**UNIVERSIDADE DE SÃO PAULO
INSTITUTO DE FÍSICA DE SÃO CARLOS**

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**Geodesic motion in the spacetime of two (un)equal mass
black holes**

São Carlos

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black holes**

Dissertation presented to the Graduate Program in Physics at the Instituto de Física de São Carlos, Universidade de São Paulo to obtain the degree of Master of Science.

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Advisor: Prof. Dr. Betti Hartmann

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ABSTRACT

TAKEDA, C. S. **Geodesic motion in the spacetime of two (un)equal mass black holes**. 2020. 60p. Dissertation (Master of Science) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2020.

One of the main approaches to study a spacetime and understand its structure is through the investigation of geodesics. This work aims to explore the Majumdar-Papapetrou metric for a binary system of black holes, describing two extreme Reissner-Nordström black holes. The name extreme comes from the fact that the absolute value of the charge is equal to the mass of the black holes. This is a situation of equilibrium, since the gravitational attraction is exactly compensated by the electrical repulsion. For a more complete analysis, timelike and lightlike geodesics were studied for the cases of equal and different masses of the black holes, separating the cases in the plane $z = 0$ and away from it. In the plane $z = 0$, the stability of the orbits with respect to the radius ρ was analyzed and for the situation of equal masses, bounded orbits were studied and their eccentricities calculated. By examining circular and bounded orbits in different scenarios, it was possible to find parameter regions with circular orbits and establish a relation between the amount of possible circular orbits and the black holes' masses. Another point was the evaluation of the behavior of orbits for extreme differences between the values of the masses. All of these studies brought some insights into the structure of this binary MP spacetime and might be a first step towards more complex systems, such as multi black holes.

Keywords: General relativity. Geodesic. Black hole. Circular orbits. Majumdar-Papapetrou metric.

RESUMO

TAKEDA, C. S. **Movimento geodésico no espaço-tempo de dois buracos negros de massas iguais e diferentes.** 2020. 60p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2020.

Uma das principais abordagens usadas para estudar um certo espaço-tempo e entender sobre a sua estrutura, é a investigação de geodésicas. Este trabalho tem como objetivo explorar a métrica de Majumdar-Papetrou para um sistema binário de buracos negros, descrevendo buracos negros extremos de Reissner-Nordström. Este nome 'extremo' é dado pelo fato de que o valor absoluto da carga é igual a massa dos buracos negros. Esta é uma situação de equilíbrio, pois a atração gravitacional é exatamente balanceada pela repulsão eletromagnética. Para uma análise mais completa, foram estudadas geodésicas tipo tempo e tipo luz para os casos de massas iguais e diferentes, também restringindo ao plano $z = 0$ e fora dele. Neste plano $z = 0$, a estabilidade das órbitas em relação ao raio ρ foi analisada e para a situação de massas iguais, foram estudadas órbitas limitadas e calculadas suas excentricidades. Através da inspeção de órbitas circulares e limitadas em diferentes cenários, foi possível encontrar regiões com órbitas circulares e estabelecer uma relação entre a quantidade dessas órbitas circulares e as massas dos buracos negros. Outra questão foi a análise do comportamento das órbitas quanto a diferenças extremas dos valores das massas. Todos esses estudos elucidaram a estrutura desse espaço-tempo binário de MP e podem se tornar o primeiro passo para compreender sistemas mais complexos, como múltiplos buracos negros.

Palavras-chave: Relatividade geral. Geodésica. Buraco negro. Órbitas circulares. Métrica de Majumdar-Papapetrou.

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LIST OF ABBREVIATIONS AND ACRONYMS

BH	Black Hole
MP	Majumdar-Papapetrou

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1 INTRODUCTION

These are exciting times for relativists. Recently, gravitational waves were detected¹ and the first image of a black hole's event horizon was presented.² The understanding of the universe has been fascinating us for a long time, and certainly is one of the main personal motivations for the present work. By seeing such amazing discoveries, the feeling of gratitude and admiration for the past scientists who made possible our progress today is inevitable.

By looking at the sky and observing orbits of celestial objects, an important mark in scientific development was made at the beginning of the modern era. One of the main contributors was Johannes Kepler (1571-1630) with his three laws of planetary motion. The main point of his discovery was the elliptical movement of the planets around the Sun. His findings were also important to reinforce the heliocentric theory and to support the future work of Isaac Newton (1643-1727). Since then, the study of orbits has brought new insights and developments to physics some of which will be discussed in what follows.

There is no doubt about the importance of the theory of gravitation. Newton made a revolution in science, by going further than explaining Kepler's law, he unified the planetary laws of motion and generalized it, explaining also the interaction between smaller massive bodies.

One of his ingenious ideas was to explain the orbital motion of planets as a result of a force proportional to one over the distance squared, which he then theorized to be an attractive force felt by every particle with mass. This is his famous theory of gravitation that later would be recognized as a basic law of nature and one of the four fundamental forces. Although Newton itself knew the limitations of his theory, it was a huge success experimentally since it could explain the motion of planets around the Sun, the tides, predict the existence and location of Neptune, and much more.

Many helped to develop Newton's Theory of Gravity during the succeeding years. For instance, Leibniz contributed with the understanding of the concept of mass and the notion of the absolute nature of space and time. Euler generalized the Newtonian Laws by formulating it in the form of differential equations, using axioms, definitions and postulates and Lagrange created a new vision of mechanics by using the calculus of variations and discovered the so called "Lagrangian points" which are points with no net force in the three body problem.³ But only about 220 years later, the theory that could answer all paradigms would come to light. This theory called General Relativity was introduced by Albert Einstein (1879-1955) in 1915.

For Einstein, mass curves the spacetime and gravity is just a consequence of this

curvature and geometry. Free particles, including light, follow geodesics in this spacetime. This simple, but brilliant idea changed the way we see the universe as a whole, and until today it is the only consistent theory that can correctly describe all gravitational phenomena.

Just as Newton's theory was reinforced by the discovery of a planet and its orbit, General Relativity had its first confirmation with the calculation of Mercury's orbit. The additional precession of 43 arc-seconds per century in its perihelion, observed in 1859 by Joseph Le Verrier, was predicted correctly by Einstein.⁴

Many other experimental and observational evidence appeared throughout the following years, such as the bending of light observed by Arthur Eddington during a solar eclipse in 1919. Einstein's fame came with good reason, as his theory not only explained all the phenomena that Newton's gravitation could not explain but correctly predicted new physical phenomena. Among them stands the existence of black holes.

A black hole (BH) is a massive body sufficiently compacted which creates a huge deformation in spacetime. Precisely, the spacetime of a mass M concentrated within a sphere of radius

$$r_g = \frac{2GM}{c^2}$$

would be that of a black hole, where G is the gravitational constant and c the speed of light. For the simplest case, the Schwarzschild metric, r_g is identified as the event horizon, which is a boundary that separates the interior from the exterior of a black hole. When a particle passing near a BH crosses the event horizon, there is no turning back. That is the reason for its name, since even light cannot escape once inside. The event horizon is a Killing horizon for a stationary, asymptotically flat spacetime which means it is a null hypersurface with vanishing Killing vector field.⁵ Additionally, a distant static observer won't see an object reach the event horizon, where usually there is an infinite gravitational redshift. In simple terms, the gravitational redshift is a photon's frequency decrease due to energy loss when moving away from a massive object.

One aspect about black holes is that they can be completely characterized with just three parameters: mass, charge and angular momentum. This became known as the "No-hair conjecture". Another feature is the presence of a singularity in their center, where there is infinite density and curvature. Their study is important to explain many astrophysical phenomena, since they are present in the center of many galaxies, and BHs with accretion disks are X-ray sources.⁶ Beyond that, they might help to unravel new physics, such as a better understanding of early stages of the Universe where e.g. primordial black holes could have formed and the development of a new theory that could unify gravitation with quantum theory.⁷ More details can be found in reference.⁶

Black holes are mathematical solutions of Einstein's equations. The first exact solution was derived by Karl Schwarzschild in 1916, describing the exterior of a spherical mass with no charge and angular momentum. His solution is spherically symmetric, static and is only valid in vacuum, therefore it does not describe the interior of a massive body. The radius r_g is also known as the Schwarzschild radius, which is the location of the event horizon in this metric.

