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**Investigations on momentum distributions
and disorder in strongly
out-of-equilibrium trapped Bose gases**

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FRANKLIN ADÁN JULCA VIVANCO

Investigations on momentum distributions and disorder in strongly out-of-equilibrium trapped Bose gases

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*The problem with the world is
that the intelligent people are
full of doubts, while the stupid
ones are full of confidence.*

Charles Bukowski.

ABSTRACT

VIVANCO, F.A.J. **Investigations on momentum distributions and disorder in strongly out-of-equilibrium trapped Bose gases.** 2017. 120 p. Thesis (Doctorate in Science) – Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2017.

From almost one century, Bose-Einstein condensation has become progressively more important especially due to its connection with superfluidity, superconductivity and many-body physics. Nowadays quantum gases are powerful experimental tools to discover new physics and to emulate systems in condensed matter due to their versatility and very high control. Despite the increasing use of quantum gases as platforms for studying many problems in physics, their comprehension is very limited if we consider systems that are out-of-equilibrium due to the lack of experimental controllability of all the parameters involved in these systems. Another limitation in the understanding of this kind of systems comes from the limitation of the theoretical frameworks used to understand non-equilibrium dynamics, although many efforts have been made in this direction. Hence, many interesting phenomena in non-equilibrium quantum systems have not yet been discovered or well understood from a theoretical and experimental point of view, and thus its physics have not been the focus of much attention, although this situation has recently changed due to the rapid development of experimental techniques which enables a better control of parameters of these systems. Motivated by this progress, we study non-equilibrium Bose gases in the search of turbulence using an oscillatory excitation performed in a Bose-Einstein condensate of ^{87}Rb atoms. In this thesis, we describe these experiments and characterize our non-equilibrium quantum system through some quantifiers. One of these quantifiers is a dimensionless value that represent the exponent γ obtained from the cascade of the transverse momentum distribution $\tilde{n}(k)$. $\tilde{n}(k)$ is obtained from absorption images of atoms in expansion using the time-of-flight technique in a well defined range of momenta. We analyze the dependence of γ with the amount of the pumped energy, and we found a steady-value which describe a well-established non-equilibrium regime. Also, it is analyzed the viability of using the fluctuations statistics in order to extract some quantifier from the power-spectrum of the fluctuations assuming that it represents an analog to the energy spectrum, due to the consideration of the time-of-flight technique. From the power-spectrum it is extracted an exponent, in the same range as for $\tilde{n}(k)$, and compared with $\gamma - 2$, that will be the exponent for the pseudo-energy spectrum in the kinetic dominated regime. Finally, we consider, again with the time-of-flight technique, the continuous Shannon entropy as quantifier that measure the disorder of the excited clouds and study their dependence with the pumped energy. These studies show us that there is an out-of-equilibrium regime that takes place when we inject a fixed quantity of energy into the system.

Keywords: Bose-Einstein condensation. Momentum distribution. Quantum turbulence.

RESUMO

VIVANCO, F. A. J. **Investigações nas distribuições de momento e na desordem em gases de Bose armadilhados fortemente fora do equilíbrio.** 2017. 120 p. Tese (Doutorado em ciências) – Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2017.

Desde há quase um século a condensação de Bose-Einstein vem se tornando cada vez mais importante, especialmente devido à sua forte conexão com superfluidez, supercondutividade e física de muitos corpos. Hoje em dia, os gases quânticos são poderosas ferramentas experimentais para descobrir-se nova física e para emular sistemas em matéria condensada devido à sua grande versatilidade e altíssimo controle. Apesar do uso crescente de gases quânticos como plataformas para se estudar diversos problemas na física, sua compreensão é muito limitada se considerarmos sistemas que estão fora de equilíbrio, devido à falta de controle experimental de todos os parâmetros envolvidos deste tipo de situações. Outra limitação na compreensão deste tipo de sistemas vem da limitação das abordagens teóricas usadas para entender a dinâmica em regimes de não equilíbrio, embora muitos esforços tem sido feitos nessa direção. Assim, muitos fenômenos interessantes em sistemas quânticos fora do equilíbrio ainda não foram descobertos ou bem compreendidos do ponto de vista teórico e experimental, e portanto, sua física não tem sido foco de muita atenção, embora esta situação tenha mudado recentemente devido ao rápido desenvolvimento de técnicas experimentais que permitem um melhor controle dos parâmetros destes sistemas. Motivados por estes progressos, estudamos aqui gases de Bose fora do equilíbrio, na busca de turbulência, através de excitações oscilatórias em um condensado de Bose-Einstein de átomos de ^{87}Rb . Nesta tese, descrevemos estes experimentos e caracterizamos o nosso sistema quântico fora do equilíbrio através de alguns quantificadores. Um desses quantificadores é o valor adimensional que representa o expoente γ obtido da de cascata na distribuição de momento transversal $\tilde{n}(k)$. $\tilde{n}(k)$ é obtido da imagem de absorção da nuvem atômica em expansão usando a técnica de tempo de voo em um intervalo de momento bem definido. É analisada a dependência de γ com energia bombeada e encontramos um valor constante o qual descreve um regime de não equilíbrio bem estabelecido. Analisamos também a viabilidade do uso da estatística das flutuações para extrair algum quantificador do espectro de potências das flutuações, supondo que ele representa um análogo ao espectro de energia, devido à consideração da técnica de tempo de voo. Do espectro de potências é extraído outro expoente, no mesmo intervalo que para $\tilde{n}(k)$, e este é comparado com $\gamma - 2$, que por sua vez, pode ser considerado como o expoente do espectro de pseudo-energia no regime cinético dominado. Finalmente, consideramos, novamente com a técnica do tempo de voo, a entropia contínua de Shannon como quantificador que mede a desordem das nuvens excitadas e estuda sua dependência com a energia bombeada. Estes estudos mostram que há um regime do fora de equilíbrio bem definido que acontece quando injetamos uma quantidade fixa de energia no sistema.

Palavras-chaves: Condensação de Bose-Einstein. Distribuição de momentos. Turbulência quântica.

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1 INTRODUCTION

More than twenty years ago the first atomic Bose-Einstein condensate was born in a vacuum chamber as the experimental realization of one of the greatest paradigms of quantum physics. (1-3) The realization, at almost the same time, by three independent research groups in the USA, using laser and evaporative cooling techniques, opened the doors to a huge amount of researches in atomic quantum gases and superfluidity, and extended its use as a platform to emulate quantum systems, as is the case for solid state physics and optical lattices. (4) The new opportunities that the experimental revolution have produced are driving a wide range of science and technology in the world. This includes an important theme of this thesis: superfluid Bose systems far-from equilibrium.

Bose-Einstein condensation (BEC) is a pure quantum mechanical effect which started its story with the statistical theory for the blackbody radiation in the 1920s by S. N. Bose (5), extended in 1924 to massive non interacting particles by A. Einstein. (6) In nature, particles can be divided into two categories: bosons, particles with integer spin (e.g. photons) and fermions, that have half-integer spin (e.g. electrons). A key difference is that fermions are limited to only one particle per state by the Pauli exclusion principle, while any number of bosons can occupy the same state. Under the Bose-Einstein statistics, the occupation of the ground state of the system diverges in the limit of zero temperature, leading a macroscopic population of this single state. This macroscopic quantum phenomena is an excellent platform for probe many effects in quantum physics, as fundamental quantum mechanics (7-9), condensed matter physics (11-12), many-body physics (13-14), and recently to simulate cosmological situations. (15)

On the other hand, the phenomenon of superfluidity observed for the first time in the 1930s in liquid Helium (16) was related to BEC phenomenon by F. London (17) in 1938. Since then, the understanding of superfluidity had received important improvements, and the prediction of elementary excitations (18) and quantum vortices (20) were probed broadly in ultracold atoms experiments.

Nowadays, atomic BEC's are produced in many laboratories around the world, in different experimental configurations, for many atomic species, typically with temperatures of $\sim 10\text{ nK} - 1\text{ }\mu\text{K}$ and atomic densities of $\sim 10^{-14}\text{ atoms/cm}^3$. These degenerate quantum gases has been achieved in different geometries (21–25) containing from few hundreds of atoms to some millions. (26) The advancing of experimental techniques has evolved in the last years due to better understanding of the physics of the processes involved and also due to the technological progress. Thus, a BEC, which in a laboratory is typically obtained in the order of many seconds, was recently obtained in the order of milliseconds using laser Raman cooling. (27)

As a consequence of the advances in the achievement of quantum gases, now it is possible to investigate in a controllable way non-equilibrium situations, and the understanding of this systems is the focus of this thesis.

1.1 Interest in out-of-equilibrium quantum gases

Investigations on non-equilibrium systems and its dynamics has made tremendous progress using cold atom systems. (28–32) While theoretical frameworks are applicable to general quantum systems and thus are not limited to the experimental capabilities, they are often motivated by the possibility of immediate experimental realizations. In recent years theoretical development in quantum turbulence have opened the door to explore this new regime in ultracold atoms. (33–38) In 2009, the first evidence of quantum turbulence in trapped dilute atomic Bose–Einstein condensates (BEC's) was presented by our research group and opened new and exciting perspectives. (39) Since then, the experimental achievement of turbulent states in different quantum systems, as low dimensional trapped quantum gases (40–41) and box potential (42), has maintained the interest in this field as a hot

topic, but despite the many progress, we are still far from a complete understanding of the phenomenon of turbulence in quantum gases.

Thus, due its high controllability, atomic superfluids continue to be good platforms for testing fundamental aspects of turbulence in quantum fluids and to explore similarities and differences between quantum turbulence and classical turbulence as realized in liquid Helium. (43) In this direction, trapped atomic superfluids in harmonical potentials continue to be excellent laboratories to test evolution and decay of the turbulence, and this is the main motivation for studying the non-equilibrium states in excited Bose gases.

Although we do not refer along this thesis to the non-equilibrium states, obtained by our excitation mechanism, as turbulent states, it is clear that our intention and efforts are directed to the study and production of well-controlled turbulent states. Thus, we are careful in distinguishing the non-equilibrium regime with the turbulent regime, simply by rigor of the definition of quantum turbulence in the last years. In a set of reviews (44-45) the actual state of art in quantum turbulence in quantum gases and superfluid helium is discussed in detail.

1.2 Structure of this thesis

In this thesis, we study three ways to quantify the non-equilibrium regimes produced by the action of a oscillatory excitation in Bose gases, with the same excitation protocol that was used in previous works. (39, 46-50) We focus on studies of momentum distribution, power-spectrum of fluctuations, and continuous entropy to probe non-equilibrium regimes in cigar-shaped trapped gases in the searching of turbulence. The apparatus for performing these experiments prepares an ultracold quantum gas of some nano-Kelvins in a magnetic trap and applies a well controlled oscillatory magnetic excitation. This excitation introduces a time dependent variation in the potential and allows to explore the evolution of the system with the increase of the injected energy. The outline of this thesis is as follows:

- In Chapter 2, we make a brief review of the well-known concepts of BEC theory on trapped BEC in harmonic potentials, and we pay special attention to the case

of weakly interacting gas in the Thomas-Fermi approximation, using the mean field approximation for the many-body phenomena known as Gross-Pitaevskii equation. Using the Gross-Pitaevskii equation, we deduce the hydrodynamics equations that describes superfluidity for atomic gases. The phenomenon of superfluidity has as its own and one of the most shocking manifestations the presence of quantum vortices, and so, we describe this vortices using the Madelug wavefunction. The presence of many quantum vortices can lead to the apparition of a new regime in superfluidity, known as quantum turbulence. Hence, we will finish this chapter, revising the phenomenology of classical turbulence, and the actual state of art of quantum turbulence.

- In Chapter 3, we present the experimental methods and the experimental apparatus used for producing atomic superfluids. Together with the description of each experimental step to obtain the degenerate gas, the physical concepts of the used techniques are presented in a succinct way. At the end of this chapter, we present the mechanism and procedure to excite the atomic gas in order to produce the non-equilibrium system that is explored in dependence of the excitation parameters.
- In Chapter 4, we focus on explore the momentum distribution of excited quantum gases through the time-of-flight technique. Assuming that absorption images in time-of-flight allows to extract the correct momentum distribution for the kinetically dominated regime, we look for the cascade behavior in the inertial range. A power-law decay in this range, show us the presence of cascades, and we use the power-law exponent as a quantifier of the non-equilibrium state. Using this exponent in dependence with the excitation parameters, we explore the regimes of the non-equilibrium, and it is found a regime of steady-value exponent, that is an indicative of a well-established non-equilibrium regime. Thereby, we are able to investigate the excitation conditions in that this well established equilibrium regime takes place.
- Chapter 5 is devoted to the statistical treatment of the absorption images of the atomic clouds to analyze the power spectrum of fluctuations in the density profile.

We present the theory for power spectrum, and the method used to extract these information from absorption images. We use as quantifier a power-law exponent in the inertial region, the same region defined in Chapter 4. With this quantifiers we explore the dependence with the excitation amplitude. Considering that the system is turbulent, we compare the cascade exponents for the power spectrum with those extracted from the momentum distribution.

- Chapter 6 explore the use of the continuous Shannon entropy as quantifier that measures disorder. We use this entropy to explore the same excitation situations presented in Chapter 4, and we demonstrate that it is a good quantifier for characterize the non equilibrium regimes.
- The conclusions and perspectives are present in Chapter 7. There we discuss how the well-established non-equilibrium regime, quantified along this thesis, may be related with the state of turbulence.

2 OVERVIEW IN BEC AND SUPERFLUIDITY

This chapter introduces the theoretical basis of the Bose-Einstein condensation of trapped gases as it is described on many atomic physics textbooks. (51–57) Then, we consider the theory of superfluidity in atomic gases and important concepts as the hydrodynamics equations and vortices. To finish, we present a brief overview about turbulence in classical fluids and the art of state on turbulence in quantum fluids.

2.1 Theoretical framework

Bose-Einstein condensation is one of the most striking features of quantum statistics. Usually BEC is defined as the macroscopic occupation of the single-particle energy state of the system. In general, a necessary condition for the macroscopic quantum phenomena to show up is that the de Broglie wavelength $\lambda_{dB} = h/\sqrt{2\pi m k_B T}$, where k_B is Boltzmann's constant, h is Planck's constant, m is the particle mass and T is the temperature (one might roughly think of it as the spatial extension of a quantum state), becomes comparable to the inter-particle spacing. Introducing the phase space density $PSD \equiv n\lambda_{dB}^3 \geq 1$ where n is the particle density, or in terms of temperature $T \leq (\hbar^2/2\pi m k_B) n^{2/3}$ we see that macroscopic quantum phenomena are inherently low temperature phenomena.

BEC's in dilute atomic gases are experimentally achieved in the lowest energy state of an external magnetic or optical potential, and in these cases, there is a coexistence with a non-condensed phase – the “thermal cloud”. In this section, we review some basic aspects of BEC's theory for trapped gases in harmonic potential, the Thomas-Fermi solution

of the Gross-Pitaevskii equation, which describes the weakly interacting BEC at zero temperature, and the expressions for the time-of-flight density distributions of the thermal cloud and BEC, that are tools used for experimentally extract information of the quantum system.

2.1.1 BEC in harmonic potential

The description of an ideal Bose gas in the grand-canonical ensemble based on the mean occupation number of the single-particle energy state ϵ is,

$$\langle n_\epsilon \rangle = \frac{1}{z^{-1}e^{\beta\epsilon} - 1}, \quad (2.1)$$

where $\beta = 1/k_B T$, $z = e^{\beta\mu}$ is the fugacity and μ is the chemical potential of the gas. The mean occupation number is bounded only from below, $\langle n_\epsilon \rangle \geq 0$; i.e. any number of bosons can occupy a given single-particle energy level. Here, we consider a non-interacting Bose gas in the external harmonic potential,

$$V(r) = \frac{1}{2}m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2), \quad (2.2)$$

which is associated with the single-particle energy state of the tridimensional harmonic oscillator,

$$\epsilon(n_x, n_y, n_z) = \left(n_x + \frac{1}{2}\right) \hbar\omega_x + \left(n_y + \frac{1}{2}\right) \hbar\omega_y + \left(n_z + \frac{1}{2}\right) \hbar\omega_z. \quad (2.3)$$

For high temperatures, μ is large and negative, and $\langle n_\epsilon \rangle \ll 1$. As T decreases, μ approaches the ground state energy of the oscillator, $\epsilon_0 = \epsilon_{000}$, and the occupation of the lowest energy levels increases until eventually the single-particle ground state ϵ_0 becomes macroscopically occupied. The ideal BEC transition temperature T_c is the highest temperature at which this macroscopic occupation takes place, and the chemical potential becomes $\mu = \epsilon_0$ when $T \rightarrow 0$, and all atoms populate the single-particle ground state.

For $T < T_c$ we can think the system as being composed of a mixture of two “phases”: a normal phase, consisting of N_{th} atoms distributed over the excited states $\epsilon > \epsilon_0$ (the “thermal cloud”); and a condensed BEC phase consisting of N_0 particles accumulated in the ground state (the “condensed cloud”).

For non-interacting bosons the condensate density distribution is,

$$n_c(\mathbf{r}) = N_0 |\varphi_0(\mathbf{r})|^2, \quad (2.4)$$

where,

$$\varphi_0(\mathbf{r}) = \frac{1}{\pi^{3/4} \sqrt{a_x a_y a_z}} e^{\left(-\frac{x^2}{2a_x^2} - \frac{y^2}{2a_y^2} - \frac{z^2}{2a_z^2}\right)}, \quad (2.5)$$

is the single-particle ground state wavefunction in the harmonic potential $V(\mathbf{r})$; and $a_i = \sqrt{\hbar/m\omega_i}$ are the harmonic oscillator lengths.

For large N and small energy level spacing with respect to the temperature $k_B T \gg \hbar\bar{\omega}$, with $\bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$ the density distribution of the thermal cloud can be calculated using a semi-classical integration of the occupation function n_ϵ over momentum degrees of freedom $\mathbf{p}$,

$$n(\mathbf{r}) = \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{1}{z^{-1} e^{\beta\epsilon(\mathbf{r},\mathbf{p})} - 1}, \quad (2.6)$$

that gives the thermal part,

$$n_{\text{th}}(\mathbf{r}) = \lambda_{\text{dB}}^{-3} g_{3/2} \left(z e^{\left[-\frac{\beta m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)\right]} \right) \quad (2.7)$$

$$= \frac{N_{\text{th}}}{g_3(z) \pi^{3/2} \sigma_x \sigma_y \sigma_z} g_{3/2} \left(z e^{\left[-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\right]} \right), \quad (2.8)$$

where $g_i(z)$ are the Bose functions.

The thermal cloud atom number below the transition temperature (for which $z = 1$) is obtained by the normalization condition,

$$N_{\text{th}} = N - N_0 = \int d\mathbf{r} n_{\text{th}}(\mathbf{r}) = \zeta(3) \left(\frac{k_B T}{\hbar\bar{\omega}} \right)^3, \quad (2.9)$$

where $\zeta(3) = g_3(1) \simeq 1.202$ and $\zeta(\alpha) = \sum_{n=1}^{\infty} n^{-\alpha}$ is the Riemann zeta function. The ideal BEC transition temperature is obtained by setting $N_0 = 0$ at $T = T_c$ in the last equation,

$$T_c = \frac{\hbar\bar{\omega}}{k_B} \left(\frac{N}{\zeta(3)} \right)^{1/3} \simeq 0.94 \frac{\hbar\bar{\omega}}{k_B} N^{1/3}. \quad (2.10)$$

Inserting the expression for T_c from (2.10) into (2.9) it results in the following expression for the temperature dependence of the condensate fraction,

$$\frac{N_0(T)}{N} = 1 - \left(\frac{T}{T_c} \right)^3. \quad (2.11)$$

The result of (2.10) is accurate in the thermodynamic limit $N \rightarrow \infty$ and $V \rightarrow \infty$ where V is the volume of the condensate. The approximation $\epsilon_0 \sim 0$ in the integrals over energy states ϵ is justified in this limit. In real BEC experiments, N is finite and the effect of the non-zero single-particle ground state energy $\epsilon_0 = \frac{\hbar}{2} (\omega_x + \omega_y + \omega_z)$ introduces an appreciable shift in the transition temperature.

The finite-size correction to T_c can be calculated by considering the effect of the zero-point energy on μ at the transition point. In a cylindrical symmetric harmonic trap with $\omega_{y,z} = \omega_r$ and $\omega_x = \omega_a$, the transition temperature T'_c that includes finite-size corrections is given by,

$$k_B T'_c \approx k_B T_c - \frac{\zeta(2)}{6\zeta(6)} \hbar(2\omega_r + \omega_a). \quad (2.12)$$

The correction in T_c (second term on the right side of (2.12)) is independent of N , but the fraction $(T'_c - T_c)/T_c$ is proportional to $N^{-1/3}$.

