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Suppressing information storage in a structured thermal bath

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"I am a sojourner and foreigner among you." - Abraham.

ABSTRACT

MARTINS, W. S. **Suppressing information storage in a structured thermal bath**. 2022. 89p. Dissertation (Master in Science) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2022.

Quantum system tend to have information lost from transmission through the correlations generated among degrees of freedom. For situations where we have a non-Markovian environment, the information contained in it may return to the system, and non-Markovian witnesses can capture such effect. In the present work, we sought how the environment structure affects the system information reachability since its finite size induces a quasi-periodic behavior in the decoherence and enables a highly non-Markovian behavior in the dynamics. To do that, we use a central qubit coupled to a spin chain with Ising interactions subject to a magnetic field, i.e., a central spin model, and solve the exact dynamics of the system. Moreover, we use two witnesses to analyze the presence of non-Markovianity: the Breuer-Laine-Piilo (BLP) trace distance-based measure and the conditional past-future correlator (CPF). On the other hand, we see how such behavior suppresses the classic plateau in Partial Information Plot (PIP) from the paradigm of quantum Darwinism and objective information from Spectrum Broadcast Structure (SBS). In addition to the system point of view, we show that the environmental structure avoids encoding accessible and distinguishable information for measurement in the environment for any model limit. Finally, he orthogonality between the density operators decodes the distinguishability between SBS states in the environment.

Keywords: Quantum non-Markovianity. Open quantum systems. Quantum information theory.

RESUMO

MARTINS, W. S. **Suprimindo o armazenamento de informação em um banho térmico estruturado**. 2022. 89p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2022.

Sistemas quânticos que interagem com ambiente tendem a ter informações perdidas na transmissão por meio das correlações geradas entre graus de liberdade do ambiente. Para situações onde temos um ambiente não-Markoviano, as informações contidas neste podem retornar ao sistema e testemunhas não-Markovianas podem capturar tal efeito. No presente trabalho, buscamos entender como a estrutura do ambiente afeta a acessibilidade da informação do sistema, uma vez que seu tamanho finito induz um comportamento quase periódico na decoerência e possibilita um comportamento altamente não-Markoviano na dinâmica. Para isso, usamos um q-bit central acoplado a uma cadeia de spin com interações de Ising sujeitas a um campo magnético, ou seja, um modelo de spin central, e resolvemos a dinâmica exata deste sistema. Além disso, usamos duas testemunhas para analisar a presença de não-Markovianidade: a medida de Breuer-Laine-Piilo(BLP) e a correlação condicional passado-futuro (CPF). Por outro lado, vemos como tal comportamento suprime o platô clássico no *Partial Information Plot* (PIP) do paradigma do darwinismo quântico e informação objetiva da *Spectrum Broadcast Structure* (SBS). Além do ponto de vista do sistema, mostramos que a estrutura do ambiente evita a codificação de informações acessíveis e distinguíveis para medição no ambiente para qualquer limite do modelo. Por fim, ortogonalidade entre os operadores densidade codifica a distinguibilidade dos estados SBS no ambiente.

Palavras-chave: Não-Markovianidade quântica. Sistemas quânticos abertos. Teoria da informação quântica.

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1 INTRODUCTION

> "A vida é um enclave de ordem num universo fadado à morte térmica."
>
> Haroldo de Campos.

In daily situations, we can not see quantum features; you can not see a superposition of a macroscopic object, for instance. (1) Interestingly, we know that particles of the world are quantum, atoms are quantum, molecules are quantum. Still, when the scale increases to an unknown point, something turns out that the objects constituted only by these quantum atoms are non-quantum objects. (2) But, we can understand important things about this issue; perhaps the first is: quantum states are fragile, i.e., they are easily destroyed by environmental interactions. (3–5)

When a quantum system is in contact with a broad surrounding environment, it entangles with its degrees of freedom and loses the quantumness. (5–7) The quantum information about the system dilutes in the largeness of a more significant object, on which other collective behaviors appear as a result of emergent properties. Here is an ultimate and fatal reality: disintegration of matter by entropy increases, annihilation of cohesion and coherence of atoms and molecules by the action of the quantum and thermal fluctuations. This process is the destination of the entire material world *. (8,9)

There is a mishmash of descriptions for the same aspect of quantum reality. Firstly, one can learn that open quantum systems can be described as stochastic processes. (10,11) It means that we can search for a stochastic equation describing systems concerning the transition maps between two different instants of time. Of course, quantum theory is a probability theory - and exceptional probabilistic theory that allows particular sorts of correlations (12) - and the Schrödinger equation gives these probability distributions. (13,14) As well as classical stochastic processes, the quantum ones are possibly classified regarding these memory effects. A quantum process in which its present state does not depend on the immediate past state is called memoryless or Markovian process, as in a classical situation. (15)

As we said previously, the coupling with many environmental degrees of freedom makes the information about the system decrease in time. The quantum process behind it is the so-called decoherence. Often, as in some finite structured environments †, the

* An exciting text about that is *The Last Question*, by Isaac Asimov.

† Here we say structured environments those in which can be composed by individual indexed Hilbert spaces, e.g., given individual quantum systems described in a Hilbert space $\mathcal{H}_{\mathrm{i}}$, one

information recovers to the system, and the process is named non-Markovian. (16) To manipulate the information about a quantum system, a possible expectation is to use the non-Markovianity to avoid information decreases; however, this scenario demand intricate engineering techniques to manipulate the environment and keep it controlled, as trapped ions. (17–19)

Conversely, another feature of the quantum systems interacting with the surrounding environment is the persistence of particular states, in which its information multiplies as copies throughout environmental components. A natural selection of some states of several particles by correlation is a vital decoherence effect; only populations - diagonal terms - survive when density operators describe these states. (3, 20) Diagonal operators concerning a quantum evolution described by dynamical maps are called pointer states, and the process is called "eisenselection". A heuristic description of this phenomenon has a suggestive name: quantum Darwinism. (4, 21)

Quantum Darwinism aims to set out a reason for objectivity emergence in the classical world. The day-to-day experience seems to show us that all systems have a property of objectivity. When one looks at an object, a pencil, a table, or a virus in an electronic microscope, it always has a well-defined and measurable quantity, e.g., position. This means that when various observers, armed with apparatus, take measures in a classical system, there will ever be an agreement between the parts. Beyond the explanation of the emergence of objectivity in quantum systems given by quantum Darwinism, another path was traced by Spectrum Broadcast Structures (SBS) paradigm. (22–24) This theory uses Bohr's non-disturbance and the characteristics of objective states to build classical ones and explain how it appears in general quantum systems.

Spectrum Broadcast Structures appears recently in the literature and indicates a strong condition to the emergence of Quantum Darwinism and, more specifically, objectivity. (22, 25–29) These structures are characterized by their possibility to be accessed and discriminate without perturbation in a measurement process. (23, 24) This framework demonstrates the structures that enable the emergence of objective states by broadcasting information between the system and environment fractions. A pretty important thing is for these states to be accessible and distinguishable concerning projective measurements.

This work aims to obtain a description of a model consisting of a qubit coupled to a structured finite-size thermal bath in an information-theoretic framework. The idea is to investigate the phenomena of non-Markovianity in two different point-of-view: (1) the system point-of-view, where we describe the information backflow caused by finite-size effects using a non-Markovianity witness and a measure; (2) the environmental perspective, where we will show how the structure of the model affects the storage of distinguishable information available to measure using SBS and quantum Darwinism, from quantum-to-

can build a bigger one as $\mathcal{H} \cong \bigotimes_i \mathcal{H}_\text{i}$.

classical transition paradigm.

This thesis is organized as follows: In the Chapter 2 we develop a theory for large systems starting from one particle quantum systems to many-particle systems and their informational and thermodynamic consequences.

In the Chapter 3 we develop the concepts of quantum channels and decoherence and their consequences in non-Markovianity. We describe the idea of non-Markovianity by information backflow and distinguishability between quantum states in the trace distance-based non-Markovianity witness and by memoryless dynamics using the conditional past-future correlator measure.

In the Chapter 4 we turn the analysis to the environment point-of-view and describe the idea of quantum Darwinism and its emergence in open quantum systems.

Finally, in the Chapter 5 we show the results, with calculations, plots, and important features of the model. Moreover, we trace the conclusions and an outlook with current developments in the area.

Throughout this text, we use $k_B = \hbar = 1$, where the first is the Boltzmann constant and the second is the normalized Planck constant. As for formal issues, the present text sought to build the problem and the tools gradually and use only the necessary formalism to attack the problem. From the introduction, which talks about probability, to the results, we try not to cite - except as a curiosity - concepts such as σ-algebra, groups, etc., to reduce the length of the text and try to make it minimally self-contained for the reader.

2 LARGE SYSTEMS AND INFORMATION

"Nature does not give all of herself in a paragraph. She is rugged and not set apart into discreet categories."

Ezra Pound.

Probability theory is present in all physics. If, on the one hand, we can use a statistical description to obtain the macroscopic behavior of a system microscopically governed by an equation of motion - such as the Schrödinger equation or the equations of classical mechanics -, (30) on the other hand, we have the intrinsically probabilistic character of quantum mechanics. (30–32) In the present chapter, we will develop the theory of statistical physics to obtain, from a microscopic description, the general behavior of physical systems. But here, a significant differentiation is needed a classical description of the macroscopic properties can explain the behavior of a plethora of systems; conversely, the existence of solids or magnetic substances, the properties of black-body radiation, or the extensivity of matter, can only be explained by a quantum-mechanical approach. That means that microscopic description is not synonymous with quantum description. (33)

2.1 Probability theory in a nutshell

Quantum mechanics is a probabilistic theory. Of course, a "special" kind of probabilistic theory, where there are correlations with peculiar features, like entanglement. Then, nothing fairer than starting with an elementary description of probability.

We can start by looking at the simplest case: two independent objects A and B. Let us assume the case in which the object A has total d_{A} possible outcomes - called "events" (31, 32, 34) - and the set of these possible outcomes can be denoted by $\Omega^{\mathrm{A}} = \{\omega_i^{\mathrm{A}}, i = 1, ..., d_{\mathrm{A}}\}$. For example, the simplest case of a coin - two possible outcomes - can be expressed as $\Omega^{\mathrm{A}} = \{\omega_0^{\mathrm{A}}, \omega_1^{\mathrm{A}}\}$, where ω_0^{A} express heads and ω_1^{A} tails. Similarly, the object B is such that it has d_{B} possible outcomes, and the set of these possible outcomes is $\Omega^{\mathrm{B}} = \{\omega_i^{\mathrm{B}}, i = 1, ..., d_{\mathrm{B}}\}$, in the same way as A.

A joint possible outcome for these two objects is $(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}})$ and forms a set that we denote by $\Omega^{A} \times \Omega^{\mathrm{B}}$, in which the symbol "$\times$" means the cartesian product between two sets. Take an example, when $\Omega^{A} = \{\omega_0^{A}, \omega_1^{\mathrm{A}}\}$ and $\Omega^{\mathrm{B}} = \{\omega_0^{\mathrm{B}}, \omega_1^{\mathrm{B}}\}$, we have $\Omega^{\mathrm{A}} \times \Omega^{\mathrm{B}} = \{(\omega_0^{A}, \omega_0^{\mathrm{B}}), (\omega_0^{A}, \omega_1^{\mathrm{B}}), (\omega_1^{\mathrm{A}}, \omega_0^{\mathrm{B}}), (\omega_1^{\mathrm{A}}, \omega_1^{\mathrm{B}})\}$. In general, the set $\Omega^{\mathrm{A}} \times \Omega^{\mathrm{B}}$ contains total $d_{\mathrm{A}} d_{\mathrm{B}}$ elements. The joint probability distribution $p_{\mathrm{AB}}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}})$ for the joint experiment realized with the objects needs to satisfy the following conditions

$$p_{\mathrm{AB}}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}}) \geq 0, \qquad \sum_{i=1}^{d_{\mathrm{A}}} \sum_{j=1}^{d_{\mathrm{B}}} p_{\mathrm{AB}}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}}) = 1. \tag{2.1}$$

These conditions comes from the definition of probability. (31) To make sense, probabilities need to be positive, and the sum of all probabilities of individual realizations sum up to one.

Then, the probability for getting the outcome $\omega_i^{\mathrm{A}} \in \Omega$ measuring the object A is given by the marginals

$$p_{\mathrm{A}}(\omega_i^{\mathrm{A}}) := \sum_{j=1}^{d_{\mathrm{B}}} p_{\mathrm{AB}}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}}) \tag{2.2}$$

and, the object B gives a analogous result

$$p_{\mathrm{B}}(\omega_j^{\mathrm{B}}) := \sum_{i=1}^{d_{\mathrm{A}}} p_{\mathrm{AB}}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}}). \tag{2.3}$$

These definitions tells that when we look just at the marginals, the functions also need to be probability functions. These marginals behave as projections of the total distributions.

With these structures in hand, we can establish the correlation relations by conditional probabilities between two observers taking measurements on A or B. For a probability distribution $p_{AB}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}})$, we can search for correlations between both objects using conditional probabilities, i.e.,

$$p_{\mathrm{A|B}}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}}) = \frac{p_{\mathrm{AB}}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}})}{p_{\mathrm{B}}(\omega_j^{\mathrm{B}})}, \tag{2.4}$$

reads "the probability to obtain ω_i^{A} given occurence of ω_j^{B}". This is an important result formalized by Andrey Kolmogorov in his theory of probabilities (35).

Analogously, the object B enjoys a symmetric situation with respect to A,

$$p_{\mathrm{B|A}}(\omega_j^{\mathrm{B}}, \omega_i^{\mathrm{A}}) = \frac{p_{\mathrm{AB}}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}})}{p_{\mathrm{A}}(\omega_i^{\mathrm{A}})}, \tag{2.5}$$

then, if the joint distribution $p_{\mathrm{AB}}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}})$ has no correlation at all, the events in a such object independ on the other. Of course, in the language of conditional probabilities this result is expressed by

$$p_{\mathrm{A|B}}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}}) = p_{\mathrm{A|B}}(\omega_i^{\mathrm{A}}, \omega_k^{\mathrm{B}}), \tag{2.6}$$

for all i, j, k integers. Similarly

$$p_{\mathrm{B|A}}(\omega_j^{\mathrm{B}}, \omega_i^{\mathrm{A}}) = p_{\mathrm{B|A}}(\omega_j^{\mathrm{B}}, \omega_k^{\mathrm{A}}), \tag{2.7}$$

for all j, i, k integers. These last two results implies that the joint probability distribution is equal to the product of the probability distribution of each party, i.e., $p_{\mathrm{AB}}(\omega_i^{\mathrm{A}}, \omega_j^{\mathrm{B}}) = p_{\mathrm{A}}(\omega_i^{\mathrm{A}}) p_{\mathrm{B}}(\omega_j^{\mathrm{B}})$ and vice versa. (11, 31)

Let us now introduce correlation functions - a statistical tool that measures correlations between events - and, for this purpose, it is necessary to use a random variable $X(\Omega_{\mathrm{A}})$ *. The average value of this random variable can be given by

$$E(X) = \sum_{i=1}^{d_{\mathrm{A}}} p_{\mathrm{A}}(\omega_i^{\mathrm{A}}) X(\omega_i^{\mathrm{A}}). \tag{2.8}$$

For simplicity, another ways to write this definition is

$$E(X) \equiv \langle X \rangle = \sum_{x \in X} p(x)x, \tag{2.9}$$

where the sum runs over all possible values in X. If we take a random variable $Y(\Omega_B)$ in the same way, i.e.,

$$E(Y) = \sum_{j=1}^{d_{\mathrm{B}}} p_{\mathrm{B}}(\omega_j^{\mathrm{B}}) Y(\omega_j^{\mathrm{B}}), \tag{2.10}$$

and also

$$E(Y) \equiv \langle Y \rangle = \sum_{y \in Y} p(y)y. \tag{2.11}$$

Here, the new notation get clearer if we explain what is a random variable.

The direct product of these two random variables gives us a random variable defined on $\Omega_A \times \Omega_B$ space (31), and then the average value for this random variable $X \times Y$ is

$$E(X, Y) \equiv \langle X, Y \rangle = \sum_{x \in X} \sum_{y \in Y} p(x, y)xy. \tag{2.12}$$

An example of paramount importance for our purposes is a random variable correspondent to the space of events $\Omega^A = \{\omega_0^A, \omega_1^A\}$ defined by $X(\omega_0^A) = 0$ and $X(\omega_1^A) = 1$. A variable like this is called a “bit”, with a notable importance for information and computation theory. Throghout the present text we will use computational basis, i.e., a basis constituted by - or a composition of - bits. For N bits, a possible value is a binary string of lenght N, i.e., $x_N x_{N-1} ... x_1$, where each x_i is a bit, i.e., $x_i \in \{0, 1\}$. There are a total 2^N possible values.

Going back to the correlation between two random variables X and Y, let us define the correlation function, which is given by

$$C(X, Y) := E(X, Y) - E(X)E(Y) \equiv \langle X, Y \rangle - \langle X \rangle \langle Y \rangle . \tag{2.13}$$

Then, we can say that two random variables does not have any correlation if and only if $C(X, Y) = 0$ for all X and Y.

* A random variable X is a measurable function that takes possible outcomes $\omega \in \Omega$ to a measurable space, in which the probability that X takes on a value in the measurable space is $p(X = x) = p(\{\omega \in \Omega | X(\omega) \text{ in the measurable space}\})$. (31, 34) The lower case letters reffers to the realizations of this variable.

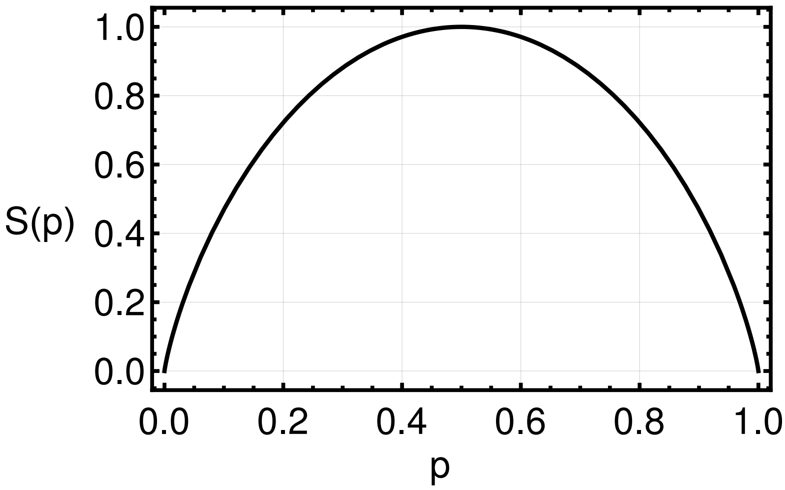

Figure 1 – Entropy S in terms of the probability p. Here is easy to see that the complete uncertainty becomes when $p = 1/2$, i.e., a coin with no bias.

Source: By the author

In the context of information theory established by Shannon, (36) a concept quantifies the degree of correlation with operational meaning between two sub-systems. The idea is the mutual information. For two random variables X and Y whose the joint probability distribution is $p(x, y)$, the mutual information between them is given by

$$I(X, Y) = \sum_{x \in X} \sum_{y \in Y} p(x, y) \log \left(\frac{p(x, y)}{p(x)p(y)} \right) \tag{2.14}$$

To understand what $I(X, Y)$ means we need to introduce the concepts of entropy and conditional entropy.

To motivate the entropy, let us suppose that one receives a message consisting in a string of symbols 0 and 1, i.e., 01110010... and suppose that 0 happens with probability p and 1 with possibility $1-p$ (following Witten approach on Ref. (37)) We want to describe how many bits of information one can extract from a long message of this kind with N symbols. For large N, the message will consist very nearly of pN occurrences of 0 and $(1-p)N$ occurrences of 1. Using combinatorics, the number of such messages is

$$\frac{N!}{(pN)!((1-p)N)!} \approx \frac{N^N}{(pN)^{pN}((1-p)N)^{(1-p)N}} = \frac{1}{p^{pN}(1-p)^{(1-p)N}} = 2^{NS}, \tag{2.15}$$

where S is the Shannon entropy per letter and we used the Stirling's approximation in the first step (38),

$$S(p) = -p \log p - (1-p) \log(1-p). \tag{2.16}$$

Then, we can understand entropy as an amount of lost information in acquiring of information. In Fig.1 is showed the behavior of for the case described above.

Considering now a most general case, in which the messages can be pursuit $k < \infty$ outcomes given by the discrete random variable that take values $x_1, ..., x_k$, and the probability to obtain x_i is $p_i \equiv p(x_i)$ for $i = 1, ..., k$. The entropy function is given by

$$\frac{N!}{(p_1 N)!...(p_k N)!} \approx \frac{N^N}{\prod_{i=1}^{k}(p_i N)^{(p_i N)}} = 2^{NS} \tag{2.17}$$

and can be explicitly writting as

$$S(\{p_i\}) := -\sum_i p_i \log p_i. \tag{2.18}$$

Another way to search for the entropy function is paying attention to general properties of a such function. Shannon postulate that S is a smooth function of the probability distribution $p(x)$ with the following properties:

1. It should be maximal when $p(x)$ is uniform, and in this case, it should increase with the number of possible values X can take;
2. It should remain the same if we reorder the probabilities assigned to different values of X;
3. The uncertainty about two independent random variables should be the sum of the uncertainties about each of them.

It was showed that the only measure of uncertainty that satisfies all these conditions is the entropy, defined as

$$S(x) := -\sum_{x\in X} p(x) \log p(x). \tag{2.19}$$

Notice this function is a generalization of the heuristic definition given by our discrete variable in the last example.

Moreover, for the joint probability distribution $p(x, y)$ of two random variables, the entropy will be

$$S(X, Y) = \sum_{x\in X}\sum_{y\in Y} p(x, y) \log p(x, y). \tag{2.20}$$

Now, we can define a new quantity, $S(X|Y)$, so-called conditional entropy, i.e., the entropy X conditional on the variable Y taking the value y, defined by

$$S(X|Y = y) = -\sum_{x\in X} p(x|y) \log p(x|y), \tag{2.21}$$

where $p(x|y) \equiv p_{A|B}(\omega_i^A, \omega_j^B)$. For most clarity in relation to the meaning of the conditional entropy,we can write down it explicitly in terms of the probability distributions, i.e.,

$$\begin{aligned} S(X|Y) &= \sum_{y\in Y} p(y) S(X|Y = y) \\ &= -\sum_{x\in X}\sum_{y\in Y} p(y)p(x|y) \log p(x|y) \\ &= -\sum_{x\in X}\sum_{y\in Y} p(x, y) \log \frac{p(x, y)}{p(y)}. \end{aligned} \tag{2.22}$$

Then, given a specific realization of the variable Y, we get the conditional entropy of a culmination of the variable X.

Finally, we can turn back to the mutual information to re-write it using entropy functions concerning our random variables, i.e.,[†]

$$\begin{aligned} I(X,Y) &= S(X) + S(Y) - S(X,Y), \\ &= S(X) - S(X|Y), \\ &= S(Y) - S(Y|X), \\ &= S(X,Y) - S(X|Y) - S(Y|X). \end{aligned} \tag{2.23}$$

Then, we can see that the mutual information quantifies the correlation of the joint probability distribution of a composite system.

All these concepts will be recovered using a proper quantum generalization. In the following sections, we present a theory of quantum probabilistic distributions by the idea of density operator, a natural extension of probability for the quantum realm. Also, from the concept of density operator, we build the definition of von Neumann entropy, widely used in the present work.

