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**Quantum fluids spatial distribution evaluation and its
characterization**

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**Quantum fluids spatial distribution evaluation and its
characterization**

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in Physics at the Instituto de Física de São
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the degree of Doctor in Science.

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“Persistence is the shortest path to success.” Charlie Chaplin

ABSTRACT

SMAIRA, A.F. **Quantum fluids spatial distribution evaluation and its characterization.** 2019. 131p. Thesis (Doctor in Science) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2019.

Bose-Einstein Condensates (BEC) are excellent macroscopic samples quantum behavior of matter study. Since it was experimentally observed in dilute atomic gases, there are important properties related to these systems that were intensely explored, as thermodynamic properties and BEC dynamics both in stationary and turbulent regime. In this work we present and discuss the results and properties determined from Bose-Einstein Condensed time-of-flight images processed using algorithms capable of extract relevant information from atomic density fluctuations observed in this kind of sample. Based on advanced data process we can go beyond the standard results, already presented, and obtain linear *momentum* maps, density fluctuations and correlations, vortices properties, as well as develop other analysis types to characterization and understanding improve results obtained in turbulent regime, besides other related properties. In this way, we developed specialized data processing library to be used in the general analyses and also to the determination of non-trivial properties, which are essential for the future study of quantum degenerate turbulent gases, with a three-dimensional atomic cloud reconstruction method from three non-orthogonal absorption images obtained from experimentally acquired time-of-flight images. In order to accomplish this we have used "three-dimensional inverse Radon Transform", analogous to tomography in the two-dimensional case, however, unlike the indetermination theorem requirement, that the number of images be large enough, we have just a few of them. This issue is lessened, but not solved, using an iterative correction method.

Keywords: Optics. Atomic physics. Condensed matter physics. Bose-Einstein condensate.

RESUMO

SMAIRA, A.F. **Avaliação da distribuição espacial de fluidos quânticos e sua caracterização**. 2019. 131p. Tese (Doutorado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2019.

Condensados de Bose-Einstein (BEC) são sistemas macroscópicos excelentes para o estudo do comportamento quântico da matéria. Desde sua obtenção experimental em gases atômicos alcalinos diluídos aprisionados por campos magnéticos, há importantes aspectos relacionados a esse sistema que foram intensamente explorados, como propriedades termodinâmicas e dinâmica do BEC tanto em regime estacionário quanto em regime turbulento. Neste trabalho apresentamos e discutimos os resultados e propriedades de turbulência em condensados de Bose-Einstein determinadas a partir de imagens de tempo de voo e processadas usando algoritmos complexos capazes de extrair informações relevantes das flutuações de densidade atômica observadas nesse tipo de amostra. Baseando-nos em processamento avançado somos capazes de ir além dos resultados padrões, já apresentados, e determinar mapas de *momentum* linear, flutuações de densidade e correlações, propriedades de vórtices, assim como desenvolver outros tipos de análise, para ajudar em uma melhor caracterização e entendimento dos resultados obtidos em regime turbulento, além de outras propriedades relacionadas. Nesse sentido, desenvolvemos uma biblioteca de processamento especializada para ser usada nas análises e na determinação de propriedades não triviais, que são essenciais para o futuro da pesquisa de gases turbulentos degenerados, com foco principal em um método para a reconstrução tridimensional de uma nuvem atômica partindo de três imagens não ortogonais de absorção capturadas em tempo de voo. Para isso usamos a "transformada inversa de Radon (princípio da tomografia computadorizada) tridimensional", porém, ao contrário do que exige o teorema da indeterminação, que estabelece que a quantidade de imagens necessárias seja suficientemente grande, temos poucas delas. Tal problema é amenizado, mas não solucionado, usando métodos de correções iterativas.

Palavras-chave: Óptica. Física atômica. Física da matéria condensada. Condensado de Bose-Einstein.

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LIST OF ABBREVIATIONS AND ACRONYMS

BEC	Bose-Einstein Condensate
MOT	Magneto-Optical Trap
GPE	Gross-Pitaevskii equation
TFA	Thomas-Fermi Approximation
TFR	Thomas-Fermi Radius
TOF	Time of Flight
CCD	Charge-coupled device
OD	Optical density
2D	Two-dimensional
3D	Three-dimensional
CT	Computed Tomography method
IBP	Iterative Back-Projection method
GPU	Graphics Processing Unit

LIST OF SYMBOLS

T_c	Condensation temperature
d	Mean distance between particles
λ	Wavelength
f^r	Occupation in state r
ε_r	Energy of state r
μ	Chemical potential
k_B	Boltzmann constant
T	System temperature
Ξ	Large partition function
ζ	System fugacity
N	Total number of particles in a system
f^0	Occupation in ground state
$\hbar$	Reduced Planck constant
n	Numerical density of particles
m	Particle mass
ρ	Quantum state degeneracy
V	Trapping potential
(x, y, z)	Spatial position in Cartesian coordinates
ω_i	Harmonic oscillator frequency in the direction i
$n_i, i \in \{x, y, z\}$	Quantum numbers
$G(\varepsilon)$	Number of particles with energy at most ε
N_0	Number of particles in ground state
n_ε	Occupancy of state with energy ε
$\Gamma(n)$	Gamma function

$\zeta(n)$	Zeta function
ψ	Time dependent Bose-Einstein condensate wave function
$\vec{r}$	Spatial position
V_{int}	Interaction potential
U_0	Effective interaction
a	Scattering length
$\delta(\vec{r} - \vec{r}_0)$	Dirac delta function
H	Hamiltonian
p	Linear momentum
i	Imaginary unit
a_{osc}	Quantum harmonic oscillator length
$\bar{\omega}$	Geometric mean of harmonic potential trapping frequencies
R_{TF}	Thomas-Fermi radius
(r, θ, φ)	Position in spherical coordinates
R_i	Thomas-Fermi radius in direction i
t	Time-of-flight
Γ	BEC circulation
h	Planck constant
ξ	Healing length
He	Chemical element Helium
I_N	Normalized image
I_{BEC}	BEC absorption image
I_L	Only laser beam image
I_B	Background image
OD	Optical density
μ_T	Attenuation coefficient

$\mathcal{R}$	Radon transform
$\vec{f}$	Trap frequencies in each direction
F	Helmholtz free energy
$\mathcal{V}$	Volume parameter
Π	Pressure parameter
σ	Standard deviation
Ω	Grand potential
$g_n(y)$	Polylogarithm function.
$\vec{j}$	Current density
S	Complex phase of wave function

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1 STATE OF THE ART AND NEED FOR ACCURATE ASSESSMENT OF $n(r)$

1.1 State of art

The Bose-Einstein Condensates (BEC) were theoretically proposed by Albert Einstein in 1925¹ based on Satyendra Nath Bose work² published an year before. If one cools a non-interacting high density gas below the critical temperature (or condensate temperature) (T_C), a macroscopic particle fraction will occupy the ground state, what characterizes the Bose-Einstein condensation.

It is important to note that this behavior can only occur with bosons, since, in the fermions case, the Pauli exclusion principle³ prevent the phenomenon occurrence as shown in figure 1. Despite that we have to remember that it is possible to create fermionic systems that behaves like bosons, as Cooper pairs, which turns possible to have BEC of molecule of fermions among other systems.⁴

Figure 1 – Difference between ground states of bosons (left) and fermions (right) systems.

Source: QUANTUM-BITS⁵

Since it is possible to obtain a large number of particles in just one quantum state, the BEC became an excellent tool to study many physics areas such as Statistical Physics,⁶ Quantum Mechanics,⁷⁻⁹ Many Body Physics,¹⁰ condensed matter¹¹ and others.

Although the BEC was foreseen in 1925, the first experimental sample was achieved only in 1995,¹²⁻¹⁴ considering that ^4He is not BEC.¹⁵ It may seem simple to get to a BEC sample (cool and trap) but the critical temperature for dilute gases is less then or about a micro-kelvin and it took much time to the development of techniques capable of cooling and trapping at these temperatures using light and magnetic field, that is, magneto-optical trap (MOT)¹⁴ and evaporative cooling.¹⁶

Gases are cooled using radiation pressure as proposed by Theodor Hänsch and Arthur Schawlow in 1975,¹⁷ and are trapped by MOT, made by a magnetic field generated by anti-Helmholtz coils, in which the gradient is linear at its geometric center. It creates a

capture environment due to photon scattering that dissipates the atoms kinetic energy because of the restoring force, which pushes them towards the trap center. Only in 1982, William D. Phillips group made the first laser cooling experiment¹⁸ and, three years after, the first magnetic trap for neutral atoms.¹⁹ However the laser cooling and the MOT are not enough to achieve such low temperatures and high densities to the point of condensation temperature and, therefore, to make a BEC. It was necessary a new cooling technique to reach these ultra cold temperatures (hundreds of nano-kelvin).

In the evaporative cooling, the most energetic atoms are removed from the sample, decreasing the mean kinetic energy of the particles. This procedure was experimentally used for the first time in 1988¹⁹ by Daniel Kleppner group, but was used to achieve the Bose-Einstein condensation only seven years later, singly, in ^{87}Rb ,²⁰ ^7Li ¹³ and ^{23}Na .²¹

Unlike early works, what were focused in BEC thermodynamics around transition, nowadays the principal studies in area are about BEC dynamics induced by external perturbation.

The first version of our current experimental setup, the BEC-1, with Rubidium 87 has begun during the PhD of Emanuel Alves de Lima Henn PhD.²² With this system it was possible to observe vortex nucleation²³ and evidences of quantum turbulence in dilute gases.²⁴

After the BEC is produced, it is necessary to obtain information from it, which is made in techniques based on atom-light interaction. One of them is the absorption image after a certain time of flight. A light beam pass through atoms, leaving a shadow, that is proportional to the cloud density at each point, in a microscope lens system, with final target in a CCD. When the atoms absorb light, the cloud is destroyed.

1.2 Need for accurate assessment of $n(\mathbf{r})$

As briefly described in the end last section, the main form to acquire information from atoms with this system is from absorption image. In this way, the main form to get more information is taking more pictures.

Generally, a few properties of the cloud are measured from these images, but it is important to remember that besides we are estimating three-dimensional system properties from two-dimensional images, they are taken after some time of flight, so it is necessary to do a regression to *in-situ* cloud after all. In this process, a few approximations are made.

It would be very interesting if, instead of the previous method, we could obtain a three-dimensional digital sample, similar or, in an ideal scenario, equal to the original *in-situ* cloud from just a few images we can acquire from the experiment. This is the main goal of our work.

1.3 Thesis summary

Basically this thesis proposes the use of an algorithm to reconstruct the three-dimensional BEC cloud and analyzes both the algorithm results and the deviations.

In chapter 2, the theoretical background necessary to understand the algorithm is presented. A brief introduction to BEC theory is shown in section 2.1. An ideal Bose gas is studied in section 2.2. After that an weak interaction is added to system in section 2.3. To understand the system in limit of strong interaction the Thomas-Fermi approximation is studied in section 2.4. To simplify the math study of BEC dynamics, quantum hydrodynamics is developed in section 2.5. As consequence of hydrodynamics equations, we get to theoretical topological defects evidenced in form of quantum vortices in section 2.6. To end the chapter, the experimental setup used in BEC-1 experiment is shown in section 2.7.

In chapter 3, it is shown theoretical topics in image storage and processing. When we have the digital image in computer, the better way to storage it is discussed in section 3.1. At last, the image is digitally processed using filtering methods to remove experimental and digitalization noise in section 3.2 to finally start the main part of the thesis.

In chapter 4, the three-dimensional tomographic method to determine the BEC density as function of spatial position is proposed. The classical tomographic method used for medicinal propose is presented in section 4.1. The analogous method, now proposed to the study of quantum gases is explained and analyzed for well behavior Thomas-Fermi clouds in section 4.2.

In chapter 5, we use the proposed method to recover the *in-situ* cloud based on absorption images from expanded cloud in time of flight.

In chapter 6, we include tomographic defects and analyze how they affect the method precision. Linear vortex is added and analyzed in section 6.1. An internal spherical bubble is added and analyzed in section 6.2. At last a ring-shaped vortex is added and analyzed in section 6.3.

In chapter 7, we use the proposed method to obtain some thermodynamic properties of BEC. We study the global parameters thermodynamic in section 7.1. After that, the values to be calculated are shown in section 7.2. At last results are shown and discussed in section 7.3.

Finally, chapter 8 presents the conclusion, summarizing the method, the results precision and the thesis importance in the quantum gases study area.

A few appendices were included after conclusions to support the reader. There is a table of physics constants in appendix A. The MATLAB® codes used in the work both for processing and analysis are in an Google Drive repository, which link address

is in appendix [B](#). An extra proof of condensate fraction in case of gas in box, which is not our experimental case, is shown in appendix [C](#). One of many ways of proving the Gross-Pitaevskii equation (GPE), which describes BEC dynamics, is shown in appendix [D](#). A proof of analytic equation of BEC ground state in an harmonic trap is shown in appendix [E](#). At last the time dependent GPE is linked to classical hydrodynamics in appendix [F](#).

2 THEORETICAL BACKGROUND

In this chapter, we will understand more about Bose-Einstein Condensation (BEC) and how one can describe them mathematically.

In nature, there are two quantum particle types, **fermions** and **bosons**:

- **Bosons:** they have integer spin and there is no limit to the particles number in each quantum state, as shown in equation 2.1, the Bose-Einstein statistics. Examples: photon, gluon, He^4 , etc.

$$f^r = \frac{1}{e^{(\varepsilon_r - \mu)/k_B T} - 1} \quad (2.1)$$

where f^r is the occupation in state r , ε_r is the energy of state r , μ is chemical potential, $k_B \approx 1.38 \times 10^{-23} J/K$ is Boltzmann constant and T is temperature.

Note that there is no maximum occupation of any state, that is, it is possible to accumulate many particles in just one state, which is the key to achieve BEC.

- **Fermions:** they have semi-integer spin and there is at most one particle in each quantum state. These particles obey the Fermi-Dirac statistics (equation 2.2) and Pauli exclusion principle. Examples: electron, proton, quarks, leptons, neutrino, etc.

$$f^r = \frac{1}{e^{(\varepsilon_r - \mu)/k_B T} + 1} \quad (2.2)$$

Note that the maximum allowed occupation of any state is 1, that is, there is no state that can be occupied by more then one fermion particle. Despite that, it is possible to prepare systems made of just fermions that present boson properties and obey Bose-Einstein statistics, such as Cooper Pairs, which can accumulate many particles in just one state, making it possible to create BECs of fermions.⁴

The Fermi-Dirac and Bose-Einstein statistics equations shown above will be proven in section 2.2.

2.1 Introduction to BEC

Before we start to describe BEC mathematically, we will qualitatively explain what the BEC is using the figure 2.

Figure 2 – Qualitative description of BEC.

Source: KETTERLE²⁵

At high temperatures, a gas can be thought of as many billiard balls with velocity v and numeric density proportional to d^{-3} where d is the mean distance between particles that collide to each other. When the temperature decreases the wave package behavior becomes relevant. Further lowering the temperature so that the De Broglie wavelength²⁶ equals to mean distance between particles $\lambda \simeq d$, most of ground state waves overlap and give rise to a large matter wave, which shows the importance of this system for macroscopic researches in quantum phenomena.

In previous situation most of particles are part of the large matter wave, but not all of them. In the hypothetical situation of zero kelvin temperature, all the particles would be in ground state, which creates a pure BEC in a giant matter wave. This temperature cannot be reached because of third law of thermodynamics, but in laboratory many cooling steps are made to decrease the temperature the maximum as possible to achieve this ideal hypothetical situation.

2.2 The ideal Bose gas

Let a trapped gas be a gas subjected to an external potential. In Statistical Physics, we have the following large partition function Ξ for non interacting bosons gas:⁶

$$\Xi(\beta, \varepsilon, \mu) = \prod_{r \in \{\text{states}\}} \frac{1}{1 - e^{-\beta(\varepsilon_r - \mu)}} \Rightarrow \ln[\Xi(\beta, \varepsilon, \mu)] = - \sum_{r \in \{\text{states}\}} \ln(1 - \zeta e^{-\beta \varepsilon_r}) \quad (2.3)$$

where $\beta = \frac{1}{k_B T}$ in which k_B is the Boltzmann constant and T is the gas temperature and $\zeta = e^{\beta\mu}$ is the system fugacity.

The total number of particles N can be obtained from:

$$N = \zeta \frac{d}{d\zeta} (\Xi) = \sum_{r \in \{\text{states}\}} \frac{1}{\zeta^{-1} e^{\beta \varepsilon_r} - 1} \quad (2.4)$$

$$\Rightarrow f^r = \frac{1}{\zeta^{-1} e^{\beta \varepsilon_r} - 1}$$

In the quantum regime, bosons have non positive chemical potential ($\mu \leq 0$) and it increases with decreasing temperature until zero at critical temperature and keeps zero below it.⁶ In this situation, we have $\zeta = 1$ and $\varepsilon \ll k_B T$ for ground state making its occupation macroscopic, which characterizes the Bose-Einstein Condensation and turn them so important because this large number of particles in just one state creates a large coherent matter wave due to superposition of many De Broglie waves of each system particle.

Since there is a macroscopic number of particles in a single state, these are excellent physical systems for study of quantum matter behavior. They allow, for example, the quantized vortex⁸ and collective modes^{27,28} study.

The figure 3 shows the particles distribution for three ξ values. See that the smaller the fugacity value (the larger the temperature), the smaller the ground state occupation, pushing the particles to the excited states. This is the classical regime, in which the number of particles in each state is microscopic, making the quantum behavior not perceived, that is, the classical regime is that with low fugacity due to high temperature in which the quantum statistic is not observable. Mathematically, in limit of low temperature, the expression for f^r in equation 2.4 goes to infinity:

$$f^0 = \lim_{r \rightarrow 0} (f^r) \rightarrow \infty \quad (2.5)$$

that is, as mentioned before, there is a macroscopic number of particles in ground state.

Figure 3 – Statistical distribution of bosons states for three fugacity values.

