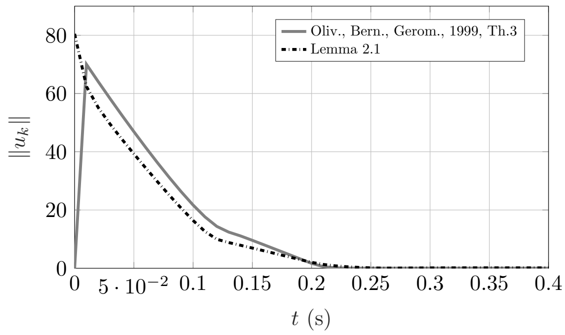


Source: author.

Figure 4 – Norms of tracking errors and control inputs obtained with the robust recursive regulator and the robust controller adopted for comparison.



(a) Norms of tracking errors.



(b) Norms of control inputs.

Source: author.

Table 2 – Averages and standard deviations of trajectory tracking errors and control inputs for Example 3.2.

Controller	$\|e_{\mathsf{v}}\|_{\mathcal{L}_2}$	σ_{v}	$\|e_{\mathsf{p}}\|_{\mathcal{L}_2}$	σ_{p}	$\|u\|_{\mathcal{L}_2}$	σ_u
Lemma 3.1	1.8996	0.0347	0.3366	0.0060	152.8521	2.7864
Oliv., Bern., Gerom., 1999, Th. 3	1.8503	0.0338	0.4185	0.0073	155.3303	2.8313

Source: author.

4 ROBUST REGULATOR FOR MARKOV JUMP SYSTEMS SUBJECT TO POLYTOPIC UNCERTAINTIES

Discrete-time Markov jump linear systems (DMJLS, for short) subject to polytopic uncertainties have attracted growing attention from researchers in recent decades. In fact, there is a vast amount of available literature enveloping robust control methods for this class of systems, and remarkable examples include, but are not limited to, Park and Kwon (2002), Ma, Zhang and Liu (2008), Zou et al. (2015), Lu, Li and Xi (2013), Gabriel, Gonçalves and Geromel (2018), Lopes et al. (2019), and Zhang, Song and Cai (2022). In such works, the synthesis of controllers relies on LMI-based problems, which grow in complexity according to the number of possible Markov modes and of vertices in each of these modes. Moreover, most of the available solutions require dedicated software packages, which might increase the computational burden and prohibit application in low-cost hardware. In this scenario, the literature lacks robust recursive approaches that can circumvent the drawbacks.

In the present chapter, we aim to find a recursive solution for the linear quadratic regulation problem regarding DMJLS with polytopic uncertainties affecting the system matrices to fill this gap. We formulate a min-max optimization problem subject to equality constraints whose solution yields the recursive regulator for this class of systems. The constraints are composed of subsystems defined on each vertex of the uncertainties. Then, by using the penalty functions method, we attain an unconstrained problem after incorporating this set into the cost function. The conditions for convergence and stability are well established based on coupled algebraic equations derived from the proposed solution. As such, once we know the parameters matrices, it is enough to check the stabilizability and detectability of the system. Furthermore, we provide numerical examples to verify the proposed robust recursive solution's effectiveness in regulation and computational effort and compare the results with the performance obtained with a robust controller borrowed from the specialized literature (GONÇALVES; FIORAVANTI; GEROMEL, 2012). We also show the results of the recursive regulator applied to an autonomous heavy-duty truck whose mathematical model was borrowed from (KIENCKE, 2005) and (RAJAMANI, 2012). The model matrices, polytope vertices, and transition probability matrix were identified based on experimental data (see details in Appendix B).

It is worth mentioning that in this chapter, we consider the transition probability matrix wholly known. The case where polytopic uncertainties affect the transition probabilities will be dealt with in Chapter 5.

4.1 Problem Formulation

A DMJLS subject to uncertainties is described by

$$x_{k+1} = (F_{\theta_k,k} + \delta F_{\theta_k,k})\, x_k + (G_{\theta_k,k} + \delta G_{\theta_k,k})\, u_k, \tag{78}$$

where $k = 0, ..., N-1$, $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input, $F_{\theta_k,k} \in \mathbb{R}^{n\times n}$ and $G_{\theta_k,k} \in \mathbb{R}^{n\times m}$ are nominal system and input matrices, respectively, $\theta = \{\theta_0, ..., \theta_{N-1}\}$ is a Markov chain with modes $\theta_k \in \Theta = \{1, ..., s\}$. The known transition probability matrix for the DMJLS (78) is defined by $\mathbb{P} = [p_{ij}] \in \mathbb{R}^{s\times s}$ with

$$\begin{gathered}\text{Prob}(\theta_{k+1} = j|\theta_k = i) = p_{ij}, \quad \text{Prob}(\theta_0 = i) = p_{i,0}, \\ \sum_{j=1}^{s} p_{ij} = 1, \quad 0 \le p_{ij} \le 1. \end{gathered} \tag{79}$$

Uncertainty matrices $\{\delta F_{\theta_k,k}, \delta G_{\theta_k,k}\}$ are modeled as

$$\begin{bmatrix}\delta F_{\theta_k,k} & \delta G_{\theta_k,k}\end{bmatrix} = \sum_{l=1}^{V} \alpha_{l,k} \begin{bmatrix} F^{(l)}_{\theta_k,k} & G^{(l)}_{\theta_k,k}\end{bmatrix}, \tag{80}$$

with known matrices (vertices) $F^{(l)}_{\theta_k,k} \in \mathbb{R}^{n\times n}$ and $G^{(l)}_{\theta_k,k} \in \mathbb{R}^{n\times m}$, and the coefficients $\alpha_k = \begin{bmatrix}\alpha_{1,k} \dots \alpha_{V,k}\end{bmatrix}^T$ belong to the unit simplex

$$\Lambda_V = \left\{ \alpha_k \in \mathbb{R}^V \,:\, \alpha_{l,k} \ge 0, \sum_{l=1}^{V} \alpha_{l,k} = 1 \right\}. \tag{81}$$

Suppose all states x_k and modes θ_k are observed at every instant k and the system evolves from $\{x_0, \theta_0\}$. Then, the objective is to determine $K_k = (K_{1,k}, \dots, K_{s,k}) \in \mathbb{H}^{m,n}$ such that $u_k = K_{\theta_k,k} x_k$, $\theta_k \in \Theta$, regulates the DMJLS (78) subject to uncertainties (80). With this in mind, let us define the following optimization problem:

$$\min_{u_k,\, x_{k+1}} \; \max_{\delta F_{\theta_k,k},\, \delta G_{\theta_k,k}} \mathbb{E}\left\{ \|x_N\|^2_{P_{\theta_N,N}} + \sum_{t=0}^{N-1} \left(\|x_t\|^2_{Q_{\theta_t,t}} + \|u_t\|^2_{R_{\theta_t,t}} \right) \,|\, \mathcal{S}_t \right\} \tag{82}$$

subject to

$$\begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} x_{k+1} = \begin{bmatrix} F_{\theta_k,k} + V\delta F^{(1)}_{\theta_k,k} \\ \vdots \\ F_{\theta_k,k} + V\delta F^{(V)}_{\theta_k,k} \end{bmatrix} x_k + \begin{bmatrix} G_{\theta_k,k} + V\delta G^{(1)}_{\theta_k,k} \\ \vdots \\ G_{\theta_k,k} + V\delta G^{(V)}_{\theta_k,k} \end{bmatrix} u_k, \tag{83}$$

where $\mathcal{S}_t = \{\theta_t, x_t\}$. Based on Bellman's Principle of Optimality, we can separate (82) into N one-step problems of the form

$$\begin{aligned} &\min_{u_k,\, x_{k+1}} \; \max_{\delta F_{\theta_k,k},\, \delta G_{\theta_k,k}} \mathcal{J}(x_{k+1}, u_k, \delta F_{\theta_k,k}, \delta G_{\theta_k,k}), \\ &\text{subject to (83)}, \end{aligned} \tag{84}$$

for $k = N-1, \dots, 0$, and quadratic cost function

$$\mathcal{J}_k(x_{k+1}, u_k, \delta F_{\theta_k,k}, \delta G_{\theta_k,k}) = x^T_{k+1}\Psi_{\theta_k,k+1}x_{k+1} + x^T_k Q_{\theta_k,k} x_k + u^T_k R_{\theta_k,k} u_k, \tag{85}$$

with

$$\Psi_{i,k+1} = \mathbb{E}\{P_{i,k+1}|\mathcal{S}_k\} = \sum_{j=1}^{s} p_{ij} P_{j,k+1},$$

where $P_{\theta_k,k+1} > 0$, $Q_{\theta_k,k} > 0$, and $R_{\theta_k,k} > 0$, $\delta F^{(l)}_{\theta_k,k} = \alpha_{l,k} F^{(l)}_{\theta_k,k}$, and $\delta G^{(l)}_{\theta_k,k} = \alpha_{l,k} G^{(l)}_{\theta_k,k}$.

By using the penalty functions method, we include the set of constraints (83) into the cost function and comprise all vertices of uncertainties at once for the actual mode at each instant k. To this end, we rewrite the constraints under the form of

$$\begin{bmatrix} x_{k+1} \\ \vdots \\ x_{k+1} \end{bmatrix} - \left(\begin{bmatrix} F_{\theta_k,k} \\ \vdots \\ F_{\theta_k,k} \end{bmatrix} + \begin{bmatrix} V\delta F^{(1)}_{\theta_k,k} \\ \vdots \\ V\delta F^{(V)}_{\theta_k,k} \end{bmatrix} \right) x_k - \left(\begin{bmatrix} G_{\theta_k,k} \\ \vdots \\ G_{\theta_k,k} \end{bmatrix} + \begin{bmatrix} V\delta G^{(1)}_{\theta_k,k} \\ \vdots \\ V\delta G^{(V)}_{\theta_k,k} \end{bmatrix} \right) u_k = 0, \tag{86}$$

and define the functions

$$g(x_{k+1}, u_k, \theta_k) = \left(\begin{bmatrix} I_n & -G_{\theta_k,k} \\ \vdots & \vdots \\ I_n & -G_{\theta_k,k} \end{bmatrix} + \begin{bmatrix} 0 & -V\delta G^{(1)}_{\theta_k,k} \\ \vdots & \vdots \\ 0 & -V\delta G^{(V)}_{\theta_k,k} \end{bmatrix} \right) \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} - \left(\begin{bmatrix} F_{\theta_k,k} \\ \vdots \\ F_{\theta_k,k} \end{bmatrix} + \begin{bmatrix} V\delta F^{(1)}_{\theta_k,k} \\ \vdots \\ V\delta F^{(V)}_{\theta_k,k} \end{bmatrix} \right) x_k. \tag{87}$$

and $\mathcal{C}(x_{k+1}, u_k, \theta_k) = g(x_{k+1}, u_k, \theta_k)^T \mu g(x_{k+1}, u_k, \theta_k)$, where $\mu > 0$ is the penalty parameter.

Then we add $\mathcal{C}(x_{k+1}, u_k, \theta_k)$ to (85) and, with some algebraic manipulation, we redefine the optimization problem (84) as

$$\min_{u_k,\, x_{k+1}} \; \max_{\delta F_{i,k},\, \delta G_{i,k}} J_k(x_{k+1}, u_k, \delta F_{i,k}, \delta G_{i,k}), \tag{88}$$

for each $i = \theta_k \in \Theta$, where $k = N-1, \ldots, 0$, with one-step cost function given by

$$J_k(x_{k+1}, u_k, \delta F_{\theta_k,k}, \delta G_{\theta_k,k}) = \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}^T \begin{bmatrix} \Psi_{i,k+1} & 0 \\ 0 & R_{i,k} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}$$
$$+ \left\{ \left(\begin{bmatrix} 0 & 0 \\ I_n & -G_{i,k} \\ \vdots & \vdots \\ I_n & -G_{i,k} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -V\delta G^{(1)}_{i,k} \\ \vdots & \vdots \\ 0 & -V\delta G^{(V)}_{i,k} \end{bmatrix} \right) \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} - \left(\begin{bmatrix} -I_n \\ F_{i,k} \\ \vdots \\ F_{i,k} \end{bmatrix} + \begin{bmatrix} 0 \\ V\delta F^{(1)}_{i,k} \\ \vdots \\ V\delta F^{(V)}_{i,k} \end{bmatrix} \right) x_k \right\}^T \begin{bmatrix} Q_{i,k} & 0 \\ 0 & \mu I_{nV} \end{bmatrix} \left\{ \bullet \right\}. \tag{89}$$

The following sections present the recursive solution for the established optimization problem and provide conditions for convergence and stability of the closed-loop system subject to polytopic uncertainties.

4.2 RLQR for DMJLS Subject to Polytopic Uncertainties

As the optimization problem (88)–(89) is a particular case of the robust regularized least-squares problem, its structure identifies with the framework outlined in Section 2.1.3 as follows:

$$J \leftarrow J_k(x_{k+1}, u_k, \delta F_{i,k}, \delta G_{i,k}),\ x \leftarrow \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix},\ Q \leftarrow \begin{bmatrix} \Psi_{i,k+1} & 0 \\ 0 & R_{i,k} \end{bmatrix},\ W \leftarrow \begin{bmatrix} Q_{i,k} & 0 \\ 0 & \mu I_{nV} \end{bmatrix},$$

$$M \leftarrow \begin{bmatrix} 0 \\ I_{nV} \end{bmatrix}, \quad A_0 \leftarrow \begin{bmatrix} 0 & 0 \\ I_n & -G_{i,k} \\ \vdots & \vdots \\ I_n & -G_{i,k} \end{bmatrix}, \delta A \leftarrow \begin{bmatrix} 0 & 0 \\ 0 & -V\delta G_{i,k}^{(1)} \\ \vdots & \vdots \\ 0 & -V\delta G_{i,k}^{(V)} \end{bmatrix}, A^{(l)} \leftarrow \begin{bmatrix} 0 & -VG_{i,k}^{(l)} \end{bmatrix}, \quad (90)$$

$$\Gamma \leftarrow \hat{\alpha}_k \otimes I_n,\ b_0 \leftarrow \begin{bmatrix} -I_n \\ F_{i,k} \\ \vdots \\ F_{i,k} \end{bmatrix} x_k,\ \delta b \leftarrow \begin{bmatrix} 0 \\ V\delta F_{i,k}^{(1)} \\ \vdots \\ V\delta F_{i,k}^{(V)} \end{bmatrix} x_k,\ b^{(l)} \leftarrow VF_{i,k}^{(l)} x_k, \quad l = 1, \ldots, V,$$

where $\hat{\alpha}_k = \textbf{diag}\{\alpha_{1,k}, \ldots, \alpha_{V,k}\}$. From (90), observe that $W > 0$. As such, the pseudo-inverse in (23) becomes an actual inverse operation, i.e., $W(\lambda) = \left(W^{-1} - \lambda^{-1} M M^T\right)^{-1}$. Bearing in mind that $\hat{\lambda} \approx \beta\mu$ for $\beta > 1$, as discussed in Section 2.1.3, we have the following relation:

$$W(\lambda) \leftarrow \begin{bmatrix} Q_{i,k} & 0 \\ 0 & \Phi^{-1} \end{bmatrix}, \quad (91)$$

where $\Phi = \mu^{-1}(1 - \beta^{-1}) I_{nV}$.

Let us now establish the main result of this chapter through the following lemma:

Lemma 4.1. *The recursive solution for the optimization problem* (88)–(89)*, for* $\mu > 0$*,* $i = \theta_k \in \Theta$ *and* $k = N-1, \ldots, 0$*, is provided by:*

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{u}_k \\ J_k(\hat{x}_{k+1}, \hat{u}_k, i) \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & x_k^T \end{bmatrix} \begin{bmatrix} L_{i,k} \\ K_{i,k} \\ P_{i,k} \end{bmatrix} x_k, \quad (92)$$

with

$$\begin{bmatrix} L_{i,k} \\ K_{i,k} \\ P_{i,k} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & 0 & \hat{F}_{i,k} \\ 0 & 0 & \hat{E}_{F_{i,k}} \\ I_n & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix}^T \begin{bmatrix} \Psi_{i,k+1}^{-1} & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & R_{i,k}^{-1} & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & Q_{i,k}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & \hat{I} & -\hat{G}_{i,k} \\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{E}_{G_{i,k}} \\ I_n & 0 & 0 & \hat{I}^T & 0 & 0 & 0 \\ 0 & I_m & 0 & -\hat{G}_{i,k}^T & -\hat{E}_{G_{i,k}}^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I_n \\ \hat{F}_{i,k} \\ \hat{E}_{F_{i,k}} \\ 0 \\ 0 \end{bmatrix}, \quad (93)$$

where $P_{i,k} > 0$*,* $Q_{i,k} > 0$*,* $R_{i,k} > 0$*,* $\Phi := \mu^{-1}(1-\beta^{-1})I_{nV}$*,* $\Sigma := (\beta\mu)^{-1}I_{nV}$*, with* $\beta > 1$*,*

$$\hat{F}_{i,k} := \begin{bmatrix} F_{i,k} \\ \vdots \\ F_{i,k} \end{bmatrix} \in \mathbb{R}^{nV\times n}, \quad \hat{G}_{i,k} := \begin{bmatrix} G_{i,k} \\ \vdots \\ G_{i,k} \end{bmatrix} \in \mathbb{R}^{nV\times m},$$

$$\hat{E}_{F_{i,k}} := V \begin{bmatrix} F_{i,k}^{(1)} \\ \vdots \\ F_{i,k}^{(V)} \end{bmatrix} \in \mathbb{R}^{nV\times n}, \quad \hat{E}_{G_k} := V \begin{bmatrix} G_{i,k}^{(1)} \\ \vdots \\ G_{i,k}^{(V)} \end{bmatrix} \in \mathbb{R}^{nV\times m}, \quad \hat{I} := \begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} \in \mathbb{R}^{nV\times n}.$$

Proof. After performing the identifications showed in (90) and (91), by applying Lemma 2.4 produces the recursive solution for (88)–(89) as follows:

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{u}_k \\ J_k(\hat{x}_{k+1},\hat{u}_k,i) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & -x_k^T I_n & x_k^T \hat{F}_{i,k}^T & x_k^T \hat{E}_{F_{i,k}}^T & 0 & 0 \end{bmatrix} \times$$
$$\begin{bmatrix} (\Psi_{i,k+1})^{-1} & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & R_{i,k}^{-1} & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & Q_{i,k}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & \hat{I} & -\hat{G}_{i,k} \\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{E}_{G_{i,k}} \\ I_n & 0 & 0 & \hat{I}^T & 0 & 0 & 0 \\ 0 & I_m & 0 & -\hat{G}_{i,k}^T & -\hat{E}_{G_{i,k}}^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I_n x_k \\ \hat{F}_{i,k} x_k \\ \hat{E}_{F_{i,k}} x_k \\ 0 \\ 0 \end{bmatrix}, \quad (94)$$

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{u}_k \\ J_k(\hat{x}_{k+1},\hat{u}_k,i) \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & x_k^T \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & -I_n & \hat{F}_{i,k}^T & \hat{E}_{F_{i,k}}^T & 0 & 0 \end{bmatrix} \times$$
$$\begin{bmatrix} (\Psi_{i,k+1})^{-1} & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & R_{i,k}^{-1} & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & Q_{i,k}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & \hat{I} & -\hat{G}_{i,k} \\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{E}_{G_{i,k}} \\ I_n & 0 & 0 & \hat{I}^T & 0 & 0 & 0 \\ 0 & I_m & 0 & -\hat{G}_{i,k}^T & -\hat{E}_{G_{i,k}}^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I_n \\ \hat{F}_{i,k} \\ \hat{E}_{F_{i,k}} \\ 0 \\ 0 \end{bmatrix} x_k. \quad (95)$$

Here we adopt the approximation $\hat{\lambda} \approx \beta\mu$, $\beta > 1$, so that $\Sigma := (\beta\mu)^{-1}I_{nV}$ and $\Phi = \mu^{-1}(1-\beta^{-1})I_{nV}$. Then, for each mode $i \in \Theta$, the closed-loop matrix $L_{i,k}$, the feedback gain $K_{i,k}$ and the weight matrix $P_{i,k}$ are computed from (95) by defining

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{u}_k \\ J_k(\hat{x}_{k+1},\hat{u}_k,i) \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & x_k^T \end{bmatrix} \begin{bmatrix} L_{i,k} \\ K_{i,k} \\ P_{i,k} \end{bmatrix} x_k.$$

□

In the sequence, we provide a reduced formulation of the solution presented by Lemma 4.1 for problem (88)–(89).

Theorem 4.1. *For a fixed $\mu > 0$, the solution for problem* (88)–(89)*, which is given by* (92)–(93)*, is equivalent to:*

$$L_{i,k} = X_{i,k+1}\bar{\mathcal{F}}_{i,k} - X_{i,k+1}\bar{\mathcal{G}}_{i,k}\left(I_m + \bar{\mathcal{G}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{G}}_{i,k}\right)^{-1}\bar{\mathcal{G}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{F}}_{i,k}, \tag{96}$$

$$\begin{aligned} K_{i,k} = -R_{i,k}^{-1}\hat{E}_{G_{i,k}}^T\left(\Sigma + \hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^T\right)^{-1}\hat{E}_{F_{i,k}} \\ - \bar{R}_{i,k}G_{i,k}^T\left(I_n + \overline{\Psi}_{i,k+1}\bar{\mathcal{G}}_{i,k}\bar{\mathcal{G}}_{i,k}^T\right)^{-1}\overline{\Psi}_{i,k+1}\bar{\mathcal{F}}_{i,k}, \end{aligned} \tag{97}$$

$$P_{i,k} = \bar{Q}_{i,k} + \bar{\mathcal{F}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{F}}_{i,k} - \bar{\mathcal{F}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{G}}_{i,k}\left(I_m + \bar{\mathcal{G}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{G}}_{i,k}\right)^{-1}\bar{\mathcal{G}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{F}}_{i,k}, \tag{98}$$

where

$$\begin{aligned}
\bar{\mathcal{F}}_{i,k} &= F_{i,k} - G_{i,k}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^T\left(\Sigma + \hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^T\right)^{-1}\hat{E}_{F_{i,k}}, \\
\bar{Q}_{i,k} &= Q_{i,k} + \hat{E}_{F_{i,k}}^T\left(\Sigma + \hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^T\right)^{-1}\hat{E}_{F_{i,k}}, \\
\bar{R}_{i,k} &= R_{i,k}^{-1}\left[I_m - \hat{E}_{G_{i,k}}^T\left(\Sigma + \hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^T\right)^{-1}\hat{E}_{G_{i,k}}R_{i,k}^{-1}\right], \\
\overline{\Psi}_{i,k+1} &= V\left(\mu^{-1}(1-\beta^{-1})I_n + V\Psi_{i,k+1}^{-1}\right)^{-1}, \quad \beta > 1, \\
X_{i,k+1} &= \Psi_{i,k+1}^{-1}\overline{\Psi}_{i,k+1}, \quad \bar{\mathcal{G}}_{i,k} = G_{i,k}\bar{R}_{i,k}^{1/2}.
\end{aligned}$$

Proof. Note that matrices $K_{i,k}$ and $L_{i,k}$ in (93) compose the solution for the following linear system:

$$\begin{bmatrix} \Psi_{i,k+1}^{-1} & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & R_{i,k}^{-1} & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & Q_{i,k}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & \hat{I} & -\hat{G}_{i,k} \\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{E}_{G_{i,k}} \\ I_n & 0 & 0 & \hat{I}^T & 0 & 0 & 0 \\ 0 & I_m & 0 & -\hat{G}_{i,k}^T & -\hat{E}_{G_{i,k}}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \Xi_4 \\ \xi_5 \\ L_{i,k} \\ K_{i,k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -I_n \\ \hat{F}_k \\ \hat{E}_{F_k} \\ 0 \\ 0 \end{bmatrix}, \tag{99}$$

with $\Xi_4 = \mathbb{1}_V \otimes \xi_4$, $\xi_4 \in \mathbb{R}^{n\times n}$, and

$$\Phi = \mu^{-1}(1-\beta^{-1})I_{nV} = I_V \otimes \varphi, \tag{100}$$

where $\varphi = \mu^{-1}(1-\beta^{-1})I_n$. By developing (99) we have the set of equations

$$\begin{cases} \Psi_{i,k+1}^{-1}\xi_1 + L_{i,k} = 0, & (101)\\ R_{i,k}^{-1}\xi_2 + K_{i,k} = 0, & (102)\\ Q_{i,k}^{-1}\xi_3 = -I_n, & (103)\\ \Phi\Xi_4 + \hat{I}L_{i,k} - \hat{G}_{i,k}K_{i,k} = \hat{F}_{i,k}, & (104)\\ \Sigma\xi_5 - \hat{E}_{G_{i,k}}K_{i,k} = \hat{E}_{F_{i,k}}, & (105)\\ I_n\xi_1 + \hat{I}^T\Xi_4 = 0, & (106)\\ I_m\xi_2 - \hat{G}_{i,k}^T\Xi_4 - \hat{E}_{G_{i,k}}^T\xi_5 = 0, & (107) \end{cases}$$

and substitute (99) into (93) to produce

$$P_{k,i} = \begin{bmatrix} 0 & 0 & -I_n & \hat{F}_{i,k}^T & \hat{E}_{F_{i,k}}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \Xi_4 \\ \xi_5 \\ L_{i,k} \\ K_{i,k} \end{bmatrix},$$

$$P_{i,k} = -\xi_3 + VF_{i,k}^T\xi_4 + \hat{E}_{F_{i,k}}^T\xi_5. \tag{108}$$

Assume that (93) holds, which implies a unique solution for the set of equations from (101) to (108). Thence, from (101) and (103) we have

$$\xi_1 = -\Psi_{i,k+1}L_{i,k}, \tag{109}$$

$$\xi_3 = -Q_{i,k}. \tag{110}$$

Next, expand (104) to yield

$$\varphi\xi_4 + L_{i,k} - G_{i,k}K_{i,k} = F_{i,k}. \tag{111}$$

From (106),

$$\xi_4 = -V^{-1}\xi_1, \tag{112}$$

and by combining it with (109) results in

$$L_{i,k} = V\Psi_{i,k+1}^{-1}\xi_4. \tag{113}$$

Now, (102) implies

$$\xi_2 = -R_{i,k}K_{i,k}, \tag{114}$$

and place it into (107) to obtain

$$K_{i,k} = -R_{i,k}^{-1} V G_{i,k}^T \xi_4 - R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T \xi_5. \tag{115}$$

By substituting (115) into (105) produces:

$$\hat{E}_{G_{i,k}} R_{i,k}^{-1} V G_{i,k}^T \xi_4 + \left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T \right) \xi_5 = \hat{E}_{F_{i,k}}. \tag{116}$$

Now, substitute (113) and (115) into (111):

$$\left(\varphi + V \Psi_{i,k+1}^{-1} + G_{i,k} R_{i,k}^{-1} V G_{i,k}^T \right) \xi_4 + G_{i,k} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T \xi_5 = F_{i,k}. \tag{117}$$

Therefore, let us combine (116) and (117) to obtain the following linear system:

$$\begin{bmatrix} \varphi + V \Psi_{i,k+1}^{-1} + G_{i,k} R_{i,k}^{-1} V G_{i,k}^T & G_{i,k} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T \\ \hat{E}_{G_{i,k}} R_{i,k}^{-1} V G_{i,k}^T & \Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T \end{bmatrix} \begin{bmatrix} \xi_4 \\ \xi_5 \end{bmatrix} = \begin{bmatrix} F_{i,k} \\ \hat{E}_{F_{i,k}} \end{bmatrix}. \tag{118}$$

Define

$$U = \begin{bmatrix} \varphi + V \Psi_{i,k+1}^{-1} + G_{i,k} R_{i,k}^{-1} V G_{i,k}^T & G_{i,k} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T \\ \hat{E}_{G_{i,k}} R_{i,k}^{-1} V G_{i,k}^T & \Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}. \tag{119}$$

Then,

$$\begin{bmatrix} \xi_4 \\ \xi_5 \end{bmatrix} = U^{-1} \begin{bmatrix} F_{i,k} \\ \hat{E}_{F_{i,k}} \end{bmatrix}, \tag{120}$$

where U^{-1} is given by the Banachiewicz formula (see Lemma A.5) as

$$U^{-1} = \begin{bmatrix} (U/\mathcal{D})^{-1} & -(U/\mathcal{D})^{-1} \mathcal{B} \mathcal{D}^{-1} \\ -\mathcal{D}^{-1} \mathcal{C} (U/\mathcal{D})^{-1} & \mathcal{D}^{-1} + \mathcal{D}^{-1} \mathcal{C} (U/\mathcal{D})^{-1} \mathcal{B} \mathcal{D}^{-1} \end{bmatrix}, \tag{121}$$

in which $U/\mathcal{D} = \mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C}$ is the Schur Complement of $\mathcal{D}$ in U. Thus,

$$U/\mathcal{D} = \varphi + V \Psi_{i,k+1}^{-1} + G_{i,k} R_{i,k}^{-1} V G_{i,k}^T - G_{i,k} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T \left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T \right)^{-1} \hat{E}_{G_{i,k}} R_{i,k}^{-1} V G_{i,k}^T,$$

$$U/\mathcal{D} = \varphi + V \Psi_{i,k+1}^{-1} + V G_{i,k} R_{i,k}^{-1} \left(I_m - \hat{E}_{G_{i,k}}^T \left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T \right)^{-1} \hat{E}_{G_{i,k}} R_{i,k}^{-1} \right) G_{i,k}^T.$$

Define

$$\bar{R}_{i,k} = R_{i,k}^{-1} \left[I_m - \hat{E}_{G_{i,k}}^T \left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T \right)^{-1} \hat{E}_{G_{i,k}} R_{i,k}^{-1} \right], \tag{122}$$

so that $U/\mathcal{D} = \varphi + V \Psi_{i,k+1}^{-1} + V G_{i,k} \bar{R}_{i,k} G_{i,k}^T$. Now define

$$\Omega_{i,k+1} = \varphi + V \Psi_{i,k+1}^{-1},$$

such that $\mathcal{U}/\mathcal{D} = \Omega_{i,k+1} + V G_{i,k} \bar{R}_{i,k} G_{i,k}^T$, and

$$(\mathcal{U}/\mathcal{D})^{-1} = \Omega_{i,k+1}^{-1} - \Omega_{i,k+1}^{-1} V G_{i,k} (\bar{R}_i^{-1} + G_{i,k}^T \Omega_{i,k+1}^{-1} V G_{i,k})^{-1} G_{i,k}^T \Omega_{i,k+1}^{-1}.$$

Also,

$$\mathcal{D}^{-1}+\mathcal{D}^{-1}\mathcal{C}(U/\mathcal{D})^{-1}\mathcal{B}\mathcal{D}^{-1}=\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1}$$
$$+\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1}V\hat{E}_{G_{i,k}}R_{i,k}^{-1}G_{i,k}^{T}\left(\Omega_{i,k+1}^{-1}-\Omega_{i,k+1}^{-1}VG_{i,k}(\bar{R}_{i}^{-1}\right.$$
$$\left.+G_{i,k}^{T}\Omega_{i,k+1}^{-1}VG_{i,k})^{-1}G_{i,k}^{T}\Omega_{i,k+1}^{-1}\right)G_{i,k}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1},$$

$$\mathcal{D}^{-1}+\mathcal{D}^{-1}\mathcal{C}(U/\mathcal{D})^{-1}\mathcal{B}\mathcal{D}^{-1}=\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1}\left[I_{nV}+V\hat{E}_{G_{i,k}}R_{i,k}^{-1}G_{i,k}^{T}\left(\Omega_{i,k+1}^{-1}\right.\right.$$
$$\left.\left.-\Omega_{i,k+1}^{-1}VG_{i,k}(\bar{R}_{i}^{-1}+G_{i,k}^{T}\Omega_{i,k+1}^{-1}VG_{i,k})^{-1}G_{i,k}^{T}\Omega_{i,k+1}^{-1}\right)G_{i,k}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1}\right].$$

From (120)–(121),

$$\xi_4=(U/\mathcal{D})^{-1}F_{i,k}-(U/\mathcal{D})^{-1}\mathcal{B}\mathcal{D}^{-1}\hat{E}_{F_{i,k}},$$
$$\xi_4=\left(\Omega_{i,k+1}^{-1}-\Omega_{i,k+1}^{-1}VG_{i,k}(\bar{R}_{i}^{-1}+G_{i,k}^{T}\Omega_{i,k+1}^{-1}VG_{i,k})^{-1}G_{i,k}^{T}\Omega_{i,k+1}^{-1}\right)$$
$$\times\left(F_{i,k}-G_{i,k}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1}\hat{E}_{F_{i,k}}\right),$$

and define

$$\bar{\mathcal{F}}_{i,k}=F_{i,k}-G_{i,k}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1}\hat{E}_{F_{i,k}},$$

such that

$$\xi_4=\left(\Omega_{i,k+1}^{-1}-\Omega_{i,k+1}^{-1}VG_{i,k}(\bar{R}_{i}^{-1}+G_{i,k}^{T}\Omega_{i,k+1}^{-1}VG_{i,k})^{-1}G_{i,k}^{T}\Omega_{i,k+1}^{-1}\right)\bar{\mathcal{F}}_{i,k},$$
$$\xi_4=\left(\Omega_{i,k+1}+VG_{i,k}\bar{R}_{i,k}G_{i,k}^{T}\right)^{-1}\bar{\mathcal{F}}_{i,k}. \qquad (123)$$

From (120)–(121),

$$\xi_5=-\mathcal{D}^{-1}\mathcal{C}(U/\mathcal{D})^{-1}F_{i,k}+\left(\mathcal{D}^{-1}+\mathcal{D}^{-1}\mathcal{C}(U/\mathcal{D})^{-1}\mathcal{B}\mathcal{D}^{-1}\right)\hat{E}_{F_{i,k}},$$
$$\xi_5=-\mathcal{D}^{-1}\mathcal{C}(U/\mathcal{D})^{-1}\left(F_{i,k}-\mathcal{B}\mathcal{D}^{-1}\hat{E}_{F_{i,k}}\right)+\mathcal{D}^{-1}\hat{E}_{F_{i,k}},$$

$$\xi_5=-\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1}V\hat{E}_{G_{i,k}}R_{i,k}^{-1}G_{i,k}^{T}$$
$$\times\left(\Omega_{i,k+1}^{-1}-\Omega_{i,k+1}^{-1}VG_{i,k}(\bar{R}_{i}^{-1}+G_{i,k}^{T}\Omega_{i,k+1}^{-1}VG_{i,k})^{-1}G_{i,k}^{T}\Omega_{i,k+1}^{-1}\right)\left(F_{i,k}\right.$$
$$\left.-G_{i,k}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1}\hat{E}_{F_{i,k}}\right)+\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1}\hat{E}_{F_{i,k}},$$

$$\xi_5=\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1}\left(\hat{E}_{F_{i,k}}-V\hat{E}_{G_{i,k}}R_{i,k}^{-1}G_{i,k}^{T}\left(\Omega_{i,k+1}+VG_{i,k}\bar{R}_{i,k}G_{i,k}^{T}\right)^{-1}\bar{\mathcal{F}}_{i,k}\right),$$
$$\xi_5=\left(\Sigma+\hat{E}_{G_{i,k}}R_{i,k}^{-1}\hat{E}_{G_{i,k}}^{T}\right)^{-1}\left(\hat{E}_{F_{i,k}}-V\hat{E}_{G_{i,k}}R_{i,k}^{-1}G_{i,k}^{T}\xi_4\right). \qquad (124)$$

Now substitute (110), (123) and (124) into (108):

$$P_{i,k} = -\xi_3 + VF_{i,k}^T\xi_4 + \hat{E}_{F_{i,k}}^T \left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T\right)^{-1} \left(\hat{E}_{F_{i,k}} - V\hat{E}_{G_{i,k}} R_{i,k}^{-1} G_{i,k}^T \xi_4\right),$$

$$P_{i,k} = Q_{i,k} + V\bar{\mathcal{F}}_{i,k}^T\xi_4 + \hat{E}_{F_{i,k}}^T \left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T\right)^{-1} \hat{E}_{F_{i,k}}.$$

Define

$$\bar{Q}_{i,k} = Q_{i,k} + \hat{E}_{F_{i,k}}^T \left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T\right)^{-1} \hat{E}_{F_{i,k}},$$

$$\bar{\mathcal{G}}_{i,k} = G_{i,k}\bar{R}_{i,k}1/2,$$

then,

$$P_{i,k} = \bar{Q}_{i,k} + V\bar{\mathcal{F}}_{i,k}^T \left(\Omega_{i,k+1} + V\bar{\mathcal{G}}_{i,k} I_m \bar{\mathcal{G}}_{i,k}^T\right)^{-1} \bar{\mathcal{F}}_{i,k},$$

$$P_{i,k} = \bar{Q}_{i,k} + V\bar{\mathcal{F}}_{i,k}^T \left(I_n + V\Omega_{i,k+1}^{-1}\bar{\mathcal{G}}_{i,k} I_m \bar{\mathcal{G}}_{i,k}^T\right)^{-1} \Omega_{i,k+1}^{-1}\bar{\mathcal{F}}_{i,k},$$

$$P_{i,k} = \bar{Q}_{i,k} + \bar{\mathcal{F}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{F}}_{i,k} - \bar{\mathcal{F}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{G}}_{i,k}\left(I_m + \bar{\mathcal{G}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{G}}_{i,k}\right)^{-1}\bar{\mathcal{G}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{F}}_{i,k},$$

which corresponds to (98), where $\overline{\Psi}_{i,k+1} = V\Omega_{i,k+1}^{-1} = V\left(\mu^{-1}(1-\beta^{-1})I_n + V\Psi_{i,k+1}^{-1}\right)^{-1}$.