The effect of inter-particle interactions, also introduces a shift of the transition temperature. Repulsive interactions reduce the peak density at the center of the trap, which in turn reduces T_c . The magnitude of this reduction is given by,

$$\frac{T_c - T'_c}{T_c} \approx -1.326 a \sqrt{\frac{m\omega}{\hbar}} N^{1/6}, \quad (2.13)$$

where a is the s-wave scattering length. The expression for T_c given in (2.13) is used to calibrate the total atom number measured in absorption imaging.

2.1.2 Effects of interactions

Elastic atom-atom interactions modify the BEC wavefunction, leading to deviations from the Gaussian density profile presented in (2.4) for the non-interacting case. In dilute ultra-cold atomic gases the effect of interactions is well-described by a mean-field approach which assumes only binary s-wave scattering.

Since elastic scattering in ultra-cold bosons only occurs in the symmetric s-wave channel, atom-atom interactions can be described by a delta-function contact interaction pseudopotential,

$$V(\mathbf{r} - \mathbf{r}') = g \delta(\mathbf{r} - \mathbf{r}'), \quad (2.14)$$

with $g = 4\pi\hbar^2 a/m$ the interaction parameter derived from the s-wave scattering length a . The dilute-gas approximation used to g is justified so long as $|a| \ll d$ where $d = n^{-1/3}$ is the average interparticle separation in a gas of average density n , i.e. $n|a|^3 \ll 1$. In typical ultracold atoms experiments, $n_c(0) \sim 10^{13} \text{ cm}^{-3}$ so that $n|a|^3 \sim 10^{-6}$. For example, interactions between two ^{87}Rb atoms are repulsive, with $a \approx 99a_0 \sim 5.2 \text{ nm}$, where a_0 is the Bohr radius.

In the dilute, “weakly interacting” regime $n|a|^3 \ll 1$, the zero-temperature properties of the non-uniform condensate can be described using the pseudopotential of (2.14) and a mean-field treatment which ignores quantum fluctuations in the condensed state. The result is the Gross-Pitaevskii equation (GPE),

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g |\psi(\mathbf{r})|^2 \right) \psi(\mathbf{r}) = \mu \psi(\mathbf{r}), \quad (2.15)$$

where $\psi(\mathbf{r}) = \sqrt{N_0} \varphi(\mathbf{r})$ is the single-particle wavefunction of the condensed state, and $n_c(\mathbf{r}) = |\psi(\mathbf{r})|^2$ the condensate density, which is subject to the normalization condition,

$$N_0 = \int d\mathbf{r} n_c(\mathbf{r}). \quad (2.16)$$

The time-independent GPE is obtained from the time-dependent version by assuming a time-dependent condensate wavefunction of the form $\psi(\mathbf{r}, t) = \psi(\mathbf{r}) e^{-i\mu t/\hbar}$. The GPE has the form of a non-linear Schrodinger equation, in which the effective potential acting on each boson is a sum of the external potential V and the mean-field $g |\psi(\mathbf{r})|^2$ produced by the other bosons. The mean-field approximation used in deriving (2.15) is valid so long as $N_0 \gg 1$.

Though the system is said to be weakly interacting, interaction effects are actually large compared with the kinetic energy of atoms in the trap in most atomic BEC experiments. The quantity $N_0 a / \bar{a}$ is a dimensionless measure of the strength of the interaction, where $\bar{a} = (a_x a_y a_z)^{1/3} = \sqrt{\hbar/m\bar{\omega}}$. For $N_0 a / \bar{a} \ll 1$, the interactions are a small perturbation, while for $N_0 a / \bar{a} \gg 1$, equilibrium is determined by the competition between potential energy and interaction energy. Most atomic BEC experiments are in this interaction-dominated regime, with $N_0 \sim 10^4$ to 10^7 .

The Thomas-Fermi approximation amounts to neglecting the kinetic energy term in the GPE equation. In this approximation the equilibrium condensate density for repulsive interactions is given by,

$$n_c(\mathbf{r}) = |\psi(\mathbf{r})|^2 = \frac{\mu - V(\mathbf{r})}{g}, \quad (2.17)$$

for $\mu > V(\mathbf{r})$, and $n_c(\mathbf{r}) = 0$ otherwise. Thus, $V(\mathbf{r}) = \mu$ defines the edges of the condensate. The spatial half-widths (Thomas-Fermi radii) of the condensate are,

$$R_i = \sqrt{\frac{2\mu}{m\omega_i}}. \quad (2.18)$$

The relationship between N and μ is established by the normalization condition for the Thomas-Fermi wavefunction. Integrating $n_c(\mathbf{r})$ over the volume defined by the Thomas-Fermi radii gives,

$$N_0 = \frac{8\pi\mu}{15g} \bar{R}^3. \quad (2.19)$$

where $\bar{R} \equiv (R_x R_y R_z)^{1/3}$ is the geometric mean Thomas-Fermi radius of the condensate. Equations (2.18) and (2.19) can be used to express the condensate density profile as,

$$n_c(\mathbf{r}) = \frac{15}{8} \frac{N_0}{R_x R_y R_z} \left(1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} - \frac{z^2}{R_z^2} \right), \quad (2.20)$$

for $\mu > V(\mathbf{r})$, and $n_c(\mathbf{r}) = 0$ otherwise. Equation (2.20) describes a parabolic cloud in three dimensions with peak density $n_c(0) = \mu/g$. From (2.19), the chemical potential is given by,

$$\mu = \frac{\hbar\bar{\omega}}{2} \left(\frac{15N_0 a}{\bar{a}} \right)^{2/5}, \quad (2.21)$$

and

$$\bar{R} = \left(\frac{15N_0 a}{\bar{a}} \right)^{1/5} \bar{a}. \quad (2.22)$$

Equation (2.22) can be used to demonstrate that, as a result of repulsive interactions, the zero-temperature size of the condensate $\bar{R}$ is much larger than that predicted by the non-interacting treatment $\bar{a}$.

The Thomas-Fermi approximation is excellent when $\mu = gn(\mathbf{r}) \gg \bar{\omega}/2$, i.e. near the center of the cloud, but not at the edges, where the density vanishes. Numerical solutions of the full GPE show that the slight Thomas-Fermi underestimate of $n_c(|\mathbf{r}| \approx R)$ is even less severe in column density distributions observed in time-of-flight absorption imaging.

2.1.3 Measuring observables in time-of-flight

We measure the atom number and the temperature of quantum gases using time-of-flight absorption imaging after abruptly switching off the trapping potential and allowing the cloud to expand for a time t_{of} . Assuming that the thermal cloud density is sufficiently low that interactions during expansion may be neglected, the time-of-flight density may be written,

$$n_{\text{th}}(\mathbf{r}, t) = \frac{N_{\text{th}}}{(2\pi)^{3/2} \sigma_x(t) \sigma_y(t) \sigma_z(t) g_3(z)} g_{3/2} \left(z e^{\left[-\frac{x^2}{2\sigma_x^2(t)} - \frac{y^2}{2\sigma_y^2(t)} - \frac{z^2}{2\sigma_z^2(t)} \right]} \right), \quad (2.23)$$

where $\sigma_i^2(t) = k_B T(1 + \omega_i t^2)/m\omega_i^2$ is the cloud size in the $i \in \{x, y, z\}$ direction after a time t of free expansion. The column density along x is obtained by integrating (2.23) along x :

$$\tilde{n}_{\text{th}}(y, z, t) = \frac{N_{\text{th}}}{2\pi \sigma_y(t) \sigma_z(t) g_3(z)} g_{3/2} \left(z e^{\left[-\frac{y^2}{2\sigma_y^2(t)} - \frac{z^2}{2\sigma_z^2(t)} \right]} \right). \quad (2.24)$$

At long times of flight the expansion of the thermal cloud becomes isotropic. When analyzing thermal clouds below the transition temperature, one must use $z = 1$ in Equations (2.23) and (2.24).

For the condensate, interparticle interactions play an important role during time-of-flight expansion. Approximate analytic solutions of the time-dependent Gross-Pitaevskii equation reveal that the parabolic shape of the condensate is preserved during time-of-flight expansion but the aspect ratio is not. In particular, for a BEC from a cylindrical symmetric anisotropic trap with $\omega_a = \lambda \omega_r$, the Thomas-Fermi radii of the condensate evolve according to,

$$R_r(t) = R_r(0) \sqrt{1 + \tau^2}, \quad (2.25)$$

$$R_a(t) = R_a(0) \left(1 + \lambda^2 \left[\tau \arctan \tau - \ln \sqrt{1 + \tau^2} \right] \right), \quad (2.26)$$

where $\tau = \omega_r t$, which is known as Castin-Dum expansion. (58) Thus the Thomas-Fermi condensate density in time-of-flight is given by (2.20) with the Thomas-Fermi radii rescaled according to equations (2.25) and (2.26). The BEC column density in the yz plane is obtained by integrating the result along x in the region bounded by $x \in [-R_x(t), R_x(t)]$,

Figure 1 – Free-expansion of a BEC. **a**, time-of-flight detection technique. **b**, image of the cloud showing the fitting to extract the thermal (green line) and condensed parts (red line).

Source: By the author.

$$\tilde{n}_c(y, z, t) = \frac{5N_0}{2\pi R_y(t)R_z(t)} \left(1 - \frac{y^2}{R_y^2(t)} - \frac{z^2}{R_z^2(t)} \right)^{3/2}. \quad (2.27)$$

Figure 1a show as is measured the density profile obtained by the time-of-flight technique and 1b shows the typical fitting to extract the thermal (gaussian) and condensed (inverted parabola) contributions by a bimodal fitting.

2.2 Superfluidity

In this section we trace the relation of the BEC dynamics described by time-dependent Gross-Pitaevskii equation in the Thomas-Fermi regime to the classical hydrodynamics equations. We discuss the time-dependent Gross-Pitaevskii equation in the Thomas-Fermi regime describes an irrotational flow of an ideal compressible fluid. This be useful to describe one of the more striking topological defects in quantum physics, the quantized vortices.

2.2.1 Hydrodynamic equations

We can write the Gross-Pitaevskii equation in the form of a hydrodynamic continuity equation. To this, we use the time-dependent Gross-Pitaevskii equation,

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g |\psi(\mathbf{r}, t)|^2 \right) \psi(\mathbf{r}, t) = i\hbar \frac{\partial \psi(\mathbf{r})}{\partial t}, \quad (2.28)$$

and multiply by $\psi^*(\mathbf{r}, t)$. From the resultant equation, and making additional calculations (52) we can obtain the continuity equation for the BEC,

$$\frac{\partial |\psi|^2}{\partial t} + \nabla \cdot \left[\frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*) \right] = 0. \quad (2.29)$$

From the equation (2.29) we identify the current density as,

$$\mathbf{j}(\mathbf{r}, t) = \frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*). \quad (2.30)$$

and we can define an expression for the velocity field of the BEC, using the Madelung form for the wavefunction (order parameter) of the superfluid,

$$\psi(\mathbf{r}, t) = \sqrt{n(\mathbf{r}, t)} e^{i\phi(\mathbf{r}, t)}, \quad (2.31)$$

from which the velocity field of the superfluid, expressed as $\mathbf{v} = \mathbf{j}/n$, is,

$$\mathbf{v}(\mathbf{r}, t) = \frac{\hbar}{m} \nabla \phi(\mathbf{r}, t). \quad (2.32)$$

Using the expression for the velocity in (2.28), results in the dynamic equation for the velocity field,

$$m \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \left(\frac{m}{2} \mathbf{v}^2 + V + gn - \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} \right) = 0, \quad (2.33)$$

where the term proportional to $(\nabla^2 \sqrt{n})/\sqrt{n}$ represents the quantum pressure in the superfluid. In this way, we can use the continuity equation (2.29) and the equation for the velocity field dynamics (2.33) as the hydrodynamic equations for the quantum fluid. These equations are equivalents to the Gross-Pitaevskii equation, and are useful to describe excitations in the superfluid as collective modes and vortices.

2.2.2 Collective modes

If we consider the Thomas Fermi approximation where the system has zero kinetic energy, we can neglect the term carrying the quantum pressure of the fluid in Equation (2.33), and due to the large number of atoms in the trap, the density profile varies smoothly. Thus, equation (2.33) comes to be (59),

$$m \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \left(\frac{m}{2} \mathbf{v}^2 + V + gn \right) = 0, \quad (2.34)$$

We can write the density as an average density $n_0(\mathbf{r})$ (that describe a coherent mode) plus the time dependent density fluctuation $\delta n(\mathbf{r}, t)$ in the form,

$$n(\mathbf{r}, t) = n_0(\mathbf{r}) + \delta n(\mathbf{r}, t).$$

For investigating the collective modes we need to consider small deviations from $n_0(\mathbf{r})$ and find the solution for the hydrodynamic equations. Thus, we consider $\delta n(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$ as small quantities, and using the hydrodynamic equations (2.29) and (2.34), we have,

$$\frac{\partial n(\mathbf{r}, t)}{\partial t} + \nabla \cdot (n_0(\mathbf{r})\mathbf{v}(\mathbf{r}, t)) = 0, \quad (2.35)$$

and,

$$m \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial t} + g \nabla \delta n(\mathbf{r}, t) = 0. \quad (2.36)$$

We can find the motion equation by taking the time derivative of (2.35) and combining with equation (2.36), that results in,

$$\frac{\partial^2 \delta n}{\partial t^2} = \nabla \cdot (c^2 \nabla \delta n), \quad (2.37)$$

where we have defined $mc^2 = gn_0$.

We obtain a solution for this last equation using the ansatz $\delta n(\mathbf{r}, t) = \delta n(\mathbf{r})e^{-i\omega t}$, that reduce equation (2.37) to,

$$\omega^2 \delta n(\mathbf{r}) = \nabla \cdot (c^2 \nabla \delta n), \quad (2.38)$$

with solutions corresponding to the low-lying frequency modes of the condensate.

For the case of an cigar-shaped harmonic potential, the collective modes are described by the equation,

$$m\omega^2 \delta n(\mathbf{r}) = \nabla \cdot \left\{ \left[\mu - \frac{m}{2} (\omega_r^2 r^2 + \omega_x^2 x^2) \right] \nabla \delta n \right\} = 0, \quad (2.39)$$

where ω_x is the axial frequency of the cigar shaped-trap.

Due the anisotropy of the equation (2.39) the density fluctuations can be expanded in spherical harmonics as $\delta n \propto P(r)Y_{\ell m}(\theta, \varphi)$, with $P(r) \sim r^\ell$ a function with radial dependence. We are interested in the cases where there are explicit expressions for solutions of the equation (2.39). Thus, when the density fluctuation are in the form $\delta n \sim$

Figure 2 – Low-lying collective modes of a cigar-shaped BEC. **a**, dipolar mode, **b**, breathing mode, **c**, quadrupolar mode. The subindex D , B , and Q is for label the order or the mode. These representations are for the fundamental frequencies of each mode.

Source: By the author.

$Y_{\ell m}(\theta, \varphi)$, there are solutions of this equation for $m = \pm\ell, \pm(\ell - 1)$, which results in dispersion laws.

In this way, we have the dipole oscillations when $\ell = 1 \rightarrow m = \pm 1, 0$, with the dipole frequencies equal to the trap frequencies $\omega_{1,\pm 1} = \omega_r$ and $\omega_{1,0} = \omega_x$, that corresponds to the translation of the center of mass of the cloud inside the trap without any deformation.

On the other hand, we have higher oscillation modes, as the quadrupolar modes when $\ell = 2 \rightarrow m = \pm 2, \pm 1, 0$, with frequencies $\omega_{2,\pm 2} = \sqrt{2}\omega_r$, $\omega_{2,\pm 1} = (\omega_r^2 + \omega_x^2)$, and $\omega_{2,0} = \pm\omega_r\sqrt{2 + (3/2)\lambda^2 + (1/2)\sqrt{9\lambda^4 - 16\lambda^2 + 16}}$, that is similar to frequency for the case $\ell = 0$, $\omega_{0,0} = \pm\omega_r\sqrt{2 + (3/2)\lambda^2 - (1/2)\sqrt{9\lambda^4 - 16\lambda^2 + 16}}$, the monopolar mode. We used in the last expressions $\lambda = \omega_x/\omega_r$. In case where $\lambda \ll 1$, as in the cigar-shaped traps, $\omega_{0,0} \sim 2\omega_r$, and $\omega_{2,0} \sim 1.58\omega_r$. The monopolar mode is commonly called as the *breathing* mode and is characterized by the change in the size of the BEC, as the same way as the quadrupolar mode. We show a pictorial representation of the collective modes in Figure 2.

2.2.3 Vortices

Topological nontrivial solutions and nonlinear stationary states rise from solutions of the Gross-Pitaevskii equation and the hydrodynamic equations. (60-61) Many of these topological excitations come in the form of vortex structures featuring quantized circulation. They are topological in the sense that they can only decay at the boundary of the system or under collision. In dilute BEC's, vortex structures have been observed in several forms,

including single vortices (62-63), vortex lattices (64) and vortex rings. (65)

Quantization of circulation

Under the Madelung transformation (2.31) the macroscopic wavefunction can be expressed in terms of a fluid density and a macroscopic phase. In order that the wavefunction remains single-valued, the phase around any closed contour C must be an integer multiple of 2π ,

$$\int_C \nabla \phi \cdot d\mathbf{l} = 2\pi q, \quad (2.40)$$

where $q = 0, \pm 1, \pm 2, \dots$, $d\mathbf{l}$ is the line element of integration, and q is an integer number that specifies the “quanta” of circulation or also known as the topological charge of the vortex. A superfluid is considered irrotational due to the relation $\nabla \times \mathbf{v} = 0$, that assures the adequate restriction to the superfluid flow. Since the gradient of the phase defines the local velocity flow via equation (2.32), this condition implies that the circulation about the contour C is given by,

$$\int_C \mathbf{v} \cdot d\mathbf{l} = q \frac{\hbar}{m}. \quad (2.41)$$

It is the same to say that the circulation of fluid is quantized in units of $\kappa = \hbar/m$. This constraint comes from the quantum mechanics associated with the macroscopic population of a single state. Any rotation of the fluid must be contained in the form of vortex lines featuring quantized circulation. Indeed, one finds that as a superfluid is rotated, the presence of vortices becomes energetically favorable above some critical rotation frequency.

Therefore, quantized vortex can be represented by the form,

$$\psi(r) = \sqrt{n_v(r)} e^{i2\pi q}, \quad (2.42)$$

where r is the distance from the vortex center. The vortex density profile $n_v(r)$, for which there is no analytic solution, is shown in Figure 3 for a singly-quantized vortex.

The circulating fluid velocity is given by $v = q\hbar/(mr)$. At the center of the vortex, the phase singularity corresponds to infinite fluid rotation speed. In order to maintain a finite energy density, the fluid density is pinned to zero, and recovers to the bulk background density n_0 at a distance of the order of several healing lengths.

The vortex energy therefore depends strongly on the size of the system. In terms of classical hydrodynamics the vortex energy per unit line length is approximated by (52),

$$E_v = \int_a^b \frac{1}{2} m n v^2 dr = \frac{m n \kappa^2}{4\pi} \ln \left(\frac{b}{a} \right), \quad (2.43)$$

where the lower cutoff is set by the vortex core size $a \sim \xi$, where ξ is the healing length, and the upper cutoff is the mean size of the system $b = \bar{R}$ (recalling that $\bar{R}$ is the mean Thomas-Fermi radius) for an harmonic trap.

Figure 3 – Vortices in a BEC. **a**, density profile for a vortex, showing the depletion in the center of the cloud. **b**, render figure of vortex in the axial direction of a cigar-shaped BEC. **c**, representation of a single vortex showing the circulation of the phase.

Source: **a**, **b**, By the author. **c**, Adapted from SAULS. (66)

2.3 Out-of-equilibrium regime: Turbulence

Turbulence in superfluids is pretty different from turbulence due viscosity as it takes place in classical fluid. Superfluids have zero viscosity, and the state of turbulence comes from the entanglement and reconnection of vortices. (67) In this section we describe an overview in classical turbulence considering the very restrictive case for the existence of the Kolmogorov law, and quantum turbulence with special considering of superfluid and wave turbulence.

Turbulence in classical fluids

A classical fluid have its dynamic well described by the Navier-Stokes equation, that shows the time evolution of a velocity field $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ (68),

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{v} + \mathbf{f}, \quad (2.44)$$

with ρ the density of the fluid, P the pressure, and $\nu = \eta/\rho$ the kinetic viscosity of the fluid.

