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INSTITUTO DE FÍSICA DE SÃO CARLOS**

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**Quantum non-Markovianity induced by classical
stochastic noise**

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José Inácio da Costa Filho

**Quantum non-Markovianity induced by classical
stochastic noise**

Dissertation presented to the Graduate Program in Physics at the Instituto de Física de São Carlos, Universidade de São Paulo to obtain the degree of Master of Science.

Concentration area: Basic Physics

Advisor: Prof. Dr. Diogo de Oliveira Soares
Pinto

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“One of the keys to happiness is a bad memory.”

Rita Mae Brown

ABSTRACT

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One of the main goals of the theory of open quantum systems is to devise methods which help preserve the quantum properties of a system interacting with its environment. One possible pathway to achieve this goal is to use non-Markovian reservoirs, characterized by information backflows and revivals of certain quantum properties. These reservoirs usually require advanced engineering techniques, which may turn their implementation impractical. In this dissertation we propose an alternative technique: the injection of a classical colored noise, which induces the desired quantum non-Markovianity. In order to do that, we investigate the dynamics of a quantum system interacting with its surrounding environment and under the injection of a classical stochastic colored noise. A time-local master equation for the system is derived by using the stochastic wave function formalism and functional calculus. Afterwards, the non-Markovianity of the evolution is detected by using the Andersson, Cresser, Hall and Li measure, which is based on the decay rates of the master equation in canonical Lindblad-like form. Finally, we evaluate the measure for three different colored noises and study the interplay between environment and noise pump necessary to induce quantum non-Markovianity, as well as the energy balance of the system.

Keywords: Quantum non-Markovianity. Open quantum systems. Quantum information theory.

RESUMO

COSTA-FILHO, J. I. **Quantum non-Markovianity induced by classical stochastic noise**. 2017. 184p. Dissertação (Mestrado em Ciências). Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2017.

Um dos objetivos principais da teoria de sistemas quânticos abertos é desenvolver métodos que ajudem a preservar as propriedades quânticas de um sistema interagindo com o ambiente. Um possível caminho para alcançar essa meta é usar reservatórios não-Markovianos, caracterizados por refluxos de informação e renascimento de certas propriedades quânticas. Esses reservatórios geralmente requerem o uso de técnicas avançadas de engenharia, o que pode tornar sua implementação impraticável. Nessa dissertação nós propomos uma técnica alternativa: a injeção de um ruído colorido clássico, o qual induz a desejada não-Markovianidade quântica. De modo a fazer isso, nós investigamos a dinâmica de um sistema quântico interagindo com o ambiente e sob a injeção de um ruído colorido clássico estocástico. Uma equação mestra local no tempo é derivada usando-se do formalismo da função de onda estocástica e de técnicas de cálculo funcional. Após isso, a não-Markovianidade da evolução é detectada através da medida de Andersson, Cresser, Hall e Li, a qual é baseada nos coeficientes da equação mestra na forma de Lindblad-like canônica. Finalmente, nós calculamos a medida para três diferentes ruídos coloridos e estudamos a relação entre o ambiente e o bombeio estocástico necessária para induzir não-Markovianidade quântica, assim como o balanço de energia do sistema.

Palavras-chave: Não-Markovianidade quântica. Sistemas quânticos abertos. Teoria da informação quântica.

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1 INTRODUCTION

Every physical system, classical or quantum, unavoidably interacts with its surrounding environment and, therefore, is affected by it. It may happen that this influence is negligible, which allows us to treat our system of interest as if it were closed, or isolated from the external world. Although this is sometimes possible in classical physics, for quantum physics an isolated system is more of an idealization than a realistic situation. Quantum systems are extremely fragile, and even the slightest of interactions can affect them in a fundamental way.

The interaction of quantum systems with the environment is the central topic of study in the theory of open quantum systems (1–6), which aims at understanding and predicting how quantum systems and their properties behave when they are coupled to the external world. This knowledge is essential if we want to use quantum mechanics in realistic scenarios, and of paramount importance to the implementation of quantum technologies.

One of the objectives of the study of open quantum systems is to understand how information leaks to the environment and whether it could be preserved or recovered^a. (7,8) This leakage is related to processes such as dissipation and decoherence (8), where quantum properties such as entanglement and coherence are not preserved and usually lost forever. These properties are quintessential for the advantages promised by quantum mechanics (9) and losing them means losing the so advocated supremacy over classical technologies; in fact, most quantum systems become classical without them. Therefore, the possibility of recovering this information from the environment opens up new avenues in the quest for harnessing quantum mechanics.

The quest for preserving quantumness explains the growing interest in quantum non-Markovianity . (10–16) Non-Markovian environments display the desired information backflows (10,17–20) that help preserve the quantum properties of a system. Quantum non-Markovianity also promises a multitude of other advantages: it is related to preservation of coherence (21), energy backflows (22), speedup of quantum speed limits (23), violations of the Landauer bound (24), formation of steady-state entanglement (25), revivals and protection of entanglement (26–32) and is an obstacle to quantum Darwinism (33), for example. It also has a plethora of applications, ranging from quantum metrology (34), superdense coding (35), quantum cryptography (36) to quantum control. (37) Recently, many experiments were conducted that verify or take advantage of non-Markovian features . (38–43) Finally, non-Markovianity is necessary for a realistic description of some quantum systems, such as strongly coupled systems (44,45), some spin baths (46), biological systems

^a The placement of citations in this text is something that I do not agree with. But rules are rules.

(47), complex nanostructures (48) and photosynthetic systems. (49) As can be seen, quantum non-Markovianity is an invaluable resource for quantum technologies, which justifies all the attention it has attracted in recent years.

The advantages of quantum non-Markovianity, however, come with a cost. Non-Markovian environments usually need to be highly structured, and require advanced reservoir engineering techniques. (34, 50–57) The main result of this dissertation is to show that one alternative to reservoir engineering is to induce quantum non-Markovianity by injection of classical noise. (58–65) A sufficiently strong noise can reverse the information flow from the system to the environment, therefore leading to the recovery of information and other quantum properties. Our procedure is based in the stochastic wave function's formalism (66, 67), where the state of the system is described by an ensemble of pure states, and the its density matrix is recovered by an averaging process. We consider the situation of a quantum system interacting with a thermal bath and subjected to a pumping, modeled by a stochastic Hamiltonian (58–60), and we use the Andersson, Cresser, Hall and Li (ACHL) measure (68) to show that we actually have quantum non-Markovianity.

But what is quantum non-Markovianity, anyways? The concept of a non-Markovian process, although a staple the theory of classical stochastic processes (69), has no straight-forward generalization to quantum processes. In classical probability theory, Markovian stochastic processes correspond to memoryless processes: the probability of going to some future state depends only on the present state and not on the previous ones, so the process retains no memory of its past states. (10) The problem is that the definition of Markovianity is based on time conditional probabilities (69), a concept which cannot be extended to the quantum realm. (10) Therefore, we must use an alternative definition in order to express quantum Markovianity. A lot of definitions of quantum non-Markovianity were proposed in the literature (1, 2, 4, 19, 70–72), but which are in most cases not equivalent to each other, showing that quantum non-Markovianity is a much more complex and varied concept than its classical counterpart.

This dissertation is organized as follows. In Chapter 2 we start with a review of the basic postulates of quantum mechanics for closed systems, which are then generalized in order to encompass open quantum systems. This leads to the important concept of a quantum channel, the most general form of a quantum evolution. After that we derive open quantum evolutions from subsystem dynamics, and we show that not all initial system-environment correlations are allowed if we want to describe the evolution as a quantum channel. Finally, we introduce time-dependent quantum channels and the related concept of divisibility, which is the backbone for the definitions of quantum non-Markovianity.

In Chapter 3 we continue with the theory of open quantum dynamics, but now working with master equations, which are equations of motion for quantum systems. We put emphasis on time-local master equations, which can be written in a slight generalization

of the well-known Lindblad form. (73) Finally, we show how to derive a master equation in Lindblad form from subsystem dynamics under the Born-Markov approximation. (1)

Chapter 4 deals with non-Markovianity, the core concept of this dissertation. Starting with the definition of non-Markovianity for classical stochastic processes, we show how it can be defined, detected and measured in the quantum realm.

Chapter 5 uses the stochastic wavefunction formalism (3,60) to derive a master equation for a system subjected to the action of a stochastic pump. The resulting evolution can be cast in the Lindblad-like form and can be non-Markovian.

Chapter 6 contains the main result of this text. We put together the Born-Markov approximation and the stochastic wavefunction formalism in order to derive a master equation for a system interacting with both an environment and a stochastic pump. We study the example of a qubit under amplitude damping and use the Andersson, Cresser, Hall and Li (ACHL) measure (68) to show that this dynamics is non-Markovian.

Finally, in Chapter 7 we present some concluding remarks and also possible future directions of this research topic. Appendices A, B, C, D and E present reviews of basic topics and some technical details which were left out of the main text.

The results presented in Chapter 6 were published in Ref. (74)

2 QUANTUM DYNAMICS

The goal of this chapter is to develop a framework for open quantum dynamics. Starting from the formalism of closed quantum systems, we expand it to encompass systems interacting with the external world. We develop, in Section 2.1, the theory of quantum channels, which are the most general class of allowed evolutions for open systems, and then, in Section 2.3, we see how they can arise from subsystem dynamics. After that, in Section 2.4 we extend the theory to time-dependent evolutions, which are represented by quantum dynamical maps.

2.1 Closed versus open quantum systems

We start this Section by briefly reviewing the usual textbook postulates of quantum mechanics, which only describe closed systems. (75–77) We then rewrite them in a more general form, and finally we extend the framework in order to encompass open quantum evolutions. The interested reader should refer to. (1, 4, 9, 75, 78–82)

Quantum mechanics is stated in the language of Hilbert spaces. (83) In this text, unless otherwise stated, all Hilbert spaces are complex and finite-dimensional, so we mostly use linear algebra, which is reviewed in Appendix A. We use the Dirac notation (84), where elements (*vectors*) of a Hilbert space $\mathcal{X}$ are denoted by kets $|\cdot\rangle$, elements of the dual space $\mathcal{X}^*$ by bras $\langle\cdot|$ and the inner product of $|\phi\rangle$ and $|\psi\rangle$ by $\langle\phi|\psi\rangle$. We also denote the set of linear operators from $\mathcal{X}_1$ to $\mathcal{X}_2$ by $\mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$ (and $\mathbf{L}(\mathcal{X})$, for brevity, when $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}$) and the set of linear operators from $\mathbf{L}(\mathcal{X}_1)$ to $\mathbf{L}(\mathcal{X}_2)$ by $\mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, whose elements are usually called *superoperators*, since they are operators that act on operators. The identity operator acting on the space $\mathcal{X}$ is written as $\mathbb{1}_{\mathcal{X}}$, and the identity superoperator acting on $\mathbf{L}^2(\mathcal{X})$ as $\mathcal{I}_{\mathcal{X}}$ (the subscripts are absent when there is no risk of confusion). We usually reserve capital letters for operators, calligraphic letters for superoperators and Hilbert spaces, and bold capital letters for spaces of operators. The usual postulates of quantum mechanics, which describe closed quantum systems, are as follows (75–77):

- 1 Representation: Every quantum system is associated with a Hilbert space $\mathcal{X}$, called its *state space*. A state is described by a unit vector $|\psi\rangle \in \mathcal{X}$, i.e., a vector with unit norm, which is called a *state vector* or *wavefunction*. The wavefunction is unique up to scalar multiplication by an arbitrary unitary complex number: if $\alpha \in \mathbb{C}$ and $|\alpha| = 1$, then $|\psi\rangle$ and $\alpha|\psi\rangle$ represent the same state.
- 2 Composition: Quantum systems can be composed by means of the tensor product. So, if $\mathcal{X}_A$ represents system A and $\mathcal{X}_B$ represents system B, the joint or composite system AB is represented by $\mathcal{X}_A \otimes \mathcal{X}_B$.

3 Measurement: a physical observable is represented by an Hermitian operator^a $A \in \mathbf{L}(\mathcal{X})$. Given its spectral decomposition,

$$A = \sum_{i=1} a_i \Pi_i, \quad (2.1)$$

a state $|\psi\rangle$, after measurement, evolves to the state

$$|\psi_i\rangle = \frac{\Pi_i |\psi\rangle}{\sqrt{\langle\psi| \Pi_i |\psi\rangle}} \quad (2.2)$$

with probability $p_i = \langle\psi| \Pi_i |\psi\rangle$, given the outcome a_i occurred.

4 Evolution: wavefunctions evolve linearly,

$$|\psi'\rangle = U |\psi\rangle, \quad (2.3)$$

where $U \in \mathbf{L}(\mathcal{X})$. Since the evolved state must be a wavefunction too, the linear operator must preserve the norm of vectors (and, consequently, the inner products between them. (85)) Linear operators $U \in \mathbf{L}(\mathcal{X})$ that preserve inner products are called *unitary operators*, and are equivalently characterized by the property (86)

$$UU^\dagger = U^\dagger U = \mathbf{1}_{\mathcal{X}}. \quad (2.4)$$

In short, we say that quantum states evolve unitarily.

Although powerful, these postulates are not general enough even for closed quantum systems. Sometimes we need to deal with ensembles (mixtures) of states, where one does not have complete knowledge of the state of the system^b. They are formally represented by density operators,

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|, \quad (2.5)$$

where $\mathbf{p} = (p_i)$ is a probability vector ($p_i \geq 0$, for every i , and $\sum_i p_i = 1$) and $|\psi_i\rangle$ are wavefunctions. In an abuse of language, we sometimes call them *quantum states*, although they are actually mixtures of quantum states. In order to distinguish them from "genuine" states, we call them *mixed states*, where wavefunctions are called *pure states*. (9) Density operators are mathematically characterized by the following properties:

$$\rho \in \mathbf{L}(\mathcal{X}), \quad \rho^\dagger = \rho, \quad \rho \geq 0, \quad \text{tr} \rho = 1, \quad (2.6)$$

i.e., they are linear, Hermitian, positive semidefinite and with unit trace^c. The set of density operators on $\mathcal{X}$ is denoted by $\mathbf{D}(\mathcal{X})$ ^d. Therefore, our new postulate is:

^a Some properties of Hermitian operators are presented in Appendix A, Section A.3.

^b These situations, as we will see in Section 2.2.6, arise naturally in some open evolutions.

^c Actually, the positive semidefiniteness property implies the Hermiticity property, since every positive semidefinite operator is Hermitian.

^d Check Appendix A, Section A.6 for more properties of density operators.

1' Representation: Every quantum system is associated with a Hilbert space $\mathcal{X}$, called its *state space*. A quantum system is completely described by a density operator $\rho \in \mathbf{D}(\mathcal{X})$.

An important property of density operators $\rho \in \mathbf{D}(\mathcal{X})$, which will be used later, is that they admit *purifications*^e, i.e.,

$$\rho = \text{tr}_{\mathcal{Y}} [|\Psi\rangle\langle\Psi|], \quad (2.7)$$

for some space $\mathcal{Y}$, where $|\Psi\rangle \in \mathcal{X} \otimes \mathcal{Y}$. That means that every system can be considered as a pure state in a larger Hilbert space. Any two different purifications $|\Psi\rangle, |\Phi\rangle \in \mathcal{X} \otimes \mathcal{Y}$ of the same system ρ can be related by^f

$$|\Psi\rangle = (\mathbb{1}_{\mathcal{X}} \otimes U) |\Phi\rangle, \quad (2.8)$$

where $U \in \mathbf{L}(\mathcal{Y})$ is an unitary operator.

The composition of density operators is performed in the same fashion as for the case of wavefunctions, so the second postulate remains unchanged.

As for measurements, note that the eigenvalues a_i of the observable A are mere labels of each outcome, and what really matters are the eigenprojections Π_i , which dictate the probabilities and the post-measurement state. Therefore, we can ignore the eigenvalues altogether and deal only with the eigenprojections. Then, a measurement is defined by a set of operators $\{\Pi_i\}$ which satisfy

$$\Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \Pi_i^\dagger = \Pi_i, \quad \sum_i \Pi_i = \mathbb{1}. \quad (2.9)$$

The first two properties state that they are *projections*^g, and the third property that they form a *resolution of the identity*, Eq. (A.20), which ensures that the probabilities sum up to one:

$$\sum_i p_i = \sum_i \langle\psi| \Pi_i |\psi\rangle = \langle\psi| \sum_i \Pi_i |\psi\rangle = \langle\psi|\psi\rangle = 1. \quad (2.10)$$

The set $\{\Pi_i\}$ is called a *projection-valued measure* (PVM) or *projective measurement*. (9)

Projective measurements are not the most general form of measurements. After one measurement is performed, posterior measurements yield the same post-measurement state, which is a consequence of the property $\Pi_i^2 = \Pi_i$. There are situations, however, in which that *repeatability*^h property (9) is not verified, or in which there is not even a post-measurement state, such as in photodetection, where the photon is destroyed in the

^e See Appendix A, Section A.6.

^f This is a special case of Theorem 19.

^g Mathematicians usually define projections as satisfying only $\Pi_i^2 = \Pi_i$, and our definition coincides with their definition of *orthogonal projections*. (86)

^h Property $\Pi_i^2 = \Pi_i$ implies that, after the first measurement, consecutive measurements yield the same post-measurement state.

measurement process. (9,87) There are also situations where different measurements are not mutually exclusive, so distinct measurement operators are not orthogonal. This leads us to drop the projection property and define general measurement operators:

3' Measurement: quantum measurements are described by a set of *measurement operators* $\{M_i\}$ which satisfy

$$\sum_i M_i^\dagger M_i = \mathbb{1}. \quad (2.11)$$

A state ρ , after measurement, evolves to the state

$$\rho_i = \frac{M_i \rho M_i^\dagger}{\text{tr}[M_i^\dagger M_i \rho]} \quad (2.12)$$

with probability $p_i = \text{tr}[M_i^\dagger M_i \rho]$, given the outcome i occurred.

Note that, for pure states,

$$p_i = \text{tr}[M_i^\dagger M_i |\psi\rangle \langle \psi|] = \langle \psi | M_i^\dagger M_i | \psi \rangle, \quad (2.13)$$

and

$$|\psi_i\rangle = \frac{M_i |\psi\rangle}{\sqrt{\langle \psi | M_i^\dagger M_i | \psi \rangle}}, \quad (2.14)$$

as would be expected. We recover projective measurements by simply setting $M_i = \Pi_i$ and noting that $\Pi_i^\dagger \Pi_i = \Pi_i^2 = \Pi_i$. Now, however, we can work with more general conditions. If we define $E_i = M_i^\dagger M_i$, we have a *positive operator-valued measure* (POVM) . (5,88,89) Since any operator of the form $M^\dagger M$ is positive semidefinite (90), we have that

$$E_i \geq 0, \quad \sum_i E_i = \mathbb{1}, \quad (2.15)$$

which defines the POVM. The probabilities for each outcome i are $p_i = \text{tr}[\rho E_i]$, but now the post-measurement state is not well definedⁱ. POVM's are useful for situations where the post-measurement state is not important, such as the already cited case of photodetection, or when we need an refined level of control over the system. (9)

Finally, we are left only to define how density operators evolve. Since they are operators themselves, evolutions are now represented by linear superoperators,

$$\rho' = \mathcal{E}(\rho), \quad (2.16)$$

where $\mathcal{E} \in \mathbf{L}^2(\mathcal{X})$, and the usual unitary evolution is recovered by

$$\rho' = \mathcal{U}(\rho) = U \rho U^\dagger. \quad (2.17)$$

The main advantage of this new set of postulates is that they move seamlessly to the open quantum domain. In fact, the only one that needs to be changed is the evolution postulate, which will be the focus of the next Section.

ⁱ We could choose $M_i = \sqrt{E_i}$, but the square root of a positive semidefinite operator is not uniquely defined. (85)

2.2 Quantum channels

How can we define quantum evolutions, or *quantum dynamics*, for open quantum systems? In the same spirit as for the usual evolution postulate, we could require an linear evolution which preserves quantum states. But now quantum systems are not represented by wavefunctions but by density operators, which are operators $\rho \in \mathbf{L}(\mathcal{X})$ such that

$$\rho^\dagger = \rho, \quad \rho \geq 0, \quad \text{tr} \rho = 1. \quad (2.18)$$

Quantum evolutions would be, therefore, linear superoperators which preserve all of these properties. The objective of this Section is to find superoperators which preserve each of these properties alone: Hermiticity, positivity and trace preserving maps. We then combine them in order to characterize the most general quantum evolutions, which are called *quantum channels*. (78) We also introduce three useful representations of these maps, the Choi, Kraus and Stinespring representations, and a few examples of common open evolutions. This Section is mostly based on the lecture notes by Watrous. (78)

2.2.1 Linear maps

Before delving into more specific cases, we develop some useful tools for general linear superoperators. First of all, instead of only dealing with "square" maps $\mathcal{E} \in \mathbf{L}^2(\mathcal{X})$, we work with maps $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$. That means that the input and output Hilbert spaces, respectively $\mathcal{X}_1$ and $\mathcal{X}_2$, need not be the same. This added generality, which is absent in some famous introductions to quantum channels (1, 9, 79) but present in others (78, 91), is not uncontournable, and in fact a map $\mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ can be produced from a map in $\mathbf{L}^2(\mathcal{X}_1)$ with the aid of *ancillas*^j and *partial traces*. (9) The reason for using them, therefore, is to encompass, in the same framework of quantum channels, operations that change the system's Hilbert space, such as the already mentioned addition of ancillas and partial traces, but also *coarse grainings* (82, 93), *assignment maps* (which will be defined in the next Section) and many others. It also has a pedagogical reason, since we can keep track of which system is the input and which is the output, since these two will be "mixed together" when we deal with Choi matrices. In order to avoid misunderstandings, it is never enough to make it clear that $\mathcal{X}_1$ and $\mathcal{X}_2$ represent a quantum system before and after some evolution, respectively, and are **not**, for example, a system $\mathcal{X}_1$ coupled to an environment $\mathcal{X}_2$. Finally, we denote by d_1 and d_2 the dimensions of $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively.

Here we will consider three useful representations of linear superoperators in $\mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$. (78) They are well known for quantum channels (9, 79), but can be easily extended to this more general case. The first one, the *Choi representation* or *Choi matrix* (94) of $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, is defined as

$$\Lambda_{\mathcal{E}} = (\mathcal{I}_{\mathcal{X}_1} \otimes \mathcal{E}) (|\Omega\rangle \langle \Omega|), \quad (2.19)$$

^j This term, which is now widely used in quantum information theory, has a very unfortunate etymology. (92)

where

$$|\Omega\rangle = \sum_i |i, i\rangle \quad (2.20)$$

is a (unnormalized) maximally entangled state in $\mathcal{X}_1 \otimes \mathcal{X}_1$, and $\{|i\rangle\}$ is an orthonormal basis for $\mathcal{X}_1$. Note that $\Lambda_{\mathcal{E}} \in \mathbf{L}(\mathcal{X}_1 \otimes \mathcal{X}_2)$: the identity operator, acting on the first half of $|\Omega\rangle$, does nothing; and the map $\mathcal{E}$, acting on the second half, leads the system from $\mathcal{X}_1$ to $\mathcal{X}_2$. Therefore, $\Lambda_{\mathcal{E}}$ is a matrix in the bipartite space $\mathcal{X}_1 \otimes \mathcal{X}_2$ consisting of the system's input and output: in this matrix, it is as if "future" and "past" coexisted together. Note that the Choi matrix is usually defined in the literature as a matrix in $\mathbf{L}(\mathcal{X}_2 \otimes \mathcal{X}_1)$ (94), but this difference is a simple matter of convention and does not lead to different results. The rank of $\Lambda_{\mathcal{E}}$ is called the *Choi rank* of $\mathcal{E}$. (78)

Equation (2.19) can be written in another two different forms. First, note that

$$\begin{aligned} \Lambda_{\mathcal{E}} &= (\mathcal{I}_{\mathcal{X}_1} \otimes \mathcal{E}) (|\Omega\rangle \langle \Omega|) \\ &= \sum_{i,j} (\mathcal{I}_{\mathcal{X}_1} \otimes \mathcal{E}) (|i\rangle \langle j| \otimes |i\rangle \langle j|) \\ &= \sum_{i,j} \mathcal{I}_{\mathcal{X}_1} (|i\rangle \langle j|) \otimes \mathcal{E}(|i\rangle \langle j|), \end{aligned} \quad (2.21)$$

so we have

$$\Lambda_{\mathcal{E}} = \sum_{i,j} |i\rangle \langle j| \otimes \mathcal{E}(|i\rangle \langle j|). \quad (2.22)$$

For the last form, we introduce the notion of vectorization of an operator. (10, 78, 95) If some operator $A \in \mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$ is written as

$$A = \sum_{i,j} A_{ij} |i\rangle \langle j|, \quad (2.23)$$

then its *vectorization* is

$$|A\rangle\rangle = \sum_{i,j} A_{ij} |j, i\rangle \in \mathcal{X}_2 \otimes \mathcal{X}_1. \quad (2.24)$$

Therefore, it basically performs the transformation $|i\rangle \langle j| \rightarrow |j\rangle \otimes |i\rangle$. Intuitively, the operation "stacks" the columns of A on top of each other, in order to form a vector. (95) This operation is also invertible: every vector $|A\rangle\rangle \in \mathcal{X}_2 \otimes \mathcal{X}_1$ corresponds to an operator $A \in \mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$. Now, since $\mathbb{1}_{\mathcal{X}_1} = \sum_i |i\rangle \langle i|$, we have that

$$|\mathbb{1}_{\mathcal{X}_1}\rangle\rangle = \sum_i |i, i\rangle = |\Omega\rangle, \quad (2.25)$$

and the Choi matrix can be written as

$$\Lambda_{\mathcal{E}} = (\mathcal{I}_{\mathcal{X}_1} \otimes \mathcal{E}) (|\mathbb{1}_{\mathcal{X}_1}\rangle\rangle \langle\langle \mathbb{1}_{\mathcal{X}_1}|). \quad (2.26)$$

Equations (2.19), (2.22) and (2.26) are useful in different contexts.

The operation of taking the Choi matrix is actually an isomorphism between linear superoperators and linear operators, i.e., between $\mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ and $\mathbf{L}(\mathcal{X}_1 \otimes \mathcal{X}_2)$, and is known as the Choi-Jamiolkowski isomorphism (94, 96–98):

Theorem 1. (*Choi-Jamolkowski Isomorphism*). The mapping

$$\begin{aligned} \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2) &\rightarrow \mathbf{L}(\mathcal{X}_1 \otimes \mathcal{X}_2) \\ \mathcal{E} &\mapsto \Lambda_{\mathcal{E}} = (\mathcal{I}_{\mathcal{X}_1} \otimes \mathcal{E}) (|\Omega\rangle \langle \Omega|), \end{aligned} \quad (2.27)$$

is an isomorphism, whose inverse is given by

$$\begin{aligned} \mathbf{L}(\mathcal{X}_1 \otimes \mathcal{X}_2) &\rightarrow \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2) \\ \Lambda_{\mathcal{E}} &\mapsto \mathcal{E}(X) = \text{tr}_1 \left[(X^T \otimes \mathbf{1}_{\mathcal{X}_2}) \Lambda_{\mathcal{E}} \right], \end{aligned} \quad (2.28)$$

where $X \in \mathbf{L}(\mathcal{X}_1)$ and X^T is the transpose^k of X .

Proof. Using the definition (2.22) of the Choi Matrix,

$$\begin{aligned} \mathcal{E}(X) &= \text{tr}_1 \left[(X^T \otimes \mathbf{1}_{\mathcal{X}_2}) \Lambda_{\mathcal{E}} \right] \\ &= \sum_{i,j} \text{tr}_1 \left[(X^T \otimes \mathbf{1}_{\mathcal{X}_2}) (|i\rangle \langle j| \otimes \mathcal{E}(|i\rangle \langle j|)) \right] \\ &= \sum_{i,j} \text{tr}_1 \left[X^T |i\rangle \langle j| \right] \mathcal{E}(|i\rangle \langle j|) \\ &= \sum_{i,j,k} \langle k| X^T |i\rangle \langle j|k\rangle \mathcal{E}(|i\rangle \langle j|) \\ &= \sum_{i,j} \langle j| X^T |i\rangle \mathcal{E}(|i\rangle \langle j|) \\ &= \mathcal{E} \left(\sum_{i,j} \langle j| X^T |i\rangle |i\rangle \langle j| \right) \\ &= \mathcal{E} \left(\sum_{i,j} \langle i| X |j\rangle |i\rangle \langle j| \right) \\ &= \mathcal{E}(X), \end{aligned} \quad (2.29)$$

so the mappings $\mathcal{E} \mapsto \Lambda_{\mathcal{E}}$ and $\Lambda_{\mathcal{E}} \mapsto \mathcal{E}(X)$ are the inverse of each other. $\square$

The next representation is the *Kraus operator sum representation*. (99) The Kraus form for $\mathcal{E}$ is defined as

$$\mathcal{E}(X) = \sum_i A_i X B_i^\dagger, \quad (2.30)$$

where $\{A_i\}$ and $\{B_i\}$ are sets of operators in $\mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$ and are called *Kraus operators*. It turns out that every linear superoperator has a Kraus form:

Theorem 2. Every $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ admits a Kraus representation.

Proof. By the singular value decomposition, Theorem 16, the Choi matrix of $\mathcal{E}$ can be written as

$$\Lambda_{\mathcal{E}} = \sum_i^r |L_i\rangle \langle R_i|, \quad (2.31)$$

^k See Appendix A, Section A.2, for the definition.

where $|L_i\rangle\rangle, |R_i\rangle\rangle \in \mathcal{X}_1 \otimes \mathcal{X}_2$ are its left and right singular vectors¹, respectively, and r is its Choi rank. Since the vectorization operation is invertible, consider $L_i, R_i \in \mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$ the operators which correspond to the left and right singular vectors (we wrote these vectors in the "double ket" notation with this association in mind). They are exactly the Kraus operators of the map:

$$A_i = L_i, \quad B_i = R_i. \quad (2.32)$$

This is not so useful for calculations, so we express the operators in matrix form. Fixing orthonormal bases $\{|\alpha\rangle\}$ and $\{|\beta\rangle\}$ in $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively, we define A_i and B_i as

$$A_i = \sum_{\alpha, \beta} \langle \alpha, \beta | L_i \rangle \rangle |\beta\rangle \langle \alpha| \quad (2.33)$$

$$B_i = \sum_{\alpha, \beta} \langle \alpha, \beta | R_i \rangle \rangle |\beta\rangle \langle \alpha|. \quad (2.34)$$

We could equivalently define A_i and B_i by their matrix elements,

$$(A_i)_{\beta\alpha} = \langle \alpha, \beta | L_i \rangle \rangle, \quad (2.35)$$

$$(B_i)_{\beta\alpha} = \langle \alpha, \beta | R_i \rangle \rangle. \quad (2.36)$$

Now, note that

$$A_i |\alpha\rangle = \langle \alpha | L_i \rangle \rangle, \quad (2.37)$$

$$B_i |\alpha\rangle = \langle \alpha | R_i \rangle \rangle. \quad (2.38)$$

Let us calculate $\langle \alpha | \Lambda_{\mathcal{E}} | \alpha' \rangle$ (Note that this is an operator on $\mathbf{L}(\mathcal{X}_2)$) using, separately, Equations (2.31) and (2.22). On the one hand,

$$\begin{aligned} \langle \alpha | \Lambda_{\mathcal{E}} | \alpha' \rangle &= \sum_i^r \langle \alpha | L_i \rangle \rangle \langle \langle R_i | \alpha' \rangle \\ &= \sum_i^r A_i |\alpha\rangle \langle \alpha' | B_i^\dagger, \end{aligned} \quad (2.39)$$

but, on the other hand,

$$\begin{aligned} \langle \alpha | \Lambda_{\mathcal{E}} | \alpha' \rangle &= \sum_{i,j}^r \langle \alpha | i \rangle \langle j | \alpha' \rangle \otimes \mathcal{E}(|i\rangle \langle j|) \\ &= \mathcal{E}(|\alpha\rangle \langle \alpha'|). \end{aligned} \quad (2.40)$$

Comparing both outcomes, we have that

$$\mathcal{E}(|\alpha\rangle \langle \alpha'|) = \sum_i^r A_i |\alpha\rangle \langle \alpha' | B_i^\dagger. \quad (2.41)$$

¹ In this case, we incorporated the singular values in the left and right singular vectors, in order to simplify the decomposition.

Since we have how the map acts on the basis elements of the space, we can find its action on any operator by linearity, and

$$\mathcal{E}(X) = \sum_i^r A_i X B_i^\dagger, \quad (2.42)$$

which completes our proof. $\square$

The minimum number of different Kraus operators is called the Kraus rank, and by the above proof it can be seen that it is equal to the Choi rank of $\mathcal{E}$, so we call it the *Choi-Kraus rank*. Also note that the Kraus representation is not unique, since any decomposition of the form of Eq. (2.31) will yield different Kraus operators.

Finally, we have the *Stinespring representation* or *Stinespring dilation* (100), which states that there is some Hilbert space $\mathcal{Y}$ such that

$$\mathcal{E}(X) = \text{tr}_{\mathcal{Y}} (A X B^\dagger), \quad (2.43)$$

where $A, B \in \mathbf{L}(\mathcal{X}_1, \mathcal{X}_2 \otimes \mathcal{Y})$. As the previous representations, the Stinespring dilation is usually known in a slightly different form, which is specific for quantum channels and has a clear physical interpretation. (9) The form (2.43) is not so intuitive, so this is the price we pay for the added generality; at least, however, it has a clearer connection with the other representations. In the end, of course, when we define quantum channels, we will recover the more usual form.

Theorem 3. *Every $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ admits a Stinespring representation.*

Proof. By theorem 2, every $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ admits a Kraus representation

$$\mathcal{E}(X) = \sum_i^N A_i X B_i^\dagger, \quad (2.44)$$

where N is the number of Kraus operators. Define $\mathcal{Y}$ as a N -dimensional Hilbert space, and fix an orthonormal basis $|i_{\mathcal{Y}}\rangle$ on it. Now, define

$$A = \sum_i^N A_i \otimes |i_{\mathcal{Y}}\rangle, \quad (2.45)$$

$$B = \sum_i^N B_i \otimes |i_{\mathcal{Y}}\rangle. \quad (2.46)$$

A few remarks must be made about the operators A and B . First, note the odd structure of a tensor product between a matrix and a vector: although mathematically there is nothing wrong about it, physically a tensor product between an operator and a state does not make sense. But remember that A and B are not "square" operators, but operators in $\mathbf{L}(\mathcal{X}_1, \mathcal{X}_2 \otimes \mathcal{Y})$: they are merely abstract mathematical constructs, and physically the

combined action of A and B represents an evolution **together** with the addition of an ancilla, i.e., an auxilliary system^m. Finally, note that what we are actually doing is "stacking" the Kraus operators on top of each other,

$$A = A_1 \otimes |1\rangle + \dots + A_N \otimes |N\rangle = \begin{bmatrix} A_1 \\ \dots \\ A_N \end{bmatrix}, \quad (2.47)$$

and that these operators act on an vector $|\psi\rangle \in \mathcal{X}_1$ as

$$A|\psi\rangle = \sum_i^N A_i |\psi\rangle \otimes |i_{\mathcal{Y}}\rangle, \quad (2.48)$$

which results in a vector in $\mathcal{X}_2 \otimes \mathcal{Y}$, as would be expected.

The summation in the Kraus representation goes until N , so the space $\mathcal{Y}$ is N -dimensional. Since the lowest possible value of N is r , the Choi-Kraus rank of $\mathcal{E}$, $\mathcal{Y}$ has to be at least r -dimensional. Proceeding with the calculations,

$$\begin{aligned} \mathcal{E}(X) &= \sum_i^N A_i X B_i^\dagger \\ &= \sum_{i,j}^N A_i X B_j^\dagger \delta_{ij} \\ &= \sum_{i,j}^N A_i X B_j^\dagger \text{tr}_{\mathcal{Y}}(|i_{\mathcal{Y}}\rangle \langle j_{\mathcal{Y}}|) \\ &= \text{tr}_{\mathcal{Y}} \left[\left(\sum_i A_i \otimes |i_{\mathcal{Y}}\rangle \right) X \left(\sum_j B_j \otimes |j_{\mathcal{Y}}\rangle \right)^\dagger \right] \\ &= \text{tr}_{\mathcal{Y}} (A X B^\dagger). \end{aligned} \quad (2.49)$$

□

The Stinespring representation is also not unique, since it depends on the Kraus operators, and the minimal dimension for the auxiliary Hilbert space $\mathcal{Y}$ is equal to the Choi-Kraus rank.

Since all of the three representations will be used throughout the text, it is interesting to know how to convert one into another.

Proposition 1. *Given a Kraus representation of $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$,*

$$\mathcal{E}(X) = \sum_i^N A_i X B_i^\dagger, \quad (2.50)$$

^m We will not have to deal with this issues when we work with the Stinespring dilation for quantum channels.

the Choi matrix of $\mathcal{E}$ is given by

$$\Lambda_{\mathcal{E}} = \sum_i^N |A_i\rangle\rangle\langle\langle B_i|, \quad (2.51)$$

and its Stinespring representation by

$$\mathcal{E}(X) = \text{tr}_{\mathcal{Y}}(AXB^\dagger), \quad (2.52)$$

where $A = \sum_i^N A_i \otimes |i_{\mathcal{Y}}\rangle$ and $B = \sum_i^N B_i \otimes |i_{\mathcal{Y}}\rangle$, for some N -dimensional Hilbert space $\mathcal{Y}$.

Proof. The Stinespring form follows from theorem 3. For the Choi matrix, we use Eq. (2.26),

$$\Lambda_{\mathcal{E}} = (\mathcal{I}_{\mathcal{X}_1} \otimes \mathcal{E})(|1_{\mathcal{X}_1}\rangle\rangle\langle\langle 1_{\mathcal{X}_1}|). \quad (2.53)$$

Now, since $\mathcal{I}_{\mathcal{X}_1}(\cdot) = 1_{\mathcal{X}_1}(\cdot)1_{\mathcal{X}_1}$ and $\mathcal{E}(\cdot) = \sum_i A_i(\cdot)B_i^\dagger$, we have that

$$(\mathcal{I}_{\mathcal{X}_1} \otimes \mathcal{E})(\cdot) = \sum_i (1_{\mathcal{X}_1} \otimes A_i)(\cdot) (1_{\mathcal{X}_1} \otimes B_i^\dagger), \quad (2.54)$$

which implies that

$$\begin{aligned} \Lambda_{\mathcal{E}} &= \sum_i (1_{\mathcal{X}_1} \otimes A_i)(|1_{\mathcal{X}_1}\rangle\rangle\langle\langle 1_{\mathcal{X}_1}|) (1_{\mathcal{X}_1} \otimes B_i^\dagger) \\ &= \sum_i [(1_{\mathcal{X}_1} \otimes A_i)|1_{\mathcal{X}_1}\rangle\rangle] [\langle\langle 1_{\mathcal{X}_1}| (1_{\mathcal{X}_1} \otimes B_i^\dagger)]. \end{aligned} \quad (2.55)$$

Finally, all we have to do is to evaluate $(1_{\mathcal{X}_1} \otimes A_i)|1_{\mathcal{X}_1}\rangle\rangle$. Fixing orthonormal bases $\{|\alpha\rangle\}$ and $\{|\beta\rangle\}$ in $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively, we can expand A_i as

$$\begin{aligned} A_i &= \sum_{\alpha, \beta} \langle\beta| A_i |\alpha\rangle |\beta\rangle \langle\alpha| \\ &\equiv \sum_{\alpha, \beta} (A_i)_{\beta\alpha} |\beta\rangle \langle\alpha|, \end{aligned} \quad (2.56)$$

and then

$$\begin{aligned} (1_{\mathcal{X}_1} \otimes A_i)|1_{\mathcal{X}_1}\rangle\rangle &= \left(\sum_{\alpha'} |\alpha'\rangle \langle\alpha'| \otimes A_i \right) \sum_{\alpha''} |\alpha'', \alpha''\rangle \\ &= \sum_{\alpha', \alpha''} (|\alpha'\rangle \langle\alpha'| \alpha''\rangle) \otimes (A_i |\alpha''\rangle) \\ &= \sum_{\alpha'} |\alpha'\rangle \otimes \left(\sum_{\alpha, \beta} (A_i)_{\beta\alpha} |\beta\rangle \langle\alpha| \alpha'\rangle \right) \\ &= \sum_{\alpha, \beta} (A_i)_{\beta\alpha} |\alpha, \beta\rangle = |A_i\rangle\rangle, \end{aligned} \quad (2.57)$$

with an analogous result for $|B_i\rangle\rangle$. Therefore,

$$\Lambda_{\mathcal{E}} = \sum_i |A_i\rangle\rangle\langle\langle B_i|. \quad (2.58)$$

□

Proposition 2. Given a Choi matrix of $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ with singular value decomposition

$$\Lambda_{\mathcal{E}} = \sum_i^r |L_i\rangle\rangle\langle\langle R_i|, \quad (2.59)$$

the Kraus representation of $\mathcal{E}$ is given by

$$\mathcal{E}(X) = \sum_i^r A_i X B_i^\dagger, \quad (2.60)$$

with Kraus operators $A_i = L_i$, $B_i = R_i$. The Stinespring representation is

$$\mathcal{E}(X) = \text{tr}_{\mathcal{Y}}(A X B^\dagger), \quad (2.61)$$

where $A = \sum_i^r A_i \otimes |i_{\mathcal{Y}}\rangle$ and $B = \sum_i^r B_i \otimes |i_{\mathcal{Y}}\rangle$, for an r -dimensional Hilbert space $\mathcal{Y}$.

Proof. The Kraus representation follows from theorem 2 and the Stinespring representation from theorem 3. $\square$

Proposition 3. Given a Stinespring representation of $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$,

$$\mathcal{E}(X) = \text{tr}_{\mathcal{Y}}(A X B^\dagger), \quad (2.62)$$

the Kraus representation of $\mathcal{E}$ is given by

$$\mathcal{E}(X) = \sum_i A_i X B_i^\dagger, \quad (2.63)$$

with Kraus operators

$$A_i = \langle i_{\mathcal{Y}}| A, \quad (2.64)$$

$$B_i = \langle i_{\mathcal{Y}}| B, \quad (2.65)$$

for some orthonormal basis $\{|i_{\mathcal{Y}}\rangle\}$ of $\mathcal{Y}$, and the Choi matrix of $\mathcal{E}$ is given by

$$\Lambda_{\mathcal{E}} = \sum_i |A_i\rangle\rangle\langle\langle B_i|. \quad (2.66)$$

Proof. For the Kraus representation, note that

$$\begin{aligned} \mathcal{E}(X) &= \text{tr}_{\mathcal{Y}}(A X B^\dagger) \\ &= \sum_i \langle i_{\mathcal{Y}}| A X B^\dagger |i_{\mathcal{Y}}\rangle \\ &= \sum_i (\langle i_{\mathcal{Y}}| A) X (\langle i_{\mathcal{Y}}| B)^\dagger \\ &\equiv \sum_i A_i X B_i^\dagger. \end{aligned} \quad (2.67)$$

Keep in mind that, since $A \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2 \otimes \mathcal{Y})$, multiplying it on the left by $\langle i_{\mathcal{Y}}|$ takes away the part in $\mathcal{Y}$, leading to an operator in $\mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, which is what is expected for A_i .

The Choi matrix follows from proposition 1. $\square$

The relationship between the three representations for linear superoperators is summarized in Figure 1. Note that all interconversions are straightforward except for the Choi $\rightarrow$ Kraus direction, where we have to find the singular value decomposition of the Choi matrixⁿ.

Figure 1 – Relationship between representations of linear superoperators.

Source: By the author.

Note that if you are only given how the map acts on its input, i.e., $\mathcal{E}(X)$, for all $X \in \mathbf{L}(\mathcal{X}_1)$, then you can easily find the Choi matrix first, by Eq. (2.19), and from there calculate the other representations. Conversely, given the Choi matrix, you can recover the map using Eq. (2.28). This is portrayed in Figure 2.

Figure 2 – Choi matrix from the action of the map and vice-versa.

Source: By the author.

Finally, how about composition of linear maps? Suppose we have $\mathcal{F}$, $\mathcal{E}$ and their composition $\mathcal{G} = \mathcal{E} \circ \mathcal{F}$. Since the composition of linear maps is a linear map, $\mathcal{G}$ should have a Choi matrix $\Lambda_{\mathcal{G}}$ too. Using the *link product* (97,98) we can express $\Lambda_{\mathcal{G}}$ as a function of $\Lambda_{\mathcal{E}}$ and $\Lambda_{\mathcal{F}}$. This result will not be used in the rest of this dissertation, so it is left for the curious reader in Appendix B.

ⁿ But recall that the Kraus decomposition is not unique, so any other decomposition which puts the Choi matrix in the same form will suffice.

2.2.2 Trace preserving maps

A superoperator $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ is *trace preserving* if $\text{tr}_2[\mathcal{E}(X)] = \text{tr}_1[X]$, $\forall X \in \mathbf{L}(\mathcal{X}_1)$. This condition imposes certain constraints on the representations of the map.

Theorem 4. (*Representations of trace preserving maps*). For $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, the following are equivalent (78):

- i) $\mathcal{E}$ is trace preserving.
- ii) (Choi) $\text{tr}_2[\Lambda_{\mathcal{E}}] = \mathbb{1}_{\mathcal{X}_1}$.
- iii) (Kraus) If $\mathcal{E}(X) = \sum_i A_i X B_i^\dagger$, then $\sum_i B_i^\dagger A_i = \mathbb{1}_{\mathcal{X}_1}$.
- iv) (Stinespring) If $\mathcal{E}(X) = \text{tr}_Y(A X B^\dagger)$, then $B^\dagger A = \mathbb{1}_{\mathcal{X}_1}$.

Proof. i) $\Leftrightarrow$ ii): if $\mathcal{E}$ is trace preserving, then

$$\begin{aligned} \text{tr}_2[\Lambda_{\mathcal{E}}] &= \sum_{i,j} |i\rangle \langle j| \text{tr}_2[\mathcal{E}(|i\rangle \langle j|)] \\ &= \sum_{i,j} |i\rangle \langle j| \text{tr}_1(|i\rangle \langle j|) \\ &= \sum_{i,j} |i\rangle \langle j| \delta_{ij} = \sum_i |i\rangle \langle i| = \mathbb{1}_{\mathcal{X}_1}. \end{aligned} \quad (2.68)$$

Conversely, if $\text{tr}_2[\Lambda_{\mathcal{E}}] = \mathbb{1}_{\mathcal{X}_1}$, then, using Equation (2.28) and defining orthonormal bases $\{|i\rangle\}$ and $\{|\alpha\rangle\}$ for $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively,

$$\begin{aligned} \text{tr}_2[\mathcal{E}(X)] &= \text{tr}[(X^T \otimes \mathbb{1}_{\mathcal{X}_2}) \Lambda_{\mathcal{E}}] \\ &= \sum_{i,\alpha} \langle i, \alpha | (X^T \otimes \mathbb{1}_{\mathcal{X}_2}) \Lambda_{\mathcal{E}} | i, \alpha \rangle \\ &= \sum_{i,\alpha} \langle i | X^T \langle \alpha | \Lambda_{\mathcal{E}} | \alpha \rangle | i \rangle \\ &= \sum_i \langle i | X^T \left(\sum_{\alpha} \langle \alpha | \Lambda_{\mathcal{E}} | \alpha \rangle \right) | i \rangle \\ &= \sum_i \langle i | X^T (\text{tr}_2[\Lambda_{\mathcal{E}}]) | i \rangle \\ &= \sum_i \langle i | X^T | i \rangle = \text{tr}_1[X^T] = \text{tr}_1[X], \end{aligned} \quad (2.69)$$

therefore $\mathcal{E}$ is trace preserving.

i) $\Leftrightarrow$ iii): note that, for every $X \in \mathcal{X}_1$, i) implies that

$$\text{tr}[\mathcal{E}(X)] = \text{tr} \left[\sum_i A_i X B_i^\dagger \right] = \text{tr} \left[\left(\sum_i B_i^\dagger A_i \right) X \right] = \text{tr}[X], \quad (2.70)$$

which means that $\sum_i B_i^\dagger A_i = \mathbb{1}_{\mathcal{X}_1}$. Conversely, we have that

$$\text{tr}[\mathcal{E}(X)] = \text{tr} \left[\left(\sum_i B_i^\dagger A_i \right) X \right] = \text{tr}[\mathbb{1}_{\mathcal{X}_1} X] = \text{tr}[X]. \quad (2.71)$$

$i) \Leftrightarrow iv)$: finally, we have that

$$\text{tr} [\mathcal{E}(X)] = \text{tr}_{1,\mathcal{Y}} [AXB^\dagger] = \text{tr} [(B^\dagger A) X], \quad (2.72)$$

and, by a similar reasoning, $i)$ and $iv)$ are equivalent. $\square$

Proposition 4. *The composite of trace preserving maps is trace preserving.*

Proof. Given $\mathcal{F} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ and $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_2, \mathcal{X}_3)$ trace preserving maps, and $\mathcal{G} = \mathcal{E} \circ \mathcal{F} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_3)$, then

$$\text{tr} [\mathcal{G}(X)] = \text{tr} [\mathcal{E}(\mathcal{F}(X))] = \text{tr} [\mathcal{F}(X)] = \text{tr} [X], \quad (2.73)$$

so $\mathcal{G}$ is trace preserving. $\square$

2.2.3 Hermiticity preserving maps

A superoperator $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ is *Hermiticity preserving* if it takes Hermitian operators to Hermitian operators.

Proposition 5. *(Representations of Hermiticity preserving maps). For $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, the following are equivalent (78, 101):*

- i) $\mathcal{E}$ is Hermiticity preserving.*
- ii) $\mathcal{E}(X)^\dagger = \mathcal{E}(X^\dagger)$, $\forall X \in \mathcal{X}_1$.*
- iii) (Choi) $\Lambda_{\mathcal{E}}$ is Hermitian.*
- iv) (Kraus) $\mathcal{E}(X) = \sum_i \lambda_i A_i X A_i^\dagger$, $\lambda_i \in \mathbb{R}$.*

Proof. $i) \Rightarrow ii)$: suppose $\mathcal{E}$ is Hermiticity preserving. Every $X \in \mathcal{X}_1$ can be decomposed as $X = H + iK$, where $H, K \in \mathcal{X}_1$ are Hermitian operators: that can be easily checked by setting $H = (X + X^\dagger)/2$ and $K = (X - X^\dagger)/2i$. Then,

$$\begin{aligned} \mathcal{E}(X)^\dagger &= (\mathcal{E}(H) + i\mathcal{E}(K))^\dagger \\ &= \mathcal{E}(H)^\dagger - i\mathcal{E}(K)^\dagger \\ &= \mathcal{E}(H) - i\mathcal{E}(K) \\ &= \mathcal{E}(H - iK) \\ &= \mathcal{E}(X^\dagger) \end{aligned} \quad (2.74)$$

$ii) \Rightarrow iii)$: assuming condition $ii)$, we have that

$$\begin{aligned} \Lambda_{\mathcal{E}}^\dagger &= \sum_{i,j} |j\rangle \langle i| \otimes \mathcal{E}(|i\rangle \langle j|)^\dagger \\ &= \sum_{i,j} |j\rangle \langle i| \otimes \mathcal{E}(|j\rangle \langle i|) \\ &= \Lambda_{\mathcal{E}}, \end{aligned} \quad (2.75)$$

since i and j are only dummy indices.

$iii) \Rightarrow iv)$: assuming that the Choi matrix is Hermitian, then it has a spectral decomposition

$$\Lambda_{\mathcal{E}} = \sum_i \lambda_i |M_i\rangle\rangle\langle\langle M_i|, \quad (2.76)$$

such that the λ_i , its eigenvalues, are real numbers. This means that we can define its Kraus operators as

$$A_i = B_i = M_i, \quad (2.77)$$

and therefore

$$\mathcal{E}(X) = \sum_i \lambda_i A_i X A_i^\dagger. \quad (2.78)$$

$iv) \Rightarrow i)$: assuming the above representation,

$$(\mathcal{E}(X))^\dagger = \sum_i \lambda_i^* A_i X^\dagger A_i^\dagger = \sum_i \lambda_i A_i X^\dagger A_i^\dagger = \mathcal{E}(X^\dagger), \quad (2.79)$$

for every $X \in \mathcal{X}_1$. □

Proposition 5 gives the Choi and Kraus representations of Hermiticity preserving maps. The Stinespring representation can be easily derived from them, but it is not so useful and does not bring nothing new.

Proposition 6. *The composite of Hermiticity preserving maps is Hermiticity preserving.*

Proof. Given $\mathcal{F} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ and $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_2, \mathcal{X}_3)$ Hermiticity preserving maps, and $\mathcal{G} = \mathcal{E} \circ \mathcal{F} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_3)$, then

$$\mathcal{G}(X)^\dagger = (\mathcal{E} \circ \mathcal{F}(X))^\dagger = \mathcal{E}(\mathcal{F}(X)^\dagger) = \mathcal{E}(\mathcal{F}(X^\dagger)) = \mathcal{G}(X^\dagger), \quad (2.80)$$

so $\mathcal{G}$ is Hermiticity preserving. □

2.2.4 Positive maps

A superoperator $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ is *positive* if it takes positive semidefinite operators to positive semidefinite operators^o. We sometimes use the notation $\rho \geq 0$ to denote that an operator is positive semidefinite. In particular, $\mathcal{E}$ is Hermiticity preserving, since positive semidefinite maps are Hermitian.

Proposition 7. *(Choi matrix for positive maps). For $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, $\mathcal{E}$ is positive if and only if $\Lambda_{\mathcal{E}}$ is block positive, i.e., $\langle\psi| \Lambda_{\mathcal{E}} |\psi\rangle \geq 0$, for every $|\psi\rangle \in \mathcal{X}_1 \otimes \mathcal{X}_2$ separable. (101, 102)*

^o See Appendix A, Section A.3 for some properties of positive semidefinite operators.

Proof. If $X \in \mathbf{L}(\mathcal{X}_1)$ is positive semidefinite, it can be written as $\sum_x \lambda_x \Pi_x$, where $\lambda_x \geq 0$ are its eigenvalues and $\Pi_x = |x\rangle\langle x|$, for some $|x\rangle \in \mathcal{X}_1$, are one-dimensional projections, also known as *rank-1 projections*. Then, $\mathcal{E}(X) = \sum_x \lambda_x \mathcal{E}(\Pi_x)$, and therefore $\mathcal{E}$ is a positive superoperator if and only if $\mathcal{E}(\Pi_x)$ is positive semidefinite for any one-dimensional projection.