2.2 States and density operator theory

More than a mere quantitative description of physical systems, states are the central objects for mathematical description of nature, with an excelsior meaning that explains the characterization of the availability of information in every phenomenon. Pure states specify all information about the systems for classical and quantum systems. In contrast, mixed states are a probability measure on the space of pure states. Its evolutions are given by general rules of the widest formulation of physical theory, e.g., Liouville equation for classical systems and von Neumann equation for the quantum ones. (14, 40) Wherefore, physical states properly tell us about the possibility of knowledge about a physical system. This overview puts the statistical description of physical systems and information theory side by side, as explained below.

The quantum theoretical correspondent of the classical probability distributions is not state vectors but density operators. A density operator ρ can be defined in the following way: (14, 40, 41)

1. ρ is Hermitian, i.e., $\rho = \rho^\dagger$.
2. ρ is positive: for any $|\psi\rangle$, $\langle\psi|\,\rho\,|\psi\rangle \geq 0$.
3. $\mathrm{Tr}\,\rho = 1$.

† This is true for classical probability theory; in quantum one this equality need to be modified using the idea of quantum discord. (39)

It follows that ρ can be diagonalized in an orthonormal basis, that the eigenvalues are all real and nonnegative, and that the eigenvalues sum to one - what recovers its probabilistic properties. Considering a situation when one prepare $|\psi_k\rangle$ with $k = 1, 2, ..., N$ with probability p_j. This is associated with the density operator

$$\rho = \sum_k^N p_k |\psi_k\rangle \langle\psi_k| . \tag{2.24}$$

and $\mathrm{Tr}\,\rho^2 \leq 1$, in which the inequality saturates only for pure states, i.e., states that can be described as

$$\rho = |\psi\rangle \langle\psi| . \tag{2.25}$$

Along this text, we just going to deal with qubit-states, thus, every state will be the form $|00...0\rangle$, $|00...1\rangle$, ..., $|11...1\rangle$, and then the sum is taking over 2^N different combinations (see the appendix B).

2.3 Physical states and information

As well as we introduce before a measure of information by the Shannon entropy, here an important quantity to measure uncertainty of a state ρ is the von Neumann entropy, defined by (14)

$$S(\rho) = -\mathrm{Tr}\rho \log \rho, \tag{2.26}$$

which is a generalization of the Shannon entropy to the quantum case. The correspondence between these two measures emerges when we written the state density operator in its spectral decomposition, i.e., $\rho = \sum_\chi p_\chi |\chi\rangle \langle\chi|$, we have $S(\rho) = S(\{p_\chi\}) = -\sum_\chi p_\chi \log p_\chi$. The von Neumann entropy is an important quantity for a quantum theory of information, (14, 39) and its features are explored bellow.

Some essential properties follow from that structure of von Neumann entropy, and now we will explain them to clarify the meaning of entropy in the quantum domain. (14, 33, 42) Let us start with a generic quantum state defined by the density matrix ρ, which acts on its association to a qubit Hilbert space $(\mathbb{C}^2)^{\otimes N}$, this is the scope of our discussion.

First, $S(\rho)$ defined in Eq. (2.26) is zero if only if ρ represents a pure state. This fact can be viewed by considering a generic state in a diagonal basis $\rho = \sum_\chi p_\chi |\chi\rangle \langle\chi|$, and one can write the entropy as $S(\{p_\chi\}) = S(p_1, ..., p_{2^N})$. For a pure state, $S(1, 0, ..., 0) = 0$, and it means that the event is a certainty, in the language of classical probability. One gains no information, when one knows in advance which message one is about to receive.

On the other hand, $S(\rho)$ is maximal and equal to $N \log 2$ for a maximally mixed state, i.e., a state defined by

$$\rho = \frac{\mathbb{1}}{2^N}. \tag{2.27}$$

that represents the situation in which, with respect of the diagonalized density matrix, one have $p_1 = ... = p_d = 2^{-N}$, or any event can occur with the same probability.

Another important property: $S(\rho)$ is invariant under changes in the basis of ρ, i.e., $S(\rho) = S(U\rho U^\dagger)$, with U a unitary transformation and also is concave. Given a collection of $\{\lambda_i\} \geq 0$ numbers sum to unit, and a collection of density operators $\{\rho_i\}$, we have the inequality

$$S\left(\sum_i \lambda_i \rho_i\right) \geq \sum_i \lambda_i S(\rho_i). \tag{2.28}$$

Its interpretation is simple: it means that if we combine two statistical ensembles relating to the same events into a single new ensemble, the uncertainty is more significant than the average of the initial uncertainties. Now, using the same definition, $S(\rho)$ always satisfies the bound

$$S\left(\sum_i \lambda_i \rho_i\right) \geq \sum_i \lambda_i S(\rho_i) - \sum_i \lambda_i \log \lambda_i. \tag{2.29}$$

where equality holds when ρ_i has orthogonal support.

$S(\rho)$ is strong subaddditive for any three systems A, B and C, i.e.,

$$S(\rho_{\mathrm{ABC}}) + S(\rho_{\mathrm{B}}) \leq S(\rho_{\mathrm{AB}}) + S(\rho_{\mathrm{BC}}). \tag{2.30}$$

From that automatically follow $S(\rho)$ is subadditive:

$$S(\rho_{\mathrm{AB}}) \leq S(\rho_{\mathrm{A}}) + S(\rho_{\mathrm{B}}). \tag{2.31}$$

This last inequality is essential to quantum mechanics: if ρ_A acts on $\mathcal{H}_{\mathrm{A}}$ and ρ_{B} acts on $\mathcal{H}_{\mathrm{B}}$, the equal sign holds only when $\rho_{\mathrm{AB}} = \rho_{\mathrm{A}} \otimes \rho_{\mathrm{B}}$ and we say that these two systems are uncorrelated. We can reformulate this idea in a most dramatic form, just saying that, in quantum mechanics, information is not defined locally.

Recovering the Eq. (2.23), if we consider two expressions which each, in the classical limit, represent the mutual information, i.e.,

$$I(A, B) = S(A) + S(B) - S(A, B) \tag{2.32}$$
$$\mathcal{I}(A, B) = S(A) - S(A|B), \tag{2.33}$$

in the nonclassical case, the quantum generalization gives us

$$I(A, B) = S(\rho_{\mathrm{A}}) + S(\rho_{\mathrm{B}}) - S(\rho_{\mathrm{AB}}) \tag{2.34}$$
$$\mathcal{I}_{\mathrm{A}}(A, B) = S(\rho_{\mathrm{B}}) - S(\rho_{\mathrm{B}}|\rho_{\mathrm{A}}), \tag{2.35}$$

and the difference between the two expressions defines the basis-dependent quantum discord (39)

$$\mathcal{D}_{\mathrm{A}}(\rho) := I(\rho) - \mathcal{I}_{\mathrm{A}}(\rho), \tag{2.36}$$

that is a measure of nonclassical correlations between two sub-systems of a quantum system, and it is assymetric, because $\mathcal{I}_{\rm A}(A,B) = S(\rho_{\rm B}) - S(\rho_{\rm B}|\rho_{\rm A}) \neq S(\rho_{\rm A}) - S(\rho_{\rm A}|\rho_{\rm B}) = \mathcal{I}_{\rm B}(A,B)$.

In the next chapter, the notion of entropy of an ensemble state will be handy. To search for equilibrium states, we need to use the assumption that these states are the result of the maximization of the entropy for some specified conditions - both entropy and the requirements results in the minimization of the *free energy*. Broadly, free energy is the portion of any first-law energy available to perform thermodynamic work at a constant temperature.

2.4 Thermodynamical equilibrium

The central statistical physics insight is the idea of "ensemble", a mental construction to describe a probabilistic treatment of complex systems that are not treatable using mere tools of few-body problems. Of course, a common picture about the problem is illustrated by the increased degrees of freedom in approaching many-body systems using individual equations of motion for each particle in it. One wants to arrive at some macroscopic description by microscopic formulations. Instead of it, statistical physics takes hypothetic copies of the same system and its possible states, constrained by energetic postulates and probabilistic assumptions.

Apart from these well-important considerations, here we are interested in an informational approach to these thermodynamical and statistical properties. Once specified a treatment for quantum and large systems, we have powerful tools to bridge between the probabilistic distribution of micro-states and macro-states, and these last ones will give the function from which we derive the thermodynamic properties.

Statistical physics builds the bridge between the macro and the micro from statistical ensembles in thermal equilibrium. (30, 43, 44) Here, a very important one for us is the Gibbs ensemble or Canonical ensemble. Roughly speaking, the Gibbs ensemble gives the energetic distribution that minimizes the Helmholtz free energy. Given a Hamiltonian H in the Gibbs formalism, we can decompose it in its energy basis, i.e.,

$$H = \sum_{\chi} E_{\chi} \left|\chi\right\rangle \left\langle\chi\right| . \tag{2.37}$$

If this system is in equilibrium, then the probability of finding it in a state $|\chi\rangle$ will be given by

$$p_{\chi} := \frac{e^{-\beta E_{\chi}}}{Z}, \tag{2.38}$$

where the normalization is given by the partition function $Z := \operatorname{Tr} e^{-\beta H} = \sum_{\chi} e^{-\beta E_{\chi}}$, where $\beta = 1/T$ is the inverse of temperature, that function encodes all necessarily thermodynamic information.

To attribute a quantum state to the system at finite temperature, we need to use the so-called thermal states, i.e., states satisfying the Gibbs distribution. The density operator of the thermal state is defined as

$$\rho := \sum_{\chi} p_{\chi} \left|\chi\right\rangle \left\langle\chi\right| , \tag{2.39}$$

that gives a complete description of the probabilistic distribution of a system in thermal equilibrium. Note that every diagonal term of the density matrix get a probability distribution of the state correspondent to they respective row and column, e.g., for a case of N qubits the eigenstate $|\chi\rangle$ has as its respective occurence probability P_{χ} in the (χ, χ) matrix entry, with $\chi = 1, ..., 2^N$.

Then, from thermal states, we can obtain the expectation value for the thermal observables. The expectation value of $\langle O\rangle$ can be written using the trace as

$$\langle O\rangle = \frac{\mathrm{Tr}\, Oe^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}, \tag{2.40}$$

And, we can write the thermal state as

$$\rho = \frac{1}{Z}\sum_{\chi} e^{-\beta E_{\chi}} \left|\chi\right\rangle \left\langle\chi\right| = \frac{e^{-\beta H}}{Z}, \tag{2.41}$$

it simply writes the states as a function of the Hamiltonian. (45)

From this, any thermodynamic observable can be calculated; to do so, we just take the trace multiplied by a given operator and divide it by the partition function, e.g., $U := Z^{-1}\mathrm{Tr}\, He^{-\beta H}$ is the internal energy with respect to a system with Hamiltonian operator H.

Here we will picture a general framework to explain some exciting and important things about information and its connection with thermodynamics. Remember that the entropy is a logarithmic measure of the number of system states

$$S(\rho) = -\mathrm{Tr}\rho \log \rho = \sum_{\chi} p_{\chi} \log p_{\chi}, \tag{2.42}$$

where we consider the density matrix in its diagonal form $\rho = \sum_{\chi} \left|\chi\right\rangle \left\langle\chi\right|$.

In what has been called the fundamental postulate in statistical mechanics (*For an isolated system with an exactly known energy and exactly known composition, the system can be found with equal probability in any microstate consistent with that knowledge*) (46), among system degenerate microstates, each microstate is assumed to be populated with equal chance. Then, for an isolated system, $p_i = 1/\Omega$ where Ω is the number of microstates whose energy equals the system's energy, and the entropy becomes

$$S(\rho) = \log \Omega \tag{2.43}$$

Barring a constant, this is the thermodynamic entropy. (44)

2.4.1 Partition function and Lee-Yang zeros

Before we talk about Lee-Yang zeros properly, let us trace the relationship of the partition function to the Helmholtz free energy. Then, taking as a definition of partition function, the sum (44)

$$Z = \sum_{\chi} e^{-\beta E_\chi}, \tag{2.44}$$

where is associated with the normalization of the probability p_χ. Notice that this sum is took over all microscopic states, thus, given such energy value, there are many coincident terms, corresponding to all the microscopic states with this particular energy value. Taking into account this factor of degenerecence, we can write

$$Z = \sum_{\chi} e^{-\beta E_\chi} = \sum_{E} \Omega(E) e^{\beta E_\chi}, \tag{2.45}$$

where $\Omega(E)$ is the number of microscopic states of the system with energy E. For a large system, we can take only the maximal term of the sum, i.e.,

$$Z = \sum_{E} e^{\log \Omega(E) - \beta E} \propto e^{-\beta \min_E \{E - TS\}} \tag{2.46}$$

where again we used the result $S(E) = \log \Omega(E)$. The minimization operation with respect to the correspondent energy is a Legendre transformation, and this argument suggest a connection between the canonical ensemble and thermodynamics given by the correspondence

$$Z \to e^{-\beta F}, \tag{2.47}$$

where $F = E - TS$ is the Helmholtz free energy. Notice that the Helmholtz free energy can be written as $F = -\beta^{-1} \log Z$, and here an exciting thing happens.

Phase transitions are characterized by a sudden change in the physical properties and, for example, in the order parameter (30, 44). In the Ising model paradigmatic case - in which will be treated here - the order parameter is the magnetization and the ordered phase happens when it is different from zero. Mathematically, a phase transition occurs when the partition function vanishes, and the free energy is singular (non-analytic). Tsung-Dao Lee and Yang Chen-Ning developed a theory for criticality based on singularities of the free energy, called the Lee-Yang theory. (47, 48) In this theory, phase transitions in large physical systems in the thermodynamic limit based on the properties of small, finite-size systems.

Let us consider the energy spectrum $\{E_\chi\}$ of finite number particles in a limited volume is discrete so the state sum the partition function Eq. (2.44), that is positive and, on the positive real axis $\beta > 0$ and in the neighborhood of it, an analytic function of its argument β. In this kind of system, there can be no sharp phase transition point. Phase transitions can thus occur only in the *thermodynamic limit*, i.e.,

$$V \to \infty \qquad \text{and} \qquad N \to \infty, \tag{2.48}$$

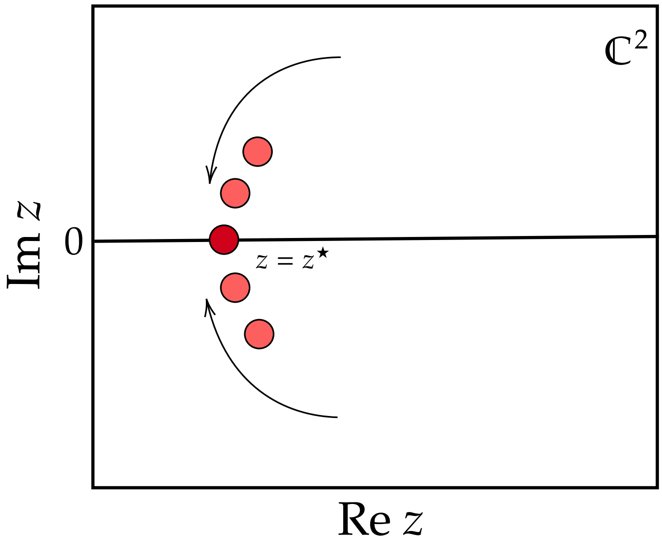

Figure 2 – General scheme of the distribution of Lee-Yang zeros in the complex plane. Considering an external parameter z, it is clear that as

Source: By the author

but with $N/V \to$ constant. The model by Lee and Yang explains how the analytic state sum develops toward non-analytic form when we approach the thermodynamic limit. (48, 49) We consider a system of hard spheres confined in the volume V. Let V_0 be the volume of one sphere, then

$$N \approx \frac{V}{V_0}, \tag{2.49}$$

is the maximum number of spheres. The state sum

$$Z_G(T, V, \mu) = \sum_{n=0}^{N} Z(T, V, N) \tag{2.50}$$

is a polynomial of degree N of the fugacity $z = e^{\beta\mu}$ and Z_G represents the grand canonical partition function. We use the shorthand notation $Z(z) = Z_G(T, V, \mu)$ and now, consider $z_1, z_2, ...z_N$ be the zeros of the polynomial $Z(z)$. Since $Z(0) = 1$, we have according to the fundamental theorem of algebra,

$$Z(z) = \prod_{n=1}^{N} \left(1 - \frac{z}{z_n}\right). \tag{2.51}$$

Because $Z(z)$ is real when z is real, the zeros must occur as conjugate pairs, and when we approach the thermodynamic limit, the number of zeros of the partition function tends to infinity. The interesting situation is given when the parameter z comes the critical point $z^{\star}$: in this case; the Lee-Yang zeros approaches the real axis at the critical point in the fugacity plane (Fig. 2).

As an example of how that theory works, let us consider the classical Ising chain with N spins and periodic boundary conditions (50, 51), i.e.,

$$H_{\text{Ising}} = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i \tag{2.52}$$

such that $\sigma_{N+1} = \sigma_i$ and $\sigma_i = \pm 1$ like in almost all textbooks of statistical physics.

Let us calculate the partition function of this N spins model with $\sigma = \{\sigma_1, ..., \sigma_N\}$ and its respective eigenvalues $\sigma_i = \pm 1$

$$\begin{aligned} Z &= \operatorname{Tr} e^{-\beta H_{\text{Ising}}}, \\ &= \sum_{\sigma_1,\ldots,\sigma_N} e^{\beta(J\sum_i \sigma_i\sigma_{i+1} + h\sum_i \sigma_i)}, \\ &= \sum_{\sigma_1,\ldots,\sigma_N} e^{\beta[J\sigma_1\sigma_2 + \frac{h}{2}(\sigma_1+\sigma_2)]}...e^{\beta[J\sigma_N\sigma_1 + \frac{h}{2}(\sigma_N+\sigma_1)]}, \end{aligned} \tag{2.53}$$

in which we write the exponent in the symmetric form for convenience. Then, defining the transfer matrix $T(\sigma_i, \sigma_{i+1}) := e^{\beta[J\sigma_i\sigma_{i+1} + \frac{h}{2}(\sigma_i+\sigma_{i+1})]} = \langle\sigma_i|\, T\, |\sigma_{i+1}\rangle$ we can re-writte the partition function as

$$Z = \sum_{\sigma_1,\ldots,\sigma_N} T(\sigma_1,\sigma_2)T(\sigma_2,\sigma_3)...T(\sigma_{N-1},\sigma_N)T(\sigma_N,\sigma_1) \tag{2.54}$$

The matrix elements of T can be computed directly from the definition

$$T = \begin{pmatrix} T(1,1) & T(1,-1) \\ T(-1,1) & T(-1,-1) \end{pmatrix} = \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{\beta J} & e^{\beta(J-h)} \end{pmatrix} \tag{2.55}$$

and finally, the partition function may be written now as

$$Z = \operatorname{Tr} T^N = \lambda_+^N + \lambda_-^N \tag{2.56}$$

where $\lambda_\pm$ are the eigenvalues of T.

Here, $\lambda_\pm$ can be computed using some linear algebra, i.e.,

$$\det(T-\lambda) = \lambda^2 - 2\lambda(2e^{\beta J}\cosh(\beta h)) + (e^{2\beta J} - e^{-2\beta J}) = 0. \tag{2.57}$$

And this characteristic vector of the linear transformation defined by the transition matrix gives us two eigenvalues

$$\lambda_\pm = e^{\beta J}\cosh(\beta h) \pm \sqrt{e^{2\beta J}\sinh^2(\beta h) + e^{-2\beta J}}. \tag{2.58}$$

In terms of these eigenvalues, we immediately can be re-write the partition function

$$Z = e^{N\beta J}\left[\left(\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right)^N + \left(\cosh(\beta h) - \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right)^N\right]. \tag{2.59}$$

The partition function can be described as a polynomial in terms of some variable. Then, we can search for the Lee-Yang zeros, i.e., the zeros of this polynomial. Make $z := e^{-2\beta h}$ and $x := e^{-2\beta J}$, to write the follow (52)

$$Z = e^{N(\beta J+\beta h)} \left[\frac{1+z}{2} + \sqrt{\left(\frac{1-z}{2}\right)^2 + x^2 z}\right]^N \left[1 + \underbrace{\left(\frac{\frac{1+z}{2} - \sqrt{\left(\frac{1-z}{2}\right)^2 + x^2 z}}{\frac{1+z}{2} + \sqrt{\left(\frac{1-z}{2}\right)^2 + x^2 z}}\right)}_{=0}\right]^N \tag{2.60}$$

and then

$$\frac{\frac{1+z}{2} - \sqrt{\left(\frac{1-z}{2}\right)^2 + x^2 z}}{\frac{1+z}{2} + \sqrt{\left(\frac{1-z}{2}\right)^2 + x^2 z}} = e^{i\pi\frac{(2n-1)}{N}} = e^{ik_n} \tag{2.61}$$

where $n = 1, 2, ..., N$. If $J > 0$, we solve the equation to obtain

$$z_n = -e^{-4\beta J} + (1 - e^{-4\beta J})\cos(k_n) \pm i\sqrt{(1 - e^{-4\beta J})[\sin^2(k_n) + e^{-4\beta J}(1 + \cos(k_n))^2]} \tag{2.62}$$

that gives to us the distribution of Lee-Yang zeros in the complex plan. Figure 3 show the behavior of the Lee-Yang zeros for that case.

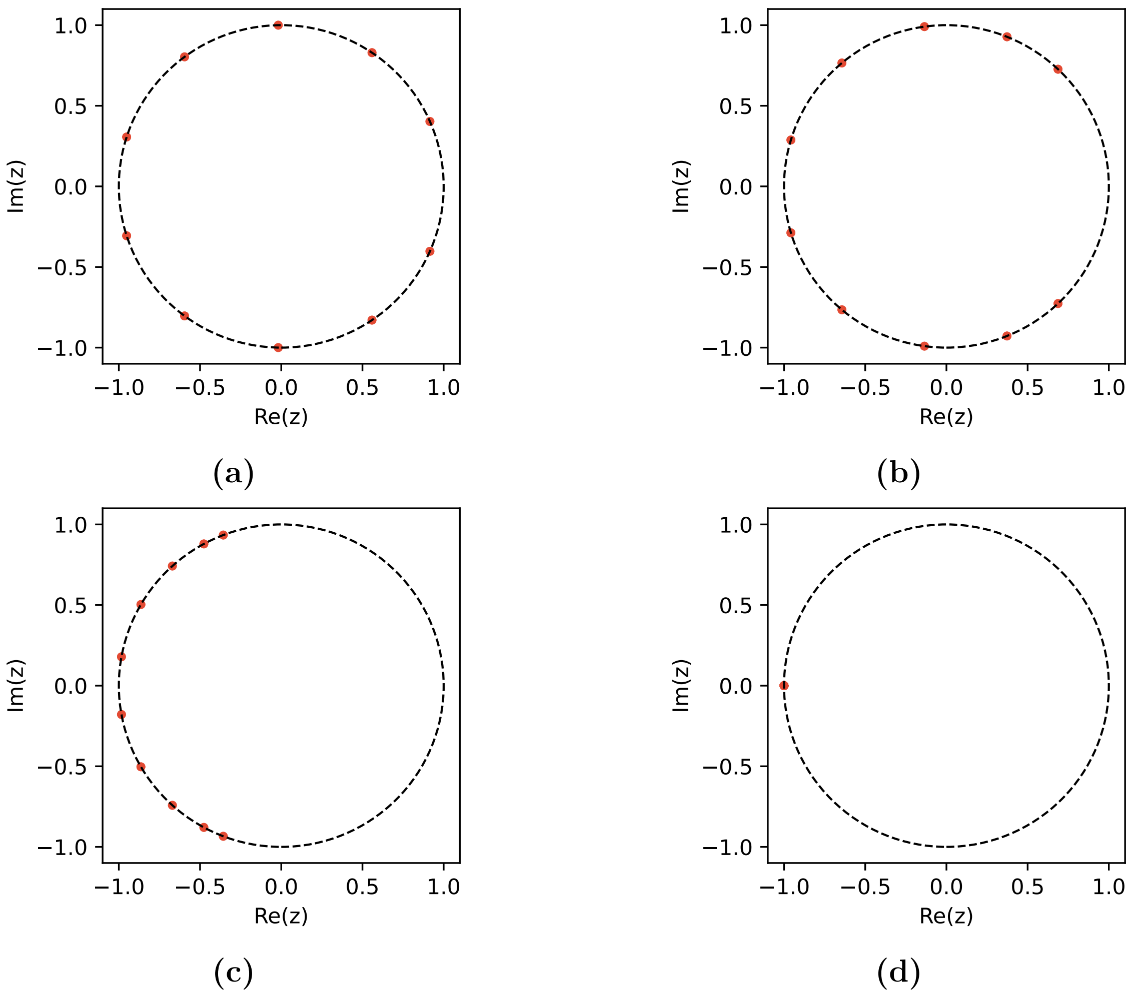

Figure 3 – Lee-Yang zeros on the fugacity plan with $N = 10$ and $J = 1$, for $\beta = 1, 0.5, 0.1, 0$, respectively showed in the Figs. (a), (b), (c), and (d). As well as the temperature increase the zeros approaches to $z = -1$. Here, changes in the variable z are caused by the magnetic field h, and zeros tend to accumulate on the real axis at the phase transition. Only real roots characterize real phase transitions in the thermodynamic limit.

