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Essays on Finance and Decision Theory

Ensaios em Finanças e Teoria de Decisão

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Resumo

Esta tese é composta por dois artigos, ambos abordando a questão da tomada de decisão em situações que envolvem risco e incerteza. O primeiro artigo aborda o efeito "Reach-for-Yield", ou seja, o aumento do apetite por risco de agentes econômicos em cenários de taxas de juro baixas. Tais situações são interpretadas como cenários com baixas taxas livres de risco, fazendo com que os agentes econômicos busquem riscos mais elevados para manter seus níveis de retorno. Mais importante ainda, contribuímos para a literatura não apenas ao identificar esse efeito no mercado de fundos de investimento brasileiros, mas também ao investigar empiricamente quais das teorias propostas para explicar esse efeito são plausíveis do ponto de vista empírico.

O segundo artigo, intitulado "Teoria da Saliência com Informação Imprecisa", propõe propriedades para funções de que ponderação de probabilidade que podem ser usadas em conjunto com teorias de decisão de risco estabelecidas (como a Teoria da Saliência de Bordalo et al., 2012). Esta abordagem visa incorporar fatos estilizados derivados do extenso crescimento desta literatura empírica e experimental no campo de decisões sob ambiguidade nas últimas duas décadas. Assim, a nossa proposta procura não apenas acomodar, mas também prever comportamentos em linha com os desenvolvimentos recentes na literatura empírica e experimental.

Abstract

This dissertation comprises two articles, both addressing the issue of decision-making in situations involving risk and uncertainty. The first article addresses the "Reach-for-Yield" effect, that is, the increase in risk-taking by economic agents in low-interest-rate scenarios. Such situations are interpreted as scenarios with low risk-free rates, causing economic agents to pursue higher risks to maintain their return levels. Most importantly, we contribute to the literature not only by identifying this effect in the Brazilian fund market but also by empirically investigating which of the proposed theories to explain this effect are plausible from an empirical standpoint.

The second article, titled "Salience Theory with Imprecise Information", proposes properties for probability weighting functions that can be used in conjunction with established risk decision theories (such as the Prominence Theory of Bordalo et al., 2012). This approach aims to incorporate stylized facts derived from the extensive growth of the literature on decision-making under ambiguity over the past two decades. Hence, our proposal seeks not only to accommodate but also to predict behaviors in line with recent developments in empirical and experimental literature.

Keywords: Decision Under Ambiguity, Salience Theory, Finance, Reach-for-Yield

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Chapter 1

Introduction

The present dissertation comprises two articles, both addressing the issue of decision-making in situations involving risk and uncertainty. The first article discusses the "Reach-for-Yield" effect, that is, the escalation in risk-taking by economic agents in low-interest rate scenarios. Such scenarios are interpreted as periods of low risk-free rates, prompting economic agents to pursue higher risks to maintain their return benchmarks. More crucially, we contribute to the literature not only by identifying this effect in the Brazilian funds market but also by empirically investigating which of the proposed theories to explain this effect are empirically plausible.

The second article, "Salience Theory with Imprecise Information" puts forward properties for probability weighting functions that can be integrated with established decision theories under risk (such as the Salience Theory by Bordalo et al., 2012), aiming to incorporate stylized facts stemming from the extensive progression of this literature over the past two decades. Thus, our proposal aims not merely to accommodate but to anticipate behaviors in line with the recent advancements in empirical and experimental literature.

Chapter 2

Shedding Light on the Causes for Reaching for Yield: Evidence From an Emerging Country

We test the Reach-for-Yield (RFY) phenomenon - agents' greater risk appetite when base interest rates are low - in the Brazilian equity fund market. We find evidence of RFY for equity funds, despite the fact that Brazilian interest rates were well above the zero-lower bound during the sample period. We also test empirically four of the latest theoretical explanations for the RFY effect, finding favourable evidence for three of them: (i) a Behavioral Hypothesis in the spirit of Lian, Ma & Wang (2019) and theories such as Salience (Bordalo, Gennaioli & Shleifer, 2012); (ii) a Manager Skill Heterogeneity Hypothesis, as in the Guerrieri & Kondor (2009) model; (iii) a Budget Constraint Hypothesis, based on the sustainable budget constraint of the Campbell & Sigalov (2022) model. These hypothesis are not mutually exclusive, indicating that a model that integrates these explanations for RFY may be a good venue for future research. To the best of our knowledge, this is the first paper to empirically test different theoretical explanations for the RFY, and also the first one to find direct evidence of this effect in an emerging economy.

2.1 Introduction

The prolonged period of near-zero interest rates in developed markets since the 2010s has aroused the interest of researchers in economics and finance regarding the effects of the extended occurrence of this scenario on the portfolio choices of economic agents. In this context, particular interest has arisen regarding the Reach for Yield effect (RFY), which can be defined as the greater risk appetite by economic agents when basic interest rates are low (Lian, Ma and Wang, 2019; Campbell and Sigalov, 2022)[1] Much of the empirical literature corroborates the occurrence of this effect in developed markets for institutional investors (Boubaker et al, 2017; Di Maggio and Kacperczyk, 2016) and for individuals (Lian et al., 2019). A notable exception, however, is the work of La Spada (2018), who, using a sample of American Money Market Funds prior to 2008, finds that RFY occurs when there is an increase in the market risk premium, but not when there is a decrease in the risk free rate.

RFY contradicts finance conventional theory (Merton, 1971), according to which the risk-taking of agents should not change with variations in the risk-free interest rate. To accommodate the existence of this effect, some of the theoretical model approaches adopted in the literature are highlighted.

First, there are approaches that start from conventional finance models, such as those of Merton (1975) and Black (1972), but changing the consumer's budget constraint, so that their consumption is linked to the expected return of their portfolio in each period. An important consequence of this approach is that agents with higher leverage - and thus, ways to stretch their expected returns in any given period - have the means to exacerbate their RFY, i.e., amplify their increase in risk-taking for a given decrease in the risk-free rate (Frazzini and Pedersen, 2014; Campbell and Sigalov, 2022). Henceforth, we will call the forecasts reported

[1] We note that the RFY term was initially used with a slightly different meaning than the most recent papers on the subject. This concept was initially associated with greater risk-taking by investment funds and financial institutions due to incentives related to fund managers' compensation rules (Rajan, 2006) and imperfections in the risk metrics used in the evaluation of managers' performance (Becker and Ivashina, 2015; Choi and Kronlund, 2018; Czech and Roberts-Sklar, 2019). Two reasons lead us to adopt the more recent concept of RFY as greater risk-taking by economic agents in environments of low basic interest rates: (i) there was consensus in this initial literature that greater risk-taking is exacerbated in environments of low interest rates; (ii) the greater risk taking in low interest rate environments was more recently documented also for individual investors, both in the portfolio compositions of various economic agents (Boubaker et al, 2018; Di Maggio and Kacperczyk, 2017; LU et al, 2019), and in randomized experiments with individuals (Lian et al., 2019).

in this approach the Budget Constraint Hypothesis (BCH).

A second approach highlights the behavioral character of RFY. In this theoretical interpretation, investors form reference points regarding the expected returns of investments. If the risk-free rate falls, for example, the individuals would interpret the new expected return scenario as a loss in comparison with the previous status quo. Assuming that the individual's preferences can be represented by a typical loss aversion function, the individual's portfolio would be located in the loss domain of this function, in which the individual would be risk-loving (Ganzach and Wohl, 2018). Therefore, a decrease in the risk free would induce economic agents to hold riskier portfolios, regardless of the existing institutional arrangements, and their eventual frictions or resulting agency problems (Lian et al., 2019). This hypothesis is also consistent with a Salience-theory like model (Bordalo et al., 2012), where low interest rates make asset returns the more salient characteristic of the assets, estimulating higher risk-taking and, consequently, RFY behavior[2]. It is noteworthy here that this approach provides an important testable prediction: agents with higher reference points (in terms of expected returns) will tend to RFY more. That is, in low interest rate environments, agents that need higher returns to consider themselves in a region of gains (in their loss aversion type value function) tend to have more pronounced RFY. From now on, we will refer to the hypothesis raised by this interpretation of the Behavioral Hypothesis (BH).

A third approach to RFY highlights the heterogeneity of investment managers, and how they may have incentives to engage in RFY behavior to maintain their reputation and investors' perception of their performance high enough to remain in the market. Guerrieri and Kondor (2009), for example, propose a principal-agent model in which investors can "fire" their investment managers if their performance is too low. Managers are heterogeneous, and may be sophisticated (S) - with superior information on asset risk - or unsophisticated (U). Investors, however, do not know if their manager is S or NS, and they only look at the results obtained by their manager in order to try to identify her type. The authors show that, in equilibrium, low interest rate scenarios would be associated with lower risks for the market as

[2] Lian et al. (2019) highlight that there are interpretations of te Salience Theory Under Risk (Bordalo et al., 2012) that could imply a "reverse RFY", i.e., less risk appetite whenever risk-free rates are low. However, the discussion of this interpretation is not the focus of this paper, since the empirical and experimental evidence for "reverse RFY" is rather scarce in the literature.

a whole (e.g., lower default risk due to low interest rates), which would encourage U managers to RFY, so that they can increase their expected returns without incurring a sharp increase in the probability of incurring losses that may imply the firing of theses managers. Thus, U managers would be able to obtain visible performance comparable to S managers, not being "fired" by investors. On the other hand, a high risk-free rate scenario would be associated with high market risks, which would encourage the opposite behavior from U managers (reduce the risk of the managed portfolios), in order to avoid large losses that could result in their "dismissal" . Thus, a prediction of the model is the prediction that more sophisticated investment managers - i.e., those with better information about the risk of assets available in the market - tend to exhibit lower RFY behavior. Throughout this article, the predictions associated with this approach to the literature will be called the Heterogeneous Manager Skill Hypothesis (HMSH).

An additional hypothesis is raised in the principal-agent model proposed by Acharya and Naqvi (2019). Their model suggests that RFY behavior is more pronounced for intermediaries in capital management that have greater liquidity available. This would occur because these capital managers would have incentives to underestimate the penalty generated by the risk of a liquidity squeeze, as this penalty would only be observable for the principal (affecting the fund's returns) in cases where liquidity falls below a certain threshold. Furthermore, expansionary monetary policies - typically associated with low basic interest rates - would exacerbate the risk-taking of financial intermediaries, by reducing the risks associated with lack of liquidity in the market. In this paper, this proposition will be referred to as the Liquidity Risk Hypothesis (LRH).

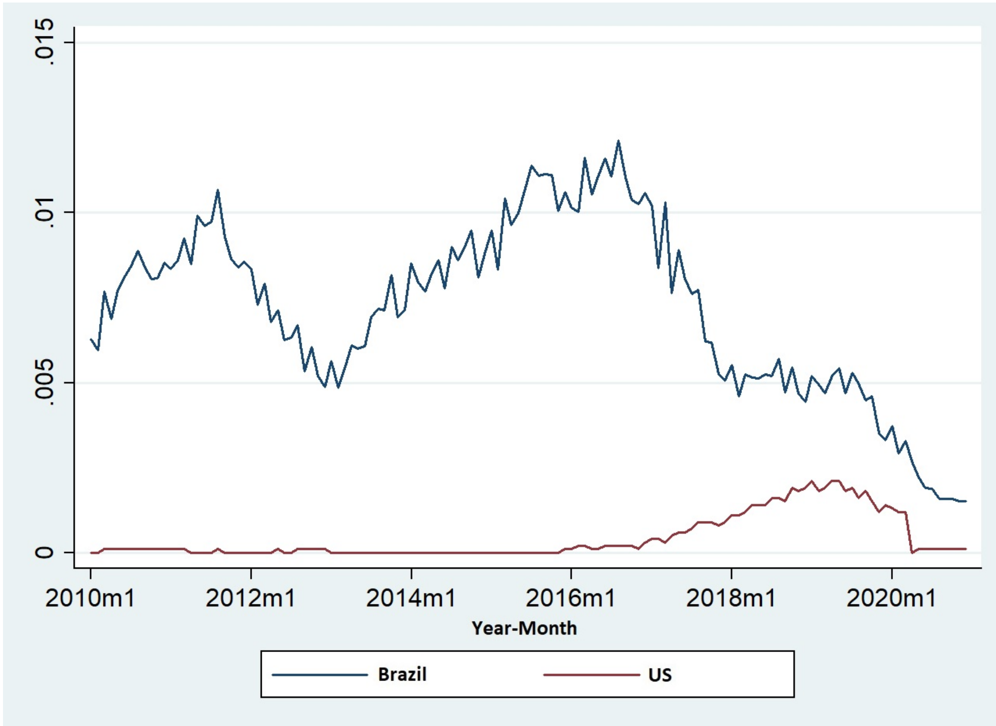
Given the described theoretical propositions, the present study contributes to the existing literature in the following ways: (i) we identify the relationship of characteristics of these funds with the RFY, indicating which models are more plausible to explain the characteristics related to the RFY effect; (ii) we verify the existence of RFY in an emerging country, taking advantage of a base with the monthly composition of equity fund portfolios.

Regarding this last contribution, we note that there are other studies that seek to identify RFY in emerging countries, especially in the sovereign debt securities market. In emerging markets, a series of papers have already specifically documented the decrease in the spread of sovereign debt of emerging countries when there is a fall in international risk-free rates (GONZALEZ-ROSADA and LEVY YEYATI, 2007; FOLEY-FISHER and GUIMARÃES, 2013), which would indicate an RFY movement of international investors, moving their capital from developed to emerging countries and thus affecting the pricing of securities in the latter. Sabbadini (2019) formally proposes a behavioral model - in line with the BH hypothesis proposed here - that seeks to explain this investor response to low risk-free rates. From simulated data, the author shows that his model's predictions are consistent with the investors' RFY.

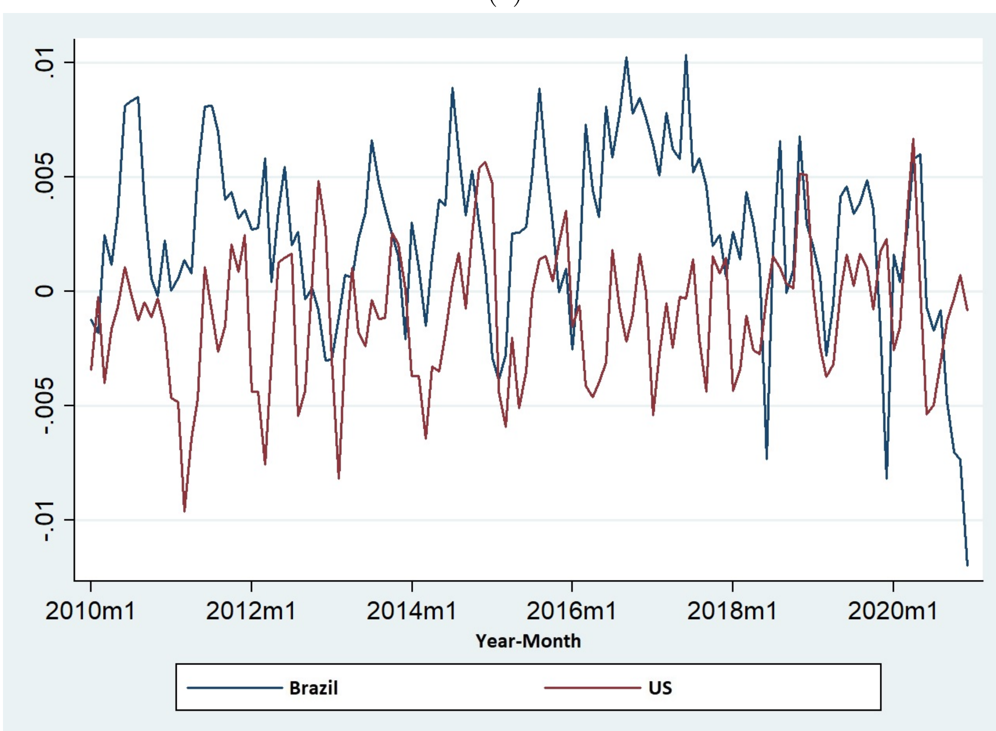
However, as far as we know, this is the first paper that directly tests the effects of changing risk-free rates on RFY behavior in fund portfolios in an emerging country - from the point of view of the local investor. We also use direct evidence from fund portfolio compositions, instead of indirect observation of this phenomenon via market prices (Battarai et al., 2021). Furthermore, it is noted that Brazil, during the sample period (2010-2021), is a particularly interesting case to test some characteristics of the RFY. Unlike developed countries (and even some emerging ones) Brazilian interest rates showed great nominal and real interest rate variability in this period, with periods of increase and decrease of these rates that do not closely follow the international market[3]. These specific characteristics of the Brazilian market allow for any asymmetries in the RFY to emerge from the data - and to differentiate to what extent the international or local interest rates are the drivers of this effect. Figure 1 illustrates some of the statements about Brazil's peculiar macroeconomic scenario in the period studied.

Furthermore, in the period studied there were periods of domestic recession that do not coincide with international recessions, which allows us to empirically disentangle the influence of two potential explanations for the eventual occurrence of the RFY: (a) the argument for the pro-cyclicality of the RFY based on at the time of the economic cycle of the argument, which argues that moments of economic boom would stimulate the RFY due to the lower need to concentrate portfolios in low-risk assets to protect against high systemic risks; (b) the RFY for the variation of the risk-free rate, which conjectures that increases in the risk-free

[3]Illustrates In this scenario, the low correlation between the Brazilian and American risk-free rates in the studied period - of 0.35.



(a)



(b)

Figure 2.1: Brazil and US monthly risk-free rates. Subfigure (a) show the US nominal rate taken from Kenneth French's website (2022), and Brazil´s equivalent, taken from the São Paulo University (USP) Nucleus for Finance Studies' (NEFIN, 2022) website. Subfigure (b) shows the same rates, inflation-adjusted by the US Consumer Price Index (CPI) and Brazil's IPCA inflation index, respectively.

rates are perceived by economic agents as opportunities for attractive returns to risks that are still relatively low, i.e., that low-risk assets have a risk-return ratio more attractive than before.

These features are used to test LRH - which is in line with argument (a) - and BH - which is in line with argument (b). Furthermore, in this article we were able to take advantage of a database that contains characteristics of the fund's regulation, to differentiate from pooled OLS and fixed effect regressions the occurrence of RFY funds by their budget constraint (BCH) and by characteristics that denote differences in the skill of the fund managers (HMSH).

Finally, this paper contributes to the literature by empirically testing different hypotheses raised by several recent theoretical models. In this section, the four theoretical hypotheses to be tested were highlighted: (i) the Budget Constraint Hypothesis (BCH); (ii) the Behavioral Hypothesis (BH); (iii) the Heterogeneous Managers Skill Hypothesis (HMSH); and (iv) the Liquidity Risk Hypothesis (LRH). These hypotheses are not mutually exclusive, i.e., an eventual RFY behavior can be derived from several causes suggested in these hypotheses. For example, it is possible that there is a behavioral factor associated with the formation of a reference point of past return and also an influence of budget constraints that require sustainable spending influencing the RFY behavior of agents. Besides, we identify whether the RFY is found for all classes of funds studied here, and what factors influence this behavior in each class of funds.

