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Oscillatory interaction in a Bose-Einstein
condensate: collective and topological
excitations

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Oscillatory interaction in a Bose-Einstein condensate: collective and topological excitations

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*The fear of the LORD is the beginning of knowledge,
but fools despise wisdom and instruction.*

Proverbs of Solomon 1:7

A little science estranges a man from God. A lot of science brings him back.

Francis Bacon

RESUMO

RAMOS, E. R. F. Interações oscilatórias em um condensado de Bose-Einstein: excitações coletivas e topológicas. 2012. 76p. Tese (Doutorado) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2012.

Neste trabalho, analisamos teoricamente o comportamento de um condensado de Bose-Einstein quando submetido a interações oscilatórias. Para tal, é aplicado um campo magnético homogêneo, sintonizado próximo a uma ressonância de Feshbach e então colocado a oscilar no tempo. Esta variação do campo magnético faz com que o comprimento de espalhamento oscile, oscilando portanto a interação entre os átomos. Com isso, estudamos as excitações coletivas e topológicas provocadas devido a oscilação da interação. Além disso, vimos o acoplamento entre modos coletivos e uma transição de fase dinâmica associada a excitação topológica.

Palavras-chave: Condensação de Bose-Einstein. Excitações coletivos. Modos topológicos.

ABSTRACT

RAMOS, E. R. F. Oscillatory interaction in a Bose-Einstein condensate: collective and topological excitations. 2012. 76p. Thesis (Doctorate) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2012.

In this work, we theoretically analyze the behavior of a Bose-Einstein condensate when it is submitted to oscillatory interactions. For that, a homogeneous magnetic field is applied, tuned near a Feshbach resonance, and then it is set to oscillate in time. This variation of the magnetic field causes a scattering length oscillation, which oscillates to interatomic interaction. Thus, we study collective and topological excitations due this oscillation in the interaction. In addition, we have seen a coupling between collective modes as well a dynamical phase transition associated to topological excitation.

Key-words: Bose-Einstein condensation. Collective excitation. Topological modes.

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1 *Introduction*

In 1997, Yukalov, Yukalova and Bagnato proposed a possible way to create a non-ground-state Bose-Einstein condensate of trapped atoms (1). Since then, it seems that this goal is a kind of Holy Grail or philosopher's stone in our research lab. With good reason, because it would be a very significant and interesting achievement. The Bose-Einstein condensate of dilute gas of trapped atoms, is, in general, produced in the ground state of a harmonic trap. The way that they proposed the generation of a non-ground-state BEC was applying an oscillatory and spatially dependent field that would couple the ground with an excited state as a two-level system.

That idea was the base of my Master's degree, where I explored, theoretically, what was the best way to excite. Also, we investigated the behavior of a BEC during this excitation and observed dynamical phase transitions (2), Rabi and Ramsey-like oscillations (3). Thus, with theory done, it was time to go to the lab.

The magnetic field where the BEC is trapped can be made by combining two coils in a anti-Helmholtz configuration, shown in Figure 1.1 as quadrupole coils, and an Ioffe coil (4). For excitation was added two quadrupole coils whose axis is aligned with the Ioffe coil axis, which is the longest axis of the BEC.

Figure 1.1 – Hypothetic coil configuration of a magnetic trap with two excitation coils.

However the real world is not as beautiful as the hypothetical one*. It is too hard to align both excitation and Ioffe coils. The result must be something like configuration shown in Figure 1.2, where the excitation coil is slightly misaligned from the condensate long axis. And this messes up everything.

Figure 1.2 – Possible real coil configuration of a magnetic trap with two excitation coils.

Because of that tilted configuration, it is possible add angular momentum in BEC cloud, generating vortices (5,6). Or, more impressively, quantum turbulence (7).

Despite the success of that the misaligned configuration, the problem of producing a BEC in an excited state of the harmonic trap persists. We have to solve the problem of alignment.

In my Master's degree defense, one of the committee members Professor Lauro Tomio asked me what happened if the scattering length were oscillating in time. The scattering length defines the strength of interatomic interaction and can be easily controlled by a homogeneous magnetic field. My answer to that question was "I don't know", but it gives us another question: would it be possible to generate a non-ground-state BEC via oscillating interactions? If yes, it would solve the problem of alignment, because the applied field is homogeneous. The pursuit of the answer to that question triggered this work, which will be presented as follows.

In Chapter 2, we present theoretical basis which is useful to understand the results. These results will be presented in two parts. First, in Chapter 3, we will show the results of generating a non-ground-state BEC by oscillating the interaction. In Chapter 4, we will present the collective excitations that we could generate and couple if the scattering length oscillates. In Chapter 5, we summarize and discuss the results. Have a nice reading.

*Or the beauty lies on the complexity and randomness of the nature?

2 *Excitations in a Bose-Einstein condensate*

2.1 Gross-Pitaevskii equation

The system that we want to describe is an atomic Bose-Einstein condensate which is a dilute gas trapped in a harmonic trap. As the gas is dilute, we can consider just binary interactions between particles, because the probability for a three-body collision to occur is much less than the two-body one. Thus, the many-body Hamiltonian (8) which describes our system is given by

$$\begin{aligned} \hat{H} = & \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \left[-\frac{\hbar^2}{2m} \nabla^2 + U_{trap}(\mathbf{r}, t) \right] \hat{\Psi}(\mathbf{r}) + \\ & + \frac{1}{2} \iint d\mathbf{r} d\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}') V_{int}(\mathbf{r}, \mathbf{r}') \hat{\Psi}(\mathbf{r}) \hat{\Psi}(\mathbf{r}'), \end{aligned} \quad (2.1)$$

where m is the atomic mass, $U_{trap}(\mathbf{r}, t)$ is the trap potential, $V_{int}(\mathbf{r}, \mathbf{r}')$ is the interaction potential, $\hat{\Psi}^\dagger(\mathbf{r})$ and $\hat{\Psi}(\mathbf{r})$ are operators of creation and annihilation of a boson in the position $\mathbf{r}$. Those operators obey the commutation relation

$$\left[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}') \right] = \delta(\mathbf{r} - \mathbf{r}'); \quad \left[\hat{\Psi}(\mathbf{r}), \hat{\Psi}(\mathbf{r}') \right] = 0; \quad \left[\hat{\Psi}^\dagger(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}') \right] = 0. \quad (2.2)$$

As we are interested in the dynamics of the system, we must know how the operator $\hat{\Psi}(\mathbf{r}, t)$ evolves in time. For that, we substitute the Equation (2.1) into the Heisenberg equation

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r}, t) = \left[\hat{\Psi}(\mathbf{r}, t), \hat{H} \right], \quad (2.3)$$

and using the commutation relation, we obtain

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r}, t) = \left[-\frac{\hbar^2}{2m} \nabla^2 + U_{trap}(\mathbf{r}, t) + \int V_{int}(\mathbf{r}, \mathbf{r}') \hat{\Psi}^\dagger(\mathbf{r}', t) \hat{\Psi}(\mathbf{r}', t) d\mathbf{r}' \right] \hat{\Psi}(\mathbf{r}, t). \quad (2.4)$$

This equation is an *exact* equation for the field operator $\hat{\Psi}(\mathbf{r}, t)$ (9). Neglecting fluctuations, this operator can be approximated by a mean field $\Phi(\mathbf{r}, t)$, normalized to unity, as

$$\int \hat{\Psi}^\dagger(\mathbf{r}, t) \hat{\Psi}(\mathbf{r}, t) d\mathbf{r} = N, \quad (2.5)$$

where N is the number of condensed atoms. Thus, Equation (2.4) can be approximated by

$$i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{r}, t) = \left[-\frac{\hbar^2}{2m} \nabla^2 + U_{trap}(\mathbf{r}, t) + N \int V_{int}(\mathbf{r}, \mathbf{r}') |\Phi(\mathbf{r}', t)|^2 d\mathbf{r}' \right] \Phi(\mathbf{r}, t). \quad (2.6)$$

Although the Equations (2.4) and (2.6) have the same structure, they represent different situations. The Equation (2.4) is an exact equation for the field operator, while the Equation (2.6) is a *mean field approximation* for a classic field, which is an order parameter of the condensate, also called *wave function of the condensate* (10).

Our system has very low temperature ($\sim 100\text{nK}$) and density ($\sim 10^{15}\text{atoms/cm}^3$), therefore, the energies are so low that the de Broglie wavelength is much larger than the range of the interactions between the atoms (11). Besides, the most important scattering process is the two-body elastic collision. Thus, the interaction potential can be represented by the Fermi contact potential (12), given by

$$V_{int}(\mathbf{r}, \mathbf{r}') = A_s \delta(\mathbf{r} - \mathbf{r}'), \quad (2.7)$$

where A_s is a constant we will derive in the next section. Substituting Equation (2.7) into (2.6), we obtain the Gross-Pitaevskii equation (GPE)

$$\hat{H}\Phi(\mathbf{r}, t) = i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{r}, t), \quad (2.8a)$$

with

$$\hat{H} = \hat{H}[\Phi] = -\frac{\hbar^2}{2m} \nabla^2 + U_{trap}(\mathbf{r}, t) + A_s N |\Phi|^2. \quad (2.8b)$$

In the next section, we will present some basic concepts of the scattering process and the meaning of the scattering length.

2.2 Interactions and the scattering length

As mentioned in the previous section, the interatomic interaction occurs in a very

special scenario. The de Broglie wavelength is much longer than the short range of the potential, which means that an atom does not see any structure of the other, just a scatter center as a hard sphere. Thus, in the lowest order, one has only spherically symmetric outgoing wave, which is so called s-wave scattering. In the following, we will derive the scattered s-wave as well as the scattering length, which defines the strength of the interaction.

First, let us consider a plane wave, traveling in the z direction, scattered by a scattering center as illustrated in Figure 2.3.

Figure 2.3 – Scattering scheme where the incoming plane wave is spherically scattered (a). Far from scattering center we can consider that the scattered wave is spherical (b).

The total wave function has the form

$$\varphi = e^{ikz} + \varphi_{sc}, \quad (2.9)$$

where k is the wave number and φ_{sc} is the scattered wave. Considering yet that the

scattering potential, $V_{sc}(r)$, is a hard sphere with radius a_s . As this potential is spherically symmetric, the Schrödinger equation admits separable solutions in the form

$$\varphi_{sc} \sim \sum R_{nl}(r) Y_l^m(\theta, \phi), \quad (2.10)$$

where $Y_l^m(\theta, \phi)$ is a spherical harmonic and $R(r)$ is a radial function which satisfy the equation

$$-\frac{\hbar^2}{2m} \frac{d^2 u(r)}{dr^2} + \left[V_{sc}(r) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u(r) = E u(r), \quad (2.11)$$

where $u(r) = r R_{nl}(r)$. We are looking for solutions far from the scattering center where the potential goes to zero (as illustrated in Fig. 2.3) and the centrifugal term is negligible. So, Equation (2.11) becomes

$$\frac{d^2 u(r)}{dr^2} \approx -k^2 u(r), \quad (2.12)$$

where

$$k^2 = \frac{2mE}{\hbar^2}. \quad (2.13)$$

The general solution can be written as

$$u(r) = C \sin(kr + \delta_s). \quad (2.14)$$

Inside the hard sphere, the solution vanishes, i. e., $u(a_s) = 0$, which means that

$$\begin{aligned} \sin(ka_s + \delta_s) &= 0 \\ \Rightarrow \delta_s &= -ka_s. \end{aligned} \quad (2.15)$$

So, the scattered wave function is given by

$$\varphi_{sc} = A \frac{\sin(kr - ka_s)}{r}. \quad (2.16)$$

Another solution can be written as

$$u(r) = A e^{ikr} + B e^{-ikr}. \quad (2.17)$$

As φ_{sc} is an outgoing wave, the incoming term in $u(r)$ must vanishes, so $B = 0$. Thus, we have

$$R_{nl}(r) \approx \frac{e^{ikr}}{r}. \quad (2.18)$$

We also have to consider that, because the scattering is a low energy one, the dominant term in the expansion (2.10) is the $l = 0$ term, i. e., it is a s-wave scattering. So, we can

write the total wave function far from the scatter center as

$$\varphi_{sc} = e^{ikz} + f(\theta, \phi) \frac{e^{ikr}}{r}, \quad (2.19)$$

where $f(\theta, \phi)$ is a scatter function. Expanding the incoming plane wave in terms of Legendre Polynomials, we have

$$e^{ikz} = e^{ikr \cos \theta} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos \theta), \quad (2.20)$$

where $j_l(x)$ is the spherical Bessel function and $P_l(x)$ are the Legendre polynomials. As we previously said, we will consider only $l = 0$ terms and the solution far from the scatter center ($r \rightarrow \infty$), which means that

$$e^{ikz} \approx j_0(kr) P_0(\cos \theta) = \frac{e^{ikr} - e^{-ikr}}{2ikr}. \quad (2.21)$$

Rewriting Equation (2.19) using (2.21), we have

$$\varphi_{sc} = \frac{1}{2ikr} [(1 + 2ikf) e^{ikr} - e^{-ikr}]. \quad (2.22)$$

Comparing Equations (2.22) and (2.16)

$$A \frac{\sin(kr - ka_s)}{r} = \frac{1}{2ikr} [(1 + 2ikf(\theta, \phi)) e^{ikr} - e^{-ikr}],$$

we obtain

$$A = \frac{e^{-ika_s}}{k} \quad (2.23)$$

and

$$f(\theta, \phi) = \frac{e^{-2ika_s} - 1}{2ik} \approx -a_s. \quad (2.24)$$

Thus, the scattered wave function (2.19) is given by

$$\varphi_{sc} = e^{ikz} - a_s \frac{e^{ikr}}{r}; \quad (2.25)$$

for low energies ($k \rightarrow 0$), we have

$$\varphi_{sc} = 1 - \frac{a_s}{r}. \quad (2.26)$$

The parameter a_s is known as *s-wave scattering length* and it can be understood as a shift in the outgoing wave. If $a_s > 0$, the shift is in the outgoing direction, as if the scatter center repealed the incoming wave. So, we have a repulsive interaction. In the other way, for $a_s < 0$, the shift is in the opposite direction, and we have an attractive interaction.