Now, place (123) into (113) to obtain

$$L_{i,k} = V\Psi_{i,k+1}^{-1}\left(\Omega_{i,k+1} + VG_{i,k}\bar{R}_{i,k}G_{i,k}^T\right)^{-1}\bar{\mathcal{F}}_{i,k},$$

$$L_{i,k} = V\Psi_{i,k+1}^{-1}\left(I_n + V\Omega_{i,k+1}^{-1}\bar{\mathcal{G}}_{i,k}^T I_m \bar{\mathcal{G}}_{i,k}^T\right)^{-1}\Omega_{i,k+1}^{-1}\bar{\mathcal{F}}_{i,k},$$

$$L_{i,k} = V\Psi_{i,k+1}^{-1}\left(I_n - V\Omega_{i,k+1}^{-1}\bar{\mathcal{G}}_{i,k}\left(I_m + \bar{\mathcal{G}}_{i,k}^T V\Omega_{i,k+1}^{-1}\bar{\mathcal{G}}_{i,k}\right)^{-1}\bar{\mathcal{G}}_{i,k}^T\right)\Omega_{i,k+1}^{-1}\bar{\mathcal{F}}_{i,k},$$

$$L_{i,k} = X_{i,k+1}\bar{\mathcal{F}}_{i,k} - X_{i,k+1}\bar{\mathcal{G}}_{i,k}\left(I_m + \bar{\mathcal{G}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{G}}_{i,k}\right)^{-1}\bar{\mathcal{G}}_{i,k}^T\overline{\Psi}_{i,k+1}\bar{\mathcal{F}}_{i,k},$$

which corresponds to (96), with $X_{i,k+1} = \Psi_{i,k+1}^{-1}\overline{\Psi}_{i,k+1}$. Finally, substitute (123) and (124) into (115), such that

$$K_{i,k} = -R_{i,k}^{-1}VG_{i,k}^T\xi_4 - R_{i,k}^{-1}\hat{E}_{G_{i,k}}^T\left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T\right)^{-1}\left(\hat{E}_{F_{i,k}} - V\hat{E}_{G_{i,k}} R_{i,k}^{-1} G_{i,k}^T\xi_4\right),$$

$$\begin{aligned}K_{i,k} = &-R_{i,k}^{-1}\hat{E}_{G_{i,k}}^T\left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T\right)^{-1}\hat{E}_{F_{i,k}} \\ &- VR_{i,k}^{-1}\left(I_m - \hat{E}_{G_{i,k}}^T\left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T\right)^{-1}\hat{E}_{G_{i,k}} R_{i,k}^{-1}\right)G_{i,k}^T\xi_4,\end{aligned}$$

$$K_{i,k} = -R_{i,k}^{-1}\hat{E}_{G_{i,k}}^T\left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T\right)^{-1}\hat{E}_{F_{i,k}} - V\bar{R}_{i,k}G_{i,k}^T\xi_4,$$

$$\begin{aligned}K_{i,k} = &-R_{i,k}^{-1}\hat{E}_{G_{i,k}}^T\left(\Sigma + \hat{E}_{G_{i,k}} R_{i,k}^{-1} \hat{E}_{G_{i,k}}^T\right)^{-1}\hat{E}_{F_{i,k}} \\ &- \bar{R}_{i,k}G_{i,k}^T\left(I_n + \overline{\Psi}_{i,k+1}\bar{\mathcal{G}}_{i,k}\bar{\mathcal{G}}_{i,k}^T\right)^{-1}\overline{\Psi}_{i,k+1}\bar{\mathcal{F}}_{i,k},\end{aligned}$$

which corresponds to (97).

□

Remark 4.1. *For any positive penalty μ, when $\beta \to 1^+$ we have*

$$L_{i,k} = \bar{\mathcal{F}}_{i,k} - \bar{\mathcal{G}}_{i,k}\left(I_m + \bar{\mathcal{G}}_{i,k}^T \Psi_{i,k+1} \bar{\mathcal{G}}_{i,k}\right)^{-1} \bar{\mathcal{G}}_{i,k}^T \Psi_{i,k+1} \bar{\mathcal{F}}_{i,k},$$

and also

$$P_{i,k} = \bar{Q}_{i,k} + \bar{\mathcal{F}}_{i,k}^T \Psi_{i,k+1} \bar{\mathcal{F}}_{i,k} - \bar{\mathcal{F}}_{i,k}^T \Psi_{i,k+1} \bar{\mathcal{G}}_{i,k}\left(I_m + \bar{\mathcal{G}}_{i,k}^T \Psi_{i,k+1} \bar{\mathcal{G}}_{i,k}\right)^{-1} \bar{\mathcal{G}}_{i,k}^T \Psi_{i,k+1} \bar{\mathcal{F}}_{i,k},$$

which is a standard coupled Riccati equation (COSTA; FRAGOSO; MARQUES, 2005).

4.3 Convergence and Stability

At this point, we can provide the conditions for convergence and stability of the closed-loop system with the recursive regulator for DMJLS subject to polytopic uncertainties. We consider the DMJLS (78) with time-invariant parameters, time-varying polytope coefficients α_k, and $\beta \to 1^+$. Moreover, we define $\bar{\mathcal{F}} = (\bar{\mathcal{F}}_1, \ldots, \bar{\mathcal{F}}_s) \in \mathbb{H}^{n,n}$, $\bar{\mathcal{G}} = (\bar{\mathcal{G}}_1, \ldots, \bar{\mathcal{G}}_s) \in \mathbb{H}^{n,m}$, $\bar{Q} = (\bar{Q}_1, \ldots, \bar{Q}_s) \in \mathbb{H}^{n,n}$, and assume $\bar{Q}_i > 0$, $\bar{R}_i > 0$, and constant p_{ij}, for $i, j \in \Theta$.

Theorem 4.2. *Consider* (98) *with initial condition $P_N > 0$, fixed $\mu > 0$, and a priori known $\bar{\mathcal{F}}$, $\bar{\mathcal{G}}$ and $\bar{Q}$. Assume the matrix pair $\{\bar{\mathcal{F}}, \bar{\mathcal{G}}\}$ is stabilizable and $\{\bar{Q}^{1/2}, \bar{\mathcal{F}}\}$ is detectable. Then, $P_k \in \mathbb{H}_+^n$ generated by* (98) *converges to a unique $P = (P_1, \ldots, P_s) \in \mathbb{H}_+^n$ for which the closed-loop matrix*

$$L_i = \bar{\mathcal{F}}_i - \bar{\mathcal{G}}_i\left(I_m + \bar{\mathcal{G}}_i^T \Psi_i \bar{\mathcal{G}}_i\right)^{-1} \bar{\mathcal{G}}_i^T \Psi_i \bar{\mathcal{F}}_i$$

of the DMJLS (78) *is mean square stable, with $\Psi_i = \sum_{j=1}^{s} p_{ij} P_j$.*

Proof. The reduced form of $P_{i,k}$ achieved in Theorem 4.1 has the structure of the coupled algebraic Riccati equations (32) when $\beta \to 1^+$, as we mentioned in Remark 4.1. As such, we make the following identifications:

$$A_i \leftarrow \bar{\mathcal{F}}_i, \quad B_i \leftarrow \bar{\mathcal{G}}_i, \quad R_i \leftarrow I_m, \quad Q_i \leftarrow \bar{Q}_i, \quad \mathcal{E}_{k+1} \leftarrow \Psi_{i,k+1}.$$

We also have that $(I_m + \bar{\mathcal{G}}_i^T \Psi_{i,k+1} \bar{\mathcal{G}}_i)$ is positive definite for any $\mu > 0$. Then, by the fundamental arguments presented in Costa, Fragoso and Marques (2005) and assuming detectability and stabilizability of the pairs $\{\bar{Q}^{1/2}, \bar{\mathcal{F}}\}$ and $\{\bar{\mathcal{F}}, \bar{\mathcal{G}}\}$, respectively, it follows that $P_k \in \mathbb{H}_+^n$ converges to $P \in \mathbb{H}_+^n$. In this case, $\Psi_{i,k+1} \to \Psi_i$, $L_{i,k} \to L_i$, $K_{i,k} \to K_i$, and the solution P ensures stability of the closed-loop matrix L_i of (78) when $u_k = K_i x_k$. □

4.4 Illustrative Examples

We provide two examples to illustrate the performance of the proposed solution for the robust regulation problem of DMJLS subject to polytopic uncertainties. For the sake of comparison, we also apply a robust Markovian H_∞ controller (GONÇALVES; FIORAVANTI; GEROMEL, 2012), which is based on an optimization problem with LMI constraints and computed via the YALMIP Toolbox (LÖFBERG, 2004).

Example 4.1. *Consider the following unstable two-mode DMJLS subject to polytopic uncertainties with randomly generated vertices:*

- *Mode 1:*

$$F_{1,k} = \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ -2.5 & 3.2 & 1.2 \\ 1.4 & 1.6 & 2.0 \end{bmatrix}, \quad G_{1,k} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

$$F_{1,k}^{(1)} = \begin{bmatrix} 0.3268 & -0.0938 & 0 \\ 0 & 0.7264 & 0.0758 \\ 0 & 0 & -0.8579 \end{bmatrix}, \quad G_{1,k}^{(1)} = \begin{bmatrix} -0.3148 & 0 & 0 \\ 0 & 0.2660 & -0.3699 \\ 0 & 0.3623 & 0 \end{bmatrix},$$

$$F_{1,k}^{(2)} = \begin{bmatrix} 0.3107 & 0 & 0 \\ 0 & -0.8650 & 0.0511 \\ 0 & -0.7575 & 0 \end{bmatrix}, \quad G_{1,k}^{(2)} = \begin{bmatrix} -0.1067 & 0 & 0 \\ 0 & -0.0500 & 0 \\ 0 & 0.0090 & 0 \end{bmatrix}.$$

- *Mode 2:*

$$F_{2,k} = \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ -2.7 & 0.4 & 2.1 \\ -3.4 & 2.5 & 4.8 \end{bmatrix}, \quad G_{2,k} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

$$F_{2,k}^{(1)} = \begin{bmatrix} 0.0367 & 0 & 0.2137 \\ 0 & 0.1028 & -0.1753 \\ 1.5460 & 0.2421 & 0 \end{bmatrix}, \quad G_{2,k}^{(1)} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 0.7893 \\ 0 & 0 & -0.1926 \end{bmatrix},$$

$$F_{2,k}^{(2)} = \begin{bmatrix} 0 & 0 & 0.2688 \\ -0.8273 & 0.0794 & 0 \\ 1.0264 & 0.2747 & 0 \end{bmatrix}, \quad G_{2,k}^{(2)} = \begin{bmatrix} -0.1681 & 0 & -0.2984 \\ 0 & 0.0159 & -0.5264 \\ 0.1500 & -0.0500 & 0 \end{bmatrix}.$$

Assume that the initial condition and the transition probability matrix are, respectively,

$$x_0 = \begin{bmatrix} 0.024 \\ 0.244 \\ 0.556 \end{bmatrix}, \quad \mathbb{P} = \begin{bmatrix} 0.60 & 0.40 \\ 0.25 & 0.75 \end{bmatrix}.$$

We choose the following parameters for (89) *to set up the penalized cost function:*

$$Q_{1,k} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}, \quad R_{1,k} = \begin{bmatrix} 10^{-4} & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

$$Q_{2,k} = \begin{bmatrix} 1.6 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}, \quad R_{2,k} = \begin{bmatrix} 10^{-4} & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

$$P_{1,0} = P_{2,0} = I_3, \quad \mu = 10^{15}, \quad \beta = 1.1.$$

With the above configuration, Lemma 4.1 provided the state feedback gains K_i and the solutions for the coupled Riccati equations as follows:

$$K_1 = \begin{bmatrix} -2.0607 & 0.0745 & -0.7042 \\ 0.1209 & -1.1493 & -0.3461 \\ 1.0366 & -0.9135 & -0.4782 \end{bmatrix}, \quad P_1 = 10^{15} \begin{bmatrix} 7.0137 & -2.5253 & 1.4391 \\ -2.5253 & 9.5378 & 1.2458 \\ 1.4391 & 1.2458 & 5.5096 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.5555 & -1.2694 & -1.4926 \\ 3.9370 & -1.1469 & -3.3779 \\ -0.4086 & 0.2399 & 0.4694 \end{bmatrix}, \quad P_2 = 10^{16} \begin{bmatrix} 1.6759 & 0.1355 & -0.0572 \\ 0.1355 & 0.1354 & 0.0965 \\ -0.0572 & 0.0965 & 0.1787 \end{bmatrix}.$$

In addition, the robust Markovian H_∞ state feedback gains are

$$K_{1,H_\infty} = \begin{bmatrix} -2.4500 & 0.3001 & -0.9001 \\ -0.4000 & -0.6001 & -0.9998 \\ 1.4500 & -1.3000 & -0.1000 \end{bmatrix}, \qquad K_{2,H_\infty} = \begin{bmatrix} -0.1500 & -1.5505 & -1.8495 \\ 4.3999 & -1.5013 & -3.7985 \\ -0.8499 & 0.5510 & 0.8488 \end{bmatrix}.$$

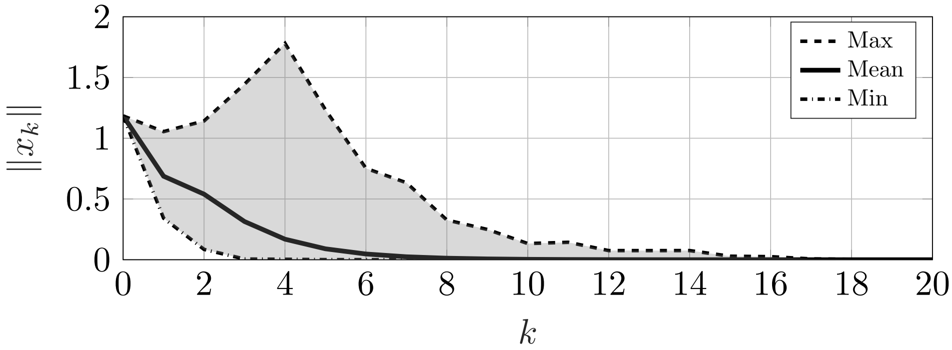
The simulation results were averaged over 5000 Monte Carlo experiments performed on a 2.50 GHz i5-3210M CPU with 8 GB of RAM. We chose, for each experiment, a time horizon of $N = 100$, and the coefficients α_k changed randomly at every iteration. In Fig. 5, we present the norms of the states vector obtained with the robust recursive regulator and averaged over all experiments. Fig. 6 shows the maximum spectral radii r_σ of the open-loop system and of the closed-loop system subject to polytopic uncertainties with the recursive regulator for different values of penalty parameter μ. It is worth pointing out that the proposed solution stabilizes the system even for small values of μ, since all the spectral radii remain lower than 1 in all experiments. As we can verify in Table 3, both the robust recursive regulator for polytopic DMJLS (PMRR for short) and the robust Markovian H_∞ controller presented equivalent performances in terms of norms of states and input vectors, denoted by $\|\bar{x}\|_{\mathcal{L}_2}$ and $\|\bar{u}\|_{\mathcal{L}_2}$ respectively. Nevertheless, the computational time $\bar{T}_c$ required to compute the feedback gains of the recursive regulator is, on average, two orders of magnitude lower when compared with the robust H_∞ controller borrowed from Gonçalves, Fioravanti and Geromel (2012).

Table 3 – Simulation results for Example 4.1.

Controller	$\|\bar{x}\|_{\mathcal{L}_2}$	$\sigma_{\bar{x}}$	$\bar{T}_c$ (ms)	$\sigma_{\bar{T}_c}$ (ms)	$\|\bar{u}\|_{\mathcal{L}_2}$	$\sigma_{\bar{u}}$
PMRR	1.3772	0.1375	3.8695	0.3694	3.1035	0.3101
Markovian H_∞	1.5175	0.1513	548.6110	57.6667	3.9510	0.3952

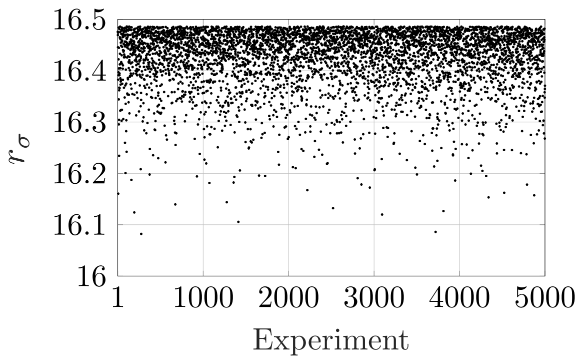
Source: author.

Figure 5 – Averaged norms of system states with the robust recursive regulator for DMJLS.

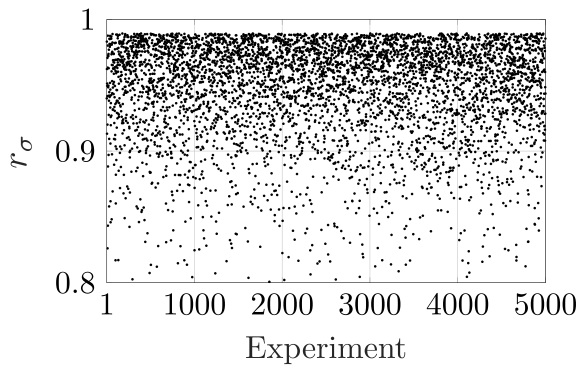


Source: author.

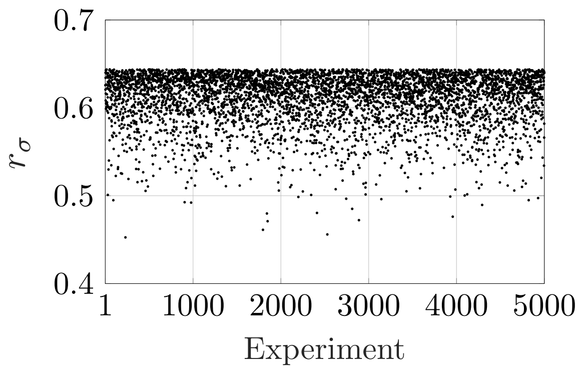
Figure 6 – Open-loop and closed-loop spectral radii, r_σ, of 5000 experiments with randomly selected coefficients $\alpha_{i,k}$.



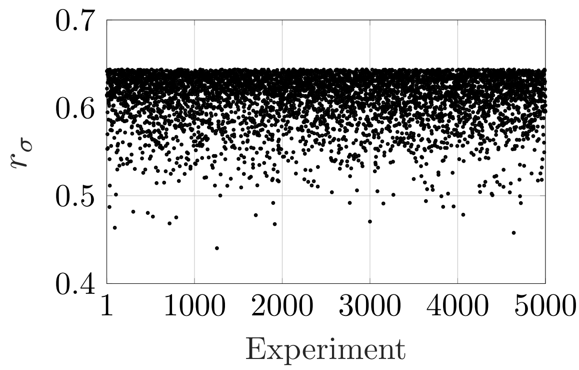
(a) Open-loop.



(b) Closed-loop, $\mu = 1$.



(c) Closed-loop, $\mu = 10^7$.



(d) Closed-loop, $\mu = 10^{15}$.

Source: author.

Example 4.2. *The following DMJLS represents the drivetrain model of an autonomous heavy-duty G 360 CB6x4HSZ Scania truck. The model matrices, polytope vertices, and transition probability matrix were identified based on experimental data acquired while driving the truck around the University of São Paulo campus at São Carlos. For this example, the model captures the drivetrain behavior regarding only the throttle inputs and has 7 Markov modes related to the transmission rates from 4th to 10th gears, hence $\theta_k \in \{1, ..., 7\}$. Each mode has three polytopic vertices to compose the uncertainties $\{\delta F_{i,k}, \delta G_{i,k}\}$, which represent different road slopes (uphill, downhill and flat). The reader can find further details about the identification process in Appendix B.*

The state errors are represented by $x_k = q_k - q_k^{ref}$, where $q_k = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T$ is composed of driveshaft torsion, engine speed, and wheel speed, respectively, and control input $u_k = \tau_k - \tau_k^{ref}$, in which τ_k is the throttle pedal position. Also, q_k^{ref} and τ_k^{ref} are the reference values for states and throttle pedal positions. The longitudinal control task consists of tracking the experimentally collected reference trajectories $q^{ref} = \begin{bmatrix} q_1^{ref} & q_2^{ref} & q_3^{ref} \end{bmatrix}^T$ via torques delivered by the engine of the autonomous truck, with initial condition $x_0 = \begin{bmatrix} 0.01 & 0.05 & 0.07 \end{bmatrix}^T$. Thus, we set up all modes of the robust recursive regulator for polytopic DMJLS (PMRR for short) with the parameters

$$Q_{i,k} = 10^7 I_3, \quad R_{i,k} = 10^{-5}, \quad P_{i,0} = I_3, \quad \mu = 10^8, \quad \beta = 1.01,$$

so Lemma 4.1 provided the following state feedback gains and solutions for the coupled algebraic Riccati equations:

$$K_1 = \begin{bmatrix} -0.6865 & 1.2587 & -1.9331 \end{bmatrix}, \quad P_1 = 10^8 \begin{bmatrix} 2.3111 & 0.6433 & -3.4075 \\ 0.6433 & 2.3121 & -1.4776 \\ -3.4075 & -1.4776 & 5.8408 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.5946 & 2.0856 & -3.7260 \end{bmatrix}, \quad P_2 = 10^8 \begin{bmatrix} 2.9142 & 1.2187 & -3.6208 \\ 1.2187 & 2.4324 & -2.0820 \\ -3.6208 & -2.0820 & 5.3026 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -1.1787 & 2.7242 & -4.9167 \end{bmatrix}, \quad P_3 = 10^8 \begin{bmatrix} 2.6205 & 0.9629 & -3.4112 \\ 0.9629 & 1.5256 & -1.2244 \\ -3.4112 & -1.2244 & 4.8664 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} -1.8830 & -1.6333 & -1.6257 \end{bmatrix}, \quad P_4 = 10^9 \begin{bmatrix} 0.1687 & 0.2244 & -0.3969 \\ 0.2244 & 0.5736 & -0.6772 \\ -0.3969 & -0.6772 & 1.1323 \end{bmatrix},$$

$$K_5 = \begin{bmatrix} -3.1022 & 2.2056 & -7.8284 \end{bmatrix}, \quad P_5 = 10^8 \begin{bmatrix} 1.8769 & 1.7349 & -3.6534 \\ 1.7349 & 2.6935 & -3.6689 \\ -3.6534 & -3.6689 & 7.9961 \end{bmatrix},$$

$$K_6 = \begin{bmatrix} -2.6115 & 2.3310 & -7.6846 \end{bmatrix}, \qquad P_6 = 10^9 \begin{bmatrix} 0.1784 & 0.2142 & -0.3951 \\ 0.2142 & 0.7585 & -0.7262 \\ -0.3951 & -0.7262 & 1.1118 \end{bmatrix},$$

$$K_7 = \begin{bmatrix} -2.4008 & 8.9769 & -14.2961 \end{bmatrix}, \qquad P_7 = 10^9 \begin{bmatrix} 0.1510 & 0.2679 & -0.4306 \\ 0.2679 & 1.4666 & -1.3874 \\ -0.4306 & -1.3874 & 1.8013 \end{bmatrix}.$$

We adopted the robust Markovian H_∞ controller proposed by Gonçalves, Fioravanti and Geromel (2012) for comparison purposes, which produced the state feedback gains

$$K_{1,H_\infty} = \begin{bmatrix} -5.9045 & -8.3354 & 5.4019 \end{bmatrix}, \qquad K_{2,H_\infty} = \begin{bmatrix} -0.5789 & -11.4083 & -5.9273 \end{bmatrix},$$

$$K_{3,H_\infty} = \begin{bmatrix} -3.3750 & -8.9377 & -3.5108 \end{bmatrix}, \quad K_{4,H_\infty} = \begin{bmatrix} -20.4683 & -45.3066 & 30.4169 \end{bmatrix},$$

$$K_{5,H_\infty} = \begin{bmatrix} -8.5321 & -21.7993 & -3.3409 \end{bmatrix}, \ K_{6,H_\infty} = \begin{bmatrix} -48.3065 & -114.8764 & 53.3493 \end{bmatrix},$$

$$K_{7,H_\infty} = \begin{bmatrix} -88.6635 & -214.5633 & 168.4129 \end{bmatrix}.$$

Notably, the robust Markovian H_∞ controller resulted in 42 LMI constraints to be satisfied to compute the state feedback gains.

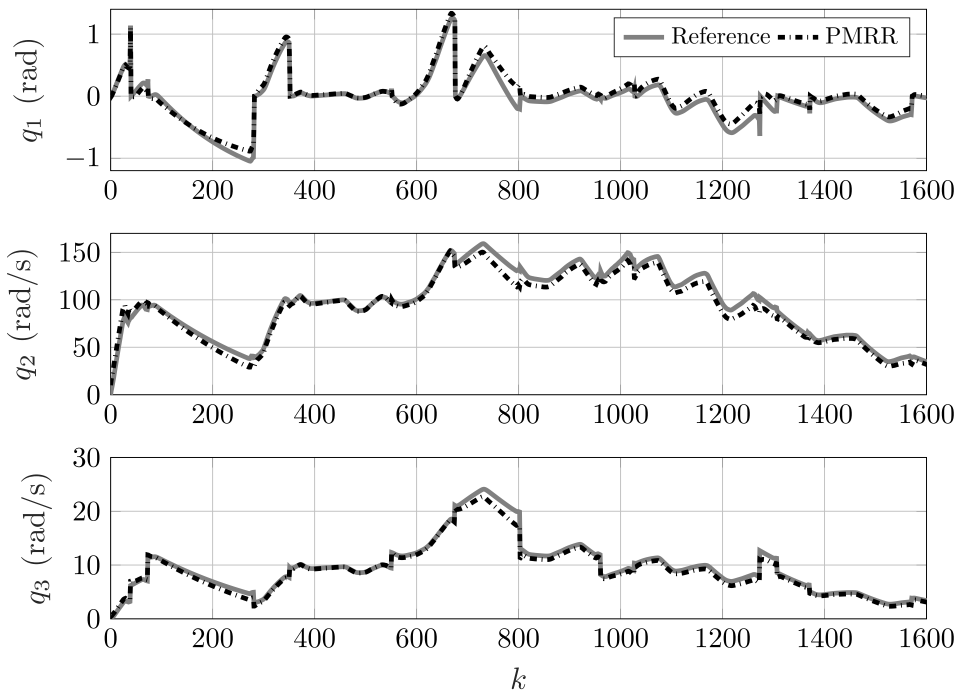
We carried out a total of 1000 Monte Carlo experiments, and during each experiment, coefficients $\alpha_{i,k}$ changed randomly a few times to emulate a more realistic scenario. Table 4 shows the averaged results regarding the norms of trajectory tracking errors and required throttle pedal position. Fig. 7 and Fig. 8 display the system states and the throttle pedal positions, respectively. Even though the LMI constraints were satisfied, the robust Markovian H_∞ strategy yielded gains that could not properly track the reference trajectories in this specific application, as seen in Fig. 9. In contrast, the proposed recursive regulator for DMJLS successfully tracked the reference trajectories with minor errors and feasible engine torques even when the system is subject to polytopic uncertainties.

Table 4 – Simulation results for Example 4.2.

Controller	$\|\bar{x}\|_{\mathcal{L}_2}$	$\sigma_{\bar{x}}$	$\|\bar{\tau}\|_{\mathcal{L}_2}$	$\sigma_{\bar{\tau}}$
PMRR	2.5992	0.0454	8.9117	0.1427
H_∞	180.8734	4.1442	42.9357	1.0703

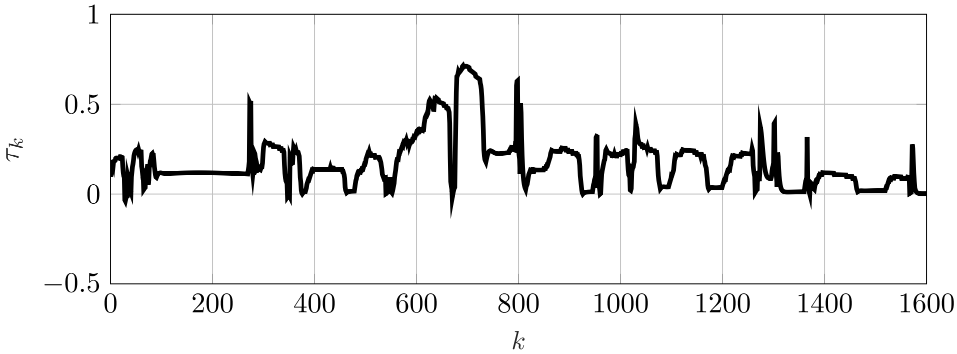
Source: author.

Figure 7 – Heavy duty vehicle states with the robust recursive regulator for DMJLS.



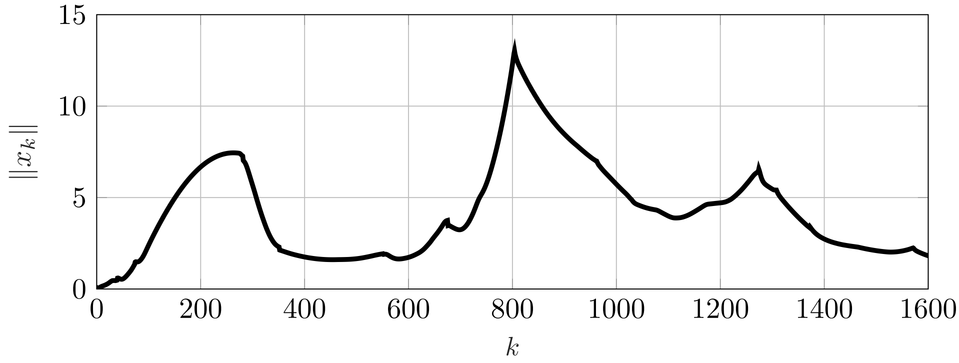
Source: author.

Figure 8 – Throttle pedal position with the robust recursive regulator for DMJLS.