Source: By the author

This phase transition occurs at critical temperature (T_c), which for uniform gas is given by:

$$T_c = C \cdot \frac{\hbar^2 n^{2/3}}{mk_B} \quad (2.6)$$

where the constant C depends on the trap, $\hbar \approx 1.055 \times 10^{-34}$ J·s is the reduced Planck constant, n is the numerical density of particles and m is the mass of each particle. To determine the constant value, we use the quantum states degeneracy ρ . We will consider a gas trapped in a three-dimensional harmonic potential V :

$$V(x, y, z) = \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \quad (2.7)$$

where ω_i is the harmonic oscillator frequency in each of the three orthogonal directions in space (x, y, z) . For this physical system, we have the quantum states, with energy $\varepsilon(n_x, n_y, n_z)$, defined by the quantum numbers n_x , n_y and n_z :

$$\varepsilon(n_x, n_y, n_z) = \left(n_x + \frac{1}{2}\right) \hbar\omega_x + \left(n_y + \frac{1}{2}\right) \hbar\omega_y + \left(n_z + \frac{1}{2}\right) \hbar\omega_z \quad (2.8)$$

From previous equation, let's determine the number of particles with energy at most $\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z$, which we will call $G(\varepsilon)$. Let's consider $\varepsilon \gg \hbar\omega_i$ and, therefore,

continuous:

$$G(\varepsilon) = \frac{1}{(\hbar\omega_x)(\hbar\omega_y)(\hbar\omega_z)} \int_0^\varepsilon d\varepsilon_x \int_0^{\varepsilon-\varepsilon_x} d\varepsilon_y \int_0^{\varepsilon-\varepsilon_x-\varepsilon_y} d\varepsilon_z \quad (2.9)$$

$$G(\varepsilon) = \frac{\varepsilon^3}{6\hbar^3\omega_x\omega_y\omega_z} \quad (2.10)$$

Now we can calculate the degeneracy, that is, the number of particles with energy between ε and $\varepsilon + d\varepsilon$:

$$\rho(\varepsilon) = \frac{d}{d\varepsilon} G(\varepsilon) = C_3 \varepsilon^2 \quad (2.11)$$

where $C_3 = \frac{1}{2\hbar^3\omega_x\omega_y\omega_z}$.

We can define this phase transition more precisely: it occurs when the excited states are saturated, forcing any new particle go to the ground state. In this situation the chemical potential is maximum ($\mu = 0$). We can describe this situation mathematically assuming the thermodynamics limit and transforming the number series in integral:

$$N = N_0 + \int_0^\infty f^r \rho(\varepsilon) d\varepsilon \Rightarrow \frac{N_0}{N} = \begin{cases} 0 & T > T_c \\ 1 - \frac{1}{N} \int_0^\infty n_\varepsilon \rho(\varepsilon) d\varepsilon & T \leq T_c \end{cases} \quad (2.12)$$

where $\frac{N_0}{N}$ is the **condensed fraction** and shows the system particles fraction that are in ground state and n_ε is the occupancy of state with energy ε .

For the condensate fraction:

$$\frac{N_0}{N} = 1 - \frac{1}{N} \int_0^\infty \left(\frac{1}{e^{\frac{\varepsilon}{k_B T}} - 1} \right) (C_3 \varepsilon^2) d\varepsilon = 1 - \frac{C_3}{N} \int_0^\infty \frac{\varepsilon^2}{e^{\frac{\varepsilon}{k_B T}} - 1} d\varepsilon = 1 - \frac{C_3}{N} (k_B T)^3 \Gamma(3) \zeta(3) \quad (2.13)$$

where $\Gamma(n) = (n-1)!$ for natural numbers is the gamma function and $\zeta(n) = \sum_{k=1}^\infty \frac{1}{k^n}$ is the zeta function. At $T = T_c$, we have $N_0 = 0$:

$$0 = 1 - \frac{C_3}{N} (k_B T_c)^3 \Gamma(3) \zeta(3) \Rightarrow T_c = \frac{\hbar}{k_B} \sqrt[3]{\frac{2N\omega_x\omega_y\omega_z}{\Gamma(3) \zeta(3)}} \approx 7.19 \times 10^{-12} \sqrt[3]{N\omega_x\omega_y\omega_z} \quad (2.14)$$

If we substitute typical number of particles and frequencies found in the laboratory, we get the condensation temperature of hundreds of nanokelvin. Since we have the

condensation temperature expression, we can simplify the condensate fraction expression and write it as function of T_c :

$$\boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^3} \quad (2.15)$$

which only depends on temperature and has limit values $\frac{N_0}{N}(T = T_c) = 0$ and $\frac{N_0}{N}(T = 0) = 1$.

If we redo all these steps for homogeneous gas (gas in a box, appendix C), we will get to:

$$\boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}} \quad (2.16)$$

which has the same behavior of the harmonic trap, but the atoms populate the ground state slower when the temperature decreases, which is shown in figure 4.

Figure 4 – Comparison between condensation fraction in harmonic trap and box.

Source: By the author

2.3 Weakly interacting gas - Gross-Pitaevskii Equation

In the previous section we described a non interacting trapped Bose-Einstein condensate, however there is a high density in a typical harmonic trap (about 10^{14} atoms/cm³), so that the interaction between the particles cannot be overlooked. Therefore, a new theory became necessary.

The called Gross-Pitaevskii equation (GPE) was developed by Eugene Gross²⁹ and Lev Pitaevskii³⁰ to describe quantized vortices in a boson gas. To get to it, it is assumed that all particles are in the same quantum state, so the system can be described by a single wave function $\psi(\vec{r}, t)$, from which the local density in each point can be obtained:

$n(\vec{r}, t) = |\psi(\vec{r}, t)|^2$. Because of that, the Gross-Pitaevskii theory is called mean field theory too. A dilute gas is still assumed, where we can consider only two-particle interaction, which can be modeled as the following interaction potential V_{int} :

$$V_{int}(\vec{r}) = U_0 \delta(\vec{r} - \vec{r}_0) \quad (2.17)$$

where $\vec{r}$ and $\vec{r}_0$ are the two particles position, $U_0 = \frac{4\pi\hbar^2}{m}a$ is the effective interaction, related to mass m and to scattering length a and $\delta(\vec{r} - \vec{r}_0)$ is Dirac delta function. The diluteness condition is:

$$n|a|^3 \ll 1 \quad (2.18)$$

With these assumptions, we can write the Hamiltonian H :

$$H = \sum_{i=1}^N \left[\frac{p_i^2}{2m_i} + V(\vec{r}_i) + U_0 \sum_{j < i} \delta(\vec{r}_i - \vec{r}_j) \right] \quad (2.19)$$

where we have kinetic energy (p_i is linear momentum of particle i and m_i is mass of particle i), external potential ($\vec{r}_i$ is the position of particle i) and two-particle interaction potential, in this order. The U_0 constant characterizes the effective contact potential between particles, where $U_0 > 0$ represents our system in which the interaction is repulsive. From this Hamiltonian we can deduce the time dependent and independent Gross-Pitaevskii equations (appendix D), also known as non linear Schrödinger equations:

$$\boxed{-\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + V(\vec{r}) \psi(\vec{r}) + U_0 |\psi(\vec{r})|^2 \psi(\vec{r}) = \mu \psi(\vec{r})} \quad (2.20)$$

$$\boxed{i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) + U_0 |\psi(\vec{r}, t)|^2 \right] \psi(\vec{r}, t)} \quad (2.21)$$

where i is the imaginary unit. Assuming harmonic potential trap, we have a quantum harmonic oscillator, which gives us the ground state wave function in non-interacting regime: Gaussian function (as shown in appendix E) with characteristic length given by the oscillator length a_{osc} :

$$a_{osc} = \sqrt{\frac{\hbar}{m\bar{\omega}}} \quad (2.22)$$

where $\bar{\omega}$ is the geometric mean of the harmonic potential trapping frequencies.

It is important to remember that this is an approximation theory, so it has restrictions. When there is significant atom loss for other states, the approximation is no longer valid. Besides that, we assumed that the inter-atomic interaction is only of

two-particle type, what happens in dilute gases ($na^3 \ll 1$) at absolute zero temperature ($T = 0K$).

2.4 Thomas-Fermi Approximation

For most of the researches, the interest system is one that can be obtained in laboratory, with high external potential due to trapping. When we cool, its kinetic energy decreases and density increases significantly, so that we can neglect kinetic energy in front of the trapping potential and of interacting energy. These are the hypotheses of Thomas-Fermi approximation (TFA): we neglect the kinetic term in front of external and interaction potential terms.

$$\mu \psi(\vec{r}) = [V(\vec{r}) + U_0 |\psi(\vec{r})|^2] \psi(\vec{r}) \quad (2.23)$$

This is an algebraic equation in density:

$$n(\vec{r}) = |\psi(\vec{r})|^2 = \begin{cases} \frac{\mu - V(\vec{r})}{U_0} & \mu > V(\vec{r}) \\ 0 & \mu \leq V(\vec{r}) \end{cases} \quad (2.24)$$

We can also define the Thomas-Fermi radius (TFR) as the cloud half-length when it obey the TFA:

$$V(R_{TF}) = \mu \Rightarrow R_{TF} = |V^{-1}(\mu)| \quad (2.25)$$

where R_{TF} is the Thomas-Fermi radius and $V^{-1}(\mu)$ is the inverse potential function in variable μ .

If we know the density, total number of particles and external potential, we normalize the wave function to obtain the chemical potential. Applying these formulas to harmonic potential, we get:

$$n(x, y, z) = \frac{\mu - \frac{1}{2}m\omega_x^2 x^2 - \frac{1}{2}m\omega_y^2 y^2 - \frac{1}{2}m\omega_z^2 z^2}{U_0} \quad (2.26)$$

When we use the expression to determine the TFR, we obtain the surface that limits the condensate region:

$$\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 = \frac{2\mu}{m} \quad (2.27)$$

which is a paraboloid. The TFR in each axis R_i can be defined as the distance between this surface and the BEC center over the axis:

$$R_i = \frac{1}{\omega_i} \sqrt{\frac{2\mu}{m}} \quad (2.28)$$

which shows that the higher the interaction potential or the smaller the trap frequency or the particle mass, the larger the radius. When we integrate the density over all the space, we get the total number of particles. Let $v_i = \omega_i i$ and $v^2 = v_x^2 + v_y^2 + v_z^2$ to simplify the calculations:

$$N = \int n(\vec{r}) d\vec{r} = \frac{1}{U_0} \int_0^{2\pi} \int_0^\pi \int_0^{\omega R} \mu - \frac{1}{2} m v^2 dv d\theta d\varphi \quad (2.29)$$

where (θ, φ) are angular positions in spherical coordinates.

$$N = \left(\frac{2\mu}{m}\right)^{3/2} \frac{8\pi\mu}{15U_0\omega_x\omega_y\omega_z} \Rightarrow \mu = \left(\frac{15N\omega_x\omega_y\omega_z}{8\pi}\right)^{2/5} \left(\frac{m}{2}\right)^{3/5} \quad (2.30)$$

Now we can determine the density as function of position:

$$n(x, y, z) = \begin{cases} \frac{m}{2U_0} \left[\left(\frac{15N\omega_x\omega_y\omega_z}{4\pi m}\right)^{2/5} - \omega_x^2 x^2 - \omega_y^2 y^2 - \omega_z^2 z^2 \right] & \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 < \left(\frac{15N\omega_x\omega_y\omega_z}{4\pi m}\right)^{2/5} \\ 0 & \text{otherwise} \end{cases} \quad (2.31)$$

2.5 Quantum hydrodynamics

We can write GPE in hydrodynamic form (step by step in appendix F), analogue to classical hydrodynamics as shown below, called Madelung equations:³¹

$$\begin{cases} m \frac{\partial \vec{v}}{\partial t} + \nabla \left[\frac{1}{2} m v^2 + V(\vec{r}) + U_0 n - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \right] = 0 & \text{(Bernoulli equation)} \\ \frac{\partial n}{\partial t} + \nabla \cdot (n \vec{v}) = 0 & \text{(Continuity equation)} \end{cases} \quad (2.32)$$

The term inside brackets of first equation is the quantum pressure term.³²

2.5.1 BEC Expansion

In 1996 Castin and Dum calculated the BEC expansion behavior³³ using Quantum hydrodynamics. They described the BEC aspect ratio after a specific TOF in a initially cigar shape BEC, which is the shape of our experimental clouds:

$$\frac{R_r}{R_{r,0}} = \sqrt{1 + \omega_r^2 t^2}$$

$$\frac{R_a}{R_{a,0}} = 1 + \frac{\omega_a^2}{\omega_r^2} \left| \omega_r t \arctg(\omega_r t) - \ln\left(\sqrt{1 + \omega_r^2 t^2}\right) \right|$$

where R_r is the radial TFR, R_a is the axial TFR, ω_r is the trap radial frequency, ω_a is the trap axial frequency and t is the time of flight.

The file *CastinDum.m* (copied in section B.1) does the simulation based in these equations.

Figure 5 – Expansion of an initially non-spherical BEC.

Source: By the author

Note that the radial dimension expands faster than the axial dimension, which shows that the BEC will be spheric in a specific moment, as shown in figure 5.

2.6 Topological defects

A topological defect occurs when the phase function is undetermined.³⁴ In case of BEC, the **vortex**, a consequence of hydrodynamics equations,³⁵ is a topological defect.

2.6.1 Vortex

The vortex is a structure that has a core with density zero and the circulation Γ , defined by line integral of velocity field around the studied point (equation 2.33), is multiple of $\frac{h}{m}$, where h is Planck constant. It is a topological defect because around its

core each point has a different phase value, from 0 to 2π , so the phase limit in core point is undefined.

$$\Gamma = \oint_C \vec{v} \cdot d\vec{l} = 2\pi l \frac{\hbar}{m} = l \frac{h}{m} \quad (2.33)$$

In,^{35–39} this and many other vortices structures are obtained from Gross-Pitaevskii equation.

The quantum vortices are topological defects whose equilibrium size is of healing length (ξ) order. This scale is defined as the one in which the kinetic energy is comparable to interaction energy:⁴⁰

$$\frac{\hbar^2}{2m} |\nabla\psi|^2 \simeq U_0 |\psi|^4 \quad (2.34)$$

Representing the spacial derivative as a division in cited scale, we have:

$$\frac{\hbar^2}{2m} \frac{|\psi|^2}{\xi^2} = U_0 |\psi|^4 \Rightarrow \xi = \frac{\hbar}{\sqrt{2mU_0n}} \quad (2.35)$$

For atomic BEC this value is of order of a few hundreds of nanometers, while for ^4He this value is much smaller (few angstroms). This difference makes BEC a much better physical system to study quantum topological defects, as the vortices.

2.7 Experimental setup

There are three experimental systems in our research group to study Bose-Einstein condensates. This work is part of called BEC-1.^{22,40–43}

As we have already discussed, it is needed very low temperatures ($\sim 100\text{nK}$) and high densities ($\sim 10^{14}\text{cm}^{-3}$) to obtain a BEC. The gas cannot be in contact with anything, that's why we use magnetic (or optical) trap inside a vacuum chamber ($\sim 10^{-10}\text{Pa}$).

The figure 6 shows the BEC-1 experimental setup, where the atoms cloud is made and trapped in the left anti-Helmholtz coils, using laser cooling with three orthogonal counter-propagating beams (MOT1), which cool the cloud to around $150\mu\text{K}$. Then the cloud is pushed to right coils, where the vacuum is better and where the sample experience evaporative cooling to turn into a BEC. At the end it is analyzed using absorption images.

Figure 6 – BEC-1 experimental setup.

Source: TAVARES⁴²

Since there is a trapped BEC, we have to get information from it. We use the absorption images in time of flight (TOF) technique, that is, we release the trapped atoms, wait for certain time tof and take the absorption image using a camera. The TOF is necessary to permit that the optical density decrease and therefore does not saturate the central part of beam, what would cause loss of information to be extracted in analyze steps. When BEC is released the repulsive interaction makes the BEC to expand, which causes the density contrast to decrease and therefore the chances of the image becoming saturated in the higher density region also decrease.

Briefly, a laser beam crosses the BEC and part of it is absorbed, casting a shadow over a CCD camera (PCO pixelfly). We will call this image I_{BEC} . Beyond this image, we get an only laser beam image (no atoms, I_L) and a background image, without atoms and beam (I_B). These images are used to generate a normalized image I_N :

$$I_N = \frac{I_{BEC} - I_B}{I_L - I_B} \quad (2.36)$$

From this normalized image we have to use the Beer-Lambert Law (equation 2.37) to determine the optical density:

$$I_N = e^{-OD} \Rightarrow OD = \mu_T z(x, y) = -\ln(I_N) = \ln\left(\frac{I_L - I_B}{I_{BEC} - I_B}\right) \quad (2.37)$$

where μ_T is the attenuation coefficient and $z(x, y)$ is thickness of BEC for each image pixel (x, y) . A set of images is shown in figure 7.

Figure 7 – Only laser beam (I_L), sample (I_{BEC}), background (I_B) and optical density (OD) images, respectively.

Source: By the author

In BEC-1, we may have up to three directions for imaging. The first one, horizontal, we call $\hat{x}$ axis, the second is 45° from first in horizontal plane and the third one, which we call $\hat{z}$ axis is orthogonal to the other two.

There are, however, some difficulties to get these images. The first one is that we cannot get the three of them at same time, so that three steps are necessary for acquisition in a case in which all the three ones are needed. That occurs because any image is made based on atoms light scattering and for better image quality we use atoms resonance frequency, destroying the sample for each time image is taken. One might say an alternative could be take the three images at the same time, but it is very difficult (maybe impossible) to make all beams reach each point of BEC at same time such as the whole three images are taken before each part is destroyed by scattering process. The second difficulty is related to first one, since we have to keep the calibration between images, that is, we have to make the experiment three times to get all three images.

3 IMAGE PREPARATION

In this work many algorithms were developed, from the most basic ones as auxiliary functions for algorithm modularization to the most complex one to reconstruct a three-dimensional BEC sample. All process was made in MATLAB® R2018a due to simplicity and language familiarity. All the codes used in this work can be found in repository for which the link is in the appendix [B](#).

At this stage it is important to clarify how computer deal with images to better understand this chapter content.