The rest of this paper is organized as follows: section 2 details the database and the methodology used; section 3 reports the main results; finally, section 4 highlights the main conclusions of the paper.

2.2 Data and Methodology

A database provided by Economática® was used in this paper. The database contains the end of the month portfolios of investment funds, from December/2009 to December/2021. We filter in the database all actively managed equity funds. To determine the class of an investment fund and if it is actively or passively managed, the following information was used: (i) the classification made by the regulatory agency (CVM, 2020), as recorded in the database; (ii) when (i) was not available, the classification made by Economática®, similar to that of the regulatory agency, was used; (iii) if none of the above information was available for any fund, the classification made by the Brazilian Association of Financial and Capital Market Entities (ANBIMA) was used. Additionally, only funds with more than 12 months of portfolio data were used - to avoid overweighting any seasonal effects. After those filters, we had an unbalanced panel of 4187 funds selected.

Once this classification was made, some additional filters were used to determine the sample of funds considered: (i) if the fund does not have portfolio composition data for December/2021, its' last month is excluded from the sample, to avoid distortions caused by portfolio compositions chosen to meet fund closure procedures [4]; (ii) mirror funds are disregarded[5], to avoid duplicating portfolio decisions of the master fund (from a master-feeder fund structure) in the database [6]; (iii) funds with assets under management of less than R$ 100 thousand [7] are disregarded; (iv) if any position in a single asset by a fund in a given month has value greater than 10 times the funds' net worth, it is also discarded. This last filter avoids that outliers - that are likely due to misreporting by investment funds - distort estimations, and represent about 0.1% of all observations after filters (i) through (iii) were applied.

After the sample was selected, each fund portfolio risk premium was estimated for each month, using the Four-Factor Carhart (1997) model, with betas estimated based on weekly sample returns from January/2001 to December/2021[8]. This model was chosen because the

[4]This treatment is necessary because investment funds that are closing typically increase their holdings of cash and low-risk, high-liquidity assets to pay their shareholders. These choices are related to the fund's closure process, not to portfolio choices that may reflect the agents' risk appetite.

[5]Mirror funds are defined by the Brazilian regulatory agency as funds with at least 95% of their portfolios allocated in other funds of the same class.

[6]To illustrate the need for this treatment, suppose that the fund i is a fund of accounts fund that invests 100% of its capital in fund j. Since the fund j is already considered in the sample, if we also include the fund i, we will be considering that two individuals from the population of investment funds have adopted exactly the same portfolio composition. However, the i fund in the example is usually created to reduce costs and take advantage of possible scale gains, and not because it represents a coincidence of portfolio choices between two different economic agents

[7]about 20 thousand USD in the average exchange rate of the sample period

[8]We take the approach of estimating risk premium for the entire sample - as opposed to rolling window estimates used in other articles (e.g. Frazzini & Pedersen, 2014) - to expand the amount of assets with a sufficient sample so that the estimation of a specific for each asset was possible. This was the same reason why weekly returns were used to estimate the model, since daily returns could distort the risk estimate of

Brazilian market is sensitive to the classic three-factors of Fama & French (1993) and also has significant premium for the momentum factor (Vuong & Vu, 2017). The estimation was made from OLS regressions of the traditional CAPM equation for each asset n with a minimum sample of 30 periods (weeks) of available return data:

$$r_{n,s} - r_{f,s} = \alpha_n + \beta_n(r_{M,s} - r_{f,s}) + \beta_n^{SMB}\ SMB_s + \beta_n^{HML}\ HML_s + \beta_n^{WML}\ WML_s \quad (2.1)$$

where $r_{n,s}$ is the asset n return, $r_{f,s}$ is the risk-free rate and $r_{M,s}$ is market portfolio return - all for the s week. The market, SMB, HML and WML portfolios are extracted from the Nucleus for Finance Studies' (NEFIN) data library (2022)[9]. Since the ex-ante premium for each factor cannot be estimated using Brazilian market data, for a lack of historical sample (Giovanetti et al., 2016), we use the Kenneth French's library factor premiums for each factor, using data from 1927 to 2021, together with the β estimated in the equation above, to get to a risk premium estimate for each asset available in the sample (i.e., $\hat{Risk}_n = \hat{\beta}_n\ R^{US} + \hat{\beta}_n^{SMB}\ SMB^{US} + \hat{\beta}_n^{HML}\ HML^{US} + \hat{\beta}_n^{WML}\ WML^{US}$).

Once this $\hat{Risk}_n$ have been estimated for all assets that meet the minimum sample size requirements, the following procedure is adopted for the other M assets that did not meet the previous estimation requirements: (i) the industry β is used of the issuer's performance of M, estimated for the sample period by the equation (2)[10] ; (ii) the average β of assets of the same type as n is used, considering possible leverage of the risk assumed in the case of derivatives. After this procedure, about 98 % of the database observations (assets in funds' portfolios for each month) had a risk estimate attributed to them. The remaining either used sectorial or asset class risk estimates (about 0.8 %), and the remainder consisted of unidentified/undeclared asset allocations.

assets due to nontrading (a common occurrence in an emerging market), and monthly returns would greatly reduce the number of assets with a representative sample for estimation.

[9]Robustness tests were also made using the Ibovespa Index as the market portfolio. The results obtained are very similar to those reported, and are in agreement with the other results here presented.

[10]The industry returns used are calculated by the Nucleus of Finance of the University of São Paulo (NEFIN, 2021) for the Brazilian market, using a similar methodology to that adopted by French (2022) for the American market. The sectoral classification of issuers of assets used is made by Economática in the database. The reduction procedure is not adopted in this case due to the lower variability of the *beta* of industry portfolios.

Once we have a $\hat{Risk}_n$ estimate for each asset available at the database, the portfolio risk for each investment fund i and month t is calculated:

$$\hat{Risk}_{i,t} = \sum_{n=1}^{N} \omega_{i,n,t} \hat{Risk}_{in} \tag{2.2}$$

where $\omega_{i,n,t}$ is the share of asset n in the composition of the portfolio of fund i at the end of month t.

The fund's portfolio liquidity is also variable of interest in this study, so we measure each asset's liquidity based on the ZEROS indicator, as suggested by Lesmond, Ogden and Trzcinka (1999):

$$ZEROS_n = \frac{1}{D} \sum_{d=1}^{D} \mathit{1}_{n,d} \tag{2.3}$$

where D is the set of days that goes from the first to the latest day when there is a valid end-of-the day price for the asset (in the January/2001 to December/2021 period) for asset n at the database. $\mathit{1}_{n,d}$ is an indicator that assumes the value one if there is a trade registered for the asset n on the day d and zero otherwise. Intuitively, equation (4) calculates the percentage of days in the sample in which the asset n was not traded. Thus, greater values of $ZEROS_n$ indicate that the asset n is more illiquid (or less liquid). From this indicator, the liquidity of the fund's portfolio i in a given month is defined in a similar way to (3):

$$ILQ_{i,t} = \sum_{n=1}^{N} \omega_{i,n,t} ZEROS_n \tag{2.4}$$

After estimating funds' risk-taking, we want to estimate how this variable responds to changes in response to the risk-free rate level - to directly test if there is a RFY effect for Brazilian investment funds. We also want to test how an eventual RFY effect interacts with proxies for the different theoretical hypothesis that explain this behavior. These proxies and the model equation estimated by pooled OLS are detailed in equation (6):

$$\hat{Risk}_{i,t} = \beta_0 + \beta_1 r_{f,t} + \beta_2(r_{f,t}R_{i,t-1}) + \beta_3(r_{f,t}ILQ_{i,t}) + \beta_4(r_{f,t}LEV_{i,t}) + \beta_5(r_{f,t}Q_{i,t}) + \beta_5\mathbf{X}_{i,t} + \epsilon_{i,t} \tag{2.5}$$

where $R_{i,t-1}$ is the i fund return in the twelve months prior to the month t, $ILQ_{i,t}$ is the illiquidity of the fund's portfolio i in t, $LEV_{i,t}$ is a dummy that takes the value 1 if the fund's regulations i allow leverage, and zero otherwise; $Q_{i,t}$ is a dummy that assumes the value one if the fund is intended exclusively for accredited or professional investors. $\mathbf{X}_{i,t}$ is the set of control variables $\mathbf{X}_{i,t} = \{R_{t,t-1}, LIQ_{i,t}, LEV_{i,t}, Q_{i,t}, Real12mGDPg_t, lnSize_{i,t}, Inflation12m_t, \Delta IVOL_t\}$. $Real12mGDPg_t$ is the growth of the Brazilian real GDP accumulated in 12 months (ending at the end of the month t); $lnSize_{i,t}$ is the natural logarithm of the market value of the fund i at the end of of the month t, measured in local currency (BRL); $Inflation12m_t$ is the accumulated inflation in 12 months, measured by the IPCA index; $\Delta IVOL_t$ is the variation of the calculated IVOLBR by NEFIN, and which is an index similar to the VIX for the American market, denoting a measure of expected market volatility [11]. This last variable helps to differentiate variations in portfolio due to changes in the risk expected by economic agents, and the effect of RFY, in which changes in risk taking are only due to the change in the risk-free rate (without this being interpreted as a change in market risk). Equation (7) is estimated by the pooled OLS method, in a similar spirit to the Choi and Kronlund (2017) RFY tests.

his estimate is related to the hypotheses raised in the following way. First, if there is RFY behavior in the studied fund class, it is expected that $\beta_1 < 0$, i.e., the increase in risk of the fund portfolio is related to low risk-free rate environments.

Second, if the Behavioral Hypothesis (BH) is true, $\beta_2 > 0$ is expected, which would indicate higher RFY of funds with a past history of low returns. The idea here is that for a fund that already has low recent returns it is more likely to already have recent returns below the reference point of its loss aversion function. If these agents suffer an additional loss in their expected return due to the fall in basic interest rates, they will tend to become even more risk-loving - thus increasing the risk of their portfolio to compensate for the losses. Here,

[11] This variable is available from August/2011 to December/2021

it is implicitly assumed in the methods that we adopt that the reference point is common for funds of the same class.

Third, if the Liquidity Risk Hypothesis (LRH) is true, $\beta_3 > 0$ is expected, indicating that less illiquid (or more liquid) funds are more prone to RFY. Here, it is assumed that funds with more liquid portfolios are less subject to the risk of their liquidity falling below the threshold that observablely affects the fund's returns (Acharya and Naqvi, 2019).

Fourth, if the Budget Constraint Hypothesis (BCH) is true, $\beta_4 < 0$ is expected, indicating that funds that have access to leverage RFY more. The logic here is that funds with access to leverage would be able to respond more to declines in expected return caused by low interest rates (CAMPBELL and SIGALOV, 2022).

Fifth, if the Heterogenous Manager Skill Hypothesis (HMSH) is true, $\beta_5 > 0$ is expected, indicating that funds intended exclusively for accredited investors have milder RFY behavior. The premise adopted here is that funds intended exclusively for accredited investors are operated by managers who are better informed about the risks of market assets, given that the fund's shareholders are investors who have a degree of qualification and high invested capital, so that fund-market competition should allocate the most qualified managers in this kind of fund. Thus, it is assumed that this type of funds is managed by more qualified and sophisticated managers, who should have a milder RFY, if the HMSH is true.

Finally, the choice of the pooled OLS - besides its previous use in the literature (Choi & Kronlund, 2001) - occurs for two main reasons. First, the estimators are asymptotically unbiased, consistent, and $\sqrt{I}$ normal for panels with large cross-section samples (large I) for a given sample size in the time series (T) (Wooldridge, 2001). This is particularly relevant given the sizes of I and T in our sample. Second, some fund characteristics of interest are not rarely constant over time for the same fund i (such as $Q_{i,t}$ $LEV_{i,t}$). This makes other panel data methods (e.g. Fixed Effects) less interesting than the methodology used here, as they mix the effects of different constant fund characteristics in the estimates. When using this method, it is also important to note that it is implicitly assumed that the intercept and the estimated angular coefficients for each independent variable are the same for funds of the same class. It is argued that this is a reasonable assumption, given that funds of the same class have the same regulatory restrictions on their capital allocation (CVM, 2020).

2.3 Results

A first glance at our sample shows that there are more than 1000 equity funds for all the months in the sample (Fig. 2). The equity funds quantity grows over time with the sample, in line with the Brazilian funds market growth during the sample period (ANBIMA, 2022). In addition, descriptive statistics of the main variables of interest of the model to be estimated are exposed in tables 1 and 2 below.

First of all, it is worth noting that the annual returns obtained by equity funds are relatively low, considering its' country risk premium is about 4.8 % [12], and returns in local currency of equity funds are about 6.9 %. This is a well-known characteristic of the Brazilian market at least in the last three decades, where fixed-income securities - including sovereign debt - have historically had returns close to the local stock exchange, without risk-adjustment. Second, as expected, the Equity funds, which by monetary authority regulation are required to hold at least 67% of their portfolio in stocks[13] keep on average high percentages of their portfolios (between 80 and 90 percent), but also keeps anything between 8 and 15 percent of their portfolios in local sovereign bonds.

Additionally, table 2 shows something that was already invoked in the introduction of this text: the Brazilian risk-free rate is significantly high - being 65bps above its American equivalent, although this rate has fluctuated greatly during the sample period. The consideration of the Brazilian risk-free rate is particularly important in the present study because, from the point of view of a Brazilian investor, the US rate may not represent a risk-free option. Currency risk, which makes the American risk-free option risky from the point of view of a Brazilian investor (assuming that the Brazilian economic agent cares about the purchasing power of its wealth in BRL) helps to explain the reason for this difference.

Moreover, Figure 3 shows graphically the evolution between the average risk (in the cross-section) taken by equity funds per month. Comparing this evolution with that of the Brazilian risk-free rate, it is noted that there are times when the relationship between the fall in the risk-free rate and the increase in the risk taken seems to be more pronounced. Notably,

[12] Data library of Damodaran (2022), considering the average country risk premium between 2001 and 2021

[13] Stock funds must allocate 67% of their portfolio among the following types of assets: actions; Brazilian Depositary Receipts (BDRs); Bonuses, subscription rights and share deposit certificates; share fund quotas (CVM, 2020).

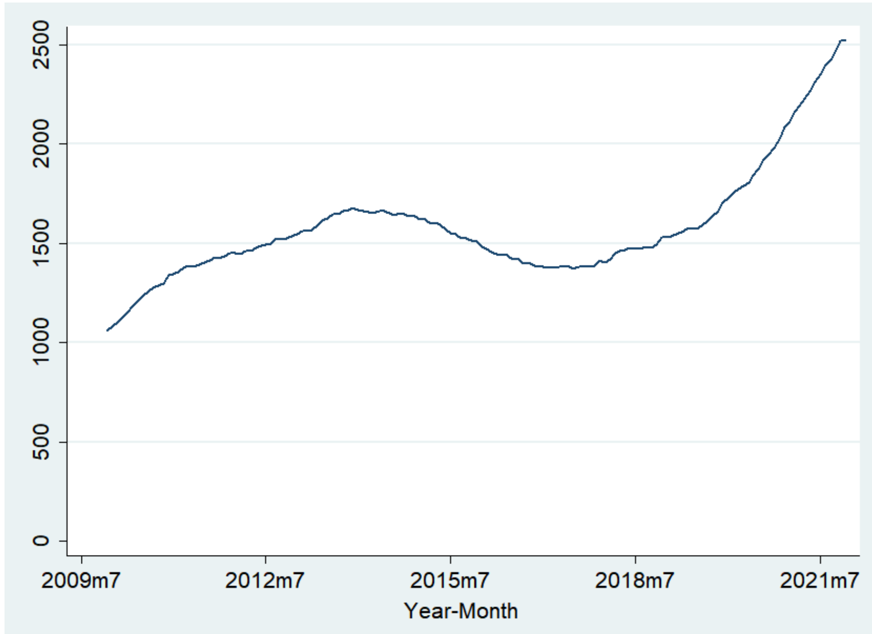


Figure 2.2: Sample Fund Quantity, by fund class and month.

the periods between 2010 and 2012, and from 2017 to 2021 seem to show an increase in risk taking by Equity Funds.

Finally, Figure 4 shows the evolution of average allocations (in the cross-section) among some of the main types of assets.

Table 2.1: Time-Series descriptive statistics. This table contains time-series descriptive statistics (over the 133 months in the sample, dec/2009 to dec/2021) of the cross-sectional mean of each variable.

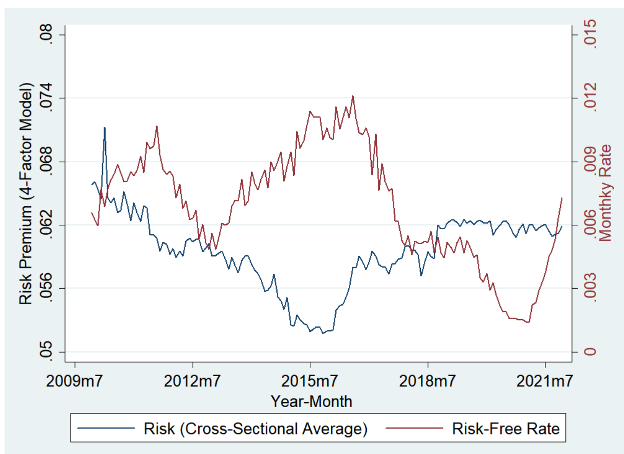
Variable	TS Mean	TS SD	TS Minimum	TS Maximum
$Risk_{i,t}$	0.059	0.004	0.052	0.071
$R_{i,t-1}$	0.067	0.179	-0.218	0.681
$ILQ_{i,t}$	0.040	0.009	0.021	0.054
$LEV_{i,t}$	0.423	0.052	0.355	0.516
$Q_{i,t}$	0.333	0.080	0.220	0.441
$lnSize_{i,t}$	17.32	0.45	16.71	18.17

(a) Equity Funds

Variable	Mean	SD	Minimum	TS Maximum
$r_{f,t}^{BR}$	0.69 %	0.27 %	0.14 %	1.21 %
$r_{f,t}^{US}$	0.04 %	0.06 %	0.00 %	0.21 %

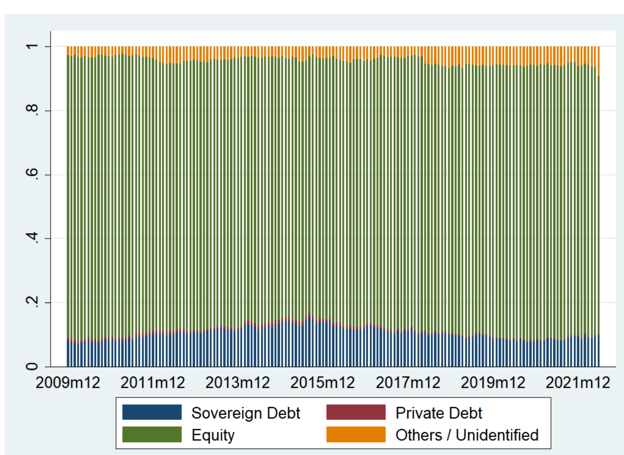
Table 2.2: Monthly Risk-Free Rate (Brazil and US) descriptive statistics. Sample period: jan/2010 to dec/2021.

