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**Quantum turbulence and thermodynamics on a
trapped Bose-Einstein condensate**

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Quantum turbulence and thermodynamics on a trapped Bose-Einstein condensate

Tese apresentada ao programa de Pós-Graduação em Física do Instituto de Física de São Carlos da Universidade de São Paulo, para obtenção do título de Doutor em Ciências.

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*“Die Bewußtheit ist die letzte und späteste
Entwicklung des Organischen und folglich auch
das Unfertigste und Unkräftigste daran.”*

- Friedrich Nietzsche (1844 - 1900)

*“Die Grenzen meiner Sprache
bedeuten die Grenzen meiner Welt.”*

- Ludwig Wittgenstein (1889 - 1951)

*“Dans la vie, rien n'est à craindre,
tout est à comprendre.”*

- Marie Curie (1867 - 1934)

Resumo

SHIOZAKI, R. F. **Turbulência quântica e termodinâmica em um condensado de Bose-Einstein aprisionado.** 2013. 137 p. Tese (Doutorado em ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2013.

Nesta tese, nós estudamos dois aspectos da condensação de Bose-Einstein (CBE) em gases diluídos aprisionados: *(i)* superfluidez e a possível ocorrência de turbulência quântica (TQ); e *(ii)* não-uniformidade, o que sugere um tratamento termodinâmico diferente pela definição de novas variáveis globais. Ambos os estudos foram realizados em amostras condensadas de átomos de ^{87}Rb aprisionados magneticamente numa armadilha do tipo QUIC. Em relação ao primeiro item, a geração de vórtices e TQ ocorreu pela aplicação de uma excitação oscilatória gerada pela adição de um campo quadrupolar ao potencial confinante do QUIC. Como dependência da amplitude e duração da excitação, diferentes regimes foram observados. Particularmente, num dos regimes, apenas vórtices bem definidos foram observados e em outro, imagens consistentes com a ocorrência de TQ foram obtidas. A transição entre estes dois regimes foi explicada em termos do tamanho finito característico de gases aprisionados. Além disto, através da análise de dados mostrando configurações com três vórtices, pudemos inferir a presença de vórtices e anti-vórtices (circulação oposta). Para explicar o mecanismo de nucleação de vórtices, analisamos, como possível causa, um movimento relativo entre as componentes térmicas e condensadas das amostras, conhecido como *contrafluxo*. Já em relação ao segundo item, a transição de fase da CBE foi descrita em termos de novas variáveis termodinâmicas globais. Um diagrama de fase foi construído ressaltando as semelhanças com a transição observada no hélio superfluido. Por fim, apresentamos resultados preliminares sobre o cálculo de uma capacidade térmica global e discutimos as vantagens desta nova abordagem em relação à alternativa usual baseada na aproximação de densidade local. Estas vantagens são particularmente relevantes no caso de nuvens condensadas que apresentam vórtices e TQ.

Palavras-chaves: Condensação de Bose-Einstein. Turbulência quântica. Termodinâmica de gases aprisionados.

Abstract

SHIOZAKI, R. F. **Quantum turbulence and thermodynamics on a trapped Bose-Einstein condensate.** 2013. 137 p. Thesis (Doctorate in Science) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2013.

In this thesis we have basically studied two aspects of Bose-Einstein condensation (BEC) in trapped dilute gases: *(i)* superfluidity with the possible onset of quantum turbulence (QT), and *(ii)* non-uniformity, which suggests the definition of new variables in order to build a global thermodynamic description. Both analyses were performed in a ^{87}Rb BEC magnetically trapped in a Quadrupole-Ioffe configuration (QUIC) trap. Concerning the first item, vortices and QT were generated by applying an oscillatory excitation formed by a quadrupole magnetic field superimposed onto the QUIC trapping potential. Scanning both the excitation amplitude and its duration allowed us to observe different regimes, particularly one with regular, well-defined vortices and another, where the onset of QT is believed to have occurred. The transition between these two regimes was explained by considering the finite-size characteristic of trapped gases. Additionally, data analyses on three vortex configurations suggested the presence of both vortices and anti-vortices (opposite circulation sign), and the vortex nucleation mechanism was proposed to be related to a relative motion between the condensate and thermal components, namely a *counterflow*. As for the second item, the BEC transition in our experiment was characterized in terms of new global thermodynamic variables. A phase diagram was constructed and compared to the superfluid helium phase transition. Finally, we provide preliminary results on the calculation of a global heat capacity, and briefly discuss the advantages of this new approach over the local density approximation alternative, particularly on BEC clouds in the presence of vortices and QT.

Keywords: Bose-Einstein condensation. Quantum turbulence. Thermodynamics of trapped gases.

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List of abbreviations and acronyms

AMO	Atomic, molecular and optical
BEC	Bose–Einstein condensation/condensate
CCD	Charge-coupled device
CT	Classical turbulence
GPE	Gross-Pitaevskii equation
LDA	Local density approximation
LIAD	Light-induced atom desorption
MOT	Magneto-optical trap
MT	Magnetic trap
QT	Quantum turbulence/turbulent
QUIC	Quadrupole–Ioffe configuration
TFA	Thomas–Fermi approximation
TOF	Time-of-flight

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1 Introduction

One of the greatest goals in science has been the description of the macroscopic world by means of its microscopic constituents. In this sense, understanding the atomic structure and how atoms interact with one another is essential. Innately related to that, the formation of molecules and light-matter interaction also play a major role. These elements basically give rise to a currently very active branch of physics, known as *atomic, molecular and optical* (AMO) physics, in which this thesis is included.

Statistical physics has astonishingly succeeded in showing that the empirical laws of thermodynamics can be explained through statistics on the huge number of particles that compose macroscopic systems. In 1924, together with the newly developed concepts of quantum mechanics, Satyendra Bose predicted the occurrence of a macroscopic ground state occupation for photons due to the statistics resulting from their indistinguishable nature (9). In 1925, Albert Einstein extended this prediction to an ideal gas of atoms that follow the same statistics as photons (10). The particles that follow this statistics were then named *bosons*, and the general phenomenon was named *Bose–Einstein condensation* (BEC).

In the same year (1938) that superfluidity was observed in ^4He liquid (11, 12), it was predicted to be related to the onset of BEC (13). Soon after 1955, when Richard Feynman proposed the occurrence of turbulence in superfluids (14), the phenomenon was observed in superfluid helium (15–18). Later, it was named quantum turbulence (QT) due to the quantum restrictions imposed to the superfluid flow, and remarkable similarities with the classical phenomenon were found in more recent experiments (19, 20).

In spite of all these advances, liquid helium presents strong interatomic interactions which precluded both an experimental observation of a large ground state occupa-

tion (BEC), and an accurate theoretical modelling. In this sense, a breakthrough took place in 1995, when three research groups independently observed (almost “textbook”) BECs in dilute alkaline gases of ^{87}Rb (21), ^7Li (22), and ^{23}Na (23). The weakly-interacting aspect of dilute gases allows an accurate mean-field theory treatment, which turns those systems into an ideal testbed for studying superfluidity and QT.

One of the reasons why those are not quite “textbook” BECs is related to the confining potential. The simplest description of BEC considers a uniform bosonic gas in a box, whereas in the experiments, the gas is trapped by a non-uniform potential, which results in many qualitative differences (24). Particularly, the volume is no longer well-defined, and the gas density is not uniform, which leads to a non-uniform pressure. Therefore, an usual global equation of state involving these thermodynamic variables is not possible. Recently (5,25,26), it was proposed a thermodynamic description in terms of new global variables that (in many senses) are analogous to the real volume and hydrostatic pressure.

The first evidence of the onset of QT in a (dilute gas) BEC were recently reported by our research group (27–31). Most of the results presented in this thesis are related to that observation. Also, we provide a description of the BEC transition in our experiment in terms of those new global thermodynamic variables.

In the following, we give a more detailed outline of this thesis. But before, we present an overview on the technical development that enabled the production of BEC in dilute gases as well as some of the great advances that succeeded this remarkable realization.

1.1 On BEC in dilute gases

The ever-increasing control over the (internal and external) atomic degrees of freedom has been the driving force of AMO physics. The story begins with the first

demonstration of the control over the internal atomic states by A. Kastler in 1957 (32). In his experiment, atoms were optically pumped to a single Zeeman sublevel and then transferred to another by applying radio-frequency (RF) radiation. At the same time, W. Paul was developing methods to control the external degrees of freedom of ions culminating in the so-called Paul trap: a combination of AC and DC electric fields that tightly confine ions (33).

In 1960 one of the most important breakthroughs in science and technology happened: the advent of lasers (34). The development of tunable sources of light enabled precise measurements of the atomic spectral lines (35). Additionally the high light intensity afforded by lasers stimulated proposals in 1975 (36,37) that atoms could be cooled down and trapped by means of the radiation force. In 1982, the first demonstration of laser cooling was done by W. Phillips and H. Metcalf, when an atomic beam of sodium atoms was decelerated by a counter-propagating laser beam as it passed through a region with a spatially varying magnetic field, the so-called *Zeeman slower* (38). Three years later, two advances were achieved: (i) by using that decelerated atomic beam, for the first time neutral atoms were magnetically trapped (39), and (ii) a 3D cooling of sodium atoms using a pair of counter-propagating laser beams in each spatial direction was performed (40), the so-called *optical molasses*. Then in 1987, by adding a quadrupole field centred in the crossing region of the six beams a (large) cooled and trapped sample was produced (41) in the so-called *magneto-optical trap* (MOT). It is important to mention that MOT has become the starting point in most ultracold atoms experiments.

To achieve BEC in all those three experiments in 1995 (21–23), the final ingredient was *evaporative cooling*, which was first performed in hydrogen in 1988 (42). Until very recently (43) evaporative cooling was the only technique that succeeded in achieving the quantum degenerate state (in dilute gases) in any species, bosonic or fermionic. In fact, the list of isotopes, of which quantum degenerate gases were obtained, is quite long and keep growing. To this date and to the best of my knowledge, BEC in dilute atomic gases has been observed in: H, ^7Li , ^{23}Na , ^{39}K , ^{41}K , ^{85}Rb , ^{87}Rb , ^{133}Cs , ^{40}Ca , ^{84}Sr , ^{86}Sr , ^{88}Sr , ^4He (metastable), ^{52}Cr , ^{164}Dy , ^{168}Er , ^{168}Yt , ^{170}Yt , ^{174}Yt , ^{176}Yt . BEC was also observed

for photons in an optical cavity (44). Quantum degenerate Fermi gases (mostly of ^6Li and ^{40}K) were also obtained (45), and molecular BECs composed of bound fermionic atoms were produced (46).

The greatest feature in this type of experiments concerns the high level of control. Particularly, by increasing the confinement in some directions, it is possible to create effectively 2D or 1D systems. Also Feshbach resonances provide broad prospects on the control of interatomic interactions. Finally we must mention the realization of *optical lattices*, which allowed a fruitful interplay between AMO and Solid State physics.

1.2 Thesis outline

In this thesis we basically study two aspects of Bose–Einstein condensation (BEC) in trapped dilute gases: *(i)* superfluidity with the possible onset of quantum turbulence (QT), and *(ii)* non–uniformity, which leads to a thermodynamic description in terms of new global variables.

We start by revisiting the basics on the BEC phenomenon in Chapter 2. First, in Section 2.1 we discuss some of the formal definitions of BEC. Then Section 2.2 presents the simplest BEC system, namely the ideal Bose gas. In Section 2.3 the more realistic interacting case is addressed. First (Subsection 2.3.1) we derive the celebrated Gross–Pitaevskii equation, which has proven to very accurately describe BEC in dilute gases. Second (Subsection 2.3.2), the hydrodynamic equations for the superfluid motion are shown, and finally (Subsection 2.3.3), we describe the BEC dynamics in terms of the Bogoliubov–de Gennes equations.

In the same way, Chapter 3 gives the basic notions on both superfluidity and turbulence. In Section 3.1, we show how the Landau criterion for superfluidity together with the Bogoliubov spectrum predicts superfluidity for an interacting BEC. Then, vortices, particularly in superfluids, are discussed in Section 3.2. And at the end, an overview on

both classical (Subsection 3.3.1) and quantum (Subsection 3.3.2) turbulence is given in Section 3.3.

In Chapter 4, we briefly discuss the main elements of the experimental setup. We start (in Section 4.1) with an overview covering: atoms sources, double-MOT configuration, mode-matching processes, QUIC magnetic trap, RF-induced evaporative cooling, and absorption imaging. Finally, in Section 4.2, we comment in more detail the designing of the new vacuum chamber, which was the main upgrade to the first version of this apparatus.

The results and analysis on the observation of QT in our BEC system are presented in Chapter 5. We start, in Section 5.1, by explaining the excitation field responsible for generating vortices and turbulence in our system. In Section 5.2, we present a diagram that summarize some of the most interesting effects observed by means of applying that excitation field. Then, Section 5.3 shows an analysis suggesting that the vortex nucleation mechanism in our experiment is due to a relative motion between the condensate and thermal components, namely *counterflow*. Finally, in Section 5.4, we discuss the onset of QT in our system as a result of its finite-size characteristic, which represents an essential difference to the large superfluid helium systems.

Chapter 6 shows the thermodynamic analysis in terms of those new global variables, namely *volume parameter* ($\mathcal{V}$) and *pressure parameter* (Π). In Section 6.1 we describe, particularly for a harmonic trap, the theoretical aspects involved in defining these new global variables. Then, Section 6.2 explains the methodology followed to extract the values of Π from the experimental data. In Section 6.3, the BEC transition in our system is described in terms of a phase diagram of Π versus temperature. At the end (Section 6.4), we present a calculation of the global heat capacity at constant $\mathcal{V}$, which represents an alternative to usual methods that rely on local density approximation.

Finally, the conclusions are summarized in Chapter 7.

2 Bose–Einstein condensation

In few words, and therefore, crudely speaking, I would generally give the following definitions for Bose–Einstein condensation (BEC):

- the macroscopic occupation of a single–particle state.
- quantum degeneracy (for bosonic systems).
- the overlap of single–particle wave packets.

Considering only the Bose statistics, one can fairly expect a large ground state occupation when the temperature T of a bosonic system is such that $k_B T \sim \epsilon_0$, where k_B is the Boltzmann constant and ϵ_0 is the ground state energy. The rather surprising aspect of BEC in this context is that it may take place at temperatures many times higher than ϵ_0/k_B , or even not happen at all (even at $T = 0$).

In this Chapter we intend to present a theoretical overview of the BEC phenomenon, and discuss some of its main concepts, particularly the ones related to the subsequent chapters.

In Section 2.1 we discuss the usual definitions of BEC. Then, in Section 2.2 we describe the ideal BEC (i.e. BEC in a non–interacting Bose gas) in an arbitrary external potential, for which a rigorous quantitative definition of the BEC transition is possible by assuming the thermodynamic limit. And finally, in Section 2.3 the weakly interacting BEC is addressed by considering a two–body contact interaction between the bosons, which leads to the celebrated Gross–Pitaevskii equation. Then some of the dynamics of the interacting BEC are developed by deriving the hydrodynamic and the Bogoliubov–de Gennes equations.

2.1 General definitions

There are many qualitative and quantitative ways to define the BEC phenomenon depending on some characteristics of the bosonic system and sometimes the “taste” of the physicist. The original description by Einstein (10) finds a macroscopic occupation of the lowest-energy single-particle state in a spinless non-interacting¹ Bose gas in thermal equilibrium below a certain temperature T_c (or above a certain density n_c). Of course, this is a very restricted description.

A broader definition was given by Penrose and Onsager² in 1956, in which they assign three different regimes based on the comparison between the eigenvalues of the single-particle density matrix $\rho(\mathbf{r}, \mathbf{r}' : t)$ ³ with the total number of bosons, N , in the system (47): if no eigenvalue is of order N , there is no BEC, and the system is considered normal; if one eigenvalue is of order N , there is a “simple BEC”; and if more than one eigenvalue is of order N , the system exhibits “fragmented BEC”. Though very general, this definition still lacks precision due to phrases like “of order N ”.

An alternative definition closely related to the one above was given by Yang in 1962. Instead of diagonalizing $\rho(\mathbf{r}, \mathbf{r}' : t)$, one must analyse its behaviour in the limit of large distances $|\mathbf{r} - \mathbf{r}'| \rightarrow \infty$. If it is non-zero at this limit, the system presents a BEC; if it vanishes, the system is normal. For the former, the system is often said to present “off-diagonal long-range order”. From a practical point of view, this definition is of little importance since in most physical systems, particularly the trapped dilute gas BEC, it is not clear how to take the $|\mathbf{r} - \mathbf{r}'| \rightarrow \infty$ limit.

Also there is an implicit definition of BEC based on the concept of a spontaneous breaking of gauge symmetry. The wave function of the condensate $\Psi_0(\mathbf{r})$ plays the role

¹At that time one could expect that in real physical systems interactions would prevent the phenomenon from happening. Although BEC is unstable against weak attractive interactions, it turns out that in many physical systems interactions may reinforce the tendency to BEC.

²The original description was restricted to the “spinless” case in which the particles have no internal degrees of freedom. But the generalization is straightforward.

