**UNIVERSIDADE DE SÃO PAULO
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Geodesic motion in the Reissner-Nordström space-time

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*This monograph is dedicated to my relatives. Mother, sister, brother and deceased father.
Also dedicated to my supportive friends and to Erwin, Enrico and Cora, beloved cats.*

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*“O estudo, a busca da verdade e da beleza são domínios
em que nos é consentido sermos crianças por toda a vida.”*

Albert Einstein

ABSTRACT

CAPOBIANCO, R. A. **Geodesic motion in the Reissner-Nordström space-time**. 2019. 53p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2019.

The motion of neutral test particles, both massive and massless, in the space time of a charged source described by the Reissner-Nordström solution is studied. This solution is characterized by two parameters, mass and charge, which defines the horizons of the source. When the mass is larger than the charge, the solution describes a black hole, with two distinct horizons. When the mass and charge are equal there is an extremal black hole, and both horizons merge to one. Finally, when the charge is larger than the mass there is a naked singularity, with no horizon. The structure and properties of these different type of solution are presented and discussed. A general solution of the equations of motion is presented in function of the Weierstrass elliptic function $\wp$. In addition, the possible orbits for test particles are discussed, and the conditions for existence of closed, circular or escape orbits are presented. The classifications is made based on the particles energy, and the mass and charge of the source. We find that all mentioned orbits are allowed for the three different type of solutions. In particular, for extremal black holes and naked singularities, we find stable circular orbits located outside the event horizon and hence being visible for an external observer.

Keywords:General relativity. Geodesics. Circular orbits. Black holes. Naked singularities.

RESUMO

CAPOBIANCO, R. A. **Movimento geodésico no espaço-tempo de Reissner-Nordström**. 2019. 53p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2019.

O movimento de partículas teste neutras, ambas massivas e sem massa, no espaço-tempo de uma fonte carregada descrita pela solução de Reissner-Nordström é estudada. Essa solução é caracterizada por dois parâmetros, massa e carga, que definem os horizontes da fonte. Quando a massa é maior que a carga tal solução descreve um buraco negro com dois horizontes distintos. Quando a massa e a carga são iguais há um buraco negro extremo, e ambos os horizontes se unem em um. Finalmente, quando a carga é maior que a massa, há uma singularidade nua, sem horizontes. A estrutura e as propriedades dessas diferentes soluções são apresentadas e discutidas. Uma solução geral da equação de movimento é apresentada em termos da função elíptica de Weierstrass, $\wp$. Além do mais as possíveis órbitas para uma partícula teste são discutidas, e as condições para existência de órbitas fechadas, circulares e de escape são apresentadas. A classificação é feita a partir da energia da partícula, e da massa e carga da fonte. Encontramos que todas as órbitas mencionadas são permitidas nos três diferentes tipos de soluções. Em particular, para buracos negros extremos e singularidades nuas, encontramos órbitas circulares estáveis localizadas fora do horizonte de eventos e, conseqüentemente, sendo visível para observadores externos.

Palavras-chave: Relatividade geral. Geodésicas. Órbitas circulares. Buracos negros. Singularidades nuas.

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LIST OF ABBREVIATIONS AND ACRONYMS

BH	Black hole
NS	Naked singularity
RN	Reissner-Nordström

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1 INTRODUÇÃO

The motion of objects in the sky has called the attention of mankind since the beginning of humanity, creating fear and questions ever since. Although some knowledge on the motion of celestial objects had been obtained by ancient cultures, the development of the modern scientific approach started with Johannes Kepler. (1)

Kepler is considered the father of modern astronomy, he was the first to describe the motion of planets by the action of a force exerted by the Sun on these planets. Based on this assumption and the analysis of the orbital data of Mars, he found that the planetary motion in the solar system was elliptical, rather than circular, as believed before.

Nevertheless, the understanding of such force as gravitation is due to Newton. Newton unified Kepler's laws using the gravitational force. The classical theory of gravitation was not only able to successfully explain the motion of the planets in the solar system, but more than this, was able to perfectly predict the existence of Neptune before direct observation, just by observing irregularities in Uranus' orbit. However, Newton's theory was not able to fully explain the motion observed in the solar system. Mercury's perihelion problem was the first hint that the non-relativistic theory of gravity cannot completely solve the problem.

The general theory of relativity changes completely how physics understand gravitation and its effects. Besides a gravitational field created by a massive source, which attracts massive bodies, in Einstein's theory of gravitation the motion of a particle is understood as an effect of the deformation of the space and time. The motion of a test particle under the influence only of a gravitational field, i.e., in a geodesic motion, provides a tool to investigate the space-time itself.

General Relativity can also explain the orbits of the planets in the solar system, as well as Kepler's planetary motion, since the description reduces to Newton's law in the weak gravitational field regime, and was able to solve Mercury's perihelion problem. One of the most fundamental differences between the theories is the fact that the curvature of space-time should also be felt by light.

This year we are celebrating the centenary of the observation of light deflection, that resulted in the breakthrough of General Relativity.

Einstein's theory of gravity opens the door for physics to enable us to explore not only the solar system, but the universe as a whole. In 1917 Einstein himself noted that his equations do not allow the universe to be static. Rather it predicts an expanding or collapsing universe evolution. The cosmological constant appears to solve this problem, allowing the universe to be static, without changing the motion on the solar system scales.

In 1929, Hubble's observations have led to the conclusion that the universe is expanding. Until 1998 the cosmological constant hypothesis was discarded. In this year by observations of high redshift supernovae it has been concluded that the expansion of the universe is accelerated.

Only one year after Einstein's publication, in 1916, an exact solution was found by Karl Schwarzschild, representing a spherically symmetric, static vacuum solution. Besides the theoretical relevance of an exact solution, the Schwarzschild space-time also allows a deeper analysis of Einstein's theory in the strong gravity regime, where peculiar objects appear, e.g. black holes(BH), where gravity is so strong that even light cannot escape from its gravitational effect.(2)

BH belongs to the most fascinating objects in astrophysics. Roughly speaking a BH is composed of a singularity and an event horizon. The event horizon is a surface with infinite redshift, and encodes all the physical information. Nowadays, it is well accepted that any astrophysical BH can be totally described by three classical parameters, the mass, angular momentum and charge. This is known as the "No hair theorem/conjecture".

Exact solutions describing BH space-times are known. A static, charged BH solution was presented in 1918 by Weyl, Reissner and Nordström (3), although it is known under the name of Reissner-Nordström. A stationary, non-charged, rotating BH was found in 1963 by Roy Patrick Kerr, named after him Kerr solution. In addition a stationary rotating, charged BH, known as Kerr-Newman solution was presented by Newman in 1965. All these space-times describe BH as well as naked singularities(NS) for appropriate choices of parameters.

BH typically possess a physical singularity, which, however, is shielded from observations by a so-called event horizon. NS possess the same type of singularity, however, there exists no horizon, i.e. observations of the singularity with infinity curvature is possible.

