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Estudo Não-Perturbativo de Teorias de Gauge
e o Confinamento de Cor

Texto que sistematiza o trabalho científico do candidato
para a obtenção do título de Livre-Docência

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Introdução

O Modelo Padrão [1] descreve as interações eletromagnéticas, fracas e fortes entre partículas elementares por meio de teorias quânticas de campos caracterizadas por invariância sob transformações de simetria — ou transformações de gauge — escolhidas independentemente para cada posição do espaço e para cada instante de tempo (teorias de gauge locais) [2].

Em particular, a interação forte é descrita por uma teoria de gauge baseada no grupo não-abeliano $SU(3)$, a Cromodinâmica Quântica (ou QCD). A QCD explica as interações fortes entre hádrons através de um modelo de partículas elementares, chamadas quarks, possuindo carga de cor e interagindo por troca de campos de gauge, os glúons [3]. As interações definidas pela QCD são fortemente não-lineares, uma característica típica de teorias de gauge não-abelianas que corresponde ao fato de a sua partícula mediadora, o glúon, possuir carga de cor (diferentemente do que acontece com o fóton no eletromagnetismo, o qual não possui carga elétrica). A complexidade das interações hadrônicas surge devido a esses efeitos não-lineares. Por outro lado, a interação entre quarks torna-se pequena no limite de pequenas distâncias, ou equivalentemente no limite de altas energias ou momentos (o limite ultravioleta), permitindo uma abordagem baseada em teoria de perturbação. Este fato, comprovado experimentalmente, é conhecido como liberdade assintótica [4].

Ao mesmo tempo, as forças eletromagnéticas e fracas são unificadas em uma teoria de gauge (a teoria eletrofraca) baseada na simetria não-abeliana $SU(2) \times U(1)$ [5]. Esta simetria é quebrada espontaneamente (mecanismo de Higgs) para explicar as massas dos bósons $W^\pm$ e Z^0 , que são as partículas mediadoras da força fraca. As grandes massas desses bósons ($m \sim 80$ GeV) implicam que, no limite de baixa energia, seus propagadores sejam dados por $1/m^2$. Assim, nesse limite, a interação fraca pode ser descrita por uma lagrangiana efetiva em que a constante de acoplamento é a chamada constante de Fermi $G_F \sim 1/m^2$, introduzida por Fermi em 1930 para explicar o decaimento β .

Apesar do grande sucesso do Modelo Padrão na descrição de experimentos a altas energias (regime perturbativo), jamais foi observado um quark isolado. Este resultado experimental levou à especulação de que quarks (e glúons) estejam permanentemente confinados no interior dos hádrons. De fato, já que a intensidade da interação entre quarks torna-se pequena a curtas distâncias, pode-se supor que ela se torne grande para distâncias grandes, resultando no confinamento de quarks e glúons. Consequentemente, a própria carga de cor encontra-se confinada (confinamento de cor). Em princípio, a QCD descreve a interação entre quarks para todas as distâncias e escalas de energia. Entretanto, o problema do confinamento de quarks — assim como os processos de baixa energia que mantêm quarks em estados ligados de hádrons e determinam portanto as massas dos hádrons, suas vidas médias, etc. — é de natureza não-perturbativa: a teoria de perturbação falha quando aplicada à QCD na escala de baixos momentos (que corresponde a grandes separações espaciais), já que a constante de acoplamento aumenta à medida que a energia diminui. Os processos na região de energias mais baixas são portanto inacessíveis ao estudo por teoria de perturbação, devendo ser estudados em maneira não-perturbativa.

É também importante lembrar que teorias de gauge, sendo invariantes sob transformações locais de simetria, são sistemas com variáveis dinâmicas redundantes, as quais não representam verdadeiros graus de liberdade. Os objetos de interesse não são os campos de gauge em si, mas sim as classes (ou órbitas) de campos relacionados por transformações de gauge. Assim, a eliminação destes graus de liberdade redundantes é essencial para o

entendimento dessas teorias e para a obtenção de informação física a partir delas. Isto é feito em geral através de um método, chamado fixação de gauge, baseado na existência de uma condição que determine de maneira unívoca um representante para o campo de gauge em cada órbita. Na Ref. [6], Gribov mostrou que as condições (bastante usuais) de gauge de Coulomb e de Landau não definem um único campo de gauge em cada órbita. Em particular, no caso do grupo de gauge $SU(2)$, ele construiu exemplos explícitos de configurações diferentes dos campos de gauge satisfazendo à mesma condição de gauge (de Coulomb ou de Landau) e que são relacionadas através de transformações de gauge. Tais configurações de gauge são chamadas de cópias de Gribov. Do ponto de vista matemático, a existência das cópias de Gribov está relacionada à não-linearidade da teoria. De fato, as equações diferenciais usadas para fixar os gauges de Coulomb e de Landau são não-lineares e em geral admitem várias soluções [6, 7]. Do ponto de vista tanto formal quanto prático, a existência das cópias de Gribov limita fortemente a simplicidade e a elegância da quantização por integrais de trajetória das teorias de campos. Ao mesmo tempo, as cópias de Gribov não afetam os resultados da teoria de perturbação, mas elas poderiam ter um papel importante no estudo de aspectos não-perturbativos, como o confinamento de quarks e de glúons na QCD [8].

A formulação de rede de teorias quânticas de campo [9, 10, 11, 12, 13, 14] oferece uma regularização não-perturbativa que preserva as propriedades essenciais da teoria, incluindo a invariância de gauge. Nesta formulação — que consiste na quantização por meio de integrais de trajetória, na continuação para tempos imaginários ou euclidianos e na regularização de rede (dada pela discretização do espaço-tempo) — a teoria torna-se equivalente a um modelo de mecânica estatística clássica. O limite do contínuo, no qual são obtidos os resultados físicos, é dado pelo ponto crítico deste modelo, que pode ser estudado através dos métodos usuais em mecânica estatística. Em particular, podem ser aplicadas simulações numéricas de Monte Carlo, que se baseiam em uma descrição estocástica dos sistemas considerados [15]. Desta forma, a QCD na rede permite um estudo não-perturbativo da interação forte a partir de primeiros princípios e sem aproximações não-controladas. A grande importância de um tratamento não-perturbativo de primeiros

princípios para problemas relacionados ao confinamento na QCD justifica o enorme esforço computacional investido em tais estudos. De fato, devido à maior complexidade da interação e ao grande número de graus de liberdade, as simulações numéricas de Monte Carlo para teorias de gauge são muito mais elaboradas do que as simulações para os modelos usuais em mecânica estatística e constituem problemas de computação de alto desempenho [16], estando incluídas na classe dos problemas-desafio [17] para a super-computação, os chamados *grand challenge problems* [18].

Nos últimos anos, resultados de QCD na rede têm causado grande impacto, devido à precisão atualmente possível em tais estudos [19, 20, 21]. Por outro lado, o entendimento do mecanismo que rege o confinamento de quarks e glúons permanece um problema em aberto [22]. Vale ressaltar que, no trabalho original para a formulação da QCD na rede, de Wilson [23], a propriedade de confinamento foi associada a uma lei de área para laços retangulares com um lado na direção temporal, os chamados laços de Wilson. Esta lei de área pode ser verificada analiticamente através de uma expansão no chamado limite de acoplamento forte — i.e. quando a constante de acoplamento de rede β é pequena — demonstrando assim o confinamento de quarks. Porém, o limite físico da teoria de rede corresponde a valores de β grandes¹ e nesse limite não é possível realizar a mesma expansão. Assim, o estudo das propriedades físicas da QCD a partir de primeiros princípios deve ser conduzido usando simulações numéricas de Monte Carlo.

Existem hoje várias tentativas de explicação do mecanismo de confinamento de cor na QCD e, mais em geral, nas teorias de Yang-Mills (ver, por exemplo, [24, 25]). Entre as ideias mais aceitas, vale relembrar a consideração de classes de configurações dos campos de gauge que deveriam dominar a integral funcional para a QCD, gerando a lei de área para os laços de Wilson e explicando assim o confinamento de quarks e glúons. Uma abordagem diferente considera as funções de Green da teoria, i.e. propagadores e vértices para os campos que entram na lagrangiana da QCD. De fato, todo o conteúdo físico e matemático de uma teoria quântica de campos pode ser recuperado através de suas funções

¹De fato, no limite de acoplamento forte (β pequeno), o confinamento pode ser demonstrado também para a QED!

de Green (ver, por exemplo, [26]). Assim, o mecanismo de confinamento deveria ser de alguma forma “codificado” no comportamento das funções de correlação para grandes separações (i.e. para pequenos momentos). As funções de Green das teorias de Yang-Mills são, porém, dependentes de gauge. Portanto, o estudo dessas funções requer a escolha de uma condição de gauge. Entre as possíveis condições de gauge, é particularmente interessante o chamado gauge de Landau (ou de Lorenz), que preserva a covariância relativística da teoria. Para este gauge existem diferentes previsões analíticas para o comportamento de propagadores e de vértices no limite infravermelho. Por outro lado, tais resultados analíticos são (em geral) obtidos usando-se aproximações não-controláveis. Claramente, a confirmação (ou não) de previsões analíticas para as funções de Green no gauge de Landau, através de simulações numéricas de primeiros princípios, seria um passo fundamental para a descrição das propriedades infravermelhas da QCD. Ao mesmo tempo, os resultados numéricos obtidos poderiam ser usados como *inputs* para cálculos analíticos (ver, por exemplo, [27, 28, 29]) e em estudos de fenomenologia hadrônica (ver, por exemplo, [30, 31]).

Os principais resultados analíticos para as funções de Green no gauge de Landau serão brevemente discutidos no Capítulo 2. No mesmo capítulo, será feita uma breve discussão do problema da cópias de Gribov e uma introdução às teorias de gauge na rede. No restante do texto, serão apresentados e discutidos resultados de nossa pesquisa.

O grupo de Física de Partículas Computacional do IFSC-USP — integrado pelos docentes Tereza Mendes e Attilio Cucchieri — foi formado em 2001, graças a um Auxílio Jovens Pesquisadores em Centros Emergentes da FAPESP, e foi o primeiro grupo ativo na área de QCD na Rede no Brasil e na América Latina. Ao longo desses anos, nosso grupo tem dado contribuições originais ao entendimento de aspectos qualitativos fundamentais da QCD, como o mecanismo de confinamento de cor e aspectos de seu desconfinamento a altas temperaturas, através do estudo de funções de Green. Em particular, realizamos o primeiro estudo numérico do vértice de glúon-fantasma [32] no gauge de Landau. Além disso, o grupo considerou propagadores em outros gauges, como o gauge de Coulomb [33]

e o gauge maximamente abeliano [34, 35] (para o qual foi realizado o primeiro estudo do propagador de fantasmas), introduziu uma nova formulação do gauge covariante linear na rede [36, 37, 38, 39] — resolvendo os problemas encontrados nas formulações anteriores [40] — assim como a primeira formulação na rede do chamado *background* gauge [41]. Mais recentemente, foram considerados os propagadores (no gauge de Landau) a temperatura finita (não-nula) [42, 43, 44, 45, 46, 47], para descrição das massas de blindagem gluônica no plasma de quarks e glúons. Também foram realizados estudos de algoritmos, seja analiticamente [48] que numericamente [49, 50].

Serão aqui apresentados os principais resultados obtidos no estudo de propagadores de glúons e de fantasmas no gauge de Landau. Mais especificamente, no Capítulo 3 serão descritos os resultados numéricos obtidos através de simulações numéricas de Monte Carlo. Ressalto que, nesse tópico, temos conseguido resultados de destaque utilizando os maiores lados de rede já considerados em simulações numéricas na área. Este trabalho computacional foi complementado pelos estudos analíticos expostos no Capítulo 4. Como explicado nesse capítulo, os resultados obtidos nesse caso permitem hoje um melhor entendimento e explicação dos dados numéricos. No Capítulo 5 serão finalmente resumidos os principais resultados aqui apresentados, e serão propostas ideias para trabalhos futuros.

Funções de Green e Confinamento de Cor

Como descrito na Introdução, a QCD é baseada em uma teoria — de Yang-Mills — com simetria (não-abeliana) de gauge local dada pelo grupo $SU(3)$. A densidade de lagrangiana da QCD é escrita como (ver, por exemplo, [3])

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^b(x) F^{b;\mu\nu}(x) + \sum_f \bar{\Psi}_f^b(x) [i \gamma^\mu D_{bc;\mu}(x) - m_f \delta_{bc}] \Psi_f^c(x). \quad (2.1)$$

Aqui,

$$F^{b;\mu\nu}(x) = \partial^\mu A^{b;\nu}(x) - \partial^\nu A^{b;\mu}(x) - ig_0 [A_\mu(x), A_\nu(x)]^b \quad (2.2)$$

$$= \partial^\mu A^{b;\nu}(x) - \partial^\nu A^{b;\mu}(x) + g_0 f^{bcd} A_\mu^c(x) A_\nu^d(x) \quad (2.3)$$

é o equivalente não-abeliano do tensor eletromagnético, escrito em termos do campo de glúons $A_\mu^b(x)$, das constantes de estrutura f^{bcd} do grupo de gauge $SU(3)$ — associadas aos comutadores dos campos gluônicos — e da constante de acoplamento nua g_0 da teoria. Ao mesmo tempo, $\Psi_f^b(x)$ é o espinor de Dirac associado aos campos de quarks de sabor f e massa nua m_f . Indicamos com

$$D_{bc;\mu}(x) = \delta_{bc} \partial_\mu - ig_0 (T^d)_{bc} A_\mu^d(x) \quad (2.4)$$

a derivada covariante, sendo $(T^d)_{bc}$ os geradores da álgebra de Lie do grupo $SU(3)$. Os índices b e c são índices de cor. Relembramos também que os campos de glúons pertencem à representação adjunta da álgebra de Lie do grupo $SU(3)$, enquanto que os campos

fermiônicos estão na representação fundamental. Pode-se verificar que a lagrangiana acima é invariante sob as transformações (locais de gauge)

$$A_\mu(x) \rightarrow g(x) A_\mu(x) g^\dagger(x) + \frac{i}{g_0} g(x) \partial_\mu g(x)^\dagger \quad (2.5)$$

$$\Psi_f(x) \rightarrow g(x) \Psi_f(x) \quad (2.6)$$

$$\bar{\Psi}_f \rightarrow \bar{\Psi}_f g^\dagger(x), \quad (2.7)$$

onde † indica a matriz transposta conjugada (ou adjunta).

A QCD, assim como as demais teorias de Yang-Mills, é geralmente quantizada usando integrais de trajetória (ver, por exemplo, [8]), o que permite uma quantização explicitamente covariante das teorias de campos. Ao mesmo tempo, no caso de teorias de gauge, as transformações de gauge podem ser interpretadas como uma simples mudança das variáveis de integração. Da mesma forma, a escolha de uma condição de gauge específica pode ser obtida adicionando uma função delta de Dirac à integração funcional, pelo preço de incluir na medida o determinante jacobiano correspondente à transformação de gauge usada para fixar o gauge (método de Faddeev-Popov; ver, por exemplo, [51]). Assim, o funcional gerador da teoria pode ser escrito no caso das teorias de Yang-Mills como

$$Z[J] = \int DA_\mu^b |det\mathcal{M}| \delta(F[A^{(g)}]) \exp \left[i \int d^4x (\mathcal{L} + J \cdot A) \right], \quad (2.8)$$

onde $F[A^{(g)}] = 0$ é o gauge escolhido, $\mathcal{M}$ é o chamado operador de Faddeev-Popov, g representa as transformações de gauge, $\mathcal{L}$ é a densidade de lagrangiana e $J_\mu^b(x)$ são as fontes externas. Dessa forma é fatorizada a constante infinita $\int Dg$, correspondente ao volume total do grupo de gauge, e torna-se possível usar a teoria de perturbação, já que a parte quadrática nos campos de gauge da ação fica inversível e pode-se definir o propagador dos campos de gauge.

2.1 C3pias de Gribov

A presena de c3pias de Gribov modifica a formula (2.8) acima. Em particular, o fator $|det\mathcal{M}|$ deve ser substituído por [8]

$$\left(\sum_m |det\mathcal{M}_m|^{-1}\right)^{-1}, \quad (2.9)$$

onde a soma em m é sobre todas as c3pias de Gribov. Claramente este fator tornaria o uso da integra3o funcional bastante complicado, at3e para a defini3o das regras de Feynman da teoria. O pr3prio Gribov, em seu trabalho [6], tentou resolver os problemas relacionados 3 exist3ncia de c3pias de Gribov. Para isso, no caso do gauge de Landau¹, ele considerou a restri3o do espao das configura3es 3 regi3o Ω de configura3es transversas — i.e. com $\partial_\mu A_\mu = 0$ — para as quais o operador de Faddeev-Popov $\mathcal{M}[A] = -\partial_\mu D_\mu[A]$ é positivo definido.² Esta regi3o est3 incluída no hiperplano Γ das configura3es transversas e é delimitada pelo chamado primeiro horizonte de Gribov, definido como o conjunto de configura3es para as quais o menor autovalor n3o-trivial³ de $\mathcal{M}$ é igual a zero.

Essa solu3o baseia-se na seguinte hip3tese: cada 3rbita intercepta a regi3o Ω uma (e somente uma) vez. Hoje sabemos que, para n3o possuir c3pias de Gribov, o espao das configura3es deve ser identificado com a chamada “regi3o modular fundamental” Λ , definida como o conjunto de m3nimos absolutos do funcional

$$E[A] = \|A^{(g)}\|^2, \quad (2.10)$$

onde $\|\cdot\|$ indica a norma L^2 . Assim, em cada 3rbita é escolhida a configura3o que est3 mais pr3xima da “origem” $A_\mu^b(x) = 0$. Nas Refs. [52, 53] foi demonstrado que todas as 3rbitas de gauge interceptam a regi3o Λ . Tamb3m foi demonstrado que a regi3o de Gribov Ω corresponde ao conjunto de m3nimos relativos do funcional $E[A]$, i.e. Ω inclui Λ e, em geral, n3o est3 livre de c3pias de Gribov. Enfim, nas Refs. [54, 55] foi demonstrado

¹Uma solu3o an3loga pode ser considerada no gauge de Coulomb em cada fatia de tempo.

²Aqui $D_\mu[A]$ é a derivada covariante.

³O operador de Faddeev-Popov $\mathcal{M}$ no gauge de Landau possui um autovalor nulo trivial, correspondente a autovetores constantes.

que somente no interior da região Λ os mínimos absolutos são não-degenerados. Assim na borda da região Λ existem cópias de Gribov. As principais propriedades das regiões Ω e Λ são apresentadas nas Refs. [7, 53, 54, 56, 57, 58, 59] e, recentemente, reunidas nas Refs. [60, 61]. Em particular, é importante lembrar que 1) o vácuo trivial $A_\mu = 0$ pertence à região Ω , 2) a região Ω é convexa e 3) a região Ω é limitada em todas as direções.

2.2 O Cenário de Confinamento de Gribov-Zwanziger

Como evidenciado na Introdução, a existência das cópias de Gribov poderia ter um papel importante na explicação do fenômeno de confinamento de cor. No chamado cenário de Gribov-Zwanziger (para o gauge de Landau) é usada a ideia de Gribov (ver seção acima e a Ref. [6]) de limitar o espaço das configurações à região Ω de configurações transversas, para as quais a matriz de Faddeev-Popov $\mathcal{M} = -\partial \cdot D$ é positiva-definida. A restrição da integral funcional (ou de trajetória) à região Ω implica uma desigualdade rigorosa [62, 63] para as componentes de Fourier do campo gluônico $A_\mu(p)$. Esta desigualdade permite que sejam feitas previsões importantes para o comportamento do propagador gluônico $\langle A_\mu^a(x) A_\nu^b(y) \rangle$ no espaço dos momentos p . Para tanto, é conveniente introduzir-se [58, 62, 63] uma espécie de campo “magnético” modulado H acoplado aos “spins de cor” $A_\mu^b(x)$. Trabalhando no espaço de momentos, Zwanziger demonstrou que a energia livre (por unidade de volume) $w(H, p)$ deste sistema de spins se anula identicamente no limite termodinâmico a momento zero ($p = 0$). Isto implica que a magnetização de cor $M(H, p) \equiv \partial w(H, p) / \partial H$, a suscetibilidade $\chi(H, p) \equiv \partial M(H, p) / \partial H$ e todas as derivadas de ordens superiores são nulas [58]. Por exemplo, no caso da suscetibilidade, pode-se escrever

$$w(H, p) = \int_0^H dy (H - y) \chi(y, p) \quad (2.11)$$

e é imediato obter-se que $\chi(H, p) = 0$ para quase todo H , no limite $p \rightarrow 0$. Neste formalismo o propagador usual de glúons $D(p)$ no espaço de momentos é dado pela suscetibilidade $\chi(H, p)$ calculada a $H = 0$. Segue que, se $\chi(H, p)$ é regular em H , então temos

também que

$$\chi(H = 0, p) = D(p) = 0 \quad (2.12)$$

no limite $p \rightarrow 0$. Entretanto, esta última hipótese não pôde ser demonstrada e portanto a natureza do comportamento infravermelho do propagador $D(p)$ no gauge de Landau não pôde ser estabelecida.⁴

De qualquer forma, os resultados acima sugerem que o propagador usual de glúons $D(p)$ seja nulo no limite de momento p tendendo a 0. Escrevendo

$$0 = D(p = 0) \propto \int d^4x D(x), \quad (2.13)$$

fica claro que neste caso o propagador de glúons no espaço-tempo usual $D(x)$ seria positivo e negativo em medidas iguais, i.e. seria evidente uma violação (máxima) da positividade por reflexão para este propagador. Como consequência, os glúons deixariam de pertencer ao espectro de partículas da teoria e seriam confinados [65]. Este tópico será discutido em mais detalhes na Seção 3.2 abaixo.

É também importante entender o que acontece à distribuição de probabilidade, que define a integração funcional no espaço euclidiano, quando a integração é limitada à região Ω [58, 66]. Para tanto, é conveniente considerar a formulação de rede da teoria no limite de volume infinito (limite termodinâmico; ver Seção 2.4 abaixo). De fato, se nesse limite a distribuição estivesse concentrada dentro da região Ω , então seria possível efetuar uma expansão perturbativa e a teoria descreveria uma fase de glúons não-confinados. Por outro lado, devido a argumentos de entropia, sabe-se que a distribuição de probabilidade euclidiana no limite termodinâmico se concentra na fronteira da região Ω , onde a matriz de Faddeev-Popov $\mathcal{M}$ adquire um autovalor (não-trivial) nulo. Poderia assim seguir que o comportamento do propagador dos fantasmas de Faddeev-Popov $G(p) \sim \mathcal{M}^{-1}$ seja fortalecido para momentos pequenos no limite termodinâmico. Este fortalecimento seria uma indicação clara de um efeito de longo alcance na teoria e, como sugerido por Gribov [6], poderia gerar um potencial linear (confinante) entre quarks. Esta ideia será mais

⁴Mais recentemente, usando essa abordagem, foi possível mostrar [64] que no caso bidimensional o propagador $D(0)$ é efetivamente nulo.

aprofundada, e parcialmente criticada, na Seção 4.1 abaixo.

2.3 Funções de Green e Confinamento

O comportamento infravermelho de propagadores e de vértices no gauge de Landau foi também estudado analiticamente usando as equações de Dyson-Schwinger (EDS) [65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94]. Existem, nesse caso, previsões diferentes e contrastantes. Em particular, no caso das previsões que estão de acordo com o cenário de confinamento de Gribov-Zwanziger descrito na seção acima — denominadas de *conformes* ou de *escala* [65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82] — o comportamento infravermelho para os propagadores de glúons $D(p)$ e de fantasmas $G(p)$ é dado respectivamente por $D(p) \sim p^{4\kappa-d/2}$ e $G(p) \sim p^{-2-2\kappa}$ (onde d é a dimensão do espaço-tempo). As previsões para o expoente infravermelho κ são [78] $\kappa \approx 0.6$ no caso quadri-dimensional, $\kappa \approx 0.4$ em três dimensões e $\kappa = 0.2$ em duas dimensões. Isto implica que o propagador de glúons é nulo no limite de momento zero ($p \rightarrow 0$). Assim, seria evidente uma violação (máxima) da positividade por reflexão para esse propagador e os glúons seriam confinados [65]. Ao mesmo tempo, o propagador de fantasmas seria fortalecido no limite de momentos pequenos, i.e. teria uma divergência com o inverso do momento p mais forte do que o propagador livre p^{-2} . A solução conforme também prevê que o vértice de fantasma-glúon [78, 82] e o propagador de quarks [79] sejam finitos no limite de momentos pequenos e que o comportamento infravermelho dos vértices de quarks-glúons, de três glúons e de quatro glúons seja dado respectivamente por $(p^2)^{-\kappa-1/2}$ (em $d = 4$), $(p^2)^{-3\kappa+(d-3)/2}$ e $(p^2)^{-4\kappa+(d-4)/2}$ (em $d = 2, 3, 4$) [78, 79]. Isto implica (por exemplo no caso $d = 4$ e $\kappa > 0.5$) um limite infinito a momento nulo para esses três vértices.

Ao contrário, as previsões chamadas de *massivas* ou de *desacoplamento* [83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94] sugerem (para $d = 3, 4$) um propagador de glúons finito

e não-nulo a momento zero e um propagador de fantasmas com um comportamento do tipo do propagador livre p^{-2} para pequenos momentos p . Ao mesmo tempo, os vértices de fantasma-glúon e de quark-glúon seriam finitos no infravermelho. A solução massiva foi obtida em três e em quatro dimensões também usando a abordagem de Gribov-Zwanziger [95, 96, 97]. Nesse caso tal solução (chamada de Gribov-Zwanziger Refinada) aparece quando é introduzido na ação um termo de massa, mantendo-se a renormalizabilidade da teoria. Em particular, o comportamento massivo estaria relacionado à condensação de um operador de dimensão $d - 2$. De fato, fixando o valor deste condensado igual a zero, é obtida novamente a solução conforme. Note-se que essa abordagem não pode ser usada no caso de dimensão dois [98], i.e. nesse caso a teoria exibiria somente a solução conforme.⁵

Como já explicado na seção acima, no caso da solução conforme, o fortalecimento infravermelho do propagador de fantasmas seria uma indicação clara de um efeito de longo alcance na teoria, explicando o confinamento de cor. De forma parecida, na Referência [79] o potencial linear é explicado usando a divergência do vértice de quark-glúon no limite de pequenos momentos. Ao contrário, no caso massivo, o confinamento de cor seria devido à condensação de vórtices [99]. Vale também ressaltar que, na Referência [100], foi formulado um critério de confinamento de quarks que é satisfeito seja no caso da solução conforme que no caso das soluções massivas.

2.4 QCD na Rede

Uma dificuldade no estudo da QCD, comum a virtualmente todas as teorias quânticas de campos, é o aparecimento de divergências ultravioleta (i.e. para altas energias ou curtas distâncias) no cálculo de quantidades físicas. Só após a remoção desses “infinitos” através de algum procedimento de renormalização são obtidos resultados finitos, que podem ser comparados aos experimentos. É necessário portanto primeiramente regularizar a teoria, escrevendo-a de modo que sejam isoladas as singularidades, para depois removê-las através

⁵Este assunto será também discutido na Seção 4.5 abaixo.

de uma redefinição dos parâmetros da lagrangiana.

A formulação da QCD na rede, introduzida em 1974 por Wilson [23], oferece uma regularização não-perturbativa conveniente, preservando a invariância de gauge da teoria [9, 10, 11, 12, 13, 14, 101]. Quarks são representados em pontos da rede e glúons nos elos entre pontos vizinhos. Os campos gluônicos são dados⁶ por matrizes do grupo SU(3). A transformação de gauge para as variáveis de elos $U_\mu(x)$ é escrita como

$$U_\mu(x) \rightarrow U_\mu^g(x) \equiv g(x) U_\mu(x) g(x + \hat{\mu})^\dagger, \quad (2.14)$$

onde $g \in \text{SU}(N_c)$, sítios da rede são indicados com x , e μ representa as 4 direções na rede.⁷ Considerando a relação acima, fica evidente que o traço do produto de variáveis de elos ao longo de qualquer percurso fechado é invariante por transformações de gauge. Assim, escrevendo a ação de rede em termos de produtos das variáveis de elos ao longo de percursos fechados (e orientados), é preservada a simetria de gauge da ação original. Na ação de Wilson para as teorias de Yang-Mills, são usados os menores percursos fechados — as plaquetas — dadas por quatro elos orientados

$$S(\{U\}) = \frac{\beta}{N_c} \sum_{\mu < \nu} \sum_x \Re \text{Tr} [\mathbb{1} - U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x)] . \quad (2.15)$$

Aqui, β é o parâmetro da rede, relacionado à constante de acoplamento nua (*bare*) g_0^2 pela formula $\beta = 2N_c/g_0^2$. Em geral são consideradas redes simétricas hipercúbicas, com condições de periodicidade na borda. Nesse caso, o número de sítios ao longo de uma direção qualquer $\mu = 1, \dots, d$ é indicado com N . Consequentemente, o volume de rede é igual a $V = N^d$. Ao mesmo tempo, indicando com a o espaçamento de rede, obtemos que $L = Na$ é o tamanho da rede em unidades físicas e $V = L^d$ é o volume físico.

Como já mencionado na Introdução, os ingredientes essenciais para a formulação de rede são:

1. O formalismo de integrais de trajetória de Feynman, em que os valores esperados dos

⁶No caso das teorias de Yang-Mills podem-se considerar grupos de gauge SU(N_c) com $N_c \geq 2$.

⁷Claramente, é possível em princípio considerar também uma rede com dimensão d qualquer.

observáveis de interesse são escritos como integrais sobre todos os graus de liberdade do problema, com um peso dado pela exponencial da ação clássica da teoria.

2. A formulação euclidiana, obtida pela continuação analítica da variável temporal a tempos imaginários. Desta forma a exponencial (complexa) oscilatória presente nas integrais descritas acima torna-se real e pode ser interpretada como uma distribuição de probabilidades.
3. A introdução da rede discreta para o espaço-tempo. Correspondentemente, operadores diferenciais são re-escritos como diferenças finitas dos campos discretizados.

A combinação dos dois primeiros ingredientes evidencia a equivalência de teorias quânticas de campo com a mecânica estatística clássica: no espaço euclidiano, uma integral de trajetória para a teoria quântica equivale a uma média térmica para o sistema estatístico correspondente. Considerando a relação escrita acima entre a constante de acoplamento g_0 da teoria de campos e o parâmetro de rede β , fica evidente que g_0 corresponde diretamente à temperatura $1/\beta$ do modelo estatístico.

O terceiro ingrediente — a discretização de rede — representa uma regularização ultravioleta. De fato, o espaçamento de rede a corresponde a um corte para momentos altos, já que não podem ser representados na rede momentos acima de $\sim 1/a$. Desta forma são suprimidos os modos causadores de divergências e a teoria é bem definida, ou seja valores esperados de observáveis são finitos. Para que seja recobrada a teoria no espaço contínuo é preciso, porém, tomar-se o limite $a \rightarrow 0$. Neste processo é necessário “sintonizar” os parâmetros nus da teoria — por exemplo a constante de acoplamento g_0 — de forma que quantidades físicas (funções de correlação, massas, etc.) convirjam para valores finitos, que podem então ser comparados a experimentos. Em particular, comprimentos de correlação ξ (correspondendo a massas inversas) medidos em unidades físicas devem tender a limites finitos à medida que o espaçamento a (também medido em unidades físicas) tende a zero. Isto significa que o comprimento de correlação medido em unidades do espaçamento de rede ξ/a deve tender a infinito. Em outras palavras, a teoria de rede considerada deve se aproximar de um ponto crítico, ou transição de fase de

segunda ordem. O estudo do limite do contínuo em teorias quânticas de campo na rede é portanto análogo ao estudo de fenômenos críticos em mecânica estatística.

Como já mencionado na Introdução, essa correspondência entre as teorias de campos euclidianas e a mecânica estatística clássica permite a aplicação de métodos usuais de física estatística ao estudo da QCD. Podem ser usadas, por exemplo, expansões de altas e baixas temperaturas, correspondendo respectivamente às expansões em acoplamentos fortes e fracos para a teoria de campos. Além disso é particularmente importante a consideração de simulações de Monte Carlo, que permitem um estudo não-perturbativo da teoria de rede considerada. Porém, a interpretação física dos dados numéricos gerados depende da correta extrapolação ao limite do contínuo, ou seja é preciso “voltar” para o espaço contínuo após a simulação na rede. Mais especificamente, é necessária a consideração de três limites para que os resultados físicos desejados possam ser obtidos a partir dos dados das simulações:

- O limite de volume infinito (ou limite termodinâmico): Assim como para a mecânica estatística, simulações da QCD são feitas em volumes finitos de rede, já que a memória do computador é finita. O volume da rede deve então ser suficientemente grande em relação à distância física relevante para o problema estudado, de forma que os efeitos de volume finito não sejam apreciáveis. Correspondentemente, não podem ser consideradas energias ou momentos pequenos demais, já que a rede finita equivale a um corte infravermelho. Como vamos ver no Capítulo 3, a consideração dos efeitos de tamanho finito da rede é de grande importância no estudo de funções de Green no gauge de Landau.
- O limite do contínuo: Para que seja recuperada a física original do contínuo é preciso tomar-se o limite $a \rightarrow 0$. Ao mesmo tempo, como explicado acima, as quantidades calculadas devem ser renormalizadas, ou seja redefinidas de forma a gerar resultados finitos no limite do contínuo. Isto pode ser feito de forma não-perturbativa utilizando-se os valores físicos de alguns observáveis, conhecidos experimentalmente. Por exemplo, escrevendo-se a massa do pión calculada na rede como $m_\pi a$ (onde m_π

é a massa física), obtém-se o valor do corte ultra-violeta a em unidades físicas. As outras quantidades calculadas podem então ser escritas da mesma forma em termos de a (que tende a zero) e “traduzidas” em unidades físicas, gerando valores (físicos) finitos. Na prática, o valor de a deve ser suficientemente pequeno quando comparado à distância relevante para o problema, por exemplo $a \approx 0.05 \text{ fm}$. É importante notar que são possíveis diversas discretizações para a ação, podendo ser usadas as chamadas ações melhoradas (*improved actions*), que convergem para a ação do contínuo mais rapidamente, ou seja para valores maiores de a . (São também possíveis diversas discretizações para os campos fermiônicos.)

- O limite quiral: Para as simulações da QCD completa (*unquenched*) é necessário considerar valores físicos para as massas dos quarks. Porém as simulações numéricas são bem mais complicadas para massas próximas a zero, o chamado limite quiral. Este problema é claramente relevante no caso dos quarks leves (up e down).

A formulação de rede da QCD oferece uma regularização que torna o grupo de gauge compacto. Desta forma a média na distribuição de Gibbs para qualquer quantidade invariante sob transformações de gauge é bem definida e em princípio não é necessária a fixação do gauge. Porém, como explicado na Introdução, o estudo de funções de Green (no contínuo e na rede) requer a fixação de uma condição de gauge. Para o gauge de Landau, considerado nos estudos numéricos descritos no Capítulo 3, esta condição pode ser obtida minimizando o funcional (ver, por exemplo, [58])

$$E[U] = 1 - \frac{a^4}{4 N_c V} \sum_{\mu} \sum_x \Re \text{Tr} U_{\mu}^g(x) \quad (2.16)$$

em relação às transformações de gauge $\{g\}$. De fato, a derivada deste funcional fornece como condição de estacionaridade a condição usual (de Landau) de quadridivergência (na rede) nula para os campos de gauge. Os campos de gauge $A_{\mu}(x)$ são escritos na rede em função das variáveis de elos, através da relação⁸

$$A_{\mu}(x) = \frac{1}{2i ag_0} [U_{\mu}(x) - U_{\mu}^{\dagger}(x)]_{\text{traço nulo}}. \quad (2.17)$$

⁸Note-se, porém, que são possíveis também outras discretizações para os campos de gauge $A_{\mu}(x)$.

As simulações numéricas para teorias de Yang-Mills no gauge de Landau são então organizadas da seguinte forma (ver, por exemplo, [102]):

1. gerar uma configuração de variáveis de elos $\{U\}$ termalizada, usando simulações de Monte Carlo;
2. achar a transformação de gauge $\{g\}$ que minimiza o funcional $E[U]$, definido acima na Eq. (2.16), mantendo fixas as variáveis de elos $\{U\}$;
3. usando a Eq. (2.14), calcular a configuração de gauge $U_\mu^g(x)$ que satisfaz à condição de gauge de Landau.

Note-se que, no processo de fixar a condição de gauge na rede, em nenhum momento é necessário introduzir os chamados fantasmas de Faddeev-Popov.

A configuração $U_\mu^g(x)$, obtida seguindo as instruções descritas acima, pode ser usada para calcular as funções de Green da teoria. Para o propagador de glúons (no gauge de Landau) isto é feito usando a relação simples

$$D_{\mu\nu}^{ab}(p) = \sum_x e^{-2i\pi k \cdot x} \langle A_\mu^a(x) A_\nu^b(0) \rangle = \delta^{ab} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) D(p^2), \quad (2.18)$$

onde a e b são índices de cor e $p(k)$ é o momento na rede com componentes $p_\mu(k) = 2 \sin(\pi k_\mu/N)$, com $k_\mu = 0, 1, \dots, N-1$. Em geral, nas simulações numéricas é calculada somente a função escalar $D(p^2)$, somando a relação aqui sobre todos os valores de a e μ com $b = a$ e $\nu = \mu$.

Como evidenciado acima, os campos de fantasmas não são diretamente considerados nas simulações numéricas de teorias de Yang-Mills no gauge de Landau. Porém, a matriz de Faddev-Popov $\mathcal{M}$ pode ser obtida (ver, por exemplo, [58]) considerando-se a segunda variação do funcional $E[U]$. Isso permite calcular o propagador de fantasmas usando a relação

$$G(p^2) = \frac{1}{N_c^2 - 1} \sum_{x, y, b} \frac{e^{-2\pi i k \cdot (x-y)}}{V} \mathcal{M}^{-1}(b, x; b, y). \quad (2.19)$$

Finalmente, o cálculo dos vértices pode ser feito a posteriori, usando os dados produzidos nas simulações para os propagadores e para as componentes de Fourier do campo de gauge.

Vale ressaltar que teorias de calibre na rede são afetadas pelo mesmo problema de cópias de Gribov encontrado no caso do contínuo. Por exemplo, no calibre de Landau a condição de calibre é imposta encontrando-se o mínimo do funcional (2.16) e nessa formulação as cópias de Gribov corresponderão a diferentes mínimos relativos de $E[U]$. (A escolha do mínimo absoluto de $E[U]$ define a “região modular fundamental” Λ da rede.) Este funcional pode ser visto como a hamiltoniana de um modelo de vidros de spin em que o espaço de estados em cada sítio é o grupo $SU(N_c)$. Como sabido, vidros de spin são sistemas multi-dimensionais caracterizados por forte desordem e frustração, levando a estruturas complexas para a energia com muitos mínimos relativos e várias escalas de barreiras [103]. É natural portanto encontrar um grande número de mínimos [104] para a funcional $E[U]$.

Resultados Numéricos no Gauge de Landau

Simulações numéricas no gauge de Landau foram realizadas desde a metade dos anos 1980 para o propagador de glúons [105] e desde a metade do anos 1990 para o propagador de fantasmas [106, 107]. Os primeiros estudos [108, 109, 110] estabeleceram que o propagador de glúons não é fortalecido para pequenos momentos.¹ Além disso, somente usando simulações no regime de acoplamento forte (i.e. com β pequeno) [112] ou em três dimensões [113] foi possível obter um propagador $D(p^2)$ suprimido no limite de momentos p pequenos, i.e. um propagador decrescente para momentos p menores do que um valor (de *turnover*) p_{to} . Ao mesmo tempo, observava-se que o propagador de fantasmas $G(p^2)$ era fortalecido no limite infravermelho [106, 107, 110, 114, 115, 116, 117], comparado com o comportamento $1/p^2$. Porém, o expoente infravermelho calculado κ parecia ficar menor quando as simulações permitiam considerar valores menores para o momento p , através do uso de redes de tamanho maior. As simulações para os propagadores de glúons e de fantasmas eram complementadas pelos estudos do vértice de três glúons [118, 119, 120], de quark-glúon [121, 122] e de fantasma-glúon [32]. Como ficou evidente mais tarde, os efeitos sistemáticos não foram devidamente considerados nas primeiras simulações numéricas para as funções de Green no gauge de Landau, limitando fortemente a eficácia das conclusões destes trabalhos. Mais especificamente, a consideração, em geral, de redes

¹Vale ressaltar que, na metade do anos 1990, um propagador de glúons no gauge de Landau menos singular do que $1/p^2$ no limite infravermelho era considerado inconsistente com a estrutura das EDS na maioria dos trabalhos analíticos (ver, por exemplo, [111]).

relativamente pequenas não permitiu controlar de forma clara a extrapolação dos dados para volume infinito.

O estudo de efeitos de volume finito para propagadores e vértices no gauge de Landau foi um dos objetivos principais de nosso grupo de pesquisa desde a sua instalação no IFSC-USP em 2001, através de um projeto Jovens Pesquisadores em Centros Emergentes da FAPESP. Neste capítulo vamos apresentar uma descrição dos resultados mais importantes obtidos por nós através de simulações numéricas com tamanhos de rede progressivamente maiores, baseadas em processamento paralelo em máquinas dedicadas [102].

3.1 Grandes Tamanhos de Rede

Após a paralelização de nossos códigos computacionais no final de 2002, usando a biblioteca MPI, foi realizado² um estudo [123] (ver artigo anexado no final desta seção) do propagador de glúons $D(p^2)$ para o grupo de gauge SU(2) em três dimensões usando o maior lado de rede já considerado (até então) em simulações numéricas de teorias de gauge na rede, i.e. $N = 140$. Este foi o principal resultado do trabalho de doutorado de André Taurines, estudante da UFRGS, do qual fui co-orientador. Para comparação, a maior rede usada na Ref. [113] era de $N = 64$, também para o caso SU(2) em três dimensões, correspondente a um volume de rede mais do que 10 vezes menor do que o volume 140^3 . Este trabalho permitiu observar de forma clara — na chamada região de escala da teoria de rede — a supressão do propagador $D(p^2)$ no limite de momentos p pequenos (ver Fig. 3.1) e determinar, pela primeira vez, o valor do momento p_{to} , que foi estimado em $p_{to} = 350^{+100}_{-50}$ MeV (ver Fig. 1 da Ref. [123]). Porém, o estudo verificou também que, mesmo usando redes de lado maior do que $24 fm$, não era ainda possível controlar completamente a extrapolação dos dados a volume infinito. De fato, como mostrado na Fig. 3.2, os dados para o propagador de glúons a momento nulo $D(0)$ podem

²Este trabalho foi possível graças ao *cluster* de PCs (com 16 nós de processamento) adquirido através do projeto FAPESP citado acima.

ser bem ajustados usando a função $d + b/L^c$, seja com $d = 0$ que com $d \neq 0$.

Vale ressaltar que o uso de dados obtidos usando diferentes valores do parâmetro de rede β , correspondendo a diferentes valores do espaçamento de rede a , requer um re-escalamento cuidadoso dos valores numéricos de $D(p^2)$, i.e. a consideração da expressão $aD(p^2)Z(a)$ (ver Figs. 3.1 e 3.2). De fato, além da inclusão do espaçamento de rede a , a fim de obter a dimensão física correta³ para $D(0)$, é necessário renormalizar os dados de rede [108], i.e. determinar numericamente o fator $Z(a)$.

³Com as definições consideradas no trabalho [123], o propagador de rede $aD(p^2)$ corresponde ao propagador do contínuo $g^2 D(p^2)/4$, que também possui, no caso tridimensional, dimensão de massa -1.

Figura 3.1: Dados para o propagador de glúons (re-escalado) como função do momento p . Os símbolos correspondem a $\beta = 4.2$ ($\times$), 5.0 ($\square$), 6.0 ($\diamond$). Erros são estimados usando propagação de erros. Este gráfico é apresentado na Figura 1 da Ref. [123].

Figura 3.2: Dados para o propagador de glúons (re-escalado), calculado a momento nulo, como função do inverso do lado de rede $1/L$. Os símbolos correspondem a $\beta = 4.2 (\times)$, $5.0 (\square)$, $6.0 (\diamond)$. Também é mostrado o ajuste dos dados usando o *Ansatz* $d + b/L^c$ para os casos com $d = 0$ e com $d \neq 0$. Erros são estimados usando propagação de erros. Este gráfico é apresentado na Figura 2 da Ref. [123].

$SU(2)$ Landau gluon propagator on a 140^3 lattice

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We present a numerical study of the gluon propagator in the lattice Landau gauge for three-dimensional pure $SU(2)$ lattice gauge theory at couplings $\beta=4.2, 5.0, 6.0$ and for lattice volumes $V=40^3, 80^3, 140^3$. In the limit of large V we observe a decreasing gluon propagator for momenta smaller than $p_{dec}=350_{-50}^{+100}$ MeV. Data are well fitted by Gribov-like formulas and seem to indicate an infrared critical exponent κ slightly above 0.6, in agreement with recent analytic results.

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I. INTRODUCTION

The study of the infrared (IR) limit of QCD is of central importance for understanding the mechanism of quark confinement and the dynamics of partons at low energy. Despite being non-gauge-invariant, the gluon propagator is a powerful tool in this (nonperturbative) investigation [1]. In particular, it would be interesting to express it in a closed form for recovering the phenomenology of Pomeron exchange from first principles [2].

Studies of the coupled set of Dyson-Schwinger equations for gluon and ghost propagators in the Landau gauge predict for the gluon propagator an IR behavior of the form $D(p) \sim p^{4\kappa-2}$ [implying $D(0)=0$ if $\kappa>0.5$]. The available predictions for the IR exponent are $\kappa \in [0.52, 1.00]$ in the four-dimensional case [3,4] and $\kappa \approx 0.648$ or $\kappa=0.75$ in three dimensions [4].

Furthermore, in the minimal Landau gauge, the gauge-fixed configurations belong to the region Ω of transverse configurations, for which the Faddeev-Popov operator is non-negative. This implies a rigorous inequality [5] for the Fourier components of the gluon field $A_\mu(x)$ and a strong suppression of the gluon propagator in the IR limit. In particular, for dimension d and infinite volume, it is proven that the (unrenormalized) gluon propagator is less singular than p^{2-d} and that, very likely, it vanishes in the IR limit [5]. A vanishing gluon propagator at $p=0$, given by the form $p^2/(p^4+\lambda^4)$, was also obtained by Gribov [6]. Here the mass scale λ arises when the configuration space is restricted to the region Ω . A generalization of this expression has been introduced in Ref. [7] as an ansatz for a nonperturbative solution of the gluon Dyson-Schwinger equation.

Numerical studies [8,9] have now established that the gluon propagator in the lattice Landau gauge shows a turnover in the IR region and attains a finite value for $p=0$.

Evidence of a decreasing gluon propagator for small p has been obtained in the 4D $SU(2)$ and $SU(3)$ cases (but only in the strong-coupling regime) [10,11], in the 3D $SU(2)$ case (also in the scaling region) [8,12,13], in the 3D $SU(2)$ adjoint Higgs model [13], in the 4D $SU(2)$ case at finite temperature [14] and for the equal-time three-dimensional transverse gluon propagator in the 4D $SU(2)$ Coulomb gauge [15]. In this last case, one obtains an excellent fit of the transverse propagator by a Gribov-like formula.

This work aims to verify the possibility of using Gribov-like formulas to fit data of the gluon propagator also in Landau gauge. At the same time, we will try to obtain a value for the IR critical exponent κ to be compared to the analytic determinations mentioned above. In order to probe the infinite-volume limit and the IR region we consider the three-dimensional case and the $SU(2)$ group, using lattice sizes up to 140^3 . Note that the study of the gluon propagator in three dimensions is also of interest in finite-temperature QCD [16].

II. NUMERICAL SIMULATIONS

We consider the standard Wilson action for $SU(2)$ lattice gauge theory in three dimensions with periodic boundary conditions. The numerical code is entirely parallelized using MPI. (Technical details and a study of the code performance are left for a subsequent work [17].) For the construction of staples we follow Ref. [18]. For the random number generator we use a double-precision implementation of RANLUX (version 2.1) with luxury level set to 2. Computations were performed on the PC cluster at the IFSC-USP. The system has 16 nodes and a server with 866 MHz Pentium III CPU and 256/512 MB RAM memory. The machines are connected with a 100 Mbps full-duplex network. The total computer time used for the runs was about 80 days on the full cluster.

In Table I we report, for each coupling β and lattice volume V , the parameters used for the simulations. All our runs start with a random gauge configuration. For thermalization we use a *hybrid overrelaxed* (HOR) algorithm [19]. Each HOR iteration consists of one heat-bath sweep over the lat-

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TABLE I. The pairs (β, V) considered for the simulations, the number of configurations, the numbers of HOR sweeps used for thermalization and between two consecutive configurations (used for evaluating the gluon propagator) and the parameter p_{sor} used by the SOR algorithm.

β	V	Configurations	Thermalization	Sweeps	p_{sor}
4.2	40^3	400	1100	100	0.70
4.2	80^3	200	2200	200	0.80
4.2	140^3	30	2750	250	0.88
5.0	40^3	400	1320	120	0.69
5.0	80^3	200	2420	220	0.80
5.0	140^3	30	3080	280	0.88
6.0	40^3	400	1540	140	0.68
6.0	80^3	200	2680	240	0.80
6.0	140^3	30	3300	300	0.87

tice followed by m micro-canonical sweeps. We did not try to find the best tuning for m ; we use $m=4$ for $V=40^3$, 80^3 and $m=5$ for $V=140^3$. In order to optimize the heat-bath code, we implement two different $SU(2)$ generators, namely methods 1 and 2 described in Ref. [20], Appendix A, with $h_{\text{cutoff}}=2$.

For the numerical gauge fixing we use the *stochastic over-relaxation* (SOR) algorithm [21,22] with even/odd update. In Table I we report the value of the tuning parameter p_{sor} used for each pair (β, V) . We stop the gauge fixing when the average value of $[(\nabla \cdot A)^b(x)]^2$ is smaller than 10^{-12} . (For a definition of the lattice gauge field $A_\mu^b(x)$ and of the lattice divergence ∇ we refer to [8].) We note that half of the configurations for $V=80^3$ were done using the so-called *Cornell method* [21,22], with tuning parameters $\alpha_{corn}=0.325, 0.32, 0.316$ respectively at $\beta=4.2, 5.0, 6.0$. In fact, the Cornell method is somewhat faster than the SOR algorithm and leads to a gauge fixing of comparable quality if one uses an even/odd update [22]. A good estimator of the quality of the gauge fixing is the quantity Σ_Q [see Eq. (6.8) in Ref. [22]], which should be zero when the configuration is gauge-fixed. By averaging over the gauge-fixed configurations, we find (at $\beta=4.2$) that the ratio between the final and the initial values of Σ_Q is (in 95% of the cases) about 5.3×10^{-10} with the Cornell method and 2.4×10^{-11} with the SOR algorithm. At the same time, the average CPU-time needed for updating each site variable is about 11% smaller for the Cornell method. In any case, the CPU-time for gauge fixing was quite significant for the large lattices. In order to go to even larger lattices one should probably implement a global algorithm such as the Fourier acceleration method [21,22] with the multigrid or conjugate gradient implementations introduced in Ref. [23], which are highly parallelizable.

We ran the 40^3 lattices on a single node, the 80^3 lattices on two nodes and the 140^3 lattices on four nodes. The parallelization of the code worked well. In fact, for runs on 2 and 4 nodes, we obtain speed-up factors 1.82 and 3.41 for the heat-bath link update. For the micro-canonical link update the factors are respectively 1.87 and 3.77 and for the SOR site update we get 1.90 and 3.72.

TABLE II. For each coupling β we report the value of the average plaquette $\langle W_{1,1} \rangle$, the string tension $\sqrt{\sigma}$ in lattice units, the lattice spacing in fm and the smallest nonzero momentum (in MeV) for the lattice volume $V=140^3$. Error bars for $\langle W_{1,1} \rangle$ have been obtained taking into account the integrated autocorrelation time of the HOR algorithm. All the other error bars come from propagation of errors.