Figure 2.3: Fund Risk-Taking and Brazil monthly risk-free rates. Each subfigure shows the cross-sectional monthly average CAPM $\hat{Risk}_{i,t}$ for a class of funds - estimated by equation (6). On the right axis (and red line in the graph), the Brazilian Montlhy Risk-Free Rate is shown.



(a)

Figure 2.4: Monthly Fund Allocation. The figure shows the cross-sectional monthly average allocation among asset types. On the right axis (and red line in the graph), the Brazilian Montlhy Risk-Free Rate is shown.



(a)

Table 2.3: Regression Estimation Results. Each column in the table reports estimates and statistics for the Pooled OLS estimates of equation (7) for the entire sample. The Long & Ervin (2001) heteroskedasticity-robust standard errors are in parentheses. $./*/**$ indicates statistical significance at the 10%, 5%, and 1% level, respectively.

Regression Results	
	All Equity Funds (Pooled OLS)
$r_{f,t}$	-0.44 (-2.25)*
$r_{f,t}$ x $R_{i,t-1}$	0.29 (0.93)
$r_{f,t}$ x $ILQ_{i,t}$	-11.02 (-1.85).
$r_{f,t}$ x LEV_i	-0.03 (-0.20)
$r_{f,t}$ x Q_i	0.21 (1.32)
$R_{i,t-1}$	-0.01 (-1.84).
$ILQ_{i,t}$	0.01 (0.05)
LEV_i	-0.01 (-2.03)*
Q_i	-0.01 (-2.38)*
$Real12mGDPg_t$	0.17 (5.49)**
$lnSize_{i,t}$	-0.01 (-13.54)**
$Inflation12m_t$	-0.04 (-4.41)**
$\Delta IVOL_t$	-0.01 (-2.27)*
$constant$	0.08 (37.31)***
$AdjustedR^2$	24.43 %
$F-stat$	1279.00**
Sample Size NxT	232445

2.3.1 Estimation Results

We then proceed to the analysis of the results of the estimation of equation (6) by Pooled OLS, as stated in section 2 of this paper. Regressions are estimated separately for each type of fund analyzed. The results are reported in Table 3 below.

The pooled OLS results already point out to the existence of the RFY effect ($\beta_1 < 0$, statistically significant). This result agrees with the extensive evidence regarding RFY in fixed income funds (BECKER and IVASHINA, 2015; CHOI and KRONLUND, 2017) and in equity funds (KIM and OLIVAN, 2015). However, none of the interactions of the risk free rate with our variables of interest has a statistically significant coefficient. The only one that is significant at 10 % is the iliquidity coefficient - even though it is in the opposite direction compared to what was expected if the LRH was not to be rejected. Otherwise, all the other coefficients of interest have a sign that agrees with their hypothesis.

Table 2.4: Fixed Effects Regression Estimation Results. Each column in the table reports estimates and statistics for fixed effects regressions that ran on subsamples of the fund portfolios panel data, separated by the criterion displayed on the top of each column. For example, the first column is a fixed effects regression for the subsample where $R_{i,t} = 1$. The Long & Ervin (2001) heteroskedasticity-robust standard errors are in parentheses. $./*/**$ indicates statistical significance at the 10%, 5%, and 1% level, respectively.

Regression Results				
	$LEV_{i,t} = 1$	$LEV_i = 0$	$Q_i = 1$	$Q_i = 0$
$r_{f,t}$	-0.50 (-1.68).	0.50 (11.10)**	0.76 (1.08)	-0.30 (-2.34)*
$r_{f,t}$ x $R_{i,t-1}$	0.28 (0.31)	-0.72 (-6.57)**	-1.70 (-1.58)	0.16 (0.40)
$r_{f,t}$ x $ILQ_{i,t}$	-1.78 (-0.83)	-24.23 (-81.10)**	-22.04 (-1.58)	-6.44 (-3.09)**
$R_{i,t-1}$	-0.01 (-0.10)	0.01 (6.49)**	0.02 (1.71).	-0.01 (-0.33)
$ILQ_{i,t}$	-0.17 (-0.79)	0.07 (35.08)**	0.06 (0.67)	0.01 (0.66)
$Real12mGDPg_t$	-0.04 (-0.60)	0.10 (6.13)**	0.14 (3.16)**	0.09 (2.66)**
$lnSize_{i,t}$	0.01 (0.08)	0.01 (8.72)**	0.01 (1.61)	0.01 (0.49)
$Inflation12m_t$	-0.03 (-2.94)**	-0.04 (-7.71)**	-0.01 (-0.25)	-0.05 (-5.40)**
$\Delta IVOL_t$	0.01 (0.63)	-0.01 (-2.31)*	-0.01 (-0.03)	-0.01 (-1.32)
$AdjustedR^2$	-1.42 %	50.21 %	52.40 %	28.00 %
$F - stat$	42.84**	5339.53**	1886.8	238.89**
Sample Size NxT	99516	132929	75648	156797

Table 2.5: Fixed Effects Regression Estimation Results. Each column in the table reports estimates and statistics for fixed effects regressions that ran on subsamples of the fund portfolios panel data, separated by the criterion displayed on the top of each column. For example, the first column is a fixed effects regression for the subsample where $R_{i,t} = 1$. The Long & Ervin (2001) heteroskedasticity-robust standard errors are in parentheses. $./*/**$ indicates statistical significance at the 10%, 5%, and 1% level, respectively.

Regression Results				
	$RWin_{i,t-1} = 1$	$RWin_{i,t-1} = 0$	H$ILQ_{i,t} = 1$	H$ILQ_{i,t} = 0$
$r_{f,t}$	-0.45 (-3.83)**	0.56 (6.72)**	-0.71 (-9.43)**	-0.15 (-0.95)
$r_{f,t}$ x $R_{i,t-1}$			-0.30 (-1.17)	0.16 (0.30)
$r_{f,t}$ x $ILQ_{i,t}$	-3.51 (-1.88).	-24.34 (-56.01)**		
$R_{i,t-1}$			0.01 (2.12)*	-0.01 (-0.22)
$ILQ_{i,t}$	-0.01 (-0.53)	0.07 (24.23)**		
$Real12mGDPg_t$	0.09 (2.52)*	0.08 (2.62)**	0.15 (6.57)**	0.02 (0.46)
$lnSize_{i,t}$	0.01 (0.70)	0.01 (5.29)**	0.01 (0.28)	0.01 (0.66)
$Inflation12m_t$	-0.08 (-7.35)**	-0.01 (-1.18)	-0.02 (-0.81)	-0.03 (-1.91)
$\Delta IVOL_t$	-0.01 (-2.13)*	-0.01 (-0.35)	-0.01 (-2.45)*	-0.01 (-0.30)
$AdjustedR^2$	3.63 %	43.22 %	-2.00 %	-5.42 %
$F - stat$	386.84**	2666.16**	115.23**	20.43**
Sample Size NxT	109880	82603	52203	52203

To investigate further, we run fixed effects models for subsamples that are formed based in the following criteria: (i) we run separate regressions for funds that have access to leverage $LEV_i = 1$ and the ones that don't $LEV_i = 0$. An analogous procedure is done in the subsampling of the Q_i variable; (ii) we define $RWin_{i,t-1}$ as a dummy variable wwith value one for all the observations where $R_{i,t-1} > R_{IBOV,t-1}$, where $R_{IBOV,t-1}$ is the inflation adjusted last twelve (from t-12 to t-1) returns of the most common benchmark used by Brazilian equity funds - the Ibovespa index. $RWin_{i,t-1} = 0$, otherwise. We then subsample the dataset into past winners ($RWin_{i,t-1} = 1$) and past losers ($RWin_{i,t-1} = 0$); (iii) we split the fund observations by portfolio liquidity: above vs. below the median liquidity of all the observations. The idea is that differences in the reaching for yield with a fixed effect for funds of each category should highlight the hypotheses that are more (and least) likely to be true.

With that in mind, we see in tables 3.4 and 3.5 the results of such estimations. As for the regression for the $LEV_i = 1$ and $LEV_i = 0$ subsamples, we see that funds that can access leveraged positions do indeed reach for yield, whilst the unleveraged ones seem to take the opposite approach, by the signal of β_1 in each estimate. However, the regression for $LEV_i = 1$ seem to fit the data poorly, by its' adjusted R^2 and F-stat results, besides the fact that β_1 is only significant at 10 % in that case. Taken altogether, we can consider the evidence that funds that can leverage their positions can be considered weakly favorable and - assuming that leverage is associated with the possibility of reaching for yield - the evidence in favor of the BCH is also only weakly favorable.

As for Q_i, we have a clearer result: funds that are not exclusively destined to accredited investors do reach for yield ($\beta_1 < 0$ and significant), while other funds don't. This is stronger evidence in favor of the HMSH - assuming that the best fund managers are allocated to funds for accredited investors. That does not mean that these managers are able to generate excess risk-adjusted returns, but only that they respond differently to the cycles in monetary policy in terms of risk-appettite. This result contrasts with experimental results such as Lian et al. (2019), that find RFY behavior from amateurs and MBA students. Further research to separate which characteristics of funds and/or managers that explain this diminished sensitivity to the monetary policy is an object for future research

From the signals and significance of β_1 an the $RWin_{i,t-1}$ regressions we actually find that funds that are winning versus the benchmark tend to reach-for-yield more - not less. This finding contrasts with the previous experimental evidence, and suggests venues that contrast with what we posed here as the Behavioral Hypothesis. However, this is not an unseen before result, as evidence of managers that are winning to take higher gambles has been previously documented, and behavioral models of "tournaments" among fund managers can explain the phenomenon (Taylor, 2003). Furthermore, as Lian et al. (2019) points out, Salience Theory of choice under risk (Bordalo et al., 2013) actually predicts reaching against yield for the loss domain (and RFY for the gain domain). It is only the interpretation of RFY coming from the "consumer-theory" version of Salience Theory (Bordalo et al., 2012) - that considers that expected returns become more salient when interest rates change and that dominates the decision-making process - that is incompatible with the results here obtained.

Finally, the fixed effects model still gives us results that go against the proposed LRH, since more illiquid funds (higher $HILQ_{i,t}$) tend to reach for yield more. However, even that statement has to be made with caution, since the F-stats and adjusted R^2 of the equation are low, so that we cannot safely say there is any evidence in the data in favor of the LRH.

2.4 Conclusion

In this paper, we studied empirically four theoretical hypothesis raised to explain the existence of reach-for-yield effect. First, we found evidence of the effect in the Brazilian equity fund market. The occurrence of this effect seem to be more strongly related to the absence of a accredited investor requirements for shareholders. Assuming that the fund manager allocation is related with skill so that more skilled managers get higher earnings, we can affirm that our results support that manager skill heterogeneity may explain partially the existence of this effect, with non-skilled managers reaching for yield trying to boost their returns - so that the fund shareholders can not differentiate if they are skilled managers or not, as in the Guerrieri & Kondor (2009) model.

We also found evidence that funds that are past winners against a benchmark tend to RFY *more*, suggesting that tournament-like behavioral models (Taylor, 2003) or models that consider that losses become more salient in decision-making when the interest rate decreases

(Bordalo et al., 2013) seem to be promising venues to explain the effect. However, models that consider that the change in the reference point of exepected returns due to a decrease in the interest rates (Lian et al., 2019) do not seem suited to explain our results.

Moreover, we found some weak evidence that factors associated with heterogenous budget constraints might relate to the RFY phenomenon. However, we alert that future research is needed to corroborate if in fact agents that heterogeneity in budget constraints can influence RFY behavior, as suggested by Campbell & Sigalov (2022). Finally, we found no evidence that liquidity risk plays a role in the RFY behavior, at least when it comes to equity funds, going against what was conjectured by the model of Acharya & Naqvi (2009), for example.

The development of plausible models that take into account the patterns empirically found in this paper is a topic for future research. Also, exploring more proxies to test the theory as here presented, and expanding this test to other markets is also a promising research subject.

Chapter 3

Salience Theory with Imprecise Information

We propose a generalization of the Salience Model for Choice Under Risk for decision under ambiguity (Bordalo et al., 2012). We achieve so by altering the the probability weighting function for valuation of alternatives in settings of decision under ambiguity. Our model is able to predict stylized facts of the literature, such as: (i) likelihood insensitivity, the fact that people tend to overestimate(underestimate) the expected probability of low(high)-likelihood events to happen in ambiguous settings - in comparison with non-ambiguous ones; (ii) the fourfold pattern of ambiguity attitudes, where decision-makers are usually ambiguity averse for bets on high probability gains and low probability losses, and ambiguity seeking otherwise. A key feature of our model is that ambiguity attitude is a result of the combination of: the outcome domain (gain or loss) of a bet; the expected probability of the outcome combinations that are relevant to the outcome of the bet; how many outcome combinations are deemed possible by the decision maker. The model may be interpreted as representing how decision-making under ambiguity is affected by bottom up attention - i.e., how the context of a decision-making setting may be in conflict with the decision-maker's goal.

3.1 Introduction

Since Ellsberg's (1961) seminal thought experiments for choice under ambiguity, decision-making behavior under ambiguity has been studied. Specifically, theoretical models that deal with decision under ambiguity try to accommodate and/or describe decision-makers (DMs) behavior when dealing with decisions involving uncertainty where assessing a unique probability distribution (even if subjective) of events relevant to the outcomes of the possible courses of action is scarce and/or imprecise. Ambiguity attitudes have received much attention as a possible explanation for economic phenomena such as the equity premium puzzle (Rieger & Wang, 2012), stock market non-participation (Antoniou et al., 2015; Dimmock et al., 2016) and home bias in portfolio choices (Dimmock et al., 2016; Ardalan, 2019).

We contribute to the theoretical literature by extending the Salience Theory for Choice Under Risk (STR) to decision under ambiguity contexts. To do so, we add to the original model a probability weighting function that reflects DMs' behavior for second-order probability representations of decision under ambiguity problems. This extended version of the model is able to predict many empirical regularities of the experimental literature, such as: (i) likelihood insensitivity, the idea that under ambiguity DMs tend to distort the weighting of events in the evaluation of an act in the direction of a naive probability distribution (for example, a 50-50 naive probability distribution for a bet with two possible outcomes) (Dimmock et al., 2013; Trautmann & van de Kuilen, 2017); (ii) the fourfold pattern of ambiguity attitude, meaning that decision-maker's are ambiguity averse for moderate-to-high probability gains and low-probability losses, but ambiguity seeking low-probability gains and moderate-to-high probability losses (see Table 1 below). For example, when comparing a risky (with objectively know probability of win) and an ambiguous bet (with imprecise information on the probability of win) that involve the possibility of gaining a positive value with small expected probability, DMs usually tend to choose the ambiguous act (be ambiguity seeking). However, if the bet now involves the same gain, but events with mid to high expected probability of the win outcome, now the DM will usually prefer the risky bet (Trautmann & van de Kuilen, 2015)

More specifically, our model can be interpreted as relating the stylized facts above in the

Table 3.1: Ambiguity Attitudes as a function of expected probability and the outcome domain (loss or gain).

	Domain	
Expected Probability	Loss	Gain
Low	Ambiguity Aversion	Ambiguity Seeking
Mid/High	Ambiguity Seeking	Ambiguity Aversion

following way: once faced with imprecise information about the probabilities of events that are relevant to the outcomes of a number of alternative courses of action, the decision-maker "fills" the information gap with other information that she is able to assess from the choice set, such as the number of possible events framed as relevant to the decision. This use of the number of possible states of the world induced by the choice set description causes likelihood insensitivity, interpreted as the DM's distortion in weighting of the events toward a naive equal probability for each relevant event deemed possible. Therefore, likelihood insensitivity causes the fourfold pattern of ambiguity attitudes: for ambiguous acts, it causes an underweight of high likelihood events, resulting in an pessimistic (optimistic) view of expected gains (losses), causing ambiguity aversion (seeking) behavior. Conversely, ambiguous acts contingent on low likelihood events are overweighted, causing ambiguity seeking (averse) behavior for gains (losses).

The central idea of our paper of the choice set outcomes shaping the DM's perception of the state-space has been applied to other decision contexts. Specifically, the psychological rationale of bottom-up attention, i.e., a stimulus caused by the specific context of a choice problem attracting the decision maker's attention "bottom up," automatically and involuntarily, is the main driver of Salience Theory of Choice Under Risk (STR) (Bordalo et al., 2021). As may be expected, our probability weighting function is then particularly well suited to be applied to extend STR for Choice Under Ambiguity. We explore as a base case this way to apply the probability weighting function as a theory of Choice Under Ambiguity that has STR as a special case - when the decision collapses to risky acts. Salience Theory has also already been tested and used to explain a variety of stylized facts, such as the tendency to take right-skewed tisks and avoid left-skewed ones (Kahneman & Tversky, 1979) and Allais paradoxes (Bordalo et al., 2012)[1].

In the theoretical literature about decision under ambiguity, our model's representation of ambiguity closely relates to previous models that incorporate the treatment of ambiguity as imprecision about objective information on events' probabilities with the subjective multiple priors approach of Gilboa & Schmeidler (1989). Gajdos et al. (2008) interpretation of ambiguity is an example that closely relates to our approach. However, in the model we propose ambiguity attitude is not a function of some combination of pessimism or optimism, but is a result of the combination of the outcome domain, expected probability of events. As we will show throughout the paper, choice set characteristics such as the number of possible relevant outcomes will also help to define how low an event likelihood has to be to be in the "low probability" row in Table 1.

To the best of our knowledge, our model is the first one that incorporates an interpretation of ambiguity as imprecise objective information that is simultaneously able to predict likelihood insensitivity and the fourfold pattern of ambiguity attitudes. We take advantage of the rapid growth in the experimental and empirical literature on decision-making under ambiguity to make our model more in line with the regularities in DMs behavior found in the literature.