3.1 Image storage

Once the image is captured by the camera, it has to be stored in a device. In this way, compression methods⁴⁴ were developed to store the maximum of information in the minimum of space, mainly for commercial purposes as personal computers, cameras, smartphones, etc. Some of these methods cause information loss, generally not visible to human eyes, but for best image analyze it is necessary to keep most of its information so in addition to a good camera as discussed in the previous section, we do not want information loss in storage process even if a lot of memory is used.

Besides that, it is important to note that we have to store three images for each run (BEC, probe and background), shown in figure [7](#). An organized way is to store all of them in only one file.

An interesting file type that satisfy both of desired properties is TIFF.⁴⁵ For compression, there are two options: uncompressed and lossless compression (none of them lose information, but both spend a lot of memory, as quoted), and for organization, TIFF files have the option to save multiple pages file. In our case the first page is the probe image, the second is the BEC image and the third is the background image.

3.2 Image processing

To continue, please see the last image (OD) from figure [7](#) from chapter [2](#) reproduced below in figure [8](#).

Figure 8 – Normalized image from took images of figure 7.

Source: By the author

There are less noisy pictures but an automated process need to work to any possible image, so we will deal with this image as example. At first, it is important to see that there are mainly two noise types in this image: periodic and impulsive.⁴⁶ They are identified in figure 9 for our example:

Figure 9 – Normalized image with identified noises: **periodic** and **impulsive**.

Source: By the author

Impulsive noise (or salt-and-pepper noise) is a type of noise where there are black pixels in white regions and/or white pixels in black regions. The main causes of this type of noise are analog-to-digital conversion and transmission errors, but in our case it is from

a coil shadow combined with the normalization process, since that division by large or small values may cause these high contrast pixels.

Periodic noise is identified as periodic structures that should not be in image, as highlighted in figure 9. Some possible causes of this one are acquisition electromagnetic interference and interference or mechanical vibration of the optical elements in between the shots. The last one is our issue.

Identified the noises, we have to remove them. The impulsive is simpler. The most known method (median filter) is sufficient to us. It assumes that the image is approximately uniform or, at least, each region is approximately uniform, that is, there is not abrupt variation of pixels intensity in original image. In this way, for each damaged pixel it is replaced by the median value of its neighborhood. This is the general algorithm, that is, the algorithm used for both salt and pepper noise, but our case is just pepper noise since that our background is white. This leads us to use a filter that consider only the black pixels as noise, that is, a maximum filter is the ideal filter in our situation. Once used the last one, we get:

Figure 10 – Original image, filtered image and difference, respectively.

Source: By the author

The two last images show that all impulsive noise was removed, but the borders of BEC was damaged too. It is important to remember this in results analysis.

If we zoom the second image in, it is possible to see on top left the periodic noise still in image (figure 11).

Figure 11 – Zoom in filtered image top left, showing the periodic noise without and with contrast enhanced, respectively.

Source: By the author

As these waves appear to have the same frequency, we can use the well known method of frequency domain filter using, for example, Fourier Transform.⁴⁷ The figure 12 shows the original image and its Fourier Transform:

Figure 12 – Original image without impulsive noise and its Fourier Transform.

Source: By the author

It is possible to see white spots in two approximately orthogonal lines (one almost vertical and other almost horizontal), from which the most significant ones are in center, so it was made an automated band-stop filter based on region intensity decay, which results are shown below in figure 13

Figure 13 – In first row are shown original image without impulsive noise at left, filtered image as described above at right and difference between them at center. In second row are shown the FFT at left, it after the frequency domain filter process at right and brightn difference between images at center.

Source: By the author

The algorithm explained can be found in file *filter_auto.m* and its dependences. The main source code is copied at [B.2](#).

After image is filtered, it is important to cut it around region of interest, because large images require much more processing time and memory. At first the BEC position is found based in center of mass position (*findCM.m* and section [B.4](#)), as shown in figure [14](#).

Figure 14 – BEC image with found center of mass highlighted.

Source: By the author

The determination of region (square) size is based in mean intensity decay in real

space domain in relation to previous found center (figure 15) and is in file *cut_BEC.m*, copied at B.3:

Figure 15 – Relative decay of mean intensity as function of cut region side.

Source: By the author

This full algorithm transform the previous original image as shown in figure 16 below:

Figure 16 – Cut algorithm result.

Source: By the author

Note that the cut processing as it is finished, the resulting image will be squared and the BEC sample will be in center of it.

4 TOMOGRAPHIC METHOD TO DETERMINE $n(r)$

Now the image preprocessing is explained, we can understand the main method used in this work: the tomographic method to make a three-dimensional reconstruction of atoms cloud. But before the main method explanation, let's understand the well know tomographic method used in medicine.

4.1 Tomographic method applied to medical diagnose

The Computed Tomography method (CT) is based on Abel Transform, that is a particular case of Radon Transform⁴⁸ (1917). These transformations deal with obtaining absorption projections from a two-dimensional (2D) sample and after obtaining the sample recover from projections.

The thesis⁴⁹ makes a complete math study of Radon transform and its generalizations. In this work we will use just the classical filtered inverse Radon transform. The Radon transform is defined mathematically⁵⁰ by:

$$\mathcal{R}(t, \theta) = \int_{t, \theta} f(x) dx \quad (4.1)$$

where $\mathcal{R}(t, \theta)$ is the $f(x)$ Radon transform in direction θ as function of t , what is nothing more than the absorption projection in direction θ . It can be rewritten as function of image to be reconstructed:

$$\mathcal{R}(t, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x, y) \delta(x \cos(\theta) + y \sin(\theta) - t) dx dy \quad (4.2)$$

where $I(x, y)$ is the image to be reconstructed and $\delta(x)$ is the Dirac delta function.

The way to recover the computational version of original sample is to invert the Radon Transform, but to emphasize this challenge difficulties, see the inverse characteristics below:

- From the **indeterminacy theorem**,^{51,52} the inverse Radon Transform is unique in the sense of being unambiguously determined by an infinite number of projections, but not for any finite number of images.
- From the **reconstruction instability issue**, small variations in projections cause large variations in inverse transform.

Computationally the reconstruction is done by for each θ summing, for each t , the value for each pixel of reconstructed image in line $x \cos(\theta) + y \sin(\theta) = t$, which is called *backprojection method*.

It was said that CT uses a 2D transform, but how it is possible to reconstruct 3D human body in the medical CT application using a 2D transform? It is done in slices, as shown in figure 17. The CT is used in each of the slices independently, recovering it. At end, they are "stacked", making the entire 3D human body up.

Figure 17 – CT for medical diagnosis.

Source: FDA⁵³

Note that for each slice, the system (X-ray source and detectors) may rotate, which makes it possible to acquire as many projections as possible to reduce the indeterminacy theorem effect. The figure 18 shows an example of the issue related to few projections use, where from the original structure, a circle, the method "recovered" an star due to using only three projections.

Figure 18 – Attempt to recover (left gray image) a circle (right gray image) using only three projections.

Source: SMITH⁵⁴

In a simple way, we can imagine this inverse transform as a discrete system modeled as a system of equations, where each slice of size $n \times n$ pixels has n^2 variables to be found.

If we take each of n_p projections of size n as n equations, we have $n_p n$ equations. If $n_p < n$, we certainly have an indeterminate system.

Now imagine the 3D BEC system, a 3D system modeled as an $n \times n \times n$ equation system, that is, n^3 variables with only $n_p n^2$ equations, with $n_p \ll n$ ($n_p \leq 3$ and $100 < n < 200$). We have a system of equations that in addition to being undetermined is difficult to estimate.

Besides all this, our system do not have slices. We have three not orthogonal crossing axes. In this way we can think of a method analogous to CT but 3D, that is, instead of one-dimensional projections of a 2D slice, we have 2D projections (absorption images) of a 3D system (atomic cloud), as shown in figure 19.

Figure 19 – 3D analogy from classical CT.

Source: CONESA-BOJ⁵⁵

We know that this small number of projections will lead to a result extremely different from original cloud. To lessen this issue, we will use an interactive method to try to correct the deviation caused by the small number of projections. This correction is made after the CT already described and starting from its result so that it is enhanced iteratively based in a comparison between original projections and the recovered sample projections in each process step.

This method is called **Iterative Back-Projection** (IBP)⁵⁶ and consists of, at each iteration, taking the original equivalent projections of recovered sample and calculate a correction factor, that may be additive (difference) or multiplicative (ratio) among other. The correction factor has the projections format and is used in same way, except for one detail: they are added to or multiplied by the recovered sample for quoted corrections type. The figure 20 shows an IBP application in medical CT.

Figure 20 – IBP application in medical CT.

Source: MA⁵⁷

Although for some iterations the image become much better, it is not always true. If we repeat the processes many times, the sample is well evidenced, but the noise is enhanced too, so we have to determine empirically the optimal number of iterations and it was done in our work.

It is important to show that the first image of this example is already with well recovered curves and the IBP is used for recover the contrast. Although this method helps us with our issue, it is not perfect for us due to small number of images.

4.2 BEC tomographic method

Now we will understand how one can use a similar method to reconstruct the BEC three-dimensional cloud. The ideal scenario would be to use the original three-dimensional atom distribution still trapped to study it but, as seen before, the cloud needs to be released to take the pictures. After image is processed, we need to extract data from it.

It is essential pay attention on how the images are taken to understand the method. When cloud is trapped and manipulated as desired, the trap is turned off. After a time of flight (TOF) interval (which may vary depending on study being done), the light beam passes through the atomic cloud towards the camera, as shown in figure 21, as to form the cloud shadow on camera, that is, the darker the higher the atom density integrated in that region.

Figure 21 – Experiment scheme in image acquisition moment.

Source: CEPOF⁵⁸

In BEC-1 experiment, we have three imaging paths, which are two in horizontal plane (plane Oxy) with an angular distance of 45° of each other and one in vertical axis (axis z). Figure 22 presents a diagram showing the allowed projections in the experiment.

Figure 22 – Example of BEC images with a vortex.

Source: VIVANCO⁵⁹

Even though the three paths cannot be used simultaneously, we can run the experiment, keeping the conditions and take the three images. With all of them, the idea of algorithm is to walk the same path (shown in figure 23), but backwards, that is, from each camera, we project the image in an spatial region so that the spatial intersection roughly describes the original experimental cloud. In essence, it corresponds to the computer tomography (or Inverse Radon Transform), already explained, but with too few images.

Figure 23 – Beams related to three cameras at same time.

Source: By the author

Now we will explain, in a simplified way, how the algorithm works. Please, find the full MATLAB[®] code in file *r3D_reconstruction.m* and its dependencies (it will not be copied in appendices due to its size). It is implemented creating an spatial region (represented by a 3D matrix) where “density beams” (light intensity proportional to atom density) are directed to so that we have the beams crossing in BEC region as shown in figure 23. Such targeting is done by rotating the reconstructed cloud inside the used matrix in order to do the projection that will be added stay parallel to one of region faces (*r3D_rotate.m* and section B.5) and then, summing image copies in all of region slices parallel to considered face as shown in figure 24.

Figure 24 – Region rotation to leave the projection parallel to one of the region faces.

Source: By the author

In order to organize source codes and optimize the program by reducing data size, two structures were made. The first one is related to each camera axis and has five properties:

- **ratio:** Number proportional to pixel size for camera

- **theta**: Azimuthal angle
- **phi**: Polar angle
- **img_a**: Image rotation angle
- **tit**: Title of plots related to this camera

This type of structure is created by *r3D_config.m* and is copied in section B.6. The second one is the BEC structure, which is made by two properties:

- **dens**: Three-dimensional matrix with values proportional to density in each point.
- **scale**: Three-entries array, where each entry shows the factor the equivalent cloud dimension must be multiplied to get the real size cloud. This implementation optimizes both processor and memory uses but loses in resolution, but it is necessary because we will study clouds of many sizes at same time. In this way we can study clouds orders of magnitude different without any problem in major part of processes.

This type of structure is created by *r3D_cloud.m* and is copied in section B.7.

Now we will use the proposed algorithm in theoretical clouds to test it. The figure 25 shows the projections (size of 181 pixels) of a hypothetical ideal three-dimensional Thomas-Fermi symmetric cloud.

Figure 25 – Hypothetical ideal three-dimensional BEC projections.

Source: By the author

We have, however, an issue. Three cylinders intersection is not even close a three-dimensional Thomas-Fermi as a BEC is. This issue is supported by the **indeterminancy theorem**. It is known that the larger the number of projections the better the result, but we have only three of them. The issue is also supported by **reconstruction instability issue**. The second line of figure 26 show result projections (in each of our three directions)

on crossing these three beams, represented by hypothetical projections for a perfectly spherical BEC, shown in its first line.

Figure 26 – Recovered projections immediately after beam crossing, with no previous clean.

Source: By the author

To improve this result, we make two more steps. The first one is the already studied IBP, where for each iteration the reconstruction is enhanced by a correction factor, which may be calculated in some ways, from which the main ones are shown in figure 27.

$$f_{mult} = \frac{\text{Image 1}}{\text{Image 2}}$$

$$f_{add} = \text{Image 1} + \text{Image 2}$$

Figure 27 – Main correction factors used in Radon inverse iterative methods.

Source: By the author

In both of them we determine a matrix of correction factors. Once calculated, it is used to enhance the current result by multiplying or summing, respectively, the matrix of factors in same way the original reconstruction, that is, the matrix goes through the 3D region performing the respective operation.

In our work we use both of them. At last, the normalization is corrected by comparing the original projections to the new ones and the cloud is multiplied by a constant

factor. A new result for same theoretical cloud using this method (*r3D_it_reconstruction.m*) is shown in figure 28.

Figure 28 – The first line shows the original projections. The second line shows the result using the iterative correction.

Source: By the author

Another way to see the result is a three-dimensional visualization. Figure 29 shows the original 3D cloud together with the reconstructed 3D cloud and the difference between them, respectively, where axes units are voxel size and color scale is proportional to cloud density. Although the 3D visualization may be an interesting way of looking at cloud (the difference is in BEC limits), it is not so convenient to analyze the result in a critical way. In this way, we will use the figure 30, which shows the mean density (over all directions) as function of cloud radial position. Note that the maximum deviation occurs precisely on border region.

Figure 29 – The original 3D cloud, the reconstructed 3D cloud and the difference between them, respectively.

Source: By the author

Figure 30 – Mean density as function of radial position for original and reconstructed clouds.

Source: By the author

By observing figure 30 above, it is possible to note that for $r < 25$, (60% of peak density) the deviation is too small in relation to original cloud. Figure 31 shows that the relevant deviation appears near the BEC edge, though the accumulated atoms relative deviation is less then 0.1% over all interval. It is important to remember that the edge region, where we have the larger deviation is the region of small density since our BECs are very close to Thomas-Fermi approximation.

Those figure shows us the first limitation of method: as we will review in next chapters, our program cannot reproduce abrupt variations of density derivatives well as shown in figures 29 and 30.

Figure 31 – Deviation between original and reconstructed clouds and accumulated atoms relative deviation, respectively.

Source: By the author

Another way to quantify the deviation is using the following math expression:

$$\varepsilon = \frac{\sum_{\vec{r}} |n_{rec}(\vec{r}) - n_{orig}(\vec{r})|}{\sum_{\vec{r}} |n_{orig}(\vec{r})|} \quad (4.3)$$

which quantifies the **relative absolute value deviation** from original cloud, that is, all positions are important using this formula. For this example the absolute relative deviation was:

$$\boxed{\varepsilon = 13\%} \quad (4.4)$$

But this is just a hypothetical cloud, used just for comparison and not filtered for noise remove since there is no noise in it. Now we will see a real result, recovering the spatial density from three absorption images from BEC-1. Using the previously explained process, the figure 32 shows the 3D plot and 33 shows the original projections and the reconstruction projections.

Figure 32 – 3D reconstruction from a ground state cigar shape BEC.

Source: By the author

Figure 33 – The first line shows the original images from experiment. The second line, for comparison, shows the reconstruction projections in the same directions.

Source: By the author

It is important to note that, the dimension differences are about border reduction due to artificial density created by beam crossing that has to be removed but is very similar to borders density. The center region keeps very similar, showing again, now for a real case, the algorithm efficiency.

5 METHOD APPLICATION TO DETERMINE $n(r)$ IN TOF

Now we will address the problem of recover the *in-situ* atomic density, that is, we wish to know the spatial distribution before it expands. We will then use the Castin-Dum expansion equations as explained in chapter 2 and copied below:

$$\frac{R_r}{R_{r,0}} = \sqrt{1 + \omega_r^2 t^2} \quad (5.1)$$

$$\frac{R_a}{R_{a,0}} = 1 + \frac{\omega_a^2}{\omega_r^2} \left| \omega_r t \operatorname{arctg}(\omega_r t) - \ln\left(\sqrt{1 + \omega_r^2 t^2}\right) \right| \quad (5.2)$$

where t is the time of flight, ω_a is the axial trapping frequency, ω_r is the radial trapping frequency, $R_{a,0}$ and $R_{r,0}$ are the in-situ axial and radial BEC dimensions and R_a and R_r are the axial and radial BEC dimensions after the time t .

Note that if we have the trap frequencies and the time of flight, we can find the original cloud dimensions from recovered cloud dimensions:

$$R_{r,0} = \frac{R_r}{\sqrt{1 + \omega_r^2 t^2}} \quad (5.3)$$

$$R_{a,0} = \frac{R_a}{1 + \frac{\omega_a^2}{\omega_r^2} \left| \omega_r t \operatorname{arctg}(\omega_r t) - \ln\left(\sqrt{1 + \omega_r^2 t^2}\right) \right|} \quad (5.4)$$

where the trap and expanded cloud parameters are given.