β	$\langle W_{1,1} \rangle$	$\sqrt{\sigma}$	a (fm)	p_{min} (MeV)
4.2	0.741861(2)	0.387(3)	0.174(1)	59.0(4)
5.0	0.786869(2)	0.314(2)	0.1407(8)	62.9(4)
6.0	0.824780(1)	0.254(1)	0.1138(5)	77.8(4)

For each β we evaluate the average plaquette $\langle W_{1,1} \rangle$ (see Table II). Results are in agreement with the data reported in Ref. [8], but we now have smaller statistical errors. (The data have been analyzed using various methods, described in footnote 4 of Ref. [8]. Here we always report the largest error found.) We also evaluate the tadpole-improved coupling $\beta_I \equiv \beta \langle W_{1,1} \rangle$. In this way, by using the fit given in Eq. (2) and Table IV of Ref. [24] we calculate the string tension $\sqrt{\sigma}$ in lattice units (see Table II) and the inverse lattice spacing a^{-1} using the input value $\sqrt{\sigma}=0.44$ GeV. The fit is valid for $\beta \gtrsim 3.0$, i.e. the couplings β considered here are well above the strong-coupling region. Let us notice that, if we compare the data for the string tension (in lattice units) with data obtained for the $SU(2)$ group in four dimensions [[25], Table 3], then our values of β correspond to $\beta \approx 2.28, 2.345, 2.41$ in the four dimensional case. Finally, in the same table we report the lattice spacing a in fm and the smallest nonzero momentum (in MeV) that can be considered for each β . Thus, with the lattice volumes and the β values used here we are able to consider momenta as small as 59 MeV (in the deep IR region) and physical lattice sides almost as large as 25 fm.

In this work we did not do a systematic study of Gribov-copy effects for the gluon propagator. However, we compared data obtained using the SOR and the Cornell gauge-fixing methods. In principle, different gauge-fixing algorithms—or even the same algorithm with different values of the tuning parameter [26]—can generate different Gribov copies starting from the same thermalized configuration. Thus, this comparison provides an estimate of the bias (Gribov noise) introduced by the gauge-fixing procedure. We found that, in most cases, the difference between the two sets of gluon-propagator data is within 1 standard deviation and that, in all cases, it is smaller than 2 standard deviations. Moreover, this difference did not show any systematic effect, suggesting that the influence of Gribov copies on the gluon propagator (if present) is of the order of magnitude of the numerical accuracy. This is in agreement with previous studies in Landau gauge for the $SU(2)$ and $SU(3)$ groups in four dimensions [10,26,27]. A similar result has also been obtained for the Coulomb gauge [28]. Note that, in the $U(1)$ case [29], Gribov copies can affect the behavior of the photon propagator, making it difficult to reproduce the known perturbative behavior in the Coulomb phase. The situation is very different for the $SU(2)$ and $SU(3)$ cases, at least when considering the lattice Landau gauge. In fact, as said in the

FIG. 1. Plot of the rescaled gluon propagator as a function of the lattice momentum for $V=80^3$ and $\beta=4.2$ ($\times$), 5.0 ($\square$), 6.0 ($\diamond$). The second plot shows only the IR region. Error bars are obtained from propagation of errors.

Introduction, in the non-Abelian case the minimal-Landau-gauge condition implies [5] the positiveness of the Faddeev-Popov matrix and a strict bound for the Fourier components of the gluon field $A_\mu(k)$. The bound applies to all Gribov copies obtained with the numerical gauge fixing. Thus, if the behavior of the gluon propagator is determined by this bound [5,6], then this behavior should be the same for all lattice Gribov copies. This would explain why we do not see Gribov-copy effects here. Clearly, the same result does not apply to the $U(1)$ theory, since in this case the Faddeev-Popov matrix is independent of the gauge field.

III. RESULTS AND CONCLUSIONS

We evaluate the lattice gluon propagator $D(k)$ and study it as a function of the lattice momentum $p^2(k)$ (see Ref. [8] for definitions). In our simulations we consider, for each gauge-fixed configuration, all vectors $k \equiv (k_x, k_y, k_z)$ with only one component different from zero and average over the three directions. For the gluon propagator we analyze the data by estimating the statistical error with three different methods: standard deviation, jack-knife with single-data elimination and bootstrap (with 10000 samples). We found that the results obtained are in agreement in all cases. Here we always use the standard-deviation error.

Let us recall that in the three-dimensional case the coupling g^2 has dimensions of mass. Thus, in order to obtain a dimensionless lattice coupling we have to set $\beta=4/(ag^2)$. Then, with our notation [8], the quantity $aD(k)$ approaches $g^2 D^{(cont)}(k)/4$ in the continuum limit, where $D^{(cont)}(k)$ is the unrenormalized continuum gluon propagator.

In order to compare lattice data at different β 's, we apply the matching technique described in [30], Sec. V B 2. (Note that we have already determined the lattice spacing a , as described above.) We start by checking for finite-size effects, comparing data at different lattice sizes and same β value. In this way, we find (for each β) a range of ultraviolet (UV) momenta for which the data are free from finite-volume corrections. We then perform the matching using data for these momenta and $V=40^3$, since for this lattice volume the errors are smallest (about 1%). In particular, when matching data obtained at two different values of β , we first interpolate the data for the larger β (the fine lattice) using a spline. Finally, we find the multiplicative factor $R_Z = Z(a_f)/Z(a_c)$ corre-

sponding to the best fit of the (multiplied) coarse-lattice data to the same spline. The error of R_Z is estimated using a procedure similar to the one described in [30]. The method works very well (see Fig. 1). Notice that we did not fix the remaining global factor Z imposing a renormalization condition, as done for example in Ref. [31]. Our case is equivalent to setting $Z(a)=1$ at $\beta=6.0$.

The data obtained after the matching are shown for $V=80^3$ in Fig. 1. Clearly, we find that the gluon propagator decreases in the IR limit for momenta smaller than p_{dec} , which corresponds to the mass scale λ in a Gribov-like propagator. From the plot we can estimate $p_{dec} = 350^{+100}_{-50}$ MeV, in agreement with Ref. [8].

In Fig. 2 we plot the rescaled gluon propagator at zero momentum, namely $aD(0)/Z(a)$, as a function of the inverse lattice side $L^{-1}=1/(aN)$ in physical units (fm^{-1}). We see that $aD(0)/Z(a)$ decreases monotonically as L increases, in agreement with Ref. [9]. It is interesting to notice that these data can be well fitted using the simple ansatz $d + b/L^c$ both with $d=0$ and $d \neq 0$ (see Fig. 2). In order to decide for one or the other result one should go to significantly larger lattice sizes. We plan [32] to extend these simulations to $\beta=3.4$ and lattice sizes up to 260^3 , allowing us to

FIG. 2. Plot of the rescaled gluon propagator at zero momentum as a function of the inverse lattice side for $\beta=4.2$ ($\times$), 5.0 ($\square$), 6.0 ($\diamond$). We also show the fit of the data using the ansatz $d + b/L^c$ both with $d=0$ and $d \neq 0$. Error bars are obtained from propagation of errors.

TABLE III. Fit of the data using Eqs. (1) and (2). In all cases χ^2/DBF was of order 1. Note that for a lattice volume $V=N^3$ we have $1+N/2$ data points and that all points have been used for the fits.

β	V	Fit 1			Fit 2	
		α	x	$ y $	α	y^2
4.2	40 ³	0.69(2)	0.17(2)	0.42(1)	0.48(4)	0.29(1)
4.2	80 ³	0.66(2)	0.17(2)	0.36(1)	0.46(4)	0.24(1)
4.2	140 ³	0.61(4)	0.19(4)	0.35(3)	0.38(6)	0.25(2)
5.0	40 ³	0.77(3)	0.11(2)	0.28(2)	0.53(6)	0.152(8)
5.0	80 ³	0.71(2)	0.11(1)	0.214(9)	0.45(3)	0.118(6)
5.0	140 ³	0.71(3)	0.10(2)	0.22(2)	0.44(6)	0.12(1)
6.0	40 ³	0.86(3)	0.069(8)	0.20(2)	0.64(6)	0.082(6)
6.0	80 ³	0.84(2)	0.032(6)	0.166(8)	0.65(5)	0.051(3)
6.0	140 ³	0.80(2)	0.037(8)	0.123(7)	0.55(6)	0.040(4)

consider a value $L^{-1} \approx 0.017 \text{ fm}^{-1}$. (This requires running in parallel on all nodes of our PC cluster.)

Following Ref. [15] we fit the data using a Gribov-like (or Stingl-like) formula

$$D(p) = \frac{s + zp^{2\alpha}}{y^2 + (p^2 + x)^2}, \quad (1)$$

where z, s, α, x and y are fitting parameters. For non-negative α this implies a finite gluon propagator in the IR limit, with a behavior given by $D(p) \propto (s + zp^{2\alpha})$. If $y^2 > 0$ this form corresponds to a propagator with poles at $m_{\pm}^2 = -x \pm iy$, while if $y^2 \leq 0$ the poles are real. Finally, note that the IR exponent κ considered in studies of Dyson-Schwinger equations is given in terms of α by $\kappa = (1 + \alpha)/2$, assuming $D(0) = 0$. Results of our fits using the un-rescaled lattice data $D(k)$ are reported in Table III. We see that α decreases when the physical lattice volume increases. Also, we always get $y^2 > 0$ and $x < |y|$, which seems to support the scenario of purely imaginary poles found in [15]. Let us recall that the poles of the gluon propagator are gauge-invariant (at all orders in perturbation theory) [33].

Notice that this fitting form leads in general to the wrong UV behavior, namely $D(p) \sim p^{2\alpha-4}$. This is not a serious problem, since the largest momentum we can consider is about 3.5 GeV, i.e. we are not really exploring the UV limit. On the other hand, the exponent α in Eq. (1) plays a role both in the IR and in the UV regimes. Thus, the values obtained for α probably correspond to averaging the behavior of the gluon propagator in these two regions. In particular, since we expect $D(p) \sim p^{-2}$ in the UV limit, it is likely that the IR behavior of the propagator be given by a smaller exponent α than the ones reported in Table III. To check this we set $x=0$ and introduce an anomalous dimension γ :

$$D(p) = \frac{s + zp^{2\alpha}}{y^2 + p^{2(1+\gamma)}}. \quad (2)$$

Results for this fitting form are also reported in Table III. Indeed, we get values of α smaller than in the previous case

(and still decreasing with increasing physical lattice volume). For the anomalous dimension we obtain $\gamma \approx 0.65$, with small volume and β dependence. The problem with Eq. (2) is that the introduction of γ compromises the pole interpretation.

Finally, one can also try the form

$$D(p) = \frac{s + zp^{2\alpha}}{(y^2 + p^4)^{\gamma}}. \quad (3)$$

This corresponds to a propagator with purely imaginary poles $m_{\pm}^2 = \pm iy$ and at the same time allows the data to select the IR and the UV behaviors separately. The problem in this case is that the fit is unstable for small lattice volumes. On the contrary, for $V=140^3$ the fit works well and we obtain $\alpha=0.27(6)$, $0.29(7)$, $0.38(8)$ respectively for $\beta=4.2$, 5.0 , 6.0 and $\gamma \approx 0.72$. Hence, our α values are of the order of 0.3 , corresponding to $\kappa \approx 0.65$. (Again, there is a decrease of α when the physical lattice volume increases.)

In order to check for possible effects from the breaking of rotational invariance [34] we redid our fits substituting $p^2(k)$ by $\tilde{p}^2(k) \equiv p^2(k) + p^{[4]}(k)/12$ (see Ref. [35]). One expects this modification to play an important role in the UV limit and to have a small effect in the IR case. For all β values and lattice volumes we obtain good fits and results similar to those reported above. In particular, using Eq. (1) one still finds $y^2 > 0$ and $x < |y|$, supporting the scenario of purely imaginary poles. At the same time, for the three fitting functions, α decreases when the physical lattice volume increases, but its value is about 20–30% larger than the one reported in Table III and above. In particular, using Eq. (3) and data for $V=140^3$ we obtain $\alpha=0.32(7)$, $0.39(8)$, $0.59(12)$ respectively for $\beta=4.2$, 5.0 , 6.0 and $\gamma \approx 0.7$. This implies slightly larger values for $\kappa = (1 + \alpha)/2$. The analysis of these discretization effects and their influence on the extrapolation of α and κ to the continuum limit will be done elsewhere [32].

We have confirmed, by numerical simulations in the scaling region, that the transverse gluon propagator in 3D $SU(2)$ Landau gauge is a decreasing function for momenta $p \lesssim 350 \text{ MeV}$ (attaining a finite value at $p=0$). Also, the data are well fitted by the Gribov-like formulas (1)–(3) and we obtain an IR critical exponent κ in agreement with recent analytic results. In order to eliminate remaining discretization and finite-volume effects [and in particular to check if $D(0)=0$], we need to simulate at larger lattice volumes and at other values of β .

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3.2 Violação da Positividade

Usando os dados apresentados na Ref. [123], descrita na seção acima, foi também possível fazer um estudo cuidadoso [124] (ver quadro abaixo) da violação da positividade por reflexão para o propagador de glúons $D(x)$, calculado no espaço-tempo usual x .

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Positivity violation for the lattice Landau gluon propagator

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We present explicit numerical evidence of reflection-positivity violation for the lattice Landau gluon propagator in three-dimensional pure $SU(2)$ gauge theory. We use data obtained at very large lattice volumes ($V = 80^3, 140^3$) and for three different lattice couplings in the scaling region ($\beta = 4.2, 5.0, 6.0$). In particular, we observe a clear oscillatory pattern in the real-space propagator $C(t)$. We also verify that the (real-space) data show good scaling in the range $t \in [0, 3]fm$ and can be fitted using a Gribov-like form. The violation of positivity is in contradiction with a stable-particle interpretation of the associated field theory and may be viewed as a manifestation of confinement.

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Relembramos que a exigência de ter uma norma positiva no espaço de Hilbert de teorias quânticas de campos se traduz, na formulação euclidiana, no chamado axioma da positividade por reflexão (ver, por exemplo, [26]). Para uma função de dois pontos $D(x - y)$ este axioma é expresso como

$$\int d^4x d^4y f^*(-x_0, \mathbf{x}) D(x - y) f(y_0, \mathbf{y}) \geq 0, \quad (3.1)$$

onde $f(x_0, \mathbf{x})$ é uma função complexa de teste qualquer. A condição acima garante a existência da chamada representação de Källen-Lehmann para a função $D(x - y)$ que, por sua vez, é necessária para interpretar campos interagentes em termos de partículas assintóticas. Assim, a violação da condição (3.1) implica que a função euclidiana de dois pontos $D(x - y)$ não representa a função de correlação para uma partícula física. Tal violação é hoje considerada uma possível indicação de confinamento [65].

Para as funções de correlação euclidianas no espaço dos momentos, geralmente estudadas nas simulações numéricas, a positividade por reflexão implica que a densidade

espectral $\rho(m^2)$, definida pela relação

$$D(p) = \int_0^\infty dm^2 \frac{\rho(m^2)}{p^2 + m^2}, \quad (3.2)$$

seja positiva. Equivalentemente, a função de correlação temporal $D(t, \vec{p} = 0)$ pode ser escrita como

$$C(t) \equiv D(t, 0) = \int_0^\infty d\omega \rho(\omega^2) e^{-\omega t}. \quad (3.3)$$

Claramente, a condição $\rho(\omega^2) > 0$ implica $C(t) > 0$ mas o contrário não é necessariamente verdadeiro. Por outro lado, se $C(t)$ ficar negativo para alguns valores de t é evidente que $\rho(\omega^2)$ não é uma função positiva.

Vale ressaltar que a verificação numérica da violação da positividade para a função de correlação temporal $C(t)$ pode ser complicada se acontecer que $C(t) < 0$ para valores grandes de t , i.e. quando a razão sinal-ruído é em geral bastante ruim. Por exemplo, este seria o caso para $C(t) = e^{-\lambda t} f(t)$, com $f(t) < 0$ para $t \gg 1/\lambda$. Assim, do ponto de vista numérico, é preferível considerar a grandeza [109]

$$G(t) \equiv \frac{d^2}{dt^2} \ln C(t) = \frac{C(t) C''(t) - [C'(t)]^2}{[C(t)]^2}, \quad (3.4)$$

que pode ser discretizada na rede como

$$G(t, a) = \frac{1}{a^2} [C(t) C(t + 2a) - C(t + a)^2], \quad (3.5)$$

onde a é o espaçamento de rede. De fato, é simples verificar que $G(t)$ é uma grandeza positiva se $\rho(\omega^2) > 0$ [65]. Ao mesmo tempo, para o caso “crítico” $C(t) = e^{-\lambda t} f(t)$ obtemos $G(t) \equiv d^2 \ln f(t)/dt^2$, que não depende mais do fator exponencial $e^{-\lambda t}$.

Para estudar a violação da positividade é possível também considerar a massa efetiva

$$m(t) = -\log [C(t + a) / C(t)], \quad (3.6)$$

e escrever

$$e^{m(t)} - e^{m(t+a)} = G(t, a) a^2 e^{m(t+a)} / C(t + a)^2. \quad (3.7)$$

Assim, para $G(t, a) \geq 0$ obtemos que $m(t) \geq m(t + a)$, i.e. a massa efetiva deveria aumentar⁴ com a separação temporal t . Porém, a Eq. (3.6) não está definida quando a função $C(t)$ muda de sinal.

Como mostrado na Fig. 3.3, a violação da positividade por reflexão foi claramente verificada pelos nossos dados numéricos. É importante ressaltar que, para melhorar a razão sinal/ruído, o cálculo do erro para a grandeza $G(t)$ não pôde ser baseado na técnica usual de propagação de erros, mas requereu o uso do método de reamostragem *bootstrap* (ver, por exemplo, [125]). Além disso, foi possível verificar que o propagador de glúons $D(t, \vec{p} = 0)$ se torna negativo para $t \approx 0.7 fm$ e obtém o seu valor mínimo para $t_{min} \approx 1 fm$. Finalmente, devido à boa qualidade dos dados, foi também possível fazer um ajuste (ver Fig. 3.4) usando a soma de duas funções do tipo

$$C(t) = c e^{-\lambda t/\sqrt{2}} \cos\left(b + \frac{\lambda t}{\sqrt{2}}\right), \quad (3.8)$$

que representa uma generalização do chamado propagador de Gribov [6, 63]

$$C(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \frac{p^2}{p^4 + M^4} e^{-ipt} = \frac{e^{-Mt/\sqrt{2}}}{2M} \cos\left(\frac{\pi}{4} + \frac{Mt}{\sqrt{2}}\right). \quad (3.9)$$

Os valores para os parâmetros do ajuste são dados na Tabela 3.1. Assim, obtemos para as massas efetivas (para o campo de glúons) os valores médios $\lambda_1 = 745(5) MeV$ e $\lambda_2 = 325(6) MeV$.

β	c_1	λ_1	c_2	b_2	λ_2
4.2	0.368(6)	3.83(4)	0.70(3)	0.099(6)	1.54(4)
5.0	0.361(6)	3.72(3)	0.56(3)	0.089(6)	1.75(5)

Tabela 3.1: Ajuste dos dados para o propagador $C(t)$ usando a soma de duas funções do tipo (3.8), fixando $b_1 = 0$. Para o $\chi^2/d.o.f.$ obtemos os valores de 0.24, no caso $\beta = 4.2$ com 13 graus de liberdade, e 0.19, no caso $\beta = 5.0$ com 17 graus de liberdade. Os valores para λ_1 e λ_2 são em fm^{-1} .

⁴Um aumento da massa efetiva $m(t)$ foi observado já no primeiro estudo numérico do propagador de glúons [105], sugerindo a violação da positividade por reflexão.

Figura 3.3: Propagador temporal de glúons $C(t)$ como função da separação temporal t para o parâmetro $\beta = 5.0$ e o volume de rede $V = 80^3$. Erros são estimados usando *bootstrap* com 1000 amostras. Todas as grandezas são em unidades de rede. Este gráfico é apresentado na Figura 1 da Ref. [124].

Figura 3.4: Ajuste para o propagador temporal de glúons $C(t)$, como função da separação temporal t (em fm), usando a soma de duas funções do tipo (3.8) para o parâmetro $\beta = 4.2$ e o volume de rede $V = 140^3$. No gráfico são também mostradas as duas funções $C_1(t)$ and $C_2(t)$ separadamente. Este gráfico é apresentado na Figura 5 da Ref. [124].

3.3 Gigantescos Tamanhos de Rede

Como mostrado na Seção 3.1, a extrapolação a volume infinito dos dados numéricos para o propagador de glúons $D(p^2)$ requer a consideração de redes realmente grandes, pelo menos no caso tridimensional. Vale ressaltar que estudos por métodos funcionais [77] (utilizando uma discretização toroidal) defendiam, em 2005, a impossibilidade de se observar em simulações de QCD na rede um propagador de glúons nulo a momento zero como extrapolação das redes finitas empregadas usualmente nas simulações. Em 2007 um estudo semelhante [126] concluiu que seriam necessárias redes de cerca de 10–15 fm , para poder observar um propagador de glúons suprimido no limite de pequenos momentos, e de cerca de 40 fm , para poder calcular com boa precisão o valor do expoente infravermelho κ . Ao mesmo tempo, na Ref. [127], considerando redes de lado até 42.7 fm para o caso SU(2) em duas dimensões, foi obtido pela primeira vez um propagador de glúons nulo a momento zero (no limite de volume infinito). O mesmo estudo obteve também um propagador de fantasmas fortalecido no infravermelho e caracterizado por um valor do expoente κ da ordem de 0.2, i.e. de acordo com as previsões das soluções conformes obtidas usando EDS [65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82]. Esses resultados desencadearam três estudos, por três grupos diferentes, usando volumes de rede “gigantescos”, inimagináveis até poucos anos antes. Os resultados preliminares desses trabalhos foram apresentados na Confência *Lattice 2007* (Regensburg, Alemanha). Em particular, a colaboração Berlin-Dubna considerou redes de até 80^4 , correspondendo a um tamanho máximo de rede⁵ de 13 fm , para o caso SU(3) [129]. O grupo da Universidade de Adelaide utilizou redes de até 112^4 , correspondendo a um tamanho de 19 fm , para o caso SU(2) [130]. Enfim, nosso grupo considerou,⁶ para o caso SU(2), redes de até 128^4 (correspondendo a um tamanho de 27 fm) e de até 320^3 (correspondendo a um tamanho de 85 fm) no caso tridimensional [131]. Relembremos que as maiores redes consideradas

⁵Em seguida [128] o mesmo grupo chegou a considerar (para o estudo do propagador de glúons) redes de volume 96^4 , correspondendo a um tamanho de 16 fm .

⁶Isto foi possível graças ao super-computador IBM, adquirido com um Programa Equipamentos Multiusuários FAPESP e instalado no início de 2007 no Laboratório de Computação Científica Avançada (LCCA) da USP.

por nós até 2007, para este tipo de estudo, eram de 140^3 para $\beta = 3.0$ (com $L \approx 24fm$) em três dimensões [123] e de 52^4 para $\beta = 2.2$ (com $L \approx 11fm$) em quatro dimensões [132]. Os principais resultados obtidos com essas novas simulações foram apresentados nos trabalhos [131] (ver artigo anexado no final desta seção), [133] (ver quadro abaixo) e [134] (discutido na próxima seção).

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Numerical test of the Gribov-Zwanziger scenario in Landau gauge

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We review the status of lattice simulations of gluon and ghost propagators in Landau gauge, testing predictions of the Gribov-Zwanziger confinement scenario.

Os resultados obtidos mostraram que, em três e em quatro dimensões, o propagador de glúons a momento zero $D(0)$ permanece não-nulo no limite de volume infinito (ver Figs. 1 e 2 da Ref. [131] e também as Figs. 3.5 e 3.6). Esses resultados estão de acordo com os dados produzidos pelo grupo de Adelaide [130] e pela colaboração Berlin-Dubna [128, 129] e também com a Ref. [135], onde foram consideradas cópias de Gribov, e com a Ref. [136], onde foram usadas ações de rede melhoradas e redes anisotrópicas.

Ao mesmo tempo, para o propagador de fantasmas $G(p^2)$, o fortalecimento observado

Figura 3.5: Propagador de glúons $D(p^2)$ como função do momento p (as duas grandezas são em unidades físicas) para o caso SU(2) em três dimensões, considerando volumes de rede até 320^3 , correspondendo a $L \approx 85 fm$. Este gráfico é apresentado na Figura 1 da Ref. [133].

Figura 3.6: Propagador de glúons $D(p^2)$ como função do momento p (as duas grandezas são em unidades físicas) para o caso SU(2) em quatro dimensões, considerando volumes de rede até 128^4 , correspondendo a $L \approx 27fm$. Este gráfico é apresentado na Figura 1 da Ref. [133].

anteriormente para momentos intermediários desaparece a momentos menores, deixando um comportamento infravermelho tipo propagador livre, i.e. tipo $1/p^2$. De fato, como aparece na Tabela 1 da Ref. [131], as estimativas numéricas fornecem para o expoente infravermelho κ (indicado nesse trabalho com α_G) valores muito pequenos e sugerem uma extrapolação nula a volume infinito. Isto fica também evidente quando é considerado o gráfico da função de renormalização $F(p^2) = p^2 G(p^2)$ (ver Figs. 3.7 e 3.8). Os dados para $F(p^2)$ são bem descritos pela função

$$F(p^2) = a - \frac{b [\log(1 + cp^2) + dp^2]}{1 + p^2}, \quad (3.10)$$

que é consistente com $\kappa = 0$. Vale ressaltar que para momentos p pequenos a função acima é equivalente à função $a - b \log(1 + cp^2)$, proposta na Ref. [87] para descrever $F(p^2)$ no limite infravermelho e na qual o parâmetro c está relacionado à massa (gerada dinamicamente) para o campo de glúons.

Figura 3.7: Função de renormalização de fantasmas $F(p^2) = p^2 G(p^2)$ como função do momento quadrado p^2 (as duas grandezas são em unidades físicas) para o caso SU(2) em três dimensões, considerando volumes de rede até 320^4 , correspondendo a $L \approx 85 fm$. Note a escala logarítmica no eixo x . Também é mostrado o ajuste descrito na Eq. (3.10) com $a = 4.75(1)$, $b = 0.491(5) GeV^2$, $c = 450(30) GeV^{-2}$ e $d = 7.1(1) GeV^{-2}$. Este gráfico é apresentado na Figura 2 da Ref. [133].

Figura 3.8: Função de renormalização de fantasmas $F(p^2) = p^2 G(p^2)$ como função do momento quadrado p^2 (as duas grandezas são em unidades físicas) para o caso SU(2) em quatro dimensões, considerando volumes de rede até 128^4 , correspondendo a $L \approx 27 fm$. Note a escala logarítmica no eixo x . Também é mostrado o ajuste descrito na Eq. (3.10) com $a = 4.32(2)$, $b = 0.38(1) GeV^2$, $c = 80(10) GeV^{-2}$ e $d = 8.2(3) GeV^{-2}$. Este gráfico é apresentado na Figura 2 da Ref. [133].

What's up with IR gluon and ghost propagators in Landau gauge? A puzzling answer from huge lattices

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Several analytic approaches predict for $SU(N_c)$ Yang-Mills theories in Landau gauge an enhanced ghost propagator $G(p^2)$ and a suppressed gluon propagator $D(p^2)$ at small momenta. This prediction applies to two, three and four space-time dimensions. Moreover, the gluon propagator is predicted to be null at $p = 0$. Numerical studies by several groups indeed support an enhanced ghost propagator when compared to the tree-level behavior $1/p^2$ and a finite infrared gluon propagator. However, the agreement between analytic and numerical studies is only at the qualitative level in three and in four dimensions. In particular, the infrared exponent of the ghost propagator seems to be smaller than the one predicted analytically and the gluon propagator seems to display a (finite) nonzero value at zero momentum. It has been argued that this discrepancy might go away once simulations are done on much larger lattice sizes than the ones used up to now. Here we present data in three and four space-time dimensions using huge lattices in the scaling region, i.e. up to 320^3 at $\beta = 3.0$ and up to 128^4 at $\beta = 2.2$, corresponding to $V \approx (85 \text{ fm})^3$ and $V \approx (27 \text{ fm})^4$. Simulations have been done on the IBM supercomputer at the University of São Paulo.

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1. Introduction

In the Gribov-Zwanziger confinement scenario for Landau gauge [1, 2] the gluon propagator $D(p^2)$ is predicted to be infrared (IR) suppressed at small momenta. In particular, one should have $D(0) = 0$, implying that reflection positivity is maximally violated. This violation of reflection positivity may be viewed as an indication of gluon confinement [3]. At the same time, the Gribov-Zwanziger [1, 4] and Kugo-Ojima [5] confinement scenarios predict (in Landau gauge) a ghost propagator $G(p^2)$ enhanced in the IR limit. This represents a long-range effect and could be related to quark confinement [1, 3, 4, 6].

Several analytic studies [7, 8, 9, 10] agree with the above scenarios predicting, for small momenta, a gluon propagator $D(p^2) \propto p^{2(a_D-1)}$ and a ghost propagator $G(p^2) \propto 1/p^{2(1+a_G)}$, with the relation $a_D = 2a_G + (4-d)/2$. Here d is the space-time dimension. Clearly, if $a_D > 1$ one has $D(0) = 0$. In the four-dimensional case, one finds [8, 9] $a_G \approx 0.59$ and $a_D = 2a_G$. Similar power behaviors have also been obtained for the various vertex functions of $SU(N_c)$ Yang-Mills theories [9, 11, 12].

These results have been confirmed at the quantitative level in the two-dimensional case, using lattices up to almost $(43\text{fm})^2$ [13]. In the three-dimensional case [14], one clearly sees an IR-suppressed gluon propagator. However, using lattice volumes up to about $(24\text{fm})^3$, it was not possible to control the extrapolation to infinite volume. In particular, the data for the rescaled gluon propagator at zero momentum $D(0)$ could be fitted [as a function of the inverse lattice side $1/L = 1/(Na)$] using the Ansatz $d + b/L^c$ both with $d = 0$ and with $d \neq 0$ [14]. Here a is the lattice spacing and N is the number of lattice points per direction. Finally, in four dimensions, the gluon propagator is clearly less divergent than in the tree-level case [15, 16, 17, 18]. On the other hand, even using lattices with a lattice side of about 10fm , one does not see a gluon propagator decreasing at small momenta [18]. One should stress, however, that the Landau gluon propagator clearly violates reflection positivity, in two, three and four space-time dimensions [13, 19, 20]. For the ghost propagator, the IR exponent a_G obtained using analytic studies has been confirmed in 2d [13], while in the 3d [21] and in the 4d [16, 17, 22, 23] cases the exponent obtained using lattice numerical simulations is always smaller than the one predicted analytically. Let us also recall that, in Ref. [24], it was shown that gluon and ghost propagator for $SU(2)$ and $SU(3)$ Yang-Mills theories are in very good agreement from momentum $p \approx 1\text{ GeV}$ to about $p \approx 10\text{ GeV}$. Similar results have been presented in [25]. These findings suggest that the IR behavior of these propagators is independent of the gauge group $SU(N_c)$, as predicted analytically [7, 8, 9].

Finally, by solving Dyson-Schwinger equations on a finite four-dimensional torus [12, 26, 27] one can show that the gluon propagator (at small momenta) seems to diverge for volumes up to about $(8\text{fm})^4$, develops a plateau for $V \approx (9\text{fm})^4$ and is IR suppressed for a lattice side larger than 10fm . Also, the extrapolation of these results to infinite volume gives a null gluon propagator at zero momentum. At the same time, one obtains that, after eliminating (for each volume) the data corresponding to the first two non-zero momenta, the ghost propagator shows a power-law behavior and, in the infinite volume-limit, one obtains the IR exponent a_G predicted by analytic studies [7, 8, 9].

From the above results, it seems necessary to extend present numerical simulations to very large lattice volumes in order to verify if the agreement obtained between lattice data and analytic

results for the two-dimensional case [13] applies to the 3d and 4d cases as well. Here we present extensive simulations in three and in four dimensions, for the $SU(2)$ case, using huge lattices. In particular, we considered lattice sides $N = 140, 200, 240$ and 320 in 3d at $\beta = 3.0$ and $N = 48, 56, 64, 80, 96$ and 128 in 4d at $\beta = 2.2$. In 3d, the number of configurations was about 630, 525, 350 and 45, respectively for the four lattice sizes, both for the gluon and for the ghost propagators. In the 4d case we have considered for the gluon propagator 168 configurations for $V = 128^4$ and about 250 configurations for the other lattice sizes. For the ghost propagator we have 21 configurations for the largest volume and about 100 in the other cases. For the inversion of the Faddeev-Popov matrix we used the so-called *point-source* method [21, 23]. Note that the lattice volumes 320^3 at $\beta = 3.0$ and 128^4 at $\beta = 2.2$ correspond, respectively, to $V \approx (85 \text{ fm})^3$ and $V \approx (27 \text{ fm})^4$. (See Refs. [14, 28] for details about how the physical lattice spacing a has been set in the two cases.) Also note that all our runs are in the scaling region.

These simulations have been done in the IBM supercomputer at USP. This machine has 112 blades with 2 dual-core PowerPC 970 2.5GHz CPU's, a Myrinet network and about 4.5 Tflops peak-performance (occupying position number 363 in the TOP500 list of November 2006).

2. Results

Considering the new data produced in the 3d case together with old data from Refs. [14, 19], we tried an extrapolation to infinite volume for the gluon propagator $D(0)$ as a function of the inverse lattice side $1/L$. The data for the propagator have been renormalized following Ref. [13]. Results are shown in Fig. 1. It is clear from the plot that there is no sign of a propagator going to zero as L goes to infinity. Moreover, the data show a behavior of the type $D(0) \sim 1/L$ and an infinite-volume extrapolation given by $D(0) \approx 2 \text{ GeV}^{-2}$. This implies $a_D = 1$.

In the 4d case (see Fig. 2), even considering very large lattice volumes and relatively large statistics, one cannot see a clear sign of a gluon propagator $D(p^2)$ decreasing at small momenta. Similar results have been presented at this conference by other groups [25, 29]. Clearly, also in this case the data suggest $a_D = 1$. On the other hand, violation of reflection positivity is confirmed and the gluon propagator, considered as a function of the spatial separation s , becomes negative at $s \approx 1 \text{ fm}$, in agreement with Ref. [30].

We have also tried to estimate the IR exponent a_G for the ghost propagator (in the 3d and 4d cases) using the Ansatz $G(p) = c/p^{2(1+a_G)}$ and considering, for each lattice volume, either the two smallest nonzero momenta or the third and fourth smallest nonzero momenta. Results are reported in Tables 1. As one can see, this IR exponent seems to go to zero as the infinite-volume limit is approached, in agreement with [16]. One should however notice that, for $p \approx 500 \text{ MeV}$, the exponent a_G is about 0.3, also in agreement with Ref. [16].

3. Conclusions

The above results leave us with several open questions. From the lattice point of view, one should of course investigate if Gribov-copy effects and/or finite-volume effects could explain our results. Let us recall that an improved gauge-fixing method [31] seems capable of reducing finite-volume effects for the gluon propagator by enlarging the set of allowed gauge transformations. At

Figure 1: Renormalized gluon propagator at zero momentum $\beta a^2 D(0)$ (in GeV^{-2}) as a function of the inverse lattice side $1/L$ (in GeV) and extrapolation to infinite volume. The fit is given by $b + c/L^e$ with $e = 1.04(5)$ and $b = 2.05(5) \text{ GeV}^{-2}$.

Figure 2: Unrenormalized gluon propagator $a^2 D(p^2)$ (in GeV^{-2}) as a function of the momentum p/a (in GeV) for lattice volumes $V = 80^4$ (left) and $V = 128^4$ (right) at $\beta = 2.2$.

N^3	a_G	N^4	a_G	N^3	a_G	N^4	a_G
140 ³	0.073(4)	48 ⁴	0.093(7)	140 ³	0.13(2)	48 ⁴	0.19(4)
200 ³	0.051(3)	56 ⁴	0.063(6)	200 ³	0.06(2)	56 ⁴	0.18(4)
240 ³	0.003(3)	64 ⁴	0.049(9)	240 ³	0.10(2)	64 ⁴	0.17(4)
320 ³	-0.021(9)	80 ⁴	0.052(5)	320 ³	0.01(5)	80 ⁴	0.15(2)
		112 ⁴	0.038(6)			112 ⁴	0.10(7)
		128 ⁴	0.016(5)			128 ⁴	0.06(3)

Table 1: Table for the ghost propagator IR exponent a_G , in the 3d and 4d cases, obtained using either the two smallest nonzero momenta (left) or the third and fourth smallest nonzero momenta (right).

the same time, one needs to reconcile the above results with the non-renormalizability of the ghost-gluon vertex [32] and with the suppression of $D(p^2)$ when considering simulations in the strong-coupling regime [33], in the interpolating gauge (or λ -gauge) [34] and in Coulomb gauge [35]. From the analytic point of view, it may seem necessary to reconsider partially the conventional confinement scenarios [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] discussed in the Introduction. One should of course recall that there are different solutions of Dyson-Schwinger equations for gluons and ghosts in Landau gauge [36, 37]. In particular, the results obtained in Ref. [36] support a finite non-zero gluon propagator and an essentially tree-level ghost propagator at small momenta. Similar results are obtained in Ref. [38]. Phenomenological tests [39] also seem to favor $D(0) > 0$.

We believe that a clarification of the present status of the Kugo-Ojima/Gribov-Zwanziger scenario will probably require new ideas and new methods, both for analytic and numerical studies, and that a key point will be a better understanding of the gauge interpolating between the Landau and the Coulomb gauge [40].

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3.4 Propagador de Glúons a Volume Infinito

Os dados produzidos por nós em 2007 para o propagador de glúons,⁷ em três e quatro dimensões, foram usados na Ref. [134] (ver artigo anexado no final desta seção) para verificar as previsões da solução de Gribov-Zwanziger Refinada (ver Seção 2.3). Nesse trabalho foram também apresentados e analisados novos dados para o propagador de glúons em duas dimensões, produzidos usando vários *clusters* de PCs instalados no IFSC-USP e adquiridos por nós através de vários auxílios de pesquisa FAPESP.

Para o caso $d = 4$, os tamanhos de rede incluídos na análise são $N = 48, 56, 64, 80, 96$ and 128 . Para as simulações foi usado o parâmetro $\beta = 2.2$, que corresponde a um espaçamento de rede de cerca de $0.210 fm$. Assim, para a rede com $N = 128$, o volume físico é de cerca de $(27 fm)^4$ e o menor momento não-nulo é de cerca de $46 MeV$. Relembramos aqui que o momento de rede $p(k)$ possui componentes $p_\mu(k) = 2 \sin(\pi k_\mu/N)$, com $k_\mu = 0, 1, \dots, N-1$, sendo N o tamanho do lado da rede. Logo, o menor momento não-nulo na rede é dado por $p_{min} = 2 \sin(\pi/N)$. Este momento é obtido para $k = k_{min}$, i.e. quando uma única componente de k_μ é não-nula e igual a um. O valor de p_{min} em unidades físicas é dado por p_{min}/a . No caso tridimensional foram considerados os tamanhos de rede $N = 140, 200, 240$ e 320 com $\beta = 3.0$, correspondendo a um espaçamento de rede de cerca de $0.268 fm$. Assim, para o volume 320^3 , o menor momento não nulo p_{min} é cerca de $14 MeV$ e o volume físico é cerca de $(85 fm)^3$. Finalmente, para $d = 2$, tivemos redes de tamanho $N = 80, 120, 160, 200, 240, 280$ e 320 com $\beta = 10.0$. Neste caso o espaçamento de rede é cerca de $0.219 fm$, o volume físico para a rede 320^2 é cerca de $(70 fm)^2$ e o valor correspondente para p_{min} é cerca de $18 MeV$. Nos três casos o espaçamento de rede a foi calculado usando como *input* o valor $\sigma^{1/2} = 0.44 GeV$ para a tensão de corda. O cálculo da tensão de corda na rede é descrito nas Refs. [137], [123] e [110], respectivamente para $d = 2, 3$ e 4 . Vale ressaltar que todas as simulações consideradas pertencem à chamada região de escala da teoria de rede.

⁷Um trabalho semelhante usando o propagador de fantasmas está em elaboração.

Com o intuito de reduzir os efeitos de discretização e, mais especificamente, os efeitos devidos à quebra da simetria rotacional [108, 138, 139], foi considerada a definição melhorada do momento [138]

$$p^2 = \sum_{\mu} p_{\mu}^2 + \frac{1}{12} \sum_{\mu} p_{\mu}^4, \quad (3.11)$$

além da definição usual

$$p^2 = \sum_{\mu} p_{\mu}^2. \quad (3.12)$$

A nova definição não afeta o valor de p^2 no limite infravermelho, mas modifica o seu valor na região ultravioleta. Isso permite reduzir fortemente as distorções introduzidas nos dados como consequência da quebra da simetria rotacional, como fica evidente comparando os dados apresentados nas Figs. 1 e 2 da Ref. [134].

Para o caso quadridimensional, foi possível um bom ajuste dos dados (ver Tabela II e Fig. 2 da Ref. [134]) usando a função

$$f_1(p^2) = C \frac{p^2 + s}{p^4 + u^2 p^2 + t^2}, \quad (3.13)$$

que pode também ser escrita usando um par de polos complexos conjugados, i.e.

$$f_2(p^2) = \frac{\alpha_+}{p^2 + \omega_+^2} + \frac{\alpha_-}{p^2 + \omega_-^2} = \frac{2a p^2 + 2(av + bw)}{p^4 + 2v p^2 + v^2 + w^2}, \quad (3.14)$$

com $\alpha_{\pm} = a \pm ib$ e $\omega_{\pm}^2 = v \pm iw$. Os resultados para os parâmetros a, b, v e w são mostrados na Tabela IV da mesma referência. Vale aqui ressaltar que para o propagador de Gribov descrito na Seção 3.2 [ver Eq. (3.9)] os polos seriam puramente imaginários.

No caso tridimensional não foi possível obter um bom ajuste (ver Fig. 3) dos dados numéricos usando a função (3.13). Porém, o resultado do ajuste melhorou bastante (ver Tabela VIII e Fig. 4) considerando a expressão mais geral

$$f_4(p^2) = C \frac{(p^2 + s)(p^2 + 1)}{(p^4 + u^2 p^2 + t^2)(p^2 + k)} \quad (3.15)$$

$$= C \frac{p^4 + (s+1)p^2 + s}{p^6 + (k+u^2)p^4 + (ku^2 + t^2)p^2 + kt^2}. \quad (3.16)$$

É importante destacar aqui que as duas expressões (3.14) e (3.16) são previsões possíveis para o propagador de glúons da solução de Gribov-Zwanziger Refinada, como descrito na Seção III da Ref. [134] [ver as Eqs. (17) e (18)]. Em analogia com o caso $d = 4$, a função (3.16) acima pode ser decomposta como soma de três polos, sendo neste caso um deles real e os outros dois complexos conjugados [ver Eq. (49) e Tabela XI].

Finalmente, para o caso $d = 2$, as previsões da solução de Gribov-Zwanziger Refinada não permitiram obter um bom ajuste dos dados numéricos. Isto era claramente esperado, sendo que no caso bidimensional os resultados estão de acordo [127] com a chamada solução conforme [65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82] enquanto a solução de Gribov-Zwanziger Refinada é de tipo massivo. Neste caso o melhor ajuste dos dados foi obtido (ver Tabela XIII e Fig. 7) usando a função

$$f_7(p^2) = C \frac{p^2 + l p^\eta + s}{p^4 + u^2 p^2 + t^2}, \quad (3.17)$$

que é uma generalização simples da Eq. (3.13). É importante notar que a função acima pode ser decomposta como

$$f_8(p^2) = \frac{\alpha_+(p^2)}{p^2 + \omega_+^2} + \frac{\alpha_-(p^2)}{p^2 + \omega_-^2} = \frac{2a p^2 + 2cw p^\eta + 2(av + bw)}{p^4 + 2v p^2 + v^2 + w^2}, \quad (3.18)$$

onde

$$\alpha_\pm(p^2) = a \pm i(b + c p^\eta), \quad \omega_\pm^2 = v \pm iw. \quad (3.19)$$

Os valores obtidos para os cinco parâmetros a, b, c, v e w são apresentados na Tabela XIV. Para o caso bidimensional foi também verificado que a extrapolação de $D(0)$ a volume infinito é nula, em acordo com a Ref. [127], e o expoente crítico κ foi estimado em 0.225, em bom acordo com a previsão analítica 0.2. Ao mesmo tempo, os dados numéricos indicam que a condição $D(0) = 0$ não é obtida com valores nulos para os parâmetros b e v na função $f_8(p^2)$ acima, i.e. com polos puramente imaginários como no propagador de Gribov, mas através da relação $a = -b$ e $v = w$.

Os ajustes descritos acima para $d = 3$ e 4 estão agora sendo usados para descrever o comportamento do propagador de glúons a temperatura finita [43, 44, 45, 46, 47]. De fato, já que a hipótese de redução dimensional associa uma rede tridimensional ao caso de uma

teoria quadridimensional em equilíbrio térmico a temperatura infinita, podemos esperar que o caso quadridimensional a temperatura finita T seja dado por uma interpolação do comportamento quadridimensional e tridimensional a temperatura zero.

Modeling the gluon propagator in Landau gauge: Lattice estimates of pole masses and dimension-two condensates

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We present an analytic description of numerical results for the Landau-gauge SU(2) gluon propagator $D(p^2)$, obtained from lattice simulations (in the scaling region) for the largest lattice sizes to date, in $d = 2, 3$ and 4 space-time dimensions. Fits to the gluon data in 3d and in 4d show very good agreement with the tree-level prediction of the refined Gribov-Zwanziger (RGZ) framework, supporting a massive behavior for $D(p^2)$ in the infrared limit. In particular, we investigate the propagator's pole structure and provide estimates of the dynamical mass scales that can be associated with dimension-two condensates in the theory. In the 2d case, fitting the data requires a noninteger power of the momentum p in the numerator of the expression for $D(p^2)$. In this case, an infinite-volume-limit extrapolation gives $D(0) = 0$. Our analysis suggests that this result is related to a particular symmetry in the complex-pole structure of the propagator and *not* to purely imaginary poles, as would be expected in the original Gribov-Zwanziger scenario.

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I. INTRODUCTION

High-precision data from lattice simulations are a key ingredient in our understanding of the low-energy aspects of Yang-Mills theories associated with color confinement. In fact, whereas new insight into the confinement mechanism may be gained by investigating the properties of gauge-field configurations produced in the simulations (see e.g. [1]), specific features of proposed confinement scenarios may be tested by comparison with lattice data. In this case, one may obtain physical values for a model's parameters by fitting the predicted expression of a given observable to its numerical realization. One may also hope to over-constrain the proposed analytic forms, if the fits can be done with a sufficiently high number and wide range of data points, from which systematic errors have been consistently eliminated. In particular, this applies to predictions for the infrared behavior of gluon and ghost propagators, formulated in Landau gauge for SU(N_c) gauge theory. Here we perform a series of fits to the gluon propagator $D(p^2)$ and test the predictions of the so-called refined Gribov-Zwanziger (RGZ) framework, which differs from the scenario originally proposed by Gribov [2] and Zwanziger [3] through the introduction of dimension-two condensates, associated with dynamical mass generation [4]. Our analysis is done for pure SU(2) gauge theory. The data have been produced previously and discussed in [5–7] (see also [8]), but they have not been systematically fitted until now. A companion paper with similar fits for the ghost propagator is under way [9]. We note that an alter-

native comparison of these data to analytic predictions was recently presented in [10].

The Gribov-Zwanziger confinement scenario is based on restricting the functional integration to the first Gribov region Ω delimited by the first Gribov horizon $\partial\Omega$, where the smallest nonzero (and positive) eigenvalue of the Faddeev-Popov matrix $\mathcal{M}$ goes to zero [2,3]. Let us recall that limiting the gauge configurations to this region was an attempt—made by Gribov in Ref. [2]—to fix the gauge completely, getting rid of spurious gauge copies, known thereafter as Gribov copies. Now, Ω is a convex region of very high dimensionality and therefore, as the infinite-volume limit is approached, the increase in entropy should favor [11] gauge configurations on the surface $\partial\Omega$. This in turn can cause the infrared enhancement of the ghost propagator (which is related to the inverse of $\mathcal{M}$), inducing long-range effects in the theory. Indeed, in Coulomb gauge, the restriction to the first Gribov region causes the appearance of a confining color-Coulomb potential [2,12]. Thus, in this scenario, formulated for momentum-space propagators, the long-range features needed to explain the color-confinement mechanism are manifest in the ghost propagator, whereas the momentum-space gluon propagator $D(p^2)$ should be *suppressed* in the infrared limit. Such a suppression is associated with violation of spectral positivity, which is commonly interpreted as gluon confinement [3,13,14]. In particular, $D(0)$ is originally expected to be zero [2,3], corresponding to maximal violation of spectral positivity. The parametrization of this behavior as a propagator having a pair of poles with purely imaginary masses has arisen in the Gribov-Zwanziger approach [2,3], in connection with the study of gauge copies.

Lattice studies (see [8] for a recent review) have confirmed the suppression of the gluon propagator in the infrared limit and the enhancement of the ghost propagator

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at intermediate momenta. However, considering lattice sizes large enough to allow the investigation of the deep infrared regime, it is clear that the results of the simulations are *not* compatible with the scenario described above. Indeed—in space-time dimension $d = 3, 4$ —the gluon propagator shows a finite value as the momentum is taken to zero and the enhancement of the ghost propagator is lost in this limit. We note the very large lattice sizes employed in order to observe such a behavior, $L \approx 20$ fm and larger [5–7,15–18]. In any case, violation of reflection positivity for the real-space gluon propagator (see e.g. [19]) is clearly observed in the data.

Recently, the quantitative description of the massive behavior for the gluon propagator has been studied by several groups [10,18,20–31], based on different proposed analytic forms. Earlier attempts of fitting gluon-propagator data can be found, for example, in Refs. [32–35]. We note that some of these studies (see e.g. [32,35]) have considered the so-called Gribov-Stingl form [36,37] for modeling the massive behavior of the gluon propagator. This form is a generalization of the Gribov propagator described above, including pairs of complex-conjugate poles with a nonzero real part. As illustrated below, the behavior predicted for $D(p^2)$ in the RGZ framework is also based on general complex-conjugate poles for the (massive) propagator. This proposed form is given for $SU(N_c)$ gauge theory and for four or three space-time dimensions [4,38–41]. On the contrary, in the $2d$ case, the RGZ approach cannot be implemented, since the dimension-two condensates would induce severe infrared singularities, precluding the restriction of the functional integration to the first Gribov region [42]. By fitting rational functions of p^2 to the *whole* range of data for the $SU(2)$ gluon propagator, we are able to obtain estimates for the values in physical units for the masses in the RGZ framework, as well as to gain a better understanding of the pole structure in the proposed expressions. In each case, we look for the best fit to the data, with the smallest number of independent parameters, and relate them to the condensates in the proposed analytic forms only at the end. Put differently, the predicted dependence of the fit parameters on the condensates is *not* imposed in the fitting form, but is obtained as a result of the fit. This allows us to use a wide fitting range, considering all data points. We note that predictions from the RGZ framework were already tested in [24], showing good fits (using a somewhat different analytic form and a smaller fitting range) to $4d$ lattice data for the $SU(3)$ case.

The paper is organized as follows. The Gribov-Zwanziger scenario is briefly reviewed in Sec. II. The introduction of condensates as part of the RGZ scenario is summarized in Sec. III, where we present the expressions to be fitted to the lattice data. The numerical results are discussed in general in Sec. IV and, in particular, for the $4d$, $3d$ and $2d$ cases, respectively, in Secs. V, VI, and VII. We present our conclusions in Sec. VIII.