³ $\rho(\mathbf{r}, \mathbf{r}' : t)$ may be described as the product of the probability amplitudes at time t to find a particular particle at $\mathbf{r}$ and at $\mathbf{r}'$ averaged over the behaviour of all the other $N - 1$ particles.

of an order parameter, i.e. it vanishes above T_c . It is a complex quantity defined up to a constant phase factor. The multiplication of $\Psi_0(\mathbf{r})$ by a uniform and constant phase factor $e^{i\varphi}$ has no physical significance, reflecting a gauge symmetry (sometimes referred to as “U(1) gauge symmetry”). Therefore an explicit choice for the value of the order parameter means a formal breaking of gauge symmetry.

Strictly speaking, a rigorous quantitative definition can be made only in certain kinds of systems in which it is possible to take an appropriate thermodynamic limit. Then the phrase “eigenvalue of order N ” becomes “eigenvalue over N constant when $N \rightarrow \infty$ ”. In the next sections we will present some of the main results for these cases.

2.2 Ideal Bose gas

There are numerous textbooks (47–50) and theses (51, 52) developing the fundamentals on ideal BEC. Therefore, here, we intend to discuss only the main aspects of an ideal spinless Bose gas in an arbitrary external potential.

The ideal BEC can be simply described by the equation below

$$\frac{N_0}{N} = \begin{cases} 0 & (T > T_c) \\ 1 - \frac{1}{N} \int_0^\infty g(\epsilon) f_{BE}(\epsilon) d\epsilon & (T \leq T_c) \end{cases}, \quad (2.1)$$

where N_0 is the number of atoms in the ground state (“condensed atoms”), and N is the total number of atoms. The function $g(\epsilon)$ is the density of states (in the ϵ -space). The function $f_{BE}(\epsilon)$ is the Bose–Einstein distribution

$$f_{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}, \quad (2.2)$$

which corresponds to the expected value of the occupation number of a state with energy ϵ for a given temperature T ($\beta = 1/(k_B T)$) and chemical potential μ . In principle $f_{BE} = f_{BE}(\epsilon_i)$ for each state i . Its continuous form and the subsequent integral in Equation 2.1 can only be taken in the **thermodynamic limit**, which is a necessary condition to rigorously quantify the phenomenon as mentioned in Section 2.1.

In order to f_{BE} be positive and finite, μ must be smaller than the lowest energy, ϵ_{min} . For a sufficiently large number of particles⁴, the zero-point energy may be neglected which, within this approximation, implies $\mu < 0$.

The interpretation of Equation 2.1 is straightforward: if temperature T is higher than the critical one, T_c , all atoms are distributed among the excited states and the integral is identically equal to the total number N ; otherwise, the integral “saturates”⁵ at a lower number and the remaining atoms accumulate in the ground state⁶. It is important to note that whether or not T_c assumes a non-zero value depends crucially on the form of $g(\epsilon)$.

In d dimensions, the density of states in a box with lengths L_i is

$$g_{box}^{(d)}(\epsilon) = k_d \frac{d}{2} \prod_i \left[\frac{L_i}{2\pi} \left(\frac{2m}{\hbar^2} \right)^{\frac{1}{2}} \right] \epsilon^{(\frac{d}{2}-1)} \quad (2.3)$$

where k_d is the volume (area or length) factor in 3D (2D or 1D) given by $4\pi/3$ (π or 2).

In the same way, for a harmonic oscillator potential with frequencies ω_i , $g(\epsilon)$ writes

$$g_{ho}^{(d)}(\epsilon) = \frac{\epsilon^{d-1}}{(d-1)! \prod_i \hbar \omega_i}. \quad (2.4)$$

One can generally write $g(\epsilon)$ in the form

$$g(\epsilon) = C_\alpha \epsilon^{\alpha-1}, \quad (2.5)$$

where C_α is a constant.

Therefore, from Equations 2.1 and 2.2 the critical condition yields

$$N = \int_0^\infty g(\epsilon) \frac{1}{e^{\epsilon/k_B T_c} - 1} d\epsilon, \quad (2.6)$$

which combined with Equation 2.5 gives the expression

$$(k_B T_c)^\alpha = \frac{N}{C_\alpha \Gamma(\alpha) \zeta(\alpha)} \quad (2.7)$$

⁴The correction ΔT_c in T_c due to the finite number of particles N is independent of N . Since T_c grows with N , the large N approximation means $\Delta T_c/T_c < 1$.

⁵The integral assumes its highest value, i.e. the chemical potential μ becomes zero.

⁶Note that the ground state does not contribute to the integral since $g(\epsilon_{gr} = 0) = 0$.

for the critical temperature. Here Γ and ζ are respectively the gamma and Riemann zeta functions.

For $T < T_c$, it follows the same expression but only for the atoms in the excited states, i.e.

$$(k_B T)^\alpha = \frac{N_{exc}}{C_\alpha \Gamma(\alpha) \zeta(\alpha)}. \quad (2.8)$$

Since $N_0/N = 1 - N_{ext}/N$, the condensate fraction is simply given by

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^\alpha. \quad (2.9)$$

To illustrate some cases of more practical importance (at least, for this thesis), in a 3D harmonic confinement, α is 3 and the critical temperature is given by

$$k_B T_c^{\{3D, ho\}} = \frac{\hbar \bar{\omega} N^{1/3}}{\zeta(3)^{1/3}} \approx 0.94 \hbar \bar{\omega} N^{1/3}, \quad (2.10)$$

where $\bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$ (geometric mean).

For a 3D box of volume V , α is 3/2, and one finds

$$k_B T_c^{\{3D, box\}} = \frac{2\pi}{\zeta(3/2)^{2/3}} \frac{\hbar^2 n^{2/3}}{m} \approx 3.31 \frac{\hbar^2 n^{2/3}}{m}, \quad (2.11)$$

where $n = N/V$ is the number density.

Rearranging Equation 2.11, we can write

$$n \left(\frac{2\pi \hbar^2}{m k_B T_c} \right)^{3/2} = \zeta(3/2) \approx 2.612, \quad (2.12)$$

which has a much deeper meaning. Now the condition for ideal BEC in a 3D box can be written as $\varpi \equiv n \lambda_{dB}^3 \geq 2.612$, where ϖ is the phase-space density and λ_{dB} is the thermal de Broglie wavelength. Also we can now understand the BEC phenomenon as the overlap of single-particle wave packets in the sense that it happens when the single-particle wave packet size, λ_{dB} , is comparable to the interparticle distance $(V/N)^{1/3}$.

2.3 Weakly interacting Bose gas

Again, since the theory on weakly interacting Bose gas was developed in so many references (see the beginning of the previous section), this section intends only to introduce the Gross-Pitaevskii equation (GPE), its hydrodynamic form and the Bogoliubov approximation.

The relatively high densities in condensed clouds make interaction effects very important in most BEC systems. However, since the usual interparticle distances are still much larger than the range of the interactions, most atomic clouds are dilute enough such that only two-body collisions are relevant. That is why the GPE succeeded in explaining BEC in dilute gases and failed (quantitatively speaking) in its original goal to explain the condensed phase of ^4He -liquid, in which interactions are much stronger.

2.3.1 Gross-Pitaevskii equation

The Gross-Pitaevskii equation (GPE) is basically the Hartree equation applied to a fully condensed large bosonic system. Assuming that all bosons occupy the same state, the many body wave function may be written in the time-independent form as

$$\Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = \prod_{i=1}^N \phi(\vec{r}_i), \quad (2.13)$$

where $\phi(\vec{r}_i)$ is some normalized single-particle wave function and not necessarily the correspondent Hamiltonian eigenfunction. In fact $\phi(\vec{r}_i)$ must be adjusted to minimize the expected value of the total energy as we shall see. The system's Hamiltonian may be written as

$$H = H_0 + H_{int}, \quad (2.14)$$

where the non-interacting Hamiltonian (kinetic energy plus external potential $V(\vec{r}_i)$) is

$$H_0 = \sum_{i=1}^N \left[\frac{\vec{p}_i^2}{2m} + V(\vec{r}_i) \right], \quad (2.15)$$

and the two-body interaction contribution is taken in the contact form

$$H_{int} = U_0 \sum_{i < j} \delta(\vec{r}_i - \vec{r}_j). \quad (2.16)$$

Here the interaction coupling constant $U_0 = 4\pi\hbar^2 a_s/m$, where a_s is the s-wave scattering length, assumed to be positive (repulsive interaction). Of course, the true interparticle potential has a much more complicated dependence on the interparticle distance. However, for low-energy bosons, it is possible to consider only the s-wave contribution to the scattering process which allows the creation of an effective potential fully described by one single quantity, namely a_s .

Therefore the interaction energy, i.e. the expected value of H_{int} in the state described by Equation 2.13, will be

$$E_{int} = \frac{N(N-1)}{2} U_0 \int |\phi(\vec{r})|^4 d\vec{r}. \quad (2.17)$$

The fraction $N(N-1)/2$ corresponds to the number of different pairs in a set of N objects. Here $N \gg 1$, which allow us to make the approximation $N(N-1)/2 \approx N^2/2$.

With the definition of the order parameter as

$$\psi(\vec{r}) = N^{1/2} \phi(\vec{r}), \quad (2.18)$$

one can rewrite Equation 2.17 as

$$E_{int} = \frac{U_0}{2} \int |\psi(\vec{r})|^4 d\vec{r}. \quad (2.19)$$

The total energy is $E = E_0 + E_{int}$, where E_0 is the correspondent expected value of H_0 . At this point one must minimize E with respect to its independent variables $\psi(\vec{r})$ and $\psi^*(\vec{r})$ (or equivalently $\Re\{\psi\}$ and $\Im\{\psi\}$), but with the restriction that N is conserved. The correspondent Lagrange multiplier to this problem is identically equal to the chemical potential, since $\delta(E) - \mu\delta(N) = 0$. Then minimizing $E - \mu N$ with respect to $\psi^*(\vec{r})$ results in

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) + U_0 |\psi(\vec{r})|^2 \psi(\vec{r}) = \mu \psi(\vec{r}), \quad (2.20)$$

which is the time-independent Gross-Pitaevskii equation (GPE) ⁷.

⁷It is possible to derive the GPE within the second quantization scheme (52–54).

The resemblance between Equation 2.20 and the Schrödinger equation is immediate. The GPE is the Schrödinger equation with the potential given by the sum of the external potential with a non-linear term $U_0 |\psi(\vec{r})|^2$, which represents the mean field created on each boson by the rest of the system. However, an important difference is the eigenvalue, which for the GPE is no longer the energy per particle ϵ but the chemical potential μ . In the absence of interactions, i.e. $U_0 = 0$, one finds $\mu = \epsilon$.

Therefore, by analogy with the time-dependent Schrödinger equation one can expect that the time-dependent GPE will be

$$i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}, t) + V(\vec{r}, t) \psi(\vec{r}, t) + U_0 |\psi(\vec{r}, t)|^2 \psi(\vec{r}, t). \quad (2.21)$$

At this point it is worth mentioning the most common approximation to analytically solve the GPE, which is based on the assumption that in Equation 2.20 the kinetic energy contribution may be neglected when compared to the other terms. This is called Thomas–Fermi approximation (TFA) and turns Equation 2.20 into the algebraic equation

$$\left(V(\vec{r}) + U_0 |\psi(\vec{r})|^2 \right) \psi(\vec{r}) = \mu \psi(\vec{r}), \quad (2.22)$$

which simply results in

$$|\psi(\vec{r})|^2 = \frac{1}{U_0} [\mu - V(\vec{r})], \quad (2.23)$$

when $\mu \geq V(\vec{r})$ and $|\psi(\vec{r})|^2 = 0$ otherwise. Therefore, within the TFA, the BEC spatial density distribution depends only on the spatial dependence of the external potential. This leads to one striking signature of BEC in most experimental systems. Since the density distribution of a thermal (non-condensed) cloud is usually almost Gaussian, the onset of condensation produces an overall bimodal profile which can be directly observed. It is interesting to note that the external potential causes this spatial phase separation, just like gravity does in an usual gas–liquid condensation.

The time-dependent GPE (Equation 2.21) will naturally rule the dynamics of a BEC. The following subsections show how to extract some of the physical content of this equation.

2.3.2 Hydrodynamic equations

A hydrodynamic continuity equation can be easily derived from the time-dependent GPE (Equation 2.21) by multiplying it by $\psi^*(\vec{r}, t)$ and subtracting the complex conjugate of the resulting equation. This operation yields

$$\frac{\partial |\psi|^2}{\partial t} + \nabla \cdot \left[\frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*) \right] = 0, \quad (2.24)$$

which can be rewritten in the form

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \vec{v}) = 0, \quad (2.25)$$

where we identified the density as $n = |\psi|^2$ and the velocity field as

$$\vec{v} = \frac{\hbar}{2mi} \frac{(\psi^* \nabla \psi - \psi \nabla \psi^*)}{|\psi|^2}. \quad (2.26)$$

Now let us redefine the order parameter in Equation 2.18 in terms of an amplitude f and a phase ϕ , i.e.

$$\psi = f e^{i\phi}. \quad (2.27)$$

Then, $n = f^2$, and the velocity field is simply reduced to

$$\vec{v} = \frac{\hbar}{m} \nabla \phi, \quad (2.28)$$

which shows an important result: the velocity field in a BEC is proportional to the gradient of its phase. Also it follows that in the absence of a singularity, the condensate must be irrotational, since

$$\nabla \times \vec{v} = \frac{\hbar}{m} \nabla \times \nabla \phi = 0. \quad (2.29)$$

In the presence of a singularity, e.g., the core of a vortex line as we shall see in Section 3.2, certainly $\nabla \times \vec{v} \neq 0$. But still one can say that a BEC has a more restricted motion than a classical fluid, which usually simplifies its description.

One can also derive an explicit function for the phase ϕ in terms of n and $\vec{v}$. By inserting the order parameter, Equation 2.27, into the GPE (Equation 2.21), we find

$$\hbar \frac{\partial}{\partial t} \phi + \left(\frac{1}{2} m \vec{v}^2 + V + U_0 n - \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} \right) = 0. \quad (2.30)$$

It is interesting to note that the Planck constant $\hbar$ appears only in the last term, which contains the gradient of the density. This term is called *quantum pressure*, and reveals that quantum aspects are more relevant in non-uniform gases.

Also from Equation 2.30 we can better understand the Thomas–Fermi approximation (TFA) introduced before (Equation 2.23). The approximation actually consists in neglecting the *quantum pressure* term in Equation 2.30 provided that density variations are small enough. In other words the approximation applies when the typical distance characterizing the density variations are large compared to a distance ξ , in which the contributions of the density variations and the interaction in Equation 2.30 are the same. Therefore

$$\xi = \frac{\hbar}{\sqrt{2mU_0n}}. \quad (2.31)$$

This length quantity has a deeper physical meaning. Consider a BEC for which the TFA applies, and then subject the gas to a localized perturbation. Close to the perturbation TFA is no longer valid. But far from it, say for distances larger than ξ , TFA applies again, and the wave function returns to its bulk value given by Equation 2.23. That is the reason why ξ is called *healing length*⁸, and it gives the characteristic size of localized perturbations in a BEC.

It is worth mentioning that all the physical content of the GPE are contained in the hydrodynamic equations, i.e. they are equivalent. Moreover, both reflect the conservation of energy and number of particles.

2.3.3 Bogoliubov–de Gennes equations

Another interesting way to analyse the dynamics of a BEC consists in linearize the time-dependent GPE by considering small perturbations in the BEC wave function. This means that one must write $\psi(\vec{r}, t)$ as $\psi(\vec{r}, t) = \psi_0 + \delta\psi$ in Equation 2.21, and keep

⁸Sometimes also referred to as *coherence length*, though in a different sense than the term employed in optics.

only the terms linear in $\delta\psi$ and its complex conjugate, which yields the following pair of equations:

$$-\frac{\hbar^2}{2m}\nabla^2\delta\psi(\vec{r},t) + V(\vec{r})\delta\psi(\vec{r},t) + U_0 \left[2|\psi_0(\vec{r},t)|^2\delta\psi(\vec{r},t) + \psi_0(\vec{r},t)^2\delta\psi^*(\vec{r},t) \right] \quad (2.32)$$

$$= i\hbar\frac{\partial\delta\psi(\vec{r},t)}{\partial t},$$

and

$$-\frac{\hbar^2}{2m}\nabla^2\delta\psi^*(\vec{r},t) + V(\vec{r})\delta\psi^*(\vec{r},t) + U_0 \left[2|\psi_0(\vec{r},t)|^2\delta\psi^*(\vec{r},t) + \psi_0^*(\vec{r},t)^2\delta\psi(\vec{r},t) \right] \quad (2.33)$$

$$= -i\hbar\frac{\partial\delta\psi^*(\vec{r},t)}{\partial t}.$$

The function $\psi_0(\vec{r},t)$ is the non-perturbed BEC wave function given by $\psi_0 = \sqrt{n(\vec{r})}e^{-i\mu t/\hbar}$. Apart from the overall phase factor $e^{-i\mu t/\hbar}$ present in the non-perturbed case, it is convenient to seek for solutions periodic in time, which suggests an ansatz in the form

$$\delta\psi(\vec{r},t) = e^{-i\mu t/\hbar}[u(\vec{r})e^{-i\omega t} - v^*(\vec{r})e^{i\omega t}]. \quad (2.34)$$

Provided that $\omega \neq 0$, after inserting this ansatz into Equations 2.32 and 2.33, one must equate to zero the coefficients of $e^{i\omega t}$ and $e^{-i\omega t}$, which results in the coupled equations for $u(\vec{r})$ and $v(\vec{r})$:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) + 2n(\vec{r})U_0 - \mu - \hbar\omega \right] u(\vec{r}) - n(\vec{r})U_0v(\vec{r}) = 0 \quad (2.35)$$

and

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) + 2n(\vec{r})U_0 - \mu + \hbar\omega \right] v(\vec{r}) - n(\vec{r})U_0u(\vec{r}) = 0. \quad (2.36)$$

known as Bogoliubov–de Gennes equations.