About two years later, Reissner and Nordström independently found another solution, but now with the addition of charge, which was named after them.^{8,9} This one is a static, spherically symmetric solution of the coupled Einstein-Maxwell equations. The Reissner-Nordström BHs have two horizons, one equivalent to the Schwarzschild event horizon and an internal one called Cauchy horizon. These two horizons become one when the mass of the BH is equal to its charge, a situation called extreme Reissner-Nordström. If the charge is greater than the mass, there is no horizon and the spacetime is that of a naked singularity. A naked singularity, as the name suggests, is a gravitational singularity unhidden by an event horizon. There is still no experimental evidence of their existence, but theoretically they would enable the observation of a singularity with infinite density if one considers the solution to be valid even at smallest distances.

The next step was to consider rotation, which was much more complicated and took around fifty years to be achieved by Roy P. Kerr in 1963.¹⁰ The Kerr metric is a stationary, axisymmetric vacuum solution, which describes the exterior of a rotating body with angular momentum and no charge. Newman generalized it in 1965, and this new solution was named Kerr-Newman metric.¹¹ It is a stationary, asymptotically flat solution to the coupled Einstein-Maxwell equations, describing an electrically charged, rotating mass with angular momentum.

There are many similarities between the Reissner-Nordström and the Kerr metric. For instance, both have two event horizons with a limiting value for the mass that separates them in two classes of solutions. If the mass is smaller than the limit, they become naked singularities and in the case of equality of mass with charge or angular momentum, respectively, the two horizons become one and it is called extreme. Therefore, although rotating, non-charged black holes are predominant in nature, the Reissner-Nordström metric is a good 'toy-model', frequently used to study the Kerr spacetime.

Later on, other exact solutions were gradually found. One of them is the Majumdar-Papapetrou metric (MP), which will be studied in this work.^{12,13} This static and axially symmetric spacetime describes the exterior of an arbitrary number of BHs in equilibrium, as interpreted by Hartle and Hawking.¹⁴ Since the value of their mass is equal to their charge, there is a compensation between the attraction of the gravitational force and the electrical repulsion. This case is called extreme Reissner-Nordström, as mentioned earlier, and a spacetime with two of these BHs will be the focus of this work.

The classical analog of this problem is integrable, but the relativistic case is non-integrable, therefore chaos arises because of the relativistic effects. Many studies explored the chaotic behavior of this spacetime, in particular, Contopoulos^{15,16} found many different orbits, some chaotic and others stable at the plane containing both black holes in the binary extreme Reissner-Nordström. For instance, chaotic geodesics were studied by references.^{17,18}

These binary systems are of great interest nowadays, since their existence was confirmed by the detection of gravitational waves produced by a binary BH merger. In 2017, a binary system of merging stars was observed through another detection of gravitational waves¹⁹ and recently, ten binary BH mergers were reported, found by the first and second observing runs of the detector network.²⁰ Considering the many binary systems that are already known and the detections that will surely increase in the future, it is of much interest to study the surroundings of such systems.

A common approach to study curved spacetimes is the analysis of a test particle's geodesic motion. A test particle is an idealized body with negligible mass and charge so that its presence won't have any external influence. The understanding of its behavior is essential to elucidate the physics and structure of spacetimes. Actually, a spacetime geometry can be constructed from the concepts of light propagation and the motion of test particles.²¹ Furthermore, this study can be useful for explaining and predicting nearby astrophysical phenomena, as was already demonstrated by past experiments such as the light deflection. Therefore, the study of geodesics will be the central point of this work.

In particular, the focus will be on closed orbits for light and massive particles, mainly circular orbits. They are interesting due to their connection with BH related phenomena. Some examples are the black hole shadow related to an unstable circular photon orbiting near the event horizon and the generation of accretion discs in regions with stable massive circular orbits.²²

Another important feature of circular orbits is the analysis of the innermost stable circular orbit (ISCO), which is the name given to the stable circular orbit with the smallest radius. Besides identifying the inner edge of an accretion disk, they determine the point where a compact binary enters the merging phase.²² There is an extensive number of studies involving the ISCO, and not just around black holes, but also other astrophysical objects such as neutron stars. For instance, gravitational waves emitted by a binary neutron star system when its circular motion is around the ISCO contain important information about the internal structure of neutron stars.²³ Thus, the study of orbits is essential for the discovery of important characteristics in a spacetime.

This specific case of two extremal Reissner-Nordström BHs might not be realistically possible, considering their excessive charge and the fact that the Majumdar-Papapetrou system is unstable. However, it is an interesting exact solution, and one of the few known

exact solutions describing binaries. For a more complex study to be possible it is necessary to start from simplified cases, with symmetries. The study of binary systems can provide a good starting point to analyze multi-black hole systems in the future.

With this in mind, the main objective of this work is to study geodesics followed by light and massive particles in the MP spacetime. Closed orbits were investigated, mainly circular orbits, in two different regimes: restricted to the $z = 0$ plane and outside this plane. Both situations of equal and different masses were explored, and also the behavior for extreme cases with similar and different masses of the BHs. The stability with respect to the radius of the orbits was analyzed for simpler cases. This thesis is an extension of the work done by Andreas Wünsch et al.²⁴ which consists of an exploration of circular orbits in the same spacetime as studied here.

The topics are divided according to the following structure: the next section starts with a very brief introduction of general relativity which gives the necessary theoretical base. The main goal is to derive the geodesic equation, since it is the center of this work. The next step is the presentation of the Reissner-Nordström metric, because the MP spacetime consists of extreme Reissner-Nordström black holes.

The next chapters contain the study of orbits in MP metric, which are classified in two main groups, circular orbits for lightlike and timelike geodesics. Each case is separated in equal and different masses and at the same time, restricted to the plane $z = 0$ and outside, since they depend on a distinct approach. At the end of each of these sections, there is also an analysis of the situation with very similar and different masses of the BHs. For the case of equal masses, the stability was evaluated for both lightlike and timelike geodesics.

After that, bounded orbits for the specific case of equal masses are explored. Their eccentricity was calculated in function of the mass and angular momentum. Lastly, there is a conclusion, including a summary of the results.

2 THEORETICAL FRAMEWORK

2.1 Geodesics

There were two main ideas which inspired the creation of a completely new theory by Einstein. One is the so called equivalence principle, a realization of the equivalence between the gravitational and inertial mass. In other words, the absence of a gravitational field is indistinguishable to a free fall accelerated body in the presence of a gravitational field. The second point was the influence of Mach's principle, which motivated Einstein to seek a theory where the structure of spacetime is influenced by the mass/energy.²⁵

Briefly, general relativity tells us that spacetime is a manifold on which is defined a metric and has a curvature related to the energy-momentum tensor by Einstein's equation. The metric tensor $g_{\mu\nu}$ describes the geometry of this spacetime, and the energy-momentum tensor $T^{\mu\nu}$ has information about the energy and momenta of mass and radiation. Thus, Einstein's equation relating the spacetime geometry to the mass/energy distribution is written as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (2.1)$$

where $R_{\mu\nu}$ is the Ricci tensor, R the scalar curvature and G the gravitational constant. Actually, this is a set of ten coupled nonlinear partial differential equations, which are not all independent due to Bianchi identities.

In vacuum, $T_{\mu\nu} = 0$, therefore (2.1) results in

$$R_{\mu\nu} = 0$$

A free falling particle will follow the curvature of the spacetime and therefore a geodesic motion. In the following, the geodesic equation will be derived through the Lagrangian formalism.

The geodesic is a curve in spacetime that extremizes the length of a curve between two events. Although it is well known that a geodesic is a straight line in a flat space, it is not that intuitive for curved spaces. It has a special importance in general relativity, since free particles are said to follow geodesics instead of being influenced by a gravitational force.²⁶ There are many ways to derive the geodesic equation, for example, by imposing

$$\delta \int_A^B \sqrt{ds^2} = 0$$

where the length of the curve between A and B is given by: $\int_A^B \sqrt{ds^2}$.