II. THE GRIBOV-ZWANZIGER ACTION

The Gribov-Zwanziger (GZ) action, introduced in 1989 [13], implements an all-order restriction of the path integral to the first Gribov region

$$\Omega \equiv \{A_\mu^a(x): \partial_\mu A_\mu^a(x) = 0, \mathcal{M}^{ab}(x, y) > 0\}, \quad (1)$$

where $A_\mu^a(x)$ is the gauge field and $\mathcal{M}^{ab}(x, y)$ is the Landau-gauge Faddeev-Popov operator

$$\begin{aligned} \mathcal{M}^{ab}(x, y) &= -\delta(x - y) \partial_\mu D_\mu^{ab} \\ &= \delta(x - y) (-\delta^{ab} \partial_\mu^2 + f^{abc} \partial_\mu A_\mu^c). \end{aligned} \quad (2)$$

By introducing auxiliary fields—a pair of complex-conjugate bosonic fields ($\bar{\varphi}_\mu^{ac}, \varphi_\mu^{ac}$) and a pair of anticommuting complex-conjugate fields ($\bar{\omega}_\mu^{ac}, \omega_\mu^{ac}$)—one is able to obtain a local renormalizable action [11,43,44]. More precisely, the generating functional for the GZ action can be written in d space-time dimensions as [13,44,45]

$$Z(J) = \int [d\Phi] e^{S_{\text{GZ}} + \int d^d x J_\mu^a(x) A_\mu^a(x)}, \quad (3)$$

where S_{GZ} is the local GZ action given by

$$S_{\text{GZ}} = S_0 + S_\gamma, \quad (4)$$

with

$$\begin{aligned} S_0 &= S_{\text{YM}} + S_{\text{gf}} + \int d^d x [\bar{\varphi}_\mu^{ac} \partial_\nu D_\nu^{ab} \varphi_\mu^{bc} - \bar{\omega}_\mu^{ac} \partial_\nu (D_\nu^{ab} \omega_\mu^{bc}) \\ &\quad - g(\partial_\nu \bar{\omega}_\mu^{an}) f^{abc} D_\nu^{bm} c^m \varphi_\mu^{cn}] \end{aligned} \quad (5)$$

and

$$\begin{aligned} S_\gamma &= -\gamma^2 g \int d^d x \left[f^{abc} A_\mu^a \varphi_\mu^{bc} + f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc} \right. \\ &\quad \left. + \frac{d}{g} (N_c^2 - 1) \gamma^2 \right]. \end{aligned} \quad (6)$$

Here, a, b, c, m and n are color indices in the adjoint representation, N_c is the number of colors, γ is the so-called Gribov parameter, S_{YM} is the classical Yang-Mills action

$$S_{\text{YM}} = \frac{1}{4} \int d^d x F_{\mu\nu}^a F_{\mu\nu}^a \quad (7)$$

and S_{gf} is the Landau-gauge-fixing action

$$S_{\text{gf}} = \int d^d x (b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu D_\mu^{ab} c^b), \quad (8)$$

where the auxiliary field b^a is a Lagrange multiplier enforcing Landau gauge and $(\bar{c}^a, c^a)$ are the Faddeev-Popov ghost fields. Also, we indicate with $[d\Phi]$ the integration over all fields $\Phi \in \{A_\mu^a, c^a, \bar{c}^a, b^a, \bar{\varphi}_\mu^{ac}, \varphi_\mu^{ac}, \omega_\mu^{ac}, \bar{\omega}_\mu^{ac}\}$. Notice that one can simplify the notation of the auxiliary fields $(\bar{\varphi}_\mu^{ac}, \varphi_\mu^{ac}, \bar{\omega}_\mu^{ac}, \omega_\mu^{ac})$ in the action S_0 using the

symmetry of this action with respect to the composite index $i \equiv (\mu, c)$. Thus, we can set

$$(\bar{\varphi}_\mu^{ac}, \varphi_\mu^{ac}, \bar{\omega}_\mu^{ac}, \omega_\mu^{ac}) = (\bar{\varphi}_i^a, \varphi_i^a, \bar{\omega}_i^a, \omega_i^a) \quad (9)$$

and write

$$S_0 = S_{\text{YM}} + S_{\text{gf}} + \int d^4x [\bar{\varphi}_i^a \partial_\nu (D_\nu^{ab} \varphi_i^b) - \bar{\omega}_i^a \partial_\nu (D_\nu^{ab} \omega_i^b) - g(\partial_\nu \bar{\omega}_i^a) f^{abc} D_\nu^{bm} c^m \varphi_i^c]. \quad (10)$$

Finally, the parameter γ in Eq. (6) is fixed by the so-called gap equation (also known as the horizon condition)

$$\langle g f^{abc} A_\mu^a \varphi_\mu^{bc} \rangle + \langle g f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc} \rangle + 2\gamma^2 d(N_c^2 - 1) = 0, \quad (11)$$

where $\langle \rangle$ indicates the expectation value in the measure defined by Eq. (3). This condition is a consequence of the restriction of the path integral to the first Gribov region.

As mentioned in the Introduction, one of the main outcomes of the GZ theory is the modification of the behavior of gluon and ghost propagators in the infrared (IR) limit in comparison with the perturbative behavior $1/p^2$ [2,3,11,13,14,46]. Indeed, the gluon propagator becomes IR-suppressed with a tree-level behavior given by

$$\begin{aligned} \langle A_\mu^a(p) A_\nu^b(-p) \rangle &\equiv \delta^{ab} D(p^2) \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \\ &= \delta^{ab} \frac{p^2}{p^4 + 2g^2 N_c \gamma^4} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right). \end{aligned} \quad (12)$$

This result is confirmed by one-loop calculations [47,48]. The above expression for the gluon propagator implies that $D(p^2)$ is null at zero momentum, which in turn indicates maximal violation of reflection positivity for the real-space gluon propagator $D(x)$ [46]. This violation is usually considered a manifestation of gluon confinement [3,13,14]. At

the same time, one finds that the ghost propagator displays an enhanced IR behavior

$$\langle c^a(p) \bar{c}^b(p) \rangle \equiv \delta^{ab} \mathcal{G}(p^2) \sim \delta^{ab} \frac{1}{p^4}. \quad (13)$$

This behavior is indicative of a long-range interaction in the theory and it should be related to quark confinement [2,14,49].

III. THE REFINED GRIBOV-ZWANZIGER FRAMEWORK

More recently, the GZ action has been “refined” by taking into account the possible existence of dimension-two condensates [4,38–41]. In the most general case [41], four different condensates are considered, i.e.

$$\begin{aligned} \langle A_\mu^a A_\mu^a \rangle &\rightarrow -m^2 & \langle \bar{\varphi}_i^a \varphi_i^a \rangle &\rightarrow M^2 \\ \langle \varphi_i^a \varphi_i^a \rangle &\rightarrow \rho & \langle \bar{\varphi}_i^a \bar{\varphi}_i^a \rangle &\rightarrow \rho^\dagger, \end{aligned} \quad (14)$$

where we have listed the dynamical mass associated to each condensate. (Note that ρ is complex, whereas $-m^2$ and M^2 are real and positive.) The condensate $-m^2$ is directly related to the gluon condensate $\langle g^2 A^2 \rangle$ (see e.g. [24]). One can show [41] that the refined Gribov-Zwanziger (RGZ) action can be renormalized. At the same time, there is clear evidence that the original GZ theory dynamically transforms into the refined theory, since the minimum of the associated effective potential favors nonvanishing condensates [41]. As displayed below, a nonzero value for these condensates has an effect on the IR behavior of gluon propagators. The influence on the ghost propagator will be discussed in a forthcoming work [9], in which one-loop results are taken into account and the role of the ghost-gluon vertex is investigated.

The gluon propagator

In the presence of the four condensates considered in Eq. (14), the GZ gluon propagator (12) is modified [41] as

$$D(p^2) = \frac{p^4 + 2M^2 p^2 + M^4 - \rho \rho^\dagger}{p^6 + p^4(m^2 + 2M^2) + p^2(2m^2 M^2 + M^4 + \lambda^4 - \rho \rho^\dagger) + m^2(M^4 - \rho \rho^\dagger) + M^2 \lambda^4 - \frac{\lambda^4}{2}(\rho + \rho^\dagger)}, \quad (15)$$

where the condensates m^2 , M^2 , ρ are described above and λ^4 is related to the Gribov parameter γ through $\lambda^4 = 2g^2 N_c \gamma^4$. Since ρ and $\rho^\dagger$ are complex-conjugate quantities, we can set

$$\rho = \rho_1 + i\rho_2 \quad \rho^\dagger = \rho_1 - i\rho_2 \quad (16)$$

and rewrite Eq. (15) as

$$D(p^2) = \frac{p^4 + 2M^2 p^2 + M^4 - (\rho_1^2 + \rho_2^2)}{p^6 + p^4(m^2 + 2M^2) + p^2[2m^2 M^2 + M^4 + \lambda^4 - (\rho_1^2 + \rho_2^2)] + m^2[M^4 - (\rho_1^2 + \rho_2^2)] + \lambda^4(M^2 - \rho_1)}. \quad (17)$$

It is interesting to notice that this propagator gets simplified if $\rho = \rho^\dagger = \rho_1$ (i.e. $\rho_2 = 0$), which corresponds to the equality $\langle \bar{\varphi} \varphi \rangle = \langle \varphi \varphi \rangle$ from (14). Indeed, in this case one can factorize the quantity $p^2 + M^2 - \rho_1$ in the numerator and in the denominator of the above formula, obtaining

$$D(p^2) = \frac{p^2 + M^2 + \rho_1}{p^4 + p^2(M^2 + m^2 + \rho_1) + m^2(M^2 + \rho_1) + \lambda^4}. \quad (18)$$

Clearly, both propagators (17) and (18) have, in principle, a finite *nonzero* value at zero momentum. Nevertheless, if the value of $D(0)$ is sufficiently small, one still finds that the real-space propagator $D(x)$ becomes negative for some (large) value of x , i.e. reflection positivity can also be violated for these propagators.

Note that both Eqs. (17) and (18) can be decomposed as sums of propagators of the type $\alpha/(p^2 + \omega^2)$. In particular, we can write Eq. (17) as

$$D(p^2) = \frac{\alpha}{p^2 + \omega_1^2} + \frac{\beta}{p^2 + \omega_2^2} + \frac{\gamma}{p^2 + \omega_3^2}. \quad (19)$$

To this end, we only need to solve the cubic equation

$$x^3 + x^2(m^2 + 2M^2) + x[2m^2M^2 + M^4 + \lambda^4 - (\rho_1^2 + \rho_2^2)] + m^2[M^4 - (\rho_1^2 + \rho_2^2)] + \lambda^4(M^2 - \rho_1) = 0, \quad (20)$$

obtained by setting $p^2 = x$ in the denominator of Eq. (17), and to find its three roots ω_1^2 , ω_2^2 and ω_3^2 . At the same time, the gluon propagator in Eq. (18) can be written as

$$D(p^2) = \frac{\alpha_+}{p^2 + \omega_+^2} + \frac{\alpha_-}{p^2 + \omega_-^2}, \quad (21)$$

where we expect to have $\alpha_- = \alpha_+^*$ if $\omega_-^2 = (\omega_+^2)^*$, i.e. if ω_+^2 and ω_-^2 are complex conjugates. Here, $\omega_\pm^2$ are the roots of the quadratic equation

$$x^2 + x(M^2 + m^2 + \rho_1) + m^2(M^2 + \rho_1) + \lambda^4 = 0, \quad (22)$$

obtained by setting $p^2 = x$ in the denominator of Eq. (18). Clearly, one finds complex-conjugate poles if $|M^2 - m^2 + \rho_1| < 2\lambda^2$.

Let us remark that rational forms such as (17) and (18) for the gluon propagator were considered by Stingl [36,37], as a way of accounting for nonperturbative effects in an extended perturbative approach to Euclidean QCD. More precisely, in his treatment, one expresses the proper vertices of the theory as an iterative sequence of functions yielding a self-consistent solution to the Dyson-Schwinger equations. In particular, for the gluon propagator, this sequence is written [see Eq. (2.10) in Ref. [37]] in terms of ratios of polynomials in the variable p^2 , of degree r in the numerator and $r+1$ in the denominator, with $r = 0, 1, 2, \dots$. This functional form is then related, via operator product expansion, to the possible existence of vacuum condensates of dimension $2n$, with $n \geq 1$. At the same time, the associated complex-conjugate poles¹ are interpreted as short-lived elementary excitations of the gluon field [3,36,37]. By comparison, in the RGZ frame-

work, one proposes specific forms for the dimension-two condensates—related to the auxiliary fields of the GZ action—and then obtains (at tree level) the rational functions in Eqs. (17) and (18), which correspond, respectively, to cases with $r = 3$ and 2 in Stingl's iterative sequence.

In Sec. V below, we show that the simplest rational form [with $r = 2$, corresponding to Eq. (18)] works well in the $4d$ SU(2) case. A similar result was obtained for the SU(3) case in [24].² It may be noted that, in Ref. [32], lattice data for the $4d$ SU(3) Landau-gauge gluon propagator were fitted using the above sequence of functions with $r = 2, 4$ and it was found that a good description of the data can be achieved only for $r = 4$. Note, however, that the fit was performed for the real-space propagator, for which the analysis is known to be complicated by several technical issues (see e.g. [19,52]). Moreover, although the lattice volume considered was rather large, the study employed asymmetric lattices, which may give rise to systematic effects [53].

IV. NUMERICAL SIMULATIONS

The data presented here for the SU(2) Landau-gauge gluon propagator were produced in 2007. The $3d$ and $4d$ cases were run on the 4.5 Tflops IBM supercomputer at LCCA-USP [54], whereas the $2d$ case was run on various PC clusters at the IFSC-USP. Most of these data have already been discussed in Refs. [5–8], but they were not systematically fitted up to now.

In the $4d$ case, we have considered lattice sides $N = 48, 56, 64, 80, 96$ and 128 , with lattice parameter $\beta = 2.2$. The corresponding lattice spacing a is approximately 0.210 fermi, implying that the smallest nonzero momentum $p_{\min} = 2 \sin(\pi/N)$ is about 46 MeV in physical units for the $N = 128$ lattice. In this case, the physical lattice volume $V = N^4$ is about (27 fermi)⁴. The number of gauge-field configurations produced was 168 for $N = 128$ and about 250 for the other lattice sizes.

In $3d$, we have $N = 140, 200, 240$ and 320 at $\beta = 3.0$, with $a \approx 0.268$ fermi. Then, for the lattice volume 320^3 the smallest nonzero momentum $p_{\min}$ is about 14 MeV and the physical volume corresponds to about (85 fermi)³. The number of configurations was 630, 525, 350 and 125, respectively, for the four lattice sizes.

Finally, in the two-dimensional case, we considered $N = 80, 120, 160, 200, 240, 280$ and 320 at $\beta = 10.0$. In this case the lattice spacing is about 0.219 fermi, the lattice volumes 320^2 correspond to $V \approx (70 \text{ fermi})^2$ and in this case $p_{\min} \approx 18$ MeV. We have about 600 configurations for each lattice volume.

In all cases we set the lattice spacing a by considering the input value $\sigma^{1/2} = 0.44$ GeV for the string tension,

¹See [37,50,51] for some considerations concerning the issue of causality for propagators with complex poles.

²Let us mention that the condensate ρ was not considered in Ref. [24]. Therefore, when discussing fit results using Eq. (18), we must compare their values for M^2 to our values for $M^2 + \rho_1$.

which is a typical value for this quantity in the $4d$ SU(3) case. The evaluation of the lattice string tension is described in [35,55,56], respectively, for $d = 2, 3$ and 4 . Note that all our runs are in the scaling region [35,56,57] and all data refer to the SU(2) case. Possible systematic effects due to Gribov copies [58–62] as well as unquenching effects [63–66] were *not* considered here. Finite-volume effects, on the other hand, are well under control. In particular, in $3d$ and in $4d$, our largest lattice volumes can be already considered as infinite. In the $2d$ case, a simple extrapolation to infinite volume needs to be considered, as done in Sec. VII to obtain the limiting value of $D(0)$.

Configurations have been generated by alternating heat-bath updates of the link variables with micro-canonical steps, in order to reduce the problem of critical slowing-down (see for example [67] and references therein). Gauge-fixing to Landau gauge was done using the stochastic overrelaxation algorithm [55,67]. Let us also recall [see Eq. (12)] that the gluon propagator $D(p^2)$ in Landau gauge is evaluated using

$$\begin{aligned} D_{\mu\nu}^{bc}(p) &= \sum_{x,y} \frac{e^{-2\pi i \hat{p} \cdot (x-y)/N}}{V} \langle A_\mu^b(x) A_\nu^c(y) \rangle \\ &= \delta^{bc} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) D(p^2). \end{aligned} \quad (23)$$

Here $A_\mu^b(x)$ is the lattice gluon field defined as³

$$A_\mu(x) = \frac{1}{2i} [U_\mu(x) - U_\mu^\dagger(x)], \quad (24)$$

where $U_\mu(x)$ are the usual link variables of the Wilson action. Also, the momentum components p_μ are given by

$$p_\mu = 2 \sin\left(\frac{\pi \hat{p}_\mu}{N}\right) \quad (25)$$

and $\hat{p}_\mu$ takes values $0, 1, N-1$.

In $2d$ we considered momenta with components $(p, 0)$ and (p, p) , plus all possible permutations of the components. Similarly, in $3d$, we have data for momenta with components $(p, 0, 0)$, $(p, p, 0)$ and (p, p, p) and all possible permutations of components. Finally, in $4d$, we evaluated the propagator for momenta with components $(p, 0, 0, 0)$, $(p, p, 0, 0)$, $(p, p, p, 0)$ and (p, p, p, p) . In this case, we considered all possible permutations of the components for momenta of the type $(p, 0, 0, 0)$. On the contrary, we did not consider permutations for the momenta $(p, p, p, 0)$ and in the case $(p, p, 0, 0)$ we allowed all permutations satisfying the constraint $p_4 = 0$. When permutations of the momentum components were available, an average over the different permutations was taken for each configura-

tion. In order to reduce discretization effects—and, in particular, those related to the breaking of rotational symmetry [33,69,70]—we have considered, in addition to the usual (unimproved) momentum defined by the squared magnitude of the lattice momenta

$$p^2 = \sum_\mu p_\mu^2, \quad (26)$$

the improved definition [69]

$$p^2 = \sum_\mu p_\mu^2 + \frac{1}{12} \sum_\mu p_\mu^4. \quad (27)$$

This definition does not affect the value of p^2 in the IR limit, but modifies its value significantly for large momenta. In particular, the largest value of p^2 —obtained when $\hat{p}_\mu = N/2$ in Eq. (25) for all directions μ —is given (in lattice units, for the d -dimensional case), by $4d$ if the unimproved definition is considered, and by $16d/3$ in the improved case. For the β values considered here, this implies that the largest momentum $p_{\max}$ is about 2.54, 2.55 and 3.75 GeV, respectively, in $2d$, $3d$ and $4d$ in the unimproved case, and about 2.94, 2.94 and 4.33 GeV using improved momenta.

In the next sections we present fits (obtained using `gnuplot`) of the $4d$, $3d$ and $2d$ data for the SU(2) gluon propagator and compare the fit results to the predictions of the RGZ action, discussed above in Sec. . We remark that the shown data for $D(p^2)$ are *not* normalized. Note that a (multiplicative) renormalization condition at a given scale μ^2 would simply correspond to a rescaling of the overall factor C in the fitting forms considered below. The condensates and the poles, on the other hand, are not affected by such a renormalization.⁴ We also note that, whenever possible, we avoid rounding off the values of the fit parameters. On the contrary, values for the associated physical quantities (i.e. condensates and poles) are rounded to show errors with one significant digit only. We refer to the `gnuplot` documentation ([71] “Statistical Overview” section) for information on the significance of the standard errors calculated for the fit parameters.

V. THE 4D CASE

As a first attempt in the $4d$ case, we consider a fitting function of the simplest Gribov-Stingl form⁵

³With this definition of the lattice gluon field, the gluon propagator evaluated on the lattice corresponds to the propagator $g^2 D(p^2)$ in the continuum, which has mass dimension $2-d$ in the d -dimensional case [68].

⁴Of course, condensates are, in general, renormalization-scale- and renormalization-scheme-dependent quantities. This dependence is not taken into account here because the fitting forms we use are inspired by a tree-level propagator. Indeed, in order to consider these effects, analytic results beyond tree-level computations must be employed.

⁵Since our largest momentum is of the order of 4 GeV, ultraviolet logarithmic corrections are not important to describe the lattice data and they are not included in the fitting functions proposed here. This also avoids the problem of having to regularize the corresponding Landau pole by hand.

TABLE I. Fits of the gluon-propagator data in the $4d$ case, for different lattice volumes, using the fitting function $f_1(p^2)$ in Eq. (28) and unimproved momenta [see Eq. (26)]. We report, besides the value of the fit parameters, the $\chi^2/\text{d.o.f.}$ obtained in each case. The whole range of momenta was considered for the fit. Errors shown in parentheses correspond to one standard deviation.

V	C	$u(\text{GeV})$	$t(\text{GeV}^2)$	$s(\text{GeV}^2)$	$\chi^2/\text{d.o.f.}$
48^4	0.567 (0.016)	0.507 (0.065)	0.607 (0.023)	2.417 (0.269)	16.47
56^4	0.572 (0.014)	0.495 (0.058)	0.602 (0.021)	2.344 (0.234)	15.46
64^4	0.566 (0.013)	0.522 (0.052)	0.612 (0.019)	2.452 (0.223)	15.00
80^4	0.562 (0.012)	0.496 (0.048)	0.612 (0.017)	2.443 (0.199)	16.59
96^4	0.567 (0.011)	0.484 (0.044)	0.604 (0.016)	2.367 (0.178)	15.96
128^4	0.560 (0.010)	0.534 (0.037)	0.621 (0.014)	2.553 (0.166)	10.65

TABLE II. Fits of the gluon-propagator data in the $4d$ case, for different lattice volumes, using the fitting function $f_1(p^2)$ in Eq. (28) and improved momenta [see Eq. (27)]. We report, besides the value of the fit parameters, the $\chi^2/\text{d.o.f.}$ obtained in each case. The whole range of momenta was considered for the fit. Errors shown in parentheses correspond to one standard deviation.

V	C	$u(\text{GeV})$	$t(\text{GeV}^2)$	$s(\text{GeV}^2)$	$\chi^2/\text{d.o.f.}$
48^4	0.791 (0.007)	0.755 (0.027)	0.707 (0.013)	2.419 (0.119)	2.09
56^4	0.801 (0.006)	0.734 (0.023)	0.696 (0.012)	2.305 (0.100)	1.92
64^4	0.791 (0.007)	0.760 (0.024)	0.710 (0.012)	2.425 (0.108)	2.35
80^4	0.785 (0.005)	0.734 (0.019)	0.708 (0.009)	2.404 (0.084)	2.04
96^4	0.795 (0.004)	0.717 (0.016)	0.694 (0.008)	2.291 (0.068)	1.66
128^4	0.784 (0.005)	0.768 (0.017)	0.720 (0.009)	2.508 (0.078)	1.63

$$f_1(p^2) = C \frac{p^2 + s}{p^4 + u^2 p^2 + t^2}, \quad (28)$$

which corresponds to the RGZ propagator in Eq. (18), modulo a global rescaling factor C . We note that, in order

FIG. 1. Plot of the $4d$ gluon propagator $D(p^2)$ (in GeV^{-2}) as a function of the (unimproved) momentum p (in GeV) for the lattice volume $V = 128^4$. As a consequence of the breaking of rotational invariance, the data do not produce a smooth curve. Note the logarithmic scale on the y axis.

to improve the stability of the fit, we impose some parameters to be positive, by setting them to be squares. The results of the fit for all lattice volumes, using unimproved and improved momenta, are reported, respectively, in Tables I and II. From the $\chi^2/\text{d.o.f.}$ values one clearly concludes that the use of improved momenta makes the behavior of the gluon propagator smoother, allowing a better fit to the data. This is also seen by comparing the data in Figs. 1 and 2, plotted, respectively, for unimproved and improved momenta. Let us stress that we are fitting the whole momentum range available and that, for the largest lattice volume, we have 257 data points.

In order to extract the value of the condensates described in Sec. above, we now consider only the fit results for the volume $V = 128^4$ (using improved momenta), reported in the last row of Table II. The corresponding plot is shown in Fig. 2. By setting $f_1(p^2)$ equal to the RGZ propagator in Eq. (18) (modulo the global factor C) and using propagation of error, we find for the condensates the values reported in the first column of Table III. (Note that, for this fitting form, the condensates M^2 and ρ_1 cannot be determined separately.) Let us mention that the values obtained here for $M^2 + \rho_1$, m^2 and λ^4 are in good quantitative agreement with the corresponding values—respectively indicated with M^2 , m^2 and $2g^2 N \gamma^4$ —reported in Ref. [24] for the SU(3) case.⁶ Also, as remarked above,

⁶For comparison with our values in Table III, the SU(3) condensates from [24] are respectively $2.15(13) \text{ GeV}^2$, $-1.81(14) \text{ GeV}^2$ and $4.16(38) \text{ GeV}^4$.

FIG. 2. Plot of the 4d gluon propagator $D(p^2)$ (in GeV^{-2}) as a function of the (improved) momentum p (in GeV) for the lattice volume $V = 128^4$. We also show the fitting function $f_1(p^2)$ [see Eq. (28)] with the parameters reported in the last row of Table II. Note the logarithmic scale on the y axis.

the condensate m^2 may be used to obtain a value for the gluon condensate $\langle g^2 A^2 \rangle$, through the relation (see e.g. [24])

$$\langle g^2 A^2 \rangle = -\frac{9}{13} \frac{N_c^2 - 1}{N_c} m^2. \quad (29)$$

In our case, the value $m^2 = -1.92(9)$ from Table III (using propagation of error) yields $\langle g^2 A^2 \rangle = 1.99(9) \text{ GeV}^2$.

Furthermore, we verify from Table III that $|M^2 - m^2 + \rho_1| < 2\lambda^2$, justifying our expectation (see end of Sec.) that the propagator may be decomposed in terms of a pair of complex-conjugate poles. We can thus write [see Eq. (21)]

TABLE III. Estimates of the parameters of the RGZ gluon propagator in Eq. (18) from fits (see last row of Table II above) to the equivalent form $f_1(p^2)$ in Eq. (28), using propagation of error. For comparison, we also report a Monte Carlo error analysis with 10 000 samples and a bootstrap analysis (fit results given in the text) with 500 samples. In all cases we considered the volume $V = 128^4$ and improved momenta. Errors shown in parentheses correspond to one standard deviation.

parameter	propagation of error	Monte Carlo analysis	bootstrap analysis
$M^2 + \rho_1$ (GeV^2)	2.51(8)	2.51(8)	2.3(3)
m^2 (GeV^2)	-1.92(9)	-1.92(9)	-1.7(2)
λ^4 (GeV^4)	5.3(9)	5.3(4)	4.5(9)

TABLE IV. Estimates of the parameters of the function $f_2(p^2)$ [see Eq. (30)] from fits (see last row of Table II above) to the equivalent form $f_1(p^2)$ in Eq. (28), using propagation of error. For comparison, we also report a Monte Carlo error analysis with 10 000 samples and a bootstrap analysis (fit results given in the text) with 500 samples. In all cases we considered the volume $V = 128^4$ and improved momenta. Errors shown in parentheses correspond to one standard deviation.

parameter	propagation of error	Monte Carlo analysis	bootstrap analysis
a	0.392(3)	0.392(2)	0.38(1)
b	1.32(7)	1.32(5)	1.20(7)
v (GeV^2)	0.29(2)	0.29(2)	0.29(3)
w (GeV^2)	0.66(2)	0.66(1)	0.64(2)

$$f_2(p^2) = \frac{\alpha_+}{p^2 + \omega_+^2} + \frac{\alpha_-}{p^2 + \omega_-^2} = \frac{2ap^2 + 2(av + bw)}{p^4 + 2vp^2 + v^2 + w^2}, \quad (30)$$

with $\alpha_{\pm} = a \pm ib$ and $\omega_{\pm}^2 = v \pm iw$. The results for the parameters a , b , v and w (again using propagation of error) are shown in the first column of Table IV. Thus, the poles are complex conjugates whose imaginary part is more than twice their real part. We recall that a Gribov propagator would have a null real part.

In order to have better control of the errors on the values of the condensates and poles, we have redone the estimates described above using a Monte Carlo error analysis (with 10 000 samples).⁷ The corresponding results are reported in the second columns of Table III and Table IV. Finally, we repeated the fit and the evaluation of the condensates and poles using a bootstrap analysis (with 500 samples). In this case, from the fit of $f_1(p^2)$ [see Eq. (28)] at $V = 128^4$, we find the parameters

$$C = 0.762 \pm 0.024 \quad (31)$$

$$u = 0.755 \pm 0.035 \text{ GeV} \quad (32)$$

$$t = 0.698 \pm 0.027 \text{ GeV}^2 \quad (33)$$

$$s = 2.292 \pm 0.253 \text{ GeV}^2. \quad (34)$$

The corresponding results for the condensates and poles of the propagator are shown in the third columns of Tables III and IV. Clearly, all results obtained agree within one standard deviation.

As a second test, we have also tried to allow for the more general form of the propagator, given in Eq. (17). To this end, we consider the fitting function

⁷To this end, we considered independent Gaussian distributions for the fit parameters. Thus, this Monte Carlo analysis may be considered as a numerical check of the analytic propagation of error.

TABLE V. Fits of the gluon-propagator data in the $4d$ case, for different lattice volumes, using the fitting function $f_3(p^2)$ in Eq. (35) and improved momenta [see Eq. (27)]. We report, besides the value of the fit parameters, the $\chi^2/\text{d.o.f.}$ obtained in each case. The whole range of momenta was considered for the fit. Errors shown in parentheses correspond to one standard deviation.

V	C	$d(\text{GeV}^4)$	$e(\text{GeV}^3)$	$b(\text{GeV}^4)$	$c(\text{GeV}^2)$	$a(\text{GeV})$	$\chi^2/\text{d.o.f.}$
48^4	0.889 (0.087)	7.742 (9.663)	2.469 (1.651)	26.613 (32.940)	9.692 (13.240)	2.085 (1.143)	1.95
56^4	0.798 (0.007)	0.495 (0.020)	0.055 (0.045)	0.014 (0.023)	0.581 (0.057)	1.093 (0.030)	2.04
64^4	0.795 (0.010)	0.625 (0.662)	0.336 (0.868)	0.543 (2.790)	0.733 (1.031)	1.121 (0.209)	2.52
80^4	0.781 (0.006)	0.514 (0.016)	0.059 (0.037)	0.016 (0.020)	0.593 (0.046)	1.122 (0.025)	2.08
96^4	0.893 (0.104)	10.539 (14.750)	3.036 (2.233)	39.289 (53.150)	16.558 (24.490)	2.720 (1.693)	1.56
128^4	0.784 (0.006)	0.578 (2.192)	0.229 (4.228)	0.253 (9.335)	0.691 (3.795)	1.143 (0.843)	1.69

TABLE VI. Fits of the gluon-propagator data in the $4d$ case, for different lattice volumes, using the fitting function $f_4(p^2)$ in Eq. (36) and improved momenta [see Eq. (27)]. We report, besides the value of the fit parameters, the $\chi^2/\text{d.o.f.}$ obtained in each case. The whole range of momenta was considered for the fit. Errors shown in parentheses correspond to one standard deviation.

V	C	$u(\text{GeV})$	$t(\text{GeV}^2)$	$s(\text{GeV}^2)$	$k(\text{GeV}^2)$	$\chi^2/\text{d.o.f.}$
48^4	0.802 (0.009)	0.686 (0.081)	0.792 (0.030)	1.662 (0.368)	0.547 (0.149)	2.02
56^4	0.809 (0.008)	0.694 (0.063)	0.761 (0.034)	1.714 (0.365)	0.622 (0.177)	1.89
64^4	0.802 (0.008)	0.701 (0.071)	0.790 (0.028)	1.716 (0.346)	0.573 (0.144)	2.28
80^4	0.793 (0.007)	0.703 (0.046)	0.767 (0.031)	1.882 (0.308)	0.668 (0.154)	2.01
96^4	0.804 (0.006)	0.673 (0.043)	0.757 (0.022)	1.694 (0.246)	0.625 (0.119)	1.62
128^4	0.793 (0.006)	0.727 (0.045)	0.791 (0.025)	1.903 (0.265)	0.631 (0.120)	1.60

$$f_3(p^2) = C \frac{p^4 + 2a^2 p^2 + b}{p^6 + c p^4 + d p^2 + e^2}, \quad (35)$$

which has six parameters. As can be seen in Table V, the values of $\chi^2/\text{d.o.f.}$ do not improve in comparison with the previous (4-parameter) fit and, with the exception of the global factor C , most of the parameters are determined with very large errors. This suggests that the above function has too many (redundant) parameters, making the fitting procedure quite unstable. Next, we reduce the number of parameters by one and introduce a general form that will prove useful in the description of the $3d$ data, in Sec. VI. More precisely, we test the fitting function

$$f_4(p^2) = C \frac{(p^2 + s)(p^2 + 1)}{(p^4 + u^2 p^2 + t^2)(p^2 + k)} \\ = C \frac{p^4 + (s + 1)p^2 + s}{p^6 + (k + u^2)p^4 + (ku^2 + t^2)p^2 + kt^2}. \quad (36)$$

This function is of the type (35) (with different parameters), but is written as a simple generalization of $f_1(p^2)$ in Eq. (28).⁸ Fit results are shown in Table VI. In this case the fits look reasonable. The corresponding values for the condensates and λ^4 in Eq. (17) are obtained by a Monte Carlo analysis (with 10 000 samples) using the data in the last row of Table VI. We note that, in this

case, the fitting form allows us to evaluate M^2 , ρ_1 and $|\rho|$ separately. We find the values

$$M^2 = 1.5 \pm 0.1 \text{ GeV}^2 \quad (37)$$

$$m^2 = -1.7 \pm 0.3 \text{ GeV}^2 \quad (38)$$

$$\lambda^4 = 4.1 \pm 1.0 \text{ GeV}^4 \quad (39)$$

$$\rho_1 = 0.5 \pm 0.1 \text{ GeV}^2 \quad (40)$$

$$\rho_1^2 + \rho_2^2 = 0.2 \pm 0.1 \text{ GeV}^2. \quad (41)$$

We see that the errors are larger, and that the value of $M^2 + \rho_1$ is incompatible with the numbers in Table III (obtained assuming $\rho_2 = 0$). Also, a comparison of the values in (40) and (41) suggests a very small (and imaginary) value for ρ_2 , implying that ρ is real and thus supporting the simpler form in Eq. (18), fitted above using the function $f_1(p^2)$. Moreover, the $\chi^2/\text{d.o.f.}$ is not better for the 5-parameter fit compared to the 4-parameter fit, indicating that the latter is more stable.

We thus conclude that our best fit is $f_1(p^2)$ in Eq. (28), i.e. the $4d$ gluon-propagator lattice data favor the simplified expression in Eq. (18), implying $\langle \bar{\varphi} \varphi \rangle = \langle \varphi \varphi \rangle$.

VI. THE 3D CASE

In this case the simplified fitting form $f_1(p^2)$ in Eq. (28) is not able to describe well the lattice data. Indeed, even using improved momenta (see Table VII), the $\chi^2/\text{d.o.f.}$

⁸Note that a fit using the more general form in Eq. (42) below is unstable in this case, yielding large errors for the fit parameters. Nevertheless, this fit suggests the factor $(p^2 + 1)$ in the numerator of (36), as adopted here.

TABLE VII. Fits of the gluon-propagator data in the $3d$ case, for different lattice volumes, using the fitting function $f_1(p^2)$ in Eq. (28) and improved momenta [see Eq. (27)]. We report, besides the value of the fit parameters, the $\chi^2/\text{d.o.f.}$ obtained in each case. The whole range of momenta was considered for the fit. Errors shown in parentheses correspond to one standard deviation.

V	$C(\text{GeV})$	$u(\text{GeV})$	$t(\text{GeV}^2)$	$s(\text{GeV}^2)$	$\chi^2/\text{d.o.f.}$
140^3	0.441 (0.003)	0.306 (0.013)	0.385 (0.009)	0.217 (0.013)	10.86
200^3	0.440 (0.002)	0.305 (0.011)	0.389 (0.008)	0.223 (0.011)	8.68
240^3	0.443 (0.002)	0.307 (0.010)	0.374 (0.007)	0.198 (0.009)	6.53
320^3	0.445 (0.002)	0.296 (0.011)	0.365 (0.006)	0.183 (0.008)	3.19

values obtained are quite large. Moreover, as can be seen in Fig. 3, the fit clearly fails in the IR region.⁹ The situation improves by considering the (5-parameter) fitting function $f_4(p^2)$ in Eq. (36) above, as can be seen from the results reported in Tables VIII and IX, obtained, respectively, using unimproved and improved momenta. Note that, as in the $4d$ case, the use of improved momenta helps to obtain a better fit to the data.

One can also try to use the more general function

$$f_5(p^2) = C \frac{(p^2 + s)(p^2 + l)}{(p^4 + u^2 p^2 + t^2)(p^2 + k)}, \quad (42)$$

obtained by introducing the extra parameter l . In this case, the results of the fit using improved momenta for the lattice volume $V = 320^3$ are

$$C = 0.405 \pm 0.003 \text{ GeV} \quad (43)$$

$$u = 0.692 \pm 0.040 \text{ GeV} \quad (44)$$

$$t = 0.635 \pm 0.018 \text{ GeV}^2 \quad (45)$$

$$s = 0.025 \pm 0.002 \text{ GeV}^2 \quad (46)$$

$$k = 0.050 \pm 0.007 \text{ GeV}^2 \quad (47)$$

$$l = 1.092 \pm 0.103 \text{ GeV}^2 \quad (48)$$

with $\chi^2/\text{d.o.f.} = 1.19$. By noticing that $l \approx 1$ and by comparing these values to the corresponding ones from the fit using $f_4(p^2)$ in Eq. (36) (reported in the last row of Table IX), it is clear that the two results are equivalent. At any rate, the values of $\chi^2/\text{d.o.f.}$ for the 6-parameter fit [using the function $f_5(p^2)$ in Eq. (42)] and for the fit in Table IX are the same, indicating that the latter is more stable.

In order to evaluate the condensates of the RGZ model, we thus consider only the results from the fit using $f_4(p^2)$, given for the lattice size $N = 320$ in the last row of Table IX. (The corresponding plot is shown in Fig. 4.) By setting $f_4(p^2)$ [see Eq. (36)] equal to the RGZ propagator (17) modulo the global factor C , we find (using propaga-

FIG. 3. Plot of the $3d$ gluon propagator $D(p^2)$ (in GeV^{-1}) as a function of the (improved) momentum p (in GeV) for the lattice volume $V = 320^3$. We also show the fitting function $f_1(p^2)$ [see Eq. (28)] with the parameters reported in the last row of Table VII. Note the logarithmic scale on both axes.

tion of error) the values for the condensates reported in the first column of Table X. Note that, using this fitting form, we are able to evaluate M^2 , ρ_1 and $|\rho|$ (and therefore ρ_2) separately. In this case, we can see that $\rho_2 \neq 0$ and ρ is indeed a complex quantity. This is consistent with the fact that the (four-parameter) fit to the simplified form $f_1(p^2)$ fails, as seen above.

Finally, we decompose the propagator as in Eq. (19) with $\beta = a + ib$, $\gamma = a - ib$, $\omega_2^2 = v + iw$ and $\omega_3^2 = v - iw$, i.e. we consider the function

$$f_6(p^2) = \frac{\alpha}{p^2 + \omega_1^2} + \frac{2ap^2 + 2(av + bw)}{p^4 + 2vp^2 + v^2 + w^2}. \quad (49)$$

We find (again using propagation of error) the results¹⁰ reported in the first column of Table XI. Note that the imaginary part w of the complex-conjugate poles is more

⁹In order to highlight the results at small momenta, here and in Fig. 4 we present the plot with a logarithmic scale on both axes.

¹⁰Clearly, we have $\omega_1^2 = k$ from $f_4(p^2)$.

TABLE VIII. Fits of the gluon-propagator data in the $3d$ case, for different lattice volumes, using the fitting function $f_4(p^2)$ in Eq. (36) and unimproved momenta [see Eq. (26)]. We report, besides the value of the fit parameters, the $\chi^2/\text{d.o.f.}$ obtained in each case. The whole range of momenta was considered for the fit. Errors shown in parentheses correspond to one standard deviation.

V	$C(\text{GeV})$	$u(\text{GeV})$	$t(\text{GeV}^2)$	$s(\text{GeV}^2)$	$k(\text{GeV}^2)$	$\chi^2/\text{d.o.f.}$
140^3	0.289 (0.002)	0.382 (0.022)	0.552 (0.006)	0.018 (0.003)	0.030 (0.006)	10.48
200^3	0.289 (0.002)	0.386 (0.019)	0.552 (0.006)	0.019 (0.003)	0.032 (0.006)	9.45
240^3	0.290 (0.002)	0.393 (0.017)	0.550 (0.005)	0.020 (0.003)	0.034 (0.005)	6.55
320^3	0.290 (0.002)	0.389 (0.017)	0.549 (0.005)	0.019 (0.003)	0.035 (0.005)	2.89

TABLE IX. Fits of the gluon-propagator data in the $3d$ case, for different lattice volumes, using the fitting function $f_4(p^2)$ in Eq. (36) and improved momenta [see Eq. (27)]. We report, besides the value of the fit parameters, the $\chi^2/\text{d.o.f.}$ obtained in each case. The whole range of momenta was considered for the fit. Errors shown in parentheses correspond to one standard deviation.

V	$C(\text{GeV})$	$u(\text{GeV})$	$t(\text{GeV}^2)$	$s(\text{GeV}^2)$	$k(\text{GeV}^2)$	$\chi^2/\text{d.o.f.}$
140^3	0.407 (0.001)	0.654 (0.008)	0.623 (0.004)	0.022 (0.002)	0.041 (0.003)	2.14
200^3	0.407 (0.001)	0.655 (0.007)	0.623 (0.004)	0.024 (0.002)	0.043 (0.003)	1.92
240^3	0.408 (0.001)	0.662 (0.007)	0.620 (0.004)	0.025 (0.002)	0.047 (0.003)	1.59
320^3	0.408 (0.001)	0.656 (0.008)	0.619 (0.005)	0.023 (0.002)	0.046 (0.004)	1.19

than twice the value of their real part v , as in the $4d$ case. Note also that the mass ω_1 and the residue α associated with the real pole are very small. Moreover, α is negative, which may be associated with violation of reflection positivity, indicating that this mass cannot correspond to a physical degree of freedom.

Also in this case we have repeated the analysis using a Monte Carlo estimate for the errors (with 10 000 samples) and the bootstrap method (with 500 samples). The results are shown, respectively, in the second and in the third columns of Tables X and XII. The values of the fit parameters for the function (36) using the bootstrap method for the lattice volume $V = 320^3$ are

$$C = 0.411 \pm 0.004 \text{ GeV} \quad (50)$$

$$u = 0.673 \pm 0.016 \text{ GeV} \quad (51)$$

$$t = 0.611 \pm 0.006 \text{ GeV}^2 \quad (52)$$

$$s = 0.025 \pm 0.002 \text{ GeV}^2 \quad (53)$$

$$k = 0.052 \pm 0.005 \text{ GeV}^2, \quad (54)$$

which should be compared to the results shown in the last row of Table IX. Again, all results obtained using the three different analyses agree within one standard deviation.

We thus conclude that, in the $3d$ case, the data support a gluon propagator given by the general RGZ form (17), in which the condensate ρ is a complex quantity. This is in contrast with the $4d$ case seen in the previous section, for which ρ was real. There are also significant differences for the values of the other condensates and of λ^4 in comparison with the $4d$ case. The masses from the complex-conjugate

poles, on the contrary, have similar values in $3d$ and $4d$ (see Tables XI and IV respectively).

VII. THE 2D CASE

In the two-dimensional case the situation is different. Indeed, we know from Refs. [6,57,72] that the gluon propagator at zero momentum $D(0)$ does go to zero in the infinite-volume limit, even though one always has $D(0) > 0$ at finite lattice volume. Moreover, the behavior of $D(p^2)$ at small momenta is of the type p^η , with some noninteger power $\eta \approx 0.8$. This makes the fitting procedure more complicated than in the above cases, for which polynomial forms were used. After trying several generalizations of the fitting functions considered in the $4d$ and $3d$ cases, we found that a good fit to the gluon data can be obtained using the function

$$f_7(p^2) = C \frac{p^2 + lp^\eta + s}{p^4 + u^2 p^2 + t^2}, \quad (55)$$

which is a simple generalization of Eq. (28). Results of the fit for the various lattice volumes, using unimproved and improved momenta, are reported in Tables XII and XIII respectively.¹¹ A plot of the fit for the lattice volume $V = 320^2$ using improved momenta can be seen in Fig. 5.

It is interesting to note that the function $f_7(p^2)$ above can be decomposed as

¹¹Note that in this case the use of improved momenta does not affect the quality of the fit significantly. Nevertheless, we choose to consider improved momenta in our analysis also in $2d$.

FIG. 4. Plot of the 3d gluon propagator $D(p^2)$ (in GeV^{-1}) as a function of the (improved) momentum p (in GeV) for the lattice volume $V = 320^3$. We also show the fitting function $f_4(p^2)$ [see Eq. (36)] with the parameters reported in the last row of Table IX. Note the logarithmic scale on both axes.

$$f_8(p^2) = \frac{\alpha_+(p^2)}{p^2 + \omega_+^2} + \frac{\alpha_-(p^2)}{p^2 + \omega_-^2} = \frac{2ap^2 + 2cwp^\eta + 2(av + bw)}{p^4 + 2vp^2 + v^2 + w^2}, \quad (56)$$

with

$$\alpha_\pm(p^2) = a \pm i(b + cp^\eta), \quad \omega_\pm^2 = v \pm iw. \quad (57)$$

An estimate for these five parameters is reported in Table XIV, using again three different analyses for the error. The average values of the fit parameters in Eq. (55) using the bootstrap method for the lattice volume $V = 320^2$ are

TABLE X. Estimates of the parameters of the RGZ propagator in Eq. (17) from fits (see last row of Table IX above) to the equivalent form $f_4(p^2)$ in Eq. (36). Errors are obtained using propagation of error, a Monte Carlo analysis with 10 000 samples and a bootstrap analysis with 500 samples. In all cases we considered the volume $V = 320^3$ and improved momenta. Errors shown in parentheses correspond to one standard deviation.

parameter	propagation of error	Monte Carlo analysis	bootstrap analysis
M^2 (GeV^2)	0.512 (1)	0.512 (1)	0.513 (1)
m^2 (GeV^2)	-0.55(1)	-0.55(1)	-0.52(2)
λ^4 (GeV^4)	0.94 (1)	0.94 (1)	0.91 (3)
ρ_1 (GeV^2)	0.479 (2)	0.479 (2)	0.477 (2)
ρ_2 (GeV^2)	0.09 (1)	0.094 (9)	0.100 (6)

TABLE XI. Estimates of the parameters of the function $f_6(p^2)$ [see Eq. (49)] from fits (see last row of Table IX above) to the equivalent form $f_4(p^2)$ in Eq. (36). Errors are obtained using propagation of error, a Monte Carlo analysis with 10 000 samples and a bootstrap analysis with 500 samples. In all cases we considered the volume $V = 320^3$ and improved momenta. Errors shown in parentheses correspond to one standard deviation.

parameter	propagation of error	Monte Carlo analysis	bootstrap analysis
α (GeV)	-0.024(5)	-0.024(5)	-0.029(4)
ω_1^2 (GeV^2)	0.046 (4)	0.046 (4)	0.046 (4)
a (GeV)	0.216 (3)	0.216 (2)	0.220 (4)
b (GeV)	0.27 (5)	0.271 (3)	0.275 (3)
v (GeV^2)	0.215 (5)	0.215 (5)	0.23 (1)
w (GeV^2)	0.580 (6)	0.580 (6)	0.57 (1)

$$C = 0.112 \pm 0.001 \text{ GeV}^2 \quad (58)$$

$$u = 0.550 \pm 0.013 \text{ GeV} \quad (59)$$

$$t = 0.255 \pm 0.006 \text{ GeV}^2 \quad (60)$$

$$s = 0.0152 \pm 0.0008 \text{ GeV}^2 \quad (61)$$

$$l = 0.326 \pm 0.033 \text{ GeV}^{2-\eta} \quad (62)$$

$$\eta = 0.859 \pm 0.026. \quad (63)$$

We also tried an extrapolation to the infinite-volume limit of the gluon propagator at zero momentum $D(0)$ using the function $A + B/N^\nu$, where N is the lattice side in lattice units. This gives¹²

$$A = -0.002 \pm 0.010 \quad (64)$$

$$B = 1.7 \pm 0.8 \quad (65)$$

$$\nu = 0.7 \pm 0.1, \quad (66)$$

with $\chi^2/\text{d.o.f.} = 1.07$ (considering seven data points). The fit improves if one sets $A = 0$. Indeed, in this case we find

$$B = 1.9 \pm 0.2 \quad (67)$$

$$\nu = 0.71 \pm 0.02 \quad (68)$$

with $\chi^2/\text{d.o.f.} = 0.87$. Thus, the value of the parameter s in $f_7(p^2)$ [see Eq. (55)] is consistent with zero at infinite-volume, implying $D(0) = 0$ in the same limit. One should note, however, that the condition $D(0) = 0$ is not obtained here with $b = 0$ and $v = 0$ [see Eqs. (56) and (57)], i.e. with purely imaginary poles as in a Gribov-like propagator

¹²Recall that, with our notation, the 2d gluon propagator is dimensionless.

TABLE XII. Fits of the gluon-propagator data in the $2d$ case, for different lattice volumes, using the fitting function $f_7(p^2)$ in Eq. (55) and unimproved momenta [see Eq. (26)]. We report, besides the value of the fit parameters, the $\chi^2/\text{d.o.f.}$ obtained in each case. The whole range of momenta was considered for the fit. Errors shown in parentheses correspond to one standard deviation.

V	$C(\text{GeV}^2)$	$u(\text{GeV})$	$t(\text{GeV}^2)$	$s(\text{GeV}^2)$	$l(\text{GeV}^{2-\eta})$	η	$\chi^2/\text{d.o.f.}$
80^2	0.073 (0.005)	0.363 (0.041)	0.265 (0.011)	0.078 (0.008)	0.403 (0.142)	1.129 (0.151)	2.92
120^2	0.069 (0.004)	0.432 (0.027)	0.252 (0.008)	0.052 (0.005)	0.566 (0.128)	1.145 (0.088)	2.75
160^2	0.067 (0.004)	0.458 (0.022)	0.250 (0.007)	0.044 (0.003)	0.665 (0.117)	1.138 (0.066)	2.68
200^2	0.068 (0.003)	0.470 (0.022)	0.254 (0.007)	0.037 (0.003)	0.653 (0.104)	1.088 (0.057)	3.22
240^2	0.069 (0.002)	0.469 (0.018)	0.252 (0.005)	0.031 (0.002)	0.626 (0.076)	1.051 (0.041)	2.61
280^2	0.069 (0.002)	0.483 (0.016)	0.261 (0.005)	0.029 (0.002)	0.648 (0.064)	0.994 (0.033)	2.34
320^2	0.070 (0.002)	0.483 (0.016)	0.260 (0.005)	0.025 (0.002)	0.630 (0.062)	0.981 (0.032)	2.77

TABLE XIII. Fits of the gluon-propagator data in the $2d$ case, for different lattice volumes, using the fitting function $f_7(p^2)$ in Eq. (55) and improved momenta [see Eq. (27)]. We report, besides the value of the fit parameters, the $\chi^2/\text{d.o.f.}$ obtained in each case. The whole range of momenta was considered for the fit. Errors shown in parentheses correspond to one standard deviation.

V	$C(\text{GeV}^2)$	$u(\text{GeV})$	$t(\text{GeV}^2)$	$s(\text{GeV}^2)$	$l(\text{GeV}^{2-\eta})$	η	$\chi^2/\text{d.o.f.}$
80^2	0.114 (0.002)	0.433 (0.031)	0.207 (0.012)	0.031 (0.004)	0.026 (0.043)	0.684 (0.594)	2.63
120^2	0.112 (0.003)	0.486 (0.024)	0.197 (0.008)	0.020 (0.002)	0.091 (0.050)	1.003 (0.186)	2.51
160^2	0.110 (0.002)	0.503 (0.020)	0.199 (0.006)	0.018 (0.001)	0.133 (0.044)	1.027 (0.111)	2.31
200^2	0.110 (0.002)	0.523 (0.020)	0.201 (0.007)	0.015 (0.001)	0.157 (0.046)	1.017 (0.092)	2.92
240^2	0.110 (0.002)	0.519 (0.018)	0.201 (0.006)	0.013 (0.001)	0.152 (0.038)	0.955 (0.073)	3.06
280^2	0.110 (0.002)	0.530 (0.016)	0.208 (0.006)	0.012 (0.001)	0.168 (0.033)	0.904 (0.055)	2.77
320^2	0.110 (0.001)	0.539 (0.015)	0.209 (0.006)	0.011 (0.001)	0.180 (0.033)	0.909 (0.049)	2.91

(12), but it seems to be due to the relations $a = -b$ and $v = w$ (see Table XIV). Thus, the behavior in the $2d$ case appears closer to the RGZ propagator (18) (with a non-integer power in the numerator) than to a GZ propagator (12). However, as mentioned in the Introduction, one cannot relate the fitting parameters to dimension-two condensates in the $2d$ case.