In a uniform system the translational invariance suggests that both $u(\vec{r})$ and $v(\vec{r})$ must be proportional to $e^{i\vec{q}\cdot\vec{r}}$, since in this case, $V(\vec{r}) = 0$ and $n(\vec{r}) = \text{const.} \equiv n$. Then, from Equation 2.23, $\mu = nU_0$. Therefore Equations 2.35 and 2.36 become

$$\left(\frac{\hbar^2 q^2}{2m} + nU_0 - \hbar\omega \right) u_q - nU_0v_q = 0 \quad (2.37)$$

and

$$\left(\frac{\hbar^2 q^2}{2m} + nU_0 + \hbar\omega \right) v_q - nU_0u_q = 0, \quad (2.38)$$

where u_q and v_q are the respective proportionality factors of $u(\vec{r})$ and $v(\vec{r})$.

The homogeneous system of Equations 2.37 and 2.38 will only present a non-trivial solution if

$$\left(\frac{\hbar^2 q^2}{2m} + nU_0 + \hbar\omega\right) \left(\frac{\hbar^2 q^2}{2m} + nU_0 - \hbar\omega\right) - n^2 U_0^2 = 0. \quad (2.39)$$

Replacing the momentum $p = \hbar q$, and solving Equation 2.39 for $\hbar\omega(p)$, it results in

$$\hbar\omega(p) = \left[\left(\frac{p^2}{2m}\right)^2 + \frac{nU_0}{m} p^2 \right]^{1/2}, \quad (2.40)$$

which is the celebrated Bogoliubov spectrum (or dispersion law) of elementary excitations. It was first derived in 1947 by Bogoliubov (55) in the context of the microscopic theory of the Bose gas, within which it becomes clear that these elementary excitations of an interacting Bose gas can be described in terms of gas of non-interacting quasi-particles. It is also possible to find Equation 2.40 by linearizing the hydrodynamic equations.

If we define a characteristic velocity as $c = \sqrt{nU_0/m}$, for the regime of small momenta, $p \ll mc$, the dispersion law takes the phonon-like form

$$\hbar\omega(p) = cp, \quad (2.41)$$

which allow us to identify c as the sound velocity. Therefore, long wavelength excitations in a BEC correspond to sound waves. For the opposite regime, $p \gg mc$, the dispersion law approaches the relation for a free particle as we find

$$\hbar\omega(p) = \frac{p^2}{2m} + nU_0. \quad (2.42)$$

The transition between the phonon and the particle regimes defines a characteristic interaction length ξ . This happens for $p = \hbar/\xi$, where ξ is the *healing length* discussed in Section 2.3.2.

3 Superfluidity and turbulence

Superfluidity is a phenomenon that actually accounts for a number of phenomena that normally go together, e.g. non-classical rotational inertia (Hess-Fairbank effect), persistent currents, quantized vortices, reduced friction on moving impurities, etc. The term was coined by Kapitza (11) after his group in Moscow (and independently Allen and Misener (12) in Cambridge) observed in 1938 that the viscosity of ^4He -liquid gets reduced to unmeasurable values when it transitions from He-I to He-II phases. So in the He-II phase, the liquid flows without any dissipation just like the electric current in the superconducting phase of a metal. In the same year London (13) proposed that this phenomenon was due to the onset of BEC in He-II phase. Although this idea was greatly accepted from the start, a macroscopic occupied quantum state in He-II is experimentally difficult to be observed in contrast with BEC in dilute gases. Some experiments and theoretical calculations (47) suggest a finite (6-8%) condensate fraction in He-II phase at $T = 0$, which, similarly to Equation 2.9, goes to zero at the critical temperature. Ironically, those superfluidity properties are usually easier to detect in ^4He -liquid than in dilute gases.

The relation between BEC and superfluidity has proved to be a subtle one. BEC does not imply superfluidity, since the ideal BEC described in Section 2.2 is a counter example (see Section 3.1). The reverse is also false since superfluidity occurs in some lower-dimensional systems which does not allow the formation of a true BEC¹. But the two phenomena are certainly related.

The first explanation for superfluidity was developed by Landau in 1941. Section 3.1 describes this theory known as Landau criterion of superfluidity, since it estab-

¹Here I refer to quasi-condensates in which there is a mesoscopic phase coherence, i.e large compared to microscopic distances but small compared to the size of the system.

lishes a critical velocity for the superfluid flow, above which the fluid ceases to exhibit superfluidity.

For the scope of this thesis, the most important superfluidity property is the occurrence of quantized vortices referred to as quantum vortices. In Section 3.2 we show the basic structure of a vortex line in a BEC, which is intrinsically related to the phenomenon of turbulence in superfluids (named quantum turbulence). Section 3.3 then gives an overview on both classical and quantum turbulence.

3.1 Landau criterion

In 1941 Landau considered that viscosity is the manifestation of elementary excitations through which a fluid may start to lose its kinetic energy when its velocity is large enough for those excitations to happen. That leads to a critical velocity above which the fluid loses its superfluidity aspect.

Let us consider a uniform fluid flowing along a capillary at constant velocity v . If it exhibits superfluidity both its energy $1/2Mv^2$ in the capillarity frame and its zero energy in a frame moving with the fluid will remain the same. But if an elementary excitation (see Section 2.3.3) takes place, the energy in the moving frame becomes $E' = \epsilon(p)$ and the momentum $\vec{P}' = \vec{p}$. In the capillarity frame, by means of a Galilean transformation, the energy must write

$$E = E' + \frac{(\vec{P}' + M\vec{v})^2}{2M} = \epsilon(p) + \vec{p} \cdot \vec{v} + \frac{1}{2}Mv^2 + \frac{p^2}{2M} \approx \epsilon(p) + \vec{p} \cdot \vec{v} + \frac{1}{2}Mv^2, \quad (3.1)$$

where M is the total mass of the system, which allows to neglect the term $p^2/(2M)$. Since the energy before the excitation was $Mv^2/2$, the term $\epsilon(p) + \vec{p} \cdot \vec{v}$ must be negative for the excitation to produce dissipation of energy and a consequent heating of the fluid. Therefore, when

$$\epsilon(p) + \vec{p} \cdot \vec{v} < 0, \quad (3.2)$$

one should not expect superfluidity effects. In other words superfluidity holds if

$$v < v_c = \min \left(\frac{\epsilon(p)}{p} \right), \quad (3.3)$$

where v_c is referred to as Landau critical velocity.

For an ideal gas $\epsilon(p) = p^2/(2m)$, which results in

$$v_c^{(ideal)} = \min \left(\frac{p}{2m} \right) = 0. \quad (3.4)$$

Therefore, although BEC is possible in an ideal (3D) Bose gas, it will not present superfluidity. In Section 2.3.3 we saw that a weakly interacting Bose gas presents, for small momenta, a Bogoliubov dispersion relation given by $\epsilon(p) = cp$, where c is the sound velocity. Then, for this case,

$$v_c^{(weak-int)} = c, \quad (3.5)$$

i.e., if the flow velocity of a weakly interacting BEC is smaller than the sound velocity in it, we could expect the flow to present superfluidity effects. However, the formation of vortex lines induce an effective viscosity, and the fluid may lose its superfluidity characteristic at velocities much smaller than the correspondent sound velocity.

3.2 Quantum vortices

In Section 2.3.2, through the hydrodynamic equations of a BEC, we saw that there are certain restrictions for the flow of a BEC compared to the one of a classical fluid. Particularly a BEC flow is irrotational in the absence of a singularity in the phase of its wave function. But vortices do happen in weakly interacting BECs, and moreover, they present quantized angular momentum as they do in helium superfluid. Also turbulence occurs in weakly interacting BECs, and just like in helium superfluid, it presents remarkable similarities to the classical phenomenon as we will see in Section 3.3.2.

The following discussion is based mostly in References (47, 50). In classical fluid dynamics the flow of an ideal fluid in local thermodynamic equilibrium and in the absence

of external forces may be fully characterized in terms of its density, temperature and velocity field $\vec{v}$. The vorticity is defined as

$$\vec{\omega} \equiv \nabla \times \vec{v}, \quad (3.6)$$

and the circulation around a closed curve $\mathcal{C}$ is

$$\Gamma \equiv \oint_{\mathcal{C}} \vec{v} \cdot d\vec{l}. \quad (3.7)$$

Of course, by Stokes's theorem, if $\vec{v}$ has no singularities within $\mathcal{C}$, $\omega = 0$ implies $\Gamma = 0$.

The vorticity of a vortex profile will depend on how $\vec{v}$ changes with the distance to the vortex line, ρ . In cylindrical coordinates (ρ, θ, z) , if the fluid has a rigid-body vortex² in the z -axis, the velocity field is $\vec{v} = \Omega \rho \hat{\theta}$, where Ω is the angular velocity, uniform in this case. Then we find $\vec{\omega} = 2\Omega \hat{z}$.

In some cases, we may have $\omega = 0$ everywhere in the fluid, except on a particular line, where it is singular in such a way that Γ has a finite value when $\mathcal{C}$ encloses this singular line, referred to as irrotational vortex. In this case, it can be shown (56) that (a) the singular line cannot terminate in the fluid, i.e. it must either close on itself (vortex ring) or terminate at the boundaries (vortex line), and (b) Γ is independent of the contour $\mathcal{C}$ provided that it encloses the singular line.

For a vortex line in the center of a cylindrically symmetric system, Equation 3.7 gives

$$\vec{v} = \frac{\Gamma}{2\pi} \frac{\hat{\theta}}{\rho}, \quad (3.8)$$

and indeed $\nabla \times \vec{v} = 0$ for $\rho \neq 0$. Since this velocity diverges when $\rho \rightarrow 0$, at this point we will postulate a vortex core radius ξ , without further explanation. Suppose the fluid has a uniform density ρ_m , and on the vortex core, the density of the fluid vanishes, and we may neglect variations on ρ_m for $\rho > \xi$, which allow us to write the vortex energy simply as the kinetic energy of the flow:

$$E_{vort} = \frac{1}{2} \rho_m \int v^2 d^3r, \quad (3.9)$$

²This happens when a reservoir containing a classical fluid rotates at constant angular velocity.

where the integral is performed all over the fluid except along the core of the vortex line. Consider that the fluid is confined in a cylindrical reservoir of radius R . Then, it can be easily shown that the vortex energy per unit length will be given by

$$\varepsilon = \frac{1}{2} \rho_m \frac{\Gamma^2}{2\pi} \ln \left(\frac{R}{\xi} \right) \quad (3.10)$$

In the same way, for a pair of parallel vortices (same sign of circulation) set a distance $r \gg \xi$ apart, we find

$$\varepsilon^{\uparrow\uparrow}(r) = \rho_m \frac{\Gamma^2}{2\pi} \ln \left(\frac{R^2}{\xi r} \right), \quad (3.11)$$

while for a pair of antiparallel vortices (opposite sign of circulation) provided that $r \ll R$, we find

$$\varepsilon^{\downarrow\uparrow}(r) = \frac{1}{2} \rho_m \frac{\Gamma^2}{2\pi} \ln \left(\frac{r}{\xi} \right). \quad (3.12)$$

Although there are many approximations in deriving Equations 3.10, 3.11 and 3.12, they provide two general qualitative results. First, a pair of parallel vortices with circulation Γ is a energetically favourable situation when compared to one vortex with circulation 2Γ , and, hence, the latter may break up into the former. Second, parallel (antiparallel) vortices repel (attract) each other.

Let us now focus in the case of a BEC. As shown in Section 2.3.2, the velocity field within a BEC is proportional to the gradient of the phase of the condensate wave function (Equation 2.28), which leads to zero vorticity unless there is a singularity in the velocity field. This is just like the case of irrotational vortices described above. Since the wave function must have a single value at each point, the phase variation $\Delta\phi$ along any closed contour must be multiple of 2π , i.e.

$$\Delta\phi = \oint \nabla\phi \cdot d\vec{l} = 2\pi\ell, \quad (3.13)$$

where ℓ is an integer number. Then, if we insert Equation 2.28 into the definition of circulation (Equation 3.7), we find

$$\Gamma = \oint \vec{v} \cdot d\vec{l} = \frac{\hbar}{m} 2\pi\ell = \ell \frac{h}{m}, \quad (3.14)$$

which shows that the circulation becomes quantized in units of h/m . Equation 3.14 is known as the Onsager–Feynman quantization condition.

From Equation 3.8, one finds that the quantized velocity field in a cylindrically symmetric BEC with a vortex line in the center is given by

$$\vec{v} = \ell \frac{\hbar}{m\rho} \hat{\theta}, \quad (3.15)$$

and, according to the approximate expression for the vortex energy per unit length in Equation 3.10, we find

$$\varepsilon = \pi n \frac{\hbar^2}{m} \ell^2 \ln \left(\frac{R}{|\ell| \xi} \right), \quad (3.16)$$

where, n is the particle density, and m is the particle mass. Both are related to the mass density as $\rho_m = nm$. Also, Equation 3.16 assumes that a vortex with ℓ quanta of circulation has a core size given by $|\ell| \xi$. The considerations concerning the size of the vortex core will become clear soon.

A more accurate calculation of the vortex energy in a BEC can be made by adding the vortex kinetic energy in the expected value of the Hamiltonian of Equation 2.14 in the state described by Equation 2.13. By the definition of the order parameter as in Equation 2.27, the vortex kinetic energy density is given by

$$u = \frac{m}{2} v^2 f^2 = \frac{\hbar^2}{2m} \ell^2 \frac{f^2}{\rho^2}, \quad (3.17)$$

and then, the expected energy writes

$$E = \int d\vec{r} \left\{ \frac{\hbar^2}{2m} \left[\left(\frac{\partial f}{\partial \rho} \right)^2 + \left(\frac{\partial f}{\partial z} \right)^2 \right] + \frac{\hbar^2}{2m} \ell^2 \frac{f^2}{\rho^2} + V(\rho, z) f^2 + \frac{U_0}{2} f^4 \right\}. \quad (3.18)$$

In the same way used to derive the time-independent GPE in Section 2.3, we find from Equation 3.18

$$-\frac{\hbar^2}{2m} \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{\partial^2 f}{\partial z^2} \right] + \frac{\hbar^2}{2m\rho^2} \ell^2 f + V(\rho, z) f + U_0 f^3 = \mu f. \quad (3.19)$$

With Equation 3.19, it is possible to calculate the energy of a vortex line for many different conditions, e.g. in uniform or harmonic oscillator external potentials. However, the results only introduce numerical coefficients close to 1 to the argument of the logarithm

of Equation 3.16. Since this argument is usually much larger than 1, we may say that Equation 3.16 gives the vortex energy to logarithmic accuracy.

Now we can address the matter with the vortex core size assumed earlier to be $|\ell| \xi$. In Equation 3.19 we find that the vortex contribution scales with ρ^{-2} . Therefore, close to the core it dominates the behaviour of the amplitude f (i.e. the condensate density), and far from the center it becomes unimportant. We define the vortex core size ξ_c as the distance in which the vortex and the interaction contributions in Equation 3.19 are the same, which leads to

$$-\frac{\hbar^2}{2m\xi_c^2} = nU_0 \quad \Rightarrow \quad \xi_c \equiv |\ell| \xi = |\ell| \frac{1}{\sqrt{8\pi n a_s}}, \quad (3.20)$$

where we used the s-wave scattering length a_s to describe the interaction instead of the interaction coupling constant U_0 ($U_0 = 4\pi\hbar^2 a_s/m$). The length quantity ξ is not only the core size of a single-charged vortex. It is the *healing length* discussed in Sections 2.3.2 and 2.3.3, and therefore a vortex can be considered a localized perturbation in the BEC wave function.

From Equation 3.20, we can see that the more interacting the quantum fluid the smaller the size of a quantum vortex. This result has important technical implications concerning the (strongly interacting) helium superfluid and the (weakly interacting) dilute BEC. From a theoretical point of view, small sized vortices in helium superfluid allow their dynamical description in terms of the *filament model* (57), in which their internal structure is neglected. In experiments, these vortices are so small that they cannot be optically observed. As for a dilute BEC, the theoretical approach through the GPE is usually much simpler, and vortices can be directly imaged. Figure 1 shows some of these images collected from References (1, 2).

Figure 1 – Direct images of vortices in dilute BECs. (a) From Reference (1) a vortex line is imaged in different directions, and (b) from Reference (2) many vortices were seen arranged in a lattice, known as Abrikosov lattice.

3.3 Turbulence

*“Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity”*

- *Poetic description of turbulence by Lewis Fry Richardson (1881 - 1953)*

Even though the first qualitative description of turbulence was made centuries ago by Leonardo da Vinci (58), the phenomenon remains, in the words of Feynman, “the most important unsolved problem of classical physics”. Nevertheless, many advances were achieved in the description of turbulent phenomena.

The following subsections present an overview on turbulence in both classical and quantum fluids.