Another possible parameter set to start with are two recovered clouds in different TOF, so that we can recover the trap parameters by solving the following equation system in variables ω_i and $R_{i,0}$:

$$\begin{cases} R_{r,1}^2 = R_{r,0}^2 (1 + \omega_r^2 t_1^2) \\ R_{r,2}^2 = R_{r,0}^2 (1 + \omega_r^2 t_2^2) \\ \frac{R_{a,1}}{R_{a,0}} = 1 + \frac{\omega_a^2}{\omega_r^2} \left| \omega_r t_1 \operatorname{arctg}(\omega_r t_1) - \ln\left(\sqrt{1 + \omega_r^2 t_1^2}\right) \right| \\ \frac{R_{a,2}}{R_{a,0}} = 1 + \frac{\omega_a^2}{\omega_r^2} \left| \omega_r t_2 \operatorname{arctg}(\omega_r t_2) - \ln\left(\sqrt{1 + \omega_r^2 t_2^2}\right) \right| \end{cases} \quad (5.5)$$

If we divide the first by the second equations, we have:

$$\omega_r^2 = \frac{R_{r,2}^2 - R_{r,1}^2}{R_{r,1}^2 t_2^2 - R_{r,2}^2 t_1^2} \quad (5.6)$$

And after dividing the third by the fourth equations, we have:

$$\omega_a^2 = \frac{\left(\frac{R_{r,2}^2}{R_{r,1}^2} - 1\right) \left(\frac{R_{a,2}}{R_{a,1}} - 1\right)}{\left(t_2^2 - \frac{R_{r,2}^2}{R_{r,1}^2} t_1^2\right) \left[\left| \omega_r t_2 \arctg(\omega_r t_2) - \ln\left(\sqrt{1 + \omega_r^2 t_2^2}\right) \right| - \frac{R_{a,2}}{R_{a,1}} \left| \omega_r t_1 \arctg(\omega_r t_1) - \ln\left(\sqrt{1 + \omega_r^2 t_1^2}\right) \right| \right]} \quad (5.7)$$

Now it is easy to determine the original cloud sizes:

$$\begin{cases} R_{r,0} = \frac{R_{r,i}}{\sqrt{1 + \omega_r^2 t_i^2}} \\ R_{a,0} = \frac{R_{a,i}}{1 + \frac{\omega_a^2}{\omega_r^2} \left| \omega_r t_i \arctg(\omega_r t_i) - \ln\left(\sqrt{1 + \omega_r^2 t_i^2}\right) \right|} \end{cases} \quad (5.8)$$

It is important to note that all the *in-situ recover* is analytic, without approximations, which makes the method exact, that is, if we had a perfect 3D expanded cloud representation, we would recover the *in-situ* cloud without error due this method, just due to the Castin Dum³³ approximations. This recover is made in file `r3D_insitu.m` and copied in section B.8.

Here it is important to remember how the cloud is represented (section B.7). It is a 3D meshgrid with local density in each position. Besides that, when the cloud is expanded, for memory and processing economy, it was added a three element vector with current scale so that the cloud is resized only when it is really needed. This method changes only the scale so that the numerical error is negligible.

The figure 34 shows the complete process, since the experiment cloud until the recovered *in-situ* cloud.

Figure 34 – Complete process, since the experiment cloud until the recovered *in-situ* cloud.

Source: By the author

When we apply it to a theoretical spherical Thomas-Fermi cloud, we obtain the results shown in figure 35 below, where dimension axes are in arbitrary units and color axis is proportional to cloud density:

Figure 35 – Theoretical spherical Thomas-Fermi cloud *in-situ* recover. The first line shows the original cloud projections and the second line shows the recovered cloud projections.

Source: By the author

The plot in figure 36 shows the deviation of each part of the algorithm, defined in equation 4.3.

The **In-Situ** part is just the the *in-situ* recover, from 3D expanded cloud for the original size. The **3D recover** part is the recover from the three images from expanded cloud to its 3D version. At least, the **complete recover** is both previous parts together.

Figure 36 – Relative deviation from each part of algorithm

Source: By the author

When we apply this process to the experimental images, the results are shown in figure 38.

Figure 37 – 3D in-situ recover of experimental sample of BEC-1.

Source: By the author

From axes profiles, using polynomial fitting, it is possible to determine the Thomas-Fermi radii in units of pixel size:

Figure 38 – Cloud radii in units of pixel size from axes profiles fitting.

Source: By the author

Which give us the following aspect ratio:

$$\frac{R_a}{R_r} \approx \frac{R_x}{\sqrt{R_y R_z}} \approx 10.2 \quad (5.9)$$

For BEC images used here, the trap frequencies were:

$$\vec{f} = (f_x, f_y, f_z) = (21, 187, 189) Hz \quad (5.10)$$

We know that radius is inversely proportional to frequency (equation 2.28). If we calculate the ratios between frequency and radius in each direction, we can verify its precision:

$$\frac{R_a}{R_r} \approx \frac{\sqrt{f_y f_z}}{f_x} \approx 9.0 \quad (5.11)$$

This result is in accordance with the deviation graph in figure 36 as expected.

6 APPLICATIONS IN CASES OF TOMOGRAPHIC DEFECTS

Until now we have applied the method to well behaved Thomas-Fermi clouds. In this chapter we will use the algorithm to analyze theoretical clouds with vortex (topological defect) and bubbles of some formats.

In this way a function was created to generate clouds in some formats and with some types of internal structures (*r3D_create_BEC.m*). It may be used to create theoretical clouds in Thomas-Fermi, Gaussian and Bimodal formats in desired size and position. Besides that, for any of these formats, it is possible to insert internal structures in any of them as bubble (spherical) and vortex (linear and ring-shaped), which we will study in this chapter.

In the same way made in chapter 5 for a well behaved Thomas-Fermi cloud, we will study these ones, but now just with well behaved theoretical clouds, made by *r3D_create_BEC* function.

6.1 Linear vortex

For linear vortex creation, the following part of function is executed:

```

121  if any(strcmpi(varargin,'vortex'))           % If 'vortex' option in input
122      i = find(strcmpi(varargin,'vortex'),1,'first'); % 'vortex' option position in input
123      if nargin<i+1 || ischar(varargin{i+1})      % If few parameters
124          q = 0;                                   % Default theta = 0
125      else                                         % Else
126          q = varargin{i+1};                       % Reads theta
127      end
128      if nargin<i+2 || ischar(varargin{i+1}) || ischar(varargin{i+2}) % If few parameters
129          phi = 0;                                 % Default phi = 0
130      else                                         % Else
131          phi = varargin{i+2};                     % Reads phi
132      end
133      [X,Y,Z] = meshgrid(-Rq(1):Rq(1),-Rq(2):Rq(2),-Rq(3):Rq(3)); % Grid
134      n.dens = r3D_rotate(n.dens,q,phi,'crop');    % Rotates BEC
135      rho = sqrt(X.^2+Y.^2)/hl;                   % Distance to axis in healing length
136      n.dens = n.dens.*rho./sqrt(2+rho.^2);        % Add vortex
137      n.dens = r3D_rotate(n.dens,q,phi,'crop','invert'); % Rotates BEC back
138      [n.dens,-] = r3D_equal_size(n.dens,zeros(s+1)); % Cuts BEC in defined size
139  end

```

In this fragment, it is assumed that there is already a BEC density created, where the vortex will be placed. Between lines 121 to 134 we have the input processing, ambient creation and BEC preprocess. In line 135, it is created a mathematical function of (x, y)

that represents the distance to vortex symmetry axis ($z = 0$) in units of healing length (hl). In line 136 it is used the following vortex approximation math expression:⁶⁰

$$f/f_0 = \frac{x}{\sqrt{2 + x^2}} \quad (6.1)$$

where f/f_0 is the function that the original density must be multiplied to obtain the desired structure and x is the distance to vortex axis.

In this way, and using the same algorithm previously explained for reconstruction of BEC from the projection in cameras axis, we have the comparison between original and reconstructed clouds projections in figure 39:

Figure 39 – Comparison between projections from original cloud (first line) and recovered projections (second line).

Source: By the author

Although a non-zero central density in recovered vortex, the result is visually very similar. For a better comparison, the figure 40 shows the original and recovered clouds in a three-dimensional view with axes in units of voxel size.

Figure 40 – Comparison between original cloud (left) and recovered (right) clouds with vortex.

Source: By the author

Another way to compare the results is using the linear (1D) profiles. The first line of figure 41 shows the orthogonal (x , y and z) axis profiles and the second line shows the central vertical profiles of projections images in cameras axis directions both in density arbitrary units versus length arbitrary units.

Figure 41 – Comparison between cloud axis profiles (first line) and projections images central profiles (second line) of a cloud with vortex.

Source: By the author

The differences seen in first line show an algorithm limit: it cannot faithfully

represent abrupt spatial derivative variations on density profile, which causes a not zero density in vortex core, as seen above.

Last but not least, for a more formal comparison, we will use the deviation expression from equation 4.3. At first the previously presented error quantitative measure:

$$\varepsilon = 15\% \quad (6.2)$$

And at second, the deviation as a function of radial position, which is big just in vortex region, since the original density in core must be almost zero but in reconstruction it is not. Out of region it is no more then 10%:

Figure 42 – Deviation between original and reconstructed clouds with vortex and accumulated atoms relative deviation, respectively.

Source: By the author

Note that the greatest deviation is located near the edge of BEC, which is not so significant, since the main structures to be studied are in the center of BEC.

6.2 Spherical bubble

For spherical bubble creation, the following part of function is executed:

```

140  if any(strcmpi(varargin,'bubble'))           % If 'bubble' option in input
141      [X,Y,Z] = meshgrid(-Rq(1):Rq(1),-Rq(2):Rq(2),-Rq(3):Rq(3)); % Grid
142      i = find(strcmpi(varargin,'bubble'),1,'first'); % 'bubble' option position in input
143      if nargin<i+1                             % If few parameters
144          f = 0.1;                             % Default bubble size ratio to BEC size
145      else                                       % Else

```

```

146         f = varargin{i+1};                                % Reads bubble size ratio to toal size
147     end
148     R = f*min(s);                                           % Bubble radius
149     rho = sqrt(X.^2+Y.^2+Z.^2);                             % Grid in spherical coordinates
150     n.dens(rho<R) = 0;                                       % No density inside bubble
151 end

```

Between lines 140 and 147 we have the input process and ambient creation. In line 148, the bubble radius is setted as a fraction of the BEC grid size. In line 149, it is created a mathematical function of (x, y, z) that represents the distance to the origin. In line 150, the sphere interior equation is used to zero the density into the sphere, that is, the densities in all position where the following condition is true is set to zero:

$$x^2 + y^2 + z^2 < R^2 \quad (6.3)$$

where R is the bubble radius. In a similar way shown, we have the comparison between original and reconstructed clouds projections in figure 43:

Figure 43 – Comparison between projections from original cloud (first line) and recovered projections (second line) both in pixel size units.

Source: By the author

Although a non-zero central density in recovered bubble, the result is visually very similar. Since bubble is an internal structure, it is impossible to see it in three-dimensional figure, so we will not analyze it. Instead, the figure 44 shows a central slice comparison.

Figure 44 – Comparison between cloud central slice in original and recovered 3D.

Source: By the author

In this representation we observe that where in original cloud there was zeros of density, in reconstructed cloud the density is greater than zero. Another way to see the same is from profiles similar to the vortex case. The first line of figure 45 shows the orthogonal (x , y and z) axis profiles and the second line shows the central linear profiles of projected images in the cameras axis directions both in density arbitrary units versus length arbitrary units.

Figure 45 – Comparison between cloud axis profiles (first line) and projections images central profiles (second line) of a cloud with an spherical bubble.

Source: By the author

Here note that although the first line profiles discrepancy are large, the second line profiles and previous figure projections discrepancies are small. This happens because the first line profiles is an internal measure (axes) and the second line profiles and absorption images are external measures (images). Since in experiment we have just external measures

(absorption images), its similarity shows the algorithm efficiency, but it shows a new algorithm limitation too: it is impossible to recover internal structures that have no external evidence in an external picture. It will be visible in deviation analysis.

Let's use math to compare clouds. At first the previously presented error quantitative measure (equation 4.3):

$$\varepsilon = 23\% \quad (6.4)$$

And at second, the deviation as function of radial position, that is so high but only in bubble region, since that the density should be zero but the reconstruction cannot represent it, as seen before:

Figure 46 – Deviation between original and reconstructed clouds with an spherical bubble and accumulated atoms relative deviation, respectively.

Source: By the author

Note that the largest deviation appears in the bubble region, which is not good, since it is the region of interest. This graph displacement of deviation graph happens precisely because of the exclusively internal structure as quoted before, but it is important to remember that the number deviation decreases significantly out of this region. It is much greater then in vortex situation precisely because the vortex has a connection to the outside, what changes the absorption images.

6.3 Ring-shaped vortex

For ring-shaped vortex creation, the following part of function is executed:

```

152  if any(strcmpi(varargin,'ring')) % If 'ring' option in input
153      [X,Y,Z] = meshgrid(-Rq(1):Rq(1),-Rq(2):Rq(2),-Rq(3):Rq(3)); % Grid
154      i = find(strcmpi(varargin,'ring'),1,'first'); % 'ring' option position in input
155      if nargin<i+1 || ischar(varargin{i+1}) % If few parameters
156          f = 0.1; % Default ring size ratio to BEC size
157      else % Else
158          f = varargin{i+1}; % Reads ring size ratio to BEC size
159      end
160      if nargin<i+2 || ischar(varargin{i+1}) || ischar(varargin{i+2}) % If few parameters
161          q = 0; % Default axis theta = 0
162      else % Else
163          q = varargin{i+2}; % Reads axis theta
164      end
165      % If few parameters
166      if nargin<i+3 || ischar(varargin{i+1}) || ischar(varargin{i+2}) || ischar(varargin{i+3})
167          phi = 0; % Default axis phi = 0
168      else % Else
169          phi = varargin{i+3}; % Reads axis phi
170      end
171      R = f*min(s); % Ring radius
172      n.dens = r3D_rotate(n.dens,q,phi,'crop'); % Rotates BEC
173      rho = sqrt(X.^2+Y.^2); % Distance to axis
174      n.dens((R-rho).^2+Z.^2<=hl^2) = 0; % Toroid of thickness healing length
175      n.dens = r3D_rotate(n.dens,q,phi,'crop','invert'); % Rotates BEC back
176  end

```

Between lines 152 and 170 we have the input process and ambient creation. In line 171, the bubble radius is set as a fraction of the BEC grid size. In line 173, it is created a mathematical function of (x, y, z) that represents the distance to ring axis ($z = 0$). In line 174, the ring interior equation is used to zero all density in the sphere, that is, all position where the following condition is true is set to zero:

$$\left(R - \sqrt{x^2 + y^2}\right)^2 + z^2 \leq \xi^2 \quad (6.5)$$

where R is the ring radius and ξ is the healing length, the vortex dimension. If we analyze this expression, it is easy to verify that for a $z = z_0$ constant, there are three possibilities:

$$\left|R - \sqrt{x^2 + y^2}\right| \leq \sqrt{\xi^2 - z_0^2} \quad (6.6)$$

- $|z_0| < |\xi| \Rightarrow \left(R - \sqrt{\xi^2 - z_0^2}\right)^2 \leq x^2 + y^2 \leq \left(R + \sqrt{\xi^2 - z_0^2}\right)^2 \Rightarrow$ Region between circumferences with radii $R \pm \sqrt{\xi^2 - z_0^2}$;
- $z_0 = \xi \Rightarrow x^2 + y^2 = R^2 \Rightarrow$ One circumference of radius R ;
- $|z_0| > |\xi| \Rightarrow$ No region.

In analogue way of previous section, we have the comparison between original and reconstructed clouds projections in figure 47:

Figure 47 – Comparison between projections from original cloud (first line) and recovered projections (second line) from cloud with ring-shaped vortex.

Source: By the author

Although there is a non-zero density in recovered vortex, the result is visually very similar. As vortex in an internal structure, it is impossible to see it in three-dimensional figure, so we will not analyze it. Instead, the figure 48 shows a central slice comparison.

Figure 48 – Comparison between cloud central slice in original and recovered 3D, respectively.

Source: By the author

In this representation we observe, as seen in bubble case, where in original cloud there was zeros of density, in reconstructed cloud the density is greater than zero. Now, please observe the linear profiles in figure 49. First line shows the orthogonal (x , y and z) axis profiles and the second line shows the central linear profiles of projected images in the cameras axis directions both in density arbitrary units versus length arbitrary units.

Figure 49 – Comparison between cloud axis profiles (first line) and projections images central profiles (second line) of a cloud with a ring-shaped vortex.

Source: By the author

Let's now use math to compare clouds. Using the previously presented error quantitative measure (equation 4.3), we have:

$$\varepsilon = 23\% \quad (6.7)$$

And at second, the deviation as function of radial position, which is high in central part but reduces significantly:

Figure 50 – Deviation between original and reconstructed clouds with a ring-shaped vortex and accumulated atoms relative deviation, respectively.

Source: By the author

Note that the largest deviation appears in the central region, which is not good, since all BEC is harmed. Note that the bubble deviation is much larger than ring vortex deviation. It happens because of structure dimension: the bubble is larger than vortex ring and in all directions. In both of them the deviation origin is the same: no BEC outside connection, as quoted before, which is shown by projections similarity.

7 THERMODYNAMIC PARAMETERS OF $n(r)$ *INSITU*

When a gas is trapped by an arbitrary external potential, the usual volume and pressure cannot be defined, since there are no walls, and the standard thermodynamics equations cannot be written.

Local density approximation is usually used for studies with inhomogeneous traps, but a global thermodynamics could be a good alternative approach.

A new theory that describe thermodynamics of gases trapped in harmonic trap macroscopically was developed in 2005.^{43,61,62} Three years later it was generalized to an arbitrary external potential.⁶³

7.1 Theory description

Although there are no walls in harmonic potential, we know that the larger the trap frequencies, the smaller the sample, that is, the smaller the “volume”. In this way we can define the **volume parameter** $\mathcal{V}$, for harmonic trap as inverse of all of trap frequencies:

$$\mathcal{V} \equiv \frac{1}{\omega_x \omega_y \omega_z} = \frac{1}{\bar{\omega}^3} \quad (7.1)$$

where $\bar{\omega} = \sqrt[3]{\omega_x \omega_y \omega_z}$ is the geometric mean of trap frequencies, remembering that condensation temperature depends on this quantity too.