Finally, taking into account the infinite-volume limit, we set $s = 0$ in $f_7(p^2)$. In this case, the gluon propagator at small momenta behaves as $D(p^2) \approx Clp^\eta/t^2$. From the conformal relation (see, for example, Ref. [57] with $d = 2$)

$$D(p^2) \sim (p^2)^{2\kappa+(4-d)/2-1} = p^{4\kappa} \quad (69)$$

we find the IR exponent $\kappa \approx 0.225$ using $\eta \approx 0.9$ (see Table XIII). This result is in reasonable agreement with the numerical estimate in Ref. [57].

VIII. CONCLUSIONS

We have performed a systematic fitting analysis of Landau-gauge gluon-propagator data for SU(2) gauge theory in 2, 3, and 4 space-time dimensions. The fit results were matched to analytic predictions from the RGZ framework, with the intent of calculating values in physical units for the dimension-two condensates in the theory, as well as to test the predictions in a neutral way. Indeed, as mentioned in the Introduction, our strategy has been firstly to find fits with convenient free parameters in simple

(rational, except for $2d$) forms and then to interpret the parameters in terms of physical quantities. In that way, we do not bias the fits by imposing relations between fit parameters from the predicted forms. In particular, a direct fit to the general RGZ formula [see Eq. (17)] would involve six parameters (corresponding to the four condensates, the parameter λ^4 and the overall normalization) with prescribed relations among them. On the contrary, our best fits in the $4d$ and $3d$ cases involve, respectively, four and five free parameters (including the overall normalization), considering all generated data points.¹³ We find that the resulting fits of the infrared gluon propagator in $3d$ and $4d$ agree remarkably well with the tree-level predictions from the RGZ scenario. Nonperturbative effects are, of course, encoded in the values obtained from the fits for the condensates and for the Gribov parameter γ .¹⁴

In particular, the $4d$ results are well described by the simplified version of the RGZ gluon propagator in Eq. (18), equivalent to the simplest Gribov-Stingl form. This corresponds to a pair of complex-conjugate poles, as opposed to the Gribov propagator, in which the poles would be purely imaginary. Our fit results—using improved momenta and the fitting function $f_1(p^2)$ in

¹³We recall, however, the importance of considering improved momenta [see Eq. (27)] for the fits.

¹⁴A continuum extrapolation of the numerical values obtained for these quantities is postponed to a future study.

FIG. 5. Plot of the 2d gluon propagator $D(p^2)$ as a function of the (improved) momentum p (in GeV) for the lattice volume $V = 320^2$. We also show the fitting function $f_7(p^2)$ [see Eq. (55)] with the parameters reported in the last row of Table XIII. Note that in 2d, with our convention, the gluon propagator $D(p^2)$ is dimensionless. Also note the logarithmic scale on the y axis.

Eq. (28)—are given in Table II. The condensates and pole masses obtained for the largest lattice size are given, respectively, in Tables III and IV. The values for the condensates $M^2 + \rho_1$, m^2 and λ^4 are in agreement with the ones obtained for the SU(3) case in Ref. [24], where the analysis was done using a slightly different notation, as previously explained. The quantitative agreement between the infrared limit of SU(2) and SU(3) theories was observed numerically before in [16,73].

In 3d, our fits support the more general form of the RGZ propagator in Eq. (17). Fit results—using improved mo-

TABLE XIV. Estimates of the parameters of the function $f_8(p^2)$ [see Eq. (56)] from fits (see last row of Table XIII above) to the equivalent form $f_7(p^2)$ in Eq. (55). Errors are obtained using propagation of error, a Monte Carlo analysis with 10 000 samples and a bootstrap analysis with 500 samples. In all cases we considered the volume $V = 320^2$ and improved momenta. Errors shown in parentheses correspond to one standard deviation. Note that the value of η can be obtained from the last row of Table XIII for propagation of error and Monte Carlo analysis, and from Eq. (63), for the bootstrap analysis.

parameter	propagation of error	Monte Carlo analysis	bootstrap analysis
a (GeV ²)	0.0550 (5)	0.0550 (5)	0.0559 (7)
b (GeV ²)	-0.049(8)	-0.049(7)	-0.037(2)
c (GeV ^{2-η)}	0.07 (1)	0.07 (1)	0.089 (8)
v (GeV ²)	0.145 (8)	0.145 (8)	0.151 (7)
w (GeV ²)	0.15 (2)	0.15 (1)	0.205 (6)

menta and the fitting function $f_4(p^2)$ in Eq. (36)—are given in Table IX, while condensates and pole masses from the largest lattice size are reported in Tables X and XI. In this case, the condensate ρ is a complex quantity and there are significant differences in the values of the other condensates and of λ^4 compared to the 4d case. Also, in 3d one has a real pole mass in addition to the pair of complex-conjugate poles. It is interesting to note that the masses from the complex poles assume similar values in 3d and 4d, with an imaginary part that is more than twice their real part. (We recall that a Gribov propagator would have a null real part.) Note also that the mass and the coefficient associated with the real pole in 3d are very small.

In the 2d case, two particular features arise in the analysis of the data. First, as known from previous lattice studies, the description of the infrared behavior of the gluon propagator $D(p^2)$ requires a noninteger power η of the momentum p . [See the fitting form $f_7(p^2)$ in Eq. (55) and fit parameters obtained using improved momenta in Table XIII.] Second, the pole structure that best fits the data is similar to the one observed in the 3d and 4d cases—i.e. complex-conjugate poles with nonzero real part—at all considered (finite) lattice volumes. In the infinite-volume limit, one finds $D(0) = 0$, as would be the case for a Gribov propagator, with purely imaginary poles. However, in our case, the real part of the poles does not seem to vanish in this limit. The null value of $D(0)$ comes, instead, from an exact cancellation of the contributions from the two complex-conjugate poles (see values of pole masses in Table XIV).

Our analysis strongly suggests—in $d = 2, 3, 4$ —a pole structure with complex-conjugate masses (with comparable real and imaginary parts) for the infrared gluon propagator in Landau gauge.¹⁵ As stressed at the end of Sec. III, one can interpret this result as describing an unstable particle [3,36,37]. In particular, by considering the position $m_g^2 - im_g\Gamma_g$ of the gluon pole, one can estimate the gluon mass m_g and its width Γ_g , which are in principle gauge-independent quantities [77,78]. In our case, if we take as a reference the pole masses in the 4d case—i.e. the values $\omega_\pm^2 = v \pm iw$ with $v \approx 0.3$ GeV² and $w \approx 0.65$ GeV² from Table IV—we obtain

$$m_g \approx 550 \text{ MeV} \quad \text{and} \quad \Gamma_g \approx 1180 \text{ MeV}. \quad (70)$$

Note that the value for the gluon mass m_g is in agreement with other determinations [24,29,79,80].¹⁶ At the same

¹⁵Let us mention that this complex-conjugate pole structure has been shown to describe also the longitudinal and transverse gluon propagators in Landau gauge at finite temperature [74–76], at least up to twice the critical temperature T_c .

¹⁶Let us remark that in Ref. [81] the authors prefer to consider, instead of the gluon mass m_g , the maximum wavelength of (confined) gluons, roughly corresponding to the inverse gluon mass $1/m_g$.

time, the very large value for the width Γ_g would correspond to a lifetime τ_g smaller than 10^{-24} s, supporting the existence of very short-lived excitations of the gluon field.

In summary, we have presented fits—inspired by an analytic prediction, i.e. the tree-level RGZ expression, which is equivalent to the Stingl formulation (see discussion at the end of Sec. III)—allowing a good description of lattice data for the Landau-gauge SU(2) gluon propagator $D(p^2)$. The data points range from about 4 GeV down to 20–40 MeV, which are the smallest simulated momenta to date. Our results thus provide an accurate modeling of $D(p^2)$ in the whole IR region using a simple rational

expression, whose parameters may be interpreted as (effective) values for the condensates in the RGZ theory. This parametrization will hopefully be a useful input in future studies of the IR sector of Yang-Mills theories.

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Resultados Analíticos no Gauge de Landau

No capítulo anterior foi considerado o estudo numérico de funções de Green no gauge de Landau. Em particular, foi evidenciado que a extrapolação a volume infinito dos resultados numéricos para os propagadores de glúons e de fantasmas não é uma tarefa simples (neste gauge). De fato, mesmo quando são usados tamanhos de rede relativamente grandes, é ainda possível obter uma boa descrição dos dados usando diferentes *Ansätze* (ver, por exemplo, a Fig. 3.2 e as Refs. [140, 141]). Como já descrito na Seção 3.1, o problema de entender melhor a extrapolação dos dados numéricos a volume infinito foi abordado analiticamente em 2007 na Ref. [126]. Apesar de suas previsões não serem completamente confirmadas pelas simulações numéricas, esse trabalho deixou evidente que estudos analíticos específicos podem ser muito valiosos não somente na análise de dados mas também no planejamento e organização de futuros trabalhos computacionais (ver segundo parágrafo da Seção 3.3).

Desde 2006, nosso grupo no IFSC-USP tentou combinar estudos analíticos aos estudos numéricos desenvolvidos por nós, a fim de permitir um maior controle da extrapolação dos dados a volume infinito e de conseguir uma melhor interpretação e um melhor entendimento dos resultados obtidos. No presente capítulo são descritos os principais resultados desses estudos analíticos. Come evidenciado nas seções abaixo, muitas vezes esses estudos foram sugeridos pelos próprios dados numéricos, representando assim uma extensão

natural do trabalho computacional.

4.1 Fortalecimento do Propagador de Fantasma

Nossa primeira tentativa de relacionar os resultados da rede com a extrapolação a volume infinito foi apresentada na Ref. [142], publicada nos anais da conferência *Quark Confinement and the Hadron Spectrum VII* (ver quadro abaixo).

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QUARK CONFINEMENT AND THE HADRON SPECTRUM VII: 7th Conference on Quark Confinement and the Hadron Spectrum - QCHS7

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We discuss recent numerical results obtained for gluon and ghost propagators in lattice Coulomb gauge and the status of the so-called Gribov-Zwanziger confinement scenario in this gauge. Particular emphasis will be given to the eigenvalue spectrum of the Faddeev-Popov matrix. © 2007 American Institute of Physics

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Nesse trabalho, além de revisar os principais resultados obtidos por vários grupos de pesquisa no estudo de propagadores no gauge de Coulomb, foi proposta uma análise mais geral (independentemente do gauge escolhido) do chamado cenário de confinamento de Gribov-Zwanziger (ver Seção 2.2 acima). Para isso foi considerada uma condição de gauge geral $\mathcal{F}[A] = 0$, obtida impondo a minimização de um funcional $E[U]$. Como no caso do gauge de Landau, a segunda variação de $E[U]$ define a matriz de Faddeev-Popov $\mathcal{M}(b, x; c, y)$ e, para todos os mínimos locais de $E[U]$, os autovalores (não triviais) λ_i de

$\mathcal{M}$ devem ser positivos. Isto permite definir a região de Gribov Ω e o correspondente primeiro horizonte de Gribov para a condição de gauge $\mathcal{F}[A] = 0$. Assim, repetindo a análise descrita na Seção 2.2, as configurações que estejam na proximidade do horizonte de Gribov deveriam ser favorecidas no limite de volume infinito. Como consequência, o menor autovalor não-trivial λ_{min} de $\mathcal{M}$ deveria tender a zero nesse limite. De fato, esta previsão já tinha sido verificada numericamente em 2006 para vários casos, i.e. para o gauge de Landau em três [117] e quatro [143] dimensões, e para os gauges de Coulomb [59] e maximamente abeliano [34] (ambos em quatro dimensões). Ao mesmo tempo, como já mostrado na Eq. (2.19), o propagador de fantasmas na rede é dado, no espaço dos momentos p , pela equação¹

$$G(p^2) = \frac{1}{N_c^2 - 1} \sum_{x, y, b} \frac{e^{-2\pi i k \cdot (x-y)}}{V} \mathcal{M}^{-1}(b, x; b, y), \quad (4.1)$$

i.e. ele depende do inverso da matriz de Faddeev-Popov.² Considerando que, no limite de volume infinito, esta matriz apresenta um autovalor não-trivial nulo, seria esperado que o propagador $G(p^2)$ fosse fortalecido, no mesmo limite, para momentos p pequenos. Tal resultado seria uma indicação clara de um efeito de longo alcance na teoria e poderia estar relacionado ao mecanismo de confinamento de cor nas teorias de Yang-Mills não-abelianas e na QCD. Vale ressaltar que este resultado não parece depender da condição de gauge considerada, i.e. o fortalecimento do propagador de fantasmas no limite de momentos pequenos seria uma característica comum a diferentes condições de gauge e poderia ser considerado um ingrediente importante na explicação do confinamento. Na época em que foi escrita a Ref. [142], vários estudos numéricos pareciam sugerir, de fato, um propagador de fantasmas fortalecido no limite infravermelho para o gauge de Landau em três [117] e em quatro [107, 110, 114, 115, 116, 144] dimensões e para o gauge de Coulomb [145] (em quatro dimensões). Por outro lado, resultados apresentados por Tereza Mendes [34] na mesma conferência para o gauge maximamente abeliano mostravam um propagador de

¹Aqui indicamos com $b = 1, 2, \dots, N_c^2 - 1$ o índice de cor — no caso do grupo de gauge $SU(N_c)$ — e com $p(k)$ os momentos na rede. [Para a definição de momento de rede $p(k)$, ver o início da Seção 3.4.] Também indicamos com $V = N^d$ o volume da rede no caso d -dimensional.

²Claramente, esta inversão é válida somente no espaço ortogonal ao *kernel* da matriz de Faddeev-Popov.

fantasmas finito no limite de pequenos momentos. Assim, a relação simples descrita acima entre a condição de gauge obtida minimizando um funcional $E[U]$, o limite de volume infinito, o anulamento do autovalor λ_{min} e o fortalecimento do propagador de fantasmas $G(p^2)$ não deveria ser correta no caso geral. Seria portanto fundamental responder à seguinte pergunta:³

Quais são as condições que a matriz de Faddeev-Popov deve satisfazer para produzir um propagador de fantasmas fortalecido no limite de pequenos momentos?

Uma possível solução ao problema descrito acima foi sugerida pelos dados numéricos disponíveis em 2006. De fato, considerando para o menor autovalor λ_{min} uma extrapolação a volume infinito do tipo $\lambda_{min} \sim 1/L^{2+\alpha}$, onde $L = aN$ indica o lado da rede (em unidades físicas), os resultados no gauge de Landau em $d = 3$ [117] e em $d = 4$ [143] e no gauge de Coulomb [59] (com $d = 4$) forneciam valores positivos para o expoente α . Assim, nesses dois gauges o autovalor λ_{min} tornar-se-ia nulo, no limite de volume infinito, a uma taxa maior que a taxa obtida no caso da discretização usual do operador laplaciano $\Delta = -\partial_\mu \partial^\mu$, para a qual $\alpha = 0$. Por outro lado, no caso do gauge maximamente abeliano, os resultados numéricos [34] apontavam para valores negativos de α . Esta observação inspirou [142] a tentativa, descrita abaixo, de relacionar o menor autovalor λ_{min} ao propagador de fantasmas $G(p^2)$. Para tanto, é útil considerar os autovalores não-triviais λ_i de $\mathcal{M}$ e os correspondentes autovetores $\psi_i(b, x)$. Desta forma a Eq. (4.1) pode ser escrita como⁴

$$G(p^2) = \frac{1}{N_c^2 - 1} \sum_{i: \lambda_i \neq 0} \sum_b \lambda_i^{-1} |\tilde{\psi}_i(b, p)|^2, \quad (4.2)$$

onde

$$\tilde{\psi}_i(b, p) = \frac{1}{\sqrt{V}} \sum_x \psi_i(b, x) e^{-2\pi i k \cdot x} \quad (4.3)$$

é a transformada de Fourier de $\psi_i(b, x)$. Ao mesmo tempo, no caso específico dos gauges de Landau, Coulomb e maximamente abeliano, o operador de Faddeev-Popov pode ser

³Uma pergunta semelhante foi formulada recentemente na Ref. [146].

⁴Esta abordagem foi inspirada pela Ref. [59].

escrito como

$$\mathcal{M}(b, x; c, y) = -\delta^{ab} \Delta(x; y) - K(b, x; c, y) . \quad (4.4)$$

Logo, usando as equações de autovalores

$$\sum_{c,y} \mathcal{M}(b, x; c, y) \psi_i(c, y) = \lambda_i \psi_i(b, y) \quad (4.5)$$

$$\sum_y \Delta(x; y) e^{-2\pi i k \cdot y} = p^2(k) e^{-2\pi i k \cdot x} \quad (4.6)$$

e a definição (4.3) de transformada de Fourier, obtemos

$$\tilde{\psi}_i(b, p) = \frac{1}{p^2 - \lambda_i} \sum_x \frac{e^{-2\pi i k \cdot x}}{\sqrt{V}} \sum_{c,y} K(b, x; c, y) \psi_\lambda(c, y) . \quad (4.7)$$

Isto implica que, no limite de volume infinito, o comportamento infravermelho do propagador de fantasmas, e em particular o valor de $G(p_{min}^2)$, depende de

$$\tilde{\psi}_{min}(b, p_{min}) = \frac{1}{p_{min}^2 - \lambda_{min}} \sum_x \frac{e^{-2\pi i k_{min} \cdot x}}{\sqrt{V}} \sum_{c,y} K(b, x; c, y) \psi_{\lambda_{min}}(c, y) \quad (4.8)$$

e claramente existe uma competição entre o menor autovalor não-nulo p_{min}^2 do laplaciano e o menor autovalor não-nulo λ_{min} da matriz de Faddeev-Popov.

O resultado acima é bastante sugestivo, mas não permite de fato obter uma previsão simples para a relação entre o limite de volume infinito e o comportamento infravermelho do propagador de fantasmas. Como vamos ver na Seção 4.3 abaixo, este problema foi resolvido na Ref. [147].

4.2 Desigualdades para o Propagador de Glúons

Um resultado analítico importante foi obtido por nós na Ref. [148] (ver artigo anexado no final desta seção) para o propagador de glúons $D(0)$ a momento nulo. Para ilustrar este trabalho, relembramos que nas simulações numéricas o valor de $D(0)$ é obtido através da relação

$$D(0) = \frac{V}{d(N_c^2 - 1)} \sum_{\mu,b} \langle |A_\mu^b(0)|^2 \rangle . \quad (4.9)$$

Aqui

$$\langle |A_\mu^b(0)| \rangle = \frac{1}{n} \sum_c |A_{\mu,c}^b(0)| \quad (4.10)$$

é a média de Monte Carlo (com n configurações) para a grandeza

$$A_\mu^b(0) = \frac{1}{V} \sum_x A_\mu^b(x), \quad (4.11)$$

que representa a transformada de Fourier (a momento nulo) do campo de gauge $A_\mu^b(x)$. Como vamos ver abaixo, em vez de considerar o propagador $D(0)$, é mais útil considerar a grandeza

$$M(0) = \frac{1}{d(N_c^2 - 1)} \sum_{\mu,b} |A_\mu^b(0)|. \quad (4.12)$$

Para um sistema de spins isto corresponde a trocar a suscetibilidade $D(0)$ pelo valor absoluto⁵ das componentes da “magnetização” $M(0)$, que é em geral uma grandeza com um ruído estatístico menor. Como descrito na Seção 2.2 acima, existe há bastante tempo um importante resultado para $M(0)$ [62, 63]. De fato, nos gauges de Landau e de Coulomb foi demonstrado que, no limite de volume infinito, $M(0)$ deve ir para zero pelo menos como $1/N$, onde N é o tamanho do lado de rede. Este resultado é uma consequência simples da positividade da matriz de Faddeev-Popov $\mathcal{M}$ para configurações de gauge que pertencem à primeira região de Gribov Ω .

Usando os dados produzidos no início de 2007 para grandes volumes de rede, descritos na Seção 3.3 acima, foi possível obter uma análise muito precisa da dependência (do valor esperado) da magnetização $\langle M(0) \rangle$ com o tamanho de rede N (ver Fig. 4.1). Em particular, para o caso quadridimensional os dados são claramente bem descritos (a menos de uma constante multiplicativa) pela função $1/N^2$. De forma semelhante, os dados no caso tridimensional são descritos pela função $1/N^{1.5}$. Relembrando que $D(0) \propto V |A_\mu^b(0)|^2$ (com $V = N^d$), parece intuitivo que uma magnetização que vai para zero como $1/N^{d/2}$ no limite de volume infinito deveria implicar um propagador de glúons $D(0)$ finito e não-nulo. Como relacionar exatamente a suscetibilidade $D(0)$ e a magnetização $\langle M(0) \rangle$ foi o

⁵Note que, após fixar a condição de gauge de Landau, a teoria ainda possui uma simetria de gauge global. Devido a esta simetria, as componentes do campo $A_\mu^b(0)$ possuem média de Monte Carlo nula se não for tomado o valor absoluto.

Figura 4.1: Magnetização $\langle M(0) \rangle$ em função do inverso do lado de rede $1/L$ (em fm^{-1}) para o caso $SU(2)$ em quatro dimensões e $\beta = 2.2$. Também é apresentado o ajuste dos dados usando a função b/N^c com $c = 1.99 \pm 0.02$. (Note que, para dados numéricos obtidos usando diferentes volumes de rede e o mesmo valor de β , a relação entre N e L é dada por $L = aN$, onde o espaçamento de rede a é constante.) Figura apresentada por mim na Conferência *Lattice 2007* (Regensburg, Alemanha, 2007).

problema abordado no trabalho [148]. Os resultados obtidos são duas desigualdades que permitem escrever limites inferior e superior para o propagador de glúons $D(0)$ em termos da magnetização $\langle M(0) \rangle$. Assim, conhecendo a dependência de $\langle M(0) \rangle$ com o volume de rede, é possível deduzir a extrapolação a volume infinito para $D(0)$.

Os limites inferior e superior para $D(0)$ são baseados na desigualdade simples

$$\left(\frac{1}{m} \sum_{i=1}^m x_i \right)^2 \leq \frac{1}{m} \sum_{i=1}^m x_i^2, \quad (4.13)$$

onde indicamos com $\vec{x}$ um vetor com m componentes x_i . Este resultado corresponde a dizer que o quadrado da média de várias grandezas é menor ou igual à média dos quadrados dessas grandezas e é equivalente à desigualdade

$$\frac{1}{m} \sum_{i=1}^m (x_i - \bar{x})^2 \geq 0, \quad \bar{x} = \frac{1}{m} \sum_{i=1}^m x_i. \quad (4.14)$$

Logo, usando a definição (4.12) para $M(0)$ e aplicando o resultado (4.13) ao vetor cujas $m = d(N_c^2 - 1)$ componentes são dadas⁶ por $\langle |A_\mu^b(0)| \rangle$, obtemos

$$\langle M(0) \rangle^2 \leq \frac{1}{d(N_c^2 - 1)} \sum_{\mu, b} \langle |A_\mu^b(0)| \rangle^2. \quad (4.15)$$

Aplicando em seguida a mesma desigualdade (4.13) à média de Monte Carlo definida na Eq. (4.10) acima, podemos escrever

$$\langle |A_\mu^b(0)| \rangle^2 \leq \langle |A_\mu^b(0)|^2 \rangle. \quad (4.16)$$

Juntando os resultados (4.15) e (4.16) à definição (4.9) para $D(0)$, é imediato chegar à desigualdade

$$V \langle M(0) \rangle^2 \leq D(0). \quad (4.17)$$

Ao mesmo tempo podemos considerar a relação (trivial)

$$\langle \sum_{\mu, b} |A_\mu^b(0)|^2 \rangle \leq \left\langle \left\{ \sum_{\mu, b} |A_\mu^b(0)| \right\}^2 \right\rangle. \quad (4.18)$$

Isto implica o limite superior

$$D(0) \leq V d(N_c^2 - 1) \langle M(0)^2 \rangle. \quad (4.19)$$

⁶Relembramos que $b = 1, 2, \dots, N_c^2 - 1$ e que $\mu = 1, 2, \dots, d$.

Como consequência das desigualdades (4.17) e (4.19) fica evidente que, se $\langle M(0) \rangle$ vai a zero como $V^{-\alpha}$, o limite de $D(0)$ para volume infinito é dado por $D(0) \rightarrow 0$, $0 < D(0) < +\infty$ ou $D(0) \rightarrow +\infty$ respectivamente se o expoente α for maior, igual ou menor do que $1/2$. Claramente, para $d = 3$ e 4 , nossos dados para $\langle M(0) \rangle$ e as desigualdades escritas acima fornecem o limite $0 < D(0) < +\infty$ para o propagador de glúons a momento zero, de acordo com os resultados apresentados no Capítulo 3. Ao mesmo tempo, no caso $d = 2$, o expoente α é aproximadamente igual a 0.68 , i.e. maior do que $1/2$. Isto implica uma extrapolação nula para $D(0)$ no limite de grandes tamanhos de rede, de novo de acordo com os dados apresentados no capítulo anterior.

Para concluir esta seção, voltando à analogia com sistemas de spins e à interpretação de $M(0)$ e $D(0)$ como magnetização e suscetibilidade, podemos observar que a solução massiva, obtida em três e quatro dimensões, corresponde ao comportamento usual esperado para grandezas extensivas (longe de possíveis pontos críticos), caracterizado por uma magnetização nula e uma suscetibilidade finita. Ao contrário, o resultado obtido em duas dimensões, i.e. magnetização e suscetibilidade nulas, desafia a nossa intuição.

Constraints on the Infrared Behavior of the Gluon Propagator in Yang-Mills Theories

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We present rigorous upper and lower bounds for the zero-momentum gluon propagator $D(0)$ of Yang-Mills theories in terms of the average value of the gluon field. This allows us to perform a controlled extrapolation of lattice data to infinite volume, showing that the infrared limit of the Landau-gauge gluon propagator in $SU(2)$ gauge theory is finite and nonzero in three and in four space-time dimensions. In the two-dimensional case, we find $D(0) = 0$, in agreement with Maas. We suggest an explanation for these results. We note that our discussion is general, although we apply our analysis only to pure gauge theory in the Landau gauge. Simulations have been performed on the IBM supercomputer at the University of São Paulo.

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Introduction.—Color confinement is a basic feature of hadron physics that still lacks a clear theoretical understanding. Among several explanations suggested in the literature (see [1] for a recent review), the so-called Landau-gauge Gribov-Zwanziger scenario [2,3] relates gluon confinement to the infrared (IR) suppression of the gluon propagator $D(p^2)$, whereas quark confinement is related to the IR enhancement of the ghost propagator $G(p^2)$. This scenario is supported by several studies using functional methods [4]. In particular, these studies [5–7] predict, for small momenta, a gluon propagator $D(p^2) \propto p^{2(a_D-1)}$ and a ghost propagator $G(p^2) \propto 1/p^{2(1+a_G)}$. The IR exponents a_D and a_G should satisfy the relation $a_D = 2a_G + (4-d)/2$, where d is the space-time dimension and a_G should have a value in the interval $[1/2, 1]$. Clearly, if $a_D > 1$, one has $D(0) = 0$, implying maximal violation of reflection positivity [3]. In the four-dimensional case, one finds [6,7] that $a_G \approx 0.59$ and $a_D = 2a_G$. Similar power behaviors have also been obtained for the various vertex functions of $SU(N_c)$ Yang-Mills theories [7,8]. As a consequence, the running coupling constants from the ghost-gluon, three-gluon, and four-gluon vertices are all finite at zero momentum, displaying a universal (qualitative) behavior [4]. Let us note that a key ingredient of these results is the nonrenormalization of the ghost-gluon vertex, i.e., $\tilde{Z}_1(p^2) = 1$, which has been verified at the nonperturbative level [9] using lattice Monte Carlo simulations.

One should stress, however, that different IR behaviors for the Landau gluon and ghost propagators have also been proposed in the literature. For example, in Ref. [10], the authors find that $D(0)$ is finite and nonzero and that $a_G \approx 0$, with a gluon propagator characterized by a dynamically generated mass. Similar results are obtained in Ref. [11]. On the other hand, in Ref. [12], using Ward-Slavnov-Taylor identities, the authors conclude that $D(p^2)$ should be (probably very weakly) divergent at small momenta and that $a_G = 0$. Recently, Chernodub and Zakharov [13] ob-

tained the relation $2a_D + a_G = 1$ for the 4D IR exponents of gluon and ghost propagators, by considering the contribution of these propagators to thermodynamic quantities of the system, such as pressure and energy density. This result, together with the previous relation between a_D and a_G , implies that $a_D = 2/5$ and $a_G = 1/5$; i.e., the ghost propagator blows up faster than p^{-2} at small momenta, while the gluon propagator diverges as $p^{-6/5}$. Very recently, in Ref. [14], it was shown that by using the Gribov-Zwanziger approach one can also obtain a finite $D(0)$ gluon propagator and $a_G = 0$. Finally, phenomenological tests [15] seem to favor a finite and nonzero $D(0)$.

Numerical studies using Monte Carlo simulations suggest that the gluon propagator is finite at zero momentum [16–20] and that the ghost propagator [16,17,21] is enhanced when compared to the tree-level behavior p^{-2} . Moreover, in 2D and in 3D [16,20] the gluon propagator $D(p^2)$ shows a maximum value for p of a few hundred MeV and decreases as p goes to 0. On the other hand, in 4D, even using lattices with a lattice side of about 10 fm, one does not see a gluon propagator decreasing at small momenta [19]. It has been argued that an IR decreasing gluon propagator can be obtained numerically only when simulations are done on large enough lattice sizes [22]. However, from recent studies in 4D using very large lattice volumes [23–25], one sees that $D(p^2)$ either displays a plateau for momenta $p \lesssim 100$ MeV or gets slightly suppressed at small momenta. Let us note that one of the main problems of the numerical studies of the gluon propagator is the lack of a simple way of extrapolating the data to infinite volume.

Here we discuss the behavior of the gluon propagator at zero momentum. We show that, instead of studying $D(0)$ directly, it is more convenient to consider the quantity

$$M(0) = \frac{1}{d(N_c^2 - 1)} \sum_{\mu,b} |\tilde{A}_\mu^b(0)|. \quad (1)$$

In a spin system, this would be equivalent to studying the average absolute value of the components of the magnetization instead of the susceptibility, which is, of course, a much noisier quantity. (Note that, by symmetry, the field components will average to zero if no absolute value is taken.) In order to relate $M(0)$ to $D(0)$, we derive rigorous lower and upper bounds for $D(0)$, which are expressed in terms of $M(0)$. Numerical data are obtained from extensive simulations in two, three, and four dimensions, for the pure $SU(2)$ case, using very large lattices in the scaling region. We show that by using these bounds for $D(0)$ and with present lattice sizes we have clear control over the extrapolation of the data to the infinite-volume limit. We suggest a possible explanation of the results obtained. Finally, we present our conclusions. We note that our discussion concerning the bounds for the gluon propagator is general, although we consider here only the Landau-gauge propagator and pure $SU(2)$ gauge theory. Note also that recent studies [25,26] have verified the analytic prediction that Landau-gauge gluon and ghost propagators in $SU(2)$ and in $SU(3)$ are rather similar. Thus, we expect that the analysis presented here should apply also to the $SU(3)$ case.

Lower and upper bounds for $D(0)$.—As noted in the introduction, interesting lower and upper bounds for the gluon propagator at zero momentum $D(0)$ can be obtained by considering the quantity $M(0)$ defined in Eq. (1), i.e., the average of the absolute value of the components of the gluon field at zero momentum. These components are given by

$$\tilde{A}_\mu^b(0) = \frac{1}{V} \sum_x A_\mu^b(x). \quad (2)$$

In Ref. [3], it was shown that in the Landau and the Coulomb gauge the quantity $M(0)$ should go to zero at least as fast as $1/N$ in the infinite-volume limit, where N is the number of lattice points per direction. This result is simply a consequence of the positivity of the Faddeev-Popov matrix; i.e., it applies to gauge-fixed configurations that belong to the interior of the first Gribov region.

In order to find the lower and upper bounds for $D(0)$, let us consider the inequality

$$\left(\frac{1}{m} \sum_{i=1}^m X_i \right)^2 \leq \frac{1}{m} \sum_{i=1}^m X_i^2, \quad (3)$$

where $\vec{X}$ is a vector with m components X_i . This result simply says that the square of the average of an observable is smaller than or equal to the average of the square of this quantity and is equivalent to the inequality

$$\frac{1}{m} \sum_{i=1}^m (X_i - \bar{X})^2 \geq 0, \quad \bar{X} = \frac{1}{m} \sum_{i=1}^m X_i. \quad (4)$$

Note that expression (3) becomes an equality when X_i = constant. We now apply (3) to the vector with $m = d(N_c^2 - 1)$ components $\langle |\tilde{A}_\mu^b(0)| \rangle$. This yields

$$\langle M(0) \rangle^2 \leq \frac{1}{d(N_c^2 - 1)} \sum_{\mu,b} \langle |\tilde{A}_\mu^b(0)|^2 \rangle. \quad (5)$$

Then we can apply the same inequality to the Monte Carlo estimate of the average value $\langle |\tilde{A}_\mu^b(0)| \rangle = n^{-1} \sum_c |\tilde{A}_{\mu,c}^b(0)|$, where n is the number of configurations. In this case, we obtain

$$\langle |\tilde{A}_\mu^b(0)| \rangle^2 \leq \langle |\tilde{A}_\mu^b(0)|^2 \rangle. \quad (6)$$

Thus, by recalling that

FIG. 1. The square of the quantity $a\langle M(0) \rangle$ and the quantity $a^2 d(N_c^2 - 1) \langle M(0)^2 \rangle$ (both in GeV^{-2}) as a function of the inverse lattice side $1/L$ (GeV) for the 2D case (top), the 3D case (center), and the 4D case (bottom). We also show the data for $a^2 D(0)/V$ (also in GeV^{-2}) and the fit of the data using the parameters reported in Table I. Note that with our notation $M(0)$, $D(0)$, and V are dimensionless quantities, while $L = aV^{1/d}$ is dimensionful.

TABLE I. Fits of $\langle M(0) \rangle$, $a^2 \langle M(0)^2 \rangle$, and $a^2 D(0)/V$, respectively, using the *Ansätze* B_l/L^l , B_u/L^u , and B/L^k . Note that B_l , B_u , and B have mass dimensions, respectively, $-l-1$, $-u-2$, and $-k-2$. Note also that, in order to obtain Fig. 1, one should multiply by $d(N_c^2 - 1)$ the data and the fit related to the fourth and fifth columns of the table. Most of the data used for the fits have a statistical error of the order of 2%–3%. For all fits we have $\chi^2/\text{d.o.f.} \approx 1$.

d	B_l	l	B_u	u	B	k
2	1.48(6)	1.367(8)	2.3(2)	2.72(1)	3.3(2)	2.73(1)
3	1.0(1)	1.48(3)	1.0(3)	2.95(5)	1.5(3)	2.96(4)
4	1.7(1)	1.99(2)	3.1(5)	3.99(4)	4.7(8)	3.99(4)

$$D(0) = \frac{V}{d(N_c^2 - 1)} \sum_{\mu,b} \langle |\tilde{A}_\mu^b(0)|^2 \rangle \quad (7)$$

and using Eqs. (5) and (6), we find that

$$V \langle M(0) \rangle^2 \leq D(0). \quad (8)$$

At the same time we can write the inequality

$$\left\langle \sum_{\mu,b} |\tilde{A}_\mu^b(0)|^2 \right\rangle \leq \left\langle \left[\sum_{\mu,b} |\tilde{A}_\mu^b(0)| \right]^2 \right\rangle. \quad (9)$$

This implies that

$$D(0) \leq V d(N_c^2 - 1) \langle M(0)^2 \rangle. \quad (10)$$

Thus, if $M(0)$ goes to zero as $V^{-\alpha}$, we find that $D(0) \rightarrow 0$, $0 < D(0) < +\infty$, or $D(0) \rightarrow +\infty$, respectively, if the exponent α is larger than, equal to, or smaller than $1/2$. Finally, let us note that the inequalities (8) and (10) can be immediately extended to the case $D(p^2)$, with $p \neq 0$.

Results.—We have considered several lattice volumes in 2D (at $\beta = 10$, up to a lattice volume $V = 320^2$), in 3D (at $\beta = 3$, up to $V = 320^3$), and in 4D (at $\beta = 2.2$, up to $V = 128^4$). Details of the simulations will be presented elsewhere [27]. We set the lattice spacing a by considering the input value $\sigma^{1/2} = 0.44$ GeV, which is a typical value for this quantity in the 4D $SU(3)$ case. Note that the lattice volumes 320^2 at $\beta = 10$, 320^3 at $\beta = 3$, and 128^4 at $\beta = 2.2$ correspond, respectively, to $a^2 V \approx (70 \text{ fm})^2$, $a^3 V \approx (85 \text{ fm})^3$, and $a^4 V \approx (27 \text{ fm})^4$. Simulations in 2D have been done on a PC cluster at the IFSC-USP (with 4 Pentium IV 2.8 GHz and 4 Pentium IV 3.0 GHz). Simulations in 3D and in 4D have been done in the 4.5 Tflops IBM supercomputer at USP [28]. The total CPU time was equivalent to about 5.7 (in 3D) and 25.9 days (in 4D) on the whole machine.

We start by considering the quantity $\langle M(0) \rangle$. We find (see Fig. 1 and Table I) that our data extrapolate very well to zero as $1/L^l$, with the values of l given in Table I. Thus,

in 3D and in 4D we have $\langle M(0) \rangle \sim 1/V^{1/2}$, implying that $D(0) > 0$. In particular, from our fits we obtain $a^2 D(0) \geq (B_l/a^l)^2$. This gives $a^2 D(0) \geq 0.4(1) \text{ GeV}^{-2}$ in 3D and $a^2 D(0) \geq 2.2(3) \text{ GeV}^{-2}$ in 4D, where we used $a = 1.35687 \text{ GeV}^{-1}$ in 3D and $a = 1.066 \text{ GeV}^{-1}$ in 4D. As for the upper bound (10), by using our fits (see again Fig. 1 and Table I) we have $a^2 D(0) \leq d(N_c^2 - 1) B_u/a^u$, yielding $a^2 D(0) \leq 4(1) \text{ GeV}^{-2}$ in 3D and $a^2 D(0) \leq 29(5) \text{ GeV}^{-2}$ in 4D. On the other hand, in 2D both the lower and the upper bounds extrapolate to zero, implying that $D(0) = 0$ in agreement with Ref. [16]. Let us note that our bounds in 3D and in 4D are in agreement with the data shown in Figs. 1 and 2 of Ref. [23]. [In the 3D case, compared to the extrapolation reported in Fig. 1 of Ref. [23], one should also include here a factor $\beta = 3.0$, i.e., $1.2(3) \leq a^2 D(0) \leq 12(3) \text{ GeV}^{-2}$.] Also note that in the three cases one finds $B_u \approx B_l^2$ and $u \approx 2l$. Indeed, one can check that $\langle M(0) \rangle^2 \leq \langle M(0)^2 \rangle$, implying that the quantity $M(0)$ is almost the same for all Monte Carlo configurations. More precisely, we verified for the three cases that $\langle M(0)^2 \rangle - \langle M(0) \rangle^2$ [i.e., the susceptibility of $M(0)$] goes to zero as $\sim 1/V$ in the infinite-volume limit.

In order to interpret these results, let us first note that, given a Gaussian random variable x with a null mean value and standard deviation σ , the random variable $|x|$ has a mean value (and standard deviation) proportional to σ . In our case, this suggests that the average value of the gluon field at zero momentum $\tilde{A}(0)$ [defined in Eq. (2)] should be zero with a standard deviation of the order of $1/L^c$, with $c \approx l$ (see Table I). This is indeed the case in 2D, 3D, and 4D. [We find, respectively, $c = 1.36(2)$, $1.47(3)$, and $1.97(1)$ for the three cases.] In 3D and in 4D, our results imply that $\sigma \propto 1/\sqrt{V}$. This property is known as *self-averaging* [29] and is the behavior expected for extensive quantities in pure phases, away from phase boundaries. (In our case, the magnetization is not extensive, because we divide by the volume, but the result holds for the relative standard deviation.) More precisely, one talks of strong self-averaging when $\sigma \propto 1/L^c$ and $c = d/2$ and of weak self-averaging when $c < d/2$. Thus, we find strong self-averaging for $M(0)$ in 3D and in 4D and some kind of *over-self-averaging* in 2D, with $c > d/2$. In simpler terms, the gluon propagator may be thought of as the susceptibility associated to the magnetization $M(0)$ [or rather to the quantity defined by Eq. (1) *without* the absolute value, which has zero average]. In 3D and 4D the system has (finite) nonzero susceptibility, while for 2D the susceptibility is zero. We do not have a simple explanation for this latter result. Here we can argue only that the 2D case is probably different since there are no propagating degrees of freedom.

Note that our results in 3D and in 4D imply only that reflection positivity is not *maximally* violated. A clear violation of reflection positivity [27,30] is still obtained in 2D, 3D, and 4D for the $SU(2)$ and $SU(3)$ cases.

Conclusions.—We have shown that the Landau-gauge gluon propagator at zero momentum $D(0)$ is finite and nonzero in 3D and in 4D. At the same time, we find that $D(0) = 0$ in 2D, in agreement with Ref. [16]. These results have been obtained by considering the inequalities in Eqs. (8) and (10), i.e., by studying the “magnetizationlike” quantity $M(0)$ instead of the “susceptibility” $D(0)$. This allows control of the extrapolation of the data to infinite volume. Moreover, the quantity $D(0)/V$ can be well fitted in this limit as a function of $1/L$. Our results in 3D and in 4D can be explained as a manifestation of strong self-averaging. As mentioned above, a similar analysis may be applied to more general cases and considering also nonzero momenta.

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4.3 Desigualdades para o Propagador de Fantasma

Logo após a conclusão do artigo [148] sobre as desigualdades para o propagador de glúons, descrito na seção anterior, pudemos obter desigualdades simples também para o propagador de fantasmas [147] (ver artigo anexado no fim desta seção). De fato, usando a relação (4.2) e as desigualdades⁷

$$0 < \lambda_{min} < \lambda_2 < \lambda_3 \dots, \quad (4.20)$$

fica claro que $G(p^2)$ deve satisfazer às relações

$$\frac{1}{N_c^2 - 1} \frac{1}{\lambda_{min}} \sum_b |\tilde{\psi}_{min}(b, p)|^2 \leq G(p^2) \quad (4.21)$$

e

$$G(p^2) \leq \frac{1}{N_c^2 - 1} \frac{1}{\lambda_{min}} \sum_{i: \lambda_i \neq 0} \sum_b |\tilde{\psi}_i(b, p)|^2. \quad (4.22)$$

Assim, após somar e subtrair na Eq. (4.22) os termos devidos às autofunções que pertencem ao *kernel* de $\mathcal{M}(b, x; c, y)$ e usando a relação de completeza

$$\sum_i \psi_i(b, x) \psi_i^*(c, y) = \delta_{bc} \delta_{xy}, \quad (4.23)$$

onde * indica conjugação complexa, é obtido o resultado

$$G(p^2) \leq \frac{1}{\lambda_{min}} \left[1 - \frac{1}{N_c^2 - 1} \sum_{j: \lambda_j = 0} \sum_b |\tilde{\psi}_j(b, p)|^2 \right]. \quad (4.24)$$

No caso específico do gauge de Landau, os autovetores com autovalor nulo são representados por modos constantes. Assim, para $p \neq 0$ obtemos as desigualdades

$$\frac{1}{N_c^2 - 1} \frac{1}{\lambda_{min}} \sum_b |\tilde{\psi}_{min}(b, p)|^2 \leq G(p^2) \leq \frac{1}{\lambda_{min}}. \quad (4.25)$$

É importante ressaltar que o limite superior na Eq. (4.25) torna-se uma igualdade se e somente se os autovalores da matriz de Faddeev-Popov forem todos degenerados e iguais a λ_{min} .

⁷Para simplificar a notação consideramos autovalores (não-nulos) não-degenerados. Esta condição é certamente satisfeita para configurações de rede não-triviais $A \neq 0$.

Os resultados descritos acima permitem relacionar o comportamento infravermelho do propagador de fantasmas $G(p^2)$ com a extrapolação do autovalor λ_{min} no limite de volume infinito. De fato, fazendo a hipótese de um comportamento do tipo $p^{-2-2\kappa}$ para $G(p^2)$ no limite de momentos p pequenos, supondo $\lambda_{min} \sim L^{-2-\alpha}$ para grandes lados de rede L e lembrando que o menor momento não-nulo é dado na rede por $p_{min} = 2 \sin(a\pi/L)$, é claro que a desigualdade (4.25) requer $2\kappa \leq \alpha$. Isto implica que $\alpha > 0$ é uma condição necessária para obter um propagador de fantasmas $G(p^2)$ fortalecido ($\kappa > 0$) no limite infravermelho. Ao mesmo tempo, o limite inferior da equação (4.25) implica que $\alpha > 0$ e $|\tilde{\psi}_{min}(a, p)|^2 \neq 0$ são condições suficientes para obter $\kappa > 0$.

Devido ao grande esforço computacional necessário para o cálculo do propagador de fantasmas $G(p^2)$ e do menor autovalor λ_{min} da matriz de Faddeev-Popov, a verificação numérica das desigualdades (4.25) foi feita na Ref. [147] usando principalmente dados já publicados, obtidos em geral para redes relativamente pequenas. Além disso, foi verificado somente o limite superior da equação (4.25). Apesar dessas limitações, o estudo evidenciou que, no limite de grandes volumes de rede, o propagador de fantasmas calculado para o menor momento p_{min} é razoavelmente bem aproximado pelo valor $1/\lambda_{min}$ (ver a Figura 1 na mesma referência). Uma verificação mais cuidadosa⁸ das desigualdades (4.25) foi recentemente completada [149] (ver quadro abaixo) usando no caso SU(2) redes de volume até 64^4 para $\beta = 2.2$. Ao mesmo tempo, foi mostrado como melhorar de forma sistemática as desigualdades demonstradas acima. De fato, considerando também o autovalor λ_2 e o correspondente autovetor $\tilde{\psi}_2(b, p)$, é simples verificar que

$$\frac{1}{N_c^2 - 1} \sum_b \left[\frac{1}{\lambda_{min}} |\tilde{\psi}_{min}(b, p)|^2 + \frac{1}{\lambda_2} |\tilde{\psi}_2(b, p)|^2 \right] \leq G(p^2) \quad (4.26)$$

e que, para momentos p não nulos,

$$G(p^2) \leq \left(\frac{1}{\lambda_{min}} - \frac{1}{\lambda_2} \right) \left(\frac{1}{N_c^2 - 1} \sum_b |\tilde{\psi}_{min}(b, p)|^2 \right) + \frac{1}{\lambda_2}. \quad (4.27)$$

⁸Isto foi possível graças ao acesso ao super-computador IBM Blue Gene/P, instalado em Abril de 2012 na Rice University nos EUA, cujo uso é compartilhado entre pesquisadores da USP e da Universidade Rice.

Ghost sector and geometry in minimal Landau gauge: further constraining the infinite-volume limit

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We present improved upper and lower bounds for the momentum-space ghost propagator of Yang-Mills theories in terms of the two smallest nonzero eigenvalues (and their corresponding eigenvectors) of the Faddeev-Popov matrix. These results are verified using data from four-dimensional numerical simulations of SU(2) lattice gauge theory in minimal Landau gauge at $\beta = 2.2$, for lattice sides $N = 16, 32, 48$ and 64 . Gribov-copy effects are discussed by considering four different sets of numerical minima. We then present a lower bound for the smallest nonzero eigenvalue of the Faddeev-Popov matrix in terms of the smallest nonzero momentum on the lattice and of a parameter characterizing the geometry of the first Gribov region Ω . This allows a simple and intuitive description of the infinite-volume limit in the ghost sector. In particular, we show how nonperturbative effects may be quantified by the rate at which typical thermalized and gauge-fixed configurations approach the boundary of Ω , known as the first Gribov horizon. Most of the simulations have been performed on the Blue Gene/P-IBM supercomputer shared by Rice University and São Paulo University.

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Também é simples verificar que o novo limite superior (4.27) é em geral uma melhoria sobre o limite superior da Eq. (4.25) e que os dois limites superiores coincidem para $\lambda_{min} = \lambda_2$. Na Ref. [149] as velhas e as novas desigualdades para o propagador de fantasmas foram verificadas numericamente considerando também diferentes cópias de Gribov. Os resultados demonstraram claramente que (no limite de volume infinito) $\lambda_{min} \approx \lambda_2$, como esperado, e que $G(p_{min}^2) \approx 1/\lambda_{min}$ (ver Fig. 4.2), confirmando os dados apresentados no trabalho original [147]. Como explicado logo abaixo da Eq. (4.25), este resultado implica que $\lambda_i \approx \lambda_{min}$ e representa uma verificação numérica do resultado “all horizons are one horizon”, discutido na Seção 3 da Ref. [66]. Ao mesmo tempo, sendo $G(p_{min}^2)$ muito maior do que os limites inferiores, fica claro que o autovetor ψ_{min} deve ser em geral muito diferente das ondas planas correspondentes ao autovalor p_{min}^2 do laplaciano discretizado. De fato, foi obtido que a projeção⁹

$$\bar{\psi}_{min}(p_{min}) = \frac{1}{N_c^2 - 1} \sum_b \frac{1}{d} \sum_\mu \left| \tilde{\psi}_{min} \left(b, \frac{2\pi}{N} e_\mu \right) \right|^2 \quad (4.28)$$

⁹Aqui e_μ indica um vetor unitário na direção μ . Relembramos que o momento p_{min} é d -vezes degenerado.

Figura 4.2: O propagador de fantasmas $G(p_{min}^2)$ (triângulos cheios), os limites inferiores nas Eqs. (4.25) e (4.26) (círculos vazios e cheios) e os limites superiores nas Eqs. (4.25) e (4.27) (quadrados vazios e cheios) como função do inverso do lado de rede $1/N$. Todas as grandezas são em unidades de rede. Para esta figura foi usado o conjunto de cópias de Gribov indicado na Ref. [149] como *first copy* (*fc*). Observe a escala logarítmica no eixo y . As barras de erro (que não são visíveis) correspondem a um desvio padrão. (Na análise dos dados foram considerados somente os erros estatísticos.) Este gráfico é apresentado na Figura 1 da Ref. [149].

é muito pequena para todas as configurações de gauge e cópias de Gribov consideradas na Ref. [149]. Enfim, foi verificado que, no limite de L grande, o menor autovalor λ_{min} da matriz de Faddeev-Popov vai a zero mais lentamente¹⁰ do que $1/L^2$, confirmando o comportamento massivo descrito no capítulo anterior para o caso quadridimensional.

¹⁰Este resultado está em desacordo com as conclusões apresentadas na Ref. [143], obtidas porém usando dados obtidos para redes relativamente pequenas.

Constraints on the infrared behavior of the ghost propagator in Yang-Mills theories

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We present rigorous upper and lower bounds for the momentum-space ghost propagator $G(p)$ of Yang-Mills theories in terms of the smallest nonzero eigenvalue (and of the corresponding eigenvector) of the Faddeev-Popov matrix. We apply our analysis to data from simulations of SU(2) lattice gauge theory in Landau gauge, using the largest lattice sizes to date. Our results suggest that, in three and in four space-time dimensions, the Landau gauge ghost propagator is *not* enhanced as compared to its tree-level behavior. This is also seen in plots and fits of the ghost dressing function. In the two-dimensional case, on the other hand, we find that $G(p)$ diverges as $p^{-2-2\kappa}$ with $\kappa \approx 0.15$, in agreement with A. Maas, Phys. Rev. D **75**, 116004 (2007). We note that our discussion is general, although we make an application only to pure gauge theory in Landau gauge. Our simulations have been performed on the IBM supercomputer at the University of São Paulo.