Also, if $|\psi\rangle$ is separable, then $|\psi\rangle = |x, y\rangle$, for $x \in \mathcal{X}_1$ and $y \in \mathcal{X}_2$. Then,

$$\begin{aligned}
 \langle \psi | \Lambda_{\mathcal{E}} | \psi \rangle &= \langle x, y | \left(\sum_{i,j} |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|) \right) |x, y\rangle \\
 &= \sum_{i,j} \langle x|i\rangle \langle j|x\rangle \langle y | \mathcal{E}(|i\rangle\langle j|) |y\rangle \\
 &= \sum_{i,j} \langle j|x\rangle \langle x|i\rangle \langle y | \mathcal{E}(|i\rangle\langle j|) |y\rangle \\
 &= \sum_{i,j} \langle j | \Pi_x |i\rangle \langle y | \mathcal{E}(|i\rangle\langle j|) |y\rangle \\
 &= \langle y | \mathcal{E} \left(\sum_{i,j} \langle j | \Pi_x |i\rangle |i\rangle\langle j| \right) |y\rangle \\
 &= \langle y | \mathcal{E} \left(\Pi_x^T \right) |y\rangle \\
 &= \langle y | \mathcal{E} (\Pi_x) |y\rangle, \tag{2.81}
 \end{aligned}$$

since the transpose of a rank-1 projection is the projection itself. Now we can prove the proposition: $\langle y | \mathcal{E} (\Pi_x) |y\rangle \geq 0$, for every $|y\rangle \in \mathcal{X}_2$ if and only if $\mathcal{E} (\Pi_x)$ is positive semidefinite, which happens if and only if $\mathcal{E}$ is a positive superoperator. $\square$

Proposition 8. *The composite of positive maps is a positive map.*

Proof. Given $\mathcal{F} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ and $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_2, \mathcal{X}_3)$ positive maps, and $\mathcal{G} = \mathcal{E} \circ \mathcal{F} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_3)$, if $X \geq 0$, then $\mathcal{F}(X) \geq 0$, since $\mathcal{F}$ is positive. Since $\mathcal{E}$ is positive, $\mathcal{E}(\mathcal{F}(X)) = \mathcal{G}(X) \geq 0$, so $\mathcal{G}$ is a positive map. $\square$

A characteristic property of positive, trace preserving maps is the contractivity with respect to the trace norm.

Proposition 9. *Let $\mathcal{E} \in \mathbf{L}^2(\mathcal{X})$ be a trace preserving map. $\mathcal{E}$ is a positive map if, and only if,*

$$\|\mathcal{E}(X)\|_1 \leq \|X\|_1, \tag{2.82}$$

for every $X \in \mathbf{L}(\mathcal{X})$ Hermitian. (90, 103)

Proof. First, recall that a Hermitian operator X has a spectral decomposition as in Eq. (A.21), $X = \sum_i \lambda_i \Pi_i$, where the eigenvalues λ_i are real numbers. It can be further

decomposed as

$$X = \sum_i \lambda_i \Pi_i = \sum_{\lambda_i > 0} \lambda_i \Pi_i - \sum_{\lambda_i < 0} (-\lambda_i) \Pi_i \equiv P - Q, \quad (2.83)$$

where P and Q are positive definite operators, since they have positive eigenvalues. The trace norm of a Hermitian operator is $\|X\|_1 = \sum_i |\lambda_i|$, being equal to its trace if and only if the operator is positive semidefinite. Since the eigenprojections Π_i are mutually orthogonal,

$$\|X\|_1 = \sum_i |\lambda_i| = \text{tr}[P] + \text{tr}[Q] = \|P\|_1 + \|Q\|_1. \quad (2.84)$$

If $\mathcal{E}$ is a positive map, then $\|\mathcal{E}(P)\|_1 = \text{tr}[\mathcal{E}(P)]$ for every positive semidefinite map P . For any Hermitian X ,

$$\|\mathcal{E}(X)\|_1 = \|\mathcal{E}(P) - \mathcal{E}(Q)\|_1 \leq \|\mathcal{E}(P)\|_1 + \|\mathcal{E}(Q)\|_1. \quad (2.85)$$

Since P and Q are positive semidefinite,

$$\|\mathcal{E}(P)\|_1 + \|\mathcal{E}(Q)\|_1 = \text{tr}[P] + \text{tr}[Q] = \|P\|_1 + \|Q\|_1 = \|X\|_1, \quad (2.86)$$

so $\|\mathcal{E}(X)\|_1 \leq \|X\|_1$ holds.

Conversely, if P is positive semidefinite and $\mathcal{E}$ is trace preserving, then

$$\|P\|_1 = \text{tr}[P] = \text{tr}[\mathcal{E}(P)] \leq \|\mathcal{E}(P)\|_1 \leq \|P\|_1, \quad (2.87)$$

since $\text{tr}[X] \leq \|X\|_1$ for any operator X . Therefore, $\|\mathcal{E}(P)\|_1 = \text{tr}[\mathcal{E}(P)]$, which holds if and only if $\mathcal{E}(P) \geq 0$. Then, $\mathcal{E}$ is a positive map. $\square$

At first sight, positive trace preserving linear superoperators should preserve all the properties of quantum states, and therefore should be the most general allowed quantum transformations. It turns out, however, that positivity of the map alone is not enough to preserve the positive semidefiniteness of the states. Why does this happen?

The answer is entanglement: if our system is entangled with an outside system, then positive maps may fail to yield positive semidefinite outputs. The classical example (9) is given by the transpose mapping. Let $\mathcal{T} \in \mathbf{L}^2(\mathbb{C}^2)$ be the transpose mapping on single qubits,

$$\mathcal{T}: \mathbf{L}(\mathbb{C}^2) \rightarrow \mathbf{L}(\mathbb{C}^2) \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & c \\ b & d \end{bmatrix}. \quad (2.88)$$

It is clearly a positive map, since it does not change the trace and the determinant of its input, which determines its eigenvalues and, therefore, its positive semidefiniteness. If this qubit is entangled with another qubit in the state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad (2.89)$$

i.e.,

$$|\psi\rangle\langle\psi| = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad (2.90)$$

which undergoes a trivial evolution, then the final state is

$$(\mathcal{T} \otimes \mathcal{I}_{\mathbb{C}^2})(|\psi\rangle\langle\psi|) = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.91)$$

which is not positive semidefinite, since it has an eigenvalue $-1/2$.

So, which are the maps that preserve positive semidefiniteness, irrespective of the presence of entanglement? Inspired by the example, we could ask that $(\mathcal{E} \otimes \mathcal{I}_k)^{\mathbf{p}}$ remained positive for every $k \in \mathbb{N}$. Therefore, any ancilla coupled to the system would not affect the positive semidefiniteness of the evolution. Evolutions of this type are called completely positive maps (9), and are our last step in the quest for quantum channels. Before studying them, however, we introduce the intermediate class of superoperators called k -positive maps.

For any $k \in \mathbb{N}$, a map is k -positive if $(\mathcal{E} \otimes \mathcal{I}_k)$ is a positive map. In particular, positive maps are 1-positive. The Choi matrix of these maps has a particularly interesting form.

Proposition 10. (*Choi matrix of k -positive maps*). For $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, $\mathcal{E}$ is k -positive if and only if $\Lambda_{\mathcal{E}}$ is k -block positive, i.e., $\langle\psi|\Lambda_{\mathcal{E}}|\psi\rangle \geq 0$, for every $|\psi\rangle \in \mathcal{X}_1 \otimes \mathcal{X}_2$ with Schmidt number^a less than or equal to k . (101, 102)

Proof. Define $\mathcal{X}_{1,k} \equiv \mathcal{X}_1 \otimes \mathbb{C}^k$ and $\mathcal{X}_{2,k} \equiv \mathcal{X}_2 \otimes \mathbb{C}^k$, so we have $\mathcal{E}_k \equiv \mathcal{E} \otimes \mathcal{I}_k \in \mathbf{L}^2(\mathcal{X}_{1,k}, \mathcal{X}_{2,k})$. By proposition 7, $\mathcal{E}_k$ is positive if and only if $\langle\psi|\Lambda_{\mathcal{E}_k}|\psi\rangle \geq 0$, for every $|\psi\rangle \in \mathcal{X}_{1,k} \otimes \mathcal{X}_{2,k}$ separable. Fix orthonormal bases $\{|i\rangle\}$ of $\mathcal{X}_1$ and $\{|\alpha\rangle\}$ of $\mathbb{C}^k$. Then,

$$\begin{aligned} \Lambda_{\mathcal{E}_k} &= \sum_{i,j} \sum_{\alpha,\beta=1}^k |i, \alpha\rangle\langle j, \beta| \otimes \mathcal{E}_k(|i, \alpha\rangle\langle j, \beta|) \\ &= \sum_{i,j} \sum_{\alpha,\beta=1}^k (|i\rangle\langle j| \otimes |\alpha\rangle\langle\beta|) \otimes \mathcal{E}_k(|i\rangle\langle j| \otimes |\alpha\rangle\langle\beta|) \\ &= \sum_{i,j} \sum_{\alpha,\beta=1}^k |i\rangle\langle j| \otimes |\alpha\rangle\langle\beta| \otimes \mathcal{E}(|i\rangle\langle j|) \otimes |\alpha\rangle\langle\beta|. \end{aligned} \quad (2.92)$$

Note that $\Lambda_{\mathcal{E}_k} \in \mathbf{L}(\mathcal{X}_1 \otimes \mathbb{C}^k \otimes \mathcal{X}_2 \otimes \mathbb{C}^k) = \mathbf{L}(\mathcal{X}_{1,k} \otimes \mathcal{X}_{2,k})$.

^p $\mathcal{I}_k$ is a notation for $\mathcal{I}_{\mathbb{C}^k}$, where $\mathbb{C}^k$ is a k -dimensional Hilbert space.

^a See Theorem 15 for the definition.

Since $|\psi\rangle$ is separable, it can be written as $|\psi\rangle = |x\rangle \otimes |y\rangle$, with $|x\rangle \in \mathcal{X}_{1,k}$ and $|y\rangle \in \mathcal{X}_{2,k}$. These can also be further decomposed as $|x\rangle = \sum_{\gamma=1}^k |x_\gamma\rangle \otimes |\gamma\rangle$ and $|y\rangle = \sum_{\epsilon=1}^k |y_\epsilon\rangle \otimes |\epsilon\rangle$, with $|x_\gamma\rangle \in \mathcal{X}_1$, $|y_\epsilon\rangle \in \mathcal{X}_2$ and $|\gamma\rangle, |\epsilon\rangle \in \mathbb{C}^k$. Now,

$$\begin{aligned}
\langle\psi| \Lambda_{\mathcal{E}_k} |\psi\rangle &= \left(\sum_{\gamma,\epsilon=1}^k \langle x_\gamma, \gamma| \otimes \langle y_\epsilon, \epsilon| \right) \Lambda_{\mathcal{E}_k} \left(\sum_{\theta,\eta=1}^k |x_\theta, \theta\rangle \otimes |y_\eta, \eta\rangle \right) \\
&= \sum_{i,j} \sum_{\alpha,\beta=1}^k \sum_{\gamma,\epsilon,\theta,\eta=1}^k \langle x_\gamma|i\rangle \langle j|x_\theta\rangle \langle \gamma|\alpha\rangle \langle \beta|\theta\rangle \langle y_\epsilon|\mathcal{E}(|i\rangle \langle j|)|y_\eta\rangle \langle \epsilon|\alpha\rangle \langle \beta|\eta\rangle \\
&= \sum_{i,j} \sum_{\alpha,\beta=1}^k \langle x_\alpha|i\rangle \langle j|x_\beta\rangle \langle y_\alpha|\mathcal{E}(|i\rangle \langle j|)|y_\beta\rangle \\
&= \left(\sum_{\alpha=1}^k \langle x_\alpha| \otimes \langle y_\alpha| \right) \left(\sum_{i,j} |i\rangle \langle j| \otimes \mathcal{E}(|i\rangle \langle j|) \right) \left(\sum_{\beta=1}^k |y_\beta\rangle \otimes |x_\beta\rangle \right) \\
&= \langle\psi| \Lambda_{\mathcal{E}} |\psi\rangle, \tag{2.93}
\end{aligned}$$

where $|\psi\rangle = \sum_{\alpha=1}^k |x_\alpha\rangle \otimes |y_\alpha\rangle$ is a vector with Schmidt number less than or equal to k . Therefore, $\langle\psi| \Lambda_{\mathcal{E}_k} |\psi\rangle \geq 0$, for every $|\psi\rangle \in \mathcal{X}_{1,k} \otimes \mathcal{X}_{2,k}$ separable, if and only if $\langle\psi| \Lambda_{\mathcal{E}} |\psi\rangle \geq 0$, for every $|\psi\rangle \in \mathcal{X}_1 \otimes \mathcal{X}_2$ with Schmidt number less than or equal to k , so the proposition is proved. $\square$

2.2.5 Completely positive maps

A superoperator $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ is *completely positive* if $(\mathcal{E} \otimes \mathcal{I}_k)$ is positive, for every $k \in \mathbb{N}$.

Proposition 11. (*Representations of completely positive maps*). For $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, the following are equivalent (78):

- i) $\mathcal{E}$ is completely positive.
- ii) $(\mathcal{E} \otimes \mathcal{I}_d)$ is positive, where $d = \min\{d_1, d_2\}$.
- iii) (Choi) $\Lambda_{\mathcal{E}}$ is positive semidefinite.
- iv) (Kraus) $\mathcal{E}(X) = \sum_i A_i X A_i^\dagger$.
- v) (Stinespring) $\mathcal{E}(X) = \text{tr}_Y (A X A^\dagger)$.

Proof. i) $\Rightarrow$ ii): trivial.

ii) $\Rightarrow$ iii): $\mathcal{E}$ is d -positive, therefore $\Lambda_{\mathcal{E}}$ is d -block positive. Since all vectors in $\mathcal{X}_1 \otimes \mathcal{X}_2$ have a Schmidt number less than d_1 and d_2 , and therefore less than d , then $\langle x| \Lambda_{\mathcal{E}} |x\rangle \geq 0$ actually holds for every $|x\rangle \in \mathcal{X}_1 \otimes \mathcal{X}_2$, and therefore $\Lambda_{\mathcal{E}}$ is positive semidefinite.

iii) $\Rightarrow$ iv): being positive semidefinite, $\Lambda_{\mathcal{E}}$ has a spectral decomposition

$$\Lambda_{\mathcal{E}} = \sum_i \lambda_i |M_i\rangle \langle M_i|, \tag{2.94}$$

where its eigenvalues λ_i are nonnegative. This means that their square roots $\sqrt{\lambda_i}$ are well defined, so we have $A_i = B_i = \sqrt{\lambda_i} M_i$, leading to $\mathcal{E}(X) = \sum_i A_i X A_i^\dagger$.

$iv) \Rightarrow v)$: follows from Proposition 1.

$v) \Rightarrow i)$: let $X \in \mathbf{L}(\mathcal{X}_1 \otimes \mathbb{C}^k)$, $k \in \mathbb{N}$ be a positive semidefinite operator. Since conjugation by a matrix preserves spectrum, we have that

$$(A \otimes \mathbf{1}_k) X (A \otimes \mathbf{1}_k)^\dagger \quad (2.95)$$

is positive too. Also, the trace operation is positive, since the trace of a positive semidefinite operator is the sum of its eigenvalues, which is a nonnegative quantity. Therefore,

$$(\mathcal{E} \otimes \mathcal{I}_k) = \text{tr}_Y \left[(A \otimes \mathbf{1}_k) X (A \otimes \mathbf{1}_k)^\dagger \right] \geq 0, \quad (2.96)$$

so $(\mathcal{E} \otimes \mathcal{I}_k)$ is a positive superoperator for every $k \in \mathbb{N}$, which means that $\mathcal{E}$ is completely positive. $\square$

As we can see, if an superoperator is at least d -positive, with $d = \min\{d_1, d_2\}$, then it is completely positive. Therefore, we do not need to check k -positivity for all values of k , but only for values up to the dimension of the system.

Proposition 12. *The composite of completely positive maps is completely positive.*

Proof. Given $\mathcal{F} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ and $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_2, \mathcal{X}_3)$ completely positive maps, they have Kraus representations $\mathcal{F}(X) = \sum_i B_i X B_i^\dagger$ and $\mathcal{E}(X) = \sum_i A_i X A_i^\dagger$. If $\mathcal{G} = \mathcal{E} \circ \mathcal{F} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_3)$, then

$$\mathcal{G}(X) = \sum_{i,j} A_i B_j X B_j^\dagger A_i^\dagger \equiv \sum_\alpha C_\alpha X C_\alpha^\dagger, \quad (2.97)$$

where $\alpha = \{i, j\}$ and $C_\alpha = A_i B_j$. Since it has a Kraus representation as above, $\mathcal{G}$ is completely positive. $\square$

Although the Kraus and Stinespring representations are not unique, for completely positive maps they are unique up to an unitary transformation.

Proposition 13. *(Unitary equivalence of Kraus operators). For $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, if $\mathcal{E}(X) = \sum_i A_i X A_i^\dagger = \sum_i B_i X B_i^\dagger$, then there is an unitary operator $U = (U_{ij})$ such that*

$$B_i = \sum_j U_{ij} A_j. \quad (2.98)$$

Proof. Suppose that $\mathcal{E}(X) = \sum_i A_i X A_i^\dagger$ and $\mathcal{F}(X) = \sum_i B_i X B_i^\dagger$, so we have that $\mathcal{E} = \mathcal{F}$. Their Choi matrices are

$$\Lambda_{\mathcal{E}} = \sum_i |A_i\rangle\rangle\langle\langle A_i|, \quad (2.99)$$

$$\Lambda_{\mathcal{F}} = \sum_i |B_i\rangle\rangle\langle\langle B_i|, \quad (2.100)$$

where $|A_i\rangle\rangle, |B_i\rangle\rangle \in \mathcal{X}_1 \otimes \mathcal{X}_2$. Since the Choi matrices are now (up to normalization) density operators, they have purifications $|\Psi_{\mathcal{E}}\rangle$ and $|\Psi_{\mathcal{F}}\rangle$ in $\mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}$, for some space $\mathcal{Y}^r$, such that

$$\mathrm{tr}_{\mathcal{Y}} [|\Psi_{\mathcal{E}}\rangle \langle \Psi_{\mathcal{E}}|] = \Lambda_{\mathcal{E}} \quad (2.101)$$

$$\mathrm{tr}_{\mathcal{Y}} [|\Psi_{\mathcal{F}}\rangle \langle \Psi_{\mathcal{F}}|] = \Lambda_{\mathcal{F}} \quad (2.102)$$

More explicitly, we have

$$\begin{aligned} |\Psi_{\mathcal{E}}\rangle &= \sum_i |A_i\rangle\rangle \otimes |i_{\mathcal{Y}}\rangle, \\ |\Psi_{\mathcal{F}}\rangle &= \sum_i |B_i\rangle\rangle \otimes |i_{\mathcal{Y}}\rangle, \end{aligned} \quad (2.103)$$

and, inversely,

$$\begin{aligned} |A_i\rangle\rangle &= \langle i_{\mathcal{Y}}| \langle \Psi_{\mathcal{E}}| \\ |B_i\rangle\rangle &= \langle i_{\mathcal{Y}}| \langle \Psi_{\mathcal{F}}|, \end{aligned} \quad (2.104)$$

which can be checked by direct computation. By the unitary equivalence of purifications, Theorem 19, there is an unitary transformation $U \in \mathbf{L}(\mathcal{Y})$ such that

$$|\Psi_{\mathcal{E}}\rangle = (\mathbf{1}_{\mathcal{X}_1 \otimes \mathcal{X}_2} \otimes U) |\Psi_{\mathcal{F}}\rangle. \quad (2.105)$$

Then, using Eqs. (2.103), (2.104) and (2.105),

$$\begin{aligned} |B_i\rangle\rangle &= \langle i_{\mathcal{Y}}| (\mathbf{1}_{\mathcal{X}_1 \otimes \mathcal{X}_2} \otimes U) \sum_j |A_j\rangle\rangle \otimes |j_{\mathcal{Y}}\rangle \\ &= \sum_j \mathbf{1}_{\mathcal{X}_1 \otimes \mathcal{X}_2} [|A_j\rangle\rangle] \otimes \langle i| U |j_{\mathcal{Y}}\rangle \\ &= \sum_j \langle i| U |j\rangle |A_j\rangle\rangle \\ &= \sum_j U_{ij} |A_j\rangle\rangle. \end{aligned} \quad (2.106)$$

□

Proposition 14. (*Unitary equivalence of Stinespring operators*). For $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, if $\mathcal{E}(X) = \mathrm{tr}_{\mathcal{Y}} [AXA^\dagger] = \mathrm{tr}_{\mathcal{Y}} [BXB^\dagger]$, then there is an unitary operator $U \in \mathbf{L}(\mathcal{Y})$ such that

$$B = (\mathbf{1}_{\mathcal{X}_2} \otimes U) A. \quad (2.107)$$

^r Note that the space $\mathcal{Y}$ need not be the same for both purifications, since the rank of both matrices are not the same (a consequence of the fact that the number of Kraus operators in both representations need not be the same). We can, however, consider as $\mathcal{Y}$ the larger of both spaces, and use it for both purifications without loss of generality.

Proof. Using the equivalence with the Kraus representation, we have that $A = \sum_i A_i \otimes |i_Y\rangle$ and $B = \sum_i B_i \otimes |i_Y\rangle$. Conversely, $A_i = \langle i_Y | A$ and $B_i = \langle i_Y | B$. By the previous proposition, $B_i = \sum_j U_{ij} A_j$, so

$$\begin{aligned}
 B &= \sum_i \left(\sum_j U_{ij} A_j \right) \otimes |i_Y\rangle \\
 &= \sum_{i,j} U_{ij} [(\mathbb{1}_{\mathcal{X}_2} \otimes \langle j_Y |) A] \otimes |i_Y\rangle \\
 &= \left(\mathbb{1}_{\mathcal{X}_2} \otimes \left(\sum_{i,j} U_{ij} |i_Y\rangle \langle j_Y| \right) \right) A \\
 &= (\mathbb{1}_{\mathcal{X}_2} \otimes U) A.
 \end{aligned} \tag{2.108}$$

□

Finally, we derive the interesting fact that every Hermiticity preserving map can be written as a difference of two completely positive maps.

Proposition 15. $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ is Hermiticity preserving if and only if $\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2$, $\mathcal{E}_1, \mathcal{E}_2$ completely positive maps.

Proof. If $\mathcal{E}$ is Hermiticity preserving, then its Choi matrix $\Lambda_{\mathcal{E}}$ is Hermitian. By the Jordan-Hahn decomposition, Theorem 18, $\Lambda_{\mathcal{E}} = \Lambda_{\mathcal{E}_1} - \Lambda_{\mathcal{E}_2}$, where $\Lambda_{\mathcal{E}_1}$ and $\Lambda_{\mathcal{E}_2}$ are positive semidefinite and, therefore, the corresponding maps $\mathcal{E}_1$ and $\mathcal{E}_2$ are completely positive. By linearity, $\Lambda_{\mathcal{E}} = \Lambda_{\mathcal{E}_1} - \Lambda_{\mathcal{E}_2} = \Lambda_{\mathcal{E}_1 - \mathcal{E}_2}$, so $\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2$.

Conversely, if $\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2$ and $X = X^\dagger$, then

$$\begin{aligned}
 (\mathcal{E}(X))^\dagger &= (\mathcal{E}_1(X) - \mathcal{E}_2(X))^\dagger \\
 &= \mathcal{E}_1^\dagger(X) - \mathcal{E}_2^\dagger(X) \\
 &= \mathcal{E}_1(X) - \mathcal{E}_2(X) = \mathcal{E}(X),
 \end{aligned} \tag{2.109}$$

since $\mathcal{E}_1$ and $\mathcal{E}_2$, being completely positive, are in particular Hermiticity preserving. □

2.2.6 Quantum channels

We are finally in position to define the most general class of quantum transformations: the completely positive, trace preserving (CPTP) maps, also known as quantum channels. Putting together propositions 4 and 11, we can characterize the three representations of quantum channels.

Theorem 5. (*Representations of quantum channels*). For $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, the following are equivalent (78):

i) $\mathcal{E}$ is a quantum channel.

- ii) (Choi) $\Lambda_{\mathcal{E}}$ is positive and $\text{tr}_2[\Lambda_{\mathcal{E}}] = \mathbb{1}_{\mathcal{X}_1}$.
- iii) (Kraus) $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$ and $\sum_i A_i^\dagger A_i = \mathbb{1}_{\mathcal{X}_1}$.
- iv) (Stinespring) $\mathcal{E}(\rho) = \text{tr}_{\mathcal{Y}}[A \rho A^\dagger]$ and $A^\dagger A = \mathbb{1}_{\mathcal{X}_1}$.

These three representations now have physical meanings. Since $\text{tr}_2[\Lambda_{\mathcal{E}}] = \mathbb{1}_{\mathcal{X}_1}$, we have that $\text{tr}[\Lambda_{\mathcal{E}}] = d_1$. Then, $d_1^{-1}\Lambda_{\mathcal{E}}$, being positive semidefinite and with unit trace, is a quantum state in $\mathcal{X}_1 \otimes \mathcal{X}_2$. Therefore, there is an isomorphism between states and channels, which is the most known form of the Choi-Jamiołkowski isomorphism. That means that we can study quantum channels with the machinery used to study states, and vice-versa. The Choi matrix has also an operational meaning. Suppose that Alice has a maximally entangled bipartite state, and sends one partition of it to Bob, through the channel $\mathcal{E}$, while keeping the other partition with her: the resulting final state is precisely the (properly normalized) Choi matrix of $\mathcal{E}$.

For the Kraus representation, note that

$$\begin{aligned}
 \mathcal{E}(\rho) &= \sum_i A_i \rho A_i^\dagger \\
 &= \sum_i \text{tr}[A_i \rho A_i^\dagger] \frac{A_i \rho A_i^\dagger}{\text{tr}[A_i \rho A_i^\dagger]} \\
 &= \sum_i \text{tr}[A_i^\dagger A_i \rho] \frac{A_i \rho A_i^\dagger}{\text{tr}[A_i^\dagger A_i \rho]} \\
 &\equiv \sum_i p_i \rho_i.
 \end{aligned} \tag{2.110}$$

Since

$$\sum_i p_i = \sum_i \text{tr}[A_i^\dagger A_i \rho] = \text{tr}\left[\left(\sum_i A_i^\dagger A_i\right) \rho\right] = \text{tr}[\rho] = 1, \tag{2.111}$$

this means that the evolution has the same effect of a non-selective quantum measurement (104), i.e., a measurement where the post-measurement state has not been "picked out" yet, so we have a mixed state of different possible outcomes. Also, it shows that open evolutions are the responsible for the creation of mixed states out of pure states, and therefore justifying the need for density operators.

Finally, for the Stinespring representation, we will first rewrite it in its most known form. (9) For simplicity, we consider $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}$. Note that the condition $A^\dagger A = \mathbb{1}_{\mathcal{X}}$ implies that $A \in \mathbf{L}(\mathcal{X}, \mathcal{X} \otimes \mathcal{Y})$ is an isometry: our objective is to turn it into an unitary operator. First, fixing an orthonormal basis $\{|i_{\mathcal{Y}}\rangle\}$ in $\mathcal{Y}$ and considering any element of this basis, say $|0_{\mathcal{Y}}\rangle$, define the operator $U \in \mathbf{L}(\mathcal{X} \otimes \mathcal{Y})$ as

$$U |0_{\mathcal{Y}}\rangle = A. \tag{2.112}$$

Then, replacing that in the Stinespring representation leads to

$$\begin{aligned}\mathrm{tr}_{\mathcal{Y}} [A\rho A^\dagger] &= \mathrm{tr}_{\mathcal{Y}} [(U|0_{\mathcal{Y}}\rangle)\rho(\langle 0_{\mathcal{Y}}|U^\dagger)] \\ &= \mathrm{tr}_{\mathcal{Y}} [U(\rho \otimes |0_{\mathcal{Y}}\rangle\langle 0_{\mathcal{Y}}|)U^\dagger].\end{aligned}\quad (2.113)$$

Finally, note that, for arbitrary states $|\psi\rangle, |\phi\rangle \in \mathcal{X}$,

$$\langle \psi, 0_{\mathcal{Y}} | U^\dagger U | \phi, 0_{\mathcal{Y}} \rangle = \langle \psi | A^\dagger A | \phi \rangle = \langle \psi | \phi \rangle, \quad (2.114)$$

which means that U preserves inner products, but only for some columns. In fact, U is only defined for that columns, and we can freely choose the other columns in a way that makes it a unitary operator. The physical interpretation is now evident: every quantum channel can be seen as a unitary evolution in an extended Hilbert space, where we initially coupled the system with an ancillary state. (9) Note a subtlety here: the initial state is in a product state. In fact, as you will see in the next Section, correlations in the initial state can lead to evolutions which do not represent quantum channels. Recalling now the relationship between the Kraus and Stinespring representations,

$$\begin{aligned}A &= \sum_i A_i \otimes |i_{\mathcal{Y}}\rangle, \\ A_i &= \langle i_{\mathcal{Y}} | A,\end{aligned}\quad (2.115)$$

we can see that

$$\langle i_{\mathcal{Y}} | U | 0_{\mathcal{Y}} \rangle = \langle i_{\mathcal{Y}} | A = A_i. \quad (2.116)$$

If we think of U as a matrix in $\mathcal{Y}$, whose elements themselves are matrices in $\mathcal{X}$, then its block matrix structure is (N is the dimension of $\mathcal{Y}$) (9)

$$U = \begin{bmatrix} [\mathbb{A}_1] & \cdot & \cdot & \cdot & \cdot & \cdots \\ [\mathbb{A}_2] & \cdot & \cdot & \cdot & \cdot & \cdots \\ [\mathbb{A}_3] & \cdot & \cdot & \cdot & \cdot & \cdots \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdots \\ [\mathbb{A}_N] & \cdot & \cdot & \cdot & \cdot & \ddots \end{bmatrix} \quad (2.117)$$

and $(\mathbb{A}_i)_{jk} = \langle j | A_i | k \rangle$, for some basis $\{|i\rangle\}$ of $\mathcal{X}$. In this way we can see that only the first column of the block structure is determined, and the only constraint in the other columns is that the matrix U , as a whole, is unitary. The relationship between the three representations for quantum channels is summarized in Figure 3. Note that the conversion Kraus $\rightarrow$ Stinespring is not as easy as in Figure 1, since this only specifies some columns of the U operator. We are free to define the other columns, as long as that preserves the unitarity of the operator.

Quantum channels represent both open and closed evolutions. Closed evolutions are represented by unitary channels, of the form

$$\mathcal{U}(X) = UXU^\dagger, \quad (2.118)$$

Figure 3 – Relationship between representations of quantum channels.

Source: By the author.

where U is an unitary operator. Most open evolutions, however, correspond to non-unitary quantum channels. A characteristic property of non-unitary dynamics is that they are irreversible: an open evolution cannot usually be "undone" for all initial states.

Theorem 6. *The inverse of a bijective quantum channel $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ is a quantum channel if and only if $\mathcal{E}$ is a unitary channel. (2)*

Proof. If $\mathcal{E}(X) = UXU^\dagger$, then $\mathcal{E}^{-1}(X) = U^\dagger XU$, which is also unitary and, in particular, a quantum channel. We now prove the converse.

First, if both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are quantum channels, and in particular positive and trace preserving maps, then proposition 9 implies that, for any Hermitian X ,

$$\|X\|_1 = \|\mathcal{E}^{-1}(\mathcal{E}(X))\|_1 \leq \|\mathcal{E}(X)\|_1 \leq \|X\|_1, \quad (2.119)$$

which means that

$$\|\mathcal{E}(X)\|_1 = \|X\|_1. \quad (2.120)$$

Second, the hypothesis also implies that $\mathcal{E}$ maps pure states (i.e., rank-1 projections) to pure states: if not, then the map would take a pure state $|\psi\rangle \in \mathcal{X}_1$ to some non-trivial convex combination of density operators,

$$\mathcal{E}(|\psi\rangle\langle\psi|) = \lambda\rho + (1 - \lambda)\sigma, \quad (2.121)$$

with $\rho, \sigma \in \mathbf{D}(\mathcal{X}_2)$ and $\lambda \in (0, 1)$. This implies that

$$|\psi\rangle\langle\psi| = \mathcal{E}^{-1}(\mathcal{E}(|\psi\rangle\langle\psi|)) = \lambda\mathcal{E}^{-1}(\rho) + (1 - \lambda)\mathcal{E}^{-1}(\sigma). \quad (2.122)$$

Since $\mathcal{E}^{-1}$ is a quantum channel, $\mathcal{E}^{-1}(\rho)$ and $\mathcal{E}^{-1}(\sigma)$ are also density operators. That means that $|\psi\rangle\langle\psi|$ is a non-trivial mixture of states, which is a contradiction, since it is a pure state.

Now, any Hermitian operator can be written as a difference of positive semidefinite operators, $X = P - Q$, so let us restrict our attention to the case where P and Q represent pure states, $P = |\psi\rangle\langle\psi|$, $Q = |\phi\rangle\langle\phi|$. Their image under $\mathcal{E}$ must be another pure state, so $\mathcal{E}(|\psi\rangle\langle\psi|) = |\psi'\rangle\langle\psi'|$ and $\mathcal{E}(|\phi\rangle\langle\phi|) = |\phi'\rangle\langle\phi'|$. Therefore, Eq. (2.120) can be rewritten as

$$\| |\psi'\rangle\langle\psi'| - |\phi'\rangle\langle\phi'| \|_1 = \| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1, \quad (2.123)$$

or, dividing both sides by 2,

$$D_1(|\psi'\rangle, |\phi'\rangle) = D_1(|\psi\rangle, |\phi\rangle). \quad (2.124)$$

Using Eq. (A.28), the above equations mean that

$$\sqrt{1 - |\langle\psi'|\phi'\rangle|^2} = \sqrt{1 - |\langle\psi|\phi\rangle|^2}, \quad (2.125)$$

which finally implies that $|\langle\psi'|\phi'\rangle| = |\langle\psi|\phi\rangle|$, i.e., ϕ is a symmetry transformation of wavefunctions. By Wigner's theorem (75), $\mathcal{E}$ can only be a unitary or an antiunitary operation. Since antiunitary operations are not quantum channels (2), $\mathcal{E}$ must be unitary. $\square$

Furthermore, the inverse of a non-unitary quantum channel is not even a positive map. (105)

2.2.7 Examples

Here we show some examples of quantum channels.

Identity channel

The most trivial quantum channel is the identity channel in $\mathbf{L}^2(\mathcal{X})$, which takes a quantum state to itself, $\mathcal{I}_{\mathcal{X}}(\rho) = \rho$. The Kraus and Stinespring representations are trivial, $\mathcal{I}_{\mathcal{X}}(\rho) = \mathbb{1}_{\mathcal{X}}\rho\mathbb{1}_{\mathcal{X}}$ ^s, and the Choi matrix is

$$\Lambda_{\mathcal{I}_{\mathcal{X}}} = (\mathcal{I}_{\mathcal{X}} \otimes \mathcal{I}_{\mathcal{X}})(|\Omega\rangle\langle\Omega|) = |\Omega\rangle\langle\Omega|, \quad (2.126)$$

which is the density operator for a maximally mixed or *Bell* state. (9)

Unitary channels

The unitary channels,

$$\mathcal{U}(\rho) = U\rho U^\dagger, \quad (2.127)$$

^s The auxiliary space in the Stinespring representation is one-dimensional, and therefore can be ignored.

for U an unitary operator in $\mathbf{L}(\mathcal{X})$, have Kraus and Stinespring representations equal to Eq. (2.127), and the Choi matrix is

$$\begin{aligned}\Lambda_U &= \sum_{i,j} |i\rangle \langle j| \otimes U |i\rangle \langle j| U^\dagger \\ &= \left(\sum_i |i\rangle \otimes U |i\rangle \right) \left(\sum_j |j\rangle \otimes U |j\rangle \right)^\dagger \equiv |\Psi\rangle \langle \Psi|,\end{aligned}\quad (2.128)$$

which is a pure state.

For convenience, we summarize in Table 1 some classes of quantum channels and their corresponding states (due to the Choi-Jamiolkowski isomorphism), i.e., its (normalized) Choi matrices.

Table 1 – Quantum channels and its corresponding states under the Choi-Jamiolkowski isomorphism.

Channel	State
Quantum channels	Quantum states
Unitary channels	Pure states
Identity channel	Bell state

Source: By the author.

Unital channels

A map $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ is unital if $\mathcal{E}(\mathbb{1}_{\mathcal{X}_1}) = \mathbb{1}_{\mathcal{X}_2}$, i.e., if it preserves the identity (or, physically, the maximally mixed state). Given a Kraus representation $\mathcal{E}(X) = \sum_i A_i X A_i^\dagger$, we have that

$$\mathcal{E}(\mathbb{1}_{\mathcal{X}_1}) = \sum_i A_i \mathbb{1}_{\mathcal{X}_1} A_i^\dagger = \sum_i A_i A_i^\dagger = \mathbb{1}_{\mathcal{X}_2}, \quad (2.129)$$

and for a Stinespring dilation $\mathcal{E}(X) = \text{tr}_Y [A X A^\dagger]$, similarly, $A^\dagger A = \mathbb{1}_{\mathcal{X}_2}$. For the Choi matrix,

$$\begin{aligned}\text{tr}_1[\Lambda_{\mathcal{E}}] &= \sum_{i,j} \text{tr}[|i\rangle \langle j|] \mathcal{E}(|i\rangle \langle j|) \\ &= \sum_i \mathcal{E}(|i\rangle \langle i|) \\ &= \mathcal{E}\left(\sum_i |i\rangle \langle i|\right) = \mathcal{E}(\mathbb{1}_{\mathcal{X}_1}) = \mathbb{1}_{\mathcal{X}_2},\end{aligned}\quad (2.130)$$

so we have that $\text{tr}_1[\Lambda_{\mathcal{E}}] = \mathbb{1}_{\mathcal{X}_2}$. A unital quantum channel is sometimes called a doubly stochastic channel. (106)

Partial trace

Suppose we have a bipartite system $\mathcal{H}_S \otimes \mathcal{H}_E$. The partial trace on E can be considered as an operator $\text{tr}_E \in \mathbf{L}^2(\mathcal{H}_S \otimes \mathcal{H}_E, \mathcal{H}_S)$. Given orthonormal bases $\{i_S\}$ and $\{i_E\}$ in $\mathcal{H}_S$ and $\mathcal{H}_E$, respectively, its Choi matrix is

$$\begin{aligned} \Lambda_{\text{tr}_E} &= \sum_{i_S, j_S, i_E, j_E} |i_S, i_E\rangle \langle j_S, j_E| \otimes \text{tr}_E [|i_S, i_E\rangle \langle j_S, j_E|] \\ &= \sum_{i_S, j_S, i_E, j_E} (|i_S\rangle \langle j_S| \otimes |i_E\rangle \langle j_E|) \otimes |i_S\rangle \langle j_S| \delta_{i_E, j_E} \\ &= \sum_{i_S, j_S} |i_S, i_S\rangle \langle j_S, j_S| \otimes \sum_{i_E} |i_E\rangle \langle i_E| \\ &\equiv |\Omega_{SS}\rangle \langle \Omega_{SS}| \otimes \mathbb{1}_{\mathcal{H}_E}, \end{aligned} \quad (2.131)$$

where $|\Omega_{SS}\rangle$ is a maximally entangled state in $\mathcal{H}_S \otimes \mathcal{H}_S$. Note that the Choi rank is equal to the dimension of the traced system, and the Choi matrix is a matrix in $\mathbf{L}(\mathcal{H}_S \otimes \mathcal{H}_E \otimes \mathcal{H}_S)$. For the Kraus representation, notice that the Choi matrix can be rewritten as

$$\Lambda_{\text{tr}_E} = \sum_{i_E} |\Omega_{SS}, i_E\rangle \langle \Omega_{SS}, i_E| \equiv \sum_i |\Psi_i\rangle \langle \Psi_i|, \quad (2.132)$$

so the Kraus operators are

$$\begin{aligned} A_i &= \sum_{j_S, k_E, l_S} \langle j_S, k_E, l_S | \Psi_i \rangle |l_S\rangle \langle j_S, k_E| \\ &= \sum_{j_S, k_E, l_S} \langle j_S, k_E, l_S | \Omega_{SS}, i_E \rangle |l_S\rangle \langle j_S, k_E| \\ &= \sum_{j_S, k_E, l_S, m_E} \langle j_S | m_S \rangle \langle k_E | i_E \rangle \langle l_S | m_S \rangle |l_S\rangle \langle j_S, k_E| \\ &= \sum_{j_S} |j_S\rangle \langle j_S, i_E| \equiv \mathbb{1}_{\mathcal{H}_S} \otimes \langle i|, \end{aligned} \quad (2.133)$$

and the Kraus decomposition is

$$\text{tr}_E(X) = \sum_i (\mathbb{1}_{\mathcal{H}_S} \otimes \langle i|) X (\mathbb{1}_{\mathcal{H}_S} \otimes \langle i|)^\dagger. \quad (2.134)$$

For the Stinespring dilation, we recover the original form of the partial trace.

Qubit amplitude damping

Finally, for a more concrete application, we consider the qubit evolution $\mathcal{E} \in \mathbf{L}^2(\mathbb{C}^2)$ with Kraus representation (9)

$$\mathcal{E}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger, \quad (2.135)$$

where

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}, \quad (2.136)$$

and $0 \leq \gamma \leq 1$. This map is known as the amplitude damping channel and is a simple model of energy dissipation in a qubit. (9) If we write A_0 and A_1 as

$$A_0 = |0\rangle \langle 0| + \sqrt{1-\gamma} |1\rangle \langle 1|, \quad (2.137)$$

$$A_1 = \sqrt{\gamma} |0\rangle \langle 1|, \quad (2.138)$$

then the Choi matrix of $\mathcal{E}$ is easily determined using proposition 1:

$$\begin{aligned} \Lambda_{\mathcal{E}} &= \sum_i |A_i\rangle\rangle \langle\langle A_i| \\ &= (|00\rangle + \sqrt{1-\gamma} |11\rangle) (\langle 00| + \sqrt{1-\gamma} \langle 11|) + \gamma |10\rangle \langle 10| \\ &= \begin{bmatrix} 1 & 0 & 0 & \sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \sqrt{1-\gamma} & 0 & 0 & 1-\gamma \end{bmatrix}. \end{aligned} \quad (2.139)$$

The Stinespring operator is

$$A = A_0 \otimes |0\rangle + A_1 \otimes |1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\gamma} \\ 0 & \sqrt{1-\gamma} \\ 0 & 0 \end{bmatrix}, \quad (2.140)$$

which clearly satisfies $A^\dagger A = \mathbb{1}$, and which partially defines the operator

$$U = \begin{bmatrix} 1 & \cdot & 0 & \cdot \\ 0 & \cdot & \sqrt{\gamma} & \cdot \\ 0 & \cdot & \sqrt{1-\gamma} & \cdot \\ 0 & \cdot & 0 & \cdot \end{bmatrix}. \quad (2.141)$$

We can easily extend it to the unitary operator

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & \sqrt{\gamma} & 0 \\ 0 & -\sqrt{\gamma} & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.142)$$

so that

$$\mathcal{E}(\rho) = \text{tr}_2 \left(U (\rho \otimes |0\rangle \langle 0|) U^\dagger \right). \quad (2.143)$$

2.2.8 The postulates revisited

We have now all that is necessary to state the postulates of quantum mechanics for open systems:

- 1' Representation: A quantum system is completely described a density operator $\rho \in \mathbf{D}(\mathcal{X})$ in the state space $\mathcal{X}$.
- 2' Composition: Quantum systems can be composed by means of the tensor product.
- 3' Measurement: quantum measurements are described by a set of measurement operators $\{M_i\}$ which satisfy

$$\sum_i M_i^\dagger M_i = \mathbf{1}. \quad (2.144)$$

A state ρ , after measurement, evolves to the state

$$\rho_i = \frac{M_i \rho M_i^\dagger}{\text{tr}[M_i^\dagger M_i \rho]} \quad (2.145)$$

with probability $p_i = \text{tr}[M_i^\dagger M_i \rho]$, given the outcome i occurred.

- 4' Evolution: evolutions are represented by quantum channels, i.e., completely positive, trace preserving linear superoperators.

It is interesting to note that every density operator can be seen as a pure state in a larger system, by means of purification; every quantum channel can be seen as part of a larger, unitary evolution, by the Stinespring dilation; and every generalized measurement can be seen as a projective measurement in a larger system too, which is the content of the Naimark dilation theorem. (9) Therefore, in this way we recover the quantum mechanics for closed systems. This procedure of enlarging the original Hilbert space to make things "simpler" was coined by John Smolin as "going to the Church of the larger Hilbert space". (107) Others may argue that such dilations are arbitrary mathematical tools and should not be worshipped or attached a real meaning, which led to the foundation of the "Church of the smaller Hilbert space", by Matthew Leifer. (108) Since this is a secular document, no belief is favoured with respect to the other, and is up to the reader to decide on what he believes.

2.3 Subsystem dynamics

Another way of seeing an open quantum system is as a subsystem of a larger "universe" (not necessarily its purification), i.e., a Hilbert space large enough to contain the system and all its surrounding environment, such that the total system is closed. More precisely, we have a system $\mathcal{H}_S$ coupled to a reservoir $\mathcal{H}_E$ such that $\mathcal{H}_S \otimes \mathcal{H}_E$ is closed. Any joint state $\rho_{SE} \in \mathbf{D}(\mathcal{H}_S \otimes \mathcal{H}_E)$ evolves unitarily, i.e., there is an unitary $U \in \mathbf{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$ such that the evolved state ρ'_{SE} is

$$\rho'_{SE} = U \rho_{SE} U^\dagger. \quad (2.146)$$

We are, however, interested in the reduced state $\rho_S = \text{tr}[\rho_{SE}]$, which evolves as

$$\rho'_S = \text{tr}_E [U \rho_{SE} U^\dagger]. \quad (2.147)$$

A question is naturally posed: is there a well-defined quantum channel $\mathcal{E}$ connecting any input $\rho_S \in \mathbf{D}(\mathcal{H}_S)$ to its output ρ'_S , i.e., such that the diagram

$$\begin{array}{ccc} \rho_{SE} & \xrightarrow{U} & U \rho_{SE} U^\dagger \\ \downarrow \text{tr}_E & & \downarrow \text{tr}_E \\ \rho_S & \xrightarrow{\mathcal{E}} & \mathcal{E}(\rho_S) \end{array} \quad (2.148)$$

commutes? In general, no. To see that, consider the simple example of a system qubit interacting with a environment qubit. (104) Consider the two initial states

$$\rho_{SE}^{(1)} = \rho_S \otimes \rho_E^{(1)}, \quad (2.149)$$

$$\rho_{SE}^{(2)} = \rho_S \otimes \rho_E^{(2)}, \quad (2.150)$$

where $\rho_E^{(1)} \neq \rho_E^{(2)}$, and the unitary *swap* operator U , which simply swaps system and bath states:

$$U |\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle, \quad |\psi\rangle \in \mathcal{H}_S, \quad |\phi\rangle \in \mathcal{H}_E. \quad (2.151)$$

In terms of the reduced state, we have the same initial state: $\text{tr}_E[\rho_{SE}^{(1)}] = \text{tr}_E[\rho_{SE}^{(2)}] = \rho_S$. The final state, however, is not unique:

$$\text{tr}_E [U \rho_{SE}^{(1)} U^\dagger] = \text{tr}_E [\rho_E^{(1)} \otimes \rho_S] = \rho_E^{(1)} \quad (2.152)$$

$$\text{tr}_E [U \rho_{SE}^{(2)} U^\dagger] = \text{tr}_E [\rho_E^{(2)} \otimes \rho_S] = \rho_E^{(2)}. \quad (2.153)$$

Therefore, we cannot even have a well-defined map for the reduced evolution, let alone a quantum channel. Then, how to proceed? Which constraints should be imposed in order to define quantum channels for the reduced system?

The solution is to restrict one, or two, of the free variables in the problem: the allowed set of initial states and the allowed set of joint unitary evolutions. To make the statement clearer, consider two examples. On the one hand, if we restrict the set of allowed initial states to

$$\mathcal{S} = \left\{ \rho_S \otimes \tilde{\rho}_E \mid \rho_S \in \mathbf{D}(\mathcal{H}_S) \right\}, \quad (2.154)$$

where $\tilde{\rho}_E$ is a fixed state in $\mathbf{D}(\mathcal{H}_E)$, then, using the spectral decomposition of $\tilde{\rho}$, $\tilde{\rho} = \sum_i \lambda_i |i\rangle \langle i|$, where the λ_i are positive real numbers,

$$\begin{aligned} \rho'_A &= \text{tr}_E [U \rho_S \otimes \tilde{\rho}_E U^\dagger] \\ &= \sum_j \langle j| \left(U \rho_S \otimes \sum_i \lambda_i |i\rangle \langle i| U^\dagger \right) |j\rangle \\ &= \sum_{i,j} \lambda_i \langle j| U |i\rangle \rho_S \langle i| U^\dagger |j\rangle \\ &\equiv \sum_{i,j} A_{ij} \rho_S A_{ij}^\dagger, \end{aligned} \quad (2.155)$$

where we defined $A_{ij} = \sqrt{\lambda_i} \langle j | U | i \rangle$. Therefore, we can define a map $\mathcal{E} \in \mathbf{L}^2(\mathcal{H}_S)$ for any unitary U by

$$\mathcal{E}(\rho) = \sum_{i,j} A_{ij} \rho A_{ij}^\dagger, \quad (2.156)$$

which is a quantum channel, since it has a Kraus decomposition. On the other hand, if we restrict the set of allowed evolutions to local unitary operations, i.e., of the form

$$U = U_S \otimes U_E, \quad (2.157)$$

then, since any ρ_{SE} can be written as $\rho_{SE} = \sum_{i,j} p_{ij} \rho_S^i \otimes \rho_E^j$ and $\rho_S = \text{tr}_E[\rho_{SE}] = \sum_i p_i \rho_S^i$, with $p_i = \sum_j p_{ij}$, we have that

$$\begin{aligned} \rho'_S &= \text{tr}_E \left[(U_S \otimes U_E) \rho_{SE} (U_S^\dagger \otimes U_E^\dagger) \right] \\ &= \sum_{i,j} p_{ij} \text{tr}_E \left[(U_S \otimes U_E) \rho_S^i \otimes \rho_E^j (U_S^\dagger \otimes U_E^\dagger) \right] \\ &= \sum_{i,j} p_{ij} \text{tr}_E \left[U_S \rho_S^i U_S^\dagger \otimes U_E \rho_E^j U_E^\dagger \right] \\ &= \sum_{i,j} p_{ij} U_S \rho_S^i U_S^\dagger \text{tr}_E \left[U_E \rho_E^j U_E^\dagger \right] \\ &= U_S \left(\sum_i p_i \rho_S^i \right) U_S^\dagger \\ &= U_S \rho_S U_S^\dagger, \end{aligned} \quad (2.158)$$

and we have a quantum channel. In fact, we have a closed evolution, as expected, since there was no interaction with the environment. To summarize, if we want to work with all possible joint unitary evolutions then we have to restrict the set of possible initial system-environment states^t, and vice-versa. In this section our goal is to find the set of initial states which guarantee well defined dynamical maps for any joint unitary evolution, which is based on the construction of a completely positive assignment map. (104)

2.3.1 Consistency

For convenience we denote the set of unitary operators and superoperators on $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ by $\mathbf{U}(\mathcal{H})$ and $\mathbf{U}^2(\mathcal{H})$, respectively, so each $U \in \mathbf{U}(\mathcal{H})$ corresponds to a $\mathcal{U}(\cdot) = U(\cdot)U^\dagger \in \mathbf{U}^2(\mathcal{H})$. Consider a set of initial states $\mathcal{S} \subset \mathbf{D}(\mathcal{H}_S \otimes \mathcal{H}_E)$ and a unitary evolution $U \in \mathbf{U}(\mathcal{H})$. If we have a reduced dynamical map $\mathcal{E}_U^S$ that makes the diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\mathcal{U}} & \mathbf{D}(\mathcal{H}_S \otimes \mathcal{H}_E) \\ \downarrow \text{tr}_E & & \downarrow \text{tr}_E \\ \text{tr}_E \mathcal{S} & \xrightarrow{\mathcal{E}_U^S} & \mathbf{D}(\mathcal{H}_S) \end{array} \quad (2.159)$$

commute, i.e.,

$$\mathcal{E}_U^S(\text{tr}_E[\rho]) = \text{tr}_E \left[U \rho U^\dagger \right], \quad (2.160)$$

^t The actual set of all possible states, as will be seen, is bigger than $\mathcal{S}$ as in Eq. (2.154). (104, 109, 110)

for every $\rho \in \mathbf{D}(\mathcal{H}_S \otimes \mathcal{H}_E)$, then it can only be well defined if, for any $\rho_1, \rho_2 \in \mathcal{S}$,

$$\mathrm{tr}_E[\rho_1] = \mathrm{tr}_E[\rho_2] \Rightarrow \mathrm{tr}_E[U\rho_1 U^\dagger] = \mathrm{tr}_E[U\rho_2 U^\dagger]. \quad (2.161)$$

Condition (2.161) is known as U -consistency. (104) Therefore, $\mathcal{E}_U^\mathcal{S}$ is well defined if $\mathcal{S}$ is U -consistent. If $\mathcal{S}$ is U -consistent for every $U \in \mathcal{G} \subset \mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$, then it is said to be $\mathcal{G}$ -consistent.

As can be seen, the properties of $\mathcal{E}_U^\mathcal{S}$ are determined by $\mathcal{S}$, so which conditions we need to impose on $\mathcal{S}$ in order to have a linear, CPTP reduced map?

In order to have a linear map, its domain must be linear space. Therefore, we expand $\mathcal{S} \in \mathbf{D}(\mathcal{H}_S \otimes \mathcal{H}_E)$ to the linear subspace $\mathcal{V} = \mathrm{span}(\mathcal{S}) \in \mathbf{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$, where $\mathrm{span}(\mathcal{S})$ is the linear subspace generated by $\mathcal{S}$, i.e., the set of all linear combinations of its elements:

$$\mathrm{span}(\mathcal{S}) = \left\{ \sum_i a_i \rho_i \mid \rho_i \in \mathcal{S}, a_i \in \mathbb{C} \right\}. \quad (2.162)$$

$\mathcal{V}$ is still $\mathcal{G}$ -consistent if $\mathcal{S}$ is, and furthermore it is generated by the density operators contained in it,

$$\mathcal{V} = \mathrm{span}(\mathbf{D}(\mathcal{H}_S \otimes \mathcal{H}_E) \cap \mathcal{V}), \quad (2.163)$$

so it is said to be spanned by states. (104) With this in mind, we forget about $\mathcal{S}$ and define a $\mathcal{G}$ -consistent subspace (104) as any linear subspace $\mathcal{V} \in \mathbf{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$ which is $\mathcal{G}$ -consistent and spanned by states. We likewise extend the reduced map from $\mathcal{E}_U^\mathcal{S} : \mathrm{tr}_E \mathcal{S} \rightarrow \mathbf{D}(\mathcal{H}_S)$ to $\mathcal{E}_U^\mathcal{V} : \mathrm{tr}_E \mathcal{V} \rightarrow \mathbf{L}(\mathcal{H}_S)$, which makes the diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\mathcal{U}} & \mathbf{L}(\mathcal{H}_S \otimes \mathcal{H}_E) \\ \downarrow \mathrm{tr}_E & & \downarrow \mathrm{tr}_E \\ \mathrm{tr}_E \mathcal{V} & \xrightarrow{\mathcal{E}_U^\mathcal{V}} & \mathbf{L}(\mathcal{H}_S) \end{array} \quad (2.164)$$

commute.