The rest of this paper is organized as follows. Section **??** presents a running Ellsberg-urn example that is going to be used throughout the paper to explain the concepts and consequences of our model. Section 3.2 gives the preliminaries of our model: how our second-order belief approach describes decision under ambiguity and some background on Salience Theory. Section 3.3 presents our Context-Weighting model, and its' implementation as a generalization of Salience Theory of Choice Under Risk (Bordalo et al, 2012). In the subsection within this section, we describe the model's postulates about how the model represents decision-making under ambiguity and, whenever possible, illustrate how theses postulates affect DM's preferences and weighting of act outcomes. For concreteness, we also provide a parametric example of a Context-weighting function, and apply it to our running example and some cho-

[1] However, we note that any theory that results in an expected value function that is not rank-dependent (such as the original Prospect Theory of Kahneman & Tversky (1979) or Regret Theory (Loomes & Sugden, 1982). The detailed development of the consequences of our proposal for the probability weighting function to these other theories is left for future research.

sen modifications of it that help to illustrate our model's properties. Section 3.4 illustrates how Choice set characteristics, DMs beliefs and the Outcome Domain of act payoffs influence ambiguity attitude in our model. We also analyze the Machina reversal problem and how events with correlated probability are dealt with. We follow with Section 3.5, where we compare our model's characteristics and results with other popular second-order belief models in the literature. Finally, we draw conclusions and give suggestions for future research.

3.2 Preliminaries

3.2.1 Decision Under Ambiguity Representation

We start setting up the preliminaries of our model, that follow objective ambiguity models that extend the multiple-prior approach of Gilboa & Schmeidler (1989), such as Gajdos et al. (2008). Let X be an arbitrary set of outcome combinations, defined on the $\mathbb{R}^N$ space, without loss of generality. Each element of X is denoted as $\mathrm{x_i}$. $\Delta(X) = \{\pi(\mathrm{x}_1), \pi(\mathrm{x}_2), ...\}$ is a set of simple distributions over X. A pure outcome is defined as $L^n = \Delta(X) \times e_n$, where e_n is the canonical basis vector in $\mathbb{R}^N$, that is, a N-dimensional vector that has its' n-th component equal to one, and all other components equal to zero. $L = \{L^1, ..., L^N\}$ is the set of pure outcomes of the choice problem. For simplicity, we denote $x^n \pi y^n$ the lottery that pays x^n with probability π and y^n otherwise. This first part is close to a von-Neumann Morgenstern setting, but we explicitly define X as vectors of outcome combinations of distinct lotteries, similarly to the Bordalo et al. (2012, p. 1253) representation.

Additionally, we define Θ as a non-empty countable finite set of states of the world, and $\mathcal{A}(\Theta)$ the family of nonempty subsets of Θ. $\Delta(\mathcal{A}(\Theta))$ is the set of probability measures defined for each $\theta \in \Delta(\mathcal{A}(\Theta))$. Let $\mathcal{P}(\theta)$ be the family of compact and convex subsets of $\Delta(\mathcal{A}(\Theta))$[2], and $\mathcal{P} = \bigcup_{\theta \in \mathcal{A}(\Theta)} \mathcal{P}(\theta)$ the family of *probability-possibility sets* occurring with positive probability[3]. Let $\mathcal{F} = \{f : \Theta \to \Delta(X)\}$ be the set of lottery acts - where each act

[2]Compactness is defined here with respect to the Euclidean space $\mathcal{R}^{\mathcal{A}(\Theta)}$

[3]For the sake of notational simplicity, we consider that all probability distributions are defined over Θ and that $p(\theta) = 0 \forall \theta \in \Theta \setminus supp(p)$, where $supp(p)$ indicates the support of the probability distribution p. In other words, any states of the world not in the support of a probability-possibility set p is a null-probability state under p. This is similar to Gajdos et al.(2008) representation.

can be viewed as a constant act which delivers that lottery regardless of the states. The decision-maker's preferences are defined over the $\mathcal{P} \times \mathcal{F} \times \{L\}$ space - the choice set.

The decision-maker then compares pairs of probability-possibility sets $p \in \mathcal{P}$ and how they affect the probabilities of the outcome of each lottery being true - conditional on all the other possible outcomes that could have been obtained if another lottery was chosen. Note that in this setting acts map states of the world into von-Neumann Morgenstern lotteries defined on the set of combinations of outcomes X^N, and not directly on outcomes. This setting follows closely other second-order belief models of objective ambiguity, such as Gajdos et al. (2008) and Olszewski (2007).

For concreteness, let's apply the definition to an Ellsberg urn example with a discrete state-space, to illustrate the model's representation. Suppose there is an urn with 3 balls. The decision-maker (DM) knows has objective information that: (i) there is 1 red ball in the urn; (ii) the remaining 2 balls are either black or yellow, in proportion unknown. We assume this information is given to the DM at the outset, and cannot be modified by her.

The DM is then offered the lotteries L^1 and L^2, contingent on the color of a single ball drawn from the urn. Table 3.2 below describes the monetary outcomes associated with each possible color of the drawn ball:

Table 3.2: Outcome matrix of example 1, an Ellsberg urn ambiguity example with two lotteries (L^1 and L^2).

	Red	Yellow	Black
L^1	100	0	0
L^2	0	100	0

It is easy to see that the outcome combination represented by "a red ball is drawn" is unambiguous, whilst "a yellow ball is drawn" and "a black ball is drawn" are ambiguous. According to our preliminaries, this problem could be described as follows: $X = \{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3\}$ is the set of outcome combinations, where $\mathrm{x}_1 = (100, 0), \mathrm{x}_2 = (0, 100), \mathrm{x}_3 = (0, 0)$ represent the outcomes of the lotteries when a red, yellow or black ball is drawn, respectively. $L = \{L^1, L^2\}$ is the set of lotteries, where $L^1 = 100\pi_r 0$, $L^2 = 100\pi_y 0$. π_r and π_y are the probabilities of a red and a yellow ball being drawn - given a true composition of the urn $\theta \in \Theta$, respectively.

$\Theta = \{\theta_1, \theta_2, \theta_3\}$ is the set of possible ball compositions of the urn, given the objective information available about it. In the example, this can be represented as $\theta_1(\pi(\mathrm{x}_1), \pi(\mathrm{x}_2), \pi(\mathrm{x}_3)) = (1/3, 0, 2/3)$, $\theta_2(\pi(\mathrm{x}_1), \pi(\mathrm{x}_2), \pi(\mathrm{x}_3)) = (1/3, 1/3, 1/3)$, $\theta_3(\pi(\mathrm{x}_1), \pi(\mathrm{x}_2), \pi(\mathrm{x}_3)) = (1/3, 2/3, 0)$. Note here that a state-space represents a true ball composition of the urn, and not an outcome realization - following a common way to represent objective ambiguity in the literature (Gajdos et al., 2008).

The probability-possibility set $\mathcal{P} = \{p(\theta) : p(\theta) \in (0, 1]\forall\theta \in \Theta \text{ and } \sum_{\theta\in\Theta} p(\theta) = 1\}$ set represents the DM's beliefs about the probability of each ball composition $\theta \in \Theta$ being the true one - representing all states θ that may happen with positive probability. For concreteness, consider a decision maker that believes that either: (i) with 50% chance, the urn has one ball of each color, and with 50% chance the urn has one red and two yellow balls; (ii) there is a 1/3 chance of each ball composition being true. Then, $\mathcal{P} = \{p_1, p_2\}$, where $p_1(\theta_1, \theta_2, \theta_3) = (0, 1/2, 1/2)$ and $p_2(\theta_1, \theta_2, \theta_3) = (1/3, 1/3, 1/3)$.

Given the DM's beliefs about possible urn composition and how likely each composition is ($\mathcal{P}$), and the objective combination probabilities of each outcome combination given by each lottery ($\Delta(X)$), the DM decides which lottery he wants to bet on.

In the next subsections, we comment on two main assumptions of the model, and on the Salience (Bordalo et al., 2012) representation that we primarily use in this paper, and how we represent the DM's valuation of a lottery through a value function.

3.2.2 Salience Representation

Following the BGS Model (Bordalo et al., 2012), we define a continuous and bounded salience function for each combination of outcomes. Denote x^n as the pure outcome that is associated with the outcome combination x if lottery L^n is chosen, and x^{-n} the vector of outcomes that would be obtained if lotteries that are not L^n were chosen. Then, define a function $\omega : X \to \mathbb{R}$ so that, $\omega(\mathrm{x}^n, \mathrm{x}^{-n})$ is considered a salience function if it satisfies:

(i) Ordering: if $\mathrm{x}^n = \max \mathrm{x}$, then for any $\epsilon, \epsilon' \geq 0$, with at least one strict inequality:

$$\omega(\mathrm{x}^n + \epsilon, \mathrm{x}^{-n} - \epsilon') > \omega(\mathrm{x}^n, \mathrm{x}^{-n})$$

if $\mathrm{x}^n = \min \mathrm{x}$, then for any $\epsilon, \epsilon' \geq 0$, with at least one strict inequality:

$$\omega(\mathrm{x}^n - \epsilon, \mathrm{x}^{-n} + \epsilon') > \omega(\mathrm{x}^n, \mathrm{x}^{-n})$$

(ii) Diminishing sensitivity: if $\mathrm{x}^n > 0$ for all n s. t. $L^n \in L$, then for any $\epsilon > 0$,

$$\omega(\mathrm{x}^n + \epsilon, \mathrm{x}^{-n} + \epsilon) < \omega(\mathrm{x}^n, \mathrm{x}^{-n})$$

(iii) Reflection: for any two outcome combinations $\mathrm{x}_i^n, \mathrm{x}_j^n$ s.t. $L^n \in L$ and $\mathrm{x}^n > 0$ for all n, we have:

$$\omega(\mathrm{x}_i^n, \mathrm{x}_i^{-n}) < \omega(\mathrm{x}_j^n, \mathrm{x}_j^{-n}) \Leftrightarrow \omega(-\mathrm{x}_i^n, -\mathrm{x}_i^{-n}) < \omega(-\mathrm{x}_j^n, -\mathrm{x}_j^{-n})$$

Heuristically, ordering affirms that states with more disperse payoffs are more salient. Diminishing Sensitivity makes sure that, all else constant, states with payoffs closer to the origin are more salient. Reflection makes sure that the ideas of ordering and diminishing sensitivity are also valid for negative payoffs.

Another important remark is that the average $\bar{\mathrm{x}}^{-n} = \sum_{m \neq n} \frac{1}{N-1} \mathrm{x}^m$ can substitute the set x^{-n} as an argument of the salience function, while keeping the properties (i), (ii) and (iii). In other words, $\omega(\mathrm{x}^n, \bar{\mathrm{x}}^{-n})$ is also a salience function (Bordalo et al., 2012).

The salience function is meant to represent how a stimulus attracts the DM's attention "bottom up", that is, how perceived characteristics of a state of the world within a specific set of possible courses of action may impact decision-making. This concept contrasts with the traditional economic approach, which views attention as either unlimited or optimally allocated "top-down" based on current goals and expectations. This approach does not highlight that "bottom up" stimulus-driven attention may compete with the DM's "top down" goals (Bordalo et al., 2022). As Kahneman (2011, p. 324) puts it, "our mind has a useful capability to focus on whatever is odd, different or unusual". Salience Theory calls the payoff combinations that draw the decision maker's attention "salient".

An example is a DM confronted with the decision of using \$10 that he has in his pocket to buy a lottery ticket for a 0.001% chance of winning a \$1,000,000 dollar prize, or investing that money for a sure outcome of \$11 (the inicial \$10 plus a \$1 return). The expected value of betting in the lottery is \$0 ($= 0.001\% \times (\$1,000,000 - \$10) + 99.999\% \times (\$ - 10)$), clearly less than the sure \$11 outcome of investing. In the standard Expected Utility Theory approach, it is evident that the risky bet's expected value being lower than the sure value of investing implies that any individual who is not risk-seeking should choose to invest. However, in reality, just the perspective of winning such a huge prize may tempt the DM to bet in the lottery instead.

That is an example of the high contrast between an outcome (receiving the lottery's prize), conditional on an outcome combination being true (the outcome combination that happens when numbers drawn in the prize draw match the lottery ticket, in the example) and a course of action chosen (buying the lottery ticket), compared to the outcome when a different course of action is taken (getting $11 for investing). Salience theory interprets this as the outcome combination associated with "the numbers drawn in the prize draw match the DM's ticket" being highly salient, because different choices lead to wildly different outcomes if this state is true. As a result, the state of the lottery prize going to the available ticket may have more weight in the DM's (called a Local Thinker by Bordalo et al. (2012)) evaluation of his options than the weight a Bayesian decision-maker would, making her more inclined to buy the ticket for a chance of winning that huge prize. That change in weights given to each state represents the DM's fear of regretting that she did not buy the lottery ticket when it is the winning ticket.

A slight change to our running example illustrates how salience may affect the weight given by the DM to each state, conditional on the acts' outcomes, in a ambiguity decision setting. Say we add $\epsilon > 0$ to the outcome combination associated with a red ball eing drawn from the urn for both lotteries (L^1, L^2) (Table 2 below).

Table 3.3: An Ellsberg-like ambiguity example (Example 2). The difference between this example and example 1 is that $\epsilon > 0$ is added to the outcomes associated with a red ball is drawn from the urn being true, for both L^1 and L^2 lotteries.

	Red	Yellow	Black
$L^{1,\epsilon}$	100+ϵ	0	0
$L^{2,\epsilon}$	0+ϵ	100	0

Then, by diminishing sensitivity, $\omega((100+\epsilon, 0+\epsilon)) < \omega((100, 0)))$. That would result in an overweight of the outcome combination $(100, 0)$ associated with a yellow ball being drawn from the urn. That is, the difference between \$100 and \$0 outcomes in favor of $L^{2,\epsilon}$ when a yellow ball is drawn seems now more attractive than the $\$100+\epsilon$ versus $\$0+\epsilon$ difference in outcomes when a red ball is drawn. That would result in a more favorable view of the $L^{2,\epsilon}$ option. If in example 1 the DM's preferences are $L^1 \succ L^2$, then for Salience Theory there is some $\epsilon > 0$ such that $L^{1,\epsilon} \preceq L^{2,\epsilon}$. This representation sharply contrasts with standard economic theory, specifically with the Sure-Thing Principle of Subjective Expected Utility

(SEU) Theory (Savage, 1954). For SEU, the addition of ϵ as portrayed necessarily does not alters preferences.

All of that considered, the value function that represents the DM's preferences and the effect of Salience in their choice is given by:

$V(L^n) = \sum_{\mathrm{x} \in X} \pi(\mathrm{x})\omega(\mathrm{x})v(\mathrm{x}^n)$

where $\omega(\mathrm{x})$ is a monotone increasing function of the salience of payoff combination x, given the set L. $v(\mathrm{x}^n)$ is the value of the outcomes associated with the choice of lottery n when the outcome combination x is true. $\pi(\mathrm{x})$ is the probability of the outcome combination x happening[4]. Since Bordalo et al. (2012) propose a theory of choice under risk - not yet considering the case of decision under ambiguity - they assume that $\pi(\mathrm{x})$ are objective and known probabilities that exist for each $\mathrm{x} \in X$. In our model, we build upon the relaxation of that assumption.

3.3 The Model

3.3.1 Main Assumptions

Next we describe two main assumptions of our model. The first one concerns the representation of preferences over constant acts - i.e., acts that do not involve risk nor ambiguity and result in the same outcome no matter what is the true state of the world. The assumptions asserts that there is a value function that represents the DM's preferences over such acts, with the standard properties of other similar models.

(A1 - Value Function on Acts) Let $\mathrm{x}^i, \mathrm{x}^j$ be constant lotteries, with outcomes $\mathrm{x}^i, \mathrm{x}^j \in \mathrm{x}$ obtaining with probability one regardless of the true state of the world, respectively. Then, there exists a value function $v : \mathbb{R} \to \mathbb{R}$, continuous, strictly increasing, and normalizable so that $v(0) = 0$ such that, $\mathrm{x}^i \succeq \mathrm{x}^j$ if and only if $v(\mathrm{x}^i) \geq v(\mathrm{x}^j)$.

Our second assumption states that the choice problem analyzed with our problem is not trivial, i.e., there is at least one act that is not constant in the choice set.

[4] that is, the n-th element of the x vector

(A2 - Nontriviality) By contrast, define a non-constant lottery L^n as one where at least two outcomes $\mathrm{x}_j^n, \mathrm{x}_k^n$ s.t. $\mathrm{x}_j^n \neq \mathrm{x}_k^n$ can obtain with probability greater than zero. Then, we assume the lottery set L contains at least two non-constant lotteries.

With that taken care of, we can move to the main definition of our model in the next section.

3.3.2 Ambiguity Adjustment Function

Given our preliminary setting, we define a Ambiguity Adjustment Function Ψ, that represents how probability weighting representing preferences may be affected by event ambiguity. We also postulate the function's properties that are sufficient for the main predictions of our model.

Definition 1 (Ambiguity Adjustment Function Ψ): Define a function $\Psi : \mathcal{P} \times \Delta(X) \to [0, 1]$. Let $\pi(\mathrm{x})$ be the probability of outcome combination x being true. Denote $p(\pi(\mathrm{x}))$ a probability distribution on $\pi(\mathrm{x})$. $\mathbb{E}[\pi(\mathrm{x})] = \sum_{\theta \in \Theta} p(\theta)\ \pi(\mathrm{x}|\theta)$ is the expected probability $\pi(\mathrm{x})$ of outcome combination x occurring, and $\sigma(\pi(\mathrm{x}))$ its' standard deviation. Denote $||X_n||$ as the cardinality of the set of outcome combinations where an outcome for lottery L^n is defined. Then, Ψ is a Salience Ambiguity Function if:

(P1) For a fixed $\sigma(\pi(\mathrm{x})) = \sigma$, $\Psi(p(\theta(\pi_{\mathrm{x}})))$ is increasing on $\mathbb{E}[\pi(\mathrm{x})]$;

(P2) Ψ is decreasing on $||X_n||$, the cardinality of X_n;

(P3) For fixed $\mathbb{E}[\pi(\mathrm{x})] = \bar{\pi}(\mathrm{x})$ and X_n, $|\Psi(\pi(\mathrm{x})) - \bar{\pi}(\mathrm{x})|$ is decreasing on $\sigma(\pi(\mathrm{x}))$, where $|a|$ denotes the absolute value of a. Moreover, if $\sigma(\pi(\mathrm{x})) = 0$, then $\Psi(\pi(\mathrm{x})) = \mathbb{E}[\pi(\mathrm{x})]$.

Considering Definition 1, we propose that a decision-maker faced with a choice under ambiguity values lotteries according to the equation below:

$$V(L^n) = \sum_{\mathrm{x} \in X_n} \Psi(\pi(\mathrm{x}), ||x_{\mathrm{x}}||)\ \omega(\mathrm{x})\ v(\mathrm{x}^n)$$

Note that the Ψ function aggregates the objective information on the probabilities π of x being the resulting outcome combination, given the subjective probabilities $p \in \mathcal{P}(\theta)$ of each state $\theta \in \Theta$ being true. Also, as we are going to see in later sections, the inclusion of the cardinality of x is crucial for our model to represent the likelihood insensitivity behavior for choice under ambiguity documented in the experimental literature (Dimmock et al., 2013).