Now we can determine the **pressure parameter**. We will use the canonical ensemble and Helmholtz free energy F :

$$F(N, \beta, \mathcal{V}) = -\frac{1}{\beta} \ln \left(\text{Tr} e^{-\beta \hat{H}} \right) \quad (7.2)$$

where $\hat{H} = \hat{K} + \hat{U} + \hat{V}_{ext}$ and $\beta = \frac{1}{k_B T}$ is the system coldness, so we have the pressure parameter Π :

$$\Pi = - \left(\frac{\partial F}{\partial \mathcal{V}} \right)_{N, \beta} = \frac{2}{3\mathcal{V}} V_{ext} \quad (7.3)$$

which may be rewritten as:

$$\Pi = \frac{m}{3\mathcal{V}} \int n(\vec{r}) (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) d^3r \quad (7.4)$$

Obviously $\mathcal{V}$ and Π do not have the same unit of the standard volume and pressure, though they have the same physical properties: the volume parameter is related to space

occupied by BEC and the pressure parameter is related to the work necessary to BEC stay in equilibrium. Another important property that is kept is that the product between these two variables is that product between then has unit of energy:

$$\Pi\mathcal{V} = \frac{2}{3}V_{ext} \quad (7.5)$$

Using the theoretical Thomas-Fermi density, we can find the product $\Pi\mathcal{V}$ analytically:⁴³

$$\Pi\mathcal{V} = \frac{1}{21}mN(\omega_x^2 R_x^2 + \omega_y^2 R_y^2 + \omega_z^2 R_z^2) \quad (7.6)$$

where R_i is the TFR in direction i . Using the theoretical Gaussian density, we can find the product $\Pi\mathcal{V}$ analytically:⁴³

$$\Pi\mathcal{V} = \frac{1}{3}mN(\omega_x^2 \sigma_x^2 + \omega_y^2 \sigma_y^2 + \omega_z^2 \sigma_z^2) \quad (7.7)$$

where σ_i is the standard deviation in direction i . Finally, we have the connection between these two variables for ideal gas, which makes the theory more consistent and give us an equation of state. Let's consider the grand potential Ω of an ideal Bose gas:

$$\Omega(\mu, T, \mathcal{V}) = -\mathcal{V}k_B T \left(\frac{k_B T}{\hbar} \right)^3 g_4(\beta\mu) \quad (7.8)$$

where $g_n(y) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{e^{x-y} - 1} dx$ is the polylogarithm function.

For number we have:

$$N_{ideal} = \left(\frac{\partial \Omega}{\partial \mu} \right)_{\mathcal{V}, T} = \mathcal{V} \left(\frac{1}{\beta \hbar} \right)^3 g_3(\beta\mu) \quad (7.9)$$

For ideal pressure parameter we have:

$$\Pi_{ideal} = - \left(\frac{\partial \Omega}{\partial \mathcal{V}} \right)_{\mu, T} = \frac{N_{ideal} k_B T}{\mathcal{V}} \frac{g_4(\beta\mu)}{g_3(\beta\mu)} \quad (7.10)$$

$$\boxed{\Pi\mathcal{V} = \frac{g_4(\beta\mu)}{g_3(\beta\mu)} N k_B T} \quad (7.11)$$

7.2 Methods

Now we will calculate some thermodynamics parameters, including these above using the 3D reconstructed BEC and compare to the values we calculate nowadays directly from images.

At first, we will see the definition used for each thermodynamics parameter:

1. **Number:**

$$N = \int n(\vec{r}) d^3\vec{r} \quad (7.12)$$

- 3D: We have to discretize the integral:

$$N = \sum_{\vec{r}} n(\vec{r}) (dr)^3 \quad (7.13)$$

where dr is linear spacial step, which is set to pixel size, that is 1 in matrices indexes:

$$N = \sum_{\vec{r}} n(\vec{r}) \quad (7.14)$$

- Images: In the same way we have to discretize the integral, but now in just two directions, since the densities are integrated over the other direction

$$N = \sum_{(x,y)} n(x,y) (dr)^2 \quad (7.15)$$

where dr is analogous to 3D case:

$$N = \sum_{(x,y)} n(x,y) \quad (7.16)$$

2. **Temperature:** The formal way to obtain it is through the thermal cloud fitting assuming Maxwell-Boltzmann distribution,⁶⁴ which lead us to the following expression for BEC temperature:⁶⁵

$$T = \frac{m}{2k_B} \cdot \frac{\omega_i^2 \sigma_i^2}{1 + \omega_i^2 t^2} \quad (7.17)$$

where ω_i is the trapping frequency in direction i , σ_i is the thermal cloud half dimension in direction i and t is the time-of-flight.

We can therefore obtain an approximate temperature to test our method (just orders of magnitude) based on time the cloud takes to expand, that is, let's take the difference between mean radius of expanded and of in-situ clouds and divide it by time of flight to get mean velocity and energy equipartition to get temperature:

$$T = \frac{m}{k_B} \left(\frac{\langle R_{tof} \rangle - \langle R_0 \rangle}{t} \right)^2 \quad (7.18)$$

where R_{tof} is the BEC dimension after the expansion, R_0 is the TFR of trapped cloud. The clouds (2D or 3D) radii are found by mean distance from its particles to BEC center of mass. Nowadays, in 2D images we assume $R_0 \approx R_{TF}$. The constants m (particle mass) and k_B (Boltzmann's constant) are shown in table 1 in appendix A. Using their values, we get to:

$$T \sim 10^{-2} \left(\frac{\langle R_{tof} \rangle - \langle R_0 \rangle}{tof} \right)^2 \quad (7.19)$$

in units of SI.

3. Pressure-Volume parameters equation:

$$\Pi\mathcal{V} = \frac{m}{3} \int n(\vec{r})(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) d^3\vec{r} \quad (7.20)$$

- 3D: We have to discretize the integral:

$$\Pi\mathcal{V} = \frac{m}{3} \sum_{\vec{r}} n(\vec{r})(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) (dr)^3 \quad (7.21)$$

where dr has already been defined:

$$\Pi\mathcal{V} = \frac{m}{3} \sum_{\vec{r}} n(\vec{r})(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \quad (7.22)$$

7.3 Results

Before starting analyze the results, it is important to remember that the length units here are pixels size and we will convert it to SI units just when it is necessary.

That said, let's start the analyses in same order as described in section 7.2 for both theoretical and experimental clouds:

1. Number:

- Theoretical Thomas-Fermi symmetric cloud
 - generated 3D

Figure 51 – Generated three-dimensional Thomas-Fermi BEC cloud.

Source: By the author

Number of particles in original sample:

$$N_0 = \sum_{\vec{r}} n(\vec{r}) = 152681 \quad (7.23)$$

– images 2D

Figure 52 – Generated three-dimensional Thomas-Fermi BEC cloud projections.

Source: By the author

Number measured directly from projections:

$$N_{2D} = \sum_{(x,y)} n(x,y) = 152681 \quad (7.24)$$

Deviation:

$$\varepsilon = \frac{N_{2D} - N_0}{N_0} = 0\% \quad (7.25)$$

– recovered 3D

Figure 53 – Recovered three-dimensional Thomas-Fermi BEC cloud.

Source: By the author

$$N_{3D} = \sum_{\vec{r}} n(\vec{r}) = 150072 \quad (7.26)$$

$$\varepsilon = \frac{N_{3D} - N_0}{N_0} = -1.7\% \quad (7.27)$$

- Experimental cloud

Figure 54 – Experimental cigar-shape cloud

Source: By the author

– images 2D: number measured directly from projections.

$$N_{2D} = \sum_{(x,y)} n(x,y) = 1707 \quad (7.28)$$

- recovered 3D: number measured from 3D reconstruction.

$$N_{3D} = \sum_{\vec{r}} n(\vec{r}) = 1775 \quad (7.29)$$

- Deviation:

$$\varepsilon = \frac{N_{3D} - N_{2D}}{N_{2D}} = 3.4\% \quad (7.30)$$

2. Temperature:

$$T \sim 10^{-2} \left(\frac{\langle R_{tof} \rangle - \langle R_{situ} \rangle}{tof} \right)^2 \quad (7.31)$$

- Experimental cloud
 - images 2D

$$T_{2D} \sim 10^{-2} \left(\frac{\langle R_{tof} \rangle - R_{TF}}{tof} \right)^2 \sim 100 \text{ nK} \quad (7.32)$$

- recovered 3D

$$T_{3D} \sim 10^{-2} \left(\frac{\langle R_{tof} \rangle - \langle R_{situ} \rangle}{tof} \right)^2 \sim 100 \text{ nK} \quad (7.33)$$

The order of magnitude is as expected.⁴²

3. Pressure-Volume parameters equation:

$$\Pi\mathcal{V} = \frac{m}{3} \int n(\vec{r}) (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) d^3\vec{r} \quad (7.34)$$

In this case we will calculate just with 3D reconstruction using an script given in file *r3D_pVNkT.m*:

$$\Pi\mathcal{V} = \frac{m}{3} \sum_{\vec{r}} n(\vec{r}) (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \approx 2.7669 \times 10^{-27} J \quad (7.35)$$

Comparing to Theoretical expressions, previously presented, we have similar values:

- Thomas-Fermi sample (equation 7.6):

$$\Pi\mathcal{V} \approx 3.6710 \times 10^{-28} \quad (7.36)$$

- Gaussian sample (equation 7.7):

$$\Pi\mathcal{V} \approx 2.5697 \times 10^{-27} \quad (7.37)$$

Our value is between the two theoretical values and in the same order of magnitude. Finally let's obtain the equation of state for our BEC:

$$\frac{\Pi\mathcal{V}}{Nk_BT} = \frac{1}{Nk_BT} \cdot \frac{m}{3} \sum_{\vec{r}} n(\vec{r})(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \approx \frac{2.7669 \times 10^{-27} J}{2.7884 \times 10^{-27} J} \approx 0.9923 \quad (7.38)$$

$$\boxed{\Pi\mathcal{V} \approx 0.9923 Nk_BT} \quad (7.39)$$

The found value is in theoretical value interval, as expected:

$$0.9 < \frac{g_4 \left(\frac{\mu}{k_BT} \right)}{g_3 \left(\frac{\mu}{k_BT} \right)} < 1 \quad (7.40)$$

8 CONCLUSION

In this work, we aimed to create a complete MATLAB[®] library for BEC absorption images processing, but always focused on making an algorithm that could recover any *in-situ* cloud from the three non-orthogonal absorption images.

In chapter 3, we needed to really get usable images from any initial large images, containing both periodic and impulsive noises. A minimum filter was used to remove the impulsive noise, and the Fourier notch filter was used for remove periodic noise satisfactorily. After an algorithm based on density profile decay was used for cutting BEC. All this process is automatic and takes just a few seconds.

Once the images were prepared, in chapter 4, the main method, based on compute tomography in analogy to 2D signals (images) method, was proposed, explained and analyzed both for theoretical clouds, where it is possible to measure method deviation, and for experimental clouds, where it is only possible to verify projections similarity. For theoretical cloud we got an “relative absolute value deviation” of just 13% and focused on BEC edges, where density is small. This edges deviation occurs because of the background removal and is reflected in experimental clouds reconstruction, where the greater dimension length is reduced.

Once the method was proposed and tested for well behaved samples, in chapter 5, we presented an application for method: an algorithm to determine the in-situ density from the 3D reconstructed atomic distribution based on Castin-Dum theory. In theoretical algorithm evaluation, the mean deviation of whole process, from images to *in-situ* 3D cloud was 14%. In experimental test, we got expected cloud size order of magnitude and aspect ratio.

In chapter 6, “tomographic defects” (linear vortex, vortex ring and spherical bubble) were inserted in theoretical BECs and the deviations were analyzed. As expected the deviation were greater then the Thomas-Fermi well behaved cloud in all of them. The linear vortex presented the smallest deviation among them (15% against 23%), what happens because of their external ends, what makes absorptions images an appropriate measure method for this defect against the other two ones studied, that do not have external contact.

At last the algorithms were applied, in chapter 7, using the global parameters thermodynamics for BECs in the experimental images. Number comparison shown 3.4% deviation of 3D measure in relation to 2D images measure. For temperature, it was used an approximated formula and for both 2D images and 3D reconstruction it was found 100 nK as order of magnitude, as expected. The product between pressure and volume

parameters (IIV) was determined using just 3D reconstruction, which when compared to $Nk_B T$ term, previously obtained, gave $g_4(\beta\mu)/g_3(\beta\mu) \approx 0.9923$ as result, which is in possible interval (between 0.9 and 1), and more: it gives us the result of same order of magnitude of theoretical values for Thomas-Fermi and Gaussian samples and between them.

Summarizing the main results, we have:

- Automatic absorption images processing;
- Automatic reconstruction process from three non-orthogonal absorption images with small theoretical deviation of 13% and mainly in borders;
- Automatic *in-situ* cloud recover with 14% mean error with expected cloud size order of magnitude and aspect ratio;
- Small deviation for linear vortex inclusion (13% in total), which have external ends, but not so small deviations for vortex ring and spherical bubble (23% in total) due to no external ends;
- Number of particles measure consistent with images measure;
- Temperature estimated in order 100 nK, as expected;
- Pressure-volume parameters state equation valid.

Now we will propose some future perspectives. First of all, it would be interesting if a math research in generalizations and optimizations of Radon transform could be done to improve method results.

Changing the focus to costs and execution time, one disadvantage of our work is the used programming language (MATLAB[®]) in the way that it is a proprietary language, which generates high costs, and it is slow if compared with other languages, so it would be interesting if the code could be ported to a language like python3, which is free of costs, faster then and of the same level of complexity of the used language.

Besides the language changing, another way to make the program faster is use GPU (Graphics Processing Unit) for processing.

Another interesting change related to programming methods is the use of machine learning to increase automation and performance of reconstruction.

After we propose more physical applications to method as calculation of 3D momentum map, correlations, detection of internal structures presence (not its properties with the results like they are now), etc.

At last, one original objective of this project application is determine if a BEC set of images might have attention or could be deleted during a day of experiments, so it would be interesting to have a graphical interface, which could be used by people who have no programming knowledg and is relatively simple to make in both MATLAB® and Python3.

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Appendix

APPENDIX A – PHYSICS CONSTANTS

Table 1 – Physics constants

Symbol	Name	Value	Unit
k_B	Boltzmann Constant	$1.38064852 \times 10^{-23}$	$\text{m}^2\text{kg}\cdot\text{s}^{-2}\text{K}^{-1}$
h	Planck Constant	$6.62607004 \times 10^{-34}$	$\text{m}^2\text{kg}\cdot\text{s}^{-1}$
$\hbar$	Reduced Planck Constant	$1.054571800 \times 10^{-34}$	$\text{m}^2\text{kg}\cdot\text{s}^{-1}$
m	Rubidium mass	$1.4192261 \times 10^{-25}$	kg

Source: By the author

APPENDIX B – MATLAB CODES

This appendix show the main codes of the work. The entire folder can be found at bit.ly/36zvTGX.

B.1 CastinDum simulation

```

1 function [Rt_x,Rt_r,wx,wr,dt,tof] = CastinDum(varargin)
2 % CastinDum - BEC dimensions evolution as function of TOF
3 % [Rt_x,Rt_y] = CastinDum(options) returns axial and radial dimensions as
4 % function of time
5 % [_ ,wx,wr] = CastinDum(options) returns trap angular frequencies
6 % [_ ,dt] = CastinDum(options) returns used time step
7 %
8 % OPTIONS
9 % 'dt',dt - time step in ms
10 % 'num',n - points number
11 % 'freq',f - trap frequencies in 3 directions (in Hz)
12 % 'show' - shows results
13 % 'size',s - BEC size
14
15 %% CopyRight
16 %{
17 Determines typical (realistic) numbers to BEC1 condensates
18 Created by Gustavo Telles, October/2013
19 Last update: Sep/2015
20 Updated by Andre Smaira, Jul/2019
21 %}
22
23 %% constants
24 hbar = cte('hbar'); % Planck constant in IS
25 mRb = cte('weight','Rb-87'); % 87Rb mass in IS
26 kB = cte('kB'); % Boltzmann cte in IS
27 ms = cte('prefix','mili'); % milisecond time unit
28 um = cte('prefix','micro'); % micrometer size unit
29 nK = cte('prefix','nano'); % nanokelvin temp. unit
30 As = cte('as','Rb-87'); % 87Rb scatt. s-wave length
31
32 %% Default values
33 f = [21.1,189.3,187.1]; % Default trap frequencies (BEC1, Hz)
34 n = 25; % Default number of points
35 T12 = [0,inf]; % Default time interval
36
37 %% Parameters process
38 if any(strcmpi(varargin,'freq')) % Trap frequencies
39     i = find(strcmpi(varargin,'freq'),1,'first');
40     f = varargin{i+1};
41 end
42 if any(strcmpi(varargin,'num')) % Number of points
43     i = find(strcmpi(varargin,'num'),1,'first');
44     n = varargin{i+1};
45 end

```

```

46 if any(strcmpi(varargin,'tof')) % Time interval
47     i = find(strcmpi(varargin,'tof'),1,'first');
48     T12 = varargin{i+1};
49 end
50 dt = (T12(2)-T12(1))/(n-1); % Default time step (in ms)
51 if any(strcmpi(varargin,'dt')) % Time step (ms)
52     i = find(strcmpi(varargin,'dt'),1,'first');
53     dt = varargin{i+1};
54 end
55 if any(strcmpi(varargin,'size')) % BEC size
56     i = find(strcmpi(varargin,'size'),1,'first');
57     R = varargin{i+1};
58 end
59 T12(2) = min(T12(1)+(n-1)*dt,T12(2));
60
61 %% Article
62 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
63 % Castin-Dum expansion of BECs
64 % According to:
65 % Y. Castin and R. Dum, Phys. Rev. Lett. 77, 5315 (1996)
66 % But prefer to use the syntax from:
67 % F. Dalfovo et al., Rev. Mod. Phys. 71, 463 (1999),
68 % see pgs 487,488
69 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
70 fx = f(1); fy = f(2); fz = f(3); % Article frequencies notation
71 lbd = fx/sqrt(fz*fy); % Lambda
72 wr = 2*pi*sqrt(fz*fy); wx = 2*pi*fx; w_ho = 2*pi*(fx*fy*fz)^(1/3); % Angular frequencies
73 tof = T12(1)*ms:dt*ms:T12(2)*ms; % TOF array
74 if tof(1)>0
75     tof = [0,tof];
76 end
77 n = numel(tof);
78 tr = wr.*tof; %tr = wx.*tof; % Non-dimensional time
79 br = sqrt(1+tr.^2); % RADIAL Castin-Dum exp term
80 bx = 1 + abs( lbd.^2.*( tr.*atan(tr) - log(br) ) ); % AXIAL Castin-Dum exp term
81 ar_ho = sqrt( hbar/(mRb*wr) ); ax_ho= sqrt( hbar/(mRb*wx) );
82 a_ho = sqrt( hbar/(mRb*w_ho) ); % HO length
83
84 %% Resultados
85 NO = 1e5; % Number of condensed Atoms
86 u = (hbar*w_ho/2)*(15*NO*As/a_ho)^(2/5); % Chemical potential
87 R0_r = sqrt( 2*u/(mRb*wr^2) ); Rt_r = R0_r*br; % Radial Thomas Fermi (BEC) radius
88 R0_x = sqrt( 2*u/(mRb*wx^2) ); Rt_x = R0_x*bx; % Axial TF radius (after tof)
89 %Vcld = 4*pi/3*Rtf_x*Rtf_r^2;
90 T0 = (hbar*w_ho/kB)*( NO/zeta(3) )^(1/3); % Standard critical temperature
91 %
92 T = (mRb/(2*kB))*( Rt_r(n).^2 + Rt_x(n).^2 )./(tof(n)^2); % Final temperature
93 res = {'N_0', 'R0_x (um)', 'R0_r (um)', 'A ratio', 'T_0 (nK)', 'T (nK)';...
94     NO, R0_x/um, R0_r/um,R0_r/R0_x, T0/nK, T/nK};
95
96 if any(strcmpi(varargin,'size'))
97     k = max([R(1)/Rt_x(1),R(2)/Rt_r(1)]); % Ratio
98     Rt_x = Rt_x*k; Rt_r = Rt_r*k; % BEC resize
99 %     wx = wx/k; wr = wr/k;
100 %     disp(size(f))
101 %     fprintf('f %.2f\nR %.2f\n',f(1),Rt_x(2));
102 %     [f] = bs(k,R,f,n,T12);