2.3.2 Assignment maps

Our goal is now restated: we want to find a $\mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$ -consistent subspace (i.e., consistent for every joint unitary evolution) for which the reduced maps $\mathcal{E}_U^\mathcal{V}$, $U \in \mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$, are linear, completely positive and trace preserving. The next proposition shows the first restriction on such sets.

Proposition 16. *A linear subspace $\mathcal{V} \in \mathbf{L}(\mathcal{H}_S \otimes \mathcal{H}_E)$, which is spanned by states, is $\mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$ -consistent if, and only if, $\mathrm{tr}_E|_{\mathcal{V}}$ is injective. (104)*

Proof. If $\mathrm{tr}_E|_{\mathcal{V}}$ is injective, then $\ker \mathrm{tr}_E|_{\mathcal{V}} = \{0\}$. This means that $\mathrm{tr}_E[X] = 0 \Rightarrow X = 0$, for every $X \in \mathcal{V}$. In particular, if $\rho_1, \rho_2 \in \mathcal{V}$, then

$$\mathrm{tr}_E[\rho_1] = \mathrm{tr}_E[\rho_2] \Rightarrow \mathrm{tr}_E[\rho_1 - \rho_2] = 0 \Rightarrow \rho_1 = \rho_2. \quad (2.165)$$

Therefore, $\mathcal{V}$ is trivially $\mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$ -consistent.

The converse statement is proved in Ref. (104), Lemma 1. $\square$

If $\text{tr}_E|_{\mathcal{V}}$ is injective then it is bijective, by the Rank-Nullity theorem. (86) Therefore, it has an inverse, which is called the assignment map (111):

$$\mathcal{A}_{\mathcal{V}} = (\text{tr}_E|_{\mathcal{V}})^{-1}. \quad (2.166)$$

Note that, in Ref. (104), this is known as the Pechukas-like assignment map. This map is not a quantum channel, but "almost" one.

Proposition 17. *If $\mathcal{V}$ is $\mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$ -consistent, then $\mathcal{A}_{\mathcal{V}}$ is linear, Hermiticity preserving and trace preserving. (104)*

Proof. Linearity follows from the fact that $\text{tr}_E|_{\mathcal{V}}$ is linear. Since

$$\text{tr}[\mathcal{A}_{\mathcal{V}}(X)] = \text{tr}_S[\text{tr}_E[\mathcal{A}_{\mathcal{V}}(X)]] = \text{tr}_S[\text{tr}_E(\text{tr}_E|_{\mathcal{V}})^{-1}(X)] = \text{tr}_S[X], \quad (2.167)$$

for every $X \in \text{tr}_E\mathcal{V}$, then $\mathcal{A}_{\mathcal{V}}$ is trace preserving.

Finally, Hermiticity preservation means that $\mathcal{A}_{\mathcal{V}}(X^\dagger) = \mathcal{A}_{\mathcal{V}}(X)^\dagger$, for every $X \in \text{tr}_E\mathcal{V}$. First, note that $X \in \text{tr}_E\mathcal{V}$ means that there is some $Y \in \mathcal{V}$ such that $X = \text{tr}_E[Y]f$ and $Y = \mathcal{A}_{\mathcal{V}}(X)$. Second, we have that $\text{tr}_E[Y^\dagger] = \text{tr}_E[Y]^\dagger$. Third, if $Y \in \mathcal{V}$, then $Y^\dagger \in \mathcal{V}$. This holds since, as $\mathcal{V}$ is spanned by states, $Y = \sum_i a_i \rho_i$, with $\rho_i \in (\mathcal{V} \cap \mathbf{D}(\mathcal{H}_S \otimes \mathcal{H}_E))$. Then,

$$Y^\dagger = \sum_i a_i^* \rho_i^\dagger = \sum_i a_i^* \rho_i. \quad (2.168)$$

Since $\mathcal{V}$ is a linear subspace, we have that $Y^\dagger \in \mathcal{V}$. Putting the pieces together,

$$\mathcal{A}_{\mathcal{V}}(X^\dagger) = \mathcal{A}_{\mathcal{V}}(\text{tr}_E[Y^\dagger]) = \mathcal{A}_{\mathcal{V}}(\text{tr}_E[Y])^\dagger = Y^\dagger = \mathcal{A}_{\mathcal{V}}(X)^\dagger. \quad (2.169)$$

$\square$

The assignment map is important because with it we can finally define the reduced maps and some of their properties.

Theorem 7. *If $\mathcal{V}$ is $\mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$ -consistent, then, for every $U \in \mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$, there is a unique reduced map $\mathcal{E}_U^\mathcal{V} : \text{tr}_E\mathcal{V} \rightarrow \mathbf{L}(\mathcal{H}_S)$ such that the diagram*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\mathcal{U}} & \mathbf{L}(\mathcal{H}_S \otimes \mathcal{H}_E) \\ \text{tr}_E \uparrow \mathcal{A}_{\mathcal{V}} & & \downarrow \text{tr}_E \\ \text{tr}_E\mathcal{V} & \xrightarrow{\mathcal{E}_U^\mathcal{V}} & \mathbf{L}(\mathcal{H}_S) \end{array} \quad (2.170)$$

commutes. It is actually given by

$$\mathcal{E}_U^\mathcal{V} = \text{tr}_E \cdot \mathcal{U} \cdot \mathcal{A}_{\mathcal{V}}, \quad (2.171)$$

and it is linear, Hermiticity and trace preserving (104)

Proof. Existence follows from Eq. (2.171) and uniqueness, linearity, Hermiticity and trace preservation follow from the same properties of tr_E , $\mathcal{U}$ and $\mathcal{A}_\mathcal{V}$, and the fact that composition preserves these properties. $\square$

The only remaining property to prove is the complete positivity of the reduced dynamics. Since tr_E and $\mathcal{U}$ are completely positive, we only need to find a $\mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$ -consistent subspace $\mathcal{V}$ for which the assignment map $\mathcal{A}_\mathcal{V}$ is completely positive. This subspace $\mathcal{V}$ possess a peculiar structure, known as the Koashi-Imoto decomposition (109), which is introduced next.

2.3.3 The Koashi-Imoto decomposition of states

Theorem 8. (*Koashi-Imoto theorem*) *For every (finite dimensional) $\mathcal{S} \subset \mathbf{D}(\mathcal{H})$ there is a unique decomposition of $\mathcal{H}$ as*

$$\mathcal{H} = \bigoplus_i \mathcal{H}_i = \bigoplus_i (\mathcal{H}_{l_i} \otimes \mathcal{H}_{r_i}) \quad (2.172)$$

such that

i) Every $\rho \in \mathcal{S}$ can be written as

$$\rho = \bigoplus_i p_i \rho_{l_i} \otimes \tilde{\omega}_{r_i}, \quad (2.173)$$

where $\mathbf{p} = (p_i)$ is a probability vector, $\rho_{l_i} \in \mathbf{D}(\mathcal{H}_{l_i})$, and $\tilde{\omega}_{r_i}$ are fixed states in $\mathbf{D}(\mathcal{H}_{r_i})$.

ii) Every quantum channel that leaves all states of $\mathcal{S}$ invariant, i.e., $\mathcal{E}(\rho) = \rho$, $\forall \rho \in \mathcal{S}$, satisfies

$$\mathcal{E}|_{\mathbf{L}(\mathcal{H}_i)} = \mathcal{I}_{l_i} \otimes \mathcal{E}|_{\mathbf{L}(\mathcal{H}_{r_i})} \equiv \mathcal{I}_{l_i} \otimes \mathcal{E}_{r_i}, \quad (2.174)$$

where the $\mathcal{E}_{r_i} \in \mathbf{L}^2(\mathcal{H}_{r_i})$ are quantum channels such that $\mathcal{E}_{r_i}(\tilde{\omega}_{r_i}) = \tilde{\omega}_{r_i}$, $\forall i$. (109, 112, 113)

This decomposition has a useful physical interpretation. (109, 113) The probability vector $\mathbf{p} = (p_i)$ is called the classical part of $\mathcal{S}$, since it can be accessed by the projective measurements Π_i of each $\mathcal{H}_i$ without disturbing all states of the set. The states $\{\rho_{l_i}\}$ are the quantum part of the set, since are inaccessible by operations that do not disturb the set. Finally, the $\{\tilde{\omega}_{r_i}\}$ are called the redundant part, since the information about them is the same for every state of the set. This decomposition is also related to the states which satisfy the strong subadditivity of the von Neumann entropy with equality. (112) We are all set to state the main theorem of the section.

Theorem 9. *Let $\mathcal{V}$ be a $\mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$ -consistent subspace. The assignment map $\mathcal{A}_\mathcal{V}$ is completely positive if, and only if, there is a decomposition*

$$\mathcal{H}_S = \bigoplus_i (\mathcal{H}_{l_i} \otimes \mathcal{H}_{r_i}) \quad (2.175)$$

of $\mathcal{H}_S$ such that every $\rho \in \mathcal{V}$ can be written as

$$\rho_{SE} = \bigoplus_i p_i \rho_{l_i} \otimes \tilde{\omega}_{r_i E}, \quad (2.176)$$

where $\mathbf{p} = (p_i)$ is a probability vector, $\rho_{l_i} \in \mathbf{D}(\mathcal{H}_{l_i})$, and $\tilde{\omega}_{r_i E}$ are fixed states in $\mathbf{D}(\mathcal{H}_{r_i} \otimes \mathcal{H}_E)$. (110, 113)

Proof. If the assignment map is completely positive, then $\mathcal{E} = \text{tr}_E \cdot \mathcal{A}_{\mathcal{V}}$ is completely positive too. Consider the Koashi-Imoto decomposition of $\text{tr}_E \mathcal{V}$,

$$\mathcal{H} = \bigoplus_i (\mathcal{H}_{l_i} \otimes \mathcal{H}_{r_i}). \quad (2.177)$$

Note that, for every $\rho \in \text{tr}_E \mathcal{V}$,

$$\mathcal{E}(\rho) = \text{tr}_E \cdot \mathcal{A}_{\mathcal{V}}(\rho) = \text{tr}_E (\text{tr}_E|_{\mathcal{V}})^{-1}(\rho) = \rho, \quad (2.178)$$

so $\mathcal{E}$ is a quantum channel that leaves all states of $\text{tr}_E \mathcal{V}$ invariant. Therefore, by the Koashi-Imoto theorem, it satisfies

$$\mathcal{E}(\rho)|_{\mathbf{L}(\mathcal{H}_i)} = \mathcal{I}_{l_i} \otimes \mathcal{E}_{r_i}, \quad (2.179)$$

where the $\mathcal{E}_{r_i} \in \mathbf{L}^2(\mathcal{H}_{r_i})$ are quantum channels such that $\mathcal{E}_{r_i}(\tilde{\omega}_{r_i}) = \tilde{\omega}_{r_i}$, $\forall i$. But, in fact,

$$\mathcal{E}(\rho)|_{\mathbf{L}(\mathcal{H}_i)} = (\text{tr}_E \cdot \mathcal{A}_{\mathcal{V}})|_{\mathbf{L}(\mathcal{H}_i)} \equiv \mathcal{I}_{l_i} \otimes \text{tr}_E \cdot (\mathcal{A}_{\mathcal{V}})_{r_i}, \quad (2.180)$$

such that $\text{tr}_E \cdot (\mathcal{A}_{\mathcal{V}})_{r_i}(\tilde{\omega}_{r_i}) = \tilde{\omega}_{r_i}$, $\forall i$. Now define

$$\tilde{\omega}_{r_i E} = (\mathcal{A}_{\mathcal{V}})_{r_i}(\tilde{\omega}_{r_i}). \quad (2.181)$$

Since tr_E is a bijection, we have that each $\rho \in \text{tr}_E \mathcal{V}$ corresponds to a unique $\rho_{SE} \in \mathcal{V}$ such that $\rho_{SE} = \mathcal{A}_{\mathcal{V}}(\rho)$. therefore,

$$\begin{aligned} \rho_{SE} &= \mathcal{A}_{\mathcal{V}} \left(\bigoplus_i p_i \rho_{l_i} \otimes \tilde{\omega}_{r_i} \right) \\ &= \bigoplus_i p_i \mathcal{I}_{l_i}(\rho_{l_i}) \otimes (\mathcal{A}_{\mathcal{V}})_{r_i}(\tilde{\omega}_{r_i}) \\ &= \bigoplus_i p_i \rho_{l_i} \otimes \tilde{\omega}_{r_i E}. \end{aligned} \quad (2.182)$$

Conversely, given the decomposition of the theorem, we can explicitly write the assignment map as

$$\mathcal{A}_{\mathcal{V}}(\rho) = \sum_i \text{tr}_{r_i} [\Pi_i \rho \Pi_i] \otimes \tilde{\omega}_{r_i E}, \quad (2.183)$$

where Π_i are the projections onto the subspaces $\mathcal{H}_i$. The map is clearly completely positive, since it is a composition of completely positive operations. $\square$

Theorem 9 has a clear physical meaning: a set of joint initial states $\mathcal{V}$ guarantees CPTP reduced dynamics if and only if their quantum part is privately owned by the system and the classical part can be accessed by local measurements without disturbance. (113) Also, the redundant part $\{\tilde{\omega}_{r_i E}\}$, which is the part that is in "contact" with the environment, is a general quantum state and can have any kind of correlations.

The fact that the information about the redundant part is the same for every state is what actually allows us to construct the CP description of the reduced dynamics, i.e., its Kraus operators. Namely, since the $\{\tilde{\omega}_{r_i E}\}$ are known, we can use their eigendecomposition

$$\tilde{\omega}_{r_i E} = \sum_{\alpha, p_i} \lambda_{\alpha, p_i} |\alpha, p_i\rangle \langle \alpha, p_i|, \quad (2.184)$$

where $\{|\alpha\rangle\}$ is a basis for $\mathcal{H}_E$ and $\{|p_i\rangle\}$ are bases for each $\mathcal{H}_{r_i}$. Therefore,

$$\begin{aligned} \mathcal{E}_U^{\mathcal{V}}(\rho) &= \text{tr}_E \cdot \mathcal{U} \cdot \mathcal{A}_{\mathcal{V}}(\rho) \\ &= \text{tr}_E \left[U \left(\sum_i \text{tr}_{r_i} [\Pi_i \rho \Pi_i] \otimes \tilde{\omega}_{r_i E} \right) U^\dagger \right] \\ &= \sum_{\alpha, i} \langle \alpha | U \sum_{p_i} \langle p_i | \Pi_i \rho \Pi_i | p_i \rangle \otimes \left(\sum_{\beta, q_i} \lambda_{\beta, q_i} |\beta, q_i\rangle \langle \beta, q_i| \right) U^\dagger | \alpha \rangle \\ &= \sum_{\alpha, \beta, i, p_i, q_i} \lambda_{\beta, q_i} \langle \alpha | U | \beta \rangle \langle p_i | \Pi_i | q_i \rangle \rho \langle q_i | \Pi_i | p_i \rangle \langle \beta | U^\dagger | \alpha \rangle \\ &\equiv \sum_{\alpha, \beta, i, p_i, q_i} A_{\alpha \beta i p_i q_i} \rho A_{\alpha \beta i p_i q_i}^\dagger, \end{aligned} \quad (2.185)$$

where

$$A_{\alpha \beta i p_i q_i} = \sqrt{\lambda_{\beta, q_i}} \langle \alpha | U | \beta \rangle \langle p_i | \Pi_i | q_i \rangle \quad (2.186)$$

are the Kraus operators of the map. In fact,

$$\begin{aligned} \sum_{\alpha, \beta, i, p_i, q_i} A_{\alpha \beta i p_i q_i}^\dagger A_{\alpha \beta i p_i q_i} &= \sum_{\alpha, \beta, i, p_i, q_i} \lambda_{\beta, q_i} \langle q_i | \Pi_i | p_i \rangle \langle \beta | U^\dagger | \alpha \rangle \langle \alpha | U | \beta \rangle \langle p_i | \Pi_i | q_i \rangle \\ &= \sum_{\beta, i, p_i, q_i} \lambda_{\beta, q_i} \langle q_i | \Pi_i | p_i \rangle \langle \beta | U^\dagger U | \beta \rangle \langle p_i | \Pi_i | q_i \rangle \\ &= \sum_{\beta, i, p_i, q_i} \lambda_{\beta, q_i} \langle q_i | \Pi_i | p_i \rangle \mathbb{1}_{\mathcal{H}_A} \langle p_i | \Pi_i | q_i \rangle \\ &= \sum_{\beta, i, q_i} \lambda_{\beta, q_i} \langle q_i | \Pi_i | q_i \rangle \mathbb{1}_{\mathcal{H}_A} \\ &= \sum_{\beta, i, q_i} \lambda_{\beta, q_i} \mathbb{1}_{\mathcal{H}_A} \\ &= \mathbb{1}_{\mathcal{H}_A}, \end{aligned} \quad (2.187)$$

so the map is indeed a quantum channel.

The simplest example of a set of initial states that guarantees reduced quantum dynamics is when the Koashi-Imoto decomposition has only one term, $i = \{1\}$, and the

space $\mathcal{H}_r$ is one-dimensional. Therefore, $\mathcal{H} = \mathcal{H}_l \otimes \mathbb{C} \simeq \mathcal{H}_l$, and $\mathcal{V}$ is simply the set

$$\mathcal{S} = \left\{ \rho_S \otimes \tilde{\rho}_E \mid \rho_S \in \mathbf{D}(\mathcal{H}_l) \right\}, \quad (2.188)$$

i.e., we recover the case of uncorrelated states. (113) If we let go of the restriction on the indices i , then $\mathcal{V}$ becomes

$$\mathcal{S} = \left\{ \bigoplus_i p_i \rho_i \otimes \tilde{\omega}_i \mid \rho_i \in \mathbf{D}(\mathcal{H}_i) \right\}, \quad (2.189)$$

where $\mathbf{p} = (p_i)$ is a probability vector and the $\tilde{\omega}_i \in \mathbf{D}(\mathcal{H}_E)$ are fixed environment states. Now we allow classical correlations (zero quantum discord) between system and environment. (113, 114) It is also possible to have other types of system-environment correlations, such as quantum entanglement. (104, 113)

Theorem 9 gives the most general set of initial states which assure that the assignment map is completely positive, for a $\mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$ -consistent subspace. The condition of consistency on the whole set of joint unitary evolutions can be relaxed to $\mathcal{G}$ -consistency, where $\mathcal{G} \subset \mathbf{U}(\mathcal{H}_S \otimes \mathcal{H}_E)$. This case is fully worked out in Refs. (104) and (110) with a generalization of the assignment map, and yields a larger set of initial states than the one in theorem 9.

2.3.4 To complete positivity and beyond

As we have seen, when we work in a consistent subspace, the reduced dynamics is linear, Hermiticity and trace preserving. In order to have complete positivity, however, the set of initial states with which we can work must be severely restricted. This raises doubts about completely positive maps in open quantum dynamics: are they really the most general class of quantum evolutions? Is complete positivity really necessary?

The debate on whether or not non-completely positive (NCP) maps are "physical" and could represent open dynamics is old and still divides quantum information scientists. (111, 115–122) It is also closely related to quantum non-Markovianity (122), a concept which will be defined in chapter 4.

Regardless of the opinion on the debate, working with NCP maps is more challenging than with CP maps, which have a simpler structure and have been extensively studied. One approach (122), which is related to the formalism developed in this section, is to restrict the domain of a Hermiticity and trace preserving map to its "positivity domain", i.e., the quantum states which are taken to other quantum states under the action of the map. This requires extreme care when handling with NCP maps, specially when composition of maps is involved. Another recent approach is the superchannel or M -map approach (15, 95, 123, 124), which works with operators of channels, called superchannels, and the state preparation procedure is considered as the input of the maps. Therefore,

this formalism avoids issues related with initial system-environment correlations and does not need the use of NCP maps, since the superchannels are always CP.

2.4 Quantum dynamical maps

Until now we studied evolutions which take an input and return an output. It is desirable to extend the formalism to allow time-dependent evolutions, so that the input is continuously changed to the output:

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle. \quad (2.190)$$

This can be easily extended to density operators: considering them as an ensemble of wavefunctions^u,

$$\rho(0) = \sum_i p_i |\psi(t_0)\rangle \langle \psi(t_0)|, \quad (2.191)$$

each of which evolves according to Eq. (2.190), then the evolved operator is

$$\begin{aligned} \rho(t) &= \sum_i p_i |\psi(t)\rangle \langle \psi(t)| \\ &= \sum_i p_i U(t, t_0) |\psi(t_0)\rangle \langle \psi(t_0)| U^\dagger(t, t_0) \\ &= U(t, t_0) \rho(t_0) U^\dagger(t, t_0) \\ &\equiv \mathcal{U}_{(t, t_0)}(\rho(t_0)), \end{aligned} \quad (2.192)$$

where $\mathcal{U}_{(t, t_0)}$ is an unitary superoperator. With this in mind, we define for open quantum systems a *quantum dynamical map* (2, 4), a quantum channel connecting an initial state $\rho(t_0)$ to a posterior state $\rho(t)$:

$$\rho(t) = \mathcal{E}_{(t, t_0)}(\rho(t_0)). \quad (2.193)$$

For a fixed initial state t_0 , we can define a family of maps $\{\mathcal{E}_{(t, t_0)}\}$ for every $t \geq t_0$. However, we cannot in general define an intermediate evolution $\mathcal{E}_{(t, t')}$, with $t' > t_0$, since it might not exist and, even if it exists, it might not be a quantum channel. (10)

2.4.1 Semigroups and divisibility

Consider the unitary dynamical map from Eq. (2.192),

$$\mathcal{U}_{(t, t_0)}(\rho(t_0)) = U(t, t_0) \rho(t_0) U^\dagger(t, t_0). \quad (2.194)$$

This evolution satisfies several useful properties. First, $\mathcal{U}_{(t, t)} = \mathcal{I}$, for any time t , since no evolution is actually taking place. Second, if we evolve some state from t_0 to t_1 ,

^u Note that pure states correspond to a trivial ensemble.

$\rho(t_1) \equiv \mathcal{U}_{(t_1, t_0)}(\rho(t_0))$, and later to t_2 , $\rho(t_2) \equiv \mathcal{U}_{(t_2, t_1)}(\rho(t_1))$, the composite evolution is $\rho(t_2) \equiv \mathcal{U}_{(t_2, t_1)}(\mathcal{U}_{(t_1, t_0)}(\rho(t_0)))$. In summary,

$$\begin{cases} \mathcal{U}_{(t, t)} = \mathcal{I} \\ \mathcal{U}_{(t_2, t_0)} = \mathcal{U}_{(t_2, t_1)} \mathcal{U}_{(t_1, t_0)}. \end{cases} \quad (2.195)$$

The conditions in Eq. (2.195) define a *two-parameter group*. (2) Now, consider the operators $U(t, t_0)$ as homogeneous in time, i.e., $U(t, t_0) = U(t - t_0)$. If we redefine $t - t_0 \rightarrow t$, Eq. (2.195) turns to

$$\begin{cases} \mathcal{U}_0 = \mathcal{I} \\ \mathcal{U}_{t+s} = \mathcal{U}_t \mathcal{U}_s, \end{cases} \quad (2.196)$$

which defines a *one-parameter group*. (2) Can open evolutions satisfy these properties too? In fact, since the inverse of a quantum channel is a quantum channel if and only if the channel is unitary, open evolutions do not form a group, but the invertibility condition can be dropped, which leaves us with a semigroup. (2) Therefore, $\mathcal{E}_{(t, t_0)}$ forms a two-parameter *semigroup* or evolution family (2) if

$$\begin{cases} \mathcal{E}_{(t, t)} = \mathcal{I} \\ \mathcal{E}_{(t_2, t_0)} = \mathcal{E}_{(t_2, t_1)} \mathcal{E}_{(t_1, t_0)}, \quad t_2 \geq t_1 \geq t_0, \end{cases} \quad (2.197)$$

and a one-parameter semigroup if

$$\begin{cases} \mathcal{E}_0 = \mathcal{I} \\ \mathcal{E}_{t+s} = \mathcal{E}_t \mathcal{E}_s, \quad \forall t, s \geq 0. \end{cases} \quad (2.198)$$

Now, when does a family of quantum dynamical maps form a semigroup? First of all, the intermediate maps $\mathcal{E}_{(t_2, t_1)}$ may not exist. If the evolution is invertible for all $t \geq 0$, then we can define (125)

$$\mathcal{E}_{(t_2, t_1)} = \mathcal{E}_{(t_2, t_0)} \mathcal{E}_{(t_1, t_0)}^{-1}, \quad (2.199)$$

but now $\mathcal{E}_{(t_2, t_1)}$ might be no longer a quantum channel, since this inverse map is in general not even positive. (105) This motivates the following definition: a quantum dynamical map $\mathcal{E}_{(t, t_0)}$ is *k-divisible* if, for every $t' \in [t, t_0]$,

$$\mathcal{E}_{(t, t_0)} = \mathcal{E}_{(t, t')} \mathcal{E}_{(t', t_0)}, \quad (2.200)$$

and $\mathcal{E}_{(t, t')}$ is a *k*-positive map. Note that this is analogous to the case of *k*-positivity:

- $\mathcal{E}$ is *k*-positive if $\mathcal{E} \otimes \mathcal{I}_k$ is positive.
- $\mathcal{E}_{(t, t_0)}$ is *k*-divisible if $\mathcal{E}_{(t, t')} \otimes \mathcal{I}_k$ is positive, for every $t' \in [t, t_0]$.

We say that a 1-divisible map is P-divisible, and a *d*-divisible map, where *d* is the dimension of the system Hilbert space (i.e., $\mathcal{E}_{(t, t')}$ is completely positive), is CP-divisible. A CP-divisible map forms a two-parameter semigroup, since the intermediate dynamics always represents quantum channels. The following theorem provides a powerful characterization of *k*-divisible maps.

Theorem 10. *An invertible quantum dynamical map $\mathcal{E}_{(t,t_0)}$ is k -divisible if, and only if,*

$$\frac{d}{dt} \left\| \left(\mathcal{E}_{(t,t_0)} \otimes \mathcal{I}_k \right) (X) \right\|_1 \leq 0, \quad (2.201)$$

for every $X \in \mathbf{L}(\mathcal{X} \otimes \mathbb{C}^k)$ Hermitian. (103)

Proof. For simplicity, define $Y = \left(\mathcal{E}_{(t,t_0)} \otimes \mathcal{I}_k \right) (X)$, so

$$\begin{aligned} \frac{d}{dt} \|Y\|_1 &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left\{ \left\| \left(\mathcal{E}_{(t+\epsilon,t_0)} \otimes \mathcal{I}_k \right) (X) \right\|_1 - \left\| \left(\mathcal{E}_{(t,t_0)} \otimes \mathcal{I}_k \right) (X) \right\|_1 \right\} \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left\{ \left\| \left(\mathcal{E}_{(t+\epsilon,t)} \mathcal{E}_{(t,t_0)} \otimes \mathcal{I}_k \right) (X) \right\|_1 - \|Y\|_1 \right\}, \end{aligned} \quad (2.202)$$

where we used that $\mathcal{E}_{(t,t_0)}$ is invertible in order to define the intermediate map. Now,

$$\left\| \left(\mathcal{E}_{(t+\epsilon,t)} \mathcal{E}_{(t,t_0)} \otimes \mathcal{I}_k \right) (X) \right\|_1 = \left\| \left(\mathcal{E}_{(t+\epsilon,t)} \otimes \mathcal{I}_k \right) \left(\mathcal{E}_{(t,t_0)} \otimes \mathcal{I}_k \right) (X) \right\|_1, \quad (2.203)$$

so

$$\frac{d}{dt} \|Y\|_1 = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left\{ \left\| \left(\mathcal{E}_{(t+\epsilon,t)} \otimes \mathcal{I}_k \right) (Y) \right\|_1 - \|Y\|_1 \right\}. \quad (2.204)$$

Since $\mathcal{E}_{(t,t_0)}$ is CPTP and, in particular, Hermiticity preserving, Hermiticity of X implies Hermiticity of Y .

Now suppose that $\mathcal{E}_{(t,t_0)}$ is k -divisible. Then, $\mathcal{E}_{(t+\epsilon,t)}$ is k -positive, so $\left(\mathcal{E}_{(t+\epsilon,t)} \otimes \mathcal{I}_k \right)$ is a positive map. By proposition 9, if X is Hermitian, then

$$\left\| \left(\mathcal{E}_{(t+\epsilon,t)} \otimes \mathcal{I}_k \right) (Y) \right\|_1 \leq \|Y\|_1, \quad (2.205)$$

so

$$\frac{d}{dt} \|Y\|_1 \leq \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \{ \|Y\|_1 - \|Y\|_1 \} = 0. \quad (2.206)$$

Conversely, if Eq. (2.201) holds, then, for every $\epsilon \geq 0$,

$$\left\| \left(\mathcal{E}_{(t+\epsilon,t)} \otimes \mathcal{I}_k \right) (Y) \right\|_1 \leq \|Y\|_1. \quad (2.207)$$

By proposition 9, $\left(\mathcal{E}_{(t+\epsilon,t)} \otimes \mathcal{I}_k \right)$ is positive, so $\mathcal{E}_{(t+\epsilon,t)}$ is k -positive, for every $t \geq t_0$ and $\epsilon \geq 0$. Therefore, $\mathcal{E}_{(t,t_0)}$ is k -divisible. $\square$

2.4.2 Matrix representation

Quantum dynamical maps, being linear superoperators, can be represented by matrices. This representation is very useful, specially when we want to study the relationship between dynamical maps and master equations. In this section, for simplicity, we restrict ourselves to superoperators in $\mathbf{L}^2(\mathcal{X})$. Consider an orthonormal basis $\{G_i\}$ of $\mathbf{D}(\mathcal{X})$,

$$\text{tr} [G_i^\dagger G_j] = \delta_{ij}, \quad \forall i, j = 0, \dots, d^2 - 1, \quad (2.208)$$

where d is the dimension of $\mathcal{X}$. We further assume that $G_0 = \mathbb{1}/\sqrt{d}$, which implies that all the other basis operators are traceless, and that the basis elements are Hermitian^v. For a density matrix $\rho(t) \in \mathbf{D}(\mathcal{X})$, we have that

$$\begin{aligned}\rho(t) &= \sum_{k=0}^{d^2-1} \text{tr}[G_k \rho(t)] G_k \\ &= \frac{1}{d} \text{tr}[\rho(t)] \mathbb{1} + \sum_{k=1}^{d^2-1} \text{tr}[G_k \rho(t)] G_k \\ &\equiv \frac{1}{d} \left(\mathbb{1} + \sum_{k=1}^{d^2-1} x_k(t) G_k \right),\end{aligned}\tag{2.209}$$

where

$$x_k(t) = d \text{tr}[G_k \rho(t)]\tag{2.210}$$

forms the vector $\vec{x}(t) = (x_k(t))$, which represents $\rho(t)$. Since $\rho(t) = \mathcal{E}_{(t,t_0)}[\rho(t_0)]$, then

$$\rho(t) = \mathcal{E}_{(t,t_0)} \left[\frac{1}{d} \left(\mathbb{1} + \sum_{k=1}^{d^2-1} x_k(t) G_k \right) \right] = \frac{1}{d} \left(\mathcal{E}_{(t,t_0)}[\mathbb{1}] + \sum_{k=1}^{d^2-1} x_k(t) [\mathcal{E}_{(t,t_0)}[G_k]] \right).\tag{2.211}$$

In terms of the vector $\vec{x}(t)$,

$$\begin{aligned}x_j(t) &= \text{tr} \left[G_j \left(\mathcal{E}_{(t,t_0)}[\mathbb{1}] + \sum_{k=1}^{d^2-1} x_k(t) [\mathcal{E}_{(t,t_0)}[G_k]] \right) \right] \\ &= \text{tr} [G_j \mathcal{E}_{(t,t_0)}[\mathbb{1}]] + \sum_{k=1}^{d^2-1} \text{tr} [G_j \mathcal{E}_{(t,t_0)}[G_k]] x_k(t) \\ &\equiv w_j(t, t_0) + \sum_{k=1}^{d^2-1} B_{jk}(t, t_0) x_k(t),\end{aligned}\tag{2.212}$$

so we have now a vector equation

$$\vec{x}(t) = \vec{w}(t, t_0) + B(t, t_0) \vec{x}(t),\tag{2.213}$$

where $B(t, t_0)$ and $\vec{w}(t, t_0)$ represent the dynamical map $\mathcal{E}_{(t,t_0)}$ and are given by

$$w_j(t, t_0) = \text{tr} [G_j \mathcal{E}_{(t,t_0)}[\mathbb{1}]],\tag{2.214}$$

$$B_{jk}(t, t_0) = \text{tr} [G_j \mathcal{E}_{(t,t_0)}[G_k]].\tag{2.215}$$

Since we separated the terms corresponding to the basis element $G_0 = \mathbb{1}/\sqrt{d}$, the dynamical map is actually represented by a matrix and a vector and not simply a matrix, but in essence the two cases are the same thing.

^v This is possible since $\mathbf{D}(\mathcal{X})$ is a subset of the set of Hermitian operators in $\mathbf{L}(\mathcal{X})$.

3 QUANTUM MASTER EQUATIONS

In this chapter we study quantum master equations, which are equations of motion for quantum systems. For closed quantum systems they correspond to the Schrödinger equation (75), for wavefunctions, and the Liouville-von Neumann equation (1), for density operators. For open quantum systems, however, master equations usually assume a complicated integrodifferential form, and therefore developing a general theory is very hard. Therefore, we mainly focus on the restricted set of *time-local* master equations (126), i.e., master equations of the form

$$\frac{d}{dt}\rho(t) = \mathcal{L}_t\rho(t), \quad (3.1)$$

which depend only on the current state of the system and not on earlier times. They have a simple form, which is a generalization of the well-known Lindblad form. (73, 127) We also introduce the concepts of positive (P) and completely positive (CP) divisibility, which will be crucial for the definition of quantum non-Markovianity, the main theme of the text. Finally, we derive a master equation from subsystem dynamics, and show that, under a specific set of assumptions and approximations, collectively known as the Born-Markov approximation (1), it corresponds to a CP-divisible quantum channel.

3.1 Closed evolution

3.1.1 The Liouville-von Neumann equation

Recall that a quantum state $|\psi\rangle \in \mathcal{X}$ in a closed system evolves under Eq. (2.190),

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle. \quad (3.2)$$

Since this evolution is unitary, we can reverse it:

$$|\psi(t_0)\rangle = U^\dagger(t, t_0) |\psi(t)\rangle. \quad (3.3)$$

Suppose, for simplicity, that the evolution is homogeneous in time, i.e., $U(t, t_0) = U(t - t_0)$, so the unitary operator depends only on the difference of final and initial times. Every unitary operator can be written as a complex exponential of an Hermitian operator (86), so

$$U(t - t_0) = e^{-iH(t-t_0)}, \quad (3.4)$$

for some Hermitian $H \in \mathbf{L}(\mathcal{X})$. Taking the time derivative of Eq. (3.2) and using Equations (3.3) and (3.4),

$$\begin{aligned} \frac{d}{dt} |\psi(t)\rangle &= \left(\frac{d}{dt} U(t-t_0) \right) |\psi(t_0)\rangle \\ &= \left(\frac{d}{dt} U(t-t_0) \right) U^\dagger(t-t_0) |\psi(t)\rangle \\ &= -i H U(t-t_0) U^\dagger(t-t_0) |\psi(t)\rangle \\ &= -i H |\psi(t)\rangle. \end{aligned} \quad (3.5)$$

Equation (3.5) is the *Schrödinger equation* (75), and is usually known in a slightly more general form, where we assume that the H operator, known as the *Hamiltonian* of the system (75), is time-dependent, and we insert the *reduced Planck constant* $\hbar$ (76):

$$\frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} H(t) |\psi(t)\rangle. \quad (3.6)$$

In this text we consider, for simplicity, $\hbar = 1$. The Schrödinger equation can be easily extended to density operators, in a procedure similar as the one performed in Section 2.4, leading to the *Liouville-von Neumann equation* (1),

$$\frac{d}{dt} \rho(t) = -i [H(t), \rho(t)]. \quad (3.7)$$

3.1.2 Dyson series

When dealing with time-dependent Hamiltonians, the unitary operator $U(t, t_0)$ does not have the simple form from Eq. (3.4). Comparing Eq. (3.5) with Eq. (3.6),

$$\frac{d}{dt} |\psi(t)\rangle = \left(\frac{d}{dt} U(t, t_0) \right) U^\dagger(t, t_0) |\psi(t)\rangle = -i H(t) |\psi(t)\rangle, \quad (3.8)$$

for every $|\psi\rangle$, so $U(t, t_0)$ obeys the differential equation

$$\left(\frac{d}{dt} U(t, t_0) \right) U^\dagger(t, t_0) = -i H(t). \quad (3.9)$$

Multiplying Eq. (3.1.2) on the right by $U(t, t_0)$ leads to

$$\frac{d}{dt} U(t, t_0) = -i H(t) U(t, t_0). \quad (3.10)$$

Integrating this equation and using the initial condition $U(t_0, t_0) = \mathbb{1}$ leads to

$$U(t, t_0) = \mathbb{1} - i \int_{t_0}^t dt_1 H(t_1) U(t_1, t_0). \quad (3.11)$$

We can substitute Eq. (3.11) in itself, leading to

$$U(t, t_0) = \mathbb{1} - i \int_{t_0}^t dt_1 H(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H(t_1) H(t_2) U(t_2, t_0), \quad (3.12)$$

and repeat this process indefinitely, arriving at (2)

$$\begin{aligned} U(t, t_0) &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n H(t_1) \cdots H(t_n) \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n H(t_1) \cdots H(t_n). \end{aligned} \quad (3.13)$$

Take a closer look at the second order integral in the above expansion,

$$\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H(t_1) H(t_2). \quad (3.14)$$

Now define the operator

$$\mathcal{T}[H(t_1)H(t_2)] = \frac{1}{2} [\theta(t_1 - t_2)H(t_1)H(t_2) + \theta(t_2 - t_1)H(t_2)H(t_1)], \quad (3.15)$$

where $\theta(t_1 - t_2)$ is the Heaviside step-function (2):

$$\theta(t_1 - t_2) = \begin{cases} 1, & t_1 \geq t_2 \\ 0, & t_1 < t_2 \end{cases} \quad (3.16)$$

Note that^a

$$\begin{aligned} \mathcal{T} \left[\left(\int_{t_0}^t ds H(s) \right)^2 \right] &= \mathcal{T} \left[\int_{t_0}^t dt_1 H(t_1) \int_{t_0}^t dt_2 H(t_2) \right] \\ &= \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \mathcal{T}[H(t_1)H(t_2)] \\ &= \frac{1}{2} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \theta(t_1 - t_2) H(t_1) H(t_2) \\ &\quad + \frac{1}{2} \int_{t_0}^t dt_2 \int_{t_0}^t dt_1 \theta(t_2 - t_1) H(t_2) H(t_1) \\ &= \frac{1}{2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H(t_1) H(t_2) \\ &\quad + \frac{1}{2} \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 H(t_2) H(t_1). \end{aligned} \quad (3.17)$$

Relabeling the indices in the last integral leads to

$$\mathcal{T} \left[\left(\int_{t_0}^t dt_1 H(t_1) \right)^2 \right] = \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H(t_1) H(t_2), \quad (3.18)$$

so we recover the second order integral. The operator $\mathcal{T}$ is called the *time-ordering operator* (1), and it can be extended to higher orders. (2) With it we can rewrite the series expansion in Eq. (3.13),

$$\begin{aligned} U(t, t_0) &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n H(t_1) \cdots H(t_n) \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \mathcal{T} \left[\left(\int_{t_0}^t ds H(s) \right)^n \right] \\ &= \mathcal{T} \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(-i \int_{t_0}^t ds H(s) \right)^n \right], \end{aligned} \quad (3.19)$$

^a Pay attention to the limits of integration.

which we can recognize as the series expansion of the exponential function:

$$U(t, t_0) = \mathcal{T} \left[e^{-i \int_{t_0}^t ds H(s)} \right]. \quad (3.20)$$

Eq. (3.20) is known as the *Dyson series* expansion (2) of $U(t, t_0)$. Note that all operators inside the argument of $\mathcal{T}$ commute, so we can treat them as numbers. For example, we can differentiate them:

$$\frac{d}{dt} U(t, t_0) = \mathcal{T} \left[-i H(t) e^{-i \int_{t_0}^t ds H(s)} \right] = -i H(t) U(t, t_0). \quad (3.21)$$

Dyson series are quite involved and will not be solved in this text. However, when the Hamiltonian commutes with itself at different times, i.e.,

$$[H(t), H(t')] = 0 \quad (3.22)$$

for $t \neq t'$, then everything acted upon by the time-ordering operator commutes and, therefore, its action is trivial. In this case, the Dyson series reduces to an ordinary exponential,

$$U(t, t_0) = e^{-i \int_{t_0}^t ds H(s)}. \quad (3.23)$$

3.2 Memory-kernel master equations

Let us assume the basic setup for an open quantum system, i.e., a bipartite system $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ where S represents the system and E the environment. The composite system has a total Hamiltonian $H(t)$ and is closed, so it evolves unitarily by the Liouville-von Neumann equation:

$$\frac{d}{dt} \rho_{SE}(t) = -i [H(t), \rho_{SE}(t)] \equiv \mathcal{L}_t \rho_{SE}(t). \quad (3.24)$$

This is a master equation for the system-environment density operator. In order to write a master equation for the reduced system operator $\rho_S = \text{tr}_E [\rho_{SE}]$, we use the Nakajima-Zwanzig projection operator technique. (1, 128, 129) Given a fixed $\rho_E \in \mathbf{L}(\mathcal{H}_E)$, define the operator $\mathcal{P} \in \mathbf{L}^2(\mathcal{H})$ by

$$\mathcal{P} \rho_{SE} = \text{tr}_E [\rho_{SE}] \otimes \rho_E. \quad (3.25)$$

It can be easily checked that $\mathcal{P}$ is a projection, i.e., $\mathcal{P}^2 = \mathcal{P}$. Define $\mathcal{Q}$ as its orthogonal projection, i.e., $\mathcal{Q} = \mathcal{I}_{\mathcal{H}} - \mathcal{P}$. Applying $\mathcal{P}$ to Eq. (3.24) leads to

$$\begin{aligned} \mathcal{P} \frac{d}{dt} \rho_{SE}(t) &= \frac{d}{dt} \mathcal{P} \rho_{SE}(t) \\ &= \mathcal{P} \mathcal{L}_t \rho_{SE}(t) \\ &= \mathcal{P} \mathcal{L}_t (\mathcal{P} + \mathcal{Q}) \rho_{SE}(t) \\ &= \mathcal{P} \mathcal{L}_t \mathcal{P} \rho_{SE}(t) + \mathcal{P} \mathcal{L}_t \mathcal{Q} \rho_{SE}(t). \end{aligned} \quad (3.26)$$

Doing the same, but instead with $\mathcal{Q}$, leads to

$$\frac{d}{dt}\mathcal{Q}\rho_{SE}(t) = \mathcal{Q}\mathcal{L}_t\mathcal{Q}\rho_{SE}(t) + \mathcal{Q}\mathcal{L}_t\mathcal{P}\rho_{SE}(t). \quad (3.27)$$

Equation (3.27) can be solved for the variable $\mathcal{Q}\rho_{SE}(t)$ and initial time t_0 (1):

$$\mathcal{Q}\rho_{SE}(t) = \mathcal{G}(t, t_0)\mathcal{Q}\rho_{SE}(t_0) + \int_{t_0}^t ds \mathcal{G}(t, s)\mathcal{Q}\mathcal{L}_s\mathcal{P}\rho_{SE}(s), \quad (3.28)$$

where

$$\mathcal{G}(t, t_0) = \mathcal{T} \left[e^{\int_{t_0}^t ds' \mathcal{Q}\mathcal{L}_{s'}} \right]. \quad (3.29)$$

Replacing in Eq. (3.26) leads to the Nakajima-Zwanzig master equation (1),

$$\frac{d}{dt}\mathcal{P}\rho_{SE}(t) = \mathcal{P}\mathcal{L}_t\mathcal{G}(t, t_0)\mathcal{Q}\rho_{SE}(t_0) + \mathcal{P}\mathcal{L}_t\mathcal{P}\rho_{SE}(t) + \int_{t_0}^t ds \mathcal{P}\mathcal{L}_t\mathcal{G}(t, s)\mathcal{Q}\mathcal{L}_s\mathcal{P}\rho_{SE}(s). \quad (3.30)$$

Equation (3.30) can be further simplified under certain hypotheses. Assuming that the system starts in a product state,

$$\rho_{SE}(t_0) = \rho_S(t_0) \otimes \rho_E, \quad (3.31)$$

for a fixed environment state ρ_E , then $\mathcal{P}\rho_{SE}(t_0) = \rho_{SE}(t_0)$. Therefore, $\mathcal{Q}\rho_{SE}(t_0) = 0$, which implies that

$$\mathcal{P}\mathcal{L}_t\mathcal{G}(t, t_0)\mathcal{Q}\rho_{SE}(t_0) = 0. \quad (3.32)$$

The master equation is then

$$\frac{d}{dt}\mathcal{P}\rho_{SE}(t) = \mathcal{P}\mathcal{L}_t\mathcal{P}\rho_{SE}(t) + \int_{t_0}^t ds \mathcal{P}\mathcal{L}_t\mathcal{G}(t, s)\mathcal{Q}\mathcal{L}_s\mathcal{P}\rho_{SE}(s). \quad (3.33)$$

If we want a master equation for the reduced system density matrix, then we have to assume that the joint state factorizes for all times:

$$\rho_{SE}(t_0) = \rho_S(t_0) \otimes \rho_E(t_0), \quad (3.34)$$

which is a reasonable assumption when we deal with systems weakly coupled to the environment. (1) We can also, under certain additional conditions (1, 2), assume that

$$\mathcal{P}\mathcal{L}_t\mathcal{P}\rho_{SE}(t) = 0, \quad (3.35)$$

so

$$\frac{d}{dt}\mathcal{P}\rho_{SE}(t) = \int_{t_0}^t ds \mathcal{P}\mathcal{L}_t\mathcal{G}(t, s)\mathcal{Q}\mathcal{L}_s\mathcal{P}\rho_{SE}(s). \quad (3.36)$$

Considering the action of the $\mathcal{P}$ operator,

$$\frac{d}{dt}\rho_S(t) \otimes \rho_E = \int_{t_0}^t ds \operatorname{tr}_E [\mathcal{L}_t\mathcal{G}(t, s)\mathcal{Q}\mathcal{L}_s(\rho_S(s) \otimes \rho_E)] \otimes \rho_E. \quad (3.37)$$

Finally, tracing out the reservoir leads to the desired memory-kernel master equation for the system density matrix,

$$\frac{d}{dt}\rho_S(t) = \int_{t_0}^t ds \mathcal{K}(t, s) [\rho_S(s)], \quad (3.38)$$

where

$$\mathcal{K}(t, s)[\cdot] = \text{tr}_E [\mathcal{L}_t \mathcal{G}(t, s) \mathcal{Q} \mathcal{L}_s([\cdot] \otimes \rho_E)] \mathcal{Q} \mathcal{L}_s \quad (3.39)$$

is the memory kernel. (1,2) It is dubbed this way because it depends on the past history of the state, so it retains a "record" of the system evolution. Not all kernels display memory effects, though: the kernel $\mathcal{K}(t, s) = \mathcal{K} \delta(t - s)$, for example, leads to a time-local master equation. Finally, as it will be seen, even local in time master equations can display memory effects, so care must be taken when handling this subject.

3.3 Time-local master equations

Now suppose that the evolution of $\rho_S(t)$ is given by a quantum dynamical map:

$$\rho_S(t) = \mathcal{E}(t, t_0) [\rho_S(t_0)]. \quad (3.40)$$

If the dynamical map $\mathcal{E}(t, t_0)$ is invertible in the interval $[t_0, t]$, then, using Eq. (3.38),

$$\begin{aligned} \frac{d}{dt}\rho_S(t) &= \int_{t_0}^t ds \mathcal{K}(t, s) \rho_S(s) \\ &= \int_{t_0}^t ds \mathcal{K}(t, s) \mathcal{E}(s, t_0) \rho_S(t_0) \\ &= \int_{t_0}^t ds \mathcal{K}(t, s) \mathcal{E}(s, t_0) [\mathcal{E}^{-1}(t, t_0) \mathcal{E}(t, t_0)] \rho_S(t_0) \\ &= \left(\int_{t_0}^t ds \mathcal{K}(t, s) \mathcal{E}(s, t_0) \mathcal{E}^{-1}(t, t_0) \right) \mathcal{E}(t, t_0) \rho_S(t_0) \\ &\equiv \mathcal{L}_t \rho_S(t), \end{aligned} \quad (3.41)$$

which is a time-local master equation. The assumption of invertibility fails if and only if two distinct states evolve to the same state during the interval $[t, t_0]$. (126) Therefore, many memory-kernel master equations can be reduced to time-local master equations^b.

What is the general form of a time-local quantum master equation (68)? Given $\rho \in \mathbf{D}(\mathcal{X})$, $\mathcal{L}_t \in \mathbf{L}^2(\mathcal{X})$ and

$$\frac{d}{dt}\rho(t) = \mathcal{L}_t \rho(t), \quad (3.42)$$

we want to find the most general $\mathcal{L}_t$ which is linear and preserves positivity and trace. In the spirit of Section 2.2, we first assume that $\mathcal{L}_t$ is linear. Since

$$\rho(t) = \rho(t_0) + \int_{t_0}^t ds \mathcal{L}_s \rho(s), \quad (3.43)$$

^b There are other methods to show that characteristics, such as the time convolutionless projection operator method. (1)

then, taking the trace,

$$\mathrm{tr} [\rho(t)] = \mathrm{tr} [\rho(t_0)] + \int_{t_0}^t ds \mathrm{tr} [\mathcal{L}_s \rho(s)]. \quad (3.44)$$

In order to preserve the trace of $\rho(t)$ for all times, we must have that

$$\int_{t_0}^t ds \mathrm{tr} [\mathcal{L}_s \rho(s)] = 0, \quad (3.45)$$

which implies that $\mathrm{tr} [\mathcal{L}_t \rho(t)] = 0$.

Taking the Hermitian conjugate,

$$\begin{aligned} \rho^\dagger(t) &= \rho^\dagger(t_0) + \int_{t_0}^t ds (\mathcal{L}_s \rho(s))^\dagger \\ &= \rho(t_0) + \int_{t_0}^t ds (\mathcal{L}_s \rho(s))^\dagger. \end{aligned} \quad (3.46)$$

Therefore, to preserve Hermiticity we must have, for all times, $(\mathcal{L}_t \rho(t))^\dagger = \mathcal{L}_t \rho(t)$. How about positivity? Finding the most general conditions guaranteeing positivity and complete positivity of time-local master equations is still an open problem (10), so we will not tackle this question here.