Geodesic motion is, maybe, one of the most important ways to study the structure of space-time. On the one hand it is the fundamental concept behind the classical tests of the general theory of relativity, namely perihelion precession, deflection of light and gravitational redshift of light. On the other hand, from an observational point of view, since only matter and light are observable (4), it provides a direct study in the strong gravitational regime. Analytical solutions of the geodesic equation using elliptic functions in Schwarzschild space-time have been found by Haginara in 1931. The same methods can be applied to other space-times. A summary of analytical solutions for geodesics in BH space-times can be found in (5)

The Reissner-Nordström (RN) space-time is the unique spherically symmetric asymptotically flat solution of the Einstein-Maxwell equations. (6) It describes the external geometry of a electrically and/or magnetically charged, nonrotating body. (7)

The RN space-time is often presented as a "toy model" to study the Kerr space-time, while arguing that it is not expected to find charged bodies in realistic distributions of matter, since the presence of plasma around the BH leads to prompt discharging.(8, 9)

Moreover, the similarity with the rotating BH appears in the properties of the space-times, when, in both space-times, for some values of the parameters, BH regions, extremal BH regions, and NS can exist. in the RN space-time, when mass is larger than charge a double horizon structure exists, an external event horizon, in the same sense as in Schwarzschild space-time, and an internal "Cauchy horizon". (8) The two horizons merge to one in the extremal case, which happens when mass and charge are equal. When the charge is larger than mass, no horizon exists and hence there is a NS.

In addition the presence of charge provides an interesting solution to the Einstein-Maxwell equations in the extremal limit. An exact solution of the Einstein-Maxwell equations describing the geometry of two (or more) BH, known as the Majumdar-Papapetrou solution. (10) Away from the extremal limit, and, in the static case, equilibrium configurations are possible only for BH and NS. (11) The stationary case has also been studied, and an equilibrium configuration for a Kerr-Newman BH and a Kerr-Newman NS has been constructed. The equilibrium configuration for two extremally rotating BH, or for NS is still an open problem.(12)

The study of binary systems have become very important nowadays, since the recent detection of gravitational waves from a binary BH system (13, 14, 15), and also a binary neutron star system. (16, 17) These discoveries establish that binary systems exist in nature, and they strongly motivate the study of the physical properties around these systems.

In this monograph the analytical solutions of the geodesic equation in the Reissner-Nordström space time are studied. In the first chapter the motion of test particles in a general curved space-time is studied using the Lagrangian formalism. A brief discussion about conserved quantities related to the properties of the space-time itself is also presented.

In the second chapter an introduction to the Reissner-Nordström space-time is presented, starting from a derivation of the solution from Einstein's field equation. Subsequently, the main physical properties of the space-time are presented, focusing on the three possible types of solutions existing in this space-time, i.e. BH, extreme BH and NS regions.

In chapter three the problem of the motion of neutral particle, both massive and massless, in the Reissner-Nordström space-time is studied using the techniques presented in the first chapter of this dissertation, a solution is presented in function of the Weierstrass elliptic function, $\wp$, as well as the classification of possible orbits for a particle in this space-time, made by the study of the equations of motion itself. The regions where closed,

circular and escape orbits are allowed to exist are presented.

The last chapter closes the dissertation, summarizing the results of the above chapters as well as an overview of ways to generalizing these results.

2 MOTION OF A TEST PARTICLE IN CURVED SPACE-TIME

The particle's motion problem remains a question that intrigues humanity since the birth of Classical Mechanics. In Newton's theory of gravitation, a set of particles feels a force created by a massive source. This force depends only on the particles distances to the massive source and their masses. The complete independence of the dynamical laws from the field equations is a direct consequence of the linearity of the field equations, moreover, interactions between masses are instantaneous.

From the General Relativity point of view, the particle motion is no longer understood as resulting from a force, but as an effect of the curvature of the space-time. A massive source, such as black hole, deforms the space and time. This deformation will be felt by test particles, massive bodies, or light rays, which will move on trajectories, called geodesics.

The physical properties of a space-time can be studied by the analysis of the motion of massive and massless particles in this space-time, being one feasible way to study the effects of objects such as black holes.

In this chapter we will present the motion of a neutral test particle in a curved space-time. The lagrangian formulation for curved space-time is presented, leading to the geodesic equation, which represents the equation of motion for a test particle, as well as a brief look at some constants of motion determined by this formulation.

2.1 Lagrangian formulation in GR

Consider a Lorentzian manifold $\mathcal{M}$, locally described by an inertial coordinate system, ξ^α , $\alpha = 0, 1, 2, 3$. The distance between two points is given by the line element:

$$ds^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta, \quad (2.1)$$

where, $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ denotes the Minkowskian metric tensor in such coordinate system.

A test particle in free fall will be following a trajectory ξ^α given by:

$$\frac{d^2 \xi^\alpha}{d\tau^2} = 0, \quad (2.2)$$

where τ is an affine parameter, which parametrizes the curve ξ^α .

Doing a local general coordinate transformation, $\xi^\alpha \rightarrow x^\alpha(\xi^\beta)$, distances will now be measured by:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu; \quad \mu, \nu = 0, 1, 2, 3, \quad (2.3)$$

where $g_{\mu\nu}$ is a symmetric tensor, called metric tensor, which determines the geometry of the space-time.

The length of a curve between two fixed points, say, a and b, is given by:

$$S = \int_a^b ds = \int_a^b \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\tau, \quad (2.4)$$

here $\dot{x}^\mu$ denotes the derivative of x^μ related to τ . In terms of x^μ the equation of motion for a test particle (2.2), becomes:

$$\ddot{x}^\mu + \Gamma_{\nu\lambda}^\mu \dot{x}^\nu \dot{x}^\lambda = 0, \quad (2.5)$$

where $\Gamma_{\nu\lambda}^\mu$ are the Christoffel symbols, or connections defined by:

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2}g^{\mu\rho} (\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}), \quad (2.6)$$

where ∂_λ denotes the derivative with respect to x^λ

From the above expression it's possible to see that all the Christoffel symbols vanish in a flat space-time.

Under a variation $x^\mu(\tau) \rightarrow x^\mu(\tau) + \delta x^\mu(\tau)$, any curve which extremizes the action $\delta S = 0$, will be a geodesic. However, deriving the equations of motion using (2.4) is complicated, the same set of equations of motion can be evaluated by extremizing the equivalent action:

$$S = \int_a^b d\tilde{s} = \frac{1}{2} \int_a^b g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu d\tau. \quad (2.7)$$

The equations governing the motion of particles will be given by the Euler-Lagrange equations:

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\lambda} \right) - \partial_\lambda \mathcal{L} = 0. \quad (2.8)$$

In conclusion, the Lagrangian, $\mathcal{L}$, for a motion in a curved space-time will be given as:

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -\frac{1}{2}\varepsilon, \quad (2.9)$$

where $\varepsilon = 0$ for null geodesics, and $\varepsilon = 1$ for massive particles.

Another advantage of the use of the Lagrangian formulation is that it makes apparent the conserved quantities. Just as in Classical Mechanics, when a coordinate, say x^λ does not appear explicitly in the Lagrangian, i.e. $\partial_\lambda \mathcal{L} = 0$ it's called a cyclic variable. The conjugate momentum

$$p_\lambda = \frac{\partial \mathcal{L}}{\partial \dot{x}^\lambda} = g_{\lambda\mu} \dot{x}^\mu \quad (2.10)$$

is then conserved in the system (Noether's Theorem).