Source: By the author

3 OPEN SYSTEMS I: SYSTEM POINT-OF-VIEW

"Between the idea
And the reality
Between the motion
And the act
Falls the Shadow"

The Hollow Man, by T.S.Eliot.

The present chapter will present an usual perspective of open quantum systems. The idea here is to describe an unitary evolution of quantum operators in closed systems and its natural generalization for open systems. Also, we use this perspective to build the idea of non-Markovian dynamics as a consequence of the evolved operator's behavior by looking at the reduced dynamics that represent the system point-of-view.

3.1 Dynamics: channels and decoherence

So far, we deal with stationary states of quantum systems (with one or many particles). But, in the present chapter, we will present quantum dynamics in open scenarios. To do that, let us start with Schrödinger's equation, valid for pure states (13)

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle\,, \tag{3.1}$$

where $|\psi\rangle \in \mathbb{C}^d$ is a pure quantum state and H the Hamiltonian operator. Of course, we are more interested in mixed states in the present work.

Given a system and an environment represented by Hilbert spaces $\mathcal{H}_{\mathrm{S}}$ and $\mathcal{H}_{\mathrm{E}}$, we can get a combined state $\rho_{\mathrm{SE}}(0)$ as the initial. For mixed density operators, the Liouville-von Neumann equation describes the evolution of the quantum state

$$\frac{\partial}{\partial t}\rho_{\mathrm{SE}}(t) = -\frac{i}{h}\left[H(t), \rho_{\mathrm{SE}}(t)\right] \tag{3.2}$$

where $H(t) = (H_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{E}} + \mathbb{1}_{\mathrm{S}} \otimes H_{\mathrm{E}} + H_{\mathrm{SB}})(t)$ is the total Hamiltonian, with H_{S} acting on $\mathcal{H}_S \cong \mathbb{C}^{d_{\mathrm{S}}}$, H_{E} on $\mathcal{H}_{\mathrm{E}} \cong \mathbb{C}^{d_{\mathrm{E}}}$ - the system and environment Hilbert space, respectively - and H_{SE} on the composite space $\mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{E}} \cong \mathbb{C}^{d_{\mathrm{S}} \times d_{\mathrm{E}}}$. The solution of that results in the unitary evolution operator (sometimes called propagator) $U(t)$

$$U(t) = \mathcal{T} e^{-\frac{i}{\hbar}\int_0^t d\tau H(\tau)}, \tag{3.3}$$

where $\mathcal{T}$ denotes path-ordering operator with respect to time, i.e., time-ordering operator, defined by

$$\mathcal{T}(A(t_1)B(t_2)) = \begin{cases} A(t_1)B(t_2) & \text{for} \quad t_1 < t_2 \\ B(t_2)A(t_2) & \text{for} \quad t_1 > t_2. \end{cases} \tag{3.4}$$

Under a closed dynamics, the combined evolution given by the interaction between system and environment is unitary, and simply reduces to

$$\rho_{\mathrm{SE}}(t) = U(t)\rho_{\mathrm{SE}}(0)U^{\dagger}(t). \tag{3.5}$$

Usually, a scenario for the theory of open quantum systems considers the effect of the vast environment in the subsystem. To do that, we trace out the environment degrees of freedom to recover the reduced dynamics. Let us consider an uncorrelated initial state of both of the system and the environment, i.e., $\rho_{\mathrm{SE}}(0) = \rho_{\mathrm{S}}(0) \otimes \rho_{\mathrm{E}}(0)$, then the effective dynamics of the reduced system can be described as

$$\mathcal{E}(\rho_{\mathrm{S}}(0)) = \rho_{\mathrm{S}}(t) = \mathrm{Tr}_{\mathrm{E}}\left[U(t)\rho_{\mathrm{S}}(0) \otimes \rho_{\mathrm{E}}(0)U^{\dagger}\right] \tag{3.6}$$

Where $\mathrm{Tr}_{\mathrm{E}}[\bullet]$ is the partial trace concerning the environment. The map $\mathcal{E}(\bullet)$* is a so-called channel, and its properties will be clear ahead.

Considering the diagonal decomposion of $\rho_{\mathrm{E}}(0) = \sum_{\chi} p_{\chi} |\chi\rangle\langle\chi|$, where $\sum_{\chi} p_{\chi} = 1$ and the states $\{|\chi\rangle\}$ spanned an orthonormal basis of the Hilbert space $\mathcal{H}_{\mathrm{E}}$. The evolution $U(t)$ will result in a system density operator evolving according to

$$\rho_{\mathrm{S}}(t) = \mathrm{Tr}_{\mathrm{E}}\left[U(t)\rho_{\mathrm{S}}(0) \otimes \left(\sum_{\chi} p_{\chi} |\chi\rangle\langle\chi|\right) U^{\dagger}(t)\right], \tag{3.7}$$

$$= \sum_{\chi\chi'} p_{\chi} \langle\chi'| U(t) |\chi\rangle \rho_{\mathrm{S}}(0) \langle\chi| U^{\dagger}(t) |\chi'\rangle . \tag{3.8}$$

Introducing the operators defined by $E_{ij}(t) = \sqrt{p_i}\langle j| U(t) |i\rangle$, we obtain

$$\rho_{\mathrm{S}}(t) = \sum_{ij} E_{ij}\rho(0)E_{ij}^{\dagger}. \tag{3.9}$$

Those operators have a very interesting structure to guarantee that density operators go to density operators and density operators result from density operators.

That dynamics given by channels define a completely positive trace preserving (CPTP) map, which can be written via Kraus operators:

$$\mathcal{E}(\rho_{\mathrm{S}}) = \sum_{i=1}^{\mathcal{K}} K_i \rho_{\mathrm{S}} K_i^{\dagger} \tag{3.10}$$

where the Kraus operators satisfies $\sum_{i=1}^{\mathcal{K}} K_i^{\dagger} K_i = \mathbb{1}$ for trace-preserving channels. (10,53) Here, we combine the two indexes i and j into a single index to write them. That representation (Kraus representation) is not unique, and the so-called Kraus representation theorem supports its existence, and $\mathcal{K} \leq d_{\mathrm{S}}^2$ is the Kraus number. (53)

Suppose a map $\mathcal{E}(\bullet)$ acting on $\mathcal{H}_{\mathrm{S}}$, with a convex set of density operators satisfying $\rho = \rho^{\dagger}$, $\rho \geq 0$ and $\mathrm{Tr}\rho = 1$, the name "complete positive trace-preserving" can be justified enumerating the properties of these maps, i.e., (40)

* Sometimes a channel is called a superoperator: an operator that takes from operators to operators.

1. Linearity. A quantum channel $\mathcal{E}(\bullet)$ is said a linear map if $\mathcal{E}(\alpha\rho_1 + \beta\rho_2) = \alpha\mathcal{E}(\rho_1) + \beta\mathcal{E}(\rho_2)$.
2. Preserves complete positivity. A quantum channel $\mathcal{E}(\bullet)$ is said completely positive (CP) if the composion $\mathcal{E} \otimes \mathbb{1}_{\mathrm{E}}$ is a positive map for any sub-system $\mathcal{H}_{\mathrm{E}}$.
3. Preserves trace. A quantum channel $\mathcal{E}(\bullet)$ preserves trace if $\operatorname{Tr}\mathcal{E}(\rho) = \operatorname{Tr}\rho = 1$.

Here, a positive map is one where $\mathcal{E}(\rho) \geq 0$, that is, $\mathcal{E}(\rho)$ is positive semi-definite $\rho \geq 0$. These criteria defines a CPTP map.

3.1.1 Dephasing

In the present text, a significant example is the dephasing channel, also called the phase-damping channel. (10, 41, 53–55) An example is the interaction of a dust particle with photons. The collision of the particle with one photon will not change the particle state. Still, if the particle was in the ground ($|0\rangle_{\mathrm{S}}$) or excited state ($|1\rangle_{\mathrm{S}}$), the photon will acquire more or less energy in the collision, thus being excited to its first or second excited state. Let us consider a most straightforward case of isometric representation of the channel, e.g.,

$$\begin{aligned} |0\rangle_{\mathrm{S}} \otimes |0\rangle_{\mathrm{E}} &\to \sqrt{1-p}\,|0\rangle_{\mathrm{S}} \otimes |0\rangle_{\mathrm{E}} + \sqrt{p}\,|0\rangle_{\mathrm{S}} \otimes |1\rangle_{\mathrm{E}} \\ |1\rangle_{\mathrm{S}} \otimes |0\rangle_{\mathrm{E}} &\to \sqrt{1-p}\,|1\rangle_{\mathrm{S}} \otimes |0\rangle_{\mathrm{E}} + \sqrt{p}\,|1\rangle_{\mathrm{S}} \otimes |2\rangle_{\mathrm{E}} \,. \end{aligned} \tag{3.11}$$

Here, notice the system qubit not make transitions in the basis $\{|0\rangle\,|1\rangle\}$ basis. Evaluating the partial trace over $\mathcal{H}_{\mathrm{E}}$ in the $\{|0\rangle_{\mathrm{E}}, |1\rangle_{\mathrm{E}}, |2\rangle_{\mathrm{E}}\}$ basis, a possible representation for the unitary transformation is

$$U = \begin{pmatrix} \sqrt{1-p} & \sqrt{p} & 0 & 0 & 0 & 0 \\ \sqrt{p} & \sqrt{1-p} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-p} & 0 & \sqrt{p} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{p} & 0 & \sqrt{1-p} \end{pmatrix}. \tag{3.12}$$

To obtain the Kraus operators of this transformation, we need to trace out the environmental degrees of freedom to get

$$K_0 = \langle 0\,|\,U\,|\,0\rangle = \sqrt{1-p}\,\mathbb{1}_{\mathrm{S}}, \qquad K_1 = \langle 1\,|\,U\,|\,0\rangle = \sqrt{p}\,|0\rangle\langle 0|\,, \qquad K_2 = \langle 2\,|\,U\,|\,0\rangle = \sqrt{p}\,|1\rangle\langle 1|\,. \tag{3.13}$$

Thus, the state evolution is given by the map

$$\mathcal{E}(\rho) = \sum_i^3 K_i \rho K_i^\dagger = (1-p)\rho + p\,|0\rangle\langle 0|\,\rho\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|\,\rho\,|1\rangle\langle 1|\,, \tag{3.14}$$

or in matrix form:

$$\mathcal{E}(\rho) = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}. \tag{3.15}$$

In the Bloch sphere, pure dephasing represents a precession around the z axis. (41,53) Let us compute how the polarization of the density operator evolves using the representation of Appendix A

$$\rho(\overrightarrow{r}) = \frac{1}{2}(\mathbb{1} + \overrightarrow{r} \cdot \overrightarrow{\sigma}) \mapsto \rho(\overrightarrow{r}'), \tag{3.16}$$

where $r'_{x,y} = (1-p)r_{x,y}$ and $r'_z = r_z$, then the Bloch ball shrinks to a prolate spheroid aligned with z axis.

To our purposes, it is interesting consider the situation $p = p\Delta t = \gamma\Delta t$, in which the probability represents a scatter event during the time Δt. Then, if we have n events in a time $t = n\Delta t$, the off-diagonal terms become $\propto (1-p)^n = (1-\gamma\Delta t)^{t/\Delta t} \approx e^{-\gamma t} := \Gamma(t)$

$$\mathcal{E}(\rho, t) = \begin{pmatrix} \rho_{00} & \Gamma(t)\rho_{01} \\ \Gamma(t)\rho_{10} & \rho_{11} \end{pmatrix}. \tag{3.17}$$

Consider for example an initial pure state $a\,|0\rangle + b\,|1\rangle$. At long times, this state reduces to:

$$\mathcal{E}(\rho, t) = \begin{pmatrix} |a|^2 & \Gamma(t)ab^\star \\ \Gamma(t)a^\star b & |\beta|^2 \end{pmatrix} \xrightarrow{t\to\infty} \begin{pmatrix} |a|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}. \tag{3.18}$$

Then, in the process of decoherence, any phase coherence is lost, and the states reduce to a classical, incoherent superposition of populations. (3, 5, 20)

3.2 Non-Markovianity: classical vs. quantum definition and its physical meaning

A crucial point for understanding the mechanism of decoherence is the study of how information flows from the system to the environment. (5) However, this is only part the story, because information can also flow in the opposite direction, that is, from the environment to the system. We call this non-Markovianity. While in a Markovian process the open system continuously loses information to the environment, a non-Markovian process can be characterized as a flow of information from the environment back into the open system. (56–59)

It is vital for us now to define stochastic process: a family of random variables $\{X(t)\}_{t\in I}$ with $I \subset \mathbb{R}$, usually representing time and can be discrete or continuous, then the stochastic process can be seen as a random variable evolving in time. (11) Now, using the language of the first section, a family of temporal joint probabilities is the joint probability of n events occurring at times $t_0 \leq t_1 \leq ... \leq t_n$ is given by

$$p(x_n, t_n; x_{n-1}, t_{n-1}; ...; x_1, t_1), \tag{3.19}$$

with the same normalization condition and positivity of usual probabilities, i.e.,

$$\begin{aligned}
&p(x_n,t_n;...;x_1,t_1)\geq 0,\\
&\sum_{x_i}p(x_n,t_n;...;x_i,t_i;...;x_1,t_1)=p(x_n,t_n;...;x_1,t_1),\\
&\sum_{x}p(x,t)=1
\end{aligned}\tag{3.20}$$

which are known as consistency conditions.

As well as in usual probability distributions, by joint probabilities, we can obtain conditional probabilities

$$p(x_n,t_n;...;x_{k+1},t_{k+1}|x_k,t_k;...;x_1,t_1)=\frac{p(x_n,t_n;...;x_1,t_1)}{p(x_k,t_k;...;x_1,t_1)}\tag{3.21}$$

such that $k<n$.

Given a stochastic process with initial time t_0, we can define the later probabilities $p(x,t|x_0,t_0)$ for any $t\geq t_0$ using form matrices with coefficients $p_{ij}=p(x_i,t|x_j,t_0)$ and they satisfy the positivity and normalization condition,

$$\begin{aligned}
&p(x,t|x_0,t_0)\geq 0,\\
&\sum_x p(x,t|x_0,t_0)=1.
\end{aligned}\tag{3.22}$$

It is important to note that the matrices satisfying the above conditions are called stochastic matrices. By the definition of conditional probability, we get

$$p(x,t)=\sum_{x_0}p(x,t|x_0,t_0)p(x_0,t_0),\tag{3.23}$$

so they can be seen as linear maps acting on the one time probabilities and evolving them. Let us take $t'\geq 0$ and the probabilities $p(x,t|x',t')$. These probabilities are not necessarily well defined, but if all the matrices $p(x,t|x_0,t_0)$ are invertible we can define

$$p(x,t|x',t')=\sum_{x_0}p(x,t|x_0,t_0)\left(p(x',t'|x_0,t_0)\right)^{-1}\tag{3.24}$$

and this matrix may not be a stochastic matrix since it may no longer satisfy the positivity condition. Also, note that

$$\begin{aligned}
&\sum_{x'}p(x,t|x',t')p(x',t'|x_0,t_0)p(x_0,t_0)\\
&=\sum_{x',x_0}p(x,t|x_0,t_0)\left(p(x',t'|x_0,t_0)\right)^{-1}p(x',t'|x_0,t_0)p(x_0,t_0)\\
&=\sum_{x_0}p(x,t|x_0,t_0)p(x_0,t_0)\\
&=p(x,t),
\end{aligned}\tag{3.25}$$

and then, they satisfy the Chapman-Kolmogorov equation (11)

$$p(x,t|x_0,t_0) = \sum_{x'} p(x,t|x',t')p(x',t'|x_0,t_0). \tag{3.26}$$

A stochastic process whose all matrices $p(x,t|x',t')$ for $(t \geq t')$ are stochastic and satisfy the equation above is called a divisible process. (11)

With this in hand, we can define a Markovian process as a process where, for any family of time-conditional probabilities, we have

$$p(x_n,t_n|x_{n-1},t_{n-1};...;x_0,t_0) = p(x_n,t_n|x_{n-1},t_{n-1}), \tag{3.27}$$

and, intuitively, this represents a process without memory of the past: the future state depends only on the state in the present. It is important to say that all Markovian process are divisible (but the return is not necessarily true).

It is opportune to obtain quantum extensions of these mathematical structures. The natural generalization of a probability distribution to the quantum realm is, as we said before, the density operator. The mixed-state density operator can be described as an ensemble of wavefunctions, i.e.,

$$\rho = \sum_x p(x) \left|\psi(x)\right\rangle \left\langle\psi(x)\right|, \tag{3.28}$$

where $\sum_x p(x) = 1$ and $p(x) \geq 0$, i.e., the $p(x)$ form a probability distribution p. Therefore, the evolution operator can be written as (58)

$$p(x,t) = \sum_{x_0} p(x,t|x_0,t_0)p(x_0,t_0) \tag{3.29}$$

and then

$$\rho(t) = \mathcal{E}(t,t_0)\rho(t_0), \tag{3.30}$$

and, for a fixed $t_0 = 0$, the quantum map forms an one-parameter subgroup with respect to the variable t.

Important recent contributions have been made to obtain definitions that mean quantum counterparts for non-Markovianity. (56–59) Classically, Markovianity is reflected in the divisibility of conditional probabilities of a stochastic process as described by the Chapmann-Kolmogorov equation. Quantumly, a definition characterized in the divisibility of quantum channels cannot simply be imported. The propose of Rivas, Huelga, and Plenio (RHP) (60) can be seen as most similar to the classical concept because consider that a quantum process $\mathcal{E}(t,t_0)$ is Markovian if it is a CP-divisible map, i.e., a trace-preserving, completely positive (CPTP) such that, for any intermediate time, it can be divisible into two CPTP maps

$$\mathcal{E}(t,t_0) = \mathcal{E}(t,t_1)\mathcal{E}(t_1,t_0), \quad t_0 \leq t_1 \leq t. \tag{3.31}$$

This composition between the operators frames a family of trace-preserving and completely positive maps - a semigroup with respect to time. Any dynamics that are Markovian according to the semigroup definition are also Markovian according to the divisibility definition, and hence according to the BLP definition, which will be presented in the next subsection.

3.2.1 Trace distance-based non-Markovianity witness

Markovian processes are memoryless processes. With this in mind, an exciting way to obtain an intuitive and consistent definition can be constructed by characterizing the Markovianity from distance measures in Hilbert space. The definition of non-Markovian dynamics proposed by Breuer, Laine, and Piilo (BLP) (61) takes into account the behavior of trace-distance. First, we will be defining this specific Markovian condition and later explain this meaning: An quantum evolution is Markovianity if the trace distance between any two states decreases monotonically with time

$$\frac{d}{dt} \left\| \rho_1(t) - \rho_2(t) \right\|_1 \leq 0, \tag{3.32}$$

where $\rho(t) = \mathcal{E}(\rho)$ and $\|X\|_1 = \mathrm{Tr}\sqrt{X^\dagger X}$ is the so-called Schatten 1-norm. (39, 53)

The trace distance measures the indistinguishability of two states, or the capacity to discriminate between two states. Then, the trace distance decreases monotonically when the system just lost information, e.g., Markovian dynamics. (61) As a result, an increase in its value indicates that some information flows back to the design and breaks the memoryless property, a natural consequence of the non-Markovian dynamics. Mathematically the trace distance is defined by

$$D(\rho_1, \rho_2) = \frac{1}{2} \left\| \rho_1 - \rho_2 \right\|_1, \tag{3.33}$$

where we using the 1-norm defined before. Then, considering two states evolving in time ($\rho_1(t) = \mathcal{E}(\rho_1)$ and $\rho_2(t) = \mathcal{E}(\rho_2)$), it is immediate that the rate change of trace distance is given by its first derivative, i.e.,

$$\sigma(t) = \frac{d}{dt} D(\rho_1(t), \rho_2(t)), \tag{3.34}$$

and, for some $t \in [0, \infty)$, the dynamics is called non-Markovian if

$$\sigma(t) \geq 0. \tag{3.35}$$

Then, from a concise definition, we have an indicator for non-Markovian dynamics in general physical systems, where the trace distance can be well defined. Such witness of non-Markovianity is widely used in the literature. Fig. 8 show the highly non-Markovian behavior for the model used in this work. The calculations are present on the Chap.5.

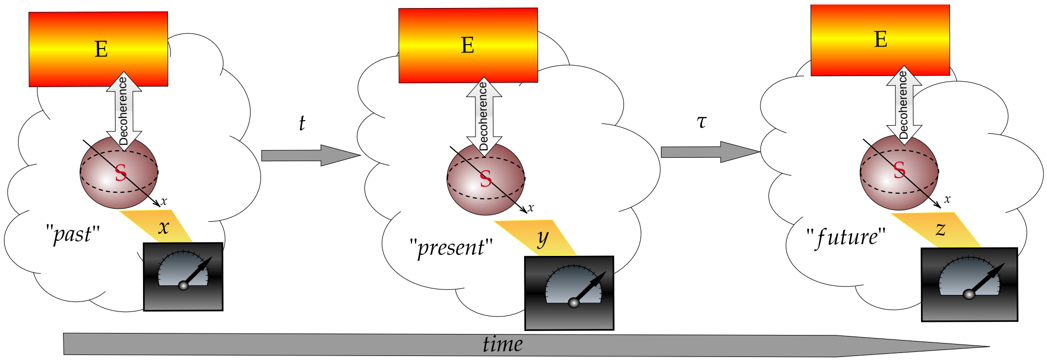

Figure 4 – The conditional past-future correlator (C_{pf}) is based on measurements made on the system - which, in the present case, interacts with a thermal bath - at a time t earlier and a time τ later in relation to a present moment. The measured correlations are related to the variables x, y, and z corresponding to the past, present, and future, respectively. Systems with strong temporal correlations retain the memory of previous states, an intuitive indication of non-Markovianity.