3.3.1 Classical turbulence

We shall only introduce the basic notions on classical turbulence (CT) following mostly References (58, 59). We start by presenting a simplified derivation of the Navier–Stokes equation. Newton’s second law applied to a volume element in a fluid readily gives

$$\rho \frac{d}{dt} \vec{u}(x, y, z, t) = f_{net} \quad \Rightarrow \quad \rho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) = f_{net}, \quad (3.21)$$

where ρ is the fluid density, and $\vec{u}$ is the velocity field. The net force per unit volume f_{net} is given by the corresponding external forces f_{ext} (commonly gravity) and internal stresses of the surrounding fluid on the volume element. Then we find

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) = \nabla \cdot \tilde{\sigma} + f_{ext}, \quad (3.22)$$

where $\tilde{\sigma}$ is the (Cauchy) *stress tensor* (also referred to as *pressure tensor*), whose diagonal (off-diagonal) elements are called normal (shear) stresses. The normal stresses define the value of pressure, p , such that

$$p = -\frac{1}{3} \text{Tr } \tilde{\sigma}. \quad (3.23)$$

Equation 3.22 is the general form of the Navier–Stokes equation. The term $(\vec{u} \cdot \nabla) \vec{u}$ is known as convective acceleration or (non-linear) inertial term. In Newtonian fluids and in absence of external forces it assumes the common “incompressible” form

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla p + \frac{\eta}{\rho} \nabla^2 \vec{u}, \quad (3.24)$$

where the last term is the viscosity one with η being the viscosity. By defining a characteristic length r_0 and velocity u_0 , one finds a set of dimensionless variables:

$$\vec{r}' = \frac{\vec{r}}{r_0}; \quad \vec{u}' = \frac{\vec{u}}{u_0}; \quad t' = \frac{tu_0}{r_0}; \quad p' = \frac{p}{\rho u_0^2}.$$

Then the Navier–Stokes equation rewrites

$$\frac{\partial \vec{u}'}{\partial t'} + (\vec{u}' \cdot \nabla') \vec{u}' = -\nabla' p' + \frac{1}{Re} \nabla'^2 \vec{u}', \quad (3.25)$$

where

$$Re = \frac{r_0 u_0 \rho}{\eta} \quad (3.26)$$

is the (classical) Reynolds number. One can see from Equation 3.25 that Re weights the importance of the viscosity term in contrast to the non-linear term. To illustrate its importance let us consider a concrete example. If a fluid flowing at a velocity v finds an obstacle with a transversal section of size L , the corresponding Reynolds number will be $Re = vL\rho/\eta$. The flow of a viscous fluid moving slowly through a small sized object has a small Reynolds number, so the viscous term dominates its dynamics, and the flow is called *laminar*. On the other hand, if Re is large, the dynamics of the flow is ruled by the non-linear term, which allows CT effects to occur.

The difficulty in solving the Navier–Stokes equation for almost all physical systems prevents from assuring that CT can be fully described by it. Nevertheless, it gives good insights on when CT is expected to occur, and moreover how it evolves. Turbulence is characterized by the formation of eddies of different sizes in a fluid. The *Richardson cascade* picture addresses this observation in terms of the Navier–Stokes equation as we explain. When the condition for turbulent flow is reached, first, unstable large eddies are formed. Then they break up into smaller structures, which keep breaking up into smaller and smaller eddies. At the length scale of very small eddies, Re becomes small, such that viscosity effects become important. This process creates a path for energy flow going from large to small eddies until it is dissipated.

Understanding how the energy is distributed among the many length scales is not a trivial task. However by making some assumptions, Kolmogorov’s theory (60,61) could successfully address this matter for a certain range of length scales. It postulates that in an intermediate range of length scales, one can consider turbulence to be homogeneous and isotropic, and the energy transferring to happen at a constant rate per unit mass, ε . Then the celebrated Kolmogorov energy spectrum is found to be

$$E(k) = C\varepsilon^{2/3}k^{-5/3}, \quad (3.27)$$

where k is the modulus of the wavevector in the Fourier transform of the velocity field,

$\vec{u}(\vec{k}) = \int \vec{u}(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d\vec{r}$. The universal constant C is experimentally found to be ≈ 1.5 (62).

The intermediate range of length scales in which Equation 3.27 is expected to be valid is known as inertial range, and corresponds to $k_K^{-1} \ll k^{-1} \ll k_D^{-1}$, where D is the length of the largest eddies, and k_K^{-1} is the Kolmogorov dissipation length. As the name suggests, at length scales $\sim k_K^{-1}$, the Reynolds number $Re \sim 1$, and viscosity dissipates the energy. Therefore, one finds

$$k_K^{-1} = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}, \quad (3.28)$$

where ν is the kinematic viscosity defined as $\nu \equiv \eta/\rho$ (see Equation 3.24).

The universality of the Kolmogorov spectrum stems from its (broad) validity regardless of boundary conditions or the mean flow field. Additionally the viscosity and density of the fluid merely define where the inertial range ends. From Reference (3) we reproduce in Figure 2 a compilation of measurements of the normalized energy spectrum of turbulent flow found in several physical systems: boundary layers, wakes, grids, ducts, pipes, jets and oceans. The plot stresses the universal power law behaviour ($\sim k^{-5/3}$). Nevertheless, in spite of this great success, there are experimental evidences (63) that the assumption of homogeneity (seen as statistical self-similarity) in the inertial range may not be fully valid. Then both the constant C and the power law would not be quite universal.

3.3.2 Quantum turbulence

Below we present some aspects of quantum turbulence (QT). For a deeper discussion on this topic, see References (64,65).

In the quantum context, turbulence is characterized by a tangle of (quantum) vortices in a superfluid as first proposed by Feynman in 1955 (14). Soon after that, between 1956 and 1958, W. Vinen and H. Hall produced QT in superfluid helium (15–18,66). In

Figure 2 – Compilation of data on the normalized energy spectrum of turbulent flow found in several physical systems: boundary layers, wakes, grids, ducts, pipes, jets and oceans. The plot demonstrates the universality of the Kolmogorov spectrum, and particularly the power law behaviour $\sim k^{-5/3}$. In the x-axis, η is the Kolmogorov dissipation length (not the viscosity). This image was taken from Reference (3).

Figure 3 – Numerical simulation of the vortex reconnection process involving two initially straight vortex lines that end up sharply kinked. This image was taken from Reference (4).

their experiment, a turbulent flow occurred by inducing a thermal counterflow in which the normal and superfluid components flow in opposite directions³.

In other experiments QT was also observed in superfluid helium by having counter-rotating disks (19), and by oscillating a grid (20) within the superfluid. Both experiments surprisingly found the Kolmogorov energy spectrum (Equation 3.27) in a certain range of length scales.

Therefore, the *Richardson cascade* picture may also apply to QT but with an important difference from its classical counterpart. The dissipation mechanism in a superfluid certainly cannot be explained in terms of the Navier–Stokes equation as in the classical case. In the latter, viscosity is responsible for energy dissipation. In a superfluid there is no viscosity, and the only other obvious dissipation mechanism would consider the normal component. However QT dissipates even at very low temperatures (normal fraction less than 2%). This aspect of QT remains an open problem.

The most accepted hypothesis on how the QT cascade is formed and dissipated is based on a process known as *vortex reconnection* (67), which can be basically explained as follows. In a superfluid the crossing of straight vortex lines results (or may result) in “kinked” vortex lines. A numerical simulation of the *vortex reconnection* process from Reference (4) is shown in Figure 3. A state heavily populated with “kinked” vortex lines will resemble the original QT pictured by Feynman.

As a consequence of the *vortex reconnection* process, the vortex lines oscillate

³In Section 5.3 we explain the vortex nucleation in our BEC system in terms of a counterflow mechanism.

forming helical waves along their extension, known as Kelvin waves. In the presence of a significant normal fraction, Kelvin waves are promptly damped configuring a dissipation mechanism. But at very low temperatures, Kelvin waves undergo non-linear interactions that transfer energy to higher and higher frequency Kelvin waves, which at some point are expected to radiate phonons. This whole picture of the energy flow in a superfluid is somewhat speculative, and further experimental support is required.

QT is better theoretically addressed by different approaches according to the type of the superfluid. Due to the small sized vortices in helium superfluid (see Section 3.2), these systems are well-described by the *filament model* (57), whereas a dilute BEC is ruled by the Gross-Pitaevskii equation (GPE), as stressed in Section 2.3.1.

In References (68, 69) the GPE was numerically solved for a trapped BEC rotating around two perpendicular axes. The authors found that this initial configuration evolves to a QT state that remarkably follows the classical Kolmogorov power law (see Equation 3.27), except for small variations in the value of the constant factor C . In this case the inertial range lies between the *healing length* and the Thomas-Fermi radius. Calculations for 2D systems based on the GPE (70, 71) also found the Kolmogorov power law.

From an experimental point of view, the first evidence of QT in dilute BECs was provided very recently by our research group (27). For this thesis, this observation generated part of the data and discussion, which is presented in Chapter 5.

4 Experimental apparatus

The experimental setup that delivered the data presented in this thesis is called *BEC-I* within our research group, and started being built in 2005. Of course, there were many upgrades since then, but the main idea remains.

Therefore in this chapter I intend to only briefly discuss our experimental setup, since it has been abundantly described in E. Henn’s Ph.D. thesis (51), articles (72, 73) and master theses (74, 75). J. Seman’s Ph.D. thesis (52) also provides information on a very similar setup.

Our apparatus basically produces BEC samples with about 10^5 - 10^6 ^{87}Rb atoms magnetically confined in a cigar–shape harmonic trap. Most of the features in our setup are commonly applied in ultracold atoms experiments.

We start in Section 4.1, by presenting a comprehensive overview on the most important experimental steps to achieve BEC in our system. Then, at the end, Section 4.2 shows how we designed the new vacuum chamber, which probably represents the main upgrade to the original setup (51).

4.1 Overview

Most ultracold atoms experiments, including ours, may be spatially divided in three parts: laser setup, science chamber, and control/acquisition computer(s). The whole process to produce a BEC sample takes roughly one minute in our experiment, and consists of many sequential steps which occur in the science chamber. These steps are programmed

Figure 4 – Scheme of the most important sequential processes that take place in the science (vacuum) chamber in order to produce a BEC sample in our setup. The typical experimental values for each step are shown.

in the control computer¹, and each one may require different laser beams. Finally data are acquired and analysed in other computer(s).

The laser setup is a crucial part, whose importance ranges from most steps in obtaining a BEC to probing the sample. It represents the main technical difference between experiments with different species. However, details on the laser setup will not be presented here, and the reader is referred to the references at beginning of this chapter.

Probably a more instructive way to describe the experiment is by discriminating all processes that occur in the science chamber. Figure 4 shows the most important ones and their typical experimental values in our system. The main parts of the vacuum system are two glass chambers at different pressures. Details on the vacuum design will be presented in Section 4.2. The subsections below follow the scheme of Figure 4.

¹In Appendix B we describe a hardware/software solution for controlling ultracold atoms experiments.

4.1.1 Atoms sources

First we “turn on” our source of Rb atoms, *dispensers* (SAES Getters), which are small metallic reservoirs containing rubidium chromate (Rb_2CrO_4) and a reducing alloy. By passing an electric current, we heat up the substances at a certain temperature ($\approx 600^\circ\text{C}$), which causes a reaction to happen, and, consequently, Rb atoms to evaporate forming a very pure vapour (76).

As a secondary source of atoms we apply the so-called *light-induced atom desorption* (LIAD) (77) technique to remove Rb atoms that stick to the glass cell internal surface. In the past, we used LIAD as the only source, provided that, before, dispensers were kept on for a few hours in order to “load” the surface. This procedure represented a significant decrease in pressure and, hence, a gain in the lifetime of our cold clouds. In our current vacuum system, that is no longer significant, and therefore, we use LIAD with dispensers still on, which allows longer working times.

4.1.2 Double-MOT configuration

Some of the atoms in the Rb vapour released by dispensers and subsequent LIAD are then trapped into a 3D magneto-optical trap (MOT), which we call MOT 1. MOT 1 traps about 10^8 ^{87}Rb atoms at $\approx 200\mu\text{K}$. Figure 5 shows pictures of MOT 1 (a) glass cell and coils, and (b) fluorescence of a trapped sample.

For the standard 3D version, a MOT basically relies on radiation pressure force created by three orthogonal pairs of counter-propagating laser beams to cool down the sample, and a quadrupole magnetic field, which together with the laser beams, spatially confine the atoms. MOTs were first experimentally realized in 1987 (41), and since then, they became widely used either as a source of cold atoms in experiments aiming to achieve

Figure 5 – Pictures of MOT 1 (a) glass cell, quadrupole and compensation coils; and (b) fluorescence of a trapped sample.

degeneracy or as cold sample itself for studying cold collisions and light–matter interaction. For a comprehensive, deep discussion on laser cooling and MOT, we suggest References (78,79) besides all theses cited at the beginning of this chapter.

Since the MOT 1 chamber is where the dispensers are placed, its pressure is too high ($\approx 10^{-8}$ Torr) to perform the next steps. Therefore we create an atomic beam by pushing the atoms with a laser beam to a second glass chamber, whose pressure is much lower ($\approx 10^{-12}$ Torr). In this cleaner region, we trap about 10^9 ^{87}Rb atoms again in a 3D MOT, which we refer to as MOT 2. Similarly to Figure 5, Figure 6 shows pictures of MOT 2 (a) glass cell and coils, and (b) fluorescence of a trapped sample.

Most experiments, either using dispensers or ovens as sources, have a too high background pressure close to the source, and somehow one must create an atomic beam to load a trap in a cleaner region. There are different approaches to circumvent this problem. Our strategy is known as “double–MOT configuration” (80), and it can also be performed using a 2D MOT as MOT 1 (81), which usually generates a higher flux than its 3D counterpart. But perhaps the most common approach is the use of a Zeeman slower (38,82) to decelerate an effusive atomic beam coming from a nozzle attached to an oven.

Figure 6 – Pictures of MOT 2 (a) glass cell, quadrupole and compensation coils; and (b) fluorescence of a trapped sample.

4.1.3 Mode-matching processes

Once MOT 2 is loaded (which takes ≈ 30 s), we must transfer the atoms to the quadrupole magnetic trap (MT) (83). An efficient transference requires the MOT sample to undergo some processes in order to have a good mode-matching with the quadrupole MT, i.e. it must “look” as similar as possible to what the sample should be in the quadrupole MT.

First, we compress the sample by detuning the trapping beams frequency further to the red, and also decrease its power. Second we perform the sub-Doppler cooling technique² by turning off the MOT quadrupole coils, and shifting even further the trapping beam frequency to the red. At this point, we estimate that the temperature of the compressed and sub-Doppler cooled sample is $\approx 50\mu\text{K}$. And finally, a particular magnetically trappable state, $|F = 2, m_F = 2\rangle$, is populated by optical pumping. All these mode-matching processes take a few milliseconds.

²Also known as polarization gradient cooling or Sisyphus cooling (78).

4.1.4 QUIC magnetic trap

We first turn on the MT quadrupole field at a low “catching value” (still for mode-matching), and then “adiabatically” increase (in $\approx 500\text{ms}$) to the final value. At this point we find that, though the efficiency concerning the number of atoms is high, we always heat up the sample to typically $250\mu\text{K}$.

In a pure quadrupole MT, BEC cannot be achieved because the bottom of the potential is a zero-field point, where the $|F = 2, m_F = 2\rangle$ spin state can undergo Majorana transitions (84) to non-trappable states, and consequently be expelled from the MT. To circumvent this problem, many different approaches can be taken, e.g. one can: (a) apply a time-averaged orbiting potential (TOP trap (85)), whose effective potential does not present a zero-field point; (b) shine a laser beam (optical plug (23)) that creates a repulsive potential around the zero-field point; or (c) place additional coils such that the potential minimum lies at a non-zero magnetic field value. Within the last approach, there are two famous configurations known as: cloverleaf (86) and QUIC (Quadrupole-Ioffe Configuration) (87).

In our system we have a QUIC MT, in which to the quadrupole anti-Helmholtz pair, a third coil, called Ioffe, is radially added, which raises the potential bottom to a non-zero magnetic field, called bias field. A drawing of our QUIC trap will be shown in Section 5.1 (Figure 10).

The full resulting trapping potential of a QUIC trap does not present an analytical expression. But, close to its minimum, it can be well-approximated to a harmonic-oscillator potential, which in our case is given by

$$V(x, y, z) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \approx \frac{1}{2}m(\omega_\rho^2 \rho^2 + \omega_z^2 z^2), \quad (4.1)$$

where we considered the cylindrical symmetry $\omega_x \approx \omega_y \equiv \omega_\rho$ ($\approx 2\pi \times 210\text{Hz}$), while ω_z ($\approx 2\pi \times 23\text{Hz}$) is the frequency of the weak confinement direction, which lies in the Ioffe axis.

4.1.5 RF-induced evaporative cooling

As soon as the quadrupole field reaches its final value, we ramp up (in $\approx 300\text{ms}$) the current in the Ioffe coil. When we finally reach the QUIC MT with a bias field of about 1G, the sample is at $\approx 300\mu\text{K}$. Then the evaporative cooling process starts.

Evaporative cooling consists in selectively remove the most energetic atoms in an interacting cloud leading to thermalization at a lower temperature (88,89). In our system, this is done by applying a radio-frequency (RF) radiation to the magnetically trapped cloud (RF-induced evaporative cooling (42)). The bigger the kinetic energy of an atom, the stronger the magnetic field in the potential it can reach, and, consequently, the larger the Zeeman splitting it can have. Therefore, RF radiation with an initially large frequency can couple the Zeeman sublevels of only the most energetic atoms, which transfers them to non-trappable states.