The right side of (2.9) shows that the Lagrangian itself is a constant of motion, which is related to invariance of S under a translation in τ , or the invariance under an affine re-parametrization. This is useful because the Lagrangian itself can be studied fulfilling a non-linear equation of motion. In the analysis below for the Reissner-Nordström space-time such property will appear to be very useful.

3 THE REISSNER-NORDSTRÖM SOLUTION

The Reissner-Nordström metric is a static solution of the coupled Einstein-Maxwell equations, representing the exterior space-time of a spherically symmetric electrically and magnetically charged source.

The presence of the charge changes the proprieties of the space-time considerably. Firstly, there is a double structure of horizon: the outer horizon, which corresponds to an event horizon (and becomes equal in value to the Schwarzschild horizon when the charge is zero). But there exists also an inner horizon, which disappears in the limit when charge goes to zero. It is important to point out that the time coordinate, t , and the radial coordinate, r , are respectively time-like and space-like inside the inner horizon, so a particle can avoid the physical singularity at $r = 0$, which is not possible in the Schwarzschild case.

Secondly, it is possible to distinguish between three types of solutions, the first one is an overcharged source, when the mass is smaller than charge, in this region there is no horizon, and the singularity is naked. The second type is the special case when the mass is exactly balanced by its charge, called extreme black hole, in this case both horizons are overlapped. Furthermore, the third case is the black hole solution, when the source mass is larger than its charge. This latter type is the astrophysically most relevant one. Some models predict some amount of charge, but not large amounts of charge. Hence, the solutions would have both horizons.

In this chapter is presented a brief introduction to the Reissner-Nordström space-time. In the first section we will present a derivation of the solution of the Einstein-Maxwell field equations. In addition, in the second section, the physical properties of the space-time types discussed above will be given.

3.1 Derivation of Reissner-Nordström solution

The fundamental idea of Einstein's theory of gravitation consists in geometrizing the effect of the gravitational field, which is created by a matter distribution. Matter in the general theory of relativity means anything that contributes to the energy momentum tensor, which appears as "source" in the Einstein equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (3.1)$$

where $R_{\mu\nu}$, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci tensor and Ricci scalar respectively, and $T_{\mu\nu}$ is the energy momentum tensor, besides we are using unites where $G = c = 1$.

Einstein's field equation (3.1) constitutes a system of 10 differential equations for the 10 possible metric functions $g_{\mu\nu}$ that exist for a 4 dimensional space-time. However, those differential equations are, in general, non linear and complicated to be solved without applying any restrictions on the symmetry of the space-time.

The RN solution is part of a special family of solutions of the Einstein equation describing the deformation of the space-time by a spherical object, for more details on how to obtain this solution, see e.g. (18, 19)

The standard form for a spherically symmetric static metric can be taken as:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (3.2)$$

for which the components of the Einstein equation read:

$$\begin{aligned} G_{tt} &= \frac{1}{2B} \left(A'' - \frac{A'B'}{2B} - \frac{(A')^2}{2A} + \frac{2A'}{r} \right) = 8\pi T_{tt}, \\ G_{rr} &= \frac{1}{2A} \left(-A'' + \frac{A'B'}{2B} + \frac{(A')^2}{2A} + \frac{2AB'}{rB} \right) = 8\pi T_{rr}, \\ G_{\theta\theta} &= 1 - \frac{1}{B} - \frac{r}{2B} \left(\frac{A'}{A} - \frac{B'}{B} \right) = 8\pi T_{\theta\theta}, \\ G_{\varphi\varphi} &= \sin^2\theta \left(1 - \frac{1}{B} - \frac{r}{2B} \left(\frac{A'}{A} - \frac{B'}{B} \right) \right) = 8\pi T_{\varphi\varphi}, \end{aligned} \quad (3.3)$$

where ' denotes the derivative with respect to r . And let's consider a coupling to an electrostatic field, using the ansatz for the gauge field:

$$A_t = -\Phi(r), \quad A_i = 0, \quad i = 1, 2, 3, \quad (3.4)$$

where $\Phi(r)$ is the usual scalar potential. The field strength tensor is given by:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (3.5)$$

For the field (3.4) we get the covariant components of the field strength tensor:

$$F_{tt} = F_{ii} = F_{ij} = 0, \quad F_{t\theta} = F_{t\varphi} = 0, \quad F_{tr} = -\partial_r \Phi(r) = E(r). \quad (3.6)$$

In addition, the contra-variant components are given by:

$$F^{tt} = F^{ii} = F^{ij} = 0, \quad F^{t\theta} = F^{t\varphi} = 0, \quad F^{tr} = -\frac{\partial_r \Phi(r)}{AB} = \frac{E(r)}{AB}. \quad (3.7)$$

Outside the source the 4-current vanishes ($j^\mu = 0$), and the Maxwell equation becomes:

$$D_\mu(\sqrt{-g}F^{\mu\nu}) = \partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0, \quad (3.8)$$

where D_μ is the usual covariant derivative and $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor. Using (3.7) we find:

$$E(r) = \frac{\sqrt{AB}}{r^2} Q^2, \quad (3.9)$$

where the integration constant can be chosen to be the total charge Q of the source when compared with the flat space-time limit.

The energy-momentum tensor for a Maxwell distribution can be calculated using:

$$T_{\mu\nu} = \frac{g_{\mu\rho}}{4\pi} \left(-F^{\sigma\rho} F_{\nu\rho} + \frac{1}{4} \delta_\nu^\rho F^{\sigma\rho} F_{\sigma\rho} \right). \quad (3.10)$$

For a static distribution of charge, using (3.6)-(3.7) we get:

$$T_{tt} = -\frac{AQ^2}{8\pi r^4}, \quad T_{rr} = \frac{Q^2}{8\pi B r^4}, \quad T_{\theta\theta} = -\frac{Q^2}{8\pi r^2}, \quad T_{\varphi\varphi} = -\frac{Q^2}{8\pi r^2} \sin^2\theta. \quad (3.11)$$

Putting (3.11) in (3.2) it is possible to solve the system of equation for the functions A and B , which is:

$$BT_{tt} + AT_{rr} = 0 \quad \rightarrow \quad A(r) = B^{-1}(r) \quad (3.12)$$

$$BG_{tt} + AG_{rr} = 0 \quad \rightarrow \quad A(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.$$

which is the RN solution.

3.2 Properties of the RN space-time

In this section we will be discussing the physical properties of the RN space-time.

Consider the Reissner-Nordström solution in the form:

$$ds^2 = -\frac{\Delta_r}{r^2} dt^2 + \frac{r^2}{\Delta_r} dr^2 + r^2 d\Omega_2^2, \quad \Delta_r = r^2 - 2Mr + Q^2, \quad (3.13)$$

where M and Q are, respectively, the gravitational mass and charge of the black hole and $d\Omega_2^2$ symbolizes the geometry of a 2-sphere, $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

In addition, let's note that (3.13) is asymptotically flat, and in the absence of charge, $Q = 0$, we recover the Schwarzschild solution.

