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Iara Naomi Nobre Ota

**Quantum Information and Relativity: Harvesting
entanglement in different setups**

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Iara Naomi Nobre Ota

**Quantum Information and Relativity: Harvesting
entanglement in different setups**

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Supervisor: Prof. Dr. Daniel Augusto Turolla Vanzella

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Comissão Julgadora

Dr(a). Daniel Augusto Turolla Vanzela

Instituição: (IFSC/USP)

Dr(a). Alberto Vazquez Saa

Instituição: (UNICAMP/Campinas)

Dr(a). George Emanuel Avraam Matsas

Instituição: (UNESP/São Paulo)

*To my parents,
André and Inês.*

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ABSTRACT

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The aim of this work is present the phenomenon denoted entanglement harvesting. We begin by introducing entanglement historically. Following, we go beyond the one particle theory in flat spacetime and introduce Quantum Field Theory in Curved Spacetime, showing two famous consequences: the Unruh effect and the Hawking radiation. Finally, we analyze entanglement harvesting for two Unruh-deWitt detectors. In the first example, we see that there is a “sudden death” point of entanglement harvesting when the detectors are near the BTZ black hole event horizon, due to redshift effect and Hawking radiation. Then, we compare the phenomenon for different scenarios, and find out that it is sensitive to the structure of spacetime. Finally, we see how detectors’ parameters affect it and find out that the smoothness of the switching of the detectors’ coupling to the field is extremely relevant. We also see how the parameters can be used to optimize entanglement harvested.

Keywords: Harvesting entanglement. Quantum field theory in curved spacetime. Entanglement.

RESUMO

OTA, I. N. N. **Informação quântica e relatividade: colhendo emaranhamento em configurações variadas**. 2018. 98p. Dissertação (Mestrado em Ciências) - Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, 2018.

O objetivo desse trabalho é apresentar o fenômeno denotado colheita de emaranhamento. Primeiramente fazemos uma introdução histórica de emaranhamento de estados quânticos. Em seguida, introduzimos a Teoria Quântica de Campos no Espaço-tempo Curvo, como um passo além da teoria quântica de uma partícula no espaço-tempo plano, e demonstramos dois resultados famosos da teoria: o efeito Unruh e a radiação Hawking. Por fim, fazemos uma análise do fenômeno de colheita de emaranhamento para dois detectores Unruh-deWitt. Nosso primeiro exemplo mostra que há um ponto de “morte súbita” do fenômeno quando os detectores se aproximam do horizonte de eventos de um buraco negro de BTZ, que é uma consequência do efeito de redshift e da radiação Hawking. Em seguida, comparamos o fenômeno em cenários diferentes, e observamos que a colheita de emaranhamento é sensível à estrutura do espaço-tempo. Por último, analisamos como os parâmetros dos detectores afetam a colheita de emaranhamento, e vemos que a suavidade em que o acoplamento dos detectores com o campo é “ligado” e “desligado” é extremamente relevante. Também analisamos como podemos usar os parâmetros dos detectores para otimizar a quantidade de emaranhamento colhida.

Palavras-chave: Colheita de emaranhamento. Teoria quântica de campos no espaço-tempo curvo. Emaranhamento.

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1 INTRODUCTION

The quantum theory, developed in the first half of the last century, aims to understand the behavior of nature in the atomic scale. It revolutionized our understanding of nature and it explains three of the four known fundamental interactions: the Electromagnetic, Weak and Strong interactions. It has important applications in areas of chemistry and biology and it is crucial for technological development. As an example, without quantum theory, transistors would not have been invented, nor would microcomputers. Therefore, the current era of information economy would not have taken place.

First identified by Einstein, Podolsky and Rosen (1) as a flaw in the theory, quantum entanglement is one of the most remarkable phenomenon of the quantum theory. It has some exhilarating applications that were once believed to be purely in the scope of science fiction. Experimentalists already successfully teleported quantum states (2,3), which was once considered impossible due to the uncertainty principle. Quantum computers are probably the most expected technological outcome. With them, information-processing time will decrease significantly. On the other hand, quantum cryptographic tasks guarantees security communication even if quantum computers are used.

Nowadays, quantum entanglement is a very active subject of research. However, most researches are restricted to the study of entanglement in inertial frames in flat spacetime. The fourth fundamental interaction is gravity, and it is described by the theory of General Relativity - a non-quantum theory. Gravity was the first fundamental interaction investigated by physicists. Newton's law of universal gravitation, which is now considered an approximation of the theory of General Relativity, was developed much earlier than the classical theory of electromagnetism. As gravity was evidently important for classical physicists, why ignore it when dealing with quantum phenomena?

General Relativity is a very successful theory. It is in accordance with every astronomical observation made so far, and was further confirmed recently by the direct detection of gravitational waves. However, it is a deterministic theory, and this means that it is incompatible with quantum theory. Although it is not expected to be proved wrong for the time being, General Relativity is believed by many to be a "large scale" approximation of some quantum gravity theory, which has not yet been developed. The Quantum Field Theory in Curved Spacetime is believed to be a good approximation of the quantum phenomena where gravity plays a role. In this theory, matter fields are quantized over a curved background geometry. Even in flat spacetime we encounter interesting results. The Unruh effect shows that an accelerated observer in the vacuum "feels" a thermal bath of particles, which means that the concept of particle is observer dependent. As entanglement requires a precise definition of state, entanglement will be dependent on the

system considered; for example, entanglement is degraded by the Unruh effect (4).

One curious phenomenon is that, contrary to entanglement degradation, entanglement can be extracted from the vacuum state of a quantum field to a pair of particle detectors that interact with the quantum field locally for a given amount of time (5,6). This phenomenon became known as entanglement harvesting. It is surprising that the pair of detectors can become entangled even if they are causally disconnected. It was shown that harvesting entanglement is sensitive to many parameters, such as spacetime background (7) and acceleration of the detectors (8). Even though this phenomenon can be used to perform entanglement tasks, the amount of entanglement that can be harvested in spacelike separation is extremely small (9). Although it is still not useful for any application, this phenomenon has the importance of demonstrating a testable violation of Bell inequalities (10).

In this work we meant to introduce the fundamental concepts and tools necessary to understand the phenomenon of entanglement harvesting. We organized the content as follows:

In chapter 2 we give a brief introduction to the postulates of Quantum Mechanics, focusing on concepts. Then, we introduce quantum entanglement with a historical approach, explaining the Einstein, Podolsky and Rosen paradox and the Bell inequalities. As an illustration, we present the quantum teleporting phenomenon. Finally, we introduce some entanglement measurements that will be used in chapter 4.

In chapter 3 we introduce the Quantum Field Theory in Curved Spacetime. Then, we demonstrate the Unruh Effect and the Hawking Radiation. The concepts of General Relativity required to understand this chapter is assumed known by the reader and a brief notation review is given in Appendix A.

In chapter 4 we present in details the phenomenon of harvesting entanglement. Then, we see how it varies when the detectors are near a black hole, where the Hawking radiation and the redshift effect take place. Later, we analyze the phenomenon for four different scenarios in order to compare the results. Finally, we inspect how harvesting entanglement is sensitive to the detectors' parameters for inertial detectors in Minkowski vacuum.

In the last chapter we present final remarks of this work.

2 QUANTUM INFORMATION THEORY

The Classical Information theory was developed in the 1940s, when Quantum Mechanics was already well established. The interest in developing a quantum theory of information came much later, when the discovery of quantum teleporting (11) motivated further researches. The fundamental aspect of the Quantum Information theory is quantum entanglement, which was named by Schrödinger when he studied it. Entanglement is a quantum correlation, thus, it is a property of two or more quantum systems. If two systems are entangled, we can obtain information about one of them by making a measurement on the other, even when they are causally disconnected. In this chapter, we will present a conceptual review on the postulates of Quantum Mechanics. Later, we will explain the EPR paradox and Bell inequalities and present quantum teleporting. We end this chapter by introducing entanglement measurements.

2.1 Postulates of Quantum Mechanics

In this section we will go over the four postulates of Quantum Mechanics - which are said to be able to describe any possible quantum system - and give a brief review of its mathematical structure. The postulates stated below are valid for pure discrete states of isolated quantum systems with finitely many degrees of freedom, and we will use them to point out important concepts and notations about Quantum Mechanics. More detailed view can be found in many Quantum Mechanics or Quantum Information books like (12–14) and their generalized versions are stated in (15, 16).

1. Quantum states of isolated physical systems are described by vectors in a Hilbert space $\mathcal{H}$.
2. Physical observables are represented by self-adjoint operators acting on $\mathcal{H}$.
3. One can mathematically relate a physical observable A with its measured values $a_i \in \mathbb{R}, i = 1, 2, \dots, n$, by the use of a set of orthogonal projection operators $\{P_{a_i}\}$ that satisfies $\sum_i^n P_{a_i} = I$, where I is the identity matrix with the same dimension as $\mathcal{H}$. If the measured value a_n is obtained from a measure A in the state $|\psi\rangle$, then the normalized quantum state after the measurement is given by

$$|\psi_i\rangle = \frac{P_{a_i} |\psi\rangle}{\sqrt{p(a_i)}},$$

where $p(a_i)$ is the probability of obtaining the measured value a_i and it is equal to the expectation value of the projection operator P_{a_i} computed in the state $|\psi\rangle$:

$$p(a_i) = \langle \psi | P_{a_i} | \psi \rangle.$$

4. The quantum state evolution is deterministic and reversible and it is described by a unitary operator $\mathcal{U}(t, t_0)$. The dynamic equation for $\mathcal{U}(t, t_0)$ is

$$i\hbar \frac{d}{dt} \mathcal{U}(t, t_0) = H(t) \mathcal{U}(t, t_0), \quad (2.1.1)$$

where $H(t)$ is the observable related to the total energy of the system - the Hamiltonian.

The first postulate tells us that a physical quantum state can be represented as a vector in a d -dimensional complex vector space - the Hilbert space:

$$\mathcal{H} = \mathbb{C}^d = \{(c_1, c_2, \dots, c_d)\},$$

where $c_\lambda, \lambda \in \{1, \dots, d\}$, are elements of a column-vector in a given basis. We shall denote vectors in $\mathcal{H}$ using Dirac-notation which reads

$$|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix}, \quad \langle\psi| = (c_1^*, c_2^*, \dots, c_d^*),$$

and the inner product of two vectors $|\psi\rangle, |\phi\rangle$ is denoted by $\langle\psi|\phi\rangle$. We take the vector states to be normalized, that is, $\sum_\lambda |c_\lambda|^2 = 1$.

The second postulate tells us how to mathematically describe observables of a quantum system. Observables are understood as *physical properties* of the quantum system that can be measured - for instance, energy and position are observables -, thus the given mathematical description must provide a way to ascribe real measurable values for quantum systems. An operator on a vector space maps vectors onto vectors, that is, if A is an operator then $A|\psi\rangle = |\phi\rangle$, where $|\psi\rangle, |\phi\rangle \in \mathcal{H}$. The *adjoint* operator $A^\dagger$ of A is defined such that it satisfies $\langle A\psi|\phi\rangle = \langle\psi|A^\dagger\phi\rangle$. If an operator A acting on a vector $|\psi\rangle$ produces the same vector multiplied by a scalar a , $A|\psi\rangle = a|\psi\rangle, a \in \mathbb{C}$, we call a an *eigenvalue* and $|\psi\rangle$ an *eigenvector* (or *eigenstate* if $|\psi\rangle$ is a quantum state) of the operator A . An operator A is *self-adjoint* if the domain of A coincides with the domain of its adjoint $A^\dagger$ and if it satisfies $\langle\phi|A|\psi\rangle = \langle\psi|A|\phi\rangle^*, \forall |\psi\rangle, |\phi\rangle \in \mathcal{H}^*$. The *Spectral Theorem* tells us that we can diagonalize any self-adjoint operator and its eigenvalues are all real. Then, we can reconstruct an observable A in a diagonal form

$$A = \sum_i a_i |\psi_i\rangle\langle\psi_i|, \quad (2.1.2)$$

* An operator that satisfies only the equality $\langle\phi|A|\psi\rangle = \langle\psi|A|\phi\rangle^*, \forall |\psi\rangle, |\phi\rangle \in \mathcal{H}$ is called Hermitian and the distinction between Hermitian and self-adjoint operators only exists in infinite-dimensional vector spaces.

where $a_i \in \mathbb{R}$ and $|\psi_i\rangle$ are its eigenvalues and eigenvector, respectively, and, as we will see below, its real eigenvalues are the possible measured values.

Before talking about the other postulates, let us consider an example. The simplest non-trivial quantum system has a two-dimensional Hilbert space and it is entitled *two-level system* or *qubit*. The electron spin, or the spin of any other spin-1/2 particle, is an observable in a two-level system. We can set a measurement apparatus in which a beam of electrons passes through an inhomogeneous magnetic field in the z direction and then, due to the interaction between the spin and the magnetic field, the electrons are deflected and detected in some position in the z direction. The spin state can be either up $|\uparrow\rangle$ or down $|\downarrow\rangle$ and the observable spin-in- z -direction matrix is written as $S_z = \frac{\hbar}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)$, where $\hbar$ is the reduced Planck constant. In the experiment described above (Stern-Gerlach), if the spin is in the state $|\uparrow\rangle$ the electron will be detected in $z > 0$, and if it is in the state $|\downarrow\rangle$ the electron will be detected in $z < 0$.

The third postulate asserts that it is possible to obtain measured values by the use of orthogonal projection operators. By definition an orthogonal projection operator P is idempotent, that is $P^2 = P$, and self-adjoint. By virtue of the Spectral Theorem, we can write $P = \sum_i p_i |i\rangle\langle i|$, $p_i \in \mathbb{R}$, where $|i\rangle$ are orthogonal-basis vectors. We have $P^2 = \sum_i p_i^2 |i\rangle\langle i| = P = \sum_i p_i |i\rangle\langle i|$, thus $p_i \in \{0, 1\}$. Therefore, P assumes the form

$$P = \sum_j |j\rangle\langle j|,$$

where $\{|j\rangle\}$ spans a subspace of the space spanned by $\{|i\rangle\}$. Considering the projection operators $P_{a_i} = |\psi_i\rangle\langle\psi_i|$, we can rewrite (2.1.2) as

$$A = \sum_i a_i P_{a_i}.$$

As each measured value is associated with a projective operator P_{a_i} , we identify the eigenvalues of A as the measured values. An arbitrary state $|\psi\rangle$ can be written as a linear combination of the basis vectors $\{|\psi_i\rangle\}$, $|\psi\rangle = \sum_i \alpha_i |\psi_i\rangle$, and, as the postulate tells us, the probability of obtaining the measured value a_i is $p(a_i) = \langle\psi|P_{a_i}|\psi\rangle = \alpha_i^2$. After the measurement is made and we obtain a_i , we will be left with the state $|\psi_i\rangle$, and we notice that, if we measure the observable A immediately after the first measurement, we are going to obtain the measured value a_i - because the probability of obtaining the value a_i from the state $|\psi_i\rangle$ equals unity. It is important to acknowledge how the consecutive measurements can fool us. The initial state $|\psi\rangle$ has some probability associated to a possible measured value and, if we obtain the value a_i , this does not mean that $|\psi\rangle$ was a state which had some property that assures us that the measured value would be a_i . The postulate tells us what is the state *after* the measurement. Therefore, we understand a measurement as an external intervention in the system - which will leave the system in the projected state - that gives us *some information* about the system. An easy way to picture this probability

Figure 1 – Stern-Gerlach experiment. Figure (a) shows a schematic picture of the experiment: due to the interaction between the magnetic field and the spin, some of the electrons are deflected upward and some downward. Figure (b) shows a diagram of sequential experiments: the beam passes through a magnetic field in the z direction (MF_z) and then the beam that deflected upward passes through a second magnetic field in the z direction and these electrons will just be deflected upward.

Source: By the author.

property is considering that the state vector is a huge collection of identical systems and the measurements are made for each system, then each measured value a_i will be obtained for $(\alpha_i^2 \times 100)\%$ of systems.

Returning to the electron spin example, the spin of a particle is an “intrinsic” angular momentum and it is not related with the standard angular momentum, as elementary particles are point particles and it is impossible to assign a mass distribution revolving around some axis. Mathematically, the spin angular momentum is regarded as rotations within its vector space (see reference (17)). This spin angular momentum is proportional to the magnetic moment μ of the particle. For the Stern-Gerlach experiment the distribution observed after detection of several electrons from the beam has two separated peaks that designates to the electrons two fixed magnetic momentum $\mu_z = -2\mu_B s_z$, where μ_B is the Bohr magneton and $s_z = \pm\hbar/2$ are the spin eigenvalues in the z direction. Thus, the measured values of the operator S_z are related to the electrons deflection due to the magnetic field. The beam of electrons is an example of the huge collection of identical systems that we pictured above. If we consider that the electrons in this beam are in the state $|\psi\rangle = (1/\sqrt{2})[|\uparrow\rangle + |\downarrow\rangle]$ we verify that half of the electrons will deflect to $z > 0$ and half to $z < 0$, as shown in Figure 1a. If we consider now that the electrons that deflected to $z > 0$ passes through another magnetic field in the same direction of the first one, and then detect the electron position, we will see that all the electrons will deflect upward in the z direction, as shown in Figure 1b. This is a consequence of the previous measurement - which was made because we already knew the first deflected position - that projected all the electrons deflected to $z > 0$ to the state $|\uparrow\rangle$.

Figure 2 – Diagram of three Stern-Gerlach sequential experiments: the beam of electrons passes through a magnetic field in the z direction (MF_z) and then the beam that deflected upward passes through a second magnetic field in the x direction (MF_x) and, for last, the beam that deflected upward x direction passes through a third magnetic field in the z direction (MF_z) and these electrons will be deflected in both directions. This happens because the measurement S_x completely destroys the S_z measurement made previously.

Source: By the author.

Someone solely familiarized with classical physics may argue that, in this experiment, the electrons were not all in state $|\psi\rangle = (1/\sqrt{2})(|\uparrow\rangle + |\downarrow\rangle)$ but rather half of them were in state $|\uparrow\rangle$ and half were in state $|\downarrow\rangle$, thus any probabilistic prevision of the measurement would be related to lack of knowledge of the initial conditions, just like the flip of a coin. This can be refuted if we consider a sequence of the Stern-Gerlach experiment such that the beam of electrons passes through an inhomogeneous magnetic field in the z direction, and then the beam of the electrons deflected to $z > 0$ passes through an inhomogeneous magnetic field in the x direction. If we detect the electrons in the x direction, we will see that half will deflect to $x > 0$ and half to $x < 0$. Finally, if we consider a third apparatus such that the electrons that deflected to $x > 0$ pass through an inhomogeneous magnetic field in the z direction the result is that half of these electrons will deflect upward z direction and half downward, as shown in Figure 2. Somehow half of the selected states $|\uparrow\rangle$ turned into the state $|\downarrow\rangle$ after going through the magnetic field in x direction. In classical physics that would not make any sense but that is in accordance with Quantum Mechanics. Mathematically, we can write the state $|\uparrow\rangle$ as a linear combination of the spin states in the x direction, that is, $|\uparrow\rangle = (1/\sqrt{2})(|\uparrow_x\rangle + |\downarrow_x\rangle)$, where $|\uparrow_x\rangle$ is the state of an electron that will be deflected $x > 0$ and $|\downarrow_x\rangle$ is the state of an electron will be deflected $x < 0$ after the measurement $S_x = \frac{\hbar}{2}(|\uparrow_x\rangle\langle\uparrow_x| - |\downarrow_x\rangle\langle\downarrow_x|) = \frac{\hbar}{2}(|\downarrow\rangle\langle\downarrow| - |\uparrow\rangle\langle\uparrow|)$. Similarly, $|\uparrow_x\rangle = (1/\sqrt{2})(|\uparrow\rangle + |\downarrow\rangle)$ and that is why these electrons will be deflected in both directions after the measurement S_z . We can see in this experiment that the measurement S_x completely destroys the S_z measurement made previously, and vice versa. This happens to all observables that do not commute, that is, $[A, B] \equiv AB - BA \neq 0$, where A and B are observables.

The last postulate tells us how quantum states evolve in time. A unitary operator $\mathcal{U}(t, t_0)$ is an operator that satisfies $\mathcal{U}(t, t_0)\mathcal{U}^\dagger(t, t_0) = \mathcal{U}^\dagger(t, t_0)\mathcal{U}(t, t_0) = I$, where I is the identity matrix. As any other operator in Hilbert space, $\mathcal{U}(t, t_0)$ maps vectors onto vectors, thus we can say that $\mathcal{U}(t, t_0)$ maps an initial state $|\psi(t_0)\rangle$ to a final state $|\psi(t)\rangle$, that is,

the dynamic evolution of a state is given by

$$|\psi(t)\rangle = \mathcal{U}(t, t_0) |\psi(t_0)\rangle. \quad (2.1.3)$$

This approach - that considers quantum states as time dependent states - is known as the *Schrödinger picture*. From (2.1.1) follows the *Schrödinger equation*:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle. \quad (2.1.4)$$

In the Schrödinger picture the probability $p(a_i)_{t_0} = \langle \psi(t_0) | P_{a_i} | \psi(t_0) \rangle$ will evolve to $p(a_i)_t = \langle \psi(t) | P_{a_i} | \psi(t) \rangle = \langle \psi(t_0) | \mathcal{U}(t, t_0)^\dagger P_{a_i} \mathcal{U}(t, t_0) | \psi(t_0) \rangle$. In other way, we can consider that the quantum state does not evolve in time but the operators $\mathcal{O}$ do, that is,

$$\mathcal{O}(t) = \mathcal{U}(t, t_0)^\dagger \mathcal{O}(t_0) \mathcal{U}(t, t_0). \quad (2.1.5)$$

This approach is called the *Heisenberg picture* and from (2.1.1) follows the *Heisenberg equation*:

$$i\hbar \frac{d}{dt} \mathcal{O}(t) = [\mathcal{O}(t), H(t)] + i\hbar \frac{\partial}{\partial t} \mathcal{O}(t). \quad (2.1.6)$$

We can easily see that the evolution of $p(a_i)$ is the same in both pictures. In short, in the Schrödinger picture the states are time dependent and the operators are time independent and in the Heisenberg picture is the opposite. A third approach is considering that both states and operators are time dependent - the *Interaction picture*. Consider the Hamiltonian can be decomposed in two parts, $H(t) = H_0 + V(t)$, where H_0 is not a function of time. A state vector $|\psi\rangle_I$ in the interaction picture is related to the state $|\psi\rangle_S$ in the Schrödinger picture by $|\psi\rangle_I \equiv e^{iH_0 t/\hbar} |\psi\rangle_S$. And an operator $\mathcal{O}_I$ in the interaction picture is related to the operator $\mathcal{O}_S$ in the Schrödinger picture as $\mathcal{O}_I \equiv e^{iH_0 t/\hbar} \mathcal{O}_S e^{-iH_0 t/\hbar}$. Therefore, it is possible to show that, in the interaction picture, state-vector evolution is determined by $V_I = e^{iH_0 t/\hbar} V(t) e^{-iH_0 t/\hbar}$, and operator evolution is determined by H_0 .