Now, we will derive the parameter A_s given in the equation (2.7). Starting with the integral Schrödinger equation

$$\varphi(\mathbf{r}) = \varphi_0(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}_0) V(\mathbf{r}_0) \varphi_0(\mathbf{r}_0) d^3 \mathbf{r}_0, \quad (2.27)$$

where

$$G(\mathbf{r} - \mathbf{r}_0) = -\frac{m_r}{2\pi\hbar^2} \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r} - \mathbf{r}_0|} \quad (2.28)$$

is the Green's function, m_r is the reduced mass of two interacting bosons, k is given by Equation (2.13) and $\varphi_0(\mathbf{r})$ is the incident wave function. We can expand Equation (2.27) in Born series, which is a iterating process like

$$\begin{aligned} \varphi(\mathbf{r}) &= \varphi_0(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}_0) V(\mathbf{r}_0) \varphi_0(\mathbf{r}_0) d^3 \mathbf{r}_0 + \\ &+ \int \int G(\mathbf{r} - \mathbf{r}_0) V(\mathbf{r}_0) G(\mathbf{r}_0 - \mathbf{r}'_0) V(\mathbf{r}'_0) \varphi_0(\mathbf{r}'_0) d^3 \mathbf{r}_0 d^3 \mathbf{r}'_0 + \dots \end{aligned} \quad (2.29)$$

Considering only the first order in the Born series, we have

$$\varphi(\mathbf{r}) = \varphi_0(\mathbf{r}) - \frac{m_r}{2\pi\hbar^2} \int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r} - \mathbf{r}_0|} V(\mathbf{r}_0) \varphi_0(\mathbf{r}_0) d^3 \mathbf{r}_0, \quad (2.30)$$

We are looking for solutions far away from the scattering center, i. e., $|\mathbf{r}| \gg |\mathbf{r}_0|$. So, we can approximate

$$|\mathbf{r} - \mathbf{r}_0|^2 = r^2 + r_0^2 - 2\mathbf{r} \cdot \mathbf{r}_0 \approx r^2 \left(1 - 2\frac{\hat{\mathbf{r}} \cdot \mathbf{r}_0}{r} \right), \quad (2.31)$$

and then

$$|\mathbf{r} - \mathbf{r}_0| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}_0. \quad (2.32)$$

Thus, considering only first order of r_0/r , we have

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r} - \mathbf{r}_0|} \approx \frac{e^{ikr}}{r} e^{-i\mathbf{k} \cdot \mathbf{r}_0}, \quad (2.33)$$

where $\mathbf{k} = k\hat{\mathbf{r}}$.

In our case, $\varphi_0(\mathbf{r})$ is the incoming wave, so

$$\varphi_0(\mathbf{r}) = e^{i\mathbf{k}' \cdot \mathbf{r}}, \quad (2.34)$$

where $\mathbf{k}' = k\hat{\mathbf{z}}$, and the Schrödinger equation, for large r and first Born approximation, is given by

$$\varphi(\mathbf{r}) = e^{ikz} - \frac{m_r}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int e^{i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}_0} V(\mathbf{r}_0) d\mathbf{r}_0. \quad (2.35)$$

Comparing with Equation (2.19), the scattering function is given by

$$f(\theta, \phi) \approx -\frac{m_r}{2\pi\hbar^2} \int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_0} V(\mathbf{r}_0) d\mathbf{r}_0. \quad (2.36)$$

In the low energy regime, $|\mathbf{k}' - \mathbf{k}|$ is small, and we obtain

$$f(\theta, \phi) \approx -\frac{m_r}{2\pi\hbar^2} \int V(\mathbf{r}_0) d\mathbf{r}_0. \quad (2.37)$$

Using the contact interaction potential given by Equation (2.7) and comparing with Equation (2.24), we have

$$A_s = \frac{2\pi\hbar^2 a_s}{m_r}. \quad (2.38)$$

Finally, if we have two identical bosons, the reduced mass is

$$m_r = \frac{m}{2},$$

where m is the mass of one single boson, then

$$A_s = \frac{4\pi\hbar^2 a_s}{m}. \quad (2.39)$$

Hence, the GPE, given in Equation (2.8), can be written as

$$i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{r}, t) = -\frac{\hbar^2}{2m} \nabla^2 \Phi(\mathbf{r}, t) + U_{trap}(\mathbf{r}, t) \Phi(\mathbf{r}, t) + \frac{4\pi\hbar^2 a_s}{m} N |\Phi|^2 \Phi(\mathbf{r}, t). \quad (2.40)$$

2.2.1 Feshbach resonance and manipulation of interaction

In this section we will see how to manipulate the strength of interactions. This is possible due to Feshbach resonance effect, which consist in coupling two interaction channels applying an external field. Here, the word *channel* refers to a set of quantum number that characterizes the internal state of an atom. Also, we will refer as an *open channel* when the kinetic energy of two atoms is higher then the threshold energy of the channel. In this way, those atoms are not allowed to form a bound state. On the other hand, we have a *closed channel* when those atoms have a kinetic energy below the threshold energy of the channel, and they are able to form a bound state. So, the Feshbach resonance occurs when the threshold energy E_{th} of the open channel is close to an energy

E_{res} of a bound state in the closed channel. Figure 2.4 illustrate those two channels and their respective energies.

Figure 2.4 – Schematic representation of difference between threshold energy of open channel and a bound state in closed channel.

The main question is how the Feshbach modifies the interaction, or in other words, how the scattering length is affected by the resonance. In order to derive an expression for the scattering length, we will follow Reference (12). First, let us consider that the total state is the sum of states in a subspace P and Q , which corresponds, respectively, to states of open and closed channel. Thus, we can write

$$|\Psi\rangle = |\Psi_P\rangle + |\Psi_Q\rangle, \quad (2.41)$$

where

$$|\Psi_P\rangle = \hat{P}|\Psi\rangle, \quad (2.42)$$

and

$$|\Psi_Q\rangle = \hat{Q}|\Psi\rangle. \quad (2.43)$$

The projection operators $\hat{P}$ and $\hat{Q}$ satisfy the conditions

$$\hat{P} + \hat{Q} = 1 \quad \text{and} \quad \hat{P}\hat{Q} = 0. \quad (2.44)$$

From these conditions, it follows that

$$\hat{P}^2|\Psi\rangle = \hat{P}|\Psi\rangle \quad \text{and} \quad \hat{Q}^2|\Psi\rangle = \hat{Q}|\Psi\rangle. \quad (2.45)$$

Applying $\hat{P}$ from left to the Schrödinger equation,

$$\hat{H}|\Psi\rangle = E|\Psi\rangle \quad (2.46)$$

and using relations (2.45), we have

$$(E - \hat{H}_{PP})|\Psi_P\rangle = \hat{H}_{PQ}|\Psi_Q\rangle, \quad (2.47)$$

where $\hat{H}_{PP} = \hat{P}\hat{H}\hat{P}$ and $\hat{H}_{PQ} = \hat{P}\hat{H}\hat{Q}$. Applying in the same way operator $\hat{Q}$, we have

$$(E - \hat{H}_{QQ})|\Psi_Q\rangle = \hat{H}_{QP}|\Psi_P\rangle, \quad (2.48)$$

where $\hat{H}_{QQ} = \hat{Q}\hat{H}\hat{Q}$ and $\hat{H}_{QP} = \hat{Q}\hat{H}\hat{P}$. From Equation (2.48), we have

$$|\Psi_Q\rangle = (E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}|\Psi_P\rangle, \quad (2.49)$$

and substituting in Equations (2.47), we obtain

$$(E - \hat{H}_{PP} - \hat{H}'_{PP})|\Psi_P\rangle = 0, \quad (2.50)$$

where

$$\hat{H}'_{PP} = \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}. \quad (2.51)$$

This term describes the Feshbach resonances which can be understood as an effective interaction in the P subspace (open channel) due to a transition to the Q subspace (closed channel) and back to P subspace.

In order to make it simpler, let us make

$$\hat{H}_{PP} = \hat{H}_0 + \hat{V}_1, \quad (2.52)$$

where H_0 is an one body operator (kinetic energy+Zeeman effect) and $\hat{V}_1$ is the interaction in the open channel. Substituting Equation (2.52) into (2.50), we have

$$(E - \hat{H}_0 - \hat{V})|\Psi_P\rangle = 0, \quad (2.53)$$

where

$$\hat{V} = \hat{V}_1 + \hat{V}_2 \quad (2.54)$$

is the total effective interaction potential in the P subspace, with $\hat{V}_2 = \hat{H}'_{PP}$.

From the first order term of Born series, given by Eq. (2.35), we can get the scatter

function, given by Eq. (2.36), which can be rewrite in bra-ket notation as

$$f = -\frac{m_r}{2\pi\hbar^2} \langle \mathbf{k}' | \hat{V} | \mathbf{k} \rangle. \quad (2.55)$$

However, here the interaction potential V is not just the contact interaction as in the previous section, but is given by Eq. (2.54). So, substituting Eq. (2.54) into (2.55), we have

$$f = -\frac{m_r}{2\pi\hbar^2} \left(\langle \mathbf{k}' | \hat{V}_1 | \mathbf{k} \rangle + \langle \mathbf{k}' | \hat{V}_2 | \mathbf{k} \rangle \right). \quad (2.56)$$

The first term, with the potential V_1 , is the scattering in the open channel, which was previously calculated and is given by Eq. (2.24). So, we have

$$\langle \mathbf{k}' | \hat{V}_1 | \mathbf{k} \rangle \approx \frac{2\pi\hbar^2}{m_r} a_{nr}, \quad (2.57)$$

where a_{nr} is the *non-resonant* scattering length, i.e., the value of the scattering length when we are far from a Feshbach resonance and the contribution of the potential $\hat{V}_2$ can be neglected.