However, here the geodesic equation will be derived using the variational principle.

Given the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad (2.2)$$

where the notation was set as $\dot{x}^\mu = dx^\mu/d\lambda$ and λ is an affine parameter that can be identified with proper time for timelike geodesics, the action is defined by

$$S = \int d\lambda \mathcal{L}$$

By the variational principle, the extremization of the action will give the equations of motion. This extremization is given by

$$\delta S = 0$$

which implies the Euler-Lagrange equation

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \frac{\partial \mathcal{L}}{\partial x^\alpha} \quad (2.3)$$

The solution of this equation gives the function that will extremize the action. The first step is

$$\bullet \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \frac{\partial}{\partial \dot{x}^\alpha} \left(\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right) = \frac{1}{2} g_{\mu\nu} \left(\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha} \dot{x}^\nu + \frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha} \dot{x}^\mu \right) = \frac{1}{2} (g_{\alpha\nu} \dot{x}^\nu + g_{\mu\alpha} \dot{x}^\mu)$$

since μ and ν are dummy indices, it is possible to change $\nu \rightarrow \mu$, resulting in

$$\frac{1}{2} (g_{\mu\alpha} \dot{x}^\mu + g_{\mu\alpha} \dot{x}^\mu) = g_{\mu\alpha} \dot{x}^\mu$$

The next step is to calculate the left-hand side of the equation (2.3)

$$\bullet \frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \right) = g_{\mu\alpha} \ddot{x}^\mu + \dot{x}^\mu \frac{dg_{\mu\alpha}}{d\lambda} = g_{\mu\alpha} \ddot{x}^\mu + \dot{x}^\mu \dot{x}^\nu \partial_\nu g_{\mu\alpha} \quad (2.4)$$

Then, the right-hand side is given by

$$\bullet \frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{1}{2} \partial_\alpha g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad (2.5)$$

Joining the equations (2.4) and (2.5) on equation (2.3) and multiplying it by two gives

$$g_{\mu\alpha} \ddot{x}^\mu + \dot{x}^\mu \dot{x}^\nu \partial_\nu g_{\mu\alpha} - \frac{1}{2} \dot{x}^\mu \dot{x}^\nu \partial_\alpha g_{\mu\nu} = 2g_{\mu\alpha} \ddot{x}^\mu + 2\dot{x}^\mu \dot{x}^\nu \partial_\nu g_{\mu\alpha} - \dot{x}^\mu \dot{x}^\nu \partial_\alpha g_{\mu\nu} = 0 \quad (2.6)$$

By symmetry, it is possible to separate the second term of (2.6)

$$2\dot{x}^\mu \dot{x}^\nu \partial_\nu g_{\mu\alpha} = \dot{x}^\mu \dot{x}^\nu (\partial_\nu g_{\mu\alpha} + \partial_\mu g_{\nu\alpha})$$

Therefore, equation (2.6) turns into

$$2g_{\mu\alpha}\ddot{x}^\mu + \dot{x}^\mu\dot{x}^\nu(\partial_\nu g_{\mu\alpha} + \partial_\mu g_{\nu\alpha} - \partial_\alpha g_{\mu\nu}) = 0$$

From the definition of the Christoffel symbol

$$\Gamma_{\mu\nu}^\gamma = \frac{1}{2}g^{\gamma\beta}(\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu}) \quad (2.7)$$

it follows the geodesic equation

$$\ddot{x}^\gamma + \Gamma_{\mu\nu}^\gamma \dot{x}^\mu \dot{x}^\nu = 0 \quad (2.8)$$

2.1.1 Symmetries in the geodesic equation

Conservation Laws are important because they reduce the order and number of the equations that have to be solved. The geodesic equation (2.8) is a set of four second order differential equations. Therefore, a spacetime with symmetries facilitates its calculation, since there is an associated conservation law for every symmetry.

A simple way to identify a symmetry is when the metric is independent of one coordinate. Every symmetry has a Killing vector ξ associated to it. For example, if the transformation $x^1 \rightarrow x^1 + \text{constant}$ leaves the metric unchanged, there is a Killing vector that lies along this unchanged direction: $\xi^\alpha = (0, 1, 0, 0)$.

Following this example, the Euler-Lagrange equation (2.3) will be written as

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^1} \right) = 0, \quad \text{since} \quad \frac{\partial \mathcal{L}}{\partial x^1} = 0$$

this implies that

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^1} = g_{\mu 1} \dot{x}^\mu = g_{\mu\alpha} \xi^\alpha u^\mu = \boldsymbol{\xi} \cdot \mathbf{u}$$

which is a conserved quantity along a geodesic in an arbitrary coordinate system⁷ and $\mathbf{u}$ is the tangent vector whose components are $u^\alpha = dx^\alpha/d\lambda$.

2.1.2 Geodesics in a static and spherically symmetric spacetime

A general static and spherically symmetric spacetime is given by²⁶

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

The Lagrangian will then be

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}\left[-f(r)\dot{t}^2 + \frac{1}{h(r)}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right] \quad (2.9)$$

and the associated momenta are

$$\begin{aligned}
\bullet p_t &= \frac{\partial \mathcal{L}}{\partial \dot{t}} = -f(r)\dot{t} & \bullet p_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\dot{r}}{h(r)} \\
\bullet p_\theta &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2\dot{\theta} & \bullet p_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \sin^2 \theta \dot{\phi}
\end{aligned}$$

With these momenta, it is possible to calculate the Hamiltonian, which will be given by

$$\mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L} = -f(r)\dot{t}^2 + \frac{1}{h(r)}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - \mathcal{L} = 2\mathcal{L} - \mathcal{L} = \mathcal{L}$$

Therefore,

$$\mathcal{H} = \mathcal{L} = \text{constant}, \quad \text{since} \quad \frac{d\mathcal{H}}{dt} = 0$$

As previously mentioned, a symmetry is associated with a conserved quantity. Being static, the metric does not depend on the coordinate t and also does not depend on ϕ because of the spherical symmetry.

The conserved quantities are the momentum associated with the condition $\partial \mathcal{L} / \partial x^\mu = 0$, which are E and L

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{t}} &= 0 \longrightarrow -p_t = f(r)\dot{t} = E \\
\frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= 0 \longrightarrow p_\phi = r^2 \sin^2 \theta \dot{\phi} = L
\end{aligned} \tag{2.10}$$

where E is the particle's energy and L its orbital angular momentum.

For θ , the equation is

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2 \sin \theta \cos \theta \dot{\phi}^2$$

if θ is chosen to be $\theta = \pi/2$ when $\dot{\theta} = 0$, then $\ddot{\theta} = 0$ and the orbit will be confined to the equatorial plane $\theta = \pi/2$.

The substitution of (2.10) with the condition above in the Lagrangian (2.9) results in

$$\frac{E^2}{f(r)} - \frac{\dot{r}^2}{h(r)} - \frac{L^2}{r^2} = -2\mathcal{L} = \delta \tag{2.11}$$

where $\delta = 0$ for null orbits since $dx^\alpha/d\lambda$ is a null vector and $\delta = 1$ for timelike orbits due to the normalization of the four-velocity $dx^\alpha/d\tau$.

Here it is possible to notice that this Lagrangian is equivalent to the normalization of the particle's four-velocity

$$\dot{x}_\mu \dot{x}^\mu = -\frac{E^2}{f(r)} + \frac{\dot{r}^2}{h(r)} + \frac{L^2}{r^2} = -\delta$$

where $\delta = 0$ for massless particles and $\delta = 1$ for massive ones.

Then (2.11) is the “effective potential” equation that gives the qualitative features of a geodesic.²⁶

$$V_r \equiv h \left(\frac{E^2}{f(r)} - \frac{L^2}{r^2} - \delta \right)$$

where $\delta = 0$ for null geodesics and $\delta = 1$ for timelike geodesics.