Finally, we can solve equation (2.1.1). Considering the initial condition $\mathcal{U}(t_0, t_0) = I$, we have $\mathcal{U}(t, t_0) = I - \frac{i}{\hbar} \int_{t_0}^t dt_1 H(t_1) \mathcal{U}(t_1, t_0)$. By iteration, we obtain:

$$\mathcal{U}(t, t_0) = \underbrace{I}_{\mathcal{U}^{(0)}} - \underbrace{\frac{i}{\hbar} \int_{t_0}^t dt_1 H(t_1)}_{\mathcal{U}^{(1)}} - \underbrace{\frac{1}{\hbar^2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H(t_1) H(t_2) + \dots}_{\mathcal{U}^{(2)}} \quad (2.1.7)$$

This expression is known as *Dyson series* of the time evolution operator.

The postulates stated above are not stated in their most general form, as they are only true for isolated systems in pure states. The most general form of quantum systems is not in a pure state but in a *mixture* of pure states. When we deal with an *ensemble* it is usually hard to prepare the whole system in the same state and then we are left with a mixture of states. The mixture is not a sum of the pure states, that is, a vector. To understand why, we first note that superposition of states is not a mixture of states.

Consider, for instance, the state of an electron to be $|\psi\rangle = (1/\sqrt{2})[|\uparrow_x\rangle + |\downarrow_x\rangle]$. The electron is in a superposition of the states $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$ and if we measure the spin in x direction the probability that it will reflect upward or downward is 0.5 for both cases. This is the main point here: the probabilities are related to the measurement results and not to the state itself - the electron is with probability 1 in the state $|\psi\rangle$ and not with probability 0.5 in the states $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$. Therefore, a mixture of states cannot be represented by a vector in the Hilbert space. However, we should be able to relate the mixture with vector states contained in the mixture. The mixture will be represented by an operator: the *density operator* or *density matrix*.

A density matrix of a pure state $|\psi\rangle$ is defined as $\rho \equiv |\psi\rangle\langle\psi|$. The four postulates of Quantum Mechanics can be rewritten in terms of density matrices. First, we have that, given an observable A and orthonormal vector basis $\{|u_i\rangle\}$ of some Hilbert space $\mathcal{H}$, the expectation value of A in a state $|\psi\rangle \in \mathcal{H}$ is

$$\begin{aligned}\langle A \rangle &= \langle \psi | A | \psi \rangle = \sum_{j,k} \langle \psi | u_j \rangle \langle u_j | A | u_k \rangle \langle u_k | \psi \rangle = \sum_{j,k} \langle u_k | \psi \rangle \langle \psi | u_j \rangle \langle u_j | A | u_k \rangle \\ &= \sum_k \langle u_k | \rho A | u_k \rangle = \text{Tr}[\rho A],\end{aligned}$$

where we inserted the identity operator $I = \sum_i |u_i\rangle\langle u_i|$. Also, if, for some operator A , there is a vector state $|\psi'\rangle = A|\psi\rangle$, then its density matrix is $\rho' = |\psi'\rangle\langle\psi'| = A|\psi\rangle\langle\psi|A^\dagger = A\rho A^\dagger$. With this properties, if we consider in the first postulate that quantum states of a physical system are described by a density matrix, all the other postulates can be rewritten in terms of the density matrices as well as the Schrödinger and Heisenberg equations. In this sense, the system described by a vector $|\psi\rangle$ is also described by its density operator ρ , thus we say that the system is in the *state* ρ . The most general state - the mixture of pure states - is given by $\rho = \sum_i p_i \rho_i$, where p_i are the probabilities for the mixed state ρ to be in the pure state $\rho_i = |\psi_i\rangle\langle\psi_i|^\dagger$.

2.2 Roots of Quantum Entanglement

In this section we will explore the historical origins of entanglement. Besides Einstein's contributions to the development of the Quantum Theory, such as the Photoelectric effect and the Bose-Einstein condensate, Einstein was known to be an opposer to Quantum Mechanics as he was unwilling to accept the probabilistic nature of the theory. In fact, Einstein criticisms turned out to be very significant to the fundamental concepts of Quantum Mechanics. In a paper (1) published in 1935 in collaboration with Podolsky and Rosen, the authors argued that Quantum Mechanics must be an incomplete theory - according to *their* concept of a complete theory. Their argument suggested the existence

[†] This interpretation for these coefficients is not true for any given density matrix as it can be given in infinitely many different forms by changing the basis vector.

of a theory of *hidden variables* that would restore the deterministic nature for Quantum Theory. But in 1964, Bell made a qualitative analysis of the problem stated by Einstein, Podolsky and Rosen (EPR) showing that no local hidden variable theory description - local in the sense that a system is unaffected by operations made in another distant system - can reproduce all the predictions of Quantum Mechanics (10).

The EPR paper analyzes multi-particle systems that are now known as entangled states. The name *entanglement* was given by Schrödinger (18) as he investigated its consequences; in fact, the “Schrödinger’s cat” is probably the most well-known thought experiment in Quantum Mechanics. In this thought experiment we consider that a cat is inside a box with a mechanism containing a radioactive atom that releases a deadly gas if the atom decays. When we open the box there can be two situations: the cat is alive and the atom has not decayed or the cat is dead and the atom has decayed. The quantum state of this system is written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\text{alive cat}\rangle \otimes |\text{unstable atom}\rangle + |\text{dead cat}\rangle \otimes |\text{stable atom}\rangle),$$

where $\otimes$ denotes the tensor product of two vectors. One of the things that most concerned Einstein was the fact that these entangled states could “send signals” faster than the speed of light. This is easy to see, rewriting the Schrödinger’s cat state in a more general form

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\psi_1\rangle \otimes |\phi_1\rangle + |\psi_2\rangle \otimes |\phi_2\rangle),$$

assuming that $|\psi_1\rangle$ and $|\psi_2\rangle$ are the possible states of a particle A, and $|\phi_1\rangle$ and $|\phi_2\rangle$ are the possible states of a particle B. If we make a measurement $\mathcal{O}$ in particle A, such that $\mathcal{O}|\psi_i\rangle = a_i|\psi_i\rangle$, and obtain the value a_1 the state of the particle A will be $|\psi_1\rangle$ and, as a consequence, the particle B is going to be in the state $|\phi_1\rangle$. We notice that the measurement made in particle A collapses particle B to the state $|\phi_1\rangle$ even if we consider that the particles are spacelike separated - their distance is such that, during the experiment, no signal of light can be emitted by one and received by the other. One of the consequences of the Einstein theory of Special Relativity is that nothing can travel faster than light, that is, there are no instantaneous signals from one particle to another. Therefore, this consequence of entanglement always bothered Einstein and in a letter he addressed to Max Born in 1947 (19) he called it “spooky actions at a distance”. This problem with causality is resolved by the fact that you must send a classical signal - limited by the speed of light - to know what measurement you should make to obtain the value associated with the state $|\phi_1\rangle$ of particle B; this will be explained with more details in the teleportation example.

In the following subsections we will give a brief explanation of the EPR paradox and the Bell Inequalities. After that, we will consider an interesting application of entangled particles that consists in teleporting one quantum state - which was considered impossible due to the third Postulate, as any measurement made aiming to characterize the state would change it before obtaining *all* the information needed to reconstruct it elsewhere.

EPR Paradox and Bell inequalities

In a paper published in 1935 (1), Einstein, Podolsky and Rosen called into question the validity of Quantum Mechanics as a complete theory. By the authors' point of view, a physical theory shall only succeed if the theory is correct and complete. For them, a theory is correct if all the conclusions of the theory agree with the experiments. But it is the completeness of the Quantum Theory they question. In the original paper, they use wave functions and the position and momentum operators to exemplify their argument. Here, as a matter of simplicity and consistency with the previous section, we will consider Bohm's version of the ERP paradox (20), which deals with spin-1/2 particles.

The crucial point of their argument is the necessary condition they impose for a complete theory: the physical world is described by *elements of physical reality* and a complete theory must have counterpart elements. As a physical theory is described by mathematics, for an unperturbed system there should be possible to predict with certainty - probability equal 1 - a theoretical value that corresponds to an element of reality. The *uncertainty principle* states that, for a particle, $\Delta x \Delta p \geq \hbar/2$, where x is position and p is the momentum of the particle, and thus it is not possible to simultaneously establish with certainty the values of the position and momentum of a particle. It follows that, according to their definition, position and momentum are not simultaneously elements of the physical reality of a particle.

Now, if we consider the eigenvalue equation for an operator A , $A|\psi\rangle = a|\psi\rangle$, we see that the physical quantity A has, with certainty, the value a whenever the physical state is given by $|\psi\rangle$, and thus there is a physical reality corresponding to the physical quantity A . Hence, two observables that do not commute (do not share eigenvectors[‡]) cannot be simultaneously elements of the physical reality of a system and if it is shown that we can find simultaneously the value of two observables that do not commute, the Quantum Mechanical description must be incomplete.

Consider a system of two particles, I and II, that interact for a certain amount of time and then they are arbitrarily separated and the interaction ceases. Assuming that before the interaction the quantum states of both systems are known, it is possible to determinate the combined state I + II for any successive time using Schrödinger's equation, but we cannot know the final state of each individual particle without making a measurement (more precisely, after the interaction, the state of each particle cannot be described by a vector in the individual Hilbert spaces). Let $\mathcal{H}_I$ and $\mathcal{H}_{II}$ be, respectively, the Hilbert spaces of particles I and II after the interaction ceased, the composite Hilbert

[‡] We can see that by considering $A|\psi\rangle = a|\psi\rangle$. If A and B commute $[A, B]|\psi\rangle = 0 \Rightarrow AB|\psi\rangle = BA|\psi\rangle = aB|\psi\rangle$, and thus $B|\psi\rangle$ is an eigenvector of A with the eigenvalue a , that is, $B|\psi\rangle = b|\psi\rangle$. If they do not commute, $B|\psi\rangle$ is not an eigenvector of A and thus $|\psi\rangle$ is not an eigenvector of B .

space being given by $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$, and we will denote the state after the interaction as $|\Psi\rangle$. Now, assuming that we want to make a measurement of an observable A of the first particle and denoting a_i as the observable's eigenvalues of its respective eigenvectors $|a_i\rangle$, we can write $|\Psi\rangle$ as an expansion in the observable's eigenvector base:

$$|\Psi\rangle = \sum_i c_i |a_i\rangle \otimes |\psi_i\rangle,$$

where $|\psi_i\rangle$ are normalized coefficients of the expansion and c_i are the normalization constants. Now, if we measure the value a_k , according to the third postulate, the first particle will be in the state $|a_k\rangle$ and, hence, the second particle will be in the state $|\psi_k\rangle$. Likewise, we can consider an observable B of the first particle with eigenvalues b_i and eigenvectors $|b_i\rangle$ and expand $|\Psi\rangle$ in this other basis:

$$|\Psi\rangle = \sum_i d_i |b_i\rangle \otimes |\phi_i\rangle.$$

If we measure the value b_r , according to the third postulate, the first particle will be in the state $|b_r\rangle$ and, hence, the second particle will be in the state $|\phi_r\rangle$.

The state $|\Psi\rangle$ represents the system when there are no interactions between the particles and thus an operation made in one of the particles, such as A and B , should not alter the state of the other particle. We showed above that it is possible to obtain two different states ($|\psi_k\rangle$ and $|\phi_r\rangle$) for the same physical reality (the second system after the interaction). Therefore, if we show that those vectors can be eigenvectors of two observables of particle II that do not commute we will conclude that particle II is not completely described by its quantum states - as stated in the first postulate.

Now we consider that the particles of the system are spin-1/2 particles and that after the interaction the total spin of the system is zero, the normalized final state being written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [|\uparrow_z\rangle_I \otimes |\downarrow_z\rangle_{II} - |\downarrow_z\rangle_I \otimes |\uparrow_z\rangle_{II}],$$

where $|\uparrow_z\rangle_{I,II}$ and $|\downarrow_z\rangle_{I,II}$ are the eigenvectors of the spin observable in z direction for each particle. If we measure the spin of the first particle in z direction, the state of the second particle will turn to $|\downarrow\rangle_{II}$, if we measure $+\hbar/2$, or it will turn to $|\uparrow\rangle_{II}$, if we measure $-\hbar/2$. Since both of these states are eigenvectors of the spin observable in z direction of the second particle, $(S_z)_{II} = \frac{\hbar}{2} [|\uparrow\rangle_{II} \langle\uparrow|_{II} - |\downarrow\rangle_{II} \langle\downarrow|_{II}]$, we conclude that the spin in z direction of the second particle is a physical reality of the system. Nevertheless, we could have written $|\Psi\rangle$ in terms of the spin in x direction basis. Knowing that $|\uparrow_x\rangle = (1/\sqrt{2})(|\uparrow_z\rangle + |\downarrow_z\rangle)$ and $|\downarrow_x\rangle = (1/\sqrt{2})(|\uparrow_z\rangle - |\downarrow_z\rangle)$, we have

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [|\uparrow_x\rangle_I \otimes |\downarrow_x\rangle_{II} - |\downarrow_x\rangle_I \otimes |\uparrow_x\rangle_{II}],$$

and, by the same argument given above, we conclude that the spin in x direction of the second particle is a physical reality of the system. But the spin operators satisfy

the commutation relation $[S_x, S_z] = -i\hbar S_y$, where S_y is the spin operator in y direction. Therefore, we are led to the conclusion that Quantum Mechanics is not a complete theory. To summarize in other words, if we measure the spin in z direction of the first particle and say we obtain the value $-\hbar/2$, the probability of obtaining the value $+\hbar/2$ of the second particle is 1 because, after the measurement, the final state of the second particle will be $|\uparrow_z\rangle$. But we know that $|\uparrow_z\rangle = (1/\sqrt{2})(|\uparrow_x\rangle + |\downarrow_x\rangle)$ and thus the probability of obtaining $\pm\hbar/2$ when we measure the spin in x direction is $1/2$. Following this argument, if instead we measure the spin in x direction of the first particle we will know for sure the value we will get when we measure the spin in x direction of the second particle, but we would not have the same certainty for the measurement value we will get if we measure the spin in z direction. We see that the measurement probabilities of the second particle have a direct dependence on the measurement made on the first particle, which, according to the authors of EPR article, could not happen because the two particles are not interacting anymore (indeed, they can even be spacelike separated).

The arguments presented above lead us to consider that Quantum Mechanics should be completed with some additional variables that are able to predict with certainty physical quantities that Quantum Mechanics could not precisely predict, such as predicting the momentum and position simultaneously. Such variables are called *hidden variables* and they are supposed to restore our classical intuition and, in such a way, we would expect such a theory of hidden variables to be *realistic* and *local*. A realistic theory is a theory in which the values measured were already predetermined. The locality implies that a measurement made in one particle cannot influence measurements made in another casually disconnected particle. In a paper published in 1964 (10) Bell showed that, in a general manner, no local realistic theory is consistent with Quantum Mechanics. There are infinitely many Bell inequalities, as it can be derived for any number of observers, which have any number of setups and outcomes. Here we will present a simple form known as the CHSH inequality (21) as it is the inequality of systems that match with the system we are considering so far: bipartite (two particles) binary (two outcomes, such as spin up and spin down) systems.

It is valid to note that the Bell inequalities are relationships of conditional probabilities under locality assumptions and, in principle, it is not a quantum mechanical property. Nevertheless, to simplify we will introduce the conditional probabilities as measurement probabilities. Suppose two particles are sent in opposite directions to Alice and Bob, who are very distant from each other and hence casually disconnected. Alice can make the measurements of the observables $\{A_i\}$ and Bob can make measurements of the observable $\{B_j\}$. The reality constraint imposes that, given the hidden variable λ , the probability of obtaining any result from measurements made either by Alice or Bob is going to be 0 or 1. Locality implies that measurements made by Alice do not influence measurements made by Bob, and vice versa. Denoting $P(a_i, b_j|A_i, B_j, \lambda)$ the probability of obtaining the

values a_i and b_j of the measurements A_i and B_j , we have

$$P(a_i, b_j | A_i, B_j, \lambda) = P(a_i | A_i, \lambda) P(b_j | B_j, \lambda).$$

The expectation value of some observable F is

$$E(F) = \sum_f \int_{\Gamma} d\lambda \rho(\lambda) P(f | F, \lambda),$$

where $\rho(\lambda)$ is the probability distribution of $\lambda \in \Gamma$. Now, we suppose that Alice and Bob can make only two measurements (A_1, A_2, B_1 , and B_2), and all the measurements are binary and the two possible measurement results are -1 and 1 , which are the normalized measurement values of spin-1/2 particles. Defining the quantity $C = E(A_1 B_1) + E(A_2 B_1) + E(A_2 B_2) - E(A_1 B_2)$, we have

$$\begin{aligned} |C| &= |E(A_1 B_1) + E(A_2 B_1) + E(A_2 B_2) - E(A_1 B_2)| \\ &= \left| \sum_{a_1, a_2, b_1, b_2} \int_{\Gamma} d\lambda \rho(\lambda) [P(a_1, b_1 | A_1, B_1, \lambda) + P(a_1, b_2 | A_1, B_2, \lambda) \right. \\ &\quad \left. + P(a_2, b_2 | A_2, B_2, \lambda) - P(a_2, b_1 | A_2, B_1, \lambda)] \right| \\ &\leq \int_{\Gamma} d\lambda |\rho(\lambda)| \left| \sum_{a_1, a_2, b_1, b_2} [P(a_1 | A_1, \lambda) P(b_1 | B_1, \lambda) + P(a_1 | A_1, \lambda) P(b_2 | B_2, \lambda) \right. \\ &\quad \left. + P(a_2 | A_2, \lambda) P(b_2 | B_2, \lambda) - P(a_2 | A_2, \lambda) P(b_1 | B_1, \lambda)] \right|. \end{aligned}$$

The reality condition implies that the probabilities $P(a_i | A_i, \lambda)$ and $P(b_j | B_j, \lambda)$ are either 1 or 0, which means that, for all observables, if the probability of obtaining the value -1 is 1, the probability of obtaining the value 1 is going to be 0, and vice versa. Hence, we have

$$\begin{aligned} \sum_{a_1, b_1} P(a_1 | A_1, \lambda) P(b_1 | B_1, \lambda) &= P(1 | A_1, \lambda) P(1 | B_1, \lambda) + P(-1 | A_1, \lambda) P(-1 | B_1, \lambda) \\ &\quad + P(-1 | A_1, \lambda) P(1 | B_1, \lambda) + P(1 | A_1, \lambda) P(-1 | B_1, \lambda) \\ &= 1 \end{aligned}$$

and, likewise, the three other sums of probabilities equals 1. Considering that the probability distribution is normalized, $\int_{\Gamma} d\lambda \rho(\lambda) = 1$, we have the CHSH inequality that all local realistic theory for bipartite binary system must satisfy:

$$|E(A_1 B_1) + E(A_2 B_1) + E(A_2 B_2) - E(A_1 B_2)| \leq 2. \quad (2.2.1)$$

Now we can determine the expected value of C for Quantum Mechanics. First we take $C = A_1 B_1 + A_2 B_1 + A_2 B_2 - A_1 B_2$ to be an observable and then take its expected

value $\langle C \rangle \equiv \langle \psi | C | \psi \rangle$, where $|\psi\rangle \in \mathcal{H}$ is an arbitrary quantum state of the composite Hilbert space. For any operator C that acts on a finite-dimensional Hilbert space, $\|\langle C \rangle\|^2 \leq \|C\|^2 = \|C^2\|$, where $\|\psi\| = 1$. Hence, we have

$$\begin{aligned} \|C^2\| &= \|(A_1 B_1 + A_2 B_1 + A_2 B_2 - A_1 B_2)^2\| \\ &= \|4I - A_2 A_1 B_1 B_2 + A_1 A_2 B_1 B_2 + A_2 A_1 B_2 B_1 - A_1 A_2 B_2 B_1\| \\ &\leq 4 + \|-A_2 A_1 B_1 B_2\| + \|A_1 A_2 B_1 B_2\| + \|A_2 A_1 B_2 B_1\| + \|-A_1 A_2 B_2 B_1\| \\ &\leq 4 + 4\|A_1\|\|A_2\|\|B_1\|\|B_2\|, \end{aligned}$$

As considered before, the operators A_1, A_2, B_1, B_2 are normalized spin operators that can have observable values -1 or 1 and, thus, $\|A_1\| = \|A_2\| = \|B_1\| = \|B_2\| = 1$. Thus, we obtain the Quantum Mechanical inequality

$$\|\langle C \rangle\| = |E(A_1 B_1) + E(A_2 B_1) + E(A_2 B_2) - E(A_1 B_2)| \leq 2\sqrt{2}. \quad (2.2.2)$$

As an example of a quantum state that violates the CHSH (2.2.1) inequalities, consider a singlet state of two spin-1/2 particles:

$$|\Psi_{-}\rangle_{\text{I,II}} = \frac{1}{\sqrt{2}}[|\uparrow\rangle_{\text{I}} \otimes |\downarrow\rangle_{\text{II}} - |\downarrow\rangle_{\text{I}} \otimes |\uparrow\rangle_{\text{II}}],$$

where I and II label particles I and II, and the vectors form a basis of the spin in the z direction. Consider the observables to be normalized measurements of spin in z and x directions, such that,

$$\begin{aligned} A_1 &= \sigma_z \otimes I \\ A_2 &= \sigma_x \otimes I \\ B_1 &= -\frac{1}{\sqrt{2}}[I \otimes (\sigma_z + \sigma_x)] \\ B_2 &= \frac{1}{\sqrt{2}}[I \otimes (\sigma_z - \sigma_x)], \end{aligned}$$

where I is the identity matrix and σ_i are the Pauli matrices. With these observables and state we have $E(A_1 B_1) = E(A_2 B_1) = E(A_2 B_2) = -E(A_1 B_2) = 1/\sqrt{2}$, and thus $\|\langle C \rangle\| = 2\sqrt{2}$.

We see that Quantum Mechanics can violate the Bell (CHSH) inequalities, and thus Quantum Mechanics is incompatible with local realistic theories. Now, it is up to the experimentalists to show if there can be systems which violate the Bell inequalities and, hence, cannot be described by a local realistic theory. Experiments were made, see for example (22–24), and they showed that not only the Bell inequalities are violated but the violation is in accordance with Quantum Mechanics; in other words, Quantum Mechanics is able to predict results that no local realistic theory is able to. From that we have learned that the notions from classical physics, such as well defined “elements of reality”, must be

taken into question when we are dealing with quantum systems that are not only described by probabilities of measurements but they also can contain long distance correlation, that is, measurements made in one subsystem can influence the results of measurements made in another arbitrarily distant subsystem.