The second term of Eq. (2.56), with the potential $\hat{V}_2$, can be calculated using the identity

$$\sum_n |n\rangle \langle n| = 1, \quad (2.58)$$

where $|n\rangle$ are states of the closed channel and form a complete set. Thus, as

$$\hat{V}_2 = \hat{H}_{PQ}(E - \hat{H}_{QP})^{-1} \hat{H}_{QP}, \quad (2.59)$$

we have

$$\begin{aligned} \langle \mathbf{k}' | \hat{V}_2 | \mathbf{k} \rangle &= \langle \mathbf{k}' | \hat{H}_{PQ}(E - \hat{H}_{QP})^{-1} \hat{H}_{QP} | \mathbf{k} \rangle \\ &= \sum_n \langle \mathbf{k}' | \hat{H}_{PQ} | n \rangle \langle n | (E - \hat{H}_{QP})^{-1} \hat{H}_{QP} | \mathbf{k} \rangle \\ &= \sum_n \frac{\langle \mathbf{k}' | \hat{H}_{PQ} | n \rangle \langle n | \hat{H}_{QP} | \mathbf{k} \rangle}{E - E_n}, \end{aligned} \quad (2.60)$$

where E_n are eigenenergies in the closed channel. Here, we will do two considerations: first, we are coupling the open channel with only one state of the closed channel, ψ_{res} ; second, the scattering does not change so much the momentum, which implies that $\mathbf{k}' \approx \mathbf{k}$. Those considerations results in

$$\langle \mathbf{k}' | \hat{V}_2 | \mathbf{k} \rangle \approx \langle \mathbf{k} | \hat{V}_2 | \mathbf{k} \rangle = \frac{|\langle \psi_{res} | \hat{H}_{QP} | \psi_0 \rangle|^2}{E_{th} - E_{res}}, \quad (2.61)$$

where ψ_0 is the state and E_{th} is the threshold energy, both of the open channel, and E_{res} is the energy of the state ψ_{res} . Substituting Equations (2.57) and (2.61) into (2.56), we have the scatter function

$$f = -a_{nr} - \frac{m_r}{2\pi\hbar^2} \frac{|\langle\psi_{res}|\hat{H}_{QP}|\psi_0\rangle|^2}{E_{th} - E_{res}}. \quad (2.62)$$

When one applies a magnetic field, due to Zeeman effect, the energies are shifted like

$$E_{th}(B) = E_{th}(0) + \mu_{th}B, \quad (2.63a)$$

$$E_{res}(B) = E_{res}(0) + \mu_{res}B, \quad (2.63b)$$

where μ_{th} and μ_{res} are the magnetic moment of two atoms in the open and closed channel respectively. For a given value of the magnetic field, $B = B_{res}$, we expect that the energies E_{th} and E_{res} are in resonance, which give us, from Equations (2.63b),

$$E_{th}(B_{res}) - E_{res}(B_{res}) = E_{th}(0) - E_{res}(0) + (\mu_{th} - \mu_{res})B_{res},$$

$$0 = E_{th}(0) - E_{res}(0) + (\mu_{th} - \mu_{res})B_{res},$$

$$E_{th}(0) - E_{res}(0) = -(\mu_{th} - \mu_{res})B_{res}. \quad (2.64)$$

Thus,

$$E_{th}(B) - E_{res}(B) = (\mu_{th} - \mu_{res})(B - B_{res}), \quad (2.65)$$

which, from Eq. (2.62), implies that

$$f = -a_{nr} \left(1 - \frac{\Delta}{B - B_{res}} \right), \quad (2.66)$$

where

$$\Delta = -\frac{m_r}{2\pi\hbar^2 a_{nr}} \frac{|\langle\psi_{res}|\hat{H}_{QP}|\psi_0\rangle|^2}{\mu_{th} - \mu_{res}}. \quad (2.67)$$

Therefore, the scattering length, in the presence of a magnetic field, is given by

$$a_s(B) = a_{nr} \left(1 - \frac{\Delta}{B - B_{res}} \right). \quad (2.68)$$

2.3 Coherent topological modes

If the trap potential $U_{trap}(\mathbf{r}, t)$ does not depend on time, i.e, $U_{trap}(\mathbf{r}, t) = U_{trap}(\mathbf{r})$, the solutions of Equation (2.8) has the form

$$\Phi_n(\mathbf{r}, t) = \phi_n(\mathbf{r})e^{-iE_nt/\hbar}, \quad (2.69)$$

where $\phi_n(\mathbf{r})$ which are solutions of the time-independent equation

$$\hat{H}[\phi_n]\phi_n(\mathbf{r}) = E_n\phi_n(\mathbf{r}). \quad (2.70)$$

Those stationary solutions $\phi_n(\mathbf{r})$ are the *coherent topological modes* (9). In the section 2.3.1 we will calculate those topological modes using the *Optimized Perturbation Theory*.

Bose-Einstein condensate are created in the ground state of the trap, but we can populate excited states by adding a time-dependent potential. In chapter 3 we will present how would be possible to excite and create a non-ground state BEC.

2.3.1 Optimized perturbation theory

We have used the *Optimized Perturbation Theory* to obtain an approximation for wave function of the problem (2.70). For that, we will follow the procedure described by Courteille *et al.* em (9).

In the usual perturbation method, we have a problem like

$$H = H_0 + H',$$

where H is the Hamiltonian that we want to solve, H_0 is the hamiltonian we know how to solve, and H' is the perturbation. In the optimized perturbation theory, we introduce variational parameters in H_0 . Thus, our problem becomes

$$\begin{aligned} H &= H_0(u, v, w, \dots) + \Delta H, \\ \Rightarrow \Delta H &= H - H_0(u, v, w, \dots), \end{aligned} \quad (2.71)$$

where we can see that $\Delta H \neq H'$.

So, the first-order correction of the energy is given by

$$E_n^{(1)}(u, v, w, \dots) = E_n^{(0)}(u, v, w, \dots) + \langle \Phi_n^{(0)} | \Delta H | \Phi_n^{(0)} \rangle, \quad (2.72)$$

where $\Phi_n^{(0)} = \Phi_n^{(0)}(u, v, w, \dots)$ are solutions of the unperturbed Hamiltonian H_0 . Once

the energies are found, we have to minimize them in terms of the variational parameters.

Thus, we have

$$\frac{\partial E_n}{\partial u} = 0 \quad ; \quad \frac{\partial E_n}{\partial v} = 0 \quad ; \quad \frac{\partial E_n}{\partial w} = 0 \quad \dots \quad (2.73)$$

Hence we obtain those variational parameters, we can get both energies and wave function.

Currently, the most part of condensate are made in cylindrically symmetric harmonic traps. So, we suppose such symmetry for the trap potential in Equation (2.8)

$$U_{trap}(\mathbf{r}) = \frac{m}{2} (\omega_r^2 r^2 + \omega_z^2 z^2), \quad (2.74)$$

where ω_r and ω_z are radial and axial frequency respectively.

Now, we will define some important parameters for our calculations. First, we have the oscillator length

$$l_r = \sqrt{\frac{\hbar}{m\omega_r}}, \quad (2.75)$$

which will be our length scale of the system. Also, we have the anisotropy parameter

$$\lambda = \frac{\omega_z}{\omega_r} \quad (2.76)$$

which give us the aspect ratio of the condensate. For $\lambda < 1$, we have an elongated cloud, as a cigar; in the case of $\lambda > 1$, the cloud is flat, as a pancake; if $\lambda = 1$, the atomic cloud is spherically symmetric. We also define the dimensionless coupled parameter

$$g = 4\pi N \frac{a_s}{l_r}, \quad (2.77)$$

which indicates the strength of interaction.

With the oscillator length, we define our dimensionless coordinates

$$x_r = \frac{r}{l_r} \quad , \quad x_z = \frac{z}{l_r}. \quad (2.78)$$

Besides, we define a dimensionless Hamiltonian,

$$\hat{H}[\psi] = \frac{\hat{H}[\phi]}{\hbar\omega_r}, \quad (2.79)$$

which means that the eigenfunctions $\psi_n(\mathbf{x})$ and eigenenergies e_n are given by

$$\psi(\mathbf{x}) = l_r^{3/2} \phi(\mathbf{r}) \quad \text{e} \quad \varepsilon_n = \frac{E_n}{\hbar\omega_r}. \quad (2.80)$$

Replacing the potential (2.74) in the Eq. (2.8), and rewriting the resulting equation

in terms of the dimensionless parameters described above, we have

$$\hat{H}\psi_n(\mathbf{x}) = \varepsilon_n\psi_n(\mathbf{x}), \quad (2.81)$$

with

$$\hat{H} = -\frac{1}{2}\nabla_x^2 + \frac{1}{2}(x_r^2 + \lambda^2 x_z^2) + g|\psi|^2. \quad (2.82)$$

Comparing Eq. (2.82) with the Hamiltonian (2.71), we identify H_0 as

$$H_0(u, v) = -\frac{1}{2}\nabla_x^2 + \frac{1}{2}(u^2 x_r^2 + v^2 x_z^2), \quad (2.83)$$

where u e v are variational parameters. We can exactly find the eigenfunctions and eigenenergies of the Hamiltonian (3.8), which are given by

$$\psi_{nmk} = \left[\frac{2n!u^{|m|+1}}{(n+|m|)!} \right]^{1/2} x_r^{|m|} e^{-ux_r^2/2} L_n^{|m|}(ux_r^2) \frac{e^{im\varphi}(v/\pi)^{1/4}}{\sqrt{2^{k+1}\pi k!}} e^{-vx_z^2/2} H_k(\sqrt{v}x_z), \quad (2.84)$$

where $L_n^{|m|}(x)$ are associated Laguerre polynomials and $H_k(x)$ are Hermite polynomials. The quantum numbers n , m and k are such that

$$n = 0, 1, 2, \dots; \quad m = 0, \pm 1, \pm 2, \dots; \quad k = 0, 1, 2, \dots$$

and eigenenergies are given by

$$\varepsilon_{nmk}^{(0)} = (2n + |m| + 1)u + (k + \frac{1}{2})v. \quad (2.85)$$

Again, comparing with the Eq. (2.71), we have

$$\begin{aligned} \Delta H &= H - H_0 \\ &= \frac{1}{2}[(1 - u^2)x_r^2 + (\lambda^2 - v^2)x_z^2] + g|\psi|^2. \end{aligned} \quad (2.86)$$

In this way, we can calculate the first-order correction for the energy

$$\begin{aligned} \varepsilon_{nmk}^{(1)} &= \varepsilon_{nmk}^{(0)} + \langle \psi_{nmk} | \Delta H | \psi_{nmk} \rangle \\ &= \frac{p}{2} \left(u + \frac{1}{u} \right) + \frac{q}{4} \left(v + \frac{\lambda^2}{v} \right) + gu\sqrt{v}I_{nmk}, \end{aligned} \quad (2.87)$$

where

$$I_{nmk} = \frac{1}{\pi^2} \left[\frac{n!}{(n+|m|)!2^k k!} \right]^2 \int_0^\infty d\rho \rho^{2|m|} e^{-2\rho} [L_n^{|m|}(\rho)]^4 \int_{-\infty}^\infty d\zeta e^{-2\zeta^2} [H_k(\zeta)]^4 \quad (2.88)$$

and

$$p = 2n + |m| + 1 \quad \text{e} \quad q = 2k + 1. \quad (2.89)$$

Minimizing the obtained energy (2.87) in terms of the variational parameters,

$$\frac{\partial}{\partial u}\varepsilon_{nmk}^{(1)} = \frac{\partial}{\partial v}\varepsilon_{nmk}^{(1)} = 0, \quad (2.90)$$

we obtain a system of equations for u e v

$$p\left(1 - \frac{1}{u^2}\right) + \frac{s}{p\lambda}\sqrt{\frac{v}{q}} = 0 \quad (2.91a)$$

$$q\left(1 - \frac{\lambda^2}{v^2}\right) + \frac{su}{p\lambda\sqrt{vq}} = 0 \quad (2.91b)$$

where

$$s = 2p\sqrt{q}I_{nmk}\lambda g. \quad (2.92)$$

The system (2.91) cannot be analytically solved, but it can be easily done numerically. Thus, with those parameters, we can build the wavefunction (2.84), and the energy spectrum (2.85). For instance, the ground state ($n=0$, $m=1$ e $k=0$) is given by

$$\psi_{000} = \left(\frac{u_{000}^2 v_{000}}{\pi^3}\right)^{1/4} e^{-(u_{000}x_r^2 + v_{000}x_z^2)/2} \quad (2.93)$$

Figure 2.5 – Density (a) and the image taken in y -direction (b) for a ^{87}Rb BEC in a cigar-shaped trap for the ground state.

Apart from the ground state, we can build other states. In the following, we will show the first three excited states,

For the vortex state, $(n=0, m=1, k=0)$, we have

$$\psi_{010} = u_{010} \left(\frac{v_{010}}{\pi^3} \right)^{1/4} x_r e^{i\varphi} e^{-(u_{010}x_r^2 + v_{010}x_z^2)/2}. \quad (2.94)$$

Figure 2.6 – Density (a) and the image taken in y -direction (b) for a ^{87}Rb BEC in a cigar-shaped trap for the vortex state.

