**UNIVERSIDADE DE SÃO PAULO
INSTITUTO DE FÍSICA DE SÃO CARLOS**

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**The superradiance-superabsorption interplay and a
PT-symmetric harmonic oscillator with complex
frequency**

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**The superradiance-superabsorption interplay and a
PT-symmetric harmonic oscillator with complex
frequency**

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Advisor: Prof. Dr. Miled Hassan Youssef Moussa

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*“Nobody exists on purpose. Nobody belongs anywhere.
Everybody’s gonna die. Come watch TV*

”

Morty

ABSTRACT

DOURADO, R. A. **The superradiance-superabsorption interplay and a $\mathcal{PT}$ -symmetric harmonic oscillator with complex frequency.** 2021. 83p.

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A superradiant pulse emitted by a sample of two-level atoms can be combined with superabsorption in order to increase the speed of the excitation interplay. We showcase this with two different experiments, first by subjecting the atomic sample to a competition between dissipation and amplification, the latter being a time-dependent multi-modal linear pumping; second, by coupling the atomic sample with a single mode of a high finesse cavity. The mean-field approximation was used in both cases. In the first, to solve the Schrödinger equation for the resulting time-dependent nonlinear Hamiltonian exactly we use the Lewis and Riesenfeld method. In the second case, we obtain a quantum master equation in the Lindblad form and solve it numerically. We observe that, in particular, the higher the number of atoms on the sample the higher will be the frequency of the interplay in comparison with that coming from the Jaynes-Cummings model. For a certain regime of parameters this system also leads to a superradiant laser, whose characteristics we discuss. These results show that many-particle effects can be further explored for the implementation of quantum information processes as well as for constructing of a superradiant laser. At last, we study non-Hermitian $\mathcal{PT}$ -symmetric bosonic nonautonomous Hamiltonians under linear and parametric amplifications. We present a generic approach to solve these Hamiltonians through the use of non-unitary similarity transformations. The choice for the metric associated with these transformations has a dramatic impact on the pseudo-Hermitian operators of the system.

Keywords: Superradiance. Superabsorption. Laser. Mean-field approximation. $\mathcal{PT}$ -symmetry.

RESUMO

DOURADO, R. A. **A interação superradiância-superabsorção e um oscilador harmônico $\mathcal{PT}$ -simétrico com frequência complexa.** 2021. 83p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2021.

Um pulso superradiante emitido por uma amostra de átomos de dois níveis pode ser combinada com superabsorção de modo a aumentar a velocidade da troca de excitações. Mostramos isso para dois experimentos diferentes. Primeiro, submetemos uma amostra atômica a uma competição entre dissipação e amplificação, esta última sendo um bombeamento multimodal dependente do tempo; segundo, acoplando a amostra atômica com um modo de uma cavidade de alta qualidade. A aproximação de campo médio é usada em ambos os casos. No primeiro, para resolver a equação de Schrödinger para o Hamiltoniano não linear e dependente do tempo resultante, utilizamos o método de Lewis e Riesenfeld. No segundo caso, obtemos uma equação mestra na forma de Lindblad e a resolvemos numericamente. Observamos que, em particular, quanto maior o número de átomos presentes na amostra, maior a frequência de troca de excitações em comparação com a mesma para o modelo de Jaynes-Cummings. Para um certo regime de parâmetros esse sistema também leva a um laser superradiante, cujas características discutimos. Esses resultados mostram que efeitos coletivos podem ser explorados para a implementação de processos de informação quântica, além de construção de lasers superradiantes. Por fim, estudamos hamiltonianos bosônicos, $\mathcal{PT}$ simétricos, não autônomos e não hermitianos sob amplificações lineares e paramétricas. Apresentamos procedimento genérico para resolver esses hamiltonianos através do uso de transformações de similaridade não unitárias. A escolha da métrica associada a essas transformações tem um impacto dramático nos operadores pseudo-hermitianos do sistema.

Palavras-chave: Superradiância. Superabsorção. Laser. Aproximação de campo médio. Simetria $\mathcal{PT}$

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1 INTRODUCTION

This work is divided on mainly three subtopics. First, the radiation-matter interplay and how coherent effects have the potential to dramatically increase the frequency of the Rabi oscillations. Second, a mean-field approach for the laser theory, and, finally, pseudo-Hermitian quantum mechanics.

1.1 Superradiance and superabsorption

In 1954 Dicke (2) showed that a two-level atom in the excited state in the presence of another atom in the ground state has its decaying rate amplitude doubled provided that the former perceives the presence of the latter. Based on this result he defined the superradiance as a collective spontaneous emission process.

Now consider an ensemble of two-level atoms that are initially prepared in the excited state interacting with a thermal bath. As the first atoms naturally revert to the ground state, the emitted radiation starts to build correlation between the dipole moments of different atoms, leading to an acceleration of the overall emission process for the atomic sample. The superradiant pulse has a duration of $\tau_c \propto 1/N$ which is much smaller than single particle spontaneous decay time, T_1 , which does not depend on the number of atoms. As a direct consequence, the intensity of the superradiant pulse is proportional to N^2 , in comparison to the uncorrelated emission of N atoms, which is proportional to N .

The superradiance was observed in experimental platforms due to the interaction with the electromagnetic field via excitation blockade, (3 - 4), and in one-dimensional lattices in ultracold atoms (5).

On the other hand, it was only in 2014 that the concept of superabsorption, the reciprocal process to the superradiance, was proposed (6). The presence of time-reversal symmetry in a system suggests that an enhanced absorption process must also take place in a quantum mechanical system where superradiance occurs. As we are going to show, there is a competition between the coherent absorption and emission processes. If the system is coupled to a reservoir, the emission rate generally dominates and the contribution of the superabsorption is negligible. However, the presence of mechanisms acting to restore the energy of the atomic sample, as for example a reservoir at high temperatures, a Glauber amplifier or a cavity mode, can make the superabsorption not negligible, and possibly the dominating phenomenon over the superradiance.

During our following discussions two assumptions will be made. The first is that the atomic sample is moderately dense, so that the atoms are distant enough from each other, allowing us to neglect the Coulomb interactions (we want only an indirect interaction

mediated by the electromagnetic field). Secondly, the size of the sample is assumed to be much smaller than the wavelength of the radiated photons, this is important in order to all the atoms to observe the same field, which, as we are going to show, is the key for the enhancement of the emission and absorption processes that we are going to explore.

We start by following the developments of Refs. (7–9) which use the mean-field approach to describe superradiance. The first goal is to characterize the superabsorption, which we are going to do by considering an atomic sample interacting with a reservoir and an external multi-modal linear pumping (often referred to as anti-reservoir). When the coupling strength of the latter dominates over the former we show that a sharp peaked, with intensity proportional to N^2 , absorption process takes place, a clear signature of the superabsorption overcoming the superradiance. To do that, we solve the Schrödinger equation for the time-dependent nonlinear Hamiltonian which generates the unitary evolution for the system.

Besides that, we also investigate the situation where the atomic sample is confined in a high-finesse cavity. We show that the superabsorption and superradiance processes increase the frequency of the excitation interplay atom-field. Evidently, we seek for a shorter period of emission and reabsorption in terms of the collective effects of both processes, which could, in principle, be used as an instrument for manipulating quantum coherence.

1.1.1 Laser theory

When dealing with large fields the semi-classical approach is adequate to describe a laser. However, whenever small fluctuations are important, together with characteristics as for example the photon-statistics (since the laser is nonequilibrium quantum statistical problem) and natural linewidth, a full quantized theory is necessary. The quantum theory of the laser was developed in the 1960s by Scully, Lamb, Haken and others (10–12).

In this work we are going to use the master equation approach, following the discussions made in Refs. (13–15). Usually, a simple description of a laser consists in consider three atomic levels. The superior two in the energy scale are called lasing levels, and the transition between these two levels are due to the interaction with a quantized electromagnetic field. The lower lasing level is assumed to have a strong coupling to the environment, in order to decay almost instantly to the ground state. Finally, the atom is excited from the ground state to the upper lasing level by an external pumping, repeating the process. If the atomic sample is not dense, each atom goes through the above described process independently.

Therefore, one of our goals is to obtain a suitable master equation to describe a superradiant laser. Such a description should have a term responsible for the unitary part of the evolution, the radiation-matter Hamiltonian which, in the rotating wave ap-

proximation, can be described by the Jaynes-Cummings model, this interaction couples the two lasing levels. Also, there must be dissipation terms regarding the coupling of the atomic sample and the cavity mode to the environment, in this work we are going to consider a two-level laser, so that the lower lasing level is also the ground state. Finally, an absorption term must also be present, connecting the ground state to the higher lasing level.

As discussed, if the atomic sample is dense enough so that the wavelength of the radiation is much greater than the size of the sample, the many body aspect of the atomic sample must be taken into consideration. With this assumption, a superradiant laser was proposed in Ref. (16), which describes the laser using three atomic levels, and the superradiant pulses occur from the lower lasing level to the ground state. As anticipated in our previous discussion, the intensity of this radiation is proportional to N^2 , and the linewidth with $1/N^2$. In our model, we are going to consider not only the superradiant effects coming from the coupling of the atomic sample with the cavity mode and the reservoir, but also the superabsorption due to an external multi-modal pumping.

Furthermore, there are several similarities between the laser and second-order phase transitions systems, as for example the phase transition of a magnet that leads to ferromagnetism. We are going to discuss this phase transition for the laser problem, defining an order parameter, the transition points and how the many body effects. At last, we show how the collective effects previously discussed should affect the laser threshold.

1.2 Pseudo-Hermitian quantum mechanics

In quantum mechanics the hermiticity requirement for observables, and, in particular, Hamiltonians, is usually implicit. This is so firstly because an Hermitian Hamiltonian ensures real spectra, and, second, unitary evolution. In other words, if a given state is normalized at a time t_0 , the time evolution, coming from the Schrödinger equation, maintains the state normalized at all times $t > t_0$ and, besides that, the expectation value for the energy of the system is a real value. Hence, it is understandable why the construction of Hamiltonians usually follows from symmetrization rules that ensure its hermiticity.

The interest in non-Hermitian, and thus less restricted, systems has been increasing in the past few decades since Bender and Boettcher (17) showed that a family of $\mathcal{PT}$ -symmetric Hamiltonians exhibit real eigenenergies. Shortly after that, the field received important contributions from Mostafazadeh, whose works (18 - 19) demonstrated that a pseudo-Hermitian Hamiltonian possesses an unbroken anti-unitary symmetry (for example, $\mathcal{PT}$ -symmetry) and, consequently, have real spectra.

In addition, Mostafazadeh also observed, Ref. (19), that unitary evolution is also preserved when taking into consideration that pseudo-Hermitian Hamiltonians have a bi-

orthogonal basis. This requires a redefinition of the inner product to include the so-called metric, Θ , as shown in the following equation

$$\langle \psi(t) | \Theta | \psi(t) \rangle = \langle \psi(t) \rangle_{\Theta} = 1, \quad (1.1)$$

where $\Theta = \eta^{\dagger} \eta$, which ensures that H enjoys self-adjointness. The operator η is the so-called Dyson map and it is, evidently, a non-unitary transformation for non-Hermitian Hamiltonians.

Now, a system whose evolution is given by the Schrödinger equation ($\hbar = 1$)

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle, \quad (1.2)$$

where $H \neq H^{\dagger}$, is solvable if we perform the following similarity transformation

$$|\Pi(t)\rangle = \eta |\psi(t)\rangle, \quad (1.3)$$

and the Schrodinger equation becomes

$$i \frac{\partial}{\partial t} |\Pi(t)\rangle = h |\Pi(t)\rangle, \quad (1.4)$$

where h , the Hermitian counterpart of H , is given by

$$h = \eta H \eta^{-1}, \quad (1.5)$$

if η is time independent. If we demand the hermiticity of h , then H must respect the following pseudo-hermiticity relation

$$H^{\dagger} \Theta = \Theta H, \quad (1.6)$$

which can be used to construct the Dyson map.

We can generalize the above relations for any observable O , as shown below

$$O = \eta^{-1} o \eta. \quad (1.7)$$

$$O^{\dagger} \Theta = \Theta O. \quad (1.8)$$

Hence, all observables become pseudo-Hermitian and, as discussed in (20), the choice of the metric can be made so that a given observable of interest, as for example the position or momentum, is Hermitian.

As already mentioned, the metric redefines the inner product of the Hilbert space spanned by the eigenvectors of the non-Hermitian Hamiltonian. Our goal is to show that the choice of the Dyson Map has a tremendous impact on the expectation values of the physical quantities related to the system described by H . This is immediately perceivable in equation (1.7), each possibility for η will result in a different pseudo-Hermitian operator. A thorough discussion of the effects of the choice of the metric on these pseudo-Hermitian operators is present in Refs. (20) and (21). It was also shown in Ref. (22) that a non-Hermitian radiation-matter interaction leads to a transition rate of states $\langle f | V_I | i \rangle$ which depends on the choice of the metric, where f and i are the final and initial states of an atom, respectively, and V_I would be the interaction term of the Hamiltonian in the interaction picture. Besides that, we mention also that non-Hermiticity gives rise to unique topological phases without Hermitian analogs (23), hence it is certainly interesting to investigate what kind of dynamical consequences that are not reachable by a Hermitian system can be actually done in a non-Hermitian setting.

If H is time-dependent (TD), it is sometimes useful to consider Dyson maps also TD. In that case, equation (1.5) no longer holds, and, as observed in Ref. (24), to guarantee the unitarity of the time evolution of the system, the following expression must be satisfied

$$h = \eta H \eta^{-1} + i \dot{\eta} \eta^{-1}. \quad (1.9)$$

One of the consequences of equation (1.9) is that the observability of the Hamiltonian H is lost, since H and h are no longer related by a similarity transformation. However, as discussed in Ref. (25), there is no compelling reason why H must be an observable, it could be treated as an auxiliary operator which generates the dynamics of the system. Furthermore, in this work we are going to be mostly interested in other quantities, such as the quadratures X and P given by

$$X = \eta^{-1} x \eta, \quad (1.10)$$

$$P = \eta^{-1} p \eta, \quad (1.11)$$

which will be further discussed in the sequence, so the time-dependency of the Hamiltonian would not hinder our analysis. To show that, we are going to solve TD non-Hermitian Hamiltonians in chapter (3).

1.2.1 Squeezed quadratures of the electromagnetic field

Consider two operators that satisfy the commutation relation

$$[x, p] = ic. \quad (1.12)$$

The Heisenberg principle imposes that there must be an uncertainty between those operators, which is given by

$$\Delta x \Delta p \geq \frac{1}{2} |\langle c \rangle|. \quad (1.13)$$

A state is called a squeezed state if one of the above variances is decreased, as for example,

$$(\Delta x)^2 < \frac{1}{2} |\langle c \rangle|. \quad (1.14)$$

This also means that the other operator, in this case p , is going to have its variance increased by at least the same amount that Δx was decreased to preserve the uncertainty principle. In other words, we diminish one variable's variance at the expense of the corresponding increase to the uncertainty of its conjugate variable. Squeezed states are extremely important in applications that require high precision measurements, such as gravitational wave detection. A further reading on the subject can be done in Ref. (13).

Now introducing the field quadratures (13)

$$x = \frac{a + a^\dagger}{2}, \quad (1.15)$$

$$p = \frac{a - a^\dagger}{2i}, \quad (1.16)$$

where $a^\dagger$ and a are the creation and destruction bosonic operators, respectively. Then, using equation (1.13),

$$\Delta x \Delta p \geq \frac{1}{4}. \quad (1.17)$$

The coherent state, often referred as $|\alpha\rangle$, presents minimum uncertainty between the quadratures in equation (1.17) and also $\Delta x = \Delta p$. This means the displacement operator,

$$D(\alpha) = e^{\alpha a - \alpha^* a^\dagger}, \quad (1.18)$$

which generates coherent states, do not result in squeezing.

We can generalize this statement for any harmonic oscillator under the influence of linear amplification,

$$H = \omega a^\dagger a + f(t)a + f^*(t)a^\dagger, \quad (1.19)$$

because, as proven in Ref. (26), the solutions for Hamiltonians of this form are stable coherent states.

One way of obtaining squeezing is by applying the following operator on the radiation field

$$S(\xi) = \exp\left(\frac{1}{2}\xi^* a^2 - \frac{1}{2}\xi a^{\dagger 2}\right), \quad (1.20)$$

where $\xi = r e^{i\phi}$ is an arbitrary complex number. The factor r is known as the compressing factor. On the other hand, ϕ is the direction of the squeezing in the phase space, hence, an axis rotation of $\phi/2$ is generally required to observe the squeezed quadratures. Finally, another interesting way of observing squeezing is through the visualization of the Wigner function in the phase space.

We conclude from this analysis that Hermitian Hamiltonians which describe harmonic oscillators under the effect of parametric amplification,

$$H = \omega a^\dagger a + f(t)a^2 + f^*(t)(a^\dagger)^2, \quad (1.21)$$

present squeezing, while linear amplification does not.

We are going to show in this work that non-Hermitian Hamiltonians do not follow that rule, the choice of the Dyson map overrules the above analysis. To do that, we show that time-independent non-Hermitian Hamiltonians under linear amplification generate, with an appropriate choice for the metric, squeezed states.

Moreover, we are going to generalize the analysis started in Ref. (27) for time-dependent non-Hermitian Hamiltonians under the effect of parametric amplification, by proposing a more generic, non-Hermitian, Dyson map. At last, we aim to obtain extremely high (virtually infinite) squeezing degrees at finite and controllable times for a specific case, a harmonic oscillator with pure imaginary frequency.

1.3 Dissertation overview

The text is organized as follows. In chapter (2), we are going to further describe mathematically the topics introduced in section (1.1). We start by analysing a system under the effect of a periodic multi-modal pumping, this will allow us to fully characterize the superradiance and the superabsorption phenomena. We use these concepts to explain how these many-body effects lead to an increase in the frequency of the interplay between an atomic sample confined to a high finesse cavity and the electromagnetic field.