Teleporting quantum states

One of the most interesting features of these long distance correlations questioned by Einstein is the possibility of teleporting quantum state. First, we have to point out what teleportation means in this case. Teleporting is well known in science fiction where, for example, one astronaut is in a spaceship and, using some machine, he dematerialize from the spaceship and appears instantaneously in a planet, without traveling in a well defined path. In physics, teleportation is a transmission of *information* and it cannot be done instantaneously, as Special Relativity imposes that nothing can travel faster than light. Now, if you want to teleport a quantum particle you would have to transfer the information of all its probabilities according to a certain measurement but, as mentioned above, the third Postulate does not allow us to fully characterize a single quantum state, as it will modify the state if any measurement is made, and thus we would not be able to transmit the information about the state so that it could be reconstruct somewhere else. One could argue that we could make copies of the state we want to teleport to make measurements to characterize the probabilities with arbitrary precision and then transmit this information. Although it seems a good solution, it is impossible to clone *unknown* quantum states, and that is not a technological restriction but a fundamental one: The Wootters and Zurek *no-cloning theorem* (25) shows that there is no way to make copies of arbitrary quantum states.

The proof of the no-cloning theorem relies on the linearity of Quantum Mechanical operators, that is, for any quantum operator T and quantum states $|\psi\rangle$ and $|\phi\rangle$, $T(|\psi\rangle + |\phi\rangle) = T(|\psi\rangle) + T(|\phi\rangle)$. Now suppose we want to clone an arbitrary state $|s\rangle = \alpha|a\rangle + \beta|b\rangle$, where $|a\rangle$ and $|b\rangle$ form a basis for a 2-dimensional Hilbert space. Let T represent a cloning machine, then $T(|s\rangle) \rightarrow |s\rangle \otimes |s\rangle = (\alpha|a\rangle + \beta|b\rangle) \otimes (\alpha|a\rangle + \beta|b\rangle)$ and, by linearity, $T(\alpha|a\rangle + \beta|b\rangle) = \alpha T(|a\rangle) + \beta T(|b\rangle) \rightarrow \alpha|a\rangle \otimes |a\rangle + \beta|b\rangle \otimes |b\rangle$. The results of the cloning operator is clearly not the same and thus arbitrary states cannot be copied.

In a paper published in 1993, Bennet et al. (11) proposed a way of teleporting an unknown quantum state using entanglement. Suppose Alice has a spin-1/2 particle in an unknown state $|\psi\rangle = a|\uparrow\rangle_\psi + b|\downarrow\rangle_\psi$ and she wants Bob to have a quantum state just like hers without physically transferring the particle. To achieve this aim, each of them possesses a spin-1/2 particle that are entangled with each other, such that the state of the

bipartite system is

$$|\phi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B],$$

where the subscript A and B labels Alice's and Bob's particles, respectively. The state of the three-particle system is

$$\begin{aligned} |\psi\rangle \otimes |\phi\rangle &= \frac{a}{\sqrt{2}}[|\uparrow\rangle_\psi \otimes |\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\uparrow\rangle_\psi \otimes |\downarrow\rangle_A \otimes |\uparrow\rangle_B] \\ &\quad + \frac{b}{\sqrt{2}}[|\downarrow\rangle_\psi \otimes |\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_\psi \otimes |\downarrow\rangle_A \otimes |\uparrow\rangle_B] \\ &= \frac{1}{2}[\Psi_-\rangle_{\psi,A} \otimes (-a|\uparrow\rangle_B - b|\downarrow\rangle_B) + \Psi_+\rangle_{\psi,A} \otimes (-a|\uparrow\rangle_B + b|\downarrow\rangle_B) \\ &\quad + \Phi_-\rangle_{\psi,A} \otimes (b|\uparrow\rangle_B + a|\downarrow\rangle_B) + \Phi_+\rangle_{\psi,A} \otimes (-b|\uparrow\rangle_B + a|\downarrow\rangle_B)], \end{aligned} \quad (2.2.3)$$

where

$$|\Psi_\pm\rangle_{i,j} = \frac{1}{\sqrt{2}}[|\uparrow\rangle_i \otimes |\downarrow\rangle_j \pm |\downarrow\rangle_i \otimes |\uparrow\rangle_j], \quad (2.2.4)$$

$$|\Phi_\pm\rangle_{i,j} = \frac{1}{\sqrt{2}}[|\uparrow\rangle_i \otimes |\uparrow\rangle_j \pm |\downarrow\rangle_i \otimes |\downarrow\rangle_j]$$

are the Bell operator basis

$$\mathcal{B} = \sum_{\Psi_\pm, \Phi_\pm} b_k |k\rangle\langle k|, \quad (2.2.5)$$

and $b_k, k \in \{\Psi_\pm, \Phi_\pm\}$, are the observable values of this operator. Now, suppose Alice makes a measurement of the particles she has with her with the observable (2.2.5). Then, her subsystem will collapse to one of the four states (2.2.4) and Bob's state will be in one of the four states below:

$$-|\psi_B\rangle \equiv -a|\uparrow\rangle_B - b|\downarrow\rangle_B \quad (2.2.6)$$

$$-\sigma_3 |\psi_B\rangle = -a|\uparrow\rangle_B + b|\downarrow\rangle_B \quad (2.2.7)$$

$$\sigma_1 |\psi_B\rangle = +b|\uparrow\rangle_B + a|\downarrow\rangle_B \quad (2.2.8)$$

$$-\sigma_3 \sigma_1 |\psi_B\rangle = -b|\uparrow\rangle_B + a|\downarrow\rangle_B, \quad (2.2.9)$$

where σ_i are the Pauli matrices. We see that the coefficients a and b were transferred to Bob's particle instantaneously, and this is independent of the distance between Alice and Bob. Finally, to obtain a perfect copy of $|\psi\rangle$, Bob needs to make a local unitary transformation in his particle. The particle (2.2.6) is exactly $|\psi\rangle$, and thus nothing is needed to be done. The particles (2.2.7)-(2.2.9) have to be rotated 180° around z (σ_3), x (σ_1) and y ($\sigma_3 \sigma_1$) axis, respectively. Bob can only know which operation he has to make after Alice tells him what is the result of her measurement and this must be done using a classical channel (limited by the speed of light), by sending a text message, for instance, and therefore quantum teleportation is limited by the speed of light.

One may argue that when Alice's subsystem collapses to $|\Psi_-\rangle$ the quantum teleportation will be instantaneous, as Bob does not need to make any operation in his particle. We argue here that the teleportation happened only if Bob can know if Alice made the measurement or not. First notice that, without any information from Alice, Bob would not be able to make any measurement in his single state to distinguish the state before and after Alice's measurement. We can see this as follows. Suppose Alice, somehow, has N identical copies of the state $|\psi\rangle$ and Alice and Bob share N entangled particles in the state (2.2). After Alice's measurement, as the four resulting states are equally probable, Bob's final mixed state of N particle will be

$$\begin{aligned}\rho_B &= \frac{1}{4} [|\psi_B\rangle\langle\psi_B| + \sigma_3 |\psi_B\rangle\langle\psi_B| \sigma_3^\dagger + \sigma_1 |\psi_B\rangle\langle\psi_B| \sigma_1^\dagger + \sigma_3 \sigma_1 |\psi_B\rangle\langle\psi_B| (\sigma_3 \sigma_1)^\dagger] \\ &= \frac{1}{2} [|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|],\end{aligned}\tag{2.2.10}$$

where we considered $|\psi\rangle$ to be normalized, i.e., $|a|^2 + |b|^2 = 1$. We see that no information about $|\psi\rangle$ probabilities is stored in the mixed state ρ_B and, therefore, Bob cannot tell if Alice made the measurement, and thus the teleportation process cannot be done faster than the speed of light.

Only four years after the theoretical experiment was proposed, an experiment done with photon polarization states (2) confirmed the existence of quantum teleportation phenomenon. Many other experiments have been done after that and they all faced technological difficulties for testing long-distance teleportation due to photon loss in the transmission channel. The reader interested in the experiments may have a look in (3) and its references.

2.3 Entanglement Characterization

In the EPR paper, entangled particles were considered as a flaw in Quantum Mechanics but nowadays entanglement is considered a very important quantum property and the heart of Quantum Information Theory. The classical information theory is a mathematical theory that studies how to process information and it has vast applications in a variety of fields, like physics and biology, and it is the fundamental theory for computer science.

But what is information and how we quantify it? Information is related to knowledge, that is, the more information we have the more knowledge about the system we possess. As an example, suppose there are four colored balls (white, black, red, and blue) and we blindly choose one. We will have some *uncertainty* about which color we have chosen (probability of 1/4 for each color). Now someone tells us that we did not choose the white ball, thus the uncertainty decreases. We see that the knowledge given (the ball is not white) decreases the uncertainty we have about the color of the ball we chose

(now it can only be black, red or blue). Indeed, uncertainty is a main concept in classical information theory and it is an information measurement: the Shannon Entropy quantifies how much uncertainty we have about the outcome of a random process and it is given by $H(X) = -\sum_x p(x) \log p(x)$ [§], where $p(x)$ is the probability of occurrence of $x \in X$. In our example, $X = \{\text{white, black, red, blue}\}$ and $p(x) = 1/4, \forall x \in X$, and after obtaining the knowledge that the color is not white, $p(\text{white}) = 0$ and $p(x) = 1/3, x \in \{\text{black, red, blue}\}$. We see that $H_{\text{before}} = \log(4) > H_{\text{after}} = \log(3)$, thus the knowledge given decreased the uncertainty H . It is also valid to note that, when we are sure about the outcome, i.e., $p(x_i) = 1$ and $p(x_j) = 0, j \neq i$, we will have no uncertainty and $H = 0$.

Consider now that we have a system of two processes X and Y (in our example X is choosing one ball and Y could be choosing a second ball). The entropy of this system is $H(X, Y) = -\sum_{x,y} p(x, y) \log p(x, y)$. We notice that if X and Y are independent (in our example X could be choosing one ball and Y flipping a coin), the entropy of the system will be the sum of the entropies of the subsystems, $H(X, Y) = H(X) + H(Y)$, otherwise $H(X, Y) < H(X) + H(Y)$. We also have $H(X, Y) = H(X|Y) + H(Y) = H(Y|X) + H(X)$, where $H(X|Y) = -\sum_{x,y} p(x|y) \log p(x|y)$ is the entropy related to the conditional outcome of X after Y already happened, and it can be understood as the remaining uncertainty of X after we already got the knowledge about Y . We can define the *mutual information* $I(X, Y) \equiv H(X) - H(X|Y)$ as the information we gain about X after we have obtained knowledge about Y . Using the relation $H(X|Y) = H(X, Y) - H(Y)$, we have

$$I(X, Y) = H(X) + H(Y) - H(X, Y) = I(Y, X) \geq 0. \quad (2.3.1)$$

As correlation means how much we can predict about one system if we have some knowledge about another system, we see that the mutual information is a good measurement of correlation between X and Y . $I(X, Y)$ is a correlation measurement of classical system and we are interested in correlations in Quantum Mechanics. Following, we will discuss the main aspects of Quantum Information Theory and some mathematical formalism, see the quantum counterpart of the Shannon entropy, the von Neumann entropy, and see that for a general mixture of quantum states this entropy is not a good correlation measurement. For more information about the Shannon entropy see chapter 5 of (15).

Pure states

In the last section we introduced some entangled states without providing a precise definition of it. We will do it now, first for pure states. Entanglement is related to correlation between two or more particles, that is, by measurements on one particle one obtains information about another particle. The best way to define mathematically what is entanglement is to define what it is not. The Hilbert space $\mathcal{H}$ of N particles is the *tensor*

[§] It is assumed that $0 \log(0) = 0$.

product of the subspaces $\mathcal{H}_i, i \in \{0, \dots, N\}$, of each particle,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_N. \quad (2.3.2)$$

A pure state $|\psi\rangle \in \mathcal{H}$ is said to be *separable* when it can be written as a tensor product of individuals states $|\psi_i\rangle \in \mathcal{H}_i$ of each particle, i.e., $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_N\rangle$. A state is said to be *entangled* when it is not separable. We see that all bipartite states considered in the previous section are entangled and, as an example, the state $(1/\sqrt{2})[|\phi_0\rangle \otimes |\psi\rangle + |\phi_1\rangle \otimes |\psi\rangle] = (1/\sqrt{2})[(|\phi_0\rangle + |\phi_1\rangle) \otimes |\psi\rangle]$ is separable.

One can always know that a state is not entangled if one can write it in a separable form but that is, in general, not an easy task. So it is interesting to seek a way to quantify entanglement. Given a function $E(|\psi\rangle)$ that measures entanglement we would expect it to obey the following axioms[¶]:

- It is zero for separable states and different from zero for entangled states;
- It does not increase under local unitary operations.
- It is maximal for maximally entangled states;

A maximally entangled state for bipartite system is given by $|\Phi_+\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle \otimes |i\rangle$, where d is the dimension of each particle Hilbert space, and we see that the Bell state in equation (2.2.4) is maximally entangled. One necessary condition for a state of N qubits to be maximally entangled is that it maximally violates the Bell inequalities (for two dimensional bipartite system the absolute value of the operator C equals $2\sqrt{2}$). Actually it can be shown that entanglement is a necessary condition for the violation of the Bell inequalities because no separable system violates it. For more information about maximally entangled qubits system see (27). The first and the third axioms state that the function $E(|\psi\rangle)$ will not only check if the system is entangled but will also give comparison about “how much” entangled the system is. The second axiom asserts that if we make a local operation in one particle subsystem the overall entanglement of the whole system should not increase, that is, $E(\mathcal{U}_1 \otimes \dots \otimes \mathcal{U}_N |\psi\rangle) \leq E(|\psi\rangle)$, where $\mathcal{U}_i$ acts on $\mathcal{H}_i, i \in 1, \dots, N$.

From now on we will only consider bipartite system: Alice and Bob have, each, one particle in the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The *Schmidt decomposition theorem* states that for any state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ there exists a set of orthonormal states $\{|k\rangle_A\} \in \mathcal{H}_A$ and $\{|\ell\rangle_B\} \in \mathcal{H}_B$ such that $|\psi\rangle = \sum_i \alpha_i |i\rangle_A \otimes |i\rangle_B$. Thus, if $\alpha_i = 1/\sqrt{d}, \forall i$, the state is maximally entangled and if $\alpha_i = 1$ and $\alpha_j = 0, j \neq i$, the state is separable. Hence, a good measurement of entanglement would be a function of α_i that is zero when

[¶] Those are not the only conditions that are said to be necessary for a good entanglement measurement. In fact, it is still an open question which are the necessary conditions. For more information about entanglement measurements see (26).

$\alpha_i = 1$ and $\alpha_j = 0$, and maximal when $\alpha_i = 1/\sqrt{d}$ and this function is the *von Neumann entropy*^{||}:

$$S(\rho) = -\text{Tr}[\rho \log \rho], \quad (2.3.3)$$

where ρ is a density matrix of a quantum system. The density matrix of our bipartite system is $\rho_{AB} = |\psi\rangle\langle\psi|$ and the entanglement measurement is the entropy of each reduced state $\rho_A = \text{Tr}_B[\rho_{AB}] = \sum_i \alpha_i |i\rangle_A \langle i|_A$, where $\text{Tr}_B[\rho_{AB}] = \sum_i \langle i|_B \rho_{AB} |i\rangle_B$ is the partial trace of the system's density matrix over system B degrees of freedom, and $\rho_B = \text{Tr}_A[\rho_{AB}] = \sum_i \alpha_i |i\rangle_B \langle i|_B$. Therefore, the entanglement measurement of a bipartite system in pure states is

$$S(\rho_A) = S(\rho_B) = -\sum_i \alpha_i \log \alpha_i. \quad (2.3.4)$$

It is clear that $S(\rho)$ is in accordance with the third axiom due to the trace operation. For qubit bipartite systems, the logarithm function is evaluated in base 2 so that $S(\rho) = 1$ for maximally entangled systems.

Mixed states

It is important to define entanglement for the most general form of quantum states: mixed states. From now on, we are going to refer the density matrix ρ of a quantum system as a state of the system. A state ρ , that acts on the Hilbert space $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$, is said to be separable when it can be written as a sum of tensor product of individuals ρ_i^j , that acts on individuals Hilbert spaces $\mathcal{H}_i$, that is, $\rho = \sum_j p_j \rho_1^j \otimes \dots \otimes \rho_N^j$, with $\sum_j p_j = 1$, otherwise the state is entangled. The first thing to notice here is that entanglement in mixed states means quantum correlations between substates ρ_i and separable states ρ can be a mixture of non-separable states ρ_i . As an example, let $\rho = \rho_+ \otimes \rho_-$, such that $\rho_{\pm} = |\Phi_{\pm}\rangle\langle\Phi_{\pm}|$, where $|\Phi_{\pm}\rangle = (1/\sqrt{2})(|\uparrow_1\uparrow_2\rangle \pm |\downarrow_1\downarrow_2\rangle)$ are the Bell basis, we can see that ρ is a separable state, but its von Neuman entropy does not equals zero. In fact, as the density matrix represents an ensemble, it has classical correlations and the von Neumann entropy does not distinguish classical and quantum correlations and we can redefine the mutual information of equation (2.3.1) in terms of the von Neumann entropy

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \quad (2.3.5)$$

where $\rho_A = \text{Tr}_B[\rho_{AB}]$ and $\rho_B = \text{Tr}_A[\rho_{AB}]$. Now the mutual information identifies quantum and classical correlations. As the von Neumann entropy is not an entanglement measurement we need to find a new way to compute entanglement for mixed states, and this is not a simple task. As we will only deal with bipartite two level systems, we will only present

^{||} Although its form is very similar to the Shannon entropy, the von Neumann entropy is associated with probabilities over quantum systems - described by the density matrix operator - and the probabilities in Shannon entropy are classical probabilities of some random variable.

ways to compute entanglement for systems with dimension 2×2 (4×4 density matrices). For more information about how to quantify quantum entanglement see (26, 28).

In 1996, Peres proposed (29) a necessary condition for separability of bipartite systems and the Horodecki family showed that Peres criterion is also a sufficient condition for systems with dimension 2×2 and 2×3 (30) - two-particle systems with both being qubits or one qubit and one three level system. The *Peres-Horodecki criterion* states that for ρ_{AB} , that acts on $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A = \mathbb{C}^2$ and $\mathcal{H}_B \in \{\mathbb{C}^2, \mathbb{C}^3\}$, the eigenvalues of the partial transposed density matrix ρ_{AB}^{PT} are all positive if ρ_{AB} is separable. The density matrix of bipartite systems can be written as

$$\rho_{AB} = \sum_{i,j,k,\ell} \rho_{ijkl} |i\rangle_A |j\rangle_B \langle k|_A \langle \ell|_B, \quad (2.3.6)$$

and the partial transposed density matrix ρ_{AB}^{PT} is defined as

$$\begin{aligned} \rho_{AB}^{PT} &= \sum_{i,j,k,\ell} \rho_{ijkl} |i\rangle_A |\ell\rangle_B \langle k|_A \langle j|_B \\ &= \sum_{i,j,k,\ell} \rho_{ijkl} |k\rangle_A |j\rangle_B \langle i|_A \langle \ell|_B. \end{aligned} \quad (2.3.7)$$

According to the Peres-Horodecki criterion, negative eigenvalues of the partial transpose give an estimate of how entangled the state is. Based on that criterion some entanglement measurements were introduced. Here we will use the *negativity* $\mathcal{N}(\rho)$ (31) which corresponds to the absolute value of the sum of the negative eigenvalues of the partial transpose:

$$\mathcal{N}(\rho) \equiv \frac{\|\rho_{AB}^{PT}\| - 1}{2} = - \sum_{\lambda_i < 0} \lambda_i, \quad (2.3.8)$$

where $\|\rho_{AB}^{PT}\| = \text{Tr} \left[\sqrt{(\rho_{AB}^{PT})^\dagger \rho_{AB}^{PT}} \right]$ and λ_i are the eigenvalues of ρ_{AB}^{PT} .

There is another simple entanglement measurement for binary qubits systems, i.e., systems with dimension 2×2 . In 1996, Bennett *et al.* (32) proposed an entanglement measurement for mixed states which they denoted *entanglement of formation*. One year later, Hill and Wootters (33) conjectured a simpler formula for the entanglement of formation and proved it for special cases. Some months later, Wootters (34) proved that the formula is valid for arbitrary states.

The density matrix of a pair of qubits, ρ_{AB} , can be written as a decomposition of pure states of the mixture, $\rho_i = |\phi_i\rangle\langle\phi_i|$, such that

$$\rho_{AB} = \sum_i p_i |\phi_i\rangle\langle\phi_i|, \quad \sum_i p_i = 1.$$

The entanglement of formation, $E(\rho_{AB})$, is defined as the average entanglement of the pure states of the decomposition, minimized over all decompositions of ρ_{AB} :

$$E(\rho_{AB}) := \min_i \sum_i p_i S(\rho_i), \quad (2.3.9)$$

where $S(\rho)$ is the von Neumann entropy (2.3.3). The equation above can be written as

$$E(\rho_{AB}) = \mathcal{E}(\mathcal{C}(\rho_{AB})),$$

where

$$\mathcal{E}(x) := -\frac{1 + \sqrt{1 - x^2}}{2} \log_2 \frac{1 + \sqrt{1 - x^2}}{2} - \frac{1 - \sqrt{1 - x^2}}{2} \log_2 \frac{1 - \sqrt{1 - x^2}}{2},$$

and

$$\mathcal{C}(\rho_{AB}) := \max[0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4], \quad (2.3.10)$$

where λ_i 's are square root of the eigenvalues, in decreasing order, of the matrix $\rho_C := \rho_{AB}(\sigma_y \otimes \sigma_y) \rho_{AB}^*(\sigma_y \otimes \sigma_y)$, and σ_y is the Pauli y matrix. $\mathcal{C}(\rho_{AB})$ is denoted *concurrence*. The function $\mathcal{E}(\mathcal{C}(\rho_{AB}))$ is monotonically increasing, and ranges from 0 to 1 as $\mathcal{C}(\rho_{AB})$ goes from 0 to 1. Therefore, we can use the concurrence as an entanglement measurement.

3 QUANTUM FIELD THEORY IN CURVED SPACETIME

The Quantum Mechanical description introduced in the last chapter is said to be incomplete since it is a one particle theory, as it does not have a precise description of interacting particles, and it is incompatible with the theory of Special Relativity. The *Quantum Field Theory* (QFT), formulated in late 1920s, emerged from efforts to quantize the electromagnetic field and, as the classical electromagnetic theory is in total accordance with Special Relativity, QFT naturally reconciles Quantum Mechanics with Special Relativity. QFT treats *quantum fields* as the fundamental objects of Nature, in contrast to particles, and it can describe three of the four known fundamental interactions: the weak nuclear and electromagnetic interactions - which were unified in the *Electroweak* theory - and the strong nuclear interaction - which is described by the theory of *Quantum Chromodynamics*.

Gravity is the fourth fundamental interaction and it remains a *classical* field theory and it is described by the theory of *General Relativity* (GR). GR is a very successful theory and all the astronomical observations made so far are in total accordance with it - the most recent confirmation of GR was the direct detection of gravitational wave made by LIGO and Virgo collaboration in 2016. Although QFT and GR are both very successful, they are not compatible as, in contrast to QFT, GR is a deterministic theory. As Classical Mechanics is a good approximation of Quantum Mechanics in a macroscopic scale, GR is believed to be a very good approximation for macroscopic scales, where macroscopic means scales of space and time greater than the Planck scales*.