2.2 The Reissner-Nordström metric

In this section, the Reissner-Nordström solution will be briefly discussed, since the spacetime of study consists of two extremal Reissner-Nordström black holes. This metric, as previously mentioned, is a static, spherically symmetric, asymptotically flat solution of the coupled Einstein-Maxwell equations, since there is the presence of electromagnetic fields. For a black hole with mass M and charge Q , where $M > |Q|$, the metric is given by

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (2.12)$$

if there is no charge $Q = 0$, then the Schwarzschild solution is recovered. By analyzing the metric above, three singularities can be noticed

$$r = 0 \quad \text{and} \quad r_{\pm} = M \pm \sqrt{M^2 - Q^2}$$

The only physical singularity is the first one $r = 0$, which is a timelike curvature singularity, while $r_{\pm}$ are coordinate singularities, meaning that they can be eliminated by another choice of coordinate system. The metric (2.12) can be rewritten as

$$ds^2 = - \frac{(r - r_+)(r - r_-)}{r^2} dt^2 + \left(\frac{(r - r_+)(r - r_-)}{r^2} \right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Since $r_+ > r_-$, r_+ is referred as outer horizon and r_- as the inner horizon. In the region outside the outer horizon $r > r_+$, r is a spacelike coordinate and t timelike. When entering this horizon $r_- < r < r_+$, r and t are inverted, r becomes timelike and t spacelike. Finally, passing through the inner horizon $0 < r < r_-$, the coordinates have the same characteristic as outside the outer horizon.

The geodesic equations for the motion of an uncharged particle can be calculated as done in section 2.1. The Euler-Lagrange equation (2.3) calculated with the Lagrangian (2.15) for the Reissner-Nordström metric (2.12) and $\theta = \pi/2$ leads to the equations of motion

$$r^2 \dot{\phi} = L, \quad \dot{r}^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) \left(\delta + \frac{L^2}{r^2}\right) = E^2$$

where E and L are constants of motion and $\delta = 0$ for null geodesics and $\delta = 1$ for timelike ones. Here, an interesting feature of the metric becomes clear, which is a repulsive gravitational force that acts on uncharged particles if $Q \neq 0$. It increases and becomes dominant as $r \rightarrow 0$.²⁷

This was all for the case of $M > |Q|$, but there are also two other cases which are called the extreme $M = |Q|$ and hyper-extreme $M < |Q|$. For the latter, there are no event horizons, so the timelike singularity is called a naked singularity, as was mentioned before.

But here, we are interested in the extreme case $M = |Q|$. In this situation, $r_{\pm} = M$, therefore both horizons 'merge' into one degenerate horizon. The main characteristic of such horizon is the vanishing of the surface gravity and hence in a semiclassical approach the vanishing of the Hawking temperature.

Then the metric is simplified to

$$ds^2 = - \left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

2.3 The Majumdar-Papapetrou metric

The Majumdar-Papapetrou (MP) is an exact solution of the Einstein-Maxwell equation, which describes two or more BHs in equilibrium. This is possible because the gravitational attraction is exactly balanced by their electric repulsion. Here, the focus will be in a binary BH system.

The MP spacetime is static and axially symmetric. Since there is this axial symmetry with respect to the z axis, it is convenient to use cylindrical coordinates. The MP metric in these coordinates containing two BHs of mass M_1 and M_2 with charge $Q_i = M_i$ is given by

$$ds^2 = - \frac{dt^2}{U^2} + U^2(d\rho^2 + \rho^2 d\phi^2 + dz^2) \quad (2.13)$$

where we choose one BH to be located at $z = a$ and the other at $z = -a$ such that U is given by

$$U(\rho, z) = 1 + \frac{M_1}{\sqrt{\rho^2 + (z - a)^2}} + \frac{M_2}{\sqrt{\rho^2 + (z + a)^2}} \quad (2.14)$$

In these coordinates, the BHs horizons are represented by points at $z = \pm a$. For more details see reference.¹⁴ The Lagrangian in the MP spacetime will then be

$$2\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -\frac{\dot{t}^2}{U^2} + U^2(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) = -\delta \quad (2.15)$$

where the dot refers to a derivative in respect to an affine parameter and the constant $\delta = 0$ for null geodesics and $\delta = 1$ for timelike geodesics.

2.4 Conserved Quantities

In this section, the two conserved quantities for the MP spacetime will be presented. Given the Lagrangian (2.15), the associated momenta are

$$\begin{aligned} * p_t &= \frac{d\mathcal{L}}{d\dot{t}} = -\frac{\dot{t}}{U^2} \quad ; \quad * p_\rho = \frac{d\mathcal{L}}{d\dot{\rho}} = U^2 \dot{\rho} \\ * p_\phi &= \frac{d\mathcal{L}}{d\dot{\phi}} = U^2 \rho^2 \dot{\phi} \quad ; \quad * p_z = \frac{d\mathcal{L}}{d\dot{z}} = U^2 \dot{z} \end{aligned}$$

The conserved quantities will be the momentum associated with the condition $\partial\mathcal{L}/\partial x^\mu = 0$. Using the Euler-Lagrange equation, which is

$$\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{d}{d\lambda} \frac{\partial\mathcal{L}}{\partial \dot{x}^\mu} = 0$$

and calculating with the condition above, two conserved quantities are found

$$\begin{aligned} * \frac{d}{d\lambda} \left(\frac{\partial\mathcal{L}}{\partial \dot{t}} \right) &= 0 \rightarrow \frac{\partial\mathcal{L}}{\partial \dot{t}} = -\frac{\dot{t}}{U^2} = \text{constant} = E \\ * \frac{d}{d\lambda} \left(\frac{\partial\mathcal{L}}{\partial \dot{\phi}} \right) &= 0 \rightarrow \frac{\partial\mathcal{L}}{\partial \dot{\phi}} = U^2 \rho^2 \dot{\phi} = \text{constant} = L_z \end{aligned}$$

where E is associated with the energy and L_z with the angular momentum of the system with respect to the z axis.

Substituting E and L_z in the Lagrangian, the expression (2.15) becomes

$$E^2 - \dot{\rho}^2 - \dot{z}^2 - \frac{L_z^2}{U^4 \rho^2} = \frac{\delta}{U^2} \quad (2.16)$$

2.5 Simplification and Stability

From now onwards, the separation of the BHs will be fixed by setting $a = 1$. By restricting to the $z = 0$ plane, the expression for U (2.14) becomes

$$U(\rho, z) = 1 + \frac{2M}{\sqrt{\rho^2 + 1}} \quad (2.17)$$

and the equation (2.16) turns into

$$E^2 - \dot{\rho}^2 - \frac{L_z^2}{U^4 \rho^2} = \frac{\delta}{U^2}$$

where the effective potential is given by

$$V_{eff}(\rho) = \frac{L_z^2}{U^4 \rho^2} + \frac{\delta}{U^2} \quad (2.18)$$

Figures 2.1 and 2.2 show some plots of the effective potential for both cases $\delta = 0$ and $\delta = 1$ respectively, for different values of L_z . Here is just an example with two fixed masses and the complete analysis will be made in the following chapters. For the lightlike case, the increase of angular momentum showed a slower decrease of the effective potential. For $M = 1$, there are no critical points, while the $M = 2.5$ case showed two critical points. The effective potential tends to zero for increasing ρ , independently of the value for the mass and angular momentum.

Figure 2.1 – $V_{eff}^0(\rho)$ with $\delta = 0$

Source: By the author.

As for the timelike case, the increase of angular momentum showed a increase of the value of the critical points in the effective potential. The quantity of critical points vary between one to three depending on the mass. The effective potential tends to one for increasing ρ , independently of the value for the mass and angular momentum.

The stability of the particle's motion will be analyzed in the following sections by calculating the second derivative of the effective potential in respect to ρ . If it is positive, the orbit is stable and unstable otherwise.²² Here we are restricting to an analysis of the

Figure 2.2 – $V_{eff}^1(\rho)$ with $\delta = 1$

Source: By the author.

stability just in ρ , at the plane $z = 0$, therefore stable orbits might not be stable in other directions.

$$\begin{aligned}
 \frac{\partial^2 V_{eff}}{\partial \rho^2} > 0 & \rightarrow \text{stable orbit} \\
 \frac{\partial^2 V_{eff}}{\partial \rho^2} < 0 & \rightarrow \text{unstable orbit}
 \end{aligned}
 \tag{2.19}$$

In the following chapters, closed orbits will be investigated in the Majumdar-Papapetrou spacetime, focusing on circular ones.