Besides that, in section (2.3) we discuss the laser theory from using the mean-field approach. Once again, we investigate how the coherent absorption and emission effects influence some of the characteristics of the laser, such as the intensity, the laser

threshold and the natural linewidth. The main results from chapter (2), and a more concise discussion for the experienced reader, can be found in Ref. (28).

In chapter (3) we come back to pseudo-Hermitian quantum mechanics. We present a mathematical procedure to obtain a Hermitian counterpart of non-Hermitian Hamiltonians under the influence of linear or parametric amplification, including time dependency. Furthermore, we investigate how the choice of the metric could lead to interesting results such as infinite squeezing degrees of the field quadratures at finite and controllable times. We close the chapter connecting a harmonic oscillator with pure imaginary frequency with the regime of strong parametric amplification. The main results from this chapter can also be found in Ref. (29), for a thorough analysis on the convergence of Dyson maps in quadratic bosonic Hamiltonians, and (30)

In chapter (4), we present the conclusions and the perspectives for future works.

2 THE INTERPLAY

2.1 Superradiance and superabsorption in an atomic sample under the competition between dissipation and amplification

As mentioned, we are going to consider a sample of two-level atoms interacting both with a thermal bath and a Glauber amplifier, the Hamiltonian which describes this system is given by

$$H = H_0 + H_I, \quad (2.1)$$

where

$$H_0 = \omega_0 S_z + \sum_{m=R,G} \sum_k \omega_{mk} b_{mk}^\dagger b_{mk}, \quad (2.2a)$$

$$H_I = \sum_k \left[\lambda_{Rk} (S_- b_{Rk}^\dagger + S_+ b_{Rk}) + \lambda_{Gk}(t) (e^{i\nu t} S_- b_{Gk} + e^{-i\nu t} S_+ b_{Gk}^\dagger) \right]. \quad (2.2b)$$

From now on, we consider the subscript S to refer the ensemble of atoms, R to the thermal bath and G to the Glauber amplifier, also

$$S_z = \sum_{i=1}^N \sigma_z^{(i)} / 2, \quad (2.3a)$$

$$S_+ = \sum_{i=1}^N \sigma_+^{(i)}, \quad (2.3b)$$

$$S_- = \sum_{i=1}^N \sigma_-^{(i)}. \quad (2.3c)$$

In the interaction picture

$$V_I = \sum_k \left[\lambda_{Rk} (S_- b_{Rk}^\dagger e^{i(\omega_{Rk} - \omega_0)t} + S_+ b_{Rk} e^{-i(\omega_{Rk} - \omega_0)t}) + \lambda_{Gk}(t) (S_- b_{Gk} e^{-i(\omega_{Gk} + \omega_0 - \nu)t} + S_+ b_{Gk}^\dagger e^{i(\omega_{Gk} + \omega_0 - \nu)t}) \right]. \quad (2.4)$$

At this point, we follow a similar procedure as done in Ref. (31), considering $\nu = 2\omega_0$. This choice is important to ensure that the interaction terms referring to the anti-reservoir, in equation (2.4), do not vanish under the rotating wave approximations. In addition, it is interesting to point out that the coupling strength of the Glauber amplifier

interaction with the sample of atoms, $\lambda_{Gk}(t)$, is time-dependent and will be described in this section.

Using the following definitions,

$$F_{m+} = \sum_k \lambda_{mk} b_{mk}^\dagger e^{i(\omega_{mk} - \omega_0)t}, \quad (2.5a)$$

$$F_{m-} = (F_{m+})^\dagger, \quad (2.5b)$$

we rewrite equation (2.4),

$$V_I = S_-[F_{R+}(t) + F_{G-}(t)] + S_+[F_{R-}(t) + F_{G+}(t)]. \quad (2.6)$$

The quantum Liouville equation in the interaction picture

$$i\dot{\rho}_I(t) = [V_I(t), \rho(t)], \quad (2.7)$$

which can also be written as, considering that the system is weakly enough coupled to the thermal bath and the Glauber amplifier to allow us to perform a second-order perturbation approximation,

$$\dot{\rho}_I(t) = -i[V_I, \rho(0)] - \int_0^t dt' [V_I(t), [V_I(t'), \rho_I(t')]]. \quad (2.8)$$

Considering both the reservoir and the Glauber amplifier in a global equilibrium, we can write

$$\rho(t) = \rho_S(t) \otimes \rho_R(0) \otimes \rho_G(0). \quad (2.9)$$

Substituting equation (2.9) into equation (2.8) and tracing over the reservoir and the anti-reservoir variables, we obtain the master equation (ME) for the system in the interaction picture

$$\dot{\rho}_S(t) = - \int_0^t dt' Tr_R Tr_G [V_I(t), [V_I(t'), \rho_R(0) \rho_G(0) \rho_S(t')]], \quad (2.10)$$

$$\dot{\rho}_S(t) = -Tr_R Tr_G \int_0^t dt' [V_I(t)V_I(t')\rho(t') - V_I(t)\rho(t')V_I(t') - V_I(t')\rho(t')V_I(t) + \rho(t')V_I(t')V_I(t)], \quad (2.11)$$

where it was also assumed that, $Tr_R(V_I \rho_R) = Tr_G(V_I \rho_G) = 0$.

We are going to use the Markov approximation, considering that the correlation time for the reservoir's and the amplifier's operators is much shorter than the relaxation time of the system,

$$\rho_S(t') \approx \rho_S(t). \quad (2.12)$$

Also,

$$\langle b_{mk} b_{m'k'} \rangle = \langle b_{mk}^\dagger b_{m'k'}^\dagger \rangle = 0, \quad m, m' = R, G, \quad (2.13)$$

and, furthermore,

$$\langle b_m^\dagger(\omega_m) b'_m(\omega_{m'}) \rangle = \delta_{mm'} N^{(m)}(\omega_m) \delta(\omega_m - \omega_{m'}), \quad (2.14)$$

with $N^{(m)}(\omega_m)$ being the thermal average photon number at the frequency ω_m . In other words, there is no correlation between modes of the reservoir and of the Glauber amplifier, expressed by $\delta_{mm'}$, also there is no correlation between different modes of either, indicated by $\delta(\omega_m - \omega_{m'})$.

With these considerations, we substitute equation (2.6) into equation (2.11) to obtain

$$\begin{aligned} \dot{\rho}_S(t) = & - \int_0^t dt' \{ [S_-, S_+ \rho_S] (Tr_R(F_{R+}(t) F_{R-}(t') \rho_R) + Tr_G(F_{G-}(t) F_{G+}(t') \rho_G)) \\ & + [S_+, S_- \rho_S] (Tr_R(F_{R-}(t) F_{R+}(t') \rho_R) + Tr_G(F_{G+}(t) F_{G-}(t') \rho_G)) \\ & - [S_-, \rho_S S_+] (Tr_R(F_{R-}(t') F_{R+}(t) \rho_R) + Tr_G(F_{G+}(t') F_{G-}(t) \rho_G)) \\ & - [S_+, \rho_S S_-] (Tr_R(F_{R+}(t') F_{R-}(t) \rho_R) + Tr_G(F_{G-}(t') F_{G+}(t) \rho_G)) \}. \end{aligned} \quad (2.15)$$

Now, defining

$$\xi_{12}^{(m)}(t - t') = Tr_m(F_{m+}(t) F_{m-}(t') \rho_m), \quad (2.16a)$$

$$\xi_{21}^{(m)}(t - t') = Tr_m(F_{m-}(t) F_{m+}(t') \rho_m), \quad (2.16b)$$

then

$$\begin{aligned} \dot{\rho}_S(t) = & - \int_0^t dt' \{ [S_-, S_+ \rho_S] (\xi_{12}^{(R)}(t - t') + \xi_{21}^{(G)}(t - t')) \\ & + [S_+, S_- \rho_S] (\xi_{21}^{(R)}(t - t') + \xi_{12}^{(G)}(t - t')) \\ & - [S_-, \rho_S S_+] (\xi_{21}^{*(R)}(t - t') + \xi_{12}^{*(G)}(t - t')) \\ & - [S_+, \rho_S S_-] (\xi_{12}^{*(R)}(t - t') + \xi_{21}^{*(G)}(t - t')) \}, \end{aligned} \quad (2.17)$$

and, finally,

$$\begin{aligned}\dot{\rho}_S(t) = & -[S_-, S_+ \rho_S] \left(\tilde{\xi}_{12}^{(R)} + \tilde{\xi}_{21}^{(G)}(t) \right) + \\ & -[S_+, S_- \rho_S] \left(\tilde{\xi}_{21}^{(R)} + \tilde{\xi}_{12}^{(G)}(t) \right) + h.c.,\end{aligned}\quad (2.18)$$

where

$$\tilde{\xi}_{12}^{(m)} = \int_0^t dt' \xi_{12}^{(m)}(t - t'), \quad (2.19a)$$

$$\tilde{\xi}_{21}^{(m)} = \int_0^t dt' \xi_{21}^{(m)}(t - t'). \quad (2.19b)$$

where $m = R, G$.

Transforming the sums over the frequency modes into integrals over a continuous frequency spectrum, following similar steps as described in refs (8, 14), we would obtain

$$\tilde{\xi}_{12}^{(m)} = \frac{1}{2} \gamma_{12}^{(m)} - i \Delta \omega_{12}^{(m)}, \quad (2.20a)$$

$$\tilde{\xi}_{21}^{(m)} = \frac{1}{2} \gamma_{21}^{(m)} + i \Delta \omega_{21}^{(m)}, \quad (2.20b)$$

with

$$\gamma_{12}^{(m)} = \pi g_m(\omega_0) |\lambda_m(\omega_0)|^2 \langle b_m^\dagger(\omega_0) b_m(\omega_0) \rangle, \quad (2.21a)$$

$$\gamma_{21}^{(m)} = \pi g_m(\omega_0) |\lambda_m(\omega_0)|^2 \langle b_m(\omega_0) b_m^\dagger(\omega_0) \rangle, \quad (2.21b)$$

and $g(\omega)$ is the density of frequencies.

In addition,

$$\Delta \omega_{12}^{(m)} = \frac{1}{2} \mathcal{P} \int_0^\infty g_m(\omega) |\lambda_m(\omega)|^2 \langle b_m(\omega) b_m^\dagger(\omega) \rangle \frac{d\omega}{\omega_0 - \omega}, \quad (2.22a)$$

$$\Delta \omega_{21}^{(m)} = \frac{1}{2} \mathcal{P} \int_0^\infty g_m(\omega) |\lambda_m(\omega)|^2 \langle b_m^\dagger(\omega) b_m(\omega) \rangle \frac{d\omega}{\omega_0 - \omega}, \quad (2.22b)$$

where $\mathcal{P}$ is the Cauchy principal value. To derive this we have used

$$\int_0^\infty d\tau e^{\pm i\epsilon\tau} = \pi \delta(\epsilon) \pm \mathcal{P} \left(\frac{1}{\epsilon} \right). \quad (2.23)$$

Equations (2.21) and (2.22) are evaluated in ω_0 and for time-independent coupling factors λ_m , which are well suited for the reservoir coefficients, $\tilde{\xi}_{12}^{(R)}$ and $\tilde{\xi}_{21}^{(R)}$. However,

we need to describe the effects of adding a time dependency on the coupling factors system-amplifier, $\lambda_G \rightarrow \lambda_G(t)$.

Considering a coupling of the form,

$$\lambda_G(\omega, t) = u(t)\lambda_G(\omega), \quad (2.24)$$

i.e., modulating the coupling factor by a real function of time $u(t)$. Now, substituting this in equation (2.19a)

$$\tilde{\xi}_{12}^{(G)}(t) = \int_0^t dt' e^{-i\omega_0(t-t')} Tr_G \left[\sum_{k,n} \lambda_{Gk}(t) \lambda_{Gn}(t') b_k^\dagger b_n \right], \quad (2.25)$$

and defining $u(t) = C_1 + C_2 \sin(\zeta t)$, the time-dependent coefficients of the Glauber amplifier are going to be described by the following equations

$$\gamma_{12}^{(G)}(t) = [C_1^2 + C_1 C_2 \sin(\zeta t)] \gamma_{12}^{(G)}, \quad (2.26a)$$

$$\gamma_{21}^{(G)}(t) = [C_1^2 + C_1 C_2 \sin(\zeta t)] \gamma_{21}^{(G)}, \quad (2.26b)$$

$$\Delta\omega_{21}^{(G)}(t) = [C_1^2 + C_1 C_2 \sin(\zeta t)] \Delta\omega_{12}^{(G)}, \quad (2.26c)$$

$$\Delta\omega_{12}^{(G)}(t) = [C_1^2 + C_1 C_2 \sin(\zeta t)] \Delta\omega_{12}^{(G)}. \quad (2.26d)$$

To obtain the above expressions, we used equation (2.23). Also, we neglected terms that oscillate with frequency ζ , for large values of ζ , in the rotating wave approximation. Having all the necessary coefficients properly obtained, we now express the ME in the Schrödinger picture

$$\begin{aligned} \dot{\rho}_N(t) = & -i[H_S, \rho_N] - [S_-, S_+ \rho_N] \left(\tilde{\xi}_{12}^{(R)} + \tilde{\xi}_{21}^{(G)}(t) \right) + \\ & - [S_+, S_- \rho_N] \left(\tilde{\xi}_{21}^{(R)} + \tilde{\xi}_{12}^{(G)}(t) \right) + h.c.. \end{aligned} \quad (2.27)$$

Because we switched to the Schrödinger picture, we also changed the subscripts from S to N to differ from the density operator in equation (2.18) and, besides that, to indicate that we have this number of atoms in our system.

In the following, we are going to use the mean-field approximation to obtain a representative atom for the system. To do that, we apply consecutive traces on the density operator until we obtain a single-particle density operator.

$$\rho_1 = Tr_2 \dots Tr_{N-1} Tr_N \{ \rho_S \}, \quad (2.28)$$

Also, assuming that there is no statistical correlation among different atoms, i.e., $\rho_2 = \rho_1 \otimes \rho_1$, one gets

$$\begin{aligned} \dot{\rho}_1 = & -i[H_{TR}, \rho_1] - \frac{\gamma^{(R)}N^{(R)} + \gamma^{(G)}(N^{(G)} + 1)}{2}(\sigma_- \sigma_+ \rho_1 - 2\sigma_+ \rho_1 \sigma_- + \rho_1 \sigma_- \sigma_+) \\ & - \frac{\gamma^{(G)}N^{(G)} + \gamma^{(R)}(N^{(R)} + 1)}{2}(\sigma_+ \sigma_- \rho_1 - 2\sigma_- \rho_1 \sigma_+ + \rho_1 \sigma_+ \sigma_-), \end{aligned} \quad (2.29)$$

where

$$\begin{aligned} H_{TR} = & \bar{\omega}(t)s_z + (N-1)[(2\Delta\omega_{eff}(t)\langle s_x \rangle - \gamma_{eff}(t)\langle s_y \rangle)s_x + \\ & + (2\Delta\omega_{eff}(t)\langle s_y \rangle + \gamma_{eff}(t)\langle s_x \rangle)s_y], \end{aligned} \quad (2.30)$$

and the $s_i = \frac{\hbar}{2}\sigma_i$ are the pseudo-spin operators.

The first term in equation (2.29) is responsible for the unitary evolution of the representative single-particle system, the subscript indicates transient regime, while the second and third terms are related to the non-unitary evolution. We will consider only the unitary evolution part of this equation because the superradiance and superabsorption occur in a time much shorter than the relaxation times, which are of the order of $1/\gamma^{(G)}$ and $1/\gamma^{(R)}$. We also consider $\gamma^{(G)}$ of the order of $\gamma^{(R)}$.

In addition, a few definitions are needed to understand equations (2.29) and (2.30).

$$\bar{\omega}(t) = \omega_0 + \Delta\omega_{21}^{(R)} + \Delta\omega_{12}^{(R)} - \Delta\omega_{21}^{(G)}(t) - \Delta\omega_{21}^{(G)}(t), \quad (2.31a)$$

$$\Delta\omega^{(m)} = \Delta\omega_{21}^{(m)} - \Delta\omega_{12}^{(m)}, \quad (2.31b)$$

$$\gamma^{(m)} = \gamma_{21}^{(m)} - \gamma_{12}^{(m)}, \quad (2.31c)$$

$$\gamma_{eff}(t) = \gamma^{(R)} - \gamma^{(G)}(t), \quad (2.31d)$$

$$\Delta\omega_{eff}(t) = \Delta\omega^{(R)} + \Delta\omega^{(G)}(t). \quad (2.31e)$$

The terms $\Delta\omega$ are usually much smaller than the natural frequency ω_0 , which allows us to neglect such terms. This approximation is further justified noticing that the shifts in the frequency cause by the reservoir and the anti-reservoir have opposite signs in equation (2.31a).

Besides that, further developing the $\gamma_{eff}(t)$ expression

$$\begin{aligned}\gamma_{eff}(t) &= \gamma^{(R)} - \gamma^{(G)} \left(C_1^2 + C_1 C_2 \sin(\zeta t) \right) \\ &= \left[\gamma^{(R)} - \gamma^{(G)} C_1^2 \right] - C_1 C_2 \gamma^{(G)} \sin(\zeta t).\end{aligned}\tag{2.32}$$

Conveniently choosing $C_1 = \left(\frac{g^{(R)}(\omega_0) |\lambda^{(R)}(\omega_0)|^2}{g^{(G)}(\omega_0) |\lambda^{(G)}(\omega_0)|^2} \right)^{1/2}$ and $C_2 = -C_1$:

$$\gamma_{eff}(t) = \gamma^{(R)} \sin(\zeta t).\tag{2.33}$$

In Ref. (7), an external periodic magnetic field was used to restore the energy of the system aiming to observe periodic superradiant pulses, in this section, we have modeled a multi-modal pumping as a Glauber amplifier to, not only restore the energy of the system but also to allow γ_{eff} to be negative, equation (2.33), so we could characterize the superabsorption phenomena. We anticipate that when $\gamma_{eff} > 0$ the ensemble of atoms will emit a superradiant pulse. On the other hand, when $\gamma_{eff} < 0$, the observed phenomena will be superabsorption.