Source: author.

Figure 9 – Norms of the state errors with the Markovian robust H_∞ controller.



Source: author.

5 ROBUST REGULATOR FOR DMJLS WITH POLYTOPIC UNCERTAIN TRANSITION PROBABILITIES

In this chapter, we focus on the class of DMJLS in which polytopic uncertainties affect not only the state space matrices but also the transition probabilities. Besides the aspects mentioned in Chapter 4, uncertain transition probabilities bring forth a higher level of complexity. A plethora of articles reported in the literature on robust control for DMJLS assume complete knowledge of these quantities; however, the transition probabilities are often estimated from experimental data and belong to an interval of uncertainties. Robust approaches are thus necessary as estimation errors might lead the systems to unstable regions or at least degrade performance (XIONG et al., 2005). There are various ways to specifically model uncertainties affecting transition probabilities. For instance, in Tzortzis, Charalambous and Hadjicostis (2021), the transition probabilities are limited by a ball. Zacchia Lun, D'Innocenzo and Di Benedetto (2019), Park and Kwon (2002), Costa, Fragoso and Todorov (2015), and Lun, Abate and D'Innocenzo (2019), to name a few, modeled the transition probabilities as quantities lying within a polytope. Li et al. (2020), and Sun, Zhang and Wu (2020) consider that some elements in the transition probabilities matrix are unknown. This approach was also found in the earlier notable works by Zhang and Boukas (2009), and Zhang and Lam (2010). It is worth mentioning that the case of unknown elements in the transition probabilities matrix can be equivalently handled by a convex combination of vertices (GONÇALVES; FIORAVANTI; GEROMEL, 2011).

That said, our main contribution in this chapter is a recursive solution for the regulation problem of DMJLS subject to polytopic uncertainties on state space matrices and transition probabilities. Our first step is to verify how the uncertain probabilities affect the expectations appearing in the cost function. We express these portions in a more suitable manner for our purposes. We formulate an optimization problem with a penalized cost function whose solution recursively returns the robust state-feedback gains, hence the name robust recursive regulator. Provided that certain positivity, stabilizability, and detectability conditions are satisfied, the associated closed-loop system is stable despite the presence of uncertainties. Finally, we validate our results in numerical and application examples, assessing the performance regarding regulation and computational burden.

5.1 Problem Formulation

Consider the following realization of a DMJLS:

$$x_{k+1} = (F_{\theta_k,k} + \delta F_{\theta_k,k})x_k + (G_{\theta_k,k} + \delta G_{\theta_k,k})u_k, \tag{125}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the input vector, $F_{\theta_k,k} \in \mathbb{R}^{n\times n}$ and $G_{\theta_k,k} \in \mathbb{R}^{n\times m}$ are system and input matrices, respectively, whereas $\theta_k \in \Theta = \{1, \ldots, s\}$ is the actual active Markov mode. $\delta F_{\theta_k,k}$ and $\delta G_{\theta_k,k}$ are convex polytopic uncertainties that depend on the time-varying coefficients $\alpha_k = \begin{bmatrix}\alpha_{1,k} & \dots & \alpha_{V_n,k}\end{bmatrix}^T \in \Lambda_{V_n}$, such that

$$\begin{aligned}\begin{bmatrix}\delta F_{\theta_k,k} & \delta G_{\theta_k,k}\end{bmatrix} &= \sum_{l=1}^{V_n} \alpha_{l,k} \begin{bmatrix}F_{\theta_k,k}^{(l)} & G_{\theta_k,k}^{(l)}\end{bmatrix}, \\ \Lambda_{V_n} &= \left\{ \alpha \in \mathbb{R}^{V_n} \;\middle|\; \sum_{l=1}^{V_n} \alpha_l = 1,\;\; 0 \leq \alpha_l \leq 1 \right\}.\end{aligned}$$

The transition probability matrix $\mathbb{P}_k \in \mathbb{R}^{s\times s}$ is assumed to be uncertain and is defined as

$$\begin{aligned}\mathbb{P}_k = \mathbb{P}_0 + \delta\mathbb{P}_k, \quad \text{Prob}(\theta_0) = \pi_i^{(0)} + \delta\pi_i, \\ \mathbb{P}_k = \begin{bmatrix}p_{ij}^{(0)} + \delta p_{ij,k}\end{bmatrix} = \text{Prob}(\theta_{k+1} = j \mid \theta_k = i), \\ \sum_{j=1}^{s}(p_{ij}^{(0)} + \delta p_{ij,k}) = 1, \quad 0 \leq p_{ij}^{(0)} + \delta p_{ij,k} \leq 1, \quad 0 \leq p_{ij}^{(0)} \leq 1,\end{aligned} \tag{126}$$

where the uncertainty $\delta\mathbb{P}_k$ is also polytopic and depends on the time-varying coefficients $\xi_k = \begin{bmatrix}\xi_{1,k} & \dots & \xi_{V_p,k}\end{bmatrix}^T \in \Lambda_{V_p}$, such that

$$\begin{aligned}\delta\mathbb{P}_k = \begin{bmatrix}\delta p_{ij,k}\end{bmatrix} = \left[\sum_{l=1}^{V_p} \xi_{l,k} p_{ij}^{(l)}\right], \\ \Lambda_{V_p} = \left\{ \xi \in \mathbb{R}^{V_p} \;\middle|\; \sum_{l=1}^{V_p} \xi_l = 1,\;\; 0 \leq \xi_l \leq 1 \right\}.\end{aligned} \tag{127}$$

Remark 5.1. *We assume $0 \leq p_{ij}^{(0)} \leq 1$. Also, from* (126)*, we have $\sum_{j=1}^{s}(p_{ij}^{(0)} + \delta p_{ij,k}) = 1$, and $0 \leq p_{ij}^{(0)} + \delta p_{ij,k} \leq 1$. Therefore, the portion $\delta p_{ij,k}$ is allowed to assume negative values to ensure that $\mathbb{P}_k$ is a transition probability matrix.*

Before investigating the robust regulation of the DMJLS (125), let us first define the s-sequences $\boldsymbol{Q}_k = (Q_{1,k}, \dots, Q_{s,k}) \in \mathbb{H}_+^n$, $\boldsymbol{R}_k = (R_{1,k}, \dots, R_{s,k}) \in \mathbb{H}_+^m$, $\boldsymbol{P}_k = (P_{1,k}, \dots, P_{s,k}) \in \mathbb{H}_+^n$, $i \in \Theta$, where $\boldsymbol{Q}_k$ and $\boldsymbol{R}_k$ are known. Then, set the following optimization problem:

$$\min_{x_{k+1},u_k} \max_{\delta F_{i,k},\delta G_{i,k},\delta\mathbb{P}_k} \Big\{\mathbb{E}\{\|x_N\|^2_{P_{\theta_N,N}} + \sum_{t=0}^{N-1}(\|x_t\|^2_{Q_{i,t}} + \|u_t\|^2_{R_{i,t}}) \mid \mathscr{I}_t\}\Big\}, \tag{128}$$

subject to

$$\begin{bmatrix}I_n \\ \vdots \\ I_n\end{bmatrix} x_{k+1} = \begin{bmatrix}F_{i,k} + V_n\delta F_{i,k}^{(1)} \\ \vdots \\ F_{i,k} + V_n\delta F_{i,k}^{(V_n)}\end{bmatrix} x_k + \begin{bmatrix}G_{i,k} + V_n\delta G_{i,k}^{(1)} \\ \vdots \\ G_{i,k} + V_n\delta G_{i,k}^{(V_n)}\end{bmatrix} u_k, \tag{129}$$

where $\delta F_{i,k}^{(l)} = \alpha_{l,k}F_{i,k}^{(l)}$, $\delta G_{i,k}^{(l)} = \alpha_{l,k}G_{i,k}^{(l)}$, $i \in \Theta$, $\theta_N \in \Theta$, and information $\mathscr{I}_t = \{\theta_t, x_t\}$. By applying Bellman's Principle of Optimality to (128), we yield

$$\min_{x_{k+1},u_k} \max_{\delta F_{i,k},\delta G_{i,k},\delta\mathbb{P}_k} \big\{J_k = \mathbb{E}\big\{\|x_{k+1}\|^2_{P_{i,k+1}} + \|x_k\|^2_{Q_{i,k}} + \|u_k\|^2_{R_{i,k}} \mid \mathscr{I}_k\big\}\big\},$$

subject to (129),

and taking the expectations in J_k we obtain:

$$\min_{x_{k+1},u_k} \max_{\delta F_{i,k},\delta G_{i,k},\delta \mathbb{P}_k} \left\{ J_k = \|x_{k+1}\|^2_{\Psi_{i,k+1}} + \|x_k\|^2_{Q_{i,k}} + \|u_k\|^2_{R_{i,k}} \right\},$$
$$\text{subject to (129)}, \tag{130}$$

where $\Psi_{i,k+1} = \mathbb{E}\{P_{i,k} \mid \mathscr{I}_k\}$. Observe that the uncertainties $\delta\mathbb{P}_k$ will reflect upon the expectation $\Psi_{i,k+1}$, given by

$$\Psi_{i,k+1} = \sum_{j=1}^{s} (p^{(0)}_{ij,k} + \delta p_{ij,k}) P_{j,k+1},$$
$$\Psi_{i,k+1} = \Psi^{(0)}_{i,k+1} + \delta\Psi_{i,k+1},$$

with

$$\Psi^{(0)}_{i,k+1} = \sum_{j=1}^{s} p^{(0)}_{ij} P_{j,k+1}, \quad \text{and} \quad \delta\Psi_{i,k+1} = \sum_{j=1}^{s} \delta p_{ij,k} P_{j,k+1}. \tag{131}$$

Remark 5.2. *The constraints* (129) *actually represent the DMJLS* (125)*. Pre-multiplying both sides of* (129) *by* $\begin{bmatrix} I_n & \cdots & I_n \end{bmatrix}$ *yields*

$$V_n x_{k+1} = V_n \left(F_{i,k} + \sum_{l=1}^{V_n} \alpha_{l,k} F^{(l)}_{i,k} \right) x_k + V_n \left(G_{i,k} + \sum_{l=1}^{V_n} \alpha_{l,k} G^{(l)}_{i,k} \right) u_k,$$

which corresponds to (125)*.*

In (131), observe that

$$\delta\Psi_{i,k+1} = \sum_{j=1}^{s} \left(\sum_{l=1}^{V_p} \xi_{l,k} p^{(l)}_{ij} \right) P_{j,k+1},$$
$$\delta\Psi_{i,k+1} = \sum_{j=1}^{s} (\xi_{1,k} p^{(1)}_{ij} + \ldots + \xi_{V_p,k} p^{(V_p)}_{ij}) P_{j,k+1},$$
$$\delta\Psi_{i,k+1} = (\xi_{1,k} p^{(1)}_{i1} + \ldots + \xi_{V_p,k} p^{(V_p)}_{i1}) P_{1,k+1} + \ldots + (\xi_{1,k} p^{(1)}_{is} + \ldots + \xi_{V_p,k} p^{(V_p)}_{is}) P_{s,k+1},$$
$$\delta\Psi_{i,k+1} = \xi_{1,k} p^{(1)}_{i1} P_{1,k+1} + \ldots + \xi_{V_p,k} p^{(V_p)}_{i1} P_{1,k+1} + \ldots + \xi_{1,k} p^{(1)}_{is} P_{s,k+1} + \ldots + \xi_{V_p,k} p^{(V_p)}_{is} P_{s,k+1},$$
$$\delta\Psi_{i,k+1} = \sqrt{\xi_{1,k} p^{(1)}_{i1}} P_{1,k+1} \sqrt{\xi_{1,k} p^{(1)}_{i1}} + \ldots + \sqrt{\xi_{V_p,k} p^{(V_p)}_{i1}} P_{1,k+1} \sqrt{\xi_{V_p,k} p^{(V_p)}_{i1}} + \ldots + \sqrt{\xi_{1,k} p^{(1)}_{is}} P_{s,k+1} \sqrt{\xi_{1,k} p^{(1)}_{is}} + \ldots + \sqrt{\xi_{V_p,k} p^{(V_p)}_{is}} P_{s,k+1} \sqrt{\xi_{V_p,k} p^{(V_p)}_{is}}.$$

Let us group the above equation in terms of vertices $l = 1, \ldots, V_p$, such that

$$\delta\Psi_{i,k+1} = \sqrt{\xi_{1,k} p^{(1)}_{i1}} P_{1,k+1} \sqrt{\xi_{1,k} p^{(1)}_{i1}} + \ldots + \sqrt{\xi_{1,k} p^{(1)}_{is}} P_{s,k+1} \sqrt{\xi_{1,k} p^{(1)}_{is}} + \ldots + \sqrt{\xi_{V_p,k} p^{(V_p)}_{i1}} P_{1,k+1} \sqrt{\xi_{V_p,k} p^{(V_p)}_{i1}} + \ldots + \sqrt{\xi_{V_p,k} p^{(V_p)}_{is}} P_{s,k+1} \sqrt{\xi_{V_p,k} p^{(V_p)}_{is}},$$
$$\delta\Psi_{i,k+1} = \delta\boldsymbol{p}^T_{i,k} \boldsymbol{\mathscr{P}}_{k+1} \delta\boldsymbol{p}_{i,k}, \tag{132}$$

where $\mathscr{P}_{k+1} = I_{V_p} \otimes \textbf{diag}(P_{1,k+1}, \ldots, P_{s,k+1})$, and

$$\delta \boldsymbol{p}_{i,k} = \begin{bmatrix} \xi_{1,k} I_{sn} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \xi_{V_p,k} I_{sn} \end{bmatrix}^{\circ 1/2} \begin{bmatrix} p_i^{(1)} \\ \vdots \\ p_i^{(V_p)} \end{bmatrix}^{\circ 1/2},$$

with vertices $p_i^{(l)} = \begin{bmatrix} p_{i1}^{(l)} I_n & \ldots & p_{is}^{(l)} I_n \end{bmatrix}^T$, for $l = 1, \ldots, V_p$. Let us now define some auxiliary matrices which we will use throughout this chapter:

$$\boldsymbol{F}_{i,k} = \mathbb{1}_{V_n} \otimes F_{i,k}, \quad \boldsymbol{G}_{i,k} = \mathbb{1}_{V_n} \otimes G_{i,k}, \quad \boldsymbol{I}_n = \mathbb{1}_{V_n} \otimes I_n,$$

$$\delta \boldsymbol{F}_{i,k} = \begin{bmatrix} \delta F_{i,k}^{(1)} \\ \vdots \\ \delta F_{i,k}^{(V_n)} \end{bmatrix}, \quad \delta \boldsymbol{G}_{i,k} = \begin{bmatrix} \delta G_{i,k}^{(1)} \\ \vdots \\ \delta G_{i,k}^{(V_n)} \end{bmatrix},$$

$$\boldsymbol{E}_{\boldsymbol{F}_{i,k}} = V_n \begin{bmatrix} F_{i,k}^{(1)} \\ \vdots \\ F_{i,k}^{(V_n)} \end{bmatrix}, \; \boldsymbol{E}_{\boldsymbol{G}_{i,k}} = V_n \begin{bmatrix} G_{i,k}^{(1)} \\ \vdots \\ G_{i,k}^{(V_n)} \end{bmatrix}, \; \boldsymbol{E}_{\mathbb{P}_i} = \begin{bmatrix} p_i^{(1)} \\ \vdots \\ p_i^{(V_p)} \end{bmatrix}^{\circ 1/2}.$$

That said, we are able to map the problem (130) into an unconstrained problem. We show this procedure in the following lemma.

Lemma 5.1. *For a given fixed penalty parameter* $\mu > 0$*, the constrained optimization problem* (130) *is equivalent to the unconstrained problem*

$$\underset{x_{k+1}, u_k}{\textit{min}} \; \underset{\boldsymbol{\delta}_{i,k}}{\textit{max}} \quad \mathcal{J}_k^{\mu}(x_{k+1}, u_k, \boldsymbol{\delta}_k), \tag{133}$$

for $k = N-1, \ldots, 0$*, where* $\boldsymbol{\delta}_{i,k} := \{\delta F_{i,k}, \delta G_{i,k}, \delta \mathbb{P}_k\}$, $i \in \Theta := \{1, \ldots, s\}$*, and cost function* $\mathcal{J}_k^{\mu}(.)$ *given by*

$$\begin{aligned} \mathcal{J}_k^{\mu}(x_{k+1}, u_k, \boldsymbol{\delta}_k) = & \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}^T \begin{bmatrix} \Psi_{i,k+1}^{(0)} & 0 \\ 0 & R_{i,k} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} + \left\{ \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \boldsymbol{I}_n & -\boldsymbol{G}_{i,k} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \delta \boldsymbol{p}_i & 0 \\ 0 & -V_n \delta \boldsymbol{G}_{i,k} \end{bmatrix} \right) \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} \right. \\ & \left. - \left(\begin{bmatrix} -I_n \\ 0 \\ \boldsymbol{F}_{i,k} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ V_n \delta \boldsymbol{F}_{i,k} \end{bmatrix} \right) x_k \right\}^T \begin{bmatrix} Q_{i,k} & 0 & 0 \\ 0 & \mathscr{P}_{k+1} & 0 \\ 0 & 0 & \mu I_{nV_n} \end{bmatrix} \left\{ \bullet \right\}. \end{aligned} \tag{134}$$

Proof. Let us reformulate the constraints in (129) as $\bar{f}(x_{k+1}, u_k, \boldsymbol{\delta}_{i,k}) = 0$, such that

$$\bar{f}(x_{k+1}, u_k, \boldsymbol{\delta}_{i,k}) = \boldsymbol{I}_n x_{k+1} - (\boldsymbol{F}_{i,k} + V_n \delta \boldsymbol{F}_{i,k}) x_k - (\boldsymbol{G}_{i,k} + V_n \delta \boldsymbol{G}_{i,k}) u_k,$$

which is equivalent to

$$\bar{f}(x_{k+1}, u_k, \boldsymbol{\delta}_{i,k}) = \left(\begin{bmatrix} \boldsymbol{I}_n & 0 \end{bmatrix} + \begin{bmatrix} 0 & -V_n \delta \boldsymbol{G}_{i,k} \end{bmatrix} \right) \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} - (\boldsymbol{F}_{i,k} + V_n \delta \boldsymbol{F}_{i,k}) x_k.$$

Let us introduce a positive penalty parameter $\mu \in \mathbb{R}$, which we shall keep fixed throughout the algorithm, and use it to design the penalty function $p_f(x_{k+1}, u_k, \boldsymbol{\delta}_{i,k}) \in \mathbb{R}$ as

$$p_f(x_{k+1}, u_k, \boldsymbol{\delta}_{i,k}) = \|\bar{f}(x_{k+1}, u_k, \boldsymbol{\delta}_{i,k})^T\|^2_{\mu I_n}.$$

Note that μ penalizes any violation of (129). Therefore, we formulate the unconstrained problem by defining the penalized cost

$$\mathcal{J}^{\mu}_k(x_{k+1}, u_k, \boldsymbol{\delta}_k) = \|x_{k+1}\|^2_{\Psi_{i,k+1}} + \|x_k\|^2_{Q_{i,k}} + \|u_k\|^2_{R_{i,k}} + p_f(x_{k+1}, u_k, \boldsymbol{\delta}_{i,k}),$$

$$\begin{aligned}\mathcal{J}^{\mu}_k(x_{k+1}, u_k, \boldsymbol{\delta}_k) = &\begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}^T \begin{bmatrix} \Psi^{(0)}_{i,k+1} & 0 \\ 0 & R_{i,k} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} + x^T_{k+1}\delta\Psi_{i,k+1}x_{k+1} + x^T_k Q_{i,k} x_k \\ &+ \left(\left(\begin{bmatrix} \boldsymbol{I}_n & 0 \end{bmatrix} + \begin{bmatrix} 0 & -V_n\delta\boldsymbol{G}_{i,k} \end{bmatrix} \right) \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} - (\boldsymbol{F}_{i,k} + V_n\delta\boldsymbol{F}_{i,k})x_k \right)^T \mu I_n(\bullet),\end{aligned}$$

$$\begin{aligned}\mathcal{J}^{\mu}_k(x_{k+1}, u_k, \boldsymbol{\delta}_k) = &\begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}^T \begin{bmatrix} \Psi^{(0)}_{i,k+1} & 0 \\ 0 & R_{i,k} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} + x^T_{k+1}(\delta\boldsymbol{p}^T_{i,k}\boldsymbol{\mathscr{P}}_{k+1}\delta\boldsymbol{p}_{i,k})x_{k+1} + x^T_k Q_{i,k} x_k \\ &+ \left(\left(\begin{bmatrix} \boldsymbol{I}_n & 0 \end{bmatrix} + \begin{bmatrix} 0 & -V_n\delta\boldsymbol{G}_{i,k} \end{bmatrix} \right) \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} - (\boldsymbol{F}_{i,k} + V_n\delta\boldsymbol{F}_{i,k})x_k \right)^T \mu I_n(\bullet),\end{aligned}$$

$$\begin{aligned}\mathcal{J}^{\mu}_k(x_{k+1}, u_k, \boldsymbol{\delta}_k) = &\begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}^T \begin{bmatrix} \Psi^{(0)}_{i,k+1} & 0 \\ 0 & R_{i,k} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} + \left\{ \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \boldsymbol{I}_n & -\boldsymbol{G}_{i,k} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \delta\boldsymbol{p}_i & 0 \\ 0 & -V_n\delta\boldsymbol{G}_{i,k} \end{bmatrix} \right) \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} \right. \\ &\left. - \left(\begin{bmatrix} -I_n \\ 0 \\ \boldsymbol{F}_{i,k} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ V_n\delta\boldsymbol{F}_{i,k} \end{bmatrix} \right) x_k \right\}^T \begin{bmatrix} Q_{i,k} & 0 & 0 \\ 0 & \boldsymbol{\mathscr{P}}_{k+1} & 0 \\ 0 & 0 & \mu I_{nV_n} \end{bmatrix} \left\{ \bullet \right\},\end{aligned}$$

which is identical to (134). □

At this point, it is convenient to restate the main goal of this chapter. Given the optimization problem (133) with $k = N-1, \ldots, 0$, we search for a recursive solution $\{\hat{x}_{k+1}, \hat{u}_k\}$, such that $\hat{u}_k = K_{i,k}x_k$. Moreover, the state-feedback gains $K_{i,k} \in \boldsymbol{K}_k$, where $\boldsymbol{K}_k = (K_{1,k}, \ldots, K_{s,k}) \in \mathbb{H}^{m,n}$, must stabilize the closed-loop DMJLS (125) in the mean-square sense regardless of uncertainties $\boldsymbol{\delta}_{i,k}$. We outline the procedure to yield the recursive solution in the next section.

5.2 RLQR for Polytopic DMJLS with Uncertain Transition Probabilities

Based upon Lemma 5.1 and Lemma 2.4, let us make the following identifications:

$$J \leftarrow \mathcal{J}^{\mu}_k, \quad x \leftarrow \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}, \quad A_0 \leftarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \boldsymbol{I}_n & -\boldsymbol{G}_{i,k} \end{bmatrix}, \quad b_0 \leftarrow \begin{bmatrix} -I_n \\ 0 \\ \boldsymbol{F}_{i,k} \end{bmatrix} x_k,$$

$$\delta A \leftarrow \begin{bmatrix} 0 & 0 \\ \delta \boldsymbol{p}_{i,k} & 0 \\ 0 & -V_n \delta \boldsymbol{G}_{i,k} \end{bmatrix}, \;\; \delta b \leftarrow \begin{bmatrix} 0 \\ 0 \\ V_n \delta \boldsymbol{F}_{i,k} \end{bmatrix} x_k, \;\; M \leftarrow \begin{bmatrix} 0 & 0 \\ I_{snV_p} & 0 \\ 0 & I_{nV_n} \end{bmatrix},$$

$$\hat{A} \leftarrow \begin{bmatrix} \boldsymbol{E}_{\mathbb{P}_i} & 0 \\ 0 & -\boldsymbol{E}_{\boldsymbol{G}_{i,k}} \end{bmatrix}, \quad \hat{b} \leftarrow \begin{bmatrix} 0 \\ \boldsymbol{E}_{\boldsymbol{F}_{i,k}} \end{bmatrix} x_k, \quad Q \leftarrow \begin{bmatrix} \Psi^{(0)}_{i,k+1} & 0 \\ 0 & R_{i,k} \end{bmatrix},$$

$$W \leftarrow \begin{bmatrix} Q_{i,k} & 0 & 0 \\ 0 & \mathscr{P}_{k+1} & 0 \\ 0 & 0 & \mu I_{nV_n} \end{bmatrix}, \quad \Gamma \leftarrow \begin{bmatrix} \bar{\xi}_k \otimes I_{sn} & 0 \\ 0 & \bar{\alpha}_k \otimes I_n \end{bmatrix}, \tag{135}$$

where $\bar{\alpha}_k = \textbf{diag}(\alpha_k)$, and $\bar{\xi}_k = \textbf{diag}(\xi_k)$.

We also have $W(\lambda) = (W^{-1} - \lambda^{-1} H H^T)^{-1}$ by the Sherman-Morrison-Woodbury inversion formula (see Lemma A.6 in Appendix A), where $\lambda = \beta \| H^T W H \|$ for some scalar $\beta > 1$, as discussed in Section 2.1.3. Therefore, we attain

$$W(\lambda) = \begin{bmatrix} Q^{-1}_{i,k} & 0 & 0 \\ 0 & \Pi_1 & 0 \\ 0 & 0 & \Pi_2 \end{bmatrix}, \quad \lambda = \beta \left\| \begin{bmatrix} \mathscr{P}_{k+1} & 0 \\ 0 & \mu I_{nV_n} \end{bmatrix} \right\|,$$

$$\Pi_1 = \mathscr{P}^{-1}_{k+1} - \lambda^{-1} I_{snV_n}, \quad \Pi_2 = (\mu^{-1} - \lambda^{-1}) I_{nV_n}. \tag{136}$$

We are now in a position to present the main result of this chapter, which is a recursive solution for the problem (133).

Theorem 5.1. *Consider known weights $\boldsymbol{Q}_k \in \mathbb{H}^n_+$, $\boldsymbol{R}_k \in \mathbb{H}^m_+$, $\boldsymbol{P}_N \in \mathbb{H}^n_+$, fixed $\mu > 0$, and $i \in \Theta$. The solution for the optimization problem* (133) *is given by*

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{u}_k \\ \mathcal{J}_k(\hat{x}_{k+1}, \hat{u}_k) \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & x^T_k \end{bmatrix} \begin{bmatrix} L_{i,k} \\ K_{i,k} \\ P_{i,k} \end{bmatrix} x_k, \quad k = 0, \ldots, N-1, \tag{137}$$

where

$$L_{i,k} = \big(I_n - \kappa(\widetilde{\Omega}_{i,k+1} + V_n \bar{G}_{i,k} \bar{G}^T_{i,k})^{-1}\big) \widetilde{\mathscr{F}}_{i,k} \\ - \bar{G}_{i,k}\big(I_m + \bar{G}^T_{i,k} \widetilde{\Psi}_{i,k+1} \bar{G}_{i,k}\big)^{-1} \bar{G}^T_{i,k} \widetilde{\Psi}_{i,k+1} \widetilde{\mathscr{F}}_{i,k}, \tag{138}$$

$$K_{i,k} = -V_n \bar{\mathscr{R}}_{i,k} G^T_{i,k} \big(\widetilde{\Omega}_{i,k+1} + V_n \bar{G}_{i,k} \bar{G}^T_{i,k}\big)^{-1} \widetilde{\mathscr{F}}_{i,k} \\ - R^{-1}_{i,k} \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}} \big(\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R^{-1}_{i,k} \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}}\big)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}}, \tag{139}$$

$$P_{i,k} = \bar{Q}_{i,k} + \widetilde{\mathscr{F}}^T_{i,k} \widetilde{\Psi}_{i,k+1} \widetilde{\mathscr{F}}_{i,k} - \widetilde{\mathscr{F}}^T_{i,k} \widetilde{\Psi}_{i,k+1} \bar{G}_{i,k} \big(I_m + \bar{G}^T_{i,k} \widetilde{\Psi}_{i,k+1} \bar{G}_{i,k}\big)^{-1} \bar{G}^T_{i,k} \widetilde{\Psi}_{i,k+1} \widetilde{\mathscr{F}}_{i,k}, \tag{140}$$

for $k = N-1, \ldots, 0$, with

$$\widetilde{\Psi}_{i,k+1} = V_n \widetilde{\Omega}_{i,k+1}^{-1}, \quad \widetilde{\Omega}_{i,k+1} = \kappa I_n + V_n \big(\Psi_{i,k+1}^{(0)}\big)^{-1} \hat{E}_{p_i,k}, \quad \kappa = (\mu^{-1} - \lambda^{-1}),$$

$$\hat{E}_{p_i,k} = I_n - \boldsymbol{E}_{\mathbb{P}_i}^T \big(\lambda^{-1} I_{sV_p n} + \boldsymbol{E}_{\mathbb{P}_i} (\Psi_{i,k+1}^{(0)})^{-1} \boldsymbol{E}_{\mathbb{P}_i}^T\big)^{-1} \boldsymbol{E}_{\mathbb{P}_i} (\Psi_{i,k+1}^{(0)})^{-1},$$

$$\widetilde{\mathscr{F}}_{i,k} = F_{i,k} - G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T \big(\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\big)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}},$$

$$\bar{\mathscr{R}}_{i,k} = R_{i,k}^{-1} \big(I_m - \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1}\big),$$

$$\bar{Q}_{i,k} = Q_{i,k} + \boldsymbol{E}_{\boldsymbol{F}_{i,k}}^T \big(\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\big)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}},$$

$$\bar{G}_{i,k} = G_{i,k} \bar{\mathscr{R}}_{i,k}^{1/2}, \quad \lambda = \beta \left\| \begin{bmatrix} \boldsymbol{\mathscr{P}}_{k+1} & 0 \\ 0 & \mu I_{nV_n} \end{bmatrix} \right\|, \quad \beta > 1.$$

Proof. Notice that the unconstrained problem (133) is a special case of the regularized least-squares with uncertain data. With the mappings given in (135), along with $W(\lambda)$ and λ as shown in (136), we design the solution (137) based upon Lemma 2.4, with $\{L_{i,k}, K_{i,k}, P_{i,k}\}$ initially obtained through the recursion

$$\begin{bmatrix} L_{i,k} \\ K_{i,k} \\ P_{i,k} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{F}_{i,k} \\ 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{E}_{\boldsymbol{F}_{i,k}} \\ I_n & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix}^T \underbrace{\begin{bmatrix} (\Psi_{i,k+1}^{(0)})^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & R_{i,k}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & Q_{i,k}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Pi_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Pi_2 & 0 & 0 & \boldsymbol{I}_n & -\boldsymbol{G}_{i,k} \\ 0 & 0 & 0 & 0 & 0 & \lambda^{-1} I_{snV_p} & 0 & \boldsymbol{E}_{\mathbb{P}_i} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^{-1} I_{nV_n} & 0 & -\boldsymbol{E}_{\boldsymbol{G}_{i,k}} \\ I_n & 0 & 0 & 0 & \boldsymbol{I}_n^T & \boldsymbol{E}_{\mathbb{P}_i}^T & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 & -\boldsymbol{G}_{i,k}^T & 0 & -\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T & 0 & 0 \end{bmatrix}}_{\mathcal{M}}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I_n \\ 0 \\ \boldsymbol{F}_{i,k} \\ 0 \\ \boldsymbol{E}_{\boldsymbol{F}_{i,k}} \\ 0 \\ 0 \end{bmatrix}, \tag{141}$$

for $k = N-1, \ldots, 0$, with $i \in \Theta$, and $\{\Pi_1, \Pi_2, \lambda\}$ as in (136). Notice that Lemma A.3 ensures $\mathcal{M} > 0$, thus $\mathcal{M}^{-1}$ exists for any $\mu > 0$. Now, from (141) we see that

$$\begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \\ \Upsilon_3 \\ \Upsilon_4 \\ \Upsilon_5 \\ \Upsilon_6 \\ \Upsilon_7 \\ L_{i,k} \\ K_{i,k} \end{bmatrix} = \mathcal{M}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I_n \\ 0 \\ \boldsymbol{F}_{i,k} \\ 0 \\ \boldsymbol{E}_{\boldsymbol{F}_{i,k}} \\ 0 \\ 0 \end{bmatrix} \implies \mathcal{M} \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \\ \Upsilon_3 \\ \Upsilon_4 \\ \Upsilon_5 \\ \Upsilon_6 \\ \Upsilon_7 \\ L_{i,k} \\ K_{i,k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -I_n \\ 0 \\ \boldsymbol{F}_{i,k} \\ 0 \\ \boldsymbol{E}_{\boldsymbol{F}_{i,k}} \\ 0 \\ 0 \end{bmatrix}, \tag{142}$$

where $\Upsilon_5 := \begin{bmatrix} \upsilon_5^T & \ldots & \upsilon_5^T \end{bmatrix}^T$, $\upsilon_5 \in \mathbb{R}^{n \times n}$. Therefore, the system of simultaneous equations given by (142), namely

$$
\begin{cases}
(\Psi_{i,k+1}^{(0)})^{-1}\Upsilon_1 + L_{i,k} = 0, & (143)\\
R_{i,k}^{-1}\Upsilon_2 + K_{i,k} = 0, & (144)\\
Q_{i,k}^{-1}\Upsilon_3 = -I_n, & (145)\\
\Pi_1\Upsilon_4 = 0, & (146)\\
\Pi_2\Upsilon_5 + \boldsymbol{I}_n L_{i,k} - \boldsymbol{G}_{i,k}K_{i,k} = \boldsymbol{F}_{i,k}, & (147)\\
\lambda^{-1}I_{snV_p}\Upsilon_6 + \boldsymbol{E}_{\mathbb{P}_i}L_{i,k} = 0, & (148)\\
\lambda^{-1}I_{nV_n}\Upsilon_7 - \boldsymbol{E}_{\boldsymbol{G}_{i,k}}K_{i,k} = \boldsymbol{E}_{\boldsymbol{F}_{i,k}}, & (149)\\
\Upsilon_1 + \boldsymbol{I}_n^T\Upsilon_5 + \boldsymbol{E}_{\mathbb{P}_i}^T\Upsilon_6 = 0, & (150)\\
\Upsilon_2 - \boldsymbol{G}_{i,k}^T\Upsilon_5 - \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\Upsilon_7 = 0, & (151)
\end{cases}
$$

has $\{L_{i,k}, K_{i,k}\}$ as elements of its unique solution. Also, combine (142) and (141) to yield

$$
P_{i,k} = \begin{bmatrix} 0 & 0 & -I_n & 0 & \boldsymbol{F}_{i,k}^T & 0 & \boldsymbol{E}_{\boldsymbol{F}_{i,k}}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \\ \Upsilon_3 \\ \Upsilon_4 \\ \Upsilon_5 \\ \Upsilon_6 \\ \Upsilon_7 \\ L_{i,k} \\ K_{i,k} \end{bmatrix},
$$

$$
\begin{aligned}
P_{i,k} &= -\Upsilon_3 + \boldsymbol{F}_{i,k}^T\Upsilon_5 + \boldsymbol{E}_{\boldsymbol{F}_{i,k}}^T\Upsilon_7,\\
P_{i,k} &= -\Upsilon_3 + V_n F_{i,k}^T\upsilon_5 + \boldsymbol{E}_{\boldsymbol{F}_{i,k}}^T\Upsilon_7. \qquad (152)
\end{aligned}
$$