3 LIGHTLIKE GEODESICS

Closed orbits possess turning points which implies that $\dot{\rho} = 0$ and $\dot{z} = 0$. For the special case of a circular orbit, there is also the condition of no acceleration, which is $\ddot{\rho} = 0$. Therefore, by applying this in equation (2.16) we obtain the conditions

$$\frac{\partial V_{eff}}{\partial \rho} = 0 \quad \text{and} \quad \frac{\partial V_{eff}}{\partial z} = 0 \quad (3.1)$$

for a circular motion. The second condition implies that $\partial U / \partial z = 0$, which results in

$$\frac{M_1}{M_2} = \frac{1+z}{1-z} \left[\frac{\rho^2 + (1-z)^2}{\rho^2 + (1+z)^2} \right]^{\frac{3}{2}} \quad (3.2)$$

This same expression can be obtained by a classical analog analysis of the forces acting on a test particle, as illustrated by the Figure 3.1.

Figure 3.1 – Schematic figure for a classical analogy.

Source: By the author.

The expression (3.2) shows that the restriction $z = 0$ implies that the masses are equal. However, all other values of the coordinate z between $0 < |z| < 1$ are possible candidates for circular orbits, as will be shown in the following section 3.1. This expression also confirms the intuitive fact that there can only exist orbits in the interval $z \in (-1, 1)$.

3.1 $z = 0$ and $M_1 = M_2 = M$

In this section, circular orbits in the MP spacetime will be analyzed for two BH with the same mass M . First, for the plane $z = 0$, which necessarily implies that the

masses are equal through expression (3.2), and then for the $z \neq 0$ case.

The first condition in (3.1) and the restriction to the plane $z = 0$ results in the following expression

$$(\rho^2 + 1)^{\frac{3}{2}} = 2M(\rho^2 - 1) \quad (3.3)$$

for $\rho \geq 1$. This equation can be solved through a substitution $\beta = \rho^2 + 1$ as done in reference,²⁴ which results in a cubic equation

$$\beta^3 - 4M^2\beta^2 + 16M^2\beta - 16M^2 = 0$$

with $\beta \geq 2$ since $\rho \geq 1$. From the roots of this equation, it follows the condition $M \geq \tilde{M}$ where

$$\tilde{M} = \sqrt{\frac{27}{8}} \approx 1.837 \quad (3.4)$$

The solution is then

$$\begin{aligned} \beta_1(M) &= \frac{4}{3} \left[M^2 + 2M\sqrt{M^2 - 3} \cos\left(\frac{\Phi}{3}\right) \right] \\ \beta_2(M) &= \frac{4}{3} \left[M^2 - 2M\sqrt{M^2 - 3} \cos\left(\frac{\pi}{3} + \frac{\Phi}{3}\right) \right] \end{aligned}$$

where

$$\Phi(M) = \arccos\left(\frac{27 - 36M^2 + 8M^4}{8M(M^2 - 3)^{3/2}}\right)$$

Finally, ρ can be found by returning to the substitution $\beta = \rho^2 + 1$.²⁴

Some plots of the effective potential (2.18) for different values of M and L_z are shown in figure 3.2. For values of $M < \tilde{M} = 1.837$ (3.4), there are no lightlike orbits, for $M = \tilde{M}$ there is only one and for $M > \tilde{M}$ there are two orbits. The orbits are located at the critical points $\partial V_{eff}/\partial \rho = 0$.

Figure 3.2 shows visually which orbit is stable or unstable. By using the condition (2.19), we verify that orbits with the smallest radius in all cases are stable, and the second orbit is always unstable.

Figure 3.2 – V_{eff} for different values of M , in the plane $z = 0$

Source: By the author.

3.2 Case $0 < |z| < 1$

Additionally, it is possible to analyze orbits outside the plane $z = 0$. It will be valid for $0 < |z| < 1$, therefore, for all other possible values of z except the already calculated $z = 0$ case. In order to do so, ρ can be written as a function of z , by isolating it from (3.2):

$$\rho^2 = \frac{4z}{\left[1 - \left(\frac{M_1}{M_2}\right)^{\frac{2}{3}} \left(\frac{1-z}{1+z}\right)^{\frac{2}{3}}\right]} - (z+1)^2 \quad (3.5)$$

For different masses, the expression for $U(\rho, z)$ is given by (2.14). The condition

(3.1) with $\delta = 0$ since it is the lightlike case, becomes

$$\frac{\partial V_{eff}}{\partial \rho}(\rho, z) = U + 2\rho \frac{\partial U}{\partial \rho} = 0$$

where

$$\frac{\partial U}{\partial \rho} = -\frac{\rho M_1}{(\rho^2 + (z-1)^2)^{\frac{3}{2}}} - \frac{\rho M_2}{(\rho^2 + (z+1)^2)^{\frac{3}{2}}}$$

Therefore,

$$\begin{aligned} \frac{\partial V_{eff}}{\partial \rho}(\rho, z) = 2\rho^2 \left[\frac{M_1}{(\rho^2 + (z-1)^2)^{3/2}} + \frac{M_2}{(\rho^2 + (z+1)^2)^{3/2}} \right] \\ - \left[1 + \frac{M_1}{\sqrt{\rho^2 + (z-1)^2}} + \frac{M_2}{\sqrt{\rho^2 + (z+1)^2}} \right] = 0 \quad (3.6) \end{aligned}$$

where ρ^2 is given by the expression (3.5).

For equal masses, we have that $M_1 = M_2 = M$ and $\frac{\partial V_{eff}}{\partial \rho}(\rho, z) = g(z)$

Figure 3.3 – $g(z)$ for different values of M , reproduced from²⁴

Source: By the author.

Figure (3.3) shows the plot of $g(z)$. The symmetry of the problem becomes clear as one can see in this figure that there are two circular orbits for

$$M < 2.598076 = \bar{M} \quad (3.7)$$

a value found numerically by reference,²⁴ with equal absolute values of height z . For M bigger than this limiting value, there are no orbits.

3.3 $M_1 = M_2 + \epsilon$

For this case, the second BH will be chosen with a slight different mass from the first one: $M_1 = M + \epsilon$. Therefore, the expression (2.14) with $a = 1$ becomes

$$U(\rho, z) = 1 + \frac{M + \epsilon}{\sqrt{\rho^2 + (z - 1)^2}} + \frac{M}{\sqrt{\rho^2 + (z + 1)^2}}$$

Once again, $\delta = 0$ and following the same steps done previously leads to

$$\begin{aligned} \frac{\partial V_{eff}}{\partial \rho}(\rho, z) = 2\rho^2 \left[\frac{M + \epsilon}{(\rho^2 + (z - 1)^2)^{3/2}} + \frac{M}{(\rho^2 + (z + 1)^2)^{3/2}} \right] \\ - \left[1 + \frac{M + \epsilon}{\sqrt{\rho^2 + (z - 1)^2}} + \frac{M}{\sqrt{\rho^2 + (z + 1)^2}} \right] = 0 \quad (3.8) \end{aligned}$$

Figure 3.4 – $f(z)$ with $M = 5$

Source: By the author.

For this case of $M_1 = M + \epsilon$, the expression (3.5) can be expanded in ϵ around $\epsilon = 0$. The final expression neglecting terms of ϵ^2 and higher is given by

$$\rho^2 = \frac{4z}{\left[1 - \left(\frac{1-z}{1+z}\right)^{\frac{2}{3}}\right]} - (z + 1)^2 + \frac{8z \left(\frac{1-z}{1+z}\right)^{\frac{2}{3}} \epsilon}{3M \left[1 - \left(\frac{1-z}{1+z}\right)^{\frac{2}{3}}\right]^2} \quad (3.9)$$

Substituting this expression of ρ^2 (3.9) in (3.8), we obtain a function of z only, which will be named $f(z)$. Figure 3.4 shows the plot of this function and figure 3.5 shows a magnification for different values of M and ϵ .