The horizons can be calculated by computing $g^{rr} = \Delta_r/r^2 = 0$. The only physical singularity is located at $r = 0$, this singularity should be interpreted as the point charge which produces the field. Aside from that, the zeros of Δ_r will be located on the surfaces $r = r_{\pm}$, where:

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \quad (3.14)$$

As a consequence, the behaviour of (3.14) will depend on the behaviour of $M^2 - Q^2$. It is interesting to distinguish three different cases:

3.2.1 1st case: $M^2 < Q^2$

Δ_r has no real roots, and the metric is completely regular in (t, r, θ, φ) . The coordinate t is always time-like, and the coordinate r is space-like.

A special feature of this case is the situation $M = 0$, describing the gravitational field of a massless charged body.

The physical singularity is time-like and it's not shielded from observation by an event horizon, such type of singularity is known as naked singularity. The existence of this situation is a problem for the theory, and considered not realistic. There is a conjecture, known as Cosmic Censorship Conjecture, which, roughly speaking, states that the collapse of any physically realistic matter configuration will not lead to a naked singularity.

3.2.2 2nd case: $M^2 = Q^2$

In this case the mass is, in some sense, balanced by charge and is usually referred to extremal Reissner-Nordström solution.

Similar to the case presented above, the singularity is time-like. The roots (3.14) are two equal roots, given by $r_{\pm} = M$, and correspond to an event horizon.

Another interesting propriety present in the Extremal case is the behavior of the geometry near the horizon. In order to understand this, let's consider the metric in the form:

$$ds^2 = - \left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2, \quad (3.15)$$

and perform the coordinate transformation:

$$t = \frac{M^2 \tau}{\epsilon}, \quad r = M + \epsilon \rho. \quad (3.16)$$

Figure 1 – Schematic representation of the function Δ_r for the three different cases discussed above in comparison with the Schwarzschild solution. The mass is fixed to be $M = 0.3$ in all cases, and the charge $Q = 0.4, 0.2, 0.3, 0$.

Source: By the author

The near horizon geometry can be studied by setting $\epsilon \rightarrow 0$, where the line element becomes:

$$ds^2 = -M^2 \left(\rho^2 d\tau^2 - \rho^{-2} d\rho^2 \right) + M^2 d\Omega_2^2, \quad (3.17)$$

which describes an $AdS_2 \times S^2$ geometry.

3.2.3 3rd case: $M^2 > Q^2$

This is considered to be the most realistic case. $r_{\pm}$ are two different real roots given by (3.14). The root r_+ is referred to as outer horizon, while the root r_- is referred to as inner horizon.

In this case, is interesting to analyze the sign of the metric in the three distinct regions. The metric function Δ_r is positive for $0 < r < r_-$ and $r > r_+$, and so the coordinate t is timelike, and r spacelike. Moreover Δ_r is negative in $r_- < r < r_+$, where hence t is spacelike, and r timelike.

4 MOTION IN RN THE SPACE-TIME

The equations of motion for a neutral, structureless test particle can be evaluated by using the Lagrangian formulation presented in the first chapter. The Lagrangian (2.9) for geodesic motion in the RN space-time is given by:

$$\mathcal{L} = \frac{1}{2} \left[-\frac{\Delta_r}{r^2} \dot{t}^2 + \frac{r^2}{\Delta_r} \dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) \right] = -\frac{1}{2} \varepsilon, \quad (4.1)$$

remembering that ε is 0 for massless particles, and 1 for massive particles.

At first inspection of the above expression it is possible to determine that (4.1) does not have an explicit dependence of the time coordinate t nor the azimuthal angle φ . This leads to the conservation of two conserved quantities along the geodesic, related to the total energy, E , and the z-component of the angular momentum, L_z , of the particle. This reads:

$$\begin{aligned} E = -\frac{\partial \mathcal{L}}{\partial \dot{t}} &\Rightarrow \dot{t} = \frac{r^2}{\Delta_r} E; \\ L_z = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} &\Rightarrow \dot{\varphi} = \frac{L_z}{r^2 \sin^2 \theta}. \end{aligned} \quad (4.2)$$

The cyclic variable can be used to reduce the Lagrangian (4.1) by introduction of (4.2) leading to:

$$\dot{r}^2 + \Delta_r \dot{\theta}^2 = E^2 - \frac{\Delta_r}{r^2} \left(\varepsilon + \frac{L_z^2}{r^2 \sin^2 \theta} \right), \quad (4.3)$$

which can be understood as the energy conservation relation for geodesics in the RN space.

The equations of motion for the (r, θ) coordinates are found by computing the Euler-Lagrange equations using (4.1)

$$\begin{aligned} \ddot{r} + \frac{1}{2\Delta_r} \left(\partial_r \Delta_r - \frac{2\Delta_r}{r} \right) (E^2 - \dot{r}^2) + \frac{\Delta_r}{r} \left(\dot{\theta}^2 + \frac{L_z^2}{r^4 \sin^2 \theta} \right) &= 0, \\ \ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} - \frac{\cos \theta}{r^4 \sin^3 \theta} L_z^2 &= 0, \end{aligned} \quad (4.4)$$

Since the angular momentum is conserved in absolute value and direction, the motion happens also in a plane with $\theta = \text{constant}$.

4.1 Motion in equatorial plane

The spherical symmetry restricts the movement of the particles to any plane which passes through the center of the spherical RN solution. In the equatorial plane, $\theta = \pi/2$, the radial geodesic becomes:

$$\ddot{r} + \frac{1}{2\Delta_r} \left(\partial_r \Delta_r - \frac{2\Delta_r}{r} \right) (E^2 - \dot{r}^2) + \frac{\Delta_r}{r^5} L_z^2 = 0. \quad (4.5)$$

Alternatively to the second order, differential equation presented above, the same physical information is contained in the energy conservation relation (4.3), which is first order, however provides a more interesting analysis. In the equatorial plane the energy conservation relation becomes:

$$\dot{r}^2 = E^2 - \varepsilon - V_{eff}(r), \quad (4.6)$$

$$V_{eff}(r) = -\frac{2M\varepsilon}{r} + \frac{Q^2\varepsilon + L_z^2}{r^2} - \frac{2ML_z^2}{r^3} + \frac{Q^2L_z^2}{r^4}. \quad (4.7)$$

where V_{eff} is the effective potential. Using the effective potential it is possible to see some expected behavior, such as the usual angular momentum barrier L_z^2/r^2 that dominates in the non-relativistic regime, as well as the familiar attractive general relativistic correction term ML_z^2/r^3 . Additionally, a repulsive correction term $Q^2L_z^2/r^4$ is present.