Source: By the author

3.2.2 Conditional past-future correlation

Before defining the conditional past-future correlator (C_{pf}), let us see some properties of correlation functions. The mean (temporal) value of a stochastic process is the quantity

$$\langle X(t)\rangle = \sum_x x(t)p(x,t), \tag{3.36}$$

and the two-time correlation function of two random variables $X(t)$ and $Y(t)$ is defined as

$$\langle X(t)Y(t')\rangle = \sum_{x,y} x(t)y(t')p(x,t;y,t'), \tag{3.37}$$

by extension. The correlation function when applied to the same random variable $X(t)$, yields the autocorrelation function, defined as

$$S(t,t') := \langle X(t)X(t')\rangle = \sum_{x,x'} x(t)x'(t')p(x,t;x',t'). \tag{3.38}$$

and we assume this function depends only on the difference $|t-t'|$, i.e.,

$$S(t,t') = S(t-t') := S(\tau), \tag{3.39}$$

where we say that the process is homogeneous in time and implies the symmetry property $S(\tau) = S(-\tau)$. (11)

The present measure is based on statistical independence of past and future system events when conditioned to a given state in the present time, proposed by Budini. (62,63) Let us consider $t_x < t_y < t_z$ yields the outcomes $x \to y \to z$ - the scheme is showed in the Fig. 4. For Markov process: $p(z, y, x) = p(z|y)p(y|x)p(x)$; from here and from Bayes rule (11)

$$p(z, x|y) = p(z|y)p(x|y) \tag{3.40}$$

$p(z, x|y)$ is the probability of y given x and z as results. Thus, past and future events become statistically independent when conditioned to a given (fixed) intermediate state for a classical Markovian process.

This property can be corroborated through a conditional past-future correlation, which is defined as

$$\begin{aligned} C_{pf} &:= \langle O_z O_x \rangle_y - \langle O_z \rangle_y \langle O_x \rangle_y \\ &= \sum_{xz} [p(z, x|y) - p(z|y)p(x|y)] O_z O_x, \end{aligned} \tag{3.41}$$

where O are observables related to each system state and such that Markovian processes lead to $C_{pf} = 0$ and non-Markovian otherwise ($C_{pf} \neq 0$). In here, indexes x and z run over all possible outcomes occurring at times t_x and t_z, respectively for fixed y at time t_y. It follows that *non-Markovian effects break conditional past-future independence and are present whenever* $C_{pf} \neq 0$.

Now, we take correspondent measurement operators $x \leftrightarrow \Pi_x$, $y \leftrightarrow \Pi_y$ and $z \leftrightarrow \Pi_z$ and satisfy $\sum_x \Pi_x^\dagger \Pi_x = \sum_y \Pi_y^\dagger \Pi_y = \sum_z \Pi_z^\dagger \Pi_z = \mathbb{1}$, where y measurement must be projective. The memory indicator can be extended to quantum regime using these measurements with respect to the events x, y and z. Taking a initial density operator $\rho_{\mathrm{SE}}(0)$ correspondent to a system-plus-environment model, subject to a dynamics given by the map $\mathcal{E} := \mathcal{E}(t_y, t_x)$ and $\mathcal{E}' := \mathcal{E}(t_z, t_y)$. Here, we can use the correspondence $\Pi_\alpha \leftrightarrow \Pi_\alpha \otimes \mathbb{1}_E$ to make clear the measurements only are doing on the system.

Then, we can follow the steps to obtain the probability distributions to obtain the C_{pf} defined in Eq. (3.41), after the first x-measurement it occurs the transformation

$$\rho_{\mathrm{SE}}^x = \frac{\Pi_x \rho_{\mathrm{SE}}(0) \Pi_x^\dagger}{\mathrm{Tr}\, \Pi_x^\dagger \Pi_x \rho_{\mathrm{SE}}(0)}. \tag{3.42}$$

The probability of each outcome is

$$p(x|0) = \mathrm{Tr}\, \Pi_x \Pi_x^\dagger \rho_{\mathrm{SE}}(0).$$

During a time interval $t = t_y - t_x$ we know that the arrangement evolves with the map $\mathcal{E}$ and, after the second y-measurement, it follows the transformation given by $\mathcal{E}(\rho_{\mathrm{SE}}^x) \to \rho_{\mathrm{SE}}^y$, where

$$\rho_{\mathrm{SE}}^y = \frac{\Pi_y \mathcal{E}(\rho_{\mathrm{SE}}^x) \Pi_y^\dagger}{\mathrm{Tr}\, \Pi_y^\dagger \Pi_y \mathcal{E}(\rho_{\mathrm{SE}}^x)}, \tag{3.43}$$

where the conditional probability of outcome y given that the previous one was x is $p(y|x) = \mathrm{Tr}\, \Pi_y^\dagger \Pi_y \mathcal{E}(\rho_{\mathrm{SE}}^x)$. Thus, the joint probability for both measurement outcomes, $p(y, x) = p(y|x)p(x|0)$, is

$$p(y, x) = \mathrm{Tr}\, \Pi_y^\dagger \Pi_y \mathcal{E}(\Pi_x \rho_{\mathrm{SE}}(0) \Pi_x^\dagger). \tag{3.44}$$

Now, using Bayes rule, the retrodicted probability can be obtained by $p(x|y) = p(y,x)/p(y)$ and reduces to

$$p(x|y) = \frac{\mathrm{Tr}\,\Pi_y^\dagger \Pi_y \mathcal{E}(\Pi_x \rho_{\mathrm{SE}}(0)\Pi_x^\dagger)}{\sum_{x'} \mathrm{Tr}\,\Pi_y^\dagger \Pi_y \mathcal{E}(\Pi_x \rho_{\mathrm{SE}}(0)\Pi_x^\dagger)}. \tag{3.45}$$

For projective measurements in y, i.e., $\Pi_y = |y\rangle\langle y|$, the state (3.43) can be separated: $\rho_{\mathrm{SE}}^y = \rho_{\mathrm{S}}^y \otimes \rho_{\mathrm{E}}^{yx}$, where the bath state can be written as

$$\rho_{\mathrm{E}}^{yx} = \mathrm{Tr}_S\, \rho_{\mathrm{SE}}^y = \frac{\mathrm{Tr}_S\, \Pi_y^\dagger \Pi_y \mathcal{E}(\rho_{\mathrm{SE}}^x)}{\mathrm{Tr}\,\Pi_y^\dagger \Pi_y \mathcal{E}(\rho_{\mathrm{SE}}^x)}, \tag{3.46}$$

and we can, finally, go to the final step, correspondent to the evolution given by the map $\mathcal{E}'$, in the time interval $\tau = t_z - t_y$. The last z-measurement, which leads to $\mathcal{E}'(\rho_{\mathrm{S}}^y \otimes \rho_{\mathrm{E}}^{yx})$ comes

$$\rho_{\mathrm{SE}}^z = \frac{\Pi_z \mathcal{E}'(\rho_{\mathrm{S}}^y \otimes \rho_{\mathrm{E}}^{yx})\Pi_z^\dagger}{\mathrm{Tr}\,\Pi_z^\dagger \Pi_z \mathcal{E}'(\rho_{\mathrm{S}}^y \otimes \rho_{\mathrm{E}}^{yx})}. \tag{3.47}$$

And it's correspondent conditional probability of outcome z given that the previous one is

$$p(z|y,x) = \mathrm{Tr}\,\Pi_z^\dagger \Pi_z \mathcal{E}'(\rho_{\mathrm{S}}^y \otimes \rho_{\mathrm{E}}^{yx}),$$

and using this and the Eq. (3.45) we obtain a final expression

$$p(z,x|y) = \mathrm{Tr}\,\Pi_z^\dagger \Pi_z \mathcal{E}'(\rho_{\mathrm{S}}^y \otimes \rho_{\mathrm{E}}^{yx}) \frac{\mathrm{Tr}\,\Pi_y^\dagger \Pi_y \mathcal{E}(\Pi_x \rho_{\mathrm{SE}}(0)\Pi_x^\dagger)}{\sum_{x'} \mathrm{Tr}\,\Pi_y^\dagger \Pi_y \mathcal{E}(\Pi_x \rho_{\mathrm{SE}}(0)\Pi_x^\dagger)}. \tag{3.48}$$

Then, for the calculation of the C_{pf} we obtain a function dependent on two-times t and τ, i.e., $C_{pf} = C_{pf}(t,\tau)$, where $p(z|y) = \sum_x p(z,x|y)$, according to the Eq. (3.41).

For the present model, the C_{pf} is given by

$$\begin{aligned} C_{pf}(t,\tau) &= \sum_{xz} [p(z,x|y) - p(z|y)p(x|y)] O_z O_x \\ &= \frac{\Gamma(t+\tau) + \Gamma^\star(t+\tau)}{4} + \frac{\Gamma(t-\tau) + \Gamma^\star(t-\tau)}{4} - \left(\frac{\Gamma(t) + \Gamma^\star(t)}{2}\right)\left(\frac{\Gamma(\tau) + \Gamma^\star(\tau)}{2}\right) \\ &= f(t,\tau) - f(t)f(\tau) \end{aligned}$$

where, in the last line, we defined $f(t,\tau) = [f(t+\tau) + f(t-\tau)]/2$ and $f(t) = \mathrm{Re}(\Gamma(t))$, and $\Gamma(t)$ is the decoherence function of the problem. The calculations are present on the Chapter 5. It is importante to note that the similar result in comparison to (63) is due to the fact that here we are also dealing with diagonal states in the σ_z basis.

4 OPEN SYSTEMS II: ENVIRONMENT POINT-OF-VIEW

> “Adianta querer saber muita coisa? O senhor sabia, lá para cima - me disseram. Mas, de repente chegou neste sertão, viu tudo diverso diferente, o que nunca tinha visto. Sabença aprendida não adiantou para nada... Serviu algum?”
>
> O Grande Sertão: Veredas, by Guimarães Rosa.

The present chapter turns out to deal with a different paradigm to investigate the non-Markovian dynamics. Here we present two ideas from the paradigm of quantum-to-classical transition or objectivity paradigms: quantum Darwinism and Spectrum Broadcast Structures (SBS). First, the objective is to seek the information about the system available in the environment by Partial Information Plots (PIPs) (64), to see how the Markovianity/non-Markovianity induced by the decoherence affects the storage of information. (65) The second idea is used to describe the structure of the density operators that represent the states. By the form of these operators, we can point out two things: (1) whether the information deposited in the environment is available for measurement or not and (2) if, once taken the measurements, these measurements can be distinguishable.

4.1 Objectivity and quantum-to-classical transition

When dealing with open systems, we reduce the degrees of freedom to analyze only a portion of the system-plus-environment configuration. (10) However, the information that at some initial moment was contained in the system will be deposited in an environment that, in general, has many more degrees of freedom. So far, we have evaluated the adequate amount of the action of the environment on the states of the system from its perspective. But instead of having all the information about the environment decoded in the decoherence function, we can look from its perspective to get the behaviour of the information flow between system-bath.

Let us now look at the explanation of classicality emergence in the point-of-view of decoherence theory, developed in the section. In a situation of pure decoherence, the interaction between the system and the environment causes the destruction of superposed states. Considering a qubit, after a certain decoherence time, one has

$$\rho_{\mathrm{S}}^{dec} \approx \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \tag{4.1}$$

and this would be the key to understanding the emergence of classicality in quantum systems, and the time of this process is typically extremely short for every day, macroscale process. (3, 20, 55) The decoherence then causes a "collapse" of the quantum state into a set of preferred states, called "pointer states". (4, 66, 67) This is the first explanation for the emergence of "classicality" or objectivity in quantum systems.

4.2 The idea of quantum Darwinism

First, let us consider quantum Darwinism, a framework based on a heuristic idea centred on the proliferation of information from a central system to a nearby environment. Being another way to analyze the same phenomenon, quantum Darwinism makes use of a more realistic platform: Instead of a monolithic structure for the environment, it is divided into fractions where there is information proliferation, and information redundancy/storage and its accessibility would be responsible for the classical emergence, from the idea of objectivity.

It is essential to say that those perspectives come from exploring the emergence of objectivity. Roughly speaking, objectivity is the standard agreement among observers about the system's state, which is not necessarily true for the quantum world. This concept is stronger than the classicality of decoherence. Let us define the idea more formally:

Definition. (Objectivity) (24, 68, 69) A system state is objective if it is (1) simultaneously accessible to many observers (2) who can all determine the state independently without perturbing it, and (3) all arrive at the same result.

Therefore, the emergence of objectivity in quantum systems means the emergence of classicality. The conditions for this vital link are given recently by the framework of quantum Darwinism but criticized, based on the importance of the possibility of information extraction, i.e., measurable, distinguishable, and accessible information, that is not necessarily taken into account in the quantum Darwinism paradigm.

Quantum Darwinism's approach was used to treat a wide range of systems, like spin (67, 70–73), and photonic (74–77) environments, harmonic oscillator (78) and Brownian (65, 79) models, and experimentally in quantum dots (80), and photonic (74, 76) setups. To study quantum Darwinism, we focus on correlations between environment and system fragments. The relevant reduced density matrix ρ_{SF} is given by

$$\rho_{\mathrm{SF}} = \mathrm{Tr}_{\mathrm{E/F}} \left|\psi_{\mathrm{SE}}\right\rangle \left\langle\psi_{\mathrm{SE}}\right|, \tag{4.2}$$

or more generally, for a mixed state

$$\rho_{\mathrm{SF}} = \mathrm{Tr}_{\mathrm{E/F}}\, \rho_{\mathrm{SE}}. \tag{4.3}$$

Above, the trace is over the space $\mathcal{H}_{\mathrm{E}}$ less $\mathcal{H}_{\mathrm{F}}$, or $\mathcal{H}_{\mathrm{E/F}}$ - all of $\mathcal{H}_{\mathrm{E}}$ except for the fragment $\mathcal{H}_{\mathrm{F}}$. The space $\mathcal{H}_{\mathrm{E}} = \mathcal{H}_1^{\mathrm{E}} \otimes \mathcal{H}_2^{\mathrm{E}} \otimes .... \otimes \mathcal{H}_N^{\mathrm{E}} = \bigotimes_{k=1}^{N} \mathcal{H}_k^{\mathrm{E}}$. Being $S(\rho_A)$ the von Neumann entropy with respect to a system A with $\mathcal{H}_{\mathrm{A}} \subseteq \mathcal{H}_{\mathrm{E}}$, quantum Darwinism gives how much F knows about S (the system, with a Hilbert space $\mathcal{H}_{\mathrm{S}}$) can be quantified using mutual information

$$I(\mathrm{S}:\mathrm{F}) = S(\rho_{\mathrm{S}}) + S(\rho_{\mathrm{F}}) - S(\rho_{\mathrm{SF}}), \tag{4.4}$$

defined as the difference between entropies of two systems treated separately and jointly. Thus, we can define quantum Darwinism from the amount of shared information that proliferates throughout the environment.

Definition. (Quantum Darwinism) There exists an environment fraction size f_0 such that all fractions larger than it, $f \geq f_0$, it holds:

$$I(\mathrm{S}:\mathrm{F}) = S(\rho_{\mathrm{S}}), \tag{4.5}$$

independently of f.

To explain this definition, we consider that the amount of information about the system is given by the von Neumann entropy about the system. Then, in a situation where this quantity coincides with the mutual information about the system and the environment fragment, we consider that it has stored sufficient information about the system to reconstruct its physical state.

The most direct way to check for this condition is via so-called partial information plots (PIPs), where $I(\mathrm{S}:\mathrm{F})$ is plotted as a function of f. The PIPs format depends on the intrinsic characteristics of the system-environment density operator. If we take, for example, a pure global state, we will have antisymmetric plots around $f = 1/2$ (67,71,79, 81), and this fact can be easily seen considering the marginal mutual information for the system, i.e., mutual information correspondents to the operators ρ_{SF} and $\rho_{\mathrm{S\bar{F}}}$, in which $\bar{\mathrm{F}} := \mathrm{B/F}$ and then

$$I(\mathrm{S}:\mathrm{F}) + I(\mathrm{S}:\bar{\mathrm{F}}) = 2S(\rho_{\mathrm{S}}) \tag{4.6}$$

which stems from the fact that the marginal entropies are equal: $S(\rho_{\mathrm{SF}}) = S(\rho_{\mathrm{S\bar{F}}})$. But for cases where we have mixed initial states (such as thermal states at finite temperature, as in this work), such symmetry is broken in the system.

To illustrate how quantum Darwinism can occur, let us assume the system is a qubit initially in a general pure state, i.e., $a\,|0\rangle_{\mathrm{S}} + b\,|1\rangle_{\mathrm{S}}$ with $|a|^2 + |b|^2 = 1$, and we assume a collection of N qubits each written in the same initial state $|\psi\rangle_{\mathrm{E_i}}$. Thus, we start with an initially factorized state

$$(a\,|0\rangle_{\mathrm{S}} + b\,|1\rangle_{\mathrm{S}}) \bigotimes_{k=1}^{N} |\psi\rangle_{\mathrm{E_k}}\,. \tag{4.7}$$

Quantum Darwinism then posits that, if after their mutual interaction, the total state of the system and environment is

$$a\,|0\rangle_{\mathrm{S}} \otimes \bigotimes_{k=1}^{N} |0\rangle_{\mathrm{E_k}} + b\,|1\rangle_{\mathrm{S}} \otimes \bigotimes_{k=1}^{N} |1\rangle_{\mathrm{E_k}} \tag{4.8}$$

then classical objectivity emerges. Taking the partial trace over N environmental qubits, the system density matrix comes

$$\rho_{\mathrm{S}} = |a|^2\,|0\rangle\,\langle 0|_{\mathrm{S}} + |b|^2\,|1\rangle\,\langle 1|_{\mathrm{S}}\,, \tag{4.9}$$

while for any single environment qubit we have

$$\rho_{\mathrm{E_k}} = |a|^2\,|0\rangle\,\langle 0|_{\mathrm{E_k}} + |b|^2\,|1\rangle\,\langle 1|_{\mathrm{E_k}}\,. \tag{4.10}$$

Then we can see that the information about the central qubit can be sought in each environment fraction.

However, a closer look at the meaning of quantum mutual information shows that things are not as straightforward as in classical information theory, as we can see in the following subsection.

4.3 Spectrum broadcast structures and strong quantum Darwinism

The focus of quantum Darwinism is on sharing information between the system and fractions of the environment from correlations, but without mentioning the character of these correlations or the accessibility of information stored in the environment. For a scenario where objectivity emerges, information about the system needs to be measured. In this way, the SBS paradigm takes into account the structure of the states formed by the system-plus-fractions set of the environment to obtain sufficient conditions for the emergence of objectivity. SBS relates to the composition of the partially traced density operator ρ_{SF}. These structures have a close relationship with the possibility of quantum Darwinism and are the keys to leading the idea of strong quantum Darwinism. (23)

Definition. (Spectrum Broadcast Structure) (23, 24) The joint state ρ_{SF} *of the system* S *and a collection of subenvironments* $\mathcal{H}_{\mathrm{F}} = \mathcal{H}_{\mathrm{E_1}} \otimes ... \otimes \mathcal{H}_{\mathrm{E_{fN}}} = \bigotimes_{k=1}^{\mathrm{fN}} \mathcal{H}_{\mathrm{E_k}}$ *has spectrum broadcast structure if it can be written as:*

$$\rho_{\mathrm{SF}} = \sum_i p_i\,|e_i\rangle\,\langle e_i| \otimes \rho_i^{\mathrm{E_1}} \otimes ... \otimes \rho_i^{\mathrm{E_{fN}}}, \tag{4.11}$$

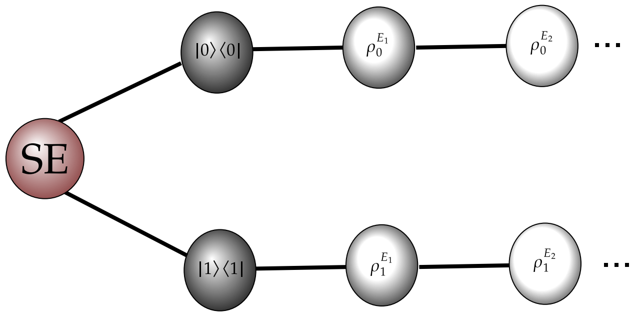

Figure 5 – Conditioned on the state $|e_i\rangle\langle e_i|$ the subenvironments take on conditional states $\rho_i^{\mathrm{E_k}}$. These conditional states must be perfectly distinguishable, i.e., $\rho_1^{\mathrm{E_k}} \perp \rho_2^{\mathrm{E_k}}$ for $k = 1, 2, \ldots$. This allows an observer with access to E_k to construct a general quantum measurement to perfectly determine index i.

Source: Adapted from LE; OLAYA-CASTRO (82)

where $\{|e_i\rangle\}$ *is the pointer basis of* S*,* p_i *are the probabilities and the operators* $\rho_i^{\mathrm{E_k}}$ *are perfectly distinguishable, i.e., two by two orthogonal considering each pair of fragment environments.*

The basic idea for such structures is to consider states that can be faithfully broadcasted satisfying Bohr non-disturbance definition:

Definition (Bohr non-disturbance) (24, 29) A measurement $\Pi_k^{\mathrm{S}'}$ *on the subsystem,* S' *is Bohr non-disturbing on the subsystem S if and only if*

$$\sum_i \mathbb{1} \otimes \Pi_i^{\mathrm{S}'} \rho_{\mathrm{SS}'} \mathbb{1} \otimes \Pi_i^{S'} = \rho_{\mathrm{SS}'} \tag{4.12}$$

Therefore, these are the states such that many observers can find out the state S independently, and without perturbing it, as assigned in the definition of objectivity.

Another important thing is that states with spectrum broadcast structure satisfy strong independence, where there are no correlations between the environment conditioned on the system's information.

Definition (Strong independence) (24, 27) Subenvironments $\{E_k\}_k$ *have strong independence relative to the system* S *if their conditional mutual information is vanishing, i.e.,*

$$I(\mathrm{E}_j : \mathrm{E}_i|\mathrm{S}) = 0, \qquad \textit{for all} \qquad i \neq j. \tag{4.13}$$

Here, we have sufficient information to answer an essential question about quantum Darwinism. Indeed, the framework of quantum Darwinism, because of the use of quantum mutual information for computing the information between the system and the environment, can not assume that the information is entirely classical in general. This situation is critical, because mutual information has both quantum and classical information, and the quantum information is responsible for the objectivity emergence. Furthermore, here we take the strategy of searching for strong independence and spectrum broadcast structure in the model to search for the emergence of objectivity.

Definition (Strong quantum Darwinism) (27, 82) A system-environment satisfies strong quantum Darwinism when the quantum discord is zero, and quantum mutual information is fully classical and equal to the information contained in the system:

$$I(\mathrm{S}:\mathrm{E}_k) = I_{acc}(\mathrm{S}:\mathrm{E}_k) = \chi(\mathrm{S}:\mathrm{E}_k) = S(\rho_{\mathrm{S}}), \qquad \mathcal{D}(\mathrm{S}:\mathrm{E}_k) = 0, \tag{4.14}$$

where $I_{acc}(\mathrm{S}:\mathrm{E}_k)$ is the classical accessible information which here is equivalent to the Holevo quantity $\chi(\mathrm{S}:\mathrm{E}_k)$.