In Equation (2.44) the left-side terms are clearly related with the acceleration of the fluid (it is clear doing a dimensional analysis), and the right-side terms describe the spatial variation of the pressure through the term $-\nabla P/\rho$, and the damping due to the viscosity as $\nu \nabla^2 \mathbf{v}$. Finally $\mathbf{f}$ represents the external force that act for unity mass.

If we take the ratio between the non linear force $(\mathbf{v} \cdot \nabla) \mathbf{v}$ and the viscous force $\nu \nabla^2 \mathbf{v}$, we find a dimensionless parameter knew as the Reynolds number (Re). When Re is large, the viscous force is not able to change the variations in the velocity field, and thus, the non linear effects dominate the flow of the fluid. When Re increases, more complex effects are introduced into the system and there is the apparition of a vortices, making the flux chaotic, that is called turbulence.

For these considerations, we can explain the evolution of turbulence through Richardson cascades. (69) This model describe the turbulent regime as coming from the division of bigger eddies in smaller ones. This smaller eddies suffer the same decay process, conserving their energies and generating eddies even smaller. This process is self-sustained until that the small scale achieved by the vortices is the order of $Re \sim 1$. At this point, the fluid viscosity is dominant and the energy is dissipated as heat without the formation of new vortices.

In 1941, Kolmogorov states that energy gets injected into a classical fluid flow at some large scale L . In this way, the the fluid motion at scale L would become unstable and lose its energy to neighboring smaller scales without dissipating it into heat. At high Reynolds numbers, this process repeat itself until reach a small scale at which no further instabilities are possible, and the energy is dissipated directly as heat. Kolmogorov assumed that the rate of energy pumped at the large scales and of energy dissipation at the

small scales are equal to each other and to the energy transfer rate across the spectrum of intermediates scales. Anisotropy and inhomogeneity at large scales are thought to diminish with decreasing scale, so that scales far smaller than L become statistically isotropic and homogeneous, this being the hypotheses of local isotropy. Also, he assumed that the energy dissipation rate remains finite in the limit of infinite Reynold numbers. (70-71)

The well-know Kolmogorov hypotheses are:

1. The scales of motion $r \ll L$ that dissipates most of the turbulent energy are locally isotropic, and their statistics are determined only by $\langle \varepsilon \rangle$, the global average of the energy dissipation rate per unit mass, ε , and the kinematic viscosity coefficient ν .
2. For the so-called inertial range scales $\eta \ll r \ll L$, the viscosity becomes irrelevant and the statistics of $\Delta \mathbf{v}$ depend only on $\langle \varepsilon \rangle$.

This inertial range has the high limit given by the expression $k_K = (\varepsilon/\nu^3)^{1/4}$, where ε is the rate with which the energy is injected in the system; and a low limit given by k_D that correspond to the largest scale of the system. In this inertial range, $k_D \ll k \ll k_K$, the Kolmogorov law to the energy spectrum is,

$$E(k) = C \varepsilon^{2/3} k^{-5/3}, \quad (2.45)$$

where C is a dimensionless constant, and k is the wavenumber obtained by the Fourier transform of the velocity field.

Some important characteristics of the so called Kolmogorov spectrum of turbulence, are the universality (the kinetic energy does no depend of the boundary conditions or the nature of the fluid), and the self-similarity of the turbulence for all the scales in the inertial range. Typical energy spectrum for classical fluids are showed in Figure 44, and the power-law behavior that characterize the Kolmogorov cascade is evident in theses cases.

Figure 4 – Energy spectrum for classical fluids showing the cascades in the inertial range following the Kolmogorov law. Here it is shown the points of injection of energy and of dissipation when the system reach the smaller scales.

Source: By the author

Turbulence in quantum fluids

Currently the state of art on the understanding of turbulence in quantum fluids is an actual very dynamic issue. Richard Feynman in 1955 defined quantum turbulence as being the quantum version of classical turbulence, characterized by the entanglement of quantum vortices in a superfluid. (72) The Feynman idea was experimentally demonstrated in superfluid helium (73), where it was produced a thermal counterflow between the normal fluid and the superfluid fraction, allowing the observation of tangle vortices in the superfluid component. (??) Most of the experiments in turbulence in superfluids have the liquid helium as a medium to generate it. The theoretical description for quantum turbulence in BEC is given by the dynamic evolution of the time dependent Gross-Pitaevskii equation, properly written in terms of atomic density $n(\mathbf{r}, t)$ and velocity field $\mathbf{v}(\mathbf{r}, t)$. (78) However, it is important to note, that this is a mean field approach and the result can be different from another approaches, as Monte Carlo simulations. (77)

Superfluid turbulence: As mentioned the most common definition of quantum turbulence is due by Feynman. The description of this kind of turbulence

The transport of the pumped energy into the system for quantum turbulence is equivalent to the classical one, and also it is based in cascades, i.e. big eddies do not stables and go to smaller ones, conserving the energy. When a certain scale is reached, the energy is dissipated as heat. However, the dissipation mechanism is not simple as in classical fluids, due to the zero viscosity of the superfluid. To explain the mechanism that takes place owing to the quantum effects, we use the concept of vortex reconnection. (74)

The interaction of two well defined vortex lines creates a re-connection at certain point for short times, and after this reconnection, the two vortices emerge with a kinked form. In a system with many vortices, the probability of re-connection is high, and the final result is a system with a big tangle of the most of the vortices. This is exactly the concept of quantum turbulence of Feynman. As a consequence of many re-connection, a vibration will be generated in the system, called Kelvin waves (75), and it is analogous to phonon radiation. This Kelvin waves transfer energy from inter-vortex scales to smaller scales, as the vortex core. Thus, one of the important mechanisms in the decay of turbulence is the presence of cascades as in the classical version. (78)

Wave turbulence

This kind of turbulence is defined as “out-of-equilibrium statistical mechanics of random nonlinear waves”. (76) This definition is limited to waves which are weakly nonlinear and dispersive, the cases when the mathematical description of wave-turbulence is most systematic and unambiguous. Generally, the treatment of wave-turbulence is as a general physical phenomenon without discarding a large number of systems which are commonly observed in nature but which have not been described rigorously yet. There are many systems that are important in a vast range of physical examples, from quantum to astrophysical scales. One of the most important predictions of the theory is the existence of waves on quantized vortex lines which are important for understanding turbulence in superfluid helium (75) and waves in BEC's. (78). In an ideal superfluid, the vortex lines have a tangle behavior like at the scales greater than the mean inter-vortex separation. But, due the zero viscosity of the quantum fluid the Kolmogorov cascade brings the energy all

Figure 5 – Wave turbulence in quantum gases.
Source: Adapted from NAZARENKO. (76)

the way down to the inter-vortex separation scale, which can be viewed as a semi-classical picture. Thus, the energy cascade to even smaller scales is carried by wave-turbulence in form of Kelvin waves, until reach a very small scale where all energy is lost of the system (phonons). In excited BEC's, initial evolution takes place as an inverse cascade driven by quartic interactions of weakly nonlinear waves and when this inverse cascade reaches large scales, the weak nonlinearity assumption breaks down and a coherent uniform condensate component appear. On the other hand, when the BEC grows strong, its evolution can be viewed as three-wave evolution of weak Bogoliubov sound. The most interesting stage occurs in between of the four-wave and the three-wave weak regimes: it comprises strong turbulence consisting of a gas of coherent vortices, which interact with each other and with the sound field, and the number of which is decreasing to zero in a finite time due to the pairwise vortex annihilation. (76)

The first experimental evidence of turbulence in Bose gases was provided by our research group (39) and since then the state of art of the topic increase in an exponential way. For example, quantum turbulence was reported in ^{87}Rb and ^{23}Na in 2D system. (40) Due the dimensionality of the systems, the phenomena of turbulence differs from 2D to 3D. A very extensive review in of the actual state of the art of the topic of quantum turbulence and turbulence in quantum fluid can be found in reference. (44)

Along this thesis we do not claim turbulence for our non-equilibrium states, in spite some evidences, due the limitations of the employed detection technique, and due that the time scales of our system are shorts ($\sim 0.5\text{s}$), it is difficulty to identify some neces-

sary conditions to probe the turbulence state. Then, in spite of many evidences that the produced states are turbulent, we call these non equilibrium states as far-from equilibrium, or strongly out-of equilibrium states, and we intent in future works to demonstrate categorically the presence of the turbulence state.

3 EXPERIMENTAL APPARATUS

This chapter is designed to provide the reader a basic understanding of the apparatus and the experimental methods to produce atomic Bose-Einstein condensates. This system is called in our laboratory as *BEC-I*, and a more detailed description of this system can be found in previous theses. (79-80) What follows is a basic description of the experimental BEC-I apparatus and the procedure for making, probing and imaging the Bose-Einstein condensate of ^{87}Rb atoms. The order of the sections try to reflect the order in which these appears in the experimental sequence.

3.1 Making a Bose-Einstein Condensate

In the last years, degenerate Bose gases have been obtained from different experimental approaches that are different depending of the final stage for achieve the condensation, the evaporative cooling process. (81) To reach the quantum degeneracy in our experiment, we use the well know technique of evaporative cooling by radio-frequency in a magnetically harmonical trap.

General vision of “BEC-1”

The main part of the experiment takes place in a science cell at ultra-high vacuum (UHV) as show in the different steps in Figure 6. Previously, a magneto-optical trap (MOT) of ^{87}Rb atoms is loaded in another chamber and the cooled atoms are pushed to the science cell by a laser beam, where the atoms are captured again in another MOT. This double MOT configuration have two advantages: first, the region for the first MOT needs

a pressure highly enough to guaranty a fast loading of the MOT, while the chamber where the BEC will be achieved needs a vapor pressure low enough to assure a long lifetime of the atoms, a requirement to an efficient evaporative cooling and a long lifetime of the BEC, which is relevant to the process of manipulation and imaging of the atomic cloud.

To achieve the BEC is necessary to cool down and compress the atoms until temperatures and densities of the order of 10^2 nK and 10^{14} atoms/cm³, respectively. To this, it is necessary to isolate the atomic gas inside the region of ultrahigh vacuum and use non-invasive techniques to manipulate it. To control the temporal sequence at any instant of the processes, we use a computerized control system which send the control parameters to two acquisition boards (PCI 6259 and PXI 6733 National Instruments ®), where the analogical and digital signals are generated. Finished the experimental sequence, that have a duration of ~ 1 min, the resultant experimental data (in special the image of the atomic cloud) is collected and analyzed by a computerized acquisition system. Figure 6 shows, in a pictorial way, each one of the main processes involved to achieve the condensation. In the next sections, each of these processes will be described in a succinct way, and for more detail, the involved physic can be consulted in references. (81-82)

3.1.1 Vacuum system

The experiment has two stages to load the atoms, first the MOT1, where is localized the atomic source (dispensers and LIAD) at high vacuum (HV) and, the second, the MOT2, which provides the necessary conditions for the manipulation of the atoms until the achievement of the BEC at UHV. Figure 7 shows a rendered picture of this system as it is disposed in the experiment. The hardware of the system can be divided in three regions of pressure: first, the region where is localized a glass cell (Pirex®) designate to contain the atomic source and the MOT1; the second, responsible by the differential pumping; and the third, where is localized a quartz cell for the MOT2 and where the BEC is achieved (the science cell).

Figure 6 – Main steps of the experimental sequence in rendered pictures. **a**, MOT. **b**, molasses. **c**, optical pumping. **d**, quadrupole magnetical trap. **e**, Quadrupole-Ioffe configuration. **f**, evaporative cooling and achievement of the BEC.

Source: By the author.

Figure 7 – Render image of the vacuum system elaborate in Keyshot® showing the principal parts of the hardware. Three ionic pump are responsible for the pumping in the regions of MOT1, differential pumping and MOT2. It can be also observed the different elements as the dispensers, titanium sublimation, etc.

Source: By the author.

The differential pumping is built, so that the pressure in the science cell is close to four orders of magnitude less than in the region of MOT1, and thus, we have a pressure less than 1×10^{-12} Torr in the region where the BEC will be achieved. In any experiment with ultracold atoms, this low pressure is essential, so that, the atom losses in the trap due to collisions with the background, will be slow enough compared with the required time for the process of evaporative cooling. In our system the lifetime in the MOT2 is ~ 90 s, whereas the necessary time for the whole evaporative process is around ~ 20 s.

3.1.2 The double MOT

A magneto-optical trap consists of three pairs of contra-propagating laser beams, red detuned and superimposed with a quadrupolar magnetic field to cool hundreds of μ K and

spatially confine the atoms. (83) The physics of the MOT is widely discussed in the most of textbooks of atomic physics, and it can be well understood in the 1997 Nobel Lecture. (84)

In Figure 8a is showed a diagram considering the energy levels for the ^{87}Rb , and the transitions relevant to all the process involved to cooling the atoms. In Figure 8b it is illustrated the MOT mechanism in one dimension, where it is considered the cooling and trapping of the state $F = 0$ using the transition $F = 0 \rightarrow F' = 1$. With the action of the magnetic field the three energy sub-levels for $F' = 1$ depends linearly with the position of the atom. In the MOT the polarization of a beam, in each of the counter-propagating beams, with respect to a fixed spatial direction is chosen to be opposite: σ^+ and σ^- , and thus, when an atom moves away from $z = 0$, the selection rules make the absorption of photons of the laser beam pushing the atoms back to the position $z = 0$, dominating over the absorption with the opposite beam. (85) This imbalance of the radiation forces, due the Zeeman effect, create a position dependent force that confine the atoms at the origin. The MOT has a high capture velocity, and thus, it captures and cools the atomic vapor atoms at the environment temperature of $\sim 20^\circ\text{C}$. The limit that the atomic cloud can be cooled by this mechanism is given by the Doppler limit, which rise due the recoil energy associated with the aleatory fluctuations around the average number of photons emitted and absorbed spontaneously. For ^{87}Rb this Doppler limit is around $146\ \mu\text{K}$.

The ^{87}Rb atoms have a complicated structure of electronic levels with a multiple ground and excited states, and several decaying paths. Fortunately, alkaline atoms have just one valence electron, that permits a good definition of the electronic levels. For ^{87}Rb the adequate transitions are found in the $D2$ line, between the ground state $5^2S_{1/2}$ and the excited state $5^2P_{3/2}^*$. The nuclear spin for ^{87}Rb is $I = 3/2$, therefore the ground state $|L = 0, J = 1/2\rangle$ is divided by the hyperfine interaction in the levels $F = I + J = 1, 2$. In the same way, the excited state $|L = 0, J = 3/2\rangle$ is divided in the levels $F = 0, 1, 2, 3$.

For the stages of MOT and molasses (Figure 6a,b), the cooling transition is the cyclic transition $|F = 2\rangle \rightarrow |F' = 3\rangle$. As the polarization of the laser with respect to the

* Here we are using the spectroscopic notation.

Figure 8 – a, Energy levels for cooling and trapping of ^{87}Rb . Here is included the transition used for imaging, optical pumping and RF cooling that is part of the subsequent processes. b, Unidimensional representation of the mechanism of the MOT. ω_ℓ is the resonant frequency for the atom, δ_+ and δ_- are the detuning for the resonance of the atoms when is considered the Zeeman splitting dependent on the position.

Source: By the author.

direction of the magnetic field is σ^- , the required transition is $|F = 2, m_F = -2\rangle \rightarrow |F' = 3, m_{F'} = -3\rangle$, which is cyclic and closed, since $|F' = 3, m_{F'} = -3\rangle$ only can decay to $|F = 2, m_F = -2\rangle$. Due to the continue absorption and emission of photons, out-of-resonance excitation to the state $|F' = 2\rangle$ may happen, which leads to decay to the ground state $|F = 1\rangle$, the dark state. To avoid populate the dark state $|F = 1\rangle$, a repumping light, resonant with the transition $|F = 1\rangle \rightarrow |F' = 2\rangle$, also needs to be present in the MOT, to continuously transfer the population of the state $|F = 1\rangle$ to the state $|F' = 2\rangle$.

Experimentally we use as an atomic source for ^{87}Rb a set of dispensers[†], and as the source of light we use three commercial lasers of Toptica®. Each of these lasers are controlled in temperature and current, and are locked-in using as reference the spectrum obtained by saturated absorption spectroscopy technique. The exact frequencies are obtained using acoustic-optical modulators (AOM)[‡], and the generated lights are used to cool and trap the atomic vapor, manipulate the atomic states, and for the diagnostic of the atoms.

For our MOT's we typically use: (i) for MOT1, 100 mW of trapping light with red detuning of 20 MHz. The diameter of the beam ($2/e^2$) is around 18 mm overlapping with

[†] This dispensers are small metallic filaments that contain rubidium chromate (Rb_2CrO_4), and a metallic alloy as reducer agent. With the action of a electric current trough it, the filaments are heating to $\sim 600^\circ\text{C}$ and a chemical reaction release the rubidium vapor in this region.

[‡] These AOM also allow to control with precision the temporal sequence for the lights in the experiment.

the repumping light of 20 mW and the with the same diameter. The magnetic field gradient is about 10 G/cm along the z axis, the typical time to load the MOT1 is about 2 s with about 3×10^8 atoms at a temperature of $\sim 320 \mu\text{K}$; (ii) for MOT2, 80 mW of cooling power with the same red detuning of 20 MHz overlapping with the repumping light of 25 mW of the same diameter, and added with a magnetic field gradient of 15 G/cm. A continuous light beam resonant with the cooling transition (called a push beam) of diameter 1 mm is applied to the MOT1, to transfer the atoms to the region of the science cell (MOT2). The flow rate of atoms going from MOT1 to MOT2 is around 10^8 atoms/s and we trap about 1.8×10^9 atoms in the MOT2 at a similar temperature that of MOT1. It takes about 20 s to load the MOT2, and after that time, we transfer the atoms into the magnetic trap.

In addition, three orthogonal coil pairs in Helmholtz configuration, are present in each of the MOT's, and are responsible for the fine positioning of MOT1 and MOT2, and for eliminate spurious magnetic fields. It is important to mention that, the alignment of the MOT1 and the push beam is done in order to maximize the transfer of atoms to MOT2. The MOT2 is prepared more carefully because here the atoms will reach the state of BEC. Thus, the optical part of the MOT2 is formed by six independent light beams, orthogonal and counter-propagating. This allows us to have a more precise alignment and better adjustment of the power and polarization of each beam.

3.2 Preparation for the transference to the magnetical trap

The transference of the atoms from the MOT to the quadrupolar magnetic trap (MT) is the most critical step before to reach the BEC, because for an optimal transference, the potential of the MOT and MT must coincide spatially in its geometries and intensities. In addition, when is the atomic cloud is cold, its size will be smaller, and in this way, for lower temperatures of the cloud at the initial stage of the transference, will be easy the capture of the atoms in the MT.

In the MOT are trapped atoms in all the Zeeman sub-levels of the hyperfine state $5^2S_{1/2} (F = 2)$, but only the states $|2, 2\rangle$ and $|2, 1\rangle$ are magnetically trappable. Therefore, to transfer most of the atoms from the MOT to the MT, we need select the adequate

Zeeman state.

The procedure to allow the atoms be captured in the MT starts increasing its density in the phase space, and cool it down to the Doppler limit, to subsequently it be optically pumped to populate the chosen magnetically trappable state.

In the experiment, preparation step is the compression of the MOT2. This is done by changing the detuning frequency of the trapping beam from $\Delta_0 = -20$ MHz to $\Delta_1 = -60$ MHz, taking it even more to the red of the resonant transition, decreasing the scattering rate of photons, and leaving the atoms to concentrate at the center of the trap. The duration of this process is 5 ms. Then we use the sub-Doppler cooling technique, where the magnetic field is switched off, and the trapping frequency is setting to a detuning of $\Delta_2 = -70$ MHz, during 3 ms. After these two processes, the temperature of the cloud is, approximately, $50 \mu\text{K}$.

Finally, we select the Zeeman state of the atoms to capture it in the magnetical trap. For this, we make an optical pumping of the atoms to a magnetically trappable state, that in our case is the state $|F = 2, m_F = 2\rangle$. This process is divided in two parts: first we make a pumping of the hyperfine state $|F = 2\rangle$, which is done turning off the the trapping light of the MOT2, leaving on just the repumping light for 1 ms. Immediately, we apply an homogeneous magnetic field of 1 G (that creates a quantization axis and open the degeneracy of the Zeeman sub-levels m_F), to perform transitions of the type $m_F \rightarrow m_{F+1}$, where our objective is to accumulate atoms in $m_F = 2$ manifold. Simultaneously, we apply two light pulses with polarization σ^+ , one resonant with the transition $5^2S_{1/2}(F = 1) \rightarrow 5^2P_{3/2}(F = 2)$, and the other with the transition $5^2S_{1/2}(F = 2) \rightarrow 5^2P_{3/2}(F = 2)$. The two pulses duration are $120 \mu\text{s}$ and $25 \mu\text{s}$, respectively.