The wavefunction for the axial dipole $(n=0, m=0, k=1)$ is given by

$$\psi_{001} = \left(\frac{4u_{001}^2 v_{001}^2}{\pi^3} \right)^{1/4} x_z e^{-(u_{001}x_r^2 + v_{001}x_z^2)/2}, \quad (2.95)$$

Finally, we have the radial dipole state $(n=1, m=0, k=0)$

$$\psi_{100} = \left(\frac{u_{100}^2 v_{100}}{\pi^3} \right)^{1/4} (1 - u_{100}x_r^2) e^{-(u_{100}x_r^2 + v_{100}x_z^2)/2}, \quad (2.96)$$

Figure 2.7 – Density (a) and the image taken in y -direction (b) for a ^{87}Rb BEC in a cigar-shaped trap for the axial dipole state.

Figures 2.5, 2.6, 2.7, and 2.8 illustrate the density (a) and the image taken in y -direction (b) for a ^{87}Rb BEC in a cigar-shaped trap, as described in Refs. (4-7), for the ground, vortex, axial dipole and radial dipole, respectively.

The first proposal of generating a non-ground state BEC was made by Yukalov, Yukalova and Bagnato in Ref. (1). In that paper, they suggested in applying an external pumping field V_p , in the form

$$V_p = V(\vec{r}) \cos \omega t. \quad (2.97)$$

So, the Hamiltonian of the system is given by

$$H = H_{GP} + V_p, \quad (2.98)$$

where H_{GP} is the Hamiltonian given by Gross-Pitaevskii equation, as described above, and V_p is treated as a perturbation. If we consider that the pumping field is resonant with the transition frequency between the ground and an excited state, it would be possible to transfer atoms from one state to another.

In chapter 3, we propose an alternative way to generate the BEC in a non-ground state. Instead of applying a spatial-dependent field, we will apply a homogeneous one,

Figure 2.8 – Density (a) and the image taken in y -direction (b) for a ^{87}Rb BEC in a cigar-shaped trap for the radial dipole state.

close to a Feshbach resonance. Oscillating this field, we are oscillating the scattering length and, if this oscillation is resonant with some transition frequency, it is also possible to generate those excited topological modes.

2.4 Collective modes

Collective excitations can be understood as small deviations of equilibrium states, which we described in previous section as the topological modes. Those modes are usually generated by the modification of the trapping potential of the condensate (12). Many experiments focused on the collective excitation mode have been reported (13-16). Theoretical works were also intense using different kinds of approximations (17,18), but here, we have chosen to follow the work of Pérez-García *et al.* (19,20), where the authors treat the problem by a variational method. In this section, we will derive the frequency and behavior of three of those collective excitations, which are dipole, quadrupole and breathing mode.

We will start constructing the Lagrangian L of the system. As we can consider the BEC cloud as continuous body, it is convenient to work with the Lagrangian density $\mathcal{L}$, which is defined as

$$L = \int \mathcal{L} d\vec{r}. \quad (2.99)$$

Also, although in this thesis we will always work with cylindrically symmetric traps, here, for convenience, we will treat the problem in cartesian coordinates. So, we start with the Lagrangian density that describes a trapped BEC, which can be build (19,20) as follows

$$\mathcal{L} = \frac{i\hbar}{2} \left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right) + \frac{\hbar^2}{2m} |\nabla \psi|^2 + U_{trap} |\psi|^2 + A_s N |\psi|^2, \quad (2.100)$$

where, U_{trap} is the harmonic trap potential and is given by

$$U_{trap} = \frac{m}{2} (\omega_x x^2 + \omega_y y^2 + \omega_z z^2) \quad (2.101)$$

Thus, in order to solve the problem, we choose the trial function as a Gaussian of the form

$$\psi(x, y, z, t) = \mathcal{C} \prod_{\eta=x,y,z} \exp \left\{ -\frac{[\eta - \eta_0(t)]^2}{2w_\eta^2(t)} + i\alpha_\eta \eta + i\beta_\eta \eta^2 \right\}, \quad (2.102)$$

where

$$\mathcal{C} = \frac{1}{\sqrt{u_x(t)u_y(t)u_z(t)} \pi^{3/4}}$$

is the normalization constant, and $\eta_0(t)$, $w_\eta(t)$, $\alpha_\eta(t)$, and $\beta_\eta(t)$ are variational parameters. Substituting Equation (2.102) into (4.6) and (4.8), we obtain the Lagrangian of the system given by

$$\begin{aligned} L = \sum_{\eta=x,y,z} & \left[\frac{\hbar^2}{2m} \left(\frac{1}{2u_\eta^2} + 2u_\eta^2 \beta_\eta^2 + \alpha_\eta^2 + 4\eta_0^2 \beta_\eta^2 + 4\eta_0 \alpha_\eta \beta_\eta \right) + \right. \\ & \left. + \frac{\hbar}{2} (2\eta_0 \dot{\alpha}_\eta + 2\eta_0^2 \dot{\beta}_\eta + u_\eta^2 \dot{\beta}_\eta) + \frac{1}{2} \omega_\eta^2 u_\eta^2 + \frac{1}{2} \omega_\eta^2 \eta_0^2 \right] \\ & + \frac{1}{\sqrt{2\pi}} \frac{Na_s \hbar^2}{u_x u_y u_z}. \end{aligned} \quad (2.103)$$

The equations for the variational parameters are obtained as solutions of Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \quad (2.104)$$

where q_j correspond to each of the variational parameters. Thus, we obtain the equations of motion for the center of mass

$$\ddot{\eta}_0 + \omega_\eta^2 \eta_0 = 0, \quad (2.105)$$

and for the widths

$$\ddot{w}_\eta + \omega_\eta^2 w_\eta - \frac{\hbar^2}{m^2 w_\eta^3} - \sqrt{\frac{2}{\pi}} \frac{N \hbar^2 a_s}{m^2 w_\eta w_x w_y w_z} = 0. \quad (2.106)$$

Remembering that $\eta = x, y, z$. For other variational parameters, we have

$$\alpha_\eta = \frac{m \dot{\eta}_0}{\hbar} - 2\eta_0 \beta_\eta, \quad \beta_\eta = \frac{m \dot{w}_\eta}{2\hbar w_\eta}. \quad (2.107)$$

For this study we want to concentrate on the behavior of the widths $w_\eta(t)$ as well as the center of mass $\eta_0(t)$. In fact, the density distribution of the BEC sample depends only on those three parameters. In Section 2.4.1, we are going to discuss the behavior of the widths, given by Equation (2.106), specially in the linear regime.

2.4.1 Linear regime, breathing and quadrupole modes

Equations (2.106) cannot be solved analytically. However, we can investigate the linear response of the system and get the natural oscillations associated to the widths. First, in order to simplify Equation (2.106), we define the dimensionless parameters

$$\tau = t\omega_r \quad \lambda = \frac{\omega_z}{\omega_r} \quad u_\eta = \frac{w_\eta}{l_r} \quad P = \sqrt{\frac{2}{\pi}} \frac{N a_s}{l_r}, \quad (2.108)$$

where

$$l_r = \sqrt{\frac{\hbar^2}{m\omega_r}}.$$

Here, we normalized in terms of ω_r because we are considering that the condensate has a cylindrical symmetry, which means that $\omega_x = \omega_y = \omega_r$. Substituting the parameters given in Equations (2.108) into Equation (2.106), we obtain the equations of motion for the dimensionless widths, given by

$$\ddot{u}_r + u_r - \frac{1}{u_r^3} - \frac{P}{u_r^3 u_z} = 0 \quad (2.109)$$

$$\ddot{u}_z + \lambda^2 u_z - \frac{1}{u_z^3} - \frac{P}{u_r^2 u_z^2} = 0. \quad (2.110)$$

Now, we assume that those widths are given by

$$u_r = u_{r0} + \delta_r(t), \quad (2.111a)$$

$$u_z = u_{z0} + \delta_z(t), \quad (2.111b)$$

where u_{r0} and u_{z0} are the equilibrium position (time independent), and $\delta_r(t)$ and $\delta_z(t)$ are small deviations around the equilibrium. We can get the equilibrium positions by substituting Equations (2.111) into (2.110) and neglecting the time dependence of u_r and u_z . Thus, we have the following equations for equilibrium

$$u_{r0} - \frac{1}{u_{r0}^3} - \frac{P}{u_{r0}^3 u_{z0}} = 0 \quad (2.112a)$$

$$\lambda^2 u_{z0} - \frac{1}{u_{z0}^3} - \frac{P}{u_{r0}^2 u_{z0}^2} = 0. \quad (2.112b)$$

As those deviations $\delta_r(t)$ and $\delta_z(t)$ are small, we will only consider linear response. Thus, substituting Equations (2.111) into Equations (2.110), and neglecting terms above first order, we have

$$\ddot{\delta}_r + 4\delta_r + \tilde{P}\delta_z = 0 \quad (2.113a)$$

$$\ddot{\delta}_z + K\delta_z + 2\tilde{P}\delta_r = 0 \quad (2.113b)$$

where

$$K = \left(3\lambda^2 + \frac{1}{u_{z0}^4} \right) \quad \text{and} \quad \tilde{P} = \frac{P}{u_{r0}^3 u_{z0}^2}. \quad (2.114)$$

Rewriting Equations (2.113) in a matricial way, we have

$$\ddot{\vec{\delta}} + M\vec{\delta} = 0, \quad (2.115)$$

where

$$\vec{\delta} = \begin{bmatrix} \delta_r \\ \delta_z \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 4 & \tilde{P} \\ 2\tilde{P} & K \end{bmatrix}. \quad (2.116)$$

Applying the Fourier transform in Equation (2.115), we obtain

$$(M - \omega^2 I) \vec{\Delta} = 0, \quad (2.117)$$

where I is the identity matrix and $\vec{\Delta}$ is the Fourier transform of $\vec{\delta}$. The nontrivial solution

occurs when

$$\det(M - \omega^2 I) = \begin{vmatrix} 4 - \omega^2 & \tilde{P} \\ 2\tilde{P} & K - \omega^2 \end{vmatrix} = 0, \quad (2.118)$$

which gives

$$\omega^4 - (4 + K)\omega^2 + 4K - 2\tilde{P}^2 = 0, \quad (2.119)$$

whose solutions are

$$\omega^2 = 2 + \frac{K}{2} + \pm \frac{\sqrt{(K-4)^2 + 8\tilde{P}^2}}{2}. \quad (2.120)$$

These angular frequencies are dimensionless, written in units of ω_r . So, rewriting ω in units of angular frequency, we have

$$\omega_b = \omega_r \sqrt{2 + \frac{K}{2} + \frac{\sqrt{(K-4)^2 + 8\tilde{P}^2}}{2}} \quad (2.121a)$$

and

$$\omega_q = \omega_r \sqrt{2 + \frac{K}{2} - \frac{\sqrt{(K-4)^2 + 8\tilde{P}^2}}{2}}. \quad (2.121b)$$

In the following we will discuss what is the meaning of those two modes.

Substituting ω_b into Equation (2.117), we obtain a solution for $\vec{\Delta}$, given by

$$\vec{\Delta}_b = \Delta_z \begin{bmatrix} \frac{4 - K + \sqrt{(K-4)^2 + 8\tilde{P}^2}}{4\tilde{P}} \\ 1 \end{bmatrix}, \quad (2.122)$$

where Δ_z is the Fourier transform of δ_z . Here we can see that

$$4 - K + \sqrt{(K-4)^2 + 8\tilde{P}^2} > 0,$$

which means that δ_r and δ_z oscillates in phase, i. e., the widths of BEC oscillates in phase. This kind of oscillation is known as *breathing mode*. In the other hand, we have another mode ω_q . Substituting it into Equation (2.117), we have

$$\vec{\Delta}_q = \Delta_z \begin{bmatrix} \frac{4 - K - \sqrt{(K-4)^2 + 8\tilde{P}^2}}{4\tilde{P}} \\ 1 \end{bmatrix}. \quad (2.123)$$

As

$$4 - K - \sqrt{(K-4)^2 + 8\tilde{P}^2} < 0,$$

here we have the widths oscillating out of phase. This kind of oscillation is known as

quadrupole mode.

In this Section, we have found three collective modes of a cylindrically trapped BEC. In Chapter 4, we will get back to this subject. In the next Chapter, we will discuss the generation of topological modes.