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I. INTRODUCTION

In the Gribov-Zwanziger scenario [1], confinement of quarks (in Landau and in Coulomb gauge) is related to a ghost propagator $G(p)$ enhanced in the infrared (IR) limit when compared to the tree-level propagator $1/p^2$. An enhanced ghost propagator is also required by the Kugo-Ojima criterion [2] in order to explain confinement of color charge (in Landau gauge).

In order to investigate the origin of this IR enhancement we consider, following Ref. [3], a generic gauge condition $\mathcal{F}[A] = 0$, where A is the gauge field. The condition is imposed on the lattice by minimizing a functional $E[U]$, where U is the (gauge) link variable. From the second variation of $E[U]$ one can define the Faddeev-Popov (FP) matrix $\mathcal{M}(a, x; b, y)$. (Here we use a and b to indicate color indices, whereas x and y represent points of the lattice.) In the SU(N_c) case, $\mathcal{M}$ is an $(N_c^2 - 1)V \times (N_c^2 - 1)V$ matrix, where $V = N^d$ is the lattice volume, given by the number of points N along each side of the lattice and by the space-time dimension d . This matrix is real and symmetric with respect to the exchange $(a, x) \leftrightarrow (b, y)$ (see Ref. [4] for a thorough discussion of the properties of the FP matrix in Landau gauge). At any (local) minimum of $E[U]$, all the eigenvalues of $\mathcal{M}$ are positive (modulo trivial null eigenvalues). The set of all minima of $E[U]$ is the so-called Gribov region Ω . On the boundary of Ω —known as the first Gribov horizon $\partial\Omega$ —the smallest nontrivial eigenvalue $\lambda_{\min}$ of the FP matrix is null. Since the configuration space has very large dimensionality we expect that,

in the limit of large lattice volumes V , entropy favors configurations near $\partial\Omega$ [5–7], i.e. $\lambda_{\min}$ should go to zero in this limit. This is indeed the case in 2d [8], 3d [9], and 4d Landau gauge [10], in 4d Coulomb gauge [7], and in 4d maximally Abelian gauge [11].

The ghost propagator is written in terms of the inverse of the FP matrix as

$$G(p) = \frac{1}{N_c^2 - 1} \sum_{x,y,a} \frac{e^{-2\pi i k \cdot (x-y)}}{V} \mathcal{M}^{-1}(a, x; a, y), \quad (1)$$

where the lattice momentum $p(k)$ has components $p_\mu(k) = 2 \sin(\pi k_\mu/N)$ with $k_\mu = 0, 1, \dots, N-1$. Also, here and in the formulas below, a path-integral average is understood. Since $\mathcal{M}(a, x; b, y)$ develops a null eigenvalue at the Gribov horizon $\partial\Omega$, one might expect that the corresponding ghost propagator diverges at small momenta in the infinite-volume limit. This could in turn introduce a long-range effect in the theory, related to the color-confinement mechanism [5,7]. Indeed, in several numerical studies (using relatively small lattice volumes), the ghost propagator $G(p)$ seems to be IR-enhanced in 3d [9] and 4d Landau gauge [12,13] and in 4d Coulomb gauge [14]. On the other hand, in maximally Abelian gauge [11] one finds an IR-finite $G(p)$. At the same time, recent results in Landau gauge using very large lattice volumes [15–17] suggest an essentially tree-level-like IR ghost propagator $G(p)$ in 3d and in 4d. (A similar result has been obtained in Refs. [18,19] using different analytic approaches.) Finally, in the 2d case, one finds [8] $G(p) \sim p^{-2.3}$ after extrapolating data for the IR exponent κ to infinite volume. Therefore, we have cases for which $G(p)$ is not IR-enhanced in the infinite-volume limit and thus the argument reported above cannot be valid in general [3]. The

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aim of this work is to understand under what conditions one should expect an IR-enhanced ghost propagator $G(p)$. This is done by investigating constraints on the behavior of $G(p)$ and thus obtaining better control over its infinite-volume extrapolation. Clearly, since the only diverging quantity (besides the lattice volume) is $1/\lambda_{\min}$, the IR enhancement of $G(p)$ may be closely related to this eigenvalue and to the projection of its corresponding eigenvector $\psi_{\min}(a, x)$ on the plane waves $e^{-2\pi i k \cdot x}/\sqrt{V}$ for small momenta $p(k)$.

II. LOWER AND UPPER BOUNDS FOR $G(p)$

In this section we obtain upper and lower bounds for the momentum-space ghost propagator $G(p)$ in terms of the smallest nonzero eigenvalue $\lambda_{\min}$ [and of the corresponding eigenvector $\psi_{\min}(a, x)$] of the FP matrix $\mathcal{M}(a, x; b, y)$. To this end, let us first introduce our notation. As usual, we define the generic eigenvalue λ_i of $\mathcal{M}(a, x; b, y)$ and the corresponding eigenvector $\psi_i(a, x)$ by using the relation $\mathcal{M}(a, x; b, y)\psi_i(b, y) = \lambda_i\psi_i(a, x)$, where the index i takes values $1, 2, \dots, (N_c^2 - 1)V$ and the sum over repeated indices is understood. As stressed in the Introduction, this matrix is positive (or semipositive) definite, whenever we use a minimizing condition in order to fix a gauge (minimal gauge). Also, since it is obtained from a second-order expansion, this matrix can always be written in a symmetric form. Then, the eigenvectors $\psi_i(a, x)$ can be assumed orthogonal to each other and normalized as $\sum_{a,x} \psi_i(a, x)\psi_j(a, x)^* = \delta_{ij}$, where $*$ indicates complex conjugation. Using this notation we can write

$$\mathcal{M}(a, x; b, y) = \sum_{i, \lambda_i \neq 0} \lambda_i \psi_i(a, x)\psi_i(b, y)^*. \quad (2)$$

Note that we are working in the space orthogonal to the kernel of the FP matrix. Then we can write the inverse of this matrix as [20]

$$\mathcal{M}^{-1}(a, x; b, y) = \sum_{i, \lambda_i \neq 0} \lambda_i^{-1} \psi_i(a, x)\psi_i(b, y)^*. \quad (3)$$

By using Eq. (1) we obtain

$$G(p) = \frac{1}{N_c^2 - 1} \sum_{i, \lambda_i \neq 0} \sum_a \lambda_i^{-1} |\tilde{\psi}_i(a, p)|^2, \quad (4)$$

where $\tilde{\psi}_i(a, p) = \sum_x \psi_i(a, x)e^{-2\pi i k \cdot x}/\sqrt{V}$ is the Fourier transform of the eigenvector $\psi_i(a, x)$. Since all the nonzero eigenvalues are positive and $0 < \lambda_{\min} \leq \lambda_i$, we can write the inequalities

$$\frac{1}{N_c^2 - 1} \frac{1}{\lambda_{\min}} \sum_a |\tilde{\psi}_{\min}(a, p)|^2 \leq G(p), \quad (5)$$

$$G(p) \leq \frac{1}{N_c^2 - 1} \frac{1}{\lambda_{\min}} \sum_{i, \lambda_i \neq 0} \sum_a |\tilde{\psi}_i(a, p)|^2. \quad (6)$$

(Note that we assume nondegenerate eigenvalues.

However, the second inequality applies also when $\lambda_{\min}$ is degenerate and the first one can be easily modified to take a degeneracy into account.) By summing and subtracting in Eq. (6) the contributions from the eigenvectors corresponding to a null eigenvalue and using the completeness relation $\sum_i \psi_i(a, x)\psi_i(b, y)^* = \delta_{ab}\delta_{xy}$, we find

$$G(p) \leq \frac{1}{\lambda_{\min}} \left[1 - \frac{1}{N_c^2 - 1} \sum_{j, \lambda_j = 0} \sum_a |\tilde{\psi}_j(a, p)|^2 \right]. \quad (7)$$

Let us stress that the above results are simply a consequence of the positivity of the FP matrix, i.e. they apply to gauge-fixed configurations that belong to the interior of the first Gribov region.

III. BOUNDS IN LANDAU GAUGE

In the Landau case the eigenvectors corresponding to the zero eigenvalue are constant modes, i.e. they contribute only to the case $p = 0$. Thus, for any nonzero momentum we have

$$\frac{1}{N_c^2 - 1} \frac{1}{\lambda_{\min}} \sum_a |\tilde{\psi}_{\min}(a, p)|^2 \leq G(p) \leq \frac{1}{\lambda_{\min}}. \quad (8)$$

As said in the Introduction, in the infinite-volume limit the measure gets concentrated on the boundary of the (first) Gribov region, i.e. $\lambda_{\min}$ goes to zero in this limit. In particular, as for the Laplacian operator, one can expect to find $\lambda_{\min} \sim N^{-\alpha}$ for large N . Now, if we make the hypothesis of a power-law behavior $p^{-2-2\kappa}$ for $G(p)$ in the IR limit and consider the smallest nonzero momentum on the lattice [i.e. $p_s = 2 \sin(\pi/N)$], we have $G(p_s) \sim N^{2+2\kappa}$ in the limit of large N . Then from Eq. (8) we get $2 + 2\kappa \leq \alpha$. Thus, assuming a power-law behavior for $G(p)$, $\alpha > 2$ is a necessary condition to obtain a ghost propagator $G(p)$ enhanced in the IR limit compared to the tree-level behavior $1/p^2$.

The lower bound in Eq. (8) clearly depends on the behavior of the quantity $|\tilde{\psi}_{\min}(a, p_s)|^2$. Note that this quantity cannot diverge in the infinite-volume limit. (Actually, since the eigenvectors are normalized, this quantity is always between 0 and 1.) Thus, if we furthermore make the assumption of a behavior at large N given by $N^{-\gamma}$, with $\gamma \geq 0$, we find the condition $\alpha - \gamma \leq 2 + 2\kappa$.

As a check of the proposed bounds, we collect here results for the ghost propagator $G(p_s)$ at the smallest nonzero momentum $p_s = 2 \sin(\pi/N)$ and for the smallest nonzero eigenvalue $\lambda_{\min}$ from Refs. [6,9,12,15] for the 3d and 4d cases. These data are shown for the smaller lattices in Tables I and II and for the larger lattices in Fig. 1. We also present new data in the two-dimensional case. (These data were obtained together with the data for the gluon propagator reported in [21].) All data refer to the SU(2) case. Let us recall that recent studies [17,22] have verified the analytic prediction that Landau gauge gluon and ghost propagators in SU(2) and in SU(3) are rather similar. Thus,

TABLE I. The ghost propagator $G(p_s)$ for the smallest nonzero momentum $p_s = 2 \sin(\pi/N)$ and the inverse of the smallest nonzero eigenvalue $\lambda_{\min}$ of the FP matrix for various lattice volumes and β values in the 3d case. Data (in physical units) are taken from Ref. [9].

N	β	a^{-1} [GeV]	$G(p_s)$ [GeV $^{-2}$]	$1/\lambda_{\min}$ [GeV $^{-2}$]
20	6.0	1.733	7.3(1)	32.5(7)
30	6.0	1.733	19.4(3)	94(2)
20	4.2	1.136	21.0(4)	107(3)
30	4.2	1.136	54.5(8)	282(6)

we expect that the analysis presented here should apply also to the SU(3) case. In all cases the quantity $1/\lambda_{\min}$ has been evaluated using the average value for $\lambda_{\min}$ and propagation of errors.

As one can see, the upper bound in Eq. (8) is always satisfied. (Of course, this is the case also when $p \neq p_s$.) The bound has also been verified using the data at $\beta = 0$ from Refs. [6,12]. We have fitted the data for $G(p_s)$ and for $\lambda_{\min}$ as a function of L for the data shown in Fig. 1 using a power-law Ansatz. It is interesting to note that (in 2d and in 4d) we actually find α smaller than $2 + 2\kappa$. The bound is still satisfied, since the multiplicative constant in the fit of $\lambda_{\min}$ is larger. It would be interesting to see how the upper bound is realized when even larger lattice volumes are considered. Also, note that the upper bound in the 4d case seems to saturate as the volume increases. This would indicate that the contribution to $G(p)$ from all the eigenvalues $\lambda_i > \lambda_{\min}$ stays finite at large volume and that the exponent γ , defined above, is zero. Let us note that this is not the case at smaller lattice volumes, since then one needs to consider the contribution to $G(p)$ from the first 150–200 smaller eigenvalues in order to reproduce the behavior of the propagator at small momenta [10]. Finally, let us recall that finite-size effects for $G(p)$ (at fixed p) are generally very small. This can be explained considering the spectral density of the FP matrix, as done in Ref. [3]. Thus, our results indicate that in 3d and in 4d the ghost propagator is likely not IR-enhanced, while in 2d

TABLE II. As in Table I for the 4d case. Data (in lattice units) are taken from Refs. [6,12] using the average called *first copy*. We find similar results when considering data obtained using the average called *absolute minimum*.

N	β	$G(p_s)$	$1/\lambda_{\min}$
8	0.8	8.94(8)	91(3)
12	0.8	22.1(6)	210(20)
16	0.8	40.6(7)	360(60)
8	1.6	7.2(1)	61(3)
16	1.6	32.1(3)	220(20)
8	2.7	3.4(1)	11.0(4)
12	2.7	7.1(3)	25(1)
16	2.7	12.9(6)	45(4)

FIG. 1. The ghost propagator $G(p_s)$ for the smallest nonzero momentum $p_s = 2 \sin(\pi/N)$ (in GeV $^{-2}$) as a function of the inverse lattice side $1/L$ (GeV) for the 2d case (top, at $\beta = 10$ with volumes up to 320^2), 3d case (center, at $\beta = 3$, with volumes up to 320^3), and the 4d case (bottom, at $\beta = 2.2$ with volumes up to 128^4). Data are taken from Ref. [15] for the 3d and the 4d cases. We also show (in 2d and in 4d) the inverse of the smallest nonzero eigenvalue $\lambda_{\min}$ of the FP matrix (in GeV $^{-2}$). In these two cases one can verify the inequality $1/\lambda_{\min} \geq G(p_s)$. The fitting parameters are reported in Table III. Note that we did not evaluate $\lambda_{\min}$ for all our configurations.

TABLE III. Fits of $G(p_s)$ and of $1/\lambda_{\min}$, respectively, using the Ansätze $bL^{2+2\kappa}$ and cL^α . In the 4d case the fit for $1/\lambda_{\min}$ has been done by considering only the three largest physical volumes. When considering the three smallest physical volumes we find $\alpha = 0.9(3)$. In the other cases all data have been considered for the fit.

d	b	$2+2\kappa$	c	α
2	0.026(1)	2.251(9)	0.12(3)	2.20(4)
3	0.11(3)	2.02(4)		
4	0.086(3)	2.043(8)	1.2(1)	1.53(2)

FIG. 2. The ghost dressing function $p^2 G(p^2)$ as a function of p^2 (in GeV) for the 2d case (top, at $\beta = 10$ with volume 320^2), 3d case (center, at $\beta = 3$, with volume 240^3), and the 4d case (bottom, at $\beta = 2.2$ with volume 80^4). The fitting functions and the corresponding fitting parameters are reported in Table V.

TABLE IV. Fits of $p^2 G(p^2)$ using the Ansätze $ap^{-2\kappa}$ in the 2d case and $a - b \log(1 + cp^2)$ in the 3d and 4d cases. In the first case we used the data in the range $p^2 \in [0, 0.5]$ for the fit. In the other two cases the data in the range $p^2 \in [0, 1]$ have been considered.

d	a	κ	b	c
2	1.134(7)	0.177(2)		
3	4.7(1)		0.579(5)	320(20)
4	4.28(1)		0.69(2)	33(3)

TABLE V. Fits of $p^2 G(p^2)$ using the Ansätze $a(p^{-2\kappa} + bp^{2e})/(1 + p^{2e})$ in the 2d case and $a - b[\log(1 + cp^2) + dp^2]/(1 + p^2)$ in the 3d and 4d cases. In the three cases we used the whole range of momenta for the fit.

d	a	κ	b	e	c	d
2	1.24(3)	0.16(2)	0.86(3)	0.75(15)		
3	4.75(1)		0.491(5)		450(30)	7.1(1)
4	4.32(2)		0.38(1)		80(10)	8.2(3)

we obtain $\kappa \approx 0.1$. We note that our result in the 2d case is essentially in agreement with Ref. [8], where however the value of κ is obtained by first fitting $G(p)$ in an appropriate momentum range for each lattice volume and then performing the extrapolation to infinite volume by a four-parameter fit. The fits for $\lambda_{\min}$ and $G(p_s)$ as a function of L are usually simpler, as shown above.

These results are confirmed if one considers the dressing function $p^2 G(p^2)$ at the largest lattice volumes for the three cases (see Fig. 2). Indeed, the data in the 2d case can be fitted using a power-law Ansatz $ap^{-2\kappa}$ (see Table IV), with $\kappa = 0.177(2)$, essentially in agreement with [8]. The fitted value for κ decreases if one considers fewer points, dropping points with larger p , and becomes 0.152(7) when considering data with $p^2 \in [0, 0.1]$. The same Ansatz does not work in 3d and in 4d. In these cases the data for the dressing function can be described by the Ansatz $a - b \log(1 + cp^2)$, inspired by the logarithmic corrections suggested in Ref. [19]. In both cases one finds (approximately) $p^2 G(p^2) \rightarrow 4.5$ in the limit $p \rightarrow 0$. This result supports the value $\kappa \approx 0$ obtained above, when considering the bound using $\lambda_{\min}$. It is interesting that one can also obtain a relatively good fit for the dressing function in the whole range of momenta by considering the fitting function $a(p^{-2\kappa} + bp^{2e})/(1 + p^{2e})$ in the 2d case and $a - b[\log(1 + cp^2) + dp^2]/(1 + p^2)$ in 3d and in 4d. These fits and the corresponding parameters are reported in Fig. 2 and in Table V.

IV. CONCLUSIONS

Our data suggest that the Landau gauge ghost propagator does not diverge faster than $1/p^2$ at small momenta in 3d and in 4d, while in 2d $G(p)$ does not diverge faster than $p^{-2-2\kappa}$ with κ between 0.1 and 0.2, in agreement with

Ref. [8]. These results have been obtained by considering the inequality in Eq. (7) and by fitting the data for $G(p_s)$ and $\lambda_{\min}$ as a function of L . In particular, we note that the use of the upper bound to constrain $G(p)$ is quite convenient, since $\lambda_{\min}$ does not depend on p . It might thus be of interest to optimize the evaluation of $\lambda_{\min}$ and of $\psi_{\min}(a, x)$.

The results obtained here in the 3d and 4d cases, together with those reported in Ref. [21] for the gluon propagator, seem to contradict the Gribov-Zwanziger confinement scenario for Landau gauge. On the other hand, one should establish if the behavior of gluon and ghost propagators at very small momenta is really so essential for the explanation of confinement. After all, when dynamical quarks are considered [23], string breaking should manifest itself at a scale of about a fermi, i.e. for an energy scale of about 200 MeV. Thus, for the physics of hadrons the behavior of the propagators at intermediate momenta is probably more important. Let us recall that for a space-time separation of about 1 F the transverse gluon propagator violates reflection positivity [24], i.e. becomes negative in real-space

coordinates, and the exponent κ for the ghost propagator is about 0.2–0.3 considering $p \sim 0.5$ GeV [9,12,13,15–17]. Thus, in the range 200–500 MeV, nonperturbative effects are clearly present in the behavior of the gluon and of the ghost propagator. For this range of momenta the use of our lower bound for the ghost propagator could probably also be valuable. Of course, the open problem is if these effects suffice to explain color confinement.

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4.4 Volume Infinito e Geometria da Região de Gribov

Na Ref. [149], já considerada na seção anterior, foi também apresentada uma nova desigualdade para o autovalor λ_{min} , escrita em função do menor momento não-nulo p_{min} e de um parâmetro que descreve a geometria da primeira região de Gribov Ω . Este resultado é baseado na concavidade da função mínimo (ver, por exemplo, a Seção 12.4 na Ref. [150]) e nas três propriedades importantes da região Ω descritas na Seção 2.1 acima. De fato, considerando uma configuração de gauge A' que pertence ao primeiro horizonte de Gribov $\partial\Omega$ e indicando com $\lambda_{min}[\mathcal{M}]$ o menor autovalor não-nulo da matriz de Faddeev-Popov $\mathcal{M}$, podemos escrever

$$\lambda_{min}[\mathcal{M}[\rho A']] = \lambda_{min}[(1 - \rho)(-\Delta) + \rho \mathcal{M}[A']] , \quad (4.29)$$

com $\rho \in \mathfrak{R}$. Pela segunda propriedade de Ω sabemos que $\rho A' \in \Omega$ para $\rho \in [0, 1]$. Ao mesmo tempo, temos que

$$\lambda_{min}[(1 - \rho)(-\Delta) + \rho \mathcal{M}[A']] = \min_{\chi \neq \text{constant}} (\chi, [(1 - \rho)(-\Delta) + \rho \mathcal{M}[A']] \chi) , \quad (4.30)$$

onde os vetores $\chi(b, x)$ são normalizados, i.e. $(\chi, \chi) = 1$. Logo, usando a concavidade da função mínimo [150] e os resultados

$$\lambda_{min}[\mathcal{M}[A']] = 0 \quad (4.31)$$

$$\lambda_{min}[\mathcal{M}[-\Delta]] = p_{min}^2 , \quad (4.32)$$

obtemos

$$\begin{aligned} \lambda_{min}[\mathcal{M}[\rho A']] &= \min_{\chi \neq \text{constant}} (\chi, [(1 - \rho)(-\Delta) + \rho \mathcal{M}[A']] \chi) \\ &\geq (1 - \rho) \min_{\chi \neq \text{constant}} (\chi, (-\Delta) \chi) + \rho \min_{\chi \neq \text{constant}} (\chi, \mathcal{M}[A'] \chi) \\ &= (1 - \rho) \lambda_{min}[-\Delta] + \rho \lambda_{min}[\mathcal{M}[A']] \\ &= (1 - \rho) p_{min}^2 . \end{aligned} \quad (4.34)$$

Assim, no limite de volume infinito, o autovalor $\lambda_{min}[\mathcal{M}[\rho A']]$ não pode decrescer mais rapidamente do que o produto $(1 - \rho) p_{min}^2$. Isto implica que λ_{min} vai para zero como

$N^{-2-\alpha}$, com $\alpha > 0$, somente se a grandeza $1 - \rho$ também vai para zero pelo menos como $N^{-\alpha}$. É interessante observar que no caso abeliano temos $\mathcal{M} = -\Delta$ e $\lambda_{min} = p_{min}^2$. Assim, todos os efeitos não-abelianos da teoria são incluídos no fator $(1 - \rho)$. É importante ressaltar que o resultado acima se aplica a qualquer cópia¹¹ de Gribov $A = \rho A' \in \Omega$ e que a nova desigualdade pode também ser escrita como

$$\lambda_{min} [\mathcal{M}[A]] \geq [1 - \rho(A)] p_{min}^2, \quad (4.35)$$

onde $\rho(A) \leq 1$ é a medida da distância entre a configuração $A \in \Omega$ e a “origem” $A = 0$, com $\rho^{-1}A \in \partial\Omega$.

Um aspecto importante do resultado (4.35) acima é que o limite inferior depende de p_{min}^2 . Desta forma, usando a desigualdade na Eq. (4.25), podemos escrever

$$G(A, p) \leq \frac{1}{\lambda_{min}(A)} \leq \frac{1}{[1 - \rho(A)] p_{min}^2}, \quad (4.36)$$

onde foi evidenciado que todas as grandezas foram calculadas para a configuração de gauge A . Assim, para obter um propagador de fantasmas $G(A, p)$ fortalecido no limite infravermelho, uma condição necessária é que o fator $1 - \rho(A)$ vá a zero de forma suficientemente rápida no limite de volume infinito. Além disso, escrevendo o propagador de fantasmas como

$$G(A, p) = \frac{1}{p^2} \frac{1}{1 - \sigma(A, p)}, \quad (4.37)$$

onde $\sigma(A, p)$ é o chamado fator de forma de Gribov, obtemos para $p = p_{min}$ a desigualdade

$$\sigma(A, p_{min}) \leq \rho(A) \quad (4.38)$$

[com $\rho(A) \leq 1$]. Relembramos que a restrição da integral funcional (ou de trajetória) — que define a função de partição $\mathcal{Z}$ da teoria — à região Ω é em geral obtida impondo a chamada *no-pole condition* [6]

$$\sigma(A, p) < 1 \quad \text{for} \quad p^2 > 0. \quad (4.39)$$

A desigualdade (4.38), que deve ser satisfeita para qualquer configuração de gauge $A \in \Omega$, representa então uma versão mais forte da condição de ausência de polos (4.39).

¹¹Claramente, cópias de Gribov diferentes serão em geral caracterizadas por valores diferentes do parâmetro ρ .

Figura 4.3: O inverso do propagador de fantasmas $1/G(p_{min}^2)$ (triângulos), calculado para o menor momento não nulo na rede p_{min} , o menor autovalor não nulo λ_{min} (quadrados) e o limite inferior na Eq. (4.35) (círculos) como função do inverso do lado de rede. Para cada grandeza é também mostrado o ajuste usando a função c/N^η . Todas as grandezas são em unidades de rede. Para esta figura foi usado o conjunto de cópias de Gribov indicado na Ref. [149] como *first copy* (*fc*). Observe a escala logarítmica nos dois eixos. As barras de erro (que não são visíveis) correspondem a um desvio padrão. (Na análise dos dados foram considerados somente os erros estatísticos.) Este gráfico é apresentado na Figura 11 da Ref. [149].

Claramente, os resultados acima podem também ser escritos como

$$1/G(A, p_{min}) \geq \lambda_{min} [\mathcal{M}[A]] \geq [1 - \rho(A)] p_{min}^2. \quad (4.40)$$

Estas desigualdades foram verificadas numericamente na Ref. [149] (ver Fig. 4.3). Como já mostrado na seção anterior, fica claro que λ_{min} se aproxima de $1/G(p_{min}^2)$ no limite de volume infinito. Ao contrário, o limite inferior $(1 - \rho) p_{min}^2$ decresce mais rapidamente do que $1/G(p_{min}^2)$ e λ_{min} . De fato, usando a função c/N^η para fazer um ajuste dos dados, foram obtidos para o expoente η os valores $\eta \approx 3.4$ para o limite inferior $(1 - \rho) p_{min}^2$, $\eta \approx 2.0$ para o inverso do propagador e $\eta \approx 1.3$ para λ_{min} . Os dados também mostram que o valor do parâmetro ρ é em geral muito próximo de 1, até para redes de tamanho relativamente pequeno, i.e. a maioria das configurações obtidas em simulações numéricas está localizada muito próxima do primeiro horizonte de Gribov $\partial\Omega$. É importante ressaltar que a desigualdade (4.35) se torna uma igualdade somente quando os autovetores correspondentes aos menores autovalores não-nulos das matrizes $-\Delta$ e $\mathcal{M}[A']$ são iguais. Assim, o resultado $\lambda_{min} \gg (1 - \rho) p_{min}^2$ confirma mais uma vez que o autovetor ψ_{min} é muito diferente das ondas planas de momento p_{min} . Ao mesmo tempo, isto fornece uma clara indicação de por que a chamada solução conforme das EDS [65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82] não é obtida nas simulações de rede (pelo menos em três e quatro dimensões), i.e. fornece uma primeira resposta para a pergunta formulada na Seção 4.1 acima. De fato, parece que o fortalecimento do propagador de fantasmas para pequenos momentos poderia acontecer se fosse $\lambda_{min} \approx (1 - \rho) p_{min}^2$, i.e. se ψ_{min} tivesse uma projeção grande nas ondas planas de momento p_{min} . Neste caso λ_{min} tenderia a zero, no limite de volume infinito, mais rapidamente do que $1/N^2$ e o limite inferior na Eq. (4.25) implicaria um propagador de fantasmas fortalecido na região infravermelha. Isto nos leva à singular conclusão de que os efeitos não-perturbativos da QCD no gauge de Landau estariam relacionados a configurações de gauge cuja matriz de Faddeev-Popov $\mathcal{M}$ é caracterizada por um autovetor ψ_{min} muito próximo de uma onda plana, i.e. próximo ao autovetor ψ_{min} do caso livre! Podemos concluir que as condições necessárias para produzir um propagador de fantasmas fortalecido no limite de pequenos momentos são tão peculiares que parece muito difícil a sua realização em simulações na

rede. Dito de outra forma, os resultados obtidos indicam que a solução massiva é, sob vários pontos de vista¹², a mais natural e que a chamada solução conforme representa uma solução *sui generis* que pode ser obtida somente em condições muito específicas.

4.5 Propagador de Glúons no Contínuo para $d = 2$

Como evidenciado no final das Seções 4.2 e 4.4, o resultado $D(0) = 0$ obtido para o propagador de glúons em duas dimensões é bastante peculiar. Além disso, o fato de os dados numéricos (no gauge de Landau) estarem de acordo, em três e em quatro dimensões, com a chamada solução massiva das EDS [83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94] — ao passo que, em duas dimensões, os dados preferem a chamada solução de escala [65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82] — representa um interessante problema teórico em aberto. Que o comportamento físico de um sistema possa depender da sua dimensionalidade é um fato bem conhecido, por exemplo no estudo dos fenômenos críticos em mecânica estatística. Porém, seria importante entender a razão das diferentes soluções achadas nas simulações numéricas para os propagadores de glúons e de fantasmas em duas, três e quatro dimensões.

No estudo das EDS as duas soluções aparecem quando são usadas diferentes condições de contorno, i.e. diferentes valores para a função de renormalização $F(p^2) = p^2 G(p^2)$ para um dado momento p [151, 152]. Em particular, considerando $p = 0$, é claro que $1/F(0) = 0$ fornece um propagador de fantasmas fortalecido no infravermelho, enquanto a condição $1/F(0) > 0$ proporciona um comportamento para $G(p^2)$ a pequenos momentos do tipo propagador livre. Assim, do ponto de vista matemático, a origem das duas soluções é bem clara. Além disso, a possibilidade de se obter uma das duas soluções não parece depender¹³ da dimensão física considerada [78]. Em particular, no estudo das EDS, a

¹²Ver também o comentário no final da Seção 4.2.

¹³Mais recentemente, na Ref. [153] foi argumentado, usando o grupo de renormalização, que em três e quatro dimensões somente a solução massiva poderia ser realizada fisicamente.

condição de escala $1/F(0) = 0$ é obtida graças a um cancelamento peculiar que acontece na EDS do campo fantasma quando a constante de acoplamento g^2 possui um valor “crítico” específico g_c^2 [154, 155]. Ao contrário, para valores $g^2 < g_c^2$ é encontrada a solução massiva.¹⁴ Deste ponto de vista, a escolha da solução dependeria do valor “físico” para a constante de acoplamento g^2 em 2, 3 e 4 dimensões. Por exemplo, no caso $d = 4$ para a teoria de Yang-Mills com o grupo de gauge SU(3) [156], o valor de g^2 no “mundo real” parece ser menor do que o valor de g_c^2 , gerando propagadores com um comportamento massivo no limite infravermelho. Estas observações, porém, ainda não diferenciam de forma substancial o caso $d = 2$ dos casos $d = 3$ e 4.

O trabalho publicado na Ref. [157] (ver artigo anexado no final desta seção), resultado de uma colaboração com o Departamento de Física da Universidade de Ghent na Bélgica, tentou acrescentar mais resultados para a explicação e o entendimento dos dados numéricos. Em particular, foram investigadas propriedades gerais do fator de forma de Gribov $\sigma(p^2)$, definido na Eq. (4.37) acima, para teorias de Yang-Mills SU(N_c) em dimensão d . Neste caso o trabalho foi inspirado pela Ref. [98], na qual foi mostrado que para a solução de Gribov-Zwanziger Refinada (descrita na Seção 2.3 acima) o fator de forma $\sigma(p^2)$, calculado a um laço, diverge logaritmicamente para $p \rightarrow 0$ no caso $d = 2$, e pela Ref. [60], na qual foi mostrado (no Apêndice B) que a função $\sigma(p^2)$ é monotonicamente decrescente, para $d = 2$. Estes dois resultados foram de fato estendidos e generalizados na Ref. [157]. Mais especificamente, foi demonstrado que para qualquer propagador de glúons $D(p^2)$ (suficientemente regular) a função $\sigma(p^2)$, calculada a um laço, diverge no limite infravermelho como $-D(0) \ln(p^2)$ no caso de dimensão dois. Assim, para $d = 2$, a condição de ausência de polos $\sigma(p^2) < 1$ (para $p^2 > 0$) pode ser satisfeita somente se o propagador de glúons for nulo a momento zero. Pelo contrário, para $d = 3$ e 4, $\sigma(p^2)$ permanece finito mesmo com $D(0) > 0$. Este foi o principal resultado de nosso trabalho: no caso bidimensional a solução massiva não é consistente com a limitação da integração funcional à região Ω , fornecendo assim uma explicação simples dos resultados encontrados

¹⁴Para valores $g^2 > g_c^2$ o propagador de fantasmas torna-se negativo para pequenos momentos e é violada a condição de ausência de polos (4.39), necessária para limitar a integração funcional à primeira região de Gribov Ω .

nas simulações na rede.¹⁵ Ao mesmo tempo, foi possível demonstrar que, para todas as dimensões $d \geq 2$, a função $\sigma(p^2)$ (a um laço) atinge o seu valor máximo para $p = 0$. Estes resultados foram também confirmados calculando explicitamente o propagador de fantasmas $G(p^2)$ a um laço, usando para o propagador de glúons as funções de ajuste descritas na Seção 3.4 para $d = 2, 3$ e 4 e o grupo de gauge $SU(2)$. Neste caso foi também obtido que, considerando a constante de acoplamento g^2 como um parâmetro livre, o propagador $G(p^2)$ apresenta uma família a um parâmetro de soluções, de acordo com a Ref. [155]. Para esta família de soluções, a condição $\sigma(0) \leq 1$ implica a condição $g^2 \leq g_c^2$, onde g_c^2 é um valor crítico. Ao mesmo tempo, foi possível verificar que, no limite de momentos pequenos, o propagador de fantasmas apresenta um comportamento tipo propagador livre quando g^2 é menor do que g_c^2 e fica fortalecido para $g^2 = g_c^2$. Finalmente, na última parte do trabalho, foi considerada a EDS para o fator de forma de Gribov $\sigma(p^2)$. Neste caso, sem tentar resolver a EDS e considerando vértices de fantasma-glúon finitos no limite infravermelho, foi possível encontrar (em $d = 2, 3, 4$) limites inferior e superior simples para $\sigma(p^2)$. Usando estas desigualdades foi então verificado novamente que a imposição da condição $\sigma(p^2) < 1$ implica o resultado $D(0) = 0$ para o caso bidimensional. Vale também ressaltar que todos os resultados obtidos na Ref. [157] se aplicam a qualquer cópia de Gribov que pertença à região Ω . Em particular, o resultado $D(0) = 0$ não depende do chamado ruído de Gribov.

Do ponto de vista técnico, os principais resultados descritos acima foram obtidos usando as propriedades da função hipergeométrica de Gauss ${}_2F_1(a, b; c; z)$, descritas no Apêndice B da Ref. [157]. Mais especificamente, estas propriedades permitiram a consideração de várias desigualdades que deixaram mais simples as demonstrações desenvolvidas no nosso trabalho. A função ${}_2F_1(a, b; c; z)$ é obtida na integral angular que aparece na expressão a um laço para $G(p^2)$, i.e.

$$\int \frac{1 - \cos^2(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu} d\Omega_d = \frac{\Omega_d}{q^2} \frac{d-1}{d} {}_2F_1\left(\nu, \nu - d/2; 1 + d/2; p^2/q^2\right), \quad (4.41)$$

válida para $p^2 < q^2$. (O resultado para $q^2 < p^2$ é obtido trocando p^2 com q^2 no lado direito

¹⁵Esta conclusão foi recentemente confirmada na Ref. [64], usando uma abordagem diferente (ver discussão na Seção 2.2 acima).

da expressão acima.) Aqui indicamos com $d\Omega_d$ a integração angular em coordenadas hiperesféricas¹⁶ r, ϕ_j (com $j = 1, 2, \dots, d-1$):

$$\begin{aligned}
 x_1 &= r \cos(\phi_1) , \\
 x_2 &= r \sin(\phi_1) \cos(\phi_2) , \\
 x_3 &= r \sin(\phi_1) \sin(\phi_2) \cos(\phi_3) , \\
 &\dots \\
 x_{d-1} &= r \sin(\phi_1) \sin(\phi_2) \dots \sin(\phi_{d-2}) \cos(\phi_{d-1}) , \\
 x_d &= r \sin(\phi_1) \sin(\phi_2) \dots \sin(\phi_{d-2}) \sin(\phi_{d-1}) ,
 \end{aligned} \tag{4.42}$$

com $r \in [0, \infty)$, $\phi_i \in [0, \pi]$ para $i = 1, 2, \dots, d-2$ e $\phi_{d-1} \in [0, 2\pi)$.

¹⁶Note-se que com esta notação o caso tridimensional usual corresponde a $x_1 = z$, $x_2 = x$ and $x_3 = y$.

No-pole condition in Landau gauge: Properties of the Gribov ghost form factor and a constraint on the $2d$ gluon propagator

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We study general properties of the Landau-gauge Gribov ghost form factor $\sigma(p^2)$ for $SU(N_c)$ Yang-Mills theories in the d -dimensional case. We find a qualitatively different behavior for $d = 3, 4$ with respect to the $d = 2$ case. In particular, considering any (sufficiently regular) gluon propagator $\mathcal{D}(p^2)$ and the one-loop-corrected ghost propagator, we prove in the $2d$ case that the function $\sigma(p^2)$ blows up in the infrared limit $p \rightarrow 0$ as $-\mathcal{D}(0)\ln(p^2)$. Thus, for $d = 2$, the no-pole condition $\sigma(p^2) < 1$ (for $p^2 > 0$) can be satisfied only if the gluon propagator vanishes at zero momentum, that is, $\mathcal{D}(0) = 0$. On the contrary, in $d = 3$ and 4 , $\sigma(p^2)$ is finite also if $\mathcal{D}(0) > 0$. The same results are obtained by evaluating the ghost propagator $\mathcal{G}(p^2)$ explicitly at one loop, using fitting forms for $\mathcal{D}(p^2)$ that describe well the numerical data of the gluon propagator in two, three and four space-time dimensions in the $SU(2)$ case. These evaluations also show that, if one considers the coupling constant g^2 as a free parameter, the ghost propagator admits a one-parameter family of behaviors (labeled by g^2), in agreement with previous works by Boucaud *et al.* In this case the condition $\sigma(0) \leq 1$ implies $g^2 \leq g_c^2$, where g_c^2 is a “critical” value. Moreover, a freelike ghost propagator in the infrared limit is obtained for any value of g^2 smaller than g_c^2 , while for $g^2 = g_c^2$ one finds an infrared-enhanced ghost propagator. Finally, we analyze the Dyson-Schwinger equation for $\sigma(p^2)$ and show that, for infrared-finite ghost-gluon vertices, one can bound the ghost form factor $\sigma(p^2)$. Using these bounds we find again that only in the $d = 2$ case does one need to impose $\mathcal{D}(0) = 0$ in order to satisfy the no-pole condition. The $d = 2$ result is also supported by an analysis of the Dyson-Schwinger equation using a spectral representation for the ghost propagator. Thus, if the no-pole condition is imposed, solving the $d = 2$ Dyson-Schwinger equations cannot lead to a massive behavior for the gluon propagator. These results apply to any Gribov copy inside the so-called first Gribov horizon; i.e., the $2d$ result $\mathcal{D}(0) = 0$ is not affected by Gribov noise. These findings are also in agreement with lattice data.

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I. INTRODUCTION

Green functions of Yang-Mills theories are gauge-dependent quantities. They can, however, be used as a starting point for the evaluation of hadronic observables (see, for example, [1–4]). Thus, the study of the infrared (IR) behavior of propagators and vertices is an important step in our understanding of QCD. In particular, the confinement mechanism for color charges [5] could reveal itself in the IR behavior of (some of) these Green functions. This IR behavior should also be relevant for the description of the deconfinement transition and of the deconfined phase of QCD. Indeed, at high temperature color charges are expected to be Debye-screened and the (electric and magnetic) screening masses should be related to the IR behavior of the gluon propagator (see, for example, [6–8]).

Among the gauge-fixing conditions employed in studies of Yang-Mills Green functions, a very popular choice is the

Landau gauge, which in momentum space reads $p_\mu A_\mu^b(p) = 0$. From the continuum perspective this gauge has various important properties, including its renormalizability, various associated nonrenormalization theorems [9] and a ghost-antighost symmetry [1]. Moreover, since it can be easily simulated on the lattice—a feature not shared with many other gauge conditions—analytic studies carried on in the Landau gauge can be compared to the nonperturbative lattice results. In the past few years many analytic studies of Green functions in the Landau gauge have focused on the solution of the Yang-Mills Dyson-Schwinger equations (DSEs), which are the exact quantum equations of motion of the theory (see, for example, [1,2,10,11]). Since the DSEs are an infinite set of coupled equations, any attempt of solving them requires a truncation scheme. Then, some Green functions (usually the gluon and the ghost propagators) are obtained self-consistently from the considered equations, while all the other Green functions entering the equations are given as an input.

For the coupled DSEs of gluon and ghost propagators two solutions have been extensively analyzed (see, for example, Chap. 10 in Ref. [5] and Ref. [12] for recent

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short reviews). The scaling solution [1,13–16] finds in $d = 2, 3$ and 4 an IR-enhanced ghost propagator $\mathcal{G}(p^2)$ and a vanishing gluon propagator $\mathcal{D}(p^2)$ at zero momentum.¹ In particular, the IR behavior of the two propagators should be given, respectively, by $\mathcal{G}(p^2) \sim (p^2)^{-\kappa_G-1}$ and by $\mathcal{D}(p^2) \sim (p^2)^{2\kappa_D+(2-d)/2}$ with $\kappa_G = \kappa_D \approx 0.2(d-1)$ [14,16]. On the other hand, the massive solution [18–26] gives (for $d = 3$ and 4) a freelike ghost propagator in the IR limit, i.e., $\kappa_G = 0$, and a massive behavior for the gluon propagator,² that is, $\mathcal{D}(0) > 0$ and $\kappa_D = (d-2)/4$.

An intriguing possibility is that the two different types of solution for the coupled gluon and ghost DSEs could be related to the use of different auxiliary *boundary conditions*.³ These conditions can be given in terms of the value of the ghost dressing function $\mathcal{F}(p^2) = p^2 \mathcal{G}(p^2)$ at a given momentum scale p [22,31]. In particular, if one considers $p = 0$, it is clear that $1/\mathcal{F}(0) = 0$ gives an IR-enhanced ghost propagator $\mathcal{G}(p^2)$ while $1/\mathcal{F}(0) > 0$ yields a freelike behavior for $\mathcal{G}(p^2)$ at small momenta. As stressed in Ref. [22] (see also the discussion in Sec. 4.2.2 of Ref. [12]), the scaling condition $1/\mathcal{F}(0) = 0$ relies on a particular cancellation in the ghost DSE which, in turn, implies a specific “critical” value g_c^2 for the coupling constant g^2 [12,32]. Thus, at least from the mathematical point of view, there is a one-parameter family of solutions for the gluon and ghost DSEs, labeled by g^2 or, equivalently, by $1/\mathcal{F}(0)$: in the case $g^2 = g_c^2$ one recovers the scaling solution while, for all cases $g^2 < g_c^2$, the solution is a massive one.⁴ In 4d, the SU(3) physical value of the coupling seems to select⁵ the massive solution [21,34].

At this point we should recall that, when considering gauge-dependent quantities in non-Abelian gauge theories, one has to deal with the existence of Gribov copies [35] (see also Ref. [36] for a recent review). Indeed, for compact non-Abelian Lie groups defined on the 4-sphere [37] or on the 4-torus [38], it is impossible to find a continuous choice of one (and only one) connection $A_\mu(x)$ on each gauge orbit. The effect of Gribov copies is not seen in perturbation theory [35]; i.e., the usual Faddeev-Popov-quantization procedure is correct at the perturbative level.

However, these copies could be relevant at the non-perturbative level, i.e., in studies of the IR properties of Yang-Mills theories.

Different approaches have been proposed in order to quantize a Yang-Mills theory while taking into account the existence of Gribov copies (see, for example, [35,39–45]). The one usually considered, both in the continuum and on the lattice, is based on restricting the functional integration to a subspace of the hyperplane of transverse configurations. The original proposal, made by Gribov [35], was based on the observation that the Landau-gauge condition $\partial_\mu A_\mu^b(x) = 0$ allows for (infinitesimally) gauge-equivalent configurations if the Landau-gauge Faddeev-Popov operator $\mathcal{M}^{bc}(x, y) = -\delta(x-y)\partial_\mu D_\mu^{bc}$ has zero modes. (Here D_μ^{bc} is the usual covariant derivative.) Indeed, since an infinitesimal gauge transformation $\delta\omega(x)$ gives $A_\mu^b(x) \rightarrow A_\mu^b(x) + D_\mu^{bc}\omega^c(x)$, it is clear that the exclusion (in the path-integral measure) of the zero modes of $\mathcal{M}^{bc}(x, y)$ implies that gauge copies connected by such infinitesimal gauge transformations are ignored in the computation of expectation values. In order to exclude these zero modes, Gribov considered a stronger condition by requiring that the functional integration be restricted to the region Ω of gauge configurations $A_\mu^b(x)$ defined as

$$\Omega \equiv \{A_\mu^b(x): \partial_\mu A_\mu^b(x) = 0, \mathcal{M}^{bc}(x, y) > 0\}. \quad (1)$$

This set, known as the (first) Gribov region, clearly includes the vacuum configuration $A_\mu^b(x) = 0$, for which the Faddeev-Popov operator is given by $-\delta(x-y)\delta^{bc}\partial_\mu^2$. The region Ω can also be defined (see, for example, [46,47]) as the whole set of local minima⁶ of the functional $\mathcal{E}[A] = \int d^d x A_\mu^b(x) A_\mu^b(x)$. Since usually each orbit allows for more than one local minimum of $\mathcal{E}[A]$, it is clear that the region Ω is not free of Gribov copies. On the contrary, in the interior of the so-called fundamental modular region Λ , given by the set of the absolute minima of the functional $\mathcal{E}[A]$, no Gribov copies occur [43,48].

The characterization of the fundamental modular region Λ , i.e., finding the absolute minima of the *energy* functional $\mathcal{E}[A]$, is a problem similar to the determination of the ground state of a spin glass system [49,50]. Thus, a local formulation of a Yang-Mills theory, with the functional measure delimited to Λ , is not available, whereas a practical way of restricting the physical configuration space to the region Ω was introduced by Gribov [35]. To this end, he required that the ghost dressing function $\mathcal{F}(p^2)$ cannot have a pole at finite nonzero momenta. After setting⁷

¹For the explanation of color confinement based on the scaling solution see, for example, Sec. 3.4 in [17] and references therein.

²The possible existence of a dynamical mass for the gluons, as well as its relation to quark confinement through vortex condensation, was discussed a long time ago in Ref. [27].

³The existence of different nonperturbative solutions for DSEs in relation to different boundary conditions has been discussed, for example, in Refs. [28,29] (see also Sec. 1.4 in [30] and Secs. 3 and 3.1 in Ref. [1]).

⁴If $g^2 > g_c^2$ one gets a negative ghost propagator at small momenta.

⁵Recently, in Ref. [33], it has also been shown—using a renormalization-group approach—that in three and four space-time dimensions only the decoupling solution is expected to be physically realized.

⁶For this reason this gauge condition is often indicated in the literature as *minimal Landau gauge*.

⁷In some works (see, for example, [51,52]) one encounters the parametrization $1/\mathcal{F}(p^2) = 1 + G(p^2) + L(p^2)$, where $G(p^2)$ corresponds to the so-called Kugo-Ojima function $u(p^2)$ [53,54] and $L(p^2)$ is a quantity presumably vanishing at $p^2 = 0$ [55]. In comparison with our notation, this implies $G(p^2) + L(p^2) = -\sigma(p^2)$ and $G(0) = -\sigma(0)$.

$$\mathcal{G}(p^2) = \frac{\mathcal{F}(p^2)}{p^2} = \frac{1}{p^2} \frac{1}{1 - \sigma(p^2)}, \quad (2)$$

this condition can be written as

$$\sigma(p^2) < 1 \quad \text{for } p^2 > 0, \quad (3)$$

where $\sigma(p^2)$ is the so-called Gribov ghost form factor [35]. Indeed, since the ghost propagator is given by

$$\mathcal{G}(p^2) = \frac{\delta^{bc}}{N_c^2 - 1} \langle p | (\mathcal{M}^{-1})^{bc} | p \rangle, \quad (4)$$

i.e., it is related to the inverse of the Faddeev-Popov matrix $\mathcal{M}^{bc}(x, y)$, the above inequality—known as the no-pole condition—should be equivalent to the restriction of the functional integration to the Gribov region Ω , i.e., to the condition $\mathcal{M}^{bc}(x, y) > 0$. One should also note that a Gribov form factor $\sigma(p^2)$ that is larger than 1 in some range of momenta p would correspond to a ghost propagator with a tachyonic mass, but tachyon propagators are usually avoided since they signal some type of vacuum instability.⁸ From this point of view, the restriction of the functional integration to the first Gribov horizon avoids the tachyon problem from the beginning.

From the discussion above, it is clear that both scaling and massive solutions of DSEs satisfy the no-pole condition, i.e., $1/\mathcal{F}(p^2) = 1 - \sigma(p^2) > 0$ for $p^2 > 0$. Indeed, in the scaling case [1, 13–16], this condition [together with the condition $1/\mathcal{F}(0) = 0$] is imposed from the beginning to the solution of the DSEs. On the contrary, for the massive solution, the no-pole condition is either verified *a posteriori*, as in Ref. [20], or used [together with the condition $1/\mathcal{F}(0) > 0$] as an input for the solution of the DSEs, as in Refs. [19, 25, 26]. In particular, in Ref. [26], the value of $1/\mathcal{F}(0)$ is fixed using lattice data.

The restriction to the first Gribov region Ω is also always implemented in lattice numerical simulations of Green functions in the Landau gauge by (numerically) finding local minima of the functional $\mathcal{E}[A]$. Results obtained using very large lattice volumes [56–59] (see also Chap. 10 in Ref. [5], Sec. 3 in Ref. [12] and Ref. [60] for recent short reviews) have shown that in $d = 3$ and 4 the gluon propagator $\mathcal{D}(p^2)$ is finite and nonzero in the limit $p \rightarrow 0$ while the ghost propagator $\mathcal{G}(p^2)$ behaves as $1/p^2$. On the contrary, for $d = 2$ the lattice data [61–64] are in quantitative agreement with the scaling solution and one finds $\kappa_D = \kappa_G \approx 0.2$.

Since the region Ω is not free of Gribov copies, their (possible) influence on the numerical evaluation of gluon and ghost propagators has been studied by various groups [65–70]. It has been found that these effects are usually observable only in the IR limit and that any attempt to

restrict the functional integration to the fundamental modular region Λ gives a stronger suppression at small momenta for both propagators, i.e., reducing the value of $\mathcal{D}(0)$ and increasing that of $1/\mathcal{F}(0)$. More recently, it has been suggested [71–74] that the one-parameter family of solutions obtained for the gluon and ghost DSEs should be related⁹ to Gribov-copy effects and that the value of $1/\mathcal{F}(0)$ could be used as a gauge-fixing parameter. This analysis finds indeed IR-enhanced ghost propagators (and sometimes a disconcerting overscaling¹⁰). On the other hand, the gluon propagator still shows a finite nonzero value at zero momentum, that is, $\mathcal{D}(0) > 0$. Moreover, this approach does not explain why the numerical results found in $d = 2$ are different from those obtained in $d = 3$ and 4, even though Gribov copies inside the first Gribov region Ω are clearly present in any space-time dimension $d > 1$.