After this steps, the atoms are ready to be trapped magnetically. The atomic cloud after all these processes is denser and colder, with $T \sim 50 \mu\text{K}$ and 2×10^8 atoms. Finally, all the lights are switched off, and begins the capture in the MT.

Figure 9 – Energy levels dependent with the magnetic field. **a**, splitting of the Zeeman levels in a homogeneous magnetic field showing the hyperfine structure of ^{87}Rb atoms. The states are labeled $|F, m_F\rangle$. The red box shows the region of field and energy levels typically used in ultracold atoms experiments. **b**, expanded view of the red box in **a**, showing the energy levels and the modification in the presence of the RF photons.

Source: By the author.

3.3 Magnetical trap

The Hamiltonian that describes the interaction of an atom with an external magnetic field $\mathbf{B}$ is $H_B = A\mathbf{I}\cdot\mathbf{J} + \frac{\mu_B}{\hbar}(g_I\mathbf{I} + g_J\mathbf{J})\cdot\mathbf{B}$, where μ_B is the Bohr magneton, A is the hyperfine structure constant, $\mathbf{J}$ is the total angular momentum operator, $\mathbf{I}$ is the total nuclear spin operator, and g_I and g_J are the correspondent Landé factors.

For ^{87}Rb , $A = h \times 3.417\text{ GHz}$, and in general for ultracold atoms experiments, we are interested in the region of low magnetic fields, where g_I is three orders of magnitude smaller than g_J , and the coupling between $\mathbf{I}$ and $\mathbf{J}$ is the dominant term. We can define $\mathbf{F} = \mathbf{I} + \mathbf{J}$, where $\mathbf{F}$ and its correspondent m_F 's are good quantum numbers. The Zeeman terms act now as a perturbation in the states $|F, m_F\rangle$, and the energy levels are,

$$E_{Az} = \frac{1}{2}A[F(F+1) - I(I+1) - J(J+1)] + g_F m_F \mu_B B_z, \quad (3.1)$$

with g_F given in terms of g_J . The first term is the usual hyperfine splitting, while the second one shows the dependence with the magnetic field. Different m_F states split with the magnetic field following $\Delta E_{|F, m_F\rangle} = g_F m_F \mu_B B_z$, hence the spacing between energy levels $E_{|F, m_F\rangle} - E_{|F, m_F \pm 1\rangle}$ is the same for all the m_F 's for a specific F [§].

[§] For high fields, the energy levels are given by $E_{PB} = A m_I m_J + \mu_B g_J m_J B_z$. For intermediates magnetic fields it exists an analytical solution called the Breit-Rabi formula. (??)

Figure 9a shows the splitting for the ground state of ^{87}Rb , $5^2S_{1/2}$, where also is showed the region for the intermediate and high fields. Figure 9b shows how this Zeeman states change with the presence of RF photons, that is relevant for the process of evaporative cooling.

Quadrupole-Ioffe configuration (QUIC): This kind of trap is produced by a coil pair in anti-Helmholtz configuration (quadrupole) with the addition of a coil perpendicular to the symmetry axis of the quadrupole.

We define the symmetry axis of the quadrupole trap as being the z -axis. The quadrupole magnetic field around the origin is given by $\mathbf{B} = B'(x, y, -2z)$, and have a local minimum with depth given by the difference between the minimum field at the origin, and the maximum field along the radial direction. The main benefit of the quadrupole trap is that there is the zero of field at the origin, where the states with different m_F are degenerate, and most of these new states are not magnetically trappable. Thereby, there is atom loss due spin flips, that reduce the lifetime of the atomic cloud. These spin flips, are called Majorana transition (86), and also induce heating in the cloud. Hence, the achievement of quantum degeneracy in a quadrupole trap is not possible.

A solution to achieved the condensation is the use of the QUIC trap. (87) The quadrupolar field is modified to has an offset from the minimum to a positive value of magnetic field, called magnetic bias, and transform the trapping potential from linear to harmonic. The main prejudice of the QUIC trap is that the confinement along the Ioffe axis is weak and, as result, the next step, the evaporative cooling, must be slow enough to allow the re-thermalization of the atoms.

Magnetic traps have the convenience that are deep enough to allow collect the atoms efficiently at the final of the laser cooling process. Another favorable point is the long lifetime in these traps, which is limited only by background collisions in ideal conditions.

In the experiment, we capture the atoms in the quadrupolar trap after the processes of preparation switching on the magnetic field with a gradient of 75 G/cm (in the axial direction), and increasing it linearly during 400 ms until the final value of 330 G/cm. The initial value of the quadrupole field to the capture the atoms is adjusted to obtain the best

Figure 10 – Quadrupole-loffe configuration (QUIC) to generate the confining harmonical potential for the atoms. **a**, render representation of the QUIC in the experiment. **b**, module of the magnetic field for the magnetical trap, where it is shown the final quadrupole field and some intermediate configuration of the transference when the current in the loffe is increased to its final value. The magnetic bias is not observed due the scale of field values.

Source: By the author.

transference from the MOT2 to the quadrupole trap. Despite the duration of this ramp of gradient be slow, when we increase it we compress the atomic cloud and increase its temperature. At the final, the atoms achieve a temperature of $\sim 300 \mu\text{K}$.

The next step is to transfer the atoms from the quadrupole trap to the QUIC. The configuration of the QUIC, as is disposed in the experiment, is showed in Figure 10a. Thereby, after the capture of the atoms in the MT, the current in the loffe is increased from 0 to 25 A in 600 ms. Initially, the position of the minimum of the magnetic field moves in the direction of the loffe, and there is the apparition of a second minimum in the resultant potential. When the current in the loffe is increased, the two minimum come together, and the effective potential is harmonic with a minimum different of zero. Figure 10b shows the module of the magnetic field in the loffe direction (x -axis for us) for many steps of the transference.

At that moment, the atoms moved 8 mm in the loffe direction, and around the minimum we have the harmonical potential,

$$V(x, y, z) = V_0 + \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2), \quad (3.2)$$

where V_0 is the value of the non-zero minimum (in our experiment that corresponds to

~ 1.45 G), called of magnetic bias. The values of the trap frequencies are, $\omega_x = 2\pi \times 21$ Hz, $\omega_y = 2\pi \times 189$ Hz and $\omega_z = 2\pi \times 187$ Hz. Note that $\omega_y \simeq \omega_z \equiv \omega_r$, what gives a time scale of ~ 5.3 ms; and in the axial axis $\omega_x \equiv \omega_a$, with a time scale of ~ 47.6 ms, a cigar-shaped trap. The atomic cloud, at the final of the transference to the QUIC has 2×10^8 atoms with a temperature of $300 \mu\text{K}$. With the atoms in this trap, we start the final step that leads to the Bose-Einstein condensation, the evaporative cooling by radio-frequency (RF).

3.4 Evaporative cooling by RF and BEC

The process of evaporative cooling consist in the selective removal of hottest atoms, allowing the remaining atoms to re-thermalize in a lower temperature. The continue repetition of this process allows to produce samples at very low temperatures (88), and it is fundamental to obtain the condensate.

Figure 11 – **a**, representation of potentials that feel atoms in the different Zeeman levels for the processes of evaporative cooling. An atom in $|2, 2\rangle$ at high temperature (more energetic) feels the RF photon and make a transition to a lower level. This steps repeats to the subsequent lower level if there are atoms that fulfill the energetic condition until reach a repulsive potential that eliminate this hot atoms. The scanning of the RF frequency allows to perform this process until achieve very low temperatures. **b**, in the Maxwell-Boltzmann distributions this look like a cut off of the atoms with high velocities, and a subsequent rethermalization in a lower average temperature.

Source: By the author.

In magnetic traps, the selective removal of the hottest atoms is made using radiative transitions between the Zeeman sub-levels. The separation of this Zeeman sub-levels is dependent of the magnetic field that the atoms experience. The hottest atoms (i.e. with more kinetic energy) reach higher values of the magnetic field in the potential, and

as consequence, the separation between the sub-levels m_F and $m_{F\pm 1}$ is bigger than for coldest atoms (with less kinetic energy, as shown in Figure 11. After remove the hottest atoms, if the process is performed in efficient way, we can reach the condensate state, and it can be detected and quantified using light.

Figure 12 – Evaporative cooling induced by RF. Evaporation ramp as used in the experiment with an exponential decay profile. The images of atomic clouds in different points of the evaporation show the cooling of the atoms until reach the condensation. Here are showed the temperatures and the condensed fraction of each cloud.

Source: By the author.

For the most energetic atoms, the separation of the Zeeman sub-levels is the order of 20 MHz. Thus, using RF radiation with this frequency, it can be induced spin transition in the trapped atoms, of the form $m_F \rightarrow m_{F-1}$, that leads the most energetic atoms to a non-trappable state, removing it from the trap. The remaining atoms will be re-thermalized to a lower temperature due elastic collisions, and therefore, these will access a minor value of the magnetic field. In this way, to remove the hottest atoms from this colder cloud, we need to change the frequency of the RF to a lower value. Thus, decreasing progressively the frequency of the RF, at a rate that allows the re-thermalization of the atomic cloud, we are going to obtain denser and colder atomic clouds.

For the optimization of the evaporative cooling process, we use as probe the phase

space density (PSD), and the elastic collisions rate (γ_{el}). For the BEC, $PSD = n\lambda_{dB}^3$, where n is the atomic density of the cloud, and λ_{dB} is the de Broglie thermal wavelength; $\gamma_{el} = n\sigma_{el}\bar{v}$, where σ_{el} is the cross section for elastic collisions and $\bar{v}$ is the average velocity of the atoms. In an harmonical potential: $PSD \propto \frac{N}{T^3}$, $n \propto \frac{N}{T^{-3/2}}$ and $\bar{v} \propto T^{1/2}$. In order to optimize the evaporative cooling process we perform an exponential ramp as showed in Figure 12a controlling the parameters of the maximum RF frequency and the duration time of the ramp.

The BEC: The quantum degeneracy will be occur if the phase space density reach $PSD \geq 2.612$. (52) Thus, a good evaporative cooling is such that allows to decrease the temperature maintain or increasing the rate of elastic collisions between the atoms, and thus, increase the density in the phase space until achieve the BEC. Figure 12b shows different absorption images for different points in the evaporation ramp. It can be observed that the critical temperature for our experiment is ~ 300 nK.

3.5 Imaging

The atoms are imaged using the standard absorption imaging in time-of-flight. Before imaging, the atomic cloud is released from the trap and allowed to expand in time-of-flight. Barring interactions, this process maps the in-trap momentum distribution of the gas into spatial separation for sufficiently long expansion times.

3.5.1 Collecting the image of the atoms

For the absorption imaging a resonant probe beam is used to illuminate the atoms and we collect the shadow in a CCD camera. Three images are taken in order to extract the correct information profile of the cloud.

- A image of the atoms, the shadow that carry information about the atomic cloud.
- A image of the probe beam, that is used to subtract the effect of the light in the atoms.
- A image of the background, that is used to remove any effect due to reminiscent background lights.

The resultant image will be a composition of this three images depending of the regime of scattering that we use. Therefore, in our experiment we usually took images after a time-of-flight < 50 ms. In this regime the cloud have a high optical density which makes necessary the use of high intensities in the probe beam (we use $I_{\text{probe}} \sim I_{\text{sat}}$). Thus, the probe beam will introduce some artificial structures in the resultant images, as speckles, that can be prejudicial for our purposes, because we are interested in the non equilibrium regime. To avoid this effect we need normalize the result image taking in count this effect, but our correction do not correct the effect coming from the multi-photon scattering.

Figure 13 – Optical arrange for collect the image of the atoms.

Source: By the author.

Figure 13 shows the image system to collect the absorption images of the clouds.

3.5.2 Extracting information from images

We can describe the process of light absorption by a medium as light on atoms undergoes spontaneous scattering events such that the intensity of the light drops:

$$\frac{dI(x, y, z)}{dz} = -\hbar\omega\gamma_{\text{sc}}n(x, y, z), \quad (3.3)$$

where I is the intensity of the light, $I_{\text{sat}} = \pi\hbar c\Gamma/3\lambda^3$ is the saturation intensity of the transition, and γ_{sc} is,

$$\gamma_{\text{sc}} = \frac{I}{I_{\text{sat}}} \frac{\frac{\Gamma}{2}}{1 + I/I_{\text{sat}} + (2\delta/\Gamma)^2}. \quad (3.4)$$

Using the definition of the scattering cross section σ_{sc} ,

$$\sigma_{\text{sc}} = \frac{\hbar\omega\Gamma}{2I_{\text{sat}}}, \quad (3.5)$$

Equation (3.3) becomes,

$$\frac{dI(x, y, z)}{dz} = -\sigma_{\text{sc}} \frac{n(x, y, z)}{1 + I/I_{\text{sat}} + (2\delta/\Gamma)^2}. \quad (3.6)$$

Figure 14 – Procedure to experimentally obtain the column density of atoms from the images taken by the CCD camera. We determine the normalized image by three images: the shadow of the atoms, the probe beam, and the background (or dark) image. Additionally if we are out of the Lambert-Beer regime (i.e we are in the regime $I_f(x, y) - I_i(x, y) \sim I_{\text{sat}}$), we make a correction of the image to avoid the probe effects in the image.

Source: By the author.

The solution of this equation is,

$$\ln \left(\frac{I_f}{I_i} \right) = \frac{1}{1 + (2\delta/\Gamma)^2} \left(\sigma_{\text{sc}} \int n(x, y, z) dz + \frac{I_f - I_i}{I_{\text{sat}}} \right), \quad (3.7)$$

where I_i is the intensity of the light before the atoms and I_f is the intensity of the light after the atoms.

In the experiment, three images are recorded by the camera. The first, A , is a “shadow” image taken of the imaging beam with the atoms casting a shadow. The second image, P , is the “light” image that is taken after the atoms have dispersed[†]. The final image, D , is a “dark” image of the background without the imaging light. $I_i = P - D$ and $I_f = A - D$. We compute the optical depth (OD_0) of the atoms,

$$OD(x, y) = \ln \left(\frac{P - D}{A - D} \right) = \ln \left(\frac{I_i}{I_f} \right). \quad (3.8)$$

Using this definition, Equation (3.6) becomes,

$$OD(x, y) = \frac{1}{1 + (2\delta/\Gamma)^2} \left(OD_0(x, y) + \frac{I_f(x, y) - I_i(x, y)}{I_{\text{sat}}} \right), \quad (3.9)$$

and

$$\tilde{n}(x, y) \equiv \int n(x, y, z) dz = \frac{1}{\sigma_{\text{sc}}} \left(OD(x, y) - \frac{I_f(x, y) - I_i(x, y)}{I_{\text{sat}}} \right), \quad (3.10)$$

[†] We use two different cameras. The PixelFly® and the Stingray®, with response time of 100 ms and 100 μ s respectively.

when we consider a resonant transition, i.e. for $\delta = 0$.

When we consider that the density of the atoms (this happen for long time-of-flight), we have $I_f(x, y) - I_i(x, y) \ll I_{\text{sat}}$, and approximate $OD(x; y) \approx \sigma_{\text{sc}} \tilde{n}(x; y)$.

It is important to note that we need to use the Equation (3.10) when $I_f(x, y) - I_i(x, y) \sim I_{\text{sat}}$, as is our case for short time-of-flight (less than 50 ms), in the other way we will have the effects of speckles of the probe beam.

With the absorption images we can extract some important parameters that characterize the BEC, by make a bimodal fitting, as the numbers of atoms, cloud sizes, and temperature. (81)

Number of atoms: From Equation (3.9) we can compute the total number of atoms, and the correspondent fractions of thermal and condensed atoms by the bimodal fitting.

- In time-of-flight, the optical density for the thermal cloud is,

$$OD_{\text{th}}(x, y) = p_0 \exp \left[-\frac{(x - x_0)^2}{2\sigma_x^2} - \frac{(y - y_0)^2}{2\sigma_y^2} \right], \quad (3.11)$$

where $p_0 = \max(OD_{\text{th}}(x, y))$ is the peak value of the distribution, x_0 and y_0 are the coordinates of the center of mass of the cloud, and $\sigma_{x,y}$ are the cloud radii. Hence, the number of thermal atoms is calculated by,

$$N_{\text{th}} = \int n(x, y, z) dx dy dz = \frac{1}{\sigma_{\text{sc}}} \int OD_{\text{th}}(x, y) dx dy, \quad (3.12)$$

which results in,

$$N_{\text{th}} = \frac{2\pi p_0}{\sigma_{\text{sc}}} \sigma_x \sigma_y. \quad (3.13)$$

Due to the ballistic expansion of a thermal sample in time-of-flight, we can extract the temperature of the gas when the consideration $\sigma_x = \sigma_y = \sigma_0$ is made for sufficiently long times. Thus, the temperature of the thermal cloud is related with the energy equipartition theorem and,

$$T = \frac{m}{k_B} \left(\frac{\sigma_0}{t_{\text{of}}} \right)^2. \quad (3.14)$$

- From the Thomas-Fermi approximation we can extract the condensed part by the relationship,

$$OD_{\text{TF}}(x, y) = q_0 \max \left\{ \left[1 - \frac{(x - x_0)^2}{R_x^2} - \frac{(y - y_0)^2}{R_y^2} \right]^{3/2}, 0 \right\}, \quad (3.15)$$

where $q_0 = \max(OD_{TF}(x, y))$ is the peak value of the distribution, x_0 and y_0 are the coordinates of the center of mass of the cloud, and $R_{x,y}$ are the Thomas-Fermi radii. Then, we have the number of condensed atoms by,

$$N_0 = \frac{2\pi}{5} \frac{q_0}{\sigma_{sc}} R_x R_y. \quad (3.16)$$

● The atomic cloud below the critical temperature has two components, one with the thermal atoms and the another with the condensate atoms. Then, the total optical density of the cloud can be write,

$$OD(x, y) = OD_{TF}(x, y) + OD_{th}(x, y),$$

and we can write the total number of atoms as,

$$N = N_{TF} + N_{th},$$

using the equations (3.13) and (3.16).

3.6 Excitation system

This thesis deals with non-equilibrium situations due oscillatory excitation in the BEC. This oscillatory excitation is an external perturbation due a time-varying magnetic field produced by an anti-Helmholtz coil pair superimposed to the trapping potential, as showed in Figure 15. A simulation of the effect caused by this excitation is reported in (89) where it is described how the perturbation produces a combination of translation, rotation and spatial deformation of the resultant trapping potential.

The two coils generate a magnetic field with an zero in the center of the condensate, increasing when we apart from it. We apply an AC current in the form, (90)

$$I(t_{exc}) = A(1 - \cos(\omega_{exc} t_{exc} + \phi)), \quad (3.17)$$

where ω_{exc} and t_{exc} are the angular frequency of the oscillation of the AC current and the duration of the excitation, respectively. ϕ is an experimental parameter that is adjusted to begin the excitation with any bias due the current in the coils, and A is the maximum

Figure 15 – Schematic drawing of the excitation coils as it is disposed in the experiment. **a**, top view showing the inclination of one of the coils. **b**, xz plane view showing the shift of the center of one coil.

Source: By the author.

current in the excitation. This kind of excitation produces a time varying magnetic field proportional to the current so that, $B' \sim A(1 - \cos(\omega_{\text{exc}} t_{\text{exc}} + \phi))$, that also is proportional to the total pumped energy. We can see this pictorial representation in Figure 16b, where the pumped energy is proportional to the area below the oscillation. Figure 16a shows an schematic drawing of the excitation system, as it is arranged in the experiment.

We excite the BEC immediately after that the evaporation process ends, we switched on the current in the excitation coils to perform the oscillatory excitation. After the excitation time, we turn off the excitation coils and wait to the atomic cloud relax and evolve in-trap for a holding time. Finally the trap is switched off and the atoms are release to expand in time-of-flight to imagine that. Experimentally, we have control of all the parameters in Equation (3.17), and we will explore the dependence of the excitation varying this parameters. We can also explore the evolution in-trap of the cloud after the excitation varying the holding time.

Figure 16 – a, sequence of the oscillatory excitation. After the achievement of the BEC is performed a sinusoidal injection of energy. The atomic cloud evolves in-trap for a holding time to be released in time-of-flight to detection. **b**, the pumped energy in the system is proportional to the area below the sinusoidal excitation and, thus, $E_{\text{pump}} \sim A t_{\text{exc}}$.

Source: By the author.

Some detail of the different considerations in the excitation procedure will be described in the next chapter where we will discuss the dependence of the quantifiers of non-equilibrium in dependence with the injected energy.