For objectivity to emerge, Eq. (4.14) should hold for sufficiently many environments $\{\mathrm{E}_k\}_k$, as well as for the observed joint environment represented by Hilbert space $\cup_{k=1}^{\mathrm{fN}} \mathcal{H}_k^{\mathrm{E}}$. Note the observed environment is never the total environment. The whole environment (provided that the total system-environment is closed and pure), retains all quantum correlations and so will always have $I(\mathrm{S}:\mathrm{E}) = 2S(\rho_{\mathrm{S}})$

As shown in Ref. (23), in addition to the notion of SBS being a formalization of the emergence of objectivity in open systems, it is also a stronger condition than quantum Darwinism for the emergence of such. There is also a proposal to witness non-objectivity in situations of strong quantum Darwinism (82) in the literature. Strong quantum Darwinism is an extension of the theory of quantum Darwinism that emphasizes the structure of states and their available information. (23, 28, 82)

An example of objective compative with all frameworks present here uses the GHZ state. (13) Considering a system initially in a general pure state, i.e., $a\,|0\rangle_{\mathrm{S}} + b\,|1\rangle_{\mathrm{S}}$ with $|a|^2 + |b|^2 = 1$ as in quantum Darwinism example, but here the N spin environment is in ground state $|0\rangle_{\mathrm{E}}$. Let us suppose a total final state after interaction

$$\rho_{\mathrm{SE}} = |a|^2\,|0\rangle\,\langle 0|_{\mathrm{S}} \otimes \bigotimes_{k=1}^{N} |0\rangle\,\langle 0|_{\mathrm{E_k}} + |b|^2\,|1\rangle\,\langle 1|_{\mathrm{S}} \otimes \bigotimes_{k=1}^{N} |1\rangle\,\langle 1|_{\mathrm{E_k}} \tag{4.15}$$

This state satisfies all frameworks of quantum Darwinism: has (invariant) spectrum broadcast structure with probabilities $|a|^2$ and $|b|^2$, the quantum mutual information condition is fulfilled, i.e., $I(\mathrm{S}:\mathrm{E}_k) = S(\rho_{\mathrm{S}}) = S(|a|^2, 1-|a|^2)$ and the state has zero discord and hence satisfies strong and “weak” quantum Darwinism.

5 RESULTS

> "Let us leave theories there and return to here's hear."
>
> Finnegans Wake, by James Joyce.

5.1 The model and its features

Let us denote $\mathcal{H}_{\rm S}$ the Hilbert space representative of the states of the central qubit and by $\mathcal{H}_{\rm E}$ the Hilbert space of the the spin environment states. The total system is then given by the tensor product space $\mathcal{H}_{\rm SE} = \mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm E}$. For the Hilbert space with respect to the central qubit, we take a 2-dimensional complex space in the computational basis, i.e., $\mathcal{H}_{\rm S} = \text{span}\{|0\rangle, |1\rangle\} \cong \mathbb{C}^{2*}$, and this states $|0\rangle, |1\rangle$ are the eigenstates of the Pauli matrix $\sigma_{\rm z}$ correspondents to the eigenvalues 1 and -1, respectively, where $\sigma_{\rm z}|0\rangle = |0\rangle$ and $\sigma_{\rm z}|1\rangle = -|1\rangle$.

Let us consider a Hilbert subspace that can be described too by a *2-dimensional* complex space, $\mathcal{H}_{\rm E_k} = \text{span}\{|0\rangle, |1\rangle\} \cong \mathbb{C}^2$. This space is the subspace descriptive of the k-th spin site in the environment. As one can see, the environment can be constructed as

* "span" of a set of vectors is the smallest linear subspace that contains the set.

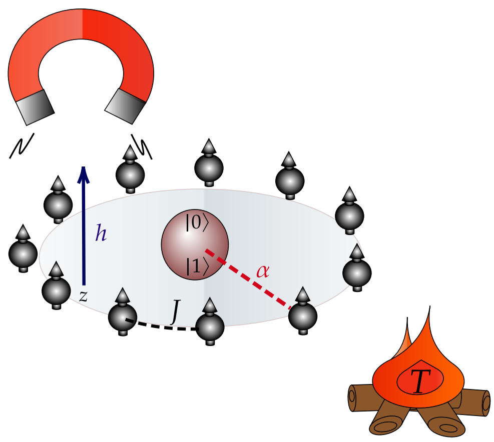

Figure 6 – Here, we use a central qubit coupled to a thermal environment structured by spins with ferromagnetic interactions between the first neighbors (with intensity $J > 0$), also subject to the action of a magnetic field h in the direction $\hat{z}$. The environment is in thermal equilibrium at the temperature $T = \frac{1}{\beta}$. Along the paper we consider $k_B = \hbar = 1$.

Source: By the author

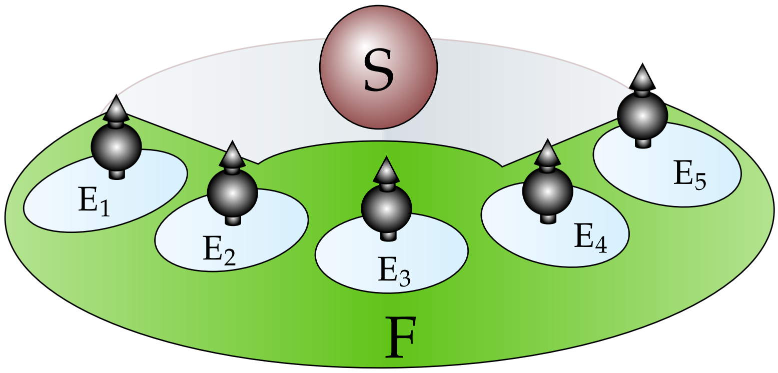

Figure 7 – For the description of the model in terms of the thermal environment point of view, we use the description given in Section, in which the central qubit is described by a Hilbert space $\mathcal{H}_{\mathrm{S}} = \mathrm{span}\{|0\rangle, |1\rangle\} \cong \mathbb{C}^2$ and each environmental spin is given by $\mathcal{H}_{\mathrm{E_k}} = \mathrm{span}\{|0\rangle, |1\rangle\} \cong \mathbb{C}^2$, from so that the fractions are compositions of fN of these spaces.

Source: By the author

the tensor product of subspace corresponding to each site

$$\mathcal{H}_{\mathrm{E}} = \bigotimes_{k=1}^{N} \mathcal{H}_{\mathrm{E_k}} \cong (\mathbb{C}^2)^{\otimes N},$$

and the notation $(\mathbb{C}^2)^{\otimes N}$ means the composition of N *2-dimensional* complex spaces.

To know how the environment acquires and records information about the central system, we will use the *environment point-of-view*. Thereby, it is essential to construct partial Hilbert spaces that will be tensorial compositions of such a number (lower than N) of subsystems. If one calls this fractional Hilbert space $\mathcal{H}_{\mathrm{F}}$, it can be written as

$$\mathcal{H}_{\mathrm{F}} = \mathcal{H}_{\mathrm{E_1}} \otimes ... \otimes \mathcal{H}_{\mathrm{E_{fN}}} = \bigotimes_{k=1}^{\mathrm{fN}} \mathcal{H}_{\mathrm{E_k}},$$

such that $fN = \#\mathcal{H}_{\mathrm{F}} \leq N$, and $\#$ represents the number of composed subsystems (or the cardinality of $\mathcal{H}_{\mathrm{F}}$ with respect to $\mathcal{H}_k$).

Now, with each Hilbert space defined, we can propose that the total Hamiltonian that describes system-environment is given by

$$H = H_S + H_{\mathrm{E}} + H_{\mathrm{SB}}, \tag{5.1}$$

where $H_{\mathrm{S}} = \frac{\omega \sigma_{\mathrm{z}}}{2}$ is the Hamiltonian of the system and H_{E} is the Ising-like environment Hamiltonian of N spin-1/2 particles, described by

$$H_{\mathrm{E}} = -J \sum_{i=1}^{N} \sigma_i^{\mathrm{z}} \otimes \sigma_{i+1}^{\mathrm{z}} - h \sum_{i=1}^{N} \sigma_i^{\mathrm{z}}, \tag{5.2}$$

that contains Pauli matrices σ_i^z acting in each space $\mathcal{H}_{\mathrm{E_k}}$, J corresponds to a nearest-neighbor coupling between the spins and h, the magnetic field along z-axis affecting the spin chain.

The initial state of the system will be considered as a general pure qubit state in the Bloch sphere

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \in \mathcal{H}_{\mathrm{S}}, \tag{5.3}$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.

Since this qubit is subject to dynamics due to the interaction with a thermal environment at temperature T, we will consider that such environment state is described by the Gibbs state

$$\rho_{\mathrm{E}} = \frac{e^{-\beta H_{\mathrm{E}}}}{Z_{\mathrm{E}}}, \tag{5.4}$$

where $Z_{\mathrm{E}} := \mathrm{Tr}[e^{-\beta H_{\mathrm{E}}}]$ the partition function.

Such Gibbs state will give us distribution in the state space and one can construct each microstate as

$$|\chi\rangle := |\chi_1 ... \chi_N\rangle \equiv |\chi_1\rangle \otimes ... \otimes |\chi_N\rangle\,, \tag{5.5}$$

that diagonalizes H_{E}, which $|\chi_k\rangle \in \mathcal{H}_{\mathrm{E_k}}$ is a eigenstate of the Pauli matrix σ^z in computational basis $\{|0\rangle\,, |1\rangle\}$, correspondent to the eigenvalues $\sigma_i = \pm 1$. Then, the diagonal Hamiltonian results in

$$H_{\mathrm{E}}\,|\chi\rangle = E(\chi)\,|\chi\rangle\,, \tag{5.6}$$

defines the configuration correspondent to the energy

$$E(\chi) = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i. \tag{5.7}$$

The diagonalization of the Ising chain Hamiltonian allows us to rewrite environment density operator in the energy basis

$$\rho_{\mathrm{E}} = \frac{e^{-\beta H_{\mathrm{E}}}}{Z_{\mathrm{E}}} = \frac{1}{Z_{\mathrm{E}}}\sum_{\chi} e^{-\beta E(\chi)}\,|\chi\rangle\langle\chi|\,. \tag{5.8}$$

This system presents ground states obtained from $\lim_{\beta\to\infty}\rho_{\mathrm{E}}$, in which for $h = 0$ emerges a $\mathbb{Z}_2$ symmetry and the ground states are $\bigotimes_{k=1}^{N}|0\rangle\langle 0|_{\mathrm{E_k}} = |\mathbf{0}\rangle\langle\mathbf{0}|$ and $\bigotimes_{k=1}^{N}|1\rangle\langle 1|_{\mathrm{E_k}} = |\mathbf{1}\rangle\langle\mathbf{1}|$, and for $h \neq 0$ the symmetry bronken and the ground states depends on the direction of the magnetic field.

The central spin interacts with the environment with an interaction strength $\alpha \in [0, 1]$ (with this interval for realistic purposes), described using the Hamiltonian

$$H_{\mathrm{SB}} = \alpha\sigma_{\mathrm{z}} \otimes \sum_i \sigma_i^{\mathrm{z}}, \tag{5.9}$$

acting in the space $\mathcal{H}_{\mathrm{SB}} \cong (\mathbb{C}^2)^{\otimes N+1}$, and inducing the unitary time evolution operator $U(t) = e^{-iH_{\mathrm{SB}}t}$ - considering here the situation of interaction picture with respect to the operator H_S; since $[H_S, H_{\mathrm{SB}}] = 0$, arises in the setup a case of pure decoherence, this does not affect $H_{\mathrm{E}} + H_{\mathrm{SB}}$ and obviously $[H_{\mathrm{E}}, \rho_{\mathrm{E}}] = 0$.

This operator H_{SB} generate a quantum map $\mathcal{E}$ and following the calculations of the Appendix, the dynamics just affects the off-diagonal terms

$$\rho_{\mathrm{S}}(t) = \begin{pmatrix} |a|^2 & a^* b\Gamma(t) \\ ab^*\Gamma^*(t) & |b|^2 \end{pmatrix}, \tag{5.10}$$

where $\Gamma(t)$ is given by

$$\begin{aligned} \Gamma(t) &= \frac{1}{Z_{\mathrm{E}}} \sum_{\sigma} e^{-\beta E(\chi)} e^{-2it\alpha \sum_i \sigma_i} \\ &= \frac{Z_{\mathrm{E}}(h - 2i\alpha t/\beta)}{Z_{\mathrm{E}}(h)}, \end{aligned} \tag{5.11}$$

and the partition function Z_{E} can be compute using the transfer matrix formalism, as described by Ref. (50) This function $\Gamma(t)$ is the so-called decoherence function, and, for the present case, is a periodic function with period $\tau = 2\pi/4\alpha$.

Here, we have a situation with finite time reversibility implied by oscillations presented in the density operator coherences and, the irreversible process is obtained by taking the continuous limit (55), e.g., considering each mode corresponding to a state $\chi = \{\chi_1, ..., \chi_N\}$ and defining a mode density operator $\Omega(\chi)$, i.e.,

$$\Omega(\chi) = \int d\chi_1 ... \int d\chi_N \prod_{i=1}^{N} D(\chi_i) \delta\left(m(\chi) - \sum_{i=1}^{N} \chi_i\right), \tag{5.12}$$

where $D(\chi_i)$ the density of states. The partition function can be written as

$$Z_{\mathrm{E}} = \int d\chi \, \Omega(\chi) e^{-\beta E(\chi)}. \tag{5.13}$$

For this case, one can speak of a decoherence rate $\gamma(t) = \log \frac{1}{\Gamma(t)}$ that describes how fast the coherence vanishes.

As a result, the discrete environment gives a dephasing process when the coherence is recovered periodically, which is a signature of non-Markovianity. Here, in a case of pure decoherence, when there is no energy dissipation in the system, for all practical purposes, it means that we have a situation without any effect on the population of the central qubit, and the impact of environment interaction in the system recover elastic scattering. (?, 9)

We will calculate the exact solution for the dynamics of the system density operator. Let us start with the initial density operator of the system $\rho_{\mathrm{S}}(0) = |\psi\rangle\langle\psi|$, with $|\psi\rangle$ as defined in Eq. (5.3), and fully uncorrelated with the thermal environment described by density operator in Eq.(5.4), i.e., $\rho = \rho_{\mathrm{S}}(0) \otimes \rho_{\mathrm{E}}$. Then the evolved density operator can be given by

$$\rho_{\mathrm{S}}(t) = \mathcal{E}(\rho_{\mathrm{S}}) = \mathrm{Tr}_{\mathrm{E}}\left[U(t)\rho_{\mathrm{S}} \otimes \rho_{\mathrm{E}} U^\dagger(t)\right], \tag{5.14}$$

that admits a representation in the Kraus form, i.e.,

$$\mathcal{E}(\rho_{\mathrm{S}}) = \sum_{\chi\chi'} K_{\chi\chi'}\rho_{\mathrm{S}}K^{\dagger}_{\chi\chi'}, \tag{5.15}$$

in which $\sum_{\chi\chi'} K^{\dagger}_{\chi\chi'}K_{\chi\chi'} = \mathbb{1}_S$. As a result, the dynamics can be calculed writting explicitly the environment operator in energy basis, like in Eq.(5.8), i.e.,

$$\begin{aligned}\rho_{\mathrm{S}}(t) &= \mathrm{Tr}_{\mathrm{E}}[U(t)\rho_{\mathrm{S}} \otimes \rho_{\mathrm{E}}U^{\dagger}(t)] \\ &= \mathrm{Tr}_{\mathrm{E}}[U(t)\rho_{\mathrm{S}} \otimes \frac{1}{Z_{\mathrm{E}}}\sum_{\chi} e^{-\beta E(\chi)} |\chi\rangle \langle\chi| U^{\dagger}(t)].\end{aligned} \tag{5.16}$$

Writting the trace operation explicity using a eigenbasis $|\chi'\rangle \in (\mathbb{C}^2)^{\otimes N}$, one can obtain the follow

$$\begin{aligned}&= \sum_{\chi'} \langle\chi'| U(t)\rho_{\mathrm{S}} \otimes \frac{1}{Z_{\mathrm{E}}}\sum_{\chi} e^{-\beta E(\chi)} |\chi\rangle \langle\chi| U^{\dagger}(t) |\chi'\rangle \\ &= \sum_{\chi\chi'} \frac{e^{-\frac{\beta E(\chi)}{2}}}{Z_{\mathrm{E}}^{\frac{1}{2}}} \langle\chi| U(t) |\chi'\rangle \rho_{\mathrm{S}} \frac{e^{-\frac{\beta E(\chi')}{2}}}{Z_{\mathrm{E}}^{\frac{1}{2}}} \langle\chi| U(t) |\chi'\rangle ,\end{aligned} \tag{5.17}$$

where we put in a form that one can identify the Kraus operators, that are

$$K_{\chi\chi'} = \frac{e^{-\frac{\beta E(\chi)}{2}}}{\sqrt{Z_{\mathrm{E}}}} \langle\chi| U(t) |\chi'\rangle . \tag{5.18}$$

Now, let us write the evolution operator at the energy eigenbasis to obtain the Kraus operators, i.e.

$$U(t) = e^{-iH_{\mathrm{SB}}t} = \sum_{n\chi} e^{-it\epsilon_n m(\chi)} |n,\chi\rangle \langle n,\chi| , \tag{5.19}$$

in which $m(\chi) = \sum_i \sigma_i$ is the total magnetization spin with respect to the state χ and $\epsilon_n = \alpha(-1)^n$ with $n = 0, 1$ is the energy gap obtained when one diagonalize the operator H_{SB} in the basis $|n,\chi\rangle = |n\rangle \otimes |\chi\rangle$. Consequently, for the Kraus operators

$$\begin{aligned}K_{\chi\chi'} &= \frac{e^{-\frac{\beta E(\chi)}{2}}}{\sqrt{Z_{\mathrm{E}}}} \langle\chi| U(t) |\chi'\rangle \\ &= \frac{e^{-\frac{\beta E(\chi)}{2}}}{\sqrt{Z_{\mathrm{E}}}} \langle\chi| \sum_{n\boldsymbol{\gamma}} e^{-it\epsilon_n m(\gamma)} |\boldsymbol{\gamma}\rangle \langle\boldsymbol{\gamma} \mid \chi'\rangle |n\rangle \langle n| \\ &= \frac{e^{-\frac{\beta E(\chi)}{2}}}{\sqrt{Z_{\mathrm{E}}}} \sum_{n\gamma} e^{-it\epsilon_n m(\gamma)} |n\rangle \langle n| \delta_{\boldsymbol{\chi\gamma}}\delta_{\boldsymbol{\gamma\chi'}} \\ &= \frac{1}{\sqrt{Z_{\mathrm{E}}}} \sum_{n\chi'} e^{-\frac{\beta E(\chi)}{2}} e^{-it\epsilon_n m(\chi')} |n\rangle \langle n| \delta_{\chi\chi'}.\end{aligned} \tag{5.20}$$

Let us decompose the initial density matrix $\rho_{\mathrm{S}} = \sum_{nm} \rho_{\mathrm{S}}^{nm} |n\rangle \langle m| $, where $\rho_{\mathrm{S}}^{nm} = \langle n| \rho_{\mathrm{S}} |m\rangle$. Then, applying these Kraus operators, an evolved state subject to the evolution take the particular form

$$\begin{aligned}\rho_{\mathrm{S}}(t) &= \frac{1}{Z_{\mathrm{E}}}\sum_{m,n,o,p,\chi,\chi'}\rho_{\mathrm{S}}^{nm}e^{-\beta E(\chi)}e^{-it(\epsilon_o-\epsilon_p)m(\chi')}\times \\ &\times |o\rangle\underbrace{\langle o,n\rangle}_{\delta_{on}}\underbrace{\langle m\,|\,p\rangle}_{\delta_{mp}}\langle p|\,\delta_{\chi\chi'} \\ &= \frac{1}{Z_{\mathrm{E}}}\sum_{m,n,\sigma}e^{-\beta E(\chi)}e^{-it(\epsilon_n-\epsilon_m)m(\chi)}\rho_{\mathrm{S}}^{nm}\,|n\rangle\,\langle m|\,.\end{aligned} \tag{5.21}$$

and, one can easily check $\epsilon_n - \epsilon_m = \alpha[(-1)^n - (-1)^m]$ results in null terms for $n = m$, then the dynamics just affects off-diagonal terms whilst the coherences (off-diagonal) are modulated by the periodic function $\Gamma(t)$, given by

$$\begin{aligned}\Gamma(t) &= \frac{1}{Z_{\mathrm{E}}}\sum_{\sigma}e^{-\beta E(\chi)}e^{-2it\alpha\sum_i\sigma_i} \\ &= \frac{Z_{\mathrm{E}}(h-2i\alpha t/\beta)}{Z_{\mathrm{E}}(h)},\end{aligned} \tag{5.22}$$

where the partition function $Z_{\mathrm{E}} := \mathrm{Tr}\, e^{-\beta H_{\mathrm{E}}}$ is the Ising partition function.

5.2 Decoherence and dynamical phase transitions: a comment

As a result, the discrete environment gives a dephasing process when the coherence is recovered periodically, as we will show, is a signature of non-Markovianity. Decoherence theory provides an archetypal mechanism for open quantum systems, which can be summarized simply as follows: interaction between the system and the environment causes decoherence and this, in turn, causes the loss of information from the system to the environment. (5) Here, a case of pure decoherence, when there is no energy dissipation in the the system, for all practical purposes, means that we have a situation without any effect on the population of the central qubit, and the result of environment interaction in the system recovers elastic scattering.

The decoherence function recalls the Loschmidt amplitude, a fundamental theory of dynamical phase transitions (DQPT). (83) The Loschmidt amplitude quantifies an overlap between an initial state and its post-quench evolution. This amplitude measures how a quantum system differs from its initial state after applying an evolution operation. One can define the Loschmidt amplitude as

$$\mathcal{G}(t) = \langle\Psi_0|\Psi_t\rangle = \langle\Psi_0|\,e^{-iHt}\,|\Psi_0\rangle\,, \tag{5.23}$$

for some initial state Ψ_0 and a general driven Hamiltonian H. Notice that the Loschmidt amplitude vanishes for the case when the states are orthogonal. Analogous to thermal phase transitions, DQPT's occurs when $t = t_c$ if $\mathcal{G}(t_c) = 0$ which results in nonanalyticity of $\log\mathcal{G}(t)$, a dynamical analog of the thermal free energy.

Accordingly, Loschmidt amplitude has a closed relation with the partition function, which can be seen considering the boundary partition function, described in the Ref. (84), represented in the following form

$$Z = \langle \Psi_1 | \, e^{-RH} \, | \Psi_2 \rangle \,, \tag{5.24}$$

in which the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ encoding the boundary conditions and H denoting the bulk Hamiltonian, R is the distance between two borders of the system, such as a situation described by our system: a qubit with two energy levels corresponding to the energy boundaries coupled to another system with a coupling strength α.

In the present text, obviously, it does not make sense to talk about phase transitions, given that our states are all orthogonal. But we can use this theory as an analogy. The coupling concerning the qubit is described by the interacting Hamiltonian, which can be re-written as $\alpha(|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes \sum_i \sigma_i^z$, and each level of the qubit is submitted to an amount α of energy concerning the environment.

Thus, we can speak of an effective Hamiltonian that computes this behavior of the system, $H(\alpha) = 2\alpha \sum_i \sigma_i^z$, introducing the overlap effect caused by the thermal environment in the central qubit, and the environment thermal state ρ_{E}, we can use the generalization for the mixed state Loschmidt amplitude given by Refs. (85) and (86) to obtain

$$\mathcal{G}(t) = \mathrm{Tr}\left[\rho_{\mathrm{E}} \exp\left(-2i\alpha t \sum_i \sigma_i^z\right)\right] = \frac{\mathrm{Tr}[e^{-\beta H_{\mathrm{E}}} e^{-iH(\alpha)t}]}{Z_{\mathrm{E}}}, \tag{5.25}$$

i.e., the same as decoherence function $\Gamma(t)$. In the same way, one can consider another way to writte the coherence function with $\Gamma(t) = \left\langle e^{-iH(\alpha)t} \right\rangle_{\mathrm{E}}$ in which $\langle \bullet \rangle_{\mathrm{E}} = \mathrm{Tr}[\bullet \rho_{\mathrm{E}}]$ denotes the thermal average with respect to the environment.