From the analytical point of view, following Gribov's approach, Zwanziger modified the usual Yang-Mills action in order to restrict the path integral to the first Gribov region Ω [75]. Although this restriction is obtained using a nonlocal term, the Gribov-Zwanziger (GZ) action¹¹ can be written as a local action and it is proven [76–78] to be renormalizable. At tree level the GZ gluon propagator is given by $\mathcal{D}(p^2) = p^2/(p^4 + \lambda^4)$, where λ is a parameter with mass-dimension 1. At the same time, the ghost propagator is given by $\mathcal{G}(p^2) \sim 1/p^4$. Thus, as in the scaling solution of the gluon and ghost DSEs, the gluon propagator is null at zero momentum¹² and the ghost propagator is IR-enhanced [42]. These tree-level results, also in agreement with the original work by Gribov [35], have been confirmed by one-loop calculations in the three- and four-dimensional cases [81–85].

More recently, the GZ action has been modified by considering (for $d = 3$ and 4) dimension-two condensates [86–89]. The corresponding action, called the refined Gribov-Zwanziger (RGZ) action, still imposes the restriction of the functional integration to the region Ω and it is renormalizable. However, the RGZ action allows for a finite nonzero value of $\mathcal{D}(0)$ and for a free-like ghost propagator $\mathcal{G}(p^2)$ in the IR limit. Thus, nonzero values for these dimension-two condensates yield for the gluon and ghost propagators an IR behavior in agreement with the massive solution of the gluon and ghost DSEs.¹³

⁹This identification is, however, based on several (unproven) hypotheses, as already stressed in Ref. [63].

¹⁰Let us recall that the scaling solution is supposed to be unique [31].

¹¹See again Ref. [36] for a comprehensive review of the GZ action.

¹²One can, however, obtain a finite nonzero value for $\mathcal{D}(0)$ within the GZ approach by considering a nonanalytic behavior for the free energy of the system [79, 80].

¹³Let us note here that a massive behavior for these two propagators has also been obtained in Refs. [90, 91] using different analytic approaches.

⁸In some cases, a tachyon instability may be resolved if the theory selects another vacuum with lower energy, through a condensation in the tachyonic channel, leading, for example, to a massive ghost. We are, however, unaware of works in this direction.

Indeed, the RGZ tree-level gluon propagator describes well the numerical data in the SU(2) case [64,92], for $d = 3$ and 4, and in the SU(3) case [93] with $d = 4$. It is also interesting to note that the fitting values for the dimension-two condensates are very similar for the SU(2) and SU(3) gauge groups in the four-dimensional case.

As stressed above, the restriction of the functional integration to the first Gribov region Ω and the no-pole condition (3) are key ingredients in the study of the IR sector of Yang-Mills theories in the Landau gauge. However, to our knowledge, a detailed investigation of the properties of the Gribov form factor $\sigma(p^2)$ as well as of the possible implications of the no-pole condition was missing up to now, although some interesting one-loop results were already presented in Refs. [88,94–96]. In particular, in Appendix B.2 of [88] it was shown that, if the gluon propagator $\mathcal{D}(p^2)$ is positive, then in the $2d$ case the derivative $\partial\sigma(p^2)/\partial p^2$ is negative for all values of p^2 ; i.e., $\sigma(p^2)$ is largest at $p^2 = 0$. Also, in Ref. [94] it was proven that in the RGZ framework the form factor $\sigma(p^2)$ presents a logarithmic IR singularity $-\ln(p^2)$ for $d = 2$. This result precluded the use of the RGZ action in the two-dimensional case, leading to a first interpretation of the different behavior found in lattice numerical simulations for the $2d$ case, compared to the $d = 3$ and 4 cases. Similar findings have been (more recently) presented in Refs. [33,97,98].

In this work we collect some general properties of the Gribov form factor $\sigma(p^2)$ and we study the consequences of imposing the no-pole condition. In particular, in Sec. II, using the expression for $\sigma(p^2)$ obtained from the evaluation of the ghost propagator at one loop, we prove that $\sigma(p^2)$ attains its maximum value at $p^2 = 0$ for any dimension $d \geq 2$. Since this expression for $\sigma(p^2)$ depends on the gluon propagator $\mathcal{D}(p^2)$, in the same section we also investigate (for a general d -dimensional space-time) which IR behavior of the gluon propagator is necessary in order to satisfy the no-pole condition $\sigma(p^2) < 1$. By considering a generic (and sufficiently regular) gluon propagator $\mathcal{D}(p^2)$, we find in the $d = 2$ case that $\sigma(p^2)$ is unbounded unless the gluon propagator is null at zero momentum. More exactly, we find $\sigma(p^2) \sim -\mathcal{D}(0) \ln(p^2)$ in the $p \rightarrow 0$ limit, in agreement with [94]. This result does not apply to the $d = 3$ and 4 cases. Indeed, in these cases one can introduce, for all values of p^2 , simple finite upper bounds for the Gribov form factor. In Sec. III we present explicit one-loop calculations for $\sigma(p^2)$ using for the gluon propagator $\mathcal{D}(p^2)$ linear combinations of Yukawa-like propagators (with real and/or complex-conjugate poles), which have been recently used to model lattice data of the gluon propagator in the SU(2) case [64,92]. Besides confirming the results obtained in Sec. II, we also find that the ghost propagator admits a one-parameter family of behaviors [21] labeled by the coupling constant g^2 , considered as a free parameter. Moreover, the massive solution

$\mathcal{G}(p^2) \sim 1/p^2$, corresponding to $\sigma(0) < 1$, is obtained for all values of g^2 smaller than a critical value g_c^2 . At the critical value g_c^2 , implying $\sigma(0) = 1$, one finds an IR-enhanced ghost propagator. [As already stressed above, the case $g^2 > g_c^2$ corresponds to $\sigma(0) > 1$ and one obtains a negative ghost propagator at small momenta.] Finally, in Sec. IV, we analyze the DSE for $\sigma(p^2)$. We stress that in this case we do not try to solve the DSE but we focus only on general properties of this equation. As we will see, considering IR-finite ghost-gluon vertices, we confirm and extend the one-loop analysis of the no-pole condition presented in Sec. II. In particular, after introducing bounds for the Gribov form factor, we show again for $d = 2$ that the gluon propagator $\mathcal{D}(p^2)$ must vanish at zero momentum in order to keep $\sigma(p^2)$ finite. On the contrary, such a constraint does not apply in the three- and four-dimensional cases. We also present alternative evidence for the $d = 2$ result using a spectral representation for the ghost propagator in the DSE.

It is important to note that all our results in Secs. II and IV apply irrespective of which set of Gribov copies (inside the region Ω) is considered; i.e., they are not affected by the so-called Gribov noise. We end with our conclusion in Sec. V. Some technical details have been collected in four appendixes. In particular, in Appendix B we present properties of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ that are relevant to prove some of our results.

II. THE ONE-LOOP-CORRECTED GHOST PROPAGATOR AND THE GRIBOV FORM FACTOR

In this section, as well as in Sec. III below, we consider the one-loop-corrected Landau-gauge ghost propagator, diagrammatically represented in Fig. 1. This propagator can be written [for the SU(N_c) gauge group in the d -dimensional case] as

$$\begin{aligned} \mathcal{G}(p^2) = & \frac{1}{p^2} - \frac{\delta^{ab}}{N_c^2 - 1} \frac{1}{p^4} g^2 f^{adc} f^{cdb} \\ & \times \int \frac{d^d q}{(2\pi)^d} (p - q)_\mu p_\nu \mathcal{D}(q^2) \mathcal{P}_{\mu\nu}(q) \frac{1}{(p - q)^2}, \end{aligned} \quad (5)$$

where $\delta^{ab} \mathcal{D}(q^2) \mathcal{P}_{\mu\nu}(q)$ is the tree-level gluon propagator [not necessarily given by $\mathcal{D}(q) = 1/q^2$] and $\mathcal{P}_{\mu\nu}(q) = (\delta_{\mu\nu} - q_\mu q_\nu / q^2)$ is the usual projector onto the transverse subspace, i.e., $q_\mu \mathcal{P}_{\mu\nu}(q) = 0$. We have also considered the tree-level ghost-gluon vertex $ig f^{adc} p_\nu$, where p is the

FIG. 1. Feynman diagrams for the one-loop-corrected Landau-gauge ghost propagator. Dashed lines represent ghosts; the curly line represents gluons.

outgoing ghost momentum. The indices a, d, c refer, respectively, to the incoming ghost, to the gluon and to the outgoing ghost. After using $f^{adc}f^{cdb} = -N_c\delta^{ab}$, valid for the adjoint representation, we obtain

$$\mathcal{G}(p^2) = \frac{1}{p^2}[1 + \sigma(p^2)], \quad (6)$$

where $\sigma(p^2)$ is the momentum-dependent function

$$\sigma(p^2) = g^2 N_c \frac{p_\mu p_\nu}{p^2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(p-q)^2} \mathcal{D}(q^2) \mathcal{P}_{\mu\nu}(q). \quad (7)$$

Finally, we can write [as in Eq. (2)]

$$\mathcal{G}(p^2) = \frac{1}{p^2} \frac{1}{1 - \sigma(p^2)}, \quad (8)$$

which corresponds to the usual resummation of an infinite set of diagrams into the self-energy. Note that this resummation makes sense only when $\sigma(k^2) < 1$, i.e., when the no-pole condition (3) is satisfied.

Clearly the function $\sigma(p^2)$ is dimensionless and it should go to zero for $p \rightarrow \infty$, modulo possible logarithmic corrections. Also, this function coincides with the so-called Gribov ghost form factor [35,95,96], even though the latter is obtained in a slightly different way.¹⁴ As discussed in the introduction, the no-pole condition $\sigma(p^2) < 1$ for $p^2 > 0$ should be equivalent to the restriction of the path integral to the first Gribov region Ω [defined in Eq. (1)]. In this section we will derive general properties of $\sigma(p^2)$ in $d \geq 2$ space-time dimensions. In particular, as we will see below, the weaker condition $\sigma(p^2) < +\infty$ is already sufficient to obtain a strong constraint on the IR behavior of the gluon propagator $\mathcal{D}(p^2)$ in the $d = 2$ case.

A. First derivative of $\sigma(p^2)$ for $d = 2$

Following Appendix B.2 of Ref. [88] one can show that, if the gluon propagator $\mathcal{D}(q^2)$ is positive, then in the $2d$ case and for all values of p^2 we have

$$\frac{\partial \sigma(p^2)}{\partial p^2} < 0, \quad (9)$$

with $\sigma(p^2)$ defined in Eq. (7). This implies that $\sigma(p^2)$ attains its maximum value at $p^2 = 0$. To this end, we choose the positive x direction parallel to the external momentum p and write (using polar coordinates)

¹⁴Note that in the Gribov ghost form factor there is usually an extra factor $1/(N_c^2 - 1)$ [95,96] compared to our Eq. (7). However, this is due to the fact that in Eq. (5) above we considered for the Landau-gauge gluon propagator the usual expression $D_{\mu\nu}^{ab}(q^2) = \delta^{ab} \mathcal{P}_{\mu\nu}(q) \mathcal{D}(q^2)$ while, in the derivation of the Gribov ghost form factor, one usually writes $A_\mu^a(q) A_\nu^a(-q) = \omega(q^2) \mathcal{P}_{\mu\nu}(q)$, as in Eq. (255) of Ref. [95].

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &= \frac{p_\mu p_\nu}{p^2} \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(p-q)^2} \mathcal{D}(q^2) \mathcal{P}_{\mu\nu}(q) \\ &= \int_0^\infty \frac{q dq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{p^2 + q^2 - 2qp \cos(\theta)}. \end{aligned} \quad (10)$$

The integral in $d\theta$ can be evaluated using contour integration on the unit circle and the residue theorem. This yields

$$\begin{aligned} \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{p^2 + q^2 - 2qp \cos(\theta)} &= \oint \frac{idz}{4pq} \frac{2 - z^2 - \bar{z}^2}{z^2 - z(p/q + q/p) + 1} \\ &= \frac{\pi}{p^2} \theta(p^2 - q^2) + \frac{\pi}{q^2} \theta(q^2 - p^2), \end{aligned} \quad (11)$$

where $\oint dz$ represents the integral on the unit circle $|z| = 1$, we indicated with $\bar{z}$ the complex conjugate of $z = e^{i\theta}$ and $\theta(x)$ is the step function. This integral is also evaluated in Eqs. (A17) and (A18) in Appendix A (for the general d -dimensional case). Considering also Eq. (B8) and (A5), with $d = 2$, we find

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{4\pi} \left[\int_0^p \frac{dq}{p^2} q \mathcal{D}(q^2) + \int_p^\infty \frac{dq}{q} \mathcal{D}(q^2) \right] \quad (12)$$

$$= \frac{1}{8\pi} \left[\int_0^{p^2} \frac{dq^2}{p^2} \mathcal{D}(q^2) + \int_{p^2}^\infty \frac{dq^2}{q^2} \mathcal{D}(q^2) \right] \quad (13)$$

$$= \frac{1}{8\pi} \int_0^\infty dq^2 \mathcal{D}(q^2) \left[\frac{\theta(p^2 - q^2)}{p^2} + \frac{\theta(q^2 - p^2)}{q^2} \right]. \quad (14)$$

Then, by using $\partial_x \theta(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta function, the derivative of $\sigma(p^2)$ with respect to p^2 yields

$$\begin{aligned} \frac{\partial \sigma(p^2)}{\partial p^2} &= -\frac{g^2 N_c}{8\pi} \int_0^\infty dq^2 \mathcal{D}(q^2) \frac{\theta(p^2 - q^2)}{p^4} \\ &= -\frac{g^2 N_c}{8\pi p^4} \int_0^{p^2} dq^2 \mathcal{D}(q^2), \end{aligned} \quad (15)$$

which is clearly negative, for any value of p^2 , if $\mathcal{D}(q^2)$ is positive. We can evaluate the limit $p^2 \rightarrow 0$ of this derivative using, for example, the trapezoidal rule¹⁵

¹⁵The trapezoidal rule gives the numerical approximation

$$\int_a^b dx f(x) = \frac{b-a}{2} [f(b) + f(a)] + \mathcal{O}(b-a)^3, \quad (16)$$

which can be obtained by integrating $f(x) \approx f(a) + (x-a) \times [f(b) - f(a)]/(b-a)$. Thus, the trapezoidal rule is equivalent to using a linear Taylor expansion $f(a) + (x-a)f'(a)$ for $f(x)$ with the first derivative $f'(a)$ approximated by a (first forward) finite difference $[f(b) - f(a)]/(b-a)$.

$$\begin{aligned}\lim_{p^2 \rightarrow 0} \frac{\partial \sigma(p^2)}{\partial p^2} &= -\lim_{p^2 \rightarrow 0} \frac{g^2 N_c}{8\pi p^4} \frac{p^2}{2} [\mathcal{D}(p^2) + \mathcal{D}(0)] \\ &= -\lim_{p^2 \rightarrow 0} \frac{g^2 N_c \mathcal{D}(0)}{8\pi p^2}.\end{aligned}\quad (17)$$

One arrives at the same result after writing Eq. (15) as

$$\begin{aligned}\frac{\partial \sigma(p^2)}{\partial p^2} &= -\frac{g^2 N_c}{8\pi p^2} \int_0^1 dx \mathcal{D}(xp^2) \\ &= -\frac{g^2 N_c}{8\pi p^2} \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2},\end{aligned}\quad (18)$$

where $\hat{D}(p^2)$ is a primitive of $\mathcal{D}(p^2)$, that is, $\hat{D}'(p^2) = \mathcal{D}(p^2)$ where we indicate with ' the first derivative with respect to the variable p^2 . The limit $p^2 \rightarrow 0$ then yields again Eq. (17).

Clearly, one finds an IR singularity at $p^2 = 0$, unless $\mathcal{D}(0) = 0$. If this condition is satisfied, using again the trapezoidal rule, we have from Eq. (17) that

$$\begin{aligned}\lim_{p^2 \rightarrow 0} \frac{\partial \sigma(p^2)}{\partial p^2} &= -\lim_{p^2 \rightarrow 0} \frac{g^2 N_c}{8\pi p^4} \frac{p^2}{2} \mathcal{D}(p^2) \\ &= -\lim_{p^2 \rightarrow 0} \frac{g^2 N_c}{16\pi} \frac{\mathcal{D}(p^2) - \mathcal{D}(0)}{p^2} \\ &= -\frac{g^2 N_c}{16\pi} \lim_{p^2 \rightarrow 0} \mathcal{D}'(p^2).\end{aligned}\quad (19)$$

For a gluon propagator $\mathcal{D}(p^2)$ that is regular at small momenta, i.e., that can be expanded as $\mathcal{D}(p^2) \approx \mathcal{D}'(0)p^2 + \mathcal{D}''(0)p^4/2$ at small p^2 , the above limit is finite. On the other hand, if the leading IR behavior of $\mathcal{D}(p^2)$ is proportional to $p^{2\eta}$ with $1 > \eta > 0$, as found, for example, in the 2d case in Refs. [14,16,61,64], then the above limit gives a singular value, due to the noninteger-power (and nonanalytic) behavior of $\mathcal{D}(p^2)$.

B. Infrared singularity of $\sigma(p^2)$ for $d = 2$

Here we prove that—for $d = 2$ and for any gluon propagator $\mathcal{D}(p^2)$ that goes to zero sufficiently fast at large momenta, e.g., as $1/p^2$, and that is reasonably regular at small momenta, e.g., that can be expanded at $p = 0$ as $\mathcal{D}(p^2) \approx \mathcal{D}(0) + Bp^{2\eta} + Cp^{2\xi}$ [with $\xi > \eta > 0$ and $\mathcal{D}(0), B$ and C finite]¹⁶—the ghost form factor (7) displays a logarithmic divergence for $p \rightarrow 0$ proportional to $\mathcal{D}(0)$. Indeed, by considering Eq. (12), one obtains

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{4\pi} \left[\int_0^p \frac{dq}{p^2} q \mathcal{D}(q^2) + \int_p^\infty \frac{dq}{q} \mathcal{D}(q^2) \right] \quad (20)$$

¹⁶For example, in Eq. (116) below, the Taylor expansion of $\mathcal{D}(p^2)$ at $p^2 = 0$ is of the type considered here with $\xi = 1$.

$$\begin{aligned}&= \frac{1}{8\pi} \lim_{\Lambda \rightarrow \infty} \left\{ \int_0^{p^2} \frac{dx}{p^2} \mathcal{D}(x) + 2 \int_p^\Lambda \frac{dq}{q} \mathcal{D}(0) \right. \\ &\quad \left. + 2 \int_p^\Lambda \frac{dq}{q} [\mathcal{D}(q^2) - \mathcal{D}(0)] \right\}\end{aligned}\quad (21)$$

$$\begin{aligned}&= \frac{1}{8\pi} \lim_{\Lambda \rightarrow \infty} \left\{ \int_0^{p^2} \frac{dx}{p^2} \mathcal{D}(x) + \mathcal{D}(0) \ln \left(\frac{\Lambda^2}{p^2} \right) \right. \\ &\quad \left. + 2 \int_p^\Lambda \frac{dq}{q} \frac{-q^{2\eta}}{q^{2\eta} + M} \left[\frac{\mathcal{D}(q^2) - \mathcal{D}(0)}{-q^{2\eta}} (q^{2\eta} + M) \right] \right\}\end{aligned}\quad (22)$$

$$\begin{aligned}&= \frac{1}{8\pi} \lim_{\Lambda \rightarrow \infty} \left\{ \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2} + \mathcal{D}(0) \ln \left(\frac{\Lambda^2}{p^2} \right) \right. \\ &\quad \left. - \int_{p^2}^{\Lambda^2} dx \frac{x^{\eta-1}}{x^\eta + M} \left[\frac{\mathcal{D}(x) - \mathcal{D}(0)}{-x^\eta} (x^\eta + M) \right] \right\},\end{aligned}\quad (23)$$

where $x = q^2$, $\hat{D}(x)$ is again a primitive of $\mathcal{D}(x)$ and $M > 0$ is a (finite) constant. If we indicate with $H(x)$ the quantity in square brackets in the last line, then we have

$$\begin{aligned}\frac{\sigma(p^2)}{g^2 N_c} &= \frac{1}{8\pi} \lim_{\Lambda \rightarrow \infty} \left\{ \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2} + \mathcal{D}(0) \ln \left(\frac{\Lambda^2}{p^2} \right) \right. \\ &\quad \left. - \frac{1}{\eta} H(x) \ln(x^\eta + M) \Big|_{p^2}^{\Lambda^2} \right. \\ &\quad \left. + \frac{1}{\eta} \int_{p^2}^{\Lambda^2} dx \ln(x^\eta + M) H'(x) \right\}.\end{aligned}\quad (24)$$

Note that for $\mathcal{D}(x) = 1/(x^\eta + M)$ we find $H(x) = 1/M = \mathcal{D}(0)$ and the last term in Eq. (24) is zero. Since $\lim_{x \rightarrow \infty} \mathcal{D}(x) = 0$ we also have that $\lim_{x \rightarrow \infty} H(x) = \mathcal{D}(0)$ and the two logarithmic singularities for infinite Λ cancel each other. Thus, we get

$$\begin{aligned}\frac{\sigma(p^2)}{g^2 N_c} &= \frac{1}{8\pi} \left\{ \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2} - \mathcal{D}(0) \ln(p^2) + \frac{1}{\eta} H(p^2) \right. \\ &\quad \times \ln(p^{2\eta} + M) + \frac{1}{\eta} \int_{p^2}^\infty dx \ln(x^\eta + M) \\ &\quad \left. \times \frac{\eta M [\mathcal{D}(x) - \mathcal{D}(0)] - x(x^\eta + M) \mathcal{D}'(x)}{x^{1+\eta}} \right\}.\end{aligned}\quad (25)$$

If $\mathcal{D}(x) \sim 1/x$ at large x , it is easy to check¹⁷ that $\sigma(p^2)$ is null for $p^2 \rightarrow \infty$, as expected. At the same time, in the limit $p^2 \rightarrow 0$ we obtain

¹⁷See details in Appendix C.

$$\begin{aligned} \frac{\sigma(0)}{g^2 N_c} = & \frac{1}{8\pi} \left\{ \mathcal{D}(0) - \lim_{p^2 \rightarrow 0} \mathcal{D}(0) \ln(p^2) + \frac{1}{\eta} H(0) \ln(M) \right. \\ & + \int_0^\infty dx \ln(x^\eta + M) \\ & \times \left. \frac{\eta M [\mathcal{D}(x) - \mathcal{D}(0)] - x(x^\eta + M) \mathcal{D}'(x)}{x^{1+\eta}} \right\}, \end{aligned} \quad (26)$$

where we used

$$\lim_{p^2 \rightarrow 0} \frac{\hat{\mathcal{D}}(p^2) - \hat{\mathcal{D}}(0)}{p^2} = \hat{\mathcal{D}}'(0) = \mathcal{D}(0) \quad (27)$$

and $H(0) = -MB$ is a finite constant.¹⁸ Finally, in Appendix C we show that, under the assumptions made for the gluon propagator,¹⁹ the last term on the right-hand side of Eq. (26) is finite. Thus, the only IR singularity in the ghost form factor $\sigma(p^2)$ is proportional to $-\mathcal{D}(0) \ln(p^2)$. This result is in qualitative agreement with [94]. An IR singularity plaguing the 2d calculation has also been recently obtained in Ref. [97].

An alternative (equivalent) proof²⁰ can be done by performing an integration by parts. Then, Eq. (12) becomes

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{8\pi} \left[\int_0^{p^2} \frac{dx}{p^2} \mathcal{D}(x) + \int_{p^2}^\infty \frac{dx}{x} \mathcal{D}(x) \right] \quad (28)$$

$$\begin{aligned} = & \frac{1}{8\pi} \left[\frac{\hat{\mathcal{D}}(p^2) - \hat{\mathcal{D}}(0)}{p^2} + \ln(x) \mathcal{D}(x) \right]_{p^2}^\infty \\ & - \int_{p^2}^\infty dx \ln(x) \mathcal{D}'(x) \end{aligned} \quad (29)$$

$$\begin{aligned} = & \frac{1}{8\pi} \left[\frac{\hat{\mathcal{D}}(p^2) - \hat{\mathcal{D}}(0)}{p^2} - \ln(p^2) \mathcal{D}(p^2) \right. \\ & \left. - \int_{p^2}^\infty dx \ln(x) \mathcal{D}'(x) \right], \end{aligned} \quad (30)$$

where we used the assumption $\mathcal{D}(x) \sim 1/x$ at large x . Note that the above result coincides with Eq. (25) when $M = 0$, which implies $H(x) = \mathcal{D}(0) - \mathcal{D}(x)$. A second integration by parts yields

¹⁸Note that here we used the IR expansion $\mathcal{D}(p^2) \approx \mathcal{D}(0) + Bp^{2\eta} + Cp^{2\xi}$ for the gluon propagator.

¹⁹In the same appendix we will also show that the hypotheses considered above for the gluon propagator $\mathcal{D}(x)$ can be relaxed.

²⁰Note that both proofs are singling out the singularity $-\mathcal{D}(0) \ln(p^2)$ by essentially doing a Taylor expansion of $\mathcal{D}(p^2)$ at $p^2 = 0$.

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} = & \frac{1}{8\pi} \left\{ \frac{\hat{\mathcal{D}}(p^2) - \hat{\mathcal{D}}(0)}{p^2} - \ln(p^2) \mathcal{D}(p^2) \right. \\ & \left. - [x \ln(x) - x] \mathcal{D}'(x) \right|_{p^2}^\infty + \int_{p^2}^\infty dx [x \ln(x) - x] \mathcal{D}''(x) \} \end{aligned} \quad (31)$$

$$\begin{aligned} = & \frac{1}{8\pi} \left\{ \frac{\hat{\mathcal{D}}(p^2) - \hat{\mathcal{D}}(0)}{p^2} - \ln(p^2) \mathcal{D}(p^2) \right. \\ & \left. + [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2) + \int_{p^2}^\infty dx [x \ln(x) - x] \mathcal{D}''(x) \right\}. \end{aligned} \quad (32)$$

Here we used the hypothesis that $\mathcal{D}'(x)$ goes to zero sufficiently fast at large momenta, e.g., as $1/x^2$. As before, one easily sees that $\sigma(p^2)$ is null for $p^2 \rightarrow \infty$ (see Appendix C). At the same time, under the assumptions made for the gluon propagator $\mathcal{D}(p^2)$, in the limit $p^2 \rightarrow 0$ we obtain

$$\begin{aligned} \frac{\sigma(0)}{g^2 N_c} = & \frac{1}{8\pi} \left\{ \mathcal{D}(0) - \lim_{p^2 \rightarrow 0} \ln(p^2) \mathcal{D}(0) \right. \\ & \left. + \int_0^\infty dx [x \ln(x) - x] \mathcal{D}''(x) \right\} \end{aligned} \quad (33)$$

and we again find²¹ an IR singularity proportional to $-\mathcal{D}(0) \ln(p^2)$, unless one has $\mathcal{D}(0) = 0$.

Thus, in the 2d case and using a generic (sufficiently regular) gluon propagator, a null value for $\mathcal{D}(0)$ is a necessary condition to obtain a finite value for $\sigma(0)$ at one loop. As a consequence, the condition $\mathcal{D}(0) = 0$ must be imposed if one wants to satisfy the no-pole condition (3) and keep the functional integration inside the first Gribov region Ω . It is important to stress again that our proofs apply to any Gribov copy inside the first Gribov horizon; i.e., the result $\mathcal{D}(0) = 0$ is not affected by the Gribov noise.

C. Properties of $\sigma(p^2)$ in d dimensions: Approximate calculation

We can easily extend the result

$$\frac{\partial \sigma(p^2)}{\partial p^2} < 0 \quad (34)$$

to the d -dimensional case by using for the integral in $d^d q$ the so-called y-max approximation or angular approximation (see, for example, [13,18,99]). The same approach allows us to show that the IR singularity $-\mathcal{D}(0) \ln(p^2)$ is present only in the two-dimensional case. Indeed, by using hyperspherical coordinates (see Appendix A) and by considering the positive x_1 direction parallel to the external momentum p , we can write the d -dimensional ghost form factor (7) as

²¹In Appendix C we will prove that the integral on the right-hand side of Eq. (33) is finite under the hypotheses made for the gluon propagator $\mathcal{D}(p^2)$. In the same appendix we will also show how these hypotheses can be relaxed in this case.

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{p_\mu p_\nu}{p^2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(p-q)^2} \mathcal{D}(q^2) \mathcal{P}_{\mu\nu}(q) \\ = \int_0^\infty dq \frac{q^{d-1}}{(2\pi)^d} \mathcal{D}(q^2) \int d\Omega_d \frac{1 - \cos^2(\phi_1)}{(p-q)^2}. \quad (35)$$

In the y-max approximation one substitutes $1/(p-q)^2$ with $1/p^2$, for $q^2 < p^2$, and with $1/q^2$, for $p^2 < q^2$. Then, we obtain

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{(2\pi)^d} \left[\int_0^p dq \frac{q^{d-1}}{p^2} \mathcal{D}(q^2) + \int_p^\infty dq \frac{q^{d-1}}{q^2} \mathcal{D}(q^2) \right] \int [1 - \cos^2(\phi_1)] d\Omega_d \quad (36)$$

$$= \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{2d} \left[\int_0^{p^2} dq^2 \frac{q^{d-2}}{p^2} \mathcal{D}(q^2) + \int_{p^2}^\infty dq^2 \frac{q^{d-2}}{q^2} \mathcal{D}(q^2) \right] \quad (37)$$

$$= \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{2d} \int_0^\infty dq^2 q^{d-2} \mathcal{D}(q^2) \times \left[\frac{\theta(p^2 - q^2)}{p^2} + \frac{\theta(q^2 - p^2)}{q^2} \right], \quad (38)$$

where we used Eq. (A12). Note that, for $d = 2$ and using Eq. (A5), one gets the exact result (14). By repeating the argument shown in Sec. II A, the proof of the inequality (34) follows directly from Eq. (38).

At the same time, we can write Eq. (37) as

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \left[\int_0^p dq q^{d-1} \frac{\mathcal{D}(q^2)}{p^2} + \int_p^\infty dq q^{d-3} \mathcal{D}(q^2) \right] = I_d(p^2, \infty), \quad (39)$$

where the integral $I_d(p^2, \ell)$ is defined in Eq. (B34). In Appendix B we have also shown that, for $d > 2$, this integral is finite when the gluon propagator $\mathcal{D}(p^2)$ is finite and nonzero at $p^2 = 0$. Thus, using the y-max approximation, we find that only in the $2d$ case the condition $\mathcal{D}(0) = 0$ is necessary in order to obtain a finite value for the Gribov form factor $\sigma(p^2)$ for all values of p^2 .

Of course, in case of ultraviolet (UV) divergences we should regularize the integral defining $\sigma(p^2)$, as done, for example, in Sec. III C below for the $4d$ case using the modified minimal subtraction ($\overline{\text{MS}}$) scheme and dimensional regularization. One can also consider a fixed momentum μ and subtract²² the value $\sigma(\mu^2)$ from the Gribov form factor $\sigma(p^2)$. Because of the use of the y-max approximation the result of the subtraction is very simple. Indeed, instead of Eq. (39) we have the relation

²²This is equivalent to a momentum-subtraction (MOM) renormalization scheme defined by the condition $\mathcal{G}(\mu^2) = 1/\mu^2$.

$$\frac{\sigma(p^2) - \sigma(\mu^2)}{g^2 N_c} = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{2d} \left[\int_0^{p^2} dx \frac{x^{d/2-1}}{p^2} \mathcal{D}(x) + \int_{p^2}^{\mu^2} dx x^{d/2-2} \mathcal{D}(x) - \int_0^{\mu^2} dx \frac{x^{d/2-1}}{\mu^2} \mathcal{D}(x) \right], \quad (40)$$

which is valid for $p^2 \leq \mu^2$ as well as for $\mu^2 < p^2$. Then, we find again

$$\frac{\partial \sigma(p^2)}{\partial p^2} = -g^2 N_c \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{2d} \int_0^{p^2} dx \frac{x^{d/2-1}}{p^4} \mathcal{D}(x) < 0 \quad (41)$$

if $\mathcal{D}(x)$ is positive. We can also easily check that, for $\mathcal{D}(0) > 0$ and $d > 2$, the Gribov form factor $\sigma(p^2)$ in Eq. (40) does not display an IR singularity.

D. Properties of $\sigma(p^2)$ in d dimensions: Exact calculation

One can improve the results obtained in the previous section by considering the formulas reported in Appendixes A and B which allow us to perform the angular integration in Eq. (35) without approximations. Indeed, we have²³

$$\frac{\sigma(p^2)}{g^2 N_c} = \int_0^\infty dq \frac{q^{d-1}}{(2\pi)^d} \mathcal{D}(q^2) \times \int d\Omega_d \frac{1 - \cos^2(\phi_1)}{p^2 + q^2 - 2pq \cos(\phi_1)} \\ = I(p^2, 1, d, \infty), \quad (42)$$

with $I(p^2, \nu, d, \ell)$ defined in Eq. (B31). Since $\nu = 1$ in this case, for $d \geq 2$ we can also make use of the inequalities (B36) and write

$$\frac{d}{2(d-1)} I_d(p^2, \infty) \leq I(p^2, 1, d, \infty) \leq I_d(p^2, \infty). \quad (43)$$

Note that $I_d(p^2, \infty)$ is the same integral obtained on the right-hand side of Eq. (39). Thus, the y-max approximation of the previous section provides, for $d = 3$ and 4 , an upper bound for the Gribov ghost factor. On the contrary, for $d = 2$, the above inequalities become equalities. At the same time, as one can see in Appendix B, the integral $I_d(p^2, \infty)$ is finite (for $d > 2$) also if $\mathcal{D}(0)$ is nonzero; i.e., we do not need to impose the condition $\mathcal{D}(0) = 0$ in order to attain a finite value for $\sigma(p^2)$ in the IR limit.

By evaluating the derivative with respect to p^2 of the result (B32) we also obtain

²³Again we make the hypothesis that a regularization is introduced in the case of UV divergences. In Appendix D we explicitly show how to extend the proof to a MOM scheme, i.e., by subtracting the value $\sigma(\mu^2)$ from $\sigma(p^2)$ where μ is a fixed momentum.

$$\frac{1}{g^2 N_c} \frac{\partial \sigma(p^2)}{\partial p^2} = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \int_0^\infty dq q^{d-1} \mathcal{D}(q^2) \left[-\frac{\theta(p^2 - q^2)}{p^4} {}_2F_1(1, 1-d/2; 1+d/2; q^2/p^2) \right. \\ \left. - \frac{q^2 \theta(p^2 - q^2)}{p^6} {}_2F_1'(1, 1-d/2; 1+d/2; q^2/p^2) + \frac{\theta(q^2 - p^2)}{q^4} {}_2F_1'(1, 1-d/2; 1+d/2; p^2/q^2) \right], \quad (44)$$

where ${}_2F_1'(a, b; c; z)$ indicates the derivative with respect to the variable z of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ (see Appendix B). Here we used again the properties of the theta and of the Dirac delta functions and Eq. (B6). For $d = 2$, the last two terms in Eq. (44) are null [see Eq. (B23)], and using the result (B8), we find again Eq. (15). In the $4d$ case one can use the expression (B9) for the Gauss hypergeometric function ${}_2F_1(1, 1-d/2; 1+d/2; z)$. Then, from Eq. (44)—or, equivalently, by evaluating the derivative with respect to p^2 of Eq. (D1) in Appendix D—we find that

$$\frac{\partial \sigma(p^2)}{\partial p^2} = \frac{g^2 N_c}{32\pi^2} \left[\int_0^p dq \mathcal{D}(q^2) \frac{2q^5 - 3p^2 q^3}{p^6} - \int_p^\infty dq \frac{\mathcal{D}(q^2)}{q} \right] \quad (45)$$

$$= \frac{g^2 N_c}{32\pi^2} \left[\int_0^1 dy \mathcal{D}(y^2 p^2) (2y^5 - 3y^3) - \int_p^\infty dq \frac{\mathcal{D}(q^2)}{q} \right], \quad (46)$$

where $y = q/p$ and we have used Eq. (A5). For $\mathcal{D}(p^2) > 0$ both terms in square brackets are negative; i.e., the derivative $\partial \sigma(p^2)/\partial p^2$ is negative for all values of the momentum p . Let us note that in the original work by Gribov [35] the same result was proven [see comment after Eq. (37) in the same reference] under the much stronger hypothesis of a gluon propagator $\mathcal{D}(q^2)$ decreasing monotonically with q^2 over the main range of integration.

A similar analysis can be done in the $3d$ case using Eq. (B17). In order to simplify the notation we define

$$\Psi(z) = {}_2F_1(1, -1/2; 5/2; z) \\ = \frac{3}{4} + \frac{3(1-z)}{8z} \left[1 - \frac{1-z}{\sqrt{z}} \operatorname{arcsinh} \left(\sqrt{\frac{z}{1-z}} \right) \right]. \quad (47)$$

This gives

$$\Psi'(z) = \frac{3}{16z^3} \left[z(z-3) + \sqrt{z}(3-2z-z^2) \operatorname{arcsinh} \left(\sqrt{\frac{z}{1-z}} \right) \right]. \quad (48)$$

Then, after setting $d = 3$ in Eq. (44) and using Eq. (A5), we obtain

$$\frac{\partial \sigma(p^2)}{\partial p^2} = \frac{g^2 N_c}{3\pi^2} \left\{ \int_0^\infty dq q^2 \mathcal{D}(q^2) \left[-\frac{\theta(p^2 - q^2)}{p^4} \Psi \left(\frac{q^2}{p^2} \right) - \frac{q^2 \theta(p^2 - q^2)}{p^6} \Psi' \left(\frac{q^2}{p^2} \right) + \frac{\theta(q^2 - p^2)}{q^4} \Psi' \left(\frac{p^2}{q^2} \right) \right] \right\} \quad (49)$$

$$= \frac{g^2 N_c}{6\pi^2} \left\{ \int_0^{p^2} dx \sqrt{x} \mathcal{D}(x) \left[-\frac{1}{p^4} \Psi \left(\frac{x}{p^2} \right) - \frac{x}{p^6} \Psi' \left(\frac{x}{p^2} \right) \right] + \int_{p^2}^\infty \frac{dx}{x^{3/2}} \mathcal{D}(x) \Psi' \left(\frac{p^2}{x} \right) \right\}, \quad (50)$$

where we also made the substitution $x = q^2$. Next, the change of variables $x = yp^2$ in the first integral and $x = p^2/y$ in the second integral yields

$$\frac{\partial \sigma(p^2)}{\partial p^2} = \frac{g^2 N_c}{6\pi^2} \left\{ - \int_0^1 dy \frac{\sqrt{y}}{p} \mathcal{D}(yp^2) [\Psi(y) + y\Psi'(y)] + \int_0^1 \frac{dy}{p} \frac{1}{\sqrt{y}} \mathcal{D}(p^2/y) \Psi'(y) \right\}. \quad (51)$$

As one can see in Fig. 2, the factor $-\Psi(y) + y\Psi'(y)$ is negative for $y \in [0, 1]$. At the same time, from Eq. (B24) we know that $\Psi'(y)$ is negative for $y \geq 0$ (see also the corresponding plot in Fig. 2). Thus, for a positive gluon propagator $\mathcal{D}(p^2)$, the $3d$ derivative $\partial \sigma(p^2)/\partial p^2$ is negative for $p^2 > 0$.

Finally, we can consider a general $d > 2$, and after suitable changes of variables (for $p^2 > 0$), we write

$$\frac{1}{g^2 N_c} \frac{\partial \sigma(p^2)}{\partial p^2} = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \frac{p^{d-4}}{2} \left\{ \int_0^1 dy y^{2-d/2} \times \mathcal{D}(p^2/y) {}_2F_1'(1, 1-d/2; 1+d/2; y) \right. \\ \left. - \int_0^1 dy y^{d/2-1} \mathcal{D}(yp^2) [{}_2F_1(1, 1-d/2; 1+d/2; y) + y {}_2F_1'(1, 1-d/2; 1+d/2; y)] \right\}. \quad (52)$$

Note that the dependence on p^2 is only in the global factor p^{d-4} and in the argument of the gluon propagator. From Appendix B we know that the derivative ${}_2F_1'(1, 1-d/2; 1+d/2; x)$ is negative for $x \in [0, 1]$ and $d > 2$ and that, under the same hypotheses, the expression in square brackets is positive. Thus, for a positive gluon propagator, both terms in the right-hand side of the above expression are negative and we have proven that, for any dimension $d \geq 2$, the Gribov form factor $\sigma(p^2)$ (at one loop) is

FIG. 2. The functions $-\Psi(y) + y\Psi'(y)$ (full line) and $\Psi'(y)$ (dashed line) for $y \in [0, 1]$.

monotonically decreasing with p^2 , i.e., it gets its maximum value at $p^2 = 0$.

III. EVALUATION OF THE ONE-LOOP CORRECTED GHOST PROPAGATOR USING (LINEAR COMBINATIONS OF) YUKAWA-LIKE GLUON PROPAGATORS

In the previous section we have proven that, at one-loop level and for a sufficiently regular gluon propagator $\mathcal{D}(p^2)$, the Gribov ghost form factor $\sigma(p^2)$ is always finite in three and four space-time dimensions while, in $d = 2$, one needs to impose $\mathcal{D}(0) = 0$ in order to avoid an IR singularity of the type $-\mathcal{D}(0)\ln(p^2)$. In this section we present an explicit calculation of $\sigma(p^2)$ at one loop for $d = 2, 3$ and 4 using, for the gluon propagator, results recently presented in Ref. [64,92] from fits to lattice data of $\mathcal{D}(p^2)$ in the SU(2) case. The expressions obtained below for the ghost propagator $\mathcal{G}(p^2)$ will be used in a subsequent work [100] to model lattice data of SU(2) ghost propagators.

In this section, besides recovering the same results reported in Sec. II, we also find that the ghost propagator $\mathcal{G}(p^2)$ admits a one-parameter family of behaviors [21,34] labeled by the coupling constant g^2 , considered as a free parameter. The no-pole condition $\sigma(0) \leq 1$ implies $g^2 \leq g_c^2$, where g_c^2 is a critical value. Moreover, for g^2 smaller than g_c^2 one has $\sigma(0) < 1$ and the ghost propagator shows a freelike behavior in the IR limit, in agreement with the so-called massive solution of gluon and ghost DSEs [18–25]. On the contrary, for $g^2 = g_c^2$ one finds $\sigma(0) = 1$ and the ghost propagator is IR-enhanced [1,13–16].

A. Yukawa-like gluon propagators and set up

In Refs. [64,92] the SU(2) gluon propagator was fitted in 2, 3 and 4 space-time dimensions using, respectively, the functions²⁴

$$\mathcal{D}(p^2) = C \frac{p^2 + lp^\eta + s}{p^4 + u^2p^2 + t^2}, \quad (53)$$

$$\mathcal{D}(p^2) = C \frac{p^4 + (s+1)p^2 + s}{p^6 + (k+u^2)p^4 + (ku^2 + t^2)p^2 + kt^2} \quad (54)$$

and

$$\mathcal{D}(p^2) = C \frac{p^2 + s}{p^4 + u^2p^2 + t^2}. \quad (55)$$

The last two propagators are tree-level gluon propagators that arise in the study of the RGZ action [86–89]. The first one is a simple generalization of the form (55) that fits the $2d$ data well. Note that these three functions can be written as a linear combination of propagators of the type $1/(p^2 + \omega^2)$, where ω^2 is in general a complex number. [In the $2d$ case we need to consider the more general form $p^\eta/(p^2 + \omega^2)$ with $\eta \geq 0$.] Thus, in order to evaluate $\sigma(p^2)$ in Eq. (7) using the above gluon propagators $\mathcal{D}(p^2)$, we first consider the integral

$$f(p, \omega^2) = \frac{p_\mu p_\nu}{p^2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(p-q)^2} \frac{1}{q^2 + \omega^2} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \quad (56)$$

The evaluation of $f(p, \omega^2)$ can be done in three and four space-time dimensions by introducing Feynman parameters and applying the usual shift in the momentum q . The integration then yields

$$\begin{aligned} f(p^2, \omega^2) = & \frac{1}{(4\pi)^{d/2}} \int_0^1 dx [\Delta^{d/2-2} \Gamma(2-d/2)] \\ & - \frac{1}{(4\pi)^{d/2}} \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{2} \Theta^{d/2-2} \Gamma(2-d/2) \right. \\ & \left. + x^2 p^2 \Theta^{d/2-3} \Gamma(3-d/2) \right] \end{aligned} \quad (57)$$

with

$$\begin{aligned} \Delta &= -x^2 p^2 + x p^2 + (1-x)\omega^2, \\ \Theta &= -x^2 p^2 + x p^2 + y \omega^2. \end{aligned} \quad (58)$$

Since the gamma function has the behavior $\Gamma(x) \approx 1/x$ for small x , it is clear that the first two integrals are UV-finite for $d < 4$ while the third one is UV-finite for $d < 6$. Below we will calculate the integral (57) for $d = 3$ and 4 . We start from the case $d = 3$, where all terms are finite, and then we evaluate the integral for the case $d = 4$, using the $\overline{\text{MS}}$ scheme. On the contrary, as stressed above, in the $2d$ case one needs to evaluate the more general function

²⁴Note that, for consistency with the notation used in Ref. [64], in this section the noninteger power of the momentum p is η and not 2η as in the rest of the manuscript.

$$f(p, \omega^2, \eta) = \frac{P_\mu P_\nu}{p^2} \int \frac{d^2 q}{(2\pi)^d} \frac{1}{(p-q)^2} \frac{q^\eta}{q^2 + \omega^2} \times \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \quad (59)$$

with $\eta \geq 0$. This case will be treated (in a slightly different way) in Sec. III D.

Most of the analytic results reported in this section have been checked using MATHEMATICA and/or MAPLE.

B. Ghost propagator in the 3d case

In the 3d case the residual x and y integrations in Eq. (57) are straightforward and give

$$f(p^2, \omega^2) = \left[\frac{1}{4\pi p} \arctan\left(\frac{p}{\sqrt{\omega^2}}\right) \right] + \left[-\frac{(p^2 - \omega^2)\sqrt{\omega^2}}{32\pi p^2 \omega^2} + \frac{\pi p}{64\pi \omega^2} - \frac{(p^2 + \omega^2)^2}{32\pi p^3 \omega^2} \arctan\left(\frac{p}{\sqrt{\omega^2}}\right) \right] + \left[\frac{3(p^2 - \omega^2)\sqrt{\omega^2}}{32\pi p^2 \omega^2} + \frac{3p^4 - 2p^2 \omega^2 + 3(\omega^2)^2}{32\pi p^3 \omega^2} \arctan\left(\frac{p}{\sqrt{\omega^2}}\right) - \frac{3\pi p}{64\pi \omega^2} \right], \quad (60)$$

where the three square brackets highlight the contribution from the three terms in Eq. (57). Here we have only made the assumption $p^2 > 0$. By simplifying the above result, we find

$$f(p^2, \omega^2) = \frac{1}{4\pi p} \arctan\left(\frac{p}{\sqrt{\omega^2}}\right) + \frac{(p^2 - \omega^2)\sqrt{\omega^2}}{16\pi p^2 \omega^2} - \frac{\pi p}{32\pi \omega^2} + \frac{(p^2 - \omega^2)^2}{16\pi p^3 \omega^2} \arctan\left(\frac{p}{\sqrt{\omega^2}}\right) = \frac{1}{32\pi p^3 \omega^2} g(p^2, \omega^2), \quad (61)$$

where

$$g(p^2, \omega^2) = -\pi p^4 + 2p^3 \sqrt{\omega^2} - 2p(\omega^2)^{3/2} + 2(p^2 + \omega^2)^2 \arctan\left(\frac{p}{\sqrt{\omega^2}}\right). \quad (62)$$

In order to use the result (61) we need to write the gluon propagator (54) as

$$\mathcal{D}(p^2) = \frac{\alpha}{p^2 + \omega_1^2} + \frac{\beta}{p^2 + \omega_2^2} + \frac{\gamma}{p^2 + \omega_3^2}. \quad (63)$$

Here ω_1^2 , ω_2^2 and ω_3^2 are the three roots of the cubic equation, with respect to the variable p^2 , obtained by setting equal to zero the denominator of Eq. (54). Thus, by combining Eqs. (7), (63), and (56) we can write for the function $\sigma(p^2)$ in the 3d case the expression

$$\sigma(p^2) = g^2 N_c [\alpha f(p^2, \omega_1^2) + \beta f(p^2, \omega_2^2) + \gamma f(p^2, \omega_3^2)] \quad (64)$$

or

$$\sigma(p^2) = \frac{g^2 N_c}{32\pi \omega^2 p^3} [\alpha g(p^2, \omega_1^2) + \beta g(p^2, \omega_2^2) + \gamma g(p^2, \omega_3^2)] \quad (65)$$

with $g(p^2, \omega^2)$ given in Eq. (62). In general, the roots ω_1^2 , ω_2^2 and ω_3^2 are all real or there is one real root, for example, ω_1^2 , and two complex-conjugate roots, i.e., $(\omega_2^2)^* = \omega_3^2$, implying also $\beta = \gamma^*$. Since the fits in Refs. [64,92] support the latter case we write

$$\beta = a + ib, \quad \gamma = a - ib \quad (66)$$

and

$$\omega_2^2 = v + iw, \quad \omega_3^2 = v - iw. \quad (67)$$

Then, following, for example, [101], we find for ω_2^2 the relations

$$\sqrt{\omega_2^2} = \sqrt{v + iw} = \frac{1}{\sqrt{2}} \sqrt{\sqrt{v^2 + w^2} + v} + \frac{i}{\sqrt{2}} \sqrt{\sqrt{v^2 + w^2} - v}, \quad (68)$$

$$(\omega_2^2)^{3/2} = (v + iw)^{3/2} = (v + iw)\sqrt{v + iw}, \quad (69)$$

$$\frac{p}{\sqrt{\omega_2^2}} = \frac{p}{\sqrt{v + iw}} = \frac{p}{\sqrt{v^2 + w^2}} \sqrt{v - iw} \quad (70)$$

and similar results for ω_3^2 . We also use the expression (see, for example, [102])

$$\arctan(z) = \frac{1}{2} \arg\left(\frac{i - z}{i + z}\right) - \frac{i}{2} \ln \left| \frac{i - z}{i + z} \right| \quad (71)$$

$\forall z \neq \{i, -i\}.$

This allows us to write the function $\sigma(p^2)$ only in terms of real quantities, i.e.,

$$\sigma(p^2) = \frac{g^2 N_c}{8} \left[\frac{\alpha g(p^2, \omega_1^2)}{4\pi \omega_1^2 p^3} + f_R(p^2) \right], \quad (72)$$

where $g(p^2, \omega^2)$ is given in Eq. (62) above. Also, we have

$$f_R(p^2) = f_1(p^2) + f_2(p^2) + f_3(p^2) + f_4(p^2) + f_5(p^2) \quad (73)$$

with

$$f_1(p^2) = -p \frac{av + bw}{2R^2}, \quad (74)$$

$$f_2(p^2) = \frac{(av + bw)\sqrt{R+v} - (bv - aw)\sqrt{R-v}}{\sqrt{2}\pi R^2}, \quad (75)$$

$$f_3(p^2) = -\frac{1}{p^2} \frac{a\sqrt{R+v} - b\sqrt{R-v}}{\sqrt{2}\pi}, \quad (76)$$

$$f_4(p^2) = A(p^2) \frac{p^4(av + bw) + 2ap^2R^2 + R^2(av - bw)}{2\pi R^2 p^3}, \quad (77)$$

$$f_5(p^2) = -L(p^2) \frac{p^4(bv - aw) + 2bp^2R^2 + R^2(bv + aw)}{2\pi R^2 p^3} \quad (78)$$

and

$$A(p^2) = \begin{cases} \arctan\left(\frac{\sqrt{2}p\sqrt{R+v}}{R-p^2}\right) & \text{if } R - p^2 > 0 \\ \pi + \arctan\left(\frac{\sqrt{2}p\sqrt{R+v}}{R-p^2}\right) & \text{if } R - p^2 < 0, \end{cases} \quad (79)$$

$$L(p^2) = \ln\left[\frac{\sqrt{p^4 + 2p^2v + R^2}}{R + p(p + \sqrt{2}\sqrt{R-v})}\right], \quad (80)$$

$$R = \sqrt{v^2 + w^2}. \quad (81)$$

One can check that $\sigma(p^2)$ is null in the limit $p \rightarrow \infty$. Finally, by expanding $\sigma(p^2)$ around $p^2 = 0$ in Eqs. (72)–(81) we find

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &= \frac{\alpha}{6\pi\sqrt{\omega_1^2}} + \sqrt{R+v} \frac{9aR^2 - (av - bw)(2v - R)}{24\sqrt{2}\pi R^3} \\ &\quad + \sqrt{R-v} \frac{9bR^2 - (bv + aw)(2v + R)}{24\sqrt{2}\pi R^3} \\ &\quad - \frac{\alpha R^2 + 2(av + bw)\omega_1^2}{32\omega_1^2 R^2} p + O(p^2), \end{aligned} \quad (82)$$

which implies $\mathcal{G}(p^2) \propto p^{-2}$ at very small momenta. However, if the constant term in the above expression is equal to $1/(g^2 N_c)$, yielding $\sigma(0) = 1$, then one gets in the IR limit $\mathcal{G}(p^2) \propto p^{-4}$ or $\mathcal{G}(p^2) \propto p^{-3}$, depending on whether the term $\alpha R^2 + 2(av + bw)\omega_1^2$ vanishes or not. In particular, in the original GZ case, i.e., when the terms containing ω_1^2 and α are absent, we do recover the usual $1/p^4$ behavior. Also note that for purely imaginary poles,

i.e., when $v = b = 0$ (and $R = w$), the condition $\sigma(0) = 1$ simplifies to

$$\frac{\alpha}{\sqrt{\omega_1^2}} + \frac{\sqrt{2}a}{\sqrt{w}} = \frac{6\pi}{g^2 N_c}. \quad (83)$$

Clearly, for a given value of N_c and with a suitable choice of g^2 , one can always set $\sigma(0) = 1$ in Eq. (82). For example, using the numerical data in the second column of Table XI of Ref. [64] and $N_c = 2$, we find from Eq. (82) the result²⁵

$$\frac{\sigma(p^2)}{2g^2} \approx 0.039(0.001) - 0.017(0.003)p + O(p^2) \quad (84)$$

and we have $\sigma(0) = 1$ if $g^2 \approx 12.82$. Thus, if one considers g^2 as a free parameter, then Eq. (82) gives a one-parameter family of behaviors, labeled by g^2 . For a specific value of $g^2 = g_c^2$ we have $\sigma(0) = 1$ and one finds an IR-enhanced ghost propagator at one loop. On the contrary, for $g^2 < g_c^2$ we obtain $\sigma(0) < 1$ and $\mathcal{G}(p^2) \propto p^{-2}$ in the IR limit. Finally, for $g^2 > g_c^2$ the no-pole condition $\sigma(0) \leq 1$ is not satisfied; i.e., the ghost propagator is negative in the IR limit. These findings are in qualitative agreement with the DSE results obtained in Refs. [21,34]. Finally, note that at small momenta the function $\sigma(p^2)$ in the above formula (84) is decreasing as p^2 increases, as expected from Sec. IID.