4 MOMENTUM DISTRIBUTION OF AN EXCITED BOSE GAS

In this chapter, the transverse momentum distribution $\tilde{n}(k)$, defined as the projected 2D momentum distribution line-in-sight, is analyzed for excited trapped Bose gases in order to study the regimes of non-equilibrium. A momentum distribution is the distribution of some physical quantity when we consider the momentum space*. In this way, energy spectrum $E(k)$, occupation number spectrum $n(k)$, vortex statistics distribution, and others, are all momentum distributions. Here, we are going to treat specifically with the 2D projection of the occupation number spectrum $n(k)$. For simplicity, we refer it in all the text as the *momentum distribution*.

Energy is pumped into the system when the BEC is driven out of equilibrium by an external oscillatory excitation. This energy can give rise to different kinds of quantum excitations in the superfluid, for example vortices, collective modes and solitons. (92) Usually, if there are some angular momentum associated with this entering energy, vortex production is the most common mechanism in which it is rearranged. Following our excitation procedure, as showed in Figure 16, the excitation initially inject some energy and vortices are created continuously with this pumped energy. This process repeats several times depending on the number of excitation cycles, and there is an increasing on the vortices number, re-connections, interactions and decaying. (36) This random and chaotic production of vortices gives rise to a regime of superfluidity that is strongly out-of-equilibrium,

* For all cases, the momentum of a quantum particle is proportional to the wavenumber, and usually is considered as being the same. (91)

and may lead to the emergence of turbulence. By other hand, the excitation procedure can gives rise to the randomic superposition of excited states of the perturbed potential and wave turbulence can take place. This two kind of turbulence can appear isolated or together depending of the excitation and other factors, understanding of the total phenomena is not trivial and there can be another effects generated by the excitation, as localization, heating, and collective modes.

In classical fluids, turbulence is defined as chaotic changes in pressure and flow velocity, and having non-zero vorticity, i.e. a turbulent medium has the presence of any type of vortices. (68) On the other hand, quantum superfluid turbulence, only be can possible in a superfluid via entanglement of quantized vortex lines, and wave turbulence looks more as a quasi-classical regime of turbulence. (78) Much of the theoretical works in these fields are still incomplete, and there are a number of areas of divergence between theoretical predictions and what has been obtained experimentally. Therefore, understand the properties of turbulence in a quantum system[†] is a great intellectual challenge and the urge to solve this problem is partially motivated by the necessity to explain how the quantum system relax the pumped energy introduced in it.

Here, we analyze far-from equilibrium excited Bose gases to studying the turbulence emergence mechanism in atomic superfluids. With respect to injected energy, we analyze the non-equilibrium regime by using $\tilde{n}(k)$ as a probe to study the evolution of the system. For that, we use the so called *time-of-flight technique*, which allows us to obtain $\tilde{n}(k)$ from absorption images of the atomic cloud. (81,93)

4.1 Density and momentum distribution of a out-of-equilibrium quantum gas

The momentum distribution $n(k)$ for a quantum gas represents how many particles with a specific moment k are distributed in the momentum space. For a pure trapped BEC with wavefunction $\psi_0(\mathbf{r})$, the momentum distribution can be obtained directly from its Fourier

[†] The reader must be noted that there is a semantic difference between turbulence in a quantum system and quantum turbulence.

Figure 17 – **a**, *in-situ* density profile of a quantum gas obtained by phase-contrast imaging. **b**, momentum distribution for the degenerate quantum gas showed in **a**, in this case, an isotropic Fermi gas.

Source: **a, b**, Adapted from MURTHY. (94)

transform, being $\psi_0(\mathbf{k}) = F\{\psi_0(\mathbf{r})\}$ the wavefunction in momentum space, where $F\{\dots\}$ represents Fourier transform. In experiments with ultracold atoms, the wavefunction of a quantum gas is not easily accessible, and it can be obtained only by interference experiments. (52) Nevertheless, the physical quantity accessible is the spatial distribution of particles occupying the real space $n(\mathbf{r}) = |\psi_0(\mathbf{r})|^2$, i.e. the atomic density. Typically, to obtain the atomic density distribution we use different techniques: *in-situ* (94-95) (Figure 17), or after free-expansion.

For a strongly out-of-equilibrium regime, the atomic system present lots of excitations and, usually, the resolution of an *in-situ* imaging is unable to represent the real density fluctuation coming from the presence of the defects in the density introduced by these excitations. On the other hand, a free-expanded cloud can show these density fluctuations more clearly, but with the disadvantage that its Fourier transform do not represent the momentum distribution of the particles as the “expanded” fluctuations do not correspond to the *in-situ* density fluctuations in a simple way.

To access the momentum distribution of a quantum gas from a time-of-flight diagnostic, we can use the regime in which the expansion is considered as ballistic, in this way the momentum is scaled with the distance that the atoms traveled after expand in time-of-flight. This is only true when the atoms are kinetically dominated, i.e., from the instant of

the trap-releasing process, the kinetic energy must be higher than the interaction energy, opposite to the case of a pure BEC in a Thomas-Fermi regime, as is the case of the initial BEC in our experiment.

Thus, we use the absorption image to extract the transverse moment distribution of the atomic superfluid $\tilde{n}(k_\rho) \equiv \tilde{n}(k_x, k_y)$ due to the line-in-sight integration. This transverse momentum distribution $\tilde{n}(k_\rho)$ is related to the 3D momentum distribution $n(\mathbf{k})$ in the kinetically dominated regime by the relation (47),

$$n(\mathbf{k}) \sim \frac{\tilde{n}(k_\rho)}{k_\rho}, \quad (4.1)$$

where $k_\rho = \sqrt{k_x^2 + k_y^2}$ is the radial transverse momentum, and $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ is the magnitude of the 3D radial momentum. When we are treating the momentum (energy) cascades, typically it is expected a power-law behavior of the momentum (energy) distribution,

$$n(k) \propto k^{-\alpha}, \quad (4.2)$$

so that,

$$\tilde{n}(k) \propto k^{-\alpha+1}. \quad (4.3)$$

In the last equation we considered that k and the in-plane projection k_ρ are proportional, and thus we are going to consider $k \equiv k_\rho$. Then, from the absorption images we can extract the exponent $\gamma = \alpha - 1$, and study its dependence with the injected energy.

4.2 Method to extract the momentum distribution

In previous works, the transverse momentum distribution of excited Bose gases was used to reconstruct the 3D momentum distribution $n(k)$ using the inverse Abel transform (47) and to study the coupling with collective modes. (49-50) Here, the same idea is applied, but our procedure allow us to explore not only the azimuthally averaged cloud, but also its angular sections, allowing us to analyze the anisotropy of $\tilde{n}(k)$.

In which follows, it will be present the method used to obtain the transverse momentum distribution of the atomic cloud using absorption images after time-of-flight.

4.2.1 Extracting $\tilde{n}(k)$ from atomic clouds by the time-of-flight technique

We start considering the absorption image of a out-of-equilibrium atomic cloud after time-of-flight t_{of} . As mentioned before, in this regime the kinetic energy is dominant over other kind of energies (interaction). Thus, the ballistic approximation can be considered as valid with some restrictions, like its imprecision to represent the lower momenta distribution. (96) Here, we are going to consider the time-of-flight $t_{\text{of}} = t$, and the expanded radius $r_{\rho}(t_{\text{of}}) \equiv r$. With these assumptions we can relate the atom position r , with this in-situ momentum k , as,

$$r = \frac{\hbar t}{m} k \longrightarrow k = \frac{m}{\hbar t} r, \quad (4.4)$$

where m is the mass of the atomic specie. Equation (4.4), allow us to explore $\tilde{n}(k)$ starting with the density distribution $\tilde{n}(r) \equiv \tilde{n}(x, y)$ obtained from the image. We consider probe beam in the z -axis, and our image in the xy -plane, from where we compute $\tilde{n}(k)$. In order to map correctly $\tilde{n}(k)$, we need to normalize the distributions by the total number of atoms, namely,

$$N = \int \tilde{n}(k_x, k_y) dk_x dk_y = \int \tilde{n}(x, y) dx dy, \quad (4.5)$$

which ensures that the correct occupation is being counted. Another relationship for N can be written in polar coordinates for the whole cloud,

$$\int \tilde{n}(k) k dk = N. \quad (4.6)$$

Equation (4.4) can be written for each one of the image axis, that is,

$$k_x = \frac{m}{\hbar t} x, k_y = \frac{m}{\hbar t} y. \quad (4.7)$$

This is equivalent to re-scale the image by a factor of $m/(\hbar t)$. We can define a function which counts the number of atoms in a polar section of the cloud as,

$$G(k, \Delta\theta) = \int_{k_r}^{k_r + \delta k_r} \int_{\theta_1}^{\theta_2} \tilde{n}(k', \theta) d\theta dk', \quad (4.8)$$

where the angular integration considers the portion of the cloud used for computing the momentum distribution, $\Delta\theta = \theta_1 - \theta_2$ (Figure 18a). When we are considering all the

Figure 18 – Procedure to obtain the momentum distribution all over polar angles for an expanded cloud. **a**, representation of the atomic density column and the procedure to extract $\tilde{n}(k, \Delta\theta)$. **b**, polar representation of the cloud showed in **a** showing the same section.

Source: By the author.

cloud ($\Delta\theta = 2\pi$) this equation is simply,

$$G(k, 2\pi) = 2\pi \int_k^{k+\delta k} \tilde{n}(k') dk'. \quad (4.9)$$

In both cases $\delta k \equiv (m/\hbar t_{\text{of}})\delta r$ is the width of the ring considered to count the atoms. For the polar transformation, δk comes from the sum of one column (Figure 18b). Thus, we obtain,

$$\tilde{n}(k, \Delta\theta) \approx C \frac{G(k, \Delta\theta)}{k}, \quad (4.10)$$

where the constant C can be determined by normalization using the equation (4.6), i.e. the integral,

$$\int G(k, 2\pi) dk = N, \quad (4.11)$$

is the total number of atoms. Therefore, we can write $k \cdot \tilde{n}(k, 2\pi) \sim G(k, 2\pi)$, as well.

With this simple procedure we obtain $\tilde{n}(k, \Delta\theta)$. Thus, once obtained the momentum distribution we can analyze how the momenta are developing when we are considering these starting from the center of mass of the distribution ($k = 0$), i.e lower momenta going to larger momenta, and analyze $\tilde{n}(k)$ along the in-plane angular sections.

We can plot $\tilde{n}(k)$ in a *log-log* graphic to investigate if there is some power-law behavior in a inertial range, as will be defined below, which is a signature of the momentum cascades. A cascade for large momenta in $\tilde{n}(k)$ means that there is a transference of population with lower momentum to large momentum when the system re-arranges the pumped energy. This

Figure 19 – Procedure to obtain $\tilde{n}(k)$ over all polar angles of an expanded cloud in $t_{\text{of}} = 24$ ms. **a**, absorption image showing the polar transformation of the cloud in the inset, that shows the anisotropy of the cloud. **b**, 3D projection of $k \cdot \tilde{n}(k)$ obtained from **a** using the equations (4.7), (4.10) and (4.11). Here it is showed the limits used to analyze the cascades.

Source: By the author.

phenomena is typically associated with turbulence (69), and signalizes that the system can develops a fully turbulent state.

Let us consider an absorption image obtained for an excited condensate as shown in Figure 19a. This image represents the density of the cloud after an expansion time t_{of} and therefore, its momentum distribution, follow the equation (4.4), inside the trap just before it be released. As mentioned before, our method is simple and explore the fact that the excited atomic cloud is a good representation of the momentum distribution for momenta larger than a minimum k , which will be established below. This hypotheses is experimentally verified in reference (42), which shows how $n(k)$, for kinetically dominated samples extracted directly from free expansion, is a good approximation when it is compared with a spectroscopy technique, one that can measure the momentum distribution directly, as in the Bragg spectroscopy. (96) Also, the authors show that $\tilde{n}(k)$ obtained from free expansion clouds is inexact for situations where the system is not kinetically dominated, as a pure BEC. In these cases the best approach is given by the Heisenberg-limited distribution. (97)

Starting from the center of mass of the excited cloud in Figure 19a, a polar transformation is performed (from the inset in 19a). We can re-scale the image using the equations (4.5) and (4.7) in order to obtain the correct momentum distribution. Then, after obtained

$\tilde{n}(k)$ we can analyze the azimuthal average $\tilde{n}(k, \Delta\theta = 2\pi) \equiv \tilde{n}(k)$ and the sections corresponding to the large and short axis of the cloud, where it can be analyzed how the momenta are developing along each direction and their contributions to the azimuthal averaged $\tilde{n}(k)$. With this, we can have an insight about how the effect of anisotropy of the system affects the validity of our method.

Figure 19b shows a 3D view of the $k \cdot \tilde{n}(k, \Delta\theta) \equiv G(k, \Delta\theta)$ along 2π (computationally we have 360 curves) in a *log-log* plot. There we observe how the momentum distribution changes for different directions. In the following, we will consider the analysis that were made for $\tilde{n}(k)$ and the local azimuthal averages $\tilde{n}(k, \Delta\theta = \pi/6)$ for the large and short axis in which can be expected a difference in the momentum spectrum due to the anisotropy of the sample.

4.2.2 Analyzing $\tilde{n}(k)$

From $\tilde{n}(k)$ it is possible to obtain a distribution from where we can analyze the global contribution of all particles, without worrying if it is more relevant in a specific direction. This is equivalent to consider our sample as being isotropic in momentum space. This assumption may be invalid when we consider highly anisotropic samples[‡].

It is possible to identify a region of momenta where we have a cascade in order to find a power-law behavior characteristic of this kind of process. Here, we call this region as *the inertial range*, in analogy to the classical turbulence. (71) Typically the dimensional units of momentum in μm^{-1} are used, but here we will use dimensionless units considering the healing length ξ_0 . We plot $\tilde{n}(k)$ vs $k \cdot \xi_0$, as shown in the horizontal axis of Figure 20. We are considering the healing length of a pure BEC because the real healing length of each atomic cloud is difficult to be obtained from an absorption image. Obviously, when we consider a pure BEC the healing length is the smallest healing length that we can expected for a system with vortices. (62)

[‡] However it is important to note that a posterior analysis considering this asymmetries will verify if this way to analyze the global contribution of the momenta is valid and represents the reality

The inertial range

In order to identify the cascades in $\tilde{n}(k)$, we need to establish a range where these cascades take place. For trapped quantum gases in harmonical potentials, it was defined the inertial range typically in the interval $k \in [2\pi/\bar{R}, 2\pi/\xi_0]$, where $\bar{R}$ is the average Thomas-Fermi radius of the trap. (36) In dimensionless units that gives the dimensionless range $[0.051, 6.283]$.

Figure 20 – Azimuthal averaged $\tilde{n}(k)$ ($\Delta\theta = 2\pi$) for a strongly out-of-equilibrium atomic cloud (showed in the inset). The inertial region is limited by $k_{\bar{R}}$ and k_{ξ_0} , the red lines. We observe a cascade with a power-law behavior in this region of $\gamma \simeq 2.1$, which can be considered as indicative of emergence of turbulence in the quantum gas. The dashed line (green) are the limits used for a linear fit in order to extract the exponent. The dashed-dotted line is a guide to the eye showing the power-law behavior in the region of cascade.

Source: By the author.

On the other hand, in the time-of-flight technique we can also define a characteristic lowest limit for the momentum as being the momentum associated to the averaged in-trap momentum for the non-excited BEC, due to the time-of-flight convolution. This lowest limit is defined as $k_{\bar{R}} = (\hbar m/t_{\text{of}})\bar{R}$, with $\bar{R} = 17.26 \mu\text{m}$ for our trap. Thus, we have $k_{\bar{R}} = 1.7578 \mu\text{m}^{-1}$, or in dimensionless units $k_{\bar{R}}\xi_0 = 0.297$. This value is different from the limit defined above, a very low value. Therefore, we need to consider our lowest limit as being 0.297, because this is the lowest value that the technique can measure with confidence.

Any variation in the spectrum comes after $k_{\bar{r}}$, and then, we choose $k_{\bar{r}}$ to indicate the low limit in our analysis. On the other hand, the higher momentum $k_{\xi_0} = 2\pi/\xi_0$, is the maximum momentum valid to expect that small scales of momentum migrate to larger scales, and in dimensionless units this have the value $k_{\xi_0} \cdot \xi_0 = 2\pi \simeq 6.283$, as mentioned before.

Figure 21 – Compensated momentum distribution with $\gamma = 2.2$. Here it can be observed that the compensated curve has a constant value inside the sub-inertial range, showing that there is a momentum cascade.

Source: By the author.

Analysis of the $\tilde{n}(k)$ curve

In Figure 20 it can be observed the region between the low and high limits, where a cascade can be developed. As an example, to extract the exponent of the cascade in $\tilde{n}(k)$ we look a case where we have a fully developed power-law behavior, and we establish a *sub-inertial range* for performing a linear fitting, that provides the exact value of γ . In all the analysis we are using this sub-inertial range as being $k \cdot \xi_0 \in [0.7, 3]$, which have the extension of $\sim 1/2$ decade, just for the cases where we treat with $\Delta\theta = 2\pi$, that will be the majority in this thesis. Also, we can observe that our inertial range have the size of ~ 1 decade, and in the sub-inertial range, $\tilde{n}(k)$ obeys a power-law behavior with exponent ~ -2.2 for Figure 20, that is showed between the green dashed lines. If we make a compensated momentum distribution in the form $k^\gamma \cdot \tilde{n}(k)$, as showed in Figure 21, the observation of the cascade is more striking in the established sub-inertial range.

Effects of anisotropy

As mentioned above, our excited cloud presents an evident anisotropy in expansion. That reflects the anisotropy in the momentum distribution. Anisotropy introduces a complication in the analysis because its effect is due to many factors and not only by the anisotropy of the trap. It is evident that a rigorous analysis of the effects coming from anisotropy, in analogy to classical fluids, as intermittency, is more complicated and here we will just consider that an azimuthal average of $\tilde{n}(k)$ represents very well the development of momenta.

Figure 22 – A comparative of the behavior for $\tilde{n}(k)$ when different sections of the cloud are considered. The black curve is the same of Figure 20, and is use as a reference. The blue curve correspond to the short axis of the cloud, as shown in superior middle inset. The red one is for the large axis (superior right inset). The exponent for each curve have the same values, ~ -2.2 .

Source: By the author.

Our arguments for that are only qualitative, and can be observed in Figure 22, which

shows three different spectra for the same excited cloud. From here, it can be deduced that an azimuthal average of the momentum distribution ($\Delta\theta = 2\pi$) represents the contributions of all the particles for the cascade. The $\tilde{n}(k)$ for an angular section in the minor axis (the blue curve) shows a small cascade with the same power-law behavior of the azimuthal averaged momentum spectrum (the black curve). This cascade starts to develop before our established lowest limit $k_{\overline{R}}$, an evidence that in this axis we have a small number of particles with smaller momentum that reach the out-of-equilibrium state developing a cascade. When we look at the major axis we can see a larger region where the cascade is developing for larger momenta. Extracting the exponent for the region of $\tilde{n}(k)$ where there is a power-law behavior in these curves, we can obtain the same value of the exponent as being ~ -2.2 . Therefore, we can say that a cascade in the azimuthal averaged momentum spectrum (the black curve) is a good physical manifestation of the contribution of all particles and reflects the correct exponent of the cascades in each spatial direction. Therefore, for all our analysis, we will use the curve of $\tilde{n}(k)$ for $\Delta\theta = 2\pi$, without worry about effects of anisotropy.

Limitations of the time-of-flight technique

Some limitations of this technique should be noted. Here, we have considered the column integrated density as being a faithful representation of the density in the system. This is true, but the same cannot be true in momentum space. Moreover, another limitation comes from the fact that a polar transformation is inaccurate from small radius of the cloud. This is not a real problem for our analysis, because we only consider the momentum being well represented above a certain lower limit. At this regime of momenta we have a good resolution because we are far from the center. However, it is important to recall that the momentum distribution for small k obtained from the time-of-flight technique do not represent the real momentum distribution in this range. One must note, however, that this will not be a problem in our analysis, which is performed in a range of large k 's.

Figure 23 – Behavior of $\tilde{n}(k)$ for several A in the sub-inertial range (green region). We can observe that curves for $A \gtrsim 0.6$ Vpp show a clear power-law dependence. For all the curves showed we use $t_{\text{exc}} = 31.66$ ms and $t_{\text{of}} = 14$ ms.

Source: By the author.

4.3 Results: exponent cascades vs pumped energy

Experimentally we can explore the quantum gas behavior with the variation of the parameters defined by the equation (3.17). In the experiment, we use the parameters A and t_{exc} to control the amount of injected energy. Thus, our results explore the dependence of the cascade exponent of $\tilde{n}(k)$ as function of these two parameters independently and its combined contribution.