The full evaporative cooling process is dynamical, which means that the RF radiation frequency must be ramped down insofar as the interacting cloud attempts to thermalize. We start at a RF radiation frequency of $\approx 20\text{MHz}$, and ramp it down (in $\approx 20\text{s}$) to $\approx 1\text{MHz}$ in five linear steps such that the whole ramp resembles an exponential decay (see Figure 7). At the end of this process we may have a ^{87}Rb almost pure (condensate fraction of 90%) BEC at ($\approx 100\text{nK}$). The condensate fraction (and temperature) can be adjusted by changing the RF radiation final frequency.

4.1.6 Absorption imaging

Before probing our sample, we turn off the QUIC trap, and let the cloud fall (under gravity) and freely expands for a time known as *time-of-flight* (TOF), which usually ranges from 3 to 30ms. Then we probe the expanded cloud by applying the absorption

Figure 7 – Typical RF-induced evaporation ramp in our setup, turning a sample of 10^9 atoms at 300 μ K into a BEC with 10^5 - 10^6 atoms at 100nK.

imaging technique, which consists in pulsing (in our case, for $\approx 50\mu$ s) a resonant beam through the atoms, and imaging its corresponding intensity profile.

According to the Beer–Lambert law, this 2D intensity profile will be given by

$$I(x, y) = I_0(x, y) \exp \left(-\sigma \int n(x, y, z) dz \right), \quad (4.2)$$

where $I_0(x, y)$ is the initial intensity profile of the probe beam, $n(x, y, z)$ is the 3D density distribution of the probed cloud, and σ is the absorption cross-section.

Therefore, the 3D projected density profile, $n_{2D}(x, y)$, is obtained as

$$n_{2D}(x, y) \equiv \int n(x, y, z) dz = -\frac{1}{\sigma} \ln \left(\frac{I(x, y)}{I_0(x, y)} \right) \equiv \frac{1}{\sigma} OD(x, y), \quad (4.3)$$

where $OD(x, y)$ is the optical density (also called absorbance). The *in situ* trapped cloud presents very high densities (of the order of 10^{14} atoms/cm³), which corresponds to very high optical densities, i.e. very low transmittances ($I/I_0 < 0.1\%$). Therefore the TOF technique allows us to avoid “imaging saturation”.

In the experiment the transmittance in Equation 4.3 is evaluated by acquiring, on a charged-coupled device (CCD) camera, three “pixelated” intensity profiles: the probe

beam that passes through the atoms, I_{atoms} ; just the probe beam in the absence of atoms, I_{probe} ; and, finally, an image of spurious light that also accounts for intrinsic noise in the camera, I_{dark} . Then, the argument of the logarithm in Equation 4.3 will be the normalized intensity profile given by

$$I_{norm} = \frac{I_{atoms} - I_{dark}}{I_{probe} - I_{dark}}, \quad (4.4)$$

and the correspondent matrix $n_{2D}^{(i,j)}$ is the raw data we usually collect from the experiment.

In order to extract information from $n_{2D}^{(i,j)}$, one often performs an appropriate fitting according to the trapping potential and the type of the cloud (thermal, partially or fully condensed). This results in $n_{2D}(x, y)$, which leads to $n_{3D}(x, y, z)$ by either assuming trapping symmetries or precisely knowing the trapping potential.

Since the TOF expansion reflects the momentum distribution of the *in situ* trapped cloud, one can estimate its temperature by knowing n_{3D} and t_{TOF} . The momentum distribution of a trapped thermal cloud can be taken in the classical limit as the (isotropic) Maxwell–Boltzmann distribution, and therefore, its expansion tends to produce an isotropic Gaussian density distribution regardless of the initial shape. Therefore, for long enough t_{TOF} ³, it is easy to show that the temperature can be found as

$$T = \frac{m}{3k_B} \left(\frac{\sigma}{t_{TOF}} \right)^2, \quad (4.5)$$

where σ is the width of the isotropic Gaussian.

Additionally, the TOF technique provides another striking signature of the occurrence of BEC, besides the bimodal profile mentioned in Section 2.3. The expanded thermal cloud will present the isotropic Gaussian density distribution mentioned above. On the other hand, the very quantum nature of a BEC relates the *in situ* spatial confinement to its momentum distribution in each direction. Heisenberg uncertainty relation states that the larger the uncertainty in position the smaller the uncertainty in momentum, and vice versa. Therefore, the expansion of an *in situ* prolate (i.e. cigar-shaped) BEC (like in our case) tends to produce an oblate (i.e. pancake-like) condensate, and vice versa. In other words, when the imaging technique allows to observe both the strong and weak confine-

³In the harmonic trap, this means longer than $2\pi/\bar{\omega}$, where $\bar{\omega}$ is the geometric mean of the frequencies.

ment axes, one can see an inversion in the aspect ratio as the condensate cloud expands. In Section 5.2.2, we will compare the expansions of a thermal cloud, a non-turbulent BEC and a turbulent BEC (Figure 17).

4.2 New vacuum system

Once one has a dilute gas loaded into any type of trap, the lowest the background pressure the better is its lifetime⁴. That is why a good vacuum is crucial in most ultracold atoms experiments, particularly the ones that require long thermalization times in cooling processes.

The new vacuum system in our experiment was constructed in 2011-2012. Figure 8 shows a 3D drawing of it. There are basically three regions corresponding to MOT 1, an intermediate, and MOT 2 regions. A gate valve placed right after the MOT 1 region allows a possible future replacement of the dispensers without “harming” the vacuum in the intermediate and in the MOT 2 regions.

To calculate the pressures in each region we will follow the results presented in Appendix A. Figure 9 shows a schematics of the vacuum system, and highlights the important parameters involved in the calculations. Basically, each region corresponds to a compartment, i , at a pressure, P_i . A large conductance tube, $C_{pump,i}$, connects each compartment to a vacuum pump whose nominal pumping speed is $V_{nom,i}$. Finally, small conductance tubes, $C_{1,2}$ and $C_{2,3}$ connect the compartments to each other.

According to Equation A.8, the effective pumping speed will be

$$\frac{1}{V_i} = \frac{1}{V_{nom,i}} + \frac{1}{C_{pump,i}}. \quad (4.6)$$

⁴Unless the background gas works as a suitable source of atoms to the trap.

Figure 8 – Drawing of the new vacuum system, which can be basically divided in three regions corresponding to MOT 1, an intermediate, and MOT 2 regions.

Figure 9 – Schematics of the vacuum system showing the important parameters involved in the calculation of the pressures.

Then the pressures can be calculated in the matrix form (see Equation A.5)

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} -C_{1,2} - V_1 & C_{1,2} & 0 \\ C_{1,2} & -C_{1,2} - C_{2,3} - V_2 & C_{2,3} \\ 0 & C_{2,3} & -C_{2,3} - V_3 \end{bmatrix}^{-1} \begin{bmatrix} -R_1 \\ 0 \\ 0 \end{bmatrix}, \quad (4.7)$$

where we consider that the only relevant source corresponds to dispensers and LIAD in the MOT 1 region.

By means of the Clausings formula shown in Equation A.9 we could calculate the conductance of the tubes. Room temperature and ^{87}Rb atoms were considered in the calculations. For the vacuum pumps we have $V_{nom1} = V_{nom2} = V_{nom3} = 75\text{L/s}$, which resulted in the effective pumping speeds $V_1 = V_2 = V_3 = 9.6\text{L/s}$. For the connecting tubes we found $C_{1,2} = 0.058\text{L/s}$ and $C_{2,3} = 0.1\text{L/s}$.

Then the differential pressure between the MOT 1 and MOT 2 regions resulted in $P_3/P_1 = 7.87 \times 10^{-5}$. These calculations were performed when designing the system. Later we found a fortunate consistence between them and the measured pressure values.

Finally it is important to mention that we performed a pre-baking procedure in the stainless steel parts of the vacuum systems (90). This technique consists in heating up these parts at a temperature of $\approx 400^\circ\text{C}$ for a few days to remove gases, mostly hydrogen, trapped in the interior of the metal. Then the outgassing is significantly reduced, justifying the assumption $R_2 = R_3 = 0$ in Equation 4.7.

5 Observation of turbulence in a BEC by oscillatory excitations

In this Chapter, we report on the observation of many effects, particularly turbulence, in our BEC system by means of an oscillatory excitation added to the QUIC trap potential discussed in Section 4.1.4. In Section 5.1, the excitation field is described. It is shown that by adding it to the QUIC trap, we are able to not only translate and deform, but also rotate the trapped cloud.

Section 5.2 summarize some of the most interesting observed effects in a simple diagram of the excitation amplitude versus its duration. In the first region (weak amplitudes and short durations), only a tilting characteristic of the *scissors mode* is observed. In the second region, regular (aligned) vortices are nucleated. Increasing further those parameters, QT is observed. Then a different aspect was observed in the last regime, referred to as granulation. Finally, we comment on numerical simulations that support our observations.

A proposal on the vortex nucleation mechanism in our experiment is presented in Section 5.3. By analysing a relative motion between the condensate and thermal components, we could infer the formation of ripples due to the onset of a frictional force. Then, in Section 5.4, the finite-size characteristic of a trapped BEC is considered to explain the transition from a state with regular vortices to QT in terms of the diagram presented in Section 5.2.

Figure 10 – Drawing of the (a) top and (b) side views of the trapping region, where it is shown the imaging beam direction and the excitation coils positions (ac-coils vertically tilted) both in reference to the QUIC coils.

5.1 Excitation field

To nucleate vortices in our BEC, we add to the QUIC trapping potential (Equation 4.1) a (temporally) oscillatory potential generated by an additional pair of coils as shown in Figure 10. We refer to this pair of coils as excitation coils or ac-coils, and they are also connected in anti-Helmholtz configuration.

Consider that the oscillatory current we pass through the ac-coils has the form $I_{ac}(t) = I_0 [1 - \cos(2\pi f_{ac} \cdot t)]$. Together with the QUIC harmonic potential (Equation 4.1), the ac-coils potential can also be approximated to a harmonic potential with time-dependent frequencies $\Omega_i(t) = \Omega_{0_i} [1 - \cos(2\pi f_{ac} \cdot t)]$, ($i = x', y', z'$), where (x', y', z') is the ac-coils coordinate system, whose origin lies at (X_0, Y_0, Z_0) in the QUIC trap coordinate system, (x, y, z) . The frequency amplitude Ω_{0_i} will depend on I_0 and the QUIC trap frequencies.

The ac-coils are placed in such way that we can consider (see Figure 10) $X_0 = 0$, $x = x'$ and

$$\begin{pmatrix} z' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta_0 & -\sin \theta_0 \\ \sin \theta_0 & \cos \theta_0 \end{pmatrix} \begin{pmatrix} z - Z_0 \\ y - Y_0 \end{pmatrix}. \quad (5.1)$$

Now we can calculate the total potential just by adding the two displaced harmonic

Figure 11 – Equipotential lines for the total time-dependent potential composed by the static QUIC potential and the oscillating one generated by our excitation coils. We show the variation for three different times (columns) in two planes (rows) in reference to the QUIC trap center (red dashed lines).

potentials corresponding to the QUIC trap and ac-coils. After considering appropriate experimental values, we could produce Figure 11, which shows the corresponding equipotential lines for different times in both yz - and xz -planes. By looking at the overall behaviour compared to the original QUIC trap potential, one can expect that the oscillatory excitation will not only displace and deforms, but also rotates the trapped BEC cloud.

Concerning the experimental sequence, at the end of the RF-evaporative cooling, we wait a few milliseconds before turning the ac-coils on, which are kept on for some time (excitation time, t_{exc}). After turning them off, we hold the cloud in the QUIC trap a little longer (t_{hold}), and then release the atoms for TOF and absorption imaging, as explained in Section 4.1.6. Therefore, the experimental knobs related to the excitation are the frequency f_{ac} , the current amplitude I_0 , and the duration t_{exc} , besides the position of the coils. The dynamics of the excitation outcome can be explored by varying t_{hold} .

5.2 Diagram of structures

By means of the oscillatory excitation explained in Section 5.1, we could observe many different effects. In this Section, we intend to summarize four of the most interesting observed effects: *scissors mode* (91), regular vortices, turbulence and granulation.

In Section 5.1, we mentioned that, besides the position of the excitation coils, the excitation parameters to be adjusted are only the frequency f_{ac} , the excitation amplitude A_{exc} (gradient of the excitation field) and its duration t_{exc} . Concerning f_{ac} most of the interesting effects were more likely to happen for values close to the strong confinement frequency of the harmonic QUIC trap. Therefore for all data presented in this section we have fixed f_{ac} in about 200Hz. Other fixed parameters are the expansion time ($t_{TOF} = 15\text{ms}$) and holding time ($t_{hold}=20\text{ms}$).

Therefore, the observed effects could be mapped into a simple diagram of A_{exc} versus t_{exc} shown in Figure 12. In the first region, at low A_{exc} and t_{exc} , we saw only a tilting of the main axes of the cloud (27, 73), which corresponds to a collective mode known as *scissors mode* (91). Concerning the formation of vortices, the most interesting aspect in the observation of the *scissors mode* is that the oscillating excitation is capable of transferring angular momentum to the cloud.

In the following subsections we explore in more detail the observation of regular vortices, turbulence and granulation in our system.

5.2.1 Regular vortices

Increasing further the amplitude (A_{exc}) and duration (t_{exc}) of the oscillatory excitation from the region where only *scissors mode* was observed, the absorption images started to present dips in the density profile as shown in Figure 13. This observation

Figure 12 – Diagram of the excitation amplitude A_{exc} versus its duration t_{exc} presenting regions where four different effects were observed. Border lines separating different regimes are just guide to the eyes.

Figure 13 – Absorption images at 15ms of TOF showing atomic clouds with different number of nucleated vortices.

is consistent with the nucleation of vortex lines, since the dips were persistent even for different t_{TOF} and their occurrence was preceded by the observation of the *scissors mode*.

Figure 13 shows atomic clouds with different number of nucleated vortices. In the experiment the number of vortices (and their positions) to be produced could not be

Figure 14 – Average number of observed vortices as a function of (a) A_{exc} for three fixed values of t_{exc} , and as a function of (b) t_{exc} for three fixed values of A_{exc} . Lines are guides to the eyes, and error bars show the standard deviation of the number of observed vortices at each condition.

deterministically chosen, but it was strongly related with A_{exc} and t_{exc} . Figure 14a(b) is a plot of the average number of observed vortices as a function of $A_{exc}(t_{exc})$ for three different $t_{exc}(A_{exc})$. It is not surprising that longer t_{exc} together with stronger A_{exc} should transfer more angular momentum to the sample, and, hence, are more likely to nucleate a higher number of vortices.

Concerning the spatial arrangement of vortex lines within a superfluid, many experiments have reported on the observation of triangular lattices known as Abrikosov

lattices (2, 92), which correspond to the most energetically favourable configurations of single-charged vortex lines with the same circulation sign. Abrikosov lattices could not be clearly seen in our system, and we believe that this fact is related to nucleation of vortices with opposite signs of circulation.

In order to have a stronger indication that the dips observed in Figure 13 indeed correspond to vortex lines, we conducted a deeper analysis on how the vortex lines were arranged when three of them were observed (52, 93). Moreover this analysis showed evidences that vortices and anti-vortices (opposite circulation sign) were nucleated.

In 52% of more than 60 images presenting three vortex lines, we observed that the vortex cores were arranged approximately as the vertices of an equilateral triangle¹, while in 36.5%, the three cores were almost aligned². Therefore, we conclude that the most stable configuration for three vortex lines in our experiment was either a (equilateral) triangular shape or a linear array.

Then, simulations were performed (93) considering three vortices with the same circulation sign in a 2D non-rotating BEC at different initial positions. The simulated results found that the triangular shape configuration is indeed more likely to be measured after some evolution time (i.e. t_{hold}), but an intermediate configuration³ between the triangular and linear arrays should be more frequently observed than the latter.

Following References (94, 95), we found (again with 2D arguments) that the linear array of three vortex lines can be a stable configuration in our system, provided that the one in the middle is an anti-vortex. Assuming an equal probably of nucleating a vortex or an anti-vortex, one could, at first analysis, expect the linear configuration to be the most frequently observed. However, there can be vortex/anti-vortex annihilation and other decay mechanisms that could occur during t_{hold} , which would destroy this linear configuration. In this way a schematic drawing of the two most stable configurations observed in our system is presented in Figure 15.

¹Largest internal angle within $[60^\circ, 100^\circ]$.

²Largest internal angle within $[140^\circ, 180^\circ]$.

³Largest internal angle within $[100^\circ, 140^\circ]$.

Figure 15 – Schematics of the most likely three vortex configurations in our system: (a) a same circulation sign triangular shape, and (b) a linear shape with an anti-vortex in between two vortices.

Figure 16 – Absorption images at 15ms of TOF showing a tangle configuration of vortices, which characterizes the quantum turbulence regime.

5.2.2 Quantum turbulence

Probably the most interesting region in the diagram of Figure 12 corresponds to the emergence of quantum turbulence (27). A turbulent superfluid is basically characterized by the presence of a tangle configuration of quantized vortices. Figure 16 shows some typical absorptions images of atomic clouds, which we believe to be turbulent. Although the images were taken at the same experimental conditions, there are large fluctuations, which are consistent with the chaotic nature of any turbulent system.