C. Ghost propagator in the 4d case

We want now to evaluate $f(p^2, \omega^2)$ in Eq. (57) for $d = 4$. As stressed above, in this case we have to deal with UV divergences. We do the calculation in the $\overline{\text{MS}}$ renormalization scheme using dimensional regularization with $d = 4 - \varepsilon$. For the first term in Eq. (57) we have

$$\begin{aligned} &\frac{(4\pi)^{2-d/2}}{16\pi^2} \int_0^1 dx [\Delta^{d/2-2} \Gamma(2 - d/2)] \\ &= \frac{1}{16\pi^2} \int_0^1 dx \left[\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln(\Delta) \right], \end{aligned} \quad (85)$$

where γ_E is the Euler constant. Then, using the usual $\overline{\text{MS}}$ prescription, we find

$$\begin{aligned} &\frac{-1}{16\pi^2} \int_0^1 dx \ln \left[\frac{-x^2 p^2 + x p^2 + (1-x)\omega^2}{\bar{\mu}^2} \right] \\ &= \frac{-1}{16\pi^2} \int_0^1 dx \left[\ln \left(\frac{p^2}{\bar{\mu}^2} \right) + \ln(1-x) + \ln \left(x + \frac{\omega^2}{p^2} \right) \right] \end{aligned} \quad (86)$$

and the dx integration yields

²⁵The errors in brackets have been evaluated using a Monte Carlo analysis with 10 000 samples (see Ref. [64] for details).

$$\begin{aligned}
& \frac{-1}{16\pi^2} \left[\ln\left(\frac{p^2}{\bar{\mu}^2}\right) - 2 - \frac{\omega^2}{p^2} \ln\left(\frac{\omega^2}{p^2}\right) + \left(1 + \frac{\omega^2}{p^2}\right) \ln\left(1 + \frac{\omega^2}{p^2}\right) \right] \\
& = \frac{-1}{16p^2\pi^2} \left[-2p^2 + p^2 \ln\left(\frac{p^2 + \omega^2}{\bar{\mu}^2}\right) + \omega^2 \ln\left(\frac{p^2 + \omega^2}{\omega^2}\right) \right],
\end{aligned} \tag{87}$$

where $\bar{\mu}$ is the renormalization scale. For the second term in Eq. (57), which is also divergent, we first perform the y integration exactly, obtaining

$$\begin{aligned}
& -\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}\omega^2(d-2)} \int_0^1 dx (-x^2 p^2 + x p^2)^{d/2-1} \\
& \times \left[\left(1 + \frac{\omega^2}{x p^2}\right)^{d/2-1} - 1 \right].
\end{aligned} \tag{88}$$

The ε expansion then gives

$$\begin{aligned}
& \frac{1}{32\pi^2} \int_0^1 dx (1-x) \left[\ln\left(\frac{-x^2 p^2 + x p^2}{\bar{\mu}^2}\right) \right. \\
& \left. + \left(1 + \frac{x p^2}{\omega^2}\right) \ln\left(1 + \frac{\omega^2}{x p^2}\right) - 1 \right],
\end{aligned} \tag{89}$$

where we have already applied the $\overline{\text{MS}}$ prescription, and after integrating in dx we find

$$\begin{aligned}
& \frac{1}{192p^4\omega^2\pi^2} \left\{ p^4(p^2 + 3\omega^2) \ln\left(\frac{\omega^2}{p^2}\right) + (p^2 + \omega^2)^3 \right. \\
& \left. \times \ln\left(\frac{p^2 + \omega^2}{\omega^2}\right) + p^2\omega^2 \left[-7p^2 - \omega^2 + 3p^2 \ln\left(\frac{p^2}{\bar{\mu}^2}\right) \right] \right\}.
\end{aligned} \tag{90}$$

Finally, the third term, which is finite, yields

$$\begin{aligned}
& -\frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy [x^2 p^2 \Theta^{-1}] \\
& = -\frac{1}{16\pi^2\omega^2} \int_0^1 dx x^2 p^2 \left[\ln\left(x + \frac{\omega^2}{p^2}\right) - \ln(x) \right] \\
& = -\frac{1}{96p^4\omega^2\pi^2} \left[p^2\omega^2(p^2 - 2\omega^2) + 2p^6 \ln\left(\frac{\omega^2}{p^2}\right) \right. \\
& \left. + 2(p^6 + \omega^6) \ln\left(\frac{p^2 + \omega^2}{\omega^2}\right) \right].
\end{aligned} \tag{92}$$

By summing the three results above we ultimately find (in the $\overline{\text{MS}}$ scheme)

$$\begin{aligned}
f(p^2, \omega^2) &= \frac{1}{64p^4\omega^2\pi^2} [f_1(p^2, \omega) + f_2(p^2, \omega^2) \\
& + f_3(p^2, \omega^2)]
\end{aligned} \tag{93}$$

with

$$f_1(p^2, \omega) = p^4(\omega^2 - p^2) \ln\left(\frac{\omega^2}{p^2}\right), \tag{94}$$

$$f_2(p^2, \omega) = -(p^6 - p^4\omega^2 + 3p^2\omega^4 + \omega^6) \ln\left(\frac{p^2 + \omega^2}{\omega^2}\right), \tag{95}$$

$$\begin{aligned}
f_3(p^2, \omega) &= p^2\omega^2 \left[5p^2 + \omega^2 + p^2 \ln\left(\frac{p^2}{\bar{\mu}^2}\right) \right. \\
& \left. - 4p^2 \ln\left(\frac{p^2 + \omega^2}{\bar{\mu}^2}\right) \right].
\end{aligned} \tag{96}$$

As shown in Refs. [64,92], in the $4d$ case the fit of the gluon-propagator data is done using the expression (55). Thus, in order to use the above result (93)–(96), we need to write the gluon propagator as

$$\mathcal{D}(p^2) = \frac{\alpha_+}{p^2 + \omega_+^2} + \frac{\alpha_-}{p^2 + \omega_-^2}, \tag{97}$$

where $\omega_{\pm}$ are the roots of the quadratic equation, with respect to the variable p^2 , obtained by setting equal to zero the denominator of Eq. (55). Then, the ghost form factor in the $\overline{\text{MS}}$ scheme is given by

$$\sigma^{\overline{\text{MS}}}(p^2) = g^2 N_c [\alpha_+ f(p^2, \omega_+^2) + \alpha_- f(p^2, \omega_-^2)] \tag{98}$$

and we have

$$\mathcal{G}^{\overline{\text{MS}}}(p^2) = \frac{1}{p^2} [1 - \sigma^{\overline{\text{MS}}}(p^2)]^{-1}. \tag{99}$$

Note that the function $\sigma^{\overline{\text{MS}}}(p^2)$ is real. From [64,92] we know that $\omega_{\pm}^2$ are complex-conjugate roots, i.e., $\omega_-^2 = (\omega_+^2)^*$ and $\alpha_- = \alpha_+^*$. By writing $\alpha_{\pm} = a \pm ib$ and $\omega_{\pm}^2 = v \pm iw$ we find

$$\begin{aligned}
\sigma^{\overline{\text{MS}}}(p^2) &= \frac{g^2 N_c}{32\pi^2 R^2} [-p^2 t_1(p^2) + R^2 t_2(p^2) \\
& + p^{-2} t_3(p^2) - p^{-4} t_4(p^2)]
\end{aligned} \tag{100}$$

with

$$\begin{aligned}
t_1(p^2) &= (av + bw)[\ell_2(p^2) + \ell_3(p^2)] \\
& - (bv - aw)[a_1(p^2) - a_2(p^2)],
\end{aligned} \tag{101}$$

$$\begin{aligned}
t_2(p^2) &= a[5 + \ell_1(p^2) + \ell_2(p^2) + \ell_3(p^2) - 4\ell_4(p^2)] \\
& - b[a_1(p^2) - a_2(p^2) - 4a_3(p^2)],
\end{aligned} \tag{102}$$

$$\begin{aligned}
t_3(p^2) &= [1 - 3\ell_3(p^2)](av^3 - bwv^2 + vaw^2 - bw^3) \\
& - 3a_2(p^2)(bv^3 + awv^2 + vbw^2 + aw^3),
\end{aligned} \tag{103}$$

$$\begin{aligned}
t_4(p^2) &= \ell_3(p^2)(av^4 - 2wbv^3 - 2vbw^3 - aw^4) \\
& + a_2(p^2)(bv^4 + 2awv^3 + 2vaw^3 - bw^4)
\end{aligned} \tag{104}$$

and

$$\ell_1(p^2) = \ln\left(\frac{p^2}{\bar{\mu}^2}\right), \quad (105)$$

$$\ell_2(p^2) = \ln\left(\frac{R}{p^2}\right), \quad (106)$$

$$\ell_3(p^2) = \ln\left(\frac{\sqrt{R^2 p^4 + R^4 + 2vR^2 p^2}}{R^2}\right), \quad (107)$$

$$\ell_4(p^2) = \ln\left(\frac{\sqrt{p^4 + 2vp^2 + R^2}}{\bar{\mu}^2}\right), \quad (108)$$

$$a_1(p^2) = \arctan\left(\frac{w}{v}\right), \quad (109)$$

$$a_2(p^2) = \arctan\left(\frac{wp^2}{R^2 + vp^2}\right), \quad (110)$$

$$a_3(p^2) = \arctan\left(\frac{w}{v + p^2}\right), \quad (111)$$

$$R = \sqrt{v^2 + w^2}. \quad (112)$$

Also note that, at large momenta, one gets

$$\sigma^{\overline{\text{MS}}}(p^2) \approx -\frac{3ag^2 N_c}{32\pi^2} \ln\left(\frac{p^2}{\bar{\mu}^2}\right). \quad (113)$$

Finally, by expanding $\sigma^{\overline{\text{MS}}}(p^2)$ around $p^2 = 0$ in Eqs. (100)–(112) we obtain

$$\frac{\sigma^{\overline{\text{MS}}}(p^2)}{g^2 N_c} = -\frac{6a \ln\left(\frac{R}{\bar{\mu}^2}\right) - 6b \arctan\left(\frac{w}{v}\right) - 5a}{64\pi^2} + \frac{[-11 + 6 \ln\left(\frac{p^2}{R}\right)](av + wb) + 6(bv - aw) \arctan\left(\frac{w}{v}\right)}{192\pi^2 R^2} p^2 + O(p^4). \quad (114)$$

Thus, if $\sigma^{\overline{\text{MS}}}(0) = 1$ we have that $\mathcal{G}^{\overline{\text{MS}}}(p^2) \sim 1/p^4$ at small momenta (plus logarithmic corrections). Clearly, also in $4d$, we obtain a one-parameter family of behaviors, labeled by the value of g^2 , and the IR-enhanced ghost propagator corresponds to the upper value of g^2 allowed by the no-pole condition (3). With the numerical values reported in the second column of Table IV of Ref. [64] and $N_c = 2$ we find²⁶

$$\begin{aligned} \frac{\sigma^{\overline{\text{MS}}}(p^2)}{2g^2} &= 0.0240(0.0007) + [-0.0082(0.0003) \\ &\quad + 0.0060(0.0002) \ln(p^2)] p^2 \end{aligned} \quad (115)$$

and the condition $\sigma^{\overline{\text{MS}}}(0) = 1$ corresponds²⁷ to $g_c^2 \approx 20.83$. Note again the negative sign of the leading order corrections at small momenta (see Sec. II D).

D. Ghost propagator in the 2d case

As stressed in Sec. III A above, Ref. [64] has shown that the fit of the gluon-propagator data in the 2d case can be done using the expression

$$\mathcal{D}(p^2) = \frac{\alpha_+ + icp^\eta}{p^2 + \omega_+^2} + \frac{\alpha_- - icp^\eta}{p^2 + \omega_-^2}, \quad (116)$$

where c is real, $\alpha_- = \alpha_+^*$, $\omega_-^2 = (\omega_+^2)^*$ and $\omega_\pm$ are the roots of the quadratic equation, with respect to the variable p^2 , obtained by setting equal to zero the denominator of Eq. (53). Thus, in order to evaluate the ghost form factor $\sigma(p^2)$ we need to consider the function $f(p, \omega^2, \eta)$,

defined in Eq. (59) above. To this end, we can choose again the positive x direction parallel to the external momentum p and consider polar coordinates. Then, after evaluating the angular integral we find

$$\begin{aligned} f(p, \omega^2, \eta) &= \frac{1}{4\pi} \left[\int_0^p \frac{dq}{p^2} \frac{q^{1+\eta}}{q^2 + \omega^2} + \int_p^\infty \frac{dq}{q^{1-\eta}} \right. \\ &\quad \left. \times \frac{1}{q^2 + \omega^2} \right], \end{aligned} \quad (117)$$

valid both for $\eta = 0$ and for $\eta > 0$.

In the case $\eta = 0$ the momentum integration is straightforward giving

$$f(p, \omega^2) = \lim_{\Lambda \rightarrow \infty} \frac{1}{4\pi} \left[\int_0^p \frac{dq}{p^2} \frac{q}{q^2 + \omega^2} + \int_p^\Lambda \frac{dq}{q} \frac{1}{q^2 + \omega^2} \right] \quad (118)$$

$$\begin{aligned} &= \lim_{\Lambda \rightarrow \infty} \frac{1}{4\pi} \left\{ \frac{1}{2p^2} \ln\left(1 + \frac{p^2}{\omega^2}\right) + \frac{1}{\omega^2} \left[\ln(\Lambda) \right. \right. \\ &\quad \left. \left. - \ln(p) - \frac{1}{2} \ln(\Lambda^2 + \omega^2) + \frac{1}{2} \ln(p^2 + \omega^2) \right] \right\} \end{aligned} \quad (119)$$

$$= \frac{1}{8\pi} \left[\frac{1}{p^2} \ln\left(1 + \frac{p^2}{\omega^2}\right) + \frac{1}{\omega^2} \ln\left(1 + \frac{\omega^2}{p^2}\right) \right]. \quad (120)$$

Note that the second term above blows up logarithmically in the IR limit $p \rightarrow 0$, in agreement with the result obtained in Sec. II B.

For $\eta > 0$ the second integral in Eq. (117) can be written, after the change of variable $t = \omega^2/(q^2 + \omega^2)$, as

$$\int_p^\infty \frac{dq}{q^{1-\eta}} \frac{1}{q^2 + \omega^2} = \frac{1}{2(\omega^2)^{1-\eta/2}} B\left(\frac{\omega^2}{p^2 + \omega^2}; 1 - \frac{\eta}{2}, \frac{\eta}{2}\right), \quad (121)$$

²⁶Again, the errors in parentheses have been evaluated using a Monte Carlo analysis with 10 000 samples.

²⁷Clearly, different renormalization schemes will modify the constant term in Eq. (115) and the value of g_c^2 .

where

$$B(x; a, b) = \int_0^x dt t^{a-1} (1-t)^{b-1} \quad (122)$$

is the incomplete beta function, which is defined for $a, b > 0$ [102], implying $2 > \eta > 0$ in our case. For the first integral in Eq. (117) we cannot use directly the changes of variable $v = 1/q$ and $t = 1/(1 + \omega^2 v^2)$ because we get an incomplete beta function (122) with $b < 0$. In this case it is convenient to introduce a Feynman parameter (using noninteger exponents) and write

$$\begin{aligned} \int_0^p \frac{dq}{p^2} \frac{q^3}{q^{2-\eta}} \frac{1}{q^2 + \omega^2} &= \frac{1}{p^2} \left(1 - \frac{\eta}{2}\right) \int_0^1 dx x^{-\eta/2} \\ &\times \int_0^p \frac{q^3 dq}{[q^2 + (1-x)\omega^2]^{2-\eta/2}} \end{aligned} \quad (123)$$

$$\begin{aligned} &= \frac{1}{2p^2} \int_0^1 dx x^{-\eta/2} \left\{ -\frac{p^2}{[p^2 + (1-x)\omega^2]^{1-\eta/2}} \right. \\ &\quad \left. + \frac{2}{\eta} [p^2 + (1-x)\omega^2]^{\eta/2} - \frac{2}{\eta} [(1-x)\omega^2]^{\eta/2} \right\}, \end{aligned} \quad (124)$$

where we have also done the integration in dq . After suitable changes of variables, the last formula can be written as

$$\begin{aligned} &\int_0^p \frac{dq}{p^2} \frac{q^3}{q^{2-\eta}} \frac{1}{q^2 + \omega^2} \\ &= -\frac{1}{2(\omega^2)^{1-\eta/2}} B\left(\frac{\omega^2}{p^2 + \omega^2}; 1 - \frac{\eta}{2}, \frac{\eta}{2}\right) \\ &\quad + \frac{p^2 + \omega^2}{\eta p^2 (\omega^2)^{1-\eta/2}} B\left(\frac{\omega^2}{p^2 + \omega^2}; 1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) \\ &\quad - \frac{1}{\eta p^2 (\omega^2)^{-\eta/2}} B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right), \end{aligned} \quad (125)$$

where $B(a, b) = B(1; a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function. Thus, by summing the two results above, we find

$$\begin{aligned} f(p, \omega^2, \eta) &= \frac{(\omega^2)^{\eta/2}}{4\pi\eta p^2} \left[\frac{p^2 + \omega^2}{\omega^2} B\left(\frac{\omega^2}{p^2 + \omega^2}; 1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) \right. \\ &\quad \left. - B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) \right] \end{aligned} \quad (126)$$

$$\begin{aligned} &= \frac{(\omega^2)^{\eta/2-1}}{4\pi\eta} B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) + \frac{(\omega^2)^{\eta/2}}{4\pi\eta p^2} \frac{p^2 + \omega^2}{\omega^2} \\ &\quad \times \left[B\left(\frac{\omega^2}{p^2 + \omega^2}; 1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) - B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) \right]. \end{aligned} \quad (127)$$

Note that for $p = 0$ the incomplete beta function $B(\omega^2/(p^2 + \omega^2); 1 - \eta/2, 1 + \eta/2)$ becomes the beta function $B(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2})$. Then, by Taylor expanding $f(p, \omega^2, \eta)$ for small momenta p , we obtain

$$\begin{aligned} f(p, \omega^2, \eta) &= \frac{(\omega^2)^{\eta/2-1}}{4\pi\eta} B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) \\ &\quad - \frac{p^\eta}{4\pi\eta(1 + \eta/2)\omega^2} [1 - O(p^2)], \end{aligned} \quad (128)$$

yielding a constant contribution at $p = 0$.

Using the expression (116), the ghost form factor in the $2d$ case is given by

$$\begin{aligned} \sigma(p^2) &= g^2 N_c [\alpha_+ f(p^2, \omega_+^2) + \alpha_- f(p^2, \omega_-^2) \\ &\quad + icf(p^2, \omega_+^2, \eta) - icf(p^2, \omega_-^2, \eta)], \end{aligned} \quad (129)$$

with $f(p^2, \omega^2)$ and $f(p^2, \omega^2, \eta)$ defined, respectively, in Eqs. (120) and (127). Of course, the function $\sigma(p^2)$ is real. By writing $\alpha_\pm = a \pm ib$ and $\omega_\pm^2 = v \pm iw$ we get for the first two terms above

$$\begin{aligned} &\alpha_+ f(p^2, \omega_+^2) + \alpha_- f(p^2, \omega_-^2) \\ &= \frac{1}{8\pi} \left\{ \frac{1}{p^2} [a\ell_3(p^2) + ba_2(p^2)] + \frac{1}{R^2} [(av + bw)\ell_5(p^2) \right. \\ &\quad \left. - (bv - aw)a_3(p^2)] \right\}, \end{aligned} \quad (130)$$

where $\ell_3(p^2)$, $a_2(p^2)$, $a_3(p^2)$ and R have already been defined in Eqs. (107) and (110)–(112) and

$$\ell_5(p^2) = \ln\left(\frac{\sqrt{p^4 + 2vp^2 + R^2}}{p^2}\right). \quad (131)$$

As shown in Sec. II B, there is a logarithmic singularity $\ell_5(p^2) \sim -\ln(p^2)$ at small momenta proportional to the gluon propagator at zero momentum, that is, $\mathcal{D}(0) = 2(av + bw)/R^2$. We also have

$$icf(p^2, \omega_+^2, \eta) - icf(p^2, \omega_-^2, \eta) = -2c\Im[f(p^2, \omega_+^2, \eta)], \quad (132)$$

where we have indicated with $\Im$ the imaginary part of the expression in square brackets.

One can easily check that $\sigma(p^2)$ is null at large momenta. Finally, the results (130) and (132), together with the expressions (128) and (129), allow us to evaluate the behavior of the ghost propagator at small momenta. We obtain

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} = & \frac{1}{8\pi} \left\{ \frac{ap^2}{2R^2} + \frac{av + bw}{R^2} \left[1 + \ln\left(\frac{R}{p^2}\right) \right] - \frac{bv - aw}{R^2} \left[\arctan\left(\frac{w}{v}\right) - \frac{wp^2}{R^2} \right] + O(p^4) \right\} \\ & - 2c \Im \left[\frac{(\omega_+^2)^{\eta/2-1}}{4\pi\eta} B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) - \frac{p^\eta}{4\pi\eta(1 + \eta/2)\omega_+^2} + O(p^{2+\eta}) \right] \end{aligned} \quad (133)$$

$$\begin{aligned} = & \frac{1}{8\pi} \left\{ \frac{ap^2}{2R^2} + \frac{av + bw}{R^2} \left[1 + \ln\left(\frac{R}{p^2}\right) \right] - \frac{bv - aw}{R^2} \left[\arctan\left(\frac{w}{v}\right) - \frac{wp^2}{R^2} \right] \right\} \\ & - 2c \sin \left[\left(\frac{\eta}{2} - 1 \right) \arctan\left(\frac{w}{v}\right) \right] \frac{R^{\eta/2-1}}{4\pi\eta} B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) - \frac{2cw p^\eta}{4\pi\eta(1 + \eta/2)R^2} + O(p^{2+\eta}). \end{aligned} \quad (134)$$

Note that, if $\sigma(0) = 1$, one finds a ghost propagator with a behavior $1/p^{2+\eta}$ in the IR limit. As in $3d$ and in $4d$ we have a one-parameter family of solutions labeled by the value of g^2 .

As explained in Ref. [64], the $2d$ data for the gluon propagator suggest the relations $a = -b$ and $v = w$, implying $av + bw = 0$ and $R^2 = 2v^2$. Then, we find

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} = & \frac{a}{32v} - 2c \sin \left[\left(\frac{\eta}{2} - 1 \right) \frac{\pi}{4} \right] \frac{(2v^2)^{\eta/2-1}}{4\pi\eta} \\ & \times B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) - \frac{cp^\eta}{4\pi\eta(1 + \eta/2)v} \\ & - \frac{ap^2}{32\pi v^2} + O(p^{2+\eta}). \end{aligned} \quad (135)$$

Using the approximate result $\eta \approx 1$ (see again Ref. [64]) this formula simplifies to

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} = & \frac{a + 4c\sqrt{1 - 1/\sqrt{2}}}{32v} - \frac{cp}{6\pi v} - \frac{ap^2}{32\pi v^2} \\ & + O(p^{2+\eta}). \end{aligned} \quad (136)$$

On the contrary, for $N_c = 2$ and with the numerical values reported in [64]—see the second column of Table XIV and, for the exponent η , the last line of Table XIII—we find for Eq. (134) the numerical results

$$\begin{aligned} \frac{\sigma(p^2)}{2g^2} \approx & 0.029(0.004) - 0.029(0.005)p^{0.909(0.049)} \\ & - 0.023(0.004)p^2. \end{aligned} \quad (137)$$

The coefficient $(av + bw)/R^2 \propto \mathcal{D}(0)$, multiplying the logarithmic IR singularity, is zero within error and we have omitted the corresponding term. Note that $\sigma(p^2)$ decreases for increasing momenta p^2 , as proven in Sec. II A above. Also note that we have $\sigma(0) = 1$ for $g_c^2 \approx 17.24$ and in this case the ghost propagator behaves as $\sim 1/p^{2.9}$ in the IR limit.

IV. THE GHOST PROPAGATOR BEYOND PERTURBATION THEORY

The one-loop analysis above has shown that, in the $2d$ case, an IR singularity $-\mathcal{D}(0) \ln(p^2)$ appears in the Gribov form factor $\sigma(p^2)$ when $p^2 \rightarrow 0$. Thus, one needs a null gluon propagator at zero momentum in order to satisfy the no-pole condition $\sigma(0) \leq 1$. On the contrary, for $d = 3$ and 4 , we found that $\sigma(p^2)$ is finite also for $\mathcal{D}(p^2) > 0$.

In this section we improve our analysis by considering the DSE for the ghost propagator $\mathcal{G}(p^2)$ (see, for example, [13,18,99]). As stressed in the introduction, here we do not try to solve the ghost propagator DSE, but instead we concentrate on general properties of this equation for different space-time dimensions. In particular, the results obtained in Sec. II are confirmed by considering a generic (sufficiently regular) gluon propagator $\mathcal{D}(p^2)$ and an IR-finite ghost-gluon vertex $igf^{adc} p_\lambda \Gamma_{\lambda\nu}(p, q)$.

Let us remark that, in the derivation of the ghost DSE, we consider the DSEs around the trivial vacuum $A_\mu^b(x) = 0$, as usually done. To the best of our knowledge, it is not clear whether such DSEs truly describe all sources of nonperturbative physics. Indeed, for example, in the case of instantons, we know that they dominate the large order behavior of the perturbation series of a general bosonic field theory [103]. Since a truncated set of DSEs contains at least a subset of the sum of all-order diagrams, one can argue that part of the nontrivial (topological) vacuum information is indeed incorporated into the DSEs. For example, Refs. [104,105] consider models for which the functional (exact) renormalization-group equations, which are closely related to the DSEs [106], are indeed able to capture tunneling effects. Recently, a first step in the direction of studying DSEs in a nontrivial background was set in Ref. [107].

A. The $2d$ case

In the $2d$ Landau gauge the DSE for the ghost propagator is written as

$$\frac{1}{\mathcal{G}(p^2)} = p^2 - g^2 N_c \int \frac{d^2 q}{(2\pi)^2} p_\lambda \Gamma_{\lambda\nu}(p, q) s_\mu \times \mathcal{D}(q^2) P_{\mu\nu}(q) \mathcal{G}(s^2), \quad (138)$$

where $s = p - q$, the gluon and the ghost propagators—respectively $\mathcal{D}(p^2)$ and $\mathcal{G}(p^2)$ —are full propagators and we indicated with $igf^{adc} p_\lambda \Gamma_{\lambda\nu}(p, q)$ the full ghost-gluon vertex. The above result implies

$$\sigma(p^2) = \frac{g^2 N_c}{p^2} \int \frac{d^2 q}{(2\pi)^2} p_\lambda \Gamma_{\lambda\nu}(p, q) s_\nu \times \mathcal{D}(q^2) P_{\mu\nu}(q) \frac{1}{s^2} \frac{1}{1 - \sigma(s^2)} \quad (139)$$

if one uses Eq. (8). For a tree-level ghost-gluon vertex $\Gamma_{\lambda\nu}(p, q) = \delta_{\lambda\nu}$ and using the transversality of the gluon propagator we finally find

$$\sigma(p^2) = g^2 N_c \frac{p_\mu p_\nu}{p^2} \int \frac{d^2 q}{(2\pi)^2} \mathcal{D}(q^2) P_{\mu\nu}(q) \frac{1}{s^2} \frac{1}{1 - \sigma(s^2)}, \quad (140)$$

which should be compared to the one-loop result (7). As in Sec. II A above, we can then choose the x direction along the external momentum p obtaining (using polar coordinates)

$$\frac{\sigma(p^2)}{g^2 N_c} = \int_0^\infty \frac{q dq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2 [1 - \sigma(s^2)]}, \quad (141)$$

with $s^2 = p^2 + q^2 - 2pq \cos(\theta)$.

This equation will be analyzed below using two different approaches. A first result can, however, be easily obtained using again the y -max approximation, as in Sec. II C above. This gives us

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{8\pi} \left\{ \int_0^{p^2} dx \frac{\mathcal{D}(x)}{p^2 [1 - \sigma(p^2)]} + \int_{p^2}^\infty dx \frac{\mathcal{D}(x)}{x [1 - \sigma(x)]} \right\}, \quad (142)$$

where we have done the angular integration and set $x = q^2$. In the limit of small momenta p^2 we then obtain

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{8\pi} \left\{ \lim_{p^2 \rightarrow 0} \frac{p^2}{2} \frac{\mathcal{D}(p^2) + \mathcal{D}(0)}{p^2 [1 - \sigma(p^2)]} + \int_0^\infty dx \frac{\mathcal{D}(x)}{x [1 - \sigma(x)]} \right\}. \quad (143)$$

In order to avoid IR singularities in the above equation we have to impose $\mathcal{D}(p^2) \approx B p^{2\eta}$; i.e., the gluon propagator should be null at zero momentum. In particular, if $\sigma(0) < 1$, i.e., for a freelike ghost propagator at small

momenta, it is sufficient to have $\eta > 0$. On the contrary, if the ghost propagator is IR-enhanced and $1 - \sigma(0) \propto x^\kappa$ for small x with $\kappa > 0$, then the condition $\eta > \kappa$ should be satisfied. Note that the predictions of the scaling solution [14–16], i.e., $\eta = 0.4$ and $\kappa = 0.2$, are consistent with the above inequality. The same results can also be obtained by setting $p^2 = 0$ directly in Eq. (141). This makes the θ integral trivial and gives

$$\frac{\sigma(0)}{g^2 N_c} = \int_0^\infty \frac{q dq}{4\pi} \frac{\mathcal{D}(q^2)}{q^2 [1 - \sigma(q^2)]}. \quad (144)$$

Note, however, that in both cases we essentially miss the logarithmic IR singularity $-\ln(p^2)$ which is found below. In the first case this is probably related to the very crude y -max approximation. On the contrary, in Eq. (144), this is due to the (improper) exchange of the q integration with the $p^2 \rightarrow 0$ limit [94].

1. Bounds on the Gribov form factor

Since the Gribov form factor is non-negative, we can easily construct a lower bound for the left-hand side of Eq. (141) by writing

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &\geq \int_0^\infty \frac{q dq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2} \\ &= I(p^2, 1, 2, \infty) = I_2(p^2, \infty) \\ &= \frac{1}{4\pi} \left[\int_0^p \frac{dq}{p^2} q \mathcal{D}(q^2) + \int_p^\infty \frac{dq}{q} \mathcal{D}(q^2) \right], \end{aligned} \quad (145)$$

where we use the definitions (B31) and (B34) and the relations (B36). The last integral in the above equation has already been analyzed in Sec. II B, where it was shown that $I_2(p^2, \infty)$ develops an IR singularity proportional to $-\ln(p^2)$ if $\mathcal{D}(0) \neq 0$. Thus, $\sigma(p^2)$ also is IR singular, unless $\mathcal{D}(0) = 0$.

One can also find an upper bound for $\sigma(p^2)$ and check that the IR singularity is indeed only logarithmic. To this end we can notice that, if $\sigma(0) < 1$, one can write²⁸

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &\leq \int_0^\infty \frac{q dq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2 [1 - \sigma(0)]} \\ &= \frac{I_2(p^2, \infty)}{1 - \sigma(0)}, \end{aligned} \quad (146)$$

²⁸Recall that, in the $2d$ case and in the one-loop approximation, the function $\sigma(p^2)$ is decreasing as p^2 increases; i.e., the maximum value of $\sigma(p^2)$ is obtained for $p^2 = 0$ (see Sec. II A). However, the proof presented here can be easily modified for the case when $\sigma(p^2) < 1$ for all momenta p and the maximum value of $\sigma(p^2)$ is not attained at $p = 0$. Finally, one should note that in the DSE (140) one uses explicitly Eq. (8). Thus, when estimating the integral in Eq. (141), we cannot simply impose $\sigma(p^2) < +\infty$ but we have to consider the stronger condition $\sigma(p^2) \leq 1$.

where we have also used Eq. (145) above. Therefore, the upper bound also blows up as $-\ln(p^2)$ in the IR limit. At the same time, if $\sigma(0) = 1$, with $\sigma(p^2) \approx 1 - cp^{2\kappa}$ at small momenta we find

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &= \int_0^\infty \frac{qdq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{cs^{2+2\kappa}} \\ &+ \int_0^\infty \frac{qdq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2} \\ &\times \left[\frac{1}{1 - \sigma(s^2)} - \frac{1}{cs^{2\kappa}} \right]. \end{aligned} \quad (147)$$

Note that the quantity in square brackets in the last integral is finite at $s = 0$ if the behavior of $\sigma(p^2)$ is given by $1 - cp^{2\kappa} + \mathcal{O}(p^\tau)$ with $\tau \geq 4\kappa$. Moreover, this quantity goes to 1 at large momenta and its absolute value is clearly bounded from above by some positive constant M if $\sigma(p^2) \in [0, 1]$. Hence, we have

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &\leq \int_0^\infty \frac{qdq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{cs^{2+2\kappa}} \\ &+ MI_2(p^2, \infty) \\ &= \frac{1}{c} I(p^2, 1 + \kappa, 2, \infty) + MI_2(p^2, \infty). \end{aligned} \quad (148)$$

For $1/2 > \kappa$ we can also use the upper bound in Eq. (B33) and write

$$\frac{\sigma(p^2)}{g^2 N_c} \leq \left(\frac{M''}{c} + M \right) I_2(p^2, \infty), \quad (149)$$

where M'' is a positive constant. Thus, we have again an IR singularity proportional to $-\ln(p^2)$ if $\mathcal{D}(0)$ is not zero. We conclude that $\sigma(p^2)$ can be finite solely if $\mathcal{D}(0) = 0$.

Let us remark that the only hypothesis considered in this case is the IR expansion $\sigma(p^2) = 1 - cp^{2\kappa} + \mathcal{O}(p^\tau)$ with $1 > 2\kappa$ and $\tau \geq 4\kappa$. Also note that the $2d$ lattice data [61] show for the ghost propagator an IR behavior in good agreement with the so-called scaling solution [14–16] that predicts $\kappa = 0.2$. Thus, the condition $1 > 2\kappa$ is verified in both cases. One can also note that, by considering in Eq. (139) the full ghost-gluon vertex $\Gamma_{\lambda\nu}(p, q)$, instead of the tree-level one $\delta_{\lambda\nu}$, the above results still apply for an IR-finite vertex. This hypothesis is usually adopted in DSE studies of gluon and ghost propagators [12,20,25,31] and it is confirmed by lattice data [108–111].

2. Analysis of the Gribov form factor using a spectral representation

In this section we analyze the DSE (141) in an alternative way, also avoiding the y -max approximation. To this

end, let us first consider the θ integral using contour integration. After setting $z = e^{i\theta}$ we find

$$\begin{aligned} \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{p^2[1 - \sigma(p^2)]} &= \frac{i}{4} \oint dz \frac{(z^2 - 1)^2}{z^2(-q + kz)(k - qz)} \\ &\times \frac{1}{1 - \sigma[(-q + kz)(k - qz)z^{-1}]}, \end{aligned} \quad (150)$$

where the integral $\oint dz$ is again taken on the unit circle $|z| = 1$. Clearly, besides the poles at $q = k/z$ and at $q = kz$ in the first denominator on the right-hand side of the above equation, one has to consider possible divergences in the function

$$f(z) \equiv \frac{1}{1 - \sigma[(-q + kz)(k - qz)z^{-1}]}. \quad (151)$$

In particular, if we assume ghost enhancement, i.e., $\sigma(0) = 1$, then $f(z)$ is divergent at $z = q/k$ and at $z = k/q$. Note that these divergences are not necessarily poles of the function $f(z)$. Indeed, $f(z)$ could display a branch cut in the unit disc or one passing through it. For example, the usual $d = 2$ DSE scaling solution has $\mathcal{G}(k^2) \sim 1/(k^2)^\nu$ in the limit $k^2 \rightarrow 0$, where ν is a fractional number. This behavior signals a nonanalyticity for $\mathcal{G}(k^2)$ at the origin and implies a nonanalyticity for the function $f(z)$ at $z = k/q$ or at $z = q/k$. Also, since the ghost is “massless” we should expect that the ghost propagator develops a branch cut along the real axis for $k^2 < 0$. Then, $z = q/k$ or $z = k/q$ would correspond to branch points of the function $f(z)$, making quite difficult the evaluation of the contour integral in the above expression.

In order to overcome this problem, we make the hypothesis that a spectral representation for the ghost propagator can be introduced; i.e., we write²⁹

$$\mathcal{G}(p^2) = \frac{1}{p^2} \frac{1}{1 - \sigma(p^2)} = \int_0^\infty dt \frac{\rho(t)}{t + p^2}, \quad (152)$$

which reproduces the branch cut in $\mathcal{G}(k^2)$ for $k^2 < 0$ (see, for example, [112]). If we assume $\sigma(\infty) = 0$ and write

$$\mathcal{G}(p^2) = \frac{1}{p^2} \int_0^\infty dt \frac{\rho(t)}{1 + t/p^2} \quad (153)$$

it is clear that the spectral density $\rho(t)$ must satisfy the normalization condition

²⁹Since we are working in the $d = 2$ case, the theory should be UV-finite and we do not need to consider renormalization factors here.

$$1 = \int_0^\infty dt \rho(t). \quad (154)$$

Also note that the tree-level ghost propagator $\mathcal{G}(p^2) = 1/p^2$ corresponds to the spectral density $\rho(t) = 2\delta(t)$, where $\delta(t)$ is the Dirac delta function. This case will be used below to recover results obtained in the one-loop analysis carried on in Secs. II A and II B. In the general case, the spectral density $\rho(t)$ is proportional to the discontinuity of the ghost propagator along the cut.³⁰

Considering Eqs. (141) and (152) we can write

$$\begin{aligned} & \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2[1 - \sigma(s^2)]} \\ &= \frac{i}{4} \int_0^\infty dt \rho(t) \oint dz \frac{z^2 + \bar{z}^2 - 2}{-pqz^2 + (p^2 + q^2 + t)z - pq} \end{aligned} \quad (155)$$

$$\begin{aligned} &= -\frac{i}{4pq} \int_0^\infty dt \rho(t) \oint \frac{dz}{z^2} \\ &\quad \times \frac{(z^2 - 1)^2}{z^2 - (p^2 + q^2 + t)z/(pq) + 1}, \end{aligned} \quad (156)$$

where we indicated with $\bar{z}$ the complex-conjugate of $z = e^{i\theta}$. Thus, using the representation (156) we can avoid dealing directly with the integral of an unknown function along the branch cut. In exchange, we have in our formulas

an extra integration of the (also unknown) spectral density $\rho(t)$. Nevertheless, as we will see below, the above equation will allow us to control the $p^2 \rightarrow 0$ limit [at least in the case $\rho(t) > 0$]. To this end, let us first note that in the contour integral (156) there is a double pole at $z = 0$ and there are single poles at

$$z_{\pm} = \frac{(p^2 + q^2 + t) \pm \sqrt{(p^2 + q^2 + t)^2 - 4p^2q^2}}{2pq}. \quad (157)$$

Since $p, q, t \geq 0$ we have that $p^2 + q^2 + t \geq 2pq \geq 0$ and one can check that the pole z_- lies within the unit disc while z_+ lies outside of it. Moreover, for $p^2 + q^2 + t = 2pq$ (which implies $t = 0$ and $p = q$) the two poles coincide and we have $z_{\pm} = 1$. It is also easy to check that the residues, inside the unit circle, for the z -integrand are

$$\mathcal{R} \text{ es}_{z=0} = -\frac{p^2 + q^2 + t}{p^2q^2}, \quad (158)$$

$$\mathcal{R} \text{ es}_{z=z_-} = \frac{\sqrt{[(p+q)^2 + t][(p-q)^2 + t]}}{p^2q^2}. \quad (159)$$

Then, using the residue theorem, we find

$$\int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2[1 - \sigma(s^2)]} = \frac{\pi}{2} \int_0^\infty dt \rho(t) \frac{p^2 + q^2 + t - \sqrt{[(p+q)^2 + t][(p-q)^2 + t]}}{p^2q^2} \quad (160)$$

and we can write the ghost DSE (141) as

$$\frac{\sigma(p^2)}{g^2N_c} = \int_0^\infty \frac{qdq}{8\pi} \mathcal{D}(q^2) \int_0^\infty dt \rho(t) \frac{p^2 + q^2 + t - \sqrt{[(p+q)^2 + t][(p-q)^2 + t]}}{p^2q^2} \quad (161)$$

$$= \int_0^\infty \frac{dx}{16\pi} \mathcal{D}(x) \int_0^\infty dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2x}. \quad (162)$$

Note that, for $\rho(t) = 2\delta(t)$ and using

$$\sqrt{[(p+q)^2][(p-q)^2]} = \begin{cases} p^2 - q^2 & \text{if } p^2 > q^2 \\ q^2 - p^2 & \text{if } q^2 > p^2, \end{cases} \quad (163)$$

we find from Eq. (160) the one-loop result (11). Also note that, by Taylor expanding the integrand at $p^2 = 0$, one finds

³⁰Note that, if $\mathcal{G}(k^2)$ has a branch cut along a curve $\mathcal{C}$ in the complex plane and if it goes to zero sufficiently fast at infinity, using Cauchy's theorem we could write down an integral relation similar to Eq. (152) with the variable t running over the curve $-\mathcal{C}$, with $z \in -\mathcal{C} \Leftrightarrow -z \in \mathcal{C}$. Also, possible poles can be included by adding δ functions to the spectral density $\rho(t)$ or, equivalently, by pulling the pole terms out of the spectral integral.

$$\frac{\sigma(0)}{g^2 N_c} = \frac{1}{8\pi} \int_0^\infty dx \mathcal{D}(x) \int_0^\infty dt \frac{\rho(t)}{t+x} = \frac{1}{8\pi} \int_0^\infty dx \frac{\mathcal{D}(x)}{x[1-\sigma(x)]}, \quad (164)$$

where we used the definition (152). As shown above [see Eq. (144)], this result can also be obtained immediately by setting $p^2 = 0$ in Eq. (141). However, as already pointed out below Eq. (144) and in Ref. [94], one should not exchange the q integration and the $p^2 \rightarrow 0$ limit. Therefore, in order to properly evaluate $\sigma(p^2)$ for small momenta p^2 , we write Eq. (162) as

$$\begin{aligned} \frac{\sigma(0)}{g^2 N_c} &= \lim_{p^2 \rightarrow 0} \int_0^{p^2} \frac{dx}{16\pi} \mathcal{D}(x) \int_0^\infty dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2 x} \\ &+ \lim_{p^2 \rightarrow 0} \int_{p^2}^\infty \frac{dx}{16\pi} \mathcal{D}(x) \int_0^\infty dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2 x}. \end{aligned} \quad (165)$$

The first integral can be estimated using the trapezoidal rule. We then obtain

$$\lim_{p^2 \rightarrow 0} \int_0^{p^2} \frac{dx}{16\pi} \mathcal{D}(x) \int_0^\infty dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2 x} \quad (166)$$

$$= \lim_{p^2 \rightarrow 0} \frac{p^2}{32\pi} \int_0^\infty dt \rho(t) \left[\mathcal{D}(p^2) \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^4} + \frac{2\mathcal{D}(0)}{p^2 + t} \right] \quad (167)$$

$$= \lim_{p^2 \rightarrow 0} \frac{1}{32\pi} \int_0^\infty dt \rho(t) \left[\mathcal{D}(p^2) \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} \right] + \lim_{p^2 \rightarrow 0} \frac{\mathcal{D}(0)}{16\pi[1-\sigma(p^2)]}, \quad (168)$$

where we used again Eq. (152). For the second integral we define

$$\mathcal{G}(x, p^2) = \frac{\mathcal{D}(x)}{16\pi} \int_0^\infty dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2} \quad (169)$$

and find

$$\int_{p^2}^\infty dx \frac{\mathcal{G}(x, p^2)}{x} = \ln(x) \mathcal{G}(x, p^2) \Big|_{p^2}^\infty - \int_{p^2}^\infty dx \ln(x) \mathcal{G}'(x, p^2) \quad (170)$$

$$= -\ln(p^2) \frac{\mathcal{D}(p^2)}{16\pi} \int_0^\infty dt \rho(t) \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} - \int_{p^2}^\infty dx \ln(x) \mathcal{G}'(x, p^2), \quad (171)$$

where $'$ refers to the derivative with respect to the x variable and we used the fact that $\mathcal{D}(x)$ goes to zero at large x . Note that, in the one-loop case $\rho(t) = 2\delta(t)$, we have $\mathcal{G}(x, p^2) = \mathcal{D}(x)/(8\pi)$ and Eq. (171) becomes

$$\int_{p^2}^\infty \frac{dx}{8\pi} \frac{\mathcal{D}(x)}{x} = -\ln(p^2) \frac{\mathcal{D}(p^2)}{8\pi} - \int_{p^2}^\infty \frac{dx}{8\pi} \ln(x) \mathcal{D}'(x), \quad (172)$$

in agreement with Eqs. (28) and (30).

By collecting the above results we can therefore write

$$\frac{\sigma(0)}{g^2 N_c} = \frac{1}{16\pi} \lim_{p^2 \rightarrow 0} \left\{ \frac{\mathcal{D}(0)}{1-\sigma(p^2)} + \mathcal{D}(p^2) \left[\frac{1}{2} - \ln(p^2) \right] \int_0^\infty dt \rho(t) \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} - \int_{p^2}^\infty dx \ln(x) \mathcal{G}'(x, p^2) \right\}. \quad (173)$$

We can now verify that the last integral in the above expression is finite. Indeed, we have

$$\begin{aligned} \mathcal{G}'(x, p^2) &= \frac{\mathcal{D}'(x)}{16\pi} \int_0^\infty dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2} + \frac{\mathcal{D}(x)}{16\pi} \int_0^\infty dt \rho(t) \left[\frac{1}{p^2} \right. \\ &\quad \left. - \frac{x + t - p^2}{p^2 \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}} \right]. \end{aligned} \quad (174)$$

Then, for large x we find

$$\mathcal{G}'(x, p^2) \sim \frac{\mathcal{D}'(x)}{8\pi} \int_0^\infty dt \rho(t) + \frac{\mathcal{D}(x)}{8\pi x^2} \int_0^\infty dt \rho(t) t = \frac{1}{8\pi} \left[\mathcal{D}'(x) + \frac{\mathcal{D}(x)}{x^2} \int_0^\infty dt \rho(t) t \right], \quad (175)$$

where we used the normalization condition (154), while for small x we have

$$\mathcal{G}'(x, p^2) \sim \frac{x\mathcal{D}'(x)}{8\pi} \int_0^\infty dt \frac{\rho(t)}{t+p^2} + \frac{\mathcal{D}(x)}{8\pi} \int_0^\infty dt \frac{\rho(t)}{t+p^2} = \frac{\mathcal{G}(p^2)}{8\pi} [x\mathcal{D}'(x) + \mathcal{D}(x)], \quad (176)$$

where we used the definition (153). Thus, the integral $\int_0^\infty dx \ln(x) \mathcal{G}'(x, p^2)$ has no IR and UV singularities if $\mathcal{D}(x)$ and $\mathcal{D}'(x)$ go to zero sufficiently fast when x goes to zero and to infinity. At the same time we need the integral

$$\int_0^\infty dt \rho(t) t \quad (177)$$

to be finite. We can also check that the integral

$$\int_0^\infty dt \rho(t) \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} \quad (178)$$

is finite and nonzero if $\rho(t)$ is non-negative.³¹ To this end let us first note that the numerator $2p^2 + t - \sqrt{t^2 + 4tp^2}$ is non-negative since $2p^2 + t \geq \sqrt{t^2 + 4tp^2}$ when $t, p^2 \geq 0$. Moreover, if we define

$$\Phi(t, p^2) = \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} \quad (182)$$

³¹Our results cannot be easily extended to the general case of a spectral density $\rho(t)$ that is negative for some values of t . However, they apply if one can explicitly verify that the integral in Eq. (178) is indeed finite and nonzero. Also note that we cannot simply consider the limit $p \rightarrow 0$ of the integral (178), since the factor multiplying $\rho(t)$ vanishes in this limit and we might erroneously conclude that the above expression is equal to zero. Indeed, already in the tree-level case, i.e., for $\rho(t) = 2\delta(t)$, one finds that the integral (178) is nonzero and equal to 2. Finally, one could also formally expand the integrand in powers of p^2 leading to

$$\frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} = -4 \sum_{n=2}^{\infty} \binom{1/2}{n} \left(\frac{4p^2}{t}\right)^{n-1}, \quad (179)$$

where

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$$

is the usual binomial coefficient. However, this series is not converging for all values of $t \in [0, +\infty)$. Moreover, for $n = 2$ we have a term proportional to

$$\int_0^\infty dt \frac{\rho(t)}{t} \quad (180)$$

and this integral is divergent. Indeed, by (formally) setting $p^2 = 0$ in Eq. (152) we find

$$\int_0^\infty dt \frac{\rho(t)}{t} = \infty. \quad (181)$$

it is clear that $2 = \Phi(0, p^2) \geq \Phi(t, p^2) \geq 0$, since the quantity $t - \sqrt{t^2 + 4tp^2}$ is negative for $t > 0$. This implies

$$\int_0^\infty dt \rho(t) \Phi(t, p^2) < 2 \int_0^\infty dt \rho(t) = 2, \quad (183)$$

where we used again the normalization condition (154). At the same time we can write

$$\begin{aligned} \int_0^\infty dt \rho(t) \Phi(t, p^2) &= \int_0^\infty dt \frac{p^2 \rho(t)}{t+p^2} \frac{(t+p^2)\Phi(t, p^2)}{p^2} \\ &> \frac{3}{2} \int_0^\infty dt \frac{p^2 \rho(t)}{t+p^2}, \end{aligned} \quad (184)$$

where we use the fact that the function $(t+p^2)\Phi(t, p^2)/p^2$ is positive and gets its minimum value, equal to $3/2$, for $t/p^2 = 1/2$. Then, using the definition (153) and the condition $\sigma(p^2) \geq 0$, we can write

$$\int_0^\infty dt \rho(t) \Phi(t, p^2) > \frac{3}{2[1 - \sigma(p^2)]} \geq \frac{3}{2}. \quad (185)$$

From the above results we conclude that in Eq. (173) we have two possible IR singularities, i.e., the term $\mathcal{D}(0)/[1 - \sigma(p^2)]$, if $\sigma(0) = 1$, and the term proportional to $-\mathcal{D}(p^2) \ln(p^2)$. In both cases we need to impose the condition $\mathcal{D}(0) = 0$ in order to avoid the singularity. Thus, we find again that a massive gluon propagator in the $d = 2$ case is not compatible with the restriction of the functional integration to the first Gribov region.