As mentioned in the Chapter 3, the excitation is performed overlapping a time dependent magnetic field in the form $B' = A[1 - \cos(\omega_{\text{exc}} t_{\text{exc}} + \phi)]$, that at the same time introduces a modulation in the trap by means of a sinusoidal potential as (90),

$$V'(r, t) \sim A \cos(\omega_{\text{exc}} t_{\text{exc}}), \quad (4.12)$$

while the energy pumped to the system have the dependence,

$$E_{\text{pump}} \sim \frac{2}{\pi} A \omega_{\text{exc}} t_{\text{exc}}. \quad (4.13)$$

The previous relation is obtained considering the pumped energy as being proportional to the integral $\int B' dt_{\text{exc}}$. Then, we can modify the amount of pumped energy through any

of parameters in equation (4.13). As in our measurements, we fixed $\omega_{exc} = 189.5 \text{ Hz}$, we modify E_{pump} changing A or t_{exc} , hence $E_{\text{pump}} \sim A t_{exc}$ for our considerations. Now, we are able to explore the cascade exponent γ varying these experimental parameters .

4.3.1 Dependence with A

When we increase A , keeping t_{exc} fixed, we are increasing the amount of injected energy into the system, which, for higher values, will lead the BEC for an out-of equilibrium state. Here, we are going to study how the power-law exponent γ for the cascade in $\tilde{n}(k)$ changes with respect to A . We scanned A from 0 to 1 Vpp (volts peak to peak). We prefer use this experimental unit to plot our results just for simplify. For our experimental system, $1 \text{ Vpp} = 471 \text{ G/cm}$, when our excitation coils are correctly aligned with respect to their centers.

Figure 24 – γ vs A for several time-of-flight. Clearly is observed that the exponent reach a steady-value for the same values of A in all cases.

Source: By the author.

A first set of measurements were taken in a sequence varying A and maintaining fixed the other excitation parameters, as holding and excitation time. For these data we use $t_{\text{hold}} = 47.6 \text{ ms}$ and 6 cycles of excitation, that correspond to $t_{exc} = 31.66 \text{ ms}$. Figure 23 shown the inertial range of $\tilde{n}(k)$ for several A . It can be observed that for $A \gtrsim 0.6 \text{ Vpp}$

Figure 25 – Average behavior of γ vs A . The gradient zone is for the cases where there is not a clear cascade behavior. The solid line is a guide to the eye following a decay power-law.

Source: By the author.

the spectrum goes to larger momenta, different from the cases with lower A . We need to remember that for lower A the system is not kinetically dominated, and thus, the $\tilde{n}(k)$ showed for these cases (A from 0 to 0.4 Vpp) are not a faithful representation of the in-situ momentum distribution.

These measurements were performed considering different time-of-flight to explore the validity of the time-of-flight technique to characterize exponent cascades, i.e. whether for each time-of-flight the behavior of γ will be the same. Therefore, we performed measures from 5 ms to 30 ms in the same conditions of excitation. We do not considered for our analysis the images with $t_{\text{of}} < 14$ ms due to saturation of the absorption image for dense samples. As we expect, γ is independent of the time-of-flight, meaning that for this time of evolution, $\tilde{n}(k)$ is a good representation of the in-situ momentum distribution. In Figure 24 we show γ for different time-of-flight when A is scanning. It is observed a similar behavior of the exponent for all t_{of} , an indicative that in all cases we are probing the same properties of the system. The most notable result here is the emergence of a stationary-value of γ for $A \gtrsim 0.6$ vpp, independently of the time-of-flight considered. Thus, we can affirm, that for these amplitudes we are in the kinetically dominated regime, and we can probe the properties of $\tilde{n}(k)$ for a considered t_{of} .

In this way, we can take the average of all the curves in Figure 24 to obtain the behavior of the exponent γ versus A . We show this in Figure 25, from where we can observe that the system reach a stationary out-of equilibrium regime. The error bars represent the standard error between the measurements for the several t_{of} . The “ $\times$ ” symbol is used for the cases where the power-law behavior is not clear, which happens to lower A . We plot also a solid curve that is a guide to the eye, obtained from an empirical fitting following a decay power-law behavior of the shape $\gamma = \max\{\gamma_0 - (sAt_{\text{exc}})^{-\eta}, 2.2\}$, where $\gamma_0 = 6.38$, s is an adjustable parameter that we fixed being $s = 1.2$, and $\eta = 1.84$ for the showed curve.

It is important to note that, in spite of γ shows us the regime of stationary non-equilibrium, there is an increase of the pumped energy that means that the system is receiving more energy, and therefore, despite of being in an equilibrium state the internal dynamic of the system change, and this is not quantified by γ .

4.3.2 Dependence with t_{exc}

Now we will consider the behavior of γ when A is fixed and t_{exc} is progressively increased. In a first instance, we considered only two different excitation amplitudes to explore how they changes the behavior of γ .

In Figure 26 it is showed the results for $A = 0.5\text{Vpp}$ and $A = 1\text{Vpp}$. A saturation of the cascade exponent is observed when the far-from equilibrium regime is reached. When A is low, this regime is achieved for larger t_{exc} (Figure 26a). When A is higher, the regime of saturation is achieved for minor t_{exc} , as showed in Figure 26b. This is another indicative that the system goes to the saturation regime when a certain amount of energy is delivered to it. On the other hand, it can be observed that the cascade exponent stop being saturated to certain t_{exc} , where $A = 0.5\text{Vpp}$. The value of γ remains constant until it starts to increase again, leaving to the stationary non-equilibrium regime.

Figure 26 – a, the value of γ reach the stationary-value regime for $t_{\text{exc}} \sim 40$ ms and leave this regime with γ increasing for $t_{\text{exc}} > 70$ ms. b, For larger $A = 1$ Vpp the steady-value γ regime is achieved for $t_{\text{exc}} > 20$ ms. For $t_{\text{exc}} > 55$ ms for this value of A , we do not have a good resolution in the absorption image due to the spreading of the atoms.

Source: By the author.

Stationary-value of γ : non-equilibrium regime diagram

Following the previous case, we took another set of measurements, and at this time, we considered more excitation amplitudes. Figure 27 shows the behavior of γ for all the amplitudes considered. The solid curves correspond to an empirical fit, based on a decaying power-law with the form $\gamma = \max\{\gamma_0 - (sAt_{\text{exc}})^{-\zeta}, 2.2\}$, where $\gamma_0 = 6.8$, $s = 1.32$ and $\zeta = 1.69$ for all curves, except for the case $A = 0.1$ Vpp, where we show an exponential decay fitting.

At the same time, a diagram for $t_{\text{exc}} - A$ shows, approximately, the region where the exponent remains constant. The fact that γ remains the same for different configurations of excitation means that the same regime is reached with different approaches, as in common phase transitions.

For clarity, we drew the diagram for the excitation parameters of Figure 28a. This diagram which is based in the cascade exponent is compared to another one (46), obtained counting vortices, for the case of oscillatory excitation, where the effect was considered as been turbulence (Figure 28b).

Figure 27 – Multi-curve plot showing γ at different excitation configurations. The solid lines are guides to the eye and follow an empirical fitting. The fitting for $A = 0.1 \text{ Vpp}$ is an exponential decay that represents the background losses. The diagram drafted in the plane show a region of stationary value for $\gamma \sim 2.2$ (red colored region).

Source: By the author.

Figure 28 – Diagram in the plane $t_{\text{exc}} - A$. **a**, for the case presented in Figure 27 following the behavior of γ . **b**, counting the number of vortices present for an oscillatory excited atomic clouds.

Source: **a**, by the author. **b**, adapted from SHIOZAKI. (46)

Thus, we have shown that there is a set of values for the excitation parameters that have dependence with the pumped energy, that lead the system to different regimes of non-equilibrium. We can essay a general strategy to characterize the non-equilibrium and the behavior of γ with respect to the pumped energy E_{pump} follow the Equation (4.13). Now, we use the same data used in Figure 27, for plotting γ vs At_{exc} . The resultant graphic is shown in Figure 29, where it is clear that exists a range of values of At_{exc}

Figure 29 – Plot of γ vs $A t_{\text{exc}}$ that shows the regimes of non-equilibrium depending of the pumped energy. The red solid line is a guide to the eye, that is for linear fitting in each region. The first part of this line was fitted by an exponential decay, the second part is the stationary-value of γ showed thicker in this region (red region), where $\gamma \sim 2.2$, and the last part is by a linear fitting.

Source: By the author.

where γ is constant. There, we identify this region between in 3 – 5 of our $A t_{\text{exc}}$ units (remembering that we use V_{pp} for A and cycles for t_{exc}). In the initial decay we identify the non-kinetically dominated regime. We can identify also another regime of non-equilibrium which has γ been different of the steady-value and increasing with $A t_{\text{exc}}$.

Thus, we have quantified the evolution of γ with respect to the injected energy by the oscillatory excitation through the excitation parameters and we could show that the cascade exponents have a well defined dependence with it. The value of the cascade exponent in the steady regime is comparable to the case of wave turbulence.

5 POWER-SPECTRUM OF FLUCTUATIONS

In the previous chapter, we analyzed the momentum distribution based on the assumption of the linear re-scaling of r to k . With this consideration, we analyzed the cascade exponent in dependence with the excitation parameters. If we claim that in this situation we are in the regime of wave turbulence – as is suggested by the cascade exponent $\gamma \sim 3.2$ for the momentum distribution, following our results in the previous chapter, we can consider the system as being in a semi-classical state turbulent dominated, where the energy distribution can be mapped through the power spectrum of the velocity field fluctuations. In classical physics, when turbulence is considered, the more relevant information about the characteristics of the system is extracted from the energy distribution (spectrum), and usually, this energy spectrum is obtained through the velocity statistics. For example, if we perturb a stationary homogeneous classic fluid, it is created a velocity flow fluctuations. When the perturbation is increased and the system coupling this energy, i.e. more energy is injected, these fluctuations will be more pronounced. In this way, can be study the coupled energy in the system due to the excitation looking at the statistics of the velocity field, using the kinetic relation $E \sim v^2$. This approach to explore the energy distribution in the search of non-equilibrium states, as turbulence, was used in the case of liquid helium turbulence (XX), and was extensively used in fluid mechanics (98), astrophysics and cosmology (99-100), oceanography (101), atmospheric sciences (102), genetics (103), and others fields of physics where non equilibrium dynamic is considered. (104-105)

In general, the method to extract the energy distribution from the velocity statistics, is a well-known statistical procedure that use two conjugated variables to mapping one of

that in the reciprocal space. In this way, we use the Fourier space to find the necessary relations to explore the statistical properties between conjugates variables, as $\mathbf{r}$ e $\mathbf{k}$. In this chapter we investigate the viability of extracting some information from the fluctuations statistics of the column density of atomic clouds to study the non-equilibrium regime, and we will compare it with the analysis using the momentum distribution. For this, we assume that each absorption image represent a pseudo-velocity (or pseudo-momentum) field in the kinetically dominated regime due to the mapping in time-of-flight. Thus, we extract the power-spectrum of the pseudo-velocity fluctuations, and discuss how it can be related with the real energy spectrum coming from the real velocity statistics. It is very important to recall, from the hydrodynamics equations, that the superfluidity velocity $\mathbf{v} \sim \nabla\phi$, is inaccessible in absorption images, therefore, the resultant spectrum from the fluctuations statistics do not represent the real energy distribution which just can be obtained from the velocity statistic of the superfluid. However, we expect that in a kinetically dominated regime, where we have access to well-represent momentum distribution, this power spectrum can be related with the real energy spectrum in the regime of wave turbulence.

5.1 Power spectrum of fluctuations

Power spectrum is given by an average over the Fourier modes of the contrast of some physical variable. In the case of configuration space characterized by the vector $\mathbf{r}$, this average extends over all Fourier modes with a wavenumber (momentum) $\mathbf{k}$, i.e. it is an average over all directions in Fourier space keeping $|\mathbf{k}| \equiv k$ constant. In simple words, Fourier modes are averaged within spherical shells of radius k . For instance, if we imagine a situation in equilibrium, where we have an homogeneous system with fluctuations (for examples, density fluctuation or mass fluctuation as happen in gases or galaxies), the internal structures can be quantified by the statistical quantifier as two-point correlation function. Figure ??a shows a representation of a velocity field and their fluctuations.

5.1.1 Correlations and spectra

TWO-POINT CORRELATIONS

Considering $\mathbf{r}_1$ and $\mathbf{r}_2$ as two points in the three dimensional Cartesian space, the spatial correlation of a function $\mathbf{u}(\mathbf{r}, t)$ is,

$$C_{ij}(\mathbf{r}_1, \mathbf{r}_2, t) = \langle \mathbf{u}_i(\mathbf{r}_1, t) \cdot \mathbf{u}_j(\mathbf{r}_2, t) \rangle, \quad (5.1)$$

where i and j denote the direction in which the vector $\mathbf{u}(\mathbf{r}, t)$ is projected.

For our purposes we can make some assumptions.

1. The system is a statistically stationary process, i.e. statistical properties independent of the position and the autocorrelation* only depends on the difference $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{s}$. Thus $C_{ij}(\mathbf{s}, t) = \langle \mathbf{u}_i(\mathbf{r}, t) \cdot \mathbf{u}_j(\mathbf{r} + \mathbf{s}, t) \rangle \equiv C_{ij}(\mathbf{r}_1, \mathbf{r}_2, t) = \langle \mathbf{u}_i(\mathbf{r}_1, t) \cdot \mathbf{u}_j(\mathbf{r}_2, t) \rangle$.
2. The system is time-stationary, i.e. the correlation functions can be uncoupled of the time: $C_{ij}(\mathbf{s}, t) = C_{ij}(\mathbf{s}) \times C_{ij}(t)$.
3. We are interested in a time-independent system, so, we analyze just $C_{ij}(\mathbf{s})$ and make the assumption that $C_{ij}(t)$ is always the same in that time for all the system.

We can define the normalized spatial autocorrelation function as,

$$c_{ij}(\mathbf{s}) = \frac{\langle \mathbf{u}_i(\mathbf{r}) \mathbf{u}_j(\mathbf{r} + \mathbf{s}) \rangle}{\langle \mathbf{u}_i \cdot \mathbf{u}_j \rangle^2}. \quad (5.2)$$

For any function $\mathbf{u}(\mathbf{r})$, it can be shown the Schwartz inequality,

$$\langle \mathbf{u}(\mathbf{r}_1) \cdot \mathbf{u}(\mathbf{r}_2) \rangle \leq \sqrt{\langle \mathbf{u}^2(\mathbf{r}_1) \rangle \langle \mathbf{u}^2(\mathbf{r}_2) \rangle}, \quad (5.3)$$

and thus for a stationary process we obtain $c \leq 1$. So, for $|\mathbf{s}| = 0$, the normalized autocorrelation is trivially unity, and it decays with increasing distance. For the limit of very distances the autocorrelation function converges to zero, since we are considering a random phenomenon.

* Autocorrelation is the correlation function between random variables representing the same quantity measured in two different points.

Now, we can consider that $\delta \mathbf{v}_i(\mathbf{r})$ represents the velocity fluctuation vector in a point. Then, the velocity fluctuation correlation is given by,

$$C_{ij}(\mathbf{s}) = \langle \delta \mathbf{v}_i(\mathbf{r}) \cdot \delta \mathbf{v}_j(\mathbf{r} + \mathbf{s}) \rangle, \quad (5.4)$$

which is the two-point correlation between $\mathbf{v}_i$ and $\mathbf{v}_j$.

POWER-SPECTRUM

Let $P_{ij}(\mathbf{k})$ denote the Fourier transform of the correlation function $C_{ij}(\mathbf{s})$:

$$P_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{-i\mathbf{k} \cdot \mathbf{s}} C_{ij}(\mathbf{s}) d\mathbf{s}, \quad (5.5)$$

where,

$$C_{ij}(\mathbf{s}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{i\mathbf{k} \cdot \mathbf{s}} P_{ij}(\mathbf{k}) d\mathbf{k}. \quad (5.6)$$

These relationships are true for any system. Assuming an isotropic distribution, i.e. $i = j$, the integral over all relative orientation between $\mathbf{s}$ and $\mathbf{k}$ can be carried out, yielding (106),

$$C(s) = \frac{1}{2\pi^2} \int_0^{\infty} E(k) \left(\frac{\sin ks}{ks} \right) k^2 dk, \quad (5.7)$$

whose inverse transform is

$$E(k) = 4\pi \int_0^{\infty} C(s) \left(\frac{\sin ks}{ks} \right) s^2 ds, \quad (5.8)$$

that indicates one way to determine the power spectrum via measuring the correlation function $C(s)$.

5.1.2 Measuring the power spectrum in 2D images

Relations (5.5) and (5.6) are very useful to obtain the conjugated functions C and P knowing any of them. However it is necessary to know how to obtain one of these function from a statistical ensemble of absorption images.

Taking p absorption images of the same experimental situation, and considering the column density distributions $\tilde{n}_i(\boldsymbol{\rho})$, with $\boldsymbol{\rho} = (x, y)$ and $i = 1, 2, \dots, p$, and $\tilde{n}_0 = \tilde{n}_0(\boldsymbol{\rho}) = \langle \tilde{n}(\boldsymbol{\rho}) \rangle_p$ being the ensemble average, the density fluctuations can be expressed as $\delta \tilde{n}_i =$

Figure 30 – Method 1: Procedure to extract $\tilde{C}(s_\rho)$ and $\tilde{P}(k_\rho)$ using the Wiener-Khintchine theorem. $\mathcal{F}\{\dots\}$ denotes Fourier transform.

Source: By the author.

$\tilde{n}_i(\rho) - \tilde{n}_0$. Using these three density fields $\tilde{n}_i$, $\tilde{n}_0$ and $\delta\tilde{n}_i$ one can have access to the transverses C and P . In this chapter we use only the mapping in the Fourier space due to the simplification of the computation in comparative to the position space[†].

Method 1: from Wiener-Khintchine theorem

Equation (5.5) give us the power spectrum from the correlation. So, we can calculate the transverse correlation function $C(s_\rho)$, with $s_\rho = (s_x, s_y)$, for an statistical ensemble, and taking its Fourier transform give access to the transverse power spectrum $P(k_\rho)$, with $k_\rho = (k_x, k_y)$.

Computationally the simplest way to compute the transverse correlation function $C(s_\rho)$

[†] For instance, for calculating the correlation function of p realizations, the Fourier space allows to reduce the computational realization in p^2 operations in comparative to the real space.

of an ensemble of absorption images is through the Wiener-Khintchine theorem. (107–108) From a 2D matrix ensemble, each with size $x_m \times y_m$ (an image), the correlation function is another 2D matrix with the size $x_m \times y_m$. From $C(s_\rho)$, we can obtain the transverse power spectrum $P(k_\rho)$ by simply taking the 2D Fourier transform. Figure 30 shows the protocol used to process the ensemble and to obtain $C(s_\rho)$ and $P(k_\rho)$.

Method 2: from fluctuations in the Fourier space

Another way to access $C(s_\rho)$ and $P(k_\rho)$ is through a direct mapping in the Fourier space of the density field. We take the 2D Fourier transform of each $\tilde{n}_i(\boldsymbol{\rho})$, i.e. we obtain $\tilde{n}_i(k_\rho)$, and use it to calculate the transverse power spectrum from the relationship (109),

$$P(k_\rho) = \frac{1}{\rho} \left[\left| \langle \tilde{n}(k_\rho) \rangle_\rho \right|^2 - \left\langle |\tilde{n}(k_\rho)|^2 \right\rangle_\rho \right]. \quad (5.9)$$

We can compute $C(s_\rho)$ taking the 2D Fourier transform of $P(k_\rho)$, as it is shown in Figure 31.

Method 3: from Fourier transform of the fluctuations

This method is very similar to the Wiener-Kintchine theorem method. We perform the Fourier transform the fluctuation field $\delta\tilde{n}_i$ (110), so that,

$$\delta\tilde{n}_i = \sum_{k \neq 0} \delta n_k e^{-ik \cdot r}$$

thus, the transverse power spectrum is,

$$P(k_\rho) = \frac{\left\langle |\delta n_k|^2 \right\rangle_\rho}{2\pi\rho^2},$$

and it is directly related to $C(s_\rho)$ by the 2D Fourier transform. Figure 32 shows as it is obtained from an ensemble of absorption images.

Figure 31 – Method 2: Procedure to extract $C(s_\rho)$ and $P(k_\rho)$ from the Fourier transform of the velocity field.

Source: By the author.