2.1.1 The superradiance and superabsorption Hamiltonian and its solution

To solve the non-linear and time-dependent, Hamiltonian shown in equation (2.30), we are going to use the Lewis and Riesenfeld method (32), which allows us to get an exact solution for the given problem. As stated in Ref. (33), the Lewis and Riesenfeld theorem states that a particular solution can be found for the Schrödinger equation

$$i \frac{\partial |\psi(t)\rangle}{\partial t} = H(t) |\psi(t)\rangle,\tag{2.34}$$

and is given by the eigenstate $|n, t\rangle$ of a time-dependent invariant, $I(t)$, defined by the equation

$$\frac{dI(t)}{dt} = \frac{\partial I(t)}{\partial t} + \frac{1}{i} [I(t), H(t)] = 0,\tag{2.35}$$

apart from a time-dependent phase factor, $\exp[i\varphi_n(t)]$, where

$$\varphi_n(t) = \frac{1}{\hbar} \int_0^t d\tau \langle n, \tau | [i\partial_t - H(t)] | n, \tau \rangle.\tag{2.36}$$

The eigenvalues of $I(t)$, which derives from the eigenvalue equation $I(t) |n, t\rangle = \lambda_n |n, t\rangle$, are time-independent.

Although a general method for the construction of Lewis and Riesenfeld TD invariants was proposed in Ref. (33), we are going to limit ourselves to the invariant proposed (7,8), as our Hamiltonian is similar to the one used in those works.

First, we rewrite the Hamiltonian in a simpler form

$$H_{TR} = s_z \bar{\omega}(t) + f(t) s_y + g(t) s_x. \quad (2.37)$$

with

$$f(t) = (N - 1)(2\Delta\omega_{eff}(t) \langle s_x \rangle - \gamma_{eff}(t) \langle s_y \rangle), \quad (2.38a)$$

$$g(t) = (N - 1)(2\Delta\omega_{eff}(t) \langle s_y \rangle + \gamma_{eff}(t) \langle s_x \rangle), \quad (2.38b)$$

and the s_i operators are the pseudo-spin operators,

$$\begin{aligned} s_x &= \frac{|e\rangle \langle g| + |g\rangle \langle e|}{2} \\ s_y &= \frac{|e\rangle \langle g| - |g\rangle \langle e|}{2i} \\ s_z &= \frac{|e\rangle \langle e| - |g\rangle \langle g|}{2}, \end{aligned} \quad (2.39)$$

which obey the following algebra,

$$[s_z, s_x] = i s_y, \quad (2.40a)$$

$$[s_x, s_y] = i s_z, \quad (2.40b)$$

$$[s_z, s_y] = -i s_x. \quad (2.40c)$$

We want the time evolution of the expected values of the pseudo-spin operators to construct our invariant. To do that, we use the Heisenberg equation,

$$\dot{\langle s_x \rangle} = i \langle [H_{TR}, s_x] \rangle, \quad (2.41)$$

$$\begin{aligned} [H_{TR}, s_x] &= [s_z \bar{\omega}(t) + f(t) s_y + g(t) s_x, s_x] = \bar{\omega}(t) [s_z, s_x] + g(t) [s_y, s_x] \\ &= i \bar{\omega}(t) s_y - i g(t) s_z. \end{aligned} \quad (2.42)$$

Substituting this commutator and the $g(t)$ expression into $\dot{\langle s_x \rangle}$, we find

$$\dot{\langle s_x \rangle} = -\bar{\omega}(t) \langle s_y \rangle + 2(N - 1)\Delta\omega_{eff}(t) \langle s_y \rangle \langle s_z \rangle + (N - 1)\gamma_{eff}(t) \langle s_x \rangle \langle s_z \rangle. \quad (2.43)$$

Repeating the process, we also obtain $\langle \dot{s}_y \rangle$ and $\langle \dot{s}_z \rangle$

$$\langle \dot{s}_y \rangle = \bar{\omega}(t) \langle s_x \rangle - 2(N-1)\Delta\omega_{eff}(t) \langle s_x \rangle \langle s_z \rangle + (N-1)\gamma_{eff}(t) \langle s_y \rangle \langle s_z \rangle, \quad (2.44)$$

$$\langle \dot{s}_z \rangle = -(N-1)\gamma_{eff}(t) (\langle s_y \rangle^2 + \langle s_z \rangle^2). \quad (2.45)$$

We are going to use the following invariant,

$$I(t) = \langle s_x \rangle s_x + \langle s_y \rangle s_y + \langle s_z \rangle s_z. \quad (2.46)$$

It is easy to verify that $I(t)$ satisfies the invariant equation (2.35), which means that its expected value is a constant of motion.

$$\langle I(t) \rangle = \langle s_x \rangle^2 + \langle s_y \rangle^2 + \langle s_z \rangle^2 = R^2. \quad (2.47)$$

The constant R defines the radius of the Bloch sphere. Defining $R = 1/2$, and introducing the angular variables θ and ϕ (8),

$$\langle s_x \rangle = \frac{1}{2} \sin \theta \cos \phi, \quad (2.48a)$$

$$\langle s_y \rangle = \frac{1}{2} \sin \theta \sin \phi, \quad (2.48b)$$

$$\langle s_z \rangle = \frac{1}{2} \cos \theta. \quad (2.48c)$$

Now applying a time derivative,

$$\langle \dot{s}_x \rangle = \frac{1}{2} (\cos \theta \cos \phi \dot{\theta} - \sin \theta \sin \phi \dot{\phi}), \quad (2.49a)$$

$$\langle \dot{s}_y \rangle = \frac{1}{2} (\cos \theta \sin \phi \dot{\theta} + \sin \theta \cos \phi \dot{\phi}), \quad (2.49b)$$

$$\langle \dot{s}_z \rangle = \frac{1}{2} (-\sin \theta \dot{\theta}). \quad (2.49c)$$

Notice that we have already obtained the above expectation values in terms of the Hamiltonian, equations (2.43), (2.44) and (2.45). Comparing equations (2.45) and (2.49c), and considering a large N such that $N-1 \approx N$,

$$\langle \dot{s}_z \rangle = \frac{1}{2} (-\sin \theta \dot{\theta}) = -N\gamma_{eff}(t) \frac{1}{4} \sin^2 \theta. \quad (2.50)$$

Also, from equations (2.49b) and (2.44)

$$\begin{aligned}\langle \dot{s}_y \rangle &= \frac{1}{2}(\cos \theta \sin \phi \dot{\theta} + \sin \theta \cos \phi \dot{\phi}) \\ &= \bar{\omega}(t) \sin \theta \cos \phi - N \Delta \omega_{eff}(t) \sin \theta \cos \theta \cos \phi + \frac{N}{2} \gamma_{eff}(t) \sin \theta \cos \theta \sin \phi,\end{aligned}\quad (2.51)$$

the system of equations reduces to

$$\dot{\theta} = \frac{N}{2} \gamma_{eff}(t) \sin \theta, \quad (2.52a)$$

$$\dot{\phi} = \bar{\omega}(t) - N \Delta \omega_{eff}(t) \cos \theta. \quad (2.52b)$$

We can immediately solve for $\theta(t)$, given that the expression of $\gamma_{eff}(t)$ is known.

$$\int_{\theta_0}^{\theta} \frac{d\theta'}{\sin \theta'} = \frac{N \gamma^{(R)}}{2} \int_0^t \sin(\zeta t') dt', \quad (2.53)$$

$$\ln \left[\frac{\tan \left(\frac{\theta}{2} \right)}{\tan \left(\frac{\theta_0}{2} \right)} \right] = \frac{N \gamma^{(R)}}{2\zeta} (1 - \cos(\zeta t)). \quad (2.54)$$

It was shown in ref (9) that, for a characteristic emission time $\tau_c = 2/\gamma^{(R)}N$, the delay time is $t_0 = \tau_c \ln N$. Also, an appropriated choice for the initial position on the Bloch sphere is $\theta(0) \sim 2/N$ because, due to fluctuations, the position $\theta(0) = 0$ is not stable. Taking these remarks into consideration, we obtain

$$\cos \theta = -\tanh \left[\frac{N \gamma^{(R)}}{2\zeta} (1 - \cos(\zeta t)) - \ln N \right], \quad (2.55)$$

which is also the population inversion when defined as

$$W = P_e - P_g = \cos \theta. \quad (2.56)$$

The state vector is, apart from the Lewis and Riesenfeld phase,

$$|\theta, \phi\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}. \quad (2.57)$$

From this point on, we are going to disregard the effects of the $\Delta \omega^{(m)}$ coefficients, because as already discussed, they will not contribute significantly in the regime where $\omega_0 \gg \Delta \omega^{(m)}$. As always, the m subscript refers to both the reservoir and the Glauber amplifier.

To obtain the phases, $\phi(t)$ and $\varphi_n(t)$, we use the developments done in Ref. (8). The phase $\varphi_n(t)$ can be represented as a sum of the geometrical phase $\varphi^g(t)$ with the dynamical phase $\varphi^d(t)$, discussed also in Ref. (34). The results are shown below.

$$\phi(t) = \phi(0) + \omega_0 t. \quad (2.58)$$

$$\varphi_n(t) = -\frac{1}{2}\omega_0 t. \quad (2.59)$$

2.1.2 Energy and intensity

Once we have the solution for $\theta(t)$, we obtain the expectation value of the energy of the atom.

$$\epsilon(t) = \langle \psi(t) | \bar{\omega}(t) s_z | \psi(t) \rangle = -\frac{1}{2}\bar{\omega}(t) \tanh \left(\frac{N\gamma^{(R)}}{2\zeta} (1 - \cos(\zeta t)) - \ln N \right), \quad (2.60)$$

which will be simplified to, as discussed in the previous section,

$$\epsilon(t) = -\frac{1}{2}\omega_0 \tanh \left(\frac{N\gamma^{(R)}}{2\zeta} (1 - \cos(\zeta t)) - \ln N \right). \quad (2.61)$$

Besides that, following the definition

$$\mathcal{I}(t) = -N \frac{d\epsilon(t)}{dt}, \quad (2.62)$$

we calculate the intensity

$$\mathcal{I}(t) = \left(\frac{N}{2} \right)^2 \omega_0 \gamma^{(R)} \sin(\zeta t) \operatorname{sech}^2 \left[\frac{N\gamma^{(R)}}{2\zeta} (1 - \cos(\zeta t)) - \ln N \right], \quad (2.63)$$

which is the expected behavior for the superradiance, with $\mathcal{I} \propto N^2$. Also, it differs from the results from Ref. (8), mainly because of the periodic functions, which will, as anticipated, superradiance and superabsorption alternating in cycles.

The present development also allows for the control of the delay time, since the peak of the first superradiant pulse is around $t = \zeta\pi/2$. Besides that, there are also some interesting properties involving the frequency ζ and the population inversion W which we will explore in the following section.

2.1.3 Population Inversion, intensity, and the modulation frequency

The choice $\zeta_1 = \frac{N\gamma^{(R)}}{2\ln N}$ gives us the desired result, periodic cycles of superradiant emissions followed by superabsorption, as shown in Figures (1) and (2).

Figure 1 – Population inversion for $\zeta_1 = \frac{N\gamma^{(R)}}{2\ln N}$. As anticipated, the system alternates between superradiance and superabsorption as the coefficient γ_{eff} changes its sign.

Source: By the author.

Figure 2 – The intensity for $\zeta_1 = \frac{N\gamma^{(R)}}{2\ln N}$.

Source: By the author.

The reason why the population only fully reverts to the ground state using the frequency ζ_1 , among the choices we have made, is because for ζ_2 and ζ_3 , Figures (3) and (4), the ensemble of atoms does not have the necessary time to completely develop the dipole-moments correlations to emit the superradiant pulses. This delay time, as already mentioned, is of the order of $2\ln N/N\gamma^{(R)}$.

At last, we end this section by emphasizing that the deciding factor for whether the observed phenomena will be superradiance or superabsorption is the sign of the γ_{eff} on the effective Hamiltonian obtained, as shown in equation (2.30). This sign comes from the competition between the emission term, γ_{21} , and the absorption term, γ_{12} , present in

Figure 3 – Population inversion for $\zeta_2 = 2 \frac{N\gamma^{(R)}}{2\ln N}$, (a) and $\zeta_3 = 10 \frac{N\gamma^{(R)}}{2\ln N}$, (b). We note that the increase in ζ diminishes the amplitude of the oscillations.

Source: By the author.

Figure 4 – The intensities for $\zeta_2 = 2 \frac{N\gamma^{(R)}}{2\ln N}$, (a) and $\zeta_3 = 10 \frac{N\gamma^{(R)}}{2\ln N}$, (b). The increase in ζ makes the system oscillate faster, but also hinders the amplitude of the curve.

Source: By the author.

the master equation.

2.2 Superradiance and superabsorption in an atomic sample coupled to a cavity field

Our next analysis focuses on observing the interplay between a sample of two-level atoms and a single mode of the electromagnetic field in a high finesse cavity. The superradiant pulse will be predominantly absorbed by the mode, which will subsequently emit the radiation back to the sample. The Hamiltonian for this system is very similar to equation (2.1), there are a few additional terms due to the presence of the cavity mode.

$$H = H_0 + H_I, \quad (2.64a)$$

$$H_0 = \omega_0 a^\dagger a + \omega_0 S_z + \sum_k \omega_k b_k^\dagger b_k, \quad (2.64b)$$

$$H_I = g(aS_+ + a^\dagger S_-) + \sum_k \lambda_k (S_- b_k^\dagger + S_+ b_k), \quad (2.64c)$$

where a , $a^\dagger$, b and $b^\dagger$ are bosonic operators. The first term in equation (2.64c) refers to the Tavis-Cummings Hamiltonian, where

$$S_\pm = \sum_{n=1}^N \sigma_\pm^{(n)} \quad (2.65)$$

are the pseudo-spin operators and S_z was already characterized by equation (2.3a).

In this work, we are going to consider homogeneous broadening in the ensemble, which means that all the atoms have the same energy gap between the ground and excited states, characterized by the frequency ω_0 .

Besides that, the consideration that the atoms are in a moderately dense sample, such that the wavelength of the emitted radiation is much greater than the size of the sample, creates the mechanism that allows all the atoms to interact with the same reservoir and cavity mode. This is what allows correlation among the atoms' dipole moments to be developed.

In the interaction picture,

$$\mathcal{H}_I = g(aS_+ + a^\dagger S_-) + V_I, \quad (2.66)$$

where V_I is the interaction of the atoms with the reservoir,

$$V_I = \sum_k \lambda_k (S_- b_k^\dagger e^{i(\omega_k - \omega_0)t} + S_+ b_k e^{-i(\omega_k - \omega_0)t}). \quad (2.67)$$

In the sequence, we write the Liouville equation in the interaction picture.

$$i\dot{\rho}_I(t) = [\mathcal{H}_I, \rho(t)]. \quad (2.68)$$

Using the usual approximations (see section 2.1), it is possible to trace out the reservoir variables to obtain the master equation,

$$\dot{\rho}_{SF} = -i \left[g(aS_+ + a^\dagger S_-), \rho_{SF} \right] - \int_0^t dt' \text{Tr}_R [V_I(t), [V_I(t'), \rho_R(0) \rho_S(t')]], \quad (2.69)$$

with the subscript SF referring to the sample-field system. The second term was already calculated in this work, plugging in the results from Section (2.1) we obtain, in the Schrödinger picture,

$$\dot{\rho}_{SF} = -i[\omega_0 a^\dagger a + \omega_0 S_z + g(aS_+ + a^\dagger S_-), \rho_{SF}] + \mathcal{L}(\rho_{SF}), \quad (2.70)$$

where

$$\begin{aligned} \mathcal{L}(\rho_{SF}) = & -\frac{\gamma_{12}}{2}(S_- S_+ \rho_{SF} - 2S_+ \rho_{SF} S_- + \rho_{SF} S_- S_+) \\ & -\frac{\gamma_{21}}{2}(S_+ S_- \rho_{SF} - 2S_- \rho_{SF} S_+ + \rho_{SF} S_+ S_-), \end{aligned} \quad (2.71)$$

and $\gamma = \gamma_{21} - \gamma_{12}$, equations (2.21).