The *Quantum Field Theory in Curved Spacetime* (QFT in CST), also known as *semi-classical gravity*, is a theory that tries to make predictions of quantum effects in curved spacetimes, where the matter fields are quantized over a curved background geometry. Quantizing gravity is not the aim of this theory and, as no satisfactory quantum gravity theory has already been developed, QFT in CST is believed to be a good approximation of the quantum phenomena where gravity plays a role, as the semi-classical approach of the electromagnetic theory once predicted phenomena that are in total accordance with Quantum Electrodynamics.

In this chapter we will give a brief review of QFT in CST and then discuss two phenomena: the *Unruh effect* and the *Hawking effect*. Although the Unruh effect is a phenomenon in flat spacetime, it is a first step beyond inertial reference frames - that

* The Planck units are written in terms of fundamental constants: the speed of light c and the Gravitational constant G , that represents the structure of spacetime of GR, and the Planck constant $\hbar$, which represents the quantum nature of the system. The *Planck length* is $\ell_p \equiv \sqrt{\frac{\hbar G}{c}} \approx 10^{-35}\text{m}$ and the *Planck time* is $t_p \equiv \frac{\ell_p}{c} \approx 10^{-43}\text{s}$.

are usually considered in standard QFT - and it shows that the concept of particles is observer dependent as an accelerated observer can detect particles in the “vacuum” state of an inertial observer. The Hawking effect is the most well known phenomenon of QFT in CST. Hawking had shown that the quantization of matter fields in the spacetime of a collapsing star results in the existence of a *thermal flux* of particles emitted from the resulting black hole and thus the mass and the area of the black hole should decrease. This evaporation would last until the black hole completely vanishes or until it reaches the Planck scale, when quantum gravity effects would be relevant.

From now on, we will consider *natural units*, in which $G = c = \hbar = k_B = 1$, and the metric signature $(+, -, -, -)$, unless stated otherwise. For a detailed study of QFT in CST see the standard books (35–37).

3.1 Canonical Formulation

We will present the canonical formalism of QFT in CST for the simplest field: the *free scalar field*[†] Φ . Although a toy model, the free scalar field introduces many features we find in real fields, like the Dirac field, which represents spin-1/2 particles. The canonical formulation of QFT is a quantization process of the classical theory of fields that intends to preserve its formal structures. To consider this canonical formulation in a generic spacetime (see Appendix A for a brief notation review) we must replace the partial derivatives ∂_μ by covariant derivatives ∇_μ , the flat spacetime Minkowski metric $\eta_{\mu\nu}$ by a general metric $g_{\mu\nu}$ and the volume element d^4x by $d^4x\sqrt{-g}$, $g := \det(g_{\mu\nu})$.

The field equations can be obtained by the *Hamilton principle* which postulates that the classical system is described by the stationary solutions of the action - that is $\delta S/\delta\Phi|_{\Phi=\phi} = 0$. The action of the free scalar field is given by

$$S[g_{\mu\nu}, \Phi] = \frac{1}{2} \int d^4x \sqrt{-g} \left[\nabla^\mu \Phi \nabla_\mu \Phi - (m^2 + \xi R) \Phi^2 \right], \quad (3.1.1)$$

where R is the Ricci scalar, m is the mass of the field and $\xi \in \mathbb{R}$. Imposing the Hamilton principle, the *Klein-Gordon equation* is given by

$$(\square + m^2 + \xi R)\phi = 0, \quad (3.1.2)$$

where $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$. We see that the field ϕ is coupled with the spacetime curvature by the covariant derivatives and the term ξR . In the canonical formulation the conjugate momentum is given by

$$\Pi := (-g)^{-1/2} \frac{\delta S}{\delta(n^\mu \nabla_\mu \Phi)} = n^\mu \nabla_\mu \Phi, \quad (3.1.3)$$

[†] It is “free” in the sense that it does not have any interactions besides the interaction with the gravitational field.

where n^μ is a vector in the direction of the future time, i.e., orthonormal to the spatial hypersurface Σ_t . Everything stated above is valid for *classical* field theory in curved spacetime. To “make” it a *quantum* theory we take Φ and Π to be self-adjoint operators that satisfy the *Heisenberg commutation relations*:

$$\begin{aligned} [\Phi(t, \mathbf{x}), \Phi(t, \mathbf{x}')] &= [\Pi(t, \mathbf{x}), \Pi(t, \mathbf{x}')] = 0, \\ [\Phi(t, \mathbf{x}), \Pi(t, \mathbf{x}')] &= i\delta^{(3)}(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (3.1.4)$$

Now we will mathematically construct a solution for the operator Φ that satisfies the Klein-Gordon equation (3.1.2) and later we will give physical interpretations. Given the Klein-Gordon inner product

$$(f, g)_{\text{KG}} \equiv i \int_{\Sigma_t} d^3x \sqrt{h} n^\mu (f^* \nabla_\mu g - g \nabla_\mu f^*), \quad (3.1.5)$$

where $h := \det(h_{ij})$ and h_{ij} is the induced metric of the hypersurface Σ_t , consider a complete set of orthonormal solutions, $\{u_\alpha(x), u_\alpha^*(x)\}_{\alpha \in \mathcal{J}}$, of the Klein-Gordon equation (3.1.2) satisfying

$$(u_\alpha, u_\beta)_{\text{KG}} = \delta(\alpha, \beta), \quad (u_\alpha^*, u_\beta^*)_{\text{KG}} = -\delta(\alpha, \beta), \quad (u_\alpha, u_\beta^*)_{\text{KG}} = 0. \quad (3.1.6)$$

This complete set is a basis for the space of complex solutions of the Klein-Gordon equation (3.1.2). The subspace spanned by $\{u_\alpha(x)\}_{\alpha \in \mathcal{J}}$, and endowed with inner product (3.1.5), can be completed to obtain a Hilbert space $\mathcal{H}$ and, with it, one can construct the associated symmetric *Fock space*

$$\mathcal{F}(\mathcal{H}) \equiv \bigoplus_{n=0}^{\infty} (\mathcal{H}^{\otimes n})_s = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})_s \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})_s \oplus \dots, \quad (3.1.7)$$

where the label s means symmetry in the tensor product. An arbitrary element of this space takes the form

$$|\Psi\rangle = (\psi_0, \psi_1, \psi_2, \psi_3, \dots), \quad (3.1.8)$$

where ψ_n is a n -point function - symmetric points in the spacetime - generated by symmetric products of the basis $u_{(\alpha_1} \otimes \dots \otimes u_{\alpha_n)}^\dagger$. We can define two operators a_α and $a_\alpha^\dagger$, $\alpha \in \mathcal{J}$, by

$$a_\alpha u_{(\alpha_1} \otimes \dots \otimes u_{\alpha_n)} \equiv \sqrt{n} (u_\alpha, u_{(\alpha_1})_{\text{KG}} u_{\alpha_2} \otimes \dots \otimes u_{\alpha_n}), \quad (3.1.9)$$

$$a_\alpha \mathbb{C} \equiv \{0\}, \quad (3.1.10)$$

$$a_\alpha^\dagger u_{(\alpha_1} \otimes \dots \otimes u_{\alpha_n)} \equiv \sqrt{n+1} u_{(\alpha} \otimes u_{\alpha_1} \otimes \dots \otimes u_{\alpha_n)}. \quad (3.1.11)$$

We call a_α and $a_\alpha^\dagger$ *annihilation* and *creation* operators, respectively. The notation is justified, i.e., $a_\alpha^\dagger$ is adjoint of a_α , as the following relation is satisfied:

$$(u_{(\alpha_1} \otimes \dots \otimes u_{\alpha_{n-1}}), a_\alpha u_{(\beta_1} \otimes \dots \otimes u_{\beta_n)})_{\text{KG}} = (a_\alpha^\dagger u_{(\alpha_1} \otimes \dots \otimes u_{\alpha_{n-1}}), u_{(\beta_1} \otimes \dots \otimes u_{\beta_n)})_{\text{KG}},$$

[‡] The parenthesis means symmetrization over the indices, that is, for an expression X that depends on the n indices $\alpha_1, \dots, \alpha_n$, $X_{(\alpha_1, \dots, \alpha_n)} = \frac{1}{n!} \sum_p X_{\alpha_{p_1} \dots \alpha_{p_n}}$, where $(p_1, \dots, p_n)$ is a permutation of $(1, \dots, n)$ and the sum is over all possible permutations.

where the inner product of the basis is defined as

$$(u_{\alpha_1} \otimes \dots \otimes u_{\alpha_n}, u_{\beta_1} \otimes \dots \otimes u_{\beta_n})_{\text{KG}} \equiv (u_{\alpha_1}, u_{\beta_1})_{\text{KG}} \dots (u_{\alpha_n}, u_{\beta_n})_{\text{KG}}.$$

From the definitions (3.1.9)-(3.1.11), one finds that

$$[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta(\alpha, \beta), \quad [\hat{a}_\alpha, \hat{a}_\beta] = [\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger] = 0. \quad (3.1.12)$$

We define the *vacuum state*, $|0\rangle$, by requiring that $a_\alpha |0\rangle = 0, \forall \alpha \in \mathcal{J}$, and it is given by

$$|0\rangle = (1, 0, 0, \dots).$$

Notice that any state $|\Psi\rangle \in \mathcal{F}(\mathcal{H})$ can be obtained from $|0\rangle$ using the operators $a_\alpha^\dagger$. Lastly, we define the *number operator* as

$$N \equiv \int_{\mathcal{J}} d\mu(\alpha) a_\alpha^\dagger a_\alpha, \quad (3.1.13)$$

where $d\mu(\alpha)$ is a measure on the index set $\mathcal{J}$. For an arbitrary state, $|\Psi\rangle \in \mathcal{F}(\mathcal{H})$, we have $N|\Psi\rangle = N(\psi_0, \psi_1, \psi_2, \dots, \psi_n, \dots) = (0, \psi_1, 2\psi_2, \dots, n\psi_n, \dots)$.

Finally, we can construct the quantum field operator as

$$\Phi \equiv \int_{\mathcal{J}} d\mu(\alpha) [\hat{a}_\alpha u_\alpha + \hat{a}_\alpha^\dagger u_\alpha^*], \quad (3.1.14)$$

where we note that $a_\alpha = (u_\alpha, \Phi)_{\text{KG}}$ and $a_\alpha^\dagger = (\Phi, u_\alpha)_{\text{KG}}$. From the annihilation and creation operators commutation relations (3.1.12), we can see that the field (3.1.17) satisfies the Heisenberg commutation relations (3.1.4).

Now we give physical interpretations to this field solution. We can recognize the commutation relations of the operators a_α and $a_\alpha^\dagger$ to be exactly the same as for the many-particle boson system creation and annihilation operators, which physically “creates” and “annihilates” particles, and the number operator N is an observable of the total particle number of the system. In fact, the operators a_α and $a_\alpha^\dagger$ of the field operator Φ can be interpreted in the same way, but that is not the general case since this interpretation depends on the form of u_α .

In Minkowski flat spacetime we take $\alpha = \mathbf{k}$ to be the total momentum of the particles of the field, $\mathcal{J} = \mathbb{R}^3$ and $d\mu(\alpha) = d^3\mathbf{k}$. Then, we can choose

$$u_{\mathbf{k}} = \frac{e^{-i(\omega_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})}}{(2\pi)^{3/2} \sqrt{2\omega_{\mathbf{k}}}}, \quad (3.1.15)$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$ and m is the field mass. We call *positive-frequency modes* those modes which are proportional to $e^{-i\omega t}$ and *negative-frequency modes* those modes which are proportional to $e^{i\omega t}$, and we see that the annihilation and creation operators are coefficients of the positive- and negative-frequency modes, respectively. We notice that

the time coordinate t in the positive- and negative-frequency modes is related to the Minkowski global timelike Killing vector field $\partial/\partial t$, which means that there is time-translation symmetry. The Killing vector field $\partial/\partial t$ is not the only timelike Killing vector field of the Minkowski spacetime and one can write the field solution as a function of the positive- and negative-frequency modes with respect to this second timelike Killing vector defining a second *distinct* vacuum state of real particles, and this means that the concept of particle is observer dependent - this problem will be presented in detail in the **Unruh Effect** section.

But how does one define particles in a generic spacetime? For static spacetimes there is a straightforward generalization of the Minkowski case. There always exists, by definition, a timelike Killing vector field $\partial/\partial t$ associated to a static spacetime, with metric that takes the form of (A.0.2), ensuring time-translation symmetry. The Klein-Gordon equation (3.1.2) of a static spacetime is given by

$$\left(\frac{\partial^2}{\partial t^2} - \mathcal{K}(x)\right)\phi = 0, \quad (3.1.16)$$

where

$$\mathcal{K}(x)\phi = f(x) \left[\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} h^{ij} \frac{\partial \phi}{\partial x^j} \right) + (m^2 + \xi R)\phi \right].$$

As in Minkowski spacetime, the coefficients a_α of the positive-frequency solutions - which are proportional to $e^{-i\omega_i t}$, where ω_i is interpreted as the energy of the particle with respect to the Killing vector field $\partial/\partial t$ - are going to define the physical vacuum, $|0\rangle$ ($a_\alpha |0\rangle = 0$), and we call it *static vacuum*. Usually, there are no straightforward generalization of the particle concept for general spacetimes, as the non-existence of time-translation symmetry prevents one to define positive- and negative-frequency solutions of the Klein-Gordon equation.

Bogoliubov transformations

Given the timelike Killing vector field, $\frac{\partial}{\partial t}$, of a spacetime, there are positive-frequency solutions $\{u_\alpha\}$, $\alpha \in \mathcal{J}$, of the Klein-Gordon equation (3.1.2), which form a basis of a Hilbert space $\mathcal{H}$ with respect to which one constructs the associated Fock space $\mathcal{F}(\mathcal{H})$, such that the quantum field Φ is given by

$$\Phi = \int_{\mathcal{J}} d\mu(\alpha) [a_\alpha u_\alpha + a_\alpha^\dagger u_\alpha^*], \quad (3.1.17)$$

where the operators a_α define the vacuum state $|0\rangle$. On the other hand, there can be another timelike Killing vector field, $\frac{\partial}{\partial t'}$, of the spacetime that gives a new set of positive-frequency solutions $\{v_{\alpha'}\}_{\alpha' \in \mathcal{J}'}$ of the Klein-Gordon equation (3.1.2), which form a basis of another Hilbert space $\widetilde{\mathcal{H}}$ with which one constructs the associated Fock space $\mathcal{F}(\widetilde{\mathcal{H}})$, such

that the quantum field $\tilde{\Phi}$ is given by

$$\tilde{\Phi} = \int_{\mathcal{J}'} d\mu(\alpha') [b_{\alpha'} v_{\alpha'} + b_{\alpha'}^\dagger v_{\alpha'}^*], \quad (3.1.18)$$

where the operators $b_{\alpha'}$ define a new vacuum state $|\tilde{0}\rangle$. The bases $\{u_{\alpha}\}$ and $\{v_{\alpha'}\}$ can be related by linear combination (including their complex conjugates, $\{u_{\alpha}^*\}$, $\{v_{\alpha'}^*\}$)

$$v_{\alpha'} \equiv \int_{\mathcal{J}} d\mu(\alpha) [c_{\alpha'\alpha} u_{\alpha} + d_{\alpha'\alpha} u_{\alpha}^*] \Leftrightarrow v_{\alpha'}^* = \int_{\mathcal{J}'} d\mu(\alpha') [c_{\alpha'\alpha'}^* u_{\alpha'}^* + d_{\alpha'\alpha'}^* u_{\alpha'}], \quad (3.1.19)$$

with the inverse transformations

$$u_{\alpha} = \int_{\mathcal{J}'} d\mu(\alpha') [c_{\alpha'\alpha}^* v_{\alpha'} - d_{\alpha'\alpha} v_{\alpha'}^*] \Leftrightarrow u_{\alpha}^* = \int_{\mathcal{J}} d\mu(\alpha) [c_{\alpha'\alpha} v_{\alpha'}^* - d_{\alpha'\alpha}^* v_{\alpha'}]. \quad (3.1.20)$$

Those are called *Bogoliubov transformations* and, as both bases satisfy the orthonormal relations (3.1.6), it follows that the *Bogoliubov coefficients* $c_{\alpha'\alpha}$ e $d_{\alpha'\alpha}$ satisfy

$$\int_{\mathcal{J}} d\mu(\alpha) [c_{\alpha'\alpha}^* c_{\beta'\alpha} - d_{\alpha'\alpha}^* d_{\beta'\alpha}] = \delta(\alpha', \beta'), \quad (3.1.21)$$

$$\int_{\mathcal{J}'} d\mu(\alpha') [c_{\alpha'\alpha} c_{\alpha'\beta}^* - d_{\alpha'\alpha}^* d_{\alpha'\beta}] = \delta(\alpha, \beta), \quad (3.1.22)$$

$$\int_{\mathcal{J}} d\mu(\alpha) [c_{\alpha'\alpha} d_{\beta'\alpha} - d_{\alpha'\alpha} c_{\beta'\alpha}] = \int_{\mathcal{J}'} d\mu(\alpha') [c_{\alpha'\alpha}^* d_{\alpha'\beta} - d_{\alpha'\alpha}^* c_{\alpha'\beta}] = 0. \quad (3.1.23)$$

Suppose there is a unitary operator, $\mathcal{U}$, such that we can relate the Fock spaces $\mathcal{F}(\mathcal{H}) \rightarrow \mathcal{F}(\tilde{\mathcal{H}})$, in such a way that, $\tilde{\Phi} = \mathcal{U}\Phi\mathcal{U}^{-1}$. With this relation and the Bogoliubov transformations, we have

$$\mathcal{U}^{-1} b_{\alpha'} \mathcal{U} = \int_{\mathcal{J}} d\mu(\alpha) [c_{\alpha'\alpha}^* a_{\alpha} - d_{\alpha'\alpha}^* a_{\alpha}^{\dagger}], \quad (3.1.24)$$

$$\mathcal{U}^{-1} b_{\alpha'}^{\dagger} \mathcal{U} = \int_{\mathcal{J}} d\mu(\alpha) [c_{\alpha'\alpha} a_{\alpha}^{\dagger} - d_{\alpha'\alpha} a_{\alpha}]. \quad (3.1.25)$$

Finally, we can evaluate the expected value of the number operator of the Fock space $\mathcal{F}(\tilde{\mathcal{H}})$ calculated in the vacuum state of $\mathcal{F}(\mathcal{H})$, and it is given by[§]

$$\langle \tilde{N} \rangle_0 = \int_{\mathcal{J}'} d\mu(\alpha') \langle 0 | \mathcal{U}^{-1} b_{\alpha'}^{\dagger} b_{\alpha'} \mathcal{U} | 0 \rangle = \int_{\mathcal{J}'} d\mu(\alpha') \int_{\mathcal{J}} d\mu(\alpha) |d_{\alpha\alpha'}|^2. \quad (3.1.26)$$

We see that if $|d_{\alpha\alpha'}|^2 \neq 0$ the vacuum state defined by one Killing vector field is populated with respect to the other Killing vector field. Therefore, in contrast with the one particle Quantum Mechanical theory, now the number of particles is observer-dependent.

3.2 Unruh Effect

We saw in the last section that the concept of particle is relative and we will see that even in Minkowski spacetime two observers can disagree with the vacuum concept.

[§] In general, when the unitary operation $\mathcal{U}$ does not exist $\langle \tilde{N} \rangle_0$ is divergent.

Figure 3 – *Right Rindler wedge*: region defined by $|t| < z$ in Minkowski spacetime. It is a static spacetime itself. The hyperboles represent the wordlines of uniformly accelerated observers, which are static in the coordinates $\{(\tau, \xi)\}$ defined in (3.2.2).

Source: By the author.