In this dissertation we are interested in a non vanishing angular momentum, i.e. we will consider $L_z \neq 0$ and $L_z > 0$ and rescale the physical quantities as:

$$x = L_z r, \quad m = M/L_z, \quad q = Q/L_z, \quad \alpha = \tau/L_z, \quad (4.8)$$

such that (4.7) becomes:

$$\left(\frac{dx}{d\alpha} \right)^2 = E^2 - \varepsilon - V_{eff}(x), \quad (4.9)$$

$$V_{eff}(x) = -\frac{2m\varepsilon}{x} + \frac{q^2\varepsilon + 1}{x^2} - \frac{2m}{x^3} + \frac{q^2}{x^4}. \quad (4.10)$$

Summarizing, the radial motion of the test particle will be described by:

$$\left(\frac{dx}{d\alpha} \right)^2 = \frac{(E^2 - \varepsilon)x^4 + 2m\varepsilon x^3 - (q^2\varepsilon + 1)x^2 + 2mx - q^2}{x^4}. \quad (4.11)$$

Defining the affine parameter as $d\lambda = d\alpha/x^2$ we get:

$$\left(\frac{dx}{d\lambda}\right)^2 = P_4(x) \quad , \quad P_4(x) = \sum_{i=0}^4 a_i x^i. \quad (4.12)$$

In order to solve the above expression, consider x_0 as a root of $P_4(x)$, and the coordinate transformation:

$$x - x_0 = \frac{1}{u}, \quad \Rightarrow \quad dx = -(x - x_0)^2 du, \quad (4.13)$$

and, for u the equation of motion becomes:

$$\left(\frac{du}{d\lambda}\right)^2 = P_3(u) \quad , \quad P_3(u) = \sum_{n=0}^3 c_n u^n. \quad (4.14)$$

Furthermore, the above expression can be put in Weierstrass form by the transformation:

$$u = \frac{1}{c_3} \left(4v - \frac{c_2}{3} \right), \quad (4.15)$$

and finally

$$\left(\frac{dv}{d\lambda}\right)^2 = 4v^3 - g_2 v - g_3 = P_3(v), \quad (4.16)$$

and so:

$$\int_{v(0)}^{v(\lambda)} \frac{dv'}{\sqrt{P_3(v')}} = \int_0^\lambda d\lambda', \quad (4.17)$$

which can be solved in function of Weierstrass elliptic function $\wp$, with:

$$\begin{aligned} g_2 &= -\frac{1}{4} \left(c_1 c_3 - \frac{c_2^2}{3} \right) \\ g_3 &= -\frac{1}{16} \left(\frac{2c_2^3}{27} + \frac{c_0 c_3^2}{16} - \frac{c_1 c_2 c_3}{3} \right). \end{aligned} \quad (4.18)$$

The above calculation is presented in more details in the appendix.

4.2 Classification of the Orbits

The orbits in the RN space-time are completely determined by the equation of motion (4.9) which can be understood as a relation between energy and effective potential, and so possible orbits for a test particle will be determined by the behavior of (4.10).

Turning points of the motion are characterized by $\dot{x} = 0$. Then different types of orbits can be distinguished as :

- a) closed, elliptical orbits have two turning points with different values of r ;
- b) circular orbits that have one turning point in r ;
- c) escape orbits which describe the motion of a particle approaching from infinity, being scattered by the gravitational object and moving off to infinity again.

The turning points of the orbits can be studied by:

$$\dot{x} = 0 \quad \Rightarrow \quad E^2 - \varepsilon = V_{eff}(x) = \frac{1}{x^4}(-2m\varepsilon x^3 + (q^2\varepsilon + 1)x^2 - 2mx + q^2). \quad (4.19)$$

Furthermore, as a special case of closed orbits consider circular orbits, which are those orbits for which both the radial velocity and acceleration vanish, leading to:

$$\ddot{x} = 0 \quad \Rightarrow \quad \frac{dV_{eff}}{dx} = \frac{2}{x^5}(m\varepsilon x^3 - (q^2\varepsilon + 1)x^2 + 3mx - 2q^2) = 0, \quad (4.20)$$

where the radii of the circular orbits, $\tilde{x}$, are the roots of the characteristic polynomial:

$$m\varepsilon x^3 - (q^2\varepsilon + 1)x^2 + 3mx - 2q^2 = 0. \quad (4.21)$$

Additionally, the stability of the orbits will be determined by the sign of the second derivative of the effective potential, being:

$$\begin{aligned} \frac{d^2V_{eff}}{dx^2}(\tilde{x}) > 0 &\rightarrow \text{unstable}, \\ \frac{d^2V_{eff}}{dx^2}(\tilde{x}) < 0 &\rightarrow \text{stable}. \end{aligned} \quad (4.22)$$

The circular unstable orbits of massless particles (photons) are relevant for the shadow of black holes as they correspond to the outer boundary of this shadow. The circular and stable orbits for massive particles are referred to as ISCOs (innermost stable circular orbits) and are relevant for planetary motion as well as accretion disc formation.

In order to discuss the possible orbits in the RN space-time it is interesting to distinguish between the geodesics for massive and massless particles, this discussion is presented below in more detail.

Figure 2 – Schematic representation of effective potential for null geodesics for two different cases discussed in comparison with the Schwarzschild case. The mass is fixed to be $m = 0.1$ in all cases, and the charge is $q = 0.09, 0.1$ and 0 .

Source: By the author

4.2.1 Null Geodesics

The equation governing the motion of light-like particles is:

$$\dot{x}^2 = E^2 - \frac{1}{x^4}(x^2 - 2mx + q^2), \quad (4.23)$$

and closed orbits are determined by the behavior of the polynomial:

$$E^2 x^4 - x^2 + 2mx - q^2 = 0. \quad (4.24)$$

The roots of the above polynomial determine the possible closed orbits, which are only allowed for $r > r_+$ or $r < r_-$. In order to study the behavior of these roots consider the polynomial discriminant:

$$\Delta_{(m,q)}(E) = -16E^2(16E^4q^6 + (27m^4 - 36m^2q^2 + 8q^4)E^2 - (m^2 - q^2)), \quad (4.25)$$

where the roots of (4.24) are all real, or all complex, for $\Delta_{(m,q)}(E) > 0$, there are two real and two complex roots for $\Delta_{(m,q)}(E) < 0$ and at least two are equal if and only if $\Delta_{(m,q)}(E) = 0$. The sign of the discriminant can be studied by the analysis of a sixth order polynomial in E , with roots:

Figure 3 – Schematic representation of the roots of the energy. Note that $E_{\pm,-}$ is positive only for $1 < q^2/m^2 < 9/8$.

Source: By the author

$$E = 0 \quad E_{\pm,\pm} = \pm \frac{m^2}{4\sqrt{2}q^3} \sqrt{-27 + 36\frac{q^2}{m^2} - 8\frac{q^4}{m^4} \pm 27 \left(1 - \frac{8q^2}{9m^2}\right)^{3/2}}. \quad (4.26)$$

The polynomial (4.24) always has at least two real roots, where one of them is negative and unphysical. Moreover, for $q^2/m^2 > 9/8$ there is no real roots, $E_{\pm,\pm}$ are always complex numbers. For $q^2/m^2 < 9/8$ it is possible to distinguish between three distinct positive roots for $E_{-,+} < E < E_{+,+}$, which can be understood as the maximum and minimum value for a closed orbit, and an escape orbit. There are at least two distinct positive roots for $E = E_{\pm,+}$, or $E = 0$, where closed orbits are allowed. There is only one positive real root for $E < E_{-,+}$ or $E > E_{+,+}$, which corresponds to an escape orbit.