To use the mean-field approximation, consider a system of N particles, and assume

$$H_N = \sum_{i=1}^N h_i + \frac{1}{N} \sum_{i \neq j}^N V_{ij}, \quad (2.72)$$

where h_i represents the free Hamiltonian of the particles, and V_{ij} the interactions. By considering that the density operator in $t = 0$ can be decomposed in a direct product, we can use the following approximation

$$\lim_{N \rightarrow \infty} \text{tr}_{n+1} \dots \text{tr}_{N-1} \text{tr}_N \left\{ e^{-iH_N t} \rho_N(0) e^{iH_N t} \right\} = \rho(t) \otimes \dots \otimes \rho(t) \quad (\text{n factors}), \quad (2.73)$$

where $\rho(t)$ represents the solution for the Hartree equation

$$\frac{d\rho(t)}{dt} = -i[h, \rho(t)] - i \text{tr}_2 [V_{12} + V_{21}, \rho_2(t)]. \quad (2.74)$$

The mean-field approximation comes from considering $\rho_2(t) \approx \rho_1(t) \otimes \rho_1(t)$, as we did in section (2.1). To apply this approximation to the interaction between the ensemble of atoms and the cavity mode, we take advantage of the following mapping, as in Ref. (35),

$$a \rightarrow \frac{1}{\sqrt{N}} \sum_i a_i, \quad (2.75a)$$

$$a^\dagger a \rightarrow \sum_i a_i^\dagger a_i, \quad (2.75b)$$

$$\tilde{g} = g\sqrt{N}. \quad (2.75c)$$

Performing the above changes, the first term in equation (2.70) can be put into the form of equation (2.72)

$$h_i = \omega_0 \left(a_i^\dagger a_i + \sigma_{z(i)}/2 \right), \quad (2.76a)$$

$$V_{ij} = \tilde{g} \left(a_i \sigma_{+(j)} + a_i^\dagger \sigma_{-(j)} \right), \quad (2.76b)$$

which allows us to use the results of equation (2.74), when performing the $N - 1$ traces in equation (2.70), to obtain a system composed by a representative atom and a representative mathematical mode, which followed from the mapping (2.75),

$$\begin{aligned} \dot{\rho}_{AF}(t) = & -i[\omega_0(a^\dagger a + \sigma_z/2), \rho_{AF}(t)] - i \text{tr}_2[a_1 \sigma_{+(2)} + a_1^\dagger \sigma_{-(2)} + a_2 \sigma_{+(1)} + a_2^\dagger \sigma_{-(1)}, \\ & \rho_{AF}(t) \otimes \rho_{AF}(t)] + \text{Tr}_2 \dots \text{Tr}_N \{ \mathcal{L}(\rho_{SF}) \}, \end{aligned} \quad (2.77)$$

and therefore,

$$\begin{aligned} \dot{\rho}_{AF}(t) = & -i \left[\omega_0(a^\dagger a + \sigma_z/2) + \tilde{g} \left(a \langle \sigma_+ \rangle + \langle a \rangle \sigma_+ + a^\dagger \langle \sigma_- \rangle + \langle a^\dagger \rangle \sigma_- \right), \rho_{AF}(t) \right] + \\ & + \text{Tr}_2 \dots \text{Tr}_N \{ \mathcal{L}(\rho_{SF}) \}. \end{aligned} \quad (2.78)$$

Noting that the mapping from equations (2.75) does not affect the interaction sample-reservoir, we can use the results from the mean-field approximation done in section (2.1).

Finally, we obtain a master equation in the Gorini-Kossakowski-Sudarshan-Lindblad form, which is shown below, already in the Schrödinger picture.

$$\begin{aligned} \frac{d\rho_{AF}}{dt} = & -i[H_{AF}, \rho_{AF}] - \frac{\gamma_{12}}{2}(\sigma_- \sigma_+ \rho_{AF} - 2\sigma_+ \rho_{AF} \sigma_- + \rho_{AF} \sigma_- \sigma_+) \\ & - \frac{\gamma_{21}}{2}(\sigma_+ \sigma_- \rho_{AF} - 2\sigma_- \rho_{AF} \sigma_+ + \rho_{AF} \sigma_+ \sigma_-), \end{aligned} \quad (2.79)$$

where

$$\begin{aligned} H_{AF} = & \omega_0 \left(a^\dagger a + \sigma_z/2 \right) + g\sqrt{N} \left(a \langle \sigma_+ \rangle + \langle a \rangle \sigma_+ + a^\dagger \langle \sigma_- \rangle + \langle a^\dagger \rangle \sigma_- \right) + \\ & + \frac{N\gamma}{2i}(\langle \sigma_- \rangle \sigma_+ - \langle \sigma_+ \rangle \sigma_-). \end{aligned} \quad (2.80)$$

The first term in equation (2.79) is responsible for the unitary evolution of this system, we are going to neglect the second and third terms in the following analysis because

the effects we are going to obtain happen in times much shorter than the relaxation times of the system. Finally, we could (and will, in the next section) incorporate a dissipation term for the cavity mode in equation (2.79), but it would also be neglected for the following transient regime analysis, as we are considering that the cavity has a high-quality factor.

2.2.1 The non-linear Hamiltonian and evolution of expectation values

The Hamiltonian expressed in equation (2.80) is non-linear and it is interesting to obtain the evolution of the expected value of a few operators, starting with σ_z

$$\langle \dot{\sigma}_z \rangle = i \langle [H_{AF}, \sigma_z] \rangle, \quad (2.81)$$

$$\langle \dot{\sigma}_z \rangle = 2i \left(g\sqrt{N} \left(\langle \sigma_- \rangle \langle a^\dagger \rangle - \langle \sigma_+ \rangle \langle a \rangle \right) - \frac{N\gamma}{i} |\langle \sigma_- \rangle|^2 \right). \quad (2.82)$$

Defining

$$\langle a \rangle = \langle a^\dagger \rangle^* = |\langle a \rangle| e^{i\phi_a}, \quad (2.83a)$$

$$\langle \sigma_- \rangle = \langle \sigma_+ \rangle^* = |\langle \sigma_- \rangle| e^{i\phi_{\sigma_-}}, \quad (2.83b)$$

then

$$\langle \dot{\sigma}_z \rangle = -4g\sqrt{N} |\langle a \rangle| |\langle \sigma_- \rangle| \sin(\phi_{\sigma_-} - \phi_a) - 2N\gamma |\langle \sigma_- \rangle|^2. \quad (2.84)$$

We also need the time evolution of $\langle a \rangle$ and $\langle \sigma_- \rangle$, which we show in the sequence.

$$\langle \dot{a} \rangle = i \langle [H_{AF}, a] \rangle = -i\omega_0 \langle a \rangle - ig\sqrt{N} \langle \sigma_- \rangle, \quad (2.85)$$

$$\langle \dot{\sigma}_- \rangle = i \langle [H_{AF}, \sigma_-] \rangle = -i\omega_0 \langle \sigma_- \rangle + ig\sqrt{N} \langle a \rangle \langle \sigma_z \rangle + \frac{N\gamma}{2} \langle \sigma_- \rangle \langle \sigma_z \rangle. \quad (2.86)$$

From the above expressions, it is possible to obtain the following system of coupled differential equations

$$|\dot{\langle \sigma_- \rangle}| = -g\sqrt{N}|\langle a \rangle| |\langle \sigma_z \rangle| \sin(\phi_a - \phi_{\sigma_-}) + \frac{N\gamma}{2} |\langle \sigma_- \rangle|, \quad (2.87a)$$

$$\dot{\phi}_{\sigma_-} = -\omega_0 + g\sqrt{N} \frac{|\langle a \rangle|}{|\langle \sigma_- \rangle|} |\langle \sigma_z \rangle| \cos(\phi_{\sigma_-} - \phi_a), \quad (2.87b)$$

$$|\dot{\langle a \rangle}| = g\sqrt{N} |\langle \sigma_- \rangle| \sin(\phi_{\sigma_-} - \phi_a), \quad (2.87c)$$

$$\dot{\phi}_a = -\omega_0 - g\sqrt{N} \frac{|\langle \sigma_- \rangle|}{|\langle a \rangle|} \cos(\phi_{\sigma_-} - \phi_a), \quad (2.87d)$$

$$\dot{\langle \sigma_z \rangle} = -4g\sqrt{N} |\langle a \rangle| |\langle \sigma_- \rangle| \sin(\phi_{\sigma_-} - \phi_a) - 2N\gamma |\langle \sigma_- \rangle|^2. \quad (2.87e)$$

Since the atom energy is $\epsilon_A(t) = \omega_0 \langle \sigma_z/2 \rangle$, we obtain the intensity using equation (2.62).

$$\mathcal{I}_A = -N \frac{d\epsilon_A}{dt} = N\omega_0 \left\{ 2g\sqrt{N} |\langle a \rangle| |\langle \sigma_- \rangle| \sin(\phi_{\sigma_-} - \phi_a) + N\gamma |\langle \sigma_- \rangle|^2 \right\} \quad (2.88)$$

The intensity for the field, whose energy is $\epsilon_F = \omega_0 \langle a^\dagger a \rangle$, is shown below.

$$\mathcal{I}_F = -N\omega_0 \left\{ 2g\sqrt{N} |\langle a \rangle| |\langle \sigma_- \rangle| \sin(\phi_{\sigma_-} - \phi_a) \right\}, \quad (2.89)$$

which is, comparing to equation (2.88), the atom intensity apart from the coherent dissipation term $N^2\omega_0\gamma |\langle \sigma_- \rangle|^2$.

2.2.2 The interplay of excitation atom-cavity mode

To obtain a numerical solution, we used the open toolbox QuTiP (36). It is shown in Figures (5) and (6) how the atom and the cavity mode exchange energy as time evolves. For this example, we used $N = 10^3$, $\omega_0 = 10^2$ and $\gamma = 10^{-3}$ and $g = 1$, the cavity mode was prepared in the vacuum state.

In this section, we also set the initial state for the representative atom described by the angles $\theta_0 = \pi/2$ and $\phi_0 = 0$. The reason for this choice is that we do not want any initial delay time, t_0 , for the atom to start the emission process. As mentioned in Ref. (9), for this value of θ_0 the dipole moments of the atoms are at their maximum. It also corroborates with the statement on Ref. (7) that for this angle the superradiant pulse has its peak.

It is a similar process to what we have obtained in section (2.1.3), in terms of the atom's variables' cyclical behavior. The main difference is that the cavity mode absorbs most of the superradiant pulses. Besides that, it also plays the role of restoring the energy of the ensemble of atoms, but not fully, since some of the energy is lost to the reservoir on each cycle.

Figure 5 – Cavity mode, represented by the dashed-dotted line, and atom excited state, indicated by the solid line, populations. There is an interplay of excitation between the atom and the cavity mode. We also note a significant decay on the energy of the system as there is dissipation to the reservoir. In this example we used $\omega_0 = 10^2$, $N = 10^3$, $\gamma = 10^{-3}$ and $g = 1$.
Source: By the author.

Figure 6 – Intensity for the atomic sample, indicated by the solid blue line, and the EM field, represented by the dashed-dotted orange line. The parameters used are the same as in Figure (5).
Source: By the author.

To better understand the superabsorption part of the cycle, we are going to refer to the results from section (2.1), more specifically equation (2.27). Neglecting $\Delta\omega$ and the

effect of the Glauber amplifier, we obtain

$$\begin{aligned} \frac{d\rho_N}{dt} = & -i[H_S, \rho_N] + \frac{\gamma_{12}}{2} (2S_+\rho_N S_- - S_- S_+ \rho_N - \rho_N S_- S_+) \\ & + \frac{\gamma_{21}}{2} (2S_-\rho_N S_+ - S_+ S_- \rho_N - \rho_N S_+ S_-), \end{aligned} \quad (2.90)$$

where γ_{12} and γ_{21} are the coefficients of the absorption and emission terms, respectively. After performing the mean-field approximation, one obtain the following Hamiltonian

$$H_{TR} = \frac{\omega_0}{2} \sigma_z + \frac{iN}{2} (\gamma_{21} - \gamma_{12}) (\sigma_- \langle \sigma_+ \rangle - \sigma_+ \langle \sigma_- \rangle). \quad (2.91)$$

In Ref. (7), the proposed system has the emission term higher than the absorption, such that $\gamma_{21} - \gamma_{12} = \gamma$, and the resulting Hamiltonian describes the superradiance. However, imposing that $\gamma_{21} - \gamma_{12} = -\gamma$ and set the initial state close to the ground state, we observe superabsorption instead, this is easily verified by substituting $\gamma_{eff}(t) \rightarrow -\gamma$ in equation (2.52a) and solving the resulting system of equations (2.52). As anticipated at the end of section (2.1.1), the deciding factor between superradiance and superabsorption is the sign of γ , which becomes evident when we express $\langle \dot{\sigma}_z \rangle$ as

$$\frac{d\langle \sigma_z \rangle}{dt} = 2N(\mp\gamma) |\langle \sigma_- \rangle|^2, \quad (2.92)$$

where the minus sign describes superradiance, and the plus sign superabsorption. Hence, the superabsorption occur due to the positive value of the derivative $\langle \dot{\sigma}_z \rangle$ which is proportional to $N\gamma |\langle \sigma_- \rangle|^2$.

We rewrite equation (2.87e) to

$$\frac{d\langle \sigma_z \rangle}{dt} = 4|\Lambda| |\langle \sigma_- \rangle|^2 \sin \phi_\Lambda, \quad (2.93)$$

where

$$|\Lambda| = \frac{N}{2} \left[\left(\frac{2g|\langle a \rangle|}{\sqrt{N}|\langle \sigma_- \rangle|} \cos(\phi_a - \phi_{\sigma_-}) \right)^2 + \left(\frac{2g|\langle a \rangle|}{\sqrt{N}|\langle \sigma_- \rangle|} \sin(\phi_a - \phi_{\sigma_-}) - \gamma \right)^2 \right]^{1/2}, \quad (2.94)$$

$$\phi_\Lambda = \arctan \left[\frac{2g|\langle a \rangle| \sin(\phi_a - \phi_{\sigma_-}) - \gamma \sqrt{N} |\langle \sigma_- \rangle|}{2g|\langle a \rangle| \cos(\phi_a - \phi_{\sigma_-})} \right], \quad (2.95)$$

we notice that, the behavior of the population, $\langle \dot{\sigma}_z \rangle$, in the above equation is very similar to the superradiance case, equation (2.92), apart from the fact that $|\Lambda|$ has a slightly more complicated form, incorporating variables of the EM field. Furthermore, the sine

Figure 7 – Comparison of the frequency for the Rabi oscillations between the Jaynes-Cummings model, $N = 1$, characterized by the dotted blue line, and a ensemble of atoms with $N = 10^3$, solid orange line. It is also shown, in the thick-solid green line the usual superradiance, and the enhanced superradiance proposed in (1), dashed red line.
Source: By the author.

Figure 8 – The figure shows how the excited state population of the representative atom evolves in time for different values of the parameter N . The dotted line represents the choice $N = 10^3$, which was also used on the previous figures. The dash-dotted line represents $N = 10^4$ and the solid line $N = 10^5$. As anticipated, the increase of N result in greater frequency of the Rabi oscillations, but also greater dissipation rate to the reservoir.
Source: By the author.

function explains the periodic behavior of the interplay, its sign indicates whether it is the superabsorption cycle, $0 < \phi_A < \pi$, or the superradiance cycle, $\pi < \phi_A < 2\pi$.

We compare the many-body Rabi oscillations of a system with $N = 10^3$ to the Jaynes-Cummings model, $N = 1$, in Figure (7). For this case, we used $N = 10^3$, $\omega_0 = 10^2$

and $\gamma = 10^{-3}$. We note that the number of atoms in the ensemble greatly increases the oscillation frequency. We also show the superradiance without the presence of the direct radiation-matter interaction ($g = 0$) for a sample-reservoir interaction, green line, and the superradiance via reservoir engineering proposed in Ref. (1), dashed red line.

In the presence of the atom-field interaction, the energy is dissipated to the reservoir at a slower rate. This behavior was already anticipated, since the cavity mode will absorb the majority of the superradiant pulses, as long as in equation (2.88) the first term is greater than the second. The second term, however, scales more with the increase of N in comparison to the first term, this means that increasing the number of atoms even further will result in higher frequencies, but also faster dissipation of the energy. This is characterized in Figure (8), where we varied the value of N .

Figure 9 – The expected value for the excited state population evolution in time. In (a) we varied the parameter γ , the solid line was used to characterize the choice $\gamma = 10^{-2}$, the dash-dotted line for $\gamma = 10^{-3}$ and the dotted line for $\gamma = 10^{-4}$. In (b), we varied the parameter ω_0 , the solid line represents the value $\omega_0 = 10^2$, the dash-dotted line $\omega_0 = 10^4$ and the dotted line $\omega_0 = 10^5$.

Source: By the author.

At last, we also vary the values of the parameters ω_0 and γ to observe its effect on the interplay. The results for these simulations are shown in Figure (9). The increase on the parameter γ , Figure (9a) result in a greater dissipation rate and a minor enhance on the oscillation frequency. The variation of ω_0 , Figure (9b), results also in a much higher dissipation rate to the reservoir, which is not obvious from equations (2.87).

2.3 A mean-field approach for a two-level superradiant laser

In this section, we are going to describe a two-level laser using the mean-field approximation. We start the analysis by rewriting equation (2.79) to incorporate a dissipation term that comes from the interaction between the cavity mode and a reservoir. Those terms responsible for the non-unitary evolution of the system were previously neglected because we were interested only in the transient regime. From this point on, our

focus will be on the steady-state of the laser, so we need to take all the terms present in the master equation into consideration.

$$\begin{aligned} \frac{d\rho_{AF}}{dt} = & -i[H_{AF}, \rho_{AF}] - \frac{\gamma_{12}}{2}(\sigma_- \sigma_+ \rho_{AF} - 2\sigma_+ \rho_{AF} \sigma_- + \rho_{AF} \sigma_- \sigma_+) \\ & - \frac{\gamma_{21}}{2}(\sigma_+ \sigma_- \rho_{AF} - 2\sigma_- \rho_{AF} \sigma_+ + \rho_{AF} \sigma_+ \sigma_-) \\ & - \kappa(a^\dagger a \rho_{AF} - 2a \rho_{AF} a^\dagger + \rho_{AF} a^\dagger a), \end{aligned} \quad (2.96)$$

where κ is the damping constant, analogous to γ_{21} , and H_{AF} described in equation (2.80).

Besides that, we are going to assume that there is an external source of energy being pumped into the system to bring the atoms from the fundamental to the excited state, in other words, we let $\gamma_{12} > \gamma_{21}$.

2.3.1 System of coupled differential equations

To obtain a system of equations similar to equations (2.87), we use the following

$$\langle \mathcal{O} \rangle = \text{Tr}(\mathcal{O} \rho_{AF}), \quad (2.97a)$$

$$\frac{d\langle \mathcal{O} \rangle}{dt} = \text{Tr}(\mathcal{O} \frac{d\rho_{AF}}{dt}), \quad (2.97b)$$

where $\mathcal{O}$ is any time-independent operator in the Schrödinger picture and $d\rho_{AF}/dt$ is given by equation (2.96).