The critical times on the Loschmidt amplitude reveal a closed relation with the zeros of the partition function, the so-called Lee-Yang zeros. For the case of equilibrium phase transitions, the theory of Lee-Yang tells us that the zeros of the partition function determine critical points in the fugacity plan. The decomposition of a partition function in the Nth order polynomial of $z := e^{-2\beta h}$ can be obtained by

$$Z_{\mathrm{E}} = \mathrm{Tr}[e^{-\beta H_{\mathrm{E}}}] = e^{\beta N h} \sum_{n=0}^{N} p_n z^n, \tag{5.26}$$

where p_n is the partition function with zero magnetic field in which $n \leq N$ spins are in the state -1 and N is the number of spins.

The N zeros of the partition function lying on the unit circle in the complex plane of z can be written as $z_n := e^{i\theta_n}$ with $n \in \mathbb{N}$. We rewrite the partition function in the function of its zeros

$$Z_{\mathrm{E}} = p_0 e^{\beta N h} \prod_{n=1}^{N} (z - z_n). \tag{5.27}$$

Then, the Lee-Yang zeros in the time domain, as proposed in Ref. (52), are

$$\Gamma(t) = e^{-2iN\alpha t}\frac{\prod_{n=1}^{N}(e^{-2\beta h+4i\alpha t} - z_n)}{\prod_{n=1}^{N}(e^{-2\beta h} - z_n)}, \tag{5.28}$$

which, of course, clarifies the one-to-one correspondence between the decoherence function (or the Loschmidt amplitude) and the Lee-Yang zeros. Notice that the numerator term is obtained simply by rewriting a new (time-dependent) magnetic field $h \to h - 2i\alpha t/\beta$. When $h = 0$, this function vanishes at the critical times given by Lee-Yang zeros in fugacity plan. Then, the situation provides a setup in which we can map an equilibrium system in a probe decoherence system. Beyond a mere theoretical result, this correspondence guarantees the possibility of observing Lee-Yang zeros experimentally, as can be seen in Ref. (87)

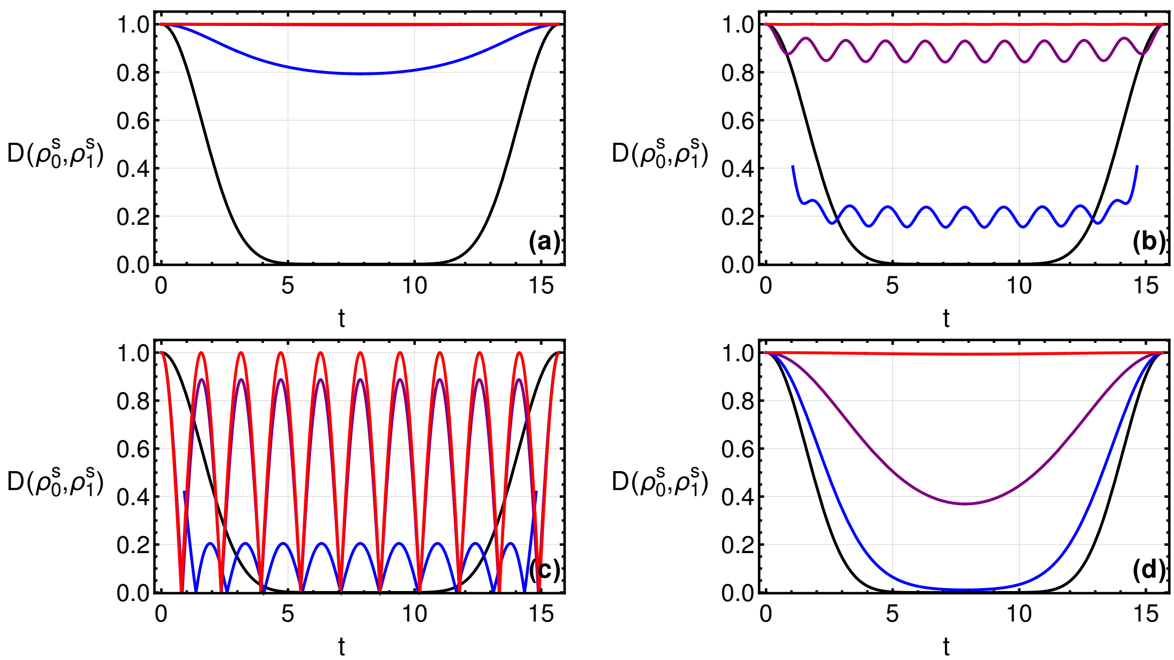

Figure 8 – Trace distance for two initial states $\rho_1 = |+\rangle\langle+|$ and $\rho_2 = |-\rangle\langle-|$ subject to the dynamics given by Eq.(5.10) for different temperatures: $\beta = 0$ (black line), $\beta = 0.75$ (blue line), $\beta = 1.75$ (purple line) and $\beta = 4$ (red line). For the case of zero magnetic field **(c)**, the trace distance oscillations correspond directly to the Lee-Yang zeros, and this correspondence is erased as the magnetic field h increases in intensity (that corresponds to **(a)** and **(b)**). For all cases, we see that the revivals in time culminates in situations in which $\sigma(\rho_1, \rho_2; t) > 0$, which would, in principle, indicate the non-Markovianity of the system, unless for those cases where the trace distance tends to remain constant $D(\rho_1, \rho_2; t) = 1$ for any t, which cover the low temperature cases (large β). For situations **(a)**, **(b)** and **(d)** one have $D(\rho_1, \rho_2) = 1$ at any time for $\beta = 4$ (red line), a situation without memory effects. We set a coupling $\alpha = 0.1$, what result in the recoherence at $t = \tau$.

Source: By the author

5.3 Non-Markovian behavior

The present section will show the results for the two aforementioned non-Markovian measures. We present a pretty detailed calculation for the trace distance-based non-Markovianity witness and the CPF measure for a specific initial system. The comparison between the trace distance-based witness and the CPF measure is given in Fig.11.

5.3.1 Trace distance-based non-Markovianity witness

In the present problem, the situation is characterized by a qubit dephasing when putting it in contact with a thermal environment. Considering two initial pure states.

$$\rho_{\mathrm{S}}^{1} = |+\rangle\langle+| = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \qquad \rho_{\mathrm{S}}^{2} = |-\rangle\langle-| = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/2$. While keeping the population terms unchanged, pure decoherence subjects the off-diagonal terms to a modulation given by the decoherence function for the model considered here. This case results in an immediate dependence of the trace distance with the decoherence function, which can be written as

$$D(\rho_{\mathrm{S}}^{1}, \rho_{\mathrm{S}}^{2}; t) = \frac{1}{2}\mathrm{Tr}\left|\rho_{\mathrm{S}}^{2}(t) - \rho_{\mathrm{S}}^{1}(t)\right| = \frac{1}{2}\mathrm{Tr}\left|\begin{pmatrix}0 & \Gamma(t)\\ \Gamma^{\star}(t) & 0\end{pmatrix}\right| = |\Gamma(t)|,$$

where $\rho_{\mathrm{S}}^{1,2}(t)$ are defined by the dynamics described in Eq.(5.20). Then, we find that

$$\frac{dD(\rho_{\mathrm{S}}^{1}, \rho_{\mathrm{S}}^{2}; t)}{dt} = \frac{\Gamma(t)}{|\Gamma(t)|}\Gamma'(t) = \mathrm{sign}[\Gamma(t)]\Gamma'(t), \tag{5.29}$$

where $\mathrm{sign}[z] := z/|z|$ with $z \in \mathbb{C}_{\neq 0}$ and $\Gamma'(t) := d\Gamma/dt$ is the function first derivative. As a result, a decrease in the trace distance mean a non-Markovian behavior as showed in the Fig. 8.

To understand Fig. 8 for the trace distance, we can analyze two different cases, where we have the interacting ($J = 1$) and noninteracting ($J = 0$) regimes. Considering two function defined by $C(t) := \cosh(\beta h - 2i\alpha t)$ and $S(t) := [e^{-4\beta J} + \sinh^2(\beta h - 2i\alpha t)]^{1/2}$, the decoherence function for the interaction regime ($J > 0$) takes the form

$$\Gamma(t) = \frac{(C(t) + S(t))^N + (C(t) - S(t))^N}{(C(0) + S(0))^N + (C(0) - S(0))^N}. \tag{5.30}$$

To search the zeros of this function, one can consider the Lee-Yang zeros, explicitly given by the formula

$$\begin{aligned} z_n = &-e^{-4\beta J} + (1 - e^{-4\beta J})\cos k_n \\ &\pm \sqrt{(e^{-4\beta J} - 1)\left[\sin^2 k_n + e^{-4\beta J}(1 + \cos k_n)^2\right]} \end{aligned} \tag{5.31}$$

with $k_n = \pi(2n - 1)/N$ and $n \in \mathbb{N}$, and hence, take a transformation in magnetic field $h \to h - 2i\alpha t/\beta$ the zeros of the decoherence function reduces to

$$t = \frac{\beta h}{2i\alpha} + \frac{1}{4i\alpha}\ln z_n \tag{5.32}$$

that, to result in real times, obviously need the conditions $h \to 0$ or $\beta \to 0$. Here, by rewriting the decoherence function in terms of Lee-Yang zeros, we have a clear correspondence for the critical times. In the case where the magnetic field is null, the decoherence function touches the time axis at an interval $t = \tau$ the same number of times that zeros of the partition function occur in the fugacity plane, as previously shown in Ref. (88)

For the condition $J = 0$, only a zero rises from the fugacity plan for $z_n = -1$. The decoherence function simply reduces to

$$\Gamma(t) = \frac{C^N(t)}{C^N(0)}. \tag{5.33}$$

Which finally results in the following zeros for the system's decoherence function

$$t = \frac{\beta h \pm i\pi/2}{2i\alpha}, \tag{5.34}$$

that results in real terms only for weak fields $h \to 0$ or high temperatures $\beta \to 0$, as the last case.

Therefore, it can be seen that this model results in a strongly non-Markovian environment, where the decoherence is not affected by low temperatures. Here, the recurrence at each interval τ appears as a finite size effect, and during short intervals, the system has its initial information restored. This effect is crucial for the storage of information in the environment, and we see here that the info deposited quickly returns to the system, preventing there from being accessible information in the environment to be measured.

We can see the trace distance behavior in the Fig. 8. For the case of zero magnetic fields Fig. 8(c), the trace distance oscillations correspond directly to the Lee-Yang zeros, and this correspondence is erased as the magnetic field h increases in intensity (that corresponds to Fig. 8(a) and Fig. 8(b)).For all cases, we see that the revivals in time culminates in situations in which $\sigma(\rho_1, \rho_2; t) > 0$, which would, in principle, indicate the non-Markovianity of the system, unless for those cases where the trace distance tends to remain constant $D(\rho_1, \rho_2; t) = 1$ for any t, which cover the low temperature cases (large β). For situations Fig. 8(a), Fig. 8(b) and Fig. 8(d) one have $D(\rho_1, \rho_2) = 1$ at any time for $\beta = 4$ (red line), a situation without memory effects. We set a coupling $\alpha = 0.1$, what result in the recoherence at $t = \tau$.

5.3.2 Conditional past-future correlation as a non-Markovianity measure

In the present case, we can take all the measures in the direction x of the qubit Bloch sphere and follow the idea proposed in the Sec. 3.2.2. Thus, the outcomes of each measurement, in successive order, are $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$, which in turn define the system operators values $O_z = z$ and $O_x = x$. The corresponding measurement operators are the same $\Pi_x = \Pi_y = \Pi_z = \Pi_{\hat{x}=\pm 1}$, where $\Pi_{\hat{x}=\pm 1} = |\hat{x}_\pm\rangle \langle \hat{x}_\pm|$, with $|\hat{x}_\pm\rangle = (|+\rangle \pm |-\rangle)/\sqrt{2}$. Here, we can summarize the steps to obtain the C_{pf}:

1. First, we take the first measure at the initial state $\rho_{\mathrm{SE}}(0)$:

$$\rho_{\mathrm{SE}}(0) \to \rho^x_{\mathrm{SE}}(0) = \frac{\Pi_{\hat{x}=x}\rho_{\mathrm{SE}}(0)\Pi_{\hat{x}=x}}{\mathrm{Tr}[\rho_{\mathrm{SE}}(0)\Pi_{\hat{x}=x}]}$$

For this purpose, we start considering the initial state of the system as $\rho_{\mathrm{S}}(0) = |0\rangle\langle 0|$, to rewritte:

$$\begin{aligned}\rho^x_{\mathrm{SE}}(0) &= \frac{\Pi_{\hat{x}=x}|0\rangle\langle 0| \otimes \rho_{\mathrm{E}}(0)\Pi_{\hat{x}=x}}{\mathrm{Tr}[|0\rangle\langle 0| \otimes \rho_{\mathrm{E}}(0)\Pi_{\hat{x}=x}]}\\ &= \frac{(|0\rangle\langle 0| + x|0\rangle\langle 1| + x|1\rangle\langle 0| + |1\rangle\langle 1|)\otimes\rho_{\mathrm{E}}(0)/4}{\frac{1}{2}}\\ &= \frac{(|0\rangle\langle 0| + x|0\rangle\langle 1| + x|1\rangle\langle 0| + |1\rangle\langle 1|)}{2}\otimes\rho_{\mathrm{E}}(0)\end{aligned}$$

2. Now, we perform a forward evolution above the density operator $\rho^x_{SB}(0)$

$$\rho^x_{SB}(t) = U(t)\rho^x_{SB}(0)U^\dagger(t)$$

to obtain $p(y|x) = \mathrm{Tr}\,\rho^x_{\mathrm{SE}}(t)\Pi_{\hat{x}=y}$. Then, we can compute:

$$\begin{aligned}\rho^x_{\mathrm{SE}}(t) &= U(t)\frac{(|0\rangle\langle 0| + x|0\rangle\langle 1| + x|1\rangle\langle 0| + |1\rangle\langle 1|)}{2}\otimes\rho_{\mathrm{E}}(0)U^\dagger(t)\\ &= \frac{(|0\rangle\langle 0| + e^{-\gamma^{01}(\sigma)t}x|0\rangle\langle 1| + e^{-\gamma^{10}(\sigma)t}x|1\rangle\langle 0| + |1\rangle\langle 1|)}{2}\otimes\rho_{\mathrm{E}}(0)\end{aligned}$$

where

$$-i\alpha t(\langle j|\sigma_z|j\rangle - \langle k|\sigma_z|k\rangle)\sum_i\sigma^z_i|\chi\rangle = -i\alpha t(\langle j|\sigma_z|j\rangle - \langle k|\sigma_z|k\rangle)\sum_i\sigma_i|\chi\rangle \equiv -i\gamma^{jk}(\chi)t|\chi\rangle \tag{5.35}$$

Thus, we obtain for the probability $p(y|x)$ calculating $\rho^x_{\mathrm{SE}}(t)\Pi_{\hat{x}=y}$:

$$\begin{aligned}\rho^x_{\mathrm{SE}}(t)\Pi_{\hat{x}=y} &= \frac{1}{4}(|0\rangle\langle 0| + |1\rangle\langle 1|)\otimes\rho_{\mathrm{E}}(0) + \frac{1}{4}(y|0\rangle\langle 1| + y|1\rangle\langle 0|)\otimes\rho_{\mathrm{E}}(0)+\\ &+\frac{1}{4}\left(xe^{-i\gamma^{01}(\chi)t}|0\rangle\langle 1| + xe^{-i\gamma^{10}(\chi)t}|1\rangle\langle 0|\right)\otimes\rho_{\mathrm{E}}(0)+\\ &+\frac{1}{4}\left(xye^{-i\gamma^{01}(\chi)t}|0\rangle\langle 0| + xye^{-i\gamma^{10}(\sigma)t}|1\rangle\langle 1|\right)\otimes\rho_{\mathrm{E}}(0)\\ &= \frac{1}{4}|0\rangle\langle 0|\left(1 + xye^{-i\gamma^{01}(\chi)t}\right)\otimes\rho_{\mathrm{E}}(0) + \frac{1}{4}|0\rangle\langle 1|\left(y + xe^{-i\gamma^{01}(\chi)t}\right)\otimes\rho_{\mathrm{E}}(0)+\\ &+\frac{1}{4}|1\rangle\langle 0|\left(1 + xye^{-i\gamma^{10}(\chi)t}\right)\otimes\rho_{\mathrm{E}}(0) + \frac{1}{4}|1\rangle\langle 1|\left(y + xe^{-i\gamma^{10}(\chi)t}\right)\otimes\rho_{\mathrm{E}}(0)\\ &= \frac{1}{4}|0\rangle\langle 0|c(t)\otimes\rho_{\mathrm{E}}(0) + \frac{1}{4}|0\rangle\langle 1|yc(t)\otimes\rho_{\mathrm{E}}(0) + \frac{1}{4}|1\rangle\langle 0|yc^\star(t)\otimes\rho_{\mathrm{E}}(0)+\\ &+\frac{1}{4}|1\rangle\langle 1|c^\star(t)\otimes\rho_{\mathrm{E}}(0)\end{aligned}$$

such that $c(t) = 1 + xye^{-i\gamma^{01}(x)t}$ (by Eq. (5.35) is easy to that $\gamma^{01} = -\gamma^{10}$) and, take the total trace we obtain:

$$\begin{aligned}
\operatorname{Tr}\rho^{x}_{\mathrm{SE}}(t)\Pi_{\hat{x}=y} &= \frac{1}{2} + \frac{1}{4}\mathrm{Tr}_{\mathrm{E}}\left(\sum_{\chi} xy\frac{e^{-\beta E(\chi)e^{-i\gamma^{10}(x)t}}}{Z_{\mathrm{E}}}\left|\chi\right\rangle\left\langle\chi\right|\right) + \\
&\quad \frac{1}{4}\mathrm{Tr}_{\mathrm{E}}\left(\sum_{\chi} xy\frac{e^{-\beta E(\chi)e^{-i\gamma^{01}(x)t}}}{Z_{\mathrm{E}}}\left|\chi\right\rangle\left\langle\chi\right|\right) \\
&= \frac{1}{2} + \frac{xy}{4}\left[\frac{Z_{\mathrm{E}}(\beta, h - 2i\alpha t/\beta)}{Z_{\mathrm{E}}(\beta, h)} + \frac{Z_{\mathrm{E}}(\beta, h + 2i\alpha t/\beta)}{Z_{\mathrm{E}}(\beta, h)}\right] \\
&= \frac{1}{2} + \frac{xy}{4}\left[\Gamma(t) + \Gamma^{\star}(t)\right] = p(y|x)
\end{aligned}$$

3. In the present step, we perform the second measure, which gives us

$$\rho^{xy}_{\mathrm{SE}}(t) = \frac{\Pi_{\hat{x}=y}\rho^{x}_{\mathrm{SE}}(t)\Pi_{\hat{x}=y}}{\mathrm{Tr}[\rho^{x}_{\mathrm{SE}}(t)\Pi_{\hat{x}=y}]}$$

and now, we can compute the numerator using the previous results

$$\Pi_{\hat{x}=y}\rho^{x}_{\mathrm{SE}}(t)\Pi_{\hat{x}=y} = \left(\frac{\left|0\right\rangle\left\langle0\right| + y\left|0\right\rangle\left\langle1\right| + y\left|1\right\rangle\left\langle0\right| + \left|1\right\rangle\left\langle1\right|}{2}\right) \otimes \rho_{\mathrm{E}}(0)\left(\frac{c^{\star}(t) + c(t)}{4}\right)$$

and finally

$$\rho^{xy}_{\mathrm{SE}}(t) = \left(\frac{\left|0\right\rangle\left\langle0\right| + y\left|0\right\rangle\left\langle1\right| + y\left|1\right\rangle\left\langle0\right| + \left|1\right\rangle\left\langle1\right|}{2}\right) \otimes \rho^{xy}_{B}(t)$$

4. In the next step, the bipartite arrangement evolves during a time interval $\tau \equiv t_z - t_y$, with the unitary dynamics dictated by the interaction Hamiltonian, $\rho^{xy}_{\mathrm{SE}}(t) \to \rho^{xy}_{\mathrm{SE}}(t+\tau)$:

$$\rho^{xy}_{\mathrm{SE}}(t+\tau) = U(\tau)\rho^{xy}_{\mathrm{SE}}(t)U^{\dagger}(\tau)$$

to obtain, finally, this resultant probability:

$$\begin{aligned}
p(z|y,x) &= \mathrm{Tr}[\rho^{xy}_{\mathrm{SE}}(t+\tau)\Pi_{\hat{x}=z}] \\
&= \frac{1}{2}\left(1 + zy\frac{[\tilde{\Gamma}(t) + \tilde{\Gamma^{\star}}(t)]}{2}\right)
\end{aligned}$$

which

$$\tilde{\Gamma}(t) = \frac{\Gamma(\tau) + xy(\Gamma(t+\tau) + \Gamma^{\star}(t-\tau))/2}{1 + xy(\Gamma(\tau) + \Gamma^{\star}(t))/2}.$$

Having these terms in hand , we can compute the conditional past-future correlation as follow:

$$\begin{aligned}
C_{pf}(t,\tau) &= \sum_{xz}[p(z,x|y) - p(z|y)p(x|y)]O_zO_x \\
&= \frac{\Gamma(t+\tau) + \Gamma^{\star}(t+\tau)}{4} + \frac{\Gamma(t-\tau) + \Gamma^{\star}(t-\tau)}{4} - \left(\frac{\Gamma(t) + \Gamma^{\star}(t)}{2}\right)\left(\frac{\Gamma(\tau) + \Gamma^{\star}(\tau)}{2}\right) \\
&= f(t,\tau) - f(t)f(\tau)
\end{aligned}$$

where, in the last line, we defined $f(t,\tau) = [f(t+\tau) + f(t-\tau)]/2$ and $f(t) = \mathrm{Re}(\Gamma(t))$. The same structure in the conditional past-future correlator in relation to (62,63) reflects the fact that the interactions present here between the baths spins are diagonal in the z basis.

Figs. 9 and 10 shows the conditional past-future correlation behavior. For large N and low temperatures (high β) one has system Markovianity limits, where information is being lost to the environment. As in trace distance-based measure, at $\beta = 4$ the memory effects are suppressed. The situations $C_{pf} \neq 0$ recover situations of finite size and temperature, which induce memory effects on the total system. The noninteracting case ($h = 0$) recaptures a quite similar situation, where there is a Markovian limit for large chain sizes (large N) and low temperature (high β). Here, we can see the same behavior of plot (d) of Fig. 8, correspondent to the noninteracting regime, in which the memory effects disappear for $\beta = 4$. We set the coupling $\alpha = 0.1$.