For simplicity, we will deal with two-dimensional Minkowski spacetime and its line element - in inertial Cartesian coordinates (t, z) - is given by

$$ds^2 = dt^2 - dz^2. \quad (3.2.1)$$

This spacetime has three independent Killing vector fields, and they represent time translation, space translation and boost - those are the familiar transformations for inertial frames in Spacial Relativity. Two of these are timelike Killing vector fields (at least in some region of the spacetime). The time translation Killing vector field is given by $\frac{\partial}{\partial t}$ and its positive-frequency modes are given by (3.1.15) in two-dimensions, where $\mathbf{x} = (z)$ and $\mathbf{k} = (k_z)$. The boost Killing vector field is given by $z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}$ and we are willing to find positive-frequency modes relative to it. First, we must try to find coordinates in which the boost Killing vector field takes the simple form $\frac{\partial}{\partial \tau}$, where τ is a time parameter. Then, the positive-frequency modes will be proportional to $e^{-i\omega\tau}$. We can achieve that by considering the coordinates (τ, ξ) such that

$$t = \frac{e^{a\xi}}{a} \sinh(a\tau), \quad z = \frac{e^{a\xi}}{a} \cosh(a\tau), \quad (3.2.2)$$

where a is a positive constant. These are called *Rindler coordinates* and they represent a set of uniformly accelerated observers, $\mathcal{A}$, and τ is the proper time of an observer with acceleration a at $\xi = 0$. The worldline of these observers are hyperboles, as shown in Figure 3, and we denote the region covered by coordinates (τ, ξ) - the region where $|t| < z$ in Minkowski spacetime - *right Rindler wedge*, and it is itself a static spacetime[¶]. The line

[¶] One could have defined the coordinates (τ', ξ') such that $t = a^{-1}e^{a\xi'} \sinh(a\tau')$, $z = -a^{-1}e^{a\xi'} \cosh(a\tau')$, and these coordinates would define a spacetime in the region $|t| < -z$ called *left Rindler wedge*.

element is now given by

$$ds^2 = e^{2a\xi} (d\tau^2 - d\xi^2) \quad (3.2.3)$$

and the boost Killing vector field takes the form $\frac{\partial}{\partial \tau}$. Hence, we expect that the particle concept of accelerated observers, $\mathcal{A}$, is given by the annihilator operators b_α which are coefficients of the positive-frequency modes $v_\alpha \propto e^{-i\omega_\alpha \tau}$. That is, in the reference frame of the set $\mathcal{A}$, the field operator can be written in the form

$$\tilde{\Phi} = \int d\mu(\alpha) [v_\alpha b_\alpha + v_\alpha^* b_\alpha^\dagger]. \quad (3.2.4)$$

We will find the Bogoliubov coefficients as follows. First, let us replace the coordinates (t, z) in equation (3.1.15) by the Rindler coordinates (3.2.2). We just need to deal with the exponential term of the positive-frequency mode:

$$e^{-i(\omega_{k_z} t - k_z z)} = e^{-i \frac{e^{a\xi}}{a} (\omega_{k_z} \sinh(a\tau) - k_z \cosh(a\tau))} = e^{-i \frac{e^{a\xi}}{a} m \sinh(a\tau - \kappa)}, \quad (3.2.5)$$

where in the last step we made the transformation $k_z = m \sinh(\kappa)$, and, thus, $\omega_{k_z} \equiv \sqrt{m^2 + k_z^2} = m \cosh(\kappa)$. Using the direct and inverse Fourier transforms in the variable τ , we have:

$$e^{i(\omega_{k_z} t - k_z z)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{-i\omega\tau} \int_{-\infty}^{+\infty} d\tau' e^{i\omega\tau'} e^{-i \frac{e^{a\xi}}{a} m \sinh(a\tau' - \kappa)}. \quad (3.2.6)$$

Making two coordinate transformations, first $\tau' := (\tilde{\tau} + \kappa)/a$ and then $\lambda := e^{\tilde{\tau}}$, we have

$$e^{i(\omega_{k_z} t - k_z z)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{-i\omega(\tau - \kappa/a)} \int_{-\infty}^{+\infty} d\lambda \lambda^{i\omega/a - 1} e^{-i \frac{e^{a\xi}}{a} m(\lambda - \lambda^{-1})/2}. \quad (3.2.7)$$

The result of the integral over λ can be found in a table of integrals^{||}, and it is given in terms of the Bessel function $K_\nu(z)$:

$$e^{i(\omega_{k_z} t - k_z z)} = \frac{1}{a\pi} \int_{-\infty}^{+\infty} d\omega e^{-i\omega(\tau - \kappa/a)} K_{\frac{i\omega}{a}} \left(\frac{m}{a} e^{a\xi} \right) e^{\frac{\pi\omega}{2a}} \quad (3.2.8)$$

$$= \frac{1}{a\pi} \int_0^\infty d\omega K_{\frac{i\omega}{a}} \left(\frac{m}{a} e^{a\xi} \right) \left[e^{\frac{\omega}{a}(\pi/2 + i\kappa)} e^{-i\omega\tau} + e^{-\frac{\omega}{a}(\pi/2 + i\kappa)} e^{i\omega\tau} \right]. \quad (3.2.9)$$

We clearly see that the positive-frequency mode of inertial observers is a mixture of the positive- and negative-frequency modes of the accelerated observers, like the Bogoliubov transformations (3.1.19), and the Bogoliubov coefficients are

$$c_{k_z\omega} = f(k_z; \omega) e^{\frac{\omega}{a}(\pi/2 + i\kappa)}, \quad (3.2.10)$$

$$d_{k_z\omega} = f(k_z; \omega) e^{-\frac{\omega}{a}(\pi/2 + i\kappa)}, \quad (3.2.11)$$

where $f(k_z; \omega)$ is some real function. Using equation (3.1.22), we have

$$\begin{aligned} & \int dk_z (c_{k_z\omega} c_{k_z\omega'}^* - d_{k_z\omega}^* d_{k_z\omega'}) = \delta(\omega - \omega') \\ \Rightarrow & \int dk_z f(k_z; \omega) f(k_z; \omega') e^{i\frac{\kappa}{a}(\omega - \omega')} \left[e^{\frac{\pi}{2a}(\omega + \omega')} - e^{-\frac{\pi}{2a}(\omega + \omega')} \right] = \delta(\omega - \omega') \end{aligned}$$

^{||} Equation EH II 82(24), or 10 from section 3.471, of (38).

$$\Rightarrow \int dk_z f(k_z; \omega) f(k_z; \omega') e^{i \frac{\kappa}{a} (\omega - \omega')} = \frac{\delta(\omega - \omega')}{e^{\frac{\pi}{2a}(\omega + \omega')} - e^{-\frac{\pi}{2a}(\omega + \omega')}}.$$

Finally, the average number of particles, between $\omega + d\omega$, the accelerated observers measure in the inertial-observer's vacuum is

$$\begin{aligned} \langle N \rangle &= \int dk_z d_{k_z \omega} d_{k_z \omega}^* = \int dk_z f(k_z; \omega) f(k_z; \omega) e^{-\frac{\pi}{a} \omega} \\ &= n_{BE}(\omega, a/2\pi) \delta(0) \end{aligned} \quad (3.2.12)$$

where $n_{BE}(\omega, T) \equiv (e^{\omega/T} - 1)^{-1}$ is the *Bose-Einstein thermal distribution* with temperature T . Notice that the average number of particles is divergent. This is due to the infinite volume of the space.

Considering a finite volume of the space, the result of the Unruh effect is exactly the same result one gets when calculating the average number of particles of non-interacting boson system at the temperature $T = \frac{a}{2\pi} = \frac{\hbar a}{2\pi k_B c}$. The interpretation of this result is that the uniformly accelerated observer in Minkowski “feels” a thermal bath of particles. This effect is called *Unruh effect* and it shows that the concept of particles is observer dependent. A detailed review of the Unruh effect can be found in (39).

3.3 Hawking Effect

The Unruh effect shows us that, in Minkowski spacetime, the concept of particles is observer dependent. But what would happen if we deal with curved spacetimes. Imagine there is an asymptotically static flat spacetime; e.g., consider the simple example where in the past and in the future the spacetime metric is Minkowski, and it is curved in mid time. In general, one would not expect the initial vacuum state, defined by the annihilation operator related to the past positive-frequency modes, to be the same vacuum defined in the future flat spacetime. This means that when the metric is not static one expects the creation of particles, codified in the Bogoliubov coefficients.

In the 1960s (40), Leonard Parker made a breakthrough discovery: the creation of particles in expanding universe. He showed that a spacetime, initially in the vacuum state, spontaneously produces particles due to the time dependence of the spacetime. His discovery not only helps to explain problems in cosmology but actually gave birth to the extension of Quantum Field Theory - that was well established in the Minkowski spacetime - to the context of General Relativity. In fact, Parker is considered, by many, the founder of QFT in CST.

A similar result, known as the Hawking effect (41), is the most influential result in QFT in CST, as it revealed a connection between thermodynamics and General Relativity. The Hawking effect shows that a black hole acts like hot bodies, as they can create and emit particles according to a thermal spectrum. The discovery of this effect is crucial for

Figure 4 – Schematic picture of black hole formation: a distribution of matter collapses into a black hole. After the collapse the frequency mode evolves, and, thus, in the future, the external observer “sees” a flux of particles from the black hole. Source: By the author.

two reasons: First, it was believed that black holes were objects that absorb everything near them, but do not emit anything. Indeed, Hawking proved that, classically, the surface area of the event horizon cannot decrease. Second, without Hawking effect, black holes pose a serious threat to the second law of thermodynamics. The reader interested in a historical approach of this problem with thermodynamics may have a look at (42).

The Hawking effect is revealed when one considers the formation of a black hole. During the process of the collapsing matter, the spacetime is not static, and one would expect the vacuum state of the past - before the collapse - to be distinct from the vacuum of the future - when the black hole has already formed. And thus, the vacuum of the past will no longer represent a system with no particle. A diagram of a collapse is pictured in figure Figure 4.

For simplicity, we will consider an unrealistic model of collapse. It consists in a spherical shell of energy, with mass M , contracting at the speed of light, and resulting in a non-rotating uncharged black hole, as pictured in Figure 4. Due to the “no hair” theorems - which state that a black hole is fully characterized by its mass, charge and angular momentum, and does not depend on any additional quantum number - this result is expected to be general, and this is confirmed by other calculations and numerical simulations.

The spherical shell separates the spacetime in two regions: flat interior solution

Figure 5 – Spacetime diagram of a spherical shell collapse. The shell separates the spacetime in two regions: Schwarzschild exterior solution and a flat interior solution. The dotted curves represent light-rays trajectories. After the collapse the past positive-frequency modes evolve to $u_{\text{future}}^{(+)} \propto \int d\tilde{\omega} (c_{\omega\tilde{\omega}} e^{-i\tilde{\omega}\tilde{t}} + d_{\omega\tilde{\omega}} e^{i\tilde{\omega}\tilde{t}})$. Source: By the author.

and Schwarzschild exterior solution^{**}. In spherical coordinates, the line element is given by

$$ds^2 = \begin{cases} dt^2 - dr^2 - r d\Omega^2, & \text{inside the shell,} \\ \left(1 - \frac{2M}{\tilde{r}}\right) d\tilde{t}^2 - \left(1 - \frac{2M}{\tilde{r}}\right)^{-1} d\tilde{r}^2 - \tilde{r}^2 d\tilde{\Omega}^2, & \text{outside the shell.} \end{cases}$$

To find the trajectory of the shell, we first obtain radial ($d\Omega^2 = 0$) trajectories of light-rays. Light-rays trajectories satisfy $ds^2 = 0$. Thus, for the inside metric, we have

$$dt^2 = dr^2 \iff t = \pm r + \text{constant},$$

where the plus and minus sign represents, respectively, light-rays outgoing to and incoming from infinity. We now take $t = 0$ at the moment of the formation of the event horizon – the moment at which the last light-ray that will not be trapped in the formed black hole passes through the center of the sphere. At this moment, the radius of the shell is $4M$ (see Figure 5), and, thus, the trajectory of the shell is given by $R_i(t) := r = -t + 4M$.

Calculating light-ray trajectories outside the shell, we have

$$d\tilde{t}^2 = \left(1 - \frac{2M}{\tilde{r}}\right)^{-2} d\tilde{r}^2 \iff \tilde{t} = \pm \left[\tilde{r} + 2M \ln \left| \frac{\tilde{r} - 2M}{2M} \right| + \text{constant} \right], \quad (3.3.1)$$

where, again, the plus and minus sign represents, respectively, light-rays outgoing to and incoming from infinity. We see that $\tilde{t} \rightarrow \infty$ as $\tilde{r}$ approaches $2M$. This means that static

^{**}This is a consequence of the Birkhoff theorem, as we take the mass $M = 0$ inside the shell. Notice that this is in total accordance with Newtonian mechanics, since flat spacetime means vanishing gravitational field.

observers outside the shell never really “see” the shell crossing the event horizon. Thus, as it is possible to see in Figure 5, outside observers in an arbitrarily far future will be receiving information (information in the sense of any signal sent at the speed of light) from moments arbitrarily close to the formation of the black hole. Thus, the trajectory of the collapsing shell near $\tilde{r} \approx 2M$ is

$$R_o(\tilde{t}_s) := \tilde{r}_s \approx 2M(1 + e^{-\tilde{t}_s/(2M)}), \quad (3.3.2)$$

where the approximation $\ln|x| \gg x$ for $|x| \ll 1$ was used.

We impose that the area of the shell is the same measured from inside and outside the shell, i.e., the radial coordinates r and $\tilde{r}$ have the same physical meaning and must be continuous through the shell (i.e., we can write $\tilde{r} = r$). This means that, as the shell approaches the event horizon, $R_i(t) = R_o(\tilde{t}_s) \Rightarrow t = 2M(1 - e^{-\tilde{t}_s/(2M)})$ at the junction of the metrics.

This relationship is valid when the radial coordinate of the exterior observer is near the event horizon. But we are interested in signals received in arbitrary coordinates $(\tilde{t}, r)$. Consider an outgoing light-ray trajectory which starts at $(\tilde{t}_s, R_o(\tilde{t}_s))$ and reaches $(\tilde{t}, r)$. Using equations (3.3.1) and (3.3.2), we have

$$\begin{aligned} \tilde{t} - \tilde{t}_s &= r + 2M \ln \left| \frac{r - 2M}{2M} \right| - R_o(\tilde{t}_s) - 2M \ln \left| \frac{R_o(\tilde{t}_s) - 2M}{2M} \right| \\ &= r + 2M \ln \left| \frac{r - 2M}{2M} \right| - 2M \left(1 + e^{-\tilde{t}_s/(2M)} \right) + \tilde{t}_s. \end{aligned} \quad (3.3.3)$$

For very large $\tilde{t}$ and r , that is, far away from the formed black hole and far in the future, $\tilde{t}_s \approx (\tilde{t} - r)/2$.

Now, let us work with the positive-frequency mode of the background quantum field. For simplicity, we will work with massless free scalar field. Consider that inside the shell, before the collapse, the positive-frequency mode $u_{\text{past}}^{(+)} \propto e^{-i\omega t}$ is stationary. We want to write the positive-frequency past mode as a linear combination of the positive- and negative-frequency of the (far away) exterior future mode, $v_{\text{future}}^{(\pm)} \propto e^{\pm i\tilde{\omega}\tilde{t}}$, and find the Bogoliubov coefficient that codifies the average number of particles in the far future.

First, a past incoming positive-frequency mode is proportional to $e^{-i\omega(t+r)}$, and then, after passing the center of the shell, it becomes an outgoing mode proportional to $e^{-i\omega(t-r)}$, and this is the mode we are interested in for the Bogoliubov coefficients. As we are dealing with free fields, there will be no interactions between the shell and the mode. Hence, at the shell, we have

$$e^{-i\omega(t-r)} \Big|_{\text{shell}} = e^{-i\omega(t-R_i(t))} \Big|_{\text{shell}} = e^{-i2\omega(t-2M)} \Big|_{\text{shell}} = e^{-i\omega 4M(e^{-\tilde{t}_s/(2M)})} \Big|_{\text{shell}},$$

Figure 6 – Contour in the complex plane to solve the integral $\int_{\Xi} dz z^{-4iM\tilde{\omega}-1} e^{-z}$.
Source: By the author.

where we made the coordinate transformation $t = 2M(1 - e^{-\tilde{t}_s/(2M)})$ in order to express the value of the mode in terms of external observer coordinates at $\tilde{t} = \tilde{t}_s$ and $r = 2M(1 + e^{-\tilde{t}_s/(2M)})$. Assume that there is no back reflection so that the phase of the mode is constant along outgoing light-ray trajectories. The mode far away from the black hole in the far future is given by $e^{4iM\omega e^{-(\tilde{t}-r)/(4M)}}$. To write this mode as a linear combination of the stationary modes $v_{\text{future}}^{(\pm)} \propto e^{\pm i\tilde{\omega}\tilde{t}}$, we make a direct and inverse Fourier transform in the variable $\tilde{t}$:

$$e^{4iM\omega e^{-(\tilde{t}-r)/(4M)}} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tilde{\omega} e^{-i\tilde{\omega}\tilde{t}} \int_{-\infty}^{+\infty} d\tilde{t}' e^{i\tilde{\omega}\tilde{t}'} e^{4iM\omega e^{r/(4M)}} e^{-\tilde{t}'/(4M)}$$

Making two coordinate transformations, first $\lambda := e^{-\tilde{t}'/(4M)}$ and then $z := -4iM\lambda\omega e^{r/(4M)}$ we have:

$$\begin{aligned} e^{4iM\omega e^{-(\tilde{t}-r)/(4M)}} &= \frac{2M}{\pi} \int_{-\infty}^{+\infty} d\tilde{\omega} e^{-i\tilde{\omega}\tilde{t}} \int_0^{+\infty} d\lambda \lambda^{-4iM\tilde{\omega}-1} e^{4iM\lambda\omega e^{r/(4M)}} \\ &= \frac{2M}{\pi} \int_{-\infty}^{+\infty} d\tilde{\omega} e^{-i\tilde{\omega}\tilde{t}} (4iM\omega e^{r/(4M)})^{4iM\tilde{\omega}} \int_{\Xi} dz z^{-4iM\tilde{\omega}-1} e^{-z}, \end{aligned}$$

where Ξ is the path in the imaginary axis that starts in 0 and goes to infinity. Knowing that $|z^{-4iM\tilde{\omega}-1} e^{-z}| = |z^{-4M\tilde{\omega}-1} e^{-z}| \leq 1/R^{(4M\tilde{\omega}+1)}$, where $z = Re^{i\theta}$, and because there are no poles in the path pictured in Figure 6, the integral in Ξ is equal to the integral in the real axis from 0 to infinity. Hence, as the gamma function is given by $\Gamma(t) = \int_0^{\infty} dx x^{t-1} e^{-x}$, we have

$$\begin{aligned} \frac{e^{-i\omega(t-r)}}{\sqrt{2\omega}} &= \frac{2M}{\pi} \int_{-\infty}^{+\infty} d\tilde{\omega} \sqrt{\frac{\tilde{\omega}}{\omega}} \frac{e^{-i\tilde{\omega}\tilde{t}}}{\sqrt{2\tilde{\omega}}} (e^{-i\pi/2})^{4iM\tilde{\omega}} (4M\omega e^{r/(4M)})^{4iM\tilde{\omega}} \Gamma(-4iM\tilde{\omega}) \\ &= \frac{2M}{\pi} \int_0^{+\infty} d\tilde{\omega} \sqrt{\frac{\tilde{\omega}}{\omega}} \left[\frac{e^{-i\tilde{\omega}(\tilde{t}-r)}}{\sqrt{2\tilde{\omega}}} e^{2\pi M\tilde{\omega}} (4M\omega)^{4iM\tilde{\omega}} \Gamma(-4iM\tilde{\omega}) \right. \\ &\quad \left. + \frac{e^{i\tilde{\omega}(\tilde{t}-r)}}{\sqrt{2\tilde{\omega}}} e^{-2\pi M\tilde{\omega}} (4M\omega)^{-4iM\tilde{\omega}} \Gamma(-4iM\tilde{\omega}) \right]. \end{aligned}$$

From that, we identify the Bogoliubov coefficients

$$d_{\omega\tilde{\omega}} = \frac{2M}{\pi} \sqrt{\frac{\tilde{\omega}}{\omega}} e^{-2\pi M\tilde{\omega}} (4M\omega)^{-4iM\tilde{\omega}} \Gamma(-4iM\tilde{\omega}), \quad c_{\omega\tilde{\omega}} = e^{4\pi M\tilde{\omega}} (4M\omega)^{8iM\tilde{\omega}} d_{\omega;\tilde{\omega}}$$

and, using the property $|\Gamma(ib)|^2 = \frac{\pi}{b \sinh(\pi b)}$, we have

$$|c_{\omega\tilde{\omega}}|^2 = e^{8\pi M\tilde{\omega}} |d_{\omega\tilde{\omega}}|^2, \quad (3.3.4)$$

$$|d_{\omega\tilde{\omega}}|^2 = \frac{4M^2}{\pi^2} \frac{\tilde{\omega}}{\omega} e^{-2\pi M\tilde{\omega}} \frac{\pi}{4M\tilde{\omega} \sinh(\pi 4M\tilde{\omega})} = \frac{2M}{\pi\omega} (e^{8\pi M\tilde{\omega}} - 1)^{-1}. \quad (3.3.5)$$

We see that the average number of particles given by (3.1.26) is divergent. This divergence is due to the infinite duration of emission of finite number of particles. If one considers observations with finite time duration, the particle number will not diverge. Using (3.3.4) and (3.1.22), we have the average number of particles between $\tilde{\omega} + d\tilde{\omega}$

$$\langle N \rangle_0 = \int_0^\infty d\omega |d_{\omega\tilde{\omega}}|^2 = n_{\text{BE}}(\tilde{\omega}, 1/(8\pi M)) \delta(0), \quad (3.3.6)$$

where $n_{\text{BE}}(\omega, T)$ is the Bose-Einstein thermal distribution.

This result tells us that an exterior observer in the future sees a *thermal flux* of particles from the black hole, that is, the black hole acts like a black body with temperature $T = \hbar c^3 / (8\pi k_B G M) \approx 6 \times 10^{-8} (M_\odot / M)$ K, where M is the black hole mass and $M_\odot \approx 2 \times 10^{30}$ kg is the solar mass. As a consequence of this thermal flux, the mass of the black hole will decrease until it vanishes or until it reaches the Plank scale; this is still an open question.

In a more general result of the Hawking effect, the black hole temperature is given by

$$T = \frac{\kappa}{2\pi}, \quad (3.3.7)$$

where κ is the surface gravity of the black hole - in the Schwarzschild case $\kappa = 1/(4M)$. In this form, the temperature is also very similar to the Unruh temperature, where κ would be the acceleration of an observer.

A different and more general derivation of the Hawking effect can be found in (35, 36).

4 HARVESTING QUANTUM ENTANGLEMENT

A quantum state may be considered entangled or not depending on the tensor-product structure chosen for the state, which is imposed by the accessible interactions and measurements (43). Therefore, entanglement is determined by particular experimental capabilities. In this sense, the vacuum state of the free scalar field in flat spacetime is entangled with respect to *local* observers, the Unruh effect is a consequence of it, as the accelerated observer is restricted to one Rindler wedge. In fact, it was showed that Bell's inequalities are maximally violated in the vacuum of free quantum fields (44).

An interesting result is that entanglement can be transferred from vacuum to a pair of particle detectors (5, 6), which is now known as *entanglement harvesting*. The hypothetical experiment is setup as follows: two detectors with two internal energy levels can interact with the field. We let the detectors interact locally with the field while they are spacelike separated. It turns out that, after the interactions ceased, the detectors become entangled. Entanglement cannot be created by local operations, thus, the detectors extracted entanglement from the field. Finally, the detectors can be used to perform quantum information tasks.

This phenomenon has been shown to be sensitive to a variety of parameters, such as the distance between the detectors, the value of the energy gap and the structure of the background spacetime.

Following, we present a study of entanglement harvesting. We begin by presenting the conditions for the phenomenon in a general context. Then, we analyze it when the detectors are near a black hole, where we show that it is in accordance with the Hawking radiation and redshift effect. Later, we make a comparison of the phenomenon in different scenarios. Finally, we analyze how the parameters of the detectors themselves affect their ability to harvest entanglement.

4.1 Entanglement from vacuum

Harvesting quantum entanglement was initially shown by Valentini (5), in 1991, and later by Reznik (6), in 2002. In this section, we will present the phenomenon in details, following the Reznik paper closely.

We will analyze the entanglement harvested from a quantum field through a local interaction with two *Unruh-DeWitt detectors*, and we will denote them A and B. Consider the Unruh-DeWitt detectors to be two-level systems. Let the excited state, $|\uparrow_i\rangle$, and the

unexcited state, $|\downarrow_i\rangle$, be eigenstates of the detector's Hamiltonian H_i :

$$H_i |\uparrow_i\rangle = \alpha_i |\uparrow_i\rangle, \quad H_i |\downarrow_i\rangle = \beta_i |\downarrow_i\rangle, \quad (4.1.1)$$

where $i \in \{A, B\}$ labels each detector. For now, we will consider the detectors to be point-like and their interaction with the field will be given, in the interaction picture, by the Hamiltonian

$$\begin{aligned} H_{\text{int}}(t) = & \varepsilon_A(t) \left[e^{+i\Omega t} \sigma_A^+ + e^{-i\Omega t} \sigma_A^- \right] \phi(x_A(t)) \\ & + \varepsilon_B(t) \left[e^{+i\Omega t} \sigma_B^+ + e^{-i\Omega t} \sigma_B^- \right] \phi(x_B(t)), \end{aligned} \quad (4.1.2)$$

where $\Omega := \alpha_i - \beta_i$ is the detectors' energy gap, $x_i(t) = (t_i(t), \mathbf{x}_i(t))$ represents the world line of the detector i , $\varepsilon_i(t)$ is the coupling function between the detector i and the field, and $\sigma_i^\pm$ are ladder operators, given by $\sigma_i^+ = |\uparrow_i\rangle\langle\downarrow_i|$ and $\sigma_i^- = |\downarrow_i\rangle\langle\uparrow_i|$, and $i \in \{A, B\}$. To treat the interactions perturbatively, we will consider the very small coupling functions, i.e., $\varepsilon_i(\tau_i) \ll 1$. Moreover, we will consider a finite interaction time T , that is, the coupling functions either vanish or go exponentially to zero outside a time interval T .