```

```

103 %      [Rt_x,Rt_r] = CastinDum('freq',f,'num',n,'tof',T12);
104 end
105
106 if any(strcmpi(varargin,'show'))
107     disp(res'); % Shows results
108     create_figure; % Creates figure
109     plot(tof/ms,Rt_x/um,'b',tof/ms,Rt_r/um,'r'); % Plots graph
110     title('Time-of-flight TF radii evolution'); % Title
111     xlabel('TOF (ms)'), ylabel('Thomas-Fermi radii (\mu m)'); % Axis labels
112     legend('Radial axis','Axial axis','Location','southeast'); % Legends
113     printable_fig;
114 end
115 end

```

B.2 Noise filter

```

1 function [I,R] = filter_auto(varargin)
2 % filter_auto filters fringe and background noise from a BEC image automatically
3 % [I,r] = filter_auto(filename) filters an image from file filename. I is
4 % the filtered image. r is the cutted rectangle
5 % [I,r] = filter_auto(img) filters the image img. I is the filtered image.
6 % r is the cutted rectangle
7 % _ = filter_auto(_,options)
8 % options:
9 % - 'notcut' : do not cut the image around the BEC
10 % - 'show' : shows the results
11 %
12 % See also OPEN_IMAGE, BLUR, CUT_BEC.
13     if nargin<0 || nargin>3 % Verify the arguments number
14         error('Invalid number of arguments');
15     end
16     if nargin>1 && ~ischar(varargin{2})
17         error('Parameters format changed') % Function parametes changed
18     end
19     show = any(strcmpi(varargin,'show')); % Flag to show results
20     cut = ~any(strcmpi(varargin,'notcut')); % Flag to cut image
21     if nargin<1 % Test options
22         filename = 'teste.tif'; % Test file
23         show = 1; % Show results
24     else % Else
25         filename = varargin{1}; % Gete parameters
26     end
27     if ischar(filename) % If the image be the filename
28         I = open_image(filename); % Open image
29     else % Else
30         I = filename; % Copy image
31     try
32         I = im2double(I); % Convert image to double class
33     catch
34     end
35     end
36     if show % If setted to show results
37         img = imadjust(I); % Copy image
38         create_figure; % Creates maximized figure
39         % First line: Original image
40         subplot(231); imshow(cut_BEC(img)); title('Original Image');

```

```

41  end
42  I = ordfilt2(I,9,ones(3,3));           % Maximum filter with 1 neighborhood
43  I = fftshift(fft2(I));                 % Image FFT
44  vI = log(abs(I)+1);                    % FFT visualization
45  vI = vI/max(vI(:));                    % Normalize
46  [sI1,sI2] = size(I);                  % FFT size
47  [X,Y] = meshgrid(1:sI2,1:sI1);        % meshgrid
48  x0 = (sI2+1)/2;                        % FFT center
49  y0 = (sI1+1)/2;
50  d2 = (X-x0).^2 + (Y-y0).^2;           % Distance to FFT center
51
52  cI = I;
53  H = ones(sI1,sI2);                    % Filter
54  ind = (d2<20^2);                       % Zeroes in center and...
55  vI(ind) = 0; cI(ind) = 0;
56  ind = (abs(cI)<0.5*trimmean(abs(cI(:)),0.5)); % ...in dark regions of visualization
57  vI(ind) = 0;
58  vI = ordfilt2(vI,1,ones(10));          % Minimum filter in visualization
59  H(vI>0) = 0;                           % Remove peaks of FFT
60  if show                                % If shown flag
61      oldI = I;                           % Copy old FFT
62  end
63
64  I = I.*H;                              % Filter FFT
65  if show                                % If show flag
66      % Filtered FFT
67      subplot(236); fftshow(I); title('Filtered FFT');
68  end
69  I = abs(ifft2(I));                     % Filtered image
70  if cut                                  % If cut flag
71      [I,r] = cut_BEC(I);                  % Cut image around BEC
72      if show
73          [img,~] = cut_BEC(img,r);
74      end
75  end
76  M = min([min(I(1,:)),min(I(:,1))]);
77  m = min(I(:));
78  I = imadjust(I,[m,M]);
79  img = imadjust(img);
80  if show                                % If show flag
81      % Filtered image
82      subplot(231); imshow(img); title('Original Image');
83      subplot(233); imshow(I); title('Filtered Image');
84      % Difference
85      I = BEC_centralize(I);
86      img = BEC_centralize(img);
87      [I,img] = r3D_equal_size(I,img);
88      subplot(232); imshow(I-img); title('Difference');
89      subplot(235); imshow(0.5*imadjust(I-img)); title('Brighten Difference');
90      clear img
91      % Second line: Original FFT
92      subplot(234); fftshow(oldI); title('Fourier Transform');
93      clear oldI
94  end
95 end

```

B.3 Cut BEC

```

1 function [F,r] = cut_BEC(F,r)
2 % cut_BEC automatically cuts an square with the BEC from a larger image
3 % I = cut_BEC(F) cut BEC from image F. I is the BEC cut.
4 % [_,r] = cut_BEC(_) r = [i_min j_min delta_i delta_j] is the rectangle
5 % cutted
6 % _ = cut_BEC(_,r) r = [i_min j_min delta_i delta_j] defines the rectangle
7 % to be cutted
8 %
9 % See also OPEN_IMAGE, FINDCENTER.
10 th = 5e-3; % Threshold
11 if nargin>2 % If more than 2 parameters
12     error('Invalid number or arguments'); % Error
13 end
14 if nargin>1 % If rectangle especified
15     [s1,s2] = size(F); % Image size
16     if r(1)<1 || r(1)>s1 || r(1)+r(3)<1 || r(1)+r(3)>s1 || ...
17         r(2)<1 || r(2)>s2 || r(2)+r(4)<1 || r(2)+r(4)>s2 || ...
18         sum(mod(r,1.0))~=0 % If invalid rectangle
19         error('Invalid rectangle'); % Error
20     end
21     F = F(r(1):(r(1)+r(3)),r(2):(r(2)+r(4))); % Else, cuts image
22     return; % Ends function
23 end
24 if nargin<1 % If no arguments
25     F = 'teste.tif'; % Test
26 end
27 if ischar(F) % If image is filename
28     F = filter_auto(F); % Opens, filters and cut image
29 end
30 [i,j] = findCM(F,'fix'); % Find center of mass
31 var = 2; % Initial half-side
32 var0 = 30; % Default half-side
33 aux = F; % Image copy
34 if ~BEC_white(aux) % If BEC is black
35     aux = BEC_complement(aux); % Complements image
36 end
37 [s1,s2] = size(aux); % Image size
38 dv = 2; % Image size variation by step
39 ds = -1; % Intensity sum variation by step
40 S = 1; % Initial sum
41 v = []; % Relative sum variation array
42 l = []; % Cut half-side array
43 while i-var-dv>0 &&... % While valid upper cut
44     i+var+dv<=s1 &&... % And valid bottom cut
45     j-var-dv>0 &&... % And valid left cut
46     j+var+dv<=s2 &&... % And valid right cut
47     -ds/S>th % And sum relative decrease larger than 0,1%
48     S = sum(sum(aux((i-var):(i+var),(j-var):(j+var)))); % Sums region density
49     ds = S-sum(sum(aux((i-var-dv):(i+var+dv),(j-var-dv):(j+var+dv)))); % Sum variation
50     l(end+1) = var; % Current half-side
51     v(end+1) = -ds/S; % Sum relative decrease
52     var = var + dv; % Increases side
53 end
54 for k=1:fix(0.01*s1) % Expands cut in 0.01*s1
55     S = sum(sum(aux(max(1,i-var):min(s1,i+var),max(1,j-var):min(s2,j+var))));

```

```

56     ds = S-sum(sum(aux(max(1,i-var-dv):min(s1,i+var+dv),max(1,j-var-dv):min(s2,j+var+dv))));
57     l(end+1) = var;
58     v(end+1) = -ds/S;
59     var = var + dv;
60 end
61 try
62     var = l(find(v<max([th*10,v(end-1)]),1,'first'));% Selects decrease upper to 1%
63 catch
64 end
65 if isempty(var)
66     var = var0;
67 else
68     var = 2*max([var,var0]);
69 end
70 clear aux
71 i1 = max(1,i-var);
72 i2 = min(s1,i+var);
73 j1 = max(1,j-var);
74 j2 = min(s2,j+var);
75 [i,j] = findCM(F(i1:i2,j1:j2),'fix');
76 i1 = max(1,i1+i-var);
77 i2 = min(s1,i1+i+var);
78 j1 = max(1,j1+j-var);
79 j2 = min(s2,j1+j+var);
80 F = F(i1:i2,j1:j2);
81 if nargout>1
82     r = [i1,j1,i2-i1,j2-j1];
83 end
84 end

```

B.4 Find center of mass

```

1 function [cy,cx,cz] = findCM(varargin)
2 % findCM calculates BEC center of mass
3 % [cy,cx] = findCM(img) calculates BEC center of mass from image img
4 % _ = findCM(_,options)
5
6 % options:
7 % - 'fix' : rounds position
8 % - 'show' : shows results plots
9 %
10 % See also OPEN_IMAGE, FILTER_AUTO, BLUR, CUT_BEC.
11
12 if nargin>3
13     error('Invalid arguments number');
14 end
15 show = any(strcmpi(varargin,'show'));
16 if nargin<1
17     img = 'teste.tif';
18     show = 1;
19 else
20     img = varargin{1};
21 end
22 tofix = any(strcmpi(varargin,'fix'));
23 if ischar(img)
24     [img,-] = filter_auto(img);

```

```

25     end
26
27     if ismatrix(img)                                % If image is 2D
28         [ny,nx] = size(img);                        % Get image size
29     end
30
31     if ~BEC_white(img)                              % If black BEC
32         nimg = BEC_complement(img);                 % Complements image
33     else                                             % If white BEC
34         nimg = img;                                % Just copy it
35     end
36
37     nimg = blur(nimg,50);                          % Blurs image
38     if ismatrix(nimg)                              % If image is 2D
39         s = regionprops(imbinarize(nimg),'Centroid'); % Get image 2D centroid
40     else                                           % Else
41         s = regionprops3(nimg>0.5*max(nimg(:)),'Centroid'); % Get 3D centroid
42     end
43     centroids = cat(1, s.Centroid);                % Concatenate centroids vector
44     try
45         cx = centroids(1,1);                      % Main centroid x
46         cy = centroids(1,2);                      % Main centroid y
47         if tofix                                  % If rounding flag
48             cx = round(cx);                      % Rounds x result
49             cy = round(cy);                      % Rounds y result
50         end
51         if nargout==3                             % If there are 3 outputs
52             cz = centroids(1,3);                  % Main centroid z
53             if tofix                              % If rounding flag
54                 cz = round(cz);                  % Rounds z result
55             end
56         end
57     catch                                           % If error
58         [cy,cx] = size(nimg);                     % Center is gotten from...
59         cx = 0.5*cx; cy = 0.5*cy;                 % ... geometric center
60         if tofix                                  % If rounding flag
61             cx = round(cx);                      % Rounds x result
62             cy = round(cy);                      % Rounds y result
63         end
64     end
65
66
67     if show && ismatrix(img)                       % If setted to show results and image is 2D
68         create_figure;                             % Creates maaximized figure
69         hold on                                    % Fixes plot axes
70         imagesc(img)                             % Draws image
71         title('Image with found Center of Mass')   % Plot title
72         colormap(gray)                           % Grayscale
73         axis off                                  % Removes axis
74         plot(cx,cy,'wo')                         % Draws center of mass position
75         set(gca, 'YDir', 'reverse');              % Reverses axis to image format
76         hold off                                  % Unfixes plot axes
77         xlim([1 nx]); ylim([1 ny]);               % Axes limits
78         printable_fig;                            % Change graph font and lines sizess
79     end
80 end

```

B.5 Rotate 3D matrix

```

1 function [V] = r3D_rotate(varargin)
2 % r3D_rotate rotates a 3D structure
3 % V = r3D_rotate(V,conf) rotates as specified in conf
4 % V = r3D_rotate(V,theta) rotates an angle theta (degrees) related to
5 % z axis in plne xz
6 % V = r3D_rotate(_,phi) rotates an angle phi (degrees) related to
7 % x axis in plane xy
8 % V = r3D_rotate(_,'crop') cuts V in original size
9 % V = r3D_rotate(_,'invert') rotates inversely
10
11 if nargin<2 % If only one argument
12     V = varargin{1}; % Output equal input
13     return; % Finish function
14 end
15 V = varargin{1}; % Matrix
16 q = varargin{2}; % Theta or struct
17 img_a = 0;
18 if isstruct(q)
19     phi = q.phi;
20     img_a = q.img_a;
21     q = q.theta;
22 elseif q==0 % If theta equal to zero
23     V = varargin{1}; % Output equal input
24     return; % Finish function
25 elseif nargin<3 % If not given phi
26     phi = 0; % phi = 0
27 else % Else
28     phi = varargin{3}; % Set phi value
29 end
30 if any(strcmpi(varargin,'crop')) % If option 'crop'
31     t = 'crop'; % Set type to 'crop'
32 else % Else
33     t = 'loose'; % Set it to 'loose'
34 end
35 if isempty(img_a)
36     img_a = 0;
37 end
38 V = inpaint_nans3(V); % Interpolate nan with neighborhood
39 if any(strcmpi(varargin,'invert')) % If option 'invert'
40     V = imrotate3(V,img_a,[0,0,1],'cubic',t); % Rotates around Z axis
41     V = imrotate3(V,q,[0,1,0],'cubic',t); % Rotates around Y axis
42     V = imrotate3(V,phi,[0,0,1],'cubic',t); % Rotates around Z axis
43 else
44     V = imrotate3(V,-phi,[0,0,1],'cubic',t); % Rotates around Z axis
45     V = imrotate3(V,-q,[0,1,0],'cubic',t); % Rotates around Y axis
46     V = imrotate3(V,-img_a,[0,0,1],'cubic',t); % Rotates around Z axis
47 end
48
49 end

```

B.6 Config structure

```

1 function [s] = r3D_config(varargin)
2 % r3D_conf creates the settings struct used in 3D built

```

```

3 % s = r3D_conf creates empty settings struct
4 % s = r3D_conf(ratio,theta,phi,img_a,tit) creates an struct with ratio,
5 % direction (theta,phi), image angular position img_a and label tit.
6 % s = r3D_conf([ratio,theta,phi,img_a])
7     if nargin==1 && strcmp(varargin{1},'default')
8         s = {r3D_conf(250,90,0,0,'Ioffe Axis (x)'),...           % X axis
9               r3D_conf(249,90,45,90,'45 Degrees Axis (x45)'),... % X45 axis
10              r3D_conf(314,0,0,45,'Vertical Axis (z)'),...      % Z axis
11            };
12     return;
13 end
14 s = struct('ratio',[],'theta',[],'phi',[],'img_a',[],'tit',''); % Empty struct
15 if nargin>=2                                     % If at least 2 arguments
16     s.ratio = varargin{1};                         % Fills the first two fields
17     s.theta = varargin{2};
18 end
19 if nargin>=3                                     % If 3 or more
20     s.phi = varargin{3};                          % Fills the 3rd field
21 end
22 if nargin>=4                                     % If 4 or more
23     s.img_a = varargin{4};                        % Fills the 4th field
24 end
25 if nargin>=5                                     % If 5 or more
26     s.tit = varargin{5};                         % Fills the 5th field
27 end
28 if nargin==1                                     % If only one
29     n = numel(varargin{1});                      % Number o elements of 1st parameter
30     if n>=1                                       % If at least one element
31         if strcmp(class(varargin{1}),class(s)) % And it is struct
32             if isfield(varargin{1},'ratio')    % If there is field ratio
33                 s.ratio = varargin{1}.ratio;   % Copies field ratio
34             end
35             if isfield(varargin{1},'theta')    % If there is field theta
36                 s.theta = varargin{1}.theta;   % Copies field theta
37             end
38             if isfield(varargin{1},'phi')      % If there is field phi
39                 s.phi = varargin{1}.phi;       % Copies field phi
40             end
41             if isfield(varargin{1},'img_a')    % If there is field img_a
42                 s.img_a = varargin{1}.img_a;   % Copies field img_a
43             end
44             if isfield(varargin{1},'tit')      % If there is field tit
45                 s.tit = varargin{1}.tit;       % Copies field tit
46             end
47             if s.theta==0
48                 s.phi = 0;
49             end
50             return;                             % Ends function
51         end
52         s.ratio = varargin{1}(1);               % Else fills 1st field
53     end
54     if n>=2                                       % If at least 2 elements
55         s.theta = varargin{1}(2);              % Fills 2nd field
56     end
57     if n>=3                                       % If at least 3 elementso
58         s.phi = varargin{1}(3);                % Fills 3rd field
59     end