Let us proceed to solve (143)–(152) to finally obtain matrices $L_{i,k}$, $K_{i,k}$, and $P_{i,k}$, $i \in \Theta$. First, from (143), (144), and (145), respectively, we have

$$\Upsilon_1 = -\Psi_{i,k+1}^{(0)}L_{i,k}, \qquad (153)$$

$$\Upsilon_2 = -R_{i,k}K_{i,k}, \qquad (154)$$

$$\Upsilon_3 = -Q_{i,k}. \qquad (155)$$

Then, from (147),

$$
\begin{aligned}
\Pi_2\Upsilon_5 + \boldsymbol{I}_n L_{i,k} - \boldsymbol{G}_{i,k}K_{i,k} &= \boldsymbol{F}_{i,k},\\
\Pi_2\upsilon_5 + L_{i,k} - G_{i,k}K_{i,k} &= F_{i,k},\\
L_{i,k} &= F_{i,k} + G_{i,k}K_{i,k} - \kappa\upsilon_5, \qquad (156)
\end{aligned}
$$

where $\kappa = \mu^{-1} - \lambda^{-1}$, hence $\kappa \in \mathbb{R}$, and substitute (153) and (156) into (150) to get

$$\begin{aligned}
-\Psi^{(0)}_{i,k+1} L_{i,k} + \boldsymbol{I}_n^T \Upsilon_5 + \boldsymbol{E}^T_{\mathbb{P}_i} \Upsilon_6 &= 0, \\
-\Psi^{(0)}_{i,k+1} L_{i,k} + V_n \upsilon_5 + \boldsymbol{E}^T_{\mathbb{P}_i} \Upsilon_6 &= 0, \\
-\Psi^{(0)}_{i,k+1} (F_{i,k} + G_{i,k} K_{i,k} - \kappa \upsilon_5) + V_n \upsilon_5 + \boldsymbol{E}^T_{\mathbb{P}_i} \Upsilon_6 &= 0, \\
-\Psi^{(0)}_{i,k+1} F_{i,k} - \Psi^{(0)}_{i,k+1} G_{i,k} K_{i,k} + \big(\kappa \Psi^{(0)}_{i,k+1} + V_n I_n\big) \upsilon_5 + \boldsymbol{E}^T_{\mathbb{P}_i} \Upsilon_6 &= 0.
\end{aligned} \tag{157}$$

From (154) and (151) we get

$$\begin{aligned}
-R_{i,k} K_{i,k} - \boldsymbol{G}^T_{i,k} \Upsilon_5 - \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}} \Upsilon_7 &= 0, \\
-R_{i,k} K_{i,k} - V_n G^T_{i,k} \upsilon_5 - \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}} \Upsilon_7 &= 0, \\
K_{i,k} &= -R^{-1}_{i,k} (V_n G^T_{i,k} \upsilon_5 + \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}} \Upsilon_7).
\end{aligned} \tag{158}$$

Place (158) into (149) to obtain

$$\begin{aligned}
\lambda^{-1} I_{nV_n} \Upsilon_7 + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R^{-1}_{i,k} (V_n G^T_{i,k} \upsilon_5 + \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}} \Upsilon_7) &= \boldsymbol{E}_{\boldsymbol{F}_{i,k}}, \\
V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R^{-1}_{i,k} G^T_{i,k} \upsilon_5 + (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R^{-1}_{i,k} \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}}) \Upsilon_7 &= \boldsymbol{E}_{\boldsymbol{F}_{i,k}}.
\end{aligned} \tag{159}$$

Substitute (158) into (157), such that

$$\begin{aligned}
-\Psi^{(0)}_{i,k+1} F_{i,k} - \Psi^{(0)}_{i,k+1} G_{i,k} K_{i,k} + \big(\kappa \Psi^{(0)}_{i,k+1} + V_n I_n\big) \upsilon_5 + \boldsymbol{E}^T_{\mathbb{P}_i} \Upsilon_6 &= 0, \\
-\Psi^{(0)}_{i,k+1} F_{i,k} + \Psi^{(0)}_{i,k+1} G_{i,k} R^{-1}_{i,k} (V_n G^T_{i,k} \upsilon_5 + \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}} \Upsilon_7) + \big(\kappa \Psi^{(0)}_{i,k+1} + V_n I_n\big) \upsilon_5 + \boldsymbol{E}^T_{\mathbb{P}_i} \Upsilon_6 &= 0,
\end{aligned}$$

and multiply both of its sides to the left by $(\Psi^{(0)}_{i,k+1})^{-1}$ to yield

$$\begin{aligned}
&-F_{i,k} + G_{i,k} R^{-1}_{i,k} (V_n G^T_{i,k} \upsilon_5 + \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}} \Upsilon_7) + \big(\kappa I_n + V_n (\Psi^{(0)}_{i,k+1})^{-1}\big) \upsilon_5 + (\Psi^{(0)}_{i,k+1})^{-1} \boldsymbol{E}^T_{\mathbb{P}_i} \Upsilon_6 = 0, \\
&- F_{i,k} + (V_n G_{i,k} R^{-1}_{i,k} G^T_{i,k} + \kappa I_n + V_n (\Psi^{(0)}_{i,k+1})^{-1}) \upsilon_5 + (\Psi^{(0)}_{i,k+1})^{-1} \boldsymbol{E}^T_{\mathbb{P}_i} \Upsilon_6 \\
&\qquad\qquad + V_n G_{i,k} R^{-1}_{i,k} \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}} \Upsilon_7 = 0.
\end{aligned}$$

Define $\Omega_{k+1} = \kappa I_n + V_n (\Psi^{(0)}_{i,k+1})^{-1}$, and the above equation becomes

$$(V_n G_{i,k} R^{-1}_{i,k} G^T_{i,k} + \Omega_{i,k+1}) \upsilon_5 + (\Psi^{(0)}_{i,k+1})^{-1} \boldsymbol{E}^T_{\mathbb{P}_i} \Upsilon_6 + G_{i,k} R^{-1}_{i,k} \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}} \Upsilon_7 = F_{i,k}. \tag{160}$$

Now, place (156) and (158) into (148) to produce

$$\begin{aligned}
\lambda^{-1} I_{snV_p} \Upsilon_6 + \boldsymbol{E}_{\mathbb{P}_i} (F_{i,k} + G_{i,k} K_{i,k} - \kappa \upsilon_5) &= 0, \\
\lambda^{-1} I_{snV_p} \Upsilon_6 + \boldsymbol{E}_{\mathbb{P}_i} F_{i,k} + \boldsymbol{E}_{\mathbb{P}_i} G_{i,k} K_{i,k} - \boldsymbol{E}_{\mathbb{P}_i} \kappa \upsilon_5 &= 0, \\
\lambda^{-1} I_{snV_p} \Upsilon_6 + \boldsymbol{E}_{\mathbb{P}_i} F_{i,k} - \boldsymbol{E}_{\mathbb{P}_i} G_{i,k} R^{-1}_{i,k} (V_n G^T_{i,k} \upsilon_5 + \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}} \Upsilon_7) - \boldsymbol{E}_{\mathbb{P}_i} \kappa \upsilon_5 &= 0, \\
-(V_n \boldsymbol{E}_{\mathbb{P}_i} G_{i,k} R^{-1}_{i,k} G^T_{i,k} + \kappa \boldsymbol{E}_{\mathbb{P}_i}) \upsilon_5 + \lambda^{-1} \Upsilon_6 - \boldsymbol{E}_{\mathbb{P}_i} G_{i,k} R^{-1}_{i,k} \boldsymbol{E}^T_{\boldsymbol{G}_{i,k}} \Upsilon_7 &= -\boldsymbol{E}_{\mathbb{P}_i} F_{i,k}.
\end{aligned} \tag{161}$$

Multiply both sides of (160) to the left by $\boldsymbol{E}_{\mathbb{P}_i}$ and add into (161) to yield

$$\begin{aligned}(V_n\boldsymbol{E}_{\mathbb{P}_i}G_{i,k}R_{i,k}^{-1}G_{i,k}^T + \boldsymbol{E}_{\mathbb{P}_i}\Omega_{i,k+1})\upsilon_5 + \boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T\Upsilon_6 + \boldsymbol{E}_{\mathbb{P}_i}G_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\Upsilon_7 \\ - (V_n\boldsymbol{E}_{\mathbb{P}_i}G_{i,k}R_{i,k}^{-1}G_{i,k}^T + \kappa\boldsymbol{E}_{\mathbb{P}_i})\upsilon_5 + \lambda^{-1}\Upsilon_6 - \boldsymbol{E}_{\mathbb{P}_i}G_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\Upsilon_7 \\ = \boldsymbol{E}_{\mathbb{P}_i}F_{i,k} - \boldsymbol{E}_{\mathbb{P}_i}F_{i,k},\end{aligned}$$

$$(\boldsymbol{E}_{\mathbb{P}_i}\Omega_{i,k+1} - \kappa\boldsymbol{E}_{\mathbb{P}_i})\upsilon_5 + (\lambda^{-1}I_{snV_p} + \boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T)\Upsilon_6 = 0. \tag{162}$$

Observe that

$$\begin{aligned}\boldsymbol{E}_{\mathbb{P}_i}\Omega_{i,k+1} - \kappa\boldsymbol{E}_{\mathbb{P}_i} &= \boldsymbol{E}_{\mathbb{P}_i}(\kappa I_n + V_n(\Psi_{i,k+1}^{(0)})^{-1}) - \kappa\boldsymbol{E}_{\mathbb{P}_i},\\ \boldsymbol{E}_{\mathbb{P}_i}\Omega_{i,k+1} - \kappa\boldsymbol{E}_{\mathbb{P}_i} &= \boldsymbol{E}_{\mathbb{P}_i}((\mu^{-1}-\lambda^{-1})I_n + V_n(\Psi_{i,k+1}^{(0)})^{-1}) - (\mu^{-1}-\lambda^{-1})\boldsymbol{E}_{\mathbb{P}_i},\\ \boldsymbol{E}_{\mathbb{P}_i}\Omega_{i,k+1} - \kappa\boldsymbol{E}_{\mathbb{P}_i} &= (\mu^{-1}-\lambda^{-1})\boldsymbol{E}_{\mathbb{P}_i} + V_n\boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1} - (\mu^{-1}-\lambda^{-1})\boldsymbol{E}_{\mathbb{P}_i},\\ \boldsymbol{E}_{\mathbb{P}_i}\Omega_{i,k+1} - \kappa\boldsymbol{E}_{\mathbb{P}_i} &= V_n\boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}.\end{aligned}$$

Therefore, (162) becomes

$$V_n\boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\upsilon_5 + (\lambda^{-1}I_{snV_p} + \boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T)\Upsilon_6 = 0. \tag{163}$$

Let us now take (159), (160) and (163) to compose the following set of equations:

$$\begin{cases} V_n\boldsymbol{E}_{\boldsymbol{G}_{i,k}}R_{i,k}^{-1}G_{i,k}^T\upsilon_5 + (\lambda^{-1}I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)\Upsilon_7 = \boldsymbol{E}_{\boldsymbol{F}_{i,k}},\\ (V_nG_{i,k}R_{i,k}^{-1}G_{i,k}^T + \Omega_{i,k+1})\upsilon_5 + (\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T\Upsilon_6 + G_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\Upsilon_7 = F_{i,k},\\ V_n\boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\upsilon_5 + (\lambda^{-1}I_{snV_p} + \boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T)\Upsilon_6 = 0.\end{cases}$$

From (163),

$$\Upsilon_6 = -(\lambda^{-1}I_{snV_p} + \boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T)^{-1}V_n\boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\upsilon_5. \tag{164}$$

Substitute (164) into (160), then

$$\begin{aligned}&(V_nG_{i,k}R_{i,k}^{-1}G_{i,k}^T + \Omega_{i,k+1})\upsilon_5\\ &\quad - (\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T(\lambda^{-1}I_{snV_p} + \boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T)^{-1}V_n\boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\upsilon_5\\ &\qquad + G_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\Upsilon_7 = F_{i,k},\\ &\Big(V_nG_{i,k}R_{i,k}^{-1}G_{i,k}^T + \Omega_{i,k+1}\\ &\quad - (\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T(\lambda^{-1}I_{snV_p} + \boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T)^{-1}V_n\boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\Big)\upsilon_5\\ &\qquad + G_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\Upsilon_7 = F_{i,k},\\ &\Big(V_nG_{i,k}R_{i,k}^{-1}G_{i,k}^T + \kappa I_n + V_n(\Psi_{i,k+1}^{(0)})^{-1}\\ &\quad - (\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T(\lambda^{-1}I_{snV_p} + \boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\boldsymbol{E}_{\mathbb{P}_i}^T)^{-1}V_n\boldsymbol{E}_{\mathbb{P}_i}(\Psi_{i,k+1}^{(0)})^{-1}\Big)\upsilon_5\\ &\qquad + G_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\Upsilon_7 = F_{i,k},\end{aligned}$$

$$\Big(V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \kappa I_n + V_n (\Psi_{i,k+1}^{(0)})^{-1} \big(I_n - \boldsymbol{E}_{\mathbb{P}_i}^T (\lambda^{-1} I_{snV_p} + \boldsymbol{E}_{\mathbb{P}_i} (\Psi_{i,k+1}^{(0)})^{-1} \boldsymbol{E}_{\mathbb{P}_i}^T)^{-1} \boldsymbol{E}_{\mathbb{P}_i} (\Psi_{i,k+1}^{(0)})^{-1}\big)\Big) \upsilon_5 + G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T \Upsilon_7 = F_{i,k}.$$

Define

$$\hat{E}_{p_i,k} = I_n - \boldsymbol{E}_{\mathbb{P}_i}^T \big(\lambda^{-1} I_{snV_p} + \boldsymbol{E}_{\mathbb{P}_i} (\Psi_{i,k+1}^{(0)})^{-1} \boldsymbol{E}_{\mathbb{P}_i}^T\big)^{-1} \boldsymbol{E}_{\mathbb{P}_i} (\Psi_{i,k+1}^{(0)})^{-1},$$

then

$$\big(V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \kappa I_n + V_n (\Psi_{i,k+1}^{(0)})^{-1} \hat{E}_{p_i,k}\big) \upsilon_5 + G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T \Upsilon_7 = F_{i,k},$$

$$\upsilon_5 = \big(V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \kappa I_n + V_n (\Psi_{i,k+1}^{(0)})^{-1} \hat{E}_{p_i,k}\big)^{-1} (F_{i,k} - G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T \Upsilon_7). \tag{165}$$

We have, from (159),

$$\Upsilon_7 = (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} (\boldsymbol{E}_{\boldsymbol{F}_{i,k}} - V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5), \tag{166}$$

and substitute Υ_7 into (165) to get

$$\upsilon_5 = \big(V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \kappa I_n + V_n (\Psi_{i,k+1}^{(0)})^{-1} \hat{E}_{p_i,k}\big)^{-1} \Big(F_{i,k} - G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} (\boldsymbol{E}_{\boldsymbol{F}_{i,k}} - V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5)\Big).$$

Define $\widetilde{\Omega}_{i,k+1} = \kappa I_n + V_n \big(\Psi_{i,k+1}^{(0)}\big)^{-1} \hat{E}_{p_i,k}$, then the above equation becomes

$$\upsilon_5 = (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} \times \Big(F_{i,k} - G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} (\boldsymbol{E}_{\boldsymbol{F}_{i,k}} - V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5)\Big),$$

$$\upsilon_5 = (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} F_{i,k} - (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \times (\boldsymbol{E}_{\boldsymbol{F}_{i,k}} - V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5),$$

$$\upsilon_5 = (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} F_{i,k} - (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}} + (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5,$$

$$\upsilon_5 = (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} \big(F_{i,k} - G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}}\big) + (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5.$$

Now, define

$$\widetilde{\mathscr{F}}_{i,k} = F_{i,k} - G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T \big(\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\big)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}},$$

such that (5.2) turns into

$$\upsilon_5 = (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} \widetilde{\mathscr{F}}_{i,k}$$
$$+ (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5,$$

$$\Big(I_n - (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T\Big) \upsilon_5$$
$$= (V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})^{-1} \widetilde{\mathscr{F}}_{i,k}.$$

Multiply both sides of the above equation to the left by $(V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1})$ to yield

$$\Big(V_n G_{i,k} R_{i,k}^{-1} G_{i,k}^T + \widetilde{\Omega}_{i,k+1} - G_{i,k} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T\Big) \upsilon_5 = \widetilde{\mathscr{F}}_{i,k},$$
$$\Big(\widetilde{\Omega}_{i,k+1} + V_n G_{i,k} R_{i,k}^{-1} (I_m - \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1}) G_{i,k}^T\Big) \upsilon_5 = \widetilde{\mathscr{F}}_{i,k}.$$

Define

$$\bar{\mathscr{R}}_{i,k} = R_{i,k}^{-1} \big(I_m - \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1}\big),$$

hence (5.2) becomes

$$(\widetilde{\Omega}_{i,k+1} + V_n G_{i,k} \bar{\mathscr{R}}_{i,k} G_{i,k}^T) \upsilon_5 = \widetilde{\mathscr{F}}_{i,k},$$
$$\upsilon_5 = (\widetilde{\Omega}_{i,k+1} + V_n G_{i,k} \bar{\mathscr{R}}_{i,k} G_{i,k}^T)^{-1} \widetilde{\mathscr{F}}_{i,k}. \tag{167}$$

Next, substitute Υ_7 from (166) into (158), such that

$$K_{i,k} = -R_{i,k}^{-1} \big(V_n G_{i,k}^T \upsilon_5 + \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} (\boldsymbol{E}_{\boldsymbol{F}_{i,k}} - V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5)\big),$$

$$K_{i,k} = -R_{i,k}^{-1} \big(V_n G_{i,k}^T \upsilon_5 + \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}}$$
$$- \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5\big),$$

$$K_{i,k} = -R_{i,k}^{-1} \big(V_n G_{i,k}^T - \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T\big) \upsilon_5$$
$$- R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}},$$

$$K_{i,k} = -V_n R_{i,k}^{-1} \big(I_m - \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1}\big) G_{i,k}^T \upsilon_5$$
$$- R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}},$$

$$K_{i,k} = -V_n \bar{\mathscr{R}}_{i,k} G_{i,k}^T \upsilon_5 - R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}},$$

and by substituting υ_5 from (167) we obtain

$$K_{i,k} = -V_n \bar{\mathscr{R}}_{i,k} G_{i,k}^T (\widetilde{\Omega}_{i,k+1} + V_n G_{i,k} \bar{\mathscr{R}}_{i,k} G_{i,k}^T)^{-1} \widetilde{\mathscr{F}}_{i,k}$$
$$- R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}}.$$

Define $\bar{G}_{i,k} = G_{i,k} \bar{\mathscr{R}}_{i,k}^{1/2}$, then the above equation becomes

$$K_{i,k} = -V_n \bar{\mathscr{R}}_{i,k} G_{i,k}^T \big(\widetilde{\Omega}_{i,k+1} + V_n \bar{G}_{i,k} \bar{G}_{i,k}^T\big)^{-1} \widetilde{\mathscr{F}}_{i,k}$$
$$- R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T \big(\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\big)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}},$$

which matches (139). Next, place (139) and (166) into (156) to produce

$$L_{i,k} = F_{i,k} - G_{i,k}R_{i,k}^{-1}(V_nG_{i,k}^T\upsilon_5 + \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\Upsilon_7) - \kappa\upsilon_5,$$
$$L_{i,k} = F_{i,k} - G_{i,k}R_{i,k}^{-1}V_nG_{i,k}^T\upsilon_5 - G_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\Upsilon_7 - \kappa\upsilon_5.$$

By using Υ_7 from (166) we have

$$L_{i,k} = F_{i,k} - G_{i,k}R_{i,k}^{-1}V_nG_{i,k}^T\upsilon_5 - \kappa\upsilon_5 - G_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T(\lambda^{-1}I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1}(\boldsymbol{E}_{\boldsymbol{F}_{i,k}} - V_n\boldsymbol{E}_{\boldsymbol{G}_{i,k}}R_{i,k}^{-1}G_{i,k}^T\upsilon_5),$$

$$L_{i,k} = F_{i,k} - G_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T(\lambda^{-1}I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1}\boldsymbol{E}_{\boldsymbol{F}_{i,k}} -\big(V_nG_{i,k}R_{i,k}^{-1}G_{i,k}^T + \kappa I_n - G_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T(\lambda^{-1}I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1}V_n\boldsymbol{E}_{\boldsymbol{G}_{i,k}}R_{i,k}^{-1}G_{i,k}^T\big)\upsilon_5,$$

$$L_{i,k} = \widetilde{\mathscr{F}}_{i,k} - \big(\kappa I_n + V_nG_{i,k}R_{i,k}^{-1}G_{i,k}^T - V_nG_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T(\lambda^{-1}I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}R_{i,k}^{-1}G_{i,k}^T\big)\upsilon_5,$$

$$L_{i,k} = \widetilde{\mathscr{F}}_{i,k} - \Big(\kappa I_n + V_nG_{i,k}R_{i,k}^{-1}\big(I_m - G_{i,k}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T(\lambda^{-1}I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}}R_{i,k}^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1}\boldsymbol{E}_{\boldsymbol{G}_{i,k}}R_{i,k}^{-1}\big)G_{i,k}^T\Big)\upsilon_5,$$
$$L_{i,k} = \widetilde{\mathscr{F}}_{i,k} - (\kappa I_n + V_nG_{i,k}\bar{\mathscr{R}}_{i,k}G_{i,k}^T)\upsilon_5,$$

and with υ_5 from (167) we produce

$$L_{i,k} = \widetilde{\mathscr{F}}_{i,k} - (\kappa I_n + V_nG_{i,k}\bar{\mathscr{R}}_{i,k}G_{i,k}^T)(\widetilde{\Omega}_{i,k+1} + V_nG_{i,k}\bar{\mathscr{R}}_{i,k}G_{i,k}^T)^{-1}\widetilde{\mathscr{F}}_{i,k},$$
$$L_{i,k} = \big(I_n - \kappa(\widetilde{\Omega}_{i,k+1} + V_nG_{i,k}\bar{\mathscr{R}}_{i,k}G_{i,k}^T)^{-1}\big)\widetilde{\mathscr{F}}_{i,k} - V_nG_{i,k}\bar{\mathscr{R}}_{i,k}G_{i,k}^T(\widetilde{\Omega}_{i,k+1} + V_nG_{i,k}\bar{\mathscr{R}}_{i,k}G_{i,k}^T)^{-1}\widetilde{\mathscr{F}}_{i,k},$$
$$L_{i,k} = \big(I_n - \kappa(\widetilde{\Omega}_{i,k+1} + V_n\bar{G}_{i,k}\bar{G}_{i,k}^T)^{-1}\big)\widetilde{\mathscr{F}}_{i,k} - V_n\bar{G}_{i,k}\bar{G}_{i,k}^T(\widetilde{\Omega}_{i,k+1} + V_n\bar{G}_{i,k}\bar{G}_{i,k}^T)^{-1}\widetilde{\mathscr{F}}_{i,k}. \tag{168}$$

Based upon Lemma A.7, we have that

$$I_m\bar{G}_{i,k}^T(\widetilde{\Omega}_{i,k+1} + V_n\bar{G}_{i,k}\bar{G}_{i,k}^T)^{-1} = (I_m + \bar{G}_{i,k}^TV_n\widetilde{\Omega}_{i,k+1}^{-1}\bar{G}_{i,k})^{-1}\bar{G}_{i,k}^T\widetilde{\Omega}_{i,k+1}^{-1},$$

and combining with (168) we obtain

$$L_{i,k} = \big(I_n - \kappa(\widetilde{\Omega}_{i,k+1} + V_n\bar{G}_{i,k}\bar{G}_{i,k}^T)^{-1}\big)\widetilde{\mathscr{F}}_{i,k} - \bar{G}_{i,k}(I_m + \bar{G}_{i,k}^TV_n\widetilde{\Omega}_{i,k+1}^{-1}\bar{G}_{i,k})^{-1}\bar{G}_{i,k}^TV_n\widetilde{\Omega}_{i,k+1}^{-1}\widetilde{\mathscr{F}}_{i,k},$$

$$L_{i,k} = \big(I_n - \kappa(\widetilde{\Omega}_{i,k+1} + V_n\bar{G}_{i,k}\bar{G}_{i,k}^T)^{-1}\big)\widetilde{\mathscr{F}}_{i,k} - \bar{G}_{i,k}(I_m + \bar{G}_{i,k}^T\widetilde{\Psi}_{i,k+1}\bar{G}_{i,k})^{-1}\bar{G}_{i,k}^T\widetilde{\Psi}_{i,k+1}\widetilde{\mathscr{F}}_{i,k},$$

which corresponds to (138), with $\widetilde{\Psi}_{i,k+1} = V_n \widetilde{\Omega}_{i,k+1}^{-1}$. Now, substitute Υ_3 and Υ_7 from (155) and (166), respectively, into (152) to yield

$$\begin{aligned}
P_{i,k} &= Q_{i,k} + V_n F_{i,k}^T \upsilon_5 + \boldsymbol{E}_{\boldsymbol{F}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} (\boldsymbol{E}_{\boldsymbol{F}_{i,k}} - V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5), \\
P_{i,k} &= Q_{i,k} + V_n F_{i,k}^T \upsilon_5 + \boldsymbol{E}_{\boldsymbol{F}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}} \\
&\quad - \boldsymbol{E}_{\boldsymbol{F}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5.
\end{aligned} \tag{169}$$

Define

$$\bar{Q}_{i,k} = Q_{i,k} + \boldsymbol{E}_{\boldsymbol{F}_{i,k}}^T \big(\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T\big)^{-1} \boldsymbol{E}_{\boldsymbol{F}_{i,k}},$$

then (169) turns into

$$\begin{aligned}
P_{i,k} &= \bar{Q}_{i,k} + V_n F_{i,k}^T \upsilon_5 - \boldsymbol{E}_{\boldsymbol{F}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} V_n \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T \upsilon_5, \\
P_{i,k} &= \bar{Q}_{i,k} + V_n \big(F_{i,k}^T - \boldsymbol{E}_{\boldsymbol{F}_{i,k}}^T (\lambda^{-1} I_{nV_n} + \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}}^T)^{-1} \boldsymbol{E}_{\boldsymbol{G}_{i,k}} R_{i,k}^{-1} G_{i,k}^T\big) \upsilon_5, \\
P_{i,k} &= \bar{Q}_{i,k} + V_n \widetilde{\mathscr{F}}_{i,k}^T \upsilon_5.
\end{aligned}$$

Finally, substitute υ_5 from (167) to obtain

$$\begin{aligned}
P_{i,k} &= \bar{Q}_{i,k} + V_n \widetilde{\mathscr{F}}_{i,k}^T (\widetilde{\Omega}_{i,k+1} + V_n G_{i,k} \bar{\mathscr{R}}_{i,k} G_{i,k}^T)^{-1} \widetilde{\mathscr{F}}_{i,k}, \\
P_{i,k} &= \bar{Q}_{i,k} + V_n \widetilde{\mathscr{F}}_{i,k}^T \Big(\widetilde{\Omega}_{i,k+1}^{-1} - \widetilde{\Omega}_{i,k+1}^{-1} V_n \bar{G}_{i,k} (I_m + \bar{G}_{i,k}^T \widetilde{\Omega}_{i,k+1}^{-1} V_n \bar{G}_{i,k})^{-1} \bar{G}_{i,k}^T \widetilde{\Omega}_{i,k+1}^{-1}\Big) \widetilde{\mathscr{F}}_{i,k}, \\
P_{i,k} &= \bar{Q}_{i,k} + \widetilde{\mathscr{F}}_{i,k}^T V_n \widetilde{\Omega}_{i,k+1}^{-1} \widetilde{\mathscr{F}}_{i,k} \\
&\quad - \widetilde{\mathscr{F}}_{i,k}^T V_n \widetilde{\Omega}_{i,k+1}^{-1} V_n \bar{G}_{i,k} (I_m + \bar{G}_{i,k}^T V_n \widetilde{\Omega}_{i,k+1}^{-1} \bar{G}_{i,k})^{-1} \bar{G}_{i,k}^T V_n \widetilde{\Omega}_{i,k+1}^{-1} \widetilde{\mathscr{F}}_{i,k}, \\
P_{i,k} &= \bar{Q}_{i,k} + \widetilde{\mathscr{F}}_{i,k}^T \widetilde{\Psi}_{i,k+1} \widetilde{\mathscr{F}}_{i,k} - \widetilde{\mathscr{F}}_{i,k}^T \widetilde{\Psi}_{i,k+1} \bar{G}_{i,k} \big(I_m + \bar{G}_{i,k}^T \widetilde{\Psi}_{i,k+1} \bar{G}_{i,k}\big)^{-1} \bar{G}_{i,k}^T \widetilde{\Psi}_{i,k+1} \widetilde{\mathscr{F}}_{i,k},
\end{aligned}$$

which matches (140) and completes the proof. □

By means of Theorem 5.1, we can compute the robust state-feedback gains $\boldsymbol{K}_k = (K_{1,k}, \ldots, K_{s,k}) \in \mathbb{H}^{m,n}$ in a recursive fashion, whereby the polytope vertices are accounted for altogether in each Markov mode. Furthermore, observe that $x_{k+1} = L_{i,k} x_k$, thus we say that $L_{i,k}$ in (138) is the correspondent closed-loop matrix associated to (125) when $u_k = K_{i,k} x_k$.

Remark 5.3. *By applying the Sherman–Morrison–Woodbury formula (see Lemma A.6), we get $\hat{E}_{p_i,k} = (I_n + \lambda \boldsymbol{E}_{\mathbb{P}_i}^T \boldsymbol{E}_{\mathbb{P}_i} (\Psi_{i,k+1}^{(0)})^{-1})^{-1}$. If $\lambda \to \infty$, then $\hat{E}_{p_i,k} \to 0$ and $P_{i,k}$ is a constant matrix for $k = 0, \ldots, N-1$. That said, we adopt some fixed $\mu > 0$ and $\beta > 1$ for which $\hat{E}_{p_i,k} > 0$ to ensure the convergence to a stabilizing solution. Moreover, as discussed in (ROCHA; TERRA, 2021), $\beta \in (1, 2]$ returns adequate numerical results regarding stability and fits as an initial candidate for any search method.*

5.2.1 Convergence and stability

Let us now consider the DMJLS described by (125) and the cost function (134) with fixed parameters, whilst allowing coefficients α_k and ξ_k to be time-varying. Thus, we analyze the realization

$$x_{k+1} = \left(F_i + \sum_{l=1}^{V_n} \alpha_{l,k} F_i^{(l)}\right) x_k + \left(G_i + \sum_{l=1}^{V_n} \alpha_{l,k} G_i^{(l)}\right) u_k, \tag{170}$$

where $u_k = K_{i,k} x_k$, with gains $K_{i,k}$, $i \in \Theta$, given by Theorem 5.1. Moreover, we assume $\hat{E}_{p_i,k} > 0$ and restate (138) and (140) with time-invariant parameters as

$$L_{i,k} = \big(I_n - \kappa(\widetilde{\Omega}_{i,k+1} + V_n \bar{G}_i \bar{G}_i^T)^{-1}\big)\widetilde{\mathscr{F}_i} - \bar{G}_i\big(I_m + \bar{G}_i^T \widetilde{\Psi}_{i,k+1} \bar{G}_i\big)^{-1} \bar{G}_i^T \widetilde{\Psi}_{i,k+1} \widetilde{\mathscr{F}_i}, \tag{171}$$

$$P_{i,k} = \bar{Q}_i + \widetilde{\mathscr{F}_i}^T \widetilde{\Psi}_{i,k+1} \widetilde{\mathscr{F}_i} - \widetilde{\mathscr{F}_i}^T \widetilde{\Psi}_{i,k+1} \bar{G}_i\big(I_m + \bar{G}_i^T \widetilde{\Psi}_{i,k+1} \bar{G}_i\big)^{-1} \bar{G}_i^T \widetilde{\Psi}_{i,k+1} \widetilde{\mathscr{F}_i}. \tag{172}$$

Notice that (172) fits into the well-known class of coupled algebraic Riccati equations (COSTA; FRAGOSO; MARQUES, 2005). Thus, the conditions for convergence and stability are established based on the classic regulation problem for DMJLS, as we outline in the following statement.