3 *Generation of nonground-state Bose-Einstein condensates*

In this chapter, we show that the temporal modulation of the scattering length can be used for generating nonground-state condensates of trapped atoms. Such states are described by nonlinear topological coherent modes and can be excited by a resonant modulation of the trapping potential (1-3,9). So, to transfer the BEC from the ground to a nonground state, it is necessary to apply a time-dependent perturbation, at a frequency close to the considered transition. As a result the resonantly excited condensate becomes an effective two-level system. The external fields considered in the previous works (1-3,9) were formed by spatially inhomogeneous alternating trapping potentials.

We advance in an alternative way for exciting the coherent modes of a trapped BEC by including an oscillatory component in the scattering length. The main idea is to superimpose onto the BEC a uniform magnetic field with a small amplitude time variation. Due to the Feshbach resonance effect, such an oscillatory field creates an external perturbation in the system, coherently transferring atoms from the ground to a chosen excited coherent state. So, now, we have a very different situation represented by a spatially homogeneous time-oscillating magnetic field, which can be easily implemented with present experimental techniques. The feasibility of the experimental implementation of this phenomenon for available atomic systems is demonstrated at the end of the chapter.

3.1 Modulation of scattering length

We start with the GPE, described previously in Eq. (2.40),

$$i\hbar \frac{\partial \Phi}{\partial t} = \left[-\frac{\hbar^2}{2m_0} \nabla^2 + U_{trap}(\mathbf{r}) + A_s N |\Phi|^2 \right] \Phi. \quad (3.1)$$

As we have seen in the section 2.2.1, in the presence of a spatially uniform magnetic field, a_s near a Feshbach resonance is given by the relation

$$a_s = a_{nr} \left(1 - \frac{\Delta}{B - B_{res}} \right). \quad (3.2)$$

Let us consider the time-dependent magnetic field

$$B(t) = B_0 + b \cos(\omega t). \quad (3.3)$$

In such a case, Eq.(3.2) becomes

$$a_s(t) = a_{nr} \left(1 - \frac{\Delta}{B_0 - B_{res} + b \cos(\omega t)} \right), \quad (3.4)$$

If $|b| \ll |B_0 - B_{res}|$, $a_s(t)$ can be expanded to first order as

$$a_s(t) \simeq a_{av} + a \cos(\omega t), \quad (3.5)$$

where

$$a_{av} = a_{nr} \left(1 - \frac{\Delta}{B_0 - B_{res}} \right), \quad a = \frac{a_{nr} b \Delta}{(B_0 - B_{res})^2}. \quad (3.6)$$

The scattering length then possesses an oscillatory component around the average value.

Combining Eq.(3.5) and Eq.(3.1), one gets the GPE with the additional oscillatory term $V = V(\mathbf{r}, t)$. With the notation $H = H_0 + V$, one has

$$H\Phi = i\hbar \frac{\partial \Phi}{\partial t}, \quad (3.7)$$

where,

$$H_0 = -\frac{\hbar^2}{2m_0} \nabla^2 + U_{trap}(\mathbf{r}) + A_{av} N |\Phi|^2, \quad (3.8)$$

and

$$V = AN \cos(\omega t) |\Phi|^2, \quad (3.9)$$

with

$$A_{av} = \frac{4\pi\hbar^2}{m_0} a_{av}, \quad A = \frac{4\pi\hbar^2}{m_0} a.$$

In order to solve Equation(3.7), we start considering as the total wavefunction a linear combination of a complete set of modes as follows

$$\Phi(\mathbf{r}, t) = \sum_j c_j(t) \phi_j(\mathbf{r}) e^{-iE_j t/\hbar}, \quad (3.10)$$

where $\phi_j(\mathbf{r})$ are stationary solutions for the equation $H_0 \phi_j = E_j \phi_j$, with eigenenergies E_j .

Here we are going to do usual time-dependent perturbation theory. Substituting Equation (3.10) into (3.7), multiplying by $\phi_m^*(\mathbf{r}) e^{-iE_m t/\hbar}$ and integrating over all space

$$\int \phi_m^* e^{-iE_m t/\hbar} (H_0 + V) \Phi d\mathbf{r} = \int \phi_m^* e^{-iE_m t/\hbar} i\hbar \frac{\partial \Phi}{\partial t} d\mathbf{r},$$

we obtain

$$\begin{aligned} i\hbar \frac{dc_m}{dt} &= A_{av} \sum_{n,k \neq n} |c_k|^2 c_n e^{i\omega_{mn}t} \int \phi_m^* (|\phi_k|^2 - |\phi_n|^2) \phi_n d\mathbf{r} + \\ &+ A_{av} \sum_{n,k,l \neq k} c_k^* c_l c_n e^{i(\omega_{mn} + \omega_{kl})t} \int \phi_m^* \phi_k^* \phi_l \phi_n d\mathbf{r} + \\ &+ A \sum_{n,k,l} c_k^* c_l c_n e^{i(\omega_{mn} + \omega_{kl})t} \cos \omega t \int \phi_m^* \phi_k^* \phi_l \phi_n d\mathbf{r}, \end{aligned} \quad (3.11)$$

where

$$w_{ij} = \frac{E_i - E_j}{\hbar}. \quad (3.12)$$

As V is a perturbation, the coefficients c_n should vary slowly in time when compared with a characteristic frequency of the unperturbed system. Thus, it is expected that

$$\left| \frac{dc_n}{dt} \right| \ll \frac{E_n}{\hbar}. \quad (3.13)$$

Therefore, we can treat c_n and its time derivative as almost invariant in time.

If a function $f(t)$ is almost invariant in time for a given period τ , it is reasonable to assume that

$$\frac{1}{\tau} \int_0^\tau f(t) e^{i\omega_{mn}t} dt \simeq \frac{1}{\tau} f_0 \int_0^\tau e^{i\omega_{mn}t} dt.$$

Using this consideration, we can get a set of Equations for c_n taking a time average of Equation (3.11). Before we do that, we must have at hand some expressions. First,

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau e^{i\omega_{mn}t} dt = \delta_{mn} \quad (3.14a)$$

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau e^{i(\omega_{mn} + \omega_{kl})t} dt = \delta_{mn} \delta_{kl} + \delta_{ml} \delta_{kn} - \delta_{mk} \delta_{ln} \delta_{nl}. \quad (3.14b)$$

Moreover, our aim is to reach in a two level system. So, we impose that the frequency ω is close to a transition frequency, ω_{p0} , between the ground and an excited state p , i.e.,

$$\Delta\omega = \omega - \omega_{p0} \rightarrow 0. \quad (3.15)$$

Thus, we have

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau e^{i\omega_{mn}t} \cos \omega t dt = \delta_{m,0} \delta_{n,p} e^{i\Delta\omega t} + \delta_{m,p} \delta_{n,0} e^{-i\Delta\omega t}. \quad (3.16)$$

Also, we assume that there is no parametric excitation. This means that we neglect any combination of frequencies that summed or subtracted is equal to the transition frequency ω_{p0} . With that, we have

$$\begin{aligned} \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau e^{i(\omega_{mn} + \omega_{kl})t} \cos \omega t dt = & \\ & (\delta_{n,p} \delta_{k,0} \delta_{m,l} + \delta_{l,p} \delta_{m,0} \delta_{n,k} + \delta_{n,p} \delta_{m,0} \delta_{l,k} + \delta_{l,p} \delta_{k,0} \delta_{m,n}) \frac{e^{i\Delta\omega t}}{2} + \\ & + (\delta_{m,p} \delta_{n,0} \delta_{k,l} + \delta_{k,p} \delta_{l,0} \delta_{n,m} + \delta_{m,p} \delta_{l,0} \delta_{n,k} + \delta_{k,p} \delta_{n,0} \delta_{m,l}) \frac{e^{-i\Delta\omega t}}{2} - \\ & - (\delta_{m,0} \delta_{n,l,k,p} + \delta_{k,0} \delta_{n,l,m,p} + \delta_{n,p} \delta_{m,l,k,0} + \delta_{l,p} \delta_{m,k,n,0}) \frac{e^{i\Delta\omega t}}{2} - \\ & - (\delta_{m,p} \delta_{n,l,k,0} + \delta_{k,p} \delta_{n,l,m,0} + \delta_{n,0} \delta_{m,l,k,p} + \delta_{l,0} \delta_{m,k,n,p}) \frac{e^{-i\Delta\omega t}}{2}. \end{aligned} \quad (3.17)$$

Now, with Equations (3.14), (3.16) and (3.17), we can take the time average of Equation (3.11), which give us

$$\begin{aligned} i\hbar \frac{dc_m}{dt} = & A_{av} \sum_{k \neq m} |c_k|^2 c_m (2I_{m,k,m} - I_{m,m,m}) + \\ & + \frac{A}{2} e^{i\Delta\omega t} \left[2c_0^* c_p c_m I_{0,m,p} - \delta_{m,p} c_0^* c_p^2 I_{0,p,p} + \delta_{m,0} \left(|c_p|^2 c_p I_{0,p,p} + \sum_{k \neq 0,k} 2|c_k|^2 c_p I_{0,k,p} \right) \right] + \\ & + \frac{A}{2} e^{-i\Delta\omega t} \left[2c_p^* c_0 c_m I_{p,m,0} - \delta_{m,0} c_p^* c_0^2 I_{p,0,0} + \delta_{m,p} \left(|c_0|^2 c_0 I_{p,0,0} + \sum_{k \neq 0,k} 2|c_k|^2 c_0 I_{p,k,0} \right) \right], \end{aligned} \quad (3.18)$$

where the integral $I_{j,k,l}$ is defined as

$$I_{j,k,l} = \int \phi_j^* |\phi_k|^2 \phi_l d\mathbf{r}. \quad (3.19)$$

Equations (3.18) give us that our system can be approached as a two-level one. It is easy to see if we take a look at the temporal evolution of population fractions, which is defined as

$$n_m(t) = |c_m(t)|^2. \quad (3.20)$$

First, let us investigate the behavior of coefficients of modes that are different from the

ground and the excited state p . Thus, we have

$$\frac{dc_q}{dt} = -\frac{iA_{av}}{\hbar} \sum_{k \neq q} |c_k|^2 c_q (2I_{q,k,q} - I_{q,q,q}) - \frac{iA}{\hbar} c_q (c_0^* c_p I_{0,q,p} e^{i\Delta\omega t} + c_p^* c_0 I_{p,q,0} e^{-i\Delta\omega t}). \quad (3.21)$$

and for population n_q ,

$$\begin{aligned} \frac{dn_q}{dt} &= c_q \frac{dc_q^*}{dt} + c_q^* \frac{dc_q}{dt} \\ &= i|c_q|^2 \left[\frac{A_{av}}{\hbar} |c_k|^2 (2I_{q,k,q} - I_{q,q,q}) + \frac{A}{\hbar} (c_0^* c_p I_{0,q,p} e^{i\Delta\omega t} + c_p^* c_0 I_{p,q,0} e^{-i\Delta\omega t}) \right] \\ &\quad i|c_q|^2 \left[\frac{A_{av}}{\hbar} |c_k|^2 (2I_{q,k,q} - I_{q,q,q}) + \frac{A}{\hbar} (c_0^* c_p I_{0,q,p} e^{i\Delta\omega t} + c_p^* c_0 I_{p,q,0} e^{-i\Delta\omega t}) \right], \\ &= 0, \quad q \neq 0, p. \end{aligned} \quad (3.22)$$

As, initially, all atoms of BEC are in the ground state, i.e, $n_m(0) = \delta_{m,0}$, we conclude that $n_q(t) = 0$ for all time, and therefore, $c_q(t) = 0$.