Now, we can find conditions for entanglement harvesting and later analyze it for particular trajectories and coupling functions. Consider that, before the interactions, the detectors and the field are in their ground state, i.e., the initial state is given by $|\Psi_{\text{in}}\rangle = |\downarrow_A \downarrow_B 0\rangle$, where we used the simplified notation $|\phi_1\rangle \otimes \dots \otimes |\phi_N\rangle = |\phi_1 \dots \phi_N\rangle$. The final state of the system is $|\Psi_{\text{out}}\rangle = \mathcal{U} |\Psi_{\text{in}}\rangle$, where $\mathcal{U}$ is the evolution operator given by the Dyson series (2.1.7). Expanding $\mathcal{U}$ to the second order of the coupling functions, we have

$$\begin{aligned} \mathcal{U} = & I - i \int_{-\infty}^{+\infty} dt H_{\text{int}}(t) - \frac{1}{2} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' T[H_{\text{int}}(t) H_{\text{int}}(t')] \\ = & I - i \int_{-\infty}^{+\infty} dt \varepsilon_A(t) \left[e^{+i\Omega t} \sigma_A^+ + e^{-i\Omega t} \sigma_A^- \right] \phi(x_A(t)) \\ & - i \int_{-\infty}^{+\infty} dt \varepsilon_B(t) \left[e^{+i\Omega t} \sigma_B^+ + e^{-i\Omega t} \sigma_B^- \right] \phi(x_B(t)) \\ & - \frac{1}{2} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' T \left[\varepsilon_A(t) \varepsilon_A(t') \left(e^{i\Omega(t+t')} (\sigma_A^+)^2 + e^{-i\Omega(t+t')} (\sigma_A^-)^2 \right. \right. \\ & \left. \left. e^{i\Omega(t-t')} \sigma_A^+ \sigma_A^- + e^{-i\Omega(t-t')} \sigma_A^- \sigma_A^+ \right) + [\varepsilon_A(t) \varepsilon_B(t') + \varepsilon_B(t) \varepsilon_A(t')] \right. \\ & \times \left(e^{i\Omega(t+t')} \sigma_A^+ \sigma_B^+ + e^{-i\Omega(t+t')} \sigma_A^- \sigma_B^- + e^{i\Omega(t-t')} \sigma_A^+ \sigma_B^- + e^{-i\Omega(t-t')} \sigma_A^- \sigma_B^+ \right) \\ & \left. + \varepsilon_B(t) \varepsilon_B(t') \left(e^{i\Omega(t+t')} (\sigma_B^+)^2 + e^{-i\Omega(t+t')} (\sigma_B^-)^2 + e^{i\Omega(t-t')} \sigma_B^+ \sigma_B^- \right. \right. \\ & \left. \left. + e^{-i\Omega(t-t')} \sigma_B^- \sigma_B^+ \right) \right], \end{aligned}$$

where $T[A(t)B(t')] \equiv \theta(t-t')A(t)B(t') + \theta(t'-t)B(t')A(t)$ is the *time ordering operator*, and $\theta(x)$ denotes the Heaviside step function. Since the operator $\mathcal{U}$ will act on $|\downarrow_i\rangle$, $i \in \{A, B\}$, the terms proportional to $(\sigma_i^\pm)^2$, with σ_i^- on the right, and $\sigma_i^- \sigma_j^\pm$, $i \neq j \in \{A, B\}$, are going to be zero. Denoting

$$\Phi_i^\pm = \int_{-\infty}^{+\infty} d\tau_i \varepsilon_i(\tau_i) e^{\pm i\Omega \tau_i} \phi(\mathbf{x}_i(\tau_i), t(\tau_i)),$$

we have

$$\begin{aligned}
|\Psi_f\rangle &= \left[I - i\Phi_A^+\sigma_A^+ - i\Phi_B^+\sigma_B^+ - \frac{1}{2}T[\Phi_A^-\Phi_A^+]\sigma_A^-\sigma_A^+ - \frac{1}{2}T[\Phi_B^-\Phi_B^+]\sigma_B^-\sigma_B^+ \right. \\
&\quad \left. - T[\Phi_A^+\Phi_B^+]\sigma_A^+\sigma_B^+ \right] |\downarrow_A\downarrow_B\rangle 0\rangle \\
&= \left[\left(1 - \frac{1}{2}T[\Phi_A^-\Phi_A^+ + \Phi_B^-\Phi_B^+] \right) |\downarrow_A\downarrow_B\rangle - T[\Phi_A^+\Phi_B^+] |\uparrow_A\uparrow_B\rangle - i\Phi_A^+ |\uparrow_A\downarrow_B\rangle \right. \\
&\quad \left. + -i\Phi_B^+ |\downarrow_A\uparrow_B\rangle \right] \otimes |0\rangle. \tag{4.1.3}
\end{aligned}$$

Up to the second order in the interaction, the density matrix of the final state is given by

$$\begin{aligned}
\rho &= (1 - T[\Phi_A^-\Phi_A^+ + \Phi_B^-\Phi_B^+]) |\downarrow_A\downarrow_B\rangle 0\rangle \langle 0| \downarrow_B\downarrow_A| - |\downarrow_A\downarrow_B\rangle 0\rangle \langle 0| \downarrow_B\downarrow_A| T[\Phi_A^+\Phi_A^- + \Phi_B^+\Phi_B^-] \\
&\quad - T[\Phi_A^+\Phi_B^+] |\uparrow_A\uparrow_B\rangle 0\rangle \langle 0| \downarrow_B\downarrow_A| - |\downarrow_A\downarrow_B\rangle 0\rangle \langle 0| \uparrow_B\uparrow_A| T[\Phi_A^-\Phi_B^-] \\
&\quad + \Phi_B^+ |\downarrow_A\uparrow_B\rangle 0\rangle \langle 0| \downarrow_B\uparrow_A| \Phi_A^- + \Phi_A^+ |\uparrow_A\downarrow_B\rangle 0\rangle \langle 0| \uparrow_B\downarrow_A| \Phi_B^- \\
&\quad - i\Phi_A^+ |\uparrow_A\downarrow_B\rangle 0\rangle \langle 0| \downarrow_B\downarrow_A| - i|\downarrow_A\downarrow_B\rangle 0\rangle \langle 0| \downarrow_B\uparrow_A| \Phi_A^- + \Phi_A^+ |\uparrow_A\downarrow_B\rangle 0\rangle \langle 0| \downarrow_B\uparrow_A| \Phi_A^- \\
&\quad - i\Phi_B^+ |\downarrow_A\uparrow_B\rangle 0\rangle \langle 0| \downarrow_B\downarrow_A| - i\Phi_B^- |\downarrow_A\downarrow_B\rangle 0\rangle \langle 0| \uparrow_B\downarrow_A| + \Phi_B^+ |\downarrow_A\uparrow_B\rangle 0\rangle \langle 0| \uparrow_B\downarrow_A| \Phi_B^-. \tag{4.1.4}
\end{aligned}$$

We want to quantify entanglement between the detectors after the interaction and, thus, we are interested only in the final state of the detector ρ_{AB} , regardless the field. Tracing over the degrees of freedom of the field, the terms proportional to $\Phi_i^\pm$ will go to zero*. Writing the state of the detectors in the matrix representation with basis $\{|\downarrow_A\downarrow_B\rangle, |\downarrow_A\uparrow_B\rangle, |\uparrow_A\downarrow_B\rangle, |\uparrow_A\uparrow_B\rangle\}$, we have

$$\rho_{AB} = \begin{pmatrix} 1 - P_{AA} - P_{BB} & 0 & 0 & X \\ 0 & P_{BB} & P_{AB} & 0 \\ 0 & P_{AB}^* & P_{AA} & 0 \\ X^* & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\varepsilon^4), \tag{4.1.5}$$

where

$$P_{ij} = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \varepsilon_i(t) \varepsilon_j(t') e^{i\Omega(t-t')} D^+(x_i(t); x_j(t')), \tag{4.1.6}$$

$$X = \int_{-\infty}^{+\infty} dt \int_{-\infty}^t dt' \varepsilon_A(t) \varepsilon_B(t') e^{i\Omega(t+t')} [D^+(x_A(t); x_B(t')) + D^+(x_B(t); x_A(t'))], \tag{4.1.7}$$

and $D^+(x_i(t); x_j(t')) := \langle 0 | \phi(x_i(t)) \phi(x_j(t')) | 0 \rangle$ is the *vacuum Wightman function* for the field ϕ , which depends on the spacetime considered. To evaluate entanglement between the detectors we can use negativity (2.3.8). The partial transpose of (4.1.5) is

$$\rho_{AB}^{\text{PT}} = \begin{pmatrix} 1 - P_{AA} - P_{BB} & 0 & 0 & P_{AB} \\ 0 & P_{BB} & X & 0 \\ 0 & X^* & P_{AA} & 0 \\ P_{AB}^* & 0 & 0 & 0 \end{pmatrix}, \tag{4.1.8}$$

* $\sum_n \langle n | \Phi_i^\pm | 0 \rangle \langle 0 | n \rangle = \sum_n \langle n | 1 \rangle \langle 0 | n \rangle = 0 = \sum_n \langle n | 0 \rangle \langle 0 | \Phi_i^\pm | n \rangle$.

and its eigenvalues are

$$\begin{aligned}\lambda_1 &= 1 - P_{AA} - P_{BB} + \mathcal{O}(\varepsilon^4), \\ \lambda_2 &= 0 + \mathcal{O}(\varepsilon^4), \\ \lambda_3 &= \frac{1}{2} \left(P_{AA} + P_{BB} + \sqrt{(P_{AA} - P_{BB})^2 + 4|X|^2} \right) + \mathcal{O}(\varepsilon^4), \\ \lambda_4 &= \frac{1}{2} \left(P_{AA} + P_{BB} - \sqrt{(P_{AA} - P_{BB})^2 + 4|X|^2} \right) + \mathcal{O}(\varepsilon^4).\end{aligned}$$

We see that λ_4 is the only possible negative eigenvalue of $\rho_{AB}^{\text{PT}\dagger}$. By Peres-Horodecki criterion, the detectors are going to be entangled if $\lambda_4 < 0$, i.e.,

$$|X|^2 > P_{AA}P_{BB}. \quad (4.1.9)$$

And the negativity (2.3.8) is given by

$$\mathcal{N} = \max \left[0, -\frac{1}{2} \left(P_{AA} + P_{BB} - \sqrt{(P_{AA} - P_{BB})^2 + 4|X|^2} \right) \right]. \quad (4.1.10)$$

One can also evaluate entanglement of the final system by computing the concurrence (2.3.10). The square root of the eigenvalues of the matrix $\rho_C = \rho_{AB}(\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y)$ are[†]

$$\begin{aligned}\lambda_1 &= \sqrt{P_{AA}P_{BB} + |X|^2 + |P_{AB}|^2} + |X| + \mathcal{O}(\varepsilon^4), \\ \lambda_2 &= \sqrt{P_{AA}P_{BB} + |X|^2 + |P_{AB}|^2} - |X| + \mathcal{O}(\varepsilon^4), \\ \lambda_3 &= \sqrt{P_{AA}P_{BB} + P_{AB}} + \mathcal{O}(\varepsilon^4), \\ \lambda_4 &= \sqrt{P_{AA}P_{BB} + P_{AB}} + \mathcal{O}(\varepsilon^4),\end{aligned}$$

By virtue of the Peres-Horodecki criterion (4.1.9)[§], λ_1 is the largest term. Therefore, the concurrence (2.3.10) is given by

$$\mathcal{C}(\rho_{AB}) = 2 \max[0, |X| - \sqrt{P_{AA}P_{BB}}]. \quad (4.1.11)$$

When $P_{AA} = P_{BB}$ the concurrence is twice the negativity.

Finally, let us give some physical interpretation for the terms that appear in the negativity. The term $|X|$ is a nonlocal term, as it contains, in the Wightman function, information about both detectors. The terms P_{AA} and P_{BB} are local terms, as they only contain information about one of the detectors. Entanglement implies that the nonlocal

[†] Apparently, λ_1 would possibly be a negativity eigenvalue, but $P_{ii} \propto \varepsilon^2 \ll 1$, thus, it is going to be positive.

[‡] Because it is a square root, the terms of $\mathcal{O}(\varepsilon^4)$ are relevant in these computations (45). Thus, the term $(P_{AA}P_{BB} + |X|^2 + |P_{AB}|^2)(|\uparrow_A\uparrow_B\rangle\langle\uparrow_A\uparrow_B| - |\downarrow_A\downarrow_B\rangle\langle\downarrow_A\downarrow_B|)$ (46) must be added to the density matrix (4.1.5).

[§] $\sqrt{P_{AA}P_{BB} + P_{AB}} < |X| + P_{AB} < |X| + \sqrt{P_{AA}P_{BB} + |X|^2 + |P_{AB}|^2}$

terms are greater than local terms. That was already expected, as entanglement is a nonlocal property of the system.

The local terms have a more direct physical interpretation: they are the transition probabilities of the detectors. This can be seen as follows. The reduced state, ρ_i , $i \in \{A, B\}$, of each individual detector is obtained by taking the trace over the degrees of freedom of the other detector, that is, $\rho_i = \text{Tr}_j \rho_{AB}$, $i \neq j$. The matrix representation of reduced states, in the basis $\{|\downarrow_i\rangle, |\uparrow_i\rangle\}$, are

$$\rho_A = \begin{pmatrix} 1 - P_{AA} & 0 \\ 0 & P_{AA} \end{pmatrix}, \quad \rho_B = \begin{pmatrix} 1 - P_{BB} & 0 \\ 0 & P_{BB} \end{pmatrix}. \quad (4.1.12)$$

By the density matrices form, the interpretation of the terms as the probabilities of the detectors to excite is clear.

The nonlocal term $|X|$ is related to the terms in the second line of the final state (4.1.4). Thus, it can be interpreted as an amplitude for virtual particle exchange. Therefore, one possible interpretation is that the detectors will get entangled if their probability of exchanging one quantum is higher than the multiplied probabilities of each emitting one quantum.

Following, we will analyze how the background spacetime affects entanglement harvesting and, later, we will see some parameter dependence of this phenomenon in Minkowski spacetime. From equations (4.1.6) and (4.1.7), we see that the spacetime dependence is contained in the vacuum Wightman function, thus, it is now a matter of determining this functions. However, analytical solutions for the vacuum Wightman function are considerably hard to obtain, and they were obtained only for a few cases. Here, we will consider the following cases: Minkowski (inertial and accelerated detectors), de Sitter and BTZ black holes, all of which have analytical solution for the vacuum Wightman function.

4.2 BTZ Black Hole

Let us begin with the Black Hole case. This case is interesting because it is related to the Hawking radiation and redshift effect. As they are generic properties of black holes, we expect the result obtained below to be general for all black holes. Here, we follow closely the reference (47).

The BTZ black hole is the solution of the Einstein equations (A.0.1) in (2+1)-dimensions with a cosmological constant $\Lambda = -\ell^{-2}$, $\ell > 0$, and its line element, in Schwarzschild-like coordinates, is given by

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 - r^2 d\varphi^2, \quad N^2 := \frac{r^2 - r_h^2}{\ell^2}, \quad r_h^2 := M\ell^2, \quad (4.2.1)$$

where M is the black hole mass. We notice that this is a static spacetime, as the metric has the form of equation (A.0.2). The Wightman function in this spacetime, outside the black hole horizon, is given by

$$D^+(x(t), x'(t')) = \frac{1}{4\pi\sqrt{2}\ell} \sum_{n=-\infty}^{\infty} \left[\frac{1}{\sqrt{\mu_n}} - \frac{\zeta}{\sqrt{\mu_n + 2}} \right], \quad (4.2.2)$$

where $\zeta \in \{-1, 0, 1\}$ are related to the anti-de Sitter's Neumann ($\zeta = -1$), Dirichlet ($\zeta = 1$), or transparent ($\zeta = 0$) boundary conditions the field should satisfy at infinity[¶] and

$$\mu_n := \frac{rr'}{r_h^2} \cosh \left[\frac{r_h}{\ell} (\Delta\varphi - 2\pi n) \right] - 1 - \frac{\sqrt{(r^2 - r_h^2)(r'^2 - r_h^2)}}{r_h^2} \cosh \left[\frac{r_h}{\ell^2} \Delta t \right], \quad (4.2.3)$$

where $\Delta\varphi \equiv \varphi - \varphi'$ and $\Delta t \equiv t - t'$.

Now, we have to define the detectors' trajectories in order to compute the amount of entanglement harvested from the field. Consider the following detectors' trajectories

$$t = \frac{\tau_i}{\gamma_i}, \quad r = R_i, \quad \varphi = \varphi_i, \quad \gamma_i := \frac{\sqrt{R_i^2 - r_h^2}}{\ell}, \quad (4.2.4)$$

where $i \in \{A, B\}$. We consider that the detectors are outside the black hole horizon, $R_A, R_B > r_h$, and, without loss of generality, we set $R_A \leq R_B$.

Let us compute the Wightman function for these coordinates. Inserting equation (4.2.4) in (4.2.3) and then inserting the result in (4.2.2), yields

$$\begin{aligned} D^+(x_i(t_i), x_j(t_j)) = & \frac{1}{4\pi\sqrt{2}\ell} \sum_{n=-\infty}^{\infty} \left\{ \left[\frac{R_i R_j}{r_h^2} \cosh \left[\frac{r_h}{\ell} (\Delta\varphi - 2\pi n) \right] - 1 \right. \right. \\ & \left. \left. - \frac{\overbrace{\sqrt{(R_i^2 - r_h^2)(R_j^2 - r_h^2)}}^{\gamma_i \gamma_j \ell^2 / r_h^2}}{r_h^2} \cosh \left[\frac{r_h}{\ell^2} \left(\frac{\tau_i}{\gamma_i} - \frac{\tau_j}{\gamma_j} \right) \right] \right]^{-\frac{1}{2}} \right. \\ & \left. \left[\frac{R_i R_j}{r_h^2} \cosh \left[\frac{r_h}{\ell} (\Delta\varphi - 2\pi n) \right] + 1 \right. \right. \\ & \left. \left. - \frac{\gamma_i \gamma_j \ell^2}{r_h^2} \cosh \left[\frac{r_h}{\ell^2} \left(\frac{\tau_i}{\gamma_i} - \frac{\tau_j}{\gamma_j} \right) \right] \right]^{-\frac{1}{2}} \right\}. \end{aligned}$$

Defining

$$\alpha_{ij,n}^{\pm} := \operatorname{arccosh} \left[\frac{r_h^2}{\gamma_i \gamma_j \ell^2} \left(\frac{R_i R_j}{r_h^2} \cosh \left(\frac{r_h}{\ell} (\Delta\varphi + 2\pi n) \right) \right) \pm 1 \right], \quad (4.2.5)$$

and making the coordinate transformation $\tau'_{i,j} = \tau_{i,j}/\gamma_{i,j}$, we have

$$D^+(x_i(t_i), x_j(t_j)) = \frac{1}{4\pi\sqrt{2}\ell} \sum_{n=-\infty}^{\infty} \left[\frac{1}{\sqrt{\cosh(\alpha_{ij,n}^-) - \cosh\left(\frac{r_h}{\ell^2}(\tau'_i - \tau'_j)\right)}} \right]$$

[¶] The BTZ black hole is asymptotically anti-de Sitter.

$$\left. \frac{\zeta}{\sqrt{\cosh(\alpha_{ij,n}^+) - \cosh\left(\frac{r_h}{\ell^2}(\tau_i' - \tau_j')\right)}} \right] = D^+(\tau_i - \tau_j). \quad (4.2.6)$$

Finally, we will consider a Gaussian coupling function, which takes the form

$$\varepsilon_i(\tau_i) = \varepsilon e^{-\frac{\tau_i^2}{2\sigma^2}}, \quad (4.2.7)$$

where $0 < \varepsilon \ll 1$ is the coupling constant, and σ is a real positive constant, which is interpreted to be proportional to the interaction time, that is, the detectors interact with the field with an amount of proper time $k\sigma$, where k is some positive constant.