Proposition 5.1. *Assume the pairs* $(\bar{Q}_i^{1/2}, \widetilde{\mathscr{F}_i})$ *and* $(\widetilde{\mathscr{F}_i}, \bar{G}_i)$, *are mean square detectable and mean square stabilizable, respectively, and* $\hat{E}_{p_i,k} > 0$, *for all* $i \in \Theta$. *Consider fixed* $\mu > 0$, $\beta > 1$, *and initial condition* $\boldsymbol{P}_N = (P_{1,N}, \ldots, P_{s,N}) \in \mathbb{H}_+^n$. *Then,* $\boldsymbol{P}_k \in \mathbb{H}_+^n$ *generated by* (172) *converges to its unique solution* $\boldsymbol{P} = (P_1, \ldots, P_s) \in \mathbb{H}_+^n$. *Moreover, the closed-loop system matrix associated with* (170) *is mean-square stable.*

Proof. Given the resemblance between (172) and the coupled algebraic Riccati equations, the proof follows based on the fundamental arguments presented in (COSTA; FRAGOSO; MARQUES, 2005) for detectability and stabilizability of DMJLS. □

In the next section, we verify the performance of the robust regulator presented in this chapter in numerical and application examples.

5.3 Illustrative Examples

We present three examples: the first two are numerical, and we investigate the regulation performance of the recursive regulator; in the third example, we assess the effectiveness of the regulator when applied to the estimated powertrain model of the autonomous Scania truck (see Appendix B. We perform the simulations in a 2.3 GHz i7-11800H CPU with 16 GB of RAM. The results are compared with those obtained with the robust H_2 controller borrowed from (COSTA; FRAGOSO; TODOROV, 2015) to verify the competitiveness and potential of our approach. For shortness, throughout the examples, we denote the robust recursive regulator for polytopic MJLS with uncertain transition probabilities by *M3R*.

Remark 5.4. *Without loss of generality, in the following examples, we produce $\alpha_k \in \Lambda_{V_n}$ and $\xi_k \in \Lambda_{V_p}$ from a uniform distribution, at each time step k.*

Example 5.1. *Let us consider the DMJLS based on (ZHANG; BOUKAS, 2009) described by the following parameters:*

- *Mode* 1*:*

$$F_{1,k} = \begin{bmatrix} 0.32 & -0.4 \\ 0.8 & -0.8 \end{bmatrix}, \quad F_{1,k}^{(1)} = \begin{bmatrix} 0.064 & -0.080 \\ 0.160 & -0.160 \end{bmatrix}, \quad F_{1,k}^{(2)} = -F_{1,k}^{(1)},$$
$$G_{1,k} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad G_{1,k}^{(1)} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, \quad G_{1,k}^{(2)} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

- *Mode* 2*:*

$$F_{2,k} = \begin{bmatrix} 0.08 & -0.26 \\ 0.80 & -1.12 \end{bmatrix}, \quad F_{2,k}^{(1)} = \begin{bmatrix} 0.016 & -0.052 \\ 0.160 & -0.224 \end{bmatrix}, \quad F_{2,k}^{(2)} = -F_{2,k}^{(1)},$$
$$G_{2,k} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad G_{2,k}^{(1)} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad G_{2,k}^{(2)} = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix}.$$

- *Mode* 3*:*

$$F_{3,k} = \begin{bmatrix} 0.16 & -0.08 \\ 0.8 & -0.96 \end{bmatrix}, \quad F_{3,k}^{(1)} = \begin{bmatrix} 0.032 & -0.016 \\ 0.160 & -0.192 \end{bmatrix}, \quad F_{3,k}^{(2)} = -F_{3,k}^{(1)},$$
$$G_{3,k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad G_{3,k}^{(1)} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad G_{3,k}^{(2)} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

- *Mode* 4*:*

$$F_{4,k} = \begin{bmatrix} 0.48 & -0.18 \\ 0.8 & -0.88 \end{bmatrix}, \quad F_{4,k}^{(1)} = \begin{bmatrix} 0.096 & -0.036 \\ 0.160 & -0.176 \end{bmatrix}, \quad F_{4,k}^{(2)} = -F_{4,k}^{(1)},$$
$$G_{4,k} = \begin{bmatrix} 0.8 \\ -1 \end{bmatrix}, \quad G_{4,k}^{(1)} = \begin{bmatrix} 0.08 \\ 0 \end{bmatrix}, \quad G_{4,k}^{(2)} = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix}.$$

The transition probabilities are assumed to be subject to uncertainties and defined by a known portion $\mathbb{P}_0$ and vertices $\{\mathbb{P}^{(1)}, \mathbb{P}^{(2)}\}$ given by

$$\mathbb{P}_0 = \begin{bmatrix} 0.3 & 0 & 0.1 & 0 \\ 0 & 0 & 0.3 & 0.2 \\ 0 & 0.1 & 0 & 0.3 \\ 0.2 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbb{P}^{(1)} = \begin{bmatrix} 0 & 0.6 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0.4 \end{bmatrix}, \quad \mathbb{P}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0.6 \\ 0 & 0.5 & 0 & 0 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & 0.4 & 0 \end{bmatrix}.$$

Also, we adopt initial conditions $x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, and $\pi_0 = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$.

We consider $N = 50$ *and compose the cost function* (134) *with weight matrices* $Q_{i,k} = I_2$, $R_{i,k} = 1$, $i \in \{1,2,3,4\}$, *and* $\mu = 10^{12}$. *Then, we compute the robust feedback gains with Theorem 5.1 with* $P_{i,N} = I_2$, $\beta = 1.001$, *hence obtaining*

$$K_1 = \begin{bmatrix} -0.2282 & 0.2655 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 0.1603 & -0.1386 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} -0.2956 & 0.2896 \end{bmatrix}, \qquad K_4 = \begin{bmatrix} -0.0149 & -0.1803 \end{bmatrix},$$

whilst the robust H_2 *gains are*

$$K_{H_2,1} = \begin{bmatrix} -0.5957 & 0.5138 \end{bmatrix}, \qquad K_{H_2,2} = \begin{bmatrix} 0.3647 & -0.4337 \end{bmatrix},$$
$$K_{H_2,3} = \begin{bmatrix} -0.4152 & 0.4576 \end{bmatrix}, \qquad K_{H_2,4} = \begin{bmatrix} 0.2314 & -0.4063 \end{bmatrix}.$$

We take 5000 *realizations of the Markov chain and allow coefficients* α_k *and* ξ_k *to change in each iteration of each experiment, which makes uncertainties* $\delta F_{i,k}$, $\delta G_{i,k}$, *and* $\delta \mathbb{P}_k$ *to be time-varying. We summarize the averaged results in Table 5. We present, in Fig. 10 and Fig. 11, respectively, the averaged norms of the state vector and the spectral radius of the closed-loop system with the robust recursive regulator. As can be seen, the regulator ensured the stability of the closed-loop system despite the effects of polytopic uncertainties on the system and transition probability matrices. Even though the results were competitive with respect to regulation, the proposed regulator demanded substantially smaller averaged computational time to return the feedback gains when compared to the robust* H_2 *controller.*

Table 5 – Averaged results of Example 5.1.

Controller	$\|x_k\|$	$\sigma_{\|x_k\|}$	$\|u_k\|$	$\sigma_{\|u_k\|}$	t_c [ms]	σ_{t_c} [ms]
M3R	1.4337	0.2009	0.0374	0.0053	0.9635	0.0976
Robust H_2	1.4403	0.2008	0.0704	0.0099	108.2582	44.397

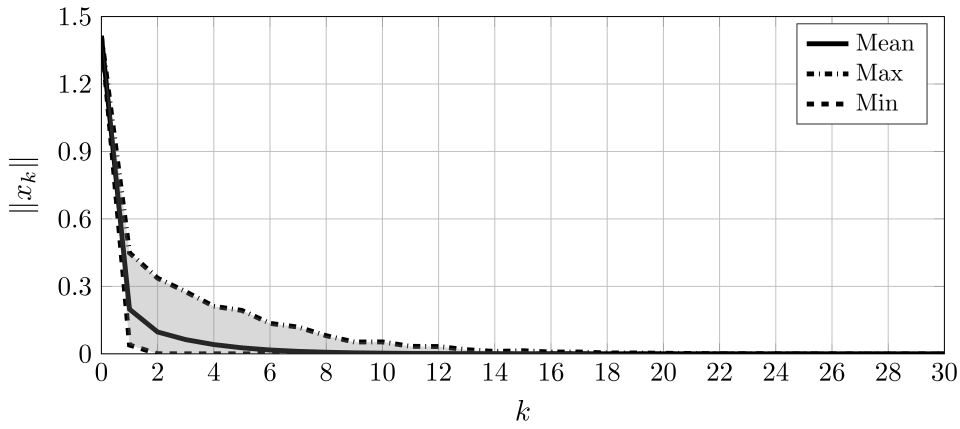
Source: author.

Example 5.2. *Consider now the DMJLS with parameters based on (COSTA et al., 1999), namely*

- *Mode 1:*

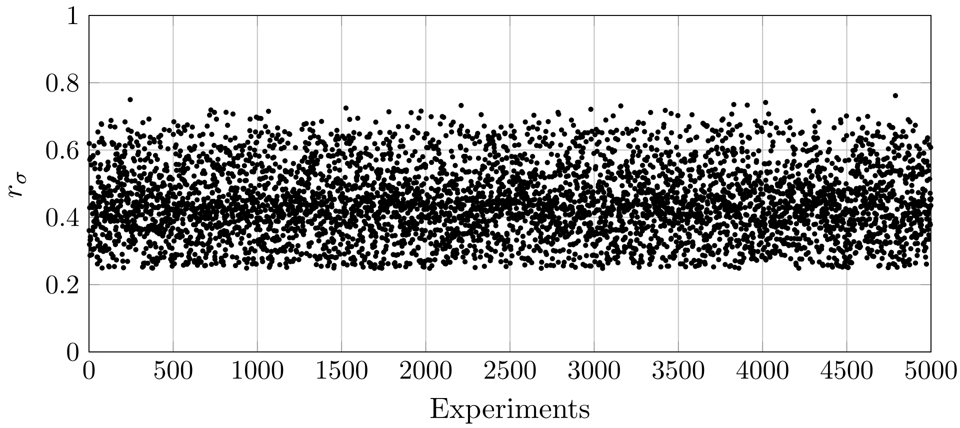
$$F_{1,k} = \begin{bmatrix} 0 & 1 \\ -2.5 & 3.2 \end{bmatrix}, \quad F_{1,k}^{(1)} = \begin{bmatrix} 0 & 0.150 \\ -0.375 & 0.480 \end{bmatrix}, \quad F_{1,k}^{(2)} = -F_{1,k}^{(1)},$$
$$G_{1,k} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G_{1,k}^{(1)} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad G_{1,k}^{(2)} = -G_{1,k}^{(1)}.$$

Figure 10 – Averaged norms of system state vector with the robust recursive regulator in Example 5.1.



Source: author.

Figure 11 – Spectral radii of the closed-loop system with the robust recursive regulator in Example 5.1.



Source: author.

- *Mode 2:*

$$F_{2,k} = \begin{bmatrix} 0 & 1 \\ -4.3 & 4.5 \end{bmatrix}, \quad F_{2,k}^{(1)} = \begin{bmatrix} 0 & 0.150 \\ -0.645 & 0.675 \end{bmatrix}, \quad F_{2,k}^{(2)} = -F_{2,k}^{(1)},$$
$$G_{2,k} = G_{1,k}, \quad G_{2,k}^{(1)} = G_{1,k}^{(1)}, \quad G_{2,k}^{(2)} = G_{1,k}^{(2)}.$$

- *Mode 3:*

$$F_{3,k} = \begin{bmatrix} 0 & 1 \\ 5.3 & -5.2 \end{bmatrix}, \quad F_{3,k}^{(1)} = \begin{bmatrix} 0 & 0.1 \\ 0.53 & -0.52 \end{bmatrix}, \quad F_{3,k}^{(2)} = -F_{3,k}^{(1)},$$
$$G_{3,k} = G_{1,k}, \quad G_{3,k}^{(1)} = G_{1,k}^{(1)}, \quad G_{3,k}^{(2)} = G_{1,k}^{(2)}.$$

We consider $x_0 = \begin{bmatrix}1 & 1\end{bmatrix}^T$*, and* $\pi_0 = \begin{bmatrix}0.16 & 0.30 & 0.54\end{bmatrix}$*, whereas the uncertain transition probability matrix is defined by the following known* $\mathbb{P}_0$ *and vertices* $\{\mathbb{P}^{(1)}, \mathbb{P}^{(2)}, \mathbb{P}^{(3)}\}$*:*

$$\mathbb{P}_0 = \begin{bmatrix}0.67 & 0.17 & 0.16\\ 0.30 & 0.47 & 0.23\\ 0.26 & 0.10 & 0.64\end{bmatrix}, \quad \mathbb{P}^{(1)} = \begin{bmatrix}-0.12 & 0 & 0.12\\ 0 & 0.07 & -0.07\\ -0.09 & 0 & 0.09\end{bmatrix},$$

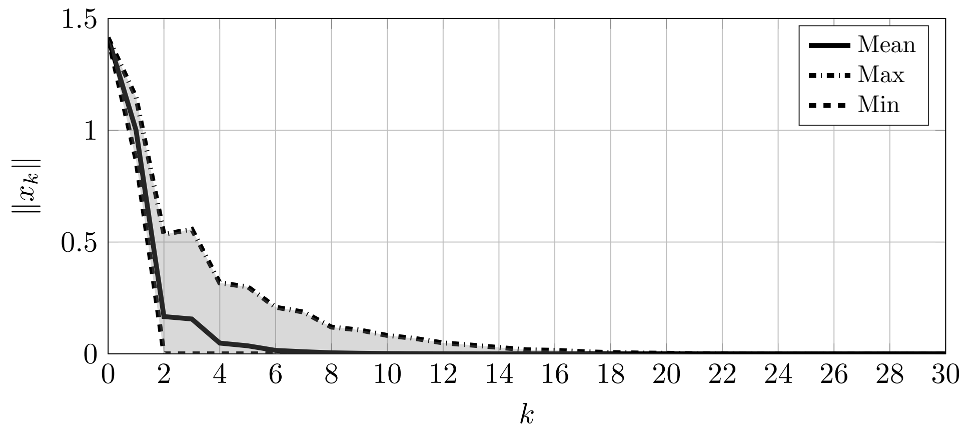
$$\mathbb{P}^{(2)} = \begin{bmatrix}0.12 & -0.12 & 0\\ -0.07 & 0.07 & 0\\ 0 & 0.09 & -0.09\end{bmatrix}, \mathbb{P}^{(3)} = \begin{bmatrix}0 & -0.12 & 0.12\\ -0.07 & 0 & 0.07\\ -0.09 & 0.09 & 0\end{bmatrix}.$$

We carry out a total of 5000 *in realizations of the Markov chain. Again, the coefficients* α_k *and* ξ_k *change from one iteration to another, as mentioned in Remark 5.4, hence* $\{\delta F_{i,k}, \delta G_{i,k}, \delta\mathbb{P}_k\}$ *are time-varying matrices. We consider* $N = 30$ *and set up* (134) *with* $Q_{i,k} = I_2$, $R_{i,k} = 1$*, and* $\mu = 10^{12}$*. Then, with* $P_{i,N} = I_2$ *and* $\beta = 1.01$*, Theorem 5.1 produces the feedback gains*

$$K_1 = \begin{bmatrix}2.7034 & -3.4232\end{bmatrix}, \quad K_2 = \begin{bmatrix}4.6743 & -4.8162\end{bmatrix}, \quad K_3 = \begin{bmatrix}-5.3000 & 5.2271\end{bmatrix}.$$

In Fig. 12, we show the averaged norms of the closed-loop state vector with the robust recursive regulator. In Fig. 13a, observe that the DMJLS with $u_k = 0$ *is highly unstable. Regardless, the proposed robust recursive regulator ensured closed-loop stability, as we depict in Fig. 13b. Furthermore, the mean computational time required to compute* $K = (K_1, K_2, K_3)$ *was* 0.8974 *ms, with standard deviation of* 67.87 μ*s. The robust* H_2 *controller, however, required on average* 61.8440 *ms, with a standard deviation of* 21.7730 *ms.*

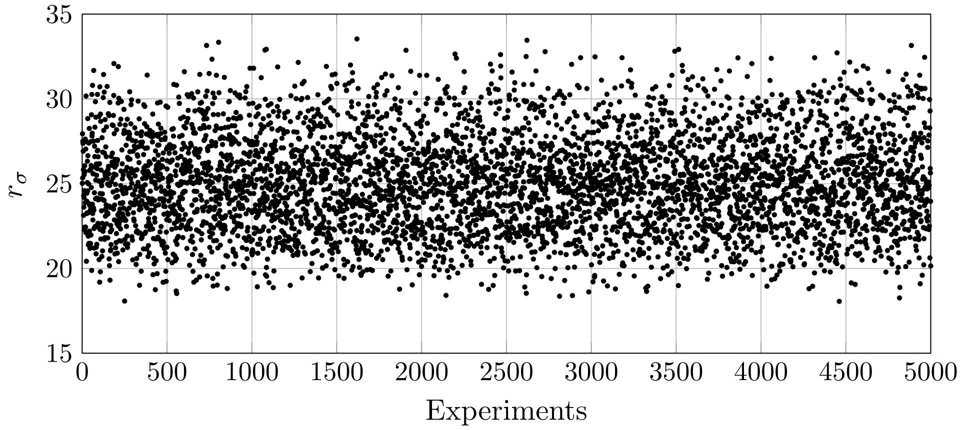
Figure 12 – Averaged norms of system state vector with the robust recursive regulator in Example 5.2.



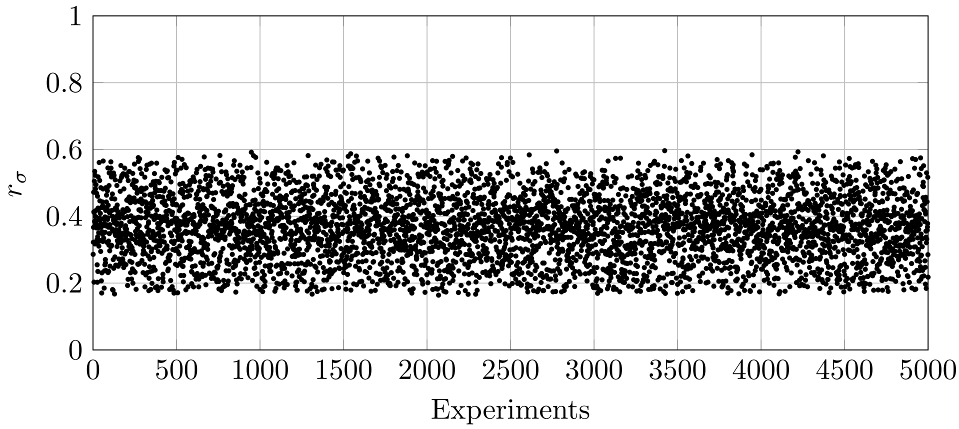
Source: author.

Example 5.3. *Let us address the powertrain control problem for the autonomous heavy-duty vehicle, whose model is described in Appendix B. The DMJLS has* 14 *operation*

Figure 13 – Spectral radii of the DMJLS in Example 5.2.



(a) Spectral radii of the open-loop DMJLS considered in Example 5.2.



(b) Spectral radii of the closed-loop system with the robust recursive regulator in Example 5.2.

Source: author.

modes, and state-space and transition probability matrices are subject to polytopic uncertainties. We recorded the Markov jumps and the reference signals for accelerator and brake pedals positions whilst driving the vehicle inside the campus of the University of São Paulo, in the city of São Carlos. To improve the controllability of the powertrain model, we must split each matrix G_i, $i \in \{1, \dots, 14\}$, *into two columns, hence* $G_i \leftarrow \begin{bmatrix} 0.5G_i & 0.5G_i \end{bmatrix}$, $u_k \leftarrow \begin{bmatrix} 0.5u_k^T & 0.5u_k^T \end{bmatrix}^T$, *and* $\tau_k \leftarrow \begin{bmatrix} 0.5\tau_k^T & 0.5\tau_k^T \end{bmatrix}^T$. *We retrieve the original required pedal positions by adding up the entries in the newly defined vector* τ_k. *We assume initial conditions* $x_0 = \begin{bmatrix} 0.10 & -0.50 & -0.07 \end{bmatrix}^T$, *and* $\theta_0 = 3$. *Therefore, the truck starts the simulation with the* 6*th gear engaged and accelerating. The recorded data comprises a time horizon* $N = 1591$, *meaning about* 5 *minutes driving.*

We select high values for Q_i *to weight the norm of tracking errors* x_k. *Meanwhile, high values for* R_i *have the purpose of diminishing energy consumption and minimizing* *acceleration and deceleration jerks, since R_i weights the norm of input u_k. That said, we build the cost function* (134) *with the parameters*

$$Q_i = 10^{10} I_3, \quad R_i = 10^{11} I_2, \quad i \in \{1, \ldots, 7\},$$
$$Q_i = 10^{10} I_3, \quad R_i = 10^{10} I_2, \quad i \in \{8, \ldots, 14\},$$
$$P_{i,N} = I_3, \quad i \in \{1, \ldots, 14\}, \quad \mu = 10^9, \quad \beta = 1.00001.$$

Given the above configuration, Theorem 5.1 returns the feedback gains

$$K_1 = \begin{bmatrix} -0.0744 & -0.5561 & 0.5460 \\ -0.0744 & -0.5561 & 0.5460 \end{bmatrix}, \qquad K_8 = \begin{bmatrix} 0 & 0 & 0.0066 \\ 0 & 0 & 0.0066 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 0.0139 & 0.0351 & -0.0773 \\ 0.0139 & 0.0351 & -0.0773 \end{bmatrix}, \qquad K_9 = \begin{bmatrix} 0 & 0 & -0.1283 \\ 0 & 0 & -0.1283 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} 0.1025 & 0.4456 & -0.5648 \\ 0.1025 & 0.4456 & -0.5648 \end{bmatrix}, \qquad K_{10} = \begin{bmatrix} 0 & 0 & 0.1284 \\ 0 & 0 & 0.1284 \end{bmatrix},$$
$$K_4 = \begin{bmatrix} -0.0552 & 2.1204 & -2.0373 \\ -0.0552 & 2.1204 & -2.0373 \end{bmatrix}, \qquad K_{11} = \begin{bmatrix} 0 & 0 & 0.0241 \\ 0 & 0 & 0.0241 \end{bmatrix},$$
$$K_5 = \begin{bmatrix} -0.0347 & 1.1205 & -1.1562 \\ -0.0347 & 1.1205 & -1.1562 \end{bmatrix}, \qquad K_{12} = \begin{bmatrix} 0 & 0 & 0.0989 \\ 0 & 0 & 0.0989 \end{bmatrix},$$
$$K_6 = \begin{bmatrix} 0.0020 & 0.0642 & -0.0676 \\ 0.0020 & 0.0642 & -0.0676 \end{bmatrix}, \qquad K_{13} = \begin{bmatrix} 0 & 0 & -0.0217 \\ 0 & 0 & -0.0217 \end{bmatrix},$$
$$K_7 = \begin{bmatrix} -0.0294 & 0.6454 & -0.6334 \\ -0.0294 & 0.6454 & -0.6334 \end{bmatrix}, \qquad K_{14} = \begin{bmatrix} 0 & 0 & -0.0782 \\ 0 & 0 & -0.0782 \end{bmatrix}.$$

We performed 1000 simulations, and allowed coefficients α_k and ξ_k to vary throughout each run conditioned by Remark 5.4. In Table 6 we summarize the results for the robust recursive regulator (M3R) and the robust H_2 controller, averaged over all runs. In Fig. 14 we depict the closed-loop trajectories of the states and the recorded references, while in Fig. 15 we show the computed accelerator and brake pedals position signals. Even affected by uncertainties $\boldsymbol{\delta}_k$, the robust recursive regulator maintained closed-loop stability and adequately tracked the reference trajectories. Under the same conditions, the robust H_2 gains could not ensure the stability of the closed-loop system.

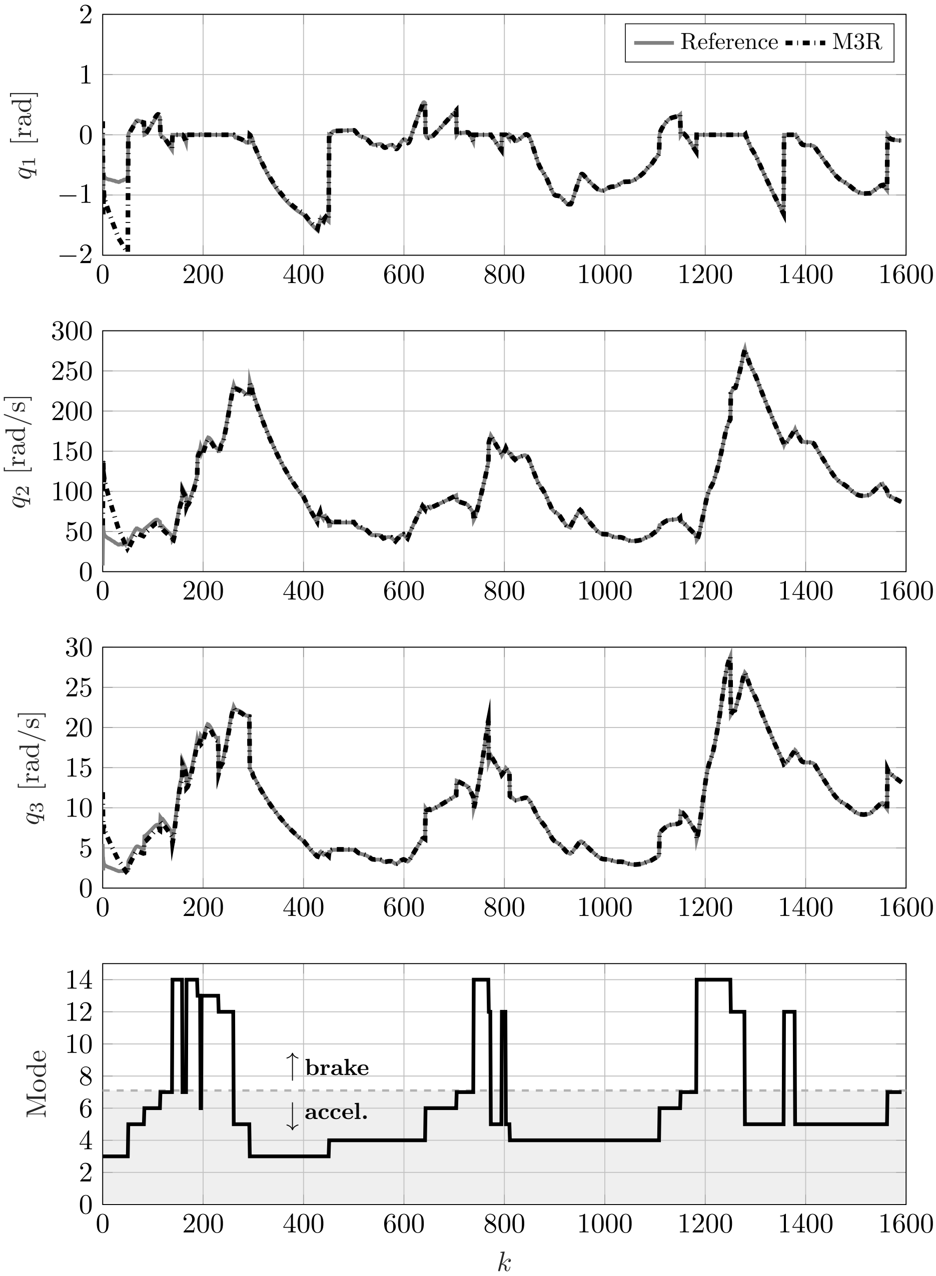
It is worth mentioning that the computational times required by the proposed recursive method are acceptable for application on the autonomous truck. In fact, we have $t_c < T_{samp} = 0.2s$, where T_{samp} is the sampling period defined for the model identification procedure (see Appendix B). Therefore, it is possible to evaluate the feedback gains within the sampling period.

Table 6 – Averaged results of the longitudinal control for the autonomous truck.

Controller	$\|x_k\|$	$\sigma_{\|x_k\|}$	$\|\tau_k\|$	$\sigma_{\|\tau_k\|}$	t_c [ms]	σ_{t_c} [ms]
M3R	3.5302	0.0825	0.8775	0.0217	17.5976	0.5690
Robust H_2	Inf	Inf	Inf	Inf	4155.2019	588.9904

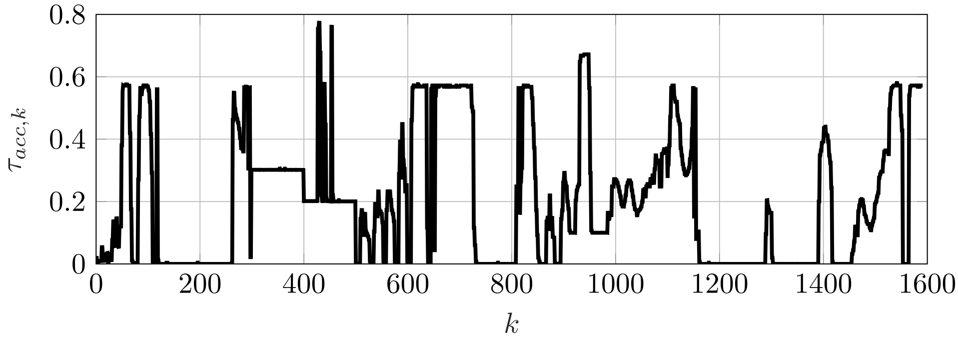
Source: author.

Figure 14 – Resulting closed-loop states trajectories with the robust recursive regulator for the shown Markov chain realization.

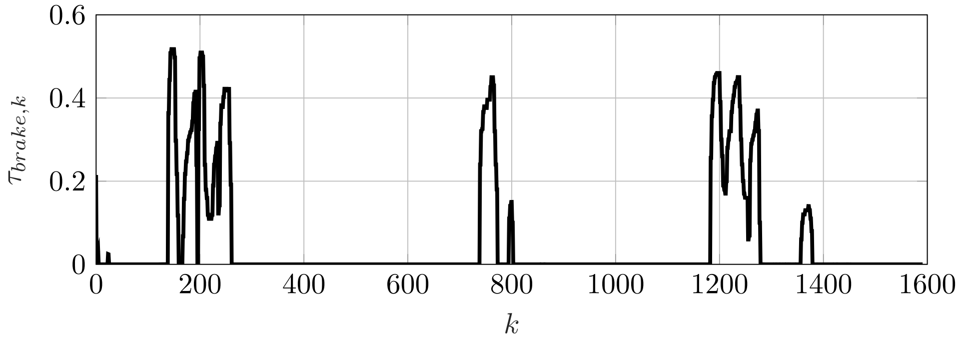


Source: author.

Figure 15 – Resulting positions of accelerator and brake pedals with the robust recursive regulator.



(a) Accelerator pedal position.



(b) Brake pedal position.

Source: author.

6 ROBUST REGULATOR FOR UNCERTAIN DMJLS WITH UNOBSERVED CHAIN

The motivation behind this chapter is related to the difficulty in detecting how Markov mode describes the dynamics of the system at a given instant. In many practical situations, the Markov chain is partially observed if the information about the jump parameter is intermittent; or totally unobserved if the information about the jump parameter is never available. Examples of physical systems with such characteristics encompass cyber security of microgrids (LIU; SIANO; WANG, 2020), robot manipulators (CHE; ZHU; ZHOU, 2021), cruise control for unmanned aerial vehicles (LI et al., 2021), and so forth. We are interested, therefore, in developing mode-independent recursive control approaches for DMJLS subject to polytopic uncertainties. Valuable results were reported in the literature to solve this class of problems. We can cite, for instance, (COSTA; FRAGOSO; TODOROV, 2015) regarding detector-based H_2 controller synthesis; (TODOROV; FRAGOSO, 2016) for mixed H_2/H_∞ synthesis; (SHEN et al., 2014) concerning the dissipative control problem; and (SOUZA, 2005) discussing the H_∞ control approach.

That said, we focus on the regulation of DMJLS with totally unobserved Markov chain and polytopic uncertainties affecting the matrices of the state-space model. We use auxiliary variables to build an augmented system and formulate a regularized least-squares problem. The related cost function is penalized to include all vertices of the DMJLS at once. We achieve a robust recursive solution, from which we obtain the feedback gains that stabilize the closed-loop augmented system. Furthermore, we assess the performance of the proposed solution in two numerical examples.