As discussed in Section 4.1.6, the expansion of a thermal cloud is qualitatively very different from a condensate one. As a consequence, it is possible to observe an aspect ratio inversion in the latter, whereas the former tends to produce an isotropic density distribution. In this sense, we observed a very distinguishing aspect of the turbulent clouds produced in our system. Figure 17a shows absorption images of turbulent clouds at three different expansion times, and the corresponding images of a thermal cloud, and a non-turbulent BEC for comparison. One can clearly see that the expansions of the thermal

Figure 17 – Different expansion behaviours of a thermal cloud, a non-turbulent BEC, and a turbulent BEC demonstrated in (a) absorption images, and (b) a plot of the aspect ratio as a function of the expansion time. Lines on (b) are guides to the eyes.

and non-turbulent BEC clouds present the respective behaviours just mentioned. As for the turbulent BEC, it apparently keeps its shape under free expansion, which we refer to as self-similar expansion behaviour. To quantify these results we present in Figure 17b, a plot of the aspect ratio as a function of the expansion time for each type of cloud.

We believe that this remarkable self-similar expansion may be a possible signature of the emergence of quantum turbulence in a trapped BEC. However, the relation between the two features is not well-understood, and is currently under investigation. Besides the different expansion behaviours, another criterion that could be used to distinguish a sample with regular vortices from a turbulent one is the simple observation of their density profiles. Figure 18 gives a 3D perspective of the density profiles for each case. One can see that regular vortices appear as valleys within the bulk density profile, whereas a turbulent cloud presents a tangle configuration of vortices, which results in partial absorptions that no longer preserve the bulk distribution.

5.2.3 Granulation

In the diagram of Figure 12, the region with the strongest amplitudes A_{exc} and the longest times t_{exc} corresponds to the possible observation of a new phase referred to

Figure 18 – Density profiles of (a) a non-turbulent cloud with well-defined vortices, and (b) a turbulent cloud reflecting partial absorption of a tangle configuration of vortices.

Figure 19 – Absorption image revealing a density profile consistent with the observation of the granular phase.

as *granulation*. This granular phase constitutes a non-equilibrium state characterized by the presence of condensate grains surrounded by a large thermal cloud (96,97). Figure 19 shows an absorption image in which we believe a granular phase was observed.

We understand the occurrence of the granular state in our system as follows (30, 52). Reference (96) find that the granular phase (also known as Bose glass) can be reached when the BEC cloud is subjected to a spatial disorder whose localization length (98,99) is larger than the interatomic distance but shorter than the BEC size. Then Reference (97) demonstrates that the action of a time-oscillating potential is, on average, equivalent to the action of a spatially random potential.

The evolution of the granular state may lead to a normal fluid in a chaotic regime (100), which may be related to the so-called weak turbulence (101). This and many other aspects, particularly the ones related to the dynamics of the granular phase, remain to be further experimentally investigated.

5.2.4 Numerical simulation

In collaboration with professors Makoto Tsubota⁴, Michikazu Kobayashi⁵ and Kenichi Kasamatsu⁶ from Japan, we were able to perform a numerical simulation of our experiment. Actually, due to limited computational power, only the corresponding 2D system was simulated, which we do not expect to provide accurate results, but rather insightful qualitative features.

Details on this simulation can be found in References (30, 52). Here, we highlight only the main assumptions and findings. First, consider the time-dependent Gross-Pitaevskii equation (GPE) in Equation 2.21. To account for dissipation mechanisms, one can show that the GPE can be rewritten as (102, 103)

$$(i - \gamma)\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, t) = \left(-\frac{\hbar^2 \nabla^2}{2m} + V(\vec{r}, t) + U_0 |\Psi(\vec{r}, t)|^2 \right) \Psi(\vec{r}, t), \quad (5.2)$$

where γ is a phenomenological constant. In the absence of dissipation, $\gamma = 0$. The external potential $V(\vec{r}, t)$ will be given by the sum of the QUIC trap potential with the oscillatory excitation as discussed in Section 5.1.

For a given excitation amplitude and $\gamma = 0.02$, the simulations found after about 15ms a dramatic increase in the expected value of the angular momentum, which reflected as wavy patterns and subsequent vortices in the density profile as shown in Figure 20.

By increasing the simulation parameter related to the excitation amplitude, the wavy patterns started to happen at earlier excitation times in agreement with the diagram of Figure 12. But probably the most interesting result concerns the dissipation parameter γ . Vortices could only be observed for non-zero values of γ . This is a strong indicative that the interplay of the condensate with the thermal cloud may be an essential ingredient for vortex nucleation in our system. In fact, the next Section will describe this interplay in terms of a relative motion between the condensate and thermal components.

⁴From Osaka City University.

⁵From University of Tokyo.

⁶From Kinki University.

Figure 20 – Snapshots of the simulated 2D BEC under our oscillatory excitation at different excitation times. 3D plots in the right column highlights the density profiles.

5.3 Counterflow

As mentioned before, there are many mechanisms to nucleate vortices in a trapped BEC, but most commonly vortices with the same circulation sign are produced. As we saw in Section 5.2.1, we have experimental evidence that we nucleate vortices and anti-vortices (opposite circulation sign) by means of the oscillatory excitation explained in Section 5.1. Moreover the presence of vortices with different circulation signs, and possibly in different directions, may be an essential ingredient to generate quantum turbulence in a BEC.

In Section 3.3.2 we mentioned that quantum turbulence was first observed in superfluid helium by inducing a thermal counterflow, i.e. a thermal gradient was created, which caused the normal and superfluid components to flow in opposite directions. In a similar way we have suggested a counterflow mechanism (104) for vortex nucleation in our experiment as a relative motion between the thermal and condensate fractions was observed. This motion corresponds to an out-of-phase oscillation mode, which has been investigated in References (105, 106).

The data presented in this section was taken at the fixed experimental conditions: condensate fraction (60%), TOF (23ms), excitation frequency ($f_{ac} = 170\text{Hz}$) and $t_{exc} = 100\text{ms}$ (see Section 5.1). The counterflow was then analysed by scanning t_{hold} . Figure 21 shows a sequence of superimposed absorption images of the excited cloud for different t_{hold} . One can see a variety of collective modes being excited: center of mass oscillation (dipole mode (107)); out-of-phase oscillations of the short and long axes (quadrupole mode (108)); and tilting of the long axis (*scissors mode* (91)). Additionally, the fact that the dipole trajectory does not close on itself indicates that the dipole mode is also being excited in the weak confinement direction (note the direction of the imaging beam in Figure 10).

A closer look to some of those absorption images reveals asymmetric density profiles as shown in Figure 22. This asymmetry means a displacement between the centers-of-mass of the condensate and the thermal cloud. Moreover this displacement changes

Figure 21 – Sequence of superimposed absorption images of the excited cloud for different t_{hold} . Arrows indicate a clockwise evolution of t_{hold} .

with t_{hold} , which indicates a relative motion between the two components.

By fitting a 1D bimodal profile (as exemplified in Figure 23) through both the longitudinal and transverse axes of the absorption images, we could measure the centers-of-mass displacement as a function of t_{hold} . The *in situ* displacement was then recovered by following the method developed in (109)). Figure 24 shows the results, in which sinusoidal fittings were performed indicating an oscillating out-of-phase motion coupled with the dipole mode. The smallest displacement was observed for holding times corresponding to the upper and lower turning points in Figure 21. At these points the counterflow velocity is maximum and estimated to be about 3 cm/s .

The data was also analysed in terms of surface ripples at the boundary region between the condensate and thermal clouds. To perform such analysis we employed the following procedure. First the fitted 2D Gaussian profile corresponding to the thermal cloud is removed from the absorption images. Then we fit the resulting image with a

Figure 22 – Asymmetric density profiles for three different t_{hold} revealing not only a displacement between the condensate (superfluid) and thermal (normal fluid) fractions, but also a relative motion between the two components.

Figure 23 – Density profile along the longitudinal axis of an absorption image with displaced thermal and condensate components. The red line is a bimodal fitting allowing different positions for the centers of the Gaussian (CM_{th}) and inverted parabola (CM_{BEC}) profiles.

Figure 24 – Longitudinal and transverse relative positions between the centers-of-mass of the condensate and thermal components as a function of t_{hold} . Dotted lines are sinusoidal fittings.

2D Thomas-Fermi profile, which is then also subtracted. The remaining image is basically a perimeter line whose variations from the regular ellipse (expected for the condensate surface) characterizes the ripples size. To establish a characteristic ripple length for each image, we take the highest peak in the Fourier transform of the ripples size over this perimeter line.

Figure 25 is a plot of the characteristic ripple length as a function of t_{hold} . One can see that fluctuations are kept low for most of the counterflow oscillation, and only close to the points of maximum relative velocity, they significantly grow.

We understand the results as follows. The frictional force between the normal and superfluid components in presence of a counterflow may be given by the phenomenological expression (per unit volume) (110)

$$\vec{F}_{ns} = -A\rho_s\rho_n(v_{ns} - v_0)^2\vec{v}_{ns}, \quad (5.3)$$

where the relative velocity between the normal and superfluid components, v_{ns} , is considered larger than the constant value, v_0 , which is usually of the order of 1cm/s. Here A is a temperature dependent constant and ρ_s (ρ_n) is the density of the superfluid (normal fluid). The interaction between the two components can be explained in terms of scattering of thermal excitations, as discussed in Reference (15).

We believe the observation of ripples is associated with the presence of the frictional force in Equation 5.3. In Figure 25 one can tell that for t_{hold} between 3.5ms and 5.5ms, the characteristic ripple length is basically zero. Therefore, v_{ns} is expected to be smaller than v_0 in this range, which allow us to estimate the latter in about 2.5cm/s. Also significant ripples require the condition $v_{ns} > v_0$, which is sustained only for a short time lapse. That explains why, in the data presented in this section, we only see indications of vortex nucleation (Figure 26), and not vortex proliferation or turbulence.

Since under similar experimental conditions we have observed the onset of turbulence (27, 28), we also performed the following analysis on the data presented in this section. Reference (111) investigates turbulence in superfluids in terms of a quantum

Figure 25 – The characteristic length (see definition in the text) of ripples on the condensate surface as a function of t_{hold} . The black dotted line is only a guide to the eye, the green dotted line displays the healing length value for reference.

Figure 26 – Absorption image close to the maximum counterflow velocity moment. Black arrows indicate large ripples on the boundary surface between the condensate and thermal clouds.

Reynolds number ⁷ defined as

$$Re_s = \frac{m\xi v}{2\pi\hbar}, \quad (5.4)$$

where ξ is the healing length (Equation 2.31), m is the atomic mass, and v is the flow velocity. By considering $v \equiv v_{ns}$, we find Re_s ranging from 0 to 1, which according to (111) is not large enough for the onset of a turbulent flow.

Of course, the counterflow friction may be neglected close to zero temperature as the thermal fraction vanishes. But as we saw in Section 5.2, the presence of a thermal component as a dissipation mechanism for the condensate may be essential to nucleate vortices, and subsequently generate turbulence in our experiment (30).

5.4 Finite-size effect

In Section 5.3 we proposed the counterflow motion between the condensate and thermal fluids as a possible vortex nucleation mechanism in our experiment. In this section, we will relate the transition from a regime with regular vortices to turbulence in terms of the finite-size characteristic of trapped gases (29), which constitutes one of the most important differences between these systems and superfluid helium.

In Figure 14 we reported on the observation of different number of vortices as a function of both the amplitude and duration of the oscillatory excitation explained in Section 5.1. As these parameters are further increased we obtain the turbulent regime. Therefore, we analysed the transition between the two regimes based on a very simple model that establishes a critical number of vortices depending on the size of the sample.

As we saw in Section 5.2, the various structures seen by means of the oscillatory excitation were mapped in a diagram in terms of the amplitude and duration of the excitation. The transition from the vortices to the turbulence regions in this diagram is shown in Figure 27.

⁷In Section 3.3 we discussed the importance of the Reynolds number for the onset of turbulence.

Figure 27 – Diagram of the excitation amplitude A_{exc} versus its duration t_{exc} presenting the transition from a non-turbulent regime with regular vortices to a turbulent one. In contrast with the guide lines in Figure 12, here the transition line is a fitting of Equation 5.9 through critical points considered to be halfway points with the same amplitude.

This model is based on the following assumptions:

- The oscillatory excitation pumps energy into the sample at a certain rate R_{pump} , which depends linearly on the excitation amplitude A_{exc} .
- During a certain period t_0 , the pumped energy is mostly “stored” in collective modes not associated with rotation. So vortex proliferation starts only after t_0 .
- After t_0 , a fraction η of the pumped energy is converted into vortices.
- Interaction between vortices are neglected. This means that the total energy contained in the vortices is equal to the number of vortices, N_{vort} , times the energy of each vortex, E_{vort} (see the vortex energy per unit length in Equation 3.16).
- The size of our harmonically trapped cloud (see Equation 4.1) is given by the cor-

responding oscillator length

$$a_{ho} = \sqrt{\frac{\hbar}{m(\omega_\rho^2 \omega_z)^{1/3}}}. \quad (5.5)$$

- Vortex/anti-vortex annihilation is not explicitly considered but it could be energetically incorporated in the value of η .

Therefore the energy balance yields

$$\eta R_{pump}(t - t_0) = N_{vort} E_{vort}, \quad (5.6)$$

which makes the expected N_{vort} be

$$N_{vort} = \frac{\eta R_{pump}}{E_{vort}}(t - t_0). \quad (5.7)$$

This means that within this model, after t_0 , N_{vort} should grow linearly with both A_{exc} and t_{exc} , which fails to predict some of the observations in Figure 14. Nevertheless, the critical line for the transition between regular vortices and turbulence can be correctly addressed if one considers the critical condition as

$$N_{vort}^{(crit)} \approx \frac{a_{ho}}{\xi}, \quad (5.8)$$

where ξ is the *healing length* (Equation 2.31). In Section 3.2, we discussed how the *healing length* characterizes the size of a vortex core. So the critical condition in Equation 5.8 reflects the maximum number of vortex lines that would fit without overlapping inside a cloud with a finite-size a_{ho} .

Finally the critical line in a diagram A_{exc} versus t_{exc} will present the form

$$A_c(t) = \frac{C}{t - t_0}. \quad (5.9)$$

By considering halfway points with the same amplitude in Figure 27 as the experimental critical points, we could fit Equation 5.9 to find our experimental transition line (also displayed in Figure 27). The fitting parameters were found in $C \approx 1.6\text{G(ms)}/(\text{cm})$ and $t_0 \approx 17\text{ms}$. Since our system presents $\xi \approx 0.06\mu\text{m}$ and $a_{ho} \approx 1.08\mu\text{m}$, we can expect $N_{vort}^{(crit)} \approx 20$.

Both values of t_0 and $N_{vort}^{(crit)}$ are in good agreement with Figure 14 concerning, respectively, the time when vortices proliferation starts and the maximum number of vortices that could be measured. So we have offered indicatives that the finite-size characteristic may play a crucial role in the onset of turbulence in trapped gases, in strong contrast with turbulence in superfluid helium.

6 Thermodynamics of trapped gases in terms of macroscopic parameters

The usual volume and hydrostatic pressure of a uniform system are no longer well defined thermodynamic variables for a fluid confined by an arbitrary external potential. In principle, for a harmonic confinement, the real volume is infinite since there is no real walls. The pressure can be only locally defined, and the usual thermodynamic equations of state in terms of global variables cannot be written.

Since experiments with ultracold atoms are usually conducted in spatially inhomogeneous traps, one usually make a local thermodynamic analysis based on the local density approximation (LDA). Therefore, a general thermodynamic approach in terms of new global variables would be quite in place. That is the subject of this Chapter. We start, in Section 6.1 by presenting an overview of this theory proposed in 2005 (5). In Section 6.2, we describe the methodology followed to experimentally obtain one of those global variables. Then, in Section 6.3 we present an analysis of the BEC transition in our system in terms of those new variables. And finally, preliminary results on the calculation of the heat capacity are presented.

6.1 Overview

By introducing new thermodynamic parameters a novel theory was developed to macroscopically describe the thermodynamics in harmonic potentials (5, 25, 112). It is possible to generalize this theory to an arbitrary external potential (26), but we shall

consider here only the harmonic case.

In Chapter. 2 we stressed the importance of taking an appropriate thermodynamic limit in order to rigorously make a quantitative definition of BEC. From Equation 2.11 the critical temperature, T_c , of an ideal Bose gas in a 3D box (homogeneous system) of volume V depends on the density N/V . So the thermodynamic limit is taken as: $N \rightarrow \infty$, $V \rightarrow \infty$, but $T_c \rightarrow \text{const}$. In a similar way, from Equation 2.10 for the harmonic case (inhomogeneous system), T_c depends on the quantity $N\bar{\omega}^3$, where $\bar{\omega}$ is the geometric mean of the frequencies. By analogy, let us define the quantity

$$\mathcal{V} \equiv \frac{1}{(\omega_x \omega_y \omega_z)} = \frac{1}{\bar{\omega}^3}, \quad (6.1)$$

hereafter known as (harmonic) *volume parameter*. In this case, the thermodynamic limit reads: $N \rightarrow \infty$, but $T_c \rightarrow \text{const}$ as long as $\mathcal{V} \rightarrow \infty$. This result was originally pointed out in 1950 by Groot *et al.* (113).