Additionally, considering the following definitions

$$\tilde{\gamma} = \frac{1}{2}(\gamma_{12} + \gamma_{21}), \quad (2.98a)$$

$$d = \frac{\gamma_{12} - \gamma_{21}}{\gamma_{12} + \gamma_{21}}, \quad (2.98b)$$

$$A(t) = \langle a \rangle e^{i\omega_0 t}, \quad (2.98c)$$

$$S(t) = \langle \sigma_- \rangle e^{i\omega_0 t}, \quad (2.98d)$$

$$D(t) = \langle \sigma_z \rangle, \quad (2.98e)$$

and using equations (2.97), we obtain

$$\frac{dA(t)}{dt} = -\kappa A(t) - ig\sqrt{N}S(t), \quad (2.99a)$$

$$\frac{dS(t)}{dt} = -[1 + dND(t)]\tilde{\gamma}S(t) + ig\sqrt{N}A(t)D(t), \quad (2.99b)$$

$$\frac{dD(t)}{dt} = 2ig\sqrt{N}(A^*(t)S(t) - A(t)S^*(t)) - 2\tilde{\gamma}(D(t) - d) + 4dN\tilde{\gamma}|S(t)|^2. \quad (2.99c)$$

Choosing the parameters of our system such that $\tilde{\gamma} > g\sqrt{N}$ we can adiabatically eliminate the variables $S(t)$ and $D(t)$. With the approximation $\dot{S}(t) = \dot{D}(t) = 0$, we obtain the equation

$$\frac{dA(t)}{dt} = -\kappa A(t) \left(1 + \frac{\tilde{\gamma}[D(t) - d][1 + dND(t)]}{2\kappa|A(t)|^2} \right), \quad (2.100)$$

where $D(t)$ can be expressed as a function of $|A(t)|$ by solving the following quadratic equation

$$D^2(t) + D(t) \left(\frac{1}{dN} + \frac{\tilde{\gamma}^2}{2dN^2g^2|A(t)|^2} \right) - \frac{\tilde{\gamma}^2}{2N^2g^2|A(t)|^2} = 0. \quad (2.101)$$

The expected behavior of the laser is to increase the cavity mode's population until a saturation point is achieved. From that point, the laser will be in a stationary state, $dA(t)/dt = 0$. For that to happen, there must be a stable equilibrium point for $A(t)$. If we consider an ensemble of atoms with N large enough such that $dN \gg 1$, we obtain

$$\frac{dA(t)}{dt} = -\kappa A(t) \left(1 - \frac{g^2}{d\tilde{\gamma}\kappa} \right). \quad (2.102)$$

Hence, there will be no stability point for large values of N besides $A(t) = 0$. In fact, increasing the value of N to large values, as for example $N = 10^3$, the interplay described in the section (2.2) takes over, as shown in Figures (10a) and (10b), where we used also $\gamma = -0.01$ and $\omega_0 = 10^2$. It is very similar behavior to what we obtained in Figures (5) and (6), apart from the fact that the energy of the system is increasing due to the external pumping. Since we are looking for a stationary state, we are going to avoid this regime of parameters, i.e., large values of N .

2.3.2 Second-order phase transition analogy

A phase transition analogy fits the laser description because it has several similarities with other out of equilibrium stationary states, such as the transition of ferromagnetic materials. As discussed in section (1.1), the superradiance comes from the alignment of the radiating dipole moments of the atoms present on the ensemble due to the electromagnetic field, which resembles the way spins align in a ferromagnetic material. The electric field operator $E = (\langle a \rangle + \langle a^\dagger \rangle)/2 = |A|$ can be seen as the average magnetization, $\langle M \rangle$, ferromagnetic order parameter.

Figure 10 – (a) The expected value for the excited state population for the representative atom, characterized by the solid line, and the cavity mode population as the dashed-dotted line. (b) The intensities for the atom, solid line, and the field, dash-dotted line. The behavior of the system resembles the results obtained in section (2.2.2), apart from the increasing energy, which comes from γ being negative.

Source: By the author.

Besides that, choosing initially $N = 1$, and considering that the $\tilde{\gamma} > g > \kappa$ we can adiabatically remove the variables $S(t)$ and $D(t)$. After setting $\dot{S}(t) = \dot{D}(t) = 0$, we obtain the following equation

$$\frac{dA(t)}{dt} = -\kappa A(t) \left(1 - \frac{C}{1 + |A(t)|^2/n_0} \right), \quad (2.103)$$

where $C = dg^2/\tilde{\gamma}\kappa$ and $n_0 = \gamma^2/2g^2$. As shown in Figure (11), the phase transition point is $C = 1$, which is referred to in Ref. (13) as the threshold point. Below threshold, $C < 1$, the only stability point is $|A| = 0$. Above threshold, $C > 1$, there will be an unstable stationary solution, $|A| = 0$, and a stable stationary solution is $|A|^2 = n_0(C - 1)$.

$$|A| = \begin{cases} 0 & \text{if } C < 1 \text{ (below threshold),} \\ \sqrt{n_0(C - 1)} & \text{if } C > 1 \text{ (above threshold).} \end{cases}$$

Comparing to the phase transition on a ferromagnetic material, the parameter C , apart from a minus sign, represents the ratio T/T_c , where T is the temperature and T_c is the Curie temperature. The threshold point $C = 1$ is also known as Hopf bifurcation, (35).

When we take the many-body effects into consideration, and solving equations (2.99) using the python library SciPy, we observe that increasing the value of N changes the threshold point from $C = 1$ to, as shown in Figure (12), values below 1.

Figure 11 – Phase transition for the laser threshold for $N = 1$, the solid line indicates the stable stability point for $|A|$, while the dashed line characterizes the unstable stability point.

Source: By the author.

Figure 12 – Laser threshold for (a) $N = 5$ and (b) $N = 20$, to obtain the curves we vary the parameter d while keeping $\tilde{\gamma} = 10$, $\kappa = 0.01$ and $g = 1$. The solid line indicates the stable stability point for $|A|$, while the dashed line characterizes the unstable stability point.

Source: By the author.

Besides that, the choice of which parameter we vary to obtain the phase transition graphs affects the form of the curve, for $N > 1$. In Figure (12) we built the curve changing the values of the parameter d , $\tilde{\gamma} = 10$, $g = 1$ and $\kappa = 0.01$. In Figure (13a) and (13b) we construct the same graphs but varying $\tilde{\gamma}$ and κ , respectively, for these two last graphs we set $d = 0.8$.

At last, we noticed a peculiar behavior for a few choices of parameters, where the phase transition curve has the two threshold points, as shown in Figure (14), varying

Figure 13 – Phase transition graph for $N = 5$ varying the parameters (a) $\tilde{\gamma}$ while keeping $\kappa = 0.01$, $d = 0.8$ and $g = 1$, and (b) varying κ while keeping $\tilde{\gamma} = 10$ and the other parameters the same as in (a). The solid line indicates the stable stability point for $|A|$, while the dashed line characterizes the unstable stability point.

Source: By the author.

Figure 14 – Phase transition graph, the parameters were chosen such that $N = 5$, $g = 1$, $\tilde{\gamma} = 10$ and $\kappa = 0.1$. A second phase transition appears at $C = 1$. The solid line indicates the stable stability point for $|A|$, while the dashed line characterizes the unstable stability point.

Source: By the author.

the parameter d . The second transition point, where the stable stability point becomes $|A(t)| = 0$ again happens at $C = 1$. For the choice of parameters we made, $N = 5$, $g = 1$, $\tilde{\gamma} = 10$ and $\kappa = 0.1$, this point coincides with $d = 1$, which is also the maximum allowed value for d , as it can be checked from equation (2.98b).

2.3.3 Population, intensity, and laser linewidth

In the sequence, we are going to set the parameters to $g = 1$, $\tilde{\gamma} = 10$, $\kappa = 0.01$, $N = 5$ and $d = 0.8$. This is a choice that allows the system to be above the threshold point. We show the time evolution for a few variables of the system in Figure (15a), more specifically we focus on the populations of the field and the atomic sample. To do that we solved the coupled differential equations (2.99) numerically.

Figure 15 – Time evolution for, (a) the variables $|A(t)|$, blue solid line, and $D(t)$, orange dashed line, and (b) the derivative $\frac{d|A(t)|}{dt}$.
Source: By the author.

The steady-state population of the field is verified to be equal to $|A(t)|^2$. The intensity of the field can also be calculated from equations (2.62), apart from a minus sign, and (2.97).

$$I_f = \omega_0 N \left\{ 2g\sqrt{N}|A(t)||S(t)|\sin(\phi_S(t) - \phi_A(t)) - 2\kappa \langle a^\dagger a \rangle \right\}, \quad (2.104)$$

where we expressed $A(t) = |A(t)|e^{i\phi_A(t)}$ and $S(t) = |S(t)|e^{i\phi_S(t)}$. The main difference between equations (2.89) and (2.104) is the second term in equation (2.104). In the interplay regime, we are interested in the transient regime, which takes place in times much shorter than the relaxation times characterized by κ^{-1} , γ_{12}^{-1} and γ_{21}^{-1} , which led us to neglect these terms. In the laser regime, however, the times of interest are much higher than that of where the interplay takes place, since we are interested in the steady state regime, making the term proportional to κ relevant. This term also indicates the saturation of the population of the field, in the stationary state the intensity goes to zero when $\langle a^\dagger a \rangle = \frac{g\sqrt{N}}{\kappa}|A(t)||S(t)|\sin(\phi_S(t) - \phi_A(t))$.

From the above equation, it is expected that the intensity will achieve higher peaks and smaller full width at half maximum when N is increased. This behavior is corroborated by the results shown in Figure (16), where we plotted the intensity curve for

values of $N = 1$, 5, and 10. We observe that the intensity scales with a factor of roughly $N^2/2$.

Figure 16 – Intensity for $N = 1$, represented by the blue dashed line, 5, orange dotted line, and 10, characterized by the green solid line.

Source: By the author.

The phase of the electromagnetic field undergoes a phase diffusion due to random spontaneous emission events. The phase will gradually become uniformly distributed over 2π , causing decay in the mean of the amplitude of the field. While a more detailed approach would require the calculation of the second-order correlation function $\langle E^-(0)E^+(\tau) \rangle$, in Refs. (13) and (14) an alternative approach allows us to calculate the linewidth using the off-diagonal terms.

$$\frac{d\langle a(t) \rangle}{dt} = \sum_n \sqrt{n} \rho_{n,n-1}, \quad (2.105)$$

If the parameters are chosen so that the laser is far above the threshold point,

$$\langle E^+(t) \rangle = \frac{d\langle a(t) \rangle}{dt} = \langle a(t) \rangle e^{-\Gamma t}, \quad (2.106)$$

and in the Markov approximation, see equation (2.12), the second order correlation will decay with a similar rate as $\langle E^+(t) \rangle$.

We notice, from equation (2.98c), that the laser field will oscillate with the frequency ω_0 and decay with the Γ factor. This is expressed in Figure (15b), after the peak of the curve $d|A(t)|/dt$, it starts to exponentially decay as the field approaches the stationary state.

Finally, we can express the correlation function as

$$\langle E^-(t)E^+(t+\tau) \rangle = |A|^2 e^{-i\omega_0\tau} e^{-\Gamma\tau}. \quad (2.107)$$

Figure 17 – Laser linewidth for $N = 1$, characterized by the blue solid line, $N = 5$, orange dotted line, and $N = 20$, green dashed line.
Source: By the author.

The Fourier transform of the above equation yields the laser linewidth.

$$S(\omega) = \frac{1}{\pi} \text{Re} \int_0^\infty \langle E^-(0) E^+(\tau) \rangle = \frac{|A|^2}{\pi} \frac{\Gamma}{(\omega - \omega_0)^2 + \Gamma^2}. \quad (2.108)$$

The results for the linewidths for $N = 1$, 5 and 20 are shown in Figure (17), where we also normalized the curves by $S(\omega_0)$. Although the linewidth increases with N , it is not a significant increase and it has a diminishing return effect, we can appreciate that the increase on the linewidth from $N = 5$ to 20 is much smaller than from $N = 1$ to 5. Moreover, we verified that increasing κ leads to a negligible increase in the linewidth for the values $N = 5$ and 20 in comparison to $N = 1$. Going back to the off-diagonal terms, it means that the collective effect present in our laser makes the reduced density matrix for the field to lose coherence faster than the usual laser.

3 NON-HERMITIAN HAMILTONIANS

To continue our discussion of section (1.2), we are going to investigate a few cases. We start with a time-independent linear Hamiltonian and, in the sequence, we advance to a quadratic Hamiltonian including time-dependency.

3.1 Time independent linear non-Hermitian Hamiltonian

From our earlier discussion, section (1.2), we showed that quadratic amplification can lead to smaller variances of the quadrature X , for example, at the expense of proportionally increasing the variance of the quadrature P , guaranteeing that the uncertainty principle is not violated, given that an appropriated rotation is made to the axis of phase space. It can also be verified that the displacement operator, $\exp(\alpha a^\dagger - \alpha^* a)$, cannot generate such effect.

Hence, to emphasize the plasticity of a non-Hermitian Hamiltonian, we are going to see how an unbalanced linear pumping can influence the squeezing of the field quadratures. We start showing the time-independent Hamiltonian

$$H = \omega(a^\dagger a + 1/2) + \gamma a + \delta a^\dagger, \quad (3.1)$$

which is non-Hermitian if we impose $\omega \in \mathbb{C}$ or $\gamma \neq \delta^*$. This Hamiltonian becomes $\mathcal{PT}$ -symmetric when demanding that $(\mathcal{PT})H(\mathcal{PT})^{-1} = H$, such that $\gamma^* = -\gamma$ and $\delta^* = -\delta$.

To obtain the Hermitian counterpart of H we are going to use equation (1.5). The choice for the Dyson map is shown below

$$\eta = \exp\left[\frac{\nu a^3 - \nu^*(a^\dagger)^3}{3}\right] \exp[\epsilon a^\dagger a + \zeta a + \zeta^* a^\dagger]. \quad (3.2)$$

Considering a small ν , such that $\nu^2 \approx 0$, we have the following transformations

$$\eta a \eta^{-1} = \left[a + \nu^*(a^\dagger)^2\right] e^{-\epsilon} - \frac{\zeta^*}{\epsilon} (1 - e^{-\epsilon}), \quad (3.3a)$$

$$\eta a^\dagger \eta^{-1} = \left(a^\dagger + \nu a^2\right) e^\epsilon - \frac{\zeta}{\epsilon} (1 - e^\epsilon). \quad (3.3b)$$

More about the perturbative construction of a Dyson map can be found in Ref. (21).

Using the transformations given by equations (3.3), we obtain the Hamiltonian counterpart of H

$$h = \omega(a^\dagger a + 1/2) + Aa^3 + B(a^\dagger)^3 + Ua^2 + V(a^\dagger)^2 + Ra + Ta^\dagger + F, \quad (3.4)$$

where

$$A = \omega\nu, \quad (3.5a)$$

$$B = \omega\nu^*, \quad (3.5b)$$

$$U = \nu \left[\frac{\omega\zeta^*}{\epsilon}(1 - e^\epsilon) + \delta e^\epsilon \right], \quad (3.5c)$$

$$V = \nu^* \left[\frac{\omega\zeta}{\epsilon}(1 - e^{-\epsilon}) + \gamma e^{-\epsilon} \right], \quad (3.5d)$$

$$R = \frac{\omega\zeta}{\epsilon}(1 - e^{-\epsilon}) + \gamma e^{-\epsilon}, \quad (3.5e)$$

$$T = \frac{\omega\zeta^*}{\epsilon}(1 - e^\epsilon) + \delta e^\epsilon, \quad (3.5f)$$

$$F = -\frac{\omega|\zeta|^2}{\epsilon^2}(2 - e^\epsilon - e^{-\epsilon}) - \gamma\frac{\zeta^*}{\epsilon}(1 - e^{-\epsilon}) - \delta\frac{\zeta}{\epsilon}(1 - e^\epsilon). \quad (3.5g)$$

To ensure that h is Hermitian a few conditions must be satisfied, which will impose constraints on the parameter of the Dyson map and the coefficients of the non-Hermitian Hamiltonian. The first restriction is $\omega \in \mathbb{R}$, this requirement guarantees that $A = B^*$. In addition, we need to impose that $F = F^*$, which represents the following constraint for the parameter ζ of the Dyson Map

$$\frac{\omega\zeta}{\epsilon} = \frac{\delta^* e^\epsilon - \gamma e^{-\epsilon}}{e^\epsilon - e^{-\epsilon}}. \quad (3.6)$$

Finally, considering $\gamma = |\gamma|e^{i\varphi_\gamma}$, $\delta = |\delta|e^{i\varphi_\delta}$ and imposing that $R = T^*$, which also guarantees that $U = V^*$, we obtain a final limitation for the phases φ_γ and φ_δ)

$$\varphi_\gamma + \varphi_\delta = n\pi, \quad n = 0, \pm 1, \pm 2, \dots \quad (3.7)$$

The above relation is automatically satisfied for the choice of a $\mathcal{PT}$ -symmetric Hamiltonian, which requires the phases φ_γ and φ_δ to be both half-integers, the sum of these phases is, therefore, an integer. However, it is evident that the $\mathcal{PT}$ symmetry is not strictly necessary for this Hamiltonian to have real eigenvalues. The important condition to be satisfied, for H to become Pseudo-Hermitian, is equation (3.7). Not only it is a far more generic requirement, but it also unveils a connection between the phases of the linear pumping that was initially clouded by the imposition of having pure imaginary coefficients γ and δ .