Figure 3.5 – $f(z)$ for different values of M and ϵ

Source: By the author.

For different values of M , the behavior of $f(z)$ is similar. The limits are given by

$$\lim_{z \rightarrow \pm 1} f(z) = \infty \quad \text{and} \quad \lim_{z \rightarrow 0} f(z) = -\infty$$

The value of the coordinate z of the circular orbits are given by the roots. The number of possible orbits depends on the value of the mass M .

It is possible to note in figure 3.5 that there are different values of M from where four orbits begin to appear for every ϵ . The limiting values of M are approximately

$$\begin{aligned} \epsilon = 0.1 &\rightarrow M \approx 2.2, & \epsilon = 0.01 &\rightarrow M \approx 2.51, \\ \epsilon = 0.001 &\rightarrow M \approx 2.59, & \epsilon = 0.0001 &\rightarrow M \approx 2.594 \end{aligned} \quad (3.10)$$

For each case, there are four circular orbits if the mass is smaller than the limiting value, and no orbits if it is greater.

Figure 3.5 also shows that as the mass and value of ϵ increases, the circular orbits value of z approaches zero.

3.4 $M_1 \neq M_2$

Since for $z = 0$, M_1 is necessary equal to M_2 , in the case of different masses, $0 < |z| < 1$.

As calculated before (3.6), the derivative of the effective potential is given by

$$\begin{aligned} \frac{\partial V_{eff}}{\partial \rho}(\rho, z) = 2\rho^2 \left[\frac{M_1}{(\rho^2 + (z-1)^2)^{3/2}} + \frac{M_2}{(\rho^2 + (z+1)^2)^{3/2}} \right] \\ - \left[1 + \frac{M_1}{\sqrt{\rho^2 + (z-1)^2}} + \frac{M_2}{\sqrt{\rho^2 + (z+1)^2}} \right] = 0 \end{aligned}$$

for $0 < |z| < 1$.

By analyzing the final equation for different masses $\frac{\partial V_{eff}}{\partial \rho}(z) = h(z) = 0$, we find a range of values for M_2 for which there is a possibility of having four circular orbits for some value of M_1

$$1.53961 < M_2 < 2.598076 = \bar{M} \quad (3.11)$$

For any other value of M_2 only two circular orbits are possible. Interestingly, the upper limit of this range is exactly the same value as found in the equal mass case (3.7).

Figure 3.6 shows the plot of the function (3.6) for values of M_2 and M_1 near these limiting values.

3.5 $M_1 \gg M_2$

In this section, the case of $M_1 \gg M_2$ will be analyzed. The equation is the same as (3.6). By fixing $M_2 = 3$ and varying $M_1 > M_2$, there are always only two circular orbits and their coordinate value of $z \rightarrow \pm 1$ as M_1 increases as shown in Figure 3.7.

Figure 3.6 – $h(z)$ near limiting values of M_1 and M_2

Source: By the author.

The limits of $h(z)$ are given by

$$\lim_{z \rightarrow \pm 1} h(z) = \infty, \quad \lim_{z \rightarrow 0^-} h(z) = -\infty, \quad \lim_{z \rightarrow z_{sing}} h(z) = -1$$

where z_{sing} is the singularity at $0 < z < 1$.

(a) $M_1 = 5$ (b) $M_1 = 10$ (c) $M_1 = 20$ (d) $M_1 = 40$ Figure 3.7 – $h(z)$ for increasing values of M_1 with $M_2 = 3$ fixed

Source: By the author.

4 TIMELIKE GEODESICS

For a timelike geodesic, the Lagrangian in expression (2.15) has the value of $\delta = 1$. Therefore, the effective potential will be given by

$$V_{eff}(\rho) = \frac{L_z^2}{U^4 \rho^2} + \frac{1}{U^2} \quad (4.1)$$

In this case, the positions of circular orbits depends on the angular momentum with respect to the coordinate z , L_z . So, it is necessary to define a direction for the velocity of the particle. As done by reference,²⁴ the initial direction will be chosen in accordance with a local tetrad, which is defined as

$$g_{ab} \mathbf{e}_\mu^a \mathbf{e}_\nu^b = \eta_{\mu\nu}$$

This gives

$$\mathbf{e}_t = U \partial_t, \quad \mathbf{e}_\rho = \frac{1}{U} \partial_\rho, \quad \mathbf{e}_\varphi = \frac{1}{U \rho} \partial_\varphi, \quad \mathbf{e}_z = \frac{1}{U} \partial_z$$

Therefore, a generic four-velocity is given by

$$\mathbf{u} = \pm A \mathbf{e}_t + B \mathbf{n}$$

where $\mathbf{n}$ is an arbitrary unit vector, as shown in Figure 4.1. This figure also illustrates how to write $\mathbf{n}$ in terms of the cylindrical coordinates unit vectors

$$\mathbf{n} = \cos \theta_1 \mathbf{e}_z + \sin \theta_1 \cos \theta_2 \mathbf{e}_\rho + \sin \theta_1 \sin \theta_2 \mathbf{e}_\varphi$$

Hence, the four-velocity can be written as

$$\mathbf{u} = AU \partial_t + B \left(\frac{\sin \theta_1 \cos \theta_2}{U} \partial_\rho + \frac{\sin \theta_1 \sin \theta_2}{U \rho} \partial_\varphi + \frac{\cos \theta_1}{U} \partial_z \right) \quad (4.2)$$

For the lightlike case, $A = B = 1$ and for a timelike geodesic,

$$A = \gamma, \quad B = \gamma\beta \quad \text{where} \quad \beta = \frac{v}{c} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

This condition comes from the normalization of the four-velocity: $u^\mu u_\mu = -1$.

Figure 4.1 – The unit vector $\mathbf{n}$ at a cylindrical coordinate system

Source: By the author.

For a timelike geodesic, there is the particle's velocity and angular momentum to consider. To remain in a circular motion, it is necessary that the velocity only has components in t and φ . Consequently,

$$u^\rho = u^z = 0 \quad \text{and} \quad \theta_1 = \theta_2 = \frac{\pi}{2}$$

The four velocity (4.2) then reads,

$$\mathbf{u} = \gamma U \partial_t + \frac{\beta \gamma}{U \rho} \partial_\varphi$$

The condition for a geodesic is that there is no acceleration. From the definition of four-acceleration²⁸

$$a^\mu = \frac{du^\mu}{d\lambda} + \Gamma_{\nu\lambda}^\mu \dot{x}^\nu \dot{x}^\lambda$$

where $\Gamma_{\nu\lambda}^\mu$ is the Christoffel symbol given by (2.7). The calculation of this acceleration results in

$$\begin{aligned} a^\rho &= -\gamma^2 U \left[\frac{\partial U}{\partial \rho} (1 + \beta^2) + \frac{U \beta^2}{\rho} \right] \\ a^z &= U \frac{\partial U}{\partial z} (1 - 2\gamma^2) \\ a^t a^\varphi &= 0 \end{aligned}$$

The condition $a^z = 0$ leads to the expression (3.2). From $a^\rho = 0$, we obtain

$$\beta = \left(\frac{U}{U + \rho \frac{\partial U}{\partial \rho}} - 1 \right)^{\frac{1}{2}} \quad (4.3)$$

which is the expression for the particle's velocity to remain in a circular orbit.²⁴

4.1 $z = 0$ and $M_1 = M_2 = M$

For equal masses in the $z = 0$ plane, the velocity (4.3) then reads

$$\beta = \left[\frac{2M\rho^2}{(\rho^2 + 1)^{3/2} + 2M} \right]^{\frac{1}{2}}$$

This is the particle's velocity required to stay on a circular orbit with radius ρ .

Figure 4.2 shows the plot of β for different masses. When the mass is equal to the critical value $\tilde{M} = 1.837$, the curve approaches $\beta = 1$, as expected since for $M > \tilde{M}$ photon orbits begin to appear (3.4). For values of M bigger than $\tilde{M}$, there is a region where no timelike orbits are possible.