Since $\mathcal{L}_t$ is a linear superoperator, it can be written, by Theorem 2, as

$$\mathcal{L}_t \rho(t) = \sum_i A_i(t) \rho(t) B_i^\dagger(t), \quad (3.47)$$

where $A_i, B_i \in \mathbf{L}(\mathcal{X})$. We now define an orthonormal basis $\{G_i\}$ on $\mathbf{L}(\mathcal{X})$,

$$\mathrm{tr} [G_i^\dagger G_j] = \delta_{ij}, \quad \forall i, j = 0, \dots, d^2 - 1, \quad (3.48)$$

where d is the dimension of $\mathcal{X}$, but such that $G_0 = \mathbf{1}/\sqrt{d}$. This definition and the orthogonality condition imply that all the other operators in the basis are traceless. Expressing $A_i(t)$ and $B_i(t)$ in this basis,

$$A_i(t) = \sum_{j=0}^{d^2-1} a_{ij}(t) G_j \quad (3.49)$$

$$B_i(t) = \sum_{k=0}^{d^2-1} b_{ik}(t) G_k. \quad (3.50)$$

Then,

$$\begin{aligned} \mathcal{L}_t \rho(t) &= \sum_{j,k=0}^{d^2-1} \left(\sum_i a_{ij}(t) b_{ik}^*(t) \right) G_j \rho(t) G_k^\dagger \\ &\equiv \sum_{j,k=0}^{d^2-1} \gamma_{jk}(t) G_j \rho(t) G_k^\dagger. \end{aligned} \quad (3.51)$$

The $\gamma_{jk}(t)$ are the coefficients of a matrix known as the *decoherence matrix*. (68) Now, note that

$$\begin{aligned} (\mathcal{L}_t \rho(t))^\dagger &= \sum_{j,k=0}^{d^2-1} \gamma_{jk}^*(t) G_k \rho(t) G_j^\dagger \\ &= \sum_{k,j=0}^{d^2-1} \gamma_{kj}^*(t) G_j \rho(t) G_k^\dagger. \end{aligned} \quad (3.52)$$

To preserve Hermiticity, that must be equal to $\mathcal{L}_t \rho(t)$, which happens if $\gamma_{jk} = \gamma_{kj}^*$, i.e., if the decoherence matrix is Hermitian. Separating the $j = 0$ and $k = 0$ terms,

$$\begin{aligned} \mathcal{L}_t \rho(t) &= \gamma_{00}(t) G_0 \rho(t) G_0^\dagger + \sum_{j=1}^{d^2-1} \gamma_{j0}(t) G_j \rho(t) G_0^\dagger + \sum_{k=1}^{d^2-1} \gamma_{0k}(t) G_0 \rho(t) G_k^\dagger + \sum_{j,k=1}^{d^2-1} \gamma_{jk}(t) G_j \rho(t) G_k^\dagger \\ &= \frac{\gamma_{00}(t)}{d} \rho(t) + \left(\sum_{j=1}^{d^2-1} \frac{\gamma_{j0}(t)}{\sqrt{d}} G_j \right) \rho(t) + \rho(t) \left(\sum_{k=1}^{d^2-1} \frac{\gamma_{0k}(t)}{\sqrt{d}} G_k^\dagger \right) + \sum_{j,k=1}^{d^2-1} \gamma_{jk}(t) G_j \rho(t) G_k^\dagger. \end{aligned} \quad (3.53)$$

Now we define

$$C(t) = \frac{1}{2} \frac{\gamma_{00}}{d} + \sum_{j=1}^{d^2-1} \frac{\gamma_{j0}(t)}{\sqrt{d}} G_j. \quad (3.54)$$

Hermiticity preservation (and a relabeling of indices) implies that

$$C^\dagger(t) = \frac{1}{2} \frac{\gamma_{00}}{d} + \sum_{k=1}^{d^2-1} \frac{\gamma_{0k}(t)}{\sqrt{d}} G_k^\dagger. \quad (3.55)$$

Putting that back in Eq. (3.53) leads to

$$\mathcal{L}_t \rho(t) = C(t) \rho(t) + \rho(t) C^\dagger(t) + \sum_{j,k=1}^{d^2-1} \gamma_{jk}(t) G_j \rho(t) G_k^\dagger. \quad (3.56)$$

Using trace preservation and the cyclic property of the trace,

$$\text{tr} [\mathcal{L}_t \rho(t)] = \text{tr} \left[\left(C(t) + C^\dagger(t) + \sum_{j,k=1}^{d^2-1} \gamma_{jk}(t) G_k^\dagger G_j \right) \rho(t) \right] = 0. \quad (3.57)$$

Since this must hold for all $\rho(t) \in \mathbf{D}(\mathcal{X})$, we have that

$$C(t) + C^\dagger(t) = - \sum_{j,k=1}^{d^2-1} \gamma_{jk}(t) G_k^\dagger G_j. \quad (3.58)$$

If we define

$$H(t) = \frac{1}{2} i \left(C(t) - C^\dagger(t) \right), \quad (3.59)$$

which is an Hermitian operator, then $C(t)$ and $C^\dagger(t)$ can be rewritten as

$$C(t) = -iH(t) - \frac{1}{2} \sum_{j,k=1}^{d^2-1} \gamma_{jk}(t) G_k^\dagger G_j \quad (3.60)$$

$$C^\dagger(t) = iH(t) - \frac{1}{2} \sum_{j,k=1}^{d^2-1} \gamma_{jk}(t) G_k^\dagger G_j. \quad (3.61)$$

Then,

$$\begin{aligned}\mathcal{L}_t \rho(t) = & -iH(t)\rho(t) - \frac{1}{2} \sum_{j,k=1}^{d^2-1} \gamma_{jk}(t) G_k^\dagger G_j \rho(t) + i\rho(t)H(t) \\ & - \frac{1}{2} \sum_{j,k=1}^{d^2-1} \gamma_{jk}(t) \rho(t) G_k^\dagger G_j + \sum_{j,k=1}^{d^2-1} \gamma_{jk}(t) G_j \rho(t) G_k^\dagger,\end{aligned}\quad (3.62)$$

so the master equation is

$$\frac{d}{dt} \rho(t) = -i [H(t), \rho(t)] + \sum_{j,k=1}^{d^2-1} \gamma_{jk}(t) \left(G_j \rho(t) G_k^\dagger - \frac{1}{2} \{G_k^\dagger G_j, \rho(t)\} \right). \quad (3.63)$$

Since the decoherence matrix is Hermitian, it has a spectral decomposition with real eigenvalues, or, equivalently, it is unitarily equivalent to a real diagonal matrix :

$$\gamma_{jk}(t) = \sum_i u_{ji}(t) \gamma_i(t) u_{ki}^*(t), \quad (3.64)$$

where the $u_{jk}(t)$ are the coefficients of some unitary matrix. Then,

$$\begin{aligned}\frac{d}{dt} \rho(t) = & -i [H(t), \rho(t)] + \sum_{i,j,k=1}^{d^2-1} u_{ji}(t) \gamma_i(t) u_{ki}^*(t) \left(G_j \rho(t) G_k^\dagger - \frac{1}{2} \{G_k^\dagger G_j, \rho(t)\} \right) \\ = & -i [H(t), \rho(t)] + \sum_{i=1}^{d^2-1} \gamma_i(t) \left(\left(\sum_{j=1}^{d^2-1} u_{ji}(t) G_j \right) \rho(t) \left(\sum_{k=1}^{d^2-1} u_{ki}^*(t) G_k^\dagger \right) \right. \\ & \left. - \frac{1}{2} \left\{ \left(\sum_{k=1}^{d^2-1} u_{ki}^*(t) G_k^\dagger \right) \left(\sum_{j=1}^{d^2-1} u_{ji}(t) G_j \right), \rho(t) \right\} \right).\end{aligned}\quad (3.65)$$

Defining

$$L_i(t) = \sum_{j=1}^{d^2-1} u_{ji}(t) G_j, \quad (3.66)$$

leads to the desired master equation,

$$\frac{d}{dt} \rho(t) = -i [H(t), \rho(t)] + \sum_{i=1}^{d^2-1} \gamma_i(t) \left(L_i(t) \rho(t) L_i^\dagger(t) - \frac{1}{2} \{L_i^\dagger(t) L_i(t), \rho(t)\} \right). \quad (3.67)$$

Also note that, since the $\{G_j\}$ are an orthonormal set of traceless operators (remember that G_0 was excluded), the $L_i(t)$ are too,

$$\text{tr} [L_i^\dagger(t) L_j(t)] = \delta_{ij}, \quad \text{tr} [L_i(t)] = 0. \quad (3.68)$$

Equation (3.67) is the most general form of a time-local master equation, with the constraints that the operators $L_i(t)$ are orthonormal (as in Eq. (3.68)), $H(t)$ is an Hermitian operator and the coefficients $\gamma_i(t)$, known as the decoherence rates or *decay rates* (68), are real. The evolution can be easily seen to be divided in two parts: an unitary part, governed by the Hermitian term $H(t)$, and an non-unitary part, which accounts for the decoherence

and dissipation, governed by the decay rates. Since we only assumed that the evolution is Hermiticity and trace preserving, there are master equations with the form of Eq. (3.67) that are not positive or completely positive, and therefore do not represent valid physical evolutions. A set of necessary and sufficient conditions to guarantee complete positivity is still an open problem (10), which was solved only for specific cases, such as single qubit evolutions (130), for example.

A differential equation can assume distinct forms and still yield the same solution (131), so the physics of a system through its master equation can be dangerous. As an example, in the next chapter we define some quantum non Markovianity measures which are based on the decay rates of the master equation of the system. But if two different master equations represent the same evolution, we will assign two different measures to the same system, which is inconsistent. Luckily, Eq. (3.67) is a unique form for time-local master equations, and therefore is called the *canonical form*. (68) It is canonical because every time-local master equation can be put in such form, and unique because the decay rates, which determine the non-unitary evolution, are invariant under unitary operations $\rho \rightarrow \tilde{\rho}(t) = U(t)\rho U^\dagger(t)$. Therefore, we can change the $L_i(t)$ and $H(t)$ operators, but the decay rates remain the same. In order to see that, note that

$$\begin{aligned} \frac{d}{dt}\tilde{\rho}(t) &= \frac{d}{dt} \left(U(t)\rho(t)U^\dagger(t) \right) \\ &= U(t) \left(\frac{d}{dt}\rho(t) \right) U^\dagger(t) + \left(\frac{dU(t)}{dt} \right) \rho(t)U^\dagger(t) + U(t)\rho(t) \left(\frac{dU^\dagger(t)}{dt} \right). \end{aligned} \quad (3.69)$$

Now, since $U^\dagger(t)U(t) = \mathbb{1}$,

$$\frac{d}{dt} \left(U^\dagger(t)U(t) \right) = \frac{dU^\dagger(t)}{dt}U(t) + U^\dagger(t)\frac{dU(t)}{dt} = 0. \quad (3.70)$$

Multiplying on the right by $U^\dagger(t)$, then

$$\frac{dU^\dagger(t)}{dt} = -U^\dagger(t)\frac{dU(t)}{dt}U^\dagger(t). \quad (3.71)$$

We can use that to evaluate the last two terms in the right hand side of Eq. (3.69),

$$\begin{aligned} & \frac{dU(t)}{dt}\rho(t)U^\dagger(t) + U(t)\rho(t)\frac{dU^\dagger(t)}{dt} \\ &= \frac{dU(t)}{dt}\rho(t)U^\dagger(t) - U(t)\rho(t)U^\dagger(t)\frac{dU(t)}{dt}U^\dagger(t) \\ &= \frac{dU(t)}{dt} \left(U^\dagger(t)U(t) \right) \rho(t)U^\dagger(t) - U(t)\rho(t)U^\dagger(t)\frac{dU(t)}{dt}U^\dagger(t) \\ &= \left(\frac{dU(t)}{dt}U^\dagger(t) \right) \tilde{\rho}(t) - \tilde{\rho}(t) \left(\frac{dU(t)}{dt}U^\dagger(t) \right) \\ &= -i \left[i \frac{dU(t)}{dt}U^\dagger(t), \tilde{\rho}(t) \right]. \end{aligned} \quad (3.72)$$

The remaining term in the right hand side of Eq. (3.69) can be similarly evaluated:

$$\begin{aligned}
U(t) \left(\frac{d}{dt} \rho(t) \right) U^\dagger(t) &= U(t) \left\{ -i [H(t), \rho(t)] \right. \\
&\quad \left. + \sum_{i=1}^{d^2-1} \gamma_i(t) \left(L_i(t) \rho(t) L_i^\dagger(t) - \frac{1}{2} \{ L_i^\dagger(t) L_i(t), \rho(t) \} \right) \right\} U^\dagger(t) \\
&= -i [U(t) H(t) U^\dagger(t), \tilde{\rho}(t)] \\
&\quad + \sum_{i=1}^{d^2-1} \gamma_i(t) \left(\tilde{L}_i(t) \tilde{\rho}(t) \tilde{L}_i^\dagger(t) - \frac{1}{2} \{ \tilde{L}_i^\dagger(t) \tilde{L}_i(t), \tilde{\rho}(t) \} \right), \quad (3.73)
\end{aligned}$$

where $\tilde{L}_i(t) = U(t) L_i(t) U^\dagger(t)$. The resulting master equation is

$$\frac{d}{dt} \tilde{\rho}(t) = -i [\tilde{H}(t), \tilde{\rho}(t)] + \sum_{i=1}^{d^2-1} \gamma_i(t) \left(\tilde{L}_i(t) \tilde{\rho}(t) \tilde{L}_i^\dagger(t) - \frac{1}{2} \{ \tilde{L}_i^\dagger(t) \tilde{L}_i(t), \tilde{\rho}(t) \} \right), \quad (3.74)$$

where

$$\tilde{H}(t) = U(t) H(t) U^\dagger(t) + i \frac{dU(t)}{dt} U^\dagger(t). \quad (3.75)$$

As we can see, the Hamiltonian $H(t)$ and the $L_i(t)$ operators changed, but the decay rates remained the same. Also note that the $L_i(t)$ are unique up to an unitary transformation.

3.3.1 Divisibility

The following theorem characterizes the divisibility of a dynamics described by a time-local master equation Eq. (3.67). It is based on the theory of (completely) positive quantum dynamical semigroups. (73, 125, 127)

Theorem 11. *The dynamics generated by Eq. (3.67) is*

i) CP-divisible if and only if $\gamma_i(t) \geq 0$, $\forall i$ and $t \geq 0$.

ii) P-divisible if and only if

$$\sum_i \gamma_i(t) |\langle m | L_i(t) | n \rangle|^2 \geq 0, \quad (3.76)$$

for every $n \neq m$, $t \geq 0$ and any orthonormal basis $\{|n\rangle\}$ of the system Hilbert space. (125)

Notice that when the evolution consists of only one channel, i.e., only one decay rate $\gamma(t)$, CP-divisibility and P-divisibility are equivalent concepts. Also, note that the P-divisibility condition alone does not guarantee that the evolution is completely positive. The CP-divisibility condition is usually known as Lindblad's theorem. (73)

3.3.2 Matrix representation

As in Section 2.4.2, we derive now a matrix representation of the master equation. The derivation is exactly the same, so we have that

$$\dot{\vec{x}}(t) = \vec{u}(t) + D(t) \vec{x}(t), \quad (3.77)$$

where

$$u_j(t) = \text{tr} [G_j \mathcal{L}_t [\mathbb{1}]], \quad (3.78)$$

$$D_{jk}(t) = \text{tr} [G_j \mathcal{L}_t [G_k]]. \quad (3.79)$$

The vector $\vec{u}(t)$ is known as the *drift vector*, and the matrix $D(t)$ as the *damping matrix*. (68)

If the master equation has the Lindblad-like form of Eq. (3.67),

$$\frac{d\rho(t)}{dt} = -i [H(t), \rho(t)] + \sum_k \gamma_k(t) \left[L_k(t) \rho(t) L_k^\dagger(t) - \frac{1}{2} \{ L_k^\dagger(t) L_k(t), \rho(t) \} \right], \quad (3.80)$$

then the drift vector components are

$$\begin{aligned} u_j(t) &= \text{tr} \left[G_j \left(-i [H(t), \mathbb{1}] + \sum_k \gamma_k(t) [L_k(t) L_k^\dagger(t) - L_k^\dagger(t) L_k(t)] \right) \right] \\ &= \sum_k \gamma_k(t) \text{tr} [G_j [L_k(t), L_k^\dagger(t)]], \end{aligned} \quad (3.81)$$

and the damping matrix coefficients are

$$\begin{aligned} D_{ij}(t) &= -i \text{tr} [G_i [H(t), G_j]] + \sum_{k=1}^{d^2-1} \gamma_k(t) \text{tr} \left[G_i L_k(t) G_j L_k^\dagger(t) - \frac{1}{2} G_i \{ L_k^\dagger(t) L_k(t), G_j \} \right] \\ &= -i \text{tr} [H(t) [G_j, G_i]] + \sum_{k=1}^{d^2-1} \gamma_k(t) \text{tr} \left[L_k^\dagger(t) G_i L_k(t) G_j - \frac{1}{2} L_k^\dagger(t) L_k(t) \{ G_j, G_i \} \right]. \end{aligned} \quad (3.82)$$

Note that they depend on the commutators and anticommutators between elements of the $\{G_i\}$ basis, so we cannot proceed without further information about the basis. We can, however, evaluate the trace of the damping matrix:

$$\begin{aligned} \text{tr}[D(t)] &= \sum_{i=1}^{d^2-1} D_{ii}(t) \\ &= \sum_{i=1}^{d^2-1} \sum_{k=1}^{d^2-1} \gamma_k(t) \text{tr} [L_k^\dagger(t) G_i L_k(t) G_i - L_k^\dagger(t) L_k(t) G_i^2] \\ &= \sum_{k=1}^{d^2-1} \gamma_k(t) \text{tr} \left[L_k^\dagger(t) \left[\left(\sum_{i=1}^{d^2-1} G_i L_k(t) G_i \right) - L_k(t) \left(\sum_{i=1}^{d^2-1} G_i \mathbb{1} G_i \right) \right] \right]. \end{aligned} \quad (3.83)$$

In order to proceed, we need to evaluate the quantities $\sum_{i=1}^{d^2-1} G_i X G_i$ for $X = L_k(t), \mathbb{1}$.

Proposition 18. *For any $X \in \mathbf{L}(\mathcal{X})$,*

$$\sum_{i=1}^{d^2-1} G_i X G_i = \text{tr}[X] \mathbb{1} - d^{-1} X. \quad (3.84)$$

Proof. Consider the orthonormal basis $\{E_i\}$, where $E_i = |i_1\rangle\langle i_2|$. Since $\{G_i\}$, for $i = 0, \dots, d^2 - 1$, is also a complete orthonormal basis^c, both bases are related by an unitary transformation^d:

$$G_i = \sum_j u_{ij} E_j, \quad (3.85)$$

where $U = (u_{ij})$ is an unitary matrix, so

$$(U^\dagger U)_{jk} = \sum_i (U^\dagger)_{ji} (U)_{ik} = \sum_i u_{ij}^* u_{ik} = (\mathbf{1})_{jk} = \delta_{jk}. \quad (3.86)$$

Also, since the G_i operators are Hermitian, we have that $G_i = \sum_j u_{ij}^* E_j^\dagger$. Therefore,

$$\begin{aligned} \sum_{i=0}^{d^2-1} G_i X G_i &= \sum_i \left(\sum_j u_{ij}^* E_j^\dagger \right) X \left(\sum_k u_{ik} E_k \right) \\ &= \sum_{j,k} \left(\sum_i u_{ij}^* u_{ik} \right) E_j^\dagger X E_k \\ &= \sum_{j,k} \delta_{jk} E_j^\dagger X E_k = \sum_j E_j^\dagger X E_j. \end{aligned} \quad (3.87)$$

This term is now easier to evaluate:

$$\sum_j E_j^\dagger X E_j = \sum_{j_1, j_2} |j_2\rangle\langle j_1| X |j_1\rangle\langle j_2| = \sum_{j_2} |j_2\rangle\langle j_2| \sum_{j_1} \langle j_1| X |j_1\rangle = \text{tr}[X] \mathbf{1}. \quad (3.88)$$

Then,

$$\sum_{i=1}^{d^2-1} G_i X G_i = \sum_{i=0}^{d^2-1} G_i X G_i - G_0 X G_0 = \text{tr}[X] \mathbf{1} - d^{-1} X. \quad (3.89)$$

□

Proposition 18 implies that

$$\sum_{i=1}^{d^2-1} G_i L_k(t) G_i = \text{tr}[L_k(t)] \mathbf{1} - d^{-1} L_k(t) = -d^{-1} L_k(t), \quad (3.90)$$

$$\sum_{i=1}^{d^2-1} G_i \mathbf{1} G_i = \text{tr}[\mathbf{1}] \mathbf{1} - d^{-1} \mathbf{1} = d \mathbf{1} - d^{-1} \mathbf{1}, \quad (3.91)$$

so

$$\begin{aligned} \text{tr}[D(t)] &= \sum_{k=1}^{d^2-1} \gamma_k(t) \text{tr} \left[L_k^\dagger(t) \left[\left(-d^{-1} L_k(t) \right) - L_k(t) (d \mathbf{1} - d^{-1} \mathbf{1}) \right] \right] \\ &= \sum_{k=1}^{d^2-1} \gamma_k(t) \left(-d^{-1} - d + d^{-1} \right) \text{tr} \left[L_k^\dagger(t) L_k(t) \right] \\ &= -d \sum_{k=1}^{d^2-1} \gamma_k(t) \delta_{kk} = -d \sum_{k=1}^{d^2-1} \gamma_k(t). \end{aligned} \quad (3.92)$$

^c Note that $G_0 = \mathbf{1}/\sqrt{d}$ is included.

^d Note that one basis has more elements than the other. This can be solved by suitably adding elements equal to zero in the smaller basis, and modifying the unitary operator accordingly.

The trace of the damping matrix is, therefore,

$$\mathrm{tr}[D(t)] = -d \sum_{k=1}^{d^2-1} \gamma_k(t). \quad (3.93)$$

The damping matrix and its trace are very useful for some quantum non-Markovianity measures, as we will see in Chapter 4.

3.3.3 Equivalence with quantum dynamical maps

We have seen two different ways of describing time-dependent and local in time quantum evolutions: via a quantum dynamical map

$$\rho(t) = \mathcal{E}_{(t,t_0)}(\rho(t_0)), \quad (3.94)$$

and via a time-local master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}_t\rho(t). \quad (3.95)$$

In this section we relate the two approaches, i.e., determine one when we are given the other.

If we have the master equation, then, to first order in ϵ ,

$$\frac{\rho(t+\epsilon, t) - \rho(t)}{\epsilon} = \mathcal{L}_t\rho(t), \quad (3.96)$$

which means that

$$\mathcal{E}_{(t+\epsilon, t)}(\rho(t)) \equiv \rho(t+\epsilon, t) = \rho(t) + \epsilon\mathcal{L}_t\rho(t), \quad (3.97)$$

so the dynamical map is

$$\mathcal{E}_{(t+\epsilon, t)} = \mathcal{I} + \epsilon\mathcal{L}_t. \quad (3.98)$$

Conversely, differentiating a (bijective) dynamical map leads to

$$\frac{d}{dt}\rho(t) = \frac{d}{dt}\mathcal{E}_{(t,t_0)}(\rho(t_0)) = \frac{d}{dt}\mathcal{E}_{(t,t_0)}\mathcal{E}_{(t,t_0)}^{-1}(\rho(t)) \equiv \mathcal{L}_t\rho(t), \quad (3.99)$$

so

$$\mathcal{L}_t = \frac{d}{dt}\mathcal{E}_{(t,t_0)}\mathcal{E}_{(t,t_0)}^{-1}. \quad (3.100)$$

The above formulas, however, are not very useful for calculations. A more "friendly" equivalence is found when we write the dynamical map and the master equation in matrix representation, Equations (2.213) and (3.77):

$$\vec{x}(t) = \vec{w}(t, t_0) + B(t, t_0)\vec{x}(t), \quad (3.101)$$

$$\dot{\vec{x}}(t) = \vec{u}(t) + D(t)\vec{x}(t). \quad (3.102)$$

In order to relate both forms, note that, on the one hand, deriving Eq. (3.101) leads to

$$\dot{\vec{x}}(t) = \dot{\vec{w}}(t, t_0) + \dot{B}(t, t_0)\vec{x}(t_0), \quad (3.103)$$

and, on the other hand, putting Eq. (3.101) in Eq. (3.102) yields

$$\begin{aligned} \dot{\vec{x}}(t) &= \vec{u}(t) + D(t) [\vec{w}(t, t_0) + B(t, t_0)\vec{x}(t_0)] \\ &= \vec{u}(t) + D(t)\vec{w}(t, t_0) + D(t)B(t, t_0)\vec{x}(t_0). \end{aligned} \quad (3.104)$$

Since this holds for every initial $\vec{x}(t_0)$ and the equations are linear, we have that

$$\dot{\vec{w}}(t, t_0) = \vec{u}(t) + D(t)\vec{w}(t, t_0) \quad (3.105)$$

$$\dot{B}(t, t_0) = D(t)B(t, t_0) \quad (3.106)$$

Equations (3.105) and (3.106) are the relation between time-local master equations and dynamical maps. If we start from the master equation, we have the formulas for $\vec{u}(t)$ and $D(t)$, so we can find $\vec{w}(t, t_0)$ and $B(t, t_0)$ by solving the corresponding differential equations, with initial conditions $\vec{w}(t_0, t_0) = 0$ and $B(t_0, t_0) = \mathbb{1}$. If we start from the dynamical map, we have $\vec{w}(t, t_0)$ and $B(t, t_0)$. The drift vector $u(t)$ can be easily determined by $\vec{u}(t) = \dot{\vec{w}}(t, t_0) - D(t)\vec{w}(t, t_0)$, but the evaluation of $D(t)$ is more complicated. In fact, the solution for $D(t)$ may not exist, since not all dynamical maps correspond to time-local master equations, or, even if it exists, it may be non unique. If $B(t, t_0)$ satisfies certain consistency conditions, then there are methods to find a unique or approximate solutions. (126) Here we will restrict ourselves to the case where $B(t, t_0)$ is invertible (this is a stronger condition than the consistency conditions): in this case, the solution is unique and given by

$$D(t) = \dot{B}(t, t_0)B^{-1}(t, t_0). \quad (3.107)$$

3.3.4 Bloch sphere representation

Consider the matrix representation of density operators in Eq. (2.209),

$$\rho(t) = \frac{1}{d} \left(\mathbb{1} + \sum_{i=1}^{d^2-1} x_i(t) G_i \right). \quad (3.108)$$

For two-dimensional systems we can use as a basis $\{G_i\}$ the Pauli matrices (76):

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (3.109)$$

These matrices can also be referred to as $\sigma_1 \equiv \sigma_x$, $\sigma_2 \equiv \sigma_y$ and $\sigma_3 \equiv \sigma_z$. This basis, however, is not normalized:

$$\text{tr} [\sigma_i, \sigma_j] = 2\delta_{ij}, \quad i, j = 1, 2, 3. \quad (3.110)$$

In order to have a matrix representation as in Eq. (3.108) we could normalize these operators and use the basis $\{\sigma_i/\sqrt{2}\}$. It is more common, however, to use the Pauli matrices and define a new matrix representation

$$\rho(t) = \frac{1}{2} \left(\mathbb{1} + \sum_{i=1}^3 n_i(t) \sigma_i \right), \quad (3.111)$$

where the new vector $\vec{n}(t)$ is related to the old one by

$$\vec{n}(t) = \frac{1}{\sqrt{2}} \vec{x}(t). \quad (3.112)$$

Eq. (3.111) is known as the *Bloch sphere representation*.⁽⁹⁾ It can be written more neatly as

$$\rho(t) = \frac{1}{2} (\mathbb{1} + \vec{n}(t) \cdot \vec{\sigma}), \quad (3.113)$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. The vector $\vec{n}(t)$, with coordinates

$$n_i(t) = \text{tr}[\sigma_i \rho(t)], \quad (3.114)$$

is known as the *Bloch vector* of the system $\rho(t)$.

Now, note that the *purity* of the density matrix, $\text{tr}[\rho^2(t)]$ ^e, is

$$\begin{aligned} \text{tr}[\rho^2(t)] &= \frac{1}{4} \text{tr} \left[\mathbb{1} + 2 \sum_{i=1}^3 n_i(t) \sigma_i + \sum_{i,j=1}^3 n_i(t) n_j(t) \sigma_i \sigma_j \right] \\ &= \frac{1}{4} \left(\text{tr}[\mathbb{1}] + 2 \sum_{i=1}^3 n_i(t) \text{tr}[\sigma_i] + \sum_{i,j=1}^3 n_i(t) n_j(t) \text{tr}[\sigma_i \sigma_j] \right) \\ &= \frac{1}{2} \left(1 + \sum_{i=1}^3 n_i^2(t) \right) = \frac{1}{2} (1 + |\vec{n}(t)|^2), \end{aligned} \quad (3.115)$$

where

$$|\vec{n}(t)| = \sqrt{n_1^2(t) + n_2^2(t) + n_3^2(t)} \quad (3.116)$$

is the norm of the Bloch vector. Since the purity of a qubit satisfies

$$\frac{1}{2} \leq \text{tr}[\rho^2(t)] \leq 1, \quad (3.117)$$

we have that $0 \leq |\vec{n}(t)| \leq 1$, i.e., the Bloch vector lies in a three dimensional sphere^f, which justifies the name of the representation. For pure states, $\text{tr}[\rho^2(t)] = 1$, the corresponding condition is $|\vec{n}(t)| = 1$, so pure states lie in the boundary of the sphere. On the other hand, the maximally mixed state, $\text{tr}[\rho^2(t)] = 1/2$, lies in the center of the sphere ($|\vec{n}(t)| = 0$).

The matrix of the density operator $\rho(t)$ in terms of the Bloch vector components is given by

$$\rho(t) = \frac{1}{2} \begin{bmatrix} 1 + n_3(t) & n_1(t) - i n_2(t) \\ n_1(t) + i n_2(t) & 1 - n_3(t) \end{bmatrix},$$

^e The purity of a density operator and its properties are presented in Appendix A, Section A.6.

^f For the matrix representation of density operators in higher dimensions, the space of vectors is no longer a sphere. (80)

so in terms of the usual matrix coefficients^g,

$$\rho = \begin{bmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{bmatrix},$$

the Bloch vector components are

$$n_1(t) = \rho_{21}(t) + \rho_{12}(t), \quad (3.118)$$

$$n_2(t) = -i(\rho_{21}(t) - \rho_{12}(t)), \quad (3.119)$$

$$n_3(t) = 2\rho_{11}(t) - 1. \quad (3.120)$$

We can also define the damping matrix and the drift vector for the Bloch representation,

$$u_j(t) = \frac{1}{2} \text{tr} [\sigma_j \mathcal{L}_t [\mathbb{1}]], \quad (3.121)$$

$$D_{jk}(t) = \frac{1}{2} \text{tr} [\sigma_j \mathcal{L}_t [\sigma_k]], \quad (3.122)$$

so we have

$$\dot{\vec{n}}(t) = \vec{u}(t) + D(t)\vec{n}(t). \quad (3.123)$$

Note that the difference from the usual damping matrix and drift vector is just a normalization constant.

3.4 The Born-Markov approximation

In this section we derive a CP-divisible time-local master equation for an open quantum system, starting from subsystem dynamics. We consider a quantum system initially decoupled from its surrounding reservoir and weakly interacting with it. Under the Born-Markov approximation (1) the memory effects of the dynamics, related to the backflow of information from the environment to the system, are neglected and the resulting evolution is local in time. Finally, the rotating wave approximation (1) guarantees that the dynamics is CP-divisible and, consequently, a CPTP map, i.e., a quantum channel. The model is illustrated with three examples which involve a quantum system interacting with some bosonic reservoir. Finally, note that this approach is not the only microscopic derivation which leads to CP-divisible dynamics: other models such as the singular coupling limit (127) and the low density limit (4) achieve the same goal.

According to Theorem 11, a CP-divisible^h master equation has the form

$$\frac{d\rho(t)}{dt} = -i[H(t), \rho(t)] + \sum_i \gamma_i(t) \left[L_i(t)\rho(t)L_i^\dagger(t) - \frac{1}{2} \{L_i^\dagger(t)L_i(t), \rho(t)\} \right], \quad (3.124)$$

where $H(t)$ is Hermitian, $L_i(t)$ are linear operators acting on the system and $\gamma_i(t) \geq 0$, for every i and t . Equation (3.124), with positive decay rates $\gamma_i(t)$, is known as the *Lindblad*

^g Note that, since $\rho(t)$ is a density operator, we have that $\rho_{11}(t) + \rho_{22}(t) = 1$ and $\rho_{21}(t) = \rho_{12}^*(t)$.

^h Recall that CP-divisibility implies, in particular, that the dynamics is CPTP.

equation or to be in *Lindblad form*. In this section we derive this equation from subsystem dynamics.

In a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, representing a open quantum system $\mathcal{H}_S$ interacting with an environment $\mathcal{H}_E$, we define the Hamiltonian (1)

$$H = H_S + H_E + H_{\text{int}}, \quad (3.125)$$

where H_S (H_E) is the system (reservoir) Hamiltonian and H_{int} accounts for the interaction between themⁱ. The interaction Hamiltonian can be written, without loss of generality, as

$$H_{\text{int}} = \sum_{\alpha} S_{\alpha} \otimes E_{\alpha}, \quad (3.126)$$

where S_{α} and E_{α} are operators acting on the system and the reservoir Hilbert space, respectively. The fact that H_{int} is Hermitian does not necessarily imply that each individual S_{α} and E_{α} must be Hermitian, but we can say that for every α there is some β such that

$$S_{\beta} \otimes E_{\beta} = S_{\alpha}^{\dagger} \otimes E_{\alpha}^{\dagger}. \quad (3.127)$$

However, it is possible, without loss of generality, to define new operators S_{α} and E_{α} such that each of them is Hermitian: each S_{α} can be written as

$$S_{\alpha} = \left(\frac{S_{\alpha}^{\dagger} + S_{\alpha}}{2} \right) + i \left(\frac{iS_{\alpha}^{\dagger} - iS_{\alpha}}{2} \right) \equiv S_{\alpha}^1 + iS_{\alpha}^2, \quad (3.128)$$

where S_{α}^1 and S_{α}^2 are Hermitian operators, as can be readily checked. The same procedure applies to each E_{α} . Since H_{int} is Hermitian, we have that

$$\sum_{\alpha} S_{\alpha} \otimes E_{\alpha} = \left(\sum_{\alpha} S_{\alpha} \otimes E_{\alpha} \right)^{\dagger} = \sum_{\alpha} S_{\alpha}^{\dagger} \otimes E_{\alpha}^{\dagger}, \quad (3.129)$$

so it can be rewritten as

$$H_{\text{int}} = \frac{1}{2} \sum_{\alpha} \left(S_{\alpha} \otimes E_{\alpha} + S_{\alpha}^{\dagger} \otimes E_{\alpha}^{\dagger} \right). \quad (3.130)$$

Using the decomposition of Eq. (3.128), we have that

$$\begin{aligned} H_{\text{int}} &= \frac{1}{2} \sum_{\alpha}^n \left[(S_{\alpha}^1 + iS_{\alpha}^2) \otimes (E_{\alpha}^1 + iE_{\alpha}^2) + (S_{\alpha}^1 - iS_{\alpha}^2) \otimes (E_{\alpha}^1 - iE_{\alpha}^2) \right] \\ &= \frac{1}{2} \sum_{\alpha}^n \left[S_{\alpha}^1 \otimes E_{\alpha}^1 + iS_{\alpha}^1 \otimes E_{\alpha}^2 + iS_{\alpha}^2 \otimes E_{\alpha}^1 - S_{\alpha}^2 \otimes E_{\alpha}^2 \right. \\ &\quad \left. + S_{\alpha}^1 \otimes E_{\alpha}^1 - iS_{\alpha}^1 \otimes E_{\alpha}^2 - iS_{\alpha}^2 \otimes E_{\alpha}^1 - S_{\alpha}^2 \otimes E_{\alpha}^2 \right] \\ &= \sum_{\alpha}^n \left(S_{\alpha}^1 \otimes E_{\alpha}^1 - S_{\alpha}^2 \otimes E_{\alpha}^2 \right) \\ &\equiv \sum_{\alpha}^{2n} S'_{\alpha} \otimes E'_{\alpha}, \end{aligned} \quad (3.131)$$

ⁱ More precisely, $H = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + H_{\text{int}}$.

where we defined

$$S'_\alpha = \begin{cases} S_\alpha^1, & \alpha = 1, \dots, n \\ -S_{\alpha-n}^2, & \alpha = n+1, \dots, 2n \end{cases} \quad (3.132)$$

$$E'_\alpha = \begin{cases} E_\alpha^1, & \alpha = 1, \dots, n \\ E_{\alpha-n}^2, & \alpha = n+1, \dots, 2n \end{cases} \quad (3.133)$$

It is trivial to see that each S'_α and each E'_α is Hermitian. We will, however, not use this approach here, and use Eq. (3.127) when necessary.

We assume that the total system $\mathcal{H}$ is closed, so its density matrix $\rho(t)$ evolves in time under the Liouville-von Neumann Equation (3.7):

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)]. \quad (3.134)$$

We now move to the interaction picture (1, 75), which is useful in dealing with changes due to interaction terms. In this framework, operators take the form

$$\tilde{O}(t) = e^{i(H_S+H_E)t} O e^{-i(H_S+H_E)t}, \quad (3.135)$$

which implies that the new Liouville-von Neumann equation for $\tilde{\rho}(t)$ is

$$\frac{d}{dt}\tilde{\rho}(t) = -i[\tilde{H}_{\text{int}}(t), \tilde{\rho}(t)], \quad (3.136)$$

as is shown in Appendix C. This equation can be integrated,

$$\tilde{\rho}(t) = \tilde{\rho}(0) - i \int_0^t dt' [\tilde{H}_{\text{int}}(t'), \tilde{\rho}(t')], \quad (3.137)$$

and Eq. (3.137) can be replaced in Eq. (3.136), resulting in

$$\frac{d}{dt}\tilde{\rho}(t) = -i[\tilde{H}_{\text{int}}(t), \rho(0)] - \int_0^t dt' [\tilde{H}_{\text{int}}(t), [\tilde{H}_{\text{int}}(t'), \tilde{\rho}(t')]], \quad (3.138)$$

where we used that $\tilde{\rho}(0) = \rho(0)$.

Since our interest lies only in the system dynamics, the reservoir can be traced out: this leaves us with the reduced density operator $\tilde{\rho}_S(t) = \text{tr}_E[\tilde{\rho}(t)]$. Then,

$$\frac{d}{dt}\tilde{\rho}_S(t) = -i\text{tr}_E[\tilde{H}_{\text{int}}(t), \rho(0)] - \int_0^t dt' \text{tr}_E[\tilde{H}_{\text{int}}(t), [\tilde{H}_{\text{int}}(t'), \tilde{\rho}(t')]]. \quad (3.139)$$

Our objective is to put Eq. (3.139) in Lindblad form. In order to perform that, a set of approximations and assumptions are necessary.

3.4.1 Initial conditions

Our first hypothesis is about the initial state of the total system: we assume that it starts in a product state, i.e.,

$$\rho(0) = \rho_S(0) \otimes \rho_E, \quad (3.140)$$

which means that initially there are no system-environment correlations, neither classical nor quantum. Physically it means that system and reservoir evolved freely with no coupling before and that at time $t = 0$ they were put together. The importance of this condition is that, as we have seen in Section 2.3, any subsystem dynamics which starts in a product state, with a fixed environment state, is a quantum channel. It is further assumed that the reservoir is in a thermal state,

$$\rho_E = \frac{e^{-\beta H_E}}{\text{tr}_E[e^{-\beta H_E}]}, \quad (3.141)$$

where $\beta = 1/k_B T$, with k_B the Boltzmann constant and T the temperature of the thermal state. From now on, we assume that $k_B = 1$. Thermal states and their properties are discussed in Appendix D.

Substituting Eq. (3.140) in Eq. (3.139),

$$\frac{d}{dt}\tilde{\rho}_S(t) = -i\text{tr}_E [\tilde{H}_{\text{int}}(t), \rho_S(0) \otimes \rho_E(0)] - \int_0^t dt' \text{tr}_E [\tilde{H}_{\text{int}}(t), [\tilde{H}_{\text{int}}(t'), \tilde{\rho}(t')]]. \quad (3.142)$$

If we assume that the mean value of the bath operators is zero, i.e.,

$$\langle E_\alpha(t) \rangle = \text{tr}_E [\tilde{E}_\alpha(t) \rho_E] = 0, \quad (3.143)$$

then

$$\begin{aligned} \text{tr}_E [\tilde{H}'_{\text{int}}(t) \rho(0)] &= \text{tr}_E \left[\sum_\alpha \tilde{S}_\alpha(t) \otimes \tilde{E}_\alpha(t) \rho_S(0) \otimes \rho_E \right] \\ &= \sum_\alpha \tilde{S}_\alpha(t) \rho_S(0) \text{tr}_E [\tilde{E}_\alpha(t) \rho_E] \\ &= \sum_\alpha \tilde{S}_\alpha(t) \rho_S(0) \langle E_\alpha(t) \rangle = 0 \end{aligned} \quad (3.144)$$

and, similarly, $\text{tr}_E [\rho(0) \tilde{H}'_{\text{int}}(t)] = 0$, so the first term in the right hand side of Eq. (3.142) is zero. It is possible, however, to get rid of the inhomogeneous term without the hypothesis of Eq. (3.143): simply redefine (132) the interaction and system Hamiltonians as

$$\begin{aligned} H'_{\text{int}} &= \sum_\alpha S_\alpha \otimes (E_\alpha - \langle E_\alpha \rangle) \\ H'_S &= H_S + \sum_\alpha \langle E_\alpha \rangle S_\alpha \otimes \mathbf{1}_E, \end{aligned} \quad (3.145)$$

where $\langle E_\alpha \rangle = \text{tr}_E [E_\alpha \rho(0)]$. That means that

$$\begin{aligned} \text{tr}_E [\tilde{H}'_{\text{int}}(t) \rho(0)] &= \text{tr}_E \left[\sum_\alpha \tilde{S}_\alpha(t) \otimes (\tilde{E}_\alpha(t) - \langle E_\alpha \rangle) \rho_S(0) \otimes \rho_E \right] \\ &= \sum_\alpha \tilde{S}_\alpha(t) \rho_S(0) \text{tr}_E [(\tilde{E}_\alpha(t) - \langle E_\alpha \rangle) \rho_E] \\ &= \sum_\alpha \tilde{S}_\alpha(t) \rho_S(0) (\langle E_\alpha \rangle - \langle E_\alpha \rangle) = 0, \end{aligned} \quad (3.146)$$

where it was used that the mean value of the bath operators is time independent, as shown in Appendix D. Equally, $\text{tr}_B [\rho(0)\tilde{H}'_{\text{int}}(t)] = 0$, so the inhomogeneous term is zero. The above procedure is simply a shift in the energy levels of the free Hamiltonian and an equal and opposite shift in the interaction Hamiltonian, thus preserving the total Hamiltonian.

Now, back to Eq. (3.142). Expanding the commutators,

$$\begin{aligned} \frac{d}{dt}\tilde{\rho}_S(t) = & - \int_0^t dt' \text{tr}_E \left[\tilde{H}_{\text{int}}(t)\tilde{H}_{\text{int}}(t')\tilde{\rho}(t') - \tilde{H}_{\text{int}}(t')\tilde{\rho}(t')\tilde{H}_{\text{int}}(t) \right. \\ & \left. + \tilde{\rho}(t')\tilde{H}_{\text{int}}(t')\tilde{H}_{\text{int}}(t) - \tilde{H}_{\text{int}}(t)\tilde{\rho}(t')\tilde{H}_{\text{int}}(t') \right]. \end{aligned} \quad (3.147)$$

Note that the last two terms inside brackets are the Hermitian conjugate of the first two, so Eq. (3.147) can be written more compactly as

$$\frac{d}{dt}\tilde{\rho}_S(t) = \int_0^t dt' \text{tr}_E \left[\tilde{H}_{\text{int}}(t')\tilde{\rho}(t')\tilde{H}_{\text{int}}(t) - \tilde{H}_{\text{int}}(t)\tilde{H}_{\text{int}}(t')\tilde{\rho}(t') + \text{h.c.} \right]. \quad (3.148)$$

At this point, Eq. (3.148) is still an exact formula, but non-Markovian, since the integrand depends on earlier times t' , i.e., the evolution is not memoryless. Approximations are needed to get rid of the memory effects.

3.4.2 Born approximation: weak coupling

Now we assume that the system-reservoir coupling is weak, so we only consider terms up to first order in the interaction strength. The *Born approximation* (1,3) states that, in this regime, the system-environment state evolves as

$$\rho(t) = \rho_S(t) \otimes \rho_E + \mathcal{O}(H_{\text{int}}). \quad (3.149)$$

Physically, this means that the interaction is so weak that the total system's density matrix, which is initially in a product state, remains decorrelated for the entire evolution, and the reservoir is not affected significantly by the system, since any excitation decays very fast. Using the Born approximation and the decomposition of the interaction Hamiltonian,

$$\tilde{H}_{\text{int}} = \sum_{\alpha} \tilde{S}_{\alpha}(t) \otimes \tilde{E}_{\alpha}(t), \quad (3.150)$$

Eq. (3.148) is now

$$\begin{aligned} \frac{d}{dt}\tilde{\rho}_S(t) = & \sum_{\alpha,\beta} \int_0^t dt' \left[\left(\tilde{S}_{\beta}(t')\tilde{\rho}_S(t')\tilde{S}_{\alpha}(t) \right) \text{tr}_E \left[\tilde{E}_{\beta}(t')\rho_E\tilde{E}_{\alpha}(t) \right] \right. \\ & \left. - \left(\tilde{S}_{\alpha}(t)\tilde{S}_{\beta}(t')\tilde{\rho}_S(t') \right) \text{tr}_E \left[\tilde{E}_{\alpha}(t)\tilde{E}_{\beta}(t')\rho_E \right] + \text{h.c.} \right]. \end{aligned} \quad (3.151)$$

Now recall that, since the interaction Hamiltonian is Hermitian, every term $S_{\alpha} \otimes E_{\alpha}$ corresponds to some term $S_{\alpha'}^{\dagger} \otimes E_{\alpha'}^{\dagger}$. Performing that substitution and, afterwards, relabeling α' back to α , leads to

$$\begin{aligned} \frac{d}{dt}\tilde{\rho}_S(t) = & \sum_{\alpha,\beta} \int_0^t dt' \left[\left(\tilde{S}_{\beta}(t')\tilde{\rho}_S(t')\tilde{S}_{\alpha}^{\dagger}(t) \right) \text{tr}_E \left[\tilde{E}_{\beta}(t')\rho_E\tilde{E}_{\alpha}^{\dagger}(t) \right] \right. \\ & \left. - \left(\tilde{S}_{\alpha}^{\dagger}(t)\tilde{S}_{\beta}(t')\tilde{\rho}_S(t') \right) \text{tr}_E \left[\tilde{E}_{\alpha}^{\dagger}(t)\tilde{E}_{\beta}(t')\rho_E \right] + \text{h.c.} \right]. \end{aligned} \quad (3.152)$$

The *reservoir two-point correlation functions* (RCFs) are defined as

$$\mathcal{R}_{\alpha\beta}(t, t') = \text{tr}_E \left[\tilde{E}_\alpha^\dagger(t) \tilde{E}_\beta(t') \rho_E \right]. \quad (3.153)$$

Since the bath is in a thermal state, the RCFs are homogeneous in time, as is proved in Appendix D, so $\mathcal{R}_{\alpha\beta}(t, t') = \mathcal{R}_{\alpha\beta}(t - t')$. Physically, the RCFs measure how an excitation of the reservoir at time t is correlated with an excitation at t' . Using the cyclic property of the trace in the first term of Eq. (3.152), we have that

$$\frac{d}{dt} \tilde{\rho}_S(t) = \sum_{\alpha, \beta} \int_0^t dt' \left\{ \mathcal{R}_{\alpha\beta}(t - t') \left[\tilde{S}_\beta(t') \tilde{\rho}_S(t') \tilde{S}_\alpha(t) - \tilde{S}_\alpha(t) \tilde{S}_\beta(t') \tilde{\rho}_S(t') \right] + \text{h.c.} \right\}. \quad (3.154)$$

It is useful to perform the substitution $t' \rightarrow t - \tau$, so the master equation is

$$\frac{d}{dt} \tilde{\rho}_S(t) = \sum_{\alpha, \beta} \int_0^t d\tau \left\{ \mathcal{R}_{\alpha\beta}(\tau) \left[\tilde{S}_\beta(t - \tau) \tilde{\rho}_S(t - \tau) \tilde{S}_\alpha(t) - \tilde{S}_\alpha(t) \tilde{S}_\beta(t - \tau) \tilde{\rho}_S(t - \tau) \right] + \text{h.c.} \right\}. \quad (3.155)$$

3.4.3 Markov approximation: short memory

It is convenient now to define two timescales: τ_E and τ_D . The former is the time over which the reservoir correlation functions decay, and the latter the system's characteristic relaxation time. More precisely (133),

$$|\mathcal{R}_{\alpha\beta}(\tau)| \sim e^{-\tau/\tau_E}, \quad (3.156)$$

and τ_D is the time over which the system's density matrix varies significantly (132),

$$\frac{\Delta \rho_S(t)}{\Delta t} \sim \frac{1}{\tau_D} \rho_S(t). \quad (3.157)$$

If we work in a timescale coarser than τ_E and finer than τ_D ,

$$\tau_E \ll t \ll \tau_D, \quad (3.158)$$

then the timescale does not resolve the reservoir correlations dynamics and they can be neglected. Since t is of the order of τ_E , we can replace $\rho_S(t - \tau)$ by $\rho_S(t)$ in Eq. (3.155), meaning that the current state of $\rho_S(t)$ depends only on the present state and not on the past states. Also, the upper integration limit can be extended to infinity, since the reservoir correlation functions decay so rapidly that after a time of the order of τ_E the integrand in Eq. (3.155) is practically zero. That is the *Markov approximation* (2): the reservoir correlation functions decay rapidly compared to damping and, consequently, the system is memoryless.

For such a approximation to be valid the condition $\tau_E \ll \tau_D$ must be satisfied. Since τ_E and τ_D are related with the inverse of the spectral width of the reservoir and system

energies $\Delta\omega_E$ and $\Delta\omega_D$, respectively, the Markov approximation or time coarse-graining condition is possible if

$$\Delta\omega_E \ll \Delta\omega_D. \quad (3.159)$$

This means that the spacing between energy levels of the reservoir must be much less than the spacing for the system. (133) That is typically achieved with a discrete spectrum system and a continuous spectrum reservoir. After the Born-Markov approximation our master equation takes the form

$$\frac{d}{dt}\tilde{\rho}_S(t) = \sum_{\alpha,\beta} \int_0^\infty d\tau \left\{ \mathcal{R}_{\alpha\beta}(\tau) \left[\tilde{S}_\beta(t-\tau)\tilde{\rho}_S(t)\tilde{S}_\alpha^\dagger(t) - \tilde{S}_\alpha^\dagger(t)\tilde{S}_\beta(t-\tau)\tilde{\rho}_S(t) \right] + \text{h.c.} \right\}. \quad (3.160)$$

Assuming that the system Hamiltonian spectrum is discrete and non-degenerate, $H_S = \sum_\varepsilon \varepsilon \Pi(\varepsilon)$, where ε are the eigenvalues and $\Pi(\varepsilon)$ the projections corresponding to each eigensubspace, then the $\tilde{S}_\alpha(t)$ operators can be expanded as

$$\tilde{S}_\alpha(t) = \sum_\omega e^{-i\omega t} S_\alpha(\omega), \quad (3.161)$$

where

$$S_\alpha(\omega) = \sum_\varepsilon \Pi(\varepsilon) S_\alpha \Pi(\varepsilon + \omega). \quad (3.162)$$

The decomposition is derived in Appendix C. This is used on the master equation by the substitution

$$\tilde{S}_\beta(t-\tau) = \sum_\omega e^{-i\omega(t-\tau)} S_\beta(\omega) \quad (3.163)$$

$$\tilde{S}_\alpha^\dagger(t) = \sum_{\omega'} e^{i\omega' t} A S_\alpha^\dagger(\omega'), \quad (3.164)$$

so

$$\frac{d}{dt}\tilde{\rho}_S(t) = \sum_{\alpha,\beta,\omega,\omega'} \left\{ e^{-i(\omega-\omega')t} \Gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega)\tilde{\rho}_S(t)S_\alpha^\dagger(\omega') - S_\alpha^\dagger(\omega')S_\beta(\omega)\tilde{\rho}_S(t) \right] + \text{h.c.} \right\}, \quad (3.165)$$

where

$$\Gamma_{\alpha\beta}(\omega) = \int_0^\infty d\tau e^{i\omega\tau} \mathcal{R}_{\alpha\beta}(\tau) \quad (3.166)$$

are the one-sided Fourier transforms of the RCFs.

3.4.4 Rotating wave approximation: complete positivity

Since we are in a coarse-grained timescale, the complex exponentials in Eq. (3.165) will oscillate very rapidly in time. Here we apply the *rotating wave approximation*^j: the

^j This approximation works very well in the optical domain. (3)

fast oscillations average to zero, and therefore only the non-oscillating terms will survive, namely when $\omega = \omega'$. (1) The master equation is then

$$\frac{d}{dt}\tilde{\rho}_S(t) = \sum_{\alpha,\beta,\omega} \left\{ \Gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega)\tilde{\rho}_S(t)S_\alpha^\dagger(\omega) - S_\alpha^\dagger(\omega)S_\beta(\omega)\tilde{\rho}_S(t) \right] + \text{h.c.} \right\}, \quad (3.167)$$

Now we split $\Gamma_{\alpha\beta}(\omega)$ as

$$\Gamma_{\alpha\beta}(\omega) = \frac{\gamma_{\alpha\beta}(\omega)}{2} + i\Upsilon_{\alpha\beta}(\omega), \quad (3.168)$$

where

$$\Upsilon_{\alpha\beta}(\omega) = \frac{1}{2i} \left(\Gamma_{\alpha\beta}(\omega) - \Gamma_{\beta\alpha}^*(\omega) \right), \quad (3.169)$$

$$\gamma_{\alpha\beta}(\omega) = \Gamma_{\alpha\beta}(\omega) + \Gamma_{\beta\alpha}^*(\omega). \quad (3.170)$$

This will allow us to separate the unitary and non-unitary parts of the master equation. Note that

$$\Gamma_{\alpha\beta}^*(\omega) = \frac{\gamma_{\alpha\beta}^*(\omega)}{2} - i\Upsilon_{\alpha\beta}^*(\omega) = \frac{\gamma_{\beta\alpha}(\omega)}{2} - i\Upsilon_{\beta\alpha}(\omega). \quad (3.171)$$

We use Equations (3.168) and (3.171) in Equation (3.167), now considering the Hermitian conjugate part:

$$\begin{aligned} \frac{d}{dt}\tilde{\rho}_S(t) &= \sum_{\alpha,\beta,\omega} \Gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega)\tilde{\rho}_S(t)S_\alpha^\dagger(\omega) - S_\alpha^\dagger(\omega)S_\beta(\omega)\tilde{\rho}_S(t) \right] \\ &\quad + \sum_{\alpha,\beta,\omega} \Gamma_{\alpha\beta}^*(\omega) \left[S_\alpha(\omega)\tilde{\rho}_S(t)S_\beta^\dagger(\omega) - \tilde{\rho}_S(t) S_\beta^\dagger(\omega)S_\alpha(\omega) \right] \\ &= \sum_{\alpha,\beta,\omega} \left(\frac{\gamma_{\alpha\beta}(\omega)}{2} + i\Upsilon_{\alpha\beta}(\omega) \right) \left[S_\beta(\omega)\tilde{\rho}_S(t)S_\alpha^\dagger(\omega) - S_\alpha^\dagger(\omega)S_\beta(\omega)\tilde{\rho}_S(t) \right] \\ &\quad + \sum_{\alpha,\beta,\omega} \left(\frac{\gamma_{\beta\alpha}(\omega)}{2} - i\Upsilon_{\beta\alpha}(\omega) \right) \left[S_\alpha(\omega)\tilde{\rho}_S(t)S_\beta^\dagger(\omega) - \tilde{\rho}_S(t) S_\beta^\dagger(\omega)S_\alpha(\omega) \right]. \end{aligned} \quad (3.172)$$

If we permute the dummy indexes in the last line, we have

$$\begin{aligned} \frac{d}{dt}\tilde{\rho}_S(t) &= \sum_{\alpha,\beta,\omega} \left(\frac{\gamma_{\alpha\beta}(\omega)}{2} + i\Upsilon_{\alpha\beta}(\omega) \right) \left[S_\beta(\omega)\tilde{\rho}_S(t)S_\alpha^\dagger(\omega) - S_\alpha^\dagger(\omega)S_\beta(\omega)\tilde{\rho}_S(t) \right] \\ &\quad + \sum_{\alpha,\beta,\omega} \left(\frac{\gamma_{\alpha\beta}(\omega)}{2} - i\Upsilon_{\alpha\beta}(\omega) \right) \left[S_\beta(\omega)\tilde{\rho}_S(t)S_\alpha^\dagger(\omega) - \tilde{\rho}_S(t) S_\alpha^\dagger(\omega)S_\beta(\omega) \right] \\ &= -i \left[\sum_{\alpha,\beta,\omega} \Upsilon_{\alpha\beta}(\omega) S_\alpha^\dagger(\omega)S_\beta(\omega), \tilde{\rho}_S(t) \right] \\ &\quad + \sum_{\alpha,\beta,\omega} \gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega)\tilde{\rho}_S(t)S_\alpha^\dagger(\omega) - \frac{1}{2} \left\{ S_\alpha^\dagger(\omega)S_\beta(\omega), \tilde{\rho}_S(t) \right\} \right], \end{aligned} \quad (3.173)$$

where the curly brackets represent an anticommutator. We define the *Lamb-shift Hamiltonian* as

$$H_{\text{LS}} = \sum_{\alpha,\beta,\omega} \Upsilon_{\alpha\beta}(\omega) S_\alpha^\dagger(\omega) S_\beta(\omega). \quad (3.174)$$

It is a Hermitian operator, since

$$\begin{aligned}
 H_{\text{LS}}^\dagger &= \sum_{\alpha, \beta, \omega} \Upsilon_{\alpha\beta}^*(\omega) S_\beta^\dagger(\omega) S_\alpha(\omega) \\
 &= \sum_{\alpha, \beta, \omega} \Upsilon_{\beta\alpha}^*(\omega) S_\alpha^\dagger(\omega) S_\beta(\omega) \\
 &= \sum_{\alpha, \beta, \omega} \Upsilon_{\alpha\beta}(\omega) S_\alpha^\dagger(\omega) S_\beta(\omega) = H_{\text{LS}}.
 \end{aligned} \tag{3.175}$$

In the second line we performed a relabeling of indices, and in the third line the relation $\Upsilon_{\beta\alpha}^*(\omega) = \Upsilon_{\alpha\beta}(\omega)$ was used. Furthermore, it commutes with the system Hamiltonian H_S . To see that, first note that

$$\begin{aligned}
 [H_S, S_\alpha(\omega)] &= \left[\sum_{\varepsilon} \varepsilon \Pi(\varepsilon), \sum_{\varepsilon', \varepsilon''} \Pi(\varepsilon') S_\alpha(\omega) \Pi(\varepsilon'') \right] \\
 &= \sum_{\varepsilon, \varepsilon', \varepsilon''} [\varepsilon \Pi(\varepsilon), \Pi(\varepsilon') S_\alpha(\omega) \Pi(\varepsilon'')] \\
 &= \sum_{\varepsilon, \varepsilon', \varepsilon''} [\varepsilon \Pi(\varepsilon) \Pi(\varepsilon') S_\alpha(\omega) \Pi(\varepsilon'') - \varepsilon \Pi(\varepsilon') S_\alpha(\omega) \Pi(\varepsilon'') \Pi(\varepsilon)] \\
 &= \sum_{\varepsilon, \varepsilon', \varepsilon''} [\varepsilon \delta_{\varepsilon, \varepsilon'} \Pi(\varepsilon') S_\alpha(\omega) \Pi(\varepsilon'') - \varepsilon \Pi(\varepsilon') S_\alpha(\omega) \delta_{\varepsilon, \varepsilon''} \Pi(\varepsilon'')] \\
 &= \sum_{\varepsilon', \varepsilon''} [\varepsilon' \Pi(\varepsilon') S_\alpha(\omega) \Pi(\varepsilon'') - \varepsilon'' \Pi(\varepsilon') S_\alpha(\omega) \Pi(\varepsilon'')] \\
 &= \sum_{\varepsilon', \varepsilon''} (\varepsilon' - \varepsilon'') \Pi(\varepsilon') S_\alpha(\omega) \Pi(\varepsilon'') \\
 &= \sum_{\varepsilon', \varepsilon''} (-\omega) \Pi(\varepsilon') S_\alpha(\omega) \Pi(\varepsilon'') \\
 &= -\omega S_\alpha(\omega)
 \end{aligned} \tag{3.176}$$

and

$$\begin{aligned}
 [H_S, S_\alpha^\dagger(\omega) S_\beta(\omega)] &= [H_S, S_\alpha^\dagger(\omega)] S_\beta(\omega) + S_\alpha^\dagger(\omega) [H_S, S_\beta(\omega)] \\
 &= \omega S_\alpha^\dagger(\omega) S_\beta(\omega) - \omega S_\alpha^\dagger(\omega) S_\beta(\omega) = 0.
 \end{aligned} \tag{3.177}$$

Therefore,

$$[H_S, H_{\text{LS}}] = \sum_{\alpha, \beta, \omega} \Upsilon_{\alpha\beta}(\omega) [H_S, S_\alpha^\dagger(\omega) S_\beta(\omega)] = 0, \tag{3.178}$$

so the Lamb shift Hamiltonian commutes with the system hamiltonian H_S . That means that its effect is only a shift in the previous energy levels of the system. Physically, it describes the "renormalization" effect the bath has on the unitary part of the system dynamics. (134)

Eq. (3.173) is now

$$\begin{aligned}
 \frac{d}{dt} \tilde{\rho}_S(t) &= -i [H_{\text{LS}}, \tilde{\rho}_S(t)] \\
 &\quad + \sum_{\alpha, \beta, \omega} \gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega) \tilde{\rho}_S(t) S_\alpha^\dagger(\omega) - \frac{1}{2} \{ S_\alpha^\dagger(\omega) S_\beta(\omega), \tilde{\rho}_S(t) \} \right].
 \end{aligned} \tag{3.179}$$

This equation can be diagonalized and, since the matrix with coefficients $\gamma_{\alpha\beta}(\omega)$ is positive semidefinite, as shown in Appendix E, the decay rates are nonnegative and, therefore, the equation has a Lindblad form, which means that the resulting evolution is Markovian. The final step is to return to the Schrodinger picture, which finally leads to^k

$$\begin{aligned} \frac{d}{dt}\rho_S(t) = & -i[H_S + H_{LS}, \rho_S(t)] \\ & + \sum_{\alpha,\beta,\omega} \gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega)\rho_S(t)S_\alpha^\dagger(\omega) - \frac{1}{2} \{S_\alpha^\dagger(\omega)S_\beta(\omega), \rho_S(t)\} \right], \end{aligned} \quad (3.180)$$

3.4.5 Summary

Here is a summary of the results of the chapter, and a quick way to derive the Lindblad equation from the microscopic model. We start from the Hamiltonian

$$H = H_S + H_E + H_{\text{int}}, \quad (3.181)$$

where $H_{\text{int}} = \sum_\alpha S_\alpha \otimes E_\alpha$. Going to the interaction picture,

$$\tilde{S}_\alpha(t) = e^{iH_S t} S_\alpha e^{-iH_S t}, \quad (3.182)$$

$$\tilde{E}_\alpha(t) = e^{iH_E t} E_\alpha e^{-iH_E t}, \quad (3.183)$$

and the system operators $\tilde{S}_\alpha(t)$ can be decomposed in frequency terms,

$$\tilde{S}_\alpha(t) = \sum_\omega e^{-i\omega t} S_\alpha(\omega). \quad (3.184)$$

The reservoir correlation functions (RCFs) are

$$\mathcal{R}_{\alpha\beta}(\tau) = \text{tr}_E [\tilde{E}_\alpha^\dagger(\tau) E_\beta \rho_E], \quad (3.185)$$

where ρ_E is the thermal state of the environment, so we have

$$\Gamma_{\alpha\beta}(\omega) = \int_0^\infty d\tau e^{i\omega\tau} \mathcal{R}_{\alpha\beta}(\tau). \quad (3.186)$$

With that, we can define the decay rates as

$$\gamma_{\alpha\beta}(\omega) = \Gamma_{\alpha\beta}(\omega) + \Gamma_{\beta\alpha}^*(\omega) = \int_{-\infty}^\infty d\tau e^{i\omega\tau} \mathcal{R}_{\alpha\beta}(\tau), \quad (3.187)$$

and

$$\Upsilon_{\alpha\beta}(\omega) = \frac{1}{2i} (\Gamma_{\alpha\beta}(\omega) - \Gamma_{\beta\alpha}^*(\omega)), \quad (3.188)$$

which is used to define the Lamb shift Hamiltonian

$$H_{LS} = \sum_{\alpha,\beta,\omega} \Upsilon_{\alpha\beta}(\omega) S_\alpha^\dagger(\omega) S_\beta(\omega). \quad (3.189)$$

Assembling all these terms together, we have the master equation

$$\begin{aligned} \frac{d}{dt}\rho_S(t) = & -i[H_S + H_{LS}, \rho_S(t)] \\ & + \sum_{\alpha,\beta,\omega} \gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega)\rho_S(t)S_\alpha^\dagger(\omega) - \frac{1}{2} \{S_\alpha^\dagger(\omega)S_\beta(\omega), \rho_S(t)\} \right]. \end{aligned} \quad (3.190)$$

^k This derivation is performed in Section C.2.

3.5 Examples

3.5.1 Harmonic oscillator

As a first application of the Born-Markov master equation, Eq. (3.180), we study the case of a harmonic oscillator coupled to a bosonic bath (1, 2), also known as the damped harmonic oscillator. (3) The total Hamiltonian is given by

$$H = \omega_0 a^\dagger a + \sum_k \omega_k b_k^\dagger b_k + \sum_k g_k (a^\dagger b_k + a b_k^\dagger), \quad (3.191)$$

where ω_0 is the natural frequency of the system oscillator, ω_k the frequencies of the bath and g_k the coupling constants. Physically, this describes a single mode of the electromagnetic field in a lossy cavity. (3) Following the previous notation, we decompose H_{int} as

$$H_{\text{int}} = a^\dagger \otimes \sum_k g_k b_k + a \otimes \sum_k g_k b_k^\dagger = S_1 \otimes E_1 + S_2 \otimes E_2. \quad (3.192)$$

Moving to the interaction picture,

$$\widetilde{H}_{\text{int}}(t) = e^{iH_0 t} H_{\text{int}}(t) e^{-iH_0 t}, \quad (3.193)$$

where $H_0 = \omega_0 a^\dagger a + \sum_k \omega_k b_k^\dagger b_k$. When evaluating $\widetilde{S}_1(t)$ and $\widetilde{S}_2(t)$, from Eq. (3.182), which are functions of the system operators, the exponentials corresponding to the bath operators in H_0 will not affect them. Since

$$[\omega_0 a^\dagger a, a^\dagger] = \omega_0 [a^\dagger a, a^\dagger] = \omega_0 a^\dagger, \quad (3.194)$$

$$[\omega_0 a^\dagger a, a] = \omega_0 [a^\dagger a, a] = (-\omega_0) a, \quad (3.195)$$

where we used that $[a, a^\dagger] = 1$, and denoting $\text{Ad}_{H_0}(\cdot) \equiv [H_0, \cdot]$ (135), we have that (12)

$$\widetilde{a}^\dagger(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \text{Ad}_{H_0}^n a^\dagger = \sum_{n=0}^{\infty} \frac{(i\omega_0)^n}{n!} a^\dagger = e^{i\omega_0 t} a^\dagger \quad (3.196)$$

$$\widetilde{a}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \text{Ad}_{H_0}^n a = \sum_{n=0}^{\infty} \frac{(-i\omega_0)^n}{n!} a = e^{-i\omega_0 t} a. \quad (3.197)$$

For the bath operators the situation is analogous, and then

$$\widetilde{S}_1(t) = e^{i\omega_0 t} a^\dagger \quad (3.198)$$

$$\widetilde{S}_2(t) = e^{-i\omega_0 t} a \quad (3.199)$$

$$\widetilde{E}_1(t) = \sum_k g_k e^{-i\omega_k t} b_k \quad (3.200)$$

$$\widetilde{E}_2(t) = \sum_k g_k e^{i\omega_k t} b_k^\dagger. \quad (3.201)$$

Therefore, we see that $S_1(\omega_0) = 0$, $S_1(-\omega_0) = a^\dagger$, $S_2(\omega_0) = a$ and $S_2(-\omega_0) = 0$. Since the reservoir is in a thermal state, we have the relations¹

$$\langle b_k b_{k'} \rangle = \langle b_k^\dagger b_{k'}^\dagger \rangle = 0 \quad (3.202)$$

$$\langle b_k b_{k'}^\dagger \rangle = \delta_{kk'} (1 + N(\omega_k)) \quad (3.203)$$

$$\langle b_k^\dagger b_{k'} \rangle = \delta_{kk'} N(\omega_k), \quad (3.204)$$

where

$$N(\omega_k) = \frac{1}{e^{\beta\omega_k} - 1} \quad (3.205)$$

is the average number of particles in a mode with frequency ω_k . (1) With them, the RCFs can be evaluated:

$$\begin{aligned} \mathcal{R}_{11}(\tau) &= \langle \tilde{E}_1^\dagger(\tau) E_1 \rangle \\ &= \left\langle \left(\sum_k g_k^* e^{i\omega_k \tau} b_k^\dagger \right) \left(\sum_{k'} g_{k'} b_{k'} \right) \right\rangle \\ &= \sum_{k,k'} g_k^* g_{k'} e^{i\omega_k \tau} \langle b_k^\dagger b_{k'} \rangle \\ &= \sum_k |g_k|^2 e^{i\omega_k \tau} N(\omega_k), \end{aligned} \quad (3.206)$$

$$\begin{aligned} \mathcal{R}_{22}(\tau) &= \langle \tilde{E}_2^\dagger(\tau) E_2 \rangle \\ &= \left\langle \left(\sum_k g_k^* e^{-i\omega_k \tau} b_k \right) \left(\sum_{k'} g_{k'} b_{k'}^\dagger \right) \right\rangle \\ &= \sum_{k,k'} g_k^* g_{k'} e^{-i\omega_k \tau} \langle b_k b_{k'}^\dagger \rangle \\ &= \sum_k |g_k|^2 e^{-i\omega_k \tau} (1 + N(\omega_k)), \end{aligned} \quad (3.207)$$

and, similarly, $\mathcal{R}_{12}(\tau) = \mathcal{R}_{21}(\tau) = 0$.

As shown in Sec. 3.4.3, the Born-Markov approximation only works if we have a continuum of reservoir frequencies. To perform the continuum limit we do the substitution (136)

$$\sum_k f(\omega_k) \rightarrow \int_0^\infty d\omega \frac{J(\omega)}{|g(\omega)|^2} f(\omega), \quad (3.208)$$

where $f(\omega_k)$ is an arbitrary function of the bath frequencies and $J(\omega)$ is the spectral density of the reservoir (1), which tells how the coupling strength varies with the frequency. The limit leads to

$$\mathcal{R}_{11}(\tau) = \int_0^\infty d\omega e^{i\omega \tau} J(\omega) N(\omega), \quad (3.209)$$

$$\mathcal{R}_{22}(\tau) = \int_0^\infty d\omega e^{-i\omega \tau} J(\omega) (1 + N(\omega)), \quad (3.210)$$

¹ These relations are derived in Appendix D.

and the other RCFs are zero. Now we are ready to calculate the decay rates $\gamma_{\alpha\beta}(\omega)$. Assuming that the system's natural frequency ω_0 is positive,

$$\begin{aligned}
\gamma_{11}(\omega_0) &= \int_{-\infty}^{+\infty} d\tau e^{i\omega_0\tau} \mathcal{R}_{11}(\tau) \\
&= \int_{-\infty}^{+\infty} d\tau e^{i\omega_0\tau} \int_0^{\infty} d\omega e^{i\omega\tau} J(\omega) N(\omega) \\
&= \int_0^{\infty} d\omega J(\omega) N(\omega) \int_{-\infty}^{+\infty} d\tau e^{i(\omega+\omega_0)\tau} \\
&= 2\pi \int_0^{\infty} d\omega J(\omega) N(\omega) \delta(\omega + \omega_0).
\end{aligned} \tag{3.211}$$

where we used an integral representation of the Dirac delta function (137),

$$\delta(\omega - \omega_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau e^{i(\omega - \omega_0)\tau}. \tag{3.212}$$

Since the frequency ω_0 is positive, the integral involving $\delta(\omega_0 + \omega)$ is zero, due to the domain of integration being the positive ω axis. Therefore, $\gamma_{11}(\omega_0) = 0$. On the other hand,

$$\gamma_{11}(-\omega_0) = 2\pi \int_0^{\infty} d\omega J(\omega) N(\omega) \delta(\omega - \omega_0) = 2\pi J(\omega_0) N(\omega_0). \tag{3.213}$$

Similarly, $\gamma_{22}(-\omega_0) = 0$, and

$$\gamma_{22}(\omega_0) = 2\pi J(\omega_0)(1 + N(\omega_0)). \tag{3.214}$$

Then, our master equation assumes the form

$$\begin{aligned}
\frac{d}{dt} \rho_S(t) &= -i [\omega_0 a^\dagger a + H_{\text{LS}}, \rho_S(t)] + 2\pi J(\omega_0)(1 + N(\omega_0)) \left[a \rho_S(t) a^\dagger - \frac{1}{2} \{a^\dagger a, \rho_S(t)\} \right] \\
&\quad + 2\pi J(\omega_0) N(\omega_0) \left[a^\dagger \rho_S(t) a - \frac{1}{2} \{a a^\dagger, \rho_S(t)\} \right],
\end{aligned} \tag{3.215}$$

and all we need to do now is evaluate the Lamb shift Hamiltonian

$$H_{\text{LS}} = \sum_{\alpha, \beta, \omega} \Upsilon_{\alpha\beta}(\omega) S_\alpha^\dagger(\omega) S_\beta(\omega). \tag{3.216}$$

Since $S_1(\omega_0) = S_2(-\omega_0) = 0$, the expression simplifies to

$$\begin{aligned}
H_{\text{LS}} &= \Upsilon_{11}(-\omega_0) S_1^\dagger(-\omega_0) S_1(-\omega_0) + \Upsilon_{22}(\omega_0) S_2^\dagger(\omega_0) S_2(\omega_0) \\
&= \Upsilon_{11}(-\omega_0) a a^\dagger + \Upsilon_{22}(\omega_0) a^\dagger a \\
&= (\Upsilon_{11}(-\omega_0) + \Upsilon_{22}(\omega_0)) a^\dagger a,
\end{aligned} \tag{3.217}$$

where, in the last step, we used the commutation relation $[a, a^\dagger] = 1$ and the fact that, since the Hamiltonian appears inside a commutator, the term proportional to the identity has a vanishing effect and therefore is irrelevant. Note that

$$\Upsilon_{11}(-\omega_0) = \frac{1}{2i} (\Gamma_{11}(-\omega_0) - \Gamma_{11}^*(-\omega_0)), \tag{3.218}$$

$$\Upsilon_{22}(\omega_0) = \frac{1}{2i} (\Gamma_{22}(\omega_0) - \Gamma_{22}^*(\omega_0)), \tag{3.219}$$

so we only have to calculate $\Gamma_{11}(-\omega_0)$ and $\Gamma_{22}(\omega_0)$. The term $\Gamma_{11}(-\omega_0)$ can be written as

$$\begin{aligned}\Gamma_{11}(-\omega_0) &= \int_0^\infty d\tau e^{-i\omega_0\tau} \mathcal{R}_{11}(\tau) \\ &= \int_0^\infty d\tau e^{-i\omega_0\tau} \int_0^\infty d\omega e^{i\omega\tau} J(\omega) N(\omega) \\ &= \int_0^\infty d\omega J(\omega) N(\omega) \int_0^{+\infty} d\tau e^{-i(\omega_0-\omega)\tau}.\end{aligned}\quad (3.220)$$

The integrals with the exponential terms are not well defined. That can be avoided by regarding them as a limit, i.e.,

$$\int_0^\infty d\tau e^{-i(\omega-\omega_0)\tau} = \lim_{\epsilon \rightarrow 0^+} \int_0^\infty d\tau e^{-i(\omega-\omega_0-i\epsilon)\tau} = -i \lim_{\epsilon \rightarrow 0^+} \frac{1}{\omega - \omega_0 - i\epsilon}, \quad (3.221)$$

where we can use Sokhotsky's formula (138),

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{x - i\epsilon} = i\pi\delta(x) + \mathcal{P}\left(\frac{1}{x}\right), \quad (3.222)$$

where $\mathcal{P}$ denotes the Cauchy Principal Value (137) and $\delta(x)$ is the Dirac delta function. Then,

$$\begin{aligned}\Gamma_{11}(\omega_0) &= \int_0^\infty d\omega J(\omega) N(\omega) \left[\pi\delta(\omega_0 - \omega) - i\mathcal{P}\left(\frac{1}{\omega_0 - \omega}\right) \right] \\ &= \pi J(\omega_0) N(\omega_0) - i\mathcal{P}\left(\int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega} N(\omega)\right),\end{aligned}\quad (3.223)$$

so we have that

$$\Upsilon_{11}(-\omega_0) = -\mathcal{P}\left(\int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega} N(\omega)\right) \equiv -\Delta'. \quad (3.224)$$

Similarly,

$$\begin{aligned}\Gamma_{22}(\omega_0) &= \int_0^\infty d\tau e^{i\omega_0\tau} \mathcal{R}_{22}(\tau) \\ &= \int_0^\infty d\omega J(\omega)(1 + N(\omega)) \int_0^{+\infty} d\tau e^{-i(\omega-\omega_0)\tau} \\ &= \int_0^\infty d\omega J(\omega)(1 + N(\omega)) \left[\pi\delta(\omega - \omega_0) + i\mathcal{P}\left(\frac{1}{\omega_0 - \omega}\right) \right] \\ &= \pi J(\omega_0)(1 + N(\omega_0)) + i\mathcal{P}\left(\int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega}\right) + i\mathcal{P}\left(\int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega} N(\omega)\right),\end{aligned}\quad (3.225)$$

so

$$\Upsilon_{22}(\omega_0) = \mathcal{P}\left(\int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega}\right) + \mathcal{P}\left(\int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega} N(\omega)\right) \equiv \Delta + \Delta' \quad (3.226)$$

Therefore, the Lamb shift Hamiltonian for the harmonic oscillator is

$$H_{\text{LS}} = \Delta a^\dagger a, \quad (3.227)$$

where

$$\Delta = \mathcal{P} \left(\int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega} \right), \quad (3.228)$$

as expected, is only a shift in the energy levels of the system Hamiltonian. Putting it all together,

$$\begin{aligned} \frac{d}{dt} \rho_S(t) = & -i \left[(\omega_0 + \Delta) a^\dagger a, \rho_S(t) \right] + \gamma(1 + N(\omega_0)) \left[a \rho_S(t) a^\dagger - \frac{1}{2} \{a^\dagger a, \rho_S(t)\} \right] \\ & + \gamma N(\omega_0) \left[a^\dagger \rho_S(t) a - \frac{1}{2} \{a a^\dagger, \rho_S(t)\} \right], \end{aligned} \quad (3.229)$$

where we defined $\gamma = 2\pi J(\omega_0)$ just for simplicity.

3.5.2 Qubit under amplitude damping

The next model is a two-level system (qubit) damped by a bosonic bath. (2) It represents, for example, an atom radiatively damped by the interaction with many modes of a radiation field. (3) The total Hamiltonian of our model is

$$H = \frac{\omega_0}{2} \sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k g_k (\sigma_+ b_k + \sigma_- b_k^\dagger), \quad (3.230)$$

with ω_0 the qubit gap and

$$\sigma_+ = \frac{\sigma_x + i\sigma_y}{2}, \quad \sigma_- = \frac{\sigma_x - i\sigma_y}{2}, \quad (3.231)$$

the raising and lowering spin operators. (75) The interaction Hamiltonian is

$$H_{\text{int}} = \sigma_+ \otimes \sum_k g_k b_k + \sigma_- \otimes \sum_k g_k b_k^\dagger \equiv S_1 \otimes E_1 + S_2 \otimes E_2. \quad (3.232)$$

Moving to the interaction picture, note that

$$\left[\frac{\omega_0}{2} \sigma_z, \sigma_\pm \right] = \pm \omega_0 \sigma_\pm, \quad (3.233)$$

and the problem is therefore equivalent the previous section, the difference being that $a^\dagger$ and a are replaced by σ_+ and σ_- , respectively. Then,

$$\tilde{S}_1(t) = e^{i\omega_0 t} \sigma_+ \quad (3.234)$$

$$\tilde{S}_2(t) = e^{-i\omega_0 t} \sigma_- \quad (3.235)$$

$$\tilde{E}_1(t) = \sum_k g_k e^{-i\omega_k t} b_k \quad (3.236)$$

$$\tilde{E}_2(t) = \sum_k g_k e^{i\omega_k t} b_k^\dagger, \quad (3.237)$$

and the master equation derivation is exactly the same as the previous example, leading to

$$\begin{aligned} \frac{d}{dt} \rho_S(t) = & -i \left[\frac{\omega_0}{2} \sigma_z + H_{\text{LS}}, \rho_S(t) \right] + \gamma(1 + N(\omega_0)) \left[\sigma_- \rho_S(t) \sigma_+ - \frac{1}{2} \{ \sigma_+ \sigma_-, \rho_S(t) \} \right] \\ & + \gamma N(\omega_0) \left[\sigma_+ \rho_S(t) \sigma_- - \frac{1}{2} \{ \sigma_- \sigma_+, \rho_S(t) \} \right]. \end{aligned} \quad (3.238)$$

The Lamb shift Hamiltonian, however, is now different, since the commutation relations changed: before we had $[a, a^\dagger] = \mathbb{1}$, but now $[\sigma_+, \sigma_-] = \sigma_z$ (and also $\{\sigma_+, \sigma_-\} = \mathbb{1}$). Then,

$$\begin{aligned}
H_{\text{LS}} &= \Upsilon_{11}(-\omega_0)\sigma_-\sigma_+ + \Upsilon_{22}(\omega_0)\sigma_+\sigma_- \\
&= \frac{1}{2}(\Upsilon_{11}(-\omega_0) - \Upsilon_{22}(\omega_0))\sigma_z + \frac{1}{2}(\Upsilon_{11}(-\omega_0) + \Upsilon_{22}(\omega_0))\mathbb{1} \\
&= \frac{1}{2}(\Upsilon_{11}(-\omega_0) - \Upsilon_{22}(\omega_0))\sigma_z \\
&= \frac{2\Delta' + \Delta}{2}\sigma_z,
\end{aligned} \tag{3.239}$$

so our desired master equation is

$$\begin{aligned}
\frac{d}{dt}\rho_S(t) &= -i \left[\frac{\omega'_0}{2}\sigma_z + H_{\text{LS}}, \rho_S(t) \right] + \gamma(1 + N(\omega_0)) \left[\sigma_-\rho_S(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_S(t)\} \right] \\
&\quad + \gamma N(\omega_0) \left[\sigma_+\rho_S(t)\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho_S(t)\} \right],
\end{aligned} \tag{3.240}$$

where $\omega'_0 = \omega_0 + 2\Delta' + \Delta$ and

$$\Delta = \mathcal{P} \left(\int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega} \right), \tag{3.241}$$

$$\Delta' = \mathcal{P} \left(\int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega} N(\omega_0) \right). \tag{3.242}$$

Note that we have two decay rates, $\gamma(1 + N(\omega_0))$ and $\gamma N(\omega_0)$. The first one is related to emission of photons and consists of two parts, stimulated emissions^m ($\gamma N(\omega_0)$) and spontaneous emissions (γ) (3); the second one is related to absorption of photons from the environment. (3) Also note that the Lamb shift consists of a temperature independent part (Δ), which is the usual Lamb shift, and a temperature dependent part (Δ'), which is a frequency shift induced by the thermal reservoir field. (3)

3.5.3 Qubit under phase damping

A qubit dynamics of pure dephasing is represented by the Hamiltonian

$$H = \frac{\omega_0}{2}\sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sigma_z \sum_k g_k (b_k + b_k^\dagger). \tag{3.243}$$

It can represent, for example, a photon randomly scattering across a waveguide. (9) Unlike the previous examples, this evolution preserves the energy of the system and only affects its coherence. Although it is exactly solvable (1), we use the formalism developed here:

$$H_{\text{int}} = \sigma_z \sum_k g_k (b_k + b_k^\dagger) = S \otimes E. \tag{3.244}$$

^m Induced by thermal photons from the reservoir. (3)

Since the interaction term commutes with the Hamiltonian, there is no dissipation and the environment only affects the coherences of the system. Moving to the interaction picture,

$$\widetilde{H}_{\text{int}}(t) = \sigma_z \otimes \sum_k g_k \left(e^{-i\omega_k t} b_k + e^{i\omega_k t} b_k^\dagger \right) = S(t) \otimes E(t), \quad (3.245)$$

which shows that $S(0) = \sigma_z$. The reservoir correlation function is

$$\begin{aligned} \mathcal{R}(\tau) &= \left\langle \sum_{k,k'} g_k^\dagger g_{k'} \left(e^{i\omega_k \tau} b_k^\dagger + e^{-i\omega_k \tau} b_k \right) \left(b_{k'} + b_{k'}^\dagger \right) \right\rangle \\ &= \sum_{k,k'} g_k^\dagger g_{k'} \left(e^{i\omega_k \tau} \langle b_k^\dagger b_{k'} \rangle + e^{i\omega_k \tau} \langle b_k^\dagger b_{k'}^\dagger \rangle + e^{-i\omega_k \tau} \langle b_k b_{k'} \rangle + e^{-i\omega_k \tau} \langle b_k b_{k'}^\dagger \rangle \right) \\ &= \sum_k |g_k|^2 \left[e^{i\omega_k \tau} N(\omega_k) + e^{-i\omega_k \tau} (1 + N(\omega_k)) \right]. \end{aligned} \quad (3.246)$$

In the continuum limit,

$$\mathcal{R}(\tau) = 2\pi \int_0^\infty d\omega' J(\omega') \left[e^{i\omega' \tau} N(\omega) + e^{-i\omega' \tau} (1 + N(\omega')) \right]. \quad (3.247)$$

The decay rate is

$$\begin{aligned} \gamma(\omega) &= \int_{-\infty}^\infty d\omega' e^{i\omega' \tau} \mathcal{R}(\tau) \\ &= 2\pi \int_0^\infty d\omega' J(\omega') N(\omega) \delta(\omega' + \omega) + 2\pi \int_0^\infty d\omega' J(\omega') (1 + N(\omega')) \delta(\omega' - \omega). \end{aligned} \quad (3.248)$$

Note that if $\omega > 0$ then $\delta(\omega' + \omega) = \delta(\omega' - (-\omega)) = 0$, since $-\omega$ is outside the domain of integration. Therefore

$$\gamma(\omega > 0) = 2\pi J(\omega) (1 + N(\omega)). \quad (3.249)$$

Equally, if $\omega < 0$ then $\omega = -|\omega|$ and $\delta(\omega' - \omega) = 0$. Then,

$$\gamma(\omega < 0) = 2\pi J(|\omega|) N(|\omega|). \quad (3.250)$$

Since $S(0) = \sigma_z$ and $S(\omega) = 0$ for every non-zero frequency, the only relevant decay rate is $\gamma(0)$, which is

$$\gamma(0) = \lim_{\omega \rightarrow 0} \gamma(\omega). \quad (3.251)$$

In our case, however, the lateral limits are

$$\lim_{\omega \rightarrow 0+} \gamma(\omega) = 2\pi J(0) (1 + N(0)) \quad (3.252)$$

$$\lim_{\omega \rightarrow 0-} \gamma(\omega) = 2\pi J(0) N(0), \quad (3.253)$$

and $\gamma(0)$ is not well defined, unless at very high temperatures, where $1 + N(|\omega|) \approx N(|\omega|)$. (139) That can be solved by assuming that

$$\lim_{\omega \rightarrow 0} J(|\omega|) = 0, \quad (3.254)$$

which is a very natural assumption, given that a mode with zero frequency does not affects the system. (139) $J(\omega)$ is also needed to approach zero linearly in ω , since $N(\omega)$ goes to infinity as $1/|\omega|$ as ω decreases. (2) That hypothesis restricts the class of spectral densities that can be used, such as the ohmic-type densities. That limitation, however, can be circumvented by using the exact solution of the problem. (1) The decay rate is, therefore

$$\gamma(0) = 2\pi \lim_{\omega \rightarrow 0} J(|\omega|)N(|\omega|), \quad (3.255)$$

and the master equation is

$$\frac{d}{dt}\rho_S(t) = -i \left[\frac{\omega_0}{2}\sigma_z + H_{\text{LS}}, \rho_S(t) \right] + \gamma(0) \left[\sigma_z \rho_S(t) \sigma_z - \frac{1}{2} \{ \sigma_z \sigma_z, \rho_S(t) \} \right]. \quad (3.256)$$

Since $\sigma_z^2 = 1$, the Lamb shift Hamiltonian is proportional to the identity (and, therefore, has no effect on the system) and the last term in the right hand side reduces to $\rho_S(t)$. The final form is

$$\frac{d}{dt}\rho_S(t) = -i \left[\frac{\omega_0}{2}\sigma_z, \rho_S(t) \right] + \gamma(0) [\sigma_z \rho_S(t) \sigma_z - \rho_S(t)]. \quad (3.257)$$

4 QUANTUM NON-MARKOVIANITY

The concept of Markovianity or of a Markovian process comes from the theory of classical stochastic processes. (69) They are basically processes with no memory of past states, where the typical example in Physics is the Brownian motion. (69) The definition of a Markovian process depends on a family of time-conditional probabilities (10), which leads to problems when we try to adapt the concept to the quantum case. The definitions of quantum Markovianity, therefore, are generalizations of other aspects of classical Markovianity, such as the memoryless property (140, 141) or the divisibility property. (10) These definitions, however, turn out to be not equivalent to each other, which shows that quantum Markovianity (and, consequently, quantum non-Markovianity) is a complex and multifaceted phenomenon. In this chapter we introduce the classical and quantum definitions of Markovian processes, methods for detecting and measuring quantum non-Markovianity and some features of system which possess this property.

4.1 Review of probability theory

In order to formulate the definition of a classical Markovian process we need to review some facts of probability theory, which will be made in a non-rigorous and brief manner. The interested reader should refer to (1, 12, 69, 142–145) in order to get acquainted with other aspects of the theory.

A *classical probability space* is a triple $(\Omega, \mathcal{A}, P)$ where Ω is the *sample space*^a, $\mathcal{A}$ the set of possible *events* and P the *probability function*. Ω is simply a set, $\mathcal{A}$ is a σ -algebra of subsets of Ω and a probability function is a function $P : \mathcal{A} \mapsto [0, 1]$ which is countably additive, i.e., the function of a pairwise disjoint countable collection of subsets is equal to the sum of the functions of each subset, and such that $P(\emptyset) = 0$ and $P(\Omega) = 1$. (1) The elements of Ω are called *outcomes*, the elements of $\mathcal{A}$ are called *events* and $P(A)$ is the probability of the event $A \in \mathcal{A}$. Since this will be a basic and conceptual introduction, we will restrict ourselves to discrete probability spaces, where the full machinery of σ -algebras and measure theory is not needed but still the relevant concepts can be appreciated. In a discrete probability space the sample space Ω is finite or countably infinite, so we can consider as a σ -algebra the power set of Ω , i.e., the set of all of its subsets. In this case, a probability function is a function over the subsets of Ω which satisfies

- a) $P(A) \geq 0$, for events $A \subset \Omega$.
- b) $P(\Omega) = 1$.
- c) If the events $\{A_i\}$ are mutually disjoint, then $P(\cup_i A_i) = \sum_i P(A_i)$.

^a Note that the sample space can consist of anything, such as, for example, bananas and apples.

The *joint probability* of events A and B , i.e., the probability of both A and B occurring, is

$$P(A; B) = P(A \cap B). \quad (4.1)$$

The *conditional probability* of A , given that B occurred (and $P(B) > 0$) is defined as

$$P(A|B) = \frac{P(A; B)}{P(B)}. \quad (4.2)$$

The condition $\mathbb{P}(A|B) = \mathbb{P}(A)$ is equivalent to $P(B|A) = P(B)$, and in this case A and B are said to be independent events.

A (real) random variable is a function $X : \Omega \mapsto \mathbb{R}$ ^b. It assigns to every outcome ω a real number $X(\omega) = x$, so we can work with real numbers instead of the original sample space^c. In this spirit, the probability distribution of X is a function $p : \mathbb{R} \mapsto [0, 1]$ defined by

$$p(x) = P(X^{-1}(\{x\})) = P(w), \quad (4.3)$$

so it is the probability function, but now defined for the values of the outcomes instead of the outcomes themselves. From the properties of the probability function we can derive that $p(x) \geq 0$ and $\sum_x p(x) = 1$, which characterize a probability distribution, and are known as the *positivity* and *normalization* conditions, respectively. When there is a discrete number of x 's, we call $\mathbf{p} = (p(x_1), \dots, p(x_n), \dots)$ a probability vector. For every vector $\mathbf{v}$ (not necessarily a probability vector) in this vector space^d, we define the 1-norm (10) as

$$\|\mathbf{v}\|_1 = \sum_i |v(x_i)|. \quad (4.4)$$

Note that for probability vectors $\|\mathbf{p}\|_1 = \sum_i p(x_i) = 1$. This 1-norm induces^e a distance function in this space,

$$d(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \sum_i |p(x_i) - q(x_i)|, \quad (4.5)$$

which is known as the Kolmogorov distance. (81)

4.1.1 Stochastic processes

A stochastic process is a family of random variables $\{X(t)\}_{t \in I}$ with $I \subset \mathbb{R}$ a real interval, which is usually understood to represent time and can be discrete or continuous. Therefore, a stochastic process can be seen as a random variable evolving in time. We can likewise extend the probability distribution to the family

$$p(x, t) = P(X(t)^{-1}(\{x\})). \quad (4.6)$$

^b A rigorous definition of a random variable requires it to be a measurable function (1, 145), but all functions defined on a discrete set are measurable (145), so this requirement is trivially satisfied for discrete probability spaces.

^c Of course, that does not make much difference if the sample space is already the real line. But for more general spaces, such as the space of bananas and apples, it provides a means to assign, to every outcome, a real number.

^d Note that the dimension of this vector space is equal to the number of x_i 's.

^e We say that distance function $d(x, y)$ is induced by the norm $\|\cdot\|$ when $d(x, y) = \|x - y\|$.

A family of (temporal) joint probabilities is the joint probability of n events occurring at times $t_0 \leq t_1 \leq \dots \leq t_n$:

$$p(x_n, t_n; x_{n-1}, t_{n-1}; \dots; x_2, t_2; x_1, t_1), \quad (4.7)$$

for a given stochastic process. As a consequence of normalization and positivity for usual probabilities, we have that

$$p(x_n, t_n; \dots; x_1, t_1) \geq 0, \quad (4.8)$$

$$\sum_{x_i} p(x_n, t_n; \dots; x_i, t_i; \dots; x_1, t_1) = p(x_n, t_n; \dots; x_1, t_1), \quad (4.9)$$

$$\sum_x p(x, t) = 1, \quad (4.10)$$

which are known as consistency conditions. (69) The complete hierarchy of joint probabilities ($n = 1, 2, \dots$) fully determines its stochastic process, and conversely any hierarchy that satisfies the consistency conditions uniquely determines some stochastic process. (69)

The joint probabilities are used, for example, to compute the conditional probabilities

$$p(x_n, t_n; \dots; x_{k+1}, t_{k+1} | x_k, t_k; \dots; x_1, t_1) = \frac{p(x_n, t_n; \dots; x_1, t_1)}{p(x_k, t_k; \dots; x_1, t_1)}, \quad (4.11)$$

where $k < n$.

For a stochastic process with initial time t_0 , the probabilities $p(x, t | x_0, t_0)$ are well defined for any $t \geq t_0$ and form matrices with coefficients $p_{ij} = p(x_i, t | x_j, t_0)$. They satisfy the positivity and *left-normalization*^f conditions,

$$p(x, t | x_0, t_0) \geq 0, \quad (4.12)$$

$$\sum_x p(x, t | x_0, t_0) = 1, \quad (4.13)$$

respectively. Any matrix that satisfies the above conditions is called a *stochastic matrix*. (10) Using the definition of conditional probability, we have that

$$p(x, t) = \sum_{x_0} p(x, t | x_0, t_0) p(x_0, t_0), \quad (4.14)$$

so they can be seen as linear maps responsible for the evolution of one-time probabilities. For $t' \geq 0$ the probabilities $p(x, t | x', t')$ are not necessarily well defined, but if all the matrices $p(x, t | x_0, t_0)$ are invertible then we can define

$$p(x, t | x', t') = \sum_{x_0} p(x, t | x_0, t_0) (p(x', t' | x_0, t_0))^{-1}, \quad (4.15)$$

^f Analogously, the right-normalization condition is given by $\sum_{x_0} p(x, t | x_0, t_0) = 1$.

but this matrix may not be a stochastic matrix, since it may no longer satisfy the positivity condition. Also, note that

$$\begin{aligned}
& \sum_{x'} p(x, t|x', t') p(x', t'|x_0, t_0) p(x_0, t_0) \\
&= \sum_{x', x_0} p(x, t|x_0, t_0) (p(x', t'|x_0, t_0))^{-1} p(x', t'|x_0, t_0) p(x_0, t_0) \\
&= \sum_{x_0} p(x, t|x_0, t_0) p(x_0, t_0) \\
&= p(x, t),
\end{aligned} \tag{4.16}$$

so they satisfy the Chapman-Kolmogorov equation:

$$p(x, t|x_0, t_0) = \sum_{x'} p(x, t|x', t') p(x', t'|x_0, t_0). \tag{4.17}$$

A stochastic process whose all matrices $p(x, t|x', t')$ ($t \geq t'$) are stochastic and satisfy Eq. (4.17) is called a *divisible process*. (10) These processes can be characterized by the monotonicity with respect to the 1-norm, Eq. (4.4):

Theorem 12. *A stochastic process is divisible if, and only if,*

$$\left\| \sum_{x'} p(x, t|x', t') v(x') \right\|_1 \leq \|v\|_1, \tag{4.18}$$

for every vector $v = (v(x))^g$ and $t \geq t'$.

Proof. If the process is divisible, then

$$\begin{aligned}
\left\| \sum_{x'} p(x, t|x', t') v(x') \right\|_1 &= \sum_x \left| \sum_{x'} p(x, t|x', t') v(x') \right| \\
&\leq \sum_x \left| \sum_{x'} p(x, t|x', t') \right| |v(x')| \\
&\leq \sum_{x'} \left(\sum_x p(x, t|x', t') \right) |v(x')| \\
&= \sum_{x'} |v(x')| = \|v\|_1,
\end{aligned} \tag{4.19}$$

where we used the positivity and the normalization of the conditional probabilities.

Conversely, if $p(x, t|x', t')$ is well defined then it automatically satisfies the normalization condition and the Chapman-Kolmogorov equation, so we only have to prove

^g Note that x labels the coordinates of the vector. Also, this vector is not necessarily a probability vector.

positivity. Using $\mathbf{v} = \mathbf{p}$ in Eq. (4.18), where $\mathbf{p}$ is any probability vector,

$$\begin{aligned}
 \|\mathbf{p}\|_1 &= \sum_{x'} p(x') \\
 &= \sum_{x'} \left(\sum_x p(x, t|x', t') \right) p(x') \\
 &= \sum_x \left(\sum_{x'} p(x, t|x', t') p(x') \right) \\
 &\leq \sum_x \left| \sum_{x'} p(x, t|x', t') p(x') \right| \\
 &= \left\| \sum_{x'} p(x, t|x', t') p(x') \right\|_1 \\
 &\leq \|\mathbf{p}\|_1.
 \end{aligned} \tag{4.20}$$

This means that all terms in the above equation are the same, since they are "sandwiched" by $\|\mathbf{p}\|_1$. In particular,

$$\sum_x \left(\sum_{x'} p(x, t|x', t') p(x') \right) = \sum_x \left| \sum_{x'} p(x, t|x', t') p(x') \right|. \tag{4.21}$$

But Eq. (4.21) holds, for every probability vector $\mathbf{p}$, if and only if

$$\sum_{x'} p(x, t|x', t') p(x') \geq 0, \tag{4.22}$$

which implies that $p(x, t|x', t') \geq 0$, since the components of a probability vector are always positive. Therefore, the process is divisible. $\square$

4.1.2 Correlation functions

The *mean (temporal) value* of a stochastic process is the quantity

$$\langle X(t) \rangle = \sum_x x(t) p(x, t), \tag{4.23}$$

and the (two-time) *correlation function* of two random variables $X(t)$ and $Y(t)$ is defined as

$$\langle X(t)Y(t') \rangle = \sum_{x,y} x(t)y(t') p(x, t; y, t'). \tag{4.24}$$

The correlation function, when applied to the same random variable $X(t)$, yields the *autocorrelation function* of it:

$$\mathcal{S}(t, t') \equiv \langle X(t)X(t') \rangle = \sum_{x,x'} x(t)x'(t') p(x, t; x', t'). \tag{4.25}$$

When the autocorrelation function depends only on the difference $|t - t'|$, i.e.,

$$\mathcal{S}(t, t') = \mathcal{S}(t - t') \equiv S(\tau), \tag{4.26}$$

we say that the process is homogeneous in time. Note that this implies the symmetry property

$$S(\tau) = S(-\tau). \quad (4.27)$$

In view of that, we usually define autocorrelation functions for homogeneous processes only for positive values of τ , and the negative case simply follows from this symmetry.

Higher-order correlation and autocorrelation functions are defined in the same fashion as above. Once again, we need the whole hierarchy of autocorrelation functions in order to describe a stochastic process; there is, however, an special class of processes for which the mean and the autocorrelation functions are sufficient: the *Gaussian random processes*. These processes are completely characterized by their (two-time) autocorrelation functions, since all higher order correlations are zero. In this text we will use the following Gaussian homogeneous processes, where the *amplitude* Ω and *correlation time* τ_c are real parameters, and $\alpha > 2$:

a) (Ornstein-Uhlenbeck) Exponential noise:

$$\mathcal{S}_{OU}(\tau) = \frac{\Omega}{2\tau_c} e^{-\tau/\tau_c} \quad (4.28)$$

b) Squared exponential noise:

$$\mathcal{S}_{SE}(\tau) = \frac{\Omega}{\sqrt{\pi}\tau_c} e^{-(\tau/\tau_c)^2} \quad (4.29)$$

c) Power law noise:

$$\mathcal{S}_{PL}(\tau) = \frac{\Omega}{\tau_c} \frac{(\alpha - 1)}{2} \frac{1}{(\tau/\tau_c + 1)^\alpha} \quad (4.30)$$

It is interesting to also consider complex random variables $Z(t)$. In this case, the autocorrelation functions are now

$$\mathcal{S}(t, t') = \langle z^*(t) z(t') \rangle, \quad (4.31)$$

and the symmetry property for homogeneous processes becomes

$$\mathcal{S}(\tau) = \mathcal{S}^*(-\tau). \quad (4.32)$$

4.2 Classical definition

A stochastic process is a Markovian process if, for any family of time-conditional probabilities, we have that

$$p(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0) = p(x_n, t_n | x_{n-1}, t_{n-1}). \quad (4.33)$$

In other words, the future state of a Markovian process only depends on the present state, so it has no memory of its past.

Theorem 13. *Markovian processes are divisible.*

Proof. The conditional probabilities $p(x, t|x', t')$ are well defined for all $t' \leq t$ and stochastic, by Eq. (4.33). For $t \geq t'' \geq t'$,

$$\begin{aligned} p(x, t; x'', t''; x', t') &= p(x, t|x'', t''; x', t')p(x'', t''; x', t') \\ &= p(x, t|x'', t''; x', t')p(x'', t''|x', t')p(x', t'). \end{aligned} \quad (4.34)$$

Using the Markov property Eq. (4.33) on $p(x, t|x'', t''; x', t')$, we have that

$$p(x, t; x'', t''; x', t') = p(x, t|x'', t'')p(x'', t''|x', t')p(x', t'). \quad (4.35)$$

Performing a summation on x'' ,

$$\begin{aligned} \sum_{x''} p(x, t|x'', t'')p(x'', t''|x', t')p(x', t') &= \sum_{x''} p(x, t; x'', t''; x', t') \\ &= p(x, t; x', t') \\ &= p(x, t|x', t')p(x', t'). \end{aligned} \quad (4.36)$$

Dividing both sides by $p(x', t')$ leads to the Chapman-Kolmogorov equation. Therefore, the process is divisible. $\square$

There are divisible processes which are non-Markovian (10), so the concepts of divisibility and Markovianity are not equivalent. They are, however, equivalent at the level of 1-point probabilities $p(x, t|x', t')$, since a such a family of matrices, if divisible, can always be seen as the conditional probabilities of some Markovian process. (10) To distinguish both concepts, therefore, we need the whole hierarchy of probabilities.

Among stochastic processes, Markovian processes are the exception, not the rule. (146) They are, however, the most studied processes in the theory, due to their striking simplicity. Namely, by the Markov condition Eq. (4.33), the whole hierarchy of time-joint probabilities can be recovered from the knowledge of $p(x_0, t_0)$ and $p(x_1, t_1|x_0, t_0)$. For example,

$$p(x_1, t_1; x_0, t_0) = p(x_1, t_1|x_0, t_0)p(x_0, t_0), \quad (4.37)$$

$$p(x_2, t_2; x_1, t_1; x_0, t_0) = p(x_2, t_2|x_1, t_1)p(x_1, t_1|x_0, t_0)p(x_0, t_0), \quad (4.38)$$

and so on. More generally,

$$p(x_n, t_n; \dots; x_0, t_0) = \prod_{i=0}^{n-1} p(x_{i+1}, t_{i+1}|x_i, t_i)p(x_0, t_0). \quad (4.39)$$

4.3 Quantum definitions

The natural generalization of a probability vector to the quantum realm is a density operator, since it represents an ensemble of wavefunctions

$$\rho = \sum_x p(x) |\psi(x)\rangle \langle \psi(x)|, \quad (4.40)$$

where $\sum_x p(x) = 1$ and $p(x) \geq 0$, i.e., the $p(x)$ form a probability vector $\mathbf{p}$. Therefore, the evolution equation

$$p(x, t) = \sum_{x_0} p(x, t | x_0, t_0) p(x_0, t_0) \quad (4.41)$$

becomes

$$\rho(t) = \mathcal{E}_{(t, t_0)} \rho(t_0), \quad (4.42)$$

where $\mathcal{E}_{(t, t_0)}$, the quantum counterpart of $p(x, t | x_0, t_0)$, is a linear superoperator. The left-normalization condition implies that it is trace-preserving, and the positivity condition implies that it is a completely positive map. Therefore, $\mathcal{E}_{(t, t_0)}$ is a quantum dynamical map.

The intermediate maps $\mathcal{E}_{(t, t')}$ are in general not well defined for a quantum evolution, but can be defined easily if $\mathcal{E}_{(t, t_0)}$ is invertible, which is analogous to the classical case. The Chapman-Kolmogorov equation becomes

$$\mathcal{E}_{(t, t')} = \mathcal{E}_{(t, t'')} \mathcal{E}_{(t'', t')}, \quad (4.43)$$

so the divisibility condition in classical probability becomes the CP-divisibility condition in quantum mechanics. Also, theorem 12 becomes theorem 10 for CP-divisible maps, since the classical 1-norm becomes the trace norm.

Finally, in order to define Markovian processes in quantum theory, we have to generalize the notion of time-conditional probabilities, i.e., the $p(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0)$. However, for $n > 1$, in order to define them we have to measure the state of the system at several different times. But in quantum mechanics measurements affect the state of the system, so the definition of the time-conditional probabilities would depend not only on the dynamics of the system, but also on the measurement scheme chosen to "sample" its state at different times. (10) Therefore, we do not have a satisfactory generalization of these time-conditional probabilities to the quantum case, which compromises the very definition of Markovianity. How to circumvent this issue? The solution is to define quantum Markovian processes in a different way, but which resembles the classical definition or has qualitatively the same features. There are several of these definitions in the literature (1, 2, 4, 19, 70–72), but we have chosen here to focus on the two most famous ones, the Rivas, Huelga and Plenio (RHP) or divisibility definition (17) and the Breuer, Lane and Piilo (BLP), indistinguishability or trace distance definition. (140, 141)

4.3.1 RHP definition

As we have seen, at the level of 1-point conditional probabilities, classical Markovian and divisible processes are equivalent. With that in mind, Rivas, Huelga and Plenio defined a quantum Markovian process as a CP-divisible map. (17) This definition is mathematically closer to the classical one, since it preserves many of its mathematical properties, such as the positivity (now complete positivity) of the intermediate dynamics and the obedience to the Chapman-Kolmogorov equation (which simply becomes the composition of dynamical maps).

A family of CP-divisible maps also forms an evolution family, reducing to a one-parameter semigroup when the maps are homogeneous in time. Some texts consider quantum non Markovian processes as quantum dynamical semigroups (4), which is a special case of the RHP definition.

4.3.2 BLP definition

Markovian processes are in practice memoryless processes, or processes where information is monotonically decreasing. Therefore, a more physical way to define quantum Markovianity is with respect to the monotonicity of some information measure. In that way, for a Markovian process this measure would only decrease, indicating that no memory of past states is being kept (or "remembered").

Breuer, Lane and Piilo defined a quantum Markovian process as any evolution where the trace distance between any two states is monotonically decreasing. (140, 141) The trace distance measures the indistinguishability of two states (125), so an increase in its value indicates that some information is flowing back to the system (which is used to tell the states apart), therefore breaking the memoryless property. More precisely, if Alice prepares two states ρ_1 and ρ_2 with equal probabilities and gives one to Bob, the maximal probability that he can correctly identify his state is equal to (141)

$$p_{max} = \frac{1}{2} (1 + D_1(\rho_1, \rho_2)), \quad (4.44)$$

where $D_1(\rho_1, \rho_2)$ is the trace distance between ρ_1 and ρ_2 ,

$$D_1(\rho_1, \rho_2) = \frac{1}{2} \|\rho_1 - \rho_2\|_1. \quad (4.45)$$

The trace distance is monotonically decreasing for quantum channels, and in particular for CP-divisible maps. Therefore, RHP Markovian processes are BLP Markovian too. The converse statement, however, is not true in general. In order to see that, remember theorem 10, which, in particular, shows that a quantum dynamical map $\mathcal{E}_{(t,t_0)} \in \mathbf{L}^2(\mathcal{H})$ is P-divisible if and only if the trace norm is monotonically decreasing with time, i.e.,

$$\frac{d}{dt} \left\| \mathcal{E}_{(t,t_0)}(X) \right\|_1 \leq 0, \quad (4.46)$$

for any Hermitian $X \in \mathbf{L}(\mathcal{H})$. Since the difference of any two quantum states $\rho_1 - \rho_2$ is Hermitian, we have that for invertible dynamical maps, P-divisibility is equivalent to

$$\frac{d}{dt} \left\| \mathcal{E}_{(t,t_0)}(\rho_1 - \rho_2) \right\|_1 \leq 0, \quad (4.47)$$

for any $\rho_1, \rho_2 \in \mathbf{D}(\mathcal{H})$. Therefore, invertible maps are BLP Markovian if and only if they are P-divisible. (103, 125) Since CP-divisibility is a special case of P-divisibility, RHP Markovian processes are BLP Markovian, but not the converse. Equivalently, BLP non-Markovianity is a special case of RHP non-Markovianity.

4.4 Detecting and measuring quantum non-Markovianity

In this section we are interested in witnesses and measures of quantum non-Markovianity. A witness of quantum non-Markovianity is a quantity that returns nonzero values only for non-Markovian processes, so it can be used to detect non-Markovian effects. When the witness is zero, however, nothing can be said about the dynamics. To that aid we introduce also non-Markovianity measures, which are witnesses that are zero if and only if the evolution is Markovian. Their value can also attest "how much" non-Markovian a process is, or at least say whether some process is "more non-Markovian" than another. Since we work with two inequivalent definitions of quantum Markovianity, namely the RHP and BLP definitions, some non-Markovianity measure related to one definition may be a witness for the other, or not even that, so care is taken to explicitly state which definition is being used.

There is a plethora of quantum non-Markovianity measures and witnesses in the literature (10, 17–19, 68, 71, 103, 140, 141, 147–150), so we choose here a few simple and elucidative ones.

4.4.1 Divisibility

As we have seen, RHP Markovianity is equivalent to CP-divisibility. Therefore, a way to measure RHP quantum non-Markovianity is by seeing how much the dynamics deviates from divisibility. First, note that a map $\mathcal{E}$ is a quantum channel if and only if its normalized Choi matrix, $d^{-1}\Lambda_{\mathcal{E}}$ (d the dimension of the system), is a quantum state, i.e., positive and with unit trace, and that the trace norm of an operator is equal to its trace if and only if it is positive. Putting together the two conditions, we have that $\|\Lambda_{\mathcal{E}}\|_1 \geq d$, with equality if and only if the map $\mathcal{E}$ is a quantum channel. Now, for a given family of bijective dynamical maps $\mathcal{E}_{(t,t_0)}$, from the initial time t_0 to the final time t_f , we have that

$$\mathcal{E}_{(t,t_0)} = \mathcal{E}_{(t,t')}\mathcal{E}_{(t',t_0)}, \quad (4.48)$$

for any $t' \in [t_0, t]$, where

$$\mathcal{E}_{(t,t')} = \mathcal{E}_{(t,t_0)}\mathcal{E}_{(t',t_0)}^{-1}, \quad (4.49)$$

which is well-defined, since the maps are bijective. The dynamics is CP-divisible if the intermediate dynamics $\mathcal{E}_{(t,t')}$ is a quantum channel, for every t' in the evolution interval. Based on the previous remark, we can measure its deviation from complete positivity by the quantity

$$\|\Lambda_{\mathcal{E}_{(t,t')}}\|_1 - d, \quad (4.50)$$

which is zero only for CPTP maps. In order to "scan" over all $t' \in [t_0, t_f]$, we introduce the quantity (17)

$$g(t) = \lim_{\epsilon \rightarrow 0^+} \frac{d^{-1} \|\Lambda_{\mathcal{E}_{(t+\epsilon, t)}}\|_1 - 1}{\epsilon}, \quad (4.51)$$

which will be positive, for some t , if and only if the evolution is non-Markovian in the RHP sense. Therefore, the RHP measure is defined as (10)

$$\mathcal{M}_{\text{RHP}} = \int_{t_0}^{t_f} g(t) dt. \quad (4.52)$$

It is a measure of RHP non-Markovianity, which implies that all BLP non-Markovian processes can be measured by it too. However, some BLP Markovian processes can return nonzero values, so it is not a suitable measure for this type of non-Markovianity.

4.4.2 Trace distance

Recall that a evolution is BLP Markovian if the trace distance between any two states is monotonically decreasing in time. Therefore, the evolution is non-Markovian if, for some ρ_1 and ρ_2 , for at least some time t during the evolution, the trace distance $D_1(\rho_1(t), \rho_2(t))$ increases, which means that

$$\frac{d}{dt} D_1(\rho_1(t), \rho_2(t)) > 0 \quad (4.53)$$

Defining

$$\sigma(\rho_1, \rho_2, t) = \max \left\{ 0, \frac{d}{dt} D_1(\rho_1(t), \rho_2(t)) \right\}, \quad (4.54)$$

this quantity is nonzero only when the derivative is positive. Then, the BLP measure is defined as (140, 141)

$$\mathcal{M}_{\text{BLP}} = \max_{\rho_1, \rho_2} \int_{t_0}^{t_f} \sigma(\rho_1, \rho_2, t) dt, \quad (4.55)$$

where the integral is taken over all intervals on which the derivative is positive, since $\sigma(\rho_1, \rho_2, t)$ is null when the derivative is negative, and the maximization procedure is over all possible initial states ρ_1 and ρ_2 . This measure is nonzero for both RHP and BLP non-Markovian processes, but it is zero for some RHP non-Markovian evolutions, since there are non CP-divisible processes which are P-divisible. Therefore, it is only a witness for RHP non-Markovianity.

The BLP condition is hard to check, since the whole set of possible initial states must be analyzed. For qubits, however, the criterion assumes a simpler form. In order to

see that, consider two infinitesimally separated density operators, ρ and $\rho + \delta\rho$. In the Bloch representation, Eq. (3.111), they assume the form

$$\rho = \frac{1}{2} \left(\mathbb{1} + \sum_i n_i \sigma_i \right), \quad (4.56)$$

$$\rho + \delta\rho = \frac{1}{2} \left(\mathbb{1} + \sum_i (n_i + \delta n_i) \sigma_i \right), \quad (4.57)$$

and the squared trace distance between them is

$$D_1^2(\rho + \delta\rho, \rho) = \frac{1}{4} \|\delta\rho\|_1^2. \quad (4.58)$$

A simple calculation shows that the eigenvalues of $\delta\rho$ are

$$\lambda_{\pm} = \pm \frac{1}{2} |\delta\vec{n}| = \pm \frac{1}{2} \sqrt{\delta n_1^2 + \delta n_2^2 + \delta n_3^2}, \quad (4.59)$$

so the trace norm is equal to

$$\|\delta\rho\|_1 = |\lambda_+| + |\lambda_-| = |\delta\vec{n}|. \quad (4.60)$$

Therefore,

$$D_1^2(\rho + \delta\rho, \rho) = \frac{1}{4} |\delta\vec{n}| |\delta\vec{n}| = \frac{1}{4} \delta\vec{n}^T \delta\vec{n} \quad (4.61)$$

If the density operators evolve in time under some (invertible) dynamical map, then they obey a time-local master equation, which can be written in the Bloch representation, Eq. (3.123),

$$\dot{\vec{n}}(t) = \vec{u}(t) + D(t)\vec{n}(t), \quad (4.62)$$

$$\dot{\vec{n}} + \delta\dot{\vec{n}}(t) = \vec{u}(t) + D(t)(\vec{n}(t) + \delta\vec{n}(t)), \quad (4.63)$$

which means that $\delta\dot{\vec{n}}(t) = D(t)\delta\vec{n}(t)$. Now, the derivative of the squared trace distance is

$$\begin{aligned} \frac{d}{dt} D_1^2(\rho + \delta\rho, \rho)(t) &= \frac{1}{4} \left[\delta\dot{\vec{n}}(t)^T \delta\vec{n}(t) + \delta\vec{n}(t)^T \delta\dot{\vec{n}}(t) \right] \\ &= \frac{1}{4} \left[\delta\vec{n}(t)^T \left(D(t) + D^T(t) \right) \delta\vec{n}(t) \right], \end{aligned} \quad (4.64)$$

which means that it can be positive only when $D(t) + D^T(t)$ has a positive eigenvalue. (68)

The trace distance between any two states can increase only when it can increase between any two infinitesimally close states, so a qubit evolution is BLP non-Markovian when (68)

$$\lambda_{\max} \left[D(t) + D^T(t) \right] > 0, \quad (4.65)$$

where $\lambda_{\max}[X]$ is the maximum eigenvalue of X .

4.4.3 Bloch volume

Remember that any finite-dimensional quantum state can be written in the matrix representation, Eq. (3.108),

$$\rho(t) = \frac{1}{d} \left(\mathbb{1} + \sum_{i=1}^{d^2-1} x_i(t) G_i \right). \quad (4.66)$$

Its master equation and dynamical map can be written as, respectively,

$$\dot{\vec{x}}(t) = \vec{u}(t) + D(t)\vec{x}(t), \quad (4.67)$$

and

$$\vec{x}(t) = \vec{w}(t, t_0) + B(t, t_0)\vec{x}(t_0), \quad (4.68)$$

and $B(t, t_0)$ and $D(t)$ are related by

$$\dot{B}(t, t_0) = D(t)B(t, t_0). \quad (4.69)$$

Equation (4.68) determines the states accessible by this dynamics, i.e., the set of all vectors $\vec{x}(t)$ which can be reached from some set of initial states $\vec{x}(t_0)$. It can be shown (147) that the volume of the set of accessible states evolves as

$$V(t) = V(t_0) \text{Det}(B(t, t_0)), \quad (4.70)$$

where Det is the determinant of a matrix. By Jacobi's formula (151),

$$\begin{aligned} \frac{d}{dt} \text{Det} B(t, t_0) &= \text{tr} \left[\frac{d}{dt} B(t, t_0) B(t, t_0)^{-1} \right] \text{Det} B(t, t_0) \\ &= \text{tr} [D(t) B(t, t_0) B(t, t_0)^{-1}] \text{Det} B(t, t_0) \\ &= \text{tr} [D(t)] \text{Det} B(t, t_0), \end{aligned} \quad (4.71)$$

so $\text{Det}(B(t, t_0))$, and therefore the volume of accessible states, can only increase if $\text{tr} [D(t)] \geq 0$. Now, for CP-divisible dynamics this volume is monotonically decreasing (12), so Lorenzo, Plastina and Paternostro (147) used this fact to define the *Bloch volume witness* of non-Markovianity: an evolution is RHP non-Markovian if the volume of the set of accessible states increases for some time. By the above considerations, that happens only when the trace of the damping matrix is positive. (68) Now we can use Eq. (3.93),

$$\text{tr}[D(t)] = -d \sum_{k=1}^{d^2-1} \gamma_k(t), \quad (4.72)$$

where $\gamma_k(t)$ are the decay rates of the master equation of the system, Eq. (3.67), and d its dimension. Therefore, we can witness RHP non-Markovianity when the sum of the decay rates is strictly negative,

$$\sum_{k=1}^{d^2-1} \gamma_k(t) < 0, \quad (4.73)$$

which is a stronger condition than the required for RHP non-Markovianity. Therefore, even if we have negative decay rates, the Bloch volume (the volume of accessible states) only increases if their contribution is larger than the contribution of the positive decay rates.

4.4.4 Decay rates

By theorem 11, a dynamics generated by a Lindblad-like master equation, Eq. (3.67), is CP-divisible, and therefore RHP Markovian, if and only if all decay rates are nonnegative. Noting that, Andersson, Cresser, Hall and Li (ACHL) defined a non-Markovianity measure based on the negativity of the decay rates. (68) For each decay rate $\gamma_i(t)$, define

$$f_i(t) = \max[0, -\gamma_i(t)] = \frac{1}{2}[|\gamma_i(t)| - \gamma_i(t)], \quad (4.74)$$

which is zero if the decay rate is positive, and equal to its absolute value if it is negative. Therefore, it measures the contribution of $\gamma_i(t)$ to the non-Markovian character of the evolution. The total contribution is measured by

$$f(t) = \sum_i f_i(t), \quad (4.75)$$

and the ACHL or *decay rates measure* is defined as (68)

$$\mathcal{M}_{\text{ACHL}} = \int_{t_0}^{t_f} f(t) dt. \quad (4.76)$$

The measure is well defined since Eq. (3.67) is in canonical form, and therefore the decay rates are uniquely defined. For some general, non-canonical master equation in Lindblad-like form, the measure can indicate non-Markovianity even when the evolution is unitary (68), so care must be taken when using it.

Since both RHP and ACHL measures are based on deviation from CP-divisibility, one could ask if they are somewhat related. In fact, the measures are proportional, being equal for two-dimensional systems.

Theorem 14. *We have that*

$$\mathcal{M}_{\text{ACHL}} = \frac{d}{2} \mathcal{M}_{\text{RHP}}, \quad (4.77)$$

where d is the dimension of the system. (68)

Proof. We will prove that $f(t) = \frac{d}{2}g(t)$, where

$$f(t) = \frac{1}{2} \sum_i [|\gamma_i(t)| - \gamma_i(t)] \quad (4.78)$$

and

$$g(t) = \lim_{\epsilon \rightarrow 0^+} \frac{d^{-1} \|\Lambda_{\mathcal{E}_{(t+\epsilon, t)}}\|_1 - 1}{\epsilon}, \quad (4.79)$$

which implies in Eq. (4.77). Recall Eq. (3.98),

$$\mathcal{E}_{(t+\epsilon, t)} = \mathcal{I} + \epsilon \mathcal{L}_t, \quad (4.80)$$

and this means that

$$g(t) = \lim_{\epsilon \rightarrow 0^+} \frac{d^{-1} \|\Lambda_{\mathcal{I}} + \epsilon \Lambda_{\mathcal{L}_t}\|_1 - 1}{\epsilon} = \lim_{\epsilon \rightarrow 0^+} \frac{d^{-1} \|S\|_1 - 1}{\epsilon}, \quad (4.81)$$

where $S = \Lambda_{\mathcal{I}} + \epsilon \Lambda_{\mathcal{L}_t} = |\Omega\rangle \langle \Omega| + \epsilon \Lambda_{\mathcal{L}_t}$ (we suppress the time dependency of S for convenience) and $|\Omega\rangle$ is a (unnormalized) maximally mixed state on two copies of the system. It is important to emphasize that $|\Omega\rangle$ is not normalized, so several factors of d^{-1} will appear due to this fact. Since S is Hermitian, we have that $\|S\|_1 = \sum_i |\lambda_i|$, where λ_i are its eigenvalues. Therefore, our first goal is to compute the spectrum of S .

First of all, note that

$$\text{tr}[S] = \text{tr}[|\Omega\rangle \langle \Omega|] + \epsilon \text{tr}[\Lambda_{\mathcal{L}_t}] = d + \epsilon \text{tr}[\Lambda_{\mathcal{L}_t}] = d, \quad (4.82)$$

which means that $\text{tr}[\Lambda_{\mathcal{L}_t}] = 0$. Now define the projection onto the direction of $|\Omega\rangle$,

$$P_{\Omega} = d^{-1} |\Omega\rangle \langle \Omega|, \quad (4.83)$$

so

$$\Lambda_{\mathcal{L}_t} = P_{\Omega} \Lambda_{\mathcal{L}_t} P_{\Omega} + (\mathbb{1} - P_{\Omega}) \Lambda_{\mathcal{L}_t} (\mathbb{1} - P_{\Omega}) \equiv P_{\Omega} \Lambda_{\mathcal{L}_t} P_{\Omega} + \Lambda_{\mathcal{L}_t}^{\perp}, \quad (4.84)$$

where $\Lambda_{\mathcal{L}_t}^{\perp} = (\mathbb{1} - P_{\Omega}) \Lambda_{\mathcal{L}_t} (\mathbb{1} - P_{\Omega})$ is the projection of $\Lambda_{\mathcal{L}_t}$ onto the subspace orthogonal to $|\Omega\rangle$. Since it is Hermitian, $\Lambda_{\mathcal{L}_t}$ has real eigenvalues, and we denote its spectral decomposition by $\Lambda_{\mathcal{L}_t} = \sum_i c_i |\omega_i\rangle \langle \omega_i|$. Since each $|\omega_i\rangle$ lies in the eigensubspace of $(\mathbb{1} - P_{\Omega})$, we have that $(\mathbb{1} - P_{\Omega}) |\omega_i\rangle = |\omega_i\rangle$, which implies that

$$\langle \omega_i | \Lambda_{\mathcal{L}_t} | \omega_i \rangle = \langle \omega_i | \Lambda_{\mathcal{L}_t}^{\perp} | \omega_i \rangle = c_i. \quad (4.85)$$

Now, since ϵ is very small, we can actually use first-order perturbation theory to compute the eigenvalues of $S = |\Omega\rangle \langle \Omega| + \epsilon \Lambda_{\mathcal{L}_t}$. The unperturbed operator $|\Omega\rangle \langle \Omega|$ has a eigenvalue of d with corresponding eigenvector $|\Omega\rangle$, so we can define the other eigenvalues as zero, with corresponding eigenvectors ω_i . The first perturbed eigenvalue is (75)

$$\lambda_1 = d + \epsilon d^{-1} \langle \Omega | \Lambda_{\mathcal{L}_t} | \Omega \rangle, \quad (4.86)$$

and the remaining are

$$\lambda_{i>1} = \epsilon \langle \omega_i | \Lambda_{\mathcal{L}_t} | \omega_i \rangle = \epsilon \langle \omega_i | \Lambda_{\mathcal{L}_t}^{\perp} | \omega_i \rangle = \epsilon c_i. \quad (4.87)$$

However, since $\text{tr}[\Lambda_{\mathcal{L}_t}] = 0$, then

$$\text{tr}[\Lambda_{\mathcal{L}_t}] = d^{-1} \langle \Omega | \Lambda_{\mathcal{L}_t} | \Omega \rangle + \sum_i \langle \omega_i | \Lambda_{\mathcal{L}_t} | \omega_i \rangle = 0, \quad (4.88)$$

so

$$d^{-1} \langle \Omega | \Lambda_{\mathcal{L}_t} | \Omega \rangle = - \langle \omega_i | \Lambda_{\mathcal{L}_t} | \omega_i \rangle = - \sum_i c_i. \quad (4.89)$$

We are ready to compute $\|S\|_1$:

$$\begin{aligned} \|S\|_1 &= \sum_{i=0} |\lambda_i| \\ &= |\lambda_1| + \sum_{i=1} |\lambda_i| \\ &= |d + \epsilon d^{-1} \langle \Omega | \Lambda_{\mathcal{L}_t} | \Omega \rangle| + \epsilon \sum_i |c_i| \\ &= d + \epsilon d^{-1} \langle \Omega | \Lambda_{\mathcal{L}_t} | \Omega \rangle + \epsilon \sum_i |c_i| \\ &= d - \epsilon \sum_i c_i + \epsilon \sum_i |c_i|. \end{aligned} \quad (4.90)$$

Therefore, $g(t)$ is

$$\begin{aligned} g(t) &= \lim_{\epsilon \rightarrow 0^+} \frac{d^{-1} \|S\|_1 - 1}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{1 - d^{-1} \epsilon \sum_i c_i + d^{-1} \epsilon \sum_i |c_i| - 1}{\epsilon} \\ &= d^{-1} \left[\sum_i |c_i| - \sum_i c_i \right]. \end{aligned} \quad (4.91)$$

The second part of the proof consists of relating the c_i with the decay rates $\gamma_i(t)$. The procedure is to arrive at a canonical master equation starting from $\Lambda_{\mathcal{L}_t}$. Since the decay rates are uniquely defined for canonical forms, the decay rates of this form will be equal to the decay rates of Eq. (3.67). To start with, note that

$$\begin{aligned} \Lambda_{\mathcal{L}_t} &= \mathbb{1} \Lambda_{\mathcal{L}_t} \mathbb{1} \\ &= (P_\Omega + (\mathbb{1} - P_\Omega)) \Lambda_{\mathcal{L}_t} (P_\Omega + (\mathbb{1} - P_\Omega)) \\ &= P_\Omega \Lambda_{\mathcal{L}_t} P_\Omega + P_\Omega \Lambda_{\mathcal{L}_t} (\mathbb{1} - P_\Omega) + (\mathbb{1} - P_\Omega) \Lambda_{\mathcal{L}_t} P_\Omega + \Lambda_{\mathcal{L}_t}^\perp \\ &= d^{-2} |\Omega\rangle (\langle \Omega | \Lambda_{\mathcal{L}_t} | \Omega \rangle) \langle \Omega | + |\Omega\rangle \left(d^{-1} \langle \Omega | \Lambda_{\mathcal{L}_t} (\mathbb{1} - P_\Omega) \right) \\ &\quad + \left(d^{-1} \Lambda_{\mathcal{L}_t} (\mathbb{1} - P_\Omega) | \Omega \rangle \right) \langle \Omega | + \Lambda_{\mathcal{L}_t}^\perp \\ &\equiv a |\Omega\rangle \langle \Omega | + |\Omega\rangle \langle \Psi | + |\Psi\rangle \langle \Omega | + \sum_i c_i |\omega_i\rangle \langle \omega_i|, \end{aligned} \quad (4.92)$$

where $a = \langle \Omega | \Lambda_{\mathcal{L}_t} | \Omega \rangle = -d \operatorname{tr} [\Lambda_{\mathcal{L}_t}^\perp]$ and $|\Psi\rangle = d^{-1} \Lambda_{\mathcal{L}_t} (\mathbb{1} - P_\Omega) | \Omega \rangle$.

Now we recover the Kraus decomposition of $\mathcal{L}_t$ from its Choi matrix, using theorem

2. The Kraus operators are

$$\sum_{\alpha,\beta} \langle \alpha, \beta | \Omega \rangle | \beta \rangle \langle \alpha | = \sum_{\alpha,\beta} \langle \alpha | \beta \rangle | \beta \rangle \langle \alpha | = \sum_{\alpha} | \alpha \rangle \langle \alpha | = \mathbb{1}, \quad (4.93)$$

$$\sum_{\alpha,\beta} \langle \alpha, \beta | \Psi \rangle | \beta \rangle \langle \alpha | \equiv W, \quad (4.94)$$

$$\sum_{\alpha,\beta} \langle \alpha, \beta | \omega_i \rangle | \beta \rangle \langle \alpha | \equiv W_i, \quad (4.95)$$

so

$$\begin{aligned} \mathcal{L}_t(\rho(t)) &= a\rho(t) + \rho(t)W^\dagger + W\rho(t) + \sum_i c_i W_i \rho(t) W_i^\dagger \\ &= \left(\frac{a}{2} + W \right) \rho(t) + \rho(t) \left(\frac{a}{2} + W \right) + \sum_i c_i W_i \rho(t) W_i^\dagger. \end{aligned} \quad (4.96)$$

The rest of the proof is very similar to the derivation of the Lindblad-like master equation Eq. (3.67). The term $\left(\frac{a}{2} + W \right)$ can be decomposed in Hermitian and anti-Hermitian parts, $\left(\frac{a}{2} + W \right) = J - \frac{i}{\hbar} K$, where J and K are Hermitian matrices. (68) Then,

$$\mathcal{L}_t(\rho(t)) = -\frac{i}{\hbar} [K, \rho(t)] + J\rho(t) + \rho(t)J + \sum_i c_i W_i \rho(t) W_i^\dagger. \quad (4.97)$$

Since $\mathcal{L}_t$ is a trace preserving map,

$$\text{tr} [\mathcal{L}_t(\rho(t))] = \text{tr} \left[\left(2J + \sum_i c_i W_i^\dagger W_i \right) \rho(t) \right] = \text{tr}[\rho(t)], \quad (4.98)$$

which implies that

$$J = -\frac{1}{2} \sum_i c_i W_i^\dagger W_i. \quad (4.99)$$

Finally, we have the canonical form (putting back all time dependences)

$$\mathcal{L}_t(\rho(t)) = -\frac{i}{\hbar} [K(t), \rho(t)] + \sum_i c_i(t) \left(W_i(t) \rho(t) W_i^\dagger(t) - \frac{1}{2} \{ W_i^\dagger(t) W_i(t), \rho(t) \} \right). \quad (4.100)$$

From the uniqueness of the decay rates in the canonical form, and comparing to Eq. (3.67), we have that $c_i(t) = \gamma_i(t)$. Therefore,

$$g(t) = \frac{1}{d} \left[\sum_i |\gamma_i(t)| - \sum_i \gamma_i(t) \right] = \frac{2}{d} f(t), \quad (4.101)$$

which completes the proof. $\square$

4.4.5 Summary

Here we recap the results of this section. Consider a d -dimensional Hilbert space X and a family of bijective dynamical maps $\mathcal{E}_{(t,t_0)} \in \mathbf{L}^2(\mathcal{X})$, from the initial time t_0 to the final time t_f , such that states evolve as

$$\rho(t) = \mathcal{E}_{(t,t_0)}[\rho(t_0)]. \quad (4.102)$$

The bijectivity implies that

$$\mathcal{E}_{(t,t_0)} = \mathcal{E}_{(t,t')}\mathcal{E}_{(t',t_0)}, \quad (4.103)$$

for any $t' \in [t_0, t]$, where

$$\mathcal{E}_{(t,t')} = \mathcal{E}_{(t,t_0)}\mathcal{E}_{(t',t_0)}^{-1}. \quad (4.104)$$

Also, it implies that the evolution can be described by a time-local master equation, which can be put in Lindblad-like form:

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H(t), \rho(t)] + \sum_{i=1}^{d^2-1} \gamma_i(t) \left(L_i(t)\rho(t)L_i^\dagger(t) - \frac{1}{2}\{L_i^\dagger(t)L_i(t), \rho(t)\} \right). \quad (4.105)$$

The Rivas, Huelga and Plenio (RHP) measure is

$$\mathcal{M}_{\text{RHP}} = \int_{t_0}^{t_f} g(t)dt, \quad (4.106)$$

where

$$g(t) = \lim_{\epsilon \rightarrow 0^+} \frac{d^{-1} \|\Lambda_{\mathcal{E}_{(t+\epsilon, t)}}\|_1 - 1}{\epsilon} \quad (4.107)$$

and $\Lambda_{\mathcal{E}_{(t+\epsilon, t)}}$ is the Choi matrix of the intermediate map $\mathcal{E}_{(t+\epsilon, t)}$. The Breuer, Lane and Piilo (BLP) measure is

$$\mathcal{M}_{\text{BLP}} = \max_{\rho_1, \rho_2} \int_{t_0}^{t_f} \sigma(\rho_1, \rho_2, t)dt, \quad (4.108)$$

where the maximization is performed over all possible initial states ρ_1 and ρ_2 and

$$\sigma(\rho_1, \rho_2, t) = \max \left\{ 0, \frac{d}{dt} D_1(\rho_1(t), \rho_2(t)) \right\}, \quad (4.109)$$

with $D_1(\rho_1(t), \rho_2(t))$ the trace distance between $\rho_1(t)$ and $\rho_2(t)$. The ACHL or decay rates measure is defined as

$$\mathcal{M}_{\text{ACHL}} = \int_{t_0}^{t_f} f(t)dt, \quad (4.110)$$

where

$$f(t) = \frac{1}{2} \sum_i [|\gamma_i(t)| - \gamma_i(t)]. \quad (4.111)$$

Note that the ACHL and RHP measures are proportional:

$$\mathcal{M}_{\text{ACHL}} = \frac{d}{2} \mathcal{M}_{\text{RHP}}. \quad (4.112)$$

Finally, the Bloch volume witness detects quantum non-Markovianity when

$$\sum_{k=1}^{d^2-1} \gamma_k(t) < 0. \quad (4.113)$$

For qubits, in the Bloch representation,

$$\rho(t) = \frac{1}{2} \left(\mathbb{1} + \sum_{i=1}^3 n_i(t) \sigma_i \right), \quad (4.114)$$

the master equation can be written as

$$\dot{\vec{n}}(t) = \vec{u}(t) + D(t)\vec{n}(t), \quad (4.115)$$

where

$$u_j(t) = \frac{1}{2} \text{tr} [\sigma_j \mathcal{L}_t [\mathbb{1}]], \quad (4.116)$$

$$D_{jk}(t) = \frac{1}{2} \text{tr} [\sigma_j \mathcal{L}_t [\sigma_k]], \quad (4.117)$$

are the drift vector and damping matrix, respectively. According to the BLP criterion, the evolution is non-Markovian when

$$\lambda_{\max} [D(t) + D^T(t)] > 0, \quad (4.118)$$

where $\lambda_{\max}[X]$ is the maximum eigenvalue of X .

5 STOCHASTIC MASTER EQUATIONS

In this chapter we study quantum dynamics under the influence of classical stochastic fields. (60,61) The stochastic wave function formalism (3) is used, where the state of an open quantum system is described by an ensemble of pure states (60)

$$\rho(t) = |\Psi(t)\rangle \langle \Psi(t)|. \quad (5.1)$$

The evolution of the system is recovered after an average over the stochastic realizations (60,69):

$$\rho_S(t) = \overline{\rho(t)}. \quad (5.2)$$

Therefore, the average of the stochastic operator $\rho(t)$ coincides with the system density operator $\rho_S(t)$.

5.1 The Gaussian noise model

In this model, we work in a Hilbert space $\mathcal{H}_S$ and with a Hamiltonian

$$H(t) = H_S + V(t), \quad (5.3)$$

where H_S is the system Hamiltonian and $V(t)$ represents the stochastic field. This term has the form

$$V(t) = \sum_{\alpha} \xi_{\alpha}(t) V_{\alpha}, \quad (5.4)$$

where V_{α} are operators on $\mathcal{H}_S$ and the $\xi_{\alpha}(t)$ are complex stochastic processes. We assume that these processes are complex Gaussian processes with zero mean value, i.e,

$$\overline{\xi_{\alpha}(t)} = 0, \quad (5.5)$$

$$\overline{\xi_{\alpha}(t) \xi_{\beta}(t')} = \overline{\xi_{\alpha}^*(t) \xi_{\beta}^*(t')} = 0, \quad (5.6)$$

$$\overline{\xi_{\alpha}^*(t) \xi_{\beta}(t')} \equiv \chi_{\alpha\beta}(t, t'), \quad (5.7)$$

for every α, β and every t and t' in the evolution interval, and where the bar denotes an average over the stochastic realizations. (60,69) The last term represents the correlation functions of the noise. Since the process is Gaussian, all higher order correlations are zero. Also, since the noise operator $V(t)$ is Hermitian, it follows that

$$\sum_{\alpha} \xi_{\alpha}(t) V_{\alpha} = \sum_{\alpha} \xi_{\alpha}^*(t) V_{\alpha}^{\dagger}. \quad (5.8)$$

The Liouville-von Neumann equation for this system is

$$\frac{d}{dt} \rho(t) = -i [H(t), \rho(t)], \quad (5.9)$$

whose solution, in terms of a family of propagators, is given by

$$\rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0), \quad (5.10)$$

where the propagator is given by

$$U(t, t_0) = \mathcal{T} \left[e^{-i \int_{t_0}^t d\tau (H_S + V(\tau))} \right]. \quad (5.11)$$

Moving to the interaction picture, Eq. (5.9) has the form

$$\frac{d}{dt}\tilde{\rho}(t) = -i [\tilde{V}(t), \tilde{\rho}(t)], \quad (5.12)$$

where

$$\tilde{V}(t) = \sum_{\alpha} \xi_{\alpha}(t) \tilde{V}_{\alpha}, \quad (5.13)$$

and the solution of Eq. (5.12) is

$$\tilde{\rho}(t) = \tilde{U}(t, t_0)\tilde{\rho}(t_0)\tilde{U}^\dagger(t, t_0), \quad (5.14)$$

where

$$\tilde{U}(t, t_0) = e^{iH_S t} U(t, t_0) e^{-iH_S t_0}. \quad (5.15)$$

The formulas for the interaction picture are demonstrated in Appendix C. Taking the average over stochastic realizations in Eq. (5.12) leads to

$$\frac{d}{dt}\tilde{\rho}_S(t) = \frac{d}{dt}\overline{\tilde{\rho}(t)} = -i \left[\overline{\tilde{V}(t)\tilde{\rho}(t)} - \overline{\tilde{\rho}(t)\tilde{V}(t)} \right]. \quad (5.16)$$

To evaluate the terms $\overline{\tilde{V}(t)\tilde{\rho}(t)}$ and $\overline{\tilde{\rho}(t)\tilde{V}(t)}$ we use Novikov's theorem. (152, 153) The average of a noise $\xi(t)$ and a functional $\mathcal{M}[\xi]$ is given by

$$\overline{\xi(t)\mathcal{M}[\xi]} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n C_{n+1}(t, t_1, \dots, t_n) \overline{\frac{\delta^n \mathcal{M}[\xi]}{\delta \xi(t_1) \dots \delta \xi(t_n)}}, \quad (5.17)$$

where the C_n are cumulants (69) of the noise and the last term is a n -th functional derivative (153) of the functional with respect to ξ . We consider a special case of Novikov's theorem, which applies to complex Gaussian noises with zero mean value (154):

$$\overline{\xi_{\alpha}(t)\mathcal{M}[\{\xi_{\gamma}(t)\}]} = \sum_{\beta} \int_{t_0}^t dt' \overline{\xi_{\alpha}(t)\xi_{\beta}^*(t')} \overline{\frac{\delta \mathcal{M}[\{\xi_{\gamma}(t)\}]}{\delta \xi_{\beta}^*(t')}}, \quad (5.18)$$

where $\mathcal{M}$ is a functional of the set of noises $\{\xi_{\gamma}(t)\}$, of which ξ_{α} is part, and the sum in the right hand side is performed over this set. Using Eq. (5.18) and Eq. (5.8),

$$\begin{aligned} \overline{\tilde{V}(t)\tilde{\rho}(t)} &= \sum_{\alpha} \tilde{V}_{\alpha}^{\dagger}(t) \overline{\xi_{\alpha}^*(t)\tilde{\rho}(t)} \\ &= \sum_{\alpha} \tilde{V}_{\alpha}^{\dagger}(t) \sum_{\beta} \int_{t_0}^t dt' \overline{\xi_{\alpha}^*(t)\xi_{\beta}(t')} \overline{\frac{\delta \tilde{\rho}(t)}{\delta \xi_{\beta}(t')}} \\ &= \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \tilde{V}_{\alpha}^{\dagger}(t) \overline{\frac{\delta \tilde{\rho}(t)}{\delta \xi_{\beta}(t')}}, \end{aligned} \quad (5.19)$$

and, similarly,

$$\overline{\tilde{\rho}(t)\tilde{V}(t)} = \sum_{\alpha,\beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \frac{\overline{\delta\tilde{\rho}(t)}}{\delta\xi_{\beta}(t')} \tilde{V}_{\alpha}^{\dagger}(t), \quad (5.20)$$

and the master equation is now

$$\frac{d}{dt}\tilde{\rho}_S(t) = -i \sum_{\alpha,\beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \left[\tilde{V}_{\alpha}^{\dagger}(t), \frac{\overline{\delta\tilde{\rho}(t)}}{\delta\xi_{\beta}(t')} \right]. \quad (5.21)$$

In order to proceed, the functional derivative must be evaluated.

5.1.1 Evaluation of the functional derivative

An integration of Eq. (5.12) leads to

$$\tilde{\rho}(t) = \tilde{\rho}(t_0) - i \int_{t_0}^t d\tau [\tilde{V}(\tau), \tilde{\rho}(\tau)]. \quad (5.22)$$

Taking the functional derivative with respect to $\xi_{\beta}(t')$, $t_0 < t' < t$,

$$\frac{\delta\tilde{\rho}(t)}{\delta\xi_{\beta}(t')} = \frac{\delta\tilde{\rho}(t_0)}{\delta\xi_{\beta}(t')} - i \int_{t_0}^t d\tau \left\{ \left[\frac{\delta\tilde{V}(\tau)}{\delta\xi_{\beta}(t')}, \tilde{\rho}(\tau) \right] + \left[\tilde{V}(\tau), \frac{\delta\tilde{\rho}(\tau)}{\delta\xi_{\beta}(t')} \right] \right\}, \quad (5.23)$$

where the chain rule for the functional derivative (153) was used. If we assume that the initial state $\rho(t_0)$ is uncorrelated with the noise, then the first term in the right hand side of the above equation is zero. Analogously, $\tilde{\rho}(\tau)$ cannot be affected by noises $\xi_{\beta}(t')$ when $\tau < t'$, and therefore the lower integration limit for the third term in the right hand side is t' . For the second term, using that (153)

$$\frac{\delta\xi_{\alpha}(\tau)}{\delta\xi_{\beta}(t')} = \delta_{\alpha\beta}\delta(\tau - t'), \quad (5.24)$$

we have

$$\frac{\delta\tilde{V}(\tau)}{\delta\xi_{\beta}(t')} = \sum_{\alpha} \left(\frac{\delta\xi_{\alpha}(\tau)}{\delta\xi_{\beta}(t')} \right) \tilde{V}_{\alpha}(\tau) = \tilde{V}_{\beta}(\tau)\delta(\tau - t'). \quad (5.25)$$

With the above results, the equation for the functional derivative is

$$\frac{\delta\tilde{\rho}(t)}{\delta\xi_{\beta}(t')} = -i \int_{t_0}^t d\tau \delta(\tau - t') [\tilde{V}_{\beta}(\tau), \tilde{\rho}(\tau)] - i \int_{t'}^t d\tau \left[\tilde{V}(\tau), \frac{\delta\tilde{\rho}(\tau)}{\delta\xi_{\beta}(t')} \right] \quad (5.26)$$

$$= -i [\tilde{V}_{\beta}(t'), \tilde{\rho}(t')] - i \int_{t'}^t d\tau \left[\tilde{V}(\tau), \frac{\delta\tilde{\rho}(\tau)}{\delta\xi_{\beta}(t')} \right]. \quad (5.27)$$

Note that, for $t = t'$,

$$\frac{\delta\tilde{\rho}(t')}{\delta\xi_{\beta}(t')} = -i [\tilde{V}_{\beta}(t'), \tilde{\rho}(t')], \quad (5.28)$$

which corresponds to our initial condition.

If we take the time derivative of Eq. (5.26), then

$$\frac{d}{dt} \frac{\delta\tilde{\rho}(t)}{\delta\xi_{\beta}(t')} = -i \left[\tilde{V}(t), \frac{\delta\tilde{\rho}(t)}{\delta\xi_{\beta}(t')} \right]. \quad (5.29)$$

which is an equation exactly like Eq. (5.12), and therefore has the same solution:

$$\begin{aligned} \frac{\delta \tilde{\rho}(t)}{\delta \xi_\beta(t')} &= \tilde{U}(t, t') \frac{\delta \tilde{\rho}(t')}{\delta \xi_\beta(t')} \tilde{U}^\dagger(t, t') \\ &= -i \tilde{U}(t, t') [\tilde{V}_\beta(t'), \tilde{\rho}(t')] \tilde{U}^\dagger(t, t'), \end{aligned} \quad (5.30)$$

where Eq. (5.28) was used in the last line. Since the $\tilde{U}(t, t')$ are unitary, i.e., $\tilde{U}^{-1}(t, t') = \tilde{U}^\dagger(t, t') = \tilde{U}(t', t)$, we can reverse the evolution in Eq. (5.14):

$$\tilde{\rho}(t') = \tilde{U}^\dagger(t, t') \tilde{\rho}(t) \tilde{U}(t, t'), \quad (5.31)$$

which leads to

$$\begin{aligned} \frac{\delta \tilde{\rho}(t)}{\delta \xi_\beta(t')} &= -i \tilde{U}(t, t') [\tilde{V}_\beta(t'), \tilde{U}^\dagger(t, t') \tilde{\rho}(t) \tilde{U}(t, t')] \tilde{U}^\dagger(t, t') \\ &= -i \tilde{U}(t, t') \tilde{V}_\beta(t') \tilde{U}^\dagger(t, t') \tilde{\rho}(t) \tilde{U}(t, t') \tilde{U}^\dagger(t, t') \\ &\quad + i \tilde{U}(t, t') \tilde{U}^\dagger(t, t') \tilde{\rho}(t) \tilde{U}(t, t') \tilde{V}_\beta(t') \tilde{U}^\dagger(t, t') \\ &= -i \tilde{U}(t, t') \tilde{V}_\beta(t') \tilde{U}^\dagger(t, t') \tilde{\rho}(t) + i \tilde{\rho}(t) \tilde{U}(t, t') \tilde{V}_\beta(t') \tilde{U}^\dagger(t, t') \\ &= -i [\tilde{U}(t, t') \tilde{V}_\beta(t') \tilde{U}^\dagger(t, t'), \tilde{\rho}(t)]. \end{aligned} \quad (5.32)$$

Now we have a formula for the functional derivative, and the master equation is

$$\frac{d}{dt} \tilde{\rho}_S(t) = - \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \left[\tilde{V}_\alpha^\dagger(t), \overline{[\tilde{U}(t, t') \tilde{V}_\beta(t') \tilde{U}^\dagger(t, t'), \tilde{\rho}(t)]} \right]. \quad (5.33)$$

5.1.2 Lindblad form

In order to evaluate the unitary operators $\tilde{U}(t, t')$, note that $\tilde{U}(t', t') = \mathbb{1}$. Therefore, assuming a weak coupling, we can retain only the zeroth-order contribution of the propagator, $\tilde{U}(t, t') \simeq \mathbb{1}$ (58):

$$\begin{aligned} \frac{d}{dt} \tilde{\rho}_S(t) &= - \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \left[\tilde{V}_\alpha^\dagger(t), \overline{[\tilde{V}_\beta(t'), \tilde{\rho}(t)]} \right] \\ &= - \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \left[\tilde{V}_\alpha^\dagger(t), [\tilde{V}_\beta(t'), \tilde{\rho}_S(t)] \right]. \end{aligned} \quad (5.34)$$

To evaluate the double commutator, two identities are used: Jacobi's identity,

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \quad (5.35)$$

and the relation (58)

$$[A, [B, C]] + [B, [A, C]] = \{\{A, B\}, C\} - 2(ACB + BCA). \quad (5.36)$$

After some algebra it can be shown that

$$2[A, [B, C]] = [[A, B], C] + \{\{A, B\}, C\} - 2(ACB + BCA), \quad (5.37)$$

and, applying it to the commutator in the master equation,

$$\begin{aligned}
2 \left[\tilde{V}_\alpha^\dagger(t), \left[\tilde{V}_\beta(t'), \tilde{\rho}_S(t) \right] \right] &= \left[\left[\tilde{V}_\alpha^\dagger(t), \tilde{V}_\beta(t') \right], \tilde{\rho}_S(t) \right] \\
&+ \left\{ \tilde{V}_\alpha^\dagger(t) \tilde{V}_\beta(t'), \tilde{\rho}_S(t) \right\} + \left\{ \tilde{V}_\beta(t') \tilde{V}_\alpha^\dagger(t), \tilde{\rho}_S(t) \right\} \\
&- 2 \left(\tilde{V}_\alpha^\dagger(t) \tilde{\rho}_S(t) \tilde{V}_\beta(t') + \tilde{V}_\beta(t') \tilde{\rho}_S(t) \tilde{V}_\alpha^\dagger(t) \right). \tag{5.38}
\end{aligned}$$

With the above manipulations, our equation is now

$$\begin{aligned}
\frac{d}{dt} \tilde{\rho}_S(t) &= -\frac{1}{2} \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \left[\left[\tilde{V}_\alpha^\dagger(t), \tilde{V}_\beta(t') \right], \tilde{\rho}_S(t) \right] \\
&+ \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \left[\tilde{V}_\beta(t') \tilde{\rho}_S(t) \tilde{V}_\alpha^\dagger(t) - \frac{1}{2} \left\{ \tilde{V}_\alpha^\dagger(t) \tilde{V}_\beta(t'), \tilde{\rho}_S(t) \right\} \right] \\
&+ \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \left[\tilde{V}_\alpha^\dagger(t) \tilde{\rho}_S(t) \tilde{V}_\beta(t') - \frac{1}{2} \left\{ \tilde{V}_\beta(t') \tilde{V}_\alpha^\dagger(t), \tilde{\rho}_S(t) \right\} \right]. \tag{5.39}
\end{aligned}$$

The first term in the right hand side of the above equation can be rewritten as

$$-i [H_{\text{eff}}(t), \tilde{\rho}_S(t)], \tag{5.40}$$

where

$$H_{\text{eff}}(t) = -\frac{i}{2} \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \left[\tilde{V}_\alpha^\dagger(t), \tilde{V}_\beta(t') \right] \tag{5.41}$$

is called the effective Hamiltonian. It is Hermitian, since

$$\begin{aligned}
H_{\text{eff}}^\dagger(t) &= \frac{i}{2} \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\beta\alpha}(t', t) \left[\tilde{V}_\beta^\dagger(t'), \tilde{V}_\alpha(t) \right] \\
&= -\frac{i}{2} \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\beta\alpha}(t', t) \left[\tilde{V}_\alpha(t), \tilde{V}_\beta^\dagger(t') \right] \\
&= -\frac{i}{2} \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \left[\tilde{V}_\alpha^\dagger(t), \tilde{V}_\beta(t') \right] = H_{\text{eff}}(t), \tag{5.42}
\end{aligned}$$

where we used the Hermiticity of $V(t)$ in the last step, i.e., the fact that for every α there is a β such that $V_\alpha^\dagger = V_\beta$ and $\xi_\alpha^* = \xi_\beta$, and

$$\overline{\xi_\beta(t) \xi_\alpha^*(t')} = \overline{\xi_\alpha^*(t') \xi_\beta(t)} = \chi_{\alpha\beta}(t', t). \tag{5.43}$$

We can also use that in the third term in the right hand side of the master equation, which leads to

$$\begin{aligned}
\frac{d}{dt} \tilde{\rho}_S(t) &= -i [H_{\text{eff}}(t), \tilde{\rho}_S(t)] \\
&+ \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \left[\tilde{V}_\beta(t') \tilde{\rho}_S(t) \tilde{V}_\alpha^\dagger(t) - \frac{1}{2} \left\{ \tilde{V}_\alpha^\dagger(t) \tilde{V}_\beta(t'), \tilde{\rho}_S(t) \right\} \right] \\
&+ \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t', t) \left[\tilde{V}_\beta(t) \tilde{\rho}_S(t) \tilde{V}_\alpha^\dagger(t') - \frac{1}{2} \left\{ \tilde{V}_\alpha^\dagger(t') \tilde{V}_\beta(t), \tilde{\rho}_S(t) \right\} \right]. \tag{5.44}
\end{aligned}$$

For the last two terms, we have to evaluate the operators $\tilde{V}_\alpha^\dagger(t)$ and $\tilde{V}_\beta(t)$. In order to do that, we shall proceed as in the previous chapter, where we assumed that the system spectrum was discrete. Then,

$$\tilde{V}_\beta(t) = \sum_{\omega} e^{-i\omega t} V_\beta(\omega) \quad (5.45)$$

$$\tilde{V}_\alpha^\dagger(t) = \sum_{\omega'} e^{i\omega' t} V_\alpha^\dagger(\omega'), \quad (5.46)$$

where $V_\beta(\omega)$ and $V_\alpha^\dagger(\omega)$ are given by Eq. (C.23). Then,

$$\begin{aligned} \frac{d}{dt} \tilde{\rho}_S(t) = & -i [H_{\text{eff}}(t), \tilde{\rho}_S(t)] \\ & + \sum_{\alpha, \beta, \omega, \omega'} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') e^{i(-\omega t' + \omega' t)} \left[V_\beta(\omega) \tilde{\rho}_S(t) V_\alpha^\dagger(\omega') - \frac{1}{2} \{ V_\alpha^\dagger(\omega') V_\beta(\omega), \tilde{\rho}_S(t) \} \right] \\ & + \sum_{\alpha, \beta, \omega, \omega'} \int_{t_0}^t dt' \chi_{\alpha\beta}(t', t) e^{i(-\omega t + \omega' t')} \left[V_\beta(\omega) \hat{\rho}_S(t) V_\alpha^\dagger(\omega') - \frac{1}{2} \{ V_\alpha^\dagger(\omega') V_\beta(\omega), \hat{\rho}_S(t) \} \right]. \end{aligned} \quad (5.47)$$

If we use the substitution $t' \rightarrow t - \tau$, then

$$\begin{aligned} \frac{d}{dt} \tilde{\rho}_S(t) = & -i [H_{\text{eff}}(t), \tilde{\rho}_S(t)] + \sum_{\alpha, \beta, \omega, \omega'} \int_{t_0}^t d\tau \left(\chi_{\alpha\beta}(t, t - \tau) e^{i(\omega' - \omega)t} e^{i\omega\tau} + \chi_{\alpha\beta}(t - \tau, t) e^{i(\omega' - \omega)t} e^{-i\omega'\tau} \right) \\ & \times \left[V_\beta(\omega) \tilde{\rho}_S(t) V_\alpha^\dagger(\omega') - \frac{1}{2} \{ V_\alpha^\dagger(\omega') V_\beta(\omega), \tilde{\rho}_S(t) \} \right]. \end{aligned} \quad (5.48)$$

We now perform the rotating wave approximation, where fast oscillating terms average to zero. Therefore, only terms with $\omega = \omega'$ survive:

$$\begin{aligned} \frac{d}{dt} \tilde{\rho}_S(t) = & -i [H_{\text{eff}}(t), \tilde{\rho}_S(t)] \\ & + \sum_{\alpha, \beta, \omega} \int_{t_0}^t d\tau \left(\chi_{\alpha\beta}(t, t - \tau) e^{i\omega\tau} + \chi_{\alpha\beta}(t - \tau, t) e^{-i\omega\tau} \right) \\ & \times \left[V_\beta(\omega) \tilde{\rho}_S(t) V_\alpha^\dagger(\omega) - \frac{1}{2} \{ V_\alpha^\dagger(\omega) V_\beta(\omega), \tilde{\rho}_S(t) \} \right]. \end{aligned} \quad (5.49)$$

Using the notation

$$\eta_{\alpha\beta\omega}(t) \equiv \int_{t_0}^t d\tau \left(\chi_{\alpha\beta}(t, t - \tau) e^{i\omega\tau} + \chi_{\alpha\beta}(t - \tau, t) e^{-i\omega\tau} \right), \quad (5.50)$$

the master equation, in the Schrodinger picture, takes the Lindblad form with time-dependent coefficients:

$$\frac{d}{dt} \rho_S(t) = -i [H_S + H_{\text{eff}}(t), \rho_S(t)] + \sum_{\alpha, \beta, \omega} \eta_{\alpha\beta\omega}(t) \left[V_\beta(\omega) \rho_S(t) V_\alpha^\dagger(\omega) - \frac{1}{2} \{ V_\alpha^\dagger(\omega) V_\beta(\omega), \rho_S(t) \} \right]. \quad (5.51)$$

The effective Hamiltonian can also be simplified by use of Eq. (C.23):

$$\begin{aligned}
H_{\text{eff}}(t) &= -\frac{i}{2} \sum_{\alpha,\beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') [\tilde{V}_{\alpha}^{\dagger}(t), \tilde{V}_{\beta}(t')] \\
&= -\frac{i}{2} \sum_{\alpha,\beta,\omega,\omega'} \int_{t_0}^t dt' e^{i(\omega't - \omega t')} \chi_{\alpha\beta}(t, t') [V_{\alpha}^{\dagger}(\omega'), V_{\beta}(\omega)] \\
&= -\frac{i}{2} \sum_{\alpha,\beta,\omega} \int_{t_0}^t dt' e^{i\omega(t-t')} \chi_{\alpha\beta}(t, t') [V_{\alpha}^{\dagger}(\omega), V_{\beta}(\omega)], \tag{5.52}
\end{aligned}$$

where we used the rotating wave approximation. In that form, it can be seen that $H_{\text{eff}}(t)$ commutes with the system Hamiltonian, and therefore represents only a shift in the energy levels. Namely,

$$[H_S, H_{\text{eff}}(t)] = -\frac{i}{2} \sum_{\alpha,\beta,\omega} \int_{t_0}^t dt' e^{i\omega(t-t')} \chi_{\alpha\beta}(t, t') [H_S, [V_{\alpha}^{\dagger}(\omega), V_{\beta}(\omega)]], \tag{5.53}$$

and the commutator is

$$\begin{aligned}
[H_S, [V_{\alpha}^{\dagger}(\omega), V_{\beta}(\omega)]] &= [H_S, V_{\alpha}^{\dagger}(\omega)V_{\beta}(\omega)] - [H_S, V_{\beta}(\omega)V_{\alpha}^{\dagger}(\omega)] \\
&= [H_S, V_{\alpha}^{\dagger}(\omega)] V_{\beta}(\omega) + V_{\alpha}^{\dagger}(\omega) [H_S, V_{\beta}(\omega)] \\
&\quad - [H_S, V_{\beta}(\omega)] V_{\alpha}^{\dagger}(\omega) - V_{\beta}(\omega) [H_S, V_{\alpha}^{\dagger}(\omega)] \\
&= \omega V_{\alpha}^{\dagger}(\omega)V_{\beta}(\omega) - \omega V_{\alpha}^{\dagger}(\omega)V_{\beta}(\omega) + \omega V_{\beta}(\omega)V_{\alpha}^{\dagger}(\omega) - \omega V_{\beta}(\omega)V_{\alpha}^{\dagger}(\omega) \\
&= 0, \tag{5.54}
\end{aligned}$$

where Eq. (3.176) was used.

5.1.3 Summary

We start from a Hamiltonian

$$H(t) = H_S + V(t) = H_S + \sum_{\alpha} \xi_{\alpha}(t) V_{\alpha}, \tag{5.55}$$

where $V(t)$ is Hermitian and the $\xi_{\alpha}(t)$ are complex Gaussian noises with zero mean, i.e.,

$$\overline{\xi_{\alpha}(t)} = 0, \tag{5.56}$$

$$\overline{\xi_{\alpha}(t)\xi_{\beta}(t')} = \overline{\xi_{\alpha}^*(t)\xi_{\beta}^*(t')} = 0, \tag{5.57}$$

$$\overline{\xi_{\alpha}^*(t)\xi_{\beta}(t')} \equiv \chi_{\alpha\beta}(t, t'). \tag{5.58}$$

The V_{α} operators, in the interaction picture, can be decomposed as

$$\tilde{V}_{\alpha}(t) = \sum_{\omega} e^{-i\omega t} V_{\alpha}(\omega). \tag{5.59}$$

Then, the decay rates are

$$\eta_{\alpha\beta\omega}(t) \equiv \int_{t_0}^t d\tau \left(\chi_{\alpha\beta}(t, t-\tau) e^{i\omega\tau} + \chi_{\alpha\beta}(t-\tau, t) e^{-i\omega\tau} \right), \tag{5.60}$$

the effective Hamiltonian is

$$H_{\text{eff}}(t) = -\frac{i}{2} \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') [\tilde{V}_{\alpha}^{\dagger}(t), \tilde{V}_{\beta}(t')], \quad (5.61)$$

and the master equation, in Lindblad form, is

$$\frac{d}{dt} \rho_S(t) = -i[H_S + H_{\text{eff}}(t), \rho_S(t)] + \sum_{\alpha, \beta, \omega} \eta_{\alpha\beta\omega}(t) \left[V_{\beta}(\omega) \rho_S(t) V_{\alpha}^{\dagger}(\omega) - \frac{1}{2} \{ V_{\alpha}^{\dagger}(\omega) V_{\beta}(\omega), \rho_S(t) \} \right]. \quad (5.62)$$

5.2 Examples

5.2.1 Harmonic oscillator

We study the dynamics of a harmonic oscillator under the action of a pumping with frequency ω . (61, 155) The Hamiltonian is

$$H(t) = H_S + V(t) = \omega_0 a^{\dagger} a + i(e^{-i\omega t} u(t) a^{\dagger} - e^{i\omega t} u^*(t) a), \quad (5.63)$$

where ω_0 is the system's natural frequency and $u(t)$, $u^*(t)$ are Gaussian noises with zero mean. The noise term is

$$V(t) = ie^{-i\omega t} u(t) a^{\dagger} - ie^{i\omega t} u^*(t) a = \xi_1(t) V_1 + \xi_2(t) V_2. \quad (5.64)$$

Moving to the interaction picture,

$$\tilde{V}(t) = \xi_1(t) e^{i\omega_0 t} a^{\dagger} + \xi_2(t) e^{-i\omega_0 t} a = \xi_1(t) \tilde{V}_1(t) + \xi_2(t) \tilde{V}_2(t). \quad (5.65)$$

From the above equation, it can be seen that $V_1(-\omega_0) = a^{\dagger}$, $V_1(\omega_0) = 0$, $V_2(\omega_0) = a$ and $V_2(-\omega_0) = 0$. Therefore, the only non zero terms in the Lindbladian will be those with $(\alpha, \beta, \omega) = (1, 1, -\omega_0)$ and $(\alpha, \beta, \omega) = (2, 2, \omega_0)$. For the noise correlation functions,

$$\chi_{11}(t, t') = \overline{\xi_1^*(t) \xi_1(t')} = e^{i\omega(t-t')} \overline{u^*(t) u(t')} \quad (5.66)$$

$$\chi_{22}(t, t') = \overline{\xi_2^*(t) \xi_2(t')} = e^{-i\omega(t-t')} \overline{u(t) u^*(t')}. \quad (5.67)$$

If we assume that the correlations are homogeneous in time, then $\overline{u^*(t) u(t')} = \overline{u(t) u^*(t')} \equiv \mathcal{S}(\tau)$, where $\tau = t - t'$. Then, the decay rates are

$$\begin{aligned} \eta_{1,1,-\omega_0}(t) &= \int_0^t d\tau (\chi_{11}(\tau) e^{-i\omega_0 \tau} + \chi_{11}(-\tau) e^{i\omega_0 \tau}) \\ &= \int_0^t d\tau \mathcal{S}(\tau) (e^{i\omega \tau} e^{-i\omega_0 \tau} + e^{-i\omega \tau} e^{i\omega_0 \tau}) \\ &= 2 \int_0^t d\tau \mathcal{S}(\tau) \cos((\omega - \omega_0)\tau) \equiv \eta(t), \end{aligned} \quad (5.68)$$

and, similarly,

$$\begin{aligned} \eta_{2,2,\omega_0}(t) &= \int_0^t d\tau \mathcal{S}(\tau) (e^{-i\omega \tau} e^{i\omega_0 \tau} + e^{i\omega \tau} e^{-i\omega_0 \tau}) \\ &= 2 \int_0^t d\tau \mathcal{S}(\tau) \cos((\omega_0 - \omega)\tau) \equiv \eta(t). \end{aligned} \quad (5.69)$$

The last step is to evaluate the effective Hamiltonian, given by

$$H_{\text{eff}}(t) = -\frac{i}{2} \sum_{\alpha, \beta} \int_0^t dt' \chi_{\alpha\beta}(t, t') [\tilde{V}_\alpha^\dagger(t), \tilde{V}_\beta(t')]. \quad (5.70)$$

The commutators that appear in the above formula are

$$\begin{aligned} [\tilde{V}_1^\dagger(t), \tilde{V}_1(t')] &= e^{-i\omega_0(t-t')} [a, a^\dagger] = e^{-i\omega_0(t-t')} \mathbb{1} \\ [\tilde{V}_2^\dagger(t), \tilde{V}_2(t')] &= e^{i\omega_0(t-t')} [a^\dagger, a] = -e^{i\omega_0(t-t')} \mathbb{1} \\ [\tilde{V}_1^\dagger(t), \tilde{V}_2(t')] &= e^{-i\omega_0(t+t')} [a, a] = 0 \\ [\tilde{V}_2^\dagger(t), \tilde{V}_1(t')] &= e^{i\omega_0(t+t')} [a^\dagger, a^\dagger] = 0, \end{aligned} \quad (5.71)$$

and the effective Hamiltonian is

$$\begin{aligned} H_{\text{eff}}(t) &= \frac{1}{2i} \left[\int_0^t dt' e^{i\Delta\omega(t-t')} \overline{u^*(t)u(t')} - \int_0^t dt' e^{-i\Delta\omega(t-t')} \overline{u(t)u^*(t')} \right] \mathbb{1} \\ &= \frac{1}{2i} \left[\int_0^t d\tau S(\tau) (e^{i\Delta\omega\tau} - e^{-i\Delta\omega\tau}) \right] \mathbb{1} = \int_0^t d\tau S(\tau) \sin(\Delta\omega\tau) \mathbb{1}, \end{aligned} \quad (5.72)$$

where $\Delta\omega = \omega - \omega_0$. The Hamiltonian is proportional to the identity and, therefore, commutes with $\rho_S(t)$ and has no effect in the master equation, which is

$$\begin{aligned} \frac{d}{dt} \rho_S(t) &= -i [\omega_0 a^\dagger a, \rho_S(t)] + \eta(t) \left[a \rho_S(t) a^\dagger - \frac{1}{2} \{a^\dagger a, \rho_S(t)\} \right] \\ &\quad + \eta(t) \left[a^\dagger \rho_S(t) a - \frac{1}{2} \{a a^\dagger, \rho_S(t)\} \right]. \end{aligned} \quad (5.73)$$

5.2.2 Qubit under amplitude damping

Now we study a qubit under the action of a pumping with frequency ω , with Hamiltonian

$$H(t) = H_S + V(t) = \frac{\omega_0}{2} \sigma_z + i (e^{-i\omega t} u(t) \sigma_+ - e^{i\omega t} u^*(t) \sigma_-). \quad (5.74)$$

The pumping is related to the operators σ_+ and σ_- , which do not commute with the system Hamiltonian, and therefore energy is not conserved. As observed in Sec. 3.5.2, the evolution is essentially the same as the harmonic oscillator evolution, with only a difference in the operators. Therefore, the evolution is

$$\begin{aligned} \frac{d}{dt} \rho_S(t) &= -i \left[\frac{\omega_0}{2} \sigma_z + H_{\text{eff}}(t), \rho_S(t) \right] + \eta(t) \left[\sigma_- \rho_S(t) \sigma_+ - \frac{1}{2} \{ \sigma_+ \sigma_-, \rho_S(t) \} \right] \\ &\quad + \eta(t) \left[\sigma_+ \rho_S(t) \sigma_- - \frac{1}{2} \{ \sigma_- \sigma_+, \rho_S(t) \} \right]. \end{aligned} \quad (5.75)$$

The effective Hamiltonian can be calculated similarly. The commutators are

$$\begin{aligned} [\tilde{V}_1^\dagger(t), \tilde{V}_1(t')] &= e^{-i\omega_0(t-t')} [\sigma_-, \sigma_+] = e^{-i\omega_0(t-t')} \sigma_z \\ [\tilde{V}_2^\dagger(t), \tilde{V}_2(t')] &= e^{i\omega_0(t-t')} [\sigma_+, \sigma_-] = -e^{i\omega_0(t-t')} \sigma_z, \end{aligned} \quad (5.76)$$

and the others are zero. Its form is

$$H_{\text{eff}}(t) = \int_0^t d\tau S(\tau) \sin(\Delta\omega\tau) \sigma_z \equiv \frac{\Theta(t)}{2} \sigma_z. \quad (5.77)$$

5.2.3 Qubit under phase damping

As a last example, we study the pure dephasing of a qubit. The Hamiltonian is

$$H = \frac{\omega_0}{2}\sigma_z + u(t)\sigma_z. \quad (5.78)$$

As in Sec. 3.5.3, the problem is exactly solvable but we use the formalism developed here. Since $V(t) = u(t)\sigma_z$ has only one term, the noise $u(t)$ must be real. Also, since it commutes with the system Hamiltonian, it does not evolve in time and, therefore, $\tilde{V}(t) = V(t)$ and $V(\omega = 0) = \sigma_z$. In particular, the effective Hamiltonian is also zero. The master equation is simply

$$\frac{d}{dt}\rho_S(t) = -i \left[\frac{\omega_0}{2}\sigma_z, \rho_S(t) \right] + \eta_0(t) [\sigma_z \rho_S(t) \sigma_z - \rho_S(t)], \quad (5.79)$$

where

$$\eta_0(t) = 2 \int_0^t d\tau \mathcal{S}(\tau) \cos(\omega_0 \tau). \quad (5.80)$$

6 QUANTUM NON-MARKOVIANITY INDUCED BY CLASSICAL NOISE

In this chapter we present the main result of the dissertation. In Section 6.1 we combine, in a single model, the system-bath evolution together with the stochastic pumping. This represents a situation where a system is subject to a classical field, e.g. a laser, but at the same time the environment interaction must be considered. The resulting evolution is non-Markovian if the pumping is strong enough to reverse the natural flow of information from the system to the reservoir. We apply the developed formalism to the case of a qubit under amplitude damping, in Section 6.2. Finally, in Sections 6.3 and 6.4 we detect and measure the quantum non-Markovianity of this example and analyze the role that the stochastic pumping has in this evolution.

6.1 Combined evolution

We start with a system-environment Hilbert space, $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, and a Hamiltonian

$$H(t) = H_S + H_E + H_{\text{int}} + V(t), \quad (6.1)$$

with H_{int} and $V(t)$ as in Eq. (3.126) and Eq. (5.4), respectively:

$$H_{\text{int}} = \sum_{\alpha} S_{\alpha} \otimes E_{\alpha}, \quad (6.2)$$

$$V(t) = \sum_{\alpha} \xi_{\alpha}(t) V_{\alpha}. \quad (6.3)$$

The joint system-environment state is represented by the operator $\rho(t) \in \mathbf{D}(\mathcal{H})$, which evolves according to

$$\frac{d}{dt}\rho(t) = -i [H(t), \rho(t)]. \quad (6.4)$$

Moving to the interaction picture, Eq. (6.4) becomes

$$\frac{d}{dt}\tilde{\rho}(t) = -i [\tilde{H}_{\text{int}}(t) + \tilde{V}(t), \tilde{\rho}(t)], \quad (6.5)$$

and its solution is

$$\begin{aligned} \tilde{\rho}(t) &= \rho(t_0) - i \int_{t_0}^t d\tau [\tilde{H}_{\text{int}}(\tau) + \tilde{V}(\tau), \tilde{\rho}(\tau)] \\ &= \tilde{U}(t, t') \tilde{\rho}(t') \tilde{U}^{\dagger}(t, t'). \end{aligned} \quad (6.6)$$

Using the stochastic wave function formalism (3), we recover the evolution with an average over the stochastic realizations, $\rho_{SE}(t) = \overline{\rho(t)}$. Then,

$$\frac{d}{dt}\tilde{\rho}_{SE}(t) = -i \overline{[\tilde{V}(t), \tilde{\rho}(t)]} - i [\tilde{H}_{\text{int}}(t), \tilde{\rho}_{SE}(t)]. \quad (6.7)$$

Since we are interested in the system, and not on the joint system-bath state, we trace out the reservoir:

$$\frac{d}{dt}\tilde{\rho}_S(t) = -i \operatorname{tr}_E \left[\widetilde{V}(t), \tilde{\rho}(t) \right] - i \operatorname{tr}_E \left[\widetilde{H}_{\text{int}}(t), \tilde{\rho}(t) \right] \equiv \mathcal{L}_t^S [\tilde{\rho}_S(t)] + \mathcal{L}_t^M [\tilde{\rho}_S(t)]. \quad (6.8)$$

The superoperators $\mathcal{L}_t^S$ and $\mathcal{L}_t^M$ are related to the stochastic and Born-Markov evolutions, respectively. We evaluate them in the next sections.

6.1.1 Stochastic evolution

Here we evaluate the superoperator

$$\mathcal{L}_t^S [\tilde{\rho}_S(t)] = -i \operatorname{tr}_E \left[\widetilde{V}(t), \tilde{\rho}(t) \right], \quad (6.9)$$

which appears on Eq. (6.8). For that, we use Eq. (5.21):

$$\left[\widetilde{V}(t), \tilde{\rho}(t) \right] = \sum_{\alpha, \beta} \int_{t_0}^t dt' \chi_{\alpha\beta}(t, t') \left[\tilde{V}_\alpha^\dagger(t), \frac{\delta \tilde{\rho}(t)}{\delta \xi_\beta(t')} \right]. \quad (6.10)$$

The functional derivative is evaluated as in Section 5.1.1:

$$\frac{\delta \tilde{\rho}(t)}{\delta \xi_\beta(t')} = -i \left[\tilde{V}_\beta(t'), \tilde{\rho}(t') \right] - i \int_{t'}^t d\tau \left[\widetilde{H}_{\text{int}}(\tau) + \tilde{V}(\tau), \frac{\delta \tilde{\rho}(\tau)}{\delta \xi_\beta(t')} \right]. \quad (6.11)$$

Taking the time derivative,

$$\frac{d}{dt} \frac{\delta \tilde{\rho}(t)}{\delta \xi_\beta(t')} = -i \left[\widetilde{H}_{\text{int}}(t) + \tilde{V}(t), \frac{\delta \tilde{\rho}(t)}{\delta \xi_\beta(t')} \right], \quad (6.12)$$

which is exactly like Eq. (6.5). The whole derivation is the same as in chapter 5, and using the same hypotheses and approximations leads us to

$$\begin{aligned} -i \left[\widetilde{V}(t), \tilde{\rho}(t) \right] &= -i \left[H_{\text{eff}}(t), \tilde{\rho}_{SE}(t) \right] \\ &+ \sum_{\alpha, \beta, \omega} \eta_{\alpha\beta\omega}(t) \left[V_\beta(\omega) \tilde{\rho}_{SE}(t) V_\alpha^\dagger(\omega) - \frac{1}{2} \left\{ V_\alpha^\dagger(\omega) V_\beta(\omega), \tilde{\rho}_{SE}(t) \right\} \right], \end{aligned} \quad (6.13)$$

where the coefficients $\bar{\eta}_{\alpha\beta\omega}(t)$ are given by Eq. (5.50):

$$\eta_{\alpha\beta\omega}(t) \equiv \int_{t_0}^t d\tau \left(\chi_{\alpha\beta}(t, t-\tau) e^{i\omega\tau} + \chi_{\alpha\beta}(t-\tau, t) e^{-i\omega\tau} \right), \quad (6.14)$$

Assuming the factorization condition,

$$\rho_{SE}(t) = \rho_S(t) \otimes \rho_E, \quad (6.15)$$

and since the operators on the right hand side lie in $\mathcal{H}_S$, we can simply trace out the operator $\rho_E(t)$, leading to

$$\begin{aligned} \mathcal{L}_t^S [\tilde{\rho}_S(t)] &= -i \left[H_{\text{eff}}(t), \tilde{\rho}_S(t) \right] \\ &+ \sum_{\alpha, \beta, \omega} \eta_{\alpha\beta\omega}(t) \left[V_\beta(\omega) \tilde{\rho}_S(t) V_\alpha^\dagger(\omega) - \frac{1}{2} \left\{ V_\alpha^\dagger(\omega) V_\beta(\omega), \tilde{\rho}_S(t) \right\} \right]. \end{aligned} \quad (6.16)$$

6.1.2 Born Markov evolution

The other superoperator is

$$\mathcal{L}_t^M [\tilde{\rho}_S(t)] = -i \text{tr}_E \left[\widetilde{H}_{\text{int}}(t), \tilde{\rho}(t) \right]. \quad (6.17)$$

Replacing Eq. (6.6) in the above equation,

$$\begin{aligned} \mathcal{L}_t^M [\tilde{\rho}_S(t)] = & -i \text{tr}_E \left[\widetilde{H}_{\text{int}}(t), \rho_{SE}(t_0) \right] \\ & - \int_{t_0}^t dt' \text{tr}_E \left[\widetilde{H}_{\text{int}}(t), \left[\widetilde{H}_{\text{int}}(t'), \tilde{\rho}_{SE}(t') \right] \right] \\ & - \int_{t_0}^t dt' \text{tr}_E \left[\widetilde{H}_{\text{int}}(t), \left[\widetilde{V}(t'), \tilde{\rho}(t') \right] \right]. \end{aligned} \quad (6.18)$$

Assuming the hypotheses and approximations of Section 3.4.1, the first term on the right hand side is zero and the second term is equal to

$$-i [H_{\text{LS}}, \tilde{\rho}_S(t)] + \sum_{\alpha, \beta, \omega} \gamma_{\alpha\beta}(\omega) \left[S_{\beta}(\omega) \tilde{\rho}_S(t) S_{\alpha}^{\dagger}(\omega) - \frac{1}{2} \left\{ S_{\alpha}^{\dagger}(\omega) S_{\beta}(\omega), \tilde{\rho}_S(t) \right\} \right]. \quad (6.19)$$

Finally, note that

$$\left[\widetilde{V}(t'), \tilde{\rho}(t') \right] = i \mathcal{L}_{t'}^S [\tilde{\rho}_S(t')] \otimes \rho_E, \quad (6.20)$$

and, therefore, the third term is equal to

$$-i \int_{t_0}^t dt' \text{tr}_E \left[\widetilde{H}_{\text{int}}(t), \mathcal{L}_{t'}^S [\tilde{\rho}_S(t')] \otimes \rho_E \right]. \quad (6.21)$$

Using

$$\widetilde{H}_{\text{int}} = \sum_{\alpha} \tilde{S}_{\alpha}(t) \otimes \tilde{E}_{\alpha}(t), \quad (6.22)$$

Eq. (6.21) becomes

$$\begin{aligned} & -i \int_{t_0}^t dt' \text{tr}_E \sum_{\alpha} \left[\tilde{S}_{\alpha}(t) \mathcal{L}_{t'}^S [\tilde{\rho}_S(t')] \otimes \left(\tilde{E}_{\alpha}(t) \rho_E \right) - \mathcal{L}_{t'}^S [\tilde{\rho}_S(t')] \tilde{S}_{\alpha}(t) \otimes \left(\rho_E \tilde{E}_{\alpha}(t) \right) \right] \\ & = -i \int_{t_0}^t dt' \sum_{\alpha} \left[\tilde{S}_{\alpha}(t), \mathcal{L}_{t'}^S [\tilde{\rho}_S(t')] \right] \langle E_{\alpha}(t) \rangle. \end{aligned} \quad (6.23)$$

Since it is assumed in Section 3.4.1 that $\langle E_{\alpha}(t) \rangle = 0$, for every α , then the term is zero. Therefore,

$$\begin{aligned} \mathcal{L}^M [\tilde{\rho}_S(t)] = & -i [H_{\text{LS}}, \tilde{\rho}_S(t)] \\ & + \sum_{\alpha, \beta, \omega} \gamma_{\alpha\beta}(\omega) \left[S_{\beta}(\omega) \tilde{\rho}_S(t) S_{\alpha}^{\dagger}(\omega) - \frac{1}{2} \left\{ S_{\alpha}^{\dagger}(\omega) S_{\beta}(\omega), \tilde{\rho}_S(t) \right\} \right]. \end{aligned} \quad (6.24)$$

Note that we dropped the time subscript in $\mathcal{L}^M$, since we can see now that it is time-independent.

6.1.3 Total evolution

With the results above, our master equation,

$$\frac{d}{dt}\tilde{\rho}_S(t) = \mathcal{L}_t^S[\tilde{\rho}_S(t)] + \mathcal{L}^M[\tilde{\rho}_S(t)], \quad (6.25)$$

moving back to the Schrodinger picture, becomes

$$\begin{aligned} \frac{d}{dt}\rho_S(t) = & -i[H_S + H_{LS} + H_{\text{eff}}(t), \rho_S(t)] \\ & + \sum_{\alpha, \beta, \omega} \eta_{\alpha\beta\omega}(t) \left[V_\beta(\omega)\rho_S(t)V_\alpha^\dagger(\omega) - \frac{1}{2} \{V_\alpha^\dagger(\omega)V_\beta(\omega), \rho_S(t)\} \right] \\ & + \sum_{\alpha, \beta, \omega} \gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega)\rho_S(t)S_\alpha^\dagger(\omega) - \frac{1}{2} \{S_\alpha^\dagger(\omega)S_\beta(\omega), \rho_S(t)\} \right]. \end{aligned} \quad (6.26)$$

If we have system-bath and pumping interactions such that $S_\alpha(\omega) = V_\alpha(\omega)$, i.e., they are expressed in the same operator basis, then the expression simplifies to

$$\begin{aligned} \frac{d}{dt}\rho_S(t) = & -i[H_S + H_{LS} + H_{\text{eff}}(t), \rho_S(t)] \\ & + \sum_{\alpha, \beta, \omega} \gamma_{\alpha\beta\omega}(t) \left[V_\beta(\omega)\rho_S(t)V_\alpha^\dagger(\omega) - \frac{1}{2} \{V_\alpha^\dagger(\omega)V_\beta(\omega), \rho_S(t)\} \right], \end{aligned} \quad (6.27)$$

where

$$\gamma_{\alpha\beta\omega}(t) \equiv \gamma_{\alpha\beta}(\omega) + \eta_{\alpha\beta\omega}(t). \quad (6.28)$$

If the master equation is in canonical form, the coefficients $\gamma_{\alpha\beta\omega}(t)$ can be used to access the non-Markovianity of the evolution: the presence of negative values indicates the breakdown of CP-divisibility and, therefore, non-Markovianity. The constant term $\gamma_{\alpha\beta}(\omega)$ corresponds to the system-bath evolution and is always positive. The time-dependent term $\eta_{\alpha\beta\omega}(t)$, related to the pumping, however, can assume negative values and, if large enough to surpass the bath term, can generate quantum non-Markovian effects, as we will see in the next sections.

6.2 Qubit under amplitude damping

For the rest of the chapter we will analyze the specific case of a qubit under amplitude damping, by both a bosonic reservoir and an external pumping. Therefore, this case is a juxtaposition of the examples in Sections 3.5.2 and 5.2.2. According to Eq. (6.27), we just have to put together both master Equations (3.240) and (5.75)^a:

$$\begin{aligned} \frac{d}{dt}\rho(t) \equiv \mathcal{L}_t[\rho(t)] = & -i \left[\frac{\omega_0(t)}{2} \sigma_z, \rho(t) \right] + \gamma_1(t) \left[\sigma_+ \rho(t) \sigma_- - \frac{1}{2} \{ \sigma_- \sigma_+, \rho(t) \} \right] \\ & + \gamma_2(t) \left[\sigma_- \rho(t) \sigma_+ - \frac{1}{2} \{ \sigma_+ \sigma_-, \rho(t) \} \right], \end{aligned} \quad (6.29)$$

^a For simplicity, we drop the system subscript from $\rho_S(t)$.

where the decay rates are

$$\gamma_1(t) = \gamma N(\omega_0) + \eta(t), \quad (6.30)$$

$$\gamma_2(t) = \gamma(1 + N(\omega_0)) + \eta(t), \quad (6.31)$$

and $\omega_0(t) = \omega_0 + 2\Delta' + \Delta + \Theta(t)$, with

$$\Delta = \mathcal{P} \left(\int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega} \right), \quad (6.32)$$

$$\Delta' = \mathcal{P} \left(\int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega} N(\omega_0) \right), \quad (6.33)$$

$$\Theta(t) = 2 \int_0^t d\tau S(\tau) \sin(\Delta\omega\tau) \sigma_z. \quad (6.34)$$

Also recall that

$$\eta(t) = 2 \int_0^t d\tau \mathcal{S}(\tau) \cos(\Delta\omega\tau), \quad (6.35)$$

where $\Delta\omega = (\omega - \omega_0)$. Note that the free parameters of Eq. (6.29) are $\omega_0(t)$, $\gamma_1(t)$ and $\gamma_2(t)$. These are the parameters which determine the characteristics of the evolution, and which are determined by the reservoir and the pumping. In this and the next section we do not concern ourselves with the nature of these parameters: we just derive everything in function of them, following a rather abstract approach. Then, in Section 6.4 we use their explicit forms to evaluate the contributions of the environment and the noise in the dynamics of the system.

The matrix coefficients of $\rho(t)$ are

$$\rho(t) = \begin{bmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{bmatrix},$$

Note that the Hermiticity of the density matrix implies that $\rho_{12}(t) = \rho_{21}^*(t)$, and the unitarity of the trace implies that $\rho_{11}(t) + \rho_{22}(t) = 1$, so $\rho_{22}(t) = 1 - \rho_{11}(t)$. Therefore, we only have to concern ourselves with, for example, $\rho_{11}(t)$ and $\rho_{21}(t)$, since the other two coefficients are determined from these. The master equation implies that these coefficients obey the following differential equations:

$$\dot{\rho}_{11}(t) = -[\gamma_1(t) + \gamma_2(t)] \rho_{11}(t) + \gamma_1(t), \quad (6.36)$$

$$\dot{\rho}_{21}(t) = \left[iw_0(t) - \frac{1}{2}(\gamma_1(t) + \gamma_2(t)) \right] \rho_{21}(t). \quad (6.37)$$

The term $\rho_{11}(t)$ represents the population of the excited state, and the term $\rho_{21}(t)$ the coherence of the system. The differential equation for the coherence is easy to solve, because it is separable:

$$\rho_{21}(t) = e^{iW(t)} e^{-\frac{1}{2}\Gamma(t)} \rho_{21}(t_0), \quad (6.38)$$

where

$$W(t) = \int_{t_0}^t w_0(s) ds, \quad (6.39)$$

$$\Gamma(t) = \int_{t_0}^t (\gamma_1(s) + \gamma_2(s)) ds. \quad (6.40)$$

We can see that the coherence oscillates with the (normalized) system frequency and decays with the decay rates. Note, however, that if the sum of the decay rates is positive, then coherence revivals may occur. In order to solve the other differential equation we have to use the technique of integrating factors (135), so

$$\rho_{11}(t) = e^{-\Gamma(t)} \int_{t_0}^t e^{\Gamma(s)} \gamma_1(s) ds + e^{-\Gamma(t)} \rho_{11}(t_0). \quad (6.41)$$

Finally, the average energy of the system is

$$\langle E(t) \rangle = \langle H_S(t) \rangle = \frac{\omega_0}{2} \langle \sigma_z(t) \rangle, \quad (6.42)$$

where

$$\langle \sigma_z(t) \rangle = \text{tr} [\sigma_z \rho(t)] = \rho_{11}(t) - \rho_{22}(t) = 2\rho_{11}(t) - 1. \quad (6.43)$$

Therefore,

$$\langle E(t) \rangle = \omega_0 \left(\rho_{11}(t) - \frac{1}{2} \right). \quad (6.44)$$

6.2.1 Canonical form

Recall that a time-local master equation is in canonical form if it can be written as Eq. (3.67),

$$\frac{d}{dt} \rho(t) = -\frac{i}{\hbar} [H(t), \rho(t)] + \sum_{i=1}^{d^2-1} \gamma_i(t) \left(L_i(t) \rho(t) L_i^\dagger(t) - \frac{1}{2} \{ L_i^\dagger(t) L_i(t), \rho(t) \} \right), \quad (6.45)$$

where $H(t)$ is Hermitian, the $\gamma_i(t)$ are real and the $L_i(t)$ are orthonormal,

$$\text{tr} [L_i^\dagger(t) L_j(t)] = \delta_{ij}, \quad \text{tr} [L_i(t)] = 0. \quad (6.46)$$

Now, one can easily check that the operators σ_+ and σ_- obey the orthonormality condition, Eq. (6.46). Also, the Hamiltonian

$$H(t) = \frac{\omega_0(t)}{2} \sigma_z \quad (6.47)$$

is Hermitian, so Eq. (6.29) is already in canonical form if we make sure to use real decay rates^b.

^b Note that we are not yet considering Eqs. (6.30) and (6.31), which show that the decay rates are, in fact, real numbers.

6.2.2 Bloch representation

Moving to the Bloch representation, Eq. (3.111)

$$\rho(t) = \frac{1}{2} \left(\mathbb{1} + \sum_{i=1}^3 n_i(t) \sigma_i \right), \quad (6.48)$$

we have that the Bloch vector components, in terms of the matrix coefficients, are

$$n_x(t) = \rho_{21}(t) + \rho_{12}(t), \quad (6.49)$$

$$n_y(t) = -i (\rho_{21}(t) - \rho_{12}(t)), \quad (6.50)$$

$$n_z(t) = 2\rho_{11}(t) - 1. \quad (6.51)$$

This means that they obey the differential equations

$$\dot{n}_x(t) = -\frac{1}{2} (\gamma_1(t) + \gamma_2(t)) n_x(t) - \omega_0(t) n_y(t), \quad (6.52)$$

$$\dot{n}_y(t) = -\frac{1}{2} (\gamma_1(t) + \gamma_2(t)) n_y(t) + \omega_0(t) n_x(t), \quad (6.53)$$

$$\dot{n}_z(t) = -[\gamma_1(t) + \gamma_2(t)] n_z(t) + (\gamma_1(t) - \gamma_2(t)). \quad (6.54)$$

The solutions of these equations can be easily obtained from Eqs. (6.41) and (6.38). In the Bloch representation, the master equation takes the form

$$\dot{\vec{n}}(t) = \vec{u}(t) + D(t)\vec{n}(t), \quad (6.55)$$

where $D(t)$ and $\vec{u}$ are the damping matrix and the drift vector, respectively. In order to calculate them, we need the following relations (126):

$$\text{tr} [\sigma_i \sigma_j] = \delta_{ij}, \quad (6.56)$$

$$\text{tr} [\sigma_i \sigma_j \sigma_k] = 2i\epsilon_{ijk}, \quad (6.57)$$

$$\text{tr} [\sigma_i \sigma_j \sigma_k \sigma_l] = 2(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \quad (6.58)$$

where ϵ_{ijk} is the Levi-Civita symbol (126), which is equal to 1 for even permutations of (i, j, k) , -1 for odd permutations, and null if there are any repeated indices. In order to use these relations, we have to expand the σ_+ and σ_- operators in terms of the Pauli matrices:

$$\sigma_{\pm} = \frac{\sigma_x \pm \sigma_y}{2}. \quad (6.59)$$

With this, we can rewrite Eq. (6.29) entirely in terms of Pauli matrices:

$$\begin{aligned} \frac{d}{dt} \rho(t) = & -i \left[\frac{\omega_0(t)}{2} \sigma_z, \rho(t) \right] + \left(\frac{\gamma_1(t) + \gamma_2(t)}{4} \right) [\sigma_x \rho(t) \sigma_x - \rho(t)] \\ & + \left(\frac{\gamma_1(t) + \gamma_2(t)}{4} \right) [\sigma_y \rho(t) \sigma_y - \rho(t)] \\ & + i \left(\frac{\gamma_1(t) - \gamma_2(t)}{4} \right) \left[\sigma_y \rho(t) \sigma_x - \frac{1}{2} \{ \sigma_x, \sigma_y, \rho(t) \} \right] \\ & - i \left(\frac{\gamma_1(t) - \gamma_2(t)}{4} \right) \left[\sigma_x \rho(t) \sigma_y - \frac{1}{2} \{ \sigma_y, \sigma_x, \rho(t) \} \right]. \end{aligned} \quad (6.60)$$

The drift vector components are

$$u_j(t) = \frac{1}{2} \text{tr} [\sigma_j \mathcal{L}_t[\mathbb{1}]] = \frac{\gamma_1(t) - \gamma_2(t)}{2} \text{tr} [\sigma_j \sigma_z] = (\gamma_1(t) - \gamma_2(t)) \delta_{j3}, \quad (6.61)$$

so $\vec{u}(t) = (0, 0, \gamma_1(t) - \gamma_2(t))$. For the damping matrix,

$$D_{jk}(t) = \frac{1}{2} \text{tr} [\sigma_j \mathcal{L}_t[\sigma_k]] = \frac{\omega_0(t)}{2} (\epsilon_{kj3} - \epsilon_{jk3}) + \frac{\gamma_1(t) + \gamma_2(t)}{2} (\delta_{j1}\delta_{k1} + \delta_{j2}\delta_{k2} - 2\delta_{jk}), \quad (6.62)$$

so it can be written as

$$D(t) = \begin{bmatrix} -\frac{\gamma_1(t) + \gamma_2(t)}{2} & -\omega_0(t) & 0 \\ \omega_0(t) & -\frac{\gamma_1(t) + \gamma_2(t)}{2} & 0 \\ 0 & 0 & -(\gamma_1(t) + \gamma_2(t)) \end{bmatrix}. \quad (6.63)$$

6.3 Witnessing quantum non-Markovianity

In this section we find which conditions the decay rates $\gamma_1(t)$ and $\gamma_2(t)$ have to satisfy in order to induce quantum non-Markovianity. We first consider the ACHL measure^c, Eq. (4.76), which measures RHP non-Markovianity. The measure is non-zero when at least one of the decay rates is negative, so the condition for non-Markovianity is that $\gamma_1(t) < 0$ or $\gamma_2(t) < 0$, for some t . This is shown graphically in Figure 4, where the blue-shaded area represents the parameter region where RHP quantum non-Markovianity is verified.

Figure 4 – RHP quantum non-Markovianity as a function of the decay rates $\gamma_1(t)$ and $\gamma_2(t)$. The non-Markovianity region corresponds to the blue-shaded area.

Source: By the author.

The BLP measure involves a complicated maximization procedure, but for qubits it yields a simple criterion for BLP non-Markovianity, Eq. (4.65):

$$\lambda_{\max} [D(t) + D^T(t)] > 0. \quad (6.64)$$

Using Eq. (6.63), we have that

$$D(t) + D^T(t) = \begin{bmatrix} -(\gamma_1(t) + \gamma_2(t)) & 0 & 0 \\ 0 & -(\gamma_1(t) + \gamma_2(t)) & 0 \\ 0 & 0 & -2(\gamma_1(t) + \gamma_2(t)) \end{bmatrix}. \quad (6.65)$$

^c We do not calculate the RHP measure, since for qubits it is equivalent to the ACHL measure.

The matrix is diagonal, so its eigenvalues are its diagonal elements. Therefore, the BLP criterion says that we have quantum non-Markovianity when

$$\gamma_1(t) + \gamma_2(t) < 0. \quad (6.66)$$

This is depicted in Figure 5, where the red-shaded area corresponds to BLP non-Markovianity.

Figure 5 – BLP quantum non-Markovianity as a function of the decay rates $\gamma_1(t)$ and $\gamma_2(t)$. The non-Markovianity region corresponds to the red-shaded area.

Source: By the author.

Finally, the Bloch volume witness states that we have quantum non-Markovianity when the sum of the decay rates is negative,

$$\gamma_1(t) + \gamma_2(t) < 0, \quad (6.67)$$

which is exactly the same condition for BLP non-Markovianity.

Figure 6 – RHP and BLP quantum non-Markovianity as a function of the decay rates $\gamma_1(t)$ and $\gamma_2(t)$. The RHP non-Markovianity region corresponds to the blue-shaded area, and the BLP quantum non-Markovianity region to the red-shaded area.

Source: By the author.

In Figure 6 we depict both conditions for RHP and BLP quantum non-Markovianity. Note that the first quadrant, where all decay rates are positive, always corresponds to Markovianity, as expected. Also, the region with blue and red shading, where the sum of the decay rates is negative, is non-Markovian by both definitions. The only-blue area, however, is controversial: it is RHP non-Markovian but BLP Markovian! It corresponds

to the situation where one of the decay rates is negative but their sum is still positive, which means that the other decay rate somehow compensates his negativity. If we think of each decay rate as corresponding to a decoherence channel, then an evolution is RHP non-Markovian when at least one of the channels displays non-Markovian effects (such as backflows of information or memory effects); but it is only BLP non-Markovian when the net effect of all channels leads to non-Markovianity. Also, that explains why the definitions coincide for master equations where there is only one decay rate.

6.4 Role of stochastic noise

In this section we measure the quantum non-Markovianity of our example from Eq. (6.29). We now use the physical model of the decay rates:

$$\gamma_1(t) = \gamma N(\omega_0) + \eta(t), \quad (6.68)$$

$$\gamma_2(t) = \gamma(1 + N(\omega_0)) + \eta(t). \quad (6.69)$$

Note that they consist of a reservoir dependent part, $\gamma N(\omega_0)$ or $\gamma(1 + N(\omega_0))$, and a part related to the stochastic pumping, $\eta(t)$. As for the reservoir, recall that $\gamma = 2\pi J(\omega_0)$, where $J(\omega_0)$ is its spectral density; from now on we assume that $\gamma = 1$. The number of states in the mode ω_0 ^d is

$$N(\omega_0) = \frac{1}{e^{\omega_0/T} - 1}, \quad (6.70)$$

where T is the temperature of the bath. We start with $N(\omega_0) = 0.05$, which is typical of some ion trapping experiments. (61) This corresponds to a temperature of

$$T = \frac{\omega_0}{\ln\left(\frac{N(\omega_0)+1}{N(\omega_0)}\right)} \simeq 0.33\omega_0. \quad (6.71)$$

As for the pump,

$$\eta(t) = 2 \int_0^t d\tau \mathcal{S}(\tau) \cos(\Delta\omega\tau), \quad (6.72)$$

where $\Delta\omega = (\omega - \omega_0)$ is the detuning between the frequency of the pump and the natural frequency of the system, and $\mathcal{S}(\tau)$ is the correlation function of the noise. We will consider three different correlation functions, which were already introduced in Chapter 4:

a) (Ornstein-Uhlenbeck) Exponential noise:

$$\mathcal{S}_{\text{OU}}(\tau) = \frac{\Omega}{2\tau_c} e^{-\tau/\tau_c}. \quad (6.73)$$

b) Squared exponential noise:

$$\mathcal{S}_{\text{SE}}(\tau) = \frac{\Omega}{\sqrt{\pi}\tau_c} e^{-(\tau/\tau_c)^2}. \quad (6.74)$$

^d We are using natural units, so $k_B = 1$.

c) Power law noise:

$$\mathcal{S}_{\text{PL}}(\tau) = \frac{\Omega}{\tau_c} \frac{(\alpha - 1)}{2} \frac{1}{(\tau/\tau_c + 1)^\alpha}. \quad (6.75)$$

The parameter Ω is the amplitude of each noise, and τ_c is the correlation time. The power law parameter α is constrained by $\alpha > 2$.

6.4.1 Exponential noise

We first consider the exponential noise from Eq. (6.73),

$$\mathcal{S}_{\text{OU}}(\tau) = \frac{\Omega}{2\tau_c} e^{-\tau/\tau_c}. \quad (6.76)$$

The exact formula for $\eta(t)$ is

$$\eta(t) = \frac{\Omega}{1 + \Delta\omega^2 \tau_c^2} \left[1 + e^{-t/\tau_c} (\Delta\omega \tau_c \sin(\Delta t) - \cos(\Delta\omega t)) \right], \quad (6.77)$$

which can be conveniently rewritten as

$$\eta(t) = \frac{\Omega}{1 + (\Delta\omega \tau_c)^2} \left[1 + \sqrt{1 + (\Delta\omega \tau_c)^2} e^{-t/\tau_c} \sin(\Delta\omega t - \phi) \right], \quad (6.78)$$

where

$$\phi = \sin^{-1} \left[\frac{1}{\sqrt{1 + (\Delta\omega \tau_c)^2}} \right]. \quad (6.79)$$

Note that the function $\eta(t)$ is composed of a constant term and a damped oscillating term. This function reaches a global minimum value when $(\Delta\omega \tau_c) = 8.5$, so we use $\Delta\omega = 2$ and $\tau_c = 4.25$. This leaves the amplitude Ω as the only free parameter.

We first analyze the Markovian to RHP non-Markovian transition^e, i.e., the transition from positive to negative decay rates. Since we have that $\gamma_2(t) \geq \gamma_1(t)$, then we analyze when the decay rate $\gamma_1(t)$ becomes negative. Suppose at first that the pump is absent, so we only have the effect of the reservoir. The decay rate is $\gamma_1(t) = N(\omega_0) = 0.05$, which means that the dynamics is Markovian. Now we turn the noise on and increase its amplitude Ω , from zero until we reach a situation where the decay rate $\gamma_1(t)$ touches the horizontal axis. This is the transition point: increasing the amplitude a little bit further means that, for some time t , the decay rate will be negative and the dynamics will become non-Markovian. The value at which it happens is $\Omega = 0.91$. The coefficients $\eta(t)$ and $\gamma_1(t)$ are plotted in Fig. 7. Now that the transition has been established, we fix the pump and analyze the situation for another two different temperatures, one in the Markovian regime, $T = 0.50\omega_0$, and another in the non-Markovian regime, $T = 0.10\omega_0$. The decay rate $\gamma_1(t)$ for these temperatures is plotted in Fig. 7(b).

^e since BLP non-Markovianity is a stronger condition, it can only be achieved after the transition to RHP non-Markovianity has happened.

Figure 7 – (a) $\eta(t)$ and (b) $\gamma_1(t)$ as a function of time using the exponential pump for three different temperatures: $T = 0.10\omega_0$, $T = 0.33\omega_0$ and $T = 0.50\omega_0$. Note that the condition to achieve RHP non-Markovianity is $\eta_{\min} < 0$. All the curves used: $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.91$, $\gamma = 1$.

Source: By the author.

Now we calculate the average energy of the system. If we assume that its initial state is

$$\rho(0) = |+\rangle \langle +|, \quad (6.80)$$

where

$$|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \quad (6.81)$$

then the population $\rho_{11}(t)$ and, consequently, the average energy of the system, Eq. (6.44), can be calculated. The average energy with (E_P) and without (E_{NP}) pump, as well as the difference

$$\Delta E = E_P - E_{NP}, \quad (6.82)$$

at $T = 0.33\omega_0$, are shown in Fig. 8(a), in units of ω_0 . Note that the average energy tends to a bigger value when the pump acts on the system, than when it is absent.

Figure 8 – Average energy of the system, in units of ω_0 , with (E_P) and without (E_{NP}) exponential pump. Inset: average energy difference, defined as $\Delta E = E_P - E_{NP}$. (a) RHP Markovian regime, with $T = 0.33\omega_0$, and (b) RHP non-Markovian regime, with $T = 0.10\omega_0$. The parameters used were: $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.91$, $\gamma = 1$.

Source: By the author.

Now the temperature is lowered to $T = 0.1\omega_0$, where RHP non-Markovian effects are present, and we compare it to the case where $T = 0.33\omega_0$. The average energy is

plotted in Fig. 8(b), and the decay rate $\gamma_1(t)$ and the ACHL measure $f(t)$ ^f are plotted in Fig. 9. We can see that the average energy increases compared to the Markovian case, due to its dependence with the density of the states, $\langle E \rangle \propto 1/N(\omega_0)$. The energy difference, however, remains small, which suggests that in this situation we can induce RHP non-Markovianity without too much energy cost. For the decay rate $\gamma_1(t)$, negative regions can be observed, which is a sign of non-Markovianity. These regions generate the peaks in the ACHL measure. The values obtained for the measure were $\mathcal{N}_{\text{ACHL}} = 0.0546$ for $T = 0.1 \omega_0$ and $\mathcal{N}_{\text{ACHL}} = 0.0046$ for $T = 0.33 \omega_0$.

Figure 9 – (a) Decay rate $\gamma_1(t)$ as a function of time in RHP non-Markovian regime using the exponential pump for two different temperatures, $T = 0.1 \omega_0$ and $T = 0.33 \omega_0$. (b) ACHL measure $f(t)$ as a function of time in the same regime, for the temperatures $T = 0.1 \omega_0$ and $T = 0.3 \omega_0$. The parameters used were: $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.91$, $\gamma = 1$.

Source: By the author.

6.4.2 Squared exponential noise

Now we repeat the previous analysis, but for the squared exponential correlation function, Eq. (6.73),

$$\mathcal{S}_{\text{SE}}(\tau) = \frac{\Omega}{\sqrt{\pi}\tau_c} e^{-(t/\tau_c)^2}, \quad (6.83)$$

and for the same parameters as in the exponential case, we reach the RHP Markovian limit with an amplitude of $\Omega = 0.47$. The average energies for $T = 0.33 \omega_0$ and $T = 0.10 \omega_0$ are plotted in Fig. 10. Note that, for this noise, the difference in the energies is almost negligible. The decay rate $\gamma_1(t)$ and the ACHL measure $f(t)$ are plotted in Fig. 11. In this case, The values for the measure were $\mathcal{N}_{\text{ACHL}} = 0.0585$ for $T = 0.10 \omega_0$ and $\mathcal{N}_{\text{ACHL}} = 0.0040$ for $T = 0.30 \omega_0$.

6.4.3 Power law noise

Finally, for the power law noise, Eq. (6.75),

$$\mathcal{S}_{\text{PL}}(\tau) = \frac{(\alpha - 1)}{2} \frac{\Omega}{\tau_c} \frac{1}{(\tau/\tau_c + 1)^\alpha}, \quad (6.84)$$

^f The ACHL measure is actually the integral of $f(t)$.

Figure 10 – Average energy of the system with (E_P) and without (E_{NP}) squared exponential pump. Inset: average energy difference, defined as $\Delta E = E_P - E_{NP}$. (a) RHP Markovian regime, with $T = 0.33\omega_0$, and (b) RHP non-Markovian regime, with $T = 0.10\omega_0$. The parameters used were: $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.47$, $\gamma = 1$.

Source: By the author.

Figure 11 – (a) Decay rate $\gamma_1(t)$ as a function of time in RHP non-Markovian regime using the square exponential pump for two different temperatures, $T = 0.1\omega_0$ and $T = 0.3\omega_0$ (b) ACHL measure $f(t)$ as a function of time in the same regime, for the temperatures $T = 0.1\omega_0$ and $T = 0.3\omega_0$. The parameters used were: $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.47$, $\gamma = 1$.

Source: By the author.

where $\alpha > 2$, we have a global minimum (using $\alpha = 3$) for $\Delta\omega\tau_c = 20$, and the RHP Markovian limit is reached with $\Omega = 1.35$. The average energies for $T = 0.33\omega_0$ (RHP Markovian regime) and $T = 0.10\omega_0$ (RHP non-Markovian regime) are plotted in Fig. 12. The decay rate $\gamma_1(t)$ and the ACHL measure $f(t)$ are plotted in Fig. 13(b). The value obtained for the measure were $\mathcal{N}_{\text{ACHL}}$ is 0.0532 for $T = 0.1\omega_0$ and $\mathcal{N}_{\text{ACHL}}$ is 0.0058 for $T = 0.3\omega_0$.

6.4.4 Comparisons

In this section we compare the three situations above. The decay rates $\gamma_1(t)$ for the three pumps are plotted in Fig. 14(a) and the average energies in Fig. 14(b), for the RHP non-Markovian case ($T = 0.1\omega_0$). The ACHL measure function $f(t)$ for the three studied noises, with the parameters as in previous sections and for the case $T = 0.1\omega_0$, is shown in Fig. 15(a). Note that the ACHL measure corresponds to the area under the functions,

Figure 12 – Average energy of the system with (E_P) and without (E_{NP}) power law pump. Inset: average energy difference, $\Delta E = E_P - E_{NP}$, in (a) RHP Markovian regime, with $T = 0.33 \omega_0$, and (b) RHP non-Markovian regime, with $T = 0.10 \omega_0$. The parameters used were: $\Delta\omega = 2$, $\tau_c = 10$, $\Omega = 1.35$, $\gamma = 1$.

Source: By the author.

Figure 13 – (a) Decay rate $\gamma_1(t)$ as a function of time in RHP non-Markovian regime using the power law pump for two different temperatures, $T = 0.1 \omega_0$ and $T = 0.3 \omega_0$ (b) ACHL measure $f(t)$ as a function of time in the same regime, for the temperatures $T = 0.1 \omega_0$ and $T = 0.3 \omega_0$. The parameters used were: $\Delta\omega = 2$, $\tau_c = 10$, $\Omega = 1.35$, $\gamma = 1$.

Source: By the author.

and its numerical values are given in Table 2. Since the ACHL measure is clearly positive, the RHP non-Markovianity of the evolution is verified. The values of the measure are approximately $\mathcal{N}_{\text{ACHL}}^{\text{OU}} = 0.0546$, $\mathcal{N}_{\text{ACHL}}^{\text{SE}} = 0.0585$ and $\mathcal{N}_{\text{ACHL}}^{\text{PL}} = 0.0532$ for the exponential, squared exponential and power law noises, respectively. In that regime, the measure has approximately the same value for the three noises.

For a better comparison among the RHP non-Markovianity generated by the noises, we plotted in Fig. 15(b) the ACHL measure using the parameters of the power law case, where we find RHP non-Markovianity for all the three cases, and the numerical values for the ACHL measure are in Table 2. It can be seen that, for the same parameters, the squared exponential noise shows a bigger value in the ACHL measure than the others. Namely, the values are $\mathcal{N}_{\text{ACHL}}^{\text{OU}} = 0.1047$, $\mathcal{N}_{\text{ACHL}}^{\text{SE}} = 0.1651$ and $\mathcal{N}_{\text{ACHL}}^{\text{PL}} = 0.0532$ for the exponential, squared exponential and power law noises, respectively.

Until now we only analyzed RHP non-Markovianity. What about BLP non-

Figure 14 – (a) Decay rate $\gamma_1(t)$ as a function of time in RHP non-Markovian regime for different noises. The parameters used were: $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.91$ (OU); $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.47$ (SE); $\Delta\omega = 2$, $\tau_c = 10$, $\Omega = 1.35$ (PL). (b) Average energy for different noises. The parameters used were: $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.91$ (OU); $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.47$ (SE); $\Delta\omega = 2$, $\tau_c = 10$, $\Omega = 1.35$ (PL). Inset: comparison of average energies on a better scale. All the curves used: $\gamma = 1$, $T = 0.1\omega_0$.

Source: By the author.

Markovianity? Note that we are in the weak coupling regime, and therefore the noise amplitude Ω must be restricted to small values. This constraint implies that the other decay rate $\gamma_2(t)$ is always positive, since the noise strength cannot be large enough to overcome the temperature term $\gamma[N(\omega) + 1]$ without violating the weak coupling condition. Moreover, the sum of the decay rates is always positive as well, since the magnitude of the positive rate results much larger than the magnitude of the negative rate. Therefore, although our evolution can be non-Markovian in the RHP sense, it can never be non-Markovian by the BLP definition, since the BLP criterion for quantum non-Markovianity (negativity of the sum of the decay rates) is never satisfied.

Table 2 – Numerical value of the ACHL measure for the three noises. The parameters for the second column were: $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.91$ (OU); $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.47$ (SE); $\Delta\omega = 2$, $\tau_c = 10$, $\Omega = 1.35$ (PL). The parameters for the third column were: $\Delta\omega = 2$, $\tau_c = 10$, $\Omega = 1.35$. In both cases, we used $\gamma = 1$, $T = 0.1\omega_0$.

Noise	ACHL Measure (different parameters)	ACHL Measure (same parameters)
Exponential (OU)	0.0546	0.1047
Squared Exponential (SE)	0.0585	0.1651
Power Law (PL)	0.0532	0.0532

Source: By the author.

Figure 15 – ACHL measure $f(t)$ for different noises. (a) The parameters used were: $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.91$ (OU); $\Delta\omega = 2$, $\tau_c = 4.25$, $\Omega = 0.47$ (SE); $\Delta\omega = 2$, $\tau_c = 10$, $\Omega = 1.35$ (PL). (b) The parameters used were: $\Delta\omega = 2$, $\tau_c = 10$, $\Omega = 1.35$. All the curves used: $\gamma = 1$, $T = 0.1\omega_0$.

Source: By the author.

7 CONCLUSION

In this dissertation we studied the evolution of a quantum system interacting with a thermal reservoir and subjected to a stochastic pump. This pump was responsible for inducing quantum non-Markovianity in the evolution, which was subsequently measured and verified.

The studied example was a two-level system (qubit) exchanging energy with a bosonic reservoir at thermal equilibrium and, at the same time, interacting with an stochastic Hamiltonian. The system was initially uncoupled to its environment and their coupling was assumed to be weak. Also, the reservoir was assumed to be infinite, which means that all information that leaks from the system propagates through all the bath degrees of freedom and is lost very fast, so the reservoir retains no memory of the past states of the system. Working in a timescale coarser than this "reservoir memory" timescale, we could use the Born Markov and the rotating wave approximations (1,2), which led the system, with the external pump turned off, to evolve under Markovian dynamics.

When we turned on the pump, however, the situation was different. The stochastic Hamiltonian also exchanged energy with the system, and was assumed to behave like a classical Gaussian noise with zero average. Using the stochastic wavefunction formalism (60), where the resulting density matrix of the system is recovered after an average over all the stochastic realizations, we could find the master equation obeyed by the system. The nature of the noise term (a Gaussian noise) allowed the evolution to be described by a Lindblad-like master equation, which now could be Markovian as well as non-Markovian.

In order to study this dynamics, we considered three different noise forms for the stochastic pump: exponential (Ornstein Uhlenbeck), squared exponential and power law noises. (63) We first subjected the system to an environment at a temperature of $T = 0.33 \omega_0$ and found the minimum values of the parameters of each noise required to reach the Markovian to non-Markovian transition. That means that we sought after the minimum noise amplitude necessary to overcome the decohering effects of the bath and reverse the energy flow. After fixing the noise parameters for this Markovianity limit, we cooled the environment to $T = 0.10 \omega_0$. Since the environment contribution to Markovianity was now smaller and the pump contribution to non-Markovianity remained the same, we should have non-Markovian dynamics. Indeed, the ACHL measure, which is based on the negativity of the decay rates of the master equation, was used and returned nonzero values for all three cases, which was a sign of non-Markovianity. After that we used the same parameters for all the three noises, in order to investigate which noise could induce more non-Markovianity. Again using the ACHL measure, the squared exponential noise return the biggest value.

We also analyzed the average energy of the system under this dynamics. The non-Markovianity caused this energy to oscillate in time, and to reach a slightly bigger value than the one which was achieved when the pump was absent. This energy difference, however, was quite small, which suggests that the energy cost necessary to implement this Markovian to non-Markovian transition is very low. Also, this value was small for all the three noises.

The ACHL measure returns nonzero values when at least one of the decay rates assumes negative values. This is equivalent to the RHP definition of quantum non-Markovianity, which was detected in our evolution. The BLP non-Markovianity, however, was not present: in our example, it requires the sum of the two decay rates to be negative. Although one of them sometimes assumed negative values, the other one was always strictly positive. The BLP non-Markovian regime could not be reached since the weak coupling assumption required the noise amplitude to be small. Using bigger noise strengths would violate the hypotheses of the model and result in nonphysical states.

Our result promises a great alternative to reservoir engineering techniques (34,50–57) in order to induce quantum non-Markovianity: instead of fine-tuning the spectral density of the environment, we could simply inject noise directly to the system and achieve the same results. This could be a cheaper, simpler and less time-consuming alternative.

Which physical experiment could be used to verify our proposals? A good candidate, suggested in Ref. (74), are single trapped ions. (156–158) In a linear Paul trap, we could choose two internal metastable electronic levels of the ion as the two-level system. The light-ion Hamiltonian, under some approximations (159), has three different terms: the Carrier, Red-Side Band and Blue-side Band transitions. For our model, the Red-Side Band interaction could be used to mimic the amplitude damping Hamiltonian and the stochastic time-dependent light field in the Carrier transition could model the noise pump.

Our results open up many possibilities for future research directions. For example, we can study if the non-Markovian features of the model are relevant to decoherence suppression or preservation of information, or if they could speed-up quantum evolutions, in the context of quantum speed limits. (23) As an possible application, we could use this model for a quantum thermal machine (160,161) and analyze if the stochastic pump could provide any thermodynamical features to this system, such as increased efficiencies of cycles.

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Appendix

APPENDIX A – LINEAR ALGEBRA REVIEW

In this Appendix we review some basic definitions and theorems from linear algebra which are used in the main text. The interested reader should refer to (9, 78, 85, 86) for a complete exposition of the subject.