In the next section, we illustrate the properties of the function with additional Ellsberg urn examples.

Illustrating Properties of the function

We proceed to give examples that illustrate why each property is important, and its' consequences to the representation of choice under ambiguity and preferences in this circumstances of decision-making. Take a modification of our Ellsberg-urn example. Take again the urn with 3 balls of example 1. Assume that the only states of the world (ball compositions of the urn) with positive probabilities (in the *probability-possibility* space) are: 1 red ball, 0 yellow balls and 2 black balls as (θ_0, with $\pi(\mathrm{x}_y|\theta_0) = 0$; 1 ball of each color as θ_1 (with $\pi(\mathrm{x}_y|\theta_1) = 1/3$); 1 red ball, 2 yellow balls and no black balls as θ_2 (with $\pi(\mathrm{x}_y|\theta_2) = 2/3$). Additionally, suppose that $\mathcal{P} = \{p_1, p_2\}$, where $p_1(\theta_0, \theta_1, \theta_2) = (1/2, 1/2, 0)$ and $p_2(\theta_0, \theta_1, \theta_2) = (0, 1/2, 1/2)$. Consider the probability of a yellow ball being drawn from the urn, $\pi(\mathrm{x}_y)$. It is easy to see that the standard deviation of $\pi(\mathrm{x}_y)$ is the same under p_1 and under p_2, but the expected value of $\pi(\mathrm{x}_y)$ is higher under p_2.

In our model, we interpret the standard deviation $\sigma_\pi(\mathrm{x}_y)$ being the same for p_1 and p_2 as the same level of ambiguity applying for both sets of beliefs. However, since one of the (second-order) probability distributions results p_2 results in a higher expected probability of the outcome combination x_y obtaining, then the weight given to that outcome combination when evaluating lotteries should be higher. That is, the value of the weighting function Ψ should be higher in the case where the DM's beliefs are represented by p_2 than in p_1.

However, we highlight that - depending on the specific form of the probability distributions p representing the DM's beliefs, it is possible that an increase in the expected probability of an outcome combination x results in a decrease in the weighting $\Psi(\mathrm{x})$ due to P3 property. We will show an example of this situation when we state our parametric example of Ψ, but for now it suffices to say that a tradeoff between more precise information about the probability of an outcome combination and the belief on a higher probability of the outcome happening may affect the weighting of that outcome combination in lottery evaluation in both directions. That is so precisely to represent that - contrary to other models of objective ambiguity (Gajdos et al., 2008) - we do not predict that DM's are always averse to imprecision in probability information, but that their attitude towards that imprecision depends on how they "fill the gap" using other information available in the choice problem.

That gets us to property P2: that Ψ is decreasing on the cardinality of $||X_n||$, the cardinality of X_n. This property represents how we expect our model to represent the "gap filling" of the imprecise information about outcome combination probabilities. Note that $||X||$ is just the quantity of outcome combinations that are possible in the problem. In our Ellsberg urn example 1, there are three outcome combinations possible: $\mathrm{x}_r = (100, 0)$, $\mathrm{x}_y = (0, 100)$ and $\mathrm{x}_b = (0, 0)$, representing the situations where a red, yellow or black ball is drawn from the urn, respectively. That is, there are three possible outcomes combinations to consider - and this information is used in the DM's weighting of combinations.

This property makes sure that, if a new outcome combination is added to the problem, the weighting of every outcome combination that has ambiguous probability of happening is diminished. To illustrate, take again the setting of example one, except for that we add one green ball to the urn, and both lotteries L^1 and L^2 pay \$150 if the green ball is drawn from the urn. Now there is a new outcome combination in the problem $\mathrm{x}_g = (150, 150)$ in X. With that, we expect likelihood sensitivity to affect differently the weighting of ambiguous outcome combinations in the DM's evaluation of the lotteries. That is so because now, with for outcome combinations possible, a naive guess of each would just be $1/4$, while with only three possible outcome combinations, the naive guess would be higher $(1/3)$. This means that we interpret likelihood insensitivity as a function of how many possible combinations of outcomes are presented, which translates into what is the naive probability of an outcome combination that the DM considers to "fill the gap" of the imprecise information on the probability of such outcomes happening.

Moreover, we specify that the effect of the quantity of outcome combinations only affects weighting for outcome combinations to which the lottery being evaluated (L^n) has a defined outcome $(X_n \subset X)$. The importance of this distinction may be seen with a two-urn Ellsberg-like example. Define a choice problem where the DM may choose between lottery L^1 from example 1, and lottery $L^{2,urn2}$, that has outcomes equivalent to L^2, but contingent on the ball being drawn from a second identical urn (call it urn 2, and the urn that determines L^1 outcomes as urn 1). Then, there is no defined outcome for L^1 if the ball drawn from urn 2 is red, for example, simply because the possible outcomes of choosing L^1 are not contingent on urn 2, nor correlated to any draw from that urn. By being more specific about the set that represents likelihood insensitivity (X_n as opposed to simply X, that contains the outcomes of $L^{2,urn2}$), we make sure that only changes in the quantity of outcome combinations in urn 1 affect the Ψ weighting function of outcome combinations. Intuitively, that means that the "naive estimate" of probabilities of each color being drawn from urn one is unaffected by the information on urn 2 when evaluating L^1, since the outcomes associated with L^1 are also unaffected by any characteristic of urn 2.

Finally, we get to property P3, that states that, for fixed expected probability of an outcome combination $\mathbb{E}[\pi(\text{x})]$ and quantity of outcome combinations X_n, an increase in the standard deviation of the probability $\sigma_\pi(\text{x})$ should amplify how far Ψ applied to x is from a rational expectations weighting - that uses $\mathbb{E}[\pi(\text{x})]$ as weighting. Again, since our model interprets the dispersion of $\sigma_\pi(\text{x})$ (i.e., the dispersion of the possible ball compositions of the urn) as the level of ambiguity associated with the outcome combination x, P3 assures that higher ambiguity level means greater distortion in the weighting function. Three things are important to note here: first, we define the Ψ function codomain as $[0, 1]$, so that there are no negative nor infinite weights possible to attribute to any outcome combination[5]. Second, we do not specify in which direction the distortions in weighting take, since we need distortions in both directions to predict the fourfold ambiguity pattern found in the empirical literature (Trautmann & van de Kuilen, 2015). Third, whenever there is no ambiguity about the probability of an outcome combination being true - as is the case with the outcome combination of a red ball being draw in example 1 - then the function collapses to a rational expectations weighting based on the objective information available (in the example, the objective information that there is one red ball in the urn).

[5]the value of the outcome combination, considered in a given lottery L^n being evaluated, on the other hand, may be negative, so that a specific outcome combination applied to L^n may make L^n seem less favorable. However, what is important here is that any outcome combination that may happen with positive probability has weight in the decision-making process.

A Parametric Example of Ψ

For concreteness, we provide below a parametric example of Ψ for a discrete state-space Θ[6], and we heuristically describe that is actually an ambiguity adjustment function[7] and then apply it to the example to draw the same conclusion as in the previous paragraph. For simplicity, we also drop the function argument in parenthesis when dealing with the parametric function below, from now on. We also pose the proposition that $\hat{\Psi}$ is in fact an ambiguity adjustment function.

$$\hat{\Psi}(\pi(\mathrm{x}), ||x_n||) = \mathbb{E}[\pi(\mathrm{x})]^{(1-\gamma\cdot\sigma_\pi} \left(\frac{1+\eta}{||X_n||}\right)^{\gamma\cdot\sigma_\pi} \tag{3.1}$$

Proposition 1 ($\hat{\Psi}$ is an Ambiguity Adjustment Function): Let $\hat{\Psi}$ be defined as in equation 3.1. Then, $\hat{\Psi}$ is an Ambiguity Adjustment Function, as per Definition 1.

Proof: in Appendix A.

here the notations follow the preliminaries section and definition 1. Since the standard deviation of $\pi(\mathrm{x})$ is bounded at $\frac{1}{2}$[8] and we assume $\gamma \in [0, 2]$, the exponents of the equation are always between zero and one. γ can be seen as the degree to which the DM "distorts" the weight given to an event as a function of the imprecision of the information about its' probability.

By our nontriviality assumption $A2$, for any ambiguous act, that is, with $\sigma_p i > 0$, $||X_n|| \geq 1$. In other words, there is at least one possible outcome considered in the outcome combinations relevant to the lottery L^n pure outcomes. Moreover, $||X_n||$ is constant for each outcome combination, given a lottery L^n considered. That means $\eta \in (-1, 0)$ assures that the second term in parenthesis in 3.1 is always greater than zero, but less than one. Here, η is the parameter that indicates at what value of expected likelihood $\mathbb{E}[\pi(\mathrm{x})]$ the DM is indifferent between being totally ignorant about the probability distribution (and

[6] We discuss continuous state-spaces in Appendix D.

[7] The complete proof is in Appendix A.

[8] To see this, note that the maximum standard deviation for $\pi(\mathrm{x})$ is obtained when the probability mass is concentrated in its' extreme points, since $\pi(\mathrm{x}) \in [0, 1]$ is bounded. That is, the maximum standard deviation of $\pi(\mathrm{x}))$ is obtained when $\pi(\mathrm{x}_1) = 0, \pi(\mathrm{x}_2) = 1$ and $p(\pi(\mathrm{x}_1)) = 1/2, p(\pi(\mathrm{x}_2)) = 1/2$. In that case, $\sigma_p i = \sqrt{1/2(1 - \mathbb{E}[\pi(\mathrm{x})]^2 + 1/2(0 - \mathbb{E}[\pi(\mathrm{x})])^2}$.

relying heavily on a naive distribution $1/||X_n||$ to determine the weight of combination x in the evaluation of the act) and knowing for sure that the probability of outcome combination x happening is $\pi(\mathrm{x}) = \mathbb{E}[\pi(\mathrm{x})]$. Higher η indicate a higher fixed point $\hat{\Psi}(\pi(\mathrm{x}), ||x_n||)$. Also, for values close to the extremes in the $(-1, 1)$ range, this indifference point may not exist at all under some distributions $p \in \mathcal{P}$.

To make it tangible, take a modified version of example 1. Let there be an Ellsberg urn with 100 balls, and consider two possible information sets: (1) the DM knows that the Ellsberg urn contains either zero or $\bar{r}$ red balls, and the remaining balls can be either, yellow, black or green, in unknown proportion; (2) the DM only knows that the urn contains either $\underline{r}$ or 100 red balls, and the remaining balls can be either, yellow, black or green, in unknown proportion[9]. In our model, assuming the DM interprets that each possible red ball composition as being equally likely, Case 1 would be described by $\pi_1(\mathrm{x}_r) = 0, \pi_2(\mathrm{x}_r) = \bar{r}/100$ and $p_1(\pi_1(\mathrm{x}_r)) = p_1(\pi_1(\mathrm{x}_r)) = 1/2$. Similarly, Case 2 would be described by $\pi_3(\mathrm{x}_r) = \underline{r}, \pi_4(\mathrm{x}_r) = 1$ and $p_2(\pi_3(\mathrm{x}_r)) = p_1(\pi_4(\mathrm{x}_r)) = 1/2$.

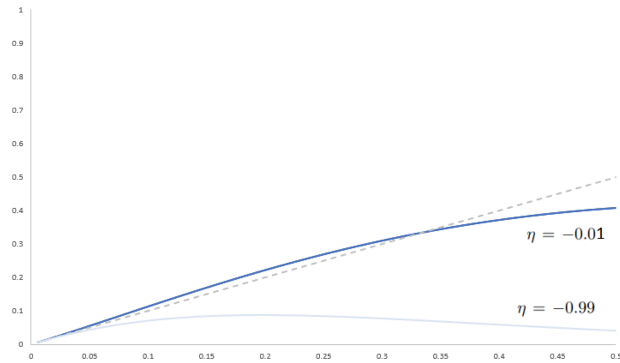
Also, note that in case (1), the expected probability of a drawn ball being red is $\bar{r}/2 \in [0, 0.5]$, while in case (2) this expected probability is $(0.5 + (1 - \underline{r})/2) \in [0.5, 1]$. Figure 3.2 shows the effect of varying η and γ in the described cases.

Figure 3.2 shows that greater values of γ indicate greater distortion in probability weighting due to ambiguity. In other words, it means that the DM is more sensitive to ambiguity for greater γ, i.e., the difference in weighting is larger for the DM, given an ambiguity level represented by the standard deviation of the probability distribution $p(\pi(\mathrm{x}_r))$. However, γ alone does not predict nor indicate ambiguity attitude, except when $\gamma = 0$, in which case the DM is ambiguity neutral for any ambiguous event.

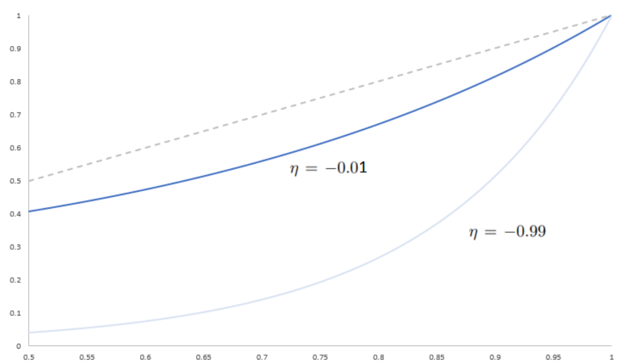
As we previously mentioned, η indicates where the $\hat{\Psi}$ function crosses the 45°curve, so that higher η dislocate that point to the right in the graph. Also, it may be that, for some extreme values of η and an act f_i that implies S^i, the two curves cross only at $\mathbb{E}[\pi(\mathrm{x}_r)] = 0$ and $\mathbb{E}[\pi(\mathrm{x}_r)]$. In subfigures (a) and (b) of Figure 3.2, we can see that for $\eta = -0.99$. In Appendix A we give a detailed assessment of the value function calculations for this modified

[9] In both cases, we assume that each possible ball color (red, yellow, black or green) is relevant relevant in determing the outcome of the evaluated act.

Figure 3.1: Effect of varying parameter η on $\hat{\Psi}$, as a function of the expected probability of the ambiguous event. Subfigures (a) and (b) show the variation of η (with fixed $\gamma = 1$) in Cases 1 and 2, respectively. Cases 1 and 2 are described in the text, in this section. Finally, the gray dashed line represents the 45°line, that is the weighting f $\hat{\Psi}$ for unambiguous events.



(a)



(b)

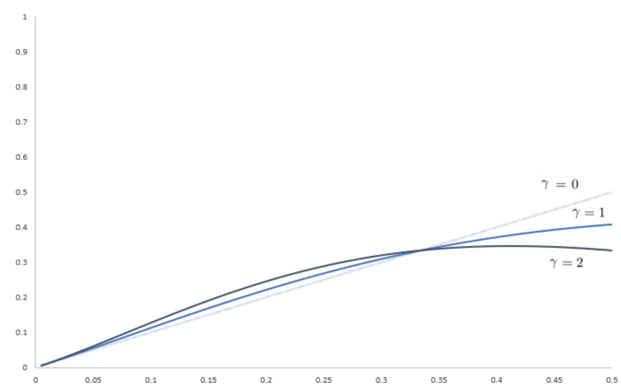
example.

Considering $\eta \in (-1, 1)$ and $\gamma \in [0, 2]$, then for any $\gamma > 0$, $V(L^2) < V(L^1)$ and the DM chooses lottery L^1 over L^2 for the given information set. That is the same ambiguity aversion result usually obtained in the literature when ambiguity choices concern mid-likelihood gains. If $\gamma = 0$, i.e., the individual is ambiguity neutral for any event, then $V(L^2) = V(L^1)$ and the DM is indifferent between L^1 and L^2.

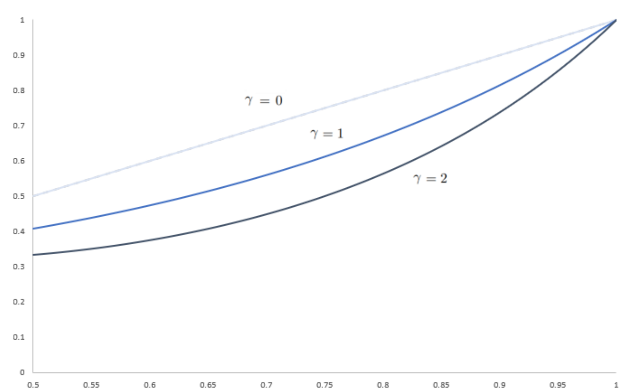
3.4 Ambiguity Attitudes

One of the main reasons that make decision under ambiguity an interesting topic is the fact that it is usual that people differentiate between acts that are contingent on unambiguous and

Figure 3.2: Effect of varying parameter γ on $\hat{\Psi}$. Subfigures (a) and (b) show the variation of γ (with fixed $\eta = -0.01$) in Cases 1 and 2, respectively. Cases 1 and 2 are described in the text, in this section. Finally, the gray dashed line represents the 45°line, that is the weighting f $\hat{\Psi}$ for unambiguous events.



(a)



(b)

ambiguous events - and also between distinct degrees of ambiguity of the events. The way people relate the imprecision of information on likelihoods of events to their choices is what we can call ambiguity attitude. So, when comparing acts with the same payoff structure, we can derive ambiguity attitude if payoffs are contingent on events that have the same expected probability of happening, but with different levels of imprecision (or vagueness) of the information about that probability. In our example 1, that is exactly what we are analyzing, since the \$100 payoff is associated with the outcome combination x_r for L^1, and x_y for L^2. Therefore, for the \$100 payoff and the $L = \{L^1, L^2\}$ lottery set, we can say that the individual is ambiguity averse if $L^1 \succeq L^2$, and ambiguity seeking if $L^2 \preceq L^1$. If both are true (i.e., $L^1 \sim L^2$), then the DM is ambiguity neutral. In different settings, both ambiguity aversion and ambiguity seeking behavior have been observed, which is why we use the more appropriate term ambiguity attitude (Trautmann & van de Kuilen, 2015).

Thus, we argue that ambiguity attitude is a result of two factors: (i) the (im)precision of available information about the probability distribution of events that are relevant to the outcomes of any act in the choice set; (ii) how people interpret this and incorporate other information about the choice set to determine their preferences.

In our model, (i) is given by $p(\pi(\mathrm{x}))$, that attaches probabilities to each distribution of events in the probability-possibility set. (ii) is described by the Ψ function that relates how that information - and complementary information about the choice set - is incorporated in the DM's choice. We discuss this in more detail in the next subsections.