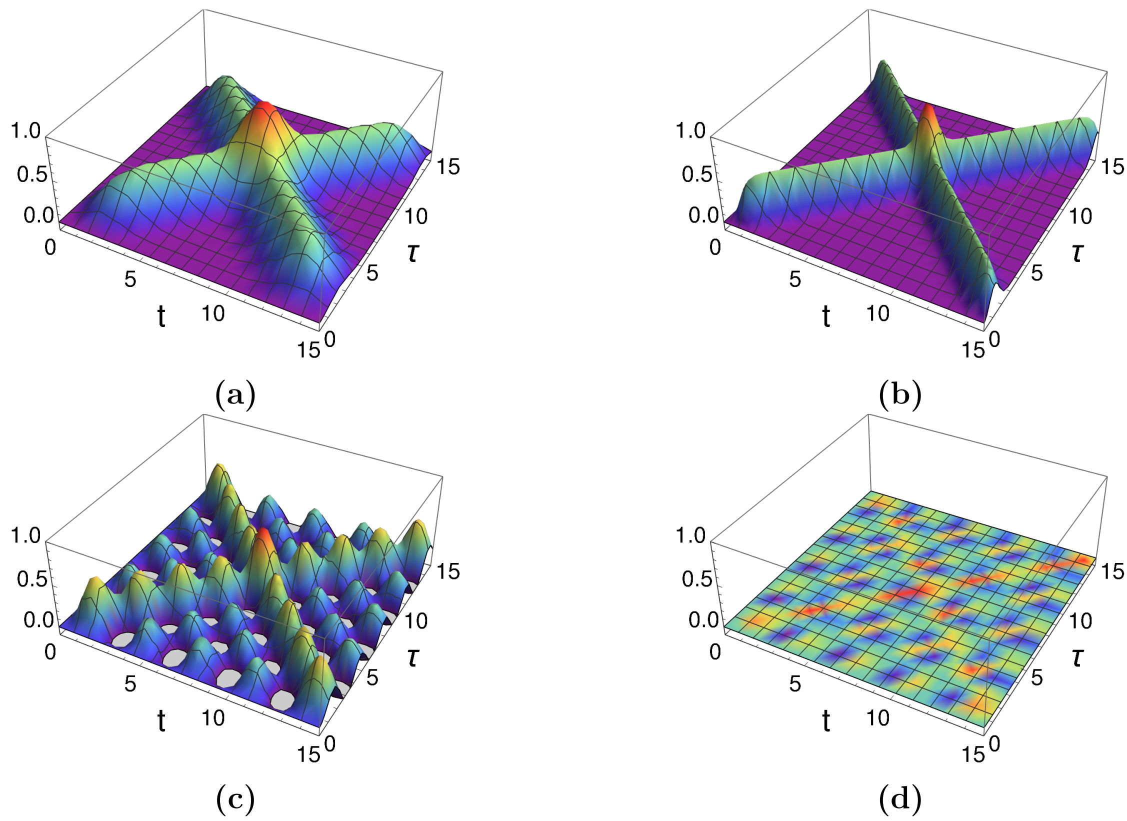

Figure 9 – Conditional past-future correlation (C_{pf}) for different chain sizes and temperatures in the interacting situation ($J = 1$). For (a), (c) and (d) the chain size is $N = 10$ and the temperature is $\beta = 0.1$, $\beta = 1$ and $\beta = 4$, respectively; $N = 50$ and $\beta = 0.1$ in (b).

Source: By the author

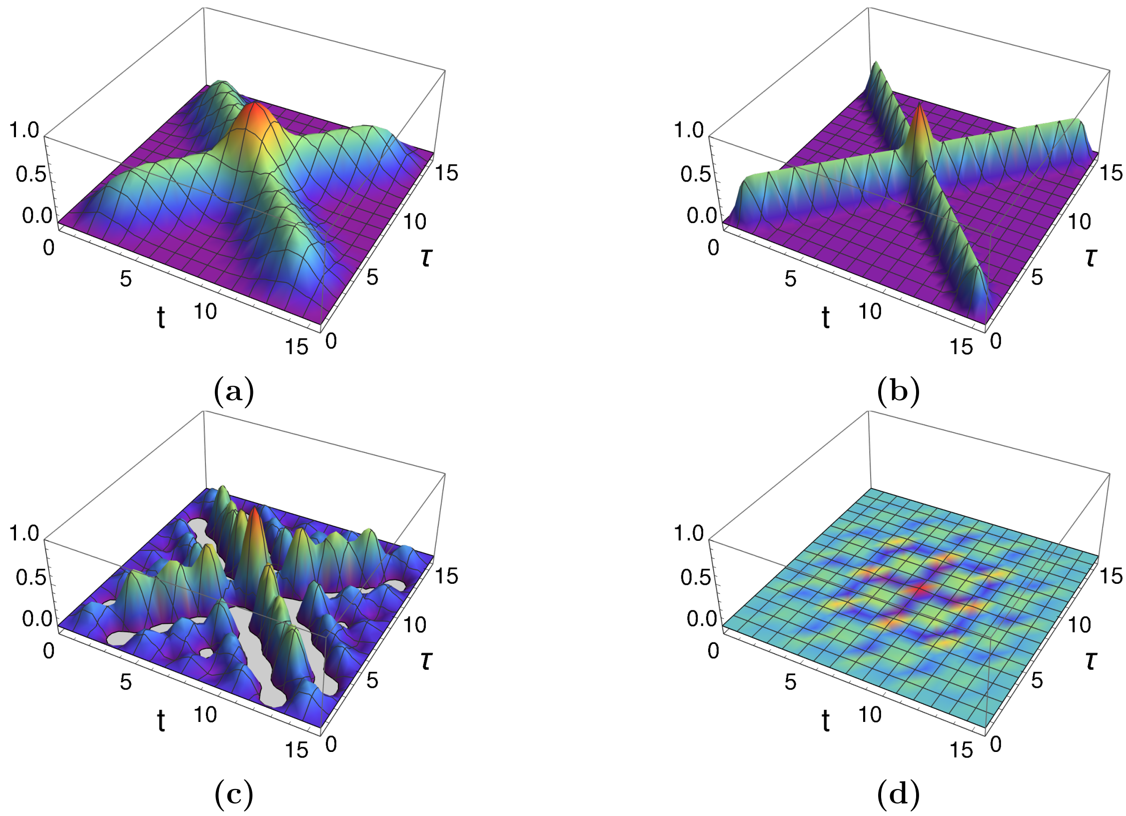

Figure 10 – Conditional past-future correlation (C_{pf}) for different chain sizes and temperatures in the non-interacting bath spins situation ($J = 0$). For (a), (c) and (d) the chain size is $N = 10$ and the temperature is $\beta = 0.1$, $\beta = 1$ and $\beta = 4$, respectively; $N = 50$ and $\beta = 0.1$ in (b).

Source: By the author

5.4 Suppressing information storage

We can summarize this session with two questions: Is there information storage in the environment? If so, is the information accessible from measurements? Such questions can be answered using the paradigms of quantum Darwinism and SBS.

For the study of quantum Darwinism, we divided the Hilbert space of the environment into fractions $\mathcal{H}_{\mathrm{E_k}} \cong \mathbb{C}^2$ (Fig. 7). The PIPs (Partial Information Plots) method quantifies the information between a set of fN fractions and the system. The idea is to consider that the complete knowledge of the environment about the system occurs when the amount of correlations is $I(\mathrm{S}:\mathrm{F}) = S(\rho_{\mathrm{S}})$. Here, we present (Fig. 12(b)) PIPs for $J = 0$ for different temperatures ($\beta = 0.1$, $\beta = 0.5$, $\beta = 1$, $\beta = 2$, $\beta = 4$) and an initial system state $\rho_{\mathrm{S}} = |+\rangle\langle+|$ and, respectively, $t = \tau/2$ (red dashed lines), and $t = \tau$ (blue solid lines). For the case where $t = \tau/2$, decoherence inhibits the storage of information in the environment more intensely as the temperature increases. This situation sheds light on the influence of information flow for the emergence of quantum Darwinism. In $t = \tau$, we have total recoherence and storage is not temperature dependent. In figure 7(a), the

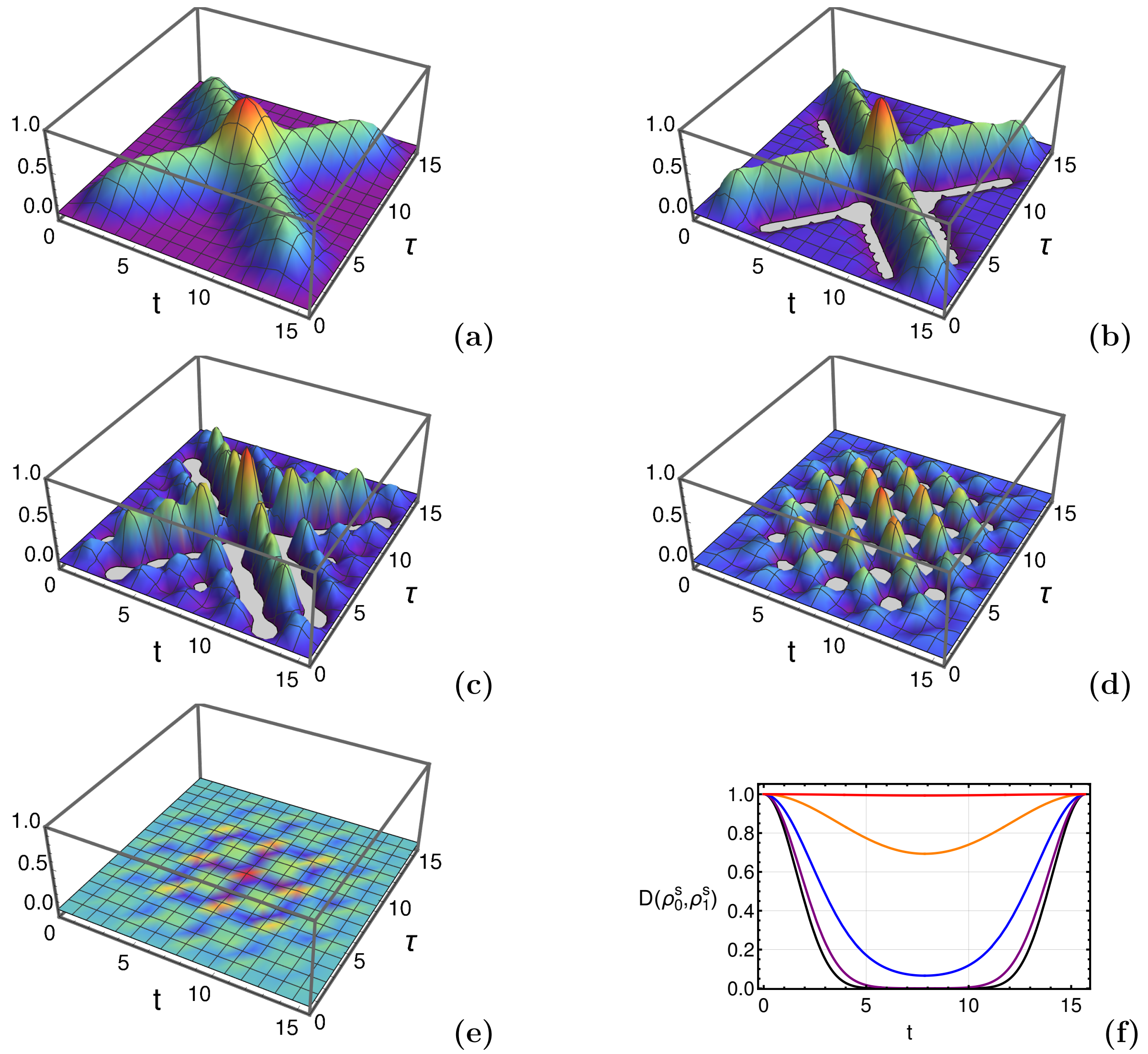

Figure 11 – For the non-interacting situation ($J = 0$), the correspondence (and agreement) between the trace-distance non-Markovianity witness and the measure based on conditional past-future independence is clear. In figure (f), the trace distance for two initial states $\rho_1 = |+\rangle\langle+|$ and $\rho_2 = |-\rangle\langle-|$ subject to the dynamics given by Eq.(5.10), for the cases $\beta = 0.1$ (black line), $\beta = 0.5$ (purple line), $\beta = 1$ (blue line), $\beta = 2$ (orange line) and $\beta = 4$ (red line), and the C_{pf} for the cases $\beta = 0.1$ (a), $\beta = 0.5$ (b), $\beta = 1$ (c), $\beta = 2$ (d) and $\beta = 4$ (e).

Source: By the author

trace distance for two initial states $\rho_1 = |+\rangle\langle+|$ and $\rho_2 = |-\rangle\langle-|$ subject to the dynamics given by Eq.(5.10) expresses the decoherence behavior for the cases $\beta = 0.1$ (black line), $\beta = 0.5$ (purple line), $\beta = 1$ (blue line), $\beta = 2$ (orange line) and $\beta = 4$ (red line). For comparison, we use the solid red line to represent an emergence of quantum Darwinism. We set the coupling $\alpha = 0.1$.

Let us assume a system-environment interaction under the decoherence action, that can be written as follows

$$\rho_{\mathrm{SF}} = \sum_n p_i \left|i\right\rangle \left\langle i\right| \otimes \bigotimes_k \rho_i^{\mathrm{E}_k} + \rho^{coh.}, \tag{5.36}$$

so that, the SBS structures emerges when $\rho^{coh.} = 0$. The partially-traced density operator for the $J > 0$,

$$\begin{aligned} \rho_{\mathrm{SF}}(t) = \sum_{ij} \rho_{\mathrm{S}}^{ij} \left|i\right\rangle \left\langle j\right| \otimes \bigotimes_{i=1}^{\mathrm{fN}} \left|\chi_i\right\rangle \left\langle \chi_i\right| \times \\ \times \sum_{\chi} e^{-\beta E(\chi)} e^{-i(\epsilon_n - \epsilon_m)\sum_i \chi_i t} \end{aligned} \tag{5.37}$$

that does not express the structure that we need to broadcast all information to the environment from the system, since $e^{\pm 2i\alpha m(\chi)t} \neq 0$ for any time and magnetization. On the other hand, in the case of our system, $\Gamma_{\mathrm{F}}(t) = 0$ (see appendix E) the non-interacting situation gives us

$$\rho_{\mathrm{SF}}(t) = \sum_i \rho_{\mathrm{S}}^{ii} \left|i\right\rangle \left\langle i\right| \otimes \bigotimes_{k=1}^{\mathrm{fN}} \frac{e^{\beta h \sigma_k^z}}{Z_{\mathrm{E}}}. \tag{5.38}$$

Notice that the partially-traced decoherence function can be written as

$$\Gamma_{\mathrm{F}}(t) = [\cos(2\alpha t) + i \tanh(\beta h) \sin(2\alpha t)]^{(1-f)N}.$$

Then, that condition is only satisfied for situations in which $\beta h \to 0$ and $t = \pi n/2\alpha - \pi/4\alpha$, such that $n \in \mathbb{Z}$. However, the second condition for these states is not satisfied, because for each pair (n, m) of states with $n \neq m$, one has no $\rho_n^{\mathrm{E}_i} \perp \rho_m^{\mathrm{E}_i}$, i.e., the states are indistinguishable. However, these are not surprising situations. The suppressing of information and its relation with the decoherence is shown in Fig. 12.

5.5 Conclusion

> "It has no explanation and no conclusion; it is, like most of the other things we encounter in life, a fragment of something else which would be intensely exciting if it were not too large to be seen."
>
> Tremendous Trifles, by G.K. Chesterton.

In the present work, we build a general description of a model with exciting features caused by the environmental structure. Here, the structure reflects the properties raised from the possibility of composing Hilbert spaces from individual objects to construct a larger interacting scheme, e.g., an Ising chain. Some of these features can be shown by the

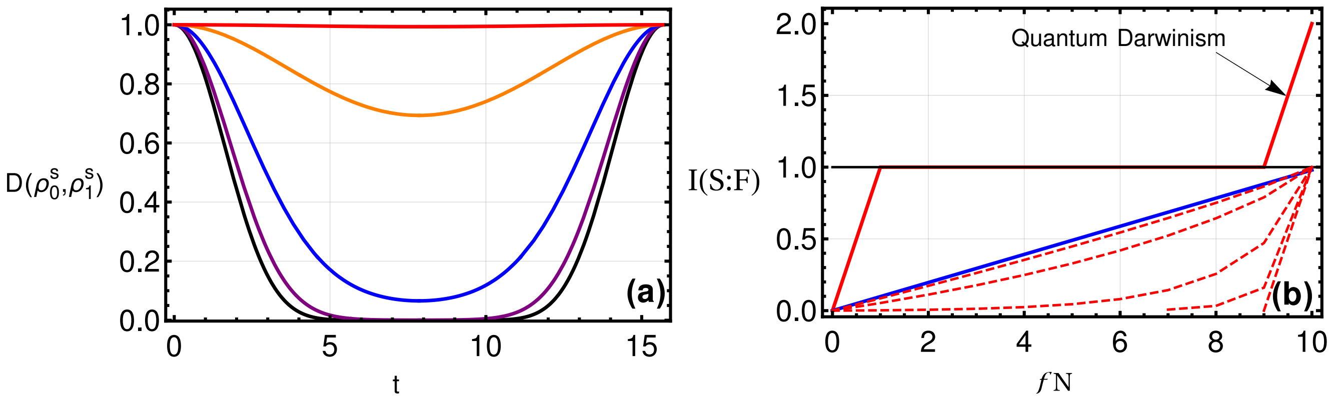

Figure 12 – In figure (a), the trace distance for two initial states $\rho_1 = |+\rangle\langle+|$ and $\rho_2 = |-\rangle\langle-|$ subject to the dynamics given by Eq.(5.10) expresses the decoherence behavior for the cases $\beta = 0.1$ (black line), $\beta = 0.5$ (purple line), $\beta = 1$ (blue line), $\beta = 2$ (orange line) and $\beta = 4$ (red line). Here, we present (plot (b)) PIPs for $J = 0$ for different temperatures ($\beta = 0.1$, $\beta = 0.5$, $\beta = 1$, $\beta = 2$, $\beta = 4$) and an initial system state $\rho_S = |+\rangle\langle+|$ and, respectively, $t = \tau/2$ (red dashed lines), and $t = \tau$ (blue solid lines).

Source: By the author

manifestation of Lee-Yang zeros in quantities like the decoherence function, $\Gamma(t)$, which has a quasiperiodic behaviour far from the thermodynamic limit (where the chain size goes to infinity, $N \to \infty$), and the trace distance $D(t) \equiv D(\rho_1, \rho_2, t)$. As described along the text, trace distance is a measure of distinguishability between two states able to point out the non-Markovian characteristics of the dynamics because when the trace distance decreases (or increases), the distance between these two states decreases (increases), signaling the return of information from the environment to the system. Also, the Lee-Yang zeros act as markers of indistinguishability of two states in finite-size chain ($N < \infty$), a pretty exciting result that is showed in Fig. 8 and demonstrates the highly non-Markovian behaviour of the system. About non-Markovianity, we have two essential things: (i) For low-temperature ($\beta \approx 4$) it was verified a Markovian dynamics at non-zero field – in the case of $h = 0$, the trace-distance behaviour, with just one vanish between recoherence times for $T \to \infty$, reveals the influence of Lee-Yang zeros that are accumulated in the real axis in this case – and a non-Markovian dynamics otherwise (Fig. 8); (ii) We can see that in the conditional past-future correlator (C_{pf}) the rise of the same behaviour, obtaining a Markovian dynamics only for low temperatures (Fig. 9 and Fig. 10), finding an agreement between the Markovianity witnesses from two different perspectives, named temporal and spatial distinguishability of density operators. All that description was focused on the phenomena captured by the decoherence function.

Next, the idea was to take the environment perspective. The previous signalized indistinguishability between two states caused by decoherence is a indication of pointer states selection, i.e., selection of classical states by monitoring the environment, a common thread of quantum Darwinism. Despite that, in this work, we can see the impossibility of Spectrum Broadcast Structure (SBS) due to the structured bath and the sort of interaction (two-level sub-systems subject to an Ising interaction). For the non-interacting regime, for example, all the individual density operators concerning the environment fragments have no orthogonal support (Eq. 5.38), a necessary condition of SBS for the emergence of objectivity. At the same time, the partial information plot (PIP) shows a vital thing: the mutual information decreases more and more as well as the temperature of the thermal bath increases, avoiding the proliferation of environmental available information and, consequently, the emergence of objectivity by the amount of redundant information about the system state (an interesting result also recently noticed in Ref. (89)). As the temperature increases, the information stored in the environment decreases, and quantum Darwinism (a weaker condition than the SBS one) is avoided (Fig. 12). In short, the text aimed to describe a system under decoherence behaviour from the dynamics vestiges sought in the environment structure. Instead of objectivity emergence, the model presents exciting relationships between quantities arising from different formalisms, such as Lee-Yang zeros and the distinguishability measure based on trace distance. Another noticeable remark is the dependence of mutual information shape with mixedness of $\rho_{\mathrm{SF}}(t)$, signalling its decay by the global temperature T increments, which defines the thermal environment state.

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APPENDIX A – CASE STUDY: SPIN-1/2 AND COMPUTATIONAL BASIS

The spin-1/2 theory has very nice features and, in general, is very simple to deal with* Using the theory described until now, let us explore the spin-1/2 formalism; it will be crucial to describe the model in this work: particles with spin one-half are prototypical setups to obtain quantum bits, the so-called qubits.

First, spin is a physical observable described by operators S_{x}, S_{y} and S_{z} acting on $\mathbb{C}^2$. Each of them describes the intensity of the quantity in some direction of the three-dimensional space. Fundamentally, a postulate of angular momentum is that these operators satisfy the algebra:

$$[S_{\mathrm{x}}, S_{\mathrm{y}}] = iS_{\mathrm{x}}, \qquad [S_{\mathrm{z}}, S_{\mathrm{x}}] = iS_{\mathrm{y}}, \qquad [S_{\mathrm{y}}, S_{\mathrm{z}}] = iS_{\mathrm{x}}. \tag{A.1}$$

This is, in fact, the algebra of angular momentum, and every property follows from these commutation relations. (13)

To obtain the spin magnitude, the total spin is given by the operator defined by $\mathbf{S}^2 = S_{\mathrm{x}}^2 + S_{\mathrm{y}}^2 + S_{\mathrm{z}}^2$, that have the eigenvalues

$$S(S+1), \qquad \text{for} \qquad S = \frac{1}{2}, 1, \frac{3}{2}, 2, ... \tag{A.2}$$

Then, the spin-1/2, like an electron, mean a system where the eigenvalue of $\mathbf{S}^2$ is $\frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{2}$. On the other hand, each operator S_α, with $\alpha = x, y, z$, have $2S+1$ eigenvalues which go from S to $-S$ in unit steps, i.e., $S, S-1, ..., -S+1, -S$. In such way, for spin 1/2 we will have a total of $2S+1 = 2$ states with eigenvalues $+1/2$ and $-1/2$.

For spin-1/2 we can label the eigenvectors in different forms, like $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$, $|+\rangle$ and $|-\rangle$, $|+1\rangle$ and $|-1\rangle$, $|0\rangle$ and $|1\rangle$. Here we will use this last labeling, according to the pre-defined computational basis. They satisfy

$$S_{\mathrm{z}} |0\rangle = \frac{1}{2} |0\rangle, \qquad S_{\mathrm{z}} |1\rangle = -\frac{1}{2} |1\rangle. \tag{A.3}$$

Here, a intelligent re-definition is given by the use of Pauli matrices, simply defined as $S_\alpha = \frac{1}{2}\sigma_\alpha$, with $\alpha = x, y, z$. The algebra - a Lie Algebra, as we said before - of the Pauli matrices is then given by

$$[\sigma_{\mathrm{x}}, \sigma_{\mathrm{y}}] = i\sigma_{\mathrm{x}}, \qquad [\sigma_{\mathrm{z}}, \sigma_{\mathrm{x}}] = i\sigma_{\mathrm{y}}, \qquad [\sigma_{\mathrm{y}}, \sigma_{\mathrm{z}}] = i\sigma_{\mathrm{x}}. \tag{A.4}$$

The eigen-equation for σ_{z} is now

$$\sigma_{\mathrm{z}} |0\rangle = |0\rangle, \qquad \sigma_{\mathrm{z}} |1\rangle = -|1\rangle. \tag{A.5}$$

* Just out of curiosity and a bit of scholarly exhibitionism, the generators of spin-1/2 operators, the famous Pauli matrices forms a basis for $\mathfrak{su}(2)$ Lie Algebra and it exponentiates forms a $SU(2)$(**S**pecial **U**nitary) Lie Group. (90)

This description is almost all the tools needed to deal with our present model, as we will show later. These two kets $|0\rangle$ and $|1\rangle$ define the quantum bits, highly used in quantum information-theoretical aim.