Using the relations obtained so far, we can rewrite R and F as

$$R = \frac{|\delta|(e^\epsilon - 1) - |\gamma|(e^{-\epsilon} - 1)}{e^\epsilon - e^{-\epsilon}} e^{i\varphi}. \quad (3.8)$$

$$F = \frac{(|\delta|e^\epsilon - |\gamma|e^{-\epsilon})(|\gamma| - |\delta|)(1 - \cosh \epsilon)}{2 \sinh^2 \epsilon}. \quad (3.9)$$

Note that the Dyson map parameter ϵ remained unconstrained during the hermitization process, which means that it is a free parameter. We have obtained a family of Hermitian Hamiltonians h in a similar way as done in (20). To show how this degree of freedom can be used to change the form of h , we are going to choose different values for ϵ and analyse the effects of such choices in the variables X and P .

3.1.1 Cubic amplification

For the next two sections, we are going to take the unbalance between the linear amplification parameters to the extreme, considering $|\delta| = 0$. Besides that, it can be verified that, by choosing $\epsilon \gg 0$, we set $|R| = |U| = F = 0$. The Hermitian counterpart, h , becomes

$$h = \omega(a^\dagger a + 1/2) + \omega\nu a^3 + \omega\nu^* (a^\dagger)^3. \quad (3.10)$$

This Hamiltonian permit us to study the impact of cubic amplification terms on the squeezing of the field quadratures.

We numerically solve the time-dependent Schrödinger equation for the above to obtain the dynamic of the variables of interest. The results of the simulation can be observed in Figure (18). For short times, as it can be seen in Figure (18a), the population of the cavity mode does not significantly increase. On the other hand, an appropriate axis rotation results in squeezed quadratures, as shown in Figure (18b).

In addition, as mentioned previously, another form of appreciating the squeezing of the quadratures is to plot the Wigner function, which we show in Figure (19). As easily noticed, the squeezing generated by the cubic amplification does not have the usual symmetric ellipsoid shape, which comes from the parametric amplification.

3.1.2 Cubic and quadratic amplifications

Next, we choose $\epsilon \ll 0$. The implication of this choice, in contrast to the previous section, is that now the other terms of the Hamiltonian, most notably the quadratic terms, are non-vanishing.

$$h = \omega(a^\dagger a + 1/2) + \omega\nu a^3 + \omega\nu^* a^{\dagger 3} + Ua^2 + U^* a^{\dagger 2} + Ra + Ta^\dagger + F, \quad (3.11)$$

Figure 18 – Using the values $\omega = 10\omega_0$, $\nu = 0.05\omega_0$, $\epsilon = 10\omega_0$, and $\omega_0 = 1$ we obtain the time evolution for the expectation value, dotted line, and variance, solid line, of the number of photons (a). In (b) we show the variance of the rotated field quadratures, $(\Delta \tilde{X})^2$, as the solid line, $(\Delta \tilde{P})^2$, as the solid line, for the cubic Hamiltonian given by equation (3.10).

Source: By the author.

Figure 19 – Wigner function for the cubic Hamiltonian given by equation (3.10).

Source: By the author.

where

$$|U| = |\nu||\gamma|. \quad (3.12)$$

The simulation for this Hamiltonian is shown in Figure (20), the effects of including the quadratic terms, which depend on the parameter γ , the linear pumping coefficient of the original non-Hermitian Hamiltonian, are easy to perceive by comparing compare

Figure 20 – Using the values $\omega = 10\omega_0$, $\gamma = 10\omega_0$, $\nu = 0.05\omega_0$, $\epsilon = -10\omega_0$, and $\omega_0 = 1$ we obtain the time evolution for the expectation value, dotted line, and variance, solid line, of the number of photons (a). In (b) we show the variance of the rotated field quadratures, $(\Delta \tilde{X})^2$, as the solid line, $(\Delta \tilde{P})^2$, as the solid line, for the Hamiltonian given by equation (3.11), which includes both cubic and quadratic amplification.

Source: By the author.

Figure 21 – Wigner function for the Hamiltonian given by equation (3.11), which is under the effect of cubic and quadratic amplification.

Source: By the author.

Figures (18) and (20).

As time evolves, there is a change in the variable whose variance is being reduced. It starts with $\tilde{p}$ being squeezed and it changes to $\tilde{x}$ around $t = 0.16$. The reason for this effect is that the cubic and the quadratic terms oscillate at different frequencies, if we analyze this system in the interaction picture. We observe that the cubic and the

quadratic terms oscillate at frequencies 3ω and 2ω , respectively. This miss-match causes an interference effect in the squeezing, when the interference is positive, we obtain higher squeezing degrees tendencies, $\dot{r}$, than what we have obtained in the previous section, where we had only the cubic terms present in the Hamiltonian.

Once again, we show the Wigner function to better visualize the squeezing, as shown in Figure (21). Once again the Wigner function presents an asymmetry in the major axis of what would be an ellipse for a quadratic Hamiltonian.

Therefore, we showed that with the appropriated choice of a Dyson map, in this case, a cubic operator constructed perturbatively, it is possible that the coefficients of the linear amplification affect the squeezing of the field quadratures. This is a rather unexpected behavior because, for Hermitian Hamiltonians, linear pumping does not generate squeezed states.

3.2 Time-dependent quadratic non-Hermitian Hamiltonian, a generic approach

Now, we are going to present an approach to solving a non-Hermitian harmonic oscillator under the effect of parametric unbalanced amplification. This problem was first solved in Ref. (20) for the time-independent case and in Ref. (27) for the time-dependent case. We propose a more generic approach than in the mentioned works by assuming a non-Hermitian Dyson map.

3.2.1 Non-Hermitian Hamiltonians and their Hermitian counterpart

We begin by defining the non-Hermitian Hamiltonian

$$H = \omega(t)(a^\dagger a + 1/2) + \alpha(t)a^2 + \beta(t)a^{\dagger 2}. \quad (3.13)$$

This Hamiltonian becomes $\mathcal{PT}$ -symmetric demanding that $\omega^*(t) = \omega(-t)$, $\alpha^*(t) = \alpha(-t)$ and $\beta^*(t) = \beta(-t)$ and non-Hermitian if $\omega(t) \neq \omega^*(t)$ or $\alpha(t) \neq \beta^*(t)$.

To obtain the Hermitian counterpart of H , we use the following Dyson map

$$\begin{aligned} \eta &= \exp \left[\epsilon(t)(a^\dagger a + 1/2) + \nu(t)(a^\dagger)^2 + \mu(t)a^2 \right] \\ &= \exp [\lambda_+(t)K_+] \exp \{ \ln[\lambda_0(t)]K_0 \} \exp [\lambda_-(t)K_-]. \end{aligned} \quad (3.14)$$

where the last equality comes from the Gauss decomposition to the $SU(1,1)$ algebra, with $K_0 = (a^\dagger a + 1/2)/2$, $K_+ = (a^\dagger)^2/2$ and $K_- = a^2/2$ satisfying the commutation relations $[K_-, K_+] = 2K_0$ and $[K_0, K_\pm] = \pm K_\pm$. The parameters for the Gauss decomposition are

$$\lambda_+(t) = \frac{2\nu(t)}{\Lambda_-(t)}, \quad (3.15)$$

$$\lambda_-(t) = \frac{2\mu(t)}{\Lambda_-(t)} = -\Phi(t)e^{i\varphi_\mu(t)}, \quad (3.16)$$

$$\lambda_0(t) = \frac{1}{\Lambda_-^2(t)} \frac{\Xi^2(t)}{\sinh^2 \Xi(t)}, \quad (3.17)$$

$$\Xi(t) = \sqrt{\epsilon^2(t) - 4\mu(t)\nu(t)}, \quad (3.18)$$

where

$$\Lambda_\pm(t) = \Xi(t) \coth \Xi(t) \pm \epsilon(t). \quad (3.19)$$

To ensure that $\lambda_0(t)$ is a real function, we impose that the parameters $\epsilon(t)$ and $\Xi(t)$ are real, which leads to the following constraints

$$\varphi_\mu + \varphi_\nu = \ell\pi, \quad \ell = 0, \pm 1, \pm 2, \dots \quad (3.20)$$

and $\epsilon^2(t) > 4\mu(t)\nu(t)$.

Defining the real parameter (we omit the time dependency on the expressions whenever possible to make the notation more compact)

$$\chi = \lambda_+\lambda_- - \lambda_0 = -\frac{\Lambda_+}{\Lambda_-}, \quad (3.21)$$

we obtain the Hermitian counterpart of H

$$\begin{aligned} h &= \eta H \eta^{-1} + i\dot{\eta}\eta^{-1}, \\ &= W(a^\dagger a + 1/2) + Ua^2 + V(a^\dagger)^2, \end{aligned} \quad (3.22)$$

where, using equations (3.15 - 3.18),

$$W = -\frac{1}{\lambda_0} \left[\omega(\chi + \lambda_+\lambda_-) + 2\alpha\lambda_+ + 2\beta\chi\lambda_- + i \left(\dot{\lambda}_-\lambda_+ - \frac{\dot{\lambda}_0}{2} \right) \right], \quad (3.23)$$

$$U = \frac{1}{\lambda_0} \left[\omega\lambda_- + \alpha + \beta\lambda_-^2 + i\frac{\dot{\lambda}_-}{2} \right], \quad (3.24)$$

$$V = \frac{1}{\lambda_0} \left[\omega\chi\lambda_+ + \alpha\lambda_+^2 + \beta\chi^2 + \frac{i}{2} \left(\dot{\lambda}_-\lambda_+^2 - \dot{\lambda}_0\lambda_+ + \dot{\lambda}_+\lambda_0 \right) \right]. \quad (3.25)$$

3.2.2 The TD parameters of the map

For the above Hamiltonian to be Hermitian, we need to ensure that $W = W^*$ and $U^* = V$, which lead to the following system of coupled nonlinear differential equations.

$$\frac{\dot{\Lambda}_+}{\Lambda_+} - \frac{\dot{\Lambda}_-}{\Lambda_-} + 2 \operatorname{Im}(\omega) = \frac{4|\Gamma|}{|\Gamma|^2 + 1} \operatorname{Im} \left[e^{i(\varphi_\alpha + \varphi_\Gamma)} \Delta \right], \quad (3.26a)$$

$$e^{2i(\varphi_\alpha + \varphi_\Gamma)} \left(\frac{|\Gamma|}{|\Gamma|^2 + 1} - \frac{|\Gamma|}{2} \right) \Delta = \left(\frac{|\Gamma|}{|\Gamma|^2 + 1} - \frac{1}{2|\Gamma|} \right) \Delta^*, \quad (3.26b)$$

where we imposed that

$$\frac{2\nu}{\Lambda_-} = \Gamma = -\frac{2\mu^*}{\Lambda_+}, \quad (3.27)$$

and Δ is given by the below equation

$$\Delta = |\alpha| \frac{\Lambda_-}{\Lambda_+} - e^{-i(\varphi_\alpha + \varphi_\beta)} |\beta| \frac{\Lambda_+}{\Lambda_-}. \quad (3.28)$$

We also verified that the condition

$$\frac{|\alpha|}{|\beta|} = \left(\frac{\Lambda_+}{\Lambda_-} \right)^2 e^{-i(\varphi_\alpha + \varphi_\beta)}, \quad (3.29)$$

automatically satisfies equation (3.26b). However, because $\Lambda_\pm$ are real functions, the phases of the parametric amplification coefficients would need to satisfy the condition

$$\varphi_\alpha + \varphi_\beta = 2n\pi, \quad n = 0, \pm 1, \pm 2, \dots \quad (3.30)$$

The above equation is a restriction on the starting Hamiltonian, H . In the sequence, we are going to show that this imposition leads to a solution for the system of equations (3.26) which presents a complete free parameter, $|\Gamma(t)|$, whose time evolution can be arbitrarily chosen.

Substituting equation (3.29) into equation (3.26a), we obtain

$$\frac{\Lambda_+}{\Lambda_-} = \Upsilon_0 \exp \left(-2 \int_0^t \operatorname{Im} [\omega(\tau)] d\tau \right) \equiv \Upsilon, \quad (3.31)$$

where $\Upsilon_0 = \phi \sqrt{\frac{|\alpha(0)|}{|\beta(0)|}}$, with $\phi = \pm 1$. Using equations (3.31) and (3.27), we are able to determine φ_Γ

$$\varphi_\Gamma = \begin{cases} \varphi_\nu - \ell\pi = -\varphi_\mu, & \text{if } \epsilon > 0, \Upsilon < -1 \\ \varphi_\nu = -\varphi_\mu + \ell\pi, & \text{if } \epsilon > 0, \Upsilon > 1 \\ \varphi_\nu = -\varphi_\mu + \ell\pi, & \text{if } \epsilon < 0, 0 < \Upsilon < 1 \\ \varphi_\nu = -\varphi_\mu, & \text{if } \epsilon < 0, -1 < \Upsilon < 0. \end{cases}$$

These conditions allow us to rewrite Υ as

$$\Upsilon = -(-1)^\ell \frac{|\alpha(0)|}{|\beta(0)|} \exp\left(-2 \int_0^t \text{Im}[\omega(\tau)] d\tau\right), \quad (3.32)$$

and to express the Dyson map parameters, ϵ , ν and μ , as

$$\epsilon = \frac{1}{2\sqrt{1 - (-1)^\ell z^2}} \ln \left[\frac{\Upsilon + 1 + [\Upsilon - 1]\sqrt{1 - (-1)^\ell z^2}}{\Upsilon + 1 - [\Upsilon - 1]\sqrt{1 - (-1)^\ell z^2}} \right], \quad (3.33)$$

$$\nu = \frac{z e^{-i\varphi_\mu}}{2\sqrt{|\Upsilon|}} \epsilon, \quad (3.34)$$

$$\mu^* = (-1)^\ell \sqrt{|\Upsilon|} \frac{z e^{-i\varphi_\mu}}{2} \epsilon, \quad (3.35)$$

where

$$z = \frac{2\sqrt{|\Upsilon|}}{|\Upsilon - 1|} |\Gamma|. \quad (3.36)$$

As anticipated, $|\Gamma(t)|$, and consequently $z(t)$, are free parameters, in the sense that their time evolution can be chosen. In Ref. (27), the freedom on the choice of the Dyson map was constrained to the initial values for the parameters $\epsilon(t)$ and $\mu(t)$. In this sense, this work presents interesting generalizations, allowing for much more control over the behavior of the coefficients W and U of the Hermitian counterpart of H .

It is important to note that a Hermitian Dyson map could not be used to obtain similar results using the present developments. One way to show that is by noticing that the definition on equation (3.27) could not have been made for a Hermitian Dyson map, i.e., $\nu = \mu^*$, since it would lead to $\Xi = 0$, as it can be easily verified. Hence, even the definition of $\Gamma(t)$ requires a non-Hermitian Dyson map.

Nevertheless, equation (3.33) lead to two necessary conditions for the existence of a non-Hermitian Dyson map

$$1 - (-1)^\ell z^2 > 0, \quad (3.37)$$

$$\frac{|\Upsilon + 1|}{|\Upsilon - 1|} > \sqrt{1 - (-1)^\ell z^2}. \quad (3.38)$$

The above equations lead to the following limits for $z(t)$

$$\frac{2\sqrt{|\Upsilon|}}{||\Upsilon| + 1|} < z < 1, \quad \ell = 0, \quad (3.39a)$$

$$0 \leq z < \frac{2\sqrt{|\Upsilon|}}{||\Upsilon| - 1|}, \quad \ell = 1. \quad (3.39b)$$

Finally, the Hamiltonian given by equation (3.13) can be rewritten as

$$H = \omega(t) \left(a^\dagger a + \frac{1}{2} \right) + \sqrt{|\alpha(t)||\beta(t)|} \left[|\Upsilon(t)| e^{i\varphi_\alpha} a^2 + \frac{1}{|\Upsilon(t)|} e^{-i\varphi_\alpha} (a^\dagger)^2 \right]. \quad (3.40)$$

The function of time $|\Upsilon(t)|$ indicates how far from the hermiticity is H . We can, for example, refer to Ref. (27) where the unbalance on the parametric pumping is taken to the extreme, for $\Upsilon = 0$ one would find the all-annihilation Hamiltonian, while $\Upsilon \rightarrow \pm\infty$ would lead to the all-creation Hamiltonian. Evidently, the hermiticity of the starting Hamiltonian relates to $|\Upsilon(t)| = 1$.

For completeness, we show also the case where condition (3.29) is not satisfied. It is easy to verify that, from equation (3.26b), that $|\Gamma| = 1$ and

$$\varphi_\alpha + \varphi_\Gamma = \arctan \left[-\frac{|\beta|\Lambda_+^2 \sin(\varphi_\alpha + \varphi_\beta)}{|\alpha|\Lambda_-^2 - \beta\Lambda_+^2 \cos(\varphi_\alpha + \varphi_\beta)} \right] + \ell\pi, \quad \ell = 0, \pm 1, \pm 2, \dots \quad (3.41)$$

Substituting these results on equation (3.26a), we obtain

$$\frac{\Lambda_+}{\Lambda_-} = \Upsilon = \Upsilon_0 \exp \left(-2 \int_0^t \text{Im} [\omega(\tau)] d\tau \right), \quad (3.42)$$

where

$$\Upsilon_0 = \frac{\Xi(0) \coth[\Xi(0)] + \epsilon(0)}{\Xi(0) \coth[\Xi(0)] - \epsilon(0)}. \quad (3.43)$$

In conclusion, equation (3.29) is a necessary condition to obtain a free parameter. Should this requirement not be satisfied, we can still solve the system of equations, although the obtained solution takes a more restricted form.