6.1 Augmented System - Problem Formulation

Let us again consider the DMJLS

$$x_{k+1} = (F_{\theta_k,k} + \delta F_{\theta_k,k})\, x_k + (G_{\theta_k,k} + \delta G_{\theta_k,k})\, u_k, \tag{173}$$

where $k = 0, ..., N-1$, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $F_{\theta_k,k} \in \mathbb{R}^{n\times n}$ and $G_{\theta_k,k} \in \mathbb{R}^{n\times m}$ are nominal system and input matrices, respectively, $\theta = \{\theta_0, ..., \theta_{N-1}\}$ is a Markov chain with modes $\theta_k \in \Theta = \{1, ..., s\}$. The transition probability matrix is known and defined by $\mathbb{P} = [p_{ij}] \in \mathbb{R}^{s\times s}$ such that

$$\begin{gathered}\text{Prob}(\theta_{k+1} = j|\theta_k = i) = p_{ij}, \quad \text{Prob}(\theta_0 = i) = p_{i,0},\\ \sum_{j=1}^{s} p_{ij} = 1, \quad 0 \le p_{ij} \le 1,\end{gathered} \tag{174}$$

whereas the uncertainties $\{\delta F_{\theta_k,k}, \delta G_{\theta_k,k}\}$ are defined as

$$\begin{bmatrix}\delta F_{\theta_k,k} & \delta G_{\theta_k,k}\end{bmatrix} = \sum_{l=1}^{V} \alpha_{i,k} \begin{bmatrix} F^{(l)}_{\theta_k,k} & G^{(l)}_{\theta_k,k}\end{bmatrix}, \tag{175}$$

with known vertices $F^{(l)}_{\theta_k,k} \in \mathbb{R}^{n\times n}$ and $G^{(l)}_{\theta_k,k} \in \mathbb{R}^{n\times m}$, and $\alpha_k = \begin{bmatrix}\alpha_{1,k} \dots \alpha_{V,k}\end{bmatrix}^T$ belong to the unit simplex

$$\Lambda_V = \left\{ \alpha_k \in \mathbb{R}^V \;:\; \alpha_{l,k} \geq 0,\, \sum_{l=1}^{V} \alpha_{l,k} = 1 \right\}. \tag{176}$$

Suppose all states x_k are observed at every instant k and the system evolves from $\{x_0, \theta_0\}$. However, now we assume that modes θ_k, $k = 1, \dots, N$, are not available to the controller. Hence we say the Markov chain is totally unobserved. That being said, the optimization problem

$$\min_{u_k,\, x_{k+1}} \; \max_{\delta F_{\theta_k,k},\, \delta G_{\theta_k,k}} \left\{ \mathbb{E}\Big\{ \|x_N\|^2_{\Psi_{\theta_N,N}} + \sum_{t=0}^{N-1} \Big(\|x_t\|^2_{Q_{\theta_t,t}} + \|u_t\|^2_{R_{\theta_t,t}} \Big) \mid \mathcal{S}_t \Big\} \right\}, \tag{177}$$

subject to (173),

where $\mathcal{S}_t = \{\theta_t = i, x_t\}$, with $i \in \Theta$, cannot be used to design a regulator for (173) since it requires the information regarding the active Markov mode through $\mathcal{S}_t$. Therefore, in the next section, we describe how to yield an augmented version of the DMJLS (173) suitable to approach the robust regulation problem.

Let us consider the Dirac measure, $\mathcal{I}_{\{\theta_k = i\}}$ defined by

$$\mathcal{I}_{\{\mathcal{V}\}}(\nu) = \begin{cases} 1, & \text{if } \nu \in \mathcal{V}, \\ 0, & \text{otherwise,} \end{cases}$$

which, in our specific case, we rewrite as

$$\mathcal{I}_{\{\theta_k = i\}} = \begin{cases} 1, & \text{if } \theta_k = i, \\ 0, & \text{otherwise.} \end{cases} \tag{178}$$

A similar deterministic approach was proposed by Costa (1994) to deal with the minimum mean-square error estimation problem. We introduce now the following auxiliary variables

$$z_{i,k} = \mathcal{I}_{\{\theta_k = i\}} x_k \in \mathbb{R}^n, \qquad v_{i,k} = \mathcal{I}_{\{\theta_k = i\}} u_k \in \mathbb{R}^m,$$

$$z_k = \begin{bmatrix} z_{1,k} \\ \vdots \\ z_{s,k} \end{bmatrix} \in \mathbb{R}^{sn}, \qquad v_k = \begin{bmatrix} v_{1,k} \\ \vdots \\ v_{s,k} \end{bmatrix} \in \mathbb{R}^{sm}. \tag{179}$$

Notice that we recover the original variables x_k and u_k from z_k and v_k, respectively, by doing

$$x_k = \sum_{i=1}^{s} z_{i,k}, \quad \text{and} \quad u_k = \sum_{i=1}^{s} v_{i,k}. \tag{180}$$

From (179) and (173) we have

$$x_{k+1} = \sum_{i=1}^{s} (F_{i,k} + \delta F_{i,k}) z_{i,k} + \sum_{i=1}^{s} (G_{i,k} + \delta G_{i,k}) v_{i,k}. \tag{181}$$

Given that $z_{j,k+1} = \mathcal{I}_{\{\theta_{k+1}=j\}} x_{k+1}$, based on (181) we also have

$$z_{j,k+1} = \left(\sum_{i=1}^{s} (F_{i,k} + \delta F_{i,k}) z_{i,k}\right) \mathcal{I}_{\{\theta_{k+1}=j\}} + \left(\sum_{i=1}^{s} (G_{i,k} + \delta G_{i,k}) v_{i,k}\right) \mathcal{I}_{\{\theta_{k+1}=j\}}. \quad (182)$$

To simplify the notation of the Dirac measure, from now on we denote $\mathcal{I}_{\{\theta_{k+1}=j\}}$ as $\mathcal{I}_{\{j\}}$. Then, by adding

$$\sum_{i=1}^{s} p_{ij} F_{i,k} z_{i,k} + \sum_{i=1}^{s} p_{ij} G_{i,k} v_{i,k}$$

to both sides of (182), we have

$$\begin{aligned} z_{j,k+1} = \sum_{i=1}^{s} p_{ij} F_{i,k} z_{i,k} + \sum_{i=1}^{s} \left((\mathcal{I}_{\{j\}} - p_{ij}) F_{i,k} + \mathcal{I}_{\{j\}} \delta F_{i,k}\right) z_{i,k} + \\ \sum_{i=1}^{s} p_{ij} G_{i,k} v_{i,k} + \sum_{i=1}^{s} \left((\mathcal{I}_{\{j\}} - p_{ij}) G_{i,k} + \mathcal{I}_{\{j\}} \delta G_{i,k}\right) v_{i,k}. \quad (183) \end{aligned}$$

Since $z_{k+1} = \begin{bmatrix} z_{1,k+1}^T & \dots & z_{s,k+1}^T \end{bmatrix}^T$, by combining (183) with (179) we obtain

$$\begin{aligned} \begin{bmatrix} z_{1,k+1} \\ \vdots \\ z_{s,k+1} \end{bmatrix} &= \begin{bmatrix} \sum_{i=1}^{s} p_{i1} F_{i,k} z_{i,k} \\ \vdots \\ \sum_{i=1}^{s} p_{is} F_{i,k} z_{i,k} \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{s} \left((\mathcal{I}_{\{1\}} - p_{i1}) F_{i,k} + \mathcal{I}_{\{1\}} \delta F_{i,k}\right) z_{i,k} \\ \vdots \\ \sum_{i=1}^{s} \left((\mathcal{I}_{\{s\}} - p_{is}) F_{i,k} + \mathcal{I}_{\{s\}} \delta F_{i,k}\right) z_{i,k} \end{bmatrix} \\ &+ \begin{bmatrix} \sum_{i=1}^{s} p_{i1} G_{i,k} v_{i,k} \\ \vdots \\ \sum_{i=1}^{s} p_{is} G_{i,k} v_{i,k} \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{s} \left((\mathcal{I}_{\{1\}} - p_{i1}) G_{i,k} + \mathcal{I}_{\{1\}} \delta G_{i,k}\right) v_{i,k} \\ \vdots \\ \sum_{i=1}^{s} \left((\mathcal{I}_{\{s\}} - p_{is}) G_{i,k} + \mathcal{I}_{\{s\}} \delta G_{i,k}\right) v_{i,k} \end{bmatrix}. \quad (184) \end{aligned}$$

Observe that

$$\begin{aligned} \begin{bmatrix} \sum_{i=1}^{s} p_{i1} F_{i,k} z_{i,k} \\ \vdots \\ \sum_{i=1}^{s} p_{is} F_{i,k} z_{i,k} \end{bmatrix} &= \begin{bmatrix} p_{11} F_{1,k} & p_{21} F_{2,k} & \dots & p_{s1} F_{s,k} \\ p_{12} F_{1,k} & p_{22} F_{2,k} & \dots & p_{s2} F_{s,k} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1s} F_{1,k} & p_{2s} F_{2,k} & \dots & p_{ss} F_{s,k} \end{bmatrix} \begin{bmatrix} z_{1,k} \\ z_{2,k} \\ \vdots \\ z_{s,k} \end{bmatrix}, \\ &= \mathcal{A}_k z_k, \end{aligned}$$

where $\mathcal{A}_k = (\mathbb{P}^T \otimes I_n)$ **diag**$\{F_{1,k}, \dots, F_{s,k}\}$, $\mathcal{A}_k \in \mathbb{R}^{sn \times sn}$. Similarly, we have

$$\begin{bmatrix} \sum_{i=1}^{s} p_{i1} G_{i,k} v_{i,k} \\ \vdots \\ \sum_{i=1}^{s} p_{is} G_{i,k} v_{i,k} \end{bmatrix} = \mathcal{B}_k v_k,$$

where $\mathcal{B}_k = (\mathbb{P}^T \otimes I_n)$ **diag**$\{G_{1,k}, \dots, G_{s,k}\}$, $\mathcal{B}_k \in \mathbb{R}^{sn \times sm}$. Moreover, we can make

$$\begin{bmatrix} \sum_{i=1}^{s} \left((\mathcal{I}_{\{1\}} - p_{i1}) F_{i,k} + \mathcal{I}_{\{1\}} \delta F_{i,k}\right) z_{i,k} \\ \vdots \\ \sum_{i=1}^{s} \left((\mathcal{I}_{\{s\}} - p_{is}) F_{i,k} + \mathcal{I}_{\{s\}} \delta F_{i,k}\right) z_{i,k} \end{bmatrix} = \delta \mathcal{A}_k z_k,$$

and

$$\begin{bmatrix} \sum_{i=1}^{s} \left((\mathcal{I}_{\{1\}} - p_{i1}) G_{i,k} + \mathcal{I}_{\{1\}} \delta G_{i,k} \right) v_{i,k} \\ \vdots \\ \sum_{i=1}^{s} \left((\mathcal{I}_{\{s\}} - p_{is}) G_{i,k} + \mathcal{I}_{\{s\}} \delta G_{i,k} \right) v_{i,k} \end{bmatrix} = \delta\mathcal{B}_k v_k.$$

Therefore, we can write (184) as the augmented version of the DMJLS (173) in terms of the auxiliary variables z_k and v_k in the following way:

$$z_{k+1} = (\mathcal{A}_k + \delta\mathcal{A}_k) z_k + (\mathcal{B}_k + \delta\mathcal{B}_k) v_k, \tag{185}$$

with $\{\delta\mathcal{A}_k, \delta\mathcal{B}_k\}$ modeled as

$$\begin{bmatrix} \delta\mathcal{A}_k & \delta\mathcal{B}_k \end{bmatrix} = \mathcal{H} \sum_{l=1}^{V} \overline{\alpha}_{l,k} \begin{bmatrix} \mathcal{A}_k^{(l)} & \mathcal{B}_k^{(l)} \end{bmatrix}, \tag{186}$$

where

$$\mathcal{A}_k^{(l)} = \textbf{diag}\left\{ \begin{bmatrix} F_{1,k} \\ F_{1,k}^{(l)} \end{bmatrix}, \ldots, \begin{bmatrix} F_{s,k} \\ F_{s,k}^{(l)} \end{bmatrix} \right\}, \quad \mathcal{B}_k^{(l)} = \textbf{diag}\left\{ \begin{bmatrix} G_{1,k} \\ G_{1,k}^{(l)} \end{bmatrix}, \ldots, \begin{bmatrix} G_{s,k} \\ G_{s,k}^{(l)} \end{bmatrix} \right\},$$

$$\overline{\alpha}_{l,k} = \alpha_{l,k} \begin{bmatrix} \textbf{diag}\{\mathcal{Y}_{11}, \mathcal{Y}_{21}, \ldots, \mathcal{Y}_{s1}\} \\ \vdots \\ \textbf{diag}\{\mathcal{Y}_{1s}, \mathcal{Y}_{2s}, \ldots, \mathcal{Y}_{ss}\} \end{bmatrix} \in \mathbb{R}^{2s^2n \times 2sn}, \quad \mathcal{Y}_{ij} = \begin{bmatrix} (\mathcal{I}_{\{j\}} - p_{ij}) I_n & 0 \\ 0 & \mathcal{I}_{\{j\}} I_n \end{bmatrix},$$

$$\mathcal{H} = \textbf{diag}\{\hat{I}_{2s}^T, \ldots, \hat{I}_{2s}^T\} \in \mathbb{R}^{sn \times 2s^2 n}.$$

We assume that $\mathcal{I}_{\{j\}}$ and the polytope coefficients $\alpha_{l,k}$ are unknown, and we grouped them into $\overline{\alpha}_{l,k}$. As we will describe in the next sections, the proposed solution is independent of $\overline{\alpha}_{l,k}$. It is possible, therefore, to compute the feedback gains and stabilize the closed-loop system irrespective of the Markov chain being unobserved.

Let us consider the min-max problem

$$\min_{z_{k+1}, v_k} \max_{\delta\mathcal{A}_k, \delta\mathcal{B}_k} \left\{ \|z_N\|_{\mathbb{X}_N}^2 + \sum_{t=0}^{N-1} \left(\|z_t\|_{\mathcal{Q}_t}^2 + \|v_t\|_{\mathcal{R}_t}^2 \right) \right\}, \tag{187}$$

subject to

$$\mathbb{1}_V \otimes z_{k+1} = \begin{bmatrix} \mathcal{A}_k + V\delta\mathcal{A}_k^{(1)} \\ \vdots \\ \mathcal{A}_k + V\delta\mathcal{A}_k^{(V)} \end{bmatrix} z_k + \begin{bmatrix} \mathcal{B}_k + V\delta\mathcal{B}_k^{(1)} \\ \vdots \\ \mathcal{B}_k + V\delta\mathcal{B}_k^{(V)} \end{bmatrix} v_k, \tag{188}$$

where $\delta\mathcal{A}_k^{(l)} = \mathcal{H}\overline{\alpha}_{l,k}\mathcal{A}_k^{(l)}$, and $\delta\mathcal{B}_k^{(l)} = \mathcal{H}\overline{\alpha}_{l,k}\mathcal{B}_k^{(l)}$. Based upon the Principle of Optimality (BERTSEKAS, 2005), we then divide (187) into $N-1$ one-step problems of the form

$$\min_{z_{k+1}, v_k} \max_{\delta\mathcal{A}_k, \delta\mathcal{B}_k} \left\{ J_k = \|z_{k+1}\|_{\mathbb{X}_{k+1}}^2 + \|z_k\|_{\mathcal{Q}_k}^2 + \|v_k\|_{\mathcal{R}_k}^2 \right\}, \tag{189}$$

subject to (188), $\quad k = N-1, \ldots, 0,$

with

$$\mathcal{Q}_k = \textbf{diag}\{Q_{1,k},\ \ldots,\ Q_{s,k}\},\ \mathcal{R}_k = \textbf{diag}\{R_{1,k},\ \ldots,\ R_{s,k}\},$$
$$\mathbb{X}_{k+1} = \textbf{diag}\{\Psi_{1,k+1},\ \ldots,\ \Psi_{s,k+1}\},\ \Psi_{i,k+1} = \sum_{j=1}^{s} p_{ij} P_{j,k+1}.$$

Remark 6.1. *Notice that*

$$\begin{aligned}
\sum_{l=1}^{V} V\delta\mathcal{A}^{(l)} z_k &= V \begin{bmatrix} \left(\sum_{i=1}^{s}(\mathcal{I}_{\{1\}} - p_{i1})F_{i,k}z_{i,k}\right)\left(\sum_{l=1}^{V}\alpha_{l,k}\right) + \sum_{i=1}^{s}\mathcal{I}_{\{1\}}\left(\sum_{l=1}^{V}\alpha_{l,k}F_{i,k}^{(l)}\right)z_{i,k} \\ \vdots \\ \left(\sum_{i=1}^{s}(\mathcal{I}_{\{s\}} - p_{is})F_{i,k}z_{i,k}\right)\left(\sum_{l=1}^{V}\alpha_{l,k}\right) + \sum_{i=1}^{s}\mathcal{I}_{\{s\}}\left(\sum_{l=1}^{V}\alpha_{l,k}F_{i,k}^{(l)}\right)z_{i,k} \end{bmatrix}, \\
&= V \begin{bmatrix} \sum_{i=1}^{s}(\mathcal{I}_{\{1\}} - p_{i1})F_{i,k}z_{i,k} + \sum_{i=1}^{s}\mathcal{I}_{\{1\}}\delta F_{i,k}z_{i,k} \\ \vdots \\ \sum_{i=1}^{s}(\mathcal{I}_{\{s\}} - p_{is})F_{i,k}z_{i,k} + \sum_{i=1}^{s}\mathcal{I}_{\{s\}}\delta F_{i,k}z_{i,k} \end{bmatrix}, \\
&= V \begin{bmatrix} \sum_{i=1}^{s}\left((\mathcal{I}_{\{1\}} - p_{i1})F_{i,k} + \mathcal{I}_{\{1\}}\delta F_{i,k}\right)z_{i,k} \\ \vdots \\ \sum_{i=1}^{s}\left((\mathcal{I}_{\{s\}} - p_{is})F_{i,k} + \mathcal{I}_{\{s\}}\delta F_{i,k}\right)z_{i,k} \end{bmatrix}, \\
&= V\delta\mathcal{A}_k z_k.
\end{aligned}$$

Similarly, we have $\sum_{l=1}^{V} V\delta\mathcal{B}^{(l)}v_k = V\delta\mathcal{B}_k v_k$. *Therefore, it is possible to retrieve* (185) *from the constraints* (188). *We multiply both sides of* (188) *on the left by* $\hat{I}_V^T$ *to obtain* $Vz_{k+1} = V(\mathcal{A}_k + \delta\mathcal{A}_k)z_k + V(\mathcal{B}_k + \delta\mathcal{B}_k)v_k$, *which corresponds to* (185).

Let us now define the following penalty function based on (188):

$$\overline{\mathcal{C}}(z_{k+1}, v_k) = \bar{g}(z_{k+1}, v_k)^T\, \mu I\, \bar{g}(z_{k+1}, v_k), \tag{190}$$

with fixed penalty $\mu > 0$, and

$$\bar{g}(z_{k+1}, v_k) = \mathbb{1}_V \otimes z_{k+1} - \begin{bmatrix} \mathcal{A}_k + V\delta\mathcal{A}_k^{(1)} \\ \vdots \\ \mathcal{A}_k + V\delta\mathcal{A}_k^{(V)} \end{bmatrix} z_k - \begin{bmatrix} \mathcal{B}_k + V\delta\mathcal{B}_k^{(1)} \\ \vdots \\ \mathcal{B}_k + V\delta\mathcal{B}_k^{(V)} \end{bmatrix} v_k.$$

By adding (190) and J_k, we attain a new penalized cost function and yield the following unconstrained counterpart of (189):

$$\min_{z_{k+1}, v_k} \max_{\delta\mathcal{A}_k, \delta\mathcal{B}_k} \quad \{\mathcal{J}_k(z_{k+1}, v_k, \delta\mathcal{A}_k, \delta\mathcal{B}_k)\}, \tag{191}$$

where

$$\begin{aligned}
\mathcal{J}_k(z_{k+1}, v_k, \delta\mathcal{A}_k, \delta\mathcal{B}_k) &= \begin{bmatrix} z_{k+1} \\ v_k \end{bmatrix}^T \begin{bmatrix} \mathbb{X}_{k+1} & 0 \\ 0 & \mathcal{R}_k \end{bmatrix} \begin{bmatrix} z_{k+1} \\ v_k \end{bmatrix} \\
&+ \left\{ \left(\begin{bmatrix} 0 & 0 \\ I_{sVn} & -\mathcal{B}_k \\ \vdots & \vdots \\ I_{sVn} & -\mathcal{B}_k \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -V\delta\mathcal{B}_k^{(1)} \\ \vdots & \vdots \\ 0 & -V\delta\mathcal{B}_k^{(V)} \end{bmatrix} \right) \begin{bmatrix} z_{k+1} \\ v_k \end{bmatrix} - \left(\begin{bmatrix} -I_{sn} \\ \mathcal{A}_k \\ \vdots \\ \mathcal{A}_k \end{bmatrix} + \begin{bmatrix} 0 \\ V\delta\mathcal{A}_k^{(1)} \\ \vdots \\ V\delta\mathcal{A}_k^{(V)} \end{bmatrix} \right) z_k \right\}^T \begin{bmatrix} \mathcal{Q}_k & 0 \\ 0 & \mu I_{sVn} \end{bmatrix} \left\{ \bullet \right\}.
\end{aligned} \tag{192}$$

The penalty parameter $\mu > 0$ can be seen as a weight related to the equality constraints (188). We shall design a recursive solution for (191), from which we extract the signals $v_k = \mathcal{K}_k z_k$ to minimize (192) and, thus, ensure the closed-loop stability of the DMJLS in (185).

Remark 6.2. *Even though* (191) *relates to the auxiliary variables* z_{k+1} *and* v_k*, we obtain* x_{k+1} *and* u_k *of the original system through* (180)*. Therefore, if* $\mathcal{K}_k$ *is such that* $v_k = \mathcal{K}_k z_k$ *makes* $z_k \to 0$*, then we expect that* $x_k \to 0$*. In this case, the closed-loop DMJLS* (78) *is stable.*

In the following section, we present the designed recursive solution for problem (191), from which it is possible to compute the feedback gains $\mathcal{K}_k$.

6.2 RLQR for DMJLS Subject to Polytopic Uncertainties and Unobserved Chain

We present the robust recursive solution for the optimization problem (191)–(192) in the next lemma.

Lemma 6.1. *Consider the problem* (191)–(192) *and assume known matrices* $Q_{i,k} > 0$*,* $R_{i,k} > 0$*, and* $P_{i,N} > 0$*,* $i \in \Theta$*. For* $k = N-1, \ldots, 0$*, and known fixed scalar* $\mu > 0$*, the solution is recursively computed by*

$$\begin{bmatrix} z^*_{k+1} \\ v^*_k \\ \mathcal{J}_k(z^*_{k+1}, v^*_k) \end{bmatrix} = \begin{bmatrix} I_{sn} & 0 & 0 \\ 0 & I_{sm} & 0 \\ 0 & 0 & z_k^T \end{bmatrix} \begin{bmatrix} \mathcal{L}_k \\ \mathcal{K}_k \\ \mathscr{P}_k \end{bmatrix} z_k, \tag{193}$$

with

$$\begin{bmatrix} \mathcal{L}_k \\ \mathcal{K}_k \\ \mathscr{P}_k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I_{sn} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{sm} \\ 0 & 0 & -I_{sn} & \hat{\mathcal{A}}_k^T & \hat{E}_{\mathcal{A}_k}^T & 0 & 0 \end{bmatrix} \times$$
$$\begin{bmatrix} \mathbb{X}_{k+1}^T & 0 & 0 & 0 & 0 & I_{sn} & 0 \\ 0 & \mathcal{R}_k^{-1} & 0 & 0 & 0 & 0 & I_{sm} \\ 0 & 0 & \mathcal{Q}_k^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & \hat{I}_{sn} & -\hat{\mathcal{B}}_k \\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{E}_{\mathcal{B}_k} \\ I_{sn} & 0 & 0 & \hat{I}_{sn}^T & 0 & 0 & 0 \\ 0 & I_{sm} & 0 & -\hat{\mathcal{B}}_k^T & -\hat{E}_{\mathcal{B}_k}^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I_{sn} \\ \hat{\mathcal{A}}_k \\ \hat{E}_{\mathcal{A}_k} \\ 0 \\ 0 \end{bmatrix}, \tag{194}$$

where $\mathscr{P}_k = \textbf{diag}\{P_{1,k}, \ldots, P_{s,k}\}$, $\Phi = \mu^{-1}(1-\beta^{-1})I_{sVn}$, $\hat{\mathcal{A}}_k = \mathbb{1}_V \otimes \mathcal{A}_k$, $\hat{\mathcal{B}}_k = \mathbb{1}_V \otimes \mathcal{B}_k$, $\Sigma = (2s\beta\mu)^{-1}I_{2sVn}$, $\beta > 1$,

$$\hat{E}_{\mathcal{A}_k} = V \begin{bmatrix} E^{(1)}_{\mathcal{A}_k} \\ \vdots \\ E^{(V)}_{\mathcal{A}_k} \end{bmatrix}, \quad \hat{E}_{\mathcal{B}_k} = V \begin{bmatrix} E^{(1)}_{\mathcal{B}_k} \\ \vdots \\ E^{(V)}_{\mathcal{B}_k} \end{bmatrix}, \quad \hat{I}_{sn} = \begin{bmatrix} I_{sn} \\ \vdots \\ I_{sn} \end{bmatrix} \in \mathbb{R}^{Vsn \times sn}.$$

Proof. We used the penalty parameter μ to translate (189) into its unconstrained counterpart (191). As (191) is a special case of the regularized least-squares problem outlined in Section 2.1.3, it is possible to make the correspondences

$$J \leftarrow \mathcal{J}_k(z_{k+1}, v_k, \delta\mathcal{A}_k, \delta\mathcal{B}_k),\ x \leftarrow \begin{bmatrix} z_{k+1} \\ v_k \end{bmatrix},\ \mathscr{Q} \leftarrow \begin{bmatrix} \mathbb{X}_{k+1} & 0 \\ 0 & \mathcal{R}_k \end{bmatrix},\ \mathscr{W} \leftarrow \begin{bmatrix} \mathcal{Q}_k & 0 \\ 0 & \mu I_{sVn} \end{bmatrix},$$

$$A_0 \leftarrow \begin{bmatrix} 0 & 0 \\ I_{sn} & -\mathcal{B}_k \\ \vdots & \vdots \\ I_{sn} & -\mathcal{B}_k \end{bmatrix},\ \delta A \leftarrow \begin{bmatrix} 0 & 0 \\ 0 & -V\delta\mathcal{B}^{(1)}_k \\ \vdots & \vdots \\ 0 & -V\delta\mathcal{B}^{(V)}_k \end{bmatrix},\ A^{(l)} \leftarrow \begin{bmatrix} 0 & -V\mathcal{B}^{(l)}_k \end{bmatrix},$$

$$b_0 \leftarrow \begin{bmatrix} -I_{sn} \\ \mathcal{A}_k \\ \vdots \\ \mathcal{A}_k \end{bmatrix} z_k,\ \delta b \leftarrow \begin{bmatrix} 0 \\ V\delta\mathcal{A}^{(1)}_k \\ \vdots \\ V\delta\mathcal{A}^{(V)}_k \end{bmatrix} z_k,\ b^{(l)} \leftarrow V\mathcal{A}^{(l)}_k z_k,\ l = 1, \ldots, V,$$

$$M \leftarrow \begin{bmatrix} 0 \\ \textbf{diag}\{\mathcal{H}, \ldots, \mathcal{H}\} \end{bmatrix},\ \Gamma \leftarrow \textbf{diag}\{\bar{\alpha}_{1,k}, \ldots, \bar{\alpha}_{V,k}\}.$$

This matching allows us to base our solution upon the matrix arrangement shown in Lemma 2.4, which yields (193) with $\mathcal{L}_k$, $\mathcal{K}_k$, and $\mathscr{P}_k$ expressed as in (194). The uniqueness of the solution is guaranteed for any positive μ, given that $\mathbb{X}_{k+1} > 0$ and $R_{i,k} > 0$. We also use the reasonable approximation $\hat{\lambda} \approx 2\beta s\mu$, for some scalar $\beta > 1$, to avoid solving an additional optimization problem, by the arguments given in 2.1.3. Finally, Lemma A.3 ensures the existence of the central block matrix in (194). □

By defining a set of simultaneous equations based upon (193)–(194) and item (*iii*) of Lemma 2.4, we produce an equivalent reduced form for $\mathcal{L}_k, \mathcal{K}_k, \mathscr{P}_k\}$ given in Lemma 6.1. We outline this procedure in the next result.

Theorem 6.1. *Consider the problem* (191)–(192) *and assume* $Q_{i,k} > 0$, $R_{i,k} > 0$, $P_{i,N} > 0$, $i \in \Theta$, $\mu > 0$, *and* $\beta > 1$. *For* $k = N-1, \ldots, 0$, *the solution in Lemma* (6.1) *can be equally expressed with matrices* $\mathcal{L}_k$, $\mathcal{K}_k$, *and* $\mathscr{P}_k$ *computed by*

$$\mathcal{L}_k = V\mathbb{X}^{-1}_{k+1}\Omega^{-1}_{k+1}\bar{\mathcal{A}}_k - V^2\mathbb{X}^{-1}_{k+1}\Omega^{-1}_{k+1}\bar{\mathcal{B}}_k\big(I_{sm} - V\bar{\mathcal{B}}^T_k\Omega^{-1}_{k+1}\bar{\mathcal{B}}_k\big)^{-1}\bar{\mathcal{B}}^T_k\Omega^{-1}_{k+1}\bar{\mathcal{A}}_k, \tag{195}$$

$$\mathcal{K}_k = -V\bar{\mathcal{R}}_k\mathcal{B}^T_k(I_{sn} + V\Omega^{-1}_{k+1}\bar{\mathcal{B}}_k\bar{\mathcal{B}}^T_k)^{-1}\Omega^{-1}_{k+1}\bar{\mathcal{A}}_k - \mathcal{R}^{-1}_k\hat{E}^T_{\mathcal{B}_k}(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}^{-1}_k\hat{E}^T_{\mathcal{B}_k})^{-1}\hat{E}_{\mathcal{A}_k}, \tag{196}$$

$$\mathscr{P}_k = \bar{\mathcal{Q}}_k + \bar{\mathcal{A}}^T_k\tilde{\Omega}_{k+1}\bar{\mathcal{A}}_k - \bar{\mathcal{A}}^T_k\tilde{\Omega}_{k+1}\bar{\mathcal{B}}_k(I_{sm} + \bar{\mathcal{B}}^T_k\tilde{\Omega}_{k+1}\bar{\mathcal{B}}_k)^{-1}\bar{\mathcal{B}}^T_k\tilde{\Omega}_{k+1}\bar{\mathcal{A}}_k, \tag{197}$$

where

$$\begin{aligned}
&\Omega_{k+1} = \varphi + V\mathbb{X}_{k+1}^{-1}, \quad \varphi = \mu^{-1}(1-\beta^{-1})I_{sn}, \quad \tilde{\Omega}_{k+1} = V\Omega_{k+1}^{-1},\\
&\bar{\mathcal{A}}_k = \mathcal{A}_k - \mathcal{B}_k\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{A}_k},\\
&\bar{\mathcal{R}}_k = \mathcal{R}_k^{-1}(I_{sm} - \hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}),\\
&\bar{\mathcal{Q}}_k = \mathcal{Q}_k + \hat{E}_{\mathcal{A}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{A}_k}, \quad \bar{\mathcal{B}}_k = \mathcal{B}_k\bar{\mathcal{R}}_k^{1/2}.
\end{aligned}$$

Proof. Based upon (194), define the matrix $\mathscr{N}$ such that

$$\underbrace{\begin{bmatrix} \mathbb{X}_{k+1}^T & 0 & 0 & 0 & 0 & I_{sn} & 0\\ 0 & \mathcal{R}_k^{-1} & 0 & 0 & 0 & 0 & I_{sm}\\ 0 & 0 & \mathcal{Q}_k^{-1} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \Phi & 0 & \hat{I}_{sn} & -\hat{\mathcal{B}}_k\\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{E}_{\mathcal{B}_k}\\ I_{sn} & 0 & 0 & \hat{I}_{sn}^T & 0 & 0 & 0\\ 0 & I_{sm} & 0 & -\hat{\mathcal{B}}_k^T & -\hat{E}_{\mathcal{B}_k}^T & 0 & 0 \end{bmatrix}}_{\mathscr{N}}^{-1} \begin{bmatrix} 0\\ 0\\ -I_{sn}\\ \hat{\mathcal{A}}_k\\ \hat{E}_{\mathcal{A}_k}\\ 0\\ 0 \end{bmatrix} = \begin{bmatrix} W_1\\ W_2\\ W_3\\ \overline{W}_4\\ W_5\\ \mathcal{L}_k\\ \mathcal{K}_k \end{bmatrix}, \tag{198}$$

with auxiliary variables W_1, W_2, W_3, $\overline{W}_4$, and W_5, where $\overline{W}_4 = \mathbb{1}_V \otimes W_4$, $W_4 \in \mathbb{R}^{sn\times sn}$. We multiply both sides of (198) on the left by $\mathscr{N}$ to yield the following set of simultaneous equations:

$$\mathbb{X}_{k+1}^{-1}W_1 + \mathcal{L}_k = 0, \tag{199a}$$
$$\mathcal{R}_k^{-1}W_2 + \mathcal{K}_k = 0, \tag{199b}$$
$$\mathcal{Q}_k^{-1}W_3 = -I_{sn}, \tag{199c}$$
$$\Phi\overline{W}_4 + \hat{I}_{sn}\mathcal{L}_k - \hat{\mathcal{B}}_k\mathcal{K}_k = \hat{\mathcal{A}}_k, \tag{199d}$$
$$\Sigma W_5 - \hat{E}_{\mathcal{B}_k}\mathcal{K}_k = \hat{E}_{\mathcal{A}_k}, \tag{199e}$$
$$W_1 + \hat{I}_{sn}^T\overline{W}_4 = 0, \tag{199f}$$
$$W_2 - \hat{\mathcal{B}}_k^T\overline{W}_4 - \hat{E}_{\mathcal{B}_k}^T W_5 = 0. \tag{199g}$$

Additionally, we substitute (198) into (194) to produce

$$\begin{aligned}
\mathscr{P}_k &= -I_{sn}W_3 + \hat{\mathcal{A}}_k^T\overline{W}_4 + \hat{E}_{\mathcal{A}_k}^T W_5,\\
\mathscr{P}_k &= -I_{sn}W_3 + V\mathcal{A}_k^T W_4 + \hat{E}_{\mathcal{A}_k}^T W_5.
\end{aligned} \tag{200}$$