```

```

60         if n>=4                                % If at least 4 elements
61             s.img_a = varargin{1}(4);            % Fills 4th field
62         end
63         if n>=5                                % If at least 5 elements
64             s.tit = varargin{1}(5);              % Fills 5th field
65         end
66     end
67     if s.theta==0
68         s.phi = 0;
69     end
70 end

```

B.7 Cloud structure

```

1 function [s] = r3D_cloud(varargin)
2 % r3D_cloud creates a cloud struct used in 3D reconstruction
3 % s = r3D_cloud creates an empty cloud struct (unit scale)
4 % s = r3D_cloud(n) creates a cloud struct of density n and unit scale
5 % s = r3D_cloud(n,scale) creates a cloud struct of density n and scale
6 % scale
7     s = struct('dens',[],'scale',[1,1,1]);      % Empty struct
8     if nargin>=1                                % If there is at least 1 parameter
9         s.dens = varargin{1};                  % Fills density field
10    end
11    if nargin>=2                                % If there are at least 2 parameters
12        s.scale = varargin{2};                  % Fills scale field
13    end
14    if numel(s.scale)==1                        % If scale is scalar
15        s.scale = [s.scale,s.scale,s.scale];    % Set as constant array
16    end
17 end

```

B.8 In-situ recover

```

1 function [V1] = r3D_insitu(varargin)
2 % r3D_insitu recovers a 3D cloud in-situ
3 % V = r3D_insitu(V1,t1,V2,t2) given two clouds and their TOFs, recovers the
4 % in-situ cloud
5 % V = r3D_insitu(V1,t1,wr,wa) given a cloud, its TOF and axial and radial
6 % frequencies, recovers the in-situ cloud
7 %
8 % See also FINDCENTER.
9
10 %% Default values
11 if nargin>5                                % Verifies arguments number
12     error('Invalid number of arguments')      % Error
13 end
14 V1 = varargin{1};                            % First argument
15 t1 = varargin{2};                            % Second argument
16 V2 = varargin{3};                            % Third argument
17 t2 = varargin{4};                            % Fourth argument
18 d1 = [NaN,NaN,NaN];                          % Mean distance to cloud center
19 da1 = Inf;                                   % Axial distance 1
20
21 %% Preprocess

```

```

22  if isa(V1,'struct') ...           % If first argument is a cloud
23      && isscalar(t1) ...           % And second is scalar
24      && isa(V2,'struct') ...       % And third is a cloud
25      && isscalar(t2) ...           % And fourth is scalar
26      d1 = d_mean(V1);              % Mean distance to axes in cloud 1
27      d2 = d_mean(V2);              % Mean distance to axes in cloud 2
28      da1 = max(d1);                % Maximum mean distance to cloud 1 axes
29      dr1 = sqrt(prod(d1)/da1);      % Cloud 1 mean radius
30      da2 = max(d2);                % Maximum mean distance to cloud 2 axes
31      dr2 = sqrt(prod(d2)/da2);      % Cloud 2 mean radius
32      wr = sqrt((dr2^2-dr1^2)/(dr1^2*t2^2-dr2^2*t1^2)); % Angular radial frequency
33      br1 = sqrt(1+wr^2*t1^2);       % Cloud 1 radial b (CastinDum article)
34      br2 = sqrt(1+wr^2*t2^2);       % Cloud 2 radial b (CastinDum article)
35      T1 = wr*t1;                   % Non-dimensional time from cloud 1
36      T2 = wr*t2;                   % Non-dimensional time from cloud 2
37      r2r1 = dr2/dr1;               % r2/r1
38      a2a1 = da2/da1;               % a2/a1
39      % Angular axial frequency
40      wa = sqrt((r2r1^2-1)*(a2a1-1)/...
41          (t2^2-r2r1^2*t1^2)/...
42          (abs(T2*atan(T2)-log(br2))-a2a1*abs(T1*atan(T1)-log(br1)))...
43      );
44  elseif isa(V1,'struct') ...       % Else if first argument is cloud
45      && isscalar(t1) ...           % And second argument is scalar
46      && isscalar(V2) ...           % And third argument is scalar
47      && isscalar(t2) ...           % And fourth argument is scalar
48      wr = V2;                      % Angular radial frequency
49      wa = t2;                      % Angular axial frequency
50  else                               % If none of these parameters settings
51      error('Invalid arguments')    % Invalid parameters
52  end
53  br = sqrt(1+wr^2*t1^2);           % Radial b (CastinDum article)
54  ba = 1+(wa/wr)^2*abs(wr*t1*atan(wr*t1)-log(br)); % Axial b (CastinDum article)
55  if da1==d1(1)                     % If axial dimension is the first one
56      V1.scale = V1.scale./[ba,br,br]; % Resizes BEC
57  elseif da1==d1(2)                 % If axial dimension is the second one
58      V1.scale = V1.scale./[br,ba,br]; % Resizes BEC
59  else                               % If axial dimension is the third one
60      V1.scale = V1.scale./[br,br,ba]; % Resizes BEC
61  end
62  V1.dens = BEC_centralize(V1.dens); % Centralizes BEC
63 end
64
65 % Mean distance to cloud center
66 function [d] = d_mean(V)
67     sc = V.scale; V = V.dens;
68     [s1,s2,s3] = size(V);          % Grid size
69     [x,y,z] = meshgrid(1:s1,1:s2,1:s3); % Creates grid
70     [i,j,k] = findCM(V,'fix');      % Cloud center
71     dx = sum(V(:).*abs(x(:)-j))/sum(V(:)); % X axis distance
72     dy = sum(V(:).*abs(y(:)-i))/sum(V(:)); % Y axis distance
73     dz = sum(V(:).*abs(z(:)-k))/sum(V(:)); % Z axis distance
74     d = [dx,dy,dz].*sc;             % Returns distances
75 end

```


APPENDIX C – CONDENSATE FRACTION FOR HOMOGENEOUS GAS (GAS IN A BOX)

In this appendix, let's consider a BEC trapped in a box of sides $2a_x$, $2a_y$ and $2a_z$, that is, the three-dimensional harmonic potential is:

$$V(x, y, z) = \begin{cases} 0 & |x| < a_x \text{ and } |y| < a_y \text{ and } |z| < a_z \\ \infty & \text{otherwise} \end{cases} \quad (\text{C.1})$$

For this physical system, we have the quantum states defined by the quantum numbers n_x , n_y and n_z :

$$\varepsilon(n_x, n_y, n_z) = \frac{\pi^2 \hbar^2}{8m} \left[\left(\frac{n_x}{a_x} \right)^2 + \left(\frac{n_y}{a_y} \right)^2 + \left(\frac{n_z}{a_z} \right)^2 \right] \quad (\text{C.2})$$

From previous equation, let's determine the number of particles G with energy at most $\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z$. Let's consider the energies gap too small in relation to the system energy, what turns possible to approximate to integral, with this coefficient:

$$\frac{d\varepsilon_i}{dn_i} = \frac{\pi^2 \hbar^2}{4ma_i^2} n_i = \frac{\pi \hbar}{(2m)^{1/2} a_i} \sqrt{\varepsilon_i} \equiv \frac{\sqrt{\varepsilon_i}}{\alpha_i} \quad (\text{C.3})$$

where $\alpha_i \equiv \frac{(2m)^{1/2}}{\pi \hbar} a_i$.

We have the integrals:

$$G(\varepsilon) = \alpha_x \alpha_y \alpha_z \int_0^\varepsilon \frac{1}{\sqrt{\varepsilon_z}} \int_0^{\varepsilon - \varepsilon_z} \frac{1}{\sqrt{\varepsilon_y}} \int_0^{\varepsilon - \varepsilon_z - \varepsilon_y} \frac{1}{\sqrt{\varepsilon_x}} d\varepsilon_x d\varepsilon_y d\varepsilon_z \quad (\text{C.4})$$

$$G(\varepsilon) = \alpha_x \alpha_y \alpha_z \int_0^\varepsilon \frac{1}{\sqrt{\varepsilon_z}} \int_0^{\varepsilon - \varepsilon_z} \frac{1}{\sqrt{\varepsilon_y}} 2(\varepsilon - \varepsilon_z - \varepsilon_y)^{1/2} d\varepsilon_y d\varepsilon_z \quad (\text{C.5})$$

Let's substitute $u = \sqrt{\varepsilon - \varepsilon_z} \sin(\theta) \Rightarrow du = \frac{1}{2\sqrt{\varepsilon - \varepsilon_z}} d\varepsilon_y$:

$$G(\varepsilon) = 4\alpha_x \alpha_y \alpha_z \int_0^\varepsilon \frac{1}{\sqrt{\varepsilon_z}} \int_0^{\sqrt{\varepsilon - \varepsilon_z}} (\varepsilon - \varepsilon_z - u^2)^{1/2} du d\varepsilon_z \quad (\text{C.6})$$

Let's substitute $u = \sqrt{\varepsilon - \varepsilon_z} \sin(\theta) \Rightarrow du = \sqrt{\varepsilon - \varepsilon_z} \cos(\theta) d\theta$:

$$G(\varepsilon) = 4\alpha_x \alpha_y \alpha_z \int_0^\varepsilon \frac{1}{\sqrt{\varepsilon_z}} \int_0^{\frac{\pi}{2}} (\varepsilon - \varepsilon_z) [1 - \sin^2(\theta)]^{1/2} \cos(\theta) d\theta d\varepsilon_z \quad (\text{C.7})$$

$$G(\varepsilon) = 4\alpha_x\alpha_y\alpha_z \int_0^\varepsilon \frac{1}{\sqrt{\varepsilon_z}} \int_0^{\frac{\pi}{2}} (\varepsilon - \varepsilon_z) \cos^2(\theta) d\theta d\varepsilon_z \quad (\text{C.8})$$

To continue, we must use the double-angle identity:

$$\cos(2\theta) = 2\cos^2(\theta) - 1 \Rightarrow \cos^2(\theta) = \frac{1 + \cos(2\theta)}{2} \quad (\text{C.9})$$

When we substitute it on integral, we get to:

$$G(\varepsilon) = 2\alpha_x\alpha_y\alpha_z \int_0^\varepsilon \frac{1}{\sqrt{\varepsilon_z}} \int_0^{\frac{\pi}{2}} (\varepsilon - \varepsilon_z) [1 + \cos(2\theta)] d\theta d\varepsilon_z \quad (\text{C.10})$$

$$G(\varepsilon) = 2\alpha_x\alpha_y\alpha_z \int_0^\varepsilon \frac{\varepsilon - \varepsilon_z}{\sqrt{\varepsilon_z}} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_0^{\frac{\pi}{2}} d\varepsilon_z \quad (\text{C.11})$$

$$G(\varepsilon) = 2\alpha_x\alpha_y\alpha_z \int_0^\varepsilon \left(\varepsilon \varepsilon_z^{-1/2} - \varepsilon_z^{1/2} \right) \cdot \frac{\pi}{2} d\varepsilon_z = \pi\alpha_x\alpha_y\alpha_z \int_0^\varepsilon \left(\varepsilon \varepsilon_z^{-1/2} - \varepsilon_z^{1/2} \right) d\varepsilon_z \quad (\text{C.12})$$

$$G(\varepsilon) = \pi\alpha_x\alpha_y\alpha_z \left(2\varepsilon \varepsilon^{1/2} - \frac{2}{3} \varepsilon^{3/2} \right) = \frac{4}{3} \pi\alpha_x\alpha_y\alpha_z \varepsilon^{3/2} \quad (\text{C.13})$$

$$G(\varepsilon) = \frac{4}{3} \pi \frac{(2m)^{1/2} a_x}{\pi \hbar} \frac{(2m)^{1/2} a_y}{\pi \hbar} \frac{(2m)^{1/2} a_z}{\pi \hbar} \varepsilon^{3/2} = \frac{4}{3} \pi \frac{(2m)^{1/2} a_x}{\pi \hbar} \frac{(2m)^{1/2} a_y}{\pi \hbar} \frac{(2m)^{1/2} a_z}{\pi \hbar} \varepsilon^{3/2} \quad (\text{C.14})$$

$$G(\varepsilon) = \frac{4}{3} \frac{(2m)^{3/2}}{\pi^2 \hbar^3} a_x a_y a_z \varepsilon^{3/2} \quad (\text{C.15})$$

Now we must calculate the degeneracy ρ , that is, the number of particles with energy between ε and $\varepsilon + d\varepsilon$:

$$\rho(\varepsilon) = \frac{d}{d\varepsilon} G(\varepsilon) = C_{3/2} \varepsilon^{1/2} \quad (\text{C.16})$$

with $C_{3/2} = \frac{2\pi(2m)^{3/2}}{h^3} (2a_x)(2a_y)(2a_z)$. For the condensate fraction:

$$\frac{N_0}{N} = 1 - \frac{1}{N} \int_0^\infty \left(\frac{1}{e^{\frac{\varepsilon}{k_B T}} - 1} \right) (C_{3/2} \varepsilon^{1/2}) d\varepsilon = 1 - \frac{C_{3/2}}{N} \int_0^\infty \frac{\varepsilon^{1/2}}{e^{\frac{\varepsilon}{k_B T}} - 1} d\varepsilon \quad (\text{C.17})$$

Let's substitute $\epsilon = \frac{\varepsilon}{k_B T} \Rightarrow d\epsilon = \frac{d\varepsilon}{k_B T}$:

$$\frac{N_0}{N} = 1 - \frac{C_{3/2}}{N} (k_B T)^{3/2} \int_0^\infty \frac{\epsilon^{1/2}}{e^\epsilon - 1} d\epsilon = 1 - \frac{C_{3/2}}{N} (k_B T)^{3/2} \Gamma(3/2) \zeta(3/2) \quad (\text{C.18})$$

where $\Gamma(n)$ is the gamma function and $\zeta(n) = \sum_{k=1}^\infty \frac{1}{k^n}$ is the zeta function. At $T = T_C$, we have $N_0 = 0$:

$$0 = 1 - \frac{C_{3/2}}{N} (k_B T_C)^{3/2} \Gamma(3/2) \zeta(3/2) \Rightarrow T_C = \frac{h^2}{4mk_B} \left[\frac{N}{2\pi a_x a_y a_z \Gamma(3/2) \zeta(3/2)} \right]^{2/3} \quad (\text{C.19})$$

$$T_C \approx 1.33 \times 10^{-45} \frac{1}{m} \left[\frac{N}{a_x a_y a_z} \right]^{2/3} \quad (\text{C.20})$$

If we consider a trap of same size as the harmonic trap, we get condensation temperature same as the other trap. Since we have the condensation temperature expression, we can simplify the condensate fraction expression and write it as function of T_C :

$$\boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_C} \right)^{3/2}} \quad (\text{C.21})$$

which has the same behavior of the harmonic trap, but the atoms populate the ground state slower when the temperature decreases.

APPENDIX D – PROOF OF GROSS-PITAEVSKII EQUATION

In this appendix, let's prove the expression of Gross-Pitaevskii Equation. Let's create the Hamiltonian operator:

$$\hat{H} = \sum_{i=1}^N \left[-\frac{\hbar^2}{2m_i} \nabla_i^2 + V(\vec{r}_i) + U_0 \sum_{j<i} \delta(\vec{r}_i - \vec{r}_j) \right] \quad (\text{D.1})$$

where there are three kinds of energy: kinetic energy, external potential and two-particle contact interaction. From that we have the functional N -particle energy:

$$E = \langle \Phi(\vec{r}_1 \cdots \vec{r}_n) | \hat{H} | \Phi(\vec{r}_1 \cdots \vec{r}_n) \rangle = \int d\vec{r}_1 \cdots \int d\vec{r}_n \Phi^*(\vec{r}_1 \cdots \vec{r}_n) \hat{H} \Phi(\vec{r}_1 \cdots \vec{r}_n) \quad (\text{D.2})$$

$$= \int d\vec{r}_1 \cdots \int d\vec{r}_n \Phi^*(\vec{r}_1 \cdots \vec{r}_n) \left\{ \sum_{i=1}^N \left[-\frac{\hbar^2}{2m_i} \nabla_i^2 + V(\vec{r}_i) + U_0 \sum_{j<i} \delta(\vec{r}_i - \vec{r}_j) \right] \right\} \Phi(\vec{r}_1 \cdots \vec{r}_n) \quad (\text{D.3})$$

$$= \sum_{i=1}^N \int d\vec{r}_1 \cdots \int d\vec{r}_n \Phi^*(\vec{r}_1 \cdots \vec{r}_n) \left[-\frac{\hbar^2}{2m_i} \nabla_i^2 + V(\vec{r}_i) + U_0 \sum_{j<i} \delta(\vec{r}_i - \vec{r}_j) \right] \Phi(\vec{r}_1 \cdots \vec{r}_n) \quad (\text{D.4})$$

where Φ is the system wave function. Using Hartree approximation,⁶⁶ that is:

$$\Phi(\vec{r}_1 \cdots \vec{r}_n) = \varphi(\vec{r}_1) \cdots \varphi(\vec{r}_n) \quad (\text{D.5})$$

where $\varphi(\vec{r}_i)$ is the wave function of i -th particle, Hamiltonian solution without interaction term:

$$E = \sum_{i=1}^N \int d\vec{r}_1 \cdots \int d\vec{r}_n [\varphi(\vec{r}_1) \cdots \varphi(\vec{r}_n)]^* \left[-\frac{\hbar^2}{2m_i} \nabla_i^2 + V(\vec{r}_i) + U_0 \sum_{j<i} \delta(\vec{r}_i - \vec{r}_j) \right] [\varphi(\vec{r}_1) \cdots \varphi(\vec{r}_n)] \quad (\text{D.6})$$

$$= \sum_{i=1}^N \int d\vec{r}_i \varphi^*(\vec{r}_i) \left[-\frac{\hbar^2}{2m_i} \nabla_i^2 + V(\vec{r}_i) + U_0 \sum_{j<i} \int \delta(\vec{r}_i - \vec{r}_j) |\varphi(\vec{r}_j)|^2 d\vec{r}_j \right] \varphi(\vec{r}_i) \quad (\text{D.7})$$

$$= \sum_{i=1}^N \int d\vec{r}_i \left[-\frac{\hbar^2}{2m_i} \varphi^*(\vec{r}_i) \nabla_i^2 \varphi(\vec{r}_i) + V(\vec{r}_i) \varphi^*(\vec{r}_i) \varphi(\vec{r}_i) + \frac{N-1}{2} U_0 |\varphi(\vec{r}_i)|^2 \varphi^*(\vec{r}_i) \varphi(\vec{r}_i) \right] \quad (\text{D.8})$$