3.2.3 Solution for the Hamiltonian and its Hermitian counterpart

So far we have solved the equations for the Dyson map that would guarantee the hermiticity of the Hamiltonian (3.22). Now, we turn to the solution for the Schrödinger

equations for $h(t)$ and $H(t)$. The eigenstates for the Hermitian counterpart of the Hamiltonian are,

$$|\Pi_n(t)\rangle = \mathcal{T}(t) |n\rangle, \quad (3.44)$$

where $\{|n\rangle\}$ defines the Fock-state base and the unitary operator $\mathcal{T}(t)$ is given by

$$\mathcal{T}(t) = \mathcal{Y}(t)S[\xi(t)]D[\theta(t)]\mathcal{R}[\varepsilon(t)], \quad (3.45)$$

where $S[\xi(t)] = \exp\left\{\left[\xi(t)(a^\dagger)^2 - \xi^*(t)a^2\right]/2\right\}$ is a squeezing operation, with $\xi(t) = r(t)e^{i\phi(t)}$, $r(t)$ being the degree of squeezing and $\phi(t)$ the direction of the squeezing in phase space. This operation transforms the quadratic Hamiltonian into a linear one, as long as the following constraints are satisfied

$$\dot{r}(t) = -2|U(t)| \sin[\varphi_U(t) + \phi(t)], \quad (3.46a)$$

$$\dot{\phi}(t) = -2W(t) - 4|U(t)| \coth[2r(t)] \cos[\varphi_U(t) + \phi(t)]. \quad (3.46b)$$

Besides that, $D[\theta(t)] = \exp[\theta(t)a^\dagger - \theta^*(t)a]$ is the displacement operator with

$$i\dot{\theta}(t) = \varepsilon(t)\theta(t), \quad (3.47)$$

$$\varepsilon(t) = W(t) + 2|U(t)| \tanh[r(t)] \cos(\varphi_U(t) + \phi(t)). \quad (3.48)$$

Also, $\mathcal{R}[\varepsilon(t)] = \exp[-i\varpi(t)a^\dagger a]$, with $\varpi(t) = \int_0^t \varepsilon(\tau)d\tau$ and $\mathcal{Y}(t) = \exp[-\varpi(t)/2]$.

The solution $|\psi(t)\rangle$ for $H(t)$ is obtained by

$$|\psi(t)\rangle = \eta^{-1}(t) |\Pi(t)\rangle, \quad (3.49)$$

and the evolution operator $\mathcal{U}(t)$ is given by

$$\mathcal{U}(t) = \mathcal{T}(t)\mathcal{T}^\dagger(0). \quad (3.50)$$

Having the evolution operator and knowing that the initial state $|\Pi(0)\rangle$ can be in a superposition of the states of the basis $\{|n\rangle\}$, we obtain the generic solution for the Schrödinger equation for the pseudo-Hermitian Hamiltonian $H(t)$,

$$|\psi(t)\rangle = \eta^{-1}(t)\mathcal{T}(t)\mathcal{T}^\dagger(0) \sum_n C_n |n\rangle. \quad (3.51)$$

3.2.4 Variances of the field quadratures

The observables $o(t)$ and $O(t)$ in the Hermitian and non-Hermitian systems modeled by $h(t)$ and $H(t)$, respectively, are related to each other as $O(t) = \eta^{-1}(t)o(t)\eta(t)$. From the field quadratures in the Hermitian system, given by $x = (a + a^\dagger)/2$ and $p = (a - a^\dagger)/2i$, we then obtain in the non-Hermitian system the observables $X = \eta^{-1}(t)x\eta(t)$ and $P = \eta^{-1}(t)p\eta(t)$, such that

$$[\Delta X]^2_\Theta = \langle X^2 \rangle_\Theta - \langle X \rangle_\Theta^2 = \langle x^2 \rangle - \langle x \rangle^2 = [\Delta x]^2, \quad (3.52a)$$

$$[\Delta P]^2_\Theta = \langle P^2 \rangle_\Theta - \langle P \rangle_\Theta^2 = \langle p^2 \rangle - \langle p \rangle^2 = [\Delta p]^2. \quad (3.52b)$$

Therefore, the variances of the observables, depend only on the variance of their Hermitian counterpart. By calculating Δp , for example, we are in fact obtaining the variance for the quadrature P .

Now performing a rotation of $\phi(t)/2$ on the axis of phase space,

$$\tilde{x} + i\tilde{p} = [x + ip]e^{-i\phi(t)/2}, \quad (3.53)$$

we obtain, considering the initial state of h is the vacuum state ($r(0) = \theta(0) = 0$),

$$[\Delta \tilde{x}]^2 = \frac{1}{4}e^{2r(t)}, \quad (3.54a)$$

$$[\Delta \tilde{p}]^2 = \frac{1}{4}e^{-2r(t)}. \quad (3.54b)$$

Choosing $\phi(t) = 3\pi/2 - \varphi_U(t)$, we obtain, from equation (3.46a), the following expression for the degree of squeezing

$$r(t) = 2 \int_0^t |U(\tau)| d\tau, \quad (3.55)$$

where the coefficient $|U|$ can be obtained using equations (3.24), (3.26) and (3.30), we show the result below,

$$|U| = \frac{1}{|\chi - 1|} \{ |\alpha|^2 + |\beta|^2 \chi^2 - 2(-1)^\ell |\alpha\beta| \chi + [\text{Im}(\omega)]^2 |\lambda_-|^2 \}^{1/2}. \quad (3.56)$$

As concluded in Ref. (27), when χ approaches the unit for a given instant t , $|U|$ and consequently r diverge, thus resulting in an infinite squeezing degree in a finite time interval.

3.3 A TD pseudo-Hermitian Hamiltonian for a cavity mode with pure imaginary frequency

In this section, we are going to particularize our problem to the case of a harmonic oscillator whose frequency has only an imaginary part.

3.3.1 Obtaining the map parameters for the harmonic oscillator with complex frequency

In the sequel, our interest relies on analyzing the effects of having a complex frequency on the starting Hamiltonian (3.13). To do that, we define $\omega(t) = |\omega(t)|e^{i\varphi_\omega(t)}$ and impose

$$\varphi_\omega = (2n + 1)\frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \pm 3, \dots \quad (3.57)$$

which sets the frequency to pure complex values.

In Ref. (27) a thorough discussion of the squeezing degrees coming from an unbalanced parametric pumping was made. We are going to show that a very similar pattern of squeezing can be obtained in the presence of a complex frequency and without any parametric pumping. To make a clear comparison, we are going to use a similar Dyson map to the one used in Ref. (27). This will require slightly modifying the map defined in equation (3.14) by setting $\nu = \mu^*$, so that the Dyson map becomes Hermitian.

Besides that, as already discussed in the end of section (3.2.2), equation (3.27) no longer holds for Hermitian Dyson maps, so that the system of differential equations coming from imposing $W = W^*$ and $U = V^*$ becomes significantly more complex. This led us to impose the following additional constraints, for mathematical convenience,

$$\varphi_\alpha + \varphi_\beta = \ell\pi, \quad \ell = 0, \pm 1, \pm 2, \pm 3, \dots \quad (3.58a)$$

$$\varphi_\mu - \varphi_\alpha = m\pi, \quad m = 0, \pm 1, \pm 2, \pm 3, \dots \quad (3.58b)$$

where we notice that equation (3.58a) is less restrictive, although sufficient, than equation (3.30) for the current analysis. Making the necessary adjustments, the system of equations (3.26) becomes

$$\dot{\Lambda}_+ \Lambda_- - \Lambda_+ \dot{\Lambda}_- = \Lambda_+ \Lambda_- \frac{d}{dt} \ln \left(\frac{\Lambda_+}{\Lambda_-} \right) = -2\Lambda_+ \Lambda_- \left(1 - \frac{4|\mu|^2}{\Lambda_+ \Lambda_-} \right) \text{Im}(\omega), \quad (3.59)$$

$$\frac{1}{|\mu|} \frac{d|\mu|}{dt} - \frac{1}{\Lambda_+ + \Lambda_-} \frac{d}{dt} (\Lambda_+ + \Lambda_-) = 2 \text{Im}(\omega), \quad (3.60)$$

$$\dot{\varphi}_\mu = (-1)^m \frac{|\alpha| \Lambda_-^2 - (-1)^\ell |\beta| \Lambda_+^2}{\Lambda_+ + \Lambda_-} \frac{1}{|\mu|} - 4(-1)^m \frac{|\alpha| - (-1)^\ell |\beta|}{\Lambda_+ + \Lambda_-} |\mu|. \quad (3.61)$$

From equation (3.60) we obtain

$$\frac{|\mu|}{\Lambda_+ + \Lambda_-} = \Psi_0 e^\Omega, \quad (3.62)$$

where $\Omega = \int_0^t 2 \operatorname{Im}[\omega(\tau)] d\tau$, and Ψ_0 is a constant of integration. We also verified that

$$\frac{\Lambda_+}{\Lambda_-} = \frac{|z| e^{-\Omega} + 4\Psi_0}{|z| e^{-\Omega} - 4\Psi_0}, \quad (3.63)$$

where $|z(t)| = 2|\mu(t)|/\epsilon(t)$. Using this definition for $|z(t)|$, equation (3.18) becomes $\Xi(t) = |\epsilon(t)|\sqrt{1 - |z(t)|^2}$. The requirement of $\Xi(t) \in \mathbb{R}$ constraints the domain of $|z(t)|$ to $0 < |z(t)| < 1$.

Now, equation (3.59) can be rewritten as

$$\frac{d|z|}{dt} = f_1(t)|z|^2 + f_2(t)|z| - f_3(t), \quad (3.64)$$

where

$$f_1(t) = \left(\frac{1}{4\Psi_0 e^\Omega} - 4\Psi_0 e^\Omega \right) \operatorname{Im}(\omega) = \left(\frac{1}{8\Psi_0 e^\Omega} - 2\Psi_0 e^\Omega \right) \dot{\Omega}, \quad (3.65)$$

$$f_2(t) = 2 \operatorname{Im}(\omega) = \dot{\Omega}, \quad (3.66)$$

$$f_3(t) = 4\Psi_0 e^\Omega \operatorname{Im}(\omega) = 2\Psi_0 e^\Omega \dot{\Omega}. \quad (3.67)$$

Equation (3.64) is a Ricatti equation, whose solution is

$$|z_\pm(\Sigma)| = \Sigma \frac{1 \mp \sin \Sigma}{1 \mp \sin \Sigma \pm \Sigma \cos \Sigma}, \quad (3.68)$$

where $\Sigma = 4\Psi_0 e^\Omega$, from the above equation it is clear that we should consider only positive values for Σ . We also assumed that the constant of integration of equation (3.64) obeys the relation $|z_0|\Psi_0 = \pm 1$, to ensure that $|z_\pm(\Sigma)| \in \mathbb{R}$.

The Dyson map parameters can be expressed as

$$\epsilon(\Sigma) = \frac{1}{2\sqrt{1 - |z_\pm(\Sigma)|^2}} \ln \left[\frac{|z_\pm(\Sigma)| + \frac{1}{2}\Sigma\sqrt{1 - |z_\pm(\Sigma)|^2}}{|z_\pm(\Sigma)| - \frac{1}{2}\Sigma\sqrt{1 - |z_\pm(\Sigma)|^2}} \right], \quad (3.69)$$

$$\mu(\Sigma) = \frac{|z_\pm(\Sigma)|\epsilon(\Sigma)e^{i\varphi_\mu}}{2}, \quad (3.70)$$

Figure 22 – Solutions for $|z_{\pm}(\Sigma)|$ and the regions of validity for each solution. It is possible to see that $|z_+|$ becomes negative and $|z_-|$ becomes greater than 1 at certain points, which limits the regime where the Dyson map is physical.
Source: By the author.

The requirement that the argument of the logarithm in the above equations must be positive and real leads to the following domain for $|z(\Sigma)|$, $\Sigma/\sqrt{1+\Sigma^2} < |z_{\pm}(\Sigma)| < 1$, the lower limit comes from imposing

$$\frac{|z_{\pm}(\Sigma)| + \frac{1}{2}\Sigma\sqrt{1-|z_{\pm}(\Sigma)|^2}}{|z_{\pm}(\Sigma)| - \frac{1}{2}\Sigma\sqrt{1-|z_{\pm}(\Sigma)|^2}} > 0. \quad (3.71)$$

Also, remembering that $\Sigma(t) = \Psi_0 \exp[\int_0^t \text{Im}(\omega(\tau))d\tau]$, this translates into a time validity for the Dyson map, as $\Sigma(t)/\sqrt{1+\Sigma^2(t)}$ reaches the value of 1 as time evolves, if $\text{Im}(\omega(t)) > 0$. In Figure (22) we show that, as $\Sigma(t)$ increases, the solution $|z_+(\Sigma)|$ reach negative values and $|z_-(\Sigma)|$ exceed the upper limit of 1.

Next, we are going to obtain another solution, using the same considerations (3.58), whose interval of existence of the Dyson map allows us to obtain higher squeezing degrees than the previous solution. We start by re-expressing the system of equations (3.26) in terms of λ_0 and Φ , as follows

$$\dot{\Phi} = \frac{2\Phi(1-\Phi^2)}{\chi-1}\text{Im}(\omega), \quad (3.72)$$

$$\frac{\dot{\lambda}_0}{\lambda_0} = -2 \left(1 + \frac{2\Phi^2}{\chi-1} \right) \text{Im}(\omega), \quad (3.73)$$

$$\varphi_{\mu} = \frac{2(-1)^m}{(1-\chi)\Phi} \left[|\alpha|(1-\Phi^2) + |\beta|(\Phi^2-\chi^2)(-1)^{\ell} \right], \quad (3.74)$$

and solving λ_0 as a function of Φ

$$\lambda_0 = \frac{\Phi(\Phi^2 - 1)}{c + \Phi}, \quad (3.75)$$

where c is a constant of integration given by

$$c = \frac{\Phi(0)}{\lambda_0(0)}(\Phi^2(0) - \lambda_0(0) - 1). \quad (3.76)$$

Now, from equation (3.72),

$$\int_{\Phi(0)}^{\Phi(t)} \frac{\Phi^2 - 1 - \lambda_0}{\Phi(1 - \Phi^2)} d\Phi = 2 \int_0^t \text{Im}(\omega(\tau)) d\tau, \quad (3.77)$$

and substituting equation (3.75) into the above equation, we obtain

$$\Phi(t) = \frac{c \Phi(0) e^{-\Omega}}{c + \Phi(0)(1 - e^{-\Omega})}. \quad (3.78)$$

Also, $|z(t)|$ follows from the equations (3.16 - 3.18)

$$|z(t)| = \frac{2\Phi(t)}{\Phi^2(t) + 1 - \lambda_0(t)}. \quad (3.79)$$

Finally, to complete our solution for the map we express the parameters ϵ and μ as

$$\epsilon(t) = \frac{1}{2\sqrt{1 - |z(t)|^2}} \ln \left[\frac{(1 + \sqrt{1 - |z(t)|^2})\Phi(t) - |z(t)|}{(1 - \sqrt{1 - |z(t)|^2})\Phi(t) - |z(t)|} \right], \quad (3.80)$$

$$\mu(t) = \frac{|z(t)|\epsilon(t)e^{i\varphi_\mu(t)}}{2}, \quad (3.81)$$

where $\varphi_\mu(t)$ is obtained from equation (3.74).

A necessary condition for this map to be physical is that $|z(t)| > 2\Phi(t)/(1 + \Phi^2(t))$, this requirement follows from imposing that the argument of the logarithm must be positive. From equation (3.79), we note that this condition is always satisfied as long as $\lambda_0(t) \neq 0$. The other requirement for the map to exist is that $|z(t)| < 1$.

3.3.2 The squeezing degree for a cavity mode with a pure imaginary frequency

It is possible to show that, from equations (3.55) and (3.56), the squeezing factor for a Hermitian Hamiltonian, i.e, $\alpha = \beta^*$, is

$$r_h(t) = 2 \int_0^t |\alpha(\tau)| d\tau. \quad (3.82)$$

We want to compare the squeezing coming from the above result to the one coming from a harmonic oscillator with a complex frequency. To do that, we are going to set the values of α and β to zero. The Hamiltonian (3.13) becomes

$$H = i|\omega(t)|(a^\dagger a + 1/2). \quad (3.83)$$

This choice will not significantly change the solutions obtained in the previous section. First of all, when we imposed the relations (3.58a) and (3.58b) for the phases φ_α , φ_β and φ_μ we made so that α and β only affect the phase φ_μ , making ϵ and $|\mu|$ independent of the parametric amplification, as it can be observed in the system of equations (3.59a - 3.59c). By removing the parametric amplification we are, in fact, setting $\dot{\varphi}_\mu = 0$, and, consequently, that the phase of the μ parameter is a constant which can be arbitrarily chosen. The solutions (3.68 - 3.70) and (3.78 - 3.81), however, remain unaltered apart from $\varphi_\mu(t) \rightarrow \varphi_\mu$.

Now, knowing that the squeezing factor follows from Eqs. (3.55) and (3.56), and that the sharp growth of the squeezing degree follows when χ approaches the unit, such that $|U|$ and consequently r diverge, then the solutions coming from both systems in Eqs. (3.59 - 3.61) and (3.72 - 3.73) should not differ significantly, as already anticipated above. Therefore, we need to evaluate $|U|$ to calculate the squeezing degree. Setting α and β to zero on equation (3.56), and using equations (3.16) and (3.21), one obtains

$$|U| = \frac{|\omega||\lambda_-|}{|\chi - 1|} = 2|\omega| \frac{|\mu|}{|\Lambda_+ + \Lambda_-|} = \frac{|\omega|\Sigma}{2}, \quad (3.84)$$

where we used, in the last step, the definition of x and equation (3.62).

Hence, the squeezing factor $r(t)$ is obtained by integrating the above result, according to equation (3.55).

$$r_{cho}(t) = \frac{1}{2}(\Sigma - 4\Psi_0), \quad (3.85)$$

where we inserted a subscript to indicate that this is a harmonic oscillator with complex frequency. The above expression is a fantastic result because it presents an exponential behavior for the growth of the squeezing factor since $\Sigma \propto \exp[\int_0^t |\omega(\tau)| d\tau]$, and, as we are going to show, it diverges in a controllable time.