Our model puts forward a non-axiomatic approach about how people interpret probabilities within the context of choice under ambiguity - leaving an eventual axiomatization of the choice criterion put here for future research. However, with that approach, we are able to make predictions that match stylized facts of the literature and relates them in a meaningful way. That contrasts with previous models, that require an additional assumption about what ambiguity attitude the DM has based on the model parameters, such as the Smooth Ambiguity Preference Model (Klibanoff et al., 2005) and the α-maxmin model (Gilboa & Schmeidler, 1989). Our model closely relates to the idea of the Gajdos et al. (2008) model that indicates that people are in general averse to information imprecision, which is granted by property P3 in our model. However, we incorporate all the information about probability distributions over outcome combinations into the DM's choice - and not only rational expectatnios and the most pessimistic scenario in terms of utility. That is, we specifically indicate how the DM's ambiguity attitude varies as a function of the expected probability of a given event and characteristics of the presented choice set, without needing to assume that the DM is always averse to imprecise information - in agreement with the empirical evidence. Of course, we are taking advantage of the fact that empirical and experimental research on ambiguity attitudes has flourished, so that we can list stylized facts and construct our model so that our predictions match the evidence - instead of having to make a model so general that it is harder to predict ambiguity attitudes in different decision settings. We expand on our model's interpretation on how this additional factors influence ambiguity attitude in the next subsections.

3.4.1 Ambiguity Attitudes and the Choice Set

As it happens with Salience Theory of Choice Under Risk - STCUR - (Bordalo et al., 2012), our model also incorporates the context of a decision, interpreted as the influence of choice set characteristics in the DM's choice. STCUR assimilates how the state-contingent outcomes and the contrast of those outcomes across acts may affect a DM's choice, as exemplified in our lottery ticket example in Section 3.2.2. We now incorporate the amount of possible outcome combinations to the context of Decision Under Ambiguity, represented by the cardinality of the outcome combination space ($||X_n||$) in our model, for discrete state-spaces[10]. This addition makes our model able to explain the widely reproduced likelihood insensitivity effect: people do not sufficiently discriminate between different levels of likelihood of an ambiguous event, transforming subjective likelihood towards a naive equal distribution among events (Dimmock et al., 2012).

Therefore, it is useful to see how changing the cardinality of the choice set may change the DM's choice in a modified version of example 1. Suppose now we get a second urn (call it urn 1), that is equal to the one in example 1 (call this urn $-g$), except that: urn 1 is equal to the original urn of example 1, except that we add one green ball to it; urn 2 is equal to the original urn of example 1, except that we add one black ball to it. All the

[10]We discuss continuous state-spaces in Appendix C

remaining assumptions remain the same as in our running example. From now on, we call this second urn the (g) urn. Moreover, suppose that the DM has three lotteries available to choose: $L^1 = 100\pi_{r,urn1}0$, $L^2 = 100\pi_{y,urn1}0$ and $L^{-g} = 100\pi_{y,urn\ -g}0$. That is, L^1 and L^2 are contingent on red and yellow balls being drawn from the urn (respectively), and L^{-g} is contingent on a yellow ball being drawn from the $-g$ urn.

Consequently, the DM knows for sure that there are no green balls in the $-g$ urn, and any payoffs contingent on the ball drawn from this second urn being green ($E_{g,-g}$) are meaningless to the pure outcomes of L^1 and L^2. We represent the new outcome combination set, that considers both urns as $X = \{\mathrm{x}_{r,urn1}, \mathrm{x}_{y,urn1}, \mathrm{x}_{b,urn1}, \mathrm{x}_{g,urn1}, \mathrm{x}_{r,urn\ -g}, \mathrm{x}_{y,urn\ -g}, \mathrm{x}_{b,urn\ -g}\}$, where all events with the $-g$ indicate events associated with the ball drawn from the second urn[11]. Thus, $L = \{L^1, L^2, L^{-g}\}$. The payoff matrix of the acts is then given below:

Table 3.4: Outcome matrix of an Ellsberg urn decision under ambiguity example (example 3), with three lotteries and results contingent on the color of the ball being drawn from two independent urns.

	Red	Yellow	Black	Green	Red - Urn $-g$	Yellow - Urn $-g$	Black - Urn $-g$)
L^1	100	0	0	0			
L^2	0	100	0	0			
L^{-g}					0	100	0

With this set of information about urn $-g$, the DM also knows that there is between 1 and 3 black balls in the urn. The plausible probabilities of the outcome combination associated with the color of the ball drawn from urn $-g$ bein red or yellow is unchanged across urns, that is, $\pi(\mathrm{x}_{r,urn1}) = \pi(\mathrm{x}_{r,-g})$ and $\pi(\mathrm{x}_{y,urn1}) = \pi(\mathrm{x}_{y,-g})$. So, the set $\Delta(X)_{-g}$ of non-null distributions for the outcome combinations associated with the second urn are:

Table 3.5: Non-null first-order probability distributions in Ellsberg urn example 3.

	$\pi(\mathrm{x}_{r,-g})$	$\pi(\mathrm{x}_{y,-g})$	$\pi(\mathrm{x}_{b,-g})$
π_1	1/4	0	3/4
π_2	1/4	1/4	2/4
π_3	1/4	2/4	1/4

[11]This example can be seen as an adaptation of a thought experiment due to Takashi Hayashi (Ahn, 2008), that - to the best of our knowledge - may be one of the first ones to explicitly pose the problem of how probability estimates of relevant events depend on the choice set.

It is easy to see that L^{-g} outcomes are contingent on ambiguous events, since there is somewhere between 0 and 1/2 probability of winning \$100 and somewhere between 1/2 and 1 probability of getting nothing. Observe that $\mathbb{E}[\pi(\mathrm{x}_{y,urn1})] = \mathbb{E}[\pi(\mathrm{x}_{y,-g})] = 1/4$. This new example is then constructed exactly in a way that the choice set changes so that the only difference in π and $\mathcal{P}$ of non-zero pure outcomes in lotteries L^1, L^2 and L^{-g} is the quantity of outcome combinations to which each lottery is defined, i.e., X_n. In other words, both L^1, L^2 and L^{-g} pay \$100 if a ball that has probability between 0 and 1/2 is drawn, and zero otherwise. The differences between the lotteries are: (i) the probability of winning in L^1 is unambiguous (it is precisely 1/4), while it is ambiguous (between 0 and 1/2) for both L^2 and L^{-g}; (ii) the amount of outcome combinations to which $L{-}g$ has a defined outcome is three (red, yellow or black ball drawn from urn $-g$) so that we can analyze concretely how the model responds to a change in the cardinality of the relevant partition of the state-space for each act.

Applying our model's valuation function of acts, we get:

$$V(L^1) = \Psi(\pi(\mathrm{x}_{r,urn1}), ||X_1||)\ \omega\ v(100)$$

$$V(L^2) = \Psi(\pi(\mathrm{x}_{y,urn1}), ||X_2||)\ \omega\ v(100)$$

$$V(L^{-g}) = \Psi(\pi(\mathrm{x}_{y,urn\ -g}), ||X_g||)\ \omega\ v(100)$$

First, note the salience is the same ω in each state of the final equation (since it involves a \$100 payoff in one of the acts and null payoffs for the other ones). Then, the DM's choice hinge on the relation between $\hat{\Psi}(\pi(\mathrm{x}_{r,urn1}), ||X_1||)$, $\hat{\Psi}(\pi(\mathrm{x}_{y,urn1}), ||X_2||)$, $\hat{\Psi}(\pi(\mathrm{x}_{y,urn\ -g}), ||X_3||)$. Note that the marginal distributions are such that $p(\pi(\mathrm{x}_{y,urn1})) = p(\pi(\mathrm{x}_{y,urn\ -g}))$. Then, property P3 is essential. Since $||X_{-g}|| < ||X_1||$, then $\hat{\Psi}(\pi(\mathrm{x}_{y,urn1}), ||X_2||) \leq \hat{\Psi}(\pi(\mathrm{x}_{y,urn\ -g}), ||X_{-g}||)$ and, thus, $V(L^{-g}) \succsim V(L^2)$. On the other hand, the preference between $V(L^{-g})$ and $V(L^1)$ depends on the specific form of the Ψ function, since two effects working in opposite directions are in play: (i) on one hand, $||X_{-g}|| < ||X_1||$, which means that the naive distribution that affects weighting for $||X_{-g}||$ is larger, by P2; (ii) on the other hand, the marginal distribution $p(\pi(\mathrm{x}_{r,urn1}))$ is a mean-preserving spread of $p(\pi(\mathrm{x}_{y,urn\ -g}))$. By P3, that means that the DM has more precise information about the probability of a red ball being drawn from urn 1 than a yellow ball being drawn from urn $-g$, and then should weight $\pi(\mathrm{x}_{r,urn1})$ more. Therefore, depending on the sensibility of the DM's weighting to information imprecision and to likelihood insensitivity, the DM's preferences may either be $L^1 \succsim L^{-g}$ or $L^1 \precsim L^{-g}$. In Appendix B, we calculate theses value functions for our parametric example of $\hat{\Psi}$, and conclude that, for any $\gamma > 0$, $\eta \in (-0.25, 0) \implies L^{-g} \succ L^1$ and $\eta \in (-1, -0.25) \implies L^1 \succ L^{-g}$.

Comparing this with the use of our parametric model in example 1, we can see that the DM is more prone to overweight L^{-g} for given η and $\gamma > 0$ parameters, since the last term of $V(L^2)$ is now $((1+\eta)/3)^{\gamma/3}$ instead of $((1+\eta)/4)^{\gamma/3}$. We can interpret that as the DM changing what is his naive probability distribution, now that only three events relevant to the acts are possible - so that the naive probability distribution would be each event occurring with 1/3 chance. That means that a given expected probability of an event is more likely to be perceived as a low-probability event, which has its' weight increased by likelihood insensitivity, in our model.

Finally, we highlight that the question whether the $(0,0)$ outcomes relative to the draw of black or a green ball should be considered as different outcome combinations or not. As suggested elsewhere (Dertwinter-Kalt & Koester, 2020), this question is probably subject to how the choice problem is framed to the DM, and the way to represent the choice set - either with both ball colors considered as the same outcome or not. The ways that framing may affect this quantity of outcome combinations is an empirical question left for future research.

To sum up, our model considers that the choice set affects probability weighting by the presentation of a set of possible combinations of outcomes, as perceived by the DM. Property P2 makes sure that, if there are more possible outcome combinations associated with the results of a lottery, then likelihood insensitivity skews the weighting of an event down (as we saw $\Psi(\pi(\mathrm{x}_{y,urn1}), ||X_1||) \leq \Psi(\pi(\mathrm{x}_{y,urn\ -g}), ||X_{-g}||), ||X_{-g}||)$ in our example). That is, the more possible outcome combinations are presented as relevant to a bet, more this information is interpreted as a lesser probability of any one of the states of the world being true, which is the essence of the likelihood insensitivity effect observed in the literature (Dimmock et al., 2013).

3.4.2 Ambiguity Attitudes and Second-Order Probabilities

In our model, as it is usual in second-order belief models of decision under ambiguity (Etner et al., 2012), second-order probabilities p represent the (im)precision of event probability information available to the DM. Since p is defined on a set of first-order probability distributions of events $\Delta(X)$, the marginal compound probability distribution has finite expected value. Its' standard deviation is taken as a measure of the imprecision of the available information. Other than that, p here is purposely defined in a broad sense, since many factors may influence the DM's beliefs about the probability of each probability distribution on outcome combinations being true, such as related historical data, the framing of the decision problem, and so forth. Furthermore, the model that we present is static, in the sense that it represents a one-shot decision under ambiguity. Eliciting second-order belief formation processes and incorporating dynamic updating of beliefs in our model is an important venue for future research.

We are also able to separate in our model how ambiguity attitudes are affected by outcome combinations with different expected probability of happening. Here, we represent the likelihood of an outcome combination assessed by the expected probability of the event, given the second-order probabilities p, for any outcome combination that has positive probability of obtaining under some p. With that, we are able to separate any over/under-weighting of states in decision-making due to lack of information about states and their probability (ambiguity itself, represented by the Ψ function) and due to salience of the known information of outcomes, represented by the salience function $\omega(\mathrm{x})$ and its covariance with the known state-contingent outcomes (Bordalo et al., 2012) - as the STCUR model already does.

However, we note that second-order distribution by itself does not elicit a DM's ambiguity attitude. Only together with $||X_n||$ and the specific parametric form of Ψ can p indicate if a state of the world is considered as a highly or lowly likely. It is this comparison of the likelihood of the outcome combination x with other characteristics of x that may pinpoint if a given expected probability of an event is considered high or low - and that in turn imply if likelihood insensitivity causes over or under-weighting of a given outcome combination, as we will see in the next section. However, once the low/high likelihood of the outcome combination is determined, the properties of our model make sure that low likelihood events are overweighted and high likelihood events are underweighted. Again, this is in agreement with the experimental evidence on likelihood insensitivity (Trautmann & Van de Kuilen, 2015).

It is useful to see how changing the assumptions on p may change the DM's choice in a modified version of our example 1. But now, assume the DM has the additional information that there are either 0 or 2 yellow balls in the urn, i.e., $p^*(\pi_2(\mathrm{x})) = 0$ - where p^* represents the DM's new beliefs considering this additional information. Now, only $p^*(\pi_1(\mathrm{x}))$ and $p^*(\pi_3(\mathrm{x}))$ are plausible (non-null) elements of $\mathcal{P}$, i.e., $\Delta(X) = \{\pi_1, \pi_3\}$, where $p^*(\pi_1(\mathrm{x}_r = 1/3)) = p^*(\pi_3(\mathrm{x}_r = 1/3)) = 1$, $p^*(\pi_1(\mathrm{x}_y = 0)) = p^*(\pi_1(\mathrm{x}_b = 2/3)) = 1/2$ and $p^*(\pi_1(\mathrm{x}_y = 2/3)) = p^*(\pi_1(\mathrm{x}_b = 0)) = 1/2$. Since information precision about π_1 and π_3 is the same, let's assume the DM beliefs them as equally likely for the sake of concreteness, i.e., $p^*(\pi_1) = p^*(\pi_3) = 1/2$.

Comparing this new p with our example 1 distribution, observe that $\mathbb{E}[\pi(\mathrm{x}_y)] = \mathbb{E}[\pi(\mathrm{x}_y)]$, that is, the expected probability of outcome combination $(0, 100)$ obtaining is the same in both cases[12], but $\sigma_{\pi|p^*} > \sigma_{\pi|p}$. So, what we can affirm based in our model is that $|\Psi(p^*(\pi(\mathrm{x}_y)), ||X_n||) - \mathbb{E}_{p^*}((\pi(\mathrm{x}_y)))| > |\Psi(p(\pi(\mathrm{x}_y)), ||X_n||) - \mathbb{E}_p((\pi(\mathrm{x}_y)))|$. That is, $\Psi(p^*(\pi(\mathrm{x}_y)))$ is farther from the expected probability of outcome x_y or, in other words, the probability weighting distortion is larger, even though it is not possible to affirm if it goes in the direction of overweighting or underweighting without further assumptions. So, even though the DM has narrowed the possible first-order probability distributions down to just two alternatives (versus 3 possibilities in our example 1), it is not obvious that outcome combination x_y becomes more over-weighted (or underweighted) under p^*. That happens because our model predicts that, given a choice set and two probability distributions with the same expected probability, more information about the second-order probability distribution only increases the weighting of the outcome combination if: (i) the expected probability of the outcome combination is equal to a parametrically defined certain likelihood insensitivity indifference point (where $\Psi(p(\pi(\mathrm{x}))) = \mathbb{E}_p[\pi(\mathrm{x})]$ for some outcome combination x); (ii) the new information imply a decrease in the standard deviation of the second-order probability of the event happening. In Appendix B.2 we illustrate how that happens in our example of parametric

[12] In fact, the expected probability is the same for all outcome combinations considered, but we focus on the one associated with a yellow ball being drawn from the urn, because it is the relevant one to our conclusions.

$\hat{\Psi}$ function.

What this section's example highlights is that - for a given expected probability of an event - new information impact probability weighting "distortions" (Ψ farther from the expected probability of the event) depend on the relationship between sensitivity to information imprecision and likelihood insensitivity. There has been recent evidence in favor of underweighting for additional information that increases standard deviation for bets involving gains (Chew et al., 2017), that can be accommodated by our model [13]. However, further empirical and experimental evidence is needed for this to be a stylized fact, and so we construct our model in a way that can accommodate both behaviours.

3.4.3 Ambiguity Attitudes and the Outcome Domain

According to the evidence in the literature (Trautmann & van de Kuilen, 2017; Di Mauro & Maffioletti, 2004; Viscusi & Chesson, 1999), one of the factors that influences ambiguity attitude is the outcome domain - interpreted as whether a given bet involves gains or losses, with respect to a reference point. Typically, this is tested experimentally comparing individuals choices when presented with a choice set involving a risky lottery of the form $x\pi(\mathrm{x}_0)0$ and an ambiguous lottery involving $x\pi(\mathrm{x}_\alpha)0$, where π_0 is the probability of a risky outcome combination x_0 (i.e., the objective probability $\pi(\mathrm{x}_0)$ is known to the decision-maker), and $\pi(\mathrm{x}_\alpha)$ is an ambiguous outcome combination. The tests usually involve the cases where $x > 0$ and $x < 0$ and the same value of expected probabilities for $\mathbb{E}_p[\pi(\mathrm{x}_0)]$ and $\mathbb{E}_p[\pi(\mathrm{x}_\alpha)]$. If an individual chooses $x\pi(\mathrm{x}_\alpha)0$ over $x\pi(\mathrm{x}_0)0$, we conclude that the individual is ambiguity seeking, and, conversely, if she chooses $x\pi(\mathrm{x}_0)0$ over $x\pi(\mathrm{x}_\alpha)0$ she is ambiguity averse.

A stylized fact drawn from this literature is that there are some regularities in the most common ambiguity attitude behavior of individuals, that are a function of the expected probability of the event for which the outcome of an act is contingent and the outcome domain of the act's results (Trautmann & Van de Kuilen, 2015). The ambiguity attitude in each case is portrayed in the table below:

One of the main advantages of our model's interpretation of the probability weighting function and the way DM's transform the available information to form their decision weights

[13] Details in Appendix B.2.

Table 3.6: Ambiguity Attitudes and Effects on Value Function - Unsureness Aversion Theory

	Domain	
Expected Probability	Loss	Gain
Low	Ambiguity Aversion	Ambiguity Seeking
Mid/High	Ambiguity Seeking	Ambiguity Aversion

on outcome combinations is that we can account for different ambiguity attitudes for different expected probabilities. That is possible because we separate how people interpret information on probabilities of events from other regularities involving how people evaluate acts and outcomes. That has similarities to how Prospect Theory axiomatizes its' probability weighting function (Kahnemann e Tversky, 1979; Prelec, 1998; Wakker, 2010), but here we can make specific predictions about how these behaviors relate to events with different levels of ambiguity, and choice sets that have different implications in terms of how bottom-up attention may affect the DM's choice.