To analyze the positions of the circular orbits, once more the condition $\partial V_{eff}/\partial \rho = 0$, (3.1) needs to be satisfied. Given the potential (4.1), the condition above becomes

$$L_z^2 = \frac{2M\rho^4 \left(1 + \frac{2M}{(\rho^2 + 1)^{1/2}} \right)^2}{(\rho^2 + 1)^{3/2} + 2M(1 - \rho^2)} \quad (4.4)$$

Figure 4.3 shows plots of L_z^2 for different values of M near limiting values for the mass, where there is a change in the behavior.

Substituting (4.4) in (4.1), it is possible to find the expression for the energy of the particle through equation (2.16), which gives

$$E^2 = \frac{2M + (\rho^2 + 1)^{3/2}}{[(\rho^2 + 1)^{3/2} + 2M(1 - \rho^2)] \left(1 + \frac{2M}{(\rho^2 + 1)^{1/2}} \right)^2} \quad (4.5)$$

Figure 4.2 – β for different values of M in the plane $z = 0$, reproduced from²⁴

Source: By the author.

(a) Limiting mass at $M = M_c$

(b) Limiting mass at $M = \tilde{M}$

Figure 4.3 – L_z^2 for increasing values of M

Source: By the author.

Expressions 4.4 and 4.5 are respectively the angular momentum and energy of a particle in a circular orbit with radius ρ . There is always at least an orbit for a given ρ , if the particle has the right amount of energy and momentum. Figure 4.4 shows some plots of E^2 for different values of M , including the limiting values for the masses. They all tend

to one for large ρ because the effective potential (4.1) tends to one as $\rho \rightarrow \infty$. They have the same behavior as L_z^2 at the critical values of M .

Figure 4.4 – E^2 for increasing values of M

Source: By the author.

The number of circular orbits for a given mass are related to the critical points, which is where the derivative is zero, in the curve of L_z^2 as can be seen in figure 4.3. The region between the critical points have three circular orbits, at exactly over the points there are two and one orbit otherwise. To find the value of these critical points, we can analyze the first derivative of L_z^2 which reads

$$D(\rho) = \frac{\partial L_z^2}{\partial \rho} = 16M^2 + 2M(\rho^2 + 1)^{1/2}(-3\rho^4 + 5\rho^2 + 8) + (\rho^2 + 1)^2(\rho^2 + 4) = 0$$

The derivative of E^2 gives the same expression. The graph of $D(\rho)$ is given in figure 4.5. The value of M where there is only one critical point is approximately²⁴

$$M_c \approx 1.02949$$

Therefore, for $M < M_c$ there is only one circular orbit and for $M > M_c$ there are regions with one, two or three orbits as can be seen in figure 4.3(a). The second limiting mass is in $M = \tilde{M}$ as already noted before in (3.4). Starting from this mass, there are singularities at the photon orbits as shown in figure 4.3(b).

Stability

For this situation of equal masses, the stability with respect to ρ can be analyzed by calculating the second derivative of the potential, as done before for the lightlike case

Figure 4.5 – $D(\rho)$ for different values of M , reproduced from²⁴

Source: By the author.

(2.19). The resulting expression, named $D^2(\rho)$ is

$$D^2(\rho) = \frac{\partial^2 V_{eff}}{\partial \rho^2} = \frac{2M \left[64M^4(3\rho^2 + 1)\sqrt{1 + \rho^2} + 8M^3(6\rho^8 - 15\rho^6 + 30\rho^4 + 43\rho^2 + 16) \right]}{\sqrt{1 + \rho^2}(2M + \sqrt{1 + \rho^2})^4[(1 + \rho^2)^{3/2} - 2M(\rho^2 - 1)]^3} \\ \frac{\left[-12M^2(2\rho^8 + 5\rho^6 - 11\rho^2 - 8)\sqrt{1 - \rho^2} + 2M(10\rho^6 - 27\rho^4 - 15\rho^2 + 16)(1 + \rho^2)^2 \right]}{\sqrt{1 + \rho^2}(2M + \sqrt{1 + \rho^2})^4[(1 + \rho^2)^{3/2} - 2M(\rho^2 - 1)]^3} \\ \frac{[+(-2\rho^4 - 13\rho^2 + 4)(1 + \rho^2)^{7/2}]}{\sqrt{1 + \rho^2}(2M + \sqrt{1 + \rho^2})^4[(1 + \rho^2)^{3/2} - 2M(\rho^2 - 1)]^3}$$

Comparing the graphs of E^2 with the plot of the second derivative, it is possible to recognize the regions with stable or unstable orbits. There are two limiting values of the mass M where there is a change in the behavior. For $M < 0.9132$ approximately, and for $M > 1.837$ when photon orbits begin to appear (3.4). Figure 4.6 shows the change of behavior near these limiting values. For $M < 0.9132$, there is always only one root, therefore orbits are stable for ρ smaller than the root and unstable otherwise. Similarly, there is also one root for $M > 1.837$, thus the interpretation is the same as the previous case.

The situation when the mass is between these values $0.9132 < M < 1.837$ is different, since there are four regions as shown in figure 4.7. Regions I and III have stable

Figure 4.6 – $D^2(\rho)$ near limiting values of M

Source: By the author.

circular orbits and the other two have unstable ones.

4.2 Case $0 < |z| < 1$

Until now, the restriction to the $z = 0$ plane was considered. For different values of z and equal masses, the velocity is given by expression (4.3) with ρ as a function of z given by (3.5). Figure 4.8 is the plot of this velocity for various values of $M_1 = M_2 = M$. The critical value of $M = 2.598$ (3.7) is the same as for the lightlike case, and the velocity reaches its limit, which is $\beta = 1$. This boundary also limits the regions with possible timelike circular orbits.

Figure 4.7 – $D^2(\rho)$ and $E^2(\rho)$ with $M = 1.4$

Source: By the author.

4.3 $M_1 \neq M_2$

The velocity for different masses is given by (4.3)

$$\beta = \left(\frac{U}{U + \rho \frac{\partial U}{\partial \rho}} - 1 \right)^{\frac{1}{2}}$$

with the expression for ρ given by (3.5) and the potential given by (2.14).

Here the analysis is similar to the lightlike case done in Section 3.4. By looking at the plots of the velocity β in figure 4.9, it is possible to note that the number of circular

Figure 4.8 – $\beta(z)$ for different values of $M_1 = M_2 = M$, reproduced from²⁴

Source: By the author.

orbits depends on the value of the masses in the same way as the lightlike case. This can also be seen by comparing this figure with figure 3.6. Therefore, the range of values for M_2 where there is the possibility of four orbits is the same as found before in (3.11)

$$1.53961 < M_2 < 2.598076 = \bar{M}$$

This is because when four lightlike circular orbits begin to exist, the curve goes through $\beta = 1$ four times, consequently leading to regions with four possible timelike orbits. The lightlike orbits are given by the points where $\beta = 1$.

4.4 $M_1 = M_2 + \epsilon$

Once more, the similar masses case with an approximation will be analyzed. The expression for the velocity is given by (4.3) but with ρ given by the approximation (3.9) done in Section 3.3. The plot of this approximated velocity named β' is found at figure 4.10 for different values of M and ϵ . These plots can be compared with those from figure 3.5. The limits of M from where four circular orbits begin to appear are the same for each value of ϵ in both cases. The reason is the same as already explained above.

Figure 4.9 – The velocity β near limiting values of M_1 and M_2

Source: By the author.

(a) $\epsilon = 0.1$	(b) $\epsilon = 0.01$
(c) $\epsilon = 0.001$	(d) $\epsilon = 0.0001$

Figure 4.10 – The velocity β' near limiting values of M

Source: By the author.

4.5 $M_1 \gg M_2$

In this section, the behavior of β (4.3) for increasing difference in the BH's masses is going to be analyzed. As done in section 3.5 the mass $M_2 = 3$ will be fixed and M_1 will be varied increasingly. In this situation, there are always only two circular orbits. The

behavior is similar as the lightlike case: as M_1 increases, the value of $z \rightarrow \pm 1$.