Before proceeding, we should address a few points. First of all, in the previous section we obtained two distinct solutions for the Dyson map, given by equations (3.69 - 3.70) and (3.80 - 3.81), which we further particularized at the beginning of this section. It is important to emphasize, though, that the two systems of equations (3.59 - 3.61) and (3.72 - 3.74) are equivalent. Thus, we are only distinguishing between the two obtained solutions when we specify which equations are being used to characterize the parameters

ϵ and μ . Therefore, the above results for $|U|$ and $r(t)$ are valid for both cases. Notice that we could also have represented $|U|$, on equation (3.84), in terms of Φ and λ_0 .

As we are going to show in the sequence, the subtlety to achieve high values of squeezing relies on delaying as much as possible the time at which the parameters $\epsilon(t)$ and $\mu(t)$ diverge, which indicates the time where the map stop having physical significance. We observe that $r_{cho}(t)$ has a rather limited maximum value for the solutions (3.69 - 3.70) because, as we showed in Figure (22), as Σ increases $|z_{\pm}(\Sigma)|$ rapidly violates the domain $\Sigma(t)/\sqrt{1 + \Sigma^2(t)} < |z_{\pm}(\Sigma)| < 1$. Therefore, it is not possible to achieve high values of squeezing with this solution.

On the other hand, we are going to show that we can obtain infinite squeezing at finite time for the Dyson map characterized by equations (3.75 - 3.81), depending on the choice of the initial parameters. We first rewrite $r_{cho}(t)$ in terms of $\Phi(t)$,

$$r_{cho}(t) = \int_0^t \frac{\Phi(\tau)|\omega(\tau)|}{|1 - \chi(\tau)|} d\tau = \frac{1}{|c|} \int_0^t d\tau \frac{\Phi(\tau)|\omega(\tau)||c + \Phi(\tau)|}{|\Phi(\tau)^2 - 1|}. \quad (3.86)$$

Setting $\Phi(0) > 1$ and $c > 0$, i.e, $\Phi^2(0) > 1 + \lambda_0(0)$, the analytical solution is

$$r_{cho}(t) = \frac{1}{2} \ln \left[\frac{\Phi(0) - 1}{\Phi(0) + 1} \frac{\Phi(t) + 1}{\Phi(t) - 1} \right]. \quad (3.87)$$

As $\Phi(t)$ approaches the unity at the critical time t_c , the squeezing degree diverges. For a linear complex frequency $\text{Im}(\omega) = \gamma^2 t$, for example, $t_c = \frac{1}{\gamma} \sqrt{\ln \left[\frac{\Phi(0)(1+c)}{\Phi(0)+c} \right]}$. However, before the critical time t_c , $|z(t)|$ becomes greater than 1 in a time t_1 , and the Dyson map on the interval $t_1 \leq t \leq t_c$ is not physical. The curve for $r_{cho}(t)$ using the initial values $\Phi(0) = 1.5$ and $\lambda_0(0) = 0.03$ is shown in Figure (23), and the parameter $\epsilon(t)$ of the Dyson map in Figure (24), which corroborates that until t_1 the parameters of the map are non-divergent.

We verify also that, as $\lambda_0(0) \rightarrow 0$, $t_1 \rightarrow t_c$, as showed in Figure (25), hence, we choose how close to the divergence our solution can get. However, to diminish the value of $\lambda_0(0)$ is to increase the value of the parameters $\epsilon(0)$ and $|\mu(0)|$, as it can be easily verified from equation (3.80), and consequently a higher energy cost on the preparation of the initial state $|\psi(0)\rangle = \eta^{-1}(0) |\Pi(0)\rangle$, if we set $|\Pi(0)\rangle$ as the vacuum state. Another possibility of diminishing the gap between t_c and t_1 is to increase the value of $\Phi(0)$, this comes at the cost of increasing t_c also.

A rich discussion about the price to pay for high squeezing degrees was made in Ref. (27). First, since the quadratures X and P are obtained performing the non-unitary transformation $\eta^{-1}x\eta$ and $\eta^{-1}p\eta$, respectively, both X and P are usually going to be linear combinations of x and p . Therefore, there will be costs involving the measurement of canonical conjugate variables. Secondly, there are also costs involving the preparation of the initial state, $|\psi(0)\rangle$, as discussed above. To obtain infinite squeezing degrees, an

Figure 23 – The squeezing degree as a function of time, r_{cho} represented by the solid line. The free parameter $|z|$ represented by the dashed line, it is shown on an inset that between t_1 and t_c the Dyson map does not exist. Source: By the author.

Figure 24 – The Dyson map parameter $\epsilon(t)$ as a function of time. Until the time t_1 it is shown that the map exists and is physical (not divergent), as t approaches t_1 a divergence starts to happen which represents the time validity of the map. Another important aspect of this curve it is the initial value, which represents the squeezing operation one must perform on the vacuum state. Source: By the author.

Figure 25 – According to the restriction that $|z| < 1$ the interval of time $t_1 \leq t \leq t_c$ is not physical, which means that the Dyson map parameter ϵ diverges. However, it is possible to move t_1 closer to t_c as much as we want, excluding the point $t_1 = t_c$, this could come at the expense of energy in the preparation of the initial state.

Source: By the author.

infinite amount of energy would be required. However, we must emphasize that the most important aspect of this analysis is the exponential behavior of the squeezing factor, which can have a huge impact on applications that require high precision measurements in a short time.

3.3.3 The relation between the harmonic oscillator with complex frequency and the regime of strong parametric amplification

In Ref. (27), an attempt to reach infinite squeezing degrees at finite times was made. However, we showed in this work that the Dyson map only has physical meaning until a time $t_1 < t_c$, with t_c being the critical time to reach the infinite squeezing degree. We also showed that only the complex part of the frequency $\omega(t)$ generates squeezing if we impose equations (3.58).

One can associate complex frequencies with strong amplitudes of parametric pumping, consider the following Hamiltonian

$$H = \omega_0(b^\dagger b + 1/2) + \gamma \cosh(\gamma t) (b^2 + (b^\dagger)^2), \quad (3.88)$$

where b and $b^\dagger$ are bosonic destruction and creation operators, respectively. Considering that $\gamma, \omega_0 \in \mathbb{R}$ this Hamiltonian is Hermitian. Performing a canonical transformation, we obtain

$$H = \xi(t)(a^\dagger a + 1/2), \quad (3.89)$$

where $\xi(t) = \sqrt{\omega_0^2 - 4\gamma^2 \cosh^2(\gamma t)} = \sqrt{\omega_0^2 - 4\gamma^2 - 4\gamma^2 \sinh^2(\gamma t)}$, this frequency becomes complex in a time $t_t = \frac{1}{\gamma} \sinh^{-1} \left[\sqrt{(\omega_0^2 - 4\gamma^2)/4\gamma^2} \right]$, in particular, on $t = 0$ if $4\gamma^2 = \omega_0^2$. In this case, the Hamiltonian would take the form

$$H = i\omega_0 \sinh(\gamma t) (a^\dagger a + 1/2), \quad (3.90)$$

which is $\mathcal{PT}$ -symmetric. Hence, the change in the behavior of the system, which could lead to extraordinary results such as high (infinite) squeezing degrees at finite times, can be associated with this transition point where the frequency becomes complex.

For $\omega_0^2 > 4\gamma^2 \cosh^2(\gamma t)$, the Hamiltonian (3.88) is Hermitian, hence, using equation (3.82) we obtain the following squeezing factor

$$r_h(t) = 2 \int_0^t |\alpha(\tau)| d\tau = 2 \sinh(\gamma t). \quad (3.91)$$

For $\omega_0^2 < 4\gamma^2 \cosh^2(\gamma t)$, the frequency $\xi(t)$ in equation (3.90) becomes complex and the

Figure 26 – Squeezing degrees for the Hermitian Hamiltonian, given by equation (3.91), characterized by the dashed line and the non-Hermitian Hamiltonian, equation (3.87), for the choice of the initial values $\Phi(0) = 2$ and $\lambda_0(0) = 0.01$, represented by the solid line. In $t = t_t$ the transition on which the Hamiltonian becomes non-Hermitian occurs. Finally, the γt_1 time derives from the choices of $\Phi(0)$ and $\lambda_0(0)$ and is, consequently, controllable.

Source: By the author.

Hamiltonian becomes non-Hermitian. The squeezing degree is obtainable using equation (3.87),

$$r_{CHO} = \frac{1}{2} \ln \left[\frac{\Phi(0) - 1}{\Phi(0) + 1} \frac{\Phi(0)[(c-1)e^{-\Omega} + 1] + c}{\Phi(0)[(c+1)e^{-\Omega} - 1] - c} \right], \quad (3.92)$$

where,

$$\Omega(t) = 4[\cosh(\gamma t) - 1]. \quad (3.93)$$

The comparison between the two squeezing factors is shown on Figure (26), where we chose $\Phi(0) = 2$ and $\lambda_0(0) = 0.01$. The divergence time is completely controllable since γt_1 and γt_c follow from the choice of the initial parameters of the map. Until the time t_t , the squeezing behaves as the r_h curve and, after that point, there is a transition and the squeezing is given by the r_{CHO} curve.

At last, we emphasize that, besides the change in squeezing factor behavior, $r_h \rightarrow r_{CHO}$, there is also a change in the angle of rotation of the phase space axis, $\phi(t)$, which is given by equation (3.46b). Hence, before the transition, the axis must be rotated following the angle $\phi_r(t)$. After the transition, this angle must also be changed to a $\phi_{CHO}(t)$.

4 CONCLUSIONS

4.1 Superradiance and superabsorption

The first objective of this work was to propose a model to describe the coherent many-body phenomena called superradiance and superabsorption. To do that, we started with a Hamiltonian which described the interaction of a system of N identical two-level atoms with a reservoir and an anti-reservoir, showed in equation (2.1). The form of the interaction term in this Hamiltonian relies on the requirement that the sample should be moderately dense, so that no interaction between atoms must be considered. Besides that, the size of the sample should be much smaller than the wavelength of the emitted radiation. The latter requirement ensures that all the atoms will interact with the same reservoir (or anti-reservoir), which resulting in the coherent effects of emission or absorption.

To describe the dynamics of the above mentioned system, we obtained the evolution for the reduced density operator for the system of N atoms, equation (2.27). To do that, we considered a few approximations such as the rotating wave approximation, the first Markov approximation, sudden coupling, etc. From that, the mean-field approximation was used to obtain a master equation for a representative atom for the system, equation (2.29). For the transient regime, it was possible to obtain an unitary evolution for this representative atom, governed by a non-linear time-dependent Hamiltonian, whose Schrödinger equation we solved using the Lewis and Riesenfeld method. From that, it was possible to calculate the intensity, equation (2.63).

Since our objective was to observe both superradiance and superabsorption, we used time-dependent coupling strength for the interaction between the atoms and the anti-reservoir. The results for the intensity, for an appropriated choice of the frequency ζ , are shown on Figures (1) and (2). As anticipated, the system goes through periodic cycles where a superradiant pulse is followed by superabsorption.

In the sequence, we coupled the ensemble of atoms previously described to a single mode of the electromagnetic field using the Tavis-Cummings Hamiltonian and a thermal bath. The Hamiltonian for this system is given by equation (2.64). Following similar methods as in section (2.1), and once again using the mean-field approximation, we obtained a Hamiltonian to describe the unitary evolution for the transient regime for this system, which is ultimately represented by an atom coupled to a cavity mode, as shown in equation (2.80).

We noticed that, for a certain regime of parameters, the cavity mode absorbs most of the superradiant pulses, and subsequently restores the energy of the atoms via super-

absorption, this periodical behavior is shown on equation (2.93). The interplay between atom and the cavity mode is shown on Figures (5) and (6).

Moreover, increasing the number of atoms result in the enhancement of the frequency of the Rabi oscillations, as shown in Figure (7), where we compared the frequency of the interplay for $N = 1$ atoms, the Jaynes-Cummings model, with $N = 10^3$ atoms, and observed a dramatic increase in the frequency, of the order of $\sqrt{N}$. Besides that, we show in the same figure that one superradiant pulse of the atoms coupled to the cavity mode is much faster than that one coming from the interaction of atoms to the reservoir, the usual superradiance.

4.2 Two-level superradiant laser

With the results discussed above we conclude the analysis of the interplay for the transient regime. In the sequence, we were interested in analysing the steady solutions for the same system, which, for a certain regime of parameters, takes the form of a superradiant laser.

To obtain that, we consider the addition of an external pumping, whose effect is to change the sign of the competition between the dissipation rate, and the absorption rate, $\gamma_{21} - \gamma_{12}$, to negative, in the Lindblad master equation (2.96). As discussed in section (2.1), by inverting this sign of the γ parameter, $\gamma_{21} - \gamma_{12} < 0$, the term proportional to N , in the Hamiltonian (2.80) describes collective effects of absorption. The second important change was to include a dissipation term for the cavity mode, which was expressed by the fourth term in equation (2.96). This term represents the radiation emitted by the laser to the environment.

We derived a system of coupled differential equations (2.99) and developed an analogy between the cavity mode reaching a saturation of its population, a steady state solution outside of the thermal equilibrium, and second-order phase transitions. We showed how the collective effects, represented by the number of atoms, would change the critical point where the phase transition occurs.

We also showed the time evolution for the population of the cavity, knowing that in the steady state $\langle a^\dagger a \rangle = |\langle a \rangle|^2 = |A|^2$, and the population of the representative atom in Figure (15a). It's also possible to see the intensity for the laser field, equation (2.104), and its increase with N in Figure (16), we observed that the intensity scales with a factor of $N^2/2$.

Another important characteristic of the laser is its linewidth, which is usually calculated through the second-order correlation function of the laser field, $\langle E(t)E(t + \tau) \rangle$. However, for systems where the Markovian approximation can be used, which is consistent with the laser problem, we can obtain an approximate result for the linewidth by

calculating the decayment of the off-diagonal terms of the reduced density operator for the field. In other words, in the steady state, the value of $|\langle a \rangle|$ exponentially decays with a coefficient Γ , as shown on Figure (15b), the Γ rate defines the laser linewidth.

We showed the results for the laser linewidth for different values of N in Figure (17). We notice that there is an increase in the linewidth when we add more atoms to the system. This also means that the off-diagonal terms, the coherence terms, decay faster if we increase N .

4.3 Pseudo-Hermitian Hamiltonians, the Dyson maps and the Hermitian counterparts

In section (1.2) and chapter (3), we focused our analysis in non-Hermitian systems. We started by showing, in equation (1.9), how to use non-unitary similarity transformations to obtain a Hermitian counterpart of the non-Hermitian Hamiltonian. Our main objective, besides showing the mathematical procedure to obtain the Hermitian counterpart and solve the following equations to obtain the Dyson map parameters, was also to show that the choice of the Dyson map can drastically change the behavior of a system.

The first analysis started from a time-independent harmonic oscillator under the influence of unbalanced linear amplification, equation (3.1). For this case, a time-independent Dyson map was sufficient, but since we wanted to prove that the linear amplification could generate squeezing on the quadratures of the field, we used a cubic Dyson map, equation (3.2), to force the Hermitian counterpart to have quadratic amplification, the resulting h was shown in equation (3.4).

Moreover, the map parameter ϵ remained unconstrained, which corroborates our previous affirmation that there is an infinite number of possible Dyson maps. In the sequence, we chose two values for the free parameter, resulting in the Hamiltonians given by equations (3.10) and (3.11).

The above mentioned Hamiltonians result in squeezing of the quadratures X and P after an appropriated phase space's axis rotation was done. We show the Wigner function for both choices of ϵ in Figures (19) and (21).

In Figure (19), the squeezing under pure cubic amplification is showcased. In comparison with the squeezing generated by quadratic amplification, whose Wigner function would be an ellipse, we notice that there is an asymmetry in the major axis. This asymmetry is also present in Figure (21), which comes from an Hamiltonian (3.11), that includes cubic and quadratic amplifications. The time evolution for the field quadratures is severely impacted when we add to (3.10) the parametric amplification. We showed this in Figures (18) and (20). We notice that there is an interference effect between the cubic and quadratic amplifications.

In the sequence, our goal was to solve the quadratic time-dependent harmonic oscillator under unbalanced parametric pumping, the described Hamiltonian can be seen on equation (3.13). The time-dependency on the Hamiltonian suggests a necessity of a time-dependent Dyson map, mainly to avoid constraints on the parameters of H .

To ensure the hermiticity of h , we obtain a system of coupled differential equations (3.26), whose solutions were discussed. For a non-Hermitian Dyson map, as shown on equation (3.14), a free parameter, was obtained.

Moreover, we eliminated the presence of the parametric pumping to show how a complex frequency, $\omega(t)$, can generate infinite squeezing degrees at finite controllable times. This result is shown in Figure (3.87).

At last, we show how a complex frequency could be obtained in the regime of strong parametric pumping, after a transition point where a Hermitian Hamiltonian, equation (3.88), could be represented by a non-hermitian harmonic oscillator, equation (3.90). This transition point represents a change in the equations of the squeezing factor $r(t)$, which is characterized on Figure (26).

4.4 Perspectives and closing thoughts

In the next stage of our work, we will be interested in exploring the collective effects in systems previously described by singles particles, but using a moderately dense atomic sample instead, obtaining from that an effective Hamiltonian for a representative atom and investigating the impact of collective effects, as we have done in this work. We also hope to investigate the states of the atom and of the field during the interplay to evaluate the viability of this system in quantum information processes.

Furthermore, we obtained a laser, in section (2.3), whose linewidth is increased in comparison to the usual laser described in ref. (35). We want to obtain solutions with smaller linewidth than that one coming from a laser with only one atom interacting with the cavity mode, which could be potentially done by adding auxiliary atomic levels to the system.

At last, a still open problem within the pseudo-Hermitian quantum mechanics field is the engineering of non-Hermitian Hamiltonians and certainly an important task to tackle. Besides that, we intent to explore the freedom that the choice of the Dyson map gives to generate other effects that are not usually present in Hermitian Hamiltonian, as we showed for the generation of squeezed states. Also, the free parameters of the form of equation (3.36) should allow us to induce specific behaviors on the Hermitian-counterparts h , and we want to explore those possibilities.

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