For $q^2/m^2 = 9/8$, the limiting case, the first region does not exist, the discriminant $\Delta_{(m,q)}$ vanishes for $E = E_{\pm}$, or $E = 0$, where closed orbits can exist, and is negative for every other value, where only escape orbits can exist.

In the naked singularity region $1 < q^2/m^2 < 9/8$ the discriminant can be positive for $E_{-,-} < E < E_{-,+}$ or $E_{+,-} < E < E_{+,+}$, where both closed and escape orbits can exist. The discriminant is null for $E = 0$ or $E = E_{\pm,\pm}$ and negative for $E < E_{-,-}$ or $E > E_{+,+}$, where only escape orbits can exist.

Figure 2 show the behavior of the effective potential for null geodesics, it's possible to see the three different regions: closed orbits are bounded by the well in the potential,

Figure 4 – Schematic representation of the discriminant (left) and the regions when the motion are allowed (right) for null geodesics in, respectively, the black hole, extremal black hole and naked singularity solution. The positive values of the discriminant represent the region when closed orbits can be found. The value of the mass is fixed as $m = 0.4$, and the charge are $q = 0.3, 0.4, 0.42$.

Source: By the author

circular orbits occur exactly at the minimal point. Figure 3 show the behavior of the roots of the discriminant, it is possible to see that the both roots for the discriminant are real only in the naked singularity region.

Figure 4 show the behavior of the discriminant and the regions where orbits can exist for null geodesics. Since one root is always negative, closed orbits can be find only for positive values of the discriminant, which exist in all the three regions. In the naked singularity solution the discriminant is positive only in a small region that is represented in the circles region. When the discriminant is negative, there is only one physical turning

point to the motion, hence only escape orbits can exist.

For $q^2/m^2 < 9/8$ two circular orbits exist, with radii given by:

$$\tilde{x}_{\pm} = \frac{3m}{2} \left(1 \pm \sqrt{1 - \frac{8q^2}{9m^2}} \right), \quad (4.27)$$

where $\tilde{x}_-$ is a stable orbit, and $\tilde{x}_+$ is unstable, and corresponds to the radius of the photon sphere around the RN source.

The unstable photon orbits determines the boundary of the shadows of the black hole. The behaviour of the radius of the unstable photon orbit in function of the event horizon can be obtained by:

$$\frac{\tilde{x}_+}{m} = \frac{3}{2} \left(1 + \frac{1}{3} \sqrt{1 + 8 \left(\frac{x_+}{m} - 1 \right)^2} \right), \quad (4.28)$$

where its possible to see that the radius of the photon sphere is a increasing function of the event horizon radius, starting in $x_+ = 1$, i.e. in the extremal case, and reaching its maximum value, $x_+ = 2$, when there is no charge.

Figure 5 shows the behavior of the roots of the derivative of the effective potential in function of the ratio q^2/m^2 . It's interesting to see that the circular orbits exist slightly beyond extremality, i.e. naked singularities with $q^2/m^2 < 9/8$ posses a photon sphere.

Putting the roots (4.27) in (4.24) it's possible to determine the energy that the test particle should have in order to be on a circular orbit. Rather than this, for practical reasons, it is interesting to use the impact parameter, $b = L/E$, which can be calculated by considering the equation without the rescaled variables, leading to:

$$b_{\pm}^2 = \frac{27M^2}{8} \frac{\left(1 \pm \sqrt{1 - \frac{8Q^2}{9M^2}} \right)^4}{1 - \frac{2Q^2}{3M^2} \pm \sqrt{1 - \frac{8Q^2}{9M^2}}}. \quad (4.29)$$

Expression 4.29 reduces to $b_{Sc} = 27m^2$ from Schwarzschild when the charge vanishes. Figure 6 shows the behaviour of the impact parameter for a photon particles enter in a circular orbit. It is interesting to note that both possibilities exists in the naked singularity region.

For the extremal case the discriminant (4.25) reduces to:

$$\Delta_{(m,m)} = -16E^4 m^4 (16m^2 E^2 - 1) \quad (4.30)$$

which is positive for $E^2 < 1/16m^2$, null for $E^2 = 1/16m^2$ and negative for $E^2 > 1/16m^2$.

Figure 5 – Roots of the derivative of the effective potential for null geodesics

Source: By the author

Figure 6 – Impact parameter for circular orbits.

Source: By the author

The circular orbits radii are:

$$\tilde{x}_+ = 2m, \quad \tilde{x}_- = m, \quad (4.31)$$

and so, in the extremal case, the stable photon orbit is on the horizon.

Moreover, the unstable photon orbit is at two times the event horizon. It is also interesting to note that unstable and stable photon orbits exist in the naked singularity regime with $q^2/m^2 < 9/8$. Since no horizons shield these orbits from observation by an external observer, both are - in principle - observable. In particular, the naked singularities possess a photon sphere (corresponding to $\tilde{x}_+$) that can be observed.

4.2.2 Time-like geodesics

The equation governing the motion of massive particles, $\varepsilon = 1$, is:

$$\dot{r}^2 = E^2 - 1 - \frac{1}{x^4}(-2mx^3 + (q^2 + 1)x^2 - 2mx + q^2). \quad (4.32)$$

The possible orbits will be determined by the turning points of the polynomial:

$$(E^2 - 1)x^4 + 2mx^3 - (q^2 + 1)x^2 + 2mx - q^2 = 0, \quad (4.33)$$

in which the behaviour of the roots will depend of the values of the energy. In order to study the possible roots of the above polynomial, consider the determinant:

$$\begin{aligned} \Delta_{(m,q)}(E) = & -16 \left[16q^6 E^6 + E^4 \left(27m^4 + 12m^2 (q^2 - 3) q^2 + 8q^8 - 32q^6 + 8q^4 \right) + \right. \\ & + E^2 \left(12m^4 (q^2 - 3) + m^2 (15q^6 - 31q^4 + 49q^2 - 1) + q^{10} - 12q^8 + 22q^6 - 12q^4 + q^2 \right) + \\ & \left. \left(16m^6 + 8m^4 (q^4 - 4q^2 + 1) + m^2 (q^2 - 1)^2 (q^4 - 10q^2 + 1) - q^{10} + 4q^8 - 6q^6 + 4q^4 - q^2 \right) \right], \end{aligned} \quad (4.34)$$

which can be studied as a biquadratic polynomial in E . The roots of (4.33) will be: four different real (or complex) roots for $\Delta_{(m,q)}(E) > 0$, at least two roots are real if $\Delta_{(m,q)}(E) = 0$, two real and two complex roots if $\Delta_{(m,q)}(E) < 0$. The physical turning points will strongly depend on the value of E^2 .

When $E^2 > 1$, the first term is always positive. Since one turning point of the motion is always negative and unphysical, and the classification of orbits is similar to the null geodesics. All the three types, closed, circular or escape orbits can exist.

For the extremal region, the discriminant becomes:

Figure 7 – Schematic representation of the discriminant for massive particles in extremal case $q = m$. The blue curve is for $m = 0.4$, the yellow is the limiting case, $m^2 = 1/8$, and the green curve is for $m = 0.1$.