Now, for ground and the excited state p , we obtain from Equations (3.18),

$$\begin{aligned} i\hbar \frac{dc_0}{dt} &= A_{av} |c_p|^2 c_0 (2I_{0,p,0} - I_{0,0,0}) \\ &+ \frac{A}{2} [e^{i\Delta\omega t} (|c_p|^2 c_p I_{0,p,p} + 2|c_0|^2 c_p I_{0,0,p}) + e^{-i\Delta\omega t} c_p^* c_0^2 I_{p,0,0}] , \end{aligned} \quad (3.23a)$$

$$\begin{aligned} i\hbar \frac{dc_p}{dt} &= A_{av} |c_0|^2 c_p (2I_{p,0,p} - I_{p,p,p}) \\ &+ \frac{A}{2} [e^{-i\Delta\omega t} (|c_0|^2 c_0 I_{p,0,0} + 2|c_p|^2 c_0 I_{p,p,0}) + e^{i\Delta\omega t} c_0^* c_p^2 I_{0,p,p}] . \end{aligned} \quad (3.23b)$$

Then, in this case, the total wavefunction (3.10) can be represented, in a good approximation, by

$$\Phi(\mathbf{r}, t) = c_0(t) \phi_0(\mathbf{r}) e^{-iE_0 t/\hbar} + c_p(t) \phi_p(\mathbf{r}) e^{-iE_p t/\hbar}. \quad (3.24)$$

In summary, to derive the latter equations, two assumptions are made. First, the time variation of $c_0(t)$ and $c_p(t)$ are to be much slower than the exponential oscillations with the transition frequency $\omega_{p0} = (E_p - E_0)/\hbar$. This condition is fulfilled, when the amplitudes $A_{av} I_{j,k,l}$ and $A I_{j,k,l}$ are smaller than $\hbar\omega_{p0}$. The second is the resonance condition, when the external alternating field connects only the two chosen nonlinear states. Another point concerns damping due to collisions between particles in the desired modes or collisions with the thermal cloud. Although the oscillation time for populations takes tens of trap periods, this time is much smaller than the lifetime of a typical BEC or a vortex state

(23,24). So, we expect that damping occurs but not as a dominant process. Thus, we have left out the damping effect for this model.

Another important aspect is that the total number of atoms does not vary in time, but the number in each state does. This variation is taking into account in Eqs. (3.23) since these equations depend on the population of each state, represented by $|c_0|^2$ and $|c_p|^2$. However, the modes ϕ_j in equation (3.10) are stationary solutions of Equation (3.8) when all atoms are in state j . Thus, if there is a variation in the atom number of some state, there is a variation in the wave function that represents this state. So, the total wave function should be written in the form

$$\Phi(\mathbf{r}, t) = \sum_j d_j(t) \phi'_j(\mathbf{r}, t), \quad (3.25)$$

where the number dependence is inserted in the time dependence. In this way, the population of a state j would be given by $|d_j(t)|^2$ and not by $|c_j(t)|^2$, since the expansions (3.10) and (3.25) are different. However, in the case of our study, the system is in a weak-coupling regime, i.e., g is small, so the variation of the wave function can be neglected and the population of a state j can be given by

$$n_j(t) \approx |c_j(t)|^2.$$

3.2 Application to a cylindrically symmetric trap

We consider a cylindrically symmetric harmonic trap

$$U_{trap} = \frac{m_0}{2}(\omega_r^2 r^2 + \omega_z^2 z^2), \quad (3.26)$$

and use the optimized perturbation theory, as discussed in Section 2.3.1, for finding the modes ϕ_0 and ϕ_p . However, it is convenient to use the dimensionless variables defined in Equations from (2.77) to (2.82). In this way, we can rewrite Equations (3.23) as

$$\begin{aligned} \frac{dc_0}{dt'} &= -ig_0 |c_p|^2 c_0 (2J_{0,p,0} - J_{0,0,0}) \\ &+ -i\frac{g_0}{2} e^{i\delta t'} \frac{a}{a_{av}} \left[|c_p|^2 c_p J_{0,p,p} + 2|c_0|^2 c_p J_{0,0,p} + c_p^* c_0^2 J_{p,0,0} e^{-2i\delta t'} \right] \end{aligned} \quad (3.27a)$$

$$\begin{aligned} \frac{dc_p}{dt'} &= -ig_0 |c_0|^2 c_p (2J_{p,0,p} - J_{p,p,p}) \\ &+ -i\frac{g_0}{2} e^{-i\delta t'} \frac{a}{a_{av}} \left[|c_0|^2 c_0 J_{p,0,0} + 2|c_p|^2 c_0 J_{p,p,0} + c_0^* c_p^2 J_{0,0,p} e^{2i\delta t'} \right], \end{aligned} \quad (3.27b)$$

where

$$t' = \omega_r t \quad \delta = \frac{\Delta\omega}{\omega_r}, \quad g_0 = \frac{4\pi N a_{av}}{l_r}. \quad (3.28)$$

In the next Section we will present numerical solutions of this set of Equations (3.27) and discuss them.

3.2.1 Time evolution of populations

Using the fourth-order Runge-Kutta method (25), we calculate the time evolution of the coefficients $c_0(t)$ and $c_p(t)$ for different values of the detuning δ and scattering length amplitude a .

For an excited mode, we take the radial dipole state $\{100\}$, which is the lowest excited mode that couples with the ground state $\{000\}$. Fig.3.9, where $\lambda = 0.2$ and $g_0 = 70$, shows the time evolution of the mode populations n_0 and n_p for different values of the detuning δ and a/a_{av} , which is given by Eq. (3.6). The chosen parameters correspond to typical experimental setups and are easily controlled in a laboratory. The solutions demonstrate different behaviors of the state populations. For $a/a_{av} = 0.7$ and $\delta = 0$, Fig.3.9(a), the populations display small oscillation amplitudes, with a considerably larger population in the ground state. Increasing the detuning to $\delta = 0.04$ results in the behavior shown in Fig.3.9(b). Although on average atoms stay longer in the ground state, for some intervals of time n_p is larger than n_0 . Changing the amplitude to $a/a_{av} = 0.72$ and maintaining $\delta = 0.04$, as in Fig.3.9(c), yields a very different temporal behavior. Atoms now stay longer in the excited state rather than in the ground state. The shape of the functions shows the inherent nonlinearity of the system. If the amplitude is increased further to $a/a_{av} = 1$, with $\delta = 0.04$, as in Fig. 3.9(d), the system shows a full population inversion. For certain times, when the mode population fully migrates from the ground $\{000\}$ to the excited $\{100\}$ state, it is possible to have a pure condensate in the coherent topological excited mode.

Figure 3.9 – Populations of the ground state n_0 (black line) and excited state $\{100\}$ n_p (red line) as a function of time for $\lambda = 0.2$ and $g_0 = 70$ with (a) $a/a_{av} = 0.7$ and $\delta = 0$; (b) $a/a_{av} = 0.7$ and $\delta = 0.04$; (c) $a/a_{av} = 0.72$ and $\delta = 0.04$; (d) $a/a_{av} = 1$ and $\delta = 0.04$

3.2.2 Dynamical order parameter

A convenient way to quantify the population behavior is through the introduction of an order parameter η , defined as the difference between the time-averaged populations for both states (2,26),

$$\eta = \bar{n}_0 - \bar{n}_p. \quad (3.29)$$

Here, the average of each population is performed over the full cycle of an oscillation.

The above order parameter η displays a nontrivial behavior when the ratio a/a_{av} is modified. For different detunings, the variation of η as a function of the ratio a/a_{av} is presented in Fig.3.10. The variation of η vs. a/a_{av} can be smooth, when $\delta \leq 0.03$, or can show sudden changes, when $\delta > 0.03$. Smaller values of a/a_{av} keep atoms preferably in the ground state. For some critical value of a/a_{av} , η becomes negative, which means that BEC stays longer with a larger population in the excited state than in the ground state. This situation is ideal for detecting the formation of topological modes.

Figure 3.10 – Order parameter η as a function of the ratio a/a_{av} , for different detunings showing a phase-transition like behavior.

An important question is the feasibility of the experimental creation of such coherent modes. The main parameter here is a/a_{av} , given by Eq.(3.6). It shows us that this value is strongly dependent on the Feshbach resonance width Δ , characteristic of each type of atoms. For systems with small Δ , the required value of a/a_{av} for obtaining the transition in η will occur only for $B_0 \approx B_{res}$. In this case, the necessary condition $b \ll B_{res} - B_0$ can only be fulfilled for very small values of b , which creates an extra difficulty with the present techniques of magnetic field control (27,28). As an experimentally realistic example, let us consider the case of $b = 0.1(B_{res} - B_0)$ and $a/a_{av} = 0.8$, which corresponds to the atomic parameters listed in Tab. 3.1. Setting $g_0 = 70$, $\omega_r = 2\pi \times 120 \text{ Hz}$, we obtain a value of b of 0.9 G for ^{85}Rb , and of 0.02 G for ^{87}Rb , which would be difficult to control. On the other hand, 10 G , for ^7Li , and 4.55 G , for ^{39}K , are the values which can be realized with the present technical capabilities (27,28).

Let us consider a condensate containing 10^5 ^7Li atoms in a trap with radial frequency $\omega_r = 2\pi \times 120 \text{ Hz}$ and $\lambda = 0.2$. With these conditions, together with the information from Tab. 3.1, the bias magnetic field is $B_0 = 632.5 \text{ G}$. The obtained result for the behavior

Table 3.1 – Amplitudes B_0 and b of the magnetic field for four different species of atoms . We set $g_0 = 70$, $\lambda = 0.2$, $a/a_{av} = 0.8$ and $b = 0.1(B_{res} - B_0)$. Scattering length is expressed in units of the Bohr radius.

Atom	$B_{res}(G)$	$\Delta(G)$	a_{av}	$B_0(G)$	$b(G)$	$N(\times 10^4)$
^{85}Rb (29)	155.0	10.7	-443	164.4	-0.9	0.2
^{87}Rb (30)	1007.34	0.17	100	1007.53	0.02	0.9
^7Li (27)	736.8	-192	-27.5	636	10	9.3
^{39}K (28)	403.4	-52	-23	357.9	4.55	4.7

of η as a function of b is shown in Figure 3.11. Considering different detunings, defined as $\delta\omega = \omega - \omega_{p0}$, where $\omega_{p0} = \omega_{100,0} = 2\pi \times 209.6 \text{ Hz}$, we observe the critical values for b ranging from 4 G ($\delta\omega = 2\pi \times 21 \text{ Hz}$) to 10.5 G ($\delta\omega = 2\pi \times 6.3 \text{ Hz}$). Such oscillating amplitudes correspond to less than 1% of the total bias field B_0 .

Figure 3.11 – Order parameter η as function of the magnetic field amplitude b , for different detunings, for a BEC with 10^5 ^7Li atoms and a field bias $B_0 = 635.2 \text{ G}$.

The behavior of η resembles that of an order parameter typical of a phase transition. Therefore, it is possible to define a critical exponent characterizing the approach of b to a critical value b_c , considering that, near b_c , $\eta \propto |(b - b_c)|^\beta$. For the case of $\delta\omega = 2\pi \times 12.6 \text{ Hz}$, we obtain $b_c \approx 6.5 \text{ G}$ for ^7Li , as is shown in Fig.3.11, and the critical exponent $\beta \approx 0.26$.

An interesting case occurs for $\delta\omega = 2\pi \times 8.4 \text{ Hz}$, where we have two different behaviors of η . If $b \rightarrow b_c^-$, $\beta \approx 0.39$; if $b \rightarrow b_c^+$, $\beta \approx -0.25$. In that case, the critical amplitude $b_c \approx 8.4 G$.