It is evident that the *volume parameter*, $\mathcal{V}$, does not have units of volume in real space. However, they share essential features besides the thermodynamic limit analogy above. As mentioned in Section 2.2, the number of atoms in this case will be given by

$$N = \int_0^\infty g_{ho}^{(3D)}(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)} - 1} d\epsilon, \quad (6.2)$$

which means that $N = N(T, \mu, \mathcal{V})$. Since N is an extensive variable, but T and μ are intensive ones, $\mathcal{V}$ must be indeed an extensive variable. Additionally they are directly related in the sense of an effective volume. At a given temperature, a volume of the order of $(k_B T / m \bar{\omega}^2)^{3/2}$ will be mostly occupied, which is proportional to $\mathcal{V}$.

In order to find the intensive variable conjugated to the *volume parameter*, which we will naturally call *pressure parameter*, let us consider the grand potential of the ideal Bose gas (still in the harmonic potential) given by (114)

$$\Omega(\mu, T, \mathcal{V}) = -\mathcal{V} k_B T \left(\frac{k_B T}{\hbar} \right)^3 g_4(\alpha), \quad (6.3)$$

where $\alpha = \mu / k_B T$, and we have introduced the Bose function

$$g_n(\alpha) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{e^{x-\alpha} - 1} dx. \quad (6.4)$$

Then the number of atoms and the *pressure parameter* will be, respectively

$$N_{ideal} = \left(\frac{\partial \Omega}{\partial \mu} \right)_{\nu, T} = \mathcal{V} \left(\frac{k_B T}{\hbar} \right)^3 g_3(\alpha), \quad (6.5)$$

and

$$\Pi_{ideal} \equiv - \left(\frac{\partial \Omega}{\partial \mathcal{V}} \right)_{\mu, T} = k_B T \left(\frac{k_B T}{\hbar} \right)^3 g_4(\alpha) = \frac{N k_B T}{\mathcal{V}} \frac{g_4(\alpha)}{g_3(\alpha)}. \quad (6.6)$$

Since $0.9 \leq g_4(\alpha)/g_3(\alpha) \leq 1$, the equation of state (in $\Pi(N/\mathcal{V}, T)$) of the ideal Bose gas is very close to the celebrated ideal classical gas, i.e. $\Pi_{cl} \mathcal{V} = N k_B T$.

In order to derive a general expression for Π , let us assume the canonical ensemble, and the Helmholtz free energy given by

$$F(N, \beta, \mathcal{V}) = -\frac{1}{\beta} \ln \left(\text{Tr} e^{-\beta \hat{H}} \right), \quad (6.7)$$

where $\hat{H} = \hat{K} + \hat{U}_{int} + \hat{V}_{ext}$ is the Hamiltonian of the system composed by the kinetic energy $\hat{K} = \sum_i \frac{\hat{p}_i^2}{2m}$, the interaction energy $\hat{U}_{int} = \sum_{i < j} \hat{U}(r_{ij})$, and the external harmonic potential $\hat{V}_{ext} = \sum_i \frac{1}{2} m (\omega_x^2 \hat{x}_i^2 + \omega_y^2 \hat{y}_i^2 + \omega_z^2 \hat{z}_i^2)$.

Then the *pressure parameter* will be given by

$$\Pi \equiv - \left(\frac{\partial F}{\partial \mathcal{V}} \right)_{N, \beta} = \frac{2}{3\mathcal{V}} \langle \hat{V}_{ext} \rangle, \quad (6.8)$$

which results in the very useful expression:

$$\Pi = \frac{2}{3\mathcal{V}} \int n(\vec{r}) \left(\frac{1}{2} m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \right) d^3 r. \quad (6.9)$$

Therefore the knowledge of the (inhomogeneous) density distribution, and the (inhomogeneous) external harmonic potential readily provides the macroscopic quantity Π .

As in the case of $\mathcal{V}$, it is clear that the Π does not have the same units as its common local (or uniform) counterpart, namely the hydrostatic pressure $p(\vec{r})$. However they are closely related in their physical meaning in the sense of being responsible for mechanical equilibrium. As seen when deriving the Navier–Stokes equations in Section 3.3.1, the condition for mechanical equilibrium in a fluid writes (56)

$$\nabla \cdot \vec{\sigma}(\vec{r}) + \vec{f}(\vec{r}) = 0, \quad (6.10)$$

where $\tilde{\sigma}(\vec{r})$ is the pressure tensor, and $\vec{f}(\vec{r})$ is the external force per unit volume, i.e. $\vec{f}(\vec{r}) = -n(\vec{r})\vec{\nabla}V_{ext}(\vec{r})$.

For a fluid at rest one finds $\tilde{\sigma}(\vec{r}) = -\tilde{1}p(\vec{r})$, where $\tilde{1}$ is the unit tensor. Therefore the integration of the virial of the forces in Equation 6.10 together with Equation 6.9 will lead to

$$\int n(\vec{r}) \vec{r} \cdot \vec{\nabla} V_{ext}(\vec{r}) d^3r = \int \vec{r} \cdot \nabla p(\vec{r}) d^3r \implies \Pi\mathcal{V} = \int \vec{r} \cdot \nabla p(\vec{r}) d^3r, \quad (6.11)$$

which shows that Π purely depends on the behaviour of $p(\vec{r})$. Additionally it is important to note that the product $\Pi\mathcal{V}$ indeed has units of energy, and the relation $\Omega = -\Pi\mathcal{V}$ holds.

Since in the presence of interactions, the density profile of a BEC strongly depends on their strength (see Equation 2.23), Equation 6.9 means that Π will also strongly depend on interactions below T_c . To stress this aspect, we reproduce from Reference (5) Figure 28, which shows calculations of Π as a function of temperature at different values of the *density parameter* $N/\mathcal{V}$ for (a) an ideal and (b) a weakly interacting Bose gas. In the latter, a self-consistent Hartree–Fock approximation was considered.

6.2 Methodology

As discussed in Section 4.1.6 the raw data provided by the experiment is a 2D absorption image of the expanded cloud. Of course, $n(\vec{r})$ in Equation 6.9 corresponds not only to a 3D function but also to the *in situ* density profile. Therefore, to determine $n(\vec{r})$ we apply the following procedure:

- Fit a 2D bimodal distribution through the raw data, which gives a 2D Gaussian and a 2D Thomas–Fermi-like profiles corresponding to the thermal and condensate components, respectively.
- By knowing the width of the 2D Gaussian and the expansion time, use Equation 4.5 to calculate the temperature.

Figure 28 – Theoretical *isodensity curves*: Pressure parameter Π as a function of temperature at different *density parameter* N/V for (a) an ideal and (b) a weakly interacting Bose gas. For the latter a Hartree–Fock approximation was considered. Image reproduced from Reference (5).

- Through trapping symmetry assumptions, find the corresponding 3D distributions of the expanded clouds.
- Assume a ballistic expansion for the thermal cloud to recover its *in situ* density.
- Apply the theory developed by Y. Castin and R. Dum (109) to calculate the *in situ* density profile of the condensate.
- Finally, $n(\vec{r})$ will be the sum of the thermal and condensate densities, i.e. $n(\vec{r}) = n_{th}(\vec{r}) + n_0(\vec{r})$.

Therefore, the integral in Equation 6.9 can be separated, and we write $\Pi = \Pi_{th} + \Pi_0$ as the sum of a thermal and a condensate contribution to the *pressure parameter*.

Let us first evaluate the thermal contribution Π_{th} . For our cigar-shape trap we find the general form of the normalized Gaussian

$$n_{th}(\vec{r}) = \frac{N_{th}}{(2\pi)^{3/2}\sigma_\rho^2\sigma_z} \exp\left(-\frac{\rho^2}{2\sigma_\rho^2} - \frac{z^2}{\sigma_z^2}\right), \quad (6.12)$$

where N_{th} is the number of thermal atoms, and σ_i are the Gaussian widths.

Therefore Equation 6.9 gives

$$\Pi_{th} = \frac{1}{3\mathcal{V}} m N_{th} (2\omega_\rho^2 \sigma_\rho^2 + \omega_z^2 \sigma_z^2). \quad (6.13)$$

However the equipartition theorem readily gives

$$\frac{1}{2} m \omega_i^2 \sigma_i^2 = \frac{1}{2} k_B T, \quad (6.14)$$

and the ideal gas result $\Pi_{th}\mathcal{V} = N_{th}k_B T$ comes as no surprise. By assuming Gaussian distributions in the harmonic trap, even for a BEC, we are indeed dealing with the non-interacting case.

Concerning the condensate contribution Π_0 , the normalized Thomas–Fermi distribution writes

$$n_0(\vec{r}) = \frac{15N_0}{8\pi R_\rho^2 R_z} \left(1 - \frac{\rho^2}{R_\rho^2} - \frac{z^2}{R_z^2}\right), \quad (6.15)$$

where N_0 is the number of condensed atoms, and R_i are the Thomas–Fermi radii. So Equation 6.9 yields

$$\Pi_0 = \frac{1}{21\mathcal{V}} m N_0 (2\omega_\rho^2 R_\rho^2 + \omega_z^2 R_z^2) \quad (6.16)$$

6.3 BEC transition

By applying the methodology above we described the BEC transition in our experiment in terms of the new macroscopic parameters. We have basically measured the

Figure 29 – *Isodensity curves: pressure parameter Π as a function of temperature for different number of atoms at a fixed volume parameter $\mathcal{V}$. Error in $\langle N \rangle$ is the standard deviation within each group. Lines are guides to the eyes.*

equation of state $\Pi = \Pi(N/\mathcal{V}, T)$. In the experiment only N and T were scanned, while $\mathcal{V}$ was kept fixed (for technical reasons¹) in $\mathcal{V} = (\omega_x \omega_y \omega_z)^{-1} \approx 4.1 \times 10^{-9} \text{ s}^3$, provided that $\omega_x = \omega_0$ and $\omega_y = \omega_z = 9\omega_0$, where $\omega_0 = 2\pi \times 23\text{Hz}$.

So each image shot presented a certain value for Π , T and N (or equivalently, a certain *density parameter* $N/\mathcal{V}$). All data was then organized in groups with similar N in order to build isodensity curves, which constitute the equation of state $\Pi(N/\mathcal{V}, T)$. Figure 29 shows five isodensity curves around the BEC transition, which is clearly seen as kinks in each curve at a corresponding critical temperature $T_c(N)$. Above T_c , Π will present a linear dependence on both T and $N/\mathcal{V}$, as discussed in Section 6.2. Below T_c the onset of BEC makes Π falls faster than linear with the decrease in T , which reveals the role of interactions as mentioned in Section 6.1. In fact, Figure 29 resembles Figure 28(b), in which Π was evaluated for a weakly interacting BEC under a Hartree–Fock approximation.

To extract the critical points (Π_c, T_c) for each $N/\mathcal{V}$ in Figure 29, we took the

¹In a magnetic trap, varying the frequencies implies a change in the bias field as well as evaporation conditions. On the other hand optical traps would be more suitable for this task.

Figure 30 – Phase diagram Π versus T of the BEC transition in our experiment. The critical points (Π_c, T_c) were extracted from Figure 29 and then fitted with Equation 6.17

crossing point between a linear fit above T_c and the extrapolation of the data points below T_c . The corresponding points allow us to build the phase diagram of Figure 30, in which we identify two domains, one corresponding to a purely thermal cloud, and the other to a superposition of a thermal and a condensate components. Therefore at the transition (or at T_c^+ , strictly speaking), one finds $\Pi_c \mathcal{V} = N k_B T_c$. In Section 2.2 we derived the T_c dependence on N for the ideal Bose gas in a harmonic trap. So considering Equation 2.10 yields

$$\Pi_c = \frac{k_B^4}{(0.94\hbar)^3} T_c^4 = (3.73 \times 10^{10}) T_c^4 \quad (\text{SI}). \quad (6.17)$$

A fitting of $\Pi_c = C T_c^4$ through our data (also shown in Figure 30) gives $C = 3.8(1) \times 10^{10}$ (SI), which agrees with the expected ideal value in Equation 6.17. Therefore, at this accuracy, interactions can be indeed neglected above T_c .

The Π versus T phase diagram of Figure 30 is remarkably similar to the celebrated

Figure 31 – Phase diagram of the saturated vapour pressure versus temperature in ^4He using data from References (6,7).

corresponding phase diagram of the (uniform!) ^4He , which we reproduce in Figure 31 (data extracted from References (6,7)). In this case we also identify one region corresponding to the presence of a pure normal fluid, and another presenting both the normal and superfluid components.

At $T = 0$ the density profile will be only given by Equation 2.23 (in the Thomas–Fermi approximation), and one finds

$$\Pi_{T=0} = \frac{1}{7} \left(15\hbar^2 a_s \sqrt{m} \right)^{2/5} \left(\frac{N}{\mathcal{V}} \right)^{7/5}. \quad (6.18)$$

The extrapolation of data points below T_c in Figure 29 allowed us to find $\Pi_{T=0}(N/\mathcal{V})$. Then a log-log plot revealed $\Pi_{T=0} \propto (N/\mathcal{V})^\delta$ with $\delta = 1.5^{+0.3}_{-0.2}$ well within the expected value 1.4. From this plot the scattering length could also be found in $a_s = 94 \pm 32 \text{\AA}$, which is a bit higher than the actual value ($\approx 50 \text{\AA}$). Due to the absence of data points at very low temperatures the extrapolation to $\Pi_{T=0}$ is subjected to a higher imprecision.

6.4 Global heat capacity

The heat capacity is certainly one of the most important thermodynamic quantities concerning phase transitions. The celebrated λ -shape in superfluid helium was found in the measurement of the heat capacity as a function of temperature. In the context of quantum gases, few experimental explorations were performed due to a lack of methods to measure the total internal energy of the system.

From the results presented in the previous section, we were able to extract the value of the heat capacity. In the following we explain how this analysis was performed.

From Equation 6.7 the internal energy will obviously be

$$U \equiv \frac{\partial}{\partial \beta}(\beta F)_{N,\mathcal{V}} = \langle \hat{H} \rangle = \langle \hat{K} \rangle + \langle \hat{U}_{int} \rangle + \langle \hat{V}_{ext} \rangle. \quad (6.19)$$

To relate the quantities in the last equation, let us make use of the virial theorem, which writes

$$\langle \hat{V}_{ext} \rangle = \langle \hat{K} \rangle + \frac{1}{2} \sum_{i,j} \langle (\vec{r}_i - \vec{r}_j) \cdot \vec{F}_{i,j} \rangle, \quad (6.20)$$

where the interparticle force will be given by $\vec{F}_{i,j} = -\vec{\nabla} U_{int}(\vec{r}_{i,j})$. By taking the interaction potential in the contact form of Equation 2.16, one finds

$$\vec{F}_{i,j} = -U_0 \vec{\nabla} \delta(\vec{r}_i - \vec{r}_j), \quad (6.21)$$

where U_0 is the interaction coupling constant.

By means of the relation $\vec{r} \cdot \vec{\nabla} \delta(\vec{r}) = -3\delta(\vec{r})$, Equation 6.20 becomes

$$\langle \hat{V}_{ext} \rangle = \langle \hat{K} \rangle + \frac{3}{2} \langle \hat{U}_{int} \rangle. \quad (6.22)$$

Solving the last equation for $\langle \hat{K} \rangle$ and introducing the result in Equation 6.19 yields

$$U = 2\langle \hat{V}_{ext} \rangle - \frac{1}{2} \langle \hat{U}_{int} \rangle = 3\Pi\mathcal{V} - \frac{1}{2} \langle \hat{U}_{int} \rangle. \quad (6.23)$$

Therefore, in the absence of interactions, we indeed find the result expected from the equipartition theorem applied to a 3D harmonic trap.

Recall that, according to the methodology described in Section 6.2, the experiment data is obtained in terms of the condensate and thermal contributions to the value of the *pressure parameter*, i.e. $\Pi = \Pi_{th} + \Pi_0$. In this sense, let us also separate the contributions to the total internal energy as

$$U = U_{th} + U_0. \quad (6.24)$$

As discussed in the previous sections, the interaction may be neglected for the thermal cloud. Then Equation 6.22 results in

$$\langle \hat{K} \rangle_{th} = \langle \hat{V}_{ext} \rangle_{th}, \quad (6.25)$$

which promptly gives the expected result

$$U_{th} = 2\langle \hat{V}_{ext} \rangle_{th} = 3\Pi_{th}\mathcal{V}. \quad (6.26)$$

As for the condensate cloud, in the Thomas–Fermi approximation, we may neglect the kinetic term in Equation 6.22. Then we find

$$\langle \hat{U}_{int} \rangle_0 = \frac{2}{3}\langle \hat{V}_{ext} \rangle_0, \quad (6.27)$$

and

$$U_0 = \left(1 + \frac{2}{3}\right) \langle \hat{V}_{ext} \rangle_0 = \frac{5}{2}\Pi_0\mathcal{V}. \quad (6.28)$$

Finally the heat capacity at constant $\mathcal{V}$ can be calculated as

$$\frac{C_{\mathcal{V}}}{\mathcal{V}} \equiv \frac{1}{\mathcal{V}} \left(\frac{\partial U}{\partial T} \right)_{\mathcal{V},N} = 3 \left(\frac{\partial \Pi_{th}}{\partial T} \right)_{\mathcal{V},N} + \frac{5}{2} \left(\frac{\partial \Pi_0}{\partial T} \right)_{\mathcal{V},N} \quad (6.29)$$

$$\frac{C_{\mathcal{V}}}{\mathcal{V}} = 3 \left(\frac{\partial \Pi}{\partial T} \right)_{\mathcal{V},N} - \frac{1}{2} \left(\frac{\partial \Pi_0}{\partial T} \right)_{\mathcal{V},N}.$$

Our experimental data shows that $\left| \frac{\partial \Pi_0}{\partial T} / \frac{\partial \Pi}{\partial T} \right| < 0.1$ even at the lowest temperature values of $T/T_c \approx 0.1$. Therefore, we may neglect the last term in Equation 6.29, which finally yields

$$C_{\mathcal{V}} = \frac{3}{\omega^3} \left(\frac{\partial \Pi}{\partial T} \right)_{N,\omega}. \quad (6.30)$$

This result shows that the sole knowledge of the equation of state $\Pi(N/\mathcal{V}, T)$ determined in the previous section allows us to obtain the heat capacity.