Concretely, by assumption A1 we have that the value function $v(x)$ can be normalized as $v(0) = 0$. By monotonicity of v, for any positive x^n, $\mathrm{x}^n > 0$ and $-(\mathrm{x}^n) < 0$. Therefore, the evaluation of acts is such that a higher value of probability weighting increases the value of the act when $x > 0$ and decreases the value of the bet when $x < 0$. To see how that works, consider a modified version of our running example, now with negative payoffs.

Table 3.7: Outcome matrix of an Ellsberg-like ambiguity example with negative payoffs (Example 4).

	Red	Yellow	Black
L^{-1}	-100	0	0
L^{-2}	0	-100	0

Now we have the following evaluations of each act, under our parametric example of the model:

$$V(L^{-1}) = \Psi(\pi(\mathrm{x}_r))\ \omega\ v(-100)$$

$$V(L^{-2}) = \Psi(\pi(\mathrm{x}_y))\ \omega\ v(-100)$$

Since $v(-100) < 0$, now $\Psi(\pi(\mathrm{x}_r)) \geq \Psi(\pi(\mathrm{x}_y))$ imply a worse evaluation of the risky act $V(L^{-1})$, reversing the result obtained in our previous example. Therefore, $\Psi(\pi(\mathrm{x}_r)) \geq \Psi(\pi(\mathrm{x}_y))$ would indicate ambiguity seeking behavior ($V(L^{-2} \succeq V(L^{-1})$, the opposite of the ambiguity aversion ($V(L^{-1} \succeq V(L^{-1})$ obtained in example 1 with payoffs in the gain domain.

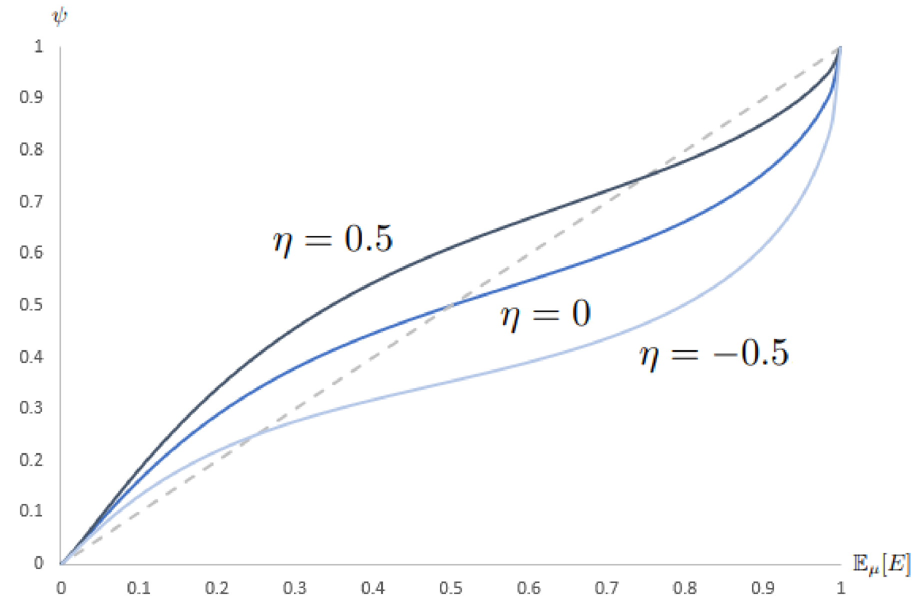
Moreover, as detailed in Section 3.4.2, our model also predicts that weighting of more ambiguous events is farther from their expected probability, but if that distortion is an over or under-weighting depends on the specific parameters of the model. Therefore, we are also able to accommodate for the stylized fact that overweighting is more typical for low expected probability events and underweighting is more typical for mid to high likelihood events, as summed up in table 3.6. We show that in our parametric $\hat{\Psi}$ function on Appendix B.3.

Since the ambiguity attitudes also change for different expected likelihoods of outcome combinations, it is also useful to consider an additional example where we change the expected likelihoods, but nothing else. Consider the following example: let there be an urn with 100 balls, that can be either yellow or black. The DM knows that there are either 100 yellow balls (and no black balls) or 100 black balls (an no yellow balls). That is, $\Delta(X) = \{\pi_1, \pi_2\}$, where $\pi_1(\mathrm{x}_y) = 1, \pi_1(\mathrm{x}_b) = 0$ and $\pi_2(\mathrm{x}_y) = 0, \pi_2(\mathrm{x}_b) = 1$. We analyze in the graphs below various values of $p'(\pi_1(\mathrm{x}_y)) = a$, with $a \in [0, 1]$. Note that our example is constructed so that $\mathbb{E}'_p(\pi_1(\mathrm{x}_y)) = p'(\pi_3(\mathrm{x}_y))$, and we assume that the quantity of possible outcomes in X_n that is relevant is $\mathcal{P} = \{\pi(\mathrm{x}_y), \pi(\mathrm{x}_b)\}$, so that $||X_n|| = 2$ for any non-constant act L^n.

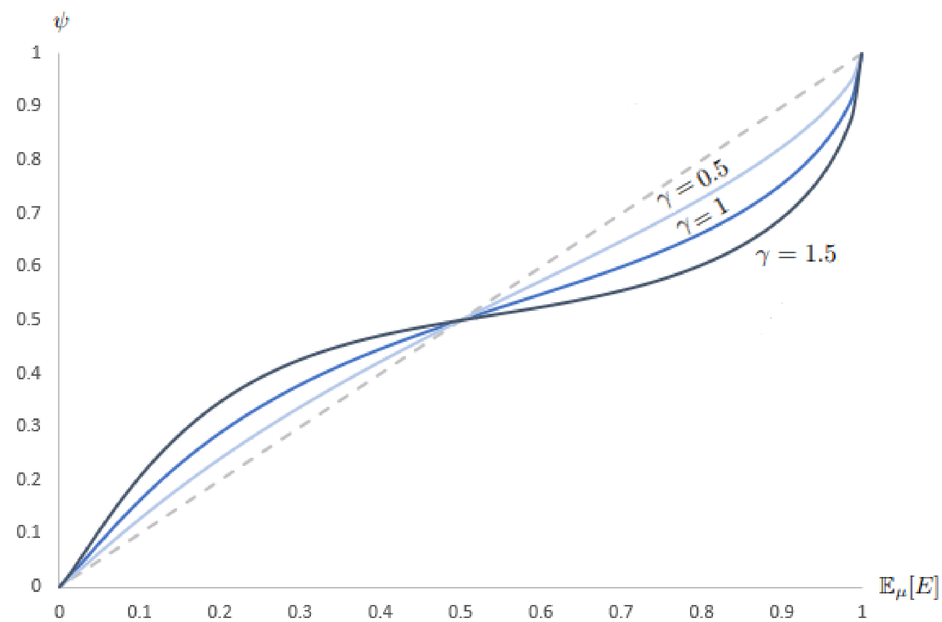
We again show the complete application of our parametric $\hat{\Psi}$ example in act valuation in Appendix B.3. We also show in the graph below in the x axis different values of $\mathbb{E}_p(\pi(\mathrm{x}_y))$ implied by different p' in agreement with our example settings, while in the y axis we have values of the $\hat{\Psi}$ function for different values of η and γ.

In graph 3.3, we can see that for lower values of $\mathbb{E}_p(\pi(\mathrm{x}_y))$, typically $\hat{\Psi}$ is above the 45° line, which means that there is an overweight of these low expected probability outcome combinations. On the other hand, for higher expected probabilities, the $\hat{\Psi}$ function has values below the 45° line, which indicate underweighting. We can interpret the point where the $\hat{\Psi}$ line and the 45° line cross as the point that determines what is a "low" and what is a "high" likelihood outcome combination. The specific value that determines that inversion in the ambiguity attitude, as we described in table 3.6, is a question for the empirical and experimental literature, and the choice of parameters and functional forms of our model will

Figure 3.3: Variations in the $\hat{\Psi}$ parametric function for this section's example, as a function of the expected probability of the ambiguous outcome combination x_y. Subfigure (a) shows the effect of varying η in the probability weighting function (with fixed $\gamma = 1$), while subfigure (b) shows the effect of varying γ (with fixed $\eta = -0.01$). Finally, the gray dashed line represents the 45°line, that is the weighting f $\hat{\Psi}$ for unambiguous events.



(a)



(b)

depend on the results obtained in future research on the matter.

It is important to highlight that, in our parametric example $\hat{\Psi}$, the point that determines what is a high or low likelihood depends on how the DM responds to likelihood insensitivity, i.e., how she considers the information on the amount of outcome combinations in her naive probability calculations, and how these affect probability weighting. That is represented in our parametric example of the model by the parameter η, where lower values of η indicate that the DM needs a really low expected probability to consider it a "low likelihood" case (for which she is typically ambiguity averse for bets on gains and ambiguity seeking for bets on losses). And, conversely, higher values of η mean that there is overweighting of events up to a higher value of $\mathbb{E}_p(\pi(\mathrm{x}_y))$. On the other hand, γ indicates the level of distortion due to the DM's sensibility to information imprecision, but not what is the indifference point that separates the region of overweighting and underweighting of probabilities. In graph 3.3 (b) we can see that the point where $\hat{\Psi}$ crosses the 45° line remains the same for different values of γ, exemplifying this statement.

The careful reader will also note that these graphs have similarities with the inverse S-shaped probability weighting curves of the original Prospect Theory of Kahnemann & Tversky (1979), designed to explain behavior for choices under risk. This is not by accident: there the prediction was also an overweighting of small probability events, and underweighting of high-probability events[14]. However, the interpretation and implications here are vastly different: in our model we are saying that more imprecise information about the probability of an event (higher σ_π, in our parametric example) mean that the DM use information about the choice set $||X_n||$ to modify the weighting that would results from a bayesian interpretation of the available information on the expected probability of the event otherwise ($\mathbb{E}_p(\pi(\mathrm{x}))$). So, it is not only the expected probability that changes the weighting of ambiguous events, as in some adaptations of the Prospect Theory and Cumulative Prospect Theory Model

[14]We note that more recent developments of both the original Prospect Theory and Cumulative Prospect Theory (Kahneman & Tversky, 1993) also make the probability weighting function flexible enough so that over or underweighting are possible for every probability value, be it high or low, depending on the specific functional form and parameters chosen (Wakker, 2010). On the other hand, that flexibility also means that meaningful predictions about the DM's behavior for choice under risk - and the model's extensions to Decision Under Ambiguity - may be prone to be too dependent on the specific parameters chosen. We argue that these models may accommodate results that are too general to give meaningful predictions about economic agents' behavior, at least in the context of Decision Under Ambiguity.

(Wakker, 2010), but the choice set and the degree of information imprecision (represented by σ_π) that determines the probability weighting for ambiguous events. Moreover, as we saw in previous examples and in Figure 3.2, it may be that for some events only high or only low probabilities are deemed plausible by the DM.

Since now we posed how the interaction between the outcome domain and expected probabilities influence the value of our Ambiguity Adjustment Function, we turn to how these same outcomes may affect salience, and that in turn may affect the results of our model.

3.4.4 Ambiguity Attitude, Salience and Context-Dependence

As we saw in section 3.2.2, the STCUR model proposes that salience is a characteristic that relates how the evaluation of a lottery L^i by a DM is affected by how it relates to outcomes, contingent on an outcome combination x obtaining and the outcomes associated with x that would have been obtained if other lotteries $L^m, m \neq n$ in the choice set were chosen. In the STCUR model interpretation, the weighting of each outcome combination in evaluating acts is dependent on how salient the combination is, where a salient outcome combination is one with highly contrasting, prominent, or surprising payoffs for the acts in the choice set. In other words, the outcomes of the choice set shape the DM's perception of the state-space (Bordalo et al., 2012).

In applying the probability weighting function as proposed in this paper to extend Salience Theory for Decision Under Ambiguity, the same principle is also present. However, it is not only the information about acts' outcomes that shape the DM's perception of the outcome combination possibilities, but also information on the probabilities of each outcome combination happening, how (im)precise is that information and the quantity of outcomes that are possible in each outcome combination. Therefore, our model does incorporate information about the choice set that does not concern the outcomes in terms of their payoffs, but the quantity of elements of outcome combination sets, in how the DMs evaluate lotteries. Again, that inclusion is crucial for models of decision under ambiguity, since the separation of an ambiguous and an unambiguous event depends on the precision of the DM's assessment of the likelihood of the event, no matter the outcomes associated. So, extending the psychological reasoning of bottom-up attention[15] influencing lottery evaluation through the information about the outcome combination is a natural extension of the same principle to deal with decision under ambiguity. Moreover, this information about the outcome combinations is induced by the available information of the choice set - in the sense that changes in the outcome set may affect the evaluation of a lottery L^n, even if those changes concern other lotteries.

We interpret the empirical evidence on likelihood insensitivity as the bottom-up attention driver of the empirical and experimental findings on ambiguity attitude (see Table 3.6). This is so because our model regards likelihood insensitivity as the DM's response to the information about which of the events that are relevant to the outcome of the acts in the choice set are plausible. Therefore, likelihood insensitivity is not separate from the fourfold pattern of ambiguity attitude, but part of what explains that attitude. Postulate P2 is instrumental for that to hold in the description of our model.

Furthermore, it is only natural that we also do not assume transitivity of preferences, since that is an assumption of the STCUR model. That is one of the main differences between Salience Theory and some of the concurring approaches, such as rank-based expectations theories (Cumulative Prospect Theory, Choquet Expected Utility (Wakker, 2010)) and other approaches based the Expected Utility framework (Klibanoff et al., 2005; Gul e Pesendorfer, 2013). However, we do imply that, for a fixed choice set, there is consistency on how the DM weights each event. Postulate P1 makes sure there is some monotonicity to that interpretation, insofar as increasing the expected probability of an event unequivocally increases its' weight, *ceteris paribus*. More, postulate P2 makes sure that only events that are relevant to the outcomes in a choice set matter, so that the way the choice set shapes the state-space is what matters for the ambiguity adjustment function.

Another question that perpasses the Salience Theory literature is whether a continuous (Bordalo et al., 2013; 2020) or a rank-based salience function (Bordalo et al., 2012) is desirable. Recently, Lanzani (2022) axiomatized STR, and argued that for representing preferences among acts with outcomes associated with correlated events, using continuous

[15] I.e., stimuli that attract the decision-maker attention automatically and involuntarily (Bordalo et al., 2021).

salience functions is desirable. Moreover, the continuous version of the salience function has been the most used one in the empirical literature (Dertwintel-Kalt et al., 2021; Nielsen et al., 2021). Considering that, and the fact that correlation of events' probabilities plays a highly important role in decision under ambiguity, we consider here the continuous version of salience weighting as the standard for the application of our probability weighting function to Salience Theory.

An important note is that we are proposing a probability weighting function applicable to act evaluation functions that are not rank-based (such as Cumulative Prospect Theory and Choquet-Expected Utility Theory), nor we require an entanglement between an optimism/pessimism criterion to ambiguity attitude directly (as Gajdos et al. (2008) and other SEU-based models typically do). Similarly to what happens to Salience Theory for Choice Under Risk, that results in a model that does not assume nor imply transitivity (Bordalo et al., 2012; Ellis & Masatlioglu, 2019; Lanzani, 2022). That fundamentally happens because of the fact that choices are not context-independent, i.e., the acts and outcomes not chosen influence act evaluation.

Finally, we note that at this moment there is no conclusive evidence in the literature that indicate salience is entangled with ambiguity weighting in a way that they could be non-separable. Therefore, we assume that salience and ambiguity weighting can be separated in our function representing the DM's preferences. Nevertheless, verifying if this kind of entanglement exists and if it is economically relevant is an important direction for future research.

3.5 Related Literature

We now briefly relate our model to other previously developed models in the literature, focusing on the most used models for decision under ambiguity and on other second-order belief models.

Smooth Ambiguity Preferences

One of the most popular second-order belief models for Decision Under Ambiguity is the Smooth Ambiguity Preferences model (Klibanoff et al., 2005). It avoids the problem of non-differentiability typical of previous models, such as the α-maxmin model (Gilboa & Schmeidler, 1989). Besides, it also considers the whole first and second-order distributions - not only the most optimistic or pessimistic scenarios - in the DM's evaluation of an act. Taking Savage's Subjective Expected Utility as a starting point, and with preliminaries that are similar to our own model, in the case of a finite set of states $s \in S$ and a finite set of second-order probabilities ("scenarii") represented by p, the value function that represents the DM's preferences is:

$$V^{SAP}(f_i) = \sum_{\theta\in\Theta} p(\theta)\Phi(\pi_\theta(s)\ u(x_s))$$

In other words, the preference criterion can be read as two-layer expected utility: first, the decision-maker evaluates the expected utility with respect to all possible priors $\pi \in \Delta(S)$, so that the DM has then a set of first-order expected utilities indexed by θ. Then, the DM takes an expectation of these utilities, "distorted" by a Φ function (Etner et al., 2012). Φ, in turn, determines the ambiguity attitude of the DM, in the folllowing sense: if Φ were linear, the compound lottery representing the decision under ambiguity would just reduce to an Expected Utility problem; for concave Φ, the DM weighs more "bad" $\pi_\theta(s)u(x_s)$ in its' evaluation of results, and thus is ambiguity averse; if Φ is convex, the DM gives more weight to "good" $\pi_\theta(s)u(x_s)$, and so she is ambiguity seeking.

In that way, the decision criterion proposed by Klibanoff et al. (2005) involves both an expected utility evaluation of the possible first-order probability distributions and a pessimistic, neutral or optimistic criterion given by the Φ function. Even though the authors allow for different Φ for different supports of Π (i.e., for different sets of first-order probability distributions), when aplying the model one still has to assume a DM ambiguity attitude through the choice of the Φ function.

Particularly, even though the Smooth Ambiguity Preferences Model provides an interesting extension of the Subjective Expected Utility framework to analyze decision under ambiguity, it still does not imply any specific prediction about DM's ambiguity attitudes, nor what may influence that ambiguity attitude. Moreover, since Φ is defined on a classic SEU-like function, one cannot use the model to assume difference in probability weighting that is independent of the outcome x_s and its' associated utility function, unless if using a rather *ad hoc* approach for defining the Φ function differently for many different supports of Π. Therefore, it is hard to use this framework to predict the fourfold pattern of ambiguity attitudes empirically observed. Concretely, we can see that this usually implies that experimenters testing the Smooth Ambiguity Preferences model put an additional assumption on the Φ function to test the theory - and therefore on ambiguity attitudes (Conte & Hey, 2013; Attanasi et al., 2014; Gneezy et al., 2015).