Source: By the author

$$\Delta_{(m,m)}(E) = -16m^4 E^2 \left(16m^2 E^4 - (1 + 20m^2 - 8m^4)E^2 + (1 + m^2)^3 \right), \quad (4.35)$$

with roots:

$$E = 0 \quad E_{\pm,\pm} = \pm \sqrt{\frac{-8m^4 + 20m^2 + 1 \pm \sqrt{-(8m^2 - 1)^3}}{32m^2}}, \quad (4.36)$$

where it is possible to note that distinct roots are allowed only for $m^2 < 1/8$, i.e. in the region where the discriminant for the extremal case can be positive.

Figure 7 show the behaviour of the discriminant in the extremal regime, the yellow curve represents the limiting case $m^2 = 1/8$. Figure 8 show the behaviour of the roots of the energy in function of mass.

The behavior of the discriminant and the regions when the orbits can happens for the black hole, extremal black hole, and naked singularity solution are shown in figure 9. Remember that, when the energy is larger than one, one turning point of the orbit is always negative. For black holes, the before mentioned graphics suggests that the discriminant can be positive only for $E^2 > 1$ hence it is possible to find two different real roots, and closed orbits can exist, as well as a escape orbit. For $E^2 < 1$, the discriminant is negative, hence it is possible to distinguish between two different real roots, since no one can be negative,

Figure 8 – Schematic representation of roots of the discriminant in the extremal case in function of the mass.

Source: By the author

only closed orbits can also be found in this region. The behavior of the discriminant changes significantly for the naked singularity, here the discriminant can be positive for $E^2 < 1$, where closed orbits can be found. For $E^2 > 1$, regions where the discriminant is positive allow closed orbits, and when it is negative only escape orbits can exist.

Since circular motion requires, $\dot{r} = 0$ and $\ddot{r} = 0$, it is possible to write:

$$1 = \frac{(m\tilde{x} - q^2)\tilde{x}^2}{\tilde{x}^2 - 3m\tilde{x} + 2q^2}, \quad (4.37)$$

$$E^2 = \frac{(\tilde{x}^2 - 2m\tilde{x} + q^2)^2}{\tilde{x}^2 - 3m\tilde{x} + 2q^2},$$

where both conditions need to be simultaneously satisfied, so circular motion is only allowed for $\tilde{x} > q^2/m$, and $\tilde{x} < \tilde{x}_-$ or $\tilde{x} > \tilde{x}_+$.

As is possible to see from figure 10 the regions for allowed circular orbits are restricted to the surfaces:

$$\tilde{x}_- \leq x_- \leq q^2/m \leq x_+ < \tilde{x}_+, \quad (4.38)$$

and so, in the black-hole case, circular motion is allowed only for $x > \tilde{x}_+$, i.e. all orbits in this region lie outside of the event horizon and can be observed. For the naked singularity case circular orbits are allowed for $q^2/m < x < \tilde{x}_-$ and $x > \tilde{x}_+$, for $q^2 > 9/8m^2$.

Figure 9 – Schematic representation of the discriminant (left) and the regions when the motion are allowed (right) for time-like geodesics in, respectively, the black hole, extremal black hole and naked singularity solution. In the naked singularity two cases was presented, being $1 < q^2/m^2 < 9/8$ (dashed) and $q^2/m^2 > 9/8$ (thick). The value of the mass is fixed as $m = 0.1$, and the charge are $q = 0.08, 0.1, 0.105, 0.115$.

By the author

Figure 10 – Regions for allowed circular orbits. Dashed lines represent the horizons.

Source: By the author

There is no distinction between $\tilde{x}_{\pm}$, and the only condition for the existence of circular orbits is $r > q^2/m$.

The stability condition for massive circular orbits is given by:

$$m\tilde{x}^3 - 6m^2\tilde{x}^2 + 9mq^2\tilde{x} - 4q^4 < 0. \quad (4.39)$$

For naked singularity solution all the orbits in the region $q^2/m \leq x < \tilde{x}_-$ are stable, also stable orbits exist in $x > \tilde{x}_+$ region. (20)

The stability analysis for non circular closed orbits leads to very complicated expressions, suggesting that its should be studied numerically.

5 SUMMARY AND CONCLUSIONS

The geodesic motion as well as the main properties of the Reissner-Nordström space-time have been presented in this thesis. Using the lagrangian formulation in curved space-time we derived the equations of motion, in terms of the two cyclic variables, originating from the properties of the space-time itself. Since the space-time is spherically symmetric and static the motion of neutral test particles is restricted to be planar, in addition the solution for the radial equation is presented using the Weierstrass elliptic function, $\wp$.

The possible orbits are also studied, the conditions for existence of closed orbits, escape and circular orbits are presented with an emphasis on deriving the conditions for existence of closed (elliptic and circular) and escape orbits in (a) the black hole case (extremal and non-extremal) and (b) the naked singularity case.

The motion of light-like particles is described by (4.23). Since one turning point for the equation of motion is always negative, the analyses reduces to study of the others three possible turning points. Closed orbits and escape orbits are allowed up to $q^2/m^2 < 9/8$, which includes, in particular, the naked singularity case. Furthermore, in this region it is possible to find two distinct circular orbits with radii given by: $\tilde{r}_{\pm} = 3M/2 \left(1 \pm \sqrt{1 - 8Q^2/9M^2}\right)$, where the unstable circular orbit $\tilde{r}_+$ lies outside the event horizon, and the stable circular orbit, $\tilde{r}_-$, is situated in the region between both horizons.

When $q^2/m^2 = 9/8$ closed orbits can exist only when the discriminant (4.25) vanishes, which happens for $E = 0$ or $E = E_{\pm}$, as a special case, an unstable circular orbit with radius $\tilde{r} = 3M/2$ is allowed in this region.

For $q^2/m^2 > 9/8$, the discriminant can only reach negative values, and there is only one positive turning point for the motion. Hence, the orbit corresponds to an escape orbit.

The motion of massive particles is determined by (4.32). The behaviour of the turning points strongly depends on the value of the energy. For $E^2 > 1$, the qualitative analysis is similar to that of the null geodesic, since the equation of motion still possesses the property to have one negative, unphysical root. When $E^2 < 1$, the discriminant has only negative values, and so, only two different real roots. Hence, bound orbits can in this case.

As a special case of closed orbits, circular motion needs to satisfy (4.37). We find that circular orbits for massive particles are only allowed for $r > Q^2/M$ and $r < \tilde{r}_-$ or $r > \tilde{r}_+$. In the black hole case, circular motion can only exist for $r > \tilde{r}_+$, so massive circular orbits are always outside the unstable photon sphere.

In the naked singularity case, there are two different regimes in which circular orbits can exist. For $1 < q^2/m^2 < 9/8$ circular orbits are allowed for $Q^2/M < r < \tilde{r}_-$ and for $r > \tilde{r}_+$. When $Q^2/M^2 > 9/8$, they can only exist since $r > Q^2/M$ is satisfied.