4 *Collective excitations*

In this chapter we will present the results about the excitation of collective modes via oscillation of the interaction. The approach is quite similar to the one presented in Section 2.4. The difference is that now the scattering length oscillates, as shown in Equation (3.5), like

$$a_s(t) = a_{av} + a \cos \omega t.$$

So, in this case, we have the Lagrangian

$$\mathcal{L} = \frac{i\hbar}{2} \left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right) + \frac{\hbar^2}{2m} |\nabla \psi|^2 + U_{trap} |\psi|^2 + (A_0 + A \cos \omega t) N |\psi|^2. \quad (4.1)$$

As the applied field is homogeneous, we expect that the BEC cloud does not change its centre of mass, i.e, there is no dipolar excitation. Also, the symmetry will not break down, so we can treat the problem in the cylindrical symmetry. This means that we can suggest a trial function that does not has the η_0 parameter and it is written in cylindrical coordinate as

$$\psi(r, z, t) = \mathcal{C} \exp \left[-\frac{r^2}{2w_r^2(t)} + i\alpha_r r + i\beta_r r^2 - \frac{z^2}{2w_z^2(t)} + i\alpha_z z + i\beta_z z^2 \right], \quad (4.2)$$

where

$$\mathcal{C} = \frac{1}{\sqrt{u_r(t)^2 u_z(t)} \pi^{3/4}}.$$

Following that same procedure, we get the same Equations for widths

$$\ddot{w}_r + \omega_r^2 w_r - \frac{\hbar^2}{m^2 w_r^3} - \sqrt{\frac{2}{\pi}} \frac{N \hbar^2 a_s}{m^2 w_r^3 w_z} = 0, \quad (4.3a)$$

$$\ddot{w}_z + \omega_z^2 w_z - \frac{\hbar^2}{m^2 w_z^3} - \sqrt{\frac{2}{\pi}} \frac{N \hbar^2 a_s}{m^2 w_r^2 w_z^2} = 0, \quad (4.3b)$$

with the difference that now a_s is time dependent. Consequently we have the same

oscillations mode as presented before; breathing mode

$$\omega_b = \omega_r \sqrt{2 + \frac{K}{2} + \frac{\sqrt{(K-4)^2 + 8\tilde{P}^2}}{2}} \quad (4.4)$$

and quadrupole mode

$$\omega_q = \omega_r \sqrt{2 + \frac{K}{2} - \frac{\sqrt{(K-4)^2 + 8\tilde{P}^2}}{2}}. \quad (4.5)$$

We solved numerically Equations (4.3), by fourth-order Runge-Kutta method, and compared with experimental data presented in Ref. (31). The experiment consists of a BEC with 3×10^5 atoms of ^7Li in the $|1, 1\rangle$ state. In this state, the atom has a Feshbach resonance at $B_{res} = 736.8 \text{ G}$, with $\Delta = 192.3 \text{ G}$ and the non-resonant scattering length $a_{nr} = -24.5a_0$, where a_0 is the Bohr radius. The BEC cloud is trapped in an optical trap with $\omega_r = 2\pi \times 235 \text{ Hz}$ and $\omega_z = 2\pi \times 4.85$. Also, it is applied a homogeneous magnetic field with a bias $B_0 = 565 \text{ G}$ and an oscillation amplitude of $B = 14 \text{ G}$, which corresponds to $a_{av} = 3a_0$ and $a = 2a_0$.

In Figure 4.12 is shown the temporal evolution of the axial width both experimental (black circles) and calculated by Equations (4.3) (red line). The experimental data was taken with a BEC submitted to an oscillatory magnetic field with a frequency of 3 Hz during 0.8 s and, after that, the excitation was turned off. The theoretical curve that best fit with data, with no adjustable parameters, was calculated with 2.4 Hz . We can see a good agreement with experiment and theory except for the excitation frequency.

Furthermore, we can see, after 0.8 s , when the excitation is turned off, that the cloud oscillates at the quadrupole frequency $\omega_q \approx 2\pi \times 8.17 \text{ Hz}$. In Figure 4.13 we present the spectrum of theoretical calculation of width oscillation during the excitation, where the driven frequency is marked with red dotted line and the quadrupole mode with blue dashed line. The presence of these two modes it is clear, but it is also noted that there are the generation of non linear modes.

Moreover, in Figure 4.14, we present the oscillation of axial width with the same parameters as before, except for frequencies. For experimental data (black circle), we have $\omega = 2\pi \times 10 \text{ Hz}$; for theoretical, $\omega = 2\pi \times 9.4 \text{ Hz}$. Again, we have a good agreement between experiment and theory, except for frequencies. Also is noted the quadrupole oscillation after the excitation was turned off.

Figure 4.12 – Temporal evolution of axial width of a BEC of ${}^7\text{Li}$ for experimental (black circle) and theoretical (red line) data with $a_s = 3a_0 + 2a_0 \cos \omega t$. Experimental was taken with $\omega = 2\pi \times 3 \text{ Hz}$ and theoretical with $\omega = 2\pi \times 2.4 \text{ Hz}$.

Figure 4.13 – Spectrum of theoretical calculation of axial width oscillation with $\omega = 2\pi \times 2.4 \text{ Hz}$. Driven frequency is marked with red dotted line and quadrupole mode with blue dashed line.

Figure 4.14 – Temporal evolution of axial width of a BEC of ^7Li for experimental (black circle) and theoretical (red line) data with $a_s = 3a_0 + 2a_0 \cos \omega t$. Experimental was taken with $\omega = 2\pi \times 10 \text{ Hz}$ and theoretical with $\omega = 2\pi \times 9.4 \text{ Hz}$.

In Figure 4.15, it is shown the spectrum of theoretical calculation of oscillation for $\omega = 2\pi 9.4 \text{ Hz}$. Again, we can see both frequencies, the driven and quadrupole, but the latter has a small shift. Also, we did the spectrum when the excitation frequency is $\omega = \omega_q$, which is presented in Figure 4.16. Here, beyond ω_q , there also are generation of nonlinear modes.

4.1 Coupling between dipole and quadrupole modes

Let us consider a Bose-Einstein condensate in a harmonic magnetic trap with a bias field near a Feshbach resonance. Suppose also that the dipole mode is excited, i. e., the center of mass of the BEC is displaced from its equilibrium position and the cloud starts to oscillate. We will treat this problem as a variational one, as presented in Section 2.4, setting a trial function with time-dependent parameters and minimize a Lagrangian with respect of these parameters.

Thus, we start with the same Lagrangian density that describes a trapped BEC

Figure 4.15 – Spectrum of theoretical calculation of axial width oscillation with $\omega = 2\pi \times 9.4$ Hz. Driven frequency is marked with red dotted line and quadrupole mode with blue dashed line.

Figure 4.16 – Spectrum of theoretical calculation of axial width oscillation with $\omega = \omega_q \approx 2\pi \times 8.17$ Hz. Quadrupole mode is marked with blue dashed line.

$$\mathcal{L} = \frac{i\hbar}{2} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) + \frac{\hbar^2}{2m} |\nabla \psi|^2 + V_{trap} |\psi|^2 + A_s N |\psi|^2, \quad (4.6)$$

where m is the atomic mass, V_{trap} is the trap potential and

$$A_s = \frac{4\pi\hbar^2 a_s}{m}, \quad (4.7)$$

where a_s is the scattering length. From Equation (4.6), we obtain the total Lagrangian of the system integrating over space coordinates,

$$L = \int \mathcal{L} d^3r. \quad (4.8)$$

As we previously said, the trap potential V_{trap} is harmonic with a bias field. So, considering a cylindrical symmetry, we can write

$$V_{trap} = \mu B_0 + \frac{m}{2} (\omega_r^2 r^2 + \omega_z^2 z^2), \quad (4.9)$$

where μ is the atomic magnetic dipole moment, B_0 is the bias field, ω_r and ω_z are, respectively, radial and axial angular frequencies. Besides, in a presence of a magnetic field, the dispersion relation of the scattering length is given by

$$a_s(B) = a_{nr} \left(1 - \frac{\Delta}{B - B_{res}} \right), \quad (4.10)$$

Here, we will do two assumptions. First, the BEC center of mass oscillates only in the z direction; second, the amplitude of that oscillation is much bigger than the size of BEC, i. e., the cloud feels the same field equally at any point. So, B can be written as

$$B = B_0 + \frac{m\omega_z^2 z_0^2(t)}{2\mu}, \quad (4.11)$$

where $z_0(t)$ is the BEC center of mass in z direction. Thus, the scattering length (4.10) is given by

$$a_s(B) = a_{nr} \left(1 - \frac{\Delta}{B_0 - B_{res} + \frac{m\omega_z^2 z_0^2(t)}{2\mu}} \right) \quad (4.12)$$

Thus, in order to solve the problem, we choose the trial function as a Gaussian of the form

$$\psi(r, z, t) = \mathcal{C} \exp \left\{ -\frac{r^2}{2w_r^2(t)} + i\alpha_r(t)r + i\beta_r(t)r^2 - \frac{[z - z_0(t)]^2}{2w_z^2(t)} + i\alpha_z z + i\beta_z z^2 \right\}, \quad (4.13)$$

where

$$\mathcal{C} = \frac{1}{u_r(t) \sqrt{u_z(t)} \pi^{3/4}}$$

is the normalization constant, $w_r(t)$, $w_z(t)$, $\alpha_r(t)$, $\alpha_z(t)$, $\beta_r(t)$, and $\beta_z(t)$ are variational parameters. Substituting the trial function (4.13) into Equations (4.6) and (4.8), we obtain

$$\begin{aligned} L = & \frac{\hbar}{2} \left(2w_r^2 \dot{\beta}_r + \sqrt{\pi} w_r \dot{\alpha}_r + w_z^2 \dot{\beta}_z + 2z_0 \alpha_z + 2z_0^2 \dot{\beta}_z \right) + \\ & \frac{\hbar^2}{2m} \left(4w_r^2 \beta_r^2 + \frac{1}{w_r^2} + \alpha_r^2 + 2\sqrt{\pi} w_r \alpha_r \beta_r + 2w_z^2 \beta_z^2 + \frac{1}{2w_z^2} + \alpha_z^2 + 4z_0 \alpha_z \beta_z + 4z_0^2 \beta_z \right) + \\ & \frac{m}{2} \left(\omega_r^2 w_r^2 + \frac{\omega_z^2 w_z^2}{2} \right) + \frac{N \hbar^2 a_{nr}}{\sqrt{2\pi m} w_r^2 w_z} \left(1 - \frac{\Delta}{B_0 + B_{res} + \frac{m \omega_z^2 z_0^2}{2\mu}} \right). \end{aligned} \quad (4.14)$$

Here, for simplicity, we suppress the explicit time dependence of the variational parameters. Also, the dots on the parameters represent time derivatives.

So, the problem lies in solving the Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \quad (4.15)$$

where q_j is a variational parameters that belongs to the set defined as $q \equiv \{w_r, w_z, \alpha_r, \alpha_z, \beta_r, \beta_z\}$.

Thus, for the set q of parameter we obtain

$$\alpha_r = 0, \quad \alpha_z = \frac{m \dot{z}_0}{\hbar} - 2z_0 \beta_z, \quad (4.16a)$$

$$\beta_r = \frac{m}{2\hbar} \frac{\dot{w}_r}{w_r}, \quad \beta_z = \frac{m}{2\hbar} \frac{\dot{w}_z}{w_z}, \quad (4.16b)$$

$$(4.16c)$$

$$\ddot{w}_r + \omega_r^2 w_r - \frac{\hbar^2}{m^2 w_r^3} - \sqrt{\frac{2}{\pi}} \frac{N \hbar^2 a_{nr}}{m^2 w_r^3 w_z} \left(1 - \frac{\Delta}{B_0 - B_{res} + \frac{m \omega_z^2 z_0^2}{2\mu}} \right) = 0, \quad (4.17a)$$

$$\ddot{w}_z + \omega_z^2 w_z - \frac{\hbar^2}{m^2 w_z^3} - \sqrt{\frac{2}{\pi}} \frac{N \hbar^2 a_{nr}}{m^2 w_r^2 w_z^2} \left(1 - \frac{\Delta}{B_0 - B_{res} + \frac{m \omega_z^2 z_0^2}{2\mu}} \right) = 0, \quad (4.17b)$$

$$\ddot{z}_0 + \omega_z^2 z_0 \left[1 + \frac{N \hbar^2 \Delta a_{nr}}{\sqrt{2\pi} m w_r^2 w_z \left(B_0 - B_{res} + \frac{m \omega_z^2 z_0^2}{2\mu} \right)} \right] = 0. \quad (4.17c)$$

We just have to know the behavior of the widths w_r and w_z , as well as the center of mass in z direction z_0 , to know the behavior of the other parameters. In fact, the density of BEC will only depends on those three parameters (widths and center of mass). In order to make Equations (4.17) simpler, we will let them dimensionless. For this, we define some parameters as follows

$$l_r = \sqrt{\frac{\hbar}{m \omega_r}} \quad \lambda = \frac{\omega_z}{\omega_r} \quad \tau = t \omega_r \quad (4.18a)$$

$$u_r = \frac{w_r}{l_r} \quad u_z = \frac{w_z}{l_r} \quad \zeta = \frac{z_0}{l_r} \quad (4.18b)$$

$$d = \frac{\Delta \mu}{\hbar \omega_r} \quad b = \frac{\mu(B_0 - B_{res})}{\hbar \omega_r} \quad P = \sqrt{\frac{2}{\pi}} \frac{N a_{nr}}{l_r}. \quad (4.18c)$$