Considerations

The amplitudes distribution of each Fourier component of the fluctuating density field is determined by the physical processes that generated the fluctuations. As long as the amplitudes are small and the linear theory applies, the evolution of these Fourier components is independent and determined by the physical processes in the in-trap atomic cloud. So, decomposing the density field into a set of Fourier components it is useful. The power spectrum then gives us the distribution of the mean square amplitudes of these components.

Figure 32 – Method 3: Procedure to extract $C(s_\rho)$ and $P(k_\rho)$ from the fluctuations in Fourier space of velocity fluctuations.

Source: By the author.

Thus, the power spectrum is the contribution of each Fourier mode to the total variance of the density fluctuations.

For example, in Method 3, the $|\delta n_k|^2$ term shows us that $P(k_\rho)$ only contains information about the amplitudes of the Fourier components of the fluctuating density field. Thus, the power spectrum $P(k_\rho)$ contains no phase information. Consequently, the two-point correlation function does not contain this information either. It is important to highlight that quite different distributions can have the same power spectrum and correlation function, because these depend on the kind of shape fluctuations.

Independent on the method used to extract $C(s_\rho)$ and $P(k_\rho)$, we have always the same behavior of the functions, but we can have some offset from one to another. We need to consider some differences in each method used, depending whether it maps directly in the

Figure 33 – Comparative of the power spectra obtained by the three methods without the normalization. In the inertial subrange it can be observed the same behavior in the curves.

Source: By the author.

Fourier space or in the real space. For example, the method 1 is more accurate to obtain the correlation function for larger wavenumbers due to the direct mapping in the Fourier space. On the other hand, the method 2 is better for low wavenumbers, because it maps the large structures in position space more efficiently. But, in the interest region we have the same behavior for all the computed $P(k_\rho)$'s. Figure 33 show us the azimuthal average of $P(k_\rho)$ obtained by the three different methods and the same behavior is observed in the inertial range. It can be observed the offset due to the different map used for each method. We are interested in the power spectrum in the inertial range, and from Figure 33 is clear that independent of the method we can quantify the same power-law exponent in that region.

5.2 Results

For our ensemble we take as many as between 15–20 absorption images of atomic clouds, in the same experimental conditions. We explore the dependence with the pumped energy varying A from 0 to 0.7 Vpp, and fixing the others excitation parameters as $\omega_{\text{exc}} = 189.5$ Hz,

Figure 34 – Comparative of the power spectrum of a pure BEC and an excited atomic cloud. For the power-law behavior in the inertial range we have an exponent of $\eta \simeq 1.1$.

Source: By the author.

$t_{\text{exc}} = 31.4 \text{ ms}$, $t_{\text{hold}} = 47.6 \text{ ms}$; and expanding in $t_{\text{of}} = 30 \text{ ms}$. From here, we compute the mean atom distribution and the fluctuating density distributions. In order to determine the spectral dependence of the density fluctuations, we use the three methods to calculate $C(s_\rho)$ and $P(k_\rho)$. Note that, assuming isotropic density fluctuations allows us to average the power spectrum in the angular dependence, yielding $C(s)$ and $P(k)$.

We take the azimuthal average of $P(k)$, so, we can analyze its dependence with the pumped energy through A . Figure 34 shows a comparative between two cases, one of them for a pure BEC and the another for an excited cloud with $A = 0.55 \text{ Vpp}$. In the last case (blue line) the power-law behavior is very clear with an exponent $\eta \sim 1.1$. Nevertheless, the power-law behavior takes place also for low amplitudes, different from the case of the momentum distribution. This can be a signature due to the fact that the power spectrum is quantifying cluster structures inside the cloud, and for low amplitudes the density fluctuations already have a cluster behavior. Hence, we have low exponents different from the case of the momentum distribution for these situations.

Now, we can analyze $P(k)$ in the inertial range for the power spectrum as we did for the momentum distribution in Chapter 4 to extract the cascade exponent η in dependence with

Figure 35 – Comparative of cascade exponents versus A . Here, it can be observed a regime of steady-value of the exponent for both cases.

Source: By the author.

A. Figure 35 shows the dependence of that exponent with the excitation amplitude A . We show a comparative with the reconstructed energy spectrum from $\tilde{n}(k)$, as $E(k) \sim k^2 \tilde{n}(k)$ for the case of the kinetically dominated regime. It is clear that for the kinetically dominated regime the exponents have the same steady-value ~ 2.1 , showing the presence of a well-established non-equilibrium regime.

It is necessary to state that the power spectrum calculated in this chapter is not the real energy spectrum coming from the superfluid velocity, but rather the power spectrum of time-of-flight expanded density fluctuations. We are relating the power spectrum of fluctuations with some energy spectrum due to the mapping of the momenta (velocities) given by the time-of-flight technique. However, if these density fluctuations actually correspond to fluctuations in populations with certain momenta, this power-spectrum somehow represents a like-energy spectrum related to the internal energy of atoms in the atomic cloud (we expected that it is totally right in the case of wave turbulence).

What follows is speculative because the only way to make sure that this mapping actually draws the energy spectrum (or a spectrum that is proportional to the real energy spectrum and from which it is extracted the same cascade exponent η) is compared to the real energy spectrum extracted from the velocity of the superfluid, which as we said before,

is only accessible through the phase of the order parameter, and from which we do not have access with our detection technique. As conclusion of this chapter, here we showed that the exponent η represent a good quantifier for the non-equilibrium regime as in the case of the exponent for the momentum distribution.

6 Quantifying the disorder in excited Bose gases

In the two previous chapter we quantified non-equilibrium states for excited Bose gases through a quantifier obtained from the momentum distribution and from the power-spectrum of fluctuations. This approach showed us that there is a stationary value for our quantifier (the cascade exponent) for a range of the injected energy, that characterize a well defined non-equilibrium state. However, the exponent do not quantify whether that regimes the degree of momentum (velocity) diversity changes with the injected energy, i.e if there is a larger amount of classes of momentum into the system.

To try quantify this degree of disorder (or diversity) we will make use of the differential Shannon entropy, the continuous version of Shannon entropy, and use it for characterize the dependence of the system with the pumped energy. So, this chapter is devoted to use the differential Shannon entropy as quantifier in order to analyze the dataset used in chapter 4, and we will try to understand the develop of the non-equilibrium states in the presence of excitation through this quantifier.

6.1 Differential Shannon entropy

This version of the Shannon entropy is an extension to the continuous distributions cases, and this entropy can be negative (ref RMP). For the case of a pure quantum state char-

acterized by a macroscopic wavefunction $\psi(x)$ in a some physical space*, the measure of the observable X (for example, it can be position or momentum) at point x by a finite resolution detector in the interval $I_{m,\delta} \in [m\delta, (m+1)\delta]$, where m can be a positive or negative integer number, and $\delta > 0$ is the minimum resolution of the detector; of the value x falls. A discrete probability function P_{X_x} can be defined with infinitely many elements. For the initial state $|\psi(x)\rangle_X$ the probability distribution in the interval $I_{m,\delta}$ is,

$$P_{X_x}(m) = \int_{m\delta}^{(m+1)\delta} |\psi(x)|^2 dx. \quad (6.1)$$

From here, we can define the discrete Shannon entropy of P_{X_x} as,

$$D(X_x) = - \sum_{m=-\infty}^{\infty} P_{X_x}(m) \cdot \log [P_{X_x}(m)]. \quad (6.2)$$

In this definition of entropy $D(X_x)$ reflect the finite dimensional properties, in particular it can not be negative, and thus, the discrete Shannon entropy can still thought of as an information measure.

Now, if we thought the probability function as being continuous, i.e, in the limit of $\delta \rightarrow 0$, the differential Shannon entropy is defined as,

$$\begin{aligned} d(X) &= \lim_{\delta \rightarrow 0} [D(X_x) + \log \delta] \\ &= \lim_{\delta \rightarrow 0} \left(- \sum_{m=-\infty}^{\infty} \delta \frac{P_{X_x}(m)}{\delta} \log \left[\frac{P_{X_x}(m)}{\delta} \right] \right). \end{aligned} \quad (6.3)$$

From equation (6.3) it can see which the differential Shannon entropy is an entropy density. When the probability distribution $P_{X_x}(m)$ is continuous $P_{X_x}(m)/\delta \rightarrow P_X(m)$, and we can write (6.3) as,

$$d(X) = - \int dx P_X(m) \log P_X(m), \quad (6.4)$$

where $P_X(m)$ is the probability density in the position X .

If we use the definition of differential Shannon entropy as equation (6.3), it is possible to write this entropy for a particular distribution of physical quantity of a quantum system. Specifically for the case of quantum gases we usually have access to the density distribution $n(\mathbf{r}) = |\psi(\mathbf{r})|^2$ and the momentum distribution $n(\mathbf{k}) = |\psi(\mathbf{k})|^2$. In this scenario

* This physical space can be correspond to the position space or momentum space in our case.

is useful define the differential Shannon entropy in position-space,

$$D_r = - \int dr n(r) \log n(r), \quad (6.5)$$

and the differential Shannon entropy in momentum-space,

$$D_k = - \int dk n(k) \log n(k). \quad (6.6)$$

One striking relation coming from equations (6.5) and (6.6), the uncertainty relationship version of entropies, stay that,

$$D_r + D_k \geq \log(e\pi), \quad (6.7)$$

and the probability distribution that minimize the uncertainty relations is a gaussian distribution.

It is important to note that there are alternatives ways to express the uncertainty relation for entropy (6.7) that can consider finite-spacing relations and contributions of the entropy of the inputs state (the state before the measure be performed). These another approaches can be extensively reviewed in references (xxx).

6.2 Extracting D_k from absorption images

We are going to adapt the definition of differential Shannon entropy for the case where we have a non equilibrium quantum gas. In our experimental situation the number of atoms varies when we perform the excitation in the Bose gas due losses with the background and atoms ejected out from the trap. Therefore it is important to quantify the normalized entropy, i.e., the normalized differential Shannon entropy. Then, we will normalize the differential Shannon entropy by the maximum probability of the distribution. Another consideration becomes from the fact that our measures map the momentum space due the time-of-flight technique, and thus, we do not have information of the in-situ density distribution of the quantum gas. From here, we are going to use the differential Shannon entropy per particle in momentum-space D_k to quantify the disorder in the atomic clouds.

We define our differential Shannon entropy D_k for the in-situ transverse momentum distribution of a cloud having N atoms as,

$$D_k = -\frac{1}{\eta_0} \int \eta_k \log \eta_k dk, \quad (6.8)$$

where η_k is the normalized transverse momentum ($\int \eta_k dk = 1$),

$$\eta_k = \frac{\tilde{n}(k_x, k_y)}{N}, \quad (6.9)$$

with $\tilde{n}(k_x, k_y)$ the usual transverse momentum distribution obtained from the expanded density distribution $\tilde{n}(x, y)$ by re-scaling it as in the equation (4.4), and η_0 being the maximum value of the probability distribution η_k , i.e. D_k represents the expected value of the normalized transverse momentum distribution. The reader must be note that our entropy carry arbitrary units.

The procedure to extract D_k from the absorption images of atomic clouds is very simple. We obtain a representation of the in-situ momentum distribution $\tilde{n}(k_x, k_y)$ by the time-of-flight technique, so, we need only to normalize it to obtain η_k , extract its logarithm $\log \eta_k$ (another image or 2×2 matrix), and perform the product of the images $\eta_k \log \eta_k$. Finally, we sum over all the points in the resultant image, and thus we compute the value of D_k . The protocol to obtain D_k is shown in Figure 36.

Our quantifier D_k , as γ in Chapter 4, can be be a good quantifier only in the cases of kinetically dominated regime because it is extracted from $\tilde{n}(k_x, k_y)$. Nonetheless, we are using this definition only for convenience in order to consider the value of D_k as uncoupled of the time-of-flight, because if we had considered $\eta_k = \tilde{n}(x, y)$, we also would obtain a value for D_k that quantify the diversity in the expanded cloud very well,. However, in this case, D_k would depend on the time-of-flight, since $\tilde{n}(x, y)$ depends on the dimensions of the imagined cloud. Choosing D_k as depending of $\tilde{n}(k_x, k_y)$ we do not need to worry about the effects introduced by the expansion, and we can have a good quantifier also in the non-kinetically dominated regime, different from in the case of $\gamma^\dagger$.

[†] We need to recall that γ is the cascade exponent in the inertial range of the azimuthally averaged $\tilde{n}(k_x, k_y)$. For the non-kinetically dominated cases, i.e. low pumped energies, the system do not develop a cascade, so, γ do not quantify in a good way these regimes.

Figure 36 – Protocol to obtain D_k . We extract the normalized transverse momentum distribution from the absorption images, the logarithm of the image and finally the quantifier D_k .

Source: By the author.

6.3 Results: differential Shannon entropy vs pumped energy

Along this chapter we use the same dataset for the results presented in Chapter 4, and we recurrently will quote the analogous situation in that for clarity. Firstly, we test the behavior of D_k for several cases of time-of-flight maintaining the same excitation parameters. We expect that our value of D_k do not change for different t_{of} , and this can be corroborated looking at Figure 37, where it is showed the behavior of D_k for different t_{of} . For clarity, we plot D_k in log scale. It is important to note from this figure that there is an increasing in the value of D_k when it goes beyond $A = 0.6$ Vpp, and continue increasing with the increase of A . This show us that there is an increasing of the quantifier D_k from a certain amount of pumped energy, and from this point D_k begins to increase as showed in Figure 37.

Then we explore D_k in dependence with t_{exc} fixing all the other excitation parameters. In Figure 38, it is shown the dependence of D_k with t_{exc} for two excitation amplitudes, for the same data presented in Figure 26 in Chapter 4. There it is clear that the quantifier D

Figure 37 – Plot of D_k vs A that shows the increase of the quantifier D when $A > 0.6$ Vpp. The solid line is a guide to the eye obtained from two exponential fits (or linear in log scale).

Source: By the author.

goes to larger values faster when A is larger, as it is seen for $A = 1$ Vpp. It is an indicative that D_k is larger when we pump the system with higher amount of energy.

Now we treat with a set of parameters of A and t_{exc} in order to explore if there is an establishment of non-equilibrium regimes using D_k . Using the same data set for the Figure 27 we obtain Figure 39 that shows that D_k increases when we are in the region of steady-value of the exponent γ in Figure 27. It is a clear indicative that D_k quantifies this regime very well. Thus, the regimes with low D_k describe situations in where the system is not in a well-defined non-equilibrium regime.

Finally, we are able to plot D_k versus the pumped energy, in the same way that we did in Chapter 4. Figure 40 shows this graphic that confirms the emergence of a well-defined non-equilibrium regime, this time it is evident by an increasing value of the quantifier D_k . Different to the case of the quantifier γ in chapter 4, that remains constant for the well-established non-equilibrium regime, our quantifier D_k in that regime has a pronounced variation. It can be related with the usual definition of entropy, that states that entropy change measures the dispersal of energy inside the system, and from Figures 37, 38 and 40 it can be seen that the regions where there is an increasing of D_k denotes a well-established

Figure 38 – Log plot of D_k vs t_{exc} for two excitation amplitudes. Here is show that for larger A the quantifier D_k grows for shorter times. The solid lines are guides to the eye.

Source: By the author.

Figure 39 – Multiplot showing the dependence of D_k with many excitation conditions. Here is observed that exist a region where the quantifier D_k increase abruptly. The solid lines are guides to the eye. The dotted line separate two well-defined regions.

Source: By the author.

Figure 40 – Log plot of D_k vs At_{exc} that shows the variation of the disorder when the energy pumped is increased. The dotted lines delimit the regions in Figure 29 in which γ is constant. Here at the same region D_k has an abrupt increase. In the other regions D_k has small variations.

Source: By the author.

regime of coupling of the system with the injected energy. So, we have demonstrated that D_k is a good quantifier for the non-equilibrium regimes.

7 CONCLUSIONS AND PERSPECTIVES

7.1 Conclusions

In this thesis, we have presented the progress on the study of non-equilibrium quantum gases excited by external oscillatory perturbations. We started with an overview in trapped BEC's, superfluidity, and quantum turbulence. After of the description of the experimental realization of the BEC, we focused on investigations of excited BEC's. We performed the excitation by a sinusoidal perturbation with controllable parameters, and use some quantifiers in order to investigate the establishment of the non-equilibrium regime. To this, we use the time-of-flight technique to probe the momentum distribution and the cascades exponents in the inertial region, through which we could investigate the dependence of these cascade exponents versus the pumped energy. On the other hand, the presence of cascades shows us that there is a transference of population with small momenta to large momenta, a signature as in the case of classical turbulence. As we showed, these cascades take place when the system is excited with a minimum amount of energy, and there is a regime where the power-law exponent is stationary, showing the presence of a well-defined non-equilibrium regime. We explored the dependence of the cascade exponent γ with the excitation parameters, and we found a range of energy values (that are obtained for a set of combination of the excitation parameters) that develop this non-equilibrium state. Complementary to this measurements, in Chapter 6, we used the continuous Shannon entropy as quantifier to show the presence of the well-defined non-equilibrium regime for the same dataset used in Chapter 4. An additional analysis through the study of statistics of the fluctuations in the density fields was implemented in Chapter 5. There, we showed the presence of cascades in the power spectrum into the inertial range, and used the exponent cascade as a quantifier for investigate the non-equilibrium regime. As in the case of momentum distributions, we demonstrate the establishment of a steady-

value for the power spectrum cascade exponent, as an indicative of the well-established non-equilibrium regime.

Assuming that our excited system develops turbulence for certain excitation configurations, we can associate the well-established non-equilibrium regime with this turbulent state. For this consideration we have some indicatives. First, the development of cascades in the momentum distribution occurs with an exponent $\gamma_{2D} \sim -2$ in the transverse momentum distribution. Recovery this exponent to the 3D situation, we have $\gamma_{3D} \sim -3$, that is an exponent theoretically found for the case of wave turbulence. (76) So, as we demonstrated in Chapter 4, there exists a set of excitation configurations that lead to this regime, or as we are speculating, to the emergence of turbulence. Again, considering turbulence, the results in chapter 5 show us a characterization of the turbulent state using the power spectrum of expanded fluctuations. In that case, we related the transverse power spectrum as being a projection of the real energy spectrum and in this way the cascade exponents have the same dependence for the kinetically dominated regime. Finally, the disorder quantifier was useful to characterize the non equilibrium state without worrying about the achievement of the kinetically dominated regime, as it shows that an expected turbulent regime is achieved when there is a change in this quantifier. In these last lines we treated the non-equilibrium phenomena as being turbulence, but we do not affirm it along this thesis because still is not clear the required conditions for the experimental achievement of the quantum turbulence state in a system with many degrees of freedom as ours. As a result of the BEC excitation, it is produced an anisotropic momentum distribution, that is different from the expected isotropic momentum distribution for classical turbulence. Another expected signature, again comparing with the classical case, is the relaxation of the turbulent state to the normal state, i.e the system after a long time must return to this initial condition when it dissipates all the coupled energy. In our system, because of the short lifetime of our condensates (~ 0.5 s), it is impossible to be observed because before reaching this regime the losses with the background dissipates all the atoms. Thus, due to these experimental limitations, we need to enhance our excitation system for a more controllable production of the non equilibrium states.

7.2 Perspectives

In order to investigate quantum turbulence using the quantifiers presented in this thesis, it is necessary to perform some improvements in our experimental system to increase the lifetime of the BEC to times $> 0.5\text{s}$. This will allow us to explore the decay of the non-equilibrium state for larger times. Another considerations are in respect to the oscillation modes in the harmonical trap. Some experimental observations show a freeze of the oscillation modes after a holding time higher than the characteristic time for each excitation situation. The study of the oscillation modes is important because its coupling with the non-equilibrium state can be one of the reason for the apparition of the anisotropic momentum distribution for a turbulent regime. However, it is desired to uncouple these oscillation modes from the excitation process. In the future, it will be implemented some different excitations protocols in the BEC-1 experiment, for example using a random perturbation instead of an oscillatory excitation, in order to explore the non-equilibrium regime without worrying about the oscillation modes, and the methods develop in this thesis will help to better understand and characterization of the non-equilibrium regimes.