This above results about circular orbits suggests that circular orbits can be a useful tool to determine differences between a black hole and a naked singularity, possibly the study of the shadows of this spaces can corroborate to this. Furthermore, a possible interesting future analysis is to investigate if there are different parameter regimes for unstable circular orbits in other space-times with extremal limits and naked singularity, such as the Kerr (-Newman) solution.

The new features developed in this studies include:

Photon sphere:

We find that close to extremality and in the naked singularity region for charges "not too large" the RN solution does have a photon sphere comparable to that of non-extremal black holes. While space-times with naked singularities are normally considered unphysical, recent results show that they might have interest in extended gravity models, where naked singularities do exist. (20, 21)

ISCOs:

Innermost stable circular orbits for massive test particles are relevant for planetary motion and many other processes in astrophysics. (15) Here we find that close to extremality and in the naked singularity region with charges "not too large" that stable circular orbits can exist.

REFERENCES

- 1 HACKMANN, E. **Geodesic equations in black hole space-times with cosmological constant**. 2010. Tese (Doutorado), 2010. Available from: <http://elib.suub.uni-bremen.de/diss/docs/00011880.pdf>. Accessible at: 01 May 2019.
- 2 RIESS, A. G.; et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. **Astronomical Journal**, v. 116, n. 3, p. 1009, 1998.
- 3 GRIFFITHS, J. B.; PODOLSKÝ, J. **Exact space-times in Einstein's general relativity**. Cambridge: Cambridge University Press, 2009.
- 4 HACKMANN, E.; KAGRAMANOVA, V.; KUNZ, J.; LÄMMERZAHN, C. Analytic solutions of the geodesic equation in higher dimensional static spherically symmetric spacetimes. **Physical Review D**, v. 78, n. 12, p. 124018, 2008.
- 5 HACKMANN, E.; LÄMMERZAHN, C. **Analytical solutions for geodesics in black hole spacetimes**. 2015. Available from: <https://arxiv.org/pdf/1506.01572.pdf>. Accessible at: 01 May 2019.
- 6 HAWKING, S. W.; ELLIS, G. F. R. **The large scale structure of space-time**. Cambridge: Cambridge University Press, 1973. v. 1.
- 7 GRUNAU, S.; KAGRAMANOVA, V. Geodesics of electrically and magnetically charged test particles in the Reissner-Nordström space-time: analytical solutions. **Physical Review D**, v. 83, n. 4, p. 044009, 2011.
- 8 CHANDRASEKHAR, S.; THORNE, K. S. **The mathematical theory of black holes**. New York: AAPT, 1985.
- 9 ZAJAČEK, M.; TURSUNOV, A. **Electric charge of black holes: is it really always negligible?** 2019. Available from: <https://arxiv.org/pdf/1904.04654.pdf>. Accessible at: 01 June 2019.
- 10 HARTLE, J. B.; HAWKING, S. W. Solutions of the Einstein-Maxwell equations with many black holes. **Communications in Mathematical Physics**, v. 26, n. 2, p. 87–101, 1972.
- 11 ALEKSEEV, G. A.; BELINSKI, V. Equilibrium configurations of two charged masses in general relativity. **Physical Review D**, v. 76, n. 2, p. 021501, 2007.
- 12 ALEKSEEV, G.; BELINSKI, V. **Superposition of fields of two rotating charged masses in General Relativity and existence of equilibrium configurations**. 2019. Available from: <https://arxiv.org/pdf/1905.05317.pdf>. Accessible at: 01 June 2019.
- 13 ABBOTT, B. P. et al. Observation of gravitational waves from a binary black hole merger. **Physical Review Letters**, v. 116, n. 6, p. 061102, 2016.
- 14 _____. Properties of the binary black hole merger gw150914. **Physical Review Letters**, v. 116, n. 24, p. 241102, 2016.

- 15 NAKASHI, K.; IGATA, T. **Innermost stable circular orbits in Majumdar–Papapetrou dihole spacetime**. 2019. Available from: <<https://arxiv.org/pdf/1903.10121.pdf>>. Accessible at: 01 june 2019.
- 16 ABBOTT, B. P. et al. Multi-messenger observations of a binary neutron star merger. **Astrophysical Journal Letters**, v. 848, n. 2, p. L12, 2017.
- 17 _____. **GWTC-1: a gravitational-wave transient catalog of compact binary mergers observed by LIGO and Virgo during the first and second observing runs**. 2018. Available from: <<https://arxiv.org/pdf/1811.12907.pdf>>. Accessible at: 10 june 2019.
- 18 BLAU, M. **Lecture notes on general relativity**. Bern: Universit at Bern, 2011. Available from: <<http://www.blau.itp.unibe.ch/newlecturesGR.pdf>>. Accessible at: 01 May 2019.
- 19 HOOFT, G. t. **Introduction to general relativity**. Utrecht: Institute for Theoretical Physics, 2012.
- 20 LIANG, E. Equatorial circular orbits of some static gravitational fields with naked singularities. **Physical Review D**, v. 9, n. 12, p. 3257, 1974.
- 21 SHAIKH, R.; KOCHERLAKOTA, P.; NARAYAN, R.; JOSHI, P. S. Shadows of spherically symmetric black holes and naked singularities. **Monthly Notices of the Royal Astronomical Society**, v. 482, n. 1, p. 52–64, 2018.

Appendix

APPENDIX A – WEIERSTRASS FORM FOR RN GEOMETRY

The equation of motion for a particle in free fall, in RN space-time can be equated as (4.11), and after redefining the affine parameter $d\lambda = d\alpha/r^2$ the equation of motion takes the form:

$$\left(\frac{dx}{d\alpha}\right)^2 = \sum_{n=0}^4 a_n x^n = P_4(x) \quad (\text{A.1})$$

which in comparison with (4.11) leads to:

$$a_4 = E^2 - \varepsilon, \quad a_3 = -2m\varepsilon, \quad a_2 = q^2 + 1, \quad a_1 = -2m, \quad a_0 = q^2. \quad (\text{A.2})$$

It is always possible to rewrite P_4 in function of its roots, consider:

$$P_4(x) = (x - x_0)(b_3x^3 + b_2x^2 + b_1x + b_0) = (x - x_0)P_3(x) \quad (\text{A.3})$$

where the constants b_n and a_n are related by:

$$\begin{aligned} b_3 &= a_4, & b_2 &= a_3 + a_4x_0, & b_1 &= a_2 + a_3x_0 + a_4x_0^2, \\ b_0 &= a_1 + a_2x_0 + a_3x_0^2 + a_4x_0^3 \end{aligned} \quad (\text{A.4})$$

After the transformation (4.13) we rewrite $P_3(u) = \sum_{n=0}^3 c_n u^n$, where the constants c_n are related to a_n by:

$$\begin{aligned} c_0 &= a_4 & c_1 &= a_3 + 4a_4x_0 & c_2 &= a_2 + 3a_3x_0 + 6a_4x_0^2 \\ c_3 &= a_1 + 2a_2x_0 + 3a_3x_0^2 + 4a_4x_0^3 \end{aligned} \quad (\text{A.5})$$

and the Weierstrass form can be obtained by the transformation (4.15).