Now we have the necessary tools to compute equations (4.1.6) and (4.1.7), which are required terms to quantify entanglement. Let us begin by computing X . Rewriting equation (4.1.7) in terms of the Heaviside step function and making the coordinate transformations $x := \tau_B/\gamma_B - \tau_A/\gamma_A$ and $y := \tau_B/\gamma_B$, we have

$$\begin{aligned} X &= -\varepsilon^2 \gamma_A \gamma_B \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy e^{-\frac{\gamma_B^2 y^2 + \gamma_A(x+y)^2}{2\sigma^2}} e^{-i\Omega(\gamma_B y + \gamma(y-x))} \\ &\quad \times [\theta(x)D^+(x) + \theta(-x)D^+(-x)] \\ &= \frac{-\varepsilon^2 \sqrt{2\pi} \sigma \gamma_A \gamma_B}{\sqrt{\gamma_A^2 + \gamma_B^2}} \exp\left[-\frac{\sigma^2 \Omega^2 (\gamma_A + \gamma_B)}{2(\gamma_A^2 + \gamma_B^2)}\right] \int_{-\infty}^{+\infty} dx \exp\left[-\frac{\gamma_A^2 \gamma_B^2 x^2}{2\sigma^2 (\gamma_A^2 + \gamma_B^2)}\right] \\ &\quad \times \exp\left[-i\Omega \gamma_A \gamma_B \frac{\gamma_B - \gamma_A}{\gamma_A^2 + \gamma_B^2} x\right] [\theta(x)D^+(x) + \theta(-x)D^+(-x)], \end{aligned} \quad (4.2.8)$$

where the integral over y was solved by completing squares. Making the coordinate transformation $z := r_h x / \ell^2$, and defining

$$\begin{aligned} a_X &:= \frac{\gamma_A^2 \gamma_B^2 \ell^2}{2\sigma^2 r_h^2 (\gamma_A^2 + \gamma_B^2)}, & \beta_X &:= -i\Omega \gamma_A \gamma_B \frac{\gamma_B - \gamma_A}{\gamma_A^2 + \gamma_B^2} \frac{\ell^2}{r_h}, \\ K_X &:= \frac{-\varepsilon^2 \sqrt{2\pi} \sigma \gamma_A \gamma_B}{\sqrt{\gamma_A^2 + \gamma_B^2}} \exp\left[-\frac{\sigma^2 \Omega^2 (\gamma_A + \gamma_B)}{2(\gamma_A^2 + \gamma_B^2)}\right], \end{aligned}$$

we obtain

$$\begin{aligned} X &= - \sum_{n=-\infty}^{\infty} \left[K_X \int_0^{\infty} dz \frac{e^{-a_X z^2} \cos(\beta_X z)}{\sqrt{\cosh(\alpha_{AB,n}^-) - \cosh(z)}} \right. \\ &\quad \left. - \zeta K_X \int_0^{\infty} dz \frac{e^{-a_X z^2} \cos(\beta_X z)}{\sqrt{\cosh(\alpha_{AB,n}^+) - \cosh(z)}} \right]. \end{aligned} \quad (4.2.9)$$

Now, let us compute P_{ii} . Inserting equation (4.2.7) in equation (4.1.6), we obtain

$$P_i = \varepsilon^2 \int_{-\infty}^{+\infty} d\tau_i \int_{-\infty}^{+\infty} d\tau_i' e^{-\frac{\tau_i^2 + \tau_i'^2}{2\sigma^2} - i\Omega(\tau_i - \tau_i')} D^+((\tau_i - \tau_i')/\gamma_i).$$

Making the coordinate transformations $x = \tau_i + \tau'_i$ and $y = \tau_i - \tau'_i$, we have

$$\begin{aligned} P_i &= \frac{\varepsilon^2}{2} \int_{-\infty}^{+\infty} dy e^{y^2/(4\sigma^2)} e^{i\Omega y} D^+(y/\gamma_i) \int_{-\infty}^{+\infty} dx e^{-x^2/(4\sigma^2)} \\ &= \frac{\varepsilon^2 \sigma \sqrt{\pi}}{2} \int_{-\infty}^{+\infty} dy e^{y^2/(4\sigma^2)} e^{i\Omega y} D^+(y/\gamma_i). \end{aligned} \quad (4.2.10)$$

Defining

$$a_i := \frac{\gamma_i^2 \ell^4}{4\sigma^2 r_h^2}, \quad \beta_i := \frac{\gamma_i \Omega \ell^2}{r_h},$$

and making the coordinate transformation $z = r_h y / (\ell^2 \gamma_i)$, we get

$$\begin{aligned} P_i &= \frac{\varepsilon^2 \sigma \sqrt{\pi} \gamma_i \ell^2}{2r_h} \left[\overbrace{\int_{-\infty}^0 dz e^{-a_i z^2 - i\beta_i z} D^+\left(\frac{z\ell^2}{r_h}\right)}^{z \rightarrow -z} + \int_0^{\infty} dz e^{-a_i z^2 - i\beta_i z} D^+\left(\frac{z\ell^2}{r_h}\right) \right] \\ &= \frac{\varepsilon^2 \sigma \sqrt{\pi} \gamma_i \ell^2}{2r_h} \int_0^{\infty} dz \left[e^{-a_i z^2 - i\beta_i z} D^+\left(\frac{z\ell^2}{r_h}\right) + \left(e^{-a_i z^2 - i\beta_i z} D^+\left(\frac{z\ell^2}{r_h}\right) \right)^* \right] \\ &= \frac{\varepsilon^2 \sigma \sqrt{\pi} \gamma_i \ell^2}{2r_h} \int_0^{\infty} dz 2 \operatorname{Re} \left[e^{-a_i z^2 - i\beta_i z} D^+\left(\frac{z\ell^2}{r_h}\right) \right] \\ &= K_i \int_0^{\infty} dz \operatorname{Re} \left[\frac{e^{-a_i z^2 - i\beta_i z} r_h}{\sqrt{1 - \cosh(z)} \sqrt{R_i^2 - r_h^2}} \right] \\ &\quad - \zeta K_i \int_0^{\infty} dz \operatorname{Re} \left[\frac{e^{-a_i z^2 - i\beta_i z}}{\sqrt{\cosh(\alpha_{ii,0}^+) - \cosh(z)}} \right] \\ &\quad 2K_i \sum_{n=1}^{\infty} \left\{ \int_0^{\infty} dz \operatorname{Re} \left[\frac{e^{-a_i z^2 - i\beta_i z}}{\sqrt{\cosh(\alpha_{ii,n}^-) - \cosh(z)}} \right] \right. \\ &\quad \left. \zeta \int_0^{\infty} dz \operatorname{Re} \left[\frac{e^{-a_i z^2 - i\beta_i z}}{\sqrt{\cosh(\alpha_{ii,n}^+) - \cosh(z)}} \right] \right\}, \end{aligned} \quad (4.2.11)$$

where $K_i := \frac{\varepsilon^2 \sigma}{2\sqrt{2\pi}}$, and, recognizing that $\cosh(x)$ is an even function, we took the sum of the negative terms to be equal the sum of the positive terms.

As these are not analytical solutions, we will use concurrence (4.1.11) to quantify entanglement, which is simpler than the negativity (4.1.10). We plot the concurrence as a function of the proper distance, which, for two points $x_1 = (t, R_1, \varphi)$ and $x_2 = (t, R_2, \varphi)$, it is given by

$$d(R_1, R_2) := \int_{R_1}^{R_2} dr \frac{\ell}{\sqrt{r^2 - r_h^2}} = \ell \ln \left[\frac{R_2 + \sqrt{R_2^2 - r_h^2}}{R_1 + \sqrt{R_1^2 - r_h^2}} \right]. \quad (4.2.12)$$

In all the plots we assign the values $\varphi_A = \varphi_B = 0$, $\sigma = 1$, $\ell = 10$, $M = 1$ and $\zeta = 1$, and the sums are truncated at $n = 50$.

Figure 7 – The concurrence of the final state of the detector as a function of the proper distance between the detectors. The concurrence is plotted for three values of the energy gap of the detectors: $\Omega = 0.01$ (solid red), $\Omega = 0.1$ (dotted blue) and $\Omega = 1$ (dashed green). Entanglement decreases when the proper distance between the detectors increases, and it decreases more slowly for larger energy gap. In this plot we took $\varphi_A = \varphi_B$, $d(r_h, R_A) = 1$, $\sigma = 1$, $\ell = 10$, $M = 1$ and $\zeta = 1$.

Source: By the author.

Figure 7 shows the concurrence of the final state of the detector as a function of the proper distance between the detectors, for three values of the energy gap Ω . We consider that detector A is at a fixed proper distance from the black hole event horizon, $d(r_h, R_A) = 1$. The entanglement harvested from the field decreases as the separation between the detector grows, going to zero at a finite proper distance. The three curves represent $\Omega = 0.01$, in solid red, $\Omega = 0.1$, in dotted blue, and $\Omega = 1$ in dashed green. We can see that entanglement decreases more slowly for larger energy gap. This is due to the fact that the negative term of the concurrence, $\sqrt{P_{AA}P_{BB}}$, is proportional to probabilities of transition - and these probabilities are smaller for a larger energy gap. Therefore, the positive term of the concurrence dominates the negative term for a larger range.

Figure 8 shows the concurrence of the final state of the detectors as a function of the proper distance between the detector A and the event horizon of the black hole, for three values of the energy gap Ω . We considered that the detectors are at a fixed proper distance, $d(R_A, R_B) = 1$. We can see that the closer the detectors get from the event horizon, less entanglement they are able to harvest. In fact, there is a critical point where there is a “sudden death” of entanglement harvesting: the detectors closer to the event horizon than the distance defined by this critical point, $d_{\text{death}}(r_h, R_A)$, cannot harvest entanglement. This result has a direct relationship with the Hawking radiation and the redshift effect. The Hawking temperature near the event horizon increases the transition probabilities of the detectors. Also, as the radii of detectors, R_i , approaches the event horizon radius, r_h , the value of γ_i (4.2.4) decreases. From equation (4.2.8), we can see that this implies that X will decrease as the detectors get near the horizon. In fact, we

Figure 8 – The concurrence of the final state of the detector as a function of the proper distance between the detector A and the event horizon of the black hole. The concurrence is plotted for three values of the energy gap of the detectors: $\Omega = 0.01$ (solid red), $\Omega = 0.1$ (dotted blue), and $\Omega = 1$ (dashed green). The proper distance between the detectors is fixed to be $d(R_A, R_B) = 1$. Entanglement decreases when the proper distance from the event horizon decreases, and it decreases more slowly for larger energy gap. In this plot we took $\varphi_A = \varphi_B$, $\sigma = 1$, $\ell = 10$, $M = 1$ and $\zeta = 1$. The noise in the curves are associated with numerical errors.

Source: By the author.

can see from the metric (4.2.1) that γ_i are redshift factors, which can be made arbitrarily small. Therefore, the proximity from the horizon, which enhances the black hole redshift effect and Hawking temperature, affects both terms of the concurrence causing its value to decrease.

Again, the three curves represent $\Omega = 0.01$, in solid red, $\Omega = 0.1$, in dotted blue, and $\Omega = 1$ in dashed green. In accordance with the first case, entanglement harvested decreases more slowly for larger energy gap - as the transition probabilities gets smaller.

In short, we determined that the amount of entanglement harvested from the field decreases as the proper distance between the detectors increases. Moreover, larger values of the energy gap of the detectors enhance the extraction of entanglement.

More notably, generic properties of black holes cause a sudden death to the phenomenon. Although we can optimize it by increasing the energy gap, no entanglement can be harvested below a critical distance between the detectors and the event horizon. As these are consequence of the Hawking radiation and the redshift effect (specially the redshift effect), it is reasonable to believe that this result is not a peculiarity of the BTZ

black hole, but it will occur in other black holes, or any other spacetime in which there is a point where the redshift factor goes to zero.

4.3 Comparison between different scenarios

Now we want to analyze how entanglement harvesting differs in distinct scenarios. The Unruh effect establishes that the concept of particle is observer dependent, as accelerated observers in Minkowski vacuum can detect particles. As entanglement is based on a well defined notion of particles (states), one would expect entanglement to be observer dependent. In fact, harvesting entanglement depends on the system considered. As we will see, the acceleration suppresses the detectors' ability to harvest entanglement. Following, we will compare four scenarios: inertial detectors in Minkowski vacuum, inertial detectors in a Minkowski thermal bath, parallel accelerated detectors in Minkowski vacuum, and comoving detectors in de Sitter spacetime^{||}. In this case, we will consider the conformal vacuum (coupling constant $\xi = 1/6$; see 5.5 of (36)). We include the latter scenario because, in 1977, Gibbons and Hawking (48) showed that a response for any inertial detector in de Sitter spacetime is the same as for an inertial detector in thermal bath of field particles with temperature $T = \kappa/2\pi$ in flat spacetime, where κ is the expansion rate of the universe. Therefore, just like accelerated observers, we can ask ourselves if entanglement harvesting in de Sitter universe is the same as heated Minkowski. We will see below that the parametric region where entanglement harvesting is possible differs in these scenarios. Here we follow the references (7, 8, 46).

Now, we will present the setup for the analysis of entanglement harvesting. Let the comoving trajectories of the detectors in the inertial cases (flat, thermal flat and de Sitter) be

$$x_A = \frac{L}{2}, \quad x_B = -\frac{L}{2}, \quad t_A = t_B = \tau, \quad y_A = y_B = z_A = z_B = 0, \quad (4.3.1)$$

where, in the Minkowski cases, L is the proper distance between the detectors and, in the de Sitter case, L is the comoving distance between the detectors. The only case in which the trajectories of the detectors are not inertial is the parallel accelerating case. In this case we will consider the following trajectories

$$\begin{aligned} x_A &= \frac{1}{\kappa}[\cosh(\kappa\tau) - 1] + \frac{L}{2}, & x_B &= \frac{1}{\kappa}[\cosh(\kappa\tau) - 1] - \frac{L}{2}, \\ t_A &= t_B = \frac{1}{\kappa} \sinh(\kappa\tau), & y_A &= y_B = z_A = z_B = 0. \end{aligned} \quad (4.3.2)$$

^{||} The de Sitter spacetime is a cosmological solution of the Einstein's equations (A.0.1), and its line element is given by $ds^2 = -dt^2 + e^{2\kappa t}(dx^2 + dy^2 + dz^2)$, where κ is an expansion rate.

where L is the distance between the detectors measured by an inertial observer. Notice that, in the acceleration and de Sitter cases, the proper distance between the detectors is not constant, that is, one is not at rest with respect to the other^{**}.

To compute the terms P_{ii} and X , we need the Wightman function $D^+(x_i(t), x_j(t'))$ for each case^{††}. We have (7), for $\eta \rightarrow 0_+$,

$$D_M^+(x_A(t), x_B(t')) = \frac{-1}{4\pi^2[(t - t' - i\eta)^2 - |\mathbf{x}_A - \mathbf{x}_B|^2]}, \quad (4.3.3)$$

for Minkowski vacuum,

$$D_{th}^+(x_A(t), x_B(t')) = \frac{T}{8\pi|\mathbf{x}_A - \mathbf{x}_B|} \{ \coth[\pi T(|\mathbf{x}_A - \mathbf{x}_B| - t + t' + i\eta)] + \coth[\pi T(|\mathbf{x}_A - \mathbf{x}_B| + t - t' - i\eta)] \}, \quad (4.3.4)$$

for Minkowski thermal state with temperature T , and

$$D_{ds}^+(x_A(t), x_B(t')) = \frac{-1}{4\pi^2} \left[\frac{4}{\kappa^2} \sinh^2 \left(\frac{\kappa}{2}(t - t' - i\eta) \right) - e^{\kappa(t+t')} |\mathbf{x}_A - \mathbf{x}_B|^2 \right]^{-1}, \quad (4.3.5)$$

for the de Sitter (conformal) vacuum. Using the trajectories (4.3.1) and (4.3.2), and defining $x := \tau + \tau'$ and $y := \tau - \tau'$, we have

$$D_{Mi}^+(x, y) = \frac{1}{4\pi^2[(y - i\eta)^2 - L^2]}, \quad (4.3.6)$$

for inertial detectors in Minkowski vacuum,

$$D_{Ma}^+(x, y) = \frac{\kappa^2}{16\pi^2} \left[\frac{L\kappa}{2} + i\eta - e^{-\frac{\kappa x}{2}} \sinh \left(\frac{\kappa y}{2} \right) \right]^{-1} \left[\frac{L\kappa}{2} - i\eta + e^{\frac{\kappa x}{2}} \sinh \left(\frac{\kappa y}{2} \right) \right]^{-1}, \quad (4.3.7)$$

for parallel accelerated detectors in Minkowski vacuum,

$$D_{th}^+(x, y) = \frac{\kappa}{16\pi^2 L} \left\{ \coth \left[\frac{\kappa}{2}(L - y + i\eta) \right] + \coth \left[\frac{\kappa}{2}(L + y - i\eta) \right] \right\}, \quad (4.3.8)$$

for inertial detectors in Minkowski thermal bath with temperature $T = \kappa/2\pi$, and

$$D_{ds}^+(x, y) = \frac{\kappa^2}{16\pi^2} \left[e^{\kappa x} \left(\frac{L\kappa}{2} \right)^2 - \sinh^2 \left(\frac{\kappa}{2}(y - i\eta) \right) \right]^{-1}, \quad (4.3.9)$$

for comoving detectors in de Sitter (conformal) vacuum. For $D^+(x_i(t), x_i(t'))$, $i \in \{A, B\}$, $L = 0$. We take the limit $\eta \rightarrow 0^+$ to evaluate the quantities P_{ii} and X .

^{**}We considered this system in order to follow references (7, 8), but emphasize that this case is very different from the other cases, because the detectors are not comoving.

^{††}Notice that in thermal case we cannot use the vacuum Wightman function. In this case, we consider that the initial state of the field is an ensemble of states which describes a system with temperature T . The terms P_{ii} and X assume the same form of (4.1.6) and (4.1.7), with the respective thermal Wightman function.

Figure 9 – Contour in the complex plane for integration of the term X .
Source: By the author.

Due to the symmetry of the detectors' trajectories, in all the cases, we have $D^+(x_A(t), x_A(t')) = D^+(x_B(t), x_B(t'))^{\ddagger\ddagger}$. This implies that $P_{AA} = P_{BB}$. Therefore, the concurrence and the negativity differ from a factor 2. As it is more usual, we will use negativity to quantify entanglement. To compute the quantities P_{ii} and X , we will use the Gaussian coupling function defined in equation (4.2.7).

Changing the variables of the integrand of equation (4.1.7) to x and y , and completing squares in the exponentials, yields to

$$X = -\frac{\varepsilon^2 e^{-(\Omega\sigma)^2}}{4} \int_0^\infty dy \int_{-\infty}^{+\infty} dx e^{-\frac{y^2 + (x - 2i\sigma^2\Omega)^2}{4\sigma^2}} \left[D^+(x, y)_{(A,B)} + D^+(x, y)_{(B,A)} \right],$$

where $D^+(x, y)_{(A,B)}$ are equations (4.3.6)-(4.3.9) and $D^+(x, y)_{(B,A)}$ are the same equations taking $L \rightarrow -L$. Making the variable transformation $x = x' - 2i\sigma^2\Omega$, gives

$$X = -\frac{\varepsilon^2 e^{-(\Omega\sigma)^2}}{4} \int_0^\infty dy \int_{-\infty + 2i\sigma^2\Omega}^{+\infty + 2i\sigma^2\Omega} dx' e^{-\frac{y^2 + x'^2}{4\sigma^2}} \times \left[D^+(x' + 2i\sigma^2\Omega, y)_{(A,B)} + D^+(x' + 2i\sigma^2\Omega, y)_{(B,A)} \right].$$

To evaluate the integral over x' , we consider the contour pictured in Figure 9. The poles come from the Wightman functions, $x'_{\text{pole}} = -2i\pi n/\kappa + i\eta$, $n \in \mathbb{Z}$. Therefore, if we restrict $\kappa\sigma^2\Omega < \pi$, the contour Γ never crosses any pole, and the contour integration vanishes, that is

$$\begin{aligned} 0 &= \lim_{R \rightarrow \infty} \oint_{\Gamma} dx' e^{-\frac{x'^2}{4\sigma^2}} \left[D^+(x' + 2i\sigma^2\Omega, y)_{(A,B)} + D^+(x' + 2i\sigma^2\Omega, y)_{(B,A)} \right] \\ &= \int_{-\infty + 2i\sigma^2\Omega}^{+\infty + 2i\sigma^2\Omega} dx' e^{-\frac{x'^2}{4\sigma^2}} \left[D^+(x' + 2i\sigma^2\Omega, y)_{(A,B)} + D^+(x' + 2i\sigma^2\Omega, y)_{(B,A)} \right] \\ &\quad + \int_{+\infty}^{-\infty} dx' e^{-\frac{x'^2}{4\sigma^2}} \left[D^+(x' + 2i\sigma^2\Omega, y)_{(A,B)} + D^+(x' + 2i\sigma^2\Omega, y)_{(B,A)} \right]. \end{aligned}$$

^{‡‡} In reference (8) they affirm that $D^+(x_A(t), x_B(t')) = D^+(x_B(t), x_A(t'))$, but that is not the case for accelerated detectors. The reader may check this by taking $L \rightarrow -L$ in equation (4.3.7). Notwithstanding, the final result will not change after we use the saddle point approximation.

Thus, we have

$$X = -\frac{\varepsilon^2 e^{-(\Omega\sigma)^2}}{4} \int_0^\infty dy \int_{-\infty}^{+\infty} dx' e^{-f(x',y)} \\ \times \left[D^+(x' + 2i\sigma^2\Omega, y)_{(A,B)} + D^+(x' + 2i\sigma^2\Omega, y)_{(B,A)} \right],$$

where $f(x', y) = \frac{y^2 + x'^2}{4\sigma^2}$. To obtain an analytical solution of X we employ a saddle point approximation. Consider x'_0 and y_0 , such that $\frac{\partial f(x'_0, y_0)}{\partial x'} = \frac{\partial f(x'_0, y_0)}{\partial y'} = 0$. These points are $x'_0 = y_0 = 0$. Expanding $f(x', y)$, $D^+(x' + 2i\sigma^2\Omega, y)_{(A,B)}$ and $D^+(x' + 2i\sigma^2\Omega, y)_{(B,A)}$, yields

$$f(x', y) = \frac{y^2 + x'^2}{4\sigma^2}, \\ D^+(x' + 2i\sigma^2\Omega, y)_{(A,B)} = D^+(x' + 2i\sigma^2\Omega, y)_{(B,A)} = D^+(2i\sigma^2\Omega, 0) + \mathcal{O}(x, y),$$

where we dropped the indices (A, B) and (B, A) because the approximation makes $D^+(x' + 2i\sigma^2\Omega, y)_{(A,B)} = D^+(x' + 2i\sigma^2\Omega, y)_{(B,A)}$. Using these results, and recognizing Gaussian integrals over both variables, we have

$$X \approx -2\pi\varepsilon^2\sigma^2 e^{-(\sigma\Omega)^2} D^+(2i\sigma^2\Omega, 0). \quad (4.3.10)$$

Making similar calculations, we obtain approximated value for equation (4.1.6):

$$P_{ii} \approx 2\pi\varepsilon^2\sigma^2 e^{-(\sigma\Omega)^2} D^+(0, 2i\sigma^2\Omega). \quad (4.3.11)$$

Using equations (4.3.7)-(4.3.9), and taking the limit $\eta \rightarrow 0$, we obtain

$$P_{iiM} \approx \frac{\varepsilon^2 e^{-(\sigma\Omega)^2}}{8\pi\sigma^2\Omega^2}, \quad (4.3.12)$$

for the inertial Minkowski case,

$$P_{ii} \approx \frac{\varepsilon^2 e^{-(\sigma\Omega)^2}}{2\pi} \left[\frac{\kappa\sigma}{2} \csc(\kappa\sigma^2\Omega) \right], \quad (4.3.13)$$

for all the other cases, and

$$X_M = X_a \approx -\frac{\varepsilon^2 e^{-(\sigma\Omega)^2} \sigma^2}{2\pi L^2}, \quad (4.3.14)$$

$$X_{ds} \approx -\frac{\varepsilon^2 e^{-(\sigma\Omega)^2} \sigma^2 e^{i2\kappa\sigma^2\Omega}}{2\pi L^2}, \quad (4.3.15)$$

$$X_{th} \approx -\frac{\varepsilon^2 e^{-(\sigma\Omega)^2} \kappa\sigma^2}{4\pi L} \coth\left(\frac{L\kappa}{2}\right). \quad (4.3.16)$$

Finally, the negativity (4.1.10) for each case is given by

$$\mathcal{N}_M \approx \max \left[0, \frac{\varepsilon^2 e^{-(\sigma\Omega)^2}}{2\pi} \left[\left(\frac{\sigma}{L} \right)^2 - \frac{1}{4(\sigma\Omega)^2} \right] \right], \quad (4.3.17)$$

$$\mathcal{N}_a = \mathcal{N}_{ds} \approx \max \left[0, \frac{\varepsilon^2 e^{-(\sigma\Omega)^2}}{2\pi} \left[\left(\frac{\sigma}{L} \right)^2 - \frac{(\kappa\sigma)^2}{4} \csc^2(\kappa\sigma^2\Omega) \right] \right], \quad (4.3.18)$$

$$\mathcal{N}_{\text{th}} \approx \max \left[0, \frac{\varepsilon^2 e^{-(\sigma\Omega)^2}}{2\pi} \left[\frac{\kappa\sigma^2}{2L} \coth\left(\frac{L\kappa}{2}\right) - \frac{(\kappa\sigma)^2}{4} \csc^2(\kappa\sigma^2\Omega) \right] \right]. \quad (4.3.19)$$

We can see that the negativity of the accelerating and de Sitter cases are the same. One could have noticed that by the fact that their X values differ only by a phase. However, we emphasize that the calculations were made with two approximations, so that this result just shows that the cases are equivalent under these approximations. Moreover, their Wightman functions and the final density matrix are different. Therefore, it is still an open question whether or not there is an equivalence in harvesting entanglement between these cases.