Let us now solve (199)–(200) for $\{W_1, W_2, W_3, W_4, W_5, \mathcal{L}_k, \mathcal{K}_k, \mathscr{P}_k\}$, to ultimately attain (195), (196) and (197). First, from (199a), (199b) and (199c), we have, respectively,

$$W_1 = -\mathbb{X}_{k+1}\mathcal{L}_k, \tag{201}$$
$$W_2 = -\mathcal{R}_k\mathcal{K}_k, \tag{202}$$
$$W_3 = -\mathcal{Q}_k. \tag{203}$$

Moreover, by combining (199f) and (201) we yield

$$-\mathbb{X}_{k+1}\mathcal{L}_k + \begin{bmatrix} I_{sn} & \dots & I_{sn} \end{bmatrix} \begin{bmatrix} W_4 \\ \vdots \\ W_4 \end{bmatrix} = 0,$$

$$-\mathbb{X}_{k+1}\mathcal{L}_k + VW_4 = 0,$$

$$\mathcal{L}_k = V\mathbb{X}_{k+1}^{-1}W_4. \tag{204}$$

Define $\varphi = \mu^{-1}(1-\beta^{-1})I_n$, such that from (199d) we have

$$\begin{bmatrix} \varphi & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \varphi \end{bmatrix} \begin{bmatrix} W_4 \\ \vdots \\ W_4 \end{bmatrix} + \begin{bmatrix} I_{sn} \\ \vdots \\ I_{sn} \end{bmatrix} \mathcal{L}_k - \begin{bmatrix} \mathcal{B}_k \\ \vdots \\ \mathcal{B}_k \end{bmatrix} \mathcal{K}_k = \begin{bmatrix} \mathcal{A}_k \\ \vdots \\ \mathcal{A}_k, \end{bmatrix},$$

$$\varphi W_4 + \mathcal{L}_k - \mathcal{B}_k\mathcal{K}_k = \mathcal{A}_k, \tag{205}$$

and substitute (204) into (205) such that

$$\varphi W_4 + V\mathbb{X}_{k+1}^{-1}W_4 - \mathcal{B}_k\mathcal{K}_k = \mathcal{A}_k,$$

$$(\varphi + V\mathbb{X}_{k+1}^{-1})W_4 - \mathcal{B}_k\mathcal{K}_k = \mathcal{A}_k,$$

$$\Omega_{k+1}W_4 - \mathcal{B}_k\mathcal{K}_k = \mathcal{A}_k, \tag{206}$$

where $\Omega_{k+1} = (\varphi + V\mathbb{X}_{k+1}^{-1})$. Now, combine (199g) and (202) to produce

$$W_2 - \begin{bmatrix} \mathcal{B}_k^T & \dots & \mathcal{B}_k^T \end{bmatrix} \begin{bmatrix} W_4 \\ \vdots \\ W_4 \end{bmatrix} - \hat{E}_{\mathcal{B}_k}^T W_5 = 0,$$

$$-\mathcal{R}_k\mathcal{K}_k - V\mathcal{B}_k^T W_4 - \hat{E}_{\mathcal{B}_k}^T W_5 = 0,$$

$$\mathcal{K}_k = -V\mathcal{R}_k^{-1}\mathcal{B}_k^T W_4 - \mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T W_5, \tag{207}$$

and substitute (207) into (206) such that

$$\Omega_{k+1}W_4 + \mathcal{B}_k V\mathcal{R}_k^{-1}\mathcal{B}_k^T W_4 + \mathcal{B}_k\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T W_5 = \mathcal{A}_k,$$

$$(\Omega_{k+1} + V\mathcal{B}_k\mathcal{R}_k^{-1}\mathcal{B}_k^T)W_4 + \mathcal{B}_k R_k^{-1}\hat{E}_{\mathcal{B}_k}^T W_5 = \mathcal{A}_k. \tag{208}$$

Place (207) into (199e) and yield

$$\Sigma W_5 + \hat{E}_{\mathcal{B}_k}V\mathcal{R}_k^{-1}\mathcal{B}_k^T W_4 + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T W_5 = \hat{E}_{\mathcal{A}_k},$$

$$(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)W_5 + \hat{E}_{\mathcal{B}_k}V\mathcal{R}_k^{-1}\mathcal{B}_k^T W_4 = \hat{E}_{\mathcal{A}_k}, \tag{209}$$

$$W_5 = (\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}(\hat{E}_{\mathcal{A}_k} - \hat{E}_{\mathcal{B}_k}V\mathcal{R}_k^{-1}\mathcal{B}_k^T W_4). \tag{210}$$

Substitute (210) into (208) to obtain

$$(\Omega_{k+1} + V\mathcal{B}_k\mathcal{R}_k^{-1}\mathcal{B}_k^T)W_4 + \mathcal{B}_k\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}(\hat{E}_{\mathcal{A}_k} - V\hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\mathcal{B}_k^T W_4) = \mathcal{A}_k,$$

$$\big(\Omega_{k+1} + V\mathcal{B}_k\mathcal{R}_k^{-1}\mathcal{B}_k^T - V\mathcal{B}_k\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\mathcal{B}_k^T\big)W_4 = \mathcal{A}_k - \mathcal{B}_k\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{A}_k},$$

$$\big(\Omega_{k+1} + V\mathcal{B}_k\mathcal{R}_k^{-1}(I_{sm} - \hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1})\mathcal{B}_k^T\big)W_4 = \mathcal{A}_k - \mathcal{B}_k\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{A}_k},$$

and define the auxiliary matrices

$$\begin{aligned}
\bar{\mathcal{A}}_k &= \mathcal{A}_k - \mathcal{B}_k\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{A}_k},\\
\bar{\mathcal{R}}_k &= \mathcal{R}_k^{-1}(I_{sm} - \hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}),\\
\bar{\mathcal{B}}_k &= \mathcal{B}_k\bar{\mathcal{R}}_k^{1/2},
\end{aligned}$$

such that

$$\begin{aligned}
(\Omega_{k+1} + V\mathcal{B}_k\bar{\mathcal{R}}_k\mathcal{B}_k^T)W_4 &= \bar{\mathcal{A}}_k,\\
(\Omega_{k+1} + V\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T)W_4 &= \bar{\mathcal{A}}_k,\\
W_4 &= (\Omega_{k+1} + V\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T)^{-1}\bar{\mathcal{A}}_k. \qquad (211)
\end{aligned}$$

Now, substitute (210) and (211) into (207) to produce

$$\begin{aligned}
\mathcal{K}_k &= -V\mathcal{R}_k^{-1}\mathcal{B}_k^T W_4 - \mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}(\hat{E}_{\mathcal{A}_k} - V\hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\mathcal{B}_k^T W_4),\\
\mathcal{K}_k &= -V\mathcal{R}_k^{-1}(I_{sm} - \hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1})\mathcal{B}_k^T W_4\\
&\quad - \mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{A}_k},\\
\mathcal{K}_k &= -V\bar{\mathcal{R}}_k\mathcal{B}_k^T W_4 - \mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{A}_k},\\
\mathcal{K}_k &= -V\bar{\mathcal{R}}_k\mathcal{B}_k^T(\Omega_{k+1} + V\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T)^{-1}\bar{\mathcal{A}}_k - \mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{A}_k}.
\end{aligned}$$

Note that

$$\begin{aligned}
(\Omega_{k+1} + V\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T)^{-1} &= \big(\Omega_{k+1}(I_{sn} + V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T)\big)^{-1},\\
&= (I_{sn} + V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T)^{-1}\Omega_{k+1}^{-1}.
\end{aligned}$$

Therefore,

$$\mathcal{K}_k = -V\bar{\mathcal{R}}_k\mathcal{B}_k^T(I_{sn} + V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T)^{-1}\Omega_{k+1}^{-1}\bar{\mathcal{A}}_k - \mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{A}_k},$$

which corresponds to (196). Next, substitute (211) into (204) and yield

$$\begin{aligned}
\mathcal{L}_k &= V\mathbb{X}_{k+1}^{-1}(\Omega_{k+1} + V\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T)^{-1}\bar{\mathcal{A}}_k,\\
\mathcal{L}_k &= V\mathbb{X}_{k+1}^{-1}(I_{sn} + V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T)^{-1}\Omega_{k+1}^{-1}\bar{\mathcal{A}}_k,\\
\mathcal{L}_k &= V\mathbb{X}_{k+1}^{-1}(I_{sn} + (V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k)I_{sm}\bar{\mathcal{B}}_k^T)^{-1}\Omega_{k+1}^{-1}\bar{\mathcal{A}}_k,\\
\mathcal{L}_k &= V\mathbb{X}_{k+1}^{-1}\big(I_{sn} - V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k(I_{sm} - \bar{\mathcal{B}}_k^T V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k)^{-1}\bar{\mathcal{B}}_k^T\big)\Omega_{k+1}^{-1}\bar{\mathcal{A}}_k,\\
\mathcal{L}_k &= V\mathbb{X}_{k+1}^{-1}\Omega_{k+1}^{-1}\bar{\mathcal{A}}_k - V^2\mathbb{X}_{k+1}^{-1}\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k(I_{sm} - V\bar{\mathcal{B}}_k^T\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k)^{-1}\bar{\mathcal{B}}_k^T\Omega_{k+1}^{-1}\bar{\mathcal{A}}_k,
\end{aligned}$$

which matches (195). Place now (203), (210) and (211) into (200), such that

$$\mathscr{P}_k = \mathcal{Q}_k + V\mathcal{A}_k^T W_4 + \hat{E}_{\mathcal{A}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}(\hat{E}_{\mathcal{A}_k} - V\hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\mathcal{B}_k^T W_4),$$

$$\mathscr{P}_k = \mathcal{Q}_k + \hat{E}_{\mathcal{A}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{A}_k} + (V\mathcal{A}_k^T - \hat{E}_{\mathcal{A}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}V\hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\mathcal{B}_k^T)W_4.$$

Define $\bar{\mathcal{Q}}_k = \mathcal{Q}_k + \hat{E}_{\mathcal{A}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}\hat{E}_{\mathcal{A}_k}$, then

$$\mathscr{P}_k = \bar{\mathcal{Q}}_k + \underbrace{(V\mathcal{A}_k^T - \hat{E}_{\mathcal{A}_k}^T(\Sigma + \hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\hat{E}_{\mathcal{B}_k}^T)^{-1}V\hat{E}_{\mathcal{B}_k}\mathcal{R}_k^{-1}\mathcal{B}_k^T)}_{\bar{\mathcal{A}}_k^T} W_4,$$

$$\mathscr{P}_k = \bar{\mathcal{Q}}_k + V\bar{\mathcal{A}}_k^T W_4,$$

$$\mathscr{P}_k = \bar{\mathcal{Q}}_k + V\bar{\mathcal{A}}_k^T(\Omega_{k+1} + V\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T)^{-1}\bar{\mathcal{A}}_k,$$

$$\mathscr{P}_k = \bar{\mathcal{Q}}_k + V\bar{\mathcal{A}}_k^T(\Omega_{k+1}(I_{sn} + V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T))^{-1}\bar{\mathcal{A}}_k^T,$$

$$\mathscr{P}_k = \bar{\mathcal{Q}}_k + V\bar{\mathcal{A}}_k^T(I_{sn} + V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k\bar{\mathcal{B}}_k^T)^{-1}\Omega_{k+1}^{-1}\bar{\mathcal{A}}_k^T,$$

$$\mathscr{P}_k = \bar{\mathcal{Q}}_k + V\bar{\mathcal{A}}_k^T\big(I_{sn} - V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k(I_{sm} + \bar{\mathcal{B}}_k^T V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k)^{-1}\bar{\mathcal{B}}_k^T\big)\Omega_{k+1}^{-1}\bar{\mathcal{A}}_k,$$

$$\mathscr{P}_k = \bar{\mathcal{Q}}_k + V\bar{\mathcal{A}}_k^T\Omega_{k+1}^{-1}\bar{\mathcal{A}}_k - V\bar{\mathcal{A}}_k^T V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k(I_{sm} + \bar{\mathcal{B}}_k^T V\Omega_{k+1}^{-1}\bar{\mathcal{B}}_k)^{-1}\bar{\mathcal{B}}_k^T\Omega_{k+1}^{-1}\bar{\mathcal{A}}_k,$$

and define $\tilde{\Omega}_{k+1} = V\Omega_{k+1}^{-1}$, such that

$$\mathscr{P}_k = \bar{\mathcal{Q}}_k + \bar{\mathcal{A}}_k^T\tilde{\Omega}_{k+1}\bar{\mathcal{A}}_k - \bar{\mathcal{A}}_k^T\tilde{\Omega}_{k+1}\bar{\mathcal{B}}_k(I_{sm} + \bar{\mathcal{B}}_k^T\tilde{\Omega}_{k+1}\bar{\mathcal{B}}_k)^{-1}\bar{\mathcal{B}}_k^T\tilde{\Omega}_{k+1}\bar{\mathcal{A}}_k,$$

which corresponds to (197). □

In Lemma 6.1, we have that $z_{k+1} = \mathcal{L}_k z_k$. Therefore, $\mathcal{L}_k$ as in either (194) or (195) expresses an equivalent form of the closed-loop system matrix with the recursive regulator. It is also noteworthy that the solution exists for any $\mu > 0$.

Observe that, whenever $\beta \to 1^+$, (197) conforms with (32), since $\tilde{\Omega}_{k+1} \to \mathbb{X}_{k+1}$. Therefore, in Theorem 6.1 we show that it is possible to write the solution for problem (191)–(192) in terms of recursive algebraic Riccati equations. We then consider time-invariant parameters and make the connections

$$P_{i,k} \leftarrow \mathscr{P}_k, \quad A_i \leftarrow \bar{\mathcal{A}}, \quad B_i \leftarrow \bar{\mathcal{B}}, \quad Q_i \leftarrow \bar{\mathcal{Q}}, \quad R_i \leftarrow I_{sm}, \quad \text{and} \quad \mathcal{E}_{k+1} \leftarrow \tilde{\Omega}_{k+1}.$$

That said, the convergence of $\mathscr{P}_k$ to the unique stabilizing solution for (197) is guaranteed based upon the paramount concepts outlined in (COSTA; FRAGOSO; MARQUES, 2005), (BERTSEKAS, 2005), and reproduced in Section 2.4.

6.3 Illustrative Examples

We present two examples to verify the performance of the solution proposed in this chapter. For comparison purposes, we adopt a robust H_∞ controller (TODOROV; FRAGOSO, 2016) and compute the feedback gain with the YALMIP Toolbox (LÖFBERG, 2004). We executed the experiments on a 2.50 GHz i5-3210M CPU with 8 GB of RAM.

Example 6.1. *Consider the longitudinal model of the G 360 CB6x4HSZ truck examined in Example 4.2 (see details in Appendix B). In this case, we account only for the acceleration modes, thus $\theta_k \in \{1, \dots, 7\}$, and each mode captures the dynamics according to the transmission rates of gears from $4th$ to $10th$. However, now we assume it is not possible to detect the actual active mode θ_k. The states are $q_k = \begin{bmatrix} q_1^T & q_2^T & q_3^T \end{bmatrix}^T$, where q_1 is the driveshaft torsion, q_2 is the engine speed, and q_3 is the wheel speed. The control input is $u_k = \tau_k - \tau_k^{ref}$, with $\tau_k \in [0, 1]$ being the throttle pedal position. We define the tracking error as $x_k = q_k - q_k^{ref}$. Moreover, q_k^{ref} is the reference trajectories for states, $\tau_k^{ref} \in [0, 1]$ is the reference throttle pedal position, and initial conditions are $\{x_0, \theta_0\} = \{ \begin{bmatrix} 0.1 \ 0.02 \ 0.02 \end{bmatrix}^T, 1\}$. We select the following parameters to compose the cost function* (192) *and to make use of Theorem 6.1: $Q_i = I_3$, $R_i = 10^{10}$, $P_{i,N} = I_3$, $\mu = 10^{15}$, $\beta = 1.001$. In this case, the system evolves according to*

$$q_{k+1} = (F_{i,k} + \delta F_{i,k})q_k + (G_{i,k} + \delta G_{i,k})\tau_k,$$

where $\tau_k = u_k + \tau_k^{ref}$, $u_k = \sum_{i=1}^{7} v_{i,k}$, and $v_k = \mathcal{K}_k z_k$, with $\{z_k, v_k\}$ defined as in (179)*. For the robust H_∞ controller, we adopted $\gamma = 450$. We executed 1000 simulations with $\alpha_k \in \Lambda_3$ varying randomly in each iteration according to a uniform distribution. In Table 7 we show the resulting norms of tracking errors $\|\bar{x}_k\|$, of throttle pedal positions $\|\bar{\tau}_k\|$, and the computational times $\bar{T}_c$, all averaged over the 1000 simulations, along with the standard deviations $\sigma_{(.)}$. We denote the robust recursive regulator for polytopic DMJLS with an unobserved chain as PMRRU for conciseness. In Fig. 16 we show the trajectories of the states q_k with the PMRRU, whereas in Fig. 17 we present the throttle pedal positions. Both approaches were capable of ensuring the stability of the closed-loop system and tracked the reference trajectories with success. Nonetheless, the computation of the recursive regulator gains was, on average, three orders of magnitude faster than the robust H_∞ controller.*

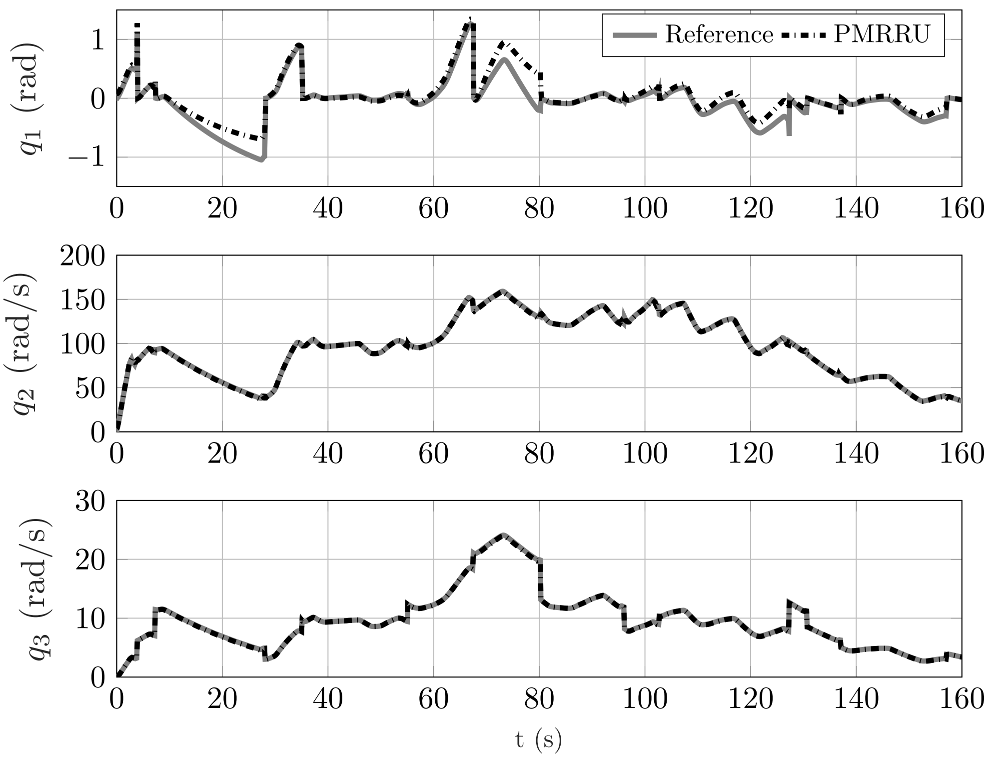
Table 7 – Simulation results for Example 4.2.

Controller	$\|\bar{x}\|_{\mathcal{L}_2}$	$\sigma_{\bar{x}}$	$\|\bar{\tau}\|_{\mathcal{L}_2}$	$\sigma_{\bar{\tau}}$	$\bar{T}_c$ (ms)	$\sigma_{\bar{T}_c}$ (ms)
PMRRU	2.3084	0.0455	9.2593	0.1461	4.178	0.6583
Robust H_∞	2.7715	0.0394	8.8008	0.1409	6.5678 $\times 10^3$	1.1280 $\times 10^3$

Source: author.

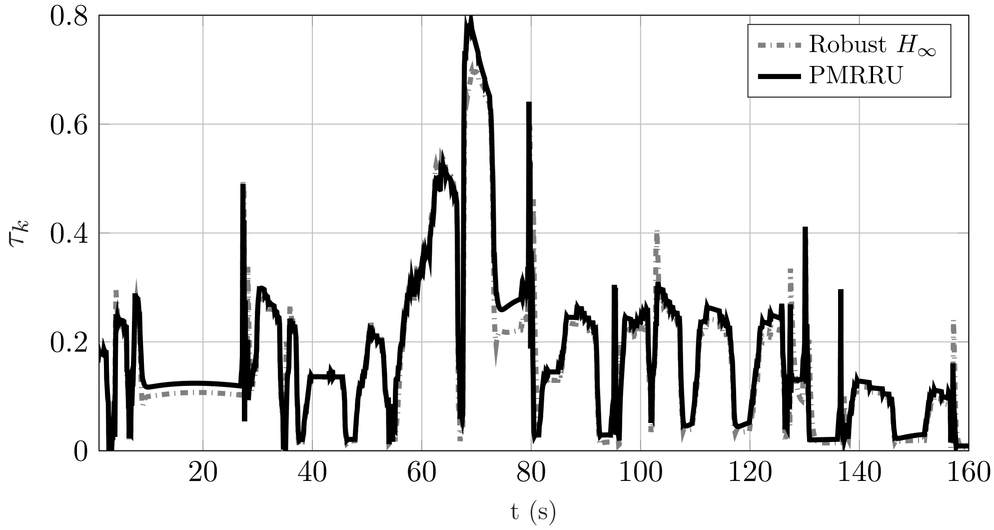
Example 6.2. *We consider the following DMJLS with three modes of an operation whose*

Figure 16 – Trajectories of the closed-loop system states with the PMRRU.



Source: author.

Figure 17 – Throttle pedal positions of the closed-loop system.



Source: author.

parameters were adapted from Boukas and Liu (2001):

$$F_{1,k} = \begin{bmatrix} 1 & 0 \\ 0 & 1.2 \end{bmatrix}, \quad F_{2,k} = \begin{bmatrix} 1.13 & 0 \\ 0.16 & 0.478 \end{bmatrix}, \quad F_{3,k} = \begin{bmatrix} 0.3 & 0.13 \\ 0.16 & 1.14 \end{bmatrix},$$

$$G_{1,k} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix}, \quad G_{2,k} = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.1 \end{bmatrix}, \quad G_{3,k} = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.1 \end{bmatrix},$$

with vertices

$$F_{1,k}^{(1)} = \begin{bmatrix} 0 & 0 \\ -0.1 & 0.1 \end{bmatrix}, \quad F_{1,k}^{(2)} = -F_{1,k}^{(1)}, \quad G_{1,k}^{(1)} = 0.1G_{1,k}, \quad G_{1,k}^{(2)} = -G_{1,k}^{(1)},$$
$$F_{2,k}^{(1)} = \begin{bmatrix} 0 & 0 \\ -0.1 & 0 \end{bmatrix}, \quad F_{2,k}^{(2)} = -F_{2,k}^{(1)}, \quad G_{2,k}^{(1)} = 0.1G_{2,k}, \quad G_{2,k}^{(2)} = -G_{2,k}^{(1)},$$
$$F_{3,k}^{(1)} = \begin{bmatrix} 0 & 0 \\ 0.1 & -0.1 \end{bmatrix}, \quad F_{3,k}^{(2)} = -F_{3,k}^{(1)}, \quad G_{3,k}^{(1)} = 0.1G_{3,k}, \quad G_{3,k}^{(2)} = -G_{3,k}^{(1)},$$

and transition probabilities given by

$$\mathbb{P} = \begin{bmatrix} 0.2 & 0.8 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.3 & 0.7 \end{bmatrix}.$$

We computed the feedback gains $\mathcal{K}_k$ of the recursive regulator, denoted by PMRRU for shortness, considering $P_{i,N} = I_2$, $Q_{i,N} = I_2$, $R_{i,N} = I_2$, $i \in \{1, 2, 3\}$, $\mu = 10^{15}$, and $\beta = 1.01$ for Lemma 6.1. In this case, we evolve $z_{k+1} = (\mathcal{A}_k + \delta\mathcal{A}_k)z_k + (\mathcal{B}_k + \delta\mathcal{B}_k)v_k$, and recover the original variables $\{x_{k+1}, u_k\}$ according to (180)*, namely $x_{k+1} = \sum_{i=1}^{3} z_{i,k+1}$ and $u_k = \sum_{i=1}^{3} v_{i,k}$, where $v_k = \mathcal{K}_k z_k$. For the robust H_∞ controller, we tuned $\gamma = 10^{-3}$. We executed 1000 experiments with time horizon $N = 20$. In Fig. 18 we show the averaged norms of states for both approaches. Table 8 summarizes the overall results of the experiments, where $\bar{T}_c$ and $\sigma_{\bar{T}_c}$ are the averaged time required to compute the feedback gains and its standard deviation, respectively. The performances were similar in terms of norms of states and control inputs, but the proposed solution required substantially lower computational effort.*

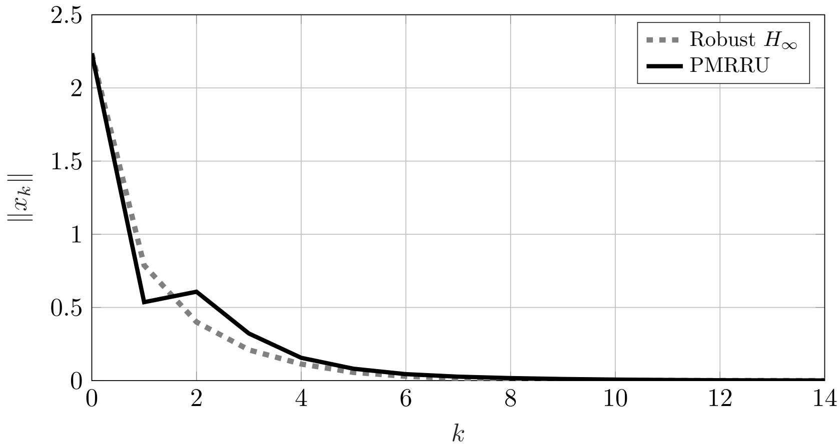
Let us consider now $x_{k+1} = (F_{i,k} + \delta F_{i,k})x_k + (G_{i,k} + \delta G_{i,k})u_k$, where $u_k = \sum_{i=1}^{3} v_{i,k}$, and $v_k = \mathcal{K}_k z_k$. Performing 1000 experiments under the same conditions mentioned above, but with $N = 100$ instead, we obtain the averaged norms of the state vector shown in Fig. 19. Observed that, even though the closed-loop system is stable, the performance degrades. Therefore, the stabilizing solution for the augmented systems might not be the most adequate solution for the original system in this example.

Table 8 – Results averaged over 1000 experiments.

Controller	$\|x_k\|$	$\|u_k\|$	$\bar{T}_c$ (ms)	$\sigma_{\bar{T}_c}$ (ms)
PMRRU	2.4071	16.7466	4.4355	0.6702
Robust H_∞	2.4172	21.9698	208.0839	32.4024

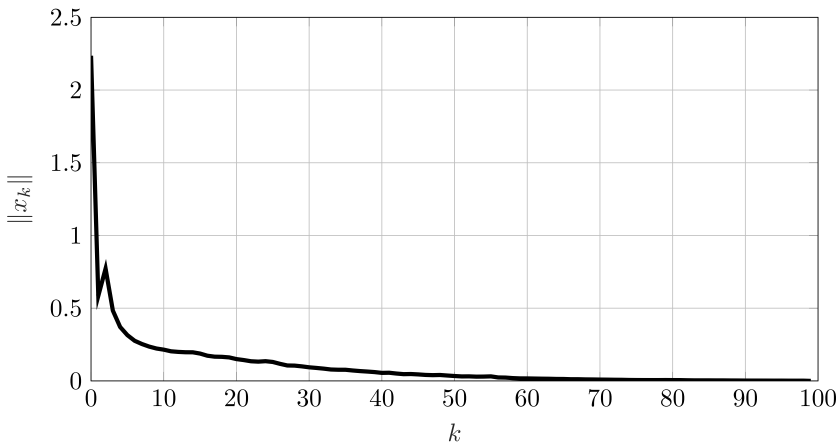
Source: author.

Figure 18 – Averaged norms of states.



Source: author.

Figure 19 – Averaged norms of states.



Source: author

7 CONCLUSION

In this thesis, we presented and discussed the overall results achieved throughout the author's Doctorate Degree Program in Electrical Engineering, as a requirement to earn the title of Doctor of Science. The theoretical contributions comprise a set of robust recursive solutions for the regulation problem of linear and Markov jump linear systems subject to polytopic uncertainties.

We began by presenting a literature review and summarizing important results on the stability and stabilizability of polytopic systems. Among these, it is noteworthy that a remarkable collection was provided by Brazilian researchers, especially regarding LMI-based optimization. Nonetheless, the number of reported recursive solutions for the regulation problem of discrete-time linear polytopic systems is scarce, which motivated our research efforts toward filling this gap. That said, this thesis also highlighted the potential of recursive methods from both theoretical and practical perspectives. We proceeded to the background theory on robust regularized least-squares with uncertain data and penalty functions method, which are bases for our approach. The first contribution of this thesis is the solution for the robust regulation problem of polytopic discrete-time linear systems. It enabled us to move forward to the regulation problem of discrete-time Markov jump linear systems subject to polytopic uncertainties on state space matrices, which is our second contribution. The third contribution involves Markov jump linear systems whose transition probabilities are also affected by polytopic uncertainties. In this case, we separated the uncertain portions in the cost function whilst keeping its quadratic structure. The fourth contribution relates to the regulation problem of polytopic DMJLS with an unobserved Markov chain, based on an augmented system where we express the information about active modes as uncertainties.

Our formulation takes all vertices of the polytopes into account in a unified manner, thus allowing us to design unconstrained min-max optimization problems in the least-squares framework. In all the aforementioned cases, the convexity of the cost functions ensured unique solutions given a selected fixed penalty parameter. The resulting state feedback gains are such that the associated closed-loop systems are stable despite the uncertainties. Furthermore, convergence and stability conditions were established in terms of algebraic Riccati equations. We provided various examples to assess the effectiveness of the robust regulators under different scenarios and adopted several robust controllers available in the literature for comparison purposes. More specifically, in the application examples we focused on trajectory tracking for unmanned aerial and ground vehicles. Our results were promising and we verified faster computation of gains without requiring any further parameter tuning during operation.

The recursive regulation of discrete-time linear systems with input saturation and state constraints are possible subjects of future research. At first, one should design solutions for the case without uncertainties, mostly to find ways of incorporating the inequality constraints into the recursive framework. Once this intermediate but nevertheless essential step is done, it would be possible to address the related robust control problems. Up to this moment, we have found several interesting approaches to address this class of problems, such as the polytopic representation of the input saturation (KIM, 2017), (HU; DUAN; TAN, 2018); approximation by saturated sine function (MRACEK; CLOUTIER, 1998); and state-dependent Riccati equations (KIM; KWON, 2017), (LIN, 2021).

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Appendix

APPENDIX A – MATRIX ANALYSIS

The purpose of this appendix is to present a compilation of results well known in the specialized literature. The reader, assumed to be familiar with the basics of Linear Algebra, might find this appendix convenient for consultation during the reading. Here we omit the demonstrations, which can be found in the presented references.

A.1 Positive (Semi)Definite Matrices

Definition A.1. *(HORN; JOHNSON, 2013) A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite, i.e., $A > 0$, if $x^T Ax > 0$ for all nonzero $x \in \mathbb{R}^n$ and $x^T Ax = 0$ if $x = 0$.*

Definition A.2. *(HORN; JOHNSON, 2013) A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive semidefinite, i.e., $A \geq 0$, if $x^T Ax \geq 0$ for all nonzero $x \in \mathbb{R}^n$.*

Definition A.3. *(HORN; JOHNSON, 2013) A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is negative definite, i.e., $A < 0$, if $x^T Ax < 0$ for all nonzero $x \in \mathbb{R}^n$ and $x^T Ax = 0$ if $x = 0$.*

Definition A.4. *(HORN; JOHNSON, 2013) A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is negative semidefinite, i.e., $A \leq 0$, if $x^T Ax \leq 0$ for all nonzero $x \in \mathbb{R}^n$.*

Proposition A.1. *(HORN; JOHNSON, 2013) Each eigenvalue of a positive definite matrix is a positive number. Each eigenvalue of a negative definite matrix is a negative number.*

Lemma A.1. *(HORN; JOHNSON, 2013) Consider matrices $A \in \mathbb{R}^{n\times n}$ symmetric positive definite and $C \in \mathbb{R}^{n\times m}$. Then,*

(i) $C^T AC$ is positive semidefinite and rank$(C^T AC) =$ rank(AC).

(ii) $rank(C^T AC) = rank(C)$ and $C^T AC$ is positive definite if and only if matrix C has full column rank.

Lemma A.2. *(ABADIR; MAGNUS, 2005) Let $A \in \mathbb{R}^{n\times n}$ be a positive semidefinite matrix. Then,*

(i) $A + B \geq 0$ if $B \geq 0$.

(ii) $A + B > 0$ if $B > 0$.

Lemma A.3 ((LUENBERGER; YE, 2010))**.** *Consider matrices $\mathscr{C} \in \mathbb{R}^{p\times p}$ and $\mathscr{B} \in \mathbb{R}^{p\times l}$. Assume $\mathscr{B}$ has rank p and $\mathscr{C} > 0$. Then, the matrix*

$$\begin{bmatrix} \mathscr{C} & \mathscr{B}^T \\ \mathscr{B} & 0 \end{bmatrix}$$

is nonsingular.