The limits of β are given by

$$\lim_{z \rightarrow \pm 1} \beta = \infty, \quad \lim_{z \rightarrow 0^-} \beta = -\infty, \quad \lim_{z \rightarrow z_{sing}} \beta = 0$$

where z_{sing} is the singularity at $0 < z < 1$.

(a) $M_1 = 5$

(b) $M_1 = 10$

(c) $M_1 = 20$

(d) $M_1 = 40$

Figure 4.11 – The velocity β for increasing values of M_1 with $M_2 = 3$ fixed

Source: By the author.

5 BOUNDED ORBITS IN THE $z = 0$ PLANE

In this section, lightlike closed bounded orbits restricted to the $z = 0$ plane will be analyzed. As already mentioned before (3.1), for the restriction $z = 0$, the masses of the BHs are the same $M = M_1 = M_2$.

These bounded orbits are found for values of energy that lie between the circular orbits, as can be seen in figure 5.1. The points where the effective potential intersects a given energy (the horizontal dashed line in the figure) are the radii of the bounded orbit.

Figure 5.1 – Example of a case with possible bounded orbit for a given energy (dashed line)

Source: By the author.

For this case, as already calculated, the potential is (2.18)

$$V_{eff}(\rho) = \frac{L^2}{U^4 \rho^2}$$

where U is given by (2.17). By varying the energy of the system and calculating the respective radii of bounded orbits, the eccentricity can be found through the expression

$$e = \sqrt{1 - \frac{b^2}{a^2}} \quad \text{with} \quad a > b$$

where a is the semi-major axis and b the semi-minor. The plot of the eccentricity with varying M and L is shown in figure 5.2.

The range of values for the eccentricity increases for increasing angular momentum L and the eccentricity increase for increasing M . This is in accordance with the characteristic of the effective potential, as can be seen in figure 3.2.

Figure 5.2 – Eccentricity varying M and L .

Source: By the author.

6 CONCLUSION

The main objective of this work was to study the Majumdar-Papapetrou spacetime, which consists of extremal Reissner-Nordström black holes in equilibrium. The case studied is the binary system of these black holes with equal and different masses. In order to do so, the circular geodesic motion of massive and lightlike particles was analyzed. For simpler cases, the stability of the orbits with respect to ρ was evaluated and for black holes with equal mass, bounded orbits were considered. Another point was the investigation of orbits in the binary spacetime with black holes of very similar (but unequal) and very different masses, respectively. Through all these studies, it was possible to understand some features of the MP spacetime and the relation between the masses of the black holes and the existence of (circular) closed orbits. A compilation of the results will be given in the following paragraphs. Additionally, the behavior of the motion of massive particles was in agreement with that of the light ones since there are no timelike orbits in the lightlike range of possible orbits. Besides that, it was interesting to compare these two cases, since for timelike geodesics the orbits depend on the angular momentum. Therefore, if the particle has sufficient velocity, there are more possibilities of closed orbits.

Firstly, a brief overview of general relativity was presented, mainly to introduce the concepts of geodesics and the Reissner-Nordström metric. Then, starting with lightlike circular orbits, a limiting value for the mass was found, denoted by $\tilde{M} = 1.837$ for orbits restricted to the $z = 0$ plane. There are three cases, and the quantity of circular orbits in each case is

$$\text{lightlike, } z = 0 \begin{cases} M < \tilde{M} & \rightarrow \text{no orbits} \\ M = \tilde{M} & \rightarrow \text{one orbit} \quad \rightarrow \text{stable} \\ M > \tilde{M} & \rightarrow \text{two orbits} \quad \rightarrow \text{one stable, one unstable} \end{cases}$$

For orbits outside the $z = 0$ plane, there is another limiting value for the mass given by $\bar{M} = 2.598076$. Under this condition, equal and different masses are two separate cases. In the first case, there are orbits only for $M < \bar{M}$

$$\text{lightlike, } 0 < |z| < 1, M_1 = M_2 = M \begin{cases} M < \bar{M} & \rightarrow \text{two orbits} \\ M = \bar{M} & \rightarrow \text{not defined} \\ M > \bar{M} & \rightarrow \text{no orbits} \end{cases}$$

If the masses are different, there is an additional lower boundary for one mass $1.53961 < M_2 < \bar{M}$, where there is a possibility of four orbits depending on the value of the other mass M_1 . Any other value of M_2 gives two orbits

$$\text{lightlike, } 0 < |z| < 1, M_1 \neq M_2 \begin{cases} 1.53961 < M_2 < \bar{M} & \rightarrow \text{four orbits possible} \\ \text{otherwise} & \rightarrow \text{two orbits} \end{cases}$$

Moving to timelike geodesics, the restriction to the $z = 0$ plane leads to a more complex analysis for the stability. First of all, there is another limiting value $M_c = 1.02949$ which separates the possible number of circular orbits as follows

$$\text{timelike, } z = 0 \begin{cases} M < M_c & \rightarrow \text{one orbit} \\ M \geq M_c & \rightarrow \text{one, two or three orbits} \end{cases}$$

The stability in this case is separated in regions, since there is an arbitrary number of orbits in each region. Therefore, the stability can be simplified as the following

$$\text{stability} \rightarrow \text{timelike, } z = 0$$

$$\begin{cases} 0.9132 < M_2 < 1.837 & \rightarrow \text{four regions} \rightarrow \text{two stable/two unstable} \\ \text{otherwise} & \rightarrow \text{two regions} \rightarrow \text{one stable/one unstable} \end{cases}$$

For orbits outside the $z = 0$ plane, the behavior of massive particles is qualitatively similar as that of light, but there are no timelike orbits in the region of height z where there are photon orbits.

$$\text{timelike, } 0 < |z| < 1, M_1 = M_2 = M \begin{cases} M < \bar{M} & \rightarrow \text{two orbits} \\ M = \bar{M} & \rightarrow \text{not defined} \\ M > \bar{M} & \rightarrow \text{no orbits} \end{cases}$$

$$\text{timelike, } 0 < |z| < 1, M_1 \neq M_2 \begin{cases} 1.53961 < M_2 < \bar{M} & \rightarrow \text{four orbits possible} \\ \text{otherwise} & \rightarrow \text{two orbits} \end{cases}$$

Regarding the analysis of the case for unequal black hole masses, there is a similar behavior between lightlike and timelike geodesics. For the approximation of very similar masses, where $M_1 = M_2 + \epsilon$, it was noticed that the orbits' values of z approach zero as the mass and ϵ increases. On the other hand, as the difference between the masses was amplified, the value of z approaches $+1$ and -1 . For the timelike and similar masses case, the decrease of ϵ showed that the heights approach zero for velocities approaching 1. As for the case of huge mass difference, there was an equal behavior, with z approaching zero as the difference between the masses increased.

Finally, bounded orbits were investigated for equal masses restricted to the $z = 0$ plane. It was found that there is a bigger range for possible values of eccentricity as the angular momentum increased. Additionally, it was noticed that the mass M is directly proportional to the eccentricity of the orbits.

After all these studies, it is possible to recognize the importance of geodesics for the study of a spacetime. Here, the investigation of circular orbits provided some insights into the structure of the binary Majumdar-Papapetrou spacetime. This work was important to learn about an exact solution of the Einstein-Maxwell equation, and might be a first step to understand more complex multi black holes systems in the future. Although it is

a simplified system, and not very likely to be found in nature, it can be helpful for the development of research about other systems and spacetimes, such as the Kerr metric as already mentioned. While in this thesis we only analyzed an uncharged test particle, it is also interesting to consider a charged one in future works. Another possible improvement would be a more complete evaluation of the stability in all cases, including outside the $z = 0$ plane. Also, non-closed orbits, so-called “fly-by” orbits are interesting, e.g. for light deflection and scattering of massive particles by these binary systems.

The new features discovered at this work were the boundaries separating the case of two from four possible circular orbits in the situation of different masses of the BHs $1.53961 < M_2 < \bar{M}$ for both lightlike and timelike cases. All analysis of approximations and huge changes in the black holes’ masses were also done by this work, including the final bounded orbits exploration. Finally, the regions of possible stability of the circular orbits with respect to ρ were also found to be a new analysis.

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