A matrix representation is usefull to make clear the meaning of these operators and state vectors. Then, one can use the follow representation for the eigen-states of the Pauli matrix σ_{z}:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.6}$$

The operators $\sigma_{\mathrm{x}}, \sigma_{\mathrm{y}}$ and σ_{z} in this basis become

$$\sigma_{\mathrm{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_{\mathrm{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_{\mathrm{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.7}$$

With its respective Pauli matrices give all Pauli operators, let us see now how the operators σ_{x} and σ_{y} act on this basis, since σ_{z} is for sure diagonal on it. When the operator σ_{x} acts on $|0\rangle$ and $|1\rangle$ the results are

$$\sigma_{\mathrm{x}} |0\rangle = |1\rangle, \qquad \sigma_{\mathrm{x}} |1\rangle = |0\rangle. \tag{A.8}$$

On the other hand, for the operator σ_{y} the action become

$$\sigma_{\mathrm{y}} |0\rangle = i\,|1\rangle, \qquad \sigma_{\mathrm{y}} |1\rangle = -i\,|0\rangle. \tag{A.9}$$

To complete our description of Pauli matrix theory, we need to consider two well critical operators: the lowering and raising operators. These operators, on the basis of σ_{z}, are given by the two matrices

$$\sigma_{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma_{-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.10}$$

respectively. In terms of these operators, the Pauli matrix can be described as

$$\sigma_{\mathrm{x}} = \sigma_{+} + \sigma_{-}, \qquad \text{and} \qquad \sigma_{\mathrm{x}} = -i(\sigma_{+} - \sigma_{-}). \tag{A.11}$$

Now, we can justify these names just by seeing how their acts on the computational basis, i.e.,

$$\sigma_{+} |1\rangle = |0\rangle, \qquad \text{and} \qquad \sigma_{-} |0\rangle = |1\rangle. \tag{A.12}$$

and when you try to raise a $|0\rangle$ state or lower a $|1\rangle$ state, the result is zero.

A vital picture will follow us throughout this text: the Bloch sphere. It is a visualization of a quantum state, i.e., a geometrical representation of a qubit. An arbitrary single-qubit state can be written as

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right), \tag{A.13}$$

where θ, ϕ and γ are real numbers. The numbers $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ define a point on a unit three-dimensional sphere, and this is the Bloch sphere.

Qubit states with arbitrary values of γ are all represented by the same point on the sphere, and the exponential factor has no observable effects. In this way, we can therefore write:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle . \tag{A.14}$$

To obtain a density operator representation of this state, we can just be doing the operation

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} \tag{A.15}$$

and that parametrization allows the description of the Bloch sphere immediately.

A meaningful interpretation comes when we put the projections of the density operator in terms of average values of observables, let us define the basis $\{\mathbb{1}, \sigma_{\mathrm{x}}, \sigma_{\mathrm{y}}, \sigma_{\mathrm{z}}\}$, one can redescribe the density operator as

$$\begin{aligned} \rho &= \frac{1}{2}(\mathbb{1} + \sigma_{\mathrm{x}}\cos\phi\sin\theta + \sigma_{\mathrm{y}}\sin\phi\cos\theta + \sigma_{\mathrm{z}}\cos\theta), \\ &= \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}), \end{aligned} \tag{A.16}$$

where $\vec{\sigma}$ is the 3-element vector of Pauli matrices $(\sigma_{\mathrm{x}}, \sigma_{\mathrm{y}}, \sigma_{\mathrm{z}})$ and $\vec{r}$ is the unit Bloch vector. Notice that this vector have, in these components, the mean values of the Pauli operators, i.e.,

$$r_x = \langle\sigma_{\mathrm{x}}\rangle, \qquad r_y = \langle\sigma_{\mathrm{y}}\rangle, \qquad r_z = \langle\sigma_{\mathrm{z}}\rangle \tag{A.17}$$

in which $\vec{r} = (r_x, r_y, r_z)$. As we see in other sections, a situation of pure dephasing affects the average values of the Pauli operator σ_{x}, for example, without perturbing the operator σ_{z}. This is an example of useful visualization given by the Bloch sphere representation.

APPENDIX B – *MORE-THAN-ONE* FORMALISM

Now we consider the case of a composite quantum system, i.e., descriptive states of systems with more than one quantum particle. We will tread the same paths as in the probabilities Sec.2.1: construct a formalism to two objects and then generalize for N objects. (13, 32, 40) Any pure state $|\psi_{\mathrm{AB}}\rangle \in \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$ can be written as

$$|\psi_{\mathrm{AB}}\rangle = \sum_{i=1}^{d_{\mathrm{A}}} \sum_{j=1}^{d_{\mathrm{B}}} c_{ij} |e_i\rangle \otimes \left|e'_j\right\rangle, \tag{B.1}$$

where $\mathcal{H}_{\mathrm{A}} \cong \mathbb{C}^{d_{\mathrm{A}}} = \mathrm{span}\{|e_i\rangle\}$ and $\mathcal{H}_{\mathrm{B}} \cong \mathbb{C}^{d_{\mathrm{B}}} = \mathrm{span}\{\left|e'_j\right\rangle\}$. This states living on the tensor product space $\mathbb{C}^{d_{\mathrm{A}}} \otimes \mathbb{C}^{d_{\mathrm{B}}}$, being the symbol "$\otimes$" appearing in the equation just poits to this structures. Actually, the total vector state is a cartesian product of both states in $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{B}}$.

For example, let us see how this structure works for spin-1/2 particles. First we attribute a set of spin operators to each particles, i.e., for the particle one $\{\sigma_1^\alpha\}$, for the particle two $\{\sigma_2^\alpha\}$, such that $\alpha = x, y$ and z. In addition to the relations already existent, we assume that operators about different particles commute, i.e.,

$$[\sigma_1^\alpha, \sigma_2^\beta] = 0 \qquad \text{for} \qquad \alpha, \beta = x, y, z. \tag{B.2}$$

Consequently, we have the two-by-two commutation relation between different indexed particles. These composition comes from the tensorial structure $\sigma_1^\alpha = \sigma^\alpha \otimes \mathbb{1}$ and $\sigma_2^\alpha = \mathbb{1} \otimes \sigma^\alpha$.

In its turn, with respect to the eigen-states of those operators, considering an index $\chi = 0, 1$ and $\mathbb{C}^2 \cong \mathrm{span}\,\{|0\rangle, |1\rangle\}$, a composition of two spin-1/2 state is given by

$$|\chi_1\rangle \otimes |\chi_2\rangle = |\chi_1\chi_2\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \left(\mathbb{C}^2\right)^{\otimes 2}. \tag{B.3}$$

To deal with the matrix elements of many-particle systems, the idea of a tensor product is also essential. Tensor products are crucial to quantum mechanics and are the idea that allows describing features like entanglement in the theory.

Considering four operators A, B, C and D, the tensor product between them will satisfy the property

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \tag{B.4}$$

then, tensor product make it clear the different structure of each space of elements, here operators. An operator $\sigma_1^\alpha \sigma_2^\beta$ is then written as

$$\sigma_1^\alpha \sigma_2^\beta = (\sigma_1^\alpha \otimes \mathbb{1})(\mathbb{1} \otimes \sigma_2^\beta) = \sigma_1^\alpha \otimes \sigma_2^\beta, \tag{B.5}$$

for $\alpha, \beta = x, y$ and z.

Considering this composition structure for states and operators, an action of an operator $\sigma_1^\alpha \sigma_2^\beta$ in a state $|\chi_1\chi_2\rangle$ is given by

$$\sigma_1^\alpha \sigma_2^\beta \left|\chi_1\chi_2\right\rangle = (\sigma_1^\alpha \otimes \sigma_2^\beta)(\left|\chi_1\right\rangle \otimes \left|\chi_2\right\rangle) = (\sigma_1^\alpha \left|\chi_1\right\rangle) \otimes (\sigma_2^\beta \left|\chi_2\right\rangle). \tag{B.6}$$

In terms of matrix objects, what the tensor product doing can be illustrated considering two operators A and B, in which the operator $A = (a_{ij})$, $i = 1, 2, ..., N$ and $j = 1, 2..., M$. The tensorial composition between them is given by

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1M}B \\ \vdots & \ddots & \vdots \\ a_{N1}B & \dots & a_{NM}B \end{pmatrix}. \tag{B.7}$$

Then, to obtain tensor products of two operators we need simply multiply the elements of one by the another one. In case of operator σ_z that we deal with, for example, we have:

$$\sigma_\mathrm{z} \otimes \sigma_\mathrm{z} = \begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & -1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.8}$$

The same is true for vector states in σ_z eigenstates basis, e.g.,

$$\left|01\right\rangle = \begin{pmatrix} 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{B.9}$$

and then, tensor products gives us a general formula to construct many-body systems using operators from the individual subsystems.

Finally, we can consider a spin operator in a specific site i, in which is written in its tensorial composition with tneighbourhoodood sites as

$$\sigma_i^\alpha = \mathbb{1} \otimes ... \otimes \mathbb{1} \otimes \sigma_\alpha \otimes \mathbb{1} \otimes ... \otimes \mathbb{1}. \tag{B.10}$$

The reader can imagine what is the structure of a N-qubit state - a binary string formed by a composition of N qubits -, i.e., $|\chi_1\chi_2...\chi_N\rangle \in \mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2 \cong (\mathbb{C}^2)^{\otimes N}$*. Then, the Hilbert space of a composite quantum system is a tensor product of the Hilbert spaces of all its subsystems of the respective individual sites.

* The object $(\mathbb{C}^2)^{\otimes N}$ is the so-called N-fold tensor product

Generally, in composite systems, we are interested in looking at a specific subsystem, then the amount of information of the remaining system can be neglected. To get a general description of the act of looking at a specific subsystem, consider the operators acting on $\mathbb{C}^{d_\mathrm{A}} \otimes \mathbb{C}^{d_\mathrm{B}}$. They will be a general structure

$$O = \sum_{i=1}^{m} A_i \otimes B_i \tag{B.11}$$

for some index i and some set of operators A_i and B_i. Starting with the operator $O = A \otimes B$, the trace of this operator is

$$\mathrm{Tr}\, O = \sum_{i,j=1}^{d_\mathrm{A},d_\mathrm{B}} \left\langle e_i, e'_j \middle| O \middle| e_i, e'_j \right\rangle, \tag{B.12}$$

and futhermore, expanding the tensor products we get

$$\begin{aligned} \mathrm{Tr}\, O &= \sum_{i,j=1}^{d_\mathrm{A},d_\mathrm{B}} (\langle e_i| \otimes \left\langle e'_j \right|)(A \otimes B)(|e_i\rangle \otimes \left| e'_j \right\rangle) \\ &= \sum_{i,j=1}^{d_\mathrm{A},d_\mathrm{B}} \langle e_i| A |e_i\rangle \otimes \left\langle e'_j \middle| B \middle| e'_j \right\rangle \\ &= \sum_{i,j=1}^{d_\mathrm{A},d_\mathrm{B}} \langle e_i| A |e_i\rangle \left\langle e'_j \middle| A \middle| e'_j \right\rangle . \end{aligned} \tag{B.13}$$

The last term represents just the product of the traces of A and B in their respective Hilbert space. Then

$$\mathrm{Tr}\, A \otimes B = \mathrm{Tr}\, A\, \mathrm{Tr} B. \tag{B.14}$$

Then, the trace of a tensor product of two operators is the product of the individual traces in each operator. In this in hand, is direct the definition of partial trace:

$$\mathrm{Tr}_\mathrm{A} A \otimes B = (\mathrm{Tr} A) B, \qquad \mathrm{Tr}_\mathrm{B} A \otimes B = A(\mathrm{Tr} B). \tag{B.15}$$

The trace over some subsystem eliminates the degrees of freedom concerning it. In general terms, the partial traces with respect to some sub-space of an operator O are

$$\mathrm{Tr}_\mathrm{A} O = \sum_{i=1}^{m} (\mathrm{Tr} A_i) B_i \qquad \text{and} \qquad \mathrm{Tr}_\mathrm{B} O = \sum_{i=1}^{m} A_i (\mathrm{Tr} B_i), \tag{B.16}$$

and expliciting the component terms one we have

$$\mathrm{Tr}_\mathrm{A} O = \sum_{i=1}^{d_\mathrm{A}} \langle e_i| O |e_i\rangle \qquad \text{and} \qquad \mathrm{Tr}_\mathrm{B} O = \sum_{j=1}^{d_\mathrm{B}} \left\langle e'_j \middle| O \middle| e'_j \right\rangle . \tag{B.17}$$

For density operators acting on $\mathcal{H}_\mathrm{AB}$, arises the idea of state subsystem by partial trace, i.e.,

$$\rho_\mathrm{A} = \mathrm{Tr}_\mathrm{B} \rho \qquad \text{and} \qquad \rho_\mathrm{B} = \mathrm{Tr}_\mathrm{A} \rho. \tag{B.18}$$

Partial traces are, of course, significant in the context of the open quantum system. In this work, we use the structure of the model to build well-defined partial traces, since we dealt with more complicated partial traces to study the emergence of quantum Darwinism and SBS structures in our model.

APPENDIX C – INTERACTION PICTURE

We begin by considering a Hamiltonian of the form

$$H = H_0 + V \tag{C.1}$$

where H denotes the free evolution Hamiltonian (that is, the unperturbed one) concerning the system, whereas V is some added external perturbation, e.g., the environmental and the interacting Hamiltonian. (10, 20, 55)

From standard quantum theory, we know that the expectation value of an operator observable $O(t)$ is given by

$$\langle A(t)\rangle = \mathrm{Tr}\, A(t)\rho(t) = \mathrm{Tr}\, A(t)e^{-iHt}\rho(0)e^{iHt}, \tag{C.2}$$

where ρ is the complete quantum state. For reasons that will immediately becohem obvious, let us rewrite this expression as

$$\langle O(t)\rangle = \mathrm{Tr}\, \left(e^{iH_0t}O(t)e^{-iH_0t}\right)\left(e^{iH_0t}e^{-iHt}\rho(0)e^{iHt}e^{-iH_0t}\right). \tag{C.3}$$

Now, we can define a new set of operators, defined by

$$O^I(t) := e^{iH_0t}O(t)e^{-iH_0t}, \tag{C.4}$$

$$\begin{aligned}\rho^I(t) &:= e^{iH_0t}e^{-iHt}\rho(0)e^{iHt}e^{-iH_0t} \\ &= e^{iH_0t}\rho(t)e^{-iH_0t},\end{aligned} \tag{C.5}$$

where the superscript I is used to denote interaction picture operators. Note that the dynamics of the interaction-picture operators are fully determined by the unperturbed operators, rather than the total Hamiltonian. Our expectation-value equation can then be written in the compact form

$$\langle O(t)\rangle = \mathrm{Tr}\, O^I(t)\rho^I(t). \tag{C.6}$$

Now, instead, consider the case of a single system subject to some perturbation, let us consider a system-environment interaction:

$$H = H_0 + V \equiv H_\mathrm{S} + H_\mathrm{E} + H_\mathrm{SE} \tag{C.7}$$

where H_S is the system Hamiltonian, H_E the environmental Hamiltonian and, finally, the interacting Hamiltonian H_SE. Using the same strategy, with $H_0 \equiv H_\mathrm{S} + H_\mathrm{E}$ and $V \equiv H_\mathrm{SE}$, we obtain:

$$\rho_S^I(t) \equiv \mathrm{Tr}_E\rho^I(t). \tag{C.8}$$

APPENDIX D – TIME EVOLUTION OF ρ_{SF}: GENERAL CASE

For the present system with non-zero coupling and magnetic field, the partially-traced density operator can be obtained expanding the density operator ρ_{E} in the energy eigenbasis (Eq. 5.8), in the same way as the exact solution. But, for present work, a better path to obtain the partially-traced density operator is decompose the time evolution operator as a tensor product, i.e., $U(t) = e^{-i\alpha\sigma^z\otimes\sigma_1^z t} \otimes ... \otimes e^{-i\alpha\sigma^z\otimes\sigma_N^z t} = \bigotimes_{i=1}^N e^{-i\alpha\sigma^z\otimes\sigma_i^z t}$. Hence, the operator comes

$$
\begin{aligned}
\rho_{\mathrm{SF}}(t) &= \mathrm{Tr}_{\mathrm{E/F}}[U(t)\rho_{\mathrm{S}}\otimes\rho_{\mathrm{E}}U^\dagger(t)] \\
&= Z_{\mathrm{E}}^{-1}\mathrm{Tr}_{\mathrm{E/F}}\left[\left(\bigotimes_{i=1}^N e^{-i\alpha\sigma^z\otimes\sigma_i^z t}\right)\rho_{\mathrm{S}}\otimes\sum_\sigma e^{-\beta E(\chi)}\left|\chi\right\rangle\left\langle\chi\right|\left(\bigotimes_{i=1}^N e^{+i\alpha\sigma^z\otimes\sigma_i^z t}\right)\right],
\end{aligned}
$$

Writing each term $\left\langle n\right|\rho_{\mathrm{SF}}\left|m\right\rangle \equiv \rho_{\mathrm{SF}}^{nm}$, one have:

$$
\begin{aligned}
\rho_{\mathrm{SF}}^{nm}(t) &= \rho_{\mathrm{S}}^{nm}Z_{\mathrm{E}}^{-1}\mathrm{Tr}_{\mathrm{E/F}}[\bigotimes_{i=1}^N e^{-i(\epsilon_n-\epsilon_m)\sigma_i^z t}\sum_\chi e^{-\beta E(\chi)}\left|\chi\right\rangle\left\langle\chi\right|] \\
&= \rho_{\mathrm{S}}^{nm}Z_{\mathrm{E}}^{-1}\sum_\chi e^{-\beta E(\chi)}\mathrm{Tr}_{\mathrm{E/F}}[\bigotimes_{i=1}^N e^{-i(\epsilon_n-\epsilon_m)\sigma_i^z t}\left|\chi_i\right\rangle\left\langle\chi_i\right|] \\
&= \rho_{\mathrm{S}}^{nm}Z_{\mathrm{E}}^{-1}\sum_\chi e^{-\beta E(\chi)}\bigotimes_{i=1}^{\mathrm{fN}} e^{-i(\epsilon_n-\epsilon_m)\sigma_i^z t}\left|\chi_i\right\rangle\left\langle\chi_i\right|\prod_{\mathrm{i=fN+1}}^N \mathrm{Tr}[e^{-i(\epsilon_n-\epsilon_m)\sigma_i^z t}\left|\chi_i\right\rangle\left\langle\chi_i\right|],
\end{aligned}
$$

and, finally:

$$
\rho_{\mathrm{SF}}(t) = \frac{1}{Z_{\mathrm{E}}}\begin{pmatrix} |a|^2\sum_\chi e^{-\beta E(\chi)}\bigotimes_{i=1}^{\mathrm{fN}}\left|\chi_i\right\rangle\left\langle\chi_i\right| & a^\star b\sum_\chi e^{-\beta E(\chi)}e^{-2i\alpha m(\chi)t}\bigotimes_{i=1}^{\mathrm{fN}}\left|\chi_i\right\rangle\left\langle\chi_i\right| \\ ab^\star\sum_\chi e^{-\beta E(\chi)}e^{2i\alpha m(\chi)t}\bigotimes_{i=1}^{\mathrm{fN}}\left|\chi_i\right\rangle\left\langle\chi_i\right| & |b|^2\sum_\chi e^{-\beta E(\chi)}\bigotimes_{i=1}^{\mathrm{fN}}\left|\chi_i\right\rangle\left\langle\chi_i\right| \end{pmatrix} \tag{D.1}
$$

that does not express the structure that we need to broadcast all information to the environment from the system. Since $e^{\pm 2i\alpha m(\chi)t} \neq 0$ for any time and magnetization.

APPENDIX E – TIME EVOLUTION OF ρ_{SF}: $J = 0$

In this appendix, we show how derivate the partially reduced state for a more simple situation (non-interacting regime). Here, the calculation is easier by the fact of the non-interacting Hamiltonian can be describe as $H_{\text{E}} = \sum_k H_{\text{E}_\text{k}}$ with $H_{\text{E}_\text{k}} = -h\mathbb{1}_1 \otimes ... \otimes \mathbb{1}_{k-1} \otimes \sigma_k^z \otimes \mathbb{1}_{k+1} \otimes ... \otimes \mathbb{1}_N$. The density operator can be re-written as

$$\rho_{\text{E}} = \frac{1}{Z_{\text{E}}} \bigotimes_{i=1}^{N} e^{\beta h \sigma_i^z}, \tag{E.1}$$

in which $Z_{\text{E}} = 2^N \cosh^N(\beta h)$. Of course, for Gibbs states, noninteragent means uncorrelated. Then, one can compute each terms of the partial reduced matrix in the follow form:

$$\begin{aligned}
\rho_{\text{SF}}^{nm}(t) &= Z_{\text{E}}^{-1} \rho_{\text{S}}^{nm} \text{Tr}_{\text{E/F}} \left[\bigotimes_{i=1}^{N} e^{-i\alpha(\epsilon_n - \epsilon_m)\sigma_i^z t} e^{\beta h \sigma_i^z} \right] \\
&= Z_{\text{E}}^{-1} \rho_{\text{S}}^{nm} \text{Tr}_{\text{fN+1}} ... \text{Tr}_{\text{N}} [e^{-i\alpha(\epsilon_n - \epsilon_m)\sigma_1^z t} e^{\beta h \sigma_1^z} \otimes ... \otimes e^{-i\alpha(\epsilon_n - \epsilon_m)\sigma_N^z t} e^{\beta h \sigma_N^z}] \\
&= Z_{\text{E}}^{-1} \rho_{\text{S}}^{nm} \bigotimes_{i=1}^{\text{fN}} e^{-i\alpha(\epsilon_n - \epsilon_m)\sigma_i^z t} e^{\beta h \sigma_i^z} \prod_{\text{fN+1}}^{N} \text{Tr} \left[e^{-i\alpha(\epsilon_n - \epsilon_m)\sigma_i^z t} e^{\beta h \sigma_i^z} \right].
\end{aligned} \tag{E.2}$$

We can make the identifications that follow

$$\rho_{\text{F}} = \bigotimes_{i=1}^{\text{fN}} \frac{e^{\beta h \sigma_i^z}}{\text{Tr}\left[e^{\beta h \sigma_i^z} \right]}, \qquad \rho'_{\text{F}}(t) = \bigotimes_{i=1}^{\text{fN}} \frac{e^{(\beta h - 2i\alpha t)\sigma_i^z}}{\text{Tr}\left[e^{\beta h \sigma_i^z} \right]}$$

and the decoherence function comes

$$\Gamma_{\text{F}}(t) = \left[\frac{\cosh(\beta h - 2i\alpha t)}{\cosh(\beta h)} \right]^{(1-f)N}.$$

Where we finally were able to write the matrix for the explicitly partially traced density operator

$$\rho_{\text{SF}}(t) = \begin{pmatrix} |a|^2 \rho_{\text{F}} & a^\star b \rho'_{\text{F}}(t) \Gamma_{\text{F}}(t) \\ a b^\star \rho'_{\text{F}}(t) \Gamma_{\text{F}}(t) & |b|^2 \rho_{\text{F}} \end{pmatrix}. \tag{E.3}$$

In this limit, the calculations for the fraction entropy can be done quickly, because the fraction density operator is a tensorial composition of fN 2×2 matrices.