But:

$$\int -\varphi^*(\vec{r}_i) d\nabla_i^2 \varphi(\vec{r}_i) d\vec{r}_i = -[\varphi^*(\vec{r}_i) \nabla_i \varphi(\vec{r}_i)]_{|\vec{r}_i| \rightarrow \infty}^0 + \int \nabla_i \varphi^*(\vec{r}_i) \nabla_i \varphi(\vec{r}_i) d\vec{r}_i \quad (\text{D.9})$$

So:

$$\boxed{E = N \int d\vec{r} \left[-\frac{\hbar^2}{2m} |\nabla \varphi(\vec{r})|^2 + V(\vec{r}) |\varphi(\vec{r})|^2 + \frac{N-1}{2} U_0 |\varphi(\vec{r})|^4 \right]} \quad (\text{D.10})$$

Let's define a new wave function ψ of whole BEC:

$$\psi(\vec{r}) = \sqrt{N} \varphi(\vec{r}) \Rightarrow n(\vec{r}) = |\psi(\vec{r})|^2 \quad (\text{D.11})$$

where $n(\vec{r})$ is the local numerical density. Let's assume N large ($N-1 \approx N$):

$$\begin{aligned} N &= \int |\psi(\vec{r})|^2 d\vec{r} \\ E &= \int -\frac{\hbar^2}{2m} |\nabla \psi(\vec{r})|^2 + V(\vec{r}) |\psi(\vec{r})|^2 + \frac{1}{2} U_0 |\psi(\vec{r})|^4 d\vec{r} \\ &= \int -\frac{\hbar^2}{2m} \psi^*(\vec{r}) \nabla^2 \psi(\vec{r}) + V(\vec{r}) \psi^*(\vec{r}) \psi(\vec{r}) + \frac{1}{2} U_0 |\psi^*(\vec{r}) \psi(\vec{r})|^2 d\vec{r} \end{aligned} \quad (\text{D.12})$$

Let's use $L = E - \mu N$, where μ is the chemical potential, that is, the energy necessary to add a new particle to the system:

$$L = \int -\frac{\hbar^2}{2m} \psi^*(\vec{r}) \nabla^2 \psi(\vec{r}) + [V(\vec{r}) - \mu] \psi^*(\vec{r}) \psi(\vec{r}) + \frac{1}{2} U_0 |\psi^*(\vec{r}) \psi(\vec{r})|^2 d\vec{r} \quad (\text{D.13})$$

At last, we use the Lagrange method to minimize the energy in relation to ψ^* , with the particles number constant:

$$\frac{\partial L}{\partial \psi^*} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\psi}^*} \right) = 0 \quad (\text{D.14})$$

$$\boxed{-\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + V(\vec{r}) \psi(\vec{r}) + U_0 |\psi(\vec{r})|^2 \psi(\vec{r}) = \mu \psi(\vec{r})} \quad (\text{D.15})$$

For the time dependent equation we can use a similar method but with the time dependent Lagrangian:

$$L = \int \frac{i}{2} \hbar \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - \left[\frac{\hbar^2}{2m} |\nabla \psi|^2 + V(\vec{r}, t) |\psi|^2 + \frac{1}{2} U_0 |\psi|^4 \right] d\vec{r} \quad (\text{D.16})$$

Using the Lagrange equation, we have the time dependent Gross-Pitaevskii Equation:

$$\frac{\partial L}{\partial \psi^*} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\psi}^*} \right) \quad (\text{D.17})$$

$$\int -\frac{i}{2} \hbar \dot{\psi} - \frac{\hbar^2}{2m} |\nabla \psi|^2 + V(\vec{r}, t) |\psi|^2 + \frac{1}{2} U_0 |\psi|^4 - \frac{i}{2} \hbar \dot{\psi} d\vec{r} = 0 \quad (\text{D.18})$$

$$\boxed{-\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}, t) + V(\vec{r}, t) \psi(\vec{r}, t) + \frac{1}{2} U_0 |\psi(\vec{r}, t)|^4 = i \hbar \frac{\partial}{\partial t} \psi(\vec{r}, t)} \quad (\text{D.19})$$

APPENDIX E – PROOF OF GROUND STATE OF HARMONIC TRAP

In this appendix, let's find the harmonic trap non-interacting system ground state solution. For the time independent GPE, we have:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) + \frac{m}{2}(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)\psi(\vec{r}) + U_0|\psi(\vec{r})|^2\psi(\vec{r}) = \mu\psi(\vec{r}) \quad (\text{E.1})$$

In case of weakly interaction, let the interaction term be neglected in front of the other terms, what turn the GPE into Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) + \frac{m}{2}(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)\psi(\vec{r}) = \mu\psi(\vec{r}) \quad (\text{E.2})$$

which is a eigenvalue problem, for which the solution is:

$$\psi_n(x, y, z) = Ae^{B_xx^2+B_yy^2+B_zz^2}(-1)^n e^{x^2}\nabla^n e^{-x^2} \quad (\text{E.3})$$

For ground state, we have $n = 0$:

$$\psi_0(x, y, z) = Ae^{B_xx^2+B_yy^2+B_zz^2} \quad (\text{E.4})$$

Now we can determine the constants using it as *ansatz* to equation [E.2](#):

$$-\frac{\hbar^2}{2m}\nabla^2[Ae^{B_xx^2+B_yy^2+B_zz^2}] + \frac{m}{2}(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)Ae^{B_xx^2+B_yy^2+B_zz^2} = \mu Ae^{B_xx^2+B_yy^2+B_zz^2} \quad (\text{E.5})$$

We can eliminate the proportional constant A :

$$-\frac{\hbar^2}{2m}\nabla^2[e^{B_xx^2+B_yy^2+B_zz^2}] + \frac{m}{2}(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)e^{B_xx^2+B_yy^2+B_zz^2} = \mu e^{B_xx^2+B_yy^2+B_zz^2} \quad (\text{E.6})$$

And the exponential function:

$$-\frac{\hbar^2}{2m}[2B_x(2B_xx^2 + 1) + 2B_y(2B_yy^2 + 1) + 2B_z(2B_zz^2 + 1)] + \frac{m}{2}(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2) = \mu \quad (\text{E.7})$$

$$\left(m\omega_x^2 - \frac{4\hbar^2}{m}B_x^2\right)x^2 + \left(m\omega_y^2 - \frac{4\hbar^2}{m}B_y^2\right)y^2 + \left(m\omega_z^2 - \frac{4\hbar^2}{m}B_z^2\right)z^2 + 2(B_x + B_y + B_z - \mu) = 0 \quad (\text{E.8})$$

Since the solution must be true for any position in space, this equation must be an identity in space variables:

$$\left\{ \begin{array}{l} m\omega_x^2 - \frac{4\hbar^2}{m}B_x^2 = 0 \\ m\omega_y^2 - \frac{4\hbar^2}{m}B_y^2 = 0 \\ m\omega_z^2 - \frac{4\hbar^2}{m}B_z^2 = 0 \\ B_x + B_y + B_z - \mu = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} B_x = \pm \frac{m\omega_x}{2\hbar} \\ B_y = \pm \frac{m\omega_y}{2\hbar} \\ B_z = \pm \frac{m\omega_z}{2\hbar} \\ \mu = B_x + B_y + B_z = \frac{m}{2\hbar}(\omega_x + \omega_y + \omega_z) \end{array} \right. \quad (\text{E.9})$$

So we have the following wave function:

$$\psi_0(x, y, z) = A e^{\pm \frac{m}{2\hbar}(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)} \quad (\text{E.10})$$

At infinity, the wave function must be zero, so the exponential signal must be negative:

$$\psi_0(x, y, z) = A e^{-\frac{m}{2\hbar}(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)} \quad (\text{E.11})$$

At least, let's normalize the wave function:

$$\int_0^\infty \int_0^\infty \int_0^\infty |\psi_0(x, y, z)|^2 dx dy dz = 1 \quad (\text{E.12})$$

$$\int_0^\infty \int_0^\infty \int_0^\infty A^2 e^{-\frac{m}{\hbar}(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)} dx dy dz = 1 \quad (\text{E.13})$$

$$A^2 \int_0^\infty e^{-\frac{m}{\hbar}\omega_z z^2} \int_0^\infty e^{-\frac{m}{\hbar}\omega_y y^2} \int_0^\infty e^{-\frac{m}{\hbar}\omega_x x^2} dx dy dz = 1 \quad (\text{E.14})$$

It is known that:

$$\int_0^\infty e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \quad (\text{E.15})$$

So:

$$A = \left(\frac{m\bar{\omega}}{\hbar\pi} \right)^{3/4} \quad (\text{E.16})$$

where $\bar{\omega}$ is the geometric mean of harmonic trap frequencies. That give us:

$$\boxed{\psi_0(x, y, z) = \left(\frac{m\bar{\omega}}{\hbar\pi}\right)^{3/4} e^{-\frac{m}{2\hbar}(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)}} \quad (\text{E.17})$$

APPENDIX F – BEC HYDRODYNAMICS

In this appendix we will link the time dependent GPE to the Classical Hydrodynamics. The equation F.1 is the time dependent GPE, already studied:

$$i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) + U_0 |\psi(\vec{r}, t)|^2 \right] \psi(\vec{r}, t) \quad (\text{F.1})$$

If we multiply GPE by $\psi^*(\vec{r}, t)$:

$$i\hbar \psi^*(\vec{r}, t) \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \psi^*(\vec{r}, t) \nabla^2 \psi(\vec{r}, t) + V(\vec{r}) |\psi(\vec{r}, t)|^2 + U_0 |\psi(\vec{r}, t)|^4 \quad (\text{F.2})$$

And subtract its complex conjugate:

$$i\hbar \left[\psi^*(\vec{r}, t) \frac{\partial \psi(\vec{r}, t)}{\partial t} + \psi(\vec{r}, t) \frac{\partial \psi^*(\vec{r}, t)}{\partial t} \right] = -\frac{\hbar^2}{2m} \left[\psi^*(\vec{r}, t) \nabla^2 \psi(\vec{r}, t) - \psi(\vec{r}, t) \nabla^2 \psi^*(\vec{r}, t) \right] \quad (\text{F.3})$$

$$i\hbar \frac{\partial |\psi(\vec{r}, t)|^2}{\partial t} = -\frac{\hbar^2}{2m} \left[\psi^*(\vec{r}, t) \nabla^2 \psi(\vec{r}, t) - \psi(\vec{r}, t) \nabla^2 \psi^*(\vec{r}, t) \right] \quad (\text{F.4})$$

$$i\hbar \frac{\partial |\psi(\vec{r}, t)|^2}{\partial t} = -\frac{\hbar^2}{2m} \nabla [\psi^*(\vec{r}, t) \nabla \psi(\vec{r}, t) + \nabla \psi^*(\vec{r}, t) \nabla \psi(\vec{r}, t) - \nabla \psi(\vec{r}, t) \nabla \psi^*(\vec{r}, t) - \psi(\vec{r}, t) \nabla^2 \psi^*(\vec{r}, t)] \quad (\text{F.5})$$

$$i\hbar \frac{\partial |\psi(\vec{r}, t)|^2}{\partial t} = -\frac{\hbar^2}{2mi} \nabla [\psi^*(\vec{r}, t) \nabla \psi(\vec{r}, t) - \psi(\vec{r}, t) \nabla \psi^*(\vec{r}, t)] \quad (\text{F.6})$$

We can rewrite it to be analogue to continuity equation. Let $\psi = \psi(\vec{r}, t)$ to simplify the notation:

$$\frac{\partial |\psi|^2}{\partial t} + \nabla \left[\frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*) \right] = 0 \quad (\text{F.7})$$

This is the **continuity equation for a BEC**, so it is possible to identify the current density as:

$$\vec{j} = \frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (\text{F.8})$$

But in classical hydrodynamics $\vec{j} = n\vec{v}$, so we can identify the velocity field:

$$\vec{v} = \frac{\hbar}{2mni} (\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (\text{F.9})$$

We can continue, writing the wave function in polar form and substituting it in previous equation:

$$\psi = \sqrt{n} e^{iS} \quad (\text{F.10})$$

$$\begin{aligned} \vec{v} &= \frac{\hbar}{2mni} \left(\sqrt{n} e^{-iS} \nabla \left(\sqrt{n} e^{iS} \right) - \sqrt{n} e^{iS} \nabla \left(\sqrt{n} e^{-iS} \right) \right) \\ &= \frac{\hbar}{2mni} \left[\sqrt{n} e^{-iS} \left(e^{iS} \nabla \sqrt{n} + i \sqrt{n} e^{iS} \nabla S \right) - \sqrt{n} e^{iS} \left(e^{-iS} \nabla \sqrt{n} - i \sqrt{n} e^{-iS} \nabla S \right) \right] \\ &= \frac{\hbar}{m} \nabla S \end{aligned} \quad (\text{F.11})$$

So the velocity field is given by:

$$\vec{v}(\vec{r}, t) = \frac{\hbar}{m} \nabla S(\vec{r}, t) \quad (\text{F.12})$$

Now let's return to equation F.2 but now we will sum the complex conjugate:

$$i\hbar \left[\psi^*(\vec{r}, t) \frac{\partial \psi(\vec{r}, t)}{\partial t} - \psi(\vec{r}, t) \frac{\partial \psi^*(\vec{r}, t)}{\partial t} \right] = -\frac{\hbar^2}{2m} [\psi^*(\vec{r}, t) \nabla^2 \psi(\vec{r}, t) + \psi(\vec{r}, t) \nabla^2 \psi^*(\vec{r}, t)] + 2V(\vec{r}) |\psi(\vec{r}, t)|^2 + 2U_0 |\psi(\vec{r}, t)|^4 \quad (\text{F.13})$$

And substitute the polar form:

$$i\hbar \left[\sqrt{n} e^{-iS} \frac{\partial \sqrt{n} e^{iS}}{\partial t} - \sqrt{n} e^{iS} \frac{\partial \sqrt{n} e^{-iS}}{\partial t} \right] = -\frac{\hbar^2}{2m} \left[\sqrt{n} e^{-iS} \nabla^2 \left(\sqrt{n} e^{iS} \right) + \sqrt{n} e^{iS} \nabla^2 \left(\sqrt{n} e^{-iS} \right) \right] + 2V(\vec{r}) n + 2U_0 n^2 \quad (\text{F.14})$$

We will start by temporal derivative:

$$i\hbar \left[\sqrt{n} e^{-iS} \left(e^{iS} \frac{\partial \sqrt{n}}{\partial t} + i \sqrt{n} e^{iS} \frac{\partial S}{\partial t} \right) - \sqrt{n} e^{iS} \left(e^{-iS} \frac{\partial \sqrt{n}}{\partial t} - i \sqrt{n} e^{-iS} \frac{\partial S}{\partial t} \right) \right] = -\frac{\hbar^2}{2m} \left[\sqrt{n} e^{-iS} \nabla^2 \left(\sqrt{n} e^{iS} \right) + \sqrt{n} e^{iS} \nabla^2 \left(\sqrt{n} e^{-iS} \right) \right] + 2V(\vec{r}) n + 2U_0 n^2 \quad (\text{F.15})$$

$$-2n\hbar \frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m} \left[\sqrt{n} e^{-iS} \nabla^2 \left(\sqrt{n} e^{iS} \right) + \sqrt{n} e^{iS} \nabla^2 \left(\sqrt{n} e^{-iS} \right) \right] + 2V(\vec{r}) n + 2U_0 n^2 \quad (\text{F.16})$$

And now, the spacial derivative:

$$-2n\hbar\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m} \left[\sqrt{n}e^{-iS}\nabla \left(e^{iS}\nabla\sqrt{n} + i\sqrt{n}e^{iS}\nabla S \right) + \sqrt{n}e^{iS}\nabla \left(e^{-iS}\nabla\sqrt{n} - i\sqrt{n}e^{-iS}\nabla S \right) \right] + 2V(\vec{r})n + 2U_0n^2 \quad (\text{F.17})$$

$$-2n\hbar\frac{\partial S}{\partial t} = -\frac{\hbar^2}{m}\sqrt{n}\nabla^2\sqrt{n} + \frac{\hbar^2}{m}n(\nabla S)^2 + 2V(\vec{r})n + 2U_0n^2 \quad (\text{F.18})$$

If we divide by $-2n$:

$$\frac{\hbar}{m}m\frac{\partial S}{\partial t} = \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} - \frac{\hbar^2}{2m}(\nabla S)^2 - V(\vec{r}) - U_0n \quad (\text{F.19})$$

$$m\frac{\partial}{\partial t}\left(\frac{\hbar}{m}S\right) = \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} - \frac{1}{2}m\left(\frac{\hbar}{m}\nabla S\right)^2 - V(\vec{r}) - U_0n \quad (\text{F.20})$$

Applying the gradient in both sides:

$$m\frac{\partial}{\partial t}\left(\frac{\hbar}{m}\nabla S\right) = -\nabla\left[\frac{1}{2}m\left(\frac{\hbar}{m}\nabla S\right)^2 + V(\vec{r}) + U_0n - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right] \quad (\text{F.21})$$

Applying the velocity field expression:

$$m\frac{\partial\vec{v}}{\partial t} + \nabla\left[\frac{1}{2}mv^2 + V(\vec{r}) + U_0n - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right] = 0 \quad (\text{F.22})$$

These are the **Madelung equations**, the hydrodynamics quantum equations:

$$\begin{cases} m\frac{\partial\vec{v}}{\partial t} + \nabla\left[\frac{1}{2}mv^2 + V(\vec{r}) + U_0n - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right] = 0 & (\text{Bernoulli equation}) \\ \frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = 0 & (\text{Continuity equation}) \end{cases} \quad (\text{F.23})$$