On the other hand, our model takes advantage of the great growth in experimental and empirical evidence in recent decades to actually predict how ambiguity attitudes change as a function of the expected probability of events the outcome domain, and other specific contextual information about a given choice set. Even though assumptions about the parametric form of our model still need to be chosen and calibrated according to empirical results - as it is usual for any such model - ambiguity attitudes result from specific properties implied in our model, instead of being just assumed *ex ante* as in the Smooth Ambiguity approach. So, not only we can accommodate for different ambiguity attitudes depending on the context, we specifically predict which factors affect ambiguity attitude in a choice problem.

We also retain the interesting properties of continuity and differentiability of the act evaluation function V, for a given choice set F and beliefs p based on the available information to the DM.

Choquet Expected Utility and Cumulative Prospect Theory

Rank-dependent theories, meaning models that rely on the valuation of acts according to the ranking of outcomes by the DM, have also been employed for decision under ambiguity problems. Choquet Expected Utility and Cumulative Prospect Theory, proposed by Schmeidler (1989) and Tversky & Kahneman (1992) respectively and later generalized and adapted for decision under ambiguity (Chateauneuf & Faro, 2009; Chateauneuf, Eichberger & Grant (2007); Wakker, 2010), are two such models.

If taken in full generality, both models can accommodate the fourfold pattern empirically observed, depending on the ambiguity parameters of the Cumulative Prospect Theory probability weighting function (Wakker, 2010). For Choquet Expected Utility, the relevance of each prior assigned in a Confidence Function such as that of Chateauneuf & Faro (2009) may also accommodate those factors. However, there may be a large number of free parameters involved, so that for empirical applications a calibration of these parameters is required. Again, the models are general enough so that calibrating their free parameters may result in the fourfold pattern, but without that specific calibration we do not have *a priori* meaningful predictions about ambiguity attitude and how they change over time.

Contraction Second-Order Belief Model

Gajdos et al. (2008) propose a model that contains an idea of how DM's use objective information on the probability of events that is similar to the one contained in our model. The authors give axiomatic foundations for a preference foundation that considers two criterion: (i) a Bayesian criterion, where information is summarized by the available information on probability distribution of events that is independent on the outcomes; (ii) a pessimistic criterion, so that the DM takes into account the distribution giving the lowest expected utility possible. The evaluation of an act can be represented by the function below:

$$V^{CM}(f_i) = \min_{\Phi \in \Phi^{CM}(p(\Pi))} \mathbb{E}_p(\Pi)\ u(f_i)$$

where $\Phi^{CM}(p(\Pi))$ is a subjective set of second-order priors estimated from the available information on event likelihood, and $\mathbb{E}_p(\Pi)$ is the vector of expected probabilities of each event associated with an outcome of act f_i. The Φ^{CM} function concept is similar to our Ψ transformation of the second-order subjective probability distribution - the idea that objective information about ambiguous event probabilities is somehow distorted in the DM's evaluation of an act. There are, however, some important differences between our model and the Contraction Model. First, we do not assume the pessimistic criterion (as we can see from the min operator) for the evaluation of acts, but consider that the state-space partition induced by the act and the choice set is what determines if a DM is "optimistic" or "pessimistic" about ambiguous prospects and events. In that way, ambiguity seeking behavior as a function of expected probability of events is easily accommodated by our model, while there is no clear effect of the expected probability of an ambiguous event on ambiguity attitude in the Contraction Model.

Second, we consider the whole set of priors in the DM's evaluation - not only the most pessimistic scenario. In that way, the dispersion of the second-order priors matter, and not only what is the subjective probabilities associated with the most pessimistic scenario. This kind of nuance in the DM's reaction for different degrees and forms of ambiguity is corroborated by recent experimental evidence (Chew et al., 2017).

3.6 Conclusion

Our article explores how contextual characteristics of decision under ambiguity may influence decision-maker's preferences, and how can that be represented through probability weighting functions. We argue that, when faced with highly imprecise information on the probabilities of each outcome combination for different courses of action involving uncertainty, then the DM uses other information from the choice set, such as the number of possible (and relevant) outcomes that can happen to "fill the gap" of information about the probabilities of each event with a naive equal probability distribution for each event. This bottom up stimuli distorts the weighting of different outcome combinations, in a phenomenon called likelihood insensitivity by the literature. Likelihood insensitivity, in turn, causes the DM to overweight (underweight) low (high) likelihood ambiguous events, causing the fourfold ambiguity attitude observed in the literature.

However, our postulates also imply some ways in which decision-maker's are consistent when dealing with ambiguous outcome combinations, once her choice set is given. Property P3 makes sure that, for a given expected probability of an event, any increase in expected probability that doesn't alter the distribution dispersion increases weighting of the x outcome combination. For example, an outcome combination that has between 10% and 11% of happening is going to be weighted more than an ambiguous outcome combination with that has between 9% and 10% chance of happening. That is, for a given level of noise in the information on probabilities, raising expected probability increases the weighting of a given outcome combination. Property P2 introduces likelihood insensitivity as a function of this amount of relevant outcome combination $||X_n||$. Finally, property P3 That is, for a given level of expected probability, adding noise to second order probabilities (i.e., making an event more ambiguous) monotonically increases (decreases) the weighting of a low (mid-to-high) probability outcome combinations. Altogether, these properties of the proposed probability weighting function Ψ introduce how a naive equal probability of each relevant outcome combination is used by the DM to classify an outcome combination as a low or high-likelihood, and then to adjust their weighting through likelihood insensitivity, to generate the fourfold pattern of Ambiguity Attitudes.

This model specification contrasts with previous ones as we define properties not about preferences over lotteries themselves, but about the DM's interpretation of available information on outcome/event probabilities and the context of the decision, given by choice set information.

For future research, some interesting questions arise, besides the ones already pointed out throughout the paper. For example, is there a limit to how many pure outcomes in an outcome combination can be considered by a decision-maker when calibrating his weighting of x with a naive probability distribution: That is, if there are 100 possible lottery outcomes relevant to lottery's L^n outcome, is the cutoff to define a "low" expected probability of an outcome combination less than the cutoff when there are 99 relevant combinations? Or is there a limit to this cutoff point? Is there an interaction between sensitivity to salience and sensitivity to distortions in probability weighting? In other words, are people who are more affected by bottom up salience are also more affected by bottom up Ambiguity Adjustment distortions in probability weighting? These questions will certainly provide great insights to calibrate and apply the proposed model many different puzzles related to decision under ambiguity.

Chapter 4

Conclusion

The present dissertation addressed two approaches regarding the phenomenon of decision-making under risk: one from a theoretical standpoint and the other from an econometric perspective, testing previously developed theories.

Economics is fundamentally a science about human decision-making. In this thesis, we aim to depict this crucial aspect of our field of study, both from a theoretical viewpoint, as discussed in Chapter 3, as well as from an empirical perspective, as elaborated in Chapter 2.

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4.1 Appendix

This appendix complements chapter 3 of this dissertation, with proofs of propositions, the expanded analysis of examples given in the main text, and notes on the model's application in specific instances of decision problems.

4.2 Appendix A: Proofs of Propositions

Proposition 1 Proof:

$\hat{\Psi}(\pi|p) = \mathbb{E}_p[\pi]^{(1-\gamma\sigma_p i)}\ (1+\eta/||X_n||)^{\gamma\ \sigma_p(pi)}$ is an Ambiguity Adjustment Function, which means properties (P1), (P2) and (P3) of Definition 1 are valid for $\hat{\Psi}(\pi|p)$. First, note that $\hat{\Psi}(\pi|p) : \Delta(X) \to [0,1]$. That is so because: (i) since $\mathbb{E}_p[\pi]$ is the expected probability of a random variable defined on the $[0,1]$ domain - therefore both also on the $[0,1]$ domain; (ii) $\sigma_p i \in [0, 1/2]$, since standard deviations are always positive or zero, and in our setting the maximum standard deviation for π(x) is obtained when the probability mass is concentrated in its' extreme points, 0 and 1, which gives $\sigma_p i = 1/2$; (iii) $\eta \in (-1, 0)$, $\gamma \in [0, 2]$ and $||X_n|| \in \mathbb{N}^*$ which, together with (i) and (ii), determine that all the terms in $\hat{\Psi}(\pi|p)$ are positive and between 0 and 1, so that $\hat{\Psi}(\pi|p) \in [0,1]$.

Now we prove that the (P1) holds for $\hat{\Psi}(\pi|p)$: Take fixed $\sigma_p i = \sigma$ and $||X_n|| = c$. Then, taking the partial derivative with respect to $\mathbb{E}_p[\pi]$ gives:

$$\frac{\delta\hat{\Psi}}{\delta\mathbb{E}_p[\pi]} = (1-\gamma\sigma)\mathbb{E}_p[\pi]^{-\gamma\sigma}\frac{1+\eta}{c}^{\gamma\sigma}$$

Given the domains of $\mathbb{E}_p[\pi] \in [0,1]$, η, $||X_n||$, that $\max(\sigma) = 1/2$, all the terms are positive, and $\frac{\delta\hat{\Psi}}{\delta\mathbb{E}_p[\pi]} \geq 0$.

Now we prove that the (P2) holds for $\hat{\Psi}(\pi|p)$. Taking the partial derivative:

$$\frac{\delta\hat{\Psi}}{\delta||X_n||} = \mathbb{E}_p[\pi]$$

Since π is only defined for an exogenously given set X_n, $\pi[\mathrm{x}], \mathbb{E}_p[\pi], \sigma_p i$ are not correlated with $||X_n||$, and the partial derivative above suffices to prove (P2).

We will now prove that (P3) holds for $\hat{\Psi}$.

First, consider the case where $\sigma_\pi = 0$. In that case:

$$\hat{\Psi} = \mathbb{E}_p(\pi(\mathrm{x}))$$

Moreover, since $|\hat{\Psi} - \bar{\pi}(\mathrm{x})| = \mathbb{E}_p(\pi(\mathrm{x}))^{1-\gamma\ \sigma_\pi}((1+\eta)/||X_n||)^{\gamma\ \sigma_\pi} = \pi(\mathrm{x})$ and $\mathbb{E}_p(\pi(\mathrm{x})) = E$, $||X_n|| = c$ and $\pi(\mathrm{x})$ are fixed in the P3 property definition, $\delta(|\hat{\Psi} - \bar{\pi}(\mathrm{x})|)/\delta(\sigma_\pi)$. Then, given the domains of the variables, both terms in the $\delta(|\hat{\Psi} - \bar{\pi}(\mathrm{x})|)/\delta(\sigma_\pi)$ will decrease with increasing σ_π.

4.3 Appendix B: Value Function Calculations of the Parametric $\hat{\Psi}$

4.3.1 Appendix B.1 section 3.4.1 example

We can observe these relations applying our parametric $\hat{\Psi}$ function to this example:

$$\begin{aligned}\hat{\Psi}(p_{\mathrm{x}_r}, ||X_{-g}||) &= \mathbb{E}_p[\mathrm{x}_r]^{(1-\gamma\cdot\sigma_p(\mathrm{x}_r))}\left(\frac{1+\eta}{||X_{-g}||}\right)^{\gamma\cdot\sigma_p(\mathrm{x}_r)} \\ &= \left(\frac{1}{4}\right)^{(1-\gamma\cdot 0)}\left(\frac{1+\eta}{4}\right)^{(\gamma\cdot 0)} = \frac{1}{4}\end{aligned} \tag{4.1}$$

$$\begin{aligned}\hat{\Psi}(p_{\mathrm{x}_y}, ||X_{-g}||) &= \mathbb{E}_p[\mathrm{x}_y]^{(1-\gamma\cdot\sigma_p(\mathrm{x}_y))}\left(\frac{1+\eta}{||X_{-g}||}\right)^{\gamma\cdot\sigma_p(\mathrm{x}_y)} \\ &= \left(\frac{1}{4}\right)^{(1-\gamma/3)}\left(\frac{1+\eta}{4}\right)^{(\gamma/3)}\end{aligned} \tag{4.2}$$

$$\begin{aligned}\hat{\Psi}(p_{E_{y,-g}}, ||X_{-g}||) &= \mathbb{E}_p[E_{y,-g}]^{(1-\gamma\cdot\sigma_p(E_{y,-g}))}\left(\frac{1+\eta}{||X_{-g}||}\right)^{\gamma\cdot\sigma_p(E_{y,-g})} \\ &= \left(\frac{1}{4}\right)^{(1-\gamma/3)}\left(\frac{1+\eta}{3}\right)^{(\gamma/3)}\end{aligned} \tag{4.3}$$

First, note that in our parametric function, for any $\gamma \in (0,2)$, $f_{\alpha,-g} \succ f_{\alpha,x_g=0}$ and $f_{0,x_g=0} \succ f_{\alpha,x_g=0}$ unequivocally. The preference relation between $f_{\alpha,-g}$ and $f_{0,x_g=0}$ than depends on the parameters for aversion to information precision γ and likelihood insensitivity η. Specifically, for any $\gamma > 0$, $\eta \in (-0.25, 1) \implies f_{\alpha,-g} \succ f_{0,x_g}$ and $\eta \in (-1, -0.25) \implies f_{0,x_g} \succ f_{\alpha,-g}$, as stated in the main text.

4.3.2 Appendix B.2: section 3.4.2 example

Applying our parametric $\hat{\Psi}$ function to section 3.4.2 example, we get:

$$\hat{\Psi}(p^*(\Pi(\mathrm{x}_r))) = \frac{1}{4} \tag{4.4}$$

$$\begin{aligned}\hat{\Psi}(p^*(\Pi(\mathrm{x}_y))) &= \mathbb{E}_p[\mathrm{x}_y]^{(1-\gamma\cdot\sigma_p(\mathrm{x}_y))}\left(\frac{1+\eta}{||X_2||}\right)^{\gamma\cdot\sigma_p(\mathrm{x}_y)} \\ &= \left(\frac{1}{4}\right)^{(1-\gamma/6)}\left(\frac{1+\eta}{4}\right)^{(\gamma/6)}\end{aligned} \tag{4.5}$$

Therefore, the probability weighting "distortions" depend on the relationship between sensitivity to information imprecision (γ) and likelihood insensitivity (η). Specifically, for any $\gamma > 0$, $\eta \in (-1, 0) \implies \hat{\Psi}(p^*(\Pi(\mathrm{x}_y))) < \hat{\Psi}(p(\Pi(\mathrm{x}_y)))$ and $\eta \in (0, 1) \implies \hat{\Psi}(p^*(\Pi(\mathrm{x}_y))) > \hat{\Psi}(p(\Pi(\mathrm{x}_y)))$.

There has been recent evidence in favor of underweighting for additional information that increases standard deviation for bets involving gains (Chew et al., 2017), that can be accommodated in our parametric example with $\eta \in (-1, 0)$.

4.3.3 Appendix B.3 section 3.4.3 example

Applying our parametric example of $\hat{\Psi}$ to the first example of the Section 3.4.3 of a bet involving losses, we get the following probability weighting functions:

$$\hat{\Psi}(p(\Pi(\mathrm{x}_r))) = \frac{1}{4} \tag{4.6}$$

$$\hat{\Psi}(p(\Pi(\mathrm{x}_y))) = \left(\frac{1}{4}\right)^{(1-\gamma/4)}\left(\frac{1+\eta}{4}\right)^{(\gamma/4)} \tag{4.7}$$

Again, in this specific example the preferences depend on the γ and η parameters. Specifically, for any $\gamma > 0$, $\eta \in (-1, 0) \implies f_{-\alpha} \succ f_{-0}$ and $\eta \in (0, 1) \implies f_{-0} \succ f_{-\alpha}$.

In our second example of the section, we explore how changes in the expected likelihoods (but nothing else) affect probability weighting of events. Specifically, applying our parametric function to the second example of Section in the main text, then the value of $\hat{\Psi}(\mathrm{x}_y)$ for an act $z\mathrm{x}_y0$ with $z \neq 0$ is given by:

$$\hat{\Psi}(p(\Pi(\mathrm{x}_y))) = (\mathbb{E}_y)^{\left(1-\gamma\cdot\sqrt{\mathbb{E}_y(1-\mathbb{E}_y)}\right)} \left(\frac{1+\eta}{2}\right)^{\left(\gamma\cdot\sqrt{\mathbb{E}_y(1-\mathbb{E}_y)}\right)} \tag{4.8}$$

where we simplify the notation so that $\mathbb{E}_y = \mathbb{E}'_p(\Pi(\mathrm{x}_y))$. We also note that, in our example, p follows a Bernoulli distribution, and since $\mathbb{E}'_p(\Pi(\mathrm{x}_y)) = p'(\Pi_1(\mathrm{x}_y))$, we can express the typical $\sqrt{p(1-p)}$ standard deviation of a Bernoulli distribution as $\sqrt{\mathbb{E}_y(1-\mathbb{E}_y)})$.

4.4 Appendix C: A Note on Continuous State-Spaces

In this paper we purposely focus on discrete outcome combination problem. When dealing with decision under ambiguity, these seem to be the most relevant cases, since the simplification of the possible combinations and their associated outcomes is a common way for individuals to deal with complex and/or incomplete information about probabilities. Even in the experimental literature, it is usual to represent continuous distributions, even some of the most known ones as the univariate normal distribution as discrete approximations (Lian et al., 2019), since the continuous distribution information itself may be too complex for the decision-maker to form meaningful scenarios that she can use in choosing from a given set of lotteries. There has also been a long-standing literature on simplifying rules used by decision-makers, when faced with complex decisions (Kahneman et al., 1982; Sundstroem, 1987), without losing significant effectiveness in decision-making (Bruce & Johnson, 1996; Hertwig & Todd, 2003).

However, we also recognize that there may be instances where considering a continuous outcome combinations may be useful, specially to relate discrete and continuous spaces through measure theory. Therefore, we alter property P2 to adapt the definitions of ambiguity adjustment function and the context-based ambiguity adjustment function, but now for continuous finite outcome combinations. Basically, now the use of the cardinality of $||X_n||$ now does not make any sense (since cardinality is not a good measure of how likely an event in a continuous outcome combination is). Instead, we use the Lebesgue Measure to measure it. Below we adapt postulate P2 considering that, which we rename as property P2'.

(P2') Ψ is decreasing on $l(X_n)$, where $l(A)$ denotes the Lebesgue measure of set A;

Analogously, we also redefine our parametric example function $\hat{\Psi}'$, but now adapted to a continuous state-space:

$$\hat{\Psi}(\pi(\mathrm{x})) = \mathbb{E}_p[\pi(\mathrm{x})]^{(1-\gamma\cdot\pi)} \left(\frac{1+\eta}{\sum_i l(\mathrm{x}^i)} \right)^{\gamma\cdot\sigma_\pi} \tag{4.9}$$

where $l(\mathrm{x}^i)$ is the Lebesgue measure of the pure outcome x^i associated with outcome combination x^i if lottery L^i is chosen. The proof that this function is an Ambiguity Adjustment Function Ψ (for continuous state-spaces) is analogous to the discrete case proof of Appendix A.