Figure 32 – Heat capacity versus normalized temperature for two average number of particles, $\langle N \rangle$. The data was extracted from Figure 29 by means of Equation 6.30. Solid lines were obtained from Equations 10 and 14 in Reference (8), in which N is the only adjustable parameter.

By performing interpolations/extrapolations of the data presented in Figure 29, we were able to calculate the heat capacity as a function of temperature for those averaged number of atoms, $\langle N \rangle$. The results for two different $\langle N \rangle$ are shown in Figure 32, in which the temperature was normalized with the correspondent critical temperature obtained in Figure 30. One can see a λ -type picture that resembles the corresponding result for liquid He (115). This qualitative aspect was found by different theories whether considering finite number effects (8, 116) and interactions (117) or not (24).

We have also analysed the behaviour of the heat capacity around the transition for each $\langle N \rangle$, i.e. we extracted the values of $C_V^- \equiv C_V(T_c^-)$, $C_V^+ \equiv C_V(T_c^+)$, and $\Delta C_V \equiv C_V^- - C_V^+$. Figure 33 shows these results, in which we can easily see a deviation from the ideal picture. For an ideal Bose gas in the large number limit $C_V^- = 10.8Nk_B$ and $\Delta C_V = 6.6Nk_B$, i.e. both have a linear dependency with $\langle N \rangle$ (8, 24). In Figures 33(a) and (b), as $\langle N \rangle$ increases we observe a deviation from linearity for higher values, which is certainly due to the presence of atomic interactions in the BEC cloud. As for C_V^+

Figure 33 – Dependence of the heat capacity around the critical temperature on the average total number of atoms: (a) the jump ΔC_V , (b) $C_V(T_c^-)$, and (c) $C_V(T_c^+)$. Dashed lines are theoretical results for the non-interacting large number limit, and solid curves are guides to the eye.

(Fig. 33(c)) we see a linear dependency with $\langle N \rangle$. Since interactions are less relevant in the thermal cloud, deviations from the ideal case are much less sensitive.

In E. Cornell's group, soon after the BEC realization, a measurement of the heat capacity was performed by considering as the internal energy, the kinetic energy released when the BEC cloud undergoes a free expansion (118). At high temperatures, their data found a linear dependency of the energy with temperature, which indicates the Maxwell-Boltzmann limit. The value obtained for ΔC_V was far from the ideal behaviour, and the differences were associated with interactions. The main field repulsion increases the

Figure 34 – Heat capacity normalized with the average number of atoms, $\langle N \rangle$, as a function of normalized temperature (T/T_c) for two different $\langle N \rangle$. Solid lines are interpolations of the data. The blue dashed line is the theoretical curve for the non-interacting large number case.

internal energy and possibly the heat capacity.

Finally we plotted the normalized heat capacity, C_V/Nk_B , versus the normalized temperature, T/T_c , for two different average number of atoms, $\langle N \rangle$ (Fig. 34). For an ideal Bose gas in the limit of large number, C_V/Nk_B is a universal curve (dashed blue line) in the sense of not depending on $\langle N \rangle$. The weakly-interacting BEC data show larger deviations from this universal curve near the critical temperature as well as for low temperatures. For the latter we find that C_V/Nk_B presents a larger value for a smaller $\langle N \rangle$, in agreement with previous calculations (117). Close to the limit $T \rightarrow 0$ one could possibly consider an “energy gap” model, in which we find (119)

$$\frac{C_V}{Nk_B} = \frac{4a_s}{\lambda_{dB}} \frac{\Delta}{k_B T} e^{-2\Delta/k_B T}, \quad (6.31)$$

where the “energy gap” is $\Delta = (\hbar^2/m)2\pi a_s \rho$, a_s is the s-wave scattering length, λ_{dB} is the thermal de Broglie wavelength, and ρ is the particle density. Unfortunately, the present data is too scarce at low temperatures to allow the investigation of such behaviour.

It is important to stress that this C_V is a global quantity that thermodynamically describes the whole trapped gas. This represents an important advantage over the calculation of a local heat capacity that relies on the validity of LDA ([120](#), [121](#)).

The results presented in this section were not yet published, but the paper was already submitted.

7 Conclusions

We organize the conclusions by discussing the results first, on quantum turbulence (QT), and second, on thermodynamics. Then we give prospects on both.

7.1 Summary on QT

We summarize the observations on QT presented in Chapter 5 as follows.

1. The oscillatory magnetic excitation field was shown to be able to translate, deform and rotate the BEC cloud trapped in the QUIC magnetic trap.
2. As a function of both the amplitude and duration of the oscillatory excitation, four related effects were observed: *scissors mode*, regular vortices, QT, and granulation.
3. Analyses on three vortices configurations indicated that vortices and anti-vortices (opposite circulation sign) are nucleated, which possibly represents an essential ingredient to produce a QT state.
4. Numerical simulations of a correspondent 2D system found the nucleation of vortices and anti-vortices, as well as the tangle configuration that characterizes QT. For the calculations, a non-zero dissipation term was necessary to produce those results.
5. In close relation to the thermal counterflow in superfluid helium, a relative motion between the condensate and thermal components was analysed as a possible vortex nucleation mechanism in our system. The importance of the normal fluid in the process is consistent with the necessity of a dissipation term in the simulations.

6. The evolution of a condition heavily populated with vortices to QT was explained by considering the finite-size of our trapped BEC. That is a distinguishing feature of QT in trapped gases when compared to QT in superfluid helium.

The first observation of vortices and QT in our system were reported in References (27, 28, 73). The results presented here were published in References (29–31, 93).

7.2 Summary on thermodynamics of a trapped BEC

The thermodynamic analyses in terms of new global variables discussed in Chapter 6 are summarized as follows.

1. The definition of the *volume parameter* as $\mathcal{V} = (\omega_x \omega_y \omega_z)^{-1}$ assures the thermodynamic limit in a harmonic trap. Additionally, $\mathcal{V}$ keeps the extensive characteristic, and is closely related to the mostly occupied (real) volume.
2. The *pressure parameter* Π defined as the intensive variable conjugated to $\mathcal{V}$ also is shown to be related to the hydrostatic pressure. In fact, Π depends solely on the overall behaviour of the local hydrostatic pressure $p(\vec{r})$.
3. From the parameters provided by the bimodal fitting, the contributions to Π due to the thermal and condensate components can be separated, and calculated by simple analytical expressions.
4. The BEC transition was characterized in terms of a diagram Π versus temperature for different *density parameters* $(N/\mathcal{V})$. In other words the equation of state $\Pi(N/\mathcal{V}, T)$ was determined. The importance of interactions below the critical temperature were clearly seen, and the critical line was shown to be similar to the helium superfluid transition.
5. Results on the calculation of the global heat capacity at constant $\mathcal{V}$ were presented. This represents a global quantity to describe the “thermal inertia” of the trapped

cloud as a whole, in contrast with calculations of a local heat capacity by means of the local density approximation (LDA).

The basic theory on this topic was first reported in References (5,25). The analyses conducted here on the BEC transition were published in Reference (112). A paper on the heat capacity measurement was prepared and submitted, and hopefully soon it will be published.

7.3 Prospects

The general prospects are listed as follows.

- It is imperative to better understand the vortex nucleation mechanism presented. At this stage, though intuitively consistent with calculations, I must stress that the mechanism is neither clear nor easily reproducible.
- Concerning the QT state, there are many important aspects to be studied. To cite a few: (i) the dependence on the superfluid fraction (i.e. temperature), (ii) the decay mechanism through high frequency Kelvin waves, and of course, (iii) the Kolmogorov spectrum itself.
- The granular phase remains a somewhat speculative observation. We believe it is related to relaxation processes, but a much deeper characterization will be necessary to allow even simple claims in this complex regime.
- Although the methodology followed to calculate the pressure parameter Π resulted in consistent results, it presents two important limitations.

1. Performing fittings means assuming a particular density profile. For a “simple” BEC plus thermal cloud, the bimodal assumption may be well-reasonable.

However, for more complicated cases, say a BEC with vortices or even QT, fitting assumptions are non-trivial and possibly nonsense.

2. Recovering the *in situ* density profile from the expanded one by using the described method is generally valid for either a pure BEC or a pure thermal cloud. Finite temperature effects may be important for more precise measurements. Again, it is far not clear how to recover the *in situ* profiles of clouds with many vortices and QT.
- Closely related to some of the previous items, a major experimental improvement would concern the imaging technique. First the ability to image *in situ* clouds would overcome the limitations of a recovering procedure. Second, a non-destructive technique would allow to promptly and more consistently study the dynamics of QT. Also, there would be no need for any assumptions concerning either trap symmetries or the nature of the imaged sample, if somehow one can directly obtain the 3D density profile. In fact, there is an ongoing project in our group that intends to reconstruct the 3D profile from a set of 2D fluorescence image “slices”.
 - Finally, and to bind the two main topics discussed in this thesis, the thermodynamic approach in terms of global variables finds in a QT trapped BEC, a system where certainly the LDA alternative is quite limited due to the large spatial density variations. Moreover, the transition to the QT state could be better described by providing a macroscopic view of the system¹ (e.g. with thermodynamic global variables) instead of a local picture of a tangle of vortices.

¹Just like the aspect ratio inversion has become a strong signature for the onset of BEC.

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Appendix

APPENDIX A – Vacuum design

In this appendix we show how to calculate the pressure throughout a vacuum system. For a general discussion on vacuum physics and technology we recommend Reference (122).

Consider a vacuum system composed of compartments and vacuum pumps, all connected by tubes. We define a quantity known as throughput:

$$Q \equiv \frac{d}{dt}(Pv), \quad (\text{A.1})$$

where P is the pressure in a given compartment, and v is the gas volume leaving it through a certain connexion. Thus, Q represents the energy per unit time required to transport the gas, and it is proportional to the mass flow for a fixed temperature.

Let us focus on the steady state, in which each compartment is at a constant pressure, and the sum of all in- and outward throughputs is zero. In this case, we find

$$Q = PS, \quad (\text{A.2})$$

where S is either a pumping speed, V , or a tube conductance, C , both usually given in units of L/s. Therefore, if the compartments are connected in series, in a given compartment n , the mass balance writes

$$R_n + (P_{n-1} - P_n)C_{n-1,n} + (P_{n+1} - P_n)C_{n,n+1} - V_n P_n = 0, \quad (\text{A.3})$$

where we introduced an additional throughput quantity R_n to account for all other sources or sinks, which can be uncontrolled (e.g. leaks, outgassing, diffusion through the walls, “spontaneous” desorption) or controlled (e.g. ovens, dispensers, LIAD).

Equation A.3 can be rewritten in matrix form

$$(\hat{C} + \hat{V}) \cdot \vec{P} = -\vec{R}, \quad (\text{A.4})$$

which solving for pressure yields

$$\vec{P} = (\hat{C} + \hat{V})^{-1} \cdot (-\vec{R}), \quad (\text{A.5})$$

Usually $\vec{R}$ can be assumed to have a single non-zero element, i.e. all sources are neglected, except for one controlled source at some compartment. In this case, independently of this non-zero value, it is possible to calculate a relative value between the elements of $\vec{P}$ allowing us to characterize the differential pumping system.

To deal with conductances and pumping speeds we provide the additional information as follows. Equation A.2 is analogous to Ohms Law $U = RI$, where $I = Q$, $U = P$, and $R = 1/S$. Following this analogy the equivalent conductance of two tubes connected either in series or in parallel is respectively,

$$\frac{1}{C_{eq}^{(series)}} = \frac{1}{C_1} + \frac{1}{C_2}, \quad (\text{A.6})$$

and

$$C_{eq}^{(parallel)} = C_1 + C_2. \quad (\text{A.7})$$

The conductance of a tube connecting a vacuum pump to some compartment can be incorporated into the value of an effective pumping speed given by

$$\frac{1}{V_{eff}} = \frac{1}{V_{nom}} + \frac{1}{C_{tube}}, \quad (\text{A.8})$$

where C_{tube} is the conductance of the tube, and V_{nom} is the nominal pumping speed of the vacuum pump.

Finally, we present the Clausings formula to calculate the conductance of a round tube in the molecular flow regime¹

$$C_{tube} = 11.4 \sqrt{\frac{T}{M}} a^2 K (\ell/a), \quad (\text{A.9})$$

¹This means that the mean free path is longer than the system length

where $T(\text{K})$ is temperature, $M(\text{a.u.})$ is the molecular mass, $a(\text{cm})$ is the round tube radius, and $K(\ell/a)$ is a function of ℓ/a , where $\ell(\text{cm})$ is the tube length. C_{tube} is then given in units of L/s. For long tubes, i.e. $\ell \gg a$, one finds $K = \frac{8a}{3\ell}$. Short tubes will present $\ell \approx a$, and then $K = 1$.

The results presented in this appendix were considered when designing our vacuum system presented in [Section 4.2](#).

APPENDIX B – Controlling experiments

In this appendix we describe a hardware/software solution for controlling ultracold atoms experiments. This solution was originally implemented by P. M. Duarte (more information can be found in his master thesis ([123](#))) in R. Hulet’s group at Rice University. It consists of a main interface in LabVIEW, in which the user selects a predefined sequence written in Python. The output of the Python code is then read by LabVIEW, which finally sends the data to *National Instruments* NI-DAQ cards.

In the following, we provide more details on both software and hardware aspects.

B.1 Software

Figure [35](#) shows a sketch on how this solution works. All relevant experimental parameters are present in the *params.INI* file. LabVIEW first calls the Python code correspondent to the selected sequence. The Python code generates the file *expseq.txt*, which contains all information for the behaviour of all digital and analog channels throughout the whole sequence. LabVIEW then loads this information to the DAQ cards. While the previous tasks are being performed, LabVIEW is already loading the MOT. Note that MOT should be the starting point of any sequence. In fact, we use a MOT fluorescence signal to trigger the sequence.

In each main Python code corresponding to a particular type of experiment (e.g.

Figure 35 – Sketch of the functioning of the hardware/software solution (described in this appendix) for controlling ultracold atoms experiments.

“magnetic trap fluorescence” or “optical dipole trap absorption imaging”) we import modules (also written in Python) corresponding to the processes presented in this type of experiment. For example, if we wish to optimize the magnetic trapping, the main code would first call the “Cooling and Compression” module. Second, the optical pumping module would transfer the atoms to a trappable state. Then the magnetic trap would be turned on by calling its module, and finally a imaging module would prepare the camera and generate the probe pulses. In this way, once a module is optimized by running some experimental realization, it gets automatically optimized for any other sequence that uses this module.

The organization of the main interface also represents a great feature of this implementation. For each measurement, a four digits number is assigned ($####$). All relevant parameters are saved in the *report####.INI* file. To the file with the channels behaviour (*exp####.txt*) and to the images, we also assign those four digits. All these files are saved in the folder of the current day, which is saved in the folder of the current month, and so on to the year. Therefore, one can always check all (controlled) experimental conditions for any data. Moreover, one can easily repeat a particular interesting experimental realization.

B.2 Hardware

One of the most important aspects of this solution relies on the **hardware timing** capabilities. This means that the whole sequence is first loaded into the cards and then the outputs of the digital channels are selected one by one by a hardware trigger signal. Then some of these digital channels will trigger cards with analog channels. In this way we can rest in the knowledge that the output of the channels will not depend on any processing speed, and certainly have more reproducible experimental realizations. Therefore, the digital channels to be used in the sequences must have **hardware timing** capabilities.

The main hardware trigger signal (that trigger all digital channels used in sequences) comes from another digital channel of one of the cards, and it will commit the whole timing engine of the digital channel. To clarify, let us give two examples: *(i)* the NI-6733 analog card has 8 hardware timing digital channels sharing a single timing engine. If we use one of them to generate the main hardware trigger signal (triggering digital channels in other cards) the remaining 7 digital channels cannot be used in sequences; *(ii)* the NI-6534 digital card has 32 digital channels (8 channels in 4 different ports). There are two timing engines, one for ports 0 and 1, and another for ports 2 and 3. If we use one of them as the main trigger, we “lose” 15 digital channels that no longer can be used in sequences. To overcome this “loss” of digital channels, we propose the use a FPGA module to generate the main hardware trigger signal. This certainly would be a great upgrade.

In Figure 35 we show a picture of the NI-PXI interface, but this is not a requirement. In fact, recently we have succeeded in implementing this solution with PCI cards. Finally we must mention that, when using GPIB interface in some devices, it is important to select modes that allow the device to be adjusted by either digital or analog signals. Since the spirit of this solution is to avoid any software timing, GPIB commands must be sent prior to the begin of the sequence (while MOT loads, for example), and after the sequence starts, only digital and analog signals control everything.