Thus, for the dimensionless widths and center of mass, we have

$$\ddot{u}_r + u_r - \frac{1}{u_r^3} - \frac{P}{u_r^3 u_z} \left(1 - \frac{2d}{2b + \lambda^2 \zeta^2} \right) = 0 \quad (4.19a)$$

$$\ddot{u}_z + \lambda^2 u_z - \frac{1}{u_z^3} - \frac{P}{u_r^2 u_z^2} \left(1 - \frac{2d}{2b + \lambda^2 \zeta^2} \right) = 0 \quad (4.19b)$$

$$\ddot{\zeta} + \lambda^2 \zeta \left[1 + \frac{2Pd}{u_r^2 u_z (2b + \lambda^2 \zeta^2)^2} \right] = 0. \quad (4.19c)$$

4.1.1 Linear response

In this section we will investigate the linear response of the widths u_r and u_z , as well as the center of mass z_0 , according to evolution equations given by (4.19). In this way, we assume that those parameter are in the form

$$u_r = u_{r0} + \delta_r(t), \quad (4.20a)$$

$$u_z = u_{z0} + \delta_z(t), \quad (4.20b)$$

$$\zeta = \zeta_0 + \delta_\zeta(t), \quad (4.20c)$$

where u_{r0} , u_{z0} , and ζ_0 are the equilibrium position (time independent), and $\delta_r(t)$, $\delta_z(t)$, and $\delta_\zeta(t)$ are small deviations around the equilibrium. We can get the equilibrium positions by Equations (4.19) neglecting the time dependence of u_r , u_z and ζ . Thus, we have the following equations for equilibrium

$$\zeta_0 = 0 \quad (4.21a)$$

$$u_{r0} - \frac{1}{u_{r0}^3} - \frac{P}{u_{r0}^3 u_{z0}} \left(1 - \frac{d}{b}\right) = 0 \quad (4.21b)$$

$$\lambda^2 u_{z0} - \frac{1}{u_{z0}^3} - \frac{P}{u_{r0}^2 u_{z0}^2} \left(1 - \frac{d}{b}\right) = 0. \quad (4.21c)$$

As those deviations $\delta_r(t)$, $\delta_z(t)$, and $\delta_\zeta(t)$ are small, we will only consider linear response. Thus, substituting Equations (4.20) into Equations (4.19), and neglecting terms above first order, we have

$$\ddot{\delta}_r + 4\delta_r + P_{eff}\delta_z = 0 \quad (4.22a)$$

$$\ddot{\delta}_z + K\delta_z + 2P_{eff}\delta_r = 0 \quad (4.22b)$$

$$\ddot{\delta}_\zeta + \lambda^2 \left(1 + \frac{Pd}{2u_{r0}^2 u_{z0} b^2}\right) \delta_\zeta = 0 \quad , \quad (4.22c)$$

where

$$K = \left(3\lambda^2 + \frac{1}{u_{z0}^4}\right) \quad \text{and} \quad P_{eff} = \frac{P}{u_{r0}^3 u_{z0}^2} \left(1 - \frac{d}{b}\right) \quad (4.23)$$

As shown in Equations (4.22), when we consider only the linear response of the system, the oscillation of the center of mass z_0 is decoupled from oscillations of widths w_r and w_z . First we will solve Equation (4.22c). Assuming as initial conditions $\zeta(0) = \zeta_{amp}$ and $\dot{\zeta}(0) = 0$, we obtain

$$\zeta = \delta_\zeta = \zeta_{amp} \cos(\lambda_{eff}\tau), \quad (4.24)$$

where

$$\lambda_{eff} = \lambda \sqrt{\left(1 + \frac{Pd}{2u_{r0}^2 u_{z0} b^2}\right)}, \quad (4.25)$$

which means that the dipole mode frequency is

$$\omega_d = \omega_z \sqrt{\left(1 + \frac{Pd}{2u_{r0}^2 u_{z0} b^2}\right)}. \quad (4.26)$$

Note that if we are far from a Feshbach resonance, b^2 is much larger than d and $\omega_d \rightarrow \omega_z$, which are the expect frequency of dipole oscillation in a harmonic trap.

Now, we will solve the equations for the widths, given by (4.22a) and (4.22b). Rewriting them, we have,

$$\ddot{\vec{\delta}} + M\vec{\delta} = 0, \quad (4.27)$$

where

$$\vec{\delta} = \begin{bmatrix} \delta_r \\ \delta_z \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 4 & P_{eff} \\ 2P_{eff} & K \end{bmatrix}. \quad (4.28)$$

This is exact the same set of Equation given in (2.115, with the difference that here we have P_{eff} instead of $\tilde{P}$. Thus, we have for breathing and quadrupole mode

$$\omega_b = \omega_r \sqrt{2 + \frac{K}{2} + \frac{\sqrt{(K-4)^2 + 8P_{eff}^2}}{2}} \quad (4.29a)$$

and

$$\omega_q = \omega_r \sqrt{2 + \frac{K}{2} - \frac{\sqrt{(K-4)^2 + 8P_{eff}^2}}{2}}. \quad (4.29b)$$

4.1.2 Numerical Results

In order to visualize the excitation of the modes, we employ a numerical integration of Eq. (4.19). Using a forth-order Runge-Kutta method and considering a ^{87}Rb BEC

in a cigar-shaped trap, as already described in (4-7), one can observe the full behavior for a large range of amplitudes imposed to the dipole oscillation. Considering a sample containing 3×10^5 condensated atoms in a trap with $\omega_r = 2\pi \times 207$ Hz along the radial direction and $\omega_z = 2\pi \times 23$ Hz along the axial direction, and the Feshbach resonance parameters $B_{res} = 1007.3$ G, $\Delta = 0.17$ G, and $a_{nr} = 100a_0$, where a_0 is the Bohr radius. We set a bias field $B_0 = 1006$ G and for different dipole oscillations amplitude (ζ_{amp}), the width oscillations are obtained. For initial conditions, we get the equilibrium widths u_{r0} and u_{z0} in the initial position, which are $r = 0$ and $z = \zeta_{amp}l_r$. In this way, the equilibrium equations that we obtain those widths is slightly different from Equations (4.21); they become

$$u_{r0} - \frac{1}{u_{r0}^3} - \frac{P_{nr}}{u_{r0}^3 u_{z0}} \left(1 - \frac{2d}{2b + \lambda^2 \zeta_{amp}^2} \right) = 0 \quad (4.30a)$$

$$\lambda^2 u_{z0} - \frac{1}{u_{z0}^3} - \frac{P_{nr}}{u_{r0}^2 u_{z0}^2} \left(1 - \frac{2d}{2b + \lambda^2 \zeta_{amp}^2} \right) = 0. \quad (4.30b)$$

In Fig. (4.17) we show the radial and axial widths as a function of time when four different conditions of dipole oscillation amplitudes are considered.

Figure 4.17 – Time evolution of u_z (thick black) and u_r (thin red) for: (a) $\zeta_{amp} = 400$; (b) $\zeta_{amp} = 600$; (c) $\zeta_{amp} = 800$; (d) $\zeta_{amp} = 1000$.

For the considered amplitudes, the radial and axial oscillations are out of phase, demonstrating the predomination of the quadrupole over the breathing mode excitation. It is also clear we are exciting more than one frequency in the oscillation as demonstrated by the presence of beating mode type oscillation. The spectral composition of such oscillation can be obtained using a Fourier transformation. Fig. (4.18) shows the frequency composition of the oscillation. The quadrupole frequency is quite clear and is indicated by the peak close to the vertical dotted line, which represents the linearized w_q solution. There is however a second frequency close to twice the trap frequency (here represented by the vertical dashed line). It is believed that this excitation comes due to the fact that as the cloud oscillates on the potential, the interactions are modulated at twice the frequency of the cloud motion and that is present on the final oscillatory behavior of the quadrupole mode. At high amplitude of oscillation, a third frequency of lower value seems to appear, but we did not identify this frequency yet.

Figure 4.18 – Spectra of oscillation of u_z (black) for: (a) $\zeta_{amp} = 400$; (b) $\zeta_{amp} = 600$; (c) $\zeta_{amp} = 800$; (d) $\zeta_{amp} = 1000$. The dotted red line is the quadrupole frequency (Eq. (4.29b)) and dashed blue line is twice the dipole frequency (Eq. (4.26)).

Rubidium has a very narrow Feshbach resonance and therefore small dipole oscillation amplitudes corresponding to severe variation on the scattering length. So, on the other hand, we present another numerical example with an atom whose Feshbach resonance is much broader: ^7Li . For this atom, we use the trap parameters as in Ref. (31), which

are a condensate with 3×10^5 atoms, $\omega_r = 2\pi \times 235$ Hz along the radial direction and $\omega_z = 2\pi \times 4.85$ Hz along the axial direction. We also set the bias field as $B_0 = 735$ G and get the parameters for its resonance given by $\Delta = -192.3$ G, $B_{res} = 736.8$ G and $a_{nr} = -24.5 a_0$ (32). The initial conditions are obtained in the same way as in the ^{87}Rb example.

Figure 4.19 – Time evolution of u_z (thick black) and u_r (thin red) for: (a) $\zeta_{amp} = 4000$; (b) $\zeta_{amp} = 5000$; (c) $\zeta_{amp} = 6000$; (d) $\zeta_{amp} = 7000$.

In Fig. 4.19, we show the numerical results for a large range of amplitudes, where we can see a beating behavior of the widths like in ^{87}Rb . However, as ^7Li has a broader resonance, it is possible to reach higher amplitudes without crossing the resonance, where the system becomes unstable. This amplifies the non-linear contribution, generating a much richer spectrum, as observed in Fig. 4.20. For small amplitudes ζ_{amp} , the spectrum presents essentially the frequencies w_q and $2w_d$, as observed in ^{87}Rb . However, as the amplitude increases, more frequencies appear, and we can clearly see a deformation of the line shape for high amplitude.

Figure 4.20 – Spectra of oscillation of u_z (black) for: (a) $\zeta_{amp} = 1000$; (b) $\zeta_{amp} = 5000$; (c) $\zeta_{amp} = 10000$; (d) $\zeta_{amp} = 13000$. The dotted red line is the quadrupole frequency (Eq. (4.29b)) and dashed blue line is twice the dipole frequency (Eq. (4.26)).

5 *Conclusions*

In conclusion, we have shown that using a magnetic modulation field, applied to a trapped Bose-condensed gas, it is possible to transfer the atomic population from the ground state to an excited state, producing a nonground-state condensate. The time-averaged population imbalance between the ground and excited states represents an order parameter, which demonstrates an interesting behavior as a function of the modulation amplitude. Depending on the detuning, the behavior of η can be either smooth or rather abrupt. This is the consequence of the strong nonlinearity of the interactions. For some range of detunings, η becomes negative above a critical value of the modulation amplitude. This occurs because of the population inversion realized during the process of the mode excitation. Larger detunings and out-of-resonance modulations keep the population in the ground state and no population inversion is observed. Numerical calculations, accomplished for ^7Li atoms, show that the values for the amplitude and modulation of the bias field are within realistic experimental conditions.

Another point to be addressed concerns losses introduced by collisions, especially near a Feshbach resonance. With an off-resonant magnetic field, the dominant loss mechanism is a three-body collision, whereas close to the resonance, the molecular formation dominates the atom-loss mechanism (33). Although the fields considered in Table 3.1 and in the ^7Li example above are within the resonance linewidth, they are far enough to the resonance, we consider three-body collision as the main loss mechanism. In the case of ^7Li , that the resonance linewidth is large, as we are in the border of this linewidth, the loss rate is very small, as shown in Ref. (27). So, we can neglect the atom loss, especially because we apply the magnetic field in a short period of time.

Also, we have demonstrated the excitation of the collective low-lying quadrupole mode of a dilute Bose gas by modulating the atomic scattering length. Using variational calculations of the time dependent Gross-Pitaevskii equation assuming a Gaussian trial wave-

function, we find good agreement with experimental results. Temporal modulation of the scattering length, as afforded by Feshbach resonances, adds an additional tool to excite collective modes of an ultracold atomic gas. This method is quite attractive in circumstances where excitation of the condensate by other means, such as trap deformation, is unavailable. In addition, this method can be used for condensates in the presence of thermal atoms where principal excitation of the condensate alone is desired, as well as in multi-component gases where excitation of only one species can be accomplished.

In addition, we have demonstrated a coupling between collective modes excitations due to the Feshbach resonance when a condensate cloud oscillates inside a magnetic trap. This type of effect may lead to important applications in situations where the cloud is spatially excited. That is the case of modulation of the trapping potential in order to generate vortices (5,6) and even turbulence (7). Another situation of interest corresponds to the case when a superposition of light and magnetic trap is applied to investigate the motion of the cloud with control of the scattering length value (31).

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