A.2 Matrix Inversion

Lemma A.4. *(HORN; JOHNSON, 2013) A matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular if and only if 0 is not an eigenvalue of A.*

Lemma A.5. *((ZHANG, 2005) - Banachiewicz inversion formula) Consider a nonsingular square matrix M and a nonsingular matrix S such that*

$$M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}.$$

Then, the Schur complement $(M/S) = P - QS^{-1}R$ is nonsingular and

$$M^{-1} = \begin{bmatrix} (M/S)^{-1} & -(M/S)^{-1}QS^{-1} \\ -S^{-1}R(M/S)^{-1} & S^{-1} + S^{-1}R(M/S)^{-1}QS^{-1} \end{bmatrix}.$$

Lemma A.6. *((HORN; JOHNSON, 2013) - Sherman–Morrison–Woodbury formula) Assume $A \in \mathbb{R}^{n\times n}$ a nonsingular matrix, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{r\times r}$ a nonsingular matrix and $D \in \mathbb{R}^{r\times n}$. If the inverse of $(C^{-1} + DA^{-1}B)$ exists, then*

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}. \tag{212}$$

Lemma A.7. *(CAMPOS, 2009) Assume $A \in \mathbb{R}^{n\times n}$ a nonsingular matrix, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{r\times r}$ a nonsingular matrix and $D \in \mathbb{R}^{r\times n}$. If the inverse of $(C^{-1} + DA^{-1}B)$ exists, then*

$$(A + BC^{-1}D)^{-1}BC^{-1} = A^{-1}B(C + DA^{-1}B)^{-1}. \tag{213}$$

A.3 Spectral Radius

Definition A.5. *(HORN; JOHNSON, 2013) The spectral radius of a matrix $A \in \mathbb{R}^{n\times n}$ is $\rho(A) = \max\{|\lambda| \,:\, \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the set of eigenvalues of A.*

APPENDIX B – IDENTIFICATION OF POLYTOPIC POWERTRAIN MODEL FOR HEAVY-DUTY VEHICLES

In this appendix, we detail the identification procedure carried out to estimate the polytopic powertrain model for the Scania G 360 CB6x4HSZ truck shown in Fig. 20. The vehicle has a 14-speed automatic gearbox and an external circuit that acquires accelerator and brake signals.

Figure 20 – Scania G 360 CB6x4HSZ vehicle.

Source: author.

B.1 General Setup

Gear shifts provoke sudden changes in the powertrain dynamics. Indeed, the literature on automotive systems, for instance (KIENCKE, 2005) and (RAJAMANI, 2012), introduce vehicle longitudinal dynamics as parameter varying models which depend on final drive and gear ratios, and on the exact knowledge of structural parameters. However, these parameters and the policies orchestrating the gear shifts are are classified as trade secrets and, therefore, usually unavailable to the general public. That said, it is reasonable to comprehend the powertrain as a DMJLS of the form

$$q_{k+1} = (F_i + \delta F_i)q_k + (G_i + \delta G_i)\tau_k, \quad i \in \Theta = \{1, \dots, s\}. \tag{214}$$

The state vector is $q_k = \begin{bmatrix} q_{1,k} & q_{2,k} & q_{3,k} \end{bmatrix}^T$, where $q_{1,k}$ is the driveshaft torsion in rad, $q_{2,k}$ is the engine angular speed in rad/s, and $q_{3,k}$ is the wheel angular speed in rad/s. The input signal τ_k is the required normalized accelerator or brake pedal positions, such that $\|\tau_k\| \leq 1$.

The system is subject to polytopic uncertainties $\{\delta F_i, \delta G_i\}$, such that

$$\begin{bmatrix}\delta F_{i,k} & \delta G_{i,k}\end{bmatrix} = \sum_{l=1}^{V_n} \alpha_{l,k} \begin{bmatrix}F_{i,k}^{(l)} & G_{i,k}^{(l)}\end{bmatrix},$$

$$\Lambda_{V_n} = \left\{ \alpha = \begin{bmatrix}\alpha_{1,k} & \dots & \alpha_{V_n,k}\end{bmatrix}^T \in \mathbb{R}^{V_n} \;\middle|\; \sum_{l=1}^{V_n} \alpha_l = 1,\;\; 0 \leq \alpha_l \leq 1 \right\}.$$

The transition probability matrix $\mathbb{P}_k \in \mathbb{R}^{s \times s}$ is defined by

$$\mathbb{P}_k = \mathbb{P}_0 + \delta\mathbb{P}_k, \quad \text{Prob}(\theta_0) = \pi_i^{(0)} + \delta\pi_i,$$
$$\mathbb{P}_k = \begin{bmatrix}p_{ij}^{(0)} + \delta p_{ij,k}\end{bmatrix} = \text{Prob}(\theta_{k+1} = j \mid \theta_k = i),$$
$$\sum_{j=1}^{s} (p_{ij}^{(0)} + \delta p_{ij,k}) = 1, \quad 0 \leq p_{ij}^{(0)} + \delta p_{ij,k} \leq 1,$$

where

$$\delta\mathbb{P}_k = \begin{bmatrix}\delta p_{ij,k}\end{bmatrix} = \begin{bmatrix}\sum_{l=1}^{V_p} \xi_{l,k} p_{ij}^{(l)}\end{bmatrix},$$

$$\Lambda_{V_p} = \left\{ \xi = \begin{bmatrix}\xi_{1,k} & \dots & \xi_{V_p,k}\end{bmatrix}^T \in \mathbb{R}^{V_p} \;\middle|\; \sum_{l=1}^{V_p} \xi_l = 1,\;\; 0 \leq \xi_l \leq 1 \right\}.$$

Remark B.1. *Throughout the identification method presented in this appendix, we assume that brake and throttle pedals are never excited simultaneously. In addition, the related signals assume values from 0 to 1, meaning 0 to 100% of the pedal position range.*

We limited the longitudinal speed at 50 km/h while driving the truck inside the university campus, and there was no extra payload connected to the bodywork. As we approach the model identification problem from a Markovian perspective, we relate each gear to two Markov modes: one for active acceleration and one for active braking. The vehicle engages gears from 4th to 10th (i.e., 7 gears) since gears 3rd and lower are used for additional payloads and gears 11th and 12th in higher speeds. That said, 14 Markov modes compose the DMJLS that describes the powertrain behavior, namely

- $i \in \{1, \dots, 7\} \implies$ accelerating in gears 4th to 10th;
- $i \in \{8, \dots, 14\} \implies$ braking in gears 4th to 10th.

Let us now define the tracking error $x_k = q_k - q_{ref,k}$, where $q_{ref,k}$ is the reference trajectory generated via accelerator/brake signals. Then, based on (214), we yield the following trajectory tracking error dynamics:

$$x_{k+1} = (F_i + \delta F_i)x_k + (G_i + \delta G_i)u_k, \;\; i \in \{1, \dots, 14\},$$

where $u_k = \tau_k - \tau_{ref,k} \in \mathbb{R}$, such that $\tau_{ref,k}$ is the reference command signal obtained from CAN readings, and u_k is the controller signal.

To acquire the real-time vehicle variables to compose q_k, we read the CAN bus at a sampling period of $T_{samp} = 0.2$ s, hence $f_{samp} = 5$ Hz.

B.2 Mode Detection

An acceleration mode is detected whenever $\tau_{ref,k} \geq 0$, and a braking mode whenever $\tau_{ref,k} < 0$. We then compute the signals to be sent to the vehicle in the following way:

$$\tau_{acc,k} = \begin{cases} \tau_k, & \text{if } \tau_{ref,k} \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\tau_{brake,k} = \begin{cases} -\tau_k, & \text{if } \tau_{ref,k} < 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\tau_{acc,k}$ and $\tau_{brake,k}$ are acceleration and brake pedals positions, respectively. Observe that, at this point, we are able to determine the active Markov mode based on the engaged gear, which we access via CAN bus, and on τ_k.

Although the road slope also affects the powertrain dynamics, the changes tend to be much slower than those caused by gear shifts. Thus, we relate the vertices $\{F_{i,k}^{(l)}, G_{i,k}^{(l)}\}$, $l = 1, 2, 3$, $i \in \Theta$, to the steepest downhill, steepest uphill, and flat terrains, respectively.

B.3 State-Space Model Identification

In this section, we adapt the least-squares identification approach presented by (YOUNG; GARNIER; GILSON, 2008) to fit our purposes. Consider a system of the form $z_k = \varphi_k \eta + w_k$, where z_k and φ_k are known vectors, η is the unknown vector with parameters that characterize the model, $k = 1, \ldots, N_S$, and N_S is the number of available samples. Define also the related noise-free process $\hat{z}_k = \varphi_k \eta$. Then, the identification problem can be stated as

$$\hat{\eta} = \arg\min_{\eta} \left\| \left[\frac{1}{N_S} \sum_{k=1}^{N_S} \varphi_k \varphi_k^T \right] \eta - \left[\frac{1}{N_S} \sum_{k=1}^{N_S} \varphi_k \hat{z}_k^T \right] \right\|^2 .$$

The solution $\hat{\eta}$ is given by (see the detailed demonstration in (YOUNG; GARNIER; GILSON, 2008))

$$\hat{\eta} = \left[\sum_{k=1}^{N_S} \varphi_k \varphi_k^T \right] \frac{1}{N_S} \sum_{k=1}^{N_S} \varphi_k \hat{z}_k^T . \tag{215}$$

To ensure numerical stability of the algorithm, we normalize the states q_k based upon constants $c_i = \begin{bmatrix} c_{i,11} & c_{i,22} & c_{i,33} \end{bmatrix}^T \in \mathbb{R}^3$, where $c_{i,11}$, $c_{i,22}$, $c_{i,33}$, $i \in \{1, \ldots, 7\}$, refer to the maximum values of the elements in q_k acquired via CAN bus while accelerating with the i-th gear engaged.

Meanwhile, for the braking modes, i.e. $i = \{8, ..., 14\}$, we have $c_i = \begin{bmatrix} 0 \; c_{i,22} \; c_{i,33} \end{bmatrix}$, where $c_{i,33}$ is the maximum value of $q_{3,k}$. Observe that $c_{i,11}$ is set to zero because the clutch relieves all driveshaft torsion as it disengages during braking. As mentioned in (KIENCKE, 2005), the relation between the angular velocities of the engine and wheels depends only on the engaged gear when the vehicle operates in a steady state. That said, we compute $c_{i,22}$ for the braking modes proportionally to the acceleration mode in the same gear:

$$c_{i,22} = \frac{c_{i,33}\, c_{j,22}}{c_{j,33}}, \;\; i \in \{8, ..., 14\}, \;\; j = i - 7,$$

where $c_{i,33}$ is the maximum value read for $q_{3,k}$. The original values are then recovered simply by multiplying q_k and x_k by c_i element-wise, for any $i \in \{1, \ldots, 14\}$. Moreover, $\{c_{i,22}, c_{i,33}\}$ can differ from $\{c_{j,22}, c_{j,33}\}$ as they normalize distinct batches of data, which generally have their own specific maximum values.

Remark B.2. *We consider a powertrain model in which the driveshaft torsion is neglected while the truck brakes. This assumption is valid because $c_{i,22}$ is relevant for control purposes when the clutch is re-engaged, while an effective clutch control must be able to re-engage the clutch with practically no torsion. This mechanical aspect is discussed with further details in (KIENCKE, 2005).*

We separate the collected batches of data with respect to engaged gears, acceleration, deceleration, and road slope. For accelerator modes, hence $i \in \{1, \ldots, 7\}$, we make the following mappings to apply (215) so as to yield the set $\{\hat{F}_i^{(l)}, \hat{G}_i^{(l)}\}$, $l = 1, 2, 3$:

$$\eta \leftarrow \begin{bmatrix} (\hat{F}_{a,i}^{(l)})^T \\ (\hat{G}_{a,i}^{(l)})^T \end{bmatrix}, \quad \varphi_k \leftarrow \begin{bmatrix} q_k^T \; \tau_k^T \end{bmatrix}, \quad \hat{z} \leftarrow q_{a,k+1},$$

where $\hat{F}_{a,i}^{(l)}$, $\hat{G}_{a,i}^{(l)}$, and $q_{a,k+1}$, are the a-th rows $\hat{F}_i^{(l)}$, $\hat{G}_i^{(l)}$, and q_{k+1}, respectively. In this case, observe that $\hat{F}_i^{(l)} = F_i + F_i^{(l)}$, and $\hat{G}_i^{(l)} = G_i + G_i^{(l)}$.

For the brake modes, thus $i \in \{8, \ldots, 14\}$, we shall consider $\hat{F}_i^{(l)}$ and $\hat{G}_i^{(l)}$ built as

$$\hat{F}_i^{(l)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & f_i^{(l)} \end{bmatrix}, \; \hat{G}_i^{(l)} = \begin{bmatrix} 0 \\ 0 \\ g_i^{(l)} \end{bmatrix},$$

for $l = 1, 2, 3$. We do so due to some mechanical characteristics that arise when the vehicle activates the braking modes. First, the driveshaft and engine disengage during the braking action, therefore there is no torsion in the driveshaft and we have $q_{1,k} = 0$. Second, since we assume normalized variables and both driveshaft and engine are disengaged, we have $q_{2,k} = q_{3,k}$ to guarantee safe re-engagement of the clutch. That said, for the braking modes we make the relations

$$\eta \leftarrow \begin{bmatrix} f_i^{(l)} \; g_i^{(l)} \end{bmatrix}^T, \quad \varphi_k \leftarrow \begin{bmatrix} q_{3,k}^T \; \tau_k^T \end{bmatrix}, \quad \hat{z} \leftarrow q_{3,k+1},$$

and apply (215) to obtain matrices $\{\hat{F}_i^{(l)}, \hat{G}_i^{(l)}\}$, $l = 1, 2, 3$.

To yield a DMJLS in the form of system (214), we consider the nominal model matrices $\{F_i, G_i\}$ as the mean of the extreme matrices $\{\hat{F}_i^{(l)}, \hat{G}_i^{(l)}\}$, whereas the vertices are given by $F_i^{(l)} = \hat{F}_i^{(l)} - F_i$, and $G_i^{(l)} = \hat{G}_i^{(l)} - G_i$, $i \in \{1, \ldots, 14\}$, $l \in \{1, 2, 3\}$. The procedure ultimately returns the following matrices to compose (214):

- Mode 1 (4th gear, accelerator):

$$F_1 = \begin{bmatrix} 1.0000 & 0.9449 & -0.9347 \\ 0.0129 & 0.4938 & 0.4688 \\ -0.0003 & 0.6135 & 0.3918 \end{bmatrix}, \quad F_1^{(1)} = \begin{bmatrix} 0 & 0.0000 & 0 \\ -0.0125 & 0.0085 & -0.0043 \\ -0.0002 & 0.0275 & -0.0276 \end{bmatrix},$$

$$F_1^{(2)} = \begin{bmatrix} 0 & 0.0000 & 0 \\ -0.0030 & 0.0646 & -0.0586 \\ 0.0023 & -0.0506 & 0.0504 \end{bmatrix}, \quad F_1^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0155 & -0.0731 & 0.0630 \\ -0.0020 & 0.0231 & -0.0227 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0 \\ 0.1757 \\ 0.0040 \end{bmatrix}, \quad G_1^{(1)} = \begin{bmatrix} 0 \\ -0.0149 \\ 0.0062 \end{bmatrix}, \quad G_1^{(2)} = \begin{bmatrix} 0 \\ 0.0142 \\ -0.0043 \end{bmatrix}, \quad G_1^{(3)} = \begin{bmatrix} 0 \\ 0.0006 \\ -0.0020 \end{bmatrix},$$

$$c_1 = \begin{bmatrix} 0.9415 & 200.6431 & 8.0915 \end{bmatrix}^T.$$

- Mode 2 (5th gear, accelerator):

$$F_2 = \begin{bmatrix} 1.0000 & 0.7998 & -0.7978 \\ 0.0051 & 0.4666 & 0.5092 \\ 0.0002 & 0.5529 & 0.4452 \end{bmatrix}, \quad F_2^{(1)} = \begin{bmatrix} 0 & 0 & -0.0000 \\ -0.0029 & -0.0277 & 0.0230 \\ -0.0022 & -0.0332 & 0.0298 \end{bmatrix},$$

$$F_2^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0005 & 0.0860 & -0.0772 \\ -0.0014 & -0.0494 & 0.0497 \end{bmatrix}, \quad F_2^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0025 & -0.0582 & 0.0542 \\ 0.0036 & 0.0825 & -0.0796 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0 \\ 0.1110 \\ 0.0233 \end{bmatrix}, \quad G_2^{(1)} = \begin{bmatrix} 0 \\ -0.0134 \\ 0.0136 \end{bmatrix}, \quad G_2^{(2)} = \begin{bmatrix} 0 \\ 0.0128 \\ -0.0004 \end{bmatrix}, \quad G_2^{(3)} = \begin{bmatrix} 0 \\ 0.0005 \\ -0.0132 \end{bmatrix},$$

$$c_2 = \begin{bmatrix} 0.8484 & 206.7168 & 10.4782 \end{bmatrix}^T.$$

- Mode 3 (6th gear, accelerator):

$$F_3 = \begin{bmatrix} 1.0000 & 1.1013 & -1.1001 \\ 0.0113 & 0.3334 & 0.6405 \\ 0.0036 & 0.6502 & 0.3457 \end{bmatrix}, \quad F_3^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0130 & -0.0428 & 0.0406 \\ 0.0043 & -0.0324 & 0.0295 \end{bmatrix},$$

$$F_3^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ -0.0220 & -0.0090 & 0.0248 \\ -0.0036 & 0.0140 & -0.0104 \end{bmatrix}, \quad F_3^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0091 & 0.0517 & -0.0655 \\ -0.0008 & 0.0184 & -0.0191 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} 0 \\ 0.0870 \\ 0.0115 \end{bmatrix}, \quad G_3^{(1)} = \begin{bmatrix} 0 \\ 0.0020 \\ 0.0031 \end{bmatrix}, \quad G_3^{(2)} = \begin{bmatrix} 0 \\ 0.0064 \\ -0.0038 \end{bmatrix}, \quad G_3^{(3)} = \begin{bmatrix} 0 \\ -0.0085 \\ 0.0008 \end{bmatrix},$$

$$c_3 = \begin{bmatrix} 2.2380 & 211.7433 & 13.2687 \end{bmatrix}^T.$$

- Mode 4 (7th gear, accelerator):

$$F_4 = \begin{bmatrix} 1.0000 & 0.8374 & -0.8363 \\ 0.0056 & 0.8110 & 0.1729 \\ 0.0043 & 0.2319 & 0.7620 \end{bmatrix}, \quad F_4^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ -0.0046 & 0.1542 & -0.1483 \\ 0 & 0.0437 & -0.0413 \end{bmatrix},$$

$$F_4^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0022 & -0.0437 & 0.0342 \\ -0.0006 & -0.0134 & 0.0098 \end{bmatrix}, \quad F_4^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0023 & -0.1105 & 0.1142 \\ 0.0005 & -0.0304 & 0.0316 \end{bmatrix},$$

$$G_4 = \begin{bmatrix} 0 \\ 0.0475 \\ 0.0180 \end{bmatrix}, \quad G_4^{(1)} = \begin{bmatrix} 0 \\ -0.0154 \\ -0.0075 \end{bmatrix}, \quad G_4^{(2)} = \begin{bmatrix} 0 \\ 0.0078 \\ 0.0032 \end{bmatrix}, \quad G_4^{(3)} = \begin{bmatrix} 0 \\ 0.0077 \\ 0.0043 \end{bmatrix},$$

$$c_4 = \begin{bmatrix} 2.0827 & 197.7109 & 15.3775 \end{bmatrix}^T.$$

- Mode 5 (8th gear, accelerator):

$$F_5 = \begin{bmatrix} 1.0000 & 1.8169 & -1.8106 \\ 0.0033 & 0.2818 & 0.7060 \\ 0.0009 & 0.5876 & 0.4104 \end{bmatrix}, \quad F_5^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ -0.0048 & 0.1694 & -0.1726 \\ -0.0017 & -0.0723 & 0.0713 \end{bmatrix},$$

$$F_5^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0061 & -0.0363 & 0.0344 \\ 0.0019 & 0.0567 & -0.0577 \end{bmatrix}, \quad F_5^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ -0.0014 & -0.1330 & 0.1382 \\ -0.0003 & 0.0155 & -0.0135 \end{bmatrix},$$

$$G_5 = \begin{bmatrix} 0 \\ 0.0386 \\ 0.0158 \end{bmatrix}, \quad G_5^{(1)} = \begin{bmatrix} 0 \\ -0.0081 \\ -0.0038 \end{bmatrix}, \quad G_5^{(2)} = \begin{bmatrix} 0 \\ 0.0023 \\ 0.0026 \end{bmatrix}, \quad G_5^{(3)} = \begin{bmatrix} 0 \\ 0.0059 \\ 0.0013 \end{bmatrix},$$

$$c_5 = \begin{bmatrix} 1.8243 & 198.2345 & 19.1594 \end{bmatrix}^T.$$

- Mode 6 (9th gear, accelerator):

$$F_6 = \begin{bmatrix} 1.0000 & 0.8161 & -0.8134 \\ 0.0083 & 0.7927 & 0.1951 \\ 0.0047 & 0.2391 & 0.7563 \end{bmatrix}, \quad F_6^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ -0.0069 & 0.0573 & -0.0489 \\ -0.0030 & 0.0260 & -0.0226 \end{bmatrix},$$

$$F_6^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ -0.0013 & -0.0546 & 0.0567 \\ -0.0008 & -0.0373 & 0.0383 \end{bmatrix}, \quad F_6^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0081 & -0.0028 & -0.0079 \\ 0.0037 & 0.0112 & -0.0156 \end{bmatrix},$$

$$G_6 = \begin{bmatrix} 0 \\ 0.0233 \\ 0.0095 \end{bmatrix}, \quad G_6^{(1)} = \begin{bmatrix} 0 \\ -0.0003 \\ -0.0004 \end{bmatrix}, \quad G_6^{(2)} = \begin{bmatrix} 0 \\ 0.0007 \\ 0.0007 \end{bmatrix}, \quad G_6^{(3)} = \begin{bmatrix} 0 \\ -0.0004 \\ -0.0004 \end{bmatrix},$$

$$c_6 = \begin{bmatrix} 3.2577 & 205.5649 & 25.1076 \end{bmatrix}^T.$$

- Mode 7 (10th gear, accelerator):

$$F_7 = \begin{bmatrix} 1.0000 & 0.9507 & -0.9469 \\ 0.0112 & 0.7144 & 0.2700 \\ 0.0085 & 0.1217 & 0.8701 \end{bmatrix}, \ F_7^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0041 & -0.0349 & 0.0352 \\ 0.0022 & -0.0574 & 0.0565 \end{bmatrix},$$

$$F_7^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0020 & -0.1244 & 0.1230 \\ 0.0037 & -0.0367 & 0.0341 \end{bmatrix}, \ F_7^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ -0.0062 & 0.1593 & -0.1583 \\ -0.0059 & 0.0942 & -0.0906 \end{bmatrix},$$

$$G_7 = \begin{bmatrix} 0 \\ 0.0190 \\ 0.0094 \end{bmatrix}, \ G_7^{(1)} = \begin{bmatrix} 0 \\ 0.0007 \\ 0.0024 \end{bmatrix}, \ G_7^{(2)} = \begin{bmatrix} 0 \\ 0.0013 \\ 0.0007 \end{bmatrix}, \ G_7^{(3)} = \begin{bmatrix} 0 \\ -0.0021 \\ -0.0030 \end{bmatrix},$$

$$c_7 = \begin{bmatrix} 3.2495 & 191.4277 & 28.9388 \end{bmatrix}^T.$$

- Mode 8 (4th gear, brake):

$$F_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0.9823 \end{bmatrix}, \ F_8^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.0054 \end{bmatrix}, \ F_8^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0048 \end{bmatrix},$$

$$F_8^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0007 \end{bmatrix}, \ G_8 = \begin{bmatrix} 0 \\ 0 \\ 0.0329 \end{bmatrix}, \ G_8^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0.0275 \end{bmatrix},$$

$$G_8^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0.0320 \end{bmatrix}, \ G_8^{(3)} = \begin{bmatrix} 0 \\ 0 \\ -0.0595 \end{bmatrix}, \ c_8 = \begin{bmatrix} 0 \\ 213.5746 \\ 8.6130 \end{bmatrix}.$$

- Mode 9 (5th gear, brake):

$$F_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0.9803 \end{bmatrix}, \ F_9^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0111 \end{bmatrix}, \ F_9^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.0149 \end{bmatrix},$$

$$F_9^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0039 \end{bmatrix}, \ G_9 = \begin{bmatrix} 0 \\ 0 \\ -0.0998 \end{bmatrix}, \ G_9^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0.0109 \end{bmatrix},$$

$$G_9^{(2)} = \begin{bmatrix} 0 \\ 0 \\ -0.0853 \end{bmatrix}, \ G_9^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 0.0745 \end{bmatrix}, \ c_9 = \begin{bmatrix} 0 \\ 192.8893 \\ 9.7773 \end{bmatrix}.$$

- Mode 10 (6th gear, brake):

$$F_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0.9915 \end{bmatrix}, \ F_{10}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0088 \end{bmatrix}, \ F_{10}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.0023 \end{bmatrix},$$
$$F_{10}^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.0066 \end{bmatrix}, \ G_{10} = \begin{bmatrix} 0 \\ 0 \\ -0.0802 \end{bmatrix}, \ G_{10}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ -0.0511 \end{bmatrix},$$
$$G_{10}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ -0.0089 \end{bmatrix}, \ G_{10}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 0.0600 \end{bmatrix}, \ c_{10} = \begin{bmatrix} 0 \\ 194.6410 \\ 12.1970 \end{bmatrix}.$$

- Mode 11 (7th gear, brake):

$$F_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0.9995 \end{bmatrix}, \ F_{11}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0021 \end{bmatrix}, \ F_{11}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0044 \end{bmatrix},$$
$$F_{11}^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.0065 \end{bmatrix}, \ G_{11} = \begin{bmatrix} 0 \\ 0 \\ -0.0772 \end{bmatrix}, \ G_{11}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ -0.0796 \end{bmatrix},$$
$$G_{11}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0.0416 \end{bmatrix}, \ G_{11}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 0.0380 \end{bmatrix}, \ c_{11} = \begin{bmatrix} 0 \\ 191.5138 \\ 14.8955 \end{bmatrix}.$$

- Mode 12 (8th gear, brake):

$$F_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0.9983 \end{bmatrix}, \ F_{12}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0114 \end{bmatrix}, \ F_{12}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.0051 \end{bmatrix},$$
$$F_{12}^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.0062 \end{bmatrix}, \ G_{12} = \begin{bmatrix} 0 \\ 0 \\ -0.0950 \end{bmatrix}, \ G_{12}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ -0.1071 \end{bmatrix},$$
$$G_{12}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0.0267 \end{bmatrix}, \ G_{12}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 0.0803 \end{bmatrix}, \ c_{12} = \begin{bmatrix} 0 \\ 197.0664 \\ 19.0465 \end{bmatrix}.$$

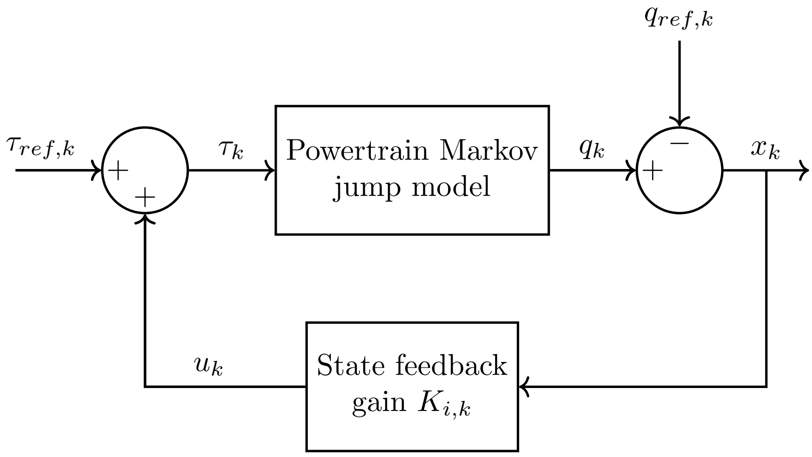

Figure 21 – Block diagram of the powertrain control loop.

- Mode 13 (9th gear, brake):

$$F_{13} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0.9890 \end{bmatrix}, \ F_{13}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0013 \end{bmatrix}, \ F_{13}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.0064 \end{bmatrix},$$

$$F_{13}^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0050 \end{bmatrix}, \ G_{13} = \begin{bmatrix} 0 \\ 0 \\ -0.0671 \end{bmatrix}, \ G_{13}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ -0.0613 \end{bmatrix},$$

$$G_{13}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0.0066 \end{bmatrix}, \ G_{13}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 0.0548 \end{bmatrix}, \ c_{13} = \begin{bmatrix} 0 \\ 201.0889 \\ 24.5609 \end{bmatrix}.$$

- Mode 14 (10th gear, brake):

$$F_{14} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1.0004 \end{bmatrix}, \ F_{14}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.0065 \end{bmatrix}, \ F_{14}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0068 \end{bmatrix},$$

$$F_{14}^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.0003 \end{bmatrix}, \ G_{14} = \begin{bmatrix} 0 \\ 0 \\ -0.0766 \end{bmatrix}, \ G_{14}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ -0.0637 \end{bmatrix},$$

$$G_{14}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0.0177 \end{bmatrix}, \ G_{14}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 0.0460 \end{bmatrix}, \ c_{14} = \begin{bmatrix} 0 \\ 168.9912 \\ 25.5470 \end{bmatrix}.$$

In Fig. 21, we show the powertrain control scheme. The controller signal is $u_k = K_k x_k$, the reference input is $\tau_{ref,k}$, the reference trajectory is $q_{ref,k}$, and τ_k is the control signal sent to the powertrain.

B.4 Transition Probabilities Identification

To estimate the elements of the transition probability matrix $\mathbb{P}_0$, we drove the Scania truck inside the campus of the university of São Paulo at São Carlos. We ensured that the vehicle went through enough gear changes, altitude variations, and transitions between acceleration and deceleration while recording the data from the CAN bus. Then, we calculated the nominal portions $p_{ij}^{(0)}$ of $\mathbb{P}_k$ as

$$p_{ij}^{(0)} = \frac{\kappa_{ij}}{\sum_{r=1}^{14} \kappa_{ir}}, \qquad i, j \in \{1, \ldots, 14\},$$

where κ_{ij} is the number of jumps from mode i to mode j, and κ_{ir} is number of jumps from mode i to mode r. Therefore, we captured the expected behavior of the Markov process and obtained the following nominal transition probability matrix $\mathbb{P}_0$:

$$\mathbb{P}_0 = \begin{bmatrix}
0.9891 & 0.0027 & 0 & 0.0055 & 0 & 0 & 0 & 0.0027 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0070 & 0.9789 & 0.0141 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0003 & 0.9956 & 0.0015 & 0.0017 & 0.0003 & 0 & 0 & 0 & 0.0006 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0040 & 0.9907 & 0.0040 & 0.0013 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0013 & 0.0009 & 0.9940 & 0.0013 & 0.0022 & 0 & 0 & 0 & 0 & 0.0003 & 0 & 0 \\
0 & 0 & 0.0007 & 0.0007 & 0.0013 & 0.9940 & 0.0007 & 0 & 0 & 0 & 0 & 0 & 0.0027 & 0 \\
0 & 0 & 0 & 0.0008 & 0.0024 & 0.0008 & 0.9927 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0033 \\
0.0174 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9739 & 0.0087 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7500 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0169 & 0 & 0 & 0 & 0 & 0.0169 & 0 & 0.9661 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0217 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9783 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0085 & 0 & 0 & 0.0042 & 0 & 0.0042 & 0.0042 & 0.9788 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0034 & 0 & 0 & 0 & 0.0034 & 0.0034 & 0.0034 & 0.9864 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0044 & 0 & 0 & 0 & 0 & 0.0132 & 0 & 0.9825
\end{bmatrix}.$$

Let us now assume a deviation of up to $\pm 30\%$ affecting the elements in the main diagonal of $\mathbb{P}_0$ to produce a more realistic model. This assumption is rather reasonable as different road scenarios, such as traveling in distinct regions of a country, provoke changes in the behavior of the Markov process. That said, the vertices $p_{ij}^{(l)}$, $l = 1, 2$, of the transition probability matrix were built in the following manner:

$$p_{ij}^{(1)} = \begin{cases} -0.3p_{ii}^{(0)}, & \text{if } j = i, \\ 0.3p_{ii}^{(0)}, & \text{if } (i < 14) \text{ and } (j = i + 1), \\ 0.3p_{ii}^{(0)}, & \text{if } (i = 14) \text{ and } (j = i - 1), \\ 0, & \text{otherwise.} \end{cases}$$

$$p_{ij}^{(2)} = \begin{cases} -0.3p_{ii}^{(0)}, & \text{if } j = i, \\ 0.3p_{ii}^{(0)}, & \text{if } (i > 1) \text{ and } (j = i - 1), \\ 0.3p_{ii}^{(0)}, & \text{if } (i = 1) \text{ and } (j = i + 1), \\ 0, & \text{otherwise.} \end{cases}$$

At this point, a couple of remarks are adequate to close the appendix.

Remark B.3. *We considered accelerator and brake pedal positions as control signals of the powertrain so as to design a more general framework. Indeed, the values of* $\tau_{acc,k}$ *and* $\tau_{brake,k}$ *can be translated into voltages or CAN bus signals to be transmitted to the related electronic modules on different ground vehicles.*

Remark B.4. *The identification procedure presented in this appendix also applies if we consider additional payloads and higher longitudinal velocities for the vehicle. If this is the case, we account for gears crawler-1, crawler-2, 1st, 2nd, 3rd, 11th, and 12th for both acceleration and brake actions. As such, the DMJLS would have 14 additional Markov modes. Additionally, we would need at least three more vertices per mode to encompass the payload variations.*