Figure 10 – Region where harvesting entanglement is possible. The green (solid curve) represents inertial detectors in Minkowski spacetime. The red (dashed curve) represents inertial detectors in a thermal bath of field particles with temperature $T = \kappa/2\pi$ in flat spacetime. The purple (dotted curve) represents parallel accelerating detectors in Minkowski spacetime and also comoving detectors in de Sitter spacetime. In all the cases we considered $\sigma = \kappa = 1$.

Source: By the author.

Figure 10 shows the parameter region where entanglement harvesting is possible, i.e., where the negativity is greater than zero. The green (solid curve) represents inertial

Figure 11 – Negativity as a function of the energy gap of the detectors. This plot represents all the cases. We considered $\sigma = \kappa = 1$ and $L = 0.01$.

Source: By the author.

detectors in Minkowski spacetime. The red (dashed curve) represents inertial detectors in a thermal bath of field particles with temperature $T = \kappa/2\pi$ in flat spacetime. The purple (dotted curve) represents parallel accelerating detectors in Minkowski spacetime and also comoving detectors in de Sitter spacetime. In all the cases we considered $\sigma = \kappa = 1$. It is clear that entanglement harvesting decreases when the distance between the detectors increase, just like the black hole case. For inertial observers in flat spacetime, larger energy gap enhances entanglement harvesting. However, in the other scenarios, this just happens until it reaches approximately 1.5.

We can see that the parameter region where entanglement harvesting is possible in the de Sitter and accelerating cases are a proper subset of the parameter region in the thermal case, which is a proper subset of the parameter region in the inertial Minkowski case. It seems that inertial detectors in Minkowski spacetime is the most efficient scenario for this phenomenon. Notice that, although one *single* detector cannot distinguish between being in a thermal bath in flat spacetime, accelerated in the vacuum of flat spacetime and inertial in expanding de Sitter universe, *two* detectors can know if they are in a thermal bath if they consider the parameter region with no intersection.

Figure 11 shows the value of negativity as a function of the energy gap. We took $L = 0.01$ to maximize the negativity. The plot represents all the cases. Although it is still possible to harvest entanglement, larger energy gap reduces the entanglement extracted from field. One might have expected a similar result to the black hole example - where larger energy gaps always increases entanglement of the final state -, as the transition probabilities would be lower for larger energy gaps - i.e., the negative contribution of the

negativity would decrease. However, we emphasize that the term X is also dependent on the energy gap, and the enlargement of Ω causes a decrease in X . The black hole case was a special case, where the value of X and P_{ii} increases and decreases, respectively, with larger energy gaps.

4.4 Parameter dependence in Minkowski spacetime

In this section, we will analyze how entanglement harvesting is affected by change in some parameters. We will analyze how distinct configurations of the detectors affect entanglement harvesting. The configurations are: shape of the detectors (Gaussian or pointlike) and the interaction switching function (sudden or smooth Gaussian). These cases result in four scenarios:

- Gaussian spatially smeared detectors and Gaussian switching function;
- Pointlike detectors and Gaussian switching function;
- Gaussian spatially smeared detectors and sudden switching function;
- Pointlike detectors and sudden switching function.

Alongside these scenarios, we will see how the distance between the detectors, their internal degrees of freedom and the time delay between each detector interaction with the field affects the phenomenon. We will follow closely the reference (9).

We will still work with Unruh-DeWitt detectors which interact with a background scalar field. For simplicity, we are going to restrict the analyses to identical (their spatial profile and energy gap are the same) inertial detectors in the same inertial reference frame in Minkowski spacetime, that is, we will consider that the detectors are at rest. The parameters we are interested in will be added to the interaction Hamiltonian, which is given by

$$H_{\text{int}}(t) = \sum_{i \in \{A, B\}} \varepsilon_i(t) \left[e^{+i\Omega t} \sigma_i^+ + e^{-i\Omega t} \sigma_i^- \right] \int d^3\mathbf{x}' F(\mathbf{x}' - \mathbf{x}_i) \phi(\mathbf{x}', t), \quad (4.4.1)$$

where $\varepsilon_i(t)$ is the coupling function, which controls the coupling strength and the interaction time, $\sigma_i^\pm$ are the ladder operators of the detectors, Ω is the energy gap of the detectors, $F(\mathbf{x})$ is spatial smearing function of the detectors, which defines the detectors shape, and $\mathbf{x}_i$ are the detectors' center of mass.

As the massless scalar field operator is given by

$$\phi(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2|\mathbf{k}|}} \left[a_{\mathbf{k}} e^{-i(|\mathbf{k}|t - \mathbf{k} \cdot \mathbf{x})} + a_{\mathbf{k}}^\dagger e^{i(|\mathbf{k}|t - \mathbf{k} \cdot \mathbf{x})} \right], \quad (4.4.2)$$

there will be two integrals inside the Hamiltonian. To simplify, and write it in terms of just one integral, we can write $F(\mathbf{x})$ in terms of its Fourier transform:

$$\tilde{F}(\mathbf{k}) := \frac{1}{\sqrt{(2\pi)^3}} \int d^3\mathbf{x} F(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Inserting the field operator (4.4.2) in the Hamiltonian (4.4.1), and making the coordinate transformation $\mathbf{x} = \mathbf{x}' - \mathbf{x}_i$, we have

$$\begin{aligned} H_{\text{int}}(t) &= \sum_{i \in \{A, B\}} \varepsilon_i(t) \left[e^{+i\Omega t} \sigma_i^+ + e^{-i\Omega t} \sigma_i^- \right] \int \frac{d^3\mathbf{k}}{\sqrt{2|\mathbf{k}|}} \\ &\quad \times \left[a_{\mathbf{k}} \left(\frac{1}{\sqrt{(2\pi)^3}} \int d^3\mathbf{x} F(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \right) e^{-i(|\mathbf{k}|t - \mathbf{k}\cdot\mathbf{x}_i)} + \text{H.c.} \right], \\ &= \sum_{i \in \{A, B\}} \varepsilon_i(t) \left[e^{+i\Omega t} \sigma_i^+ + e^{-i\Omega t} \sigma_i^- \right] \int \frac{d^3\mathbf{k} \tilde{F}(\mathbf{k})}{\sqrt{2|\mathbf{k}|}} \\ &\quad \times \left[a_{\mathbf{k}}^\dagger e^{i(|\mathbf{k}|t - \mathbf{k}\cdot\mathbf{x}_i)} + a_{\mathbf{k}} e^{-i(|\mathbf{k}|t - \mathbf{k}\cdot\mathbf{x}_i)} \right] \end{aligned}$$

where H.c. denotes the Hermitian conjugate, and we considered $F(\mathbf{x})$ a real even function, and therefore $\tilde{F}(-\mathbf{k}) = \tilde{F}(\mathbf{k})^{\S\S}$.

As before, we will consider that the detectors and the field are in their ground states, and, thus, entanglement is harvested from the vacuum. The density matrix of the initial state is given by

$$\rho_0 = \rho_{AB,0} \otimes |0\rangle\langle 0|,$$

where $|0\rangle$ is the scalar field vacuum state and $\rho_{AB,0} = |\downarrow_A \downarrow_B\rangle\langle \downarrow_A \downarrow_B|$. The matrix representation of the final state of the detector, in the basis $\{|\downarrow_A \downarrow_B\rangle, |\downarrow_A \uparrow_B\rangle, |\uparrow_A \downarrow_B\rangle, |\uparrow_A \uparrow_B\rangle\}$, is given by

$$\rho_{AB} = \begin{pmatrix} 1 - \mathcal{L}_{AA} - \mathcal{L}_{BB} & 0 & 0 & \mathcal{M} \\ 0 & \mathcal{L}_{BB} & \mathcal{L}_{AB} & 0 \\ 0 & \mathcal{L}_{AB}^* & \mathcal{L}_{AA} & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\varepsilon^4). \quad (4.4.3)$$

The expressions of $\mathcal{L}_{ij}$ and $\mathcal{M}$, obtained in Appendix C, are given by

$$\mathcal{L}_{ij} = \int d^3\mathbf{k} L_i(\mathbf{k}) L_j(\mathbf{k})^*, \quad (4.4.4)$$

$$\mathcal{M} = \int d^3\mathbf{k} M(\mathbf{k}), \quad (4.4.5)$$

where

$$L_i(\mathbf{k}) = \frac{e^{-i\mathbf{k}\cdot\mathbf{x}_i} \tilde{F}(\mathbf{k})}{\sqrt{2|\mathbf{k}|}} \int_{-\infty}^{+\infty} dt \varepsilon_i(t) e^{i(|\mathbf{k}| + \Omega)t}, \quad (4.4.6)$$

^{\S\S}If $F(\mathbf{x})$ is real and even, then $\tilde{F}(\mathbf{k})$ is real. Therefore, $\tilde{F}(\mathbf{k}) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3\mathbf{x} F(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{\sqrt{(2\pi)^3}} \int d^3\mathbf{x} F(\mathbf{x})^* e^{i\mathbf{k}\cdot\mathbf{x}} = \left(\frac{1}{\sqrt{(2\pi)^3}} \int d^3\mathbf{x} F(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right)^* = \tilde{F}(-\mathbf{k})^* = \tilde{F}(-\mathbf{k})$.

$$M(\mathbf{k}) = -e^{i\mathbf{k}\cdot(\mathbf{x}_A - \mathbf{x}_B)} \frac{[\tilde{F}(\mathbf{k})]^2}{2|k|} \int_{-\infty}^{+\infty} dt \int_{-\infty}^t dt' e^{-i|\mathbf{k}|(t-t')} e^{i\Omega(t+t')} \\ \times [\varepsilon_A(t)\varepsilon_B(t') + \varepsilon_B(t)\varepsilon_A(t')]. \quad (4.4.7)$$

It is interesting to notice here that the final state (4.4.3) has the same form of the final state (4.1.5), obtained in the first section of this chapter. This consistency is very important. Although the functions $\mathcal{M}$ and $\mathcal{L}_{ij}$ are distinct from X and P_{ij} , the only operators inside the interaction Hamiltonian are, in both cases, the ladder operators. Thus, the final states should not differ from the form of these functions.

Now, let us define the shape of the detectors as a Gaussian smearing function:

$$F(\mathbf{x}) = \frac{1}{(\sqrt{\pi}\sigma)^3} e^{-\mathbf{x}^2/\sigma^2},$$

whose Fourier transform is given by

$$\tilde{F}(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^3} e^{-\frac{1}{4}\mathbf{k}^2\sigma^2}. \quad (4.4.8)$$

The pointlike case is obtained from this function by taking the limit $\sigma \rightarrow 0$:

$$F(\mathbf{x}) = \delta^{(3)}(\mathbf{x}) \Rightarrow \tilde{F}(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^3}.$$

Finally, let us compute the terms $\mathcal{M}$ and $\mathcal{L}_{ij}$ for each scenario stated above.

Table 1 – Dimensionless parameters used to compute $\mathcal{M}$ and $\mathcal{L}_{ij}$, where T is the characteristic time scale of the switching function.

Dimensionless variable	Expression	Physical meaning
α	ΩT	Energy gap
β_i	$\mathbf{x}_i/T$	Detectors' position
$\beta = \beta_B - \beta_A $	$ \mathbf{x}_B - \mathbf{x}_A /T$	Detectors spatial distance
γ	$(t_B - t_A)/T$	Time delay
δ	σ/T	Detectors' size
$\kappa, \kappa = \kappa $	$\mathbf{k}T$	Momentum
τ	t/T	Time parameter

Source: POZAS-KERSTJENS.; MARTÍN-MARTÍNEZ. (9)

Gaussian spatially smeared detectors and Gaussian switching function

Consider a smooth Gaussian switching coupling function given by

$$\varepsilon_i(t) = \varepsilon e^{-(t-t_i)^2/T^2}, \quad (4.4.9)$$

where T is the total time duration of the interaction, the constant t_i indicates the center of the time interval T of the detector i , and $\varepsilon \ll 1$ is the positive coupling constant. To simplify, we will rewrite everything in terms of the following dimensionless variables: $\alpha := \Omega T$, $\beta_i := \mathbf{x}_i/T$, $\delta := \sigma/T$, $\boldsymbol{\kappa} := \mathbf{k}T$ and $\tau_i := t_i/T$ (these parameters are summarized in Table 1). In terms of these variables, we have

$$\varepsilon_i(t) = \varepsilon e^{-(t-\tau_i)^2}, \quad (4.4.10)$$

$$\tilde{F}(\boldsymbol{\kappa}) = \frac{1}{\sqrt{(2\pi)^3}} e^{-\frac{1}{4}\boldsymbol{\kappa}^2\delta^2}, \quad (4.4.11)$$

where $\kappa \equiv |\boldsymbol{\kappa}|$. Using the relations above and the dimensionless parameters, we can evaluate equation (4.4.6), yielding

$$L_i(\boldsymbol{\kappa}) = \varepsilon \sqrt{T} \frac{e^{-i\boldsymbol{\kappa} \cdot \beta_i} e^{-\frac{1}{4}\boldsymbol{\kappa}^2\delta^2}}{\sqrt{2\kappa(2\pi)^3}} G_1(\kappa, \tau_i), \quad (4.4.12)$$

where

$$\begin{aligned} G_1(\kappa, \tau_i) &= T \int_{-\infty}^{+\infty} d\tau e^{-(\tau-\tau_i)^2} e^{i(\kappa+\alpha)\tau} \\ &= T e^{-\frac{1}{4}(\kappa+\alpha)^2 - i(\kappa+\alpha)\tau_i} \int_{-\infty}^{+\infty} d\tau e^{-[\tau - (\tau_i - \frac{i}{2}(\kappa+\alpha))]^2} \\ &= \sqrt{\pi} T e^{-\frac{1}{4}(\kappa+\alpha)^2 - i(\kappa+\alpha)\tau_i}. \end{aligned} \quad (4.4.13)$$

To evaluate $\mathcal{L}_{ij}$, we insert the results above in equation (4.4.4). First, for $i = j$, we have

$$\begin{aligned} \mathcal{L}_{ii} &= \frac{1}{T^3} \int_{-\infty}^{+\infty} d^3\boldsymbol{\kappa} |L_i(\boldsymbol{\kappa})|^2 \\ &= \frac{\varepsilon^2 \pi}{2(2\pi)^3} \int_{-\infty}^{+\infty} d^3\boldsymbol{\kappa} e^{-\frac{1}{2}\boldsymbol{\kappa}^2\delta^2} e^{-\frac{1}{2}(\kappa+\alpha)^2} \\ &= \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{4\pi} \int_0^\infty d\kappa \kappa e^{-\left[\frac{1}{\sqrt{2}}\kappa\sqrt{1+\delta^2} + \frac{\alpha}{\sqrt{2(1+\delta^2)}}\right]^2} e^{\frac{\alpha^2}{2(1+\delta^2)}}. \end{aligned}$$

Making the coordinate transformation $x = \kappa\sqrt{\frac{1+\delta^2}{2}} + \frac{\alpha}{\sqrt{2(1+\delta^2)}}$, we get

$$\begin{aligned} \mathcal{L}_{ii} &= \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{4\pi} \frac{\sqrt{2}}{\sqrt{1+\delta^2}} \left[\int_{\frac{\alpha}{\sqrt{2(1+\delta^2)}}}^\infty dx \frac{x\sqrt{2}}{\sqrt{1+\delta^2}} e^{-x^2} \right. \\ &\quad \left. - \frac{\alpha}{1+\delta^2} \int_{\frac{\alpha}{\sqrt{2(1+\delta^2)}}}^\infty dx e^{-x^2} \right] e^{\frac{\alpha^2}{2(1+\delta^2)}} \\ &= \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{8\pi(1+\delta^2)} \left[2 - \frac{\alpha\sqrt{2\pi} e^{\frac{\alpha^2}{2(1+\delta^2)}} \operatorname{erfc}\left(\frac{\alpha}{\sqrt{2(1+\delta^2)}}\right)}{\sqrt{1+\delta^2}} \right], \end{aligned} \quad (4.4.14)$$

where

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt, \quad (4.4.15)$$

is the complementary error function and

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (4.4.16)$$

is the error function. Notice that this result is independent of the indice i and, therefore, $\mathcal{L}_{AA} = \mathcal{L}_{BB}$.

Now, let us compute $\mathcal{L}_{AB}$. Using (4.4.12) and (4.4.13), we have

$$L_A(\boldsymbol{\kappa})L_B(\boldsymbol{\kappa})^* = \frac{\varepsilon^2 \pi T^3}{2\kappa(2\pi)^3} e^{-i\boldsymbol{\kappa} \cdot (\boldsymbol{\beta}_A - \boldsymbol{\beta}_B)} e^{-\frac{1}{2}\kappa^2 \delta^2} e^{-\frac{1}{2}(\kappa + \alpha)^2} e^{-i(\kappa + \alpha)(\tau_A - \tau_B)}.$$

Denoting $\gamma := \tau_B - \tau_A$ and $\beta := |\boldsymbol{\beta}_B - \boldsymbol{\beta}_A|$, and inserting the result above in equation (4.4.4), we get

$$\begin{aligned} \mathcal{L}_{AB} &= \frac{1}{T^3} \int d^3\boldsymbol{\kappa} L_A(\boldsymbol{\kappa})L_B(\boldsymbol{\kappa})^* \\ &= \frac{\varepsilon^2 e^{i\alpha\gamma}}{8\pi} \int_0^\infty d\kappa \kappa e^{-\frac{1}{2}\kappa^2(\delta^2+1)} e^{-\frac{\alpha^2}{2}} e^{\kappa(i\gamma-\alpha)} \int_{-1}^1 d(\cos\theta) e^{i\kappa\beta \cos\theta} \\ &= \frac{\varepsilon^2 e^{i\alpha\gamma} e^{-\frac{\alpha^2}{2}}}{i8\pi\beta} \left[\int_0^\infty d\kappa \kappa e^{\left[\kappa \frac{\sqrt{1+\delta^2}}{2} + \frac{\alpha-i\gamma-i\beta}{\sqrt{2(1+\delta^2)}}\right]^2 + \frac{(\alpha-i\gamma-i\beta)^2}{2(1+\delta^2)}} \right. \\ &\quad \left. - \int_0^\infty d\kappa \kappa e^{\left[\kappa \frac{\sqrt{1+\delta^2}}{2} + \frac{\alpha-i\gamma+i\beta}{\sqrt{2(1+\delta^2)}}\right]^2 + \frac{(\alpha-i\gamma+i\beta)^2}{2(1+\delta^2)}} \right]. \end{aligned}$$

Recognizing the equality $(\alpha - i\gamma \pm i\beta)^2 = -(\mp i\alpha \mp \gamma + \beta)^2$ and making the coordinate transformations $x = \kappa \sqrt{\frac{1+\delta^2}{2}} + \frac{\alpha-i\gamma-i\beta}{\sqrt{2(1+\delta^2)}}$, for the first integral, and $y = \kappa \sqrt{\frac{1+\delta^2}{2}} + \frac{\alpha-i\gamma+i\beta}{\sqrt{2(1+\delta^2)}}$, for the second integral, we obtain

$$\begin{aligned} \mathcal{L}_{AB} &= \frac{i\varepsilon^2 e^{-\frac{\alpha^2}{2}} e^{i\alpha\gamma}}{8\sqrt{2\pi}\beta\sqrt{1+\delta^2}} \left[e^{-\frac{(\beta-\gamma-i\alpha)^2}{2(1+\delta^2)}} \operatorname{erfc}\left(i \frac{\beta-\gamma-i\alpha}{\sqrt{2(1+\delta^2)}}\right) \right. \\ &\quad \left. - e^{-\frac{(\beta+\gamma+i\alpha)^2}{2(1+\delta^2)}} \operatorname{erfc}\left(-i \frac{\beta+\gamma+i\alpha}{\sqrt{2(1+\delta^2)}}\right) \right]. \end{aligned} \quad (4.4.17)$$

Let us now compute $\mathcal{M}$. Considering the dimensionless parameters, we use equations (4.4.10) and (4.4.11) to evaluate equation (4.4.7), yielding

$$M(\boldsymbol{\kappa}) = -\varepsilon^2 T e^{i\boldsymbol{\kappa} \cdot (\boldsymbol{\beta}_A - \boldsymbol{\beta}_B)} \frac{e^{-\frac{1}{2}\kappa^2 \delta^2}}{2\kappa(2\pi)^3} G_2(\kappa), \quad (4.4.18)$$

where

$$\begin{aligned} G_2(\kappa) &= T^2 \int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 e^{-i\kappa(\tau_1-\tau_2)} e^{i\alpha(\tau_1+\tau_2)} \\ &\quad \times \left[e^{-(\tau_1-\tau_B)^2} e^{-(\tau_2-\tau_A)^2} + e^{-(\tau_2-\tau_B)^2} e^{-(\tau_1-\tau_A)^2} \right]. \end{aligned}$$

This integral is a bit involved to solve. The solution is presented in details in Appendix B, and the result is given by

$$G_2(\kappa) = \frac{\pi}{2} T^2 e^{-\frac{1}{2}(\alpha^2 + \kappa^2 - 2i\gamma\alpha - 4i\alpha\tau_A)} [E(\kappa, \gamma) + E(\kappa, -\gamma)], \quad (4.4.19)$$

where

$$E(\kappa, \gamma) := e^{i\gamma\kappa} \left[1 - \operatorname{erf} \left(\frac{\gamma + i\kappa}{\sqrt{2}} \right) \right].$$

Finally, we can evaluate $\mathcal{M}$. Inserting equation (4.4.19) in (4.4.18), we have

$$\begin{aligned} \mathcal{M} &= \frac{1}{T^3} \int d^3\kappa M(\kappa) \\ &= \frac{-\varepsilon^2}{8(2\pi)^2} \int_0^\infty d\kappa e^{-\frac{1}{2}(\alpha^2 + \kappa^2(1+\delta^2) - 2i\gamma\alpha - 4i\alpha\tau_A)} [E(\kappa, \gamma) + E(\kappa, -\gamma)] \\ &\quad \times \int_{-1}^1 d(\cos\theta) e^{-i\kappa\beta \cos\theta} \\ &= - \frac{\varepsilon^2 e^{-\frac{1}{2}(\alpha^2 - 2i\gamma\alpha - 4i\alpha\tau_A)}}{i8\pi\beta} \\ &\quad \times \int_0^\infty d\kappa e^{-\frac{1}{2}\kappa^2(1+\delta^2)} \sin(\beta\kappa) [E(\kappa, \gamma) + E(\kappa, -\gamma)]. \end{aligned} \quad (4.4.20)$$

And its absolute value is given by

$$|\mathcal{M}| = \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{8\pi\beta} \left| \int_0^\infty d\kappa e^{-\frac{1}{2}\kappa^2(1+\delta^2)} \sin(\beta\kappa) [E(\kappa, \gamma) + E(\kappa, -\gamma)] \right|. \quad (4.4.21)$$

Considering the limit $\gamma \gg 1$, we can obtain an analytical solution for equation (4.4.21). $\gamma = (t_B - t_A)/T$ is the separation between the center of the Gaussian functions (4.4.10), that is, it is interpreted as the time delay between detectors' interaction with the field. