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Matheus do Carmo Teodoro

Perturbations of black holes pierced by cosmic strings.

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Comissão Julgadora

Dr(a). Betti Hartmann

Instituição: (IFSC/USP)

Dr(a). Alex Eduardo de Bernardini

Instituição: (UFSCar/São Carlos)

Dr(a). Samuel Rocha de Oliveira

Instituição: (UNICAMP/Campinas)

This thesis is dedicated to my nephew, Felipe Guimarães Teodoro, for his little evergreen heart can always keep mine full of hope.

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ABSTRACT

TEODORO, M. **Perturbations of black holes pierced by cosmic strings.**. 2018. 58p. Dissertation (Master in Science) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2018.

The present-day interest in gravitational waves, justified by the recent direct detections made by LIGO, is opening the exciting possibility to answer many questions regarding General Relativity in extreme situations. One of these questions is whether black hole are – indeed – described totally by their mass, charge and angular momentum or whether they can have additional long-range hair. This project is concerned with this question. We aim at studying the influence of additional structure on the black hole horizon in the form of long-range hair by studying linearized Einstein equation the solutions when perturbed. More precisely, we will study the Schwarzschild solution, pierced by an infinitely long and thin cosmic string such that the space-time possesses a global deficit angle. Quasi-normal modes are believed to dominate the gravitational wave emission during the ring down phase of an excited black hole that would e.g. be the result of a merger of two ultra-compact objects, therefore linearized perturbations can be considered. With the advent of gravitational wave astronomy the proposed study will be very important when reconstructing the source of the detected gravitational wave signals.

Key words: Black holes. Cosmic string. General relativity. Quasi-normal modes. Gravitational waves.

RESUMO

TEODORO, M. **Perturbações de buracos negros atravessados por cordas cósmicas.**. 2018. 58p. Dissertação (Mestre em Ciência) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2018.

O atual interesse em ondas gravitacionais, justificado pelas detecções diretas feitas pela colaboração LIGO recentemente, está abrindo a excitante possibilidade de responder várias questões a respeito da Relatividade Geral em condições estremas. Uma dessas questões é se buracos negros são – realmente – totalmente discritos apenas por sua massa, carga e momento angular ou se eles podem ter os chamados "cabelos de longo alcance" adicionais. Nosso projeto se preocupa em responder esta pergunta. Nosso objetivo está em estudar a influência de uma estrutura adicional no horizonte de eventos de um buraco negro através do comportamento da equação linearizada de Einstein quando a solução é perturbada. Mais precisamente, nós estudaremos a solução de Schwarzschild atravessada por uma corda cósmica infinitamente fina, tal corda faz com que o espaço-tempo tenha um hiato angular em seu plano equatorial. Acredita-se que modos quasi-normais dominem a emissão de ondas gravitacionais durante a fase de "ringing down" de buracos negros excitados que podem, por exemplo, se originar da colisão de objetos ultra compactos, portanto perturbações lineares podem ser consideradas. Com o advento da astronomia através de ondas gravitacionais o estudo proposto será importante para que se possa reconstruir a origem de sinais detectados.

Palavras-chave: Buracos negros. Cordas cósmicas. Relatividade geral. Modos quasi-normais. Ondas gravitacionais.

LIST OF ABBREVIATIONS AND ACRONYMS

BH	Black Hole;
BH+CS	Black holes pierced by a cosmic string;
CS	Cosmic String;
GR	General Relativity;
QNM	Quasi-normal modes

CONTENTS

1 INTRODUCTION

Almost every single astronomical object oscillates when perturbed and General Relativity (GR) predicts that such oscillations must spread out in the form of, among others (e.g. radiowaves, $\gamma-$rays), Gravitational Waves (GW). Such waves are described as evolving deformations of the space-time. The intriguing possibility that, in the same way one can deduce the existence of electromagnetic waves from the Maxwell equations, gravitational waves could be deduced from the Einstein equations made the subject a focus of curiosity. Indeed, countless effort was put into making such prediction proved and nowadays GW can be measured and can be seen as a prove that, at least in its proposed range of phenomena, GR is correct. Yet black holes are solutions for strong gravitational fields, GW appear in a weak limit and are often studied with linearization of the Einstein equation, which provides in some cases analytical results. Yet measurable, these waves are extremely weak and need both, extreme phenomena as source and big interferometers as detectors.

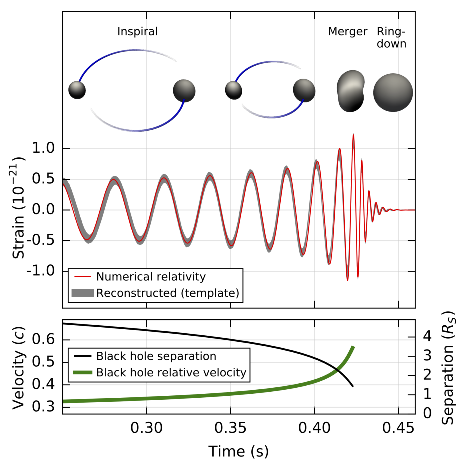


Figure 1 - Stages of a BH merge.
Source: ABBOTT[1].

Indeed, such waves are easier to measure if emitted from extreme events involving compact objects, as the merging of two black holes, the phenomenon from which the first detections of such waves was recently made by LIGO[1,2]. The merging can be better understood if described through three distinct stages[3,4](see figure 1): the inspiral stage, which can be well approximated by post-Newtonian theory, the merge phase itself, which can only be expressed through numerical simulations, and finally the ringing down phase, when the black hole resulting from the merge relaxes into an equilibrium solution of the field equations .The wave signal linked to this last phase is believed to be dominated by quasi-normal modes (QNM) expansion[5–8].

The ringing down stage can be treated with perturbative theory, linearizing then the Einstein field equation, which inspired us to present a perturbative approach to a Schwarzschild black hole pierced by a cosmic string (see figure 2). This system is indeed a solution of Einstein's equation[9] and has been studied due its importance for being a candidate for probing the non-hair conjecture. This conjecture makes a rather unexpected claim:

> *Black holes are simple objects, described by only three parameters called "long range hairs", these are: its mass (M), angular momentum (J) and electrical charge (Q).*

This is very counterintuitive since, for any other astronomical object, the gravitational field contains information of all its density distributions. Still, the existence of an event horizon changes the scenario for a Black Hole (BH), being, for instance, a simple Schwarzschild metric (which depends only of M) sufficient to describe it completely. The question then is, could a black hole have more than these three proprieties? If yes, would it be possible to measure them from far away? The former question we try to answer by adding structure to the Schwarzschild solution: a cosmic string piercing the black hole. This structure gives the BH a new parameter in the plane orthogonal to the string, β, related to a deficit angle caused by a change from asymptotically flat to now conical geometry. The latter question will be answered by looking at the waves emitted from the system, if the β parameter affects these we have indeed a new long range hair. Therefore our project title **"Perturbations of black holes pierced by cosmic strings".**

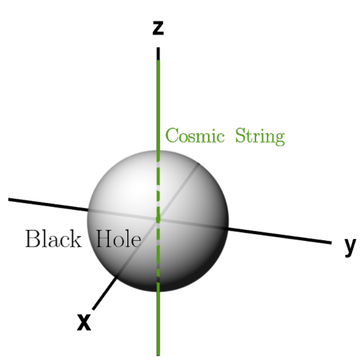


Figure 2 - BH+CS.
Source: By the author.

Also, the study of cosmic strings has its own importance, being closely related to String Theory and the physics of the early universe. If strings could be detected that would provide important information from the opaque stage of our universe and for physics beyond the Standard Model. Thus the idea of the possibility of finding such objects through the gravitational waves of a system such as the one proposed in this project also is a good motivation for our study. Finally, we are also interested in testing if the solution of a Black Hole pierced by a Cosmic String is indeed stable under small perturbations.

In the following we will briefly review some aspects of the background necessary for the work done in this thesis.

2 BACKGROUND

2.1 General Relativity Basics: Einstein equation and geodesic equation.

The aim of this section is to discuss the study of geodesics around black holes. To do that we shall begin with a quick review of the Schwarzschild solution and its effective potential for a massive test particle. The plots shown in the following were made using Mathematica as a pedagogical exercise to get used to the topic and the Schwarzschild metric. Also, we were interested in, later, studying the so-called eikonal limit* for there are studies relating the Quasi-Normal Modes (QNM) of black holes[10] to null geodesics around them. Such limit is useful for when the tensor perturbations can not be calculated explicitly.

2.1.1 Review of the Schwarzschild solution.

The Schwarzschild solution is an asymptotically flat, static and spherically symmetric vacuum solution of the Einstein equation that can represent a black hole with these characteristics. The solution can be described by the following metric.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta + r^2 \sin^2(\theta) d\phi^2, \tag{2.1}$$

choosing $c = G = 1$, M is the mass of the black hole, (r, θ, ϕ) are the usual spherical coordinates and t is the temporal coordinate. Containing differentiable symmetries, this space-time is expected to have also conserved quantities through Noether's theorem. Two of these symmetries are translations in time and the azimuthal angle, $t \to t + c_1$ and $\phi \to \phi + c_2$. They infer the existence of the following Killing vectors:

$$\xi_t = (1, 0, 0, 0) \text{ and } \xi_\phi = (0, 0, 0, 1)$$

Letting u be a velocity vector of a test particle in this geometry, $u = \frac{d}{d\tau}(x^t, x^r, x^\theta, x^\phi)$ we can use the Killing vectors to obtain the invariants quantities,

$$\epsilon \doteq -\xi_t u = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}$$
$$l \doteq -\xi_\phi u = r^2 \sin^2(\theta) \frac{d\phi}{d\tau}$$

For a particle with mass ($m = 1$), the first one is associated to the energy measured from a far away observer at rest, the second to the angular momentum, which could be expected to be conserved. Taking advantage of the angular momentum conservation we

* short wavelengths or high multipole number

can, without loss of generality, assume the particle movement bound to the plane $\theta = \frac{\pi}{2}$. Also, for a real massive particle we have the normalization[†] $g_{\mu\nu}u^{\mu}u^{\nu} = 1$, leading to the equation:

$$-\left(1 - \frac{2M}{r}\right)(u^t)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(u^r)^2 + r^2(u^\theta)^2 + r^2(u^\phi)^2 = -1.$$

Using the conserved quantities, l and ϵ,

$$u^t = \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}\epsilon, \qquad u^\phi = \frac{d\phi}{d\tau} = \frac{l}{r^2\sin^2(\frac{\pi}{2})} = \frac{l}{r^2}.$$

and, after manipulating the normalization equation, we get:

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{l^2}{r^2} - \frac{2Ml^2}{r^3} - \frac{2M}{r} = \epsilon^2 - 1.$$

Defining,

$$V_{eff} \doteq \frac{l^2}{r^2} - \frac{2Ml^2}{r^3} - \frac{2M}{r}$$

and

$$E \doteq \frac{(\epsilon^2 - 1)}{2}$$

we get the following equation,

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff} = E$$

This way we get an effective potential equation, describing the possible orbits and stable points for a particle in the Schwarzschild space-time.

[†] This one should follow a time-like curve always.

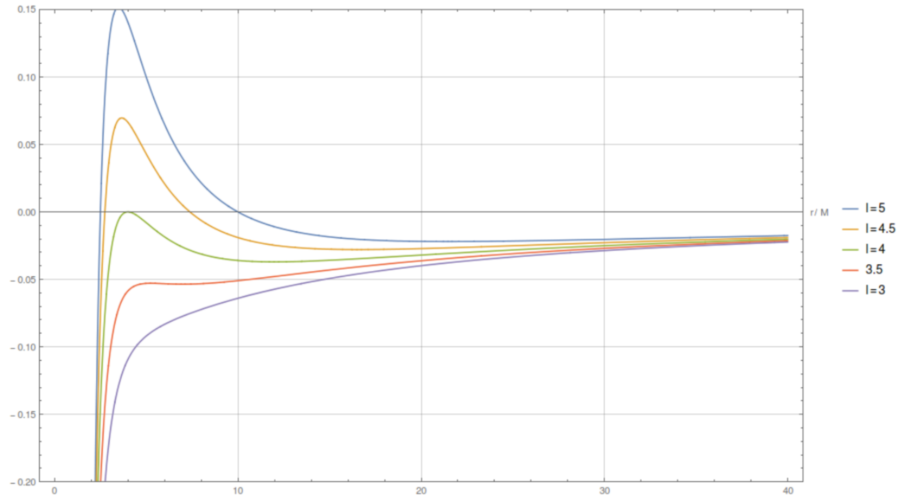


Figure 3 - Potentials for different angular momenta l, the $x-$axis represent r/M and the $y-$axis the effective potential.
Source: By the author

For small l's, which here represent the modulus of the particle angular momentum, the particle is likely to fall into the black hole. On the other hand, for bigger l's the behavior of the potential becomes such that closed orbits around the black holes are allowed due to a stable point for certain energies.

2.1.2 Possible orbits: example

Consider the potential for $l = 5$ and keeping $\theta = \pi/2$, we can write

$$\frac{d\phi}{dr} = \frac{l}{\sqrt{2}r^2}(V_{eff} - E)^{1/2}$$

Taking $x \doteq \frac{1}{r} \implies dx = -r^{-2}dr$,

$$\frac{d\phi}{dx} = -\frac{l}{\sqrt{2}}(V_{eff}(x) - E)^{1/2}$$

Now we have to integrate the equation with respect to x. To do that it is useful first to specify E and study the extremes of V_{eff}. By taking the first and second derivatives of V_{eff} one finds

- $V_{max} = 0.151615$ at $x = 0.0464816$
- $V_{min} = -0.021985$ at $x = 0.151615$

To get a stable orbit we have to choose an energy greater or equal to V_{min}. To calculate the orbit it is also necessary to give the particle initial position (remember that the velocity is automatically calculated since we specified l). Setting these variables we can have four different kinds of orbits:

1. Circular orbits,
2. Bound precessing orbits,
3. Scattered orbits,
4. Plugging orbits.

To integrate the equation for ϕ it is essential to identify which of these types of orbits will appear once the initial conditions are set. This has to be done because at the points where $E = V_{eff}(x)$ the integral will have singularities. To avoid this issue we implemented a simple logic test on Mathematica that automatically moves the singularities slightly alway.

```
testa = (Eng < 0) && (ust < tp2)
If[testa, u2 = tp1 (1 + eps); u2 = tp2 (1 - eps)]
testb = (Eng > 0) && (Eng < vmax) && (ust < tp2)
If[testb, u1 = ust; u2 = ust (1 - esp)]
testc = (Eng < vmax) && (ust > tp3)
If[testc, u1 = .5; u2 = tp3 (1 + eps)]
testd = (Eng > vmax)
If[testd, u1 = .5 ; u2 = tp3 (1 + eps)]
```

Here the value $tp1$, $tp2$ and $tp3$ are the three (possible) values of x for which $E = V_{eff}(x)$. Eng is the stated initial energy and u_1 and u_2 will be the limits of integration. Also ust is the starting position of the particle and eps is a small parameter (10^{-8}). This test selects the following orbits,

1. testa: Tests if the particle stays bound around $tp1$ and $tp2$, precessing and moves the singularities of the integral singularities. The energy also tests $tp3$, but since the movement is bound we do not need to worry about this one.
2. testb: Tests if the particle has enough energy and distance to move away, but can not pass through the major potential barrier, characterizing a scattering orbit.
3. testc: Tests if the particle does not have enough energy to pass trough the potential barrier and is also too close to the black hole, being then pulled into the black hole.
4. testd: Tests if the particle has enough energy to cross the potential barrier and fall into the black hole;

To exemplify this running of the whole code we will show an example in detail. The example chosen is a precessing orbit. This example is important since the Schwarzschild solution can be used not only for black holes but also (in a approximation scenario) for spherically symmetric mass distributions, such as the sun. The precessing of Mercury's orbit was one of the first experimental achievements of General Relativity, since such precession does not occur in the Newtonian theory of gravity.

The initial conditions of our example are,

```
ust = 0.05
eps = 0.00000001
Eng = -0.01
norbit = 3
```

where *norbit* is just a parameter to tell the code how many orbits have to be calculated. For these parameters the effective potential and the energy are given in the following plot:

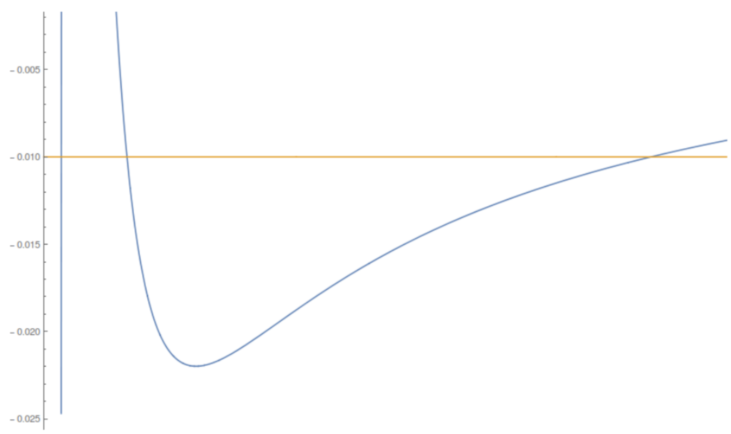


Figure 4 - $x-$axis represent r/M and the $y-$axis is in units of energy. The yellow line represents the initial energy and the blue one the effective potential for $l = 5$.
Source: By the author.

here the line parallel to the x-axis, which indicates the energy, intersects the potential three times and, depending of the initial position, the orbit will indeed be bounded. The implemented test responds as follows:

```
In[31]:= testa = (Eng < 0) && (ust < tp2)
         If[testa, u2 = tp1 (1 + eps); u2 = tp2 (1 - eps)]
         testb = (Eng > 0) && (Eng < vmax) && (ust < tp2)
         If[testb, u1 = ust; u2 = ust (1 - esp)]
         testc = (Eng < vmax) && (ust > tp3)
         If[testc, u1 = .5; u2 = tp3 (1 + eps)]
         testd = (Eng > vmax)
         If[testd, u1 = .5 ; u2 = tp3 (1 + eps)]
Out[31]= True

Out[33]= False

Out[35]= False

Out[37]= False

In[39]:= u1
Out[39]= 0.0116597

In[40]:= u2
Out[40]= 0.0850696
```

Since the singularities were removed we can do the integration and plot the orbit. However to actually display the orbit we ought to define some new variables. We start defining a theta function depending on the numerical integration result between $u1$ and a free parameter u. Also we define the variable delphi as the angle swept by ϕ between the end points of the orbit:

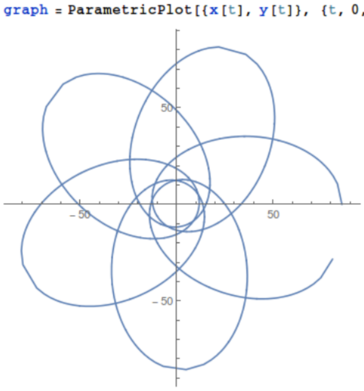
```
theta[u_, Eng_, l_, u1_] :=
 NIntegrate[l / 2 ^ (1 / 2) (Eng - V[w, l]) ^ (-1 / 2), {w, u1, u}]

delphi = theta[u2, Eng, l, u1]
3.73604
```

Now, to display the result we will need some more auxiliary functions (remember we want to display the orbit in the plane xy where the particle is moving). The way to do this is the following: first we have to define a variable z that runs from 0 to norbit and make the following procedure,

```
n[z_] := IntegerPart[z]
zf[z_] := FractionalPart[z]
ua[z_] := u1 (1 - 2 zf[z]) + u2 2 zf[z]
ub[z_] := u1 (2 zf[z] - 1) + 2 u2 (1 - zf[z])
u[z_] := If[zf[z] < .5, ua[z], ub[z]]
phia[z_] := 2 (n[z]) delphi + theta[u[z], Eng, l, u1]
phib[z_] := 2 (n[z] + 1) delphi - theta[u[z], Eng, l, u1]
accphi[z_] := If[zf[z] < .5, phia[z], phib[z]]
x[z_] := Cos[accphi[z]] / u[z]
y[z_] := Sin[accphi[z]] / u[z]
```

Finally we can plot the orbit.

```
graph = ParametricPlot[{x[t], y[t]}, {t, 0, norbit}]
```



2.2 Gravitational waves.

In contrast to black holes, gravitational waves appear in the weak gravitational field limit as the linearized perturbations propagating through vacuum. Here we shall demonstrate how GR predicts the existence of such waves in Minkowski space-time. Thus let us consider the metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.2}$$

with $\eta_{\mu\nu} = diag(-1,1,1,1)$ and $\|h_{\mu\nu}\| \ll 1$, $h_{\mu\nu}$ which will represents the perturbation. this way it reads,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu},$$

while the Christoffel symbols are:

$$\Gamma^{\mu}_{\nu\lambda} = \eta^{\mu\rho}\tfrac{1}{2}(\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\nu\lambda}), \tag{2.3}$$

the Riemann tensor is:

$$R_{\mu\nu\rho\sigma} = \tfrac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\rho\nu} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\rho\mu}\right),$$

while the Ricci tensor reads:

$$R_{\mu\nu} = \tfrac{1}{2}(\partial_\sigma\partial_\nu h^\sigma_\mu + \partial_\sigma\partial_\mu h^\sigma_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}),$$

where we use the following definitions:

$$h \equiv h^\mu_\mu \qquad \Box \equiv \partial^\mu\partial_\mu.$$

We are interested in vacuum Einstein equation solutions, i.e. solutions to $R_{\mu\nu} = 0$. This will be assumed throughout the entire thesis. For this specific case, the equation becomes:

$$R_{\mu\nu} = 0 \implies \partial_\sigma\partial_\nu h^\sigma_\mu + \partial_\sigma\partial_\mu h^\sigma_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} = 0, \tag{2.4}$$

which is still a bit of an obscure equation. To make it simpler we have to remember that in the same way that the Maxwell equations have a gauge freedom, so should the Einstein equation. General relativity tells us that the Einstein equation is locally invariant under a general coordinate transformation, but since we are dealing with a linearized version of such equations it is natural to assume invariance under linearized general transformations. Indeed the following equality will hold,

$$h'_{\mu\nu} = h_{\mu\nu} + L_V\eta_{\mu\nu},$$

where V is any differentiable vector field, for $\eta_{\mu\nu} + L_V\eta_{\mu\nu}$ is nothing but an infinitesimal coordinate transformation of the Minkowski metric.‡ Therefore we can always add a term like the following to the perturbation,

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu. \tag{2.5}$$

Now we have the freedom to rewrite the final $g_{\mu\nu}$ in a more suitable way as long as we follow the above formula. The so-called "harmonic gauge", defined by

$$g^{\mu\nu}\Gamma^\rho_{\mu\nu}$$

will be of later use, for now we can already see why the name "harmonic" is justified, since the d'Alembertian of the coordinates vanishes,

$$\Box x^\mu = g^{\nu\rho}\nabla_\rho\partial_\nu x^\mu = -g^{\nu\rho}\Gamma^\mu_{\nu\rho} = 0,$$

‡ L_V denotes the Lie derivative operator with respect to V

which reminds of the Lorenz gauge in Maxwell theory, $\partial_\mu A^\mu = 0$.

The gauge condition will then become, using equation (2.3) and linearizing the outcome,

$$\partial_\mu h^\mu_\lambda - \tfrac{1}{2}\partial_\lambda h = 0.$$

That said we can find the field V_μ that can achieve this condition easily,

$$\Box V_\mu = -(\partial_\mu h^\mu_\lambda - \tfrac{1}{2}\partial_\lambda h). \tag{2.6}$$

So the harmonic gauge can be achieved. But we still did not explore all the gauge freedom the linearized Einstein equation possesses. In the harmonic gauge we can always add another vector field to the coordinates $x^\mu + \xi^\mu$ as long as $\Box\xi^\mu$ vanishes. This last gauge freedom can provide a easy way to understand the two independent polarizations of a GW.

With the harmonic gauge the Einstein equation looks much simpler,

$$\Box h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\Box h = 0$$

and, therefore, in the vacuum, adding the gauge condition,

$$\Box h_{\mu\nu} = 0, \qquad \partial_\mu h^\mu_\lambda - \tfrac{1}{2}\partial_\lambda h = 0, \tag{2.7}$$

we find the above to be the set of equations which evolve tensor perturbations in the vacuum. It already looks like a wave equation, but it is usual to define,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h.$$

The (2.7) becomes:

$$\Box\bar{h}_{\mu\nu} = 0, \qquad \partial_\mu \bar{h}^\mu_\nu = 0.$$

2.3 Quasi-normal modes.

During the ringing down phase of a BH merger, the space-time geometry is only slightly disturbed from the final stationary solution, thus the system can be treated as perturbative and one can linearize the Einstein equation arriving into a system of coupled PDEs. Depending of the black hole geometry (e.g. Kerr, Schwarzschild, etc.), these PDEs can be decoupled and by treating the resulting equations analytically one arrives naturally at the QNM expansion. The QNM frequencies, in contrast with Normal Modes frequencies, are complex numbers, the imaginary part being usually (that depends of the convention) an oscillation frequency and the real part a damping. Two valuable reviews which summerize the 1990's and 2000's application of QNM as well as its formalism can be found in [7,8].

This treatment is intrinsically related to the boundary conditions which, for this case, are very unique due to the presence of an event horizon. Thus, selecting the QNM frequencies from gravitational wave signals can be used to add precision to the black hole mass and spin measurements. [11] But we can go even further. QNM have been considered a good tool to probe the existence of event horizons and the no-hair conjuncture [12,13]. Apart from that, it is necessary to mention that these modes are also useful tools in the context of the AdS/CFT correspondence. For instance, equilibrium proprieties of strongly coupled thermal gauge theories can be linked through the correspondence to higher dimensional black holes and black branes. In these dual gravitational backgrounds, the QNM give the location of the poles of the retarded correlations in the gauge theory[14]. Thus we fell motivated to study QNM.

2.3.1 The eikonal limit and null geodesics.

In [10] it was shown that in the so-called eikonal limit the real and imaginary part of the gravitational emission from a black hole with spherical symmetry or spherical sections can be given in terms of the effective potential in which a point-like particle moves on a light-like geodesic. The important thing in this limit is that these light-like geodesics are circular and unstable.

This result motivated us to redo the analysis of geodesics around a Schwarzschild BH and discuss the case of a Schwarzschild BH+CS[15]. This was done both as a GR exercise and as well in order to understand the QNM of the BH+CS geometry.

2.3.2 Time-dependence of perturbations, the quasi-normal modes.

During the next sections an important consideration will be implicitly made about the time dependence of perturbations around black hole geometries. In order not only to formally understand these assumptions but as well to later be able to interpret gravitational wave signals one must be familiar with the quasi-normal modes concept. To introduce this mathematical tool we shall show a pedagogical example first [16].

Consider a one dimensional string of length π, fixed ends and wave velocity $c = 1$ located on the x-axis. Let $y(t, x)$ be the displacement of the string on the $y-$axis. This displacement will be described by the wave equation,

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \qquad for \;\; x \in [0, \pi], \tag{2.8}$$

where the boundary conditions are $y(0, t) = y(\pi, t) = 0$. The well known solution for this differential equation is given by,

$$y(t, x) = \sum_{n=1}^{\infty} (C_n \cos(nt) + C'_n \sin(nt)) sin(nx), \tag{2.9}$$

a superposition of normal modes, which are labeled by n, with harmonic dependence and frequency $\omega = n$. The coefficients C_n and C'_n can be obtained by using the initial configuration of the string, say $y(0,x) = y_0(x)$ and $\partial_t y(0,x) = v_0(x)$, by integration,

$$C_n = \frac{2}{\pi}\int_0^\pi y_0(x)\sin(nx)dx,$$
$$C'_n = \frac{2}{n\pi}\int_0^\pi v_0(c)\sin(nx)dx.$$

But this is a special case, i.e. the system is conservative which will not be the case for gravitational waves since they can spread away to infinity or be pulled into a black hole. For this type of wave a better toy model would be a inhomogeneous wave equation coupled to a time-independent potential,

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} - V(x)y = \mathcal{S}, \tag{2.10}$$

where the $\mathcal{S}$ term can be physically understood as a source. In this case, the normal mode expansion is not that straightforward and yet it would be possible solving this. Let us use Green's function to extract the expansion. Considering the Laplace transform to be,

$$\mathcal{L}y(t,x) \doteq \hat{y}(\omega,t) = \int_{t_0}^{\infty} y(t,x)e^{i\omega t}dt$$

and its inverse,

$$y(t,x) = \frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} \hat{y}(\omega,x)e^{-i\omega t}d\omega$$

we can eliminate the temporal dependence from (2.10),

$$\frac{\partial^2 \hat{y}}{\partial x^2} + \left[\omega^2 - V\right]\hat{y} = I(\omega,x), \tag{2.11}$$

where

$$I(\omega,x) \doteq e^{i\omega t_0}\left[i\omega y(t,x) - \frac{\partial y(t,x)}{\partial t}\right]_{t=t_0} + \mathcal{L}[\mathcal{S}]. \tag{2.12}$$

If one finds the Green function for (2.11), i.e

$$\frac{\partial^2 G(x,x')}{\partial t^2} + \left[\omega^2 - V\right]G(x,x') = \delta(x-x'),$$

the solution will simply be,

$$\hat{y} = \int I(\omega,x')G(x,x')dx'.$$

To find the Green function however we can use two linearly independent solutions of the homogeneous part of (2.11), each satisfying one boundary condition, i.e. $\hat{y}_1(\omega, 0) = 0$ and $\hat{y}(\omega, \phi) = 0$. With these two solutions we can build G as follows,

$$G(x, x') = \frac{1}{W} \begin{cases} \hat{y}_1(x)\hat{y}_2(x') & if\, x \leq x' \\ \hat{y}_1(x')\hat{y}_2(x) & if\, x' \leq x \end{cases}$$

were W is the Wronskian between $\hat{y}_1$ and $\hat{y}_2$. For (2.11) we can set $\hat{y}_1 = sin(\omega x)$ and $\hat{y}_2 = sin(\omega(x - \pi))$, so we get,

$$G(x, x') = \begin{cases} -\frac{\sin(\omega x)\sin(\omega(x'-\pi))}{\omega \sin(\omega \pi)} & if\, x \leq x' \\ -\frac{\sin(\omega x')\sin(\omega(x-\pi))}{\omega \sin(\omega \pi)} & if\, x' \leq x, \end{cases} \tag{2.13}$$

and here we get in to the major point of this example. It is easy to see from (2.13) that the normal modes are the poles of the Green function $G(x, x')$. This way, saying that, for instance $t_0 = \mathcal{S} = 0$, the final solution will be given by the integral,

$$y(x, t) = \frac{1}{2\pi} \int dx' d\omega \left[i\omega u_0(x') - y_0(x') \right] G(x, x') e^{-i\omega t} \tag{2.14}$$

where the integral over ω can be done in the complex plane closing the contour shown in Figure 5 and using Cauchy's theorem. This way indeed the final result will become a summation over the poles of the Green function.

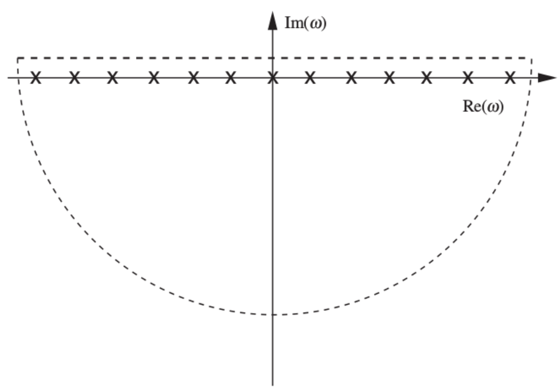


Figure 5 - Poles of the Green function.
Source: BERTI[16]

Finally, for dissipative systems, perturbations which obey a wave equation similar to (2.10) will have the temporal dependence solved in a similar way but for this case the poles of the Green function will not necessarily be real nor simple poles. Yet, for the case in which they are simple poles, they are called quasi-normal modes. Yet the quasi-normal mode expansion is often used, there is no prove yet that the later converges. However for data analysis, e.g. for gravitational waves, just a small set of modes is necessary.

There are some important points to highlight here. The first one is that the boundary conditions for a black hole are quite unique. At the event horizon there are no outgoing waves and that assumption ends up imprinted in the QNMs. Therefore by extracting these modes one can probe the existence of event horizons. We shall not extend too much the discussion about the modes calculation since it was not actually necessary to calculate them at the end of the presented project.

At least we shall state that since the QNM expansion is possible all our perturbations time-dependence will behave as follows:

$$e^{i\sigma t}$$

where σ is the QNM. Remember that since this mode can be complex, the perturbation can (and usually it does) suffer a damping as well as the oscillation itself. If the amplitude of the oscillation is increasing, the perturbed solution is not stable. That is why QNMs are also a good way of testing the stability of geometries.

2.4 Cosmic Strings.

Cosmic strings are topological defects which may have been formed in the early universe due to phase transitions. These objects have been studied during the last decades in much detail [17,18].

The first systematic approach to describe cosmic strings has been done using the so-called thin string limit. In this limit, the string thickness, which is much smaller than all other dimensions, is assumed to be zero. The string is hence simply a 1-dimensional object – a line – that can be described by its energy per unit length μ, which is equal to its tension T. Strings formed at the Grand Unification (GUT) scale can have up to $\mu \sim 10^{23} kg/m$, while strings formed at the electroweak scale would have $\mu \sim 10^{-5} kg/m$. In the thin string limit, analytical studies are possible, especially a Nambu-Goto action can be used to describe the string. This approach corresponds hence to a macroscopic description of cosmic strings that allows for analytic solutions, that – however – does not take the underlying field theory into account. If we assume the symmetry breaking scale of the phase transition at which the cosmic strings form to be much smaller than the Planck scale, one can linearize the Einstein field equation and determine the gravitational effects of strings. This has been done for straight and static strings in the thin string limit [19]. In this case, the metric can be given explicitly and it was shown that it can be matched to an exact cylindrically symmetric vacuum solution of the Einstein equation. The main observation is that the space-time around the string is locally flat, but is globally conical.

The angular coordinate φ do not vary from 0 to 2π, but only from 0 to $2\pi(1-4G\mu)$, where G is Newtons constant and μ the energy per unit length of the string. The result is a deficit angle $\Delta = 8\pi G\mu \sim (\eta/M_{pl})^2$, where η is the symmetry breaking scale and M_{pl} the Planck mass. The first step towards a non-perturbative treatment of cosmic strings was done in [20–22]. In [20,21], the Einstein equation has been solved exactly for the exterior and the interior of the string, respectively. For the interior the energy-momentum was assumed to describe a uniform density string, while the exterior was the vacuum. It has been shown that the interior space-time is that of a spherical cap, while the exterior space-time is conical confirming the result of [19], but now giving the non-perturbative result correct in all orders of μ.

Different other space-times containing cosmic strings have also been discussed. This study has mainly been motivated by the pioneering work of Bach and Weyl [23] describing a pair of black holes held apart by an infinitely thin strut. This solution has later been reinterpreted in terms of cosmic strings describing a pair of black holes held apart by two cosmic strings extending to infinity in opposite direction. Consequently, cosmic string piercing a static black hole given by the Schwarzschild solution have also been discussed. Interestingly, the solution found in [9] is a Schwarzschild solution which however differs from the standard spherically symmetric case by the replacement of the angular variable φ by $\beta\varphi$, where the parameter β is related to the deficit angle by $\Delta = 2\pi(1-\beta)$. In this sense, the space-time is thus not uniquely determined by the mass, but is described by the mass and deficit angle parameter β. As pointed out in [17,18], any solution with a symmetry axis can be generalized to incorporate cosmic strings – at least in the thin string limit. One simply assumes the angular coordinate in the plane perpendicular to the string not to vary from 0 to 2π, but from 0 to $2\pi - \Delta$, where Δ is the deficit angle. In such a way, analytic solutions have been generalized to include cosmic strings, e.g. stationary black hole solutions i.e. Kerr solutions. Static and stationary black holes pierced by cosmic strings are of interest since they could have formed in phase transitions in the early universe. The metric describing a Schwarzschild solution pierced by a cosmic string has also been used to describe the exterior space–time of the sun taking into account departures from perfect spherical symmetry [24].

The simplest field theoretical model that possesses a string solution, a so called *global string*, that of is a complex scalar field $\phi(x)$ with the following Lagrangian density:

$$\mathcal{L} = (\partial_\mu\phi)^*\partial^\mu\phi - V(|\phi|^2)$$

with a kinetic term and a potential:

$$V(|\phi|^2) = \tfrac{1}{2}\lambda(|\phi|^2 - \tfrac{1}{2}\eta^2)^2.$$

One can see that this Lagrangian has a global $U(1)$ symmetry, i.e. the Lagrangian density is invariant under,

$$\phi \to \phi e^{i\alpha},$$

where α is a constant.

The Euler-Lagrange equation of the fields[§] reads

$$[\partial^{\mu}\partial_{\mu} + \lambda(|\phi|^2 - \tfrac{1}{2}\eta^2)]\phi = 0$$

The ground state solution, i.e. the vacuum, is given by

$$\phi = \frac{\eta}{\sqrt{2}} e^{i\alpha_0},$$

where α_0 is a "zero mode". This solution is not $U(1)$ invariant, the symmetry is then "broken" by the vacuum.

Now imagine that this sort of symmetry breaking happens in different and casually disconnected parts of the space-time in the early universe. This is, in a very simplified way, the so called Kibble mechanism. A zoo of defects can be created, such as as monopoles, strings, walls, textures and even strings with higher winding numbers. The most stable of these are simple strings (with winding number equals to one). Different regions of space-time that are not casually connected, the symmetry breaking leads, in general, to different ground states. On the boundaries between these regions with different ground states defects are created, in the case shown below (figure 6) a string.

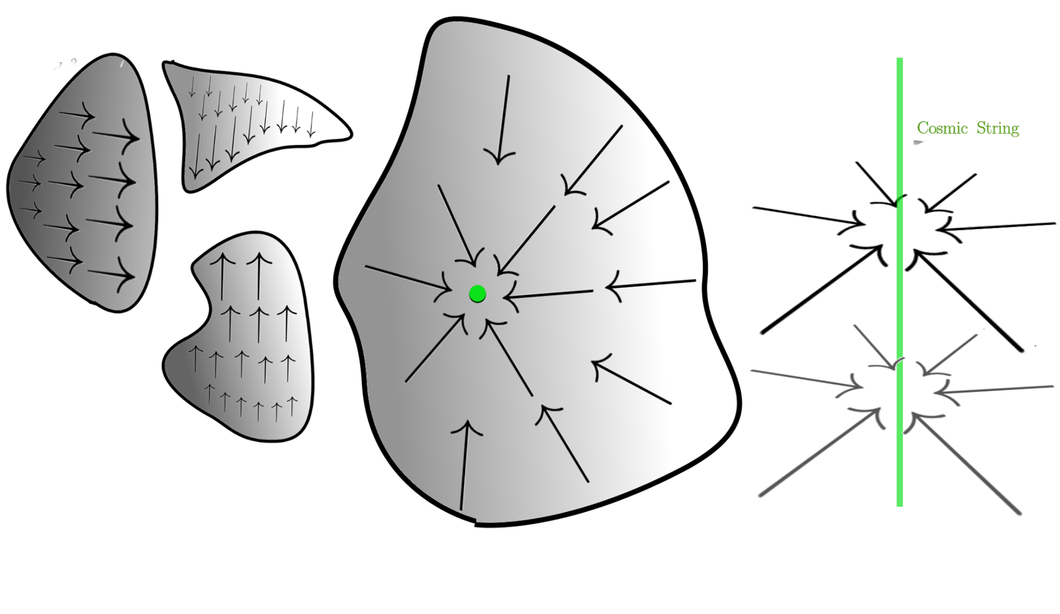


Figure 6 - Different regions of space creating a defect through symmetry breaking.
Source: By the author.

§ For complex fields this can be done independently for the ϕ^* and ϕ part.

3 RESULTS

3.1 Scalar perturbations.

Let Ψ be a free massless scalar field in a Schwarzschild black hole pierced by a cosmic string (BH+CS) space-time. It is well known that such field obeys the following wave equation,

$$\tfrac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\Psi = 0$$

and we wish to find solutions which, similarly to the pure Schwarzschild case decouple with the solution,

$$\Psi(t,r,\theta,\phi) = e^{-i\omega t}R(r)Y_l^m(\theta,\phi),$$

where Y_l^m are the spherical harmonics. Using the BH+CS background metric,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\theta + r^2\beta^2\sin^2(\theta)d\phi^2,$$

the wave equation become,

$$\frac{1}{\beta r^2\sin(\theta)}\partial_\mu[g^{\mu\nu}\beta r^2\sin(\theta)\partial_\nu]\Psi = 0$$

Assuming separation of variables is possible, i.e. $\Psi(t,r,\theta,\phi) = e^{-i\omega t}R(r)\Theta(\theta)\Phi(\phi)$, we find (see Appendix A),

$$\tfrac{1}{R(r)}\partial_r[\left(1-\tfrac{2M}{r}\right)^{-1}r^2\partial_r R(r)] + \tfrac{1}{\Theta(\theta)\sin(\theta)}\partial_\theta[\sin(\theta)\partial_\theta\Theta(\theta)] +$$

$$\tfrac{1}{\Phi(\phi)\beta^2\sin^2(\theta)}\partial_{\phi\phi}\Phi(\phi) - \tfrac{1}{e^{-it\omega}}r^2\left(1-\tfrac{2M}{r}\right)\partial_{tt}e^{-it\omega} = 0.(3.1)$$

Following the usual calculations for separation of variable, we can use,

$$\Phi(\phi) = e^{im\phi}, \qquad n \in Z$$

such as the r and θ derivatives can be separated as follows,

$$\tfrac{1}{R(r)}\partial_r[g^{rr}r^2\partial_r R(r)] + r^2(1-\tfrac{2M}{r})^{-1}\omega^2 = -C$$

as long as,

$$\tfrac{1}{\Theta(\theta)\sin(\theta)}\partial_\theta[\sin(\theta)\partial_\theta\Theta(\theta)] - \tfrac{m^2}{\beta^2} = C$$

where C is the separation constant. The angular equation,

$$\tfrac{1}{\sin(\theta)}\partial_\theta[\sin(\theta)\partial_\theta\Theta(\theta)] - (\tfrac{m^2}{\beta^2} + C)\Theta(\theta) = 0$$

can be solved by making the substitution,

$\xi = \cos(\theta)$,

$(1\text{-}\xi^2)\frac{d^2\Theta(\xi)}{d\xi^2} - 2\xi\frac{d\Theta(\xi)}{d\xi} - (C + \frac{m^2}{\beta^2\xi^2(1-\xi^2)})\Theta(\xi) = 0.$

This is a general Legendre equation. The solutions of this equations are the Legendre polynomials only if $\frac{m^2}{\beta^2}$ is an integer. Since β is a real number, we can not use the polynomials expansion as a solution, but rather the so called Legendre functions,

$$P^{l_\beta}_{m_\beta}(\xi), \quad \text{where} \quad C = (l_\beta + 1)l_\beta, \;\; m_\beta = \frac{m}{\beta}.$$

The radial equation on the other hand has no β terms. The calculations is then the same one as for the usual Schwarzschild case, but since we can not use the Legendre polynomials anymore it is not possible to use the spherical harmonics. The complete pure Schwarzschild case is recovered by taking $\beta = 1$.

3.2 Tensor perturbations.

For gravitational waves are perturbations of the space-time itself, the core for studying such waves lies in adding a small perturbation on a background space-time followed by the study of its propagation. Namely,

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu},$$

where $g^0_{\mu\nu}$ is the known background geometry, e.g. a Schwarzschild BH and, $h_{\mu\nu}$ is the perturbation. Our aim is to use the Einstein equation to study the propagation of $h_{\mu\nu}$. Remembering that the GW analysis is restrict to the vacuum media around the matter generating the space-time geometry, the stress-energy tensor vanishes and the Einstein equation is reduced to,

$$G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0.$$

Therefore most of the work done can be presented within in the following steps,

1. Find the most suitable way to write $g^0_{\mu\nu}$ and add $h_{\mu\nu}$,
2. Calculate the corresponding tensor $R_{\mu\nu}$ (and $G_{\mu\nu}$, if necessary),
3. Find enough independent differential equations for $h_{\mu\nu}$ through the Einstein equation,

4. Decouple the components of the equations if possible.

It is important for the reader to keep in mind that the crucial first step, i.e. a suitable way to write the metric makes all the calculations much easier since the number of variables of $h_{\mu\nu}$ can decrease drastically. For instance, considering a specific geometry in which some components of $g_{\mu\nu}$ vanish, the same components of $h_{\mu\nu}$ will vanish as well for the perturbations should not alter the symmetries of the problem.

3.2.1 Axisymmetric and time-independent space-times.

Even though perturbations on axisymmetric space-times have time in dependence, we shall start with the simpler case that includes time dependence. Only after exploring all the important results, this example can provide us will, we will introduce time-dependence. Let us first introduce the notation we will use:

$$\left(x^0, x^1, x^2, x^3\right) = (t, \phi, r, \theta),$$

and, since we aim at a axisymmetric, stationary, space-time, we can request that the metric components may be x^0, x^1-independent,

$$g_{\mu\nu} = g_{\mu\nu}(x^2, x^3),$$

We also shall consider that the space-time is invariant under the transformations $t \to -t$ and $\phi \to -\phi$. The former consideration can also be applied for asymptotically conical space-times (such as the black hole pierced by a cosmic string), but later we will also eliminate this requirement. It is quite clear that with these two last requirements we find,

$$g_{02} = g_{03} = g_{12} = g_{13} = 0,$$

leaving us with

$$ds^2 = g_{00}(dx^0)^2 + 2g_{01}dx^1dx^0 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + 2g_{32}dx^2dx^3 + g_{33}(dx^3)^2 \quad (3.2)$$

To simplify this metric we can state the following theorem,

Theorem 1

Any metric of a 2D space (y^1, y^2), with constant signature, can always be put in the form,

$$ds^2 = \pm e^{2\mu}((dy^1)^2 + (dy^2)^2),$$

where $e^{2\mu}$ is a function of (y^1, y^2).

Proof

Take into consideration the coordinate transformation,

$$y^{1'} = \phi(y^1, y^2) \qquad y^{2'} = \psi(y^1, y^2),$$

and require that the metric in this new coordinate system to be diagonal, for this it is sufficient that the new metric tensor satisfies the following conditions:

(a) $g^{1'2'} = g^{11}\phi_{,1}\psi_{,1} + g^{12}(\phi_{,1}\psi, 2 + \phi_{,2}\psi_{,1}) + g^{22}\phi_{,2}\psi_{,2} = 0,$

(b) $g^{1'1'} - g^{2'2'} = g^{11}(\phi_{,1}^2 - \psi_{,1}^2) + 2g^{12}(\phi_{,1}\phi_{,2} - \psi_{,1}\psi_{,2}) + g^{22}(\phi_{,2}^2 - \psi_{,2}^2) = 0.$

Now we shall prove that this transformation indeed exists. It is easy to see that the condition (a) can be satisfied if

$$\phi_{,1} = \kappa(g^{21}\psi_{,1} + g^{22}\psi_{,2}) \quad \text{and} \quad \phi_{,2} = -\kappa(g^{11}\psi_{,1} + g^{12}\psi_{,2}),$$

where κ is an arbitrary function. That stated we can go back to condition (b) and verify that this one becomes,

$$(\kappa^2(g^{11}g^{22} - (g^{12})^2 - 1)(g^{11}\psi_{,1}^2 + 2g^{12}\psi_{,1}\psi_{,2} + g^{22}\psi_{,2}^2) = 0,$$

noting that the second multiplication term can not vanish since the metric is supposed to be negative or positive-definite, the only way to satisfy this equation is by choosing,

$$\kappa^2 = \frac{1}{g^{11}g^{22} - (g^{12})^2} = g_{11}g_{22} - (g_{12})^2 = g,$$

where g is, as usual, the determinant of the metric. Therefore we can summarize this result imposing simply,

$$\phi_{,1} = g^{\frac{1}{2}}g^{2k}\psi_{,k},$$

and

$$\phi_{,2} = -g^{\frac{1}{2}}g^{2k}\psi_{,k}.$$

By the integrability condition we finally arrive at,

$$(g^{\frac{1}{2}}g^{ik}\psi_{,k})_{,i} = 0,$$

which means that ψ is any solution of the Laplace equation and therefore exists, as well as ϕ. This proves the theorem.

Now we can write the final metric of the time-independent axisymmetric space-time in a simpler form, rewriting the coefficients of the metric (3.2):

- $g_{00} = (e^{2\nu} - 2e^{2\psi}\omega^2)$
- $g_{11} = -e^{2\psi}$
- $g_{10} = e^{2\psi}\omega$
- $g_{23} = 0$
- $g_{22} = e^{2\mu_2}$
- $g_{33} = e^{2\mu_3}$

Here, instead of having the 6 independent functions, of (x^2, x^3) in (3.2), we have 5 independent functions of (x^2, x^3), ω, ν, ψ, μ_2 and μ_3. The g_{23} can be chosen to vanish since the $(x^2, x^3)-$space has constant signature and the theorem proved above applies. We could also have chosen $\mu_2 = \mu_3$ due to the same reason, but for now we shall not use this choice yet. The final metric is written as

$$ds^2 = e^{2\nu}(dx^0)^2 - e^{2\psi}(dx^1 - \omega dx^0)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2 \tag{3.3}$$

In this form the metric coefficients have clearer meaning. As an example let us make some considerations about the dragging of the inertial frame.

3.2.2 The dragging of the inertial frame

The inertial frame of reference is an old and useful concept in physics. Inertial forces are those which appear to change an object from its previous inertial frame. Before GR, inertial frames where always constant-velocity frames following straight lines, which naturally do not experience forces. But in the GR context this scenario is different, the mass distribution around an object may change the locally natural inertial frames of it, creating rather counterintuitive frames for which a body does not experience forces. This often happens around some mass distributions with dynamical (even stationary) properties. For instance, Earth itself, due its rotation, drags the inertial frames of references around it, this is a very well known fact and used to the calibration of GPS nowadays.

The question we want to investigate here is how to identify such dragging for a geometry. To do that we can choose a suitable tetrad frame for the metric (3.3) which have the propriety $\eta_{(a)(b)} = diag(-1, 1, 1, 1)$*. This way we will have a frame with a Minkowskian metric and therefore a locally inertial one. The procedure to find such basis is the following. We take the condition

$$e^i_{(a)} e_{(b)i} = \eta_{(a)(b)} = diag(-1, 1, 1, 1),$$

using the metric to raise the indices without parameters we can find $e_{1(a)}$. For the metric (3.3) we find,

$$\begin{cases} e_{(0)i} = (e^{\nu}, \ 0, \ 0, \ 0), \\ e_{(1)i} = (\omega e^{\psi}, \ -e^{\psi}, \ 0, \ 0), \\ e_{(2)i} = (e^{\nu}, \ 0, \ -e^{\mu_2}, \ 0), \\ e_{(3)i} = (e^{\nu}, \ 0, \ 0, \ -e^{\mu_3}). \end{cases} \tag{3.4}$$

* $i, a, b \in \{0, 1, 2, 3\}$

Now we can consider a point with a four-velocity (u^0, u^1, u^2, u^3), being s a affine parameter along the worldline of a particle, in the standard coordinates we have,

$u^0 = \frac{dt}{ds} = \frac{e^{-\nu}}{\sqrt{1-V^2}}$,

$u^1 = \frac{d\phi}{ds} = u^0\Omega$, with $\Omega = \frac{d\phi}{dt}$

$u^i = \frac{dx^i}{ds} = u^0 v^i$ $(i = 2, 3)$, with $v^i = \frac{dx^i}{dt}$, $(i = 2, 3)$

and $V^2 = e^{2\psi-2\nu}(\Omega - \omega)^2 + e^{2\mu_2-2\nu}(v^2)^2 + e^{2\mu-2\nu}(v^3)^2$.

These calculations are straightforward and the definitions above just help the final interpretation to be clearer. Now let us write the four-velocity expanded on the tetrad basis by using

$$u^{(a)} = e_i^{(a)} u^i = \eta^{(a)(b)} e_{(b)i} u^i,$$

this way we find:

$$u^{(0)} = \frac{1}{\sqrt{1-V^2}}, \quad u^{(1)} = \frac{e^{\psi-\nu}(\Omega - \omega)}{\sqrt{1-V^2}} \;\; and \;\; u^{(i)} = \frac{e^{\mu_i-\nu} v^i}{\sqrt{1-V^2}} \;\; (i = 2, 3).$$

So, it is clear that a point particle at relative rest, $(u^0, 0, 0, 0)$ needs to have an angular velocity $\Omega = \omega$ in a circular orbit around the mass distribution. That is why we call the inertial frame to be dragged. The use of tetrad coordinates is useful since this coordinates enable us to project four-vectors at every point of the space-time into a suitable (in this case Minkowskian) space-time.

This discussion is important because, as we shall see later, parameters such as ω have a different nature than ν, μ_1, μ_2 and ψ. But there can be other parameters similar to ω when we consider time-dependent axisymmetic space-times. Identifying these new parameters as well as writing down a general metric for these space-times will be discussed in the following.

3.2.3 Axisymmetric and time-dependent space-times

When considering perturbations, the time-independence condition does not hold. Therefore from now on we shall consider axisymmetric non-stationary space-times, in this case

$$g_{\mu\nu} = g_{\mu\nu}(x^0, x^2, x^3).$$

By the Cotton-Darboux theorem, which is a generalization of the theorem proved in the last section, we can consider the metric on the 3D space (x^0, x^2, x^3) to be diagonal, what means that $g^{02} = g^{03} = g^{32} = 0$. Similarly to the considerations in the last section the final metric can be written as

$$ds^2 = e^{2\nu}(dx^0)^2 - e^{2\psi}(dx^1 - q_2 dx^2 - q_3 dx^3 - \omega dx^0)^2 - e^{\mu_2}(dx^2)^2 - e^{\mu_3}(dx^3)^2. \quad (3.5)$$

where q_2 and q_3 are functions of (x^0, x^2, x^3)

The next step is to calculate the Ricci tensor for this metric. To do this we went through two methods, the first one was done by Chandrasekhar [25], using Cartan's equations of structure. This is a rather tedious calculation, so we shall simply write down the contractions in Appendix B, where it was have used the useful definitions,

$$Q_{AB} \doteq q_{A,B} - q_{B,A} \quad and \quad Q_{A0} \doteq q_{A,0} - \omega_{,A}, \quad (A, B = 2, 3). \quad (3.6)$$

For now the indices 2 and 3 have no difference, so R_{33}, R_{03}, R_{13} and G_{33} can be obtained by changing 2 by 3 in the correspondent R_{22}, R_{02}, R_{12} and G_{22}. This calculation was done by hand and there is some evidence that it might be wrong[26]. Yet, since it is still used largely nowadays we keep the Chandrasekhar calculations as the basis of our considerations for the majority of our project.

The first step towards perturbing the BH+CS metric, $g^0_{\mu\nu}$, is to write down how the metric coefficients discussed above, $g_{\mu\nu}(x^0, x^2, x^3)$, should behave in the non-perturbed scenario:

$$g_{\mu\nu}(x^0, x^2, x^3) \to g^0_{\mu\nu}(x^0, x^2, x^3) \implies$$

$$e^{2\nu} = e^{-2\mu_2} = \Delta/r^2, \;\; e^{2\mu_3} = \beta r, \;\; e^{2\psi} = \beta r \sin\theta, \quad (3.7)$$

$$\omega = q_2 = q_3 = 0 \quad (3.8)$$

where $\Delta = r^2 - 2Mr$.

We can then introduce the perturbation as follows,

1. $q_2, \; q_3, \; \omega \to$ these functions will behave like small parameters to be considered up to first order.

2. $\nu \;, \mu_2 \;, \mu_3, \; \psi \to$ these functions will experience small increments to be considered up to first order, $\delta\nu \;, \delta\mu_2 \;, \delta\mu_3, \; \delta\psi$.

The first set of functions will have a different meaning than the second. As hinted from the analysis of the dragging of the inertial frame caused by ω discussed above, the first set of functions will be related to the so called axial perturbations. The second set will be, on the other hand, related to the so called polar perturbations. Also, having different natures, these two kinds of perturbations can be considered independent from each other, as we shall show.

3.2.4 Axial perturbation.

Since axial perturbations are described by ω, q_1, q_3, either with or without the cosmic string, we expected that the axial perturbations would be the same as the ones for the Schwarzschild black hole alone.

The easiest way to study it is to use the Riemann tensor components R_{12} and R_{13}. That is because these are the only quantities in which all the variables we wish to analyze appear. We shall then consider ω, q_1, q_3 as small quantities and add the unperturbed values of ν, μ_2, μ_3, ψ to the equations

$$R_{12} = R_{13} = 0$$

we have,

$$R_{12} = -\frac{1}{2}e^{-2\psi-\nu-\mu_3}[(e^{3\psi+\nu-\mu_2-\mu_3}Q_{32})_{,3} - (e^{3\psi-\nu+\mu_3-\mu_2}Q_{02})_{,0}] = 0,$$

and since ν, μ_2, μ_3, ψ will become the unperturbed values which do not depend on time, we have[†]

$$(e^{3\psi+\nu-\mu_2-\mu_3}Q_{23})_{,3} = -e^{3\psi-\nu+\mu_3-\mu_2}Q_{02}{}_{,0} \tag{3.9}$$

From $R_{13} = 0$ we find:

$$(e^{3\psi+\nu-\mu_2-\mu_3}Q_{23})_{,2} = e^{3\psi-\nu+\mu_2-\mu_3}Q_{03}{}_{,0}\,. \tag{3.10}$$

Inserting the values of equation (3.7) in (3.9) and (3.10), we get,

$$\frac{\beta}{r^4\sin^3\theta}(\Delta\sin^3\theta Q_{23})_{,2} = -\beta Q_{02,0}$$

and

$$\frac{\beta\Delta}{r^4\sin^3\theta}(\Delta\sin^3\theta Q_{23})_{,3} = \beta Q_{03,0}$$

† we used the identity,$Q_{23} = -Q_{32}$, in order to have a similar equation as the one applied for $R_{13} = 0$

Now, we can simplify the coefficient β such that the equations do not depend of the string parameter. Also, defining $Q = \Delta \sin^3\theta Q_{23}$, we can simplify both equations,

$$\frac{1}{r^4 \sin^3\theta}\frac{\partial Q}{\partial \theta} = -(\omega_{,2} - q_{2,0})_{,0} \tag{3.11}$$

$$\frac{\Delta}{r^4 \sin^3\theta}\frac{\partial Q}{\partial r} = (\omega_{,3} - q_{3,0})_{,0} \tag{3.12}$$

Now let us make the reasonable assumption that the time-dependence of the perturbations is $e^{i\sigma t}$, which means that we are separating already the frequencies of the perturbations, as one also may do for electromagnetic waves for instance. With this assumption, equations (3.11) and (3.12) become:

$$\frac{1}{r^4 \sin^3\theta}\frac{\partial Q}{\partial \theta} = -i\sigma\omega_{,2} - \sigma^2 q_2, \tag{3.13}$$

$$\frac{\Delta}{r^4 \sin^3\theta}\frac{\partial Q}{\partial r} = i\sigma\omega_{,3} + \sigma^2 q_3. \tag{3.14}$$

Now, taking ∂_3 on (3.13) and ∂_2 on (3.14) we can both eliminate ω and use the definition of Q in order to obtain,

$$r^4\frac{\partial}{\partial r}\left(\frac{\Delta}{r^4} + \frac{\partial Q}{\partial r}\right) + \sin^3\theta\frac{\partial}{\partial \theta}\left(\frac{1}{\sin^3\theta}\frac{\partial Q}{\partial \theta}\right) + \sigma^2\frac{r^4 Q}{\Delta} = 0 \tag{3.15}$$

The natural question now is how to find the solutions which decouple $x^2 = \theta$ and $x^3 = r$. The answer can be found in an extension of the Legendre polynomials, the Gegenbauer polynomials, following the calculations done in [25]. These polynomials are the eigenfunctions of the following operator,

$$\left[\frac{d}{d\theta}\sin^{2\nu}\frac{d}{d\theta} + n(n+2\nu)\sin^{2\nu}\theta\right]C_n^{\nu}(\theta) = 0 \tag{3.16}$$

Therefore, to decouple equation (3.15) we can chose $Q(r,\theta) = Q(r)C_{l+2}^{-3/2}(\theta)$. We obtain the radial differential equation,

$$\Delta\frac{d}{dr}\frac{\Delta}{r^4}\frac{dQ}{dr} + \mu^2\frac{\Delta}{r^4}Q + \sigma^2 Q = 0, \tag{3.17}$$

where $\mu^2 = (l+2)(l-2)$. Now we change to the tortoise coordinate r_*, given by

$$r_* \doteq r + 2M\ln(r/M - 1),$$

$$\frac{d}{dr_*} = \frac{\Delta}{r^2}\frac{d}{dr}.$$

It will be also useful to introduce the function, $Q(r) = rZ^{(-)}$ in order to obtain a Schrödinger-like equation,

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right) Z^{(-)} = V^{(-)} Z^{(-)}, \tag{3.18}$$

where the potential is given by

$$V^{(-)} = \frac{\Delta}{r^5}\left[(\mu^2+2)r - 6M\right]. \tag{3.19}$$

This is the so called Regge-Wheeler equation and governs the axial perturbations. Clearly the cosmic string has no effect on the equation.

Somehow the lack of the string influence was expected for this sort of perturbation, since the string does not imprint a dragging of the inertial frame, as one can verify by the constraints (3.8), which are the same as the ones for the pure Schwarzschild geometry.

3.2.5 Polar perturbations.

The polar perturbations are characterized by the non-vanishing of ν, ψ, μ_2 and μ_3. Now the Ricci tensors to be studied are R_{02}, R_{03}, R_{23}, R_{11} together with G_{11}. Indeed, in the expression for these tensors, Q_{AB} appear only quadratically, justifying why we can treat axial and polar perturbations separately. In this section the explicit calculation will be done only for R_{02}, since the mechanism that makes β drop from the equations will be analogous for all the other components. Writing the perturbations as follows,

$$\psi \to \psi + \delta\psi \quad \nu \to \nu + \delta\nu \tag{3.20}$$
$$\mu_2 \to \mu_2 + \delta\mu_2 \quad \mu_3 \to \mu_3 + \delta\mu_3$$

And adding (3.7) to get the derivatives,

$\psi_{,2} = r^{-1}$, $\psi_{,3} = \cot(\theta)$, $\mu_{3,2} = r^{-1}$(3.21)we find (keeping in mind that ν is independent of β):

$$R_{02} = e^{-\nu-\delta\nu-\mu_2-\delta\mu_2}\left[(\psi+\delta\psi+\mu_3+\delta\mu_3)_{,2,0} + (\psi+\delta\psi)_{,2}(\psi+\delta\psi-\mu_2-\delta\mu_2)_{,0}\right.$$
$$\left.+(\mu_3+\delta\mu_3)_{,2}(\mu_3+\delta\mu_3-\mu_2-\delta\mu_2)_{,0} - (\psi+\delta\psi+\mu_3+\delta\mu_3)_{,0}(\nu+\delta\nu)_{,2} = 0\right.$$

Since only the perturbations are time dependent, we have:

$$(\delta\psi+\delta\mu_3)_{,2,0} + (\psi+\delta\psi)_{,2}(\delta\psi-\delta\mu_2)_{,0} + (\mu_3+\delta\mu_3)_{,2}(\delta\mu_3-\delta\mu_2)_{,0}$$

$$-(\delta\psi + \delta\mu_3)_{,0}(\nu + \delta\nu)_{,2} = 0$$

Now, taking all the perturbations to have the same time-dependence, $e^{-it\sigma}$, this becomes

$$e^{-it\sigma}\left[(\delta\psi + \delta\mu_3)_{,2} + (\psi + \delta\psi)_{,2}(\delta\psi - \delta\mu_2) + (\mu_3 + \delta\mu_3)_{,2}(\delta\mu_3 - \delta\mu_2)\right.$$
$$-(\delta\psi + \delta\mu_3)(\nu + \delta\nu)_{,2} = 0,$$

and successively,

$$(\delta\psi + \delta\mu_3)_{,2} + (\psi + \delta\psi)_{,2}(\delta\psi - \delta\mu_2) + (\mu_3 + \delta\mu_3)_{,2}(\delta\mu_3 - \delta\mu_2)$$

$$-(\delta\psi + \delta\mu_3)(\nu + \delta\nu)_{,2} = 0.$$

Using the results (3.21), this equation can be rearranged as follows:

$$(\delta\psi + \delta\mu_3)_{,r} + (r^{-1} - \nu_{,r})(\delta\psi + \delta\mu_3) - \frac{2\delta\mu_2}{r} = 0. \tag{3.22}$$

We can see from this calculation and from the calculations related to the axial perturbations that β will only appear in the final result if we have non-quadratic terms multiplying ψ (without derivatives) or e^{ψ} (as long as this term is not multiplying all linear terms, since in this case the equality to zero required for the vacuum solution will force it to drop). A closer look at the Ricci and Einstein tensors for an arbitrary symmetric space-time will show that none of these terms appear, showing that the cosmic string does not imprint a change in the perturbations on the model we are studying.This way, all the GW calculations, including quasi-normal modes for the Schwarzschild black hole are unchanged. Thus we will nor show the next steps of the calculations here, e.g.the separation of the differential equations, since they will not present any new result. Yet the reader should keep in mind that all the time derivatives of the perturbation will be substituted by a $i\sigma$ term, and then the separation will be done only in space coordinates.

That is a rather unexpected result. Yet, there are two independent alternative ways to rethink the tensors perturbations around the BH+CS background we explored as well. These were the cases in which,

- The cosmic string parameter β is not time-independent.
- The Chandrasekhar contractions are not correct[26]

A brief study of the former case will done in the next section.

3.3 The time-dependent cosmic string

Any physical system can evolve in time. Indeed, various simulations of the evolution of cosmic strings have been done using numerical methods . But, evolving a string in time and space is rather complicated, and there is no analytical research for such evolution in a black hole background. Therefore, we chose not to discard all our hypothesis to do this analysis. Let us still consider an infinitely thin and straight string on the z axis of a Schwarzschild BH. Besides that, now, we shall consider $\beta(x^0)$. The intuitive reason for this is the fact that β is proportional to a deficit angle δ in the ϕ coordinate. Remembering that ϕ will be perturbed with a time dependence, it is natural to believe that β will be perturbed as well. Other dependencies are discarded due to the hypothesis mentioned.

Consider then the most general form of the desired metric,

$$\mathrm{ds}^2 = e^{2\nu}(dx^0)^2 - e^{2\psi}(dx^1 - q_2 dx^2 - q_3 dx^3 - \omega dx^0)^2 - e^{\mu_2}(dx^2)^2 - e^{\mu_3}\beta(t)(dx^3)^2$$

We can see that in the limits (3.7) and (3.8) we recover the usual BH+CS metric but with $\beta \to \beta(t)$. Further we will see that the difficulty on the decoupling of the new equations for polar and axial perturbations arrives from the fact that now it is impossible to cancel the time dependence $e^{it\sigma}$ since the term $\frac{\beta_{,t}}{\beta}$ has no explicit time-dependence. Yet an interesting result appears if we propose the relation

$$\frac{\beta_{,t}}{\beta} = \delta\beta i\sigma e^{it\sigma},$$

Where $\delta\beta$ is a small (positive or negative) constant, β_0 is the initial state of the string and σ is the QNM of all the other perturbations, already expanded. The solution of the above equation is,

$$\beta(t) = \beta_0 e^{\delta\beta e^{it\sigma}}.$$

This solution is not real, but we can take in account on its real part, which can be approximated from an harmonic function with or without a decaying amplitude for the case of a stable black hole.

Starting with the axial perturbations, in which the differential equations are much simpler, we will have from $R_{12} = 0$.

$$(e^{3\psi+\nu-\mu_2-\mu_3}Q_{23})_{,3} = -e^{3\psi-\nu+\mu_3-\mu_2}Q_{02,0} \tag{3.23}$$

and then

$$\frac{Q_{,3}}{r^4 \sin^3\theta} = -(\frac{\beta'}{\beta} Q_{02,0} + Q_{02}). \tag{3.24}$$

and from $R_{13} = 0$,

$$(e^{3\psi+\nu-\mu_3-\mu_2} Q_{23})_{,2} = -e^{3\psi-\nu+\mu_2-\mu_3} Q_{03,0} \tag{3.25}$$

$$\frac{\Delta Q_{,2}}{r^4 \sin^3\theta} = +(\frac{\beta'}{\beta} Q_{03,0} + Q_{03}) \tag{3.26}$$

since we wish $\delta\beta$ to be considered only up to first order, the terms $\frac{\beta'}{\beta} Q_{02,0}$ and $\frac{\beta'}{\beta} Q_{03,0}$ drop, so we get the same equations as before. Now, for polar perturbations we will have a more interesting result, since we are now taking $\psi \to \psi + \delta\psi$, we can make the substitution

$$e^{\psi+\delta\psi} \to e^{\psi+\delta\psi+\delta\beta e^{\sigma t}}$$

since we will take $q_3 = 0$ and the only term left involving both ψ and β in the metric is multiplying both. This will work nice because then the time dependence of $\delta\psi + \delta\beta e^{\sigma t}$ will still be $e^{\sigma t}$, which means that all results from polar perturbations (including the decoupling) can be recovered with a small shift on the $\delta\beta$ perturbation.

4 CONCLUSION

In this thesis we have shown the effects of a cosmic string piercing a Schwarzschild black hole with respect to perturbation in this geometry. The string parameter β, related to the deficit angle on the equatorial plane, has been shown to affect only scalar perturbations, having no influence in tensorial perturbations, the latter being the ones related to gravitational waves. The effect on scalar perturbations was that was not possible to find a solution in terms of the spherical harmonics, what can be seen as expected since the string breaks the spherical symmetry of the black hole. On the other hand, the formalism used for tensor perturbations took the axial symmetry into account. From these two results we can conclude that the BH+CS solution is stable under small linear tensor perturbations but does not imprint differences on gravitational wave emission during the ringing down of a Schwarzschild solution. Since the tensor perturbations could be done explicitly, there was no need to work with the eikonal limit, but a further work can be done with this limit to check our result.

We also analyzed the case in which a cosmic string could oscillate in time, concluding that in the case that this string parameter β oscillates with the same modes as the polar oscillations, we also find no effect on first order axial oscillations and only a small shift, which can be ignored since it is a constant in space, on the polar oscillations.

We suggest as a future work to study the perturbations of the Kerr solution pierced by a cosmic string in its symmetry axis. We guess the result can be similar to ours since the Kerr geometry has already axial symmetry. A further analysis in the Schwarzschild case with a time-dependent string or with the correction done in [26] would be of interest as well.

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Appendix

APPENDIX A

Details of scalar perturbations (3.1).

since the metric is diagonal, we only have four terms,

$$\tfrac{1}{r^2\sin(\theta)}\partial_r[g^{rr}r^2\sin(\theta)\partial_r]\Psi + \tfrac{1}{r^2\sin(\theta)}\partial_\theta[g^{\theta\theta}r^2\sin(\theta)\partial_\theta]\Psi +$$

$$\tfrac{1}{r^2\sin(\theta)}\partial_\phi[g^{\phi\phi}r^2\sin(\theta)\partial_\phi]\Psi + \tfrac{1}{r^2\sin(\theta)}\partial_t[g^{tt}r^2\sin(\theta)\partial_t]\Psi = 0$$

$\Downarrow$

$$\partial_r[g^{rr}r^2\partial_r]\Psi + \tfrac{1}{\sin(\theta)}\partial_\theta[\sin(\theta)\partial_\theta]\Psi + \tfrac{1}{\beta^2\sin^2(\theta)}\partial_{\phi\phi}\Psi + -r^2g^{tt}\partial_{tt}\Psi = 0$$

APPENDIX B

$$-\mathrm{R}_{00} = e^{-2\nu}\left[(\psi+\mu_2+\mu_3)_{,0,0} + \psi_{,0}(\psi-\mu)_{,0} + \mu_{2,0}(\mu_2-\nu)_{,0} + \mu_{3,0}(\mu_3-\nu)_{,0}\right]$$
$$- e^{-2\mu_2}\left[\nu_{,2,2} + \nu_{,2}(\psi+\nu-\mu_2+\mu_3)_{,2)}\right] - e^{-2\mu_3}\left[\nu_{,3,3} + \nu_{,3}(\psi+\nu+\mu_2-\mu_3)_{,3)}\right]$$
$$- \tfrac{1}{2}e^{2\psi-2\nu}\left[e^{-2\mu_2}Q_{20}^2 + e^{-2\mu_3}Q_{03}^2\right]$$

$$-\mathrm{R}_{11} = e^{-2\mu_2}\left[\psi_{,2,2} + \psi_{,2}(\psi+\nu-\mu_2+\mu_3)_{,2}\right] + e^{-\mu_3}\left[\psi_{,3,3} + \psi_{,3}(\psi+\nu+\mu_2-\mu_3)_{,3}\right]$$
$$-e^{-2\nu}\left[\psi_{,0,0} + \psi_{,0}(\psi-\nu+\mu_2+\mu_3)_{,0}\right] - \tfrac{1}{2}e^{2\psi-2\mu_2-2\mu_3}Q_{23}^2 + \tfrac{1}{2}e^{2\psi-2\nu}\left[e^{-2\mu_3}Q_{30}^2 + e^{-2\mu_2}Q_{20}^2\right]$$

$$-\mathrm{R}_{22} = e^{-2\mu_2}\left[(\psi+\nu+\mu_3)_{,2,2} + \psi_{,2}(\psi-\mu_2)_{,2} + \mu_{3,2}(\mu_3-\mu_2)_{,2} + \nu_{,2}(\nu-\mu_2)_{,2}\right]$$
$$+ e^{2\mu_3}\left[\mu_{2,3,3} + \mu_{2,3}(\psi+\nu+\mu_2-\mu_3)_{,3}\right]$$
$$- e^{-2\nu}\left[\mu_{2,0,0} + \mu_{2,2}(\psi-\nu+\mu_2+\mu_3)_{,0}\right]$$
$$+ \tfrac{1}{2}e^{2\psi-2\mu_2}\left[e^{-2\mu_3}Q_{23}^2 - e^{-2\nu}Q_{20}^2\right.$$

$$-\mathrm{R}_{01} = \tfrac{1}{2}e^{2\psi-\mu_2-\mu_3}\left[(e^{2\psi-\nu-\mu_2+\mu_3}Q_{20})_{,2} + (e^{3\psi-\nu+\mu_2-\mu_3}Q_{30})_{,3}\right]$$

$$-\mathrm{R}_{12} = \tfrac{1}{2}e^{2\psi-\nu-\mu_3}\left[(e^{2\psi+\nu-\mu_2-\mu_3}Q_{32})_{,3} + (e^{3\psi-\nu-\mu_2+\mu_3}Q_20)_{,0}\right]$$

$$-\mathrm{R}_{02} = e^{-\mu_2-\nu}\left[(\psi+\mu_3)_{,2,0} + \psi_{,2}(\psi-\mu_2)_0 + \mu_{3,2}(\mu_3-\mu_2)_{,0}\right.$$

$$-\nu_{,2}(\psi-\mu_3)_0 - \tfrac{1}{2}e^{2\psi-\nu-2\mu_2-2\mu_3}Q_{23}Q_{30}$$

$$-\mathrm{R}_{23} = e^{-\mu_2-\mu_3}\left[(\psi+\nu)_{,2,3} - \mu_{2,3}(\psi+\nu)_2 - \mu_{3,2}(\psi+\nu)_{,3} + \psi_{,2}\psi_{,3}\right.$$

$$-\nu_{,2}\nu_{,3} - \tfrac{1}{2}e^{2\psi-2\nu-\mu_2-\mu_3}Q_{20}Q_{30}$$

$$\mathrm{G}_{00} = e^{-2\mu_2}\left[(\psi+\mu_3)_{,2,2} + \psi_{,2}(\psi-\mu_2+\mu_3)_{,2} + \mu_{3,2}(\mu_3-\mu_2)_{,2}\right]$$
$$- e^{-2\mu_3}\left[(\psi+\mu_2)_{,3,3} + \psi_{,3}(\psi-\mu_3+\mu_2)_{,3} + \mu_{2,3}(\mu_2-\mu_3)_{,3}\right]$$
$$+ e^{-2\nu}\left[\psi_{,0}(\mu_2+\mu_3) + \mu_{3,0}\mu_{2,0}\right] - \tfrac{1}{4}e^{2\psi-\nu}\left[e^{-2\mu_2}Q_{02}^0 + e^{-2\mu_3}Q_{03}^0\right] - \tfrac{1}{4}e^{2\psi-2\mu_2-2\mu_3Q_{23}^2}$$

$$\mathrm{G}_{11} = e^{2\mu_2}\left[(\nu+\mu_3)_{,2,2} + \mu_{,2}(\nu-\mu_2+\mu_3)_{,2} + \mu_{3,2}(\mu_3-\mu_2)_{,2}\right]$$
$$+ e^{2\mu_3}\left[(\nu+\mu_2)_{,3,3} + \mu_{,3}(\nu-\mu_3+\mu_2)_{,3} + \mu_{2,3}(\mu_2-\mu_3)_{,3}\right]$$
$$- e^{2\nu}\left[(\mu_2+\mu_3)_{,0,0} + \mu_{2,0}(\mu_2-\nu)_{,0} + \mu_{,3,0}(\mu_3-\nu)_{,0} + \mu_{2,0}\mu_{3,0}\right] +$$
$$\tfrac{3}{4}e^{2\psi}\left[e^{-2\mu_2-2\mu_3}Q_{23}^2 - e^{2\mu_2-2\nu}Q_{20} - e^{2\mu_3-2\nu}Q_{30}\right]$$

$$\mathrm{G}_{22} = e^{2\mu_3}\left[(\psi+\nu)_{,3,3} + (\psi+\nu)_{,3}(\nu-\mu_3)_{,3} + \psi_{,3}\psi_{,3}\right]$$
$$+ e^{-2\mu_2}\left[\nu_{,2}(\psi+\mu_3)_{,2} + \psi_{,2}\mu_{3,2}\right]$$
$$- e^{-2\nu}\left[(\psi+\mu_3)_{,0,0} + (\psi+\mu_3)_{,0}(\mu_3-\nu)_{,0} + \psi_0\psi_0\right]$$
$$- \tfrac{1}{4}e^{2\psi}\left[e^{-2\mu_2-2\mu_3}Q_{23}^2 - e^{2\mu_2-\nu}Q_{20}^2 + e^{-2\mu_3-2\nu}Q_{30}^2\right]$$

where we have used the useful definitions,

$$Q_{AB} \doteq q_{A,B} - q_{B,A} \quad and \quad Q_{A0} \doteq q_{A,0} - \omega_{,A}, \quad (A, B = 2, 3).$$