B. The 3d case

In the 3d case we expect no UV divergences when using dimensional regularization³² and the DSE for the Gribov form factor is simply

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &= \frac{1}{p^2} \int \frac{d^3 q}{(2\pi)^3} p_\lambda \Gamma_{\lambda\nu}(p, q) s_\nu \mathcal{D}(q^2) P_{\mu\nu}(q) \frac{1}{s^2} \\ &\times \frac{1}{1 - \sigma(s^2)} \end{aligned} \quad (186)$$

³²As shown in Sec. III B above at one-loop level, the evaluation of the ghost propagator in 3d usually involves gamma functions with half-integer arguments, which do not generate infinities. Indeed, for non-negative values of n with n integer, one has [102] $\Gamma(n + 1/2) = \sqrt{\pi} 2^{-n} (2n-1)!!$ and $\Gamma(-n + 1/2) = (-2)^n \times \sqrt{\pi}/(2n-1)!!$, where $n!!$ denotes the double factorial.

$$= \int_0^\infty dq \frac{q^2}{(2\pi)^3} \mathcal{D}(q^2) \int d\Omega_3 \frac{1 - \cos^2(\phi_1)}{s^2[1 - \sigma(s^2)]}, \quad (187)$$

where we used the tree-level ghost-gluon vertex $\Gamma_{\lambda\nu}(p, q) = \delta_{\lambda\nu}$ and $s^2 = p^2 + q^2 - 2pq \cos(\phi_1)$. We can now work as in Sec. IV A 1 and use the results of Appendix B. In this way we obtain the upper bounds

$$\frac{\sigma(p^2)}{g^2 N_c} \leq \frac{I_3(p^2, \infty)}{1 - \sigma(0)}, \quad (188)$$

if $\sigma(p^2) \leq \sigma(0) < 1$, and

$$\frac{\sigma(p^2)}{g^2 N_c} \leq \left(\frac{M''}{c} + M \right) I_3(p^2, \infty), \quad (189)$$

if $\sigma(p^2) \leq \sigma(0) = 1$ with $\sigma(p^2) \approx 1 - cp^{2\kappa} + \mathcal{O}(p^\tau)$ for small p^2 . In the latter case we also need the conditions $1 > \kappa$ and $\tau \geq 4\kappa$. As we saw in Eq. (B35), under simple assumptions for the gluon propagator $\mathcal{D}(q^2)$, the integral $I_3(p^2, \infty)$ is finite in the IR limit $p \rightarrow 0$. Thus, in both cases the upper bound of $\sigma(p^2)$ is also finite³³ and, in order to have a finite value for $\sigma(0)$ in the 3d case we do not need to set $\mathcal{D}(0) = 0$. This result also applies when an IR-finite ghost-gluon vertex is included in the ghost DSE (186). Let us also note that the scaling solution predicts in the 3d case [14–16] a value $\kappa \approx 0.4$ for which the condition $1 > \kappa > 0$ is satisfied.

C. The 4d case

In 4d the DSE for $\sigma(p^2)$ is given by (see, for example, [99])

$$\begin{aligned} \sigma(p^2) &= 1 - \tilde{Z}_3 + \tilde{Z}_1 g^2 N_c \int_0^\infty dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \\ &\quad \times \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2[1 - \sigma(s^2)]}, \end{aligned} \quad (191)$$

where $\tilde{Z}_3$ and $\tilde{Z}_1$ are the renormalization constants for the ghost propagator and the ghost-gluon vertex, respectively,³⁴ and $s^2 = p^2 + q^2 - 2pq \cos(\phi_1)$. In order to eliminate these constants from the expression for $\sigma(p^2)$ we can subtract³⁵ the same equation for some fixed value

³³Using the fact that $\sigma(p^2)$ is non-negative and Eqs. (B31) and (B36), the lower bound

$$\frac{3}{4} I_3(p^2, \infty) \leq I(p^2, 1, 3, \infty) \leq \frac{\sigma(p^2)}{g^2 N_c} \quad (190)$$

clearly applies. However, since $\sigma(p^2)$ is finite, this bound does not add any relevant information to our analysis. Note that this observation can be made also for the 4d case described in Sec. IV C.

³⁴Note that we are again considering a tree-level ghost-gluon vertex $\Gamma_{\lambda\nu}(p, q) = \delta_{\lambda\nu}$.

³⁵Again, this corresponds to a MOM scheme with the condition $\mathcal{G}(\mu^2) = 1/\mu^2$.

$p^2 = \mu^2$ and set $\tilde{Z}_1 = 1$, using the nonrenormalization of the ghost-gluon vertex in the Landau gauge [113]. This gives

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &= \frac{\sigma(\mu^2)}{g^2 N_c} + \int_0^\infty dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \\ &\quad \times \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2[1 - \sigma(s^2)]} - \int_0^\infty dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \\ &\quad \times \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{t^2[1 - \sigma(t^2)]}, \end{aligned} \quad (192)$$

with $t^2 = \mu^2 + q^2 - 2\mu q \cos(\phi_1)$. Clearly, considering $\mathcal{D}(q^2) \sim 1/q^2$ at large momenta, the UV (logarithmic) divergence of the two integrals cancels out. In order to derive upper bounds for the above expression without spoiling the cancellation of UV divergences and since we are interested in the IR limit $p \rightarrow 0$, we can consider a momentum scale ℓ sufficiently large and write

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &= \frac{\sigma(\mu^2)}{g^2 N_c} + \int_0^\ell dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \\ &\quad \times \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2[1 - \sigma(s^2)]} + \int_\ell^\infty dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \\ &\quad \times \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2[1 - \sigma(s^2)]} - \int_0^\infty dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \\ &\quad \times \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{t^2[1 - \sigma(t^2)]}. \end{aligned} \quad (193)$$

For small momenta p only the first integral on the right-hand side of the above equation can produce an IR singularity. Following the analysis presented in the 3d case above we can then write

$$\int_0^\ell dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2[1 - \sigma(s^2)]} \leq \frac{I_4(p^2, \ell)}{1 - \sigma(0)}, \quad (194)$$

if $\sigma(p^2) \leq \sigma(0) < 1$, and

$$\begin{aligned} &\int_0^\ell dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2[1 - \sigma(s^2)]} \\ &\leq \left(\frac{M''}{c} + M \right) I_4(p^2, \ell), \end{aligned} \quad (195)$$

if $\sigma(p^2) = 1 - cp^{2\kappa} + \mathcal{O}(p^\tau)$ with $\tau \geq 4\kappa$ and $3/2 > \kappa$. Again, thanks to the result (B35), both upper bounds are finite in the IR limit $p \rightarrow 0$ also for $\mathcal{D}(0) > 0$.

An alternative proof can be given by working directly with Eq. (191) and using dimensional regularization, i.e., considering a dimension $d = 4 - \varepsilon$. In this case we can write

$$\sigma(p^2) = 1 - \tilde{Z}_3 + \tilde{Z}_1 \sigma_d(p^2) \quad (196)$$

with³⁶

$$\frac{\sigma_d(p^2)}{g^2 N_c} = \int_0^\infty dq \frac{q^{d-1}}{(2\pi)^d} \mathcal{D}(q^2) \int d\Omega_d \frac{1 - \cos^2(\phi_1)}{s^2[1 - \sigma_d(s^2)]}. \quad (197)$$

Then, if we can show that no IR singularities occur for $2 < d \leq 4$, the UV infinity that appears for $d \rightarrow 4$ is taken care of by the renormalization factors. In order to show that $\sigma_d(p^2)$ is IR-finite we can work as done above and write

$$\frac{\sigma_d(p^2)}{g^2 N_c} \leq \frac{I_4(p^2, \infty)}{1 - \sigma(0)}, \quad (198)$$

or

$$\frac{\sigma_d(p^2)}{g^2 N_c} \leq \left(\frac{M''}{c} + M \right) I_4(p^2, \ell), \quad (199)$$

depending on the value of $\sigma_d(0)$. In the latter case we considered again the IR expansion $\sigma_d(p^2) \approx 1 - c p^{2\kappa} + \mathcal{O}(p^7)$ and the conditions $\tau \geq 4\kappa$ and $3/2 > \kappa$. We conclude that also in the $4d$ case, $\sigma(0)$ is finite if $\mathcal{D}(0)$ is also finite (but not necessarily null).

Let us remark that the IR exponent usually obtained in the scaling solution [14–16] is $\kappa \approx 0.6$ in the $4d$ case; i.e., the condition $\kappa < 3/2$ is satisfied. Also note that when $(d-1)/2 \leq \kappa$ the hypergeometric function ${}_2F_1(1 + \kappa, 1 + \kappa - d/2; 1 + d/2; z)$ is not convergent at $z = 1$ and we cannot use the above proofs in order to derive properties of the Gribov form factor. However, these large values of κ imply for the ghost propagator $\mathcal{G}(p^2)$ a very strong IR enhancement with a behavior at least as singular as $1/k^5$ in $4d$ and at least as singular as $1/k^4$ for $d = 3$.

V. CONCLUSION

Summarizing, in this manuscript we have considered general properties of the Landau-gauge Gribov ghost form factor $\sigma(p^2)$ for $SU(N_c)$ Euclidean Yang-Mills theories in $d \geq 2$ space-time dimensions. This form factor is in a one-to-one correspondence with the ghost propagator $\mathcal{G}(p^2)$ via Eq. (2). Also, as explained in the introduction, $\sigma(p^2)$ is bounded by 1 if the no-pole condition (3) is imposed, i.e., if one restricts the functional integration to the first Gribov region Ω . The main result of this work is an

exact proof of the qualitatively different behavior of $\sigma(p^2)$ for $d = 3, 4$ with respect to $d = 2$. In particular, for $d = 2$, the gluon propagator $\mathcal{D}(p^2)$ needs to vanish at zero momentum in order to avoid in $\sigma(p^2)$ an IR singularity proportional to $-\mathcal{D}(0) \ln(p^2)$. On the contrary, for $d = 3$ and 4, an IR-finite ghost form factor $\sigma(p^2)$ is obtained also when $\mathcal{D}(0) > 0$. These results were proven, in Sec. II, using perturbation theory at one loop and, in Sec. IV, by considering the DSE for the ghost propagator. Let us stress again that in DSE studies of correlation functions in the minimal Landau gauge, besides using the no-pole condition, a specific boundary condition is usually imposed on the Gribov ghost form factor at zero momentum. Here, instead, we have tried to prove general properties of the Gribov ghost form factor $\sigma(p^2)$ when the restriction to the first Gribov horizon is considered.

At the same time, in Sec. III, we have presented closed analytic expressions for the Gribov form factor $\sigma(p^2)$ at one loop, considering for the gluon propagator linear combinations of Yukawa-like propagators (with real and/or complex-conjugate poles). These functional forms, briefly described in Eqs. (53)–(55), were recently used to fit lattice data of the gluon propagator in the $SU(2)$ case [64,92]. The expressions obtained for $\sigma(p^2)$ confirm the results presented in Sec. II. These expressions also show that, for the ghost propagator $\mathcal{G}(p^2)$, there is a one-parameter family of behaviors [21,34] labeled by the coupling constant g^2 , when it is considered as a free parameter. The no-pole condition $\sigma(0) \leq 1$ then implies $g^2 \leq g_c^2$, where g_c^2 is a critical value. For g^2 smaller than g_c^2 one has $\sigma(0) < 1$ and the ghost propagator is a massive one. On the contrary, at the critical value g_c^2 , i.e., for $\sigma(0) = 1$, one finds an IR-enhanced ghost propagator. As stressed in the introduction, the physical value of the coupling is expected to select the actual value of $\sigma(0)$. Present results [21,34] give $\sigma(0) < 1$ in the four-dimensional $SU(3)$ case.

Our findings imply that a massive gluon propagator cannot be obtained in the two-dimensional case, in disagreement with some of the results presented in Ref. [16] (see their Table 2).³⁷ A possible massive behavior for the gluon propagator in the $2d$ case was also explicitly conjectured in Ref. [69] as a Gribov-copy effect. However, since our $2d$ result is valid for any Gribov copy inside the first Gribov region, we have shown that, at least for the $2d$ gluon propagator $\mathcal{D}(p^2)$ in the minimal Landau gauge, Gribov-copy effects do not alter our conclusion for the value of the gluon propagator at zero momentum; i.e., $\mathcal{D}(0)$ must vanish. This observation also represents an explicit counterexample to the identification of the one-parameter family of solutions for the gluon and ghost DSEs

³⁶Of course, with $d = 4 - \varepsilon$ and $\varepsilon > 0$, the integral in Eq. (197) is no longer dimensionless. To keep the dimensionality correct we should, as always, scale out a dimensional factor m^{4-d} where m is a mass scale, which could then be combined with the coupling constant g^2 , making $\sigma(p^2)$ dimensionless also for $\varepsilon > 0$. This is important when evaluating the ε expansion in order to single out UV divergencies. Since here we are mainly interested in the IR behavior of $\sigma(p^2)$, we do not keep track explicitly of all the terms depending on ε and we simply consider the coupling g^2 dimensionful.

³⁷A massive solution in the $2d$ case was also obtained in Ref. [14]. On the other hand, the author of [14] stressed that a full understanding of the $2d$ case would require a more detailed investigation.

[22,31] with different Gribov copies, as suggested in [69,71,72].

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APPENDIX A: ANGULAR INTEGRATION IN THE d -DIMENSIONAL CASE

Following Appendix B in Ref. [15] one can easily perform, using hyperspherical coordinates, the angular integrations necessary for our calculations. To this end let us recall that, in d dimensions, one has the following

relations between Cartesian coordinates x_i (with $i = 1, 2, \dots, d$) and hyperspherical coordinates r, ϕ_j (with $j = 1, 2, \dots, d-1$):

$$\begin{aligned} x_1 &= r \cos(\phi_1), & x_2 &= r \sin(\phi_1) \cos(\phi_2), \\ x_3 &= r \sin(\phi_1) \sin(\phi_2) \cos(\phi_3), & \dots \\ x_{d-1} &= r \sin(\phi_1) \sin(\phi_2) \dots \sin(\phi_{d-2}) \cos(\phi_{d-1}), \\ x_d &= r \sin(\phi_1) \sin(\phi_2) \dots \sin(\phi_{d-2}) \sin(\phi_{d-1}). \end{aligned} \quad (\text{A1})$$

The hyperspherical coordinates take values, respectively, $r \in [0, \infty)$, $\phi_i \in [0, \pi]$ for $i = 1, 2, \dots, d-2$ and $\phi_{d-1} \in [0, 2\pi)$. At the same time, the volume measure is given by

$$dV = r^{d-1} dr d\Omega_d \quad (\text{A2})$$

with

$$d\Omega_d = \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) \dots \sin(\phi_{d-2}) d\phi_1 d\phi_2 \dots d\phi_{d-2} d\phi_{d-1}. \quad (\text{A3})$$

Note that, in the usual three-dimensional case, this notation corresponds to $x_1 = z$, $x_2 = x$ and $x_3 = y$.

Here we want to evaluate the integral

$$\int d\Omega_d f(\phi_1). \quad (\text{A4})$$

If we indicate with Ω_d the well-known result

$$\Omega_d = \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}, \quad (\text{A5})$$

where $\Gamma(x)$ is the gamma function, then we have

$$\begin{aligned} \int f(\phi_1) d\Omega_d &= \frac{\Omega_d}{\int_0^\pi \sin^{d-2}(\phi_1) d\phi_1} \\ &\times \int_0^\pi \sin^{d-2}(\phi_1) f(\phi_1) d\phi_1, \end{aligned} \quad (\text{A6})$$

where all other angular integrations have already been evaluated. The integral in the denominator can be written as

$$\begin{aligned} \int_0^\pi \sin^{d-2}(\phi_1) d\phi_1 &= \int_{-1}^1 (1-z^2)^{(d-3)/2} dz \\ &= \int_0^1 t^{-1/2} (1-t)^{(d-3)/2} dt \\ &= B\left(\frac{d-1}{2}, \frac{1}{2}\right), \end{aligned} \quad (\text{A7})$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function.

In our calculations we consider two different functions $f(\phi_1)$, i.e.,

$$f(\phi_1) = 1 - \cos^2(\phi_1) \quad (\text{A8})$$

and

$$f(\phi_1) = \frac{1 - \cos^2(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^p}. \quad (\text{A9})$$

In the first case the integration gives

$$\begin{aligned} &\int [1 - \cos^2(\phi_1)] d\Omega_d \\ &= \frac{\Omega_d}{B(\frac{d-1}{2}, \frac{1}{2})} \int_0^\pi \sin^{d-2}(\phi_1) [1 - \cos^2(\phi_1)] d\phi_1 \end{aligned} \quad (\text{A10})$$

and the integral in the numerator is

$$\begin{aligned} \int_{-1}^1 (1-z^2)^{(d-1)/2} dz &= \int_0^1 t^{-1/2} (1-t)^{(d-1)/2} dt \\ &= B\left(\frac{d+1}{2}, \frac{1}{2}\right). \end{aligned} \quad (\text{A11})$$

Collecting these results we find

$$\int [1 - \cos^2(\phi_1)] d\Omega_d = \Omega_d \frac{B(\frac{d+1}{2}, \frac{1}{2})}{B(\frac{d-1}{2}, \frac{1}{2})} = \Omega_d \frac{d-1}{d}, \quad (\text{A12})$$

where we used $x\Gamma(x) = \Gamma(x+1)$. In the second case, i.e., when considering the integral

$$\int \frac{1 - \cos^2(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu} d\Omega_d, \quad (\text{A13})$$

we have

$$\frac{\Omega_d}{B(\frac{d-1}{2}, \frac{1}{2})} \int_0^\pi \frac{\sin^d(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu} d\phi_1. \quad (\text{A14})$$

We can now use the result (see, for example, formula 3.665.2 in [102])

$$\begin{aligned} & \int_0^\pi \frac{\sin^{2\mu-1}(\theta)}{[1 + a^2 \pm 2a \cos(\theta)]^\nu} d\theta \\ &= B(\mu, 1/2) {}_2F_1(\nu, \nu - \mu + 1/2; \mu + 1/2; a^2), \end{aligned} \quad (\text{A15})$$

which is valid for $|a| < 1$ and $\text{Re}(\mu) > 0$. Here ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function (see Appendix B). Then, we find

$$\begin{aligned} & \int \frac{1 - \cos^2(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu} d\Omega_d \\ &= \frac{\Omega_d}{B(\frac{d-1}{2}, \frac{1}{2})} \frac{1}{q^2} B\left(\frac{d+1}{2}, \frac{1}{2}\right) {}_2F_1(\nu, \nu - d/2; 1 + d/2; 1 + d/2; p^2/q^2) \end{aligned} \quad (\text{A16})$$

$$= \frac{\Omega_d}{q^2} \frac{d-1}{d} {}_2F_1(\nu, \nu - d/2; 1 + d/2; p^2/q^2) \quad (\text{A17})$$

if $p^2 < q^2$ and

$$\begin{aligned} & \int \frac{1 - \cos^2(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu} d\Omega_d \\ &= \frac{\Omega_d}{p^2} \frac{d-1}{d} {}_2F_1(\nu, \nu - d/2; 1 + d/2; q^2/p^2) \end{aligned} \quad (\text{A18})$$

if $q^2 < p^2$.

APPENDIX B: PROPERTIES OF THE GAUSS HYPERGEOMETRIC FUNCTION

Let us recall that the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined [102] for $|z| < 1$ by the series

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \\ &= 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2} \dots, \end{aligned} \quad (\text{B1})$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (\text{B2})$$

is a so-called Pochhammer symbol. This series is converging for $c \neq 0, -1, -2, \dots$. It is also converging for $|z| = 1$ if $\Re(c - a - b) > 0$, where $\Re$ indicates the real part. When this condition is satisfied one can use, for $z = 1$, the result (see formula 9.122.1 in Ref. [102])

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}. \quad (\text{B3})$$

In Eqs. (A17) and (A18) the hypergeometric function appears with $c = 1 + d/2$. Therefore, the corresponding series is converging inside the unit circle for any dimension $d > 0$. At the same time we have $c - a - b = d + 1 - 2\nu$ and ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$ is finite on the unit circle for $(d+1)/2 > \nu$. Also, using the relation (see Eq. 9.131.1 in Ref. [102])

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z) \quad (\text{B4})$$

we can write

$$\begin{aligned} & {}_2F_1(\nu, \nu - d/2; 1 + d/2; z) \\ &= (1-z)^{d+1-2\nu} {}_2F_1(1 + d/2 - \nu, 1 + d - \nu; 1 + d/2; z) \end{aligned} \quad (\text{B5})$$

and for $(d+1)/2 - \nu > 0$ we have that $1 + d/2 - \nu, 1 + d - \nu > 0$; i.e., the hypergeometric function ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$ is clearly positive for $z \in [0, 1]$.

In the case $\nu = 1$ the above results simplify. In particular, we have convergence of the series on the unit circle for any dimension $d > 1$. Then Eq. (B3) yields

$$\begin{aligned} {}_2F_1(1, 1 - d/2; 1 + d/2; 1) &= \frac{\Gamma(1 + d/2)\Gamma(d-1)}{\Gamma(d)\Gamma(d/2)} \\ &= \frac{d/2}{d-1}, \end{aligned} \quad (\text{B6})$$

which is positive. Note that for $d = 2$ the above result gives ${}_2F_1(1, 0; 2; 1) = 1$. One can actually check that, when $d = 2$, the function ${}_2F_1(1, 1 - d/2; 1 + d/2; z) = {}_2F_1(1, 0; 2; z)$ is simply equal to 1 for any value $|z| \leq 1$. Indeed, from the series representation (B1) it is obvious that ${}_2F_1(a, b; c; z) = 1$ for $b = 0$ (and/or for $a = 0$). The same result can be obtained by considering Eq. (B4) and the relation (see Eq. 9.121.1 in Ref. [102])

$${}_2F_1(n, c; c; z) = (1-z)^{-n}, \quad (\text{B7})$$

yielding

$$\begin{aligned} {}_2F_1(a, 0; c; z) &= (1-z)^{c-a} {}_2F_1(c-a, c; c; z) \\ &= (1-z)^{c-a} (1-z)^{a-c} = 1. \end{aligned} \quad (\text{B8})$$

The hypergeometric function ${}_2F_1(1, 1 - d/2; 1 + d/2; z)$ is actually very simple for any even dimension $d = 2n$, with $n \geq 1$. Indeed, in this case we are considering the function ${}_2F_1(1, 1 - n; 1 + n; z)$ and the coefficient b is either zero or negative. This implies that the series (B1) is actually a polynomial in z . For example, for $d = 4$ we have the very simple expression

$${}_2F_1(1, -1; 3; z) = 1 - \frac{z}{3}. \quad (\text{B9})$$

Note that it is also possible to find a closed form for the hypergeometric function ${}_2F_1(1, 1 - d/2; 1 + d/2; z)$ in the $3d$ case. Indeed, by considering the relations (see Eqs. 9.137.8 and 9.137.1 in Ref. [102])

$$0 = {}_2F_1(a, b; c; z)c + {}_2F_1(a + 1, b; c + 1; z)(b - c) - {}_2F_1(a + 1, b + 1; c + 1; z)b(1 - z), \quad (\text{B10})$$

$$0 = {}_2F_1(a, b; c; z)[c - 1 - (2c - a - b - 1)z] + {}_2F_1(a, b; c + 1; z)(c - a)(c - b)z + {}_2F_1(a, b; c - 1; z)c(c - 1)(z - 1) \quad (\text{B11})$$

and (see Eqs. 9.131.1 in [102])

$${}_2F_1(a, b; c; z) = (1 - z)^{-b} {}_2F_1\left(b, c - a; c; \frac{z}{z - 1}\right), \quad (\text{B12})$$

we can write

$${}_2F_1(1, -1/2; 5/2; z) = \frac{3}{4} {}_2F_1(0, -1/2; 3/2; z) + \frac{1 - z}{4} {}_2F_1(1, 1/2; 5/2; z) \quad (\text{B13})$$

$$= \frac{3}{4} + \frac{1 - z}{4} \frac{3(1 - z)}{2z} [{}_2F_1(1, 1/2; 1/2; z) - {}_2F_1(1, 1/2; 3/2; z)] \quad (\text{B14})$$

$$= \frac{3}{4} + \frac{3(1 - z)}{8z} [1 - (1 - z) {}_2F_1(1, 1/2; 3/2; z)] \quad (\text{B15})$$

$$= \frac{3}{4} + \frac{3(1 - z)}{8z} \left[1 - \sqrt{1 - z} {}_2F_1\left(1/2, 1/2; 3/2; \frac{z}{z - 1}\right) \right] \quad (\text{B16})$$

$$= \frac{3}{4} + \frac{3(1 - z)}{8z} \left[1 - \frac{1 - z}{\sqrt{z}} \operatorname{arcsinh}\left(\sqrt{\frac{z}{1 - z}}\right) \right], \quad (\text{B17})$$

where we have also used Eq. (B7), the relations ${}_2F_1(0, b; c; z) = 0$ and (see 9.121.27 in [102])

$${}_2F_1(1/2, 1/2; 3/2; -z^2) = \frac{\operatorname{arcsinh} z}{z}. \quad (\text{B18})$$

From the expression (B17) it is easy to check that ${}_2F_1(1, -1/2; 5/2; 1) = 3/4$ and that ${}_2F_1(1, -1/2; 5/2; 0) = 1$, as expected.

Using the above series (B1) one can verify that the derivative of ${}_2F_1(a, b; c; z)$, with respect to the variable z , is given by

$$\frac{\partial}{\partial z} {}_2F_1(a, b; c; z) = \frac{ab}{c} {}_2F_1(a + 1, b + 1; c + 1; z). \quad (\text{B19})$$

Thus, in the case of interest for us, we have

$$\frac{\partial}{\partial z} {}_2F_1(\nu, \nu - d/2; 1 + d/2; z) = \frac{\nu(\nu - d/2)}{1 + d/2} {}_2F_1(\nu + 1, \nu + 1 - d/2; 2 + d/2; z). \quad (\text{B20})$$

These results can be written as

$$\frac{\partial}{\partial z} {}_2F_1(\nu, \nu - d/2; 1 + d/2; z) = \frac{\nu(\nu - d/2)}{1 + d/2} (1 - z)^{d-2\nu} {}_2F_1(1 + d/2 - \nu, 1 + d - \nu; 2 + d/2; z), \quad (\text{B21})$$

where we made use of Eq. (B4). When the hypergeometric function ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$ is finite in the unit circle, i.e., for $(d + 1)/2 > \nu$, it is clear that this derivative is positive for $\nu > d/2$ and ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$ attains its maximum value at $z = 1$, equal to

$$\begin{aligned} & {}_2F_1(\nu, \nu - d/2; 1 + d/2; 1) \\ &= \frac{\Gamma(1 + d/2)\Gamma(1 + d - 2\nu)}{\Gamma(1 + d - \nu)\Gamma(1 + d/2 - \nu)}, \end{aligned} \quad (\text{B22})$$

and its minimum value, equal to 1, at $z = 0$. On the contrary, the same derivative is negative when $\nu < d/2$. In this case ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$ is largest at $z = 0$, with ${}_2F_1(\nu, \nu - d/2; 1 + d/2; 0) = 1$, and smallest at $z = 1$ with a value given by Eq. (B22). Finally, for $\nu = d/2$ the derivative is null and the hypergeometric function ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$ is equal to 1 for all values of $z \in [0, 1]$. Thus, for $(d + 1)/2 > \nu$ we can always write $M' < {}_2F_1(\nu, \nu - d/2; 1 + d/2; z) < M''$, for some positive constants M' and M'' and with z taking values in the interval $[0, 1]$.

For $\nu = 1$ these results again simplify, yielding

$$\begin{aligned} & \frac{\partial}{\partial z} {}_2F_1(1, 1 - d/2; 1 + d/2; z) \\ &= \frac{2 - d}{2 + d} {}_2F_1(2, 2 - d/2; 2 + d/2; z). \end{aligned} \quad (\text{B23})$$

As expected, for $d = 2$ this derivative is zero since ${}_2F_1(1, 0; 2; z) = 1$. The result above also simplifies for $d = 4$, for which we find on the right-hand side of Eq. (B23) the value $-{}_2F_1(2, 0; 4; z)/3 = -1/3$, as already known from Eq. (B9). At the same time, for $d = 3$ we can write

$$\begin{aligned} \frac{\partial}{\partial z} {}_2F_1(1, -1/2; 5/2; z) &= -\frac{1}{5} {}_2F_1(2, 1/2; 7/2; z) \\ &= -\frac{1}{5} \left(1 + \frac{2z}{7} + \frac{z^2}{14} \dots \right) \end{aligned} \quad (\text{B24})$$

and the derivative is clearly negative for any value $z \geq 0$. The same result can actually be proven for any dimension d larger than 2. Indeed, the hypergeometric function ${}_2F_1(2, 2 - d/2; 2 + d/2; z)$ is finite in the unit circle for $d > 2$ and using Eq. (B4) we can easily verify that it is also positive for $z \in [0, 1]$. Thus, the derivative in Eq. (B23) is negative for $d > 2$ (and $z \in [0, 1]$). As a consequence, under the same hypotheses, we have that the hypergeometric function ${}_2F_1(1, 1 - d/2; 1 + d/2; z)$ has its maximum value, equal to 1, at $z = 0$, and its minimum value, equal to $d/(2(d - 1))$, at $z = 1$ [see Eq. (B6)].

Using the definition (B2) of the Pochhammer symbol $(a)_n$ it is also easy to verify that

$$(a)_n = a(a + 1)(a + 2) \dots (a + n - 1) \quad (\text{B25})$$

which implies

$$n(a)_n = a[(a + 1)_n - (a)_n]. \quad (\text{B26})$$

Then, using Eq. (B1), one can prove the relation [114]

$$z \frac{\partial}{\partial z} {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} \quad (\text{B27})$$

$$= a \left[\sum_{n=0}^{\infty} \frac{(a + 1)_n(b)_n}{(c)_n} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} \right] \quad (\text{B28})$$

$$= a[{}_2F_1(a + 1, b; c; z) - {}_2F_1(a, b; c; z)]. \quad (\text{B29})$$

In particular, we can write

$$\begin{aligned} & z \frac{\partial}{\partial z} {}_2F_1(1, 1 - d/2; 1 + d/2; z) \\ &+ {}_2F_1(1, 1 - d/2; 1 + d/2; z) \\ &= {}_2F_1(2, 1 - d/2; 1 + d/2; z). \end{aligned} \quad (\text{B30})$$

Note that the right-hand side in the above relation is finite in the unit circle for $d > 2$ and, using again Eq. (B4), we can verify that it is also positive for $z \in [0, 1]$.

The above results, together with Eqs. (A17) and (A18), allow us to write a lower and an upper bound for the integral

$$\begin{aligned} I(p^2, \nu, d, \ell) &= \int_0^\ell dq \frac{q^{d-1}}{(2\pi)^d} \mathcal{D}(q^2) \\ &\times \int d\Omega_d \frac{1 - \cos^2(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu}. \end{aligned} \quad (\text{B31})$$

Indeed, after considering the angular integration, we have (for $\ell > p$)

$$\begin{aligned} I(p^2, \nu, d, \ell) &= \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \left[\int_0^p dq q^{d-1} \right. \\ &\times \frac{\mathcal{D}(q^2)}{p^2} {}_2F_1(\nu, \nu - d/2; 1 + d/2; q^2/p^2) \\ &+ \int_p^\ell dq q^{d-3} \mathcal{D}(q^2) \\ &\times {}_2F_1(\nu, \nu - d/2; 1 + d/2; p^2/q^2) \left. \right]. \end{aligned} \quad (\text{B32})$$

Then, for $1 + d - 2\nu > 0$ we obtain

$$M' I_d(p^2, \ell) \leq I(p^2, \nu, d, \ell) \leq M'' I_d(p^2, \ell) \quad (\text{B33})$$

with³⁸

$$I_d(p^2, \ell) = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \left[\int_0^p dq q^{d-1} \frac{\mathcal{D}(q^2)}{p^2} + \int_p^\ell dq q^{d-3} \mathcal{D}(q^2) \right]. \quad (\text{B34})$$

Also, in the limit $p \rightarrow 0$, we find (for $d > 1$)

$$\lim_{p \rightarrow 0} I_d(p^2, \ell) = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \left[\lim_{p \rightarrow 0} \frac{p^{d-2} \mathcal{D}(p^2)}{2} + \int_0^\ell dq q^{d-3} \mathcal{D}(q^2) \right], \quad (\text{B35})$$

where we used the trapezoidal rule. Clearly, for $\mathcal{D}(0) > 0$, the first term is IR-finite if $d \geq 2$ while the second term is finite for $d > 2$. Finally, note that for $\nu = 1$ the condition $1 + d - 2\nu > 0$ simplifies to $d > 1$. At the same time, for $d \geq 2$ the inequalities (B33) become

$$\frac{d}{2(d-1)} I_d(p^2, \ell) \leq I(p^2, \nu, d, \ell) \leq I_d(p^2, \ell) \quad (\text{B36})$$

and for $d = 2$ we have $I(p^2, \nu, 2, \ell) = I_2(p^2, \ell)$.

APPENDIX C: HYPOTHESES ON THE $2d$ GLUON PROPAGATOR

In Sec. II B we have proven, using two different approaches, that in the $2d$ case one needs to set $\mathcal{D}(0) = 0$ in order to have $\sigma(p^2) < +\infty$. The assumptions made for the gluon propagator were rather general. Indeed, for the first proof one needs, for small momenta p^2 , an expansion of the gluon propagator of the type $\mathcal{D}(p^2) \approx \mathcal{D}(0) + Bp^{2\eta} + Cp^{2\xi}$, with $\xi > \eta > 0$ and $\mathcal{D}(0)$, B and C finite. At the same time, for large momenta p^2 , we required

$$\lim_{p^2 \rightarrow \infty} \mathcal{D}(p^2) = \lim_{p^2 \rightarrow \infty} \frac{\hat{D}(p^2)}{p^2} = 0. \quad (\text{C1})$$

Let us recall that we are indicating with $\hat{D}(p^2)$ a primitive of $\mathcal{D}(p^2)$ and that $\mathcal{D}'(p^2)$ is the first derivative with respect to the variable p^2 . In Sec. II B above we considered for $\mathcal{D}(p^2)$ a large p^2 behavior of the type $1/p^2$. However, it is clear that a weaker condition can also be used. Indeed, the behavior $\mathcal{D}(p^2) \sim 1/p^{2\epsilon}$ with $1 > \epsilon > 0$ also allows us to satisfy the above conditions. In order to check this, one should recall that $\mathcal{D}(p^2) \sim 1/p^{2\epsilon}$ implies³⁹ $\hat{D}(p^2) \sim p^{2-2\epsilon} + \text{constant}$ and these two asymptotic behaviors yield

³⁸Note that, for a gluon propagator $\mathcal{D}(q^2)$ with a behavior $1/q^2$ at large momenta, the second integral in Eq. (B34) is UV divergent if $d \geq 4$ and $\ell = \infty$.

³⁹Let us recall that, while this (Abelian theorem) is a correct statement, the converse (also called the Tauberian theorem), i.e., $\hat{D}(p^2) \sim p^{2-2\epsilon}$ implies $\mathcal{D}(p^2) \sim 1/p^{2\epsilon}$, is not always true (see, for example, Ref. [115] and Sec. 7.3 in Ref. [116]). This is why the so-called de l'Hôpital's rule does not always apply (see also footnote 41).

the limits in Eq. (C1). Under the same hypothesis we can also verify that the integral [see Eq. (26)]

$$\int_0^\infty dx \ln(x^\eta + M) \times \frac{\eta M[\mathcal{D}(x) - \mathcal{D}(0)] - x(x^\eta + M)\mathcal{D}'(x)}{x^{1+\eta}} \quad (\text{C2})$$

is finite. To this end, let us first consider the integral

$$\int_0^{p^2} dx \ln(x^\eta + M) \times \frac{\eta M[\mathcal{D}(x) - \mathcal{D}(0)] - x(x^\eta + M)\mathcal{D}'(x)}{x^{1+\eta}} \quad (\text{C3})$$

with $0 < p$ and p small. In the limit $x \rightarrow 0$ the integrand behaves as

$$\ln(M)[MC(\eta - \xi)x^{\xi-\eta-1} - B\eta x^{\eta-1}] \quad (\text{C4})$$

and no singularity arises in the integration from $x = 0$ to $x = p^2$ (if $\xi > \eta > 0$). At the same time, for large x the same integrand behaves as

$$\ln(x) \left[\frac{\eta M \mathcal{D}(0)}{x^{1+\eta}} - \mathcal{D}'(x) \right]. \quad (\text{C5})$$

Thus, for sufficiently large ℓ^2 , the integral

$$\int_{\ell^2}^\infty dx \ln(x^\eta + M) \times \frac{\eta M[\mathcal{D}(x) - \mathcal{D}(0)] - x(x^\eta + M)\mathcal{D}'(x)}{x^{1+\eta}} \quad (\text{C6})$$

can be approximated by

$$I_a(\ell^2) = \int_{\ell^2}^\infty dx \ln(x) \left[\frac{\eta M \mathcal{D}(0)}{x^{1+\eta}} - \mathcal{D}'(x) \right]. \quad (\text{C7})$$

After integrating by parts⁴⁰ we then find

$$I_a(\ell^2) = -\ln(x) \left[\frac{M \mathcal{D}(0)}{x^\eta} + \mathcal{D}(x) \right] \Big|_{\ell^2}^\infty + \int_{\ell^2}^\infty dx \left[\frac{M \mathcal{D}(0)}{x^{1+\eta}} + \frac{\mathcal{D}(x)}{x} \right], \quad (\text{C8})$$

which is clearly finite under the assumptions made for the gluon propagator $\mathcal{D}(p^2)$. Finally, the remaining term

⁴⁰Of course, one could also make hypotheses on the first derivative $\mathcal{D}'(x)$ and avoid the partial integration.

$$\int_{p^2}^{\ell^2} dx \ln(x^\eta + M) \times \frac{\eta M [\mathcal{D}(x) - \mathcal{D}(0)] - x(x^\eta + M) \mathcal{D}'(x)}{x^{1+\eta}}, \quad (\text{C9})$$

with $p \ll \ell$, can be easily bounded by making the assumption that neither $\mathcal{D}(x)$ nor $\mathcal{D}'(x)$ displays a singularity for $x \in [p^2, \ell^2]$. Thus, we can conclude that the integral (C2) is indeed finite. We further notice that $I_a(\ell^2)$ is null in the limit $\ell^2 \rightarrow \infty$. Going back to the first proof in Sec. II B, one can verify that the above conditions allow us to show that the Gribov form factor $\sigma(p^2)$ goes to zero for p^2 going to infinity and that the term $-\mathcal{D}(0) \ln(p^2)$ is the only singularity of $\sigma(p^2)$ in the IR limit $p^2 \rightarrow 0$.

The situation is, of course, very similar in the second proof. In this case we considered a finite value for $\mathcal{D}(0)$ and the limit

$$\lim_{p^2 \rightarrow 0} [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2) = 0. \quad (\text{C10})$$

We also imposed, for large momenta p^2 , the limits

$$\begin{aligned} \lim_{p^2 \rightarrow \infty} [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2) &= \lim_{p^2 \rightarrow \infty} \ln(p^2) \mathcal{D}(p^2) \\ &= \lim_{p^2 \rightarrow \infty} \frac{\hat{D}(p^2)}{p^2} = 0. \end{aligned} \quad (\text{C11})$$

Clearly, any gluon propagator with an IR behavior of the type $\mathcal{D}(p^2) \approx \mathcal{D}(0) + B p^{2\eta}$, with $\eta > 0$ and with $\mathcal{D}(0)$ and B finite, satisfies the limit (C10). Also, if we make the hypothesis $\mathcal{D}'(p^2) \sim 1/p^{2+2\epsilon}$, with $1 > \epsilon > 0$ for large values of p^2 we have, in the same limit,⁴¹ $\mathcal{D}(p^2) \sim$

⁴¹As before, one needs to be careful and make hypotheses on $\mathcal{D}'(p^2)$ and not on $\mathcal{D}(p^2)$. For example, after making assumptions on the UV behavior of $\mathcal{D}(p^2)$ one could employ de l'Hôpital's rule in order to obtain

$$\begin{aligned} 0 &= \lim_{p^2 \rightarrow \infty} \mathcal{D}(p^2) \ln(p^2) = \lim_{p^2 \rightarrow \infty} \frac{\mathcal{D}(p^2)}{\frac{1}{\ln(p^2)}} = \lim_{p^2 \rightarrow \infty} \frac{\mathcal{D}'(p^2)}{-\frac{1}{p^2 \ln^2(p^2)}} \\ &= - \lim_{p^2 \rightarrow \infty} p^2 \ln^2(p^2) \mathcal{D}'(p^2) \end{aligned} \quad (\text{C12})$$

and conclude that $\lim_{p^2 \rightarrow \infty} [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2) = 0$ follows from the condition $\lim_{p^2 \rightarrow \infty} \ln(p^2) \mathcal{D}(p^2) = 0$. However, it is easy to find counterexamples to this result. A classical one is $\mathcal{D}(p^2) = \sin(p^2)/p^2$ for which $\lim_{p^2 \rightarrow \infty} [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2)$ is undetermined even though $\lim_{p^2 \rightarrow \infty} \ln(p^2) \mathcal{D}(p^2) = 0$ clearly holds. Thus, in order to apply the result in Eq. (C12), additional auxiliary (also called Tauberian) conditions have to be imposed to the function $\mathcal{D}(p^2)$, for example, on $\mathcal{D}'(x)$ or on $\mathcal{D}''(x)$ (see again Ref. [116]). In particular, imposing $\mathcal{D}(p^2) > 0$ is not sufficient since $\mathcal{D}(p^2) = [2 + \sin(p^2)]/p^2$ is also a counterexample. Here we find it simpler to make hypotheses directly on the asymptotic behavior of $\mathcal{D}'(p^2)$. However, one should also notice that the large- x behavior $\mathcal{D}'(p^2) \sim 1/p^{2+2\epsilon}$, with $1 > \epsilon > 0$, could also imply $\mathcal{D}(p^2) \sim 1/p^{2\epsilon} + \text{constant}$ (see, for example, Ref. [115]). Thus, we also have to impose explicitly the condition $\mathcal{D}(p^2) \rightarrow 0$ for $p^2 \rightarrow \infty$.

$1/p^{2\epsilon}$ and $\hat{D}(p^2) \sim p^{2-2\epsilon}$ and one can easily prove the results in Eq. (C11). As a consequence, we can also verify that the integral [see Eq. (33)]

$$I_b(p^2) = \int_{p^2}^{\infty} dx [x \ln(x) - x] \mathcal{D}''(x) \quad (\text{C13})$$

is finite for any $p^2 \geq 0$. Indeed, we can integrate by parts⁴² obtaining

$$I_b(p^2) = [x \ln(x) - x] \mathcal{D}'(x) \Big|_{p^2}^{\infty} - \int_{p^2}^{\infty} dx \ln(x) \mathcal{D}'(x) \quad (\text{C14})$$

$$= [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2) - \int_{p^2}^{\infty} dx \ln(x) \mathcal{D}'(x). \quad (\text{C15})$$

Note that the integral on the right-hand side of Eq. (C15) also appears in the second term of Eq. (C7). Thus, using Eqs. (C8) and (C11) we have that, for large p^2 , the integral $I_b(p^2)$ is finite and $\lim_{p^2 \rightarrow \infty} I_b(p^2) = 0$. At the same time, for p^2 going to zero we have $\mathcal{D}''(p^2) \sim p^{2\eta-4}$ and the integrand in Eq. (C13) behaves as $[\ln(x) - 1]x^{\eta-1}$. Thus, with $\eta > 0$, no singularity arises⁴³ from the integration at $x = 0$. This result completes the conditions necessary (in the second proof) to show that $\sigma(p^2)$ goes to zero at large momenta and that the IR singularity $-\mathcal{D}(0) \ln(p^2)$ appears in the limit $p^2 \rightarrow 0$.

Finally, due to the well-known results

$$\lim_{x \rightarrow 0} x^\epsilon \ln^a(x) = \lim_{x \rightarrow \infty} \frac{\ln^a(x)}{x^\epsilon} = 0 \quad (\text{C16})$$

for $\epsilon > 0$, it is clear that the above proofs can also be generalized to asymptotic behaviors that include logarithmic functions. At large momenta these logarithmic corrections could be present, for example, if one uses, as an input in the evaluation of $\sigma(p^2)$, a gluon propagator obtained in perturbation theory beyond the tree-level term.

APPENDIX D: THE $d = 4$ CASE USING A MOM SCHEME

Let us start from Eq. (B32), with $\nu = 1$ and $\ell = \infty$, and subtract $\sigma(\mu^2)$ from $\sigma(p^2)$, where μ is a fixed momentum. Then we can write

⁴²Again, instead of integrating by parts, we could also make hypotheses on the second derivative $\mathcal{D}''(x)$.

⁴³For the first term this can be easily shown by integrating by parts.

$$\begin{aligned}
\frac{\sigma(p^2) - \sigma(\mu^2)}{g^2 N_c} = & \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \left\{ \int_0^p dq q^{d-1} \mathcal{D}(q^2) \frac{{}_2F_1(1, 1-d/2; 1+d/2; q^2/p^2)}{p^2} \right. \\
& - \int_0^\mu dq q^{d-1} \mathcal{D}(q^2) \frac{{}_2F_1(1, 1-d/2; 1+d/2; q^2/\mu^2)}{\mu^2} \\
& + \int_p^\mu dq q^{d-3} \mathcal{D}(q^2) {}_2F_1(1, 1-d/2; 1+d/2; p^2/q^2) \\
& + \int_\mu^\infty dq q^{d-3} \mathcal{D}(q^2) [{}_2F_1(1, 1-d/2; 1+d/2; p^2/q^2) \\
& \left. - {}_2F_1(1, 1-d/2; 1+d/2; \mu^2/q^2)] \right\}. \tag{D1}
\end{aligned}$$

In the case $d = 4$ we can use the result (B9) in Appendix B. Thus, the last integral in the above expression becomes

$$\frac{\mu^2 - p^2}{3} \int_\mu^\infty dq \frac{\mathcal{D}(q^2)}{q}, \tag{D2}$$

which is UV-finite for any gluon propagator that goes to zero at large momenta. The apparent linear divergence for large p^2 in the above integral is of course canceled by the third integral in Eq. (D1) above, i.e., by

$$\int_p^\mu dq q \mathcal{D}(q^2) \left(1 - \frac{p^2}{3q^2}\right) = \int_\mu^p dq \frac{\mathcal{D}(q^2)}{3q} (p^2 - q^2). \tag{D3}$$

Then, for $p^2 \rightarrow \infty$ and for $\mathcal{D}(q^2) \sim 1/q^2$ at large momenta one only gets a logarithmic contribution $-\ln(p)$, as expected.

Using the above formula (D1), the proof that $\sigma(p^2)$ is IR-finite for $\mathcal{D}(p^2) > 0$ can be obtained as in Sec. IID.

Indeed, for p^2 going to zero we have to consider only the first and the third integrals⁴⁴ in Eq. (D1). Then, using again the result (B9) we can write

$$\begin{aligned}
& \int_0^p dq q^3 \frac{\mathcal{D}(q^2)}{p^2} \left(1 - \frac{q^2}{3p^2}\right) + \int_p^\mu dq q \mathcal{D}(q^2) \left(1 - \frac{p^2}{3q^2}\right) \\
& \leq \int_0^p dq q^3 \frac{\mathcal{D}(q^2)}{p^2} + \int_p^\mu dq q \mathcal{D}(q^2) \tag{D4}
\end{aligned}$$

and no singularity arises in the limit $p^2 \rightarrow 0$ if $\mathcal{D}(0) > 0$. One arrives at the same result by setting $\mathcal{D}(q^2) = \mathcal{D}(0)$ and by integrating explicitly the left-hand side in the above equation.

⁴⁴Clearly, the second integral does not depend on p^2 and the last one is regular at $p^2 = 0$ [see Eq. (D2)].

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Conclusão

Os dez artigos descritos nos Capítulos 3 e 4 acima foram desenvolvidos, por mim e por meus colaboradores, nos últimos dez anos. Como foi evidenciado ao longo desses dois capítulos, nosso conhecimento sobre o comportamento infravermelho das funções de Green no gauge de Landau e, em específico, dos propagadores de glúons e de fantasmas, evoluiu muito neste período de tempo. Isso foi devido, em parte, à possibilidade de usar máquinas com um poder computacional muito maior. Também contribuiu para isso um melhor entendimento da relação existente entre as soluções analíticas propostas, em geral obtidas no estudo das EDS, e as simulações numéricas na rede. De fato, hoje sabemos que, em três e em quatro dimensões, as funções de Green das teorias de Yang-Mills são bem descritas pela chamada solução massiva, ao passo que, em duas dimensões, é possível considerar somente a solução conforme. Existem porém questões ainda em aberto. A mais importante, do meu ponto de vista, é a explicação dos resultados numéricos no caso $d = 2$. De fato, se por um lado foi demonstrado na Ref. [157] que a solução massiva é incompatível com a condição de ausência de pólos para o caso bidimensional, por outro lado a Ref. [149] sugere que a solução conforme possa ser obtida nas simulações numéricas somente para configurações de rede com propriedades muito peculiares. Deveria então ser possível entender analiticamente por que as configurações de rede para $d = 2$ satisfazem (naturalmente) a essas condições. Isto poderia levar a uma solução analítica do caso bidimensional no gauge de Landau. Ao mesmo tempo, deveria ser melhor entendida a

boa descrição (qualitativa) dos dados de rede obtida, para $d = 4$, usando um mapeamento (a nível clássico) das teorias de Yang-Mills na teoria de um campo escalar sem massa (com um potencial tipo $\lambda\phi^4$) [158, 159]. Além destes aspectos teóricos, é ainda necessário considerar a extrapolação a volume infinito das funções de vértice e estudar suas estrutura tensorial, incluir nos estudos numéricos a extrapolação para o limite do contínuo e estender as simulações ao caso da QCD completa (*unquenched*). Ao mesmo tempo, deveria ser analisado o efeito das cópias de Gribov sobre os resultados obtidos. De fato, foi feita recentemente a hipótese [160, 161, 162, 163] de que a família a um parâmetro de soluções encontradas para as EDS dos propagadores de glúons e de fantasmas (ver discussão na Seção 4.5 acima) poderia estar relacionada à presença das cópias de Gribov e que o valor da função de renormalização de fantasmas $F(0)$ a momento nulo poderia ser usado como parâmetro de fixação de gauge. Seria também de grande importância o desenvolvimento de estudos numéricos sobre a quebra da chamada simetria BRST, induzida pelo parâmetro de Gribov γ , quando a integração funcional é limitada à região Ω (ver, por exemplo, Refs. [60, 61, 96]). Vale ressaltar que um estudo cuidadoso destes últimos dois tópicos requereria provavelmente o desenvolvimento de novos métodos computacionais (no caso dos efeitos das cópias de Gribov) e a introdução de novas ideias (no caso da quebra da simetria BRST). Finalmente, deveria ser entendido como estender os resultados obtidos no gauge de Landau a outras condições de gauge, como por exemplo o gauge de Coulomb, o gauge maximamente abeliano e o gauge covariante linear.

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