A.1 Metric spaces

Given a set X , a *metric* is a function $d: X \times X \rightarrow \mathbb{R}$ which is faithful, symmetric and satisfies the triangle inequality, i.e., $\forall x, y, z \in X$,

$$d(x, y) = 0 \iff x = y, \quad (\text{A.1})$$

$$d(x, y) = d(y, x), \quad (\text{A.2})$$

$$d(x, y) \leq d(x, z) + d(z, y). \quad (\text{A.3})$$

If X is a vector space over the field $\mathbb{F}$ of real or complex numbers, a *norm* is a function $\|\cdot\|: X \rightarrow \mathbb{R}_+$ which is faithful, homogeneous and subadditive, i.e., $\forall x, y \in X, \forall \alpha \in \mathbb{F}$,

$$\|x\| = 0 \iff x = 0, \quad (\text{A.4})$$

$$\|\alpha x\| = |\alpha| \|x\|, \quad (\text{A.5})$$

$$\|x + y\| \leq \|x\| + \|y\|. \quad (\text{A.6})$$

Finally, an *inner product* is a function $(\cdot, \cdot): X \times X \rightarrow \mathbb{C}$ which is positive-definite, faithful, linear in the second argument and conjugate-linear, i.e., $\forall x, y, z \in X, \forall \alpha, \beta \in \mathbb{F}$,

$$(x, x) \geq 0, \quad (\text{A.7})$$

$$(x, x) = 0 \iff x = 0, \quad (\text{A.8})$$

$$(x, \alpha y + \beta z) = \alpha(x, y) + \beta(x, z), \quad (\text{A.9})$$

$$(x, y)^* = (y, x). \quad (\text{A.10})$$

The ordered pairs (X, d) , $(X, \|\cdot\|)$ and $(X, (\cdot, \cdot))$ are, respectively, a *metric space*, a *normed space* and an *inner product space*. Every inner product space is a normed space with the induced norm $\|x\| = \sqrt{(x, x)}$, and every normed space is a metric space with the induced metric $d(x, y) = \|x - y\|$. Conversely, a metric (on a vector space) induces a norm if it is homogeneous and translationally invariant, i.e., $\forall x, y, z \in X, \forall \alpha \in \mathbb{F}$,

$$d(\alpha x, \alpha y) = |\alpha| d(x, y), \quad (\text{A.11})$$

$$d(x + z, y + z) = d(x, y), \quad (\text{A.12})$$

and $\|x\| = d(x, 0)$, and a norm induces an inner product if it satisfies the parallelogram law,

$$2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2, \quad (\text{A.13})$$

$\forall x, y \in X$. The induced inner product, for a complex vector space, is given by the polarization identity,

$$(x, y) = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 - i\|x + iy\|^2 + i\|x - iy\|^2 \right). \quad (\text{A.14})$$

A metric space is *complete* if every Cauchy sequence (85) converges in it; a normed space is a *Banach space* if it is complete with respect to the metric induced by the norm; and an inner product space is a *Hilbert space* if it is complete with respect to the metric induced by the inner product.

A.2 Linear operators

In this dissertation, all vector spaces are, unless otherwise stated, finite-dimensional vector spaces over the complex field. We use the Dirac notation, which was already introduced in Chapter 2. The *linear span* of a set $S \in \mathcal{X}$ is the set composed of all finite linear combinations of vectors from S . A linearly independent set of vectors $\{|i\rangle\}_{i=1,\dots,n}$ is a *basis* for the Hilbert space $\mathcal{X}$ if the space is equal to its linear span. Furthermore, it is an *orthonormal basis* if $\langle i|j\rangle = \delta_{i,j}$ for every i and j , i.e., the vectors are mutually orthogonal and normalized. Given $\mathcal{X}_A$ and $\mathcal{X}_B$, the tensor product of them is $\mathcal{X}_A \otimes \mathcal{X}_B$, whose elements are generated by the basis $\{|i\rangle_A \otimes |j\rangle_B\}$, $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ bases of $\mathcal{X}_A$ and $\mathcal{X}_B$, respectively.

Theorem 15. (*Schmidt decomposition*) For every nonzero $|\psi\rangle \in \mathcal{X}_A \otimes \mathcal{X}_B$ there is a unique positive number r , a set of positive real numbers $\{s_i\}$ and orthonormal sets $\{|x_i\rangle\}$ of $\mathcal{X}_A$ and $\{|y_i\rangle\}$ of $\mathcal{X}_B$ such that

$$|\psi\rangle = \sum_{i=1}^r s_i |x_i\rangle \otimes |y_i\rangle, \quad (\text{A.15})$$

which is called the *Schmidt decomposition* of $|\psi\rangle$, and r its *Schmidt number*. (9, 78)

If $r = 1$, $|\psi\rangle$ is called *separable*, because it can be written as a tensor product of two vectors, one in $\mathcal{X}_A$ and the other in $\mathcal{X}_B$. Otherwise, if $r > 1$ it is called *entangled*, because such a decomposition is not possible.

The set of linear operators from $\mathcal{X}_1$ to $\mathcal{X}_2$ is denoted by $\mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$, and is itself a vector space. Given $\{|i_1\rangle\}$ and $\{|i_2\rangle\}$ orthonormal bases of $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively, the set $\{E_{ij}\}$, $E_{ij} = |i_2\rangle\langle j_1|$ is an orthonormal basis for $\mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$. Also, any $T \in \mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$ can be represented as a $d_2 \times d_1$ matrix (d_2 and d_1 the dimensions of $\mathcal{X}_2$ and $\mathcal{X}_1$, respectively) whose elements T_{ij} are given by

$$T_{ij} = \langle i_2|T|j_1\rangle. \quad (\text{A.16})$$

With the matrix representation we can define the *transpose operator* $T^T \in \mathbf{L}(\mathcal{X}_2, \mathcal{X}_1)$ of $T \in \mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$, defined by $T_{ij}^T = T_{ji}$. For multipartite linear operators we can define the

partial transpose. For example, for $T \in \mathbf{L}(\mathcal{X}_1 \otimes \mathcal{X}_2)$, we can define the partial transpose with respect to 2, $T^{T_2} = (\mathbb{1}_{\mathcal{X}_1} \otimes \Phi)T$, and with respect to 1, $T^{T_1} = (\Phi \otimes \mathbb{1}_{\mathcal{X}_2})T$, where Φ denotes the transposition operation.

Theorem 16. (*Singular value decomposition*) For every $T \in \mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$, with rank (the dimension of its image set) equal to r , there is a unique set of positive real numbers $\{s_i\}$ (called singular values) and orthonormal sets $\{|R_i\rangle\}$ (right singular vectors) of $\mathcal{X}_1$ and $\{|L_i\rangle\}$ (left singular vectors) of $\mathcal{X}_2$ such that

$$T = \sum_{i=1}^r s_i |L_i\rangle \langle R_i|, \quad (\text{A.17})$$

which is called the singular value decomposition of T . (78)

A.3 Normal operators

The *adjoint* of $T \in \mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$ is the unique operator $T^\dagger \in \mathbf{L}(\mathcal{X}_2, \mathcal{X}_1)$ which satisfies

$$\langle \phi | T^\dagger | \psi \rangle^* = \langle \psi | T | \phi \rangle, \quad (\text{A.18})$$

for every $|\psi\rangle \in \mathcal{X}_1$, $|\phi\rangle \in \mathcal{X}_2$. In terms of the matrix elements, we have that $T_{ij}^\dagger = T_{ji}^*$. A map $T \in \mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$ is a (linear) *isometry* if $T^\dagger T = \mathbb{1}_{\mathcal{X}_1}$ or, equivalently, if it preserves the inner product between any two vectors. An operator $T \in \mathbf{L}(\mathcal{X}, \mathcal{X}) \equiv \mathbf{L}(\mathcal{X})$ is *normal* if $[T, T^\dagger] = 0$, *Hermitian* if $T = T^\dagger$ and *unitary* if $T^\dagger T = TT^\dagger = \mathbb{1}_{\mathcal{X}}$, where $\mathbb{1}_{\mathcal{X}}$ is the identity operator on $\mathcal{X}$. In particular, Hermitian and unitary operators are normal. An Hermitian operator is a *projection* if $T^2 = T$, and positive semidefinite (denoted $T \geq 0$) if $T = A^\dagger A$, for some $A \in \mathbf{L}(\mathcal{X})$. Equivalently, it is positive semidefinite if, for every $|\psi\rangle \in \mathcal{X}$,

$$\langle \psi | T | \psi \rangle \geq 0. \quad (\text{A.19})$$

An operator which satisfies $\langle \psi | T | \psi \rangle > 0$ for every $|\psi\rangle \in \mathcal{X}$ is called *positive definite*. For every projection T , $T^T = T$.

A few remarks: a unitary operator is a surjective isometry, which is seen by the additional relation $TT^\dagger = \mathbb{1}_{\mathcal{X}}$. In particular, an operator in $\mathbf{L}(\mathcal{X}_1, \mathcal{X}_2)$ can be an isometry, but it can only be unitary if $\mathcal{X}_1 = \mathcal{X}_2$. A projection operator which has rank equal to one is called an *one-dimensional* or *rank-1 projection*, and can be represented as $|\psi\rangle \langle \psi|$, where $|\psi\rangle \in \mathcal{X}$.

Theorem 17. (*Spectral decomposition*) For every normal operator $T \in \mathbf{L}(\mathcal{X})$ there is a positive integer m , a unique set of distinct complex numbers $\{\lambda_i\}$ (eigenvalues) and a set of non-zero projection operators $\{\Pi_i\}$ (eigenprojections) which form a resolution of the identity, i.e.,

$$\sum_{i=1}^m \Pi_i = \mathbb{1}_{\mathcal{X}}, \quad (\text{A.20})$$

such that T can be decomposed as

$$T = \sum_{i=1}^m \lambda_i \Pi_i, \quad (\text{A.21})$$

which is called its spectral decomposition. If we consider repeated eigenvalues, the eigenprojections can be further decomposed into rank-1 projections and therefore we can write

$$T = \sum_{i=1} \lambda_i |\psi_i\rangle \langle \psi_i|, \quad (\text{A.22})$$

where $\{|\psi_i\rangle\}$ is its eigenbasis, which is orthogonal.

If T is Hermitian, its eigenvalues are real numbers; if it is positive semidefinite, the eigenvalues are nonnegative; and if it is unitary, the eigenvalues are complex numbers with unit norm. Every complex function f can be extended to normal operators, by using the spectral decomposition:

$$f(T) = \sum_{i=1}^m f(\lambda_i) \Pi_i. \quad (\text{A.23})$$

Theorem 18. (*Jordan-Hahn decomposition*) Every Hermitian operator $H \in \mathcal{X}$ can be uniquely written as $H = P - Q$, where P and Q are positive semidefinite operators such that $PQ = 0$. (78)

A.4 Trace norms

Given an orthonormal basis $\{|i\rangle\}$ of $\mathcal{X}$, the *trace* of an operator $T \in \mathbf{L}(\mathcal{X})$ is defined as^a

$$\text{tr}[T] = \sum_i \langle i|T|i\rangle. \quad (\text{A.24})$$

It is basis independent, linear and cyclic, i.e., $\text{tr}[AB] = \text{tr}[BA]$. With the trace we can define the *Schatten p -norms*,

$$\|T\|_p = \left(\text{tr} \left[\left(\sqrt{T^\dagger T} \right)^p \right] \right)^{\frac{1}{p}}, \quad (\text{A.25})$$

where $p \geq 1$ is real. These are norms which turn $\mathbf{L}(\mathcal{X})$ into a Banach space. The 1-norm is known as the *trace norm*,

$$\|T\|_1 = \text{tr} \left[\sqrt{T^\dagger T} \right]. \quad (\text{A.26})$$

We have that, for Hermitian operators, $\|T\|_1 = \sum_i |\lambda_i|$, where λ_i are its eigenvalues. Therefore, $\|T\|_1 = \text{tr}[T]$ if and only if T is a positive semidefinite operator. Also, for any operator T , $\text{tr}[T] \leq \|T\|_1$. The distance function induced by the trace norm is the *trace distance*,

$$D_1(A, B) = \frac{1}{2} \|A - B\|_1, \quad (\text{A.27})$$

^a We do not concern ourselves with convergence issues since we are only working with finite dimensional vector spaces.

which for rank-1 projections assumes the simpler form

$$D_1(|\psi\rangle, |\phi\rangle) = \sqrt{1 - |\langle\psi|\phi\rangle|^2}. \quad (\text{A.28})$$

The 2-norm is the *Hilbert-Schmidt norm*,

$$\|T\|_2 = \sqrt{\text{tr}[T^\dagger T]}. \quad (\text{A.29})$$

Since the 2-norm satisfies the parallelogram law, $(\mathbf{L}(\mathcal{H}), \|\cdot\|_2)$ is also a Hilbert space, with the *Hilbert-Schmidt inner product*

$$\langle A, B \rangle = \text{tr}[A^\dagger B]. \quad (\text{A.30})$$

A.5 Linear superoperators

The set of linear operators from $\mathbf{L}(\mathcal{X}_1)$ to $\mathbf{L}(\mathcal{X}_2)$ is denoted by $\mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$ ^b, and its elements are usually called *superoperators*, since they are operators that act on operators. To avoid confusion, the identity operator in $\mathbf{L}(\mathcal{X})$ will be denoted as $\mathcal{I}_{\mathcal{X}}$ instead of $\mathbb{1}_{\mathbf{L}(\mathcal{X})}$. The trace can be seen as a superoperator

$$\begin{aligned} \text{tr}: \mathbf{L}(\mathcal{X}) &\rightarrow \mathbb{C} \\ T &\mapsto \text{tr}[T], \end{aligned} \quad (\text{A.31})$$

and we can define, on a composite system $\mathbf{L}(\mathcal{X}_A \otimes \mathcal{X}_B)$, the partial traces $\text{tr}_A = \text{tr} \otimes \mathcal{I}_{\mathcal{X}_B}$ and $\text{tr}_B = \mathcal{I}_{\mathcal{X}_A} \otimes \text{tr}$.

A.6 Density operators

An operator $\rho \in \mathbf{L}(\mathcal{X})$ is a density operator if it is positive semidefinite and with unit trace:

$$\rho \geq 0, \quad \text{tr}\rho = 1. \quad (\text{A.32})$$

In particular, positive semidefinite operators are also Hermitian, so $\rho^\dagger = \rho$. The set of density operators on $\mathcal{X}$ is denoted by $\mathbf{D}(\mathcal{X})$. Since they are positive semidefinite, they can be decomposed as

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|, \quad (\text{A.33})$$

where the $|\psi_i\rangle$ are mutually orthogonal and normalized, and $p_i \geq 0$ for every i . The trace constraint imposes that $\sum_i p_i = 1$. The set $\mathbf{D}(\mathcal{X})$ is convex, i.e., if ρ and σ are density operators and $\lambda \in [0, 1]$, then $\lambda\rho + (1 - \lambda)\sigma$ is a density operator too: that means that $\mathbf{D}(\mathcal{X})$ is closed under "mixing" of operators. An extremal element of a convex set is an element which cannot be written as a nontrivial combination of two other states of the set, and is sometimes called "pure", opposed to general "mixed" states. The pure states

^b This has nothing to do with square integrable functions, despite the similarity in notation.

of $\mathbf{D}(\mathcal{X})$ are projections, which, since have unit trace, are rank-1 projections^c. Rank-1 projections can be written as $|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is a unit vector in $\mathcal{X}$ which is unique up to scalar multiplication. The trace of the square of a density operator, $\text{tr}[\rho^2]$, is called its purity, and we have that $d^{-1} \geq \text{tr}[\rho^2] \leq 1$, where d is the dimension of $\mathcal{X}$. The upper bound is achieved with pure states (since they are projections and, therefore, $\rho^2 = \rho$) and the lower bound with the normalized identity, $d^{-1}\mathbb{1}_{\mathcal{X}}$.

If $\rho \in \mathbf{D}(\mathcal{X})$ and $\mathcal{Y}$ is some Hilbert space, a vector $|\Psi\rangle \in \mathcal{X} \otimes \mathcal{Y}$ is a *purification* of ρ if

$$\rho = \text{tr}_{\mathcal{Y}} [|\Psi\rangle\langle\Psi|]. \quad (\text{A.34})$$

A purification of ρ exists if the dimension of $\mathcal{Y}$ is bigger than or equal the rank of ρ . (78) If $\{|i_{\mathcal{Y}}\rangle\}$ is a basis for $\mathcal{Y}$ and ρ has the decomposition of Eq. (A.33), the vector

$$|\Psi\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle \otimes |i_{\mathcal{Y}}\rangle \quad (\text{A.35})$$

is a purification of ρ , which can be easily verified. Although the purification of a density operator is not uniquely defined, all of them are related by a partial isometry:

Theorem 19. (*Equivalence of purifications*) Let $\rho \in \mathbf{D}(\mathcal{X})$ be a density operator and $|\Psi_1\rangle \in \mathcal{X} \otimes \mathcal{Y}_1$, $|\Psi_2\rangle \in \mathcal{X} \otimes \mathcal{Y}_2$ purifications of it (suppose, without loss of generality, that $\dim(\mathcal{Y}_2) \geq \dim(\mathcal{Y}_1)$). Then, there is an isometry $V \in \mathbf{L}(\mathcal{Y}_1, \mathcal{Y}_2)$ such that

$$|\Psi_2\rangle = (\mathbb{1}_{\mathcal{X}} \otimes V) |\Psi_1\rangle. \quad (\text{A.36})$$

If $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Y}$, then $V \in \mathbf{L}(\mathcal{Y})$ is an unitary operator. (91)

^c This is an important remark. A density operator is a projection if, and only if, it is a rank-1 projection.

APPENDIX B – COMPOSITION OF CHOI MATRICES

In this appendix we show how Choi matrices are sequentially composed.

Theorem 20. (*Composition of Choi matrices*). Given $\mathcal{F} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_2)$, $\mathcal{E} \in \mathbf{L}^2(\mathcal{X}_2, \mathcal{X}_3)$ and $\mathcal{G} = \mathcal{E} \circ \mathcal{F} \in \mathbf{L}^2(\mathcal{X}_1, \mathcal{X}_3)$, $\Lambda_{\mathcal{G}}$ is given by the link product (97, 98)

$$\Lambda_{\mathcal{G}} = \Lambda_{\mathcal{E}} \star \Lambda_{\mathcal{F}} \equiv \text{tr}_2 \left[\left(\Lambda_{\mathcal{F}}^{T_2} \otimes \mathbb{1}_{\mathcal{X}_3} \right) (\mathbb{1}_{\mathcal{X}_1} \otimes \Lambda_{\mathcal{E}}) \right], \quad (\text{B.1})$$

where $\Lambda_{\mathcal{F}}^{T_2}$ denotes the partial transpose of $\Lambda_{\mathcal{F}}$ with respect to $\mathcal{X}_2$ (See Appendix A).

Proof. Using Eq. (2.28), we have that

$$\mathcal{F}(X) = \text{tr}_1 \left[\left(X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_2} \right) \Lambda_{\mathcal{F}} \right], \quad X \in \mathbf{L}(\mathcal{X}_1) \quad (\text{B.2})$$

$$\mathcal{E}(Y) = \text{tr}_2 \left[\left(Y^{T_2} \otimes \mathbb{1}_{\mathcal{X}_3} \right) \Lambda_{\mathcal{E}} \right], \quad Y \in \mathbf{L}(\mathcal{X}_2) \quad (\text{B.3})$$

$$\mathcal{G}(X) = \text{tr}_1 \left[\left(X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_3} \right) \Lambda_{\mathcal{G}} \right], \quad X \in \mathbf{L}(\mathcal{X}_1), \quad (\text{B.4})$$

where we used the partial transpose (even though, in this case, it is a "total" transpose) just to keep track of which space is being acted upon. Since $\mathcal{G} = \mathcal{E} \circ \mathcal{F}$, composing Eqs. (B.3) and (B.4),

$$\begin{aligned} \mathcal{G}(X) &= \mathcal{E} \circ \mathcal{F}(X) \\ &= \text{tr}_2 \left[\left(\{ \mathcal{F}(X) \}^{T_2} \otimes \mathbb{1}_{\mathcal{X}_3} \right) \Lambda_{\mathcal{E}} \right] \\ &= \text{tr}_2 \left[\left(\left\{ \text{tr}_1 \left[\left(X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_2} \right) \Lambda_{\mathcal{F}} \right] \right\}^{T_2} \otimes \mathbb{1}_{\mathcal{X}_3} \right) \Lambda_{\mathcal{E}} \right] \\ &= \text{tr}_2 \left[\left(\left\{ \text{tr}_1 \left[\left(X^{T_1, T_2} \otimes \mathbb{1}_{\mathcal{X}_2}^{T_2} \right) \Lambda_{\mathcal{F}}^{T_2} \right] \right\} \otimes \mathbb{1}_{\mathcal{X}_3} \right) \Lambda_{\mathcal{E}} \right] \\ &= \text{tr}_2 \left[\left(\left\{ \text{tr}_1 \left[\left(X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_2} \right) \Lambda_{\mathcal{F}}^{T_2} \right] \right\} \otimes \mathbb{1}_{\mathcal{X}_3} \right) \Lambda_{\mathcal{E}} \right], \end{aligned} \quad (\text{B.5})$$

since $X^{T_2} = X$ (because X lives in $\mathcal{X}_1$) and $\mathbb{1}_{\mathcal{X}_2}^{T_2} = \mathbb{1}_{\mathcal{X}_2}$ (the identity is equal to its transpose). Putting the trace over $\mathcal{X}_1$ outside the expression,

$$\begin{aligned} \mathcal{G}(X) &= \text{tr}_{1,2} \left[\left(\left\{ \left(X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_2} \right) \Lambda_{\mathcal{F}}^{T_2} \right\} \otimes \mathbb{1}_{\mathcal{X}_3} \right) (\mathbb{1}_{\mathcal{X}_1} \otimes \Lambda_{\mathcal{E}}) \right] \\ &= \text{tr}_{1,2} \left[\left(X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_2} \otimes \mathbb{1}_{\mathcal{X}_3} \right) \left(\Lambda_{\mathcal{F}}^{T_2} \otimes \mathbb{1}_{\mathcal{X}_3} \right) (\mathbb{1}_{\mathcal{X}_1} \otimes \Lambda_{\mathcal{E}}) \right]. \end{aligned} \quad (\text{B.6})$$

Note that we have three terms under parentheses, and each of them lives in $\mathbf{L}(\mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3)$. The first one, $X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_2} \otimes \mathbb{1}_{\mathcal{X}_3}$, however, is the only one which is in a product state with

respect to $\mathcal{X}_2$, and furthermore the term in that space is the identity, so we have

$$\begin{aligned}
\mathcal{G}(X) &= \text{tr}_1 \left[\sum_i \langle i_2 | \left[(X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_2} \otimes \mathbb{1}_{\mathcal{X}_3}) (\Lambda_{\mathcal{F}}^{T_2} \otimes \mathbb{1}_{\mathcal{X}_3}) (\mathbb{1}_{\mathcal{X}_1} \otimes \Lambda_{\mathcal{E}}) \right] | i_2 \rangle \right] \\
&= \text{tr}_1 \left[(X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_3}) \sum_i \langle i_2 | \mathbb{1}_{\mathcal{X}_2} \left[(\Lambda_{\mathcal{F}}^{T_2} \otimes \mathbb{1}_{\mathcal{X}_3}) (\mathbb{1}_{\mathcal{X}_1} \otimes \Lambda_{\mathcal{E}}) \right] | i_2 \rangle \right] \\
&= \text{tr}_1 \left[(X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_3}) \sum_i \langle i_2 | \left[(\Lambda_{\mathcal{F}}^{T_2} \otimes \mathbb{1}_{\mathcal{X}_3}) (\mathbb{1}_{\mathcal{X}_1} \otimes \Lambda_{\mathcal{E}}) \right] | i_2 \rangle \right] \\
&= \text{tr}_1 \left[(X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_3}) \text{tr}_2 \left[(\Lambda_{\mathcal{F}}^{T_2} \otimes \mathbb{1}_{\mathcal{X}_3}) (\mathbb{1}_{\mathcal{X}_1} \otimes \Lambda_{\mathcal{E}}) \right] \right]. \tag{B.7}
\end{aligned}$$

Now, comparing it with Equation (B.4),

$$\mathcal{G}(X) = \text{tr}_1 \left[(X^{T_1} \otimes \mathbb{1}_{\mathcal{X}_3}) \Lambda_{\mathcal{G}} \right], \tag{B.8}$$

we can easily see that

$$\Lambda_{\mathcal{G}} = \text{tr}_2 \left[(\Lambda_{\mathcal{F}}^{T_2} \otimes \mathbb{1}_{\mathcal{X}_3}) (\mathbb{1}_{\mathcal{X}_1} \otimes \Lambda_{\mathcal{E}}) \right]. \tag{B.9}$$

□

APPENDIX C – INTERACTION PICTURE

Consider a system with Hamiltonian

$$H(t) = H_0 + H_{\text{int}}(t), \quad (\text{C.1})$$

and evolving under the Liouville-von Neumann equation

$$\frac{d}{dt}\rho(t) = -i [H(t), \rho(t)]. \quad (\text{C.2})$$

The term H_0 is the free Hamiltonian of the system, which is time-independent and generally well understood, while the interaction term $H_{\text{int}}(t)$ is responsible for the (possible) time dependence of the Hamiltonian and is usually a perturbation of the original system. A useful way to study this problem is to move to the *interaction picture* (75), where the free evolution is neglected and we deal only with changes due to the interaction Hamiltonian. In this picture, operators take the form

$$\tilde{O}(t) = V^\dagger(t) O V(t), \quad (\text{C.3})$$

where $V(t) = e^{-iH_0 t}$ is a unitary operator, i.e., $V^\dagger(t)V(t) = V(t)V^\dagger(t) = \mathbb{1}$. How the new density matrix $\tilde{\rho}(t)$ evolves in time? We have that (for clarity we omit the time-dependence of the operators)

$$\frac{d}{dt}\tilde{\rho} = \frac{d}{dt}(V^\dagger \rho V) = V^\dagger \left(\frac{d}{dt}\rho \right) V + \left(\frac{d}{dt}V^\dagger \right) \rho V + V^\dagger \rho \left(\frac{d}{dt}V \right). \quad (\text{C.4})$$

The first term on the right hand side of Eq. (C.4) is, using Eq. (C.2) and the unitarity of $V(t)$,

$$\begin{aligned} V^\dagger \left(\frac{d}{dt}\rho \right) V &= -i (V^\dagger H \rho V - V^\dagger \rho H V(t)) \\ &= -i (V^\dagger H V V^\dagger \rho V - V^\dagger \rho V V^\dagger H V) \\ &= -i (\tilde{H} \tilde{\rho} - \tilde{\rho} \tilde{H}) = -i [\tilde{H}, \tilde{\rho}]. \end{aligned} \quad (\text{C.5})$$

Now, since $V^\dagger V = \mathbb{1}$,

$$\frac{d}{dt}(V^\dagger V) = \left(\frac{d}{dt}V^\dagger \right) V + V^\dagger \left(\frac{d}{dt}V \right) = 0. \quad (\text{C.6})$$

Multiplying on the right by $V^\dagger$, then

$$\frac{d}{dt}V^\dagger = -V^\dagger \left(\frac{d}{dt}V \right) V^\dagger. \quad (\text{C.7})$$

We can use that to evaluate the remaining terms in Eq. (C.4):

$$\begin{aligned}
\left(\frac{d}{dt}V^\dagger\right)\rho V + V^\dagger\rho\left(\frac{d}{dt}V\right) &= -V^\dagger\left(\frac{d}{dt}V\right)V^\dagger\rho V + V^\dagger\rho(VV^\dagger)\left(\frac{d}{dt}V\right) \\
&= -V^\dagger\frac{d}{dt}V\tilde{\rho} + \tilde{\rho}V^\dagger\frac{d}{dt}V \\
&= -\left[V^\dagger\frac{d}{dt}V, \tilde{\rho}\right].
\end{aligned} \tag{C.8}$$

Therefore, Eq. (C.4) is now

$$\frac{d}{dt}\tilde{\rho}(t) = -i\left[\tilde{H}(t) - iV^\dagger(t)\frac{d}{dt}V(t), \tilde{\rho}(t)\right]. \tag{C.9}$$

Using $V(t) = e^{-iH_0t}$, then

$$-iV^\dagger(t)\frac{d}{dt}V(t) = -ie^{iH_0t}\left(-iH_0e^{-iH_0t}\right) = -H_0, \tag{C.10}$$

since an operator commutes with a differentiable function of it. (85) For the same reason, we have that $\tilde{H}_0(t) = H_0$. Therefore, since $\tilde{H}(t) - \tilde{H}_0(t) = \tilde{H}_{\text{int}}(t)$, the Liouville-von Neumann equation in the interaction picture is

$$\frac{d}{dt}\tilde{\rho}(t) = -i\left[\tilde{H}_{\text{int}}(t), \tilde{\rho}(t)\right], \tag{C.11}$$

where it can be seen that the effects of the free Hamiltonian H_0 are neglected.

For the propagators, since

$$\rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0), \tag{C.12}$$

then

$$\begin{aligned}
\tilde{\rho}(t) &= V^\dagger(t)\rho(t)V(t) \\
&= V^\dagger(t)U(t, t_0)\rho(t_0)U^\dagger(t, t_0)V(t) \\
&= V^\dagger(t)U(t, t_0)\left(V(t_0)V^\dagger(t_0)\right)\rho(t_0)\left(V(t_0)V^\dagger(t_0)\right)U^\dagger(t, t_0)V(t) \\
&= \tilde{U}(t, t_0)\tilde{\rho}(t_0)\tilde{U}^\dagger(t, t_0),
\end{aligned} \tag{C.13}$$

and therefore the propagators take the form

$$\begin{aligned}
\tilde{U}(t, t_0) &= V^\dagger(t)U(t, t_0)V(t) \\
&= e^{iH_0t}U(t, t_0)e^{-iH_0t_0}.
\end{aligned} \tag{C.14}$$

C.1 Interaction term

If we are in a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, where $H_0 = H_S + H_E$ and there is an interaction term of the form

$$V = \sum_{\alpha} S_{\alpha} \otimes E_{\alpha}, \tag{C.15}$$

then its form in the interaction picture is

$$\begin{aligned}\tilde{V}(t) &= e^{i(H_S+H_E)t} V e^{-i(H_S+H_E)t} \\ &= \sum_{\alpha} \tilde{S}_{\alpha}(t) \otimes \tilde{E}_{\alpha}(t),\end{aligned}\tag{C.16}$$

where

$$\tilde{S}_{\alpha}(t) = e^{i(H_S+H_E)t} S_{\alpha} e^{-i(H_S+H_E)t} = e^{iH_S t} S_{\alpha} e^{-iH_S t},\tag{C.17}$$

$$\tilde{E}_{\alpha}(t) = e^{i(H_S+H_E)t} E_{\alpha} e^{-i(H_S+H_E)t} = e^{iH_E t} E_{\alpha} e^{-iH_E t},\tag{C.18}$$

since H_S and H_E act on different Hilbert spaces and, therefore, the operators themselves commute,

$$[H_S, H_E] = [H_S \otimes \mathbf{1}_E, \mathbf{1}_S \otimes H_E] = H_S \otimes H_E - H_S \otimes H_E = 0,\tag{C.19}$$

as well as their exponentials $e^{iH_S t}$ and $e^{iH_E t}$. (86) Assuming that the system Hamiltonian spectrum is discrete and non-degenerate, i.e.,

$$H_S = \sum_{\varepsilon} \varepsilon \Pi(\varepsilon),\tag{C.20}$$

where ε are the eigenvalues and $\Pi(\varepsilon)$ the projections corresponding to each eigensubspace, then this decomposition on Eq. (C.17) leads to

$$\begin{aligned}\tilde{S}_{\alpha}(t) &= e^{iH_S t} S_{\alpha} e^{-iH_S t} \\ &= e^{it \sum_{\varepsilon} \varepsilon \Pi(\varepsilon)} S_{\alpha} e^{-it \sum_{\varepsilon'} \varepsilon' \Pi(\varepsilon')}.\end{aligned}$$

The function of an operator is equal to the function applied to its eigenvalues, with the operator in the diagonal form. (86) Therefore,

$$\begin{aligned}\tilde{S}_{\alpha}(t) &= \sum_{\varepsilon} e^{i\varepsilon t} \Pi(\varepsilon) S_{\alpha} \sum_{\varepsilon'} e^{-i\varepsilon' t} \Pi(\varepsilon') \\ &= \sum_{\varepsilon, \varepsilon'} e^{-i(\varepsilon' - \varepsilon)t} \Pi(\varepsilon) S_{\alpha} \Pi(\varepsilon').\end{aligned}\tag{C.21}$$

Defining $\omega = \varepsilon' - \varepsilon$, we have that

$$\Pi(\varepsilon') = \sum_{\omega} \delta_{\varepsilon', \varepsilon + \omega} \Pi(\varepsilon + \omega).\tag{C.22}$$

Inserting that into the expression for $\tilde{S}_{\alpha}(t)$ leads to

$$\begin{aligned}\tilde{S}_{\alpha}(t) &= \sum_{\varepsilon, \varepsilon'} e^{-i(\varepsilon' - \varepsilon)t} \Pi(\varepsilon) S_{\alpha} \sum_{\omega} \delta_{\varepsilon', \varepsilon + \omega} \Pi(\varepsilon + \omega) \\ &= \sum_{\omega} e^{-i\omega t} \sum_{\varepsilon, \varepsilon'} \delta_{\varepsilon', \varepsilon + \omega} \Pi(\varepsilon) S_{\alpha} \Pi(\varepsilon + \omega) \\ &= \sum_{\omega} e^{-i\omega t} \sum_{\varepsilon} \Pi(\varepsilon) S_{\alpha} \Pi(\varepsilon + \omega) \equiv \sum_{\omega} e^{-i\omega t} S_{\alpha}(\omega),\end{aligned}\tag{C.23}$$

where we defined $S_\alpha(\omega)$ as

$$S_\alpha(\omega) = \sum_{\varepsilon} \Pi(\varepsilon) S_\alpha \Pi(\varepsilon + \omega). \quad (\text{C.24})$$

Taking the Hermitian conjugate, we have also this useful alternative decomposition,

$$\tilde{S}_\alpha^\dagger(t) = \sum_{\omega} e^{i\omega t} S_\alpha^\dagger(\omega). \quad (\text{C.25})$$

C.2 Back to the Schrodinger picture

How about the reverse way? We now want to recover the master equation in the original Schrodinger picture, i.e., for

$$\rho(t) = V(t) \tilde{\rho}(t) V^\dagger(t). \quad (\text{C.26})$$

Writing the master equation as

$$\frac{d}{dt} \tilde{\rho}(t) = \mathcal{L}_t [\tilde{\rho}(t)], \quad (\text{C.27})$$

where $\mathcal{L}_t$ is a (possibly time-dependent) superoperator, we can basically do the same steps of the previous sections, but using $V^\dagger(t)$ instead of $V(t)$, and vice-versa. That leads to

$$\frac{d}{dt} \rho(t) = -i \left[-iV(t) \frac{dV^\dagger(t)}{dt}, \rho(t) \right] + \left(V(t) \mathcal{L}_t V^\dagger(t) \right) [\rho(t)]. \quad (\text{C.28})$$

Since $V(t) = e^{-iH_0 t}$, we have that

$$\frac{d}{dt} \rho(t) = -i [H_0, \rho(t)] + \left(V(t) \mathcal{L}_t V^\dagger(t) \right) [\rho(t)]. \quad (\text{C.29})$$

What is the form of the operators $S_\alpha(\omega)$ in the Schrodinger picture? Doing the reverse transformation, we have that

$$\begin{aligned} V(t) S_\alpha(\omega) V^\dagger(t) &= e^{-iH_0 t} S_\alpha(\omega) e^{iH_0 t} \\ &= \sum_{\omega'} e^{i\omega' t} \sum_{\varepsilon'} \Pi(\varepsilon') S_\alpha(\omega) \Pi(\varepsilon' + \omega') \\ &= \sum_{\omega'} e^{i\omega' t} \sum_{\varepsilon'} \Pi(\varepsilon') \left(\sum_{\varepsilon} \Pi(\varepsilon) S_\alpha \Pi(\varepsilon + \omega) \right) \Pi(\varepsilon' + \omega') \\ &= \sum_{\omega', \varepsilon', \varepsilon} e^{i\omega' t} \Pi(\varepsilon') \Pi(\varepsilon) S_\alpha \Pi(\varepsilon + \omega) \Pi(\varepsilon' + \omega') \\ &= \sum_{\omega', \varepsilon', \varepsilon} e^{i\omega' t} \delta_{\varepsilon', \varepsilon} \Pi(\varepsilon) S_\alpha \delta_{\varepsilon' + \omega', \varepsilon + \omega} \Pi(\varepsilon + \omega) \\ &= \sum_{\varepsilon} e^{i\omega t} \Pi(\varepsilon) S_\alpha \Pi(\varepsilon + \omega) = e^{i\omega t} S_\alpha(\omega), \end{aligned} \quad (\text{C.30})$$

and, similarly,

$$V(t) S_\alpha^\dagger(\omega) V^\dagger(t) = e^{-i\omega t} S_\alpha^\dagger(\omega). \quad (\text{C.31})$$

With this we can calculate the Schrodinger representation of Eq. (3.179),

$$\begin{aligned} \frac{d}{dt}\tilde{\rho}_S(t) &= -i [H_{\text{LS}}, \tilde{\rho}_S(t)] \\ &+ \sum_{\alpha,\beta,\omega} \gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega)\tilde{\rho}_S(t)S_\alpha^\dagger(\omega) - \frac{1}{2} \{S_\alpha^\dagger(\omega)S_\beta(\omega), \tilde{\rho}_S(t)\} \right]. \end{aligned} \quad (\text{C.32})$$

Note that

$$\begin{aligned} V(t)H_{\text{LS}}V^\dagger(t) &= \sum_{\alpha,\beta,\omega} S_{\alpha\beta}(\omega)V(t)S_\alpha^\dagger(\omega)S_\beta(\omega)V^\dagger(t) \\ &= \sum_{\alpha,\beta,\omega} S_{\alpha\beta}(\omega)V(t)S_\alpha^\dagger(\omega)V^\dagger(t)V(t)S_\beta(\omega)V^\dagger(t) \\ &= \sum_{\alpha,\beta,\omega} S_{\alpha\beta}(\omega)e^{-i\omega t}S_\alpha^\dagger(\omega)e^{i\omega t}S_\beta(\omega) \\ &= \sum_{\alpha,\beta,\omega} S_{\alpha\beta}(\omega)S_\alpha^\dagger(\omega)S_\beta(\omega) = H_{\text{LS}}, \end{aligned} \quad (\text{C.33})$$

so the Lamb-shift Hamiltonian remains unchanged. Likewise, the rest of the right hand side of Eq. (3.179) remains unchanged, so the master equation is

$$\begin{aligned} \frac{d}{dt}\rho_S(t) &= -i [H_S + H_{\text{LS}}, \rho_S(t)] \\ &+ \sum_{\alpha,\beta,\omega} \gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega)\rho_S(t)S_\alpha^\dagger(\omega) - \frac{1}{2} \{S_\alpha^\dagger(\omega)S_\beta(\omega), \rho_S(t)\} \right]. \end{aligned} \quad (\text{C.34})$$

APPENDIX D – THERMAL STATES

D.1 Properties

Thermal states are defined as (9)

$$\rho_E = \frac{e^{-\beta H_E}}{\text{tr}_E [e^{-\beta H_E}]}, \quad (\text{D.1})$$

where $\beta = 1/T^{\text{a}}$ is the inverse temperature of the bath. Since ρ_E is a function of H_E , it follows that (139)

$$[\rho_E, H_E] = 0. \quad (\text{D.2})$$

This implies that the mean value of the bath operators is time independent:

$$\begin{aligned} \text{tr}_E [\tilde{E}_\alpha(t) \rho_E] &= \text{tr}_E [e^{iH_E t} E_\alpha e^{-iH_E t} \rho_E] \\ &= \text{tr}_E [E_\alpha e^{-iH_E t} \rho_E e^{iH_E t}] \\ &= \text{tr}_E [E_\alpha \rho_E e^{-iH_E t} e^{iH_E t}] \\ &= \text{tr}_E [E_\alpha \rho_E] = \langle E_\alpha \rangle, \end{aligned} \quad (\text{D.3})$$

where Eq. (D.2) and the cyclic property of the trace were used. Also, the RCFs are homogeneous in time:

$$\begin{aligned} \text{tr}_E [\tilde{E}_\alpha^\dagger(t) \tilde{E}_\beta(t') \rho_E] &= \text{tr}_E [e^{iH_E t} E_\alpha^\dagger e^{-iH_E t} e^{iH_E t'} E_\beta e^{-iH_E t'} \rho_E] \\ &= \text{tr}_E [e^{iH_E t} E_\alpha^\dagger e^{-iH_E t} e^{iH_E t'} E_\beta \rho_E e^{-iH_E t'}] \\ &= \text{tr}_E [e^{-iH_E t'} e^{iH_E t} E_\alpha^\dagger e^{-iH_E t} e^{iH_E t'} E_\beta \rho_E] \\ &= \text{tr}_E [e^{iH_E(t-t')} E_\alpha^\dagger e^{-iH_E(t-t')} E_\beta \rho_E] \\ &= \text{tr}_E [\tilde{E}_\alpha^\dagger(t-t') E_\beta \rho_E] = \mathcal{R}_{\alpha\beta}(t-t'). \end{aligned} \quad (\text{D.4})$$

D.2 Average of boson operators

Consider a bosonic bath described by the Hamiltonian

$$H_E = \sum_k \omega_k b_k^\dagger b_k, \quad (\text{D.5})$$

where ω_k are the modes' frequencies and $b_k^\dagger$, b_k the creation and annihilation operators for the mode k . This is a bath with an infinite number of bosons, each in one of the modes with frequency ω_k . The eigenvectors and eigenvalues of H_E , defined by the eigenvalue equation

$$H_E |\{n_k\}\rangle = \varepsilon_{\{n_k\}} |\{n_k\}\rangle, \quad (\text{D.6})$$

^a We use units such that Boltzmann's constant, k_B , is equal to one.

are

$$|\{n_k\}\rangle = |n_1, n_2, n_3, \dots\rangle, \quad (\text{D.7})$$

where $\{n_k\}$ is the set of occupation numbers, and

$$\varepsilon_{\{n_k\}} = \sum_k \omega_k n_k, \quad (\text{D.8})$$

where we disregarded the constant energy terms. The thermal state in Eq. (D.1) is now

$$\rho_E = \frac{1}{\mathcal{Z}} \sum_{\{n_k\}} e^{-\beta \varepsilon_{\{n_k\}}} |\{n_k\}\rangle \langle \{n_k\}|, \quad (\text{D.9})$$

where

$$\mathcal{Z} = \text{tr}_E [e^{-\beta H_E}] = \sum_{\{n_k\}} e^{-\beta \varepsilon_{\{n_k\}}} \quad (\text{D.10})$$

is called the partition function. (1) The average value of any operator O is given by

$$\begin{aligned} \langle O \rangle &= \text{tr}_E [O \rho_E] \\ &= \frac{1}{\mathcal{Z}} \sum_{\{n_k\}} e^{-\beta \varepsilon_{\{n_k\}}} \text{tr}_E [O |\{n_k\}\rangle \langle \{n_k\}|] \\ &= \frac{1}{\mathcal{Z}} \sum_{\{n_k\}} e^{-\beta \varepsilon_{\{n_k\}}} \langle \{n_k\} | O | \{n_k\} \rangle. \end{aligned} \quad (\text{D.11})$$

Our objective is to evaluate the average values $\langle b_k^\dagger b_{k'}^\dagger \rangle$, $\langle b_k b_{k'} \rangle$, $\langle b_k^\dagger b_{k'} \rangle$ and $\langle b_k b_{k'}^\dagger \rangle$. Since the operators $b_k^\dagger b_{k'}^\dagger$ and $b_k b_{k'}$ do not conserve the occupation numbers, we have that

$$\langle \{n_k\} | b_k^\dagger b_{k'}^\dagger | \{n_k\} \rangle = 0, \quad \langle \{n_k\} | b_k^\dagger b_{k'} | \{n_k\} \rangle = 0, \quad (\text{D.12})$$

and therefore the corresponding average values are zero, too. For the operator $b_k^\dagger b_{k'}$, note that it only conserves the occupation numbers if $k = k'$, and in that case it corresponds to the number operator for the mode k :

$$\langle \{n_k\} | b_k^\dagger b_{k'} | \{n_k\} \rangle = \delta_{kk'} n_k \langle \{n_k\} | \{n_k\} \rangle = \delta_{kk'} n_k. \quad (\text{D.13})$$

Then,

$$\langle b_k^\dagger b_{k'} \rangle = \delta_{kk'} \frac{1}{\mathcal{Z}} \sum_{\{n_k\}} n_k e^{-\beta \varepsilon_{\{n_k\}}}. \quad (\text{D.14})$$

In order to evaluate Eq. (D.14), note that

$$\frac{\partial (e^{-\beta \varepsilon_{\{n_k\}}})}{\partial (\beta \omega_k)} = -n_k e^{-\beta \varepsilon_{\{n_k\}}}, \quad (\text{D.15})$$

since the partial derivative is nonzero only when applied to the exponential $e^{-\beta \omega_k n_k}$. For all other exponentials $e^{-\beta \omega_{k'} n_{k'}}$ with $k' \neq k$, the partial derivative is zero. Then,

$$\langle b_k^\dagger b_{k'} \rangle = -\delta_{kk'} \frac{1}{\mathcal{Z}} \sum_{\{n_k\}} \frac{\partial (e^{-\beta \varepsilon_{\{n_k\}}})}{\partial (\beta \omega_k)} = -\delta_{kk'} \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial (\beta \omega_k)}. \quad (\text{D.16})$$

Now we have to calculate the partition function:

$$\begin{aligned}
\mathcal{Z} &= \sum_{\{n_k\}} e^{-\beta \varepsilon_{\{n_k\}}} \\
&= \sum_{n_1, n_2, \dots} e^{-\beta \omega_1 n_1} e^{-\beta \omega_2 n_2} \dots \\
&= \prod_j \left(\sum_{n_j} e^{-\beta \omega_j n_j} \right) = \prod_j \frac{1}{1 - e^{-\beta \omega_j}}.
\end{aligned} \tag{D.17}$$

Its derivative is, then,

$$\begin{aligned}
\frac{\partial \mathcal{Z}}{\partial (\beta \omega_k)} &= \frac{\partial}{\partial (\beta \omega_k)} \prod_j \frac{1}{1 - e^{-\beta \omega_j}} \\
&= \frac{\partial}{\partial (\beta \omega_k)} \left(\frac{1}{1 - e^{-\beta \omega_k}} \right) \prod_{j \neq k} \frac{1}{1 - e^{-\beta \omega_j}} \\
&= -\frac{e^{\beta \omega_k}}{(e^{\beta \omega_k} - 1)^2} (1 - e^{-\beta \omega_k}) \mathcal{Z} \\
&= -\frac{\mathcal{Z}}{e^{\beta \omega_k} - 1}.
\end{aligned} \tag{D.18}$$

Finally, the average value can be evaluated:

$$\langle b_k^\dagger b_{k'} \rangle = \delta_{kk'} \frac{1}{\mathcal{Z}} \frac{\mathcal{Z}}{e^{\beta \omega_k} - 1} = \delta_{kk'} \frac{1}{e^{\beta \omega_k} - 1} \equiv \delta_{kk'} N(\omega_k), \tag{D.19}$$

where $N(\omega_k)$, the average number of particles in mode k , was defined. For the last average value, $\langle b_k b_{k'}^\dagger \rangle$, from the commutation relation $[b_k^\dagger, b_{k'}] = 1$, we have that

$$\langle \{n_k\} | b_k b_{k'}^\dagger | \{n_k\} \rangle = \delta_{kk'} (1 + n_k), \tag{D.20}$$

and

$$\begin{aligned}
\langle b_k b_{k'}^\dagger \rangle &= \delta_{kk'} \frac{1}{\mathcal{Z}} \sum_{\{n_k\}} e^{-\beta \varepsilon_{\{n_k\}}} (1 + n_k) \\
&= \delta_{kk'} \left(\frac{1}{\mathcal{Z}} \sum_{\{n_k\}} e^{-\beta \varepsilon_{\{n_k\}}} + \frac{1}{\mathcal{Z}} \sum_{\{n_k\}} n_k e^{-\beta \varepsilon_{\{n_k\}}} \right) = \delta_{kk'} (1 + N(\omega_k)).
\end{aligned} \tag{D.21}$$

In summary, the average values are

$$\langle b_k b_{k'} \rangle = \langle b_k^\dagger b_{k'}^\dagger \rangle = 0, \tag{D.22}$$

$$\langle b_k b_{k'}^\dagger \rangle = \delta_{kk'} (1 + N(\omega_k)), \tag{D.23}$$

$$\langle b_k^\dagger b_{k'} \rangle = \delta_{kk'} N(\omega_k). \tag{D.24}$$

APPENDIX E – POSITIVITY OF THE DECAY RATES

In this Appendix it is shown that the matrix with the decay rates $\gamma_{\alpha\beta}(\omega)$ as coefficients is positive semidefinite and, therefore, Eq. (3.180) is in Lindblad form. (2) First, note that the $\gamma_{\alpha\beta}(\omega)$ can be expressed as the Fourier transforms of the reservoir correlation functions,

$$\begin{aligned}\gamma_{\alpha\beta}(\omega) &= \Gamma_{\alpha\beta}(\omega) + \Gamma_{\beta\alpha}^*(\omega) \\ &= \int_0^\infty d\tau e^{i\omega\tau} \text{tr} \left\{ E_\alpha^\dagger(\tau) E_\beta \rho_E \right\} + \int_0^\infty d\tau e^{-i\omega\tau} \text{tr} \left\{ E_\alpha^\dagger(-\tau) E_\beta \rho_E \right\} \\ &= \int_{-\infty}^\infty d\tau e^{i\omega\tau} \text{tr} \left\{ E_\alpha^\dagger(\tau) E_\beta \rho_E \right\}.\end{aligned}\tag{E.1}$$

A complex matrix O is positive semidefinite if $v^\dagger O v \geq 0$, for every vector v . (86) We denote the decay rates matrix as $\gamma(\omega)$, so we must have $v^\dagger \gamma(\omega) v \geq 0$ for every ω . But

$$\begin{aligned}v^\dagger \gamma(\omega_0) v &= \sum_{\alpha,\beta} v_\alpha^* \gamma_{\alpha\beta}(\omega) v_\beta \\ &= \sum_{\alpha,\beta} v_\alpha^* \int_{-\infty}^\infty d\tau e^{i\omega\tau} \text{tr} \left\{ E_\alpha^\dagger(\tau) E_\beta \rho_E \right\} v_\beta \\ &= \int_{-\infty}^\infty d\tau e^{i\omega\tau} \text{tr} \left\{ \sum_{\alpha,\beta} v_\alpha^* e^{iH_E\tau} E_\alpha^\dagger e^{-iH_E\tau} E_\beta \rho_E v_\beta \right\} \\ &= \int_{-\infty}^\infty d\tau e^{i\omega\tau} \text{tr} \left\{ e^{iH_E\tau} \left(\sum_\alpha v_\alpha^* E_\alpha^\dagger \right) e^{-iH_E\tau} \left(\sum_\beta E_\beta v_\beta \right) \rho_E \right\} \\ &\equiv \int_{-\infty}^\infty d\tau e^{i\omega\tau} \text{tr} \left\{ e^{iH_E\tau} A^\dagger e^{-iH_E\tau} A \rho_E \right\},\end{aligned}\tag{E.2}$$

where we defined $A = (\sum_\alpha v_\alpha E_\alpha)$. (2) According to Bochner's theorem (162), the Fourier transform of a function of positive type, i.e., a function $f(t)$ such that the matrix with coefficients $f_{ij} = f(t_i - t_j)$ is positive semidefinite for every t_i and t_j , is a positive number. Therefore, we achieve our goal if we prove that $f(t) = \text{tr} \left\{ e^{iH_E t} A^\dagger e^{-iH_E t} A \rho_E \right\}$ is of positive type. Using a basis $\{|\{n_k\}\rangle\}$ of H_E eigenvectors, we have that $e^{iH_E t} |\{n_k\}\rangle = e^{i\varepsilon_{\{n_k\}} t} |\{n_k\}\rangle$ and

$$\rho_E |\{n_k\}\rangle = \frac{1}{\mathcal{Z}} \sum_{\{n'_k\}} e^{-\beta\varepsilon_{\{n'_k\}}} |\{n'_k\}\rangle \langle \{n'_k\} | \{n_k\} \rangle = \frac{1}{\mathcal{Z}} e^{-\beta\varepsilon_{\{n_k\}}} |\{n_k\}\rangle \equiv p_{\{n_k\}} |\{n_k\}\rangle, \tag{E.3}$$

where $p_{\{n_k\}}$ is a positive quantity. Therefore

$$\begin{aligned}
f(t) &= \text{tr} \left\{ e^{iH_E t} A^\dagger e^{-iH_E t} A \rho_E \right\} \\
&= \sum_{\{n_k\}} \langle \{n_k\} | e^{iH_E t} A^\dagger e^{-iH_E t} A \rho_E | \{n_k\} \rangle \\
&= \sum_{\{n_k\}} p_{\{n_k\}} e^{i\varepsilon_{\{n_k\}} t} \langle \{n_k\} | A^\dagger e^{-iH_E t} A | \{n_k\} \rangle \\
&= \sum_{\{n_k\}, \{n'_k\}} p_{\{n_k\}} e^{i\varepsilon_{\{n_k\}} t} \langle \{n_k\} | A^\dagger e^{-iH_E t} | \{n'_k\} \rangle \langle \{n'_k\} | A | \{n_k\} \rangle \\
&= \sum_{\{n_k\}, \{n'_k\}'} p_{\{n_k\}} e^{i(\varepsilon_{\{n_k\}} - \varepsilon_{\{n'_k\}}) t} \langle \{n_k\} | A^\dagger | \{n'_k\} \rangle \langle \{n'_k\} | A | \{n_k\} \rangle \\
&= \sum_{\{n_k\}, \{n'_k\}} p_{\{n_k\}} e^{i(\varepsilon_{\{n_k\}} - \varepsilon_{\{n'_k\}}) t} |\langle \{n'_k\} | A | \{n_k\} \rangle|^2. \tag{E.4}
\end{aligned}$$

Defining $\Omega = \varepsilon_{\{n_k\}} - \varepsilon_{\{n'_k\}}$, the matrix f_{ij} is positive semidefinite if, for an arbitrary vector w ,

$$\begin{aligned}
\sum_{i,j} w_i^* f_{ij}(\omega) w_j &= \sum_{i,j} w_i^* f(t_i - t_j) w_j \\
&= \sum_{i,j} w_i^* \sum_{\{n_k\}, \{n'_k\}} p_{\{n_k\}} e^{i\Omega(t_i - t_j)} |\langle \{n'_k\} | A | \{n_k\} \rangle|^2 w_j \\
&= \sum_{\{n_k\}, \{n'_k\}} \left(\sum_i w_i^* e^{i\Omega t_i} \right) \left(\sum_j w_j e^{-i\Omega t_j} \right) p_{\{n_k\}} |\langle \{n'_k\} | A | \{n_k\} \rangle|^2 \\
&= \sum_{\{n_k\}, \{n'_k\}} \left| \sum_i w_i^* e^{i\Omega t_i} \right|^2 p_{\{n_k\}} |\langle \{n'_k\} | A | \{n_k\} \rangle|^2 \geq 0, \tag{E.5}
\end{aligned}$$

since all quantities in the right hand side are positive. Therefore, $f(t)$ is of positive type, which implies, by Bochner's theorem, that the decay rates matrix is positive semidefinite.