References

- 1 ANDERSON, M. H. et al. Observation of Bose-Einstein condensation in a dilute atomic vapor. *Science*, v. 269, n. 5221, p. 198-201, 1995.
- 2 DAVIS, K. B. et al. Bose-Einstein condensation in a gas of sodium atoms. *Physical Review Letters*, v. 75, n. 22, p. 3969-3973, 1995.
- 3 BRADLEY, C. C. et al. Evidence of Bose-Einstein condensation in an atomic gas with attractive interactions. *Physical Review Letters*, v. 75, n. 9, p. 1687-1690, 1995.
- 4 BLOCH, I. Ultracold gases in optical lattices. *Nature Physics*, v. 1, n. 1, p. 23-30, 2005.
- 5 BOSE, S. N. Plancks gesetz und lichtquantenhypothese. *Zeitschrift für Physik*, v. 26, n. 1, p. 178-181, 1924.
- 6 EINSTEIN, A. Quantentheorie de einatomigen idealen gases-zweite. *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, v. 1, p. 3-14, 1925.
- 7 KAPUSTA, J. I. Bose-Einstein condensation, spontaneous symmetry breaking, and gauge theories. *Physical Review D*, v. 24, n.2, p. 426, 1981.
- 8 HERRMANN, S. et al. Testing fundamental physics with degenerate quantum gases in microgravity. *Microgravity Science and Technology*, v. 22, n. 4, p. 529-538, 2010.
- 9 ESTEVE, J. et al. Squeezing and entanglement in a Bose-Einstein condensate. *Nature*, v. 455, n. 7217, p. 1216-1219, 2008
- 10 BLOCH, I.; DALIBARD, J. ; NASCIMBÈNE, S. Quantum simulations with ultracold quantum gases. *Nature Physics*, v. 8, n. 4, p. 267-276, 2012.
- 11 STEINHAEUER, J. et al. Excitation spectrum of a Bose-Einstein condensate. *Physical Review Letters*, v. 88, n. 12, p. 120407, 2002.
- 12 GREINER, M. et al., Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms, *Nature*, v. 415, n. 6867, p. 39-44, 2002.
- 13 BLOCH, I.; DALIBARD, J.; ZWERGER, W. Many-body physics with ultracold gases. *Reviews of Modern Physics*, v. 80, n. 3, p. 885-964, 2008.
- 14 ISLAM, R. et al. Measuring entanglement entropy in a quantum many-body system. *Nature*, v. 528, n. 7580, p. 77-83, 2015.

- 15 STEINHAEUER, J. Observation of quantum Hawking radiation and its entanglement in an analogue black hole. *Nature Physics*, v. 2, n. 10, p. 959–965, 2016.
- 16 KAPITZA, P. Viscosity of liquid Helium below the λ -point. *Nature*, v. 141, n. 3558, p. 74, 1938.
- 17 LONDON, F. On the Bose–Einstein condensation. *Physical Review Letters*, v. 54, n. 11, p. 947–954, 1938.
- 18 JIN, D. S. et al. Collective excitations of a Bose–Einstein condensate in a dilute gas. *Physical Review Letters*, v. 77, n. 3, p. 420–423, 1996.
- 19 MARAGO, O. M. et al. Observation of the scissors mode and evidence for superfluidity of a trapped Bose–Einstein condensed gas. *Physical Review Letters*, v. 84, n. 10, p. 2056–2059, 2000.
- 20 MATTHEWS, M. R. et al. Vortices in a Bose–Einstein condensate. *Physical Review Letters*, v. 83, n. 13, p. 2498–2501, 1999.
- 21 RAMANATHAN, A. et al. Superflow in a toroidal Bose–Einstein condensate: an atom circuit with a tunable weak link. *Physical Review Letters*, v. 106, n. 13, p. 130401, 2011.
- 22 GUPTA, S. et al. Bose–Einstein condensation in a circular waveguide. *Physical Review Letters*, v. 95, n. 14, p. 143201, 2005.
- 23 MEYRATH, P. et al. Bose–Einstein condensate in a box. *Physical Review A*, v. 71, n. 4, p. 041604, 2005.
- 24 GAUNT, A. L. et al. Bose–Einstein condensation of atoms in a uniform potential. *Physical Review Letters*, v. 110, n. 20, p. 200406, 2013.
- 25 BELL, T. A. et al. Bose–Einstein condensation in large time-averaged optical ring potentials. *New Journal of Physics*, v. 18, n. 3, p. 035003, 2016.
- 26 VAN DER STAM, K. M. et al. Large atom number Bose–Einstein condensate of sodium. *Review of Scientific Instruments*, v. 78, n. 1, p. 013102, 2007.
- 27 HU, J. et al. Creation of a Bose-condensed gas of rubidium 87 by laser cooling. 2017. Available from: <<https://arxiv.org/abs/1705.03421>>. Accessible at: 12 May 2017.
- 28 SCHMIEDMAYER, J. Non-equilibrium quantum gas: how not to boil. *Nature Physics*, v. 9, n. 5, p. 266–267, 2013.
- 29 PROUKAKIS, N. et al. *Quantum gases: finite temperature and non-equilibrium dynamics*. London: Imperial College Press, 2013. v. 1.

- 30 ZHANG, X. et al. Observation of quantum criticality with ultracold atoms in optical lattices. *Science*, v. 335, n. 6072, p. 1070-1072, 2012.
- 31 NAVON, N. et al. Critical dynamics of spontaneous symmetry breaking in a homogeneous Bose gas. *Science*, v. 347, n. 6218, p. 167-170, 2015.
- 32 ROATI, G. et al. Anderson localization of a non-interacting Bose-Einstein condensate. *Nature*, v. 453, n. 7197, p. 895-898, 2008.
- 33 KOBAYASHI, M.; TSUBOTA, M. Kolmogorov spectrum of quantum turbulence. *Journal of the Physical Society of Japan*, v. 74, n. 12, p. 3248-3258, 2005.
- 34 BARENGHI, C.; DONELLY, R.; VINEN, W. *Quantized vortex dynamics and superfluid turbulence*. New York: Springer Verlag, 2001. (Lecture notes in physics, v. 571).
- 35 TSUBOTA, M. Quantum turbulence – from superfluid helium to atomic Bose-Einstein condensates. *Journal of Physics: condensed matter*, v. 21, n. 16, p. 164207, 2009.
- 36 KOBAYASHI, M.; TSUBOTA, M. Quantum turbulence in a trapped Bose-Einstein condensate. *Physical Review A*, v. 76, n. 4, p. 045603, 2007.
- 37 BARENGHI, C. et al. Introduction to quantum turbulence. *Proceedings of the National Academy of Sciences of the United States of America*, v. 111, p. 4647-4652, 2014. Supplement 1. doi: 10.1073/pnas.1400033111.
- 38 YUKALOV, V. I. Turbulent superfluid as continuous vortex mixture. *Laser Physics Letters*, v. 7, n. 6, p. 467-476, 2010.
- 39 HENN, E. A. L. et al. Emergence of turbulence in an oscillating Bose-Einstein condensate. *Physical Review Letters*, v. 103, n. 4, p. 045301, 2009.
- 40 REEVES, M. T. et al. Classical and quantum regimes of two-dimensional turbulence in trapped Bose-Einstein condensates. *Physical Review A*, v. 86, n. 5, p. 053621, 2012.
- 41 KWON, W. J. et al. Relaxation of superfluid turbulence in highly oblate Bose-Einstein condensates. *Physical Review A*, v. 90, n. 6, p. 063627, 2014.
- 42 NAVON, N. et al. Emergence of a turbulent cascade in a quantum gas. *Nature*, v. 539, n. 7627, p. 72-75, 2016.
- 43 PAOLETTI, M. S. et al. Velocity statistics distinguish quantum turbulence from classical turbulence. *Physical Review Letters*, v. 101, n. 15, p. 154501, 2008.
- 44 TSATSOS, M. C. et al. Quantum turbulence in trapped atomic Bose-Einstein condensates. *Physics Reports*, v. 622, p. 1-52, 2016.

- 45 WHITE, A.; ANDERSON, B. P.; BAGNATO, V. S. Vortices and turbulence in trapped atomic condensates. *Proceedings of the National Academy of Sciences of the United States of America*, v. 111, p. 4719-4726, 2014. Supplement 1. doi: 10.1073/pnas.1312737110.
- 46 SHIOZAKI, R. F. et al. Transition to quantum turbulence in finite-size superfluids. *Laser Physics Letters*, v. 8, n. 5, p. 393, 2011. doi: 10.1016/j.physrep.2016.02.003.
- 47 THOMPSON, K. J. et al. Evidence of power law behavior in the momentum distribution of a turbulent Bose-Einstein condensate. *Laser Physics Letters*, v. 11, n. 1, p. 015501, 2014.
- 48 TAVARES, P. E. S. et al. Out-of-phase oscillation between superfluid and thermal components for a trapped Bose condensate under oscillatory excitation. *Laser Physics Letters*, v. 10, n. 4, p. 045501, 2013.
- 49 FRITSCH, A. R. et al. Nonlinear dependence observed in quadrupolar collective excitation of a trapped BEC. *Journal of Low Temperatures Physics*, v. 180, n. 1, p. 144-152, 2015.
- 50 BAHRAMI, A. et al. Investigation of the momentum distribution of an excited BEC by free expansion: coupling with the collective modes. *Journal of Low Temperatures Physics*, v. 180, n. 1, p. 126-132, 2015.
- 51 FOOT, C. J. *Atomic physics*. New York: Oxford University Press, 2005. 331 p.
- 52 PITAEVSKII, L.; STRINGARI, S. *Bose-Einstein condensation*. New York: Oxford University Press, 2003. 382 p.
- 53 PETHICK, C. J.; SMITH, H. *Bose-Einstein condensation in dilute gases*. 2nd ed. Cambridge: Cambridge University Press, 2008. 569 p.
- 54 UEDA, M. *Fundamentals and new frontiers of Bose-Einstein condensation*. Singapore: World Scientific Publishing, 2010. 368 p.
- 55 DALFOVO, F.; GIORGINI, S.; PITAEVSKII, L. P.; STRINGARI, S. Theory of Bose-Einstein condensation in trapped gases. *Review of Modern Physics*, v. 71, n. 3, p. 463-512, 1999.
- 56 PROUKAKIS, N. P.; BURNETT, K. Theory of Bose-Einstein condensation for trapped atoms. *Philosophical Transactions: mathematical, physical and engineering sciences*, v. 355, n. 1733, p. 2235-2245, 1997.
- 57 COURTEILLE, P. W.; BAGNATO, V. S.; YUKALOV, V. I. Bose-Einstein condensation in trapped atomic gases. *Laser Physics*, v. 11, n. 6, p. 659-800, 2001.

- 58 CASTIN, Y.; DUM, R. Bose-Einstein condensates in time dependent traps. *Physical Review Letters*, v. 77, n. 27, p. 5315–5319, 1996.
- 59 PITAEVSKII, L. P. Collective excitations of a trapped Bose-condensed gas. *Physical Review Letters*, v. 77, n. 12, p. 2360–2363, 1996.
- 60 PITAEVSKII, L. P. Vortex lines in an imperfect Bose gas. *Soviet Physics Journal of Experimental and Theoretical Physics*, v. 13, n. 2, p. 451–454, 1961.
- 61 GROSS, E. P. Structure of a quantized vortex in boson system. *Il Nuovo Cimento*, v. 81, n. 3, p. 454–477, 1961.
- 62 LUNDH, E.; PETHICK, C. J.; SMITH, H. Vortices in Bose-Einstein-condensed atomic clouds. *Physical Review A*, v. 58, n. 6, p. 4816–4823, 1998.
- 63 JIN, D. S. et al. Vortices in a Bose-Einstein condensate. *Physical Review Letters*, v. 83, n. 3, p. 420–423, 1996.
- 64 ABO-SHAER, J. R. et al. Observation of vortex lattices in Bose-Einstein condensates. *Science*, v. 292, n. 5516, p. 476–479, 2001.
- 65 SHOMRONI, I. et al. Evidence for an oscillating soliton/vortex ring by density engineering of a Bose-Einstein condensate. *Nature Physics*, v. 5, n. 5, p. 193–197, 2009.
- 66 SAULS, J. A. Viewpoint: half-quantum vortices in superfluid helium. *Physics*, v. 9, n. 148, 2016. doi: 10.1103/Physics.9.148.
- 67 VINEN, W. F.; NIEMELA, J. J. Quantum turbulence. *Journal of Low Temperature Physics*, v. 128, n. 516, p. 167–231, 2002.
- 68 FRITSCH, U. *Turbulence: the legacy of A. N. Kolmogorov*. Cambridge: Cambridge University Press, 1995.
- 69 RICHARDSON, L. F. *Weather prediction by numerical process*. Cambridge: Cambridge University Press, 1992.
- 70 KOLMOGOROV, A. N. Dissipation of energy in the locally isotropic turbulence. *Doklady Akademii Nauk SSSR*, v. 32, p. 16–18, 1941. Available from: <<http://rspa.royalsocietypublishing.org/content/434/1890/15>>. Accessible at: 12 Oct. 2016.
- 71 KOLMOGOROV, A. N. The local structure of turbulence in incompressible viscous fluid for very large Reynold numbers. *Doklady Akademii Nauk SSSR*, v. 30, p. 299–303, 1941. Available from: <<http://rspa.royalsocietypublishing.org/content/434/1890/9>>. Accessible at: 12 Oct. 2016.

- 72 FEYNMAN, R. P. Application of quantum mechanics to liquid helium. *Progress in Low Temperatures Physics*, v. 1, n. 1, p. 17-53, 1955.
- 73 MAURER, J.; TABELING, P. Local investigation of superfluid turbulence. *Europhysics Letters*, v. 43, n. 1, p. 29-34, 1998.
- 74 FONDA, E. et al. Direct observation of Kelvin Waves excited by quantized vortex reconnection. *Proceedings of the National Academy of Sciences of the United States of America*, v. 111, n. 1, p. 4707-4710, 2014.
- 75 THOMSON, W. On gravitational oscillations of rotating water. *Proceedings of the Royal Society of Edinburgh*, v. 10, p. 92-100, 1879.
- 76 NAZARENKO, S. *Wave turbulence*. Heidelberg: Springer-Verlag, 2011. 279 p. (Lecture notes in physics, v. 825).
- 77 NORRIE, A. A. Quantum turbulence in condensate collisions: an application of the classical field method. *Physical Review Letters*, v. 94, n. 4, p. 040401, 2005.
- 78 PAOLETTI, M. S.; LATHROP, D. P. Quantum turbulence. *Annual Review of Condensed Matter Physics*, v. 2, n. 1, p. 213-234, 2011.
- 79 HENN, E. A. L. *Produção experimental de excitações topológicas em um condensado de Bose-Einstein*. 2008. 126 p. Tese (Doutorado) — Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2008.
- 80 TAVARES, P. E. S. *Consequências das excitações oscilatórias em condensados de Bose-Einstein*. 2012. 103 p. Tese (Doutorado) — Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2012.
- 81 KETTERLE, W.; DURFEE, D. S.; STAMPERR-KURN, D. M. Making, probing and understanding Bose-Einstein condensates. In: INGUSCIO, M.; STRINGARI, S.; WIEMAN, C. E. (Ed.). *Bose-Einstein condensation in atomic gases*. Amsterdam: IOS Press, 1999. p. 67-176. (International School of Physics 'Enrico Fermi', 1998).
- 82 STAMPER-KURN, D. M.; UEDA, M. Experiments and theory in cold and ultracold collisions. *Review of Modern Physics*, v. 71, n. 1, p. 1-85, 1999.
- 83 RAAB, E. L. et al. Trapping of neutral sodium atoms with radiation pressure. *Physical Review Letters*, v. 59, n. 23, p. 2631, 1987.
- 84 PHILLIPS, W. D. Nobel lecture: laser cooling and trapping of neutral atoms. *Review of Modern Physics*, v. 70, n. 3, p. 721-741, 1998.

- 85 TANNODJI, C. C.; GRYNBERG, G; DUPONT-ROE, J. *Atom-photon interactions*. New York: John Wiley and Sons, 1992.
- 86 MAJORANA, E. Atomi orientati in campo magnetico variabile. *Il Nuovo Cimento* (1924-1942), v. 9, n. 2, p. 43-50, 1932.
- 87 ESSLINGER, T.; BLOCH, I.; HANSCH, T. W. Bose-Einstein condensation in a quadrupole-Ioffe configuration trap. *Physical Review A*, v. 58, n. 4, p. 2664, 1998.
- 88 DAVIS, K. B. et al. Evaporative cooling of sodium atoms. *Physical Review Letters*, v. 74, n. 26, p. 5202-5205, 1995.
- 89 HENN, E. A. L. et al. Observation of vortex formation in an oscillating trapped Bose-Einstein condensate. *Physical Review A*, v. 79, n. 4, p. 43618, 2009.
- 90 YUKALOV, V. I.; NOVIKOV, A. N. BAGNATO, V. S. Strongly nonequilibrium Bose-condensed atomic systems. *Journal of Low Temperatures Physics*, v. 180, n. 1, p. 53-67, 2015.
- 91 SAKURAI, J. J. *Modern quantum mechanics*. 2nd ed. Boston: Addison Wesley, 1993. 708 p.
- 92 BURGER, S. et al. Dark solitons in Bose-Einstein condensates. *Physical Review Letters*, v. 83, n. 25, p. 5198-5201, 1999.
- 93 HABER, K. S.; ALBRETCH, A. C. Time-of-flight technique for mobility measurements in the condensed phase. *Journal of Physical Chemistry*, v. 88, n. 24, p. 6025-6030, 1984.
- 94 MURTHY, P. A. et al. Matter-wave Fourier optics with a strongly interacting two-dimensional Fermi gas. *Physical Review A*, v. 90, n. 4, p. 043611, 2014.
- 95 MEPELINK, R. et al. Phase contrast imaging of Bose condensed clouds. 2009. Available from: <<https://arxiv.org/abs/0909.4429>>. Accessible at: 10 Dec. 2015.
- 96 STENGER, J. et al. Bragg spectroscopy of a Bose-Einstein condensate. *Physical Review Letters*, v. 82, n. 23, p. 4569-4573, 2013.
- 97 BOUYER, P.; KASEVICH, M. A. Heisenberg-limited spectroscopy with degenerate Bose-Einstein gases. *Physical Review A*, v. 56, n. 2, p. R1083(R) , 1997.
- 98 MONIN, A. S.; YAGLOM, A. N. *Statistical fluid mechanics: mechanics of turbulence*. Cambridge: The MIT Press, 1975. v. 2.
- 99 ELMEGREEN, B. G. Interstellar turbulence I: observations and processes. *Annual Review of Astronomy and Astrophysics*, v. 42, n. 1, p. 211-273, 2004.

- 100 JACKSON, A. P. et al. Power-law wrinkling turbulence-flame interaction model for astrophysical flames. *Astrophysical Journal*, v. 784, n. 2, p. 174-194 ,2014.
- 101 KIMAL, J. C. et al. Spectral characteristics of surface-layer turbulence. *Quarterly Journal of the Royal Meteorological Society*, v. 98, n. 417, p. 563-589, 1972.
- 102 RAO, C. et al. Measuring the power-law exponent of an atmospheric turbulence phase power spectrum with a Shack Hartmann wave-front sensor. *Optics Letters*, v. 24, n. 15, p. 1008-1010, 1999.
- 103 SUCCI, S. Lattice Boltzmann across scales: from turbulence to DNA translocation. *The European Physical Journal B*, v. 64, n. 3, p. 471-479, 2008.
- 104 GYR, A.; KINZELBACH, W.; TSINOBER, A. *Fundamental problematic issues in turbulence*. Basel: Springer Basel AG, 1999. 489 p.
- 105 BRATANOV, V. et al. Nonuniversal power-law spectra in turbulent systems. *Physical Review Letters*, v. 111 n. 7, p. 075001, 2013.
- 106 STANISIC, M. M. *The Mathematical theory of turbulence*. New York: Springer-Verlag, 1988. 501 p.
- 107 WIENER, N. Generalized harmonic analysis. *Acta Mathematica*, v. 55, n. 1, p. 117-258, 1930.
- 108 KHINTCHINE, A. Korrelationstheorie der stationären stochastischen Prozesse. *Mathematische Annalen*, v. 109, n. 1, p. 604-615, 1934.
- 109 BLUMKIN, A. et al. Observing atom bunching by the Fourier slice theorem. *Physical Review Letters*, v. 110, n. 26, p. 265301 , 2013.
- 110 RODRIGUES, J. D. et al. Photon bubble turbulence in cold atomic gases. 2016. Available from: <<https://arxiv.org/abs/1604.08114>>. Accessible at: 11 Jan. 2017.
- 111 ARAKI, H.; LIEB, E. H. Entropy inequalities. *Communications in Mathematical Physics*, v. 18, n. 2, p. 160-170, 1970.
- 112 DURÃO, L. M. M.; CALDEIRA, A. O. Statistical entropy of open quantum systems. *Physical Review E*, v. 94, n. 6, p. 062147 , 2016.
- 113 CATTANI, F. et al. Measuring and engineering entropy and spin squeezing in weakly linked Bose-Einstein condensates. *New Journal of Physics*, v. 15, n. 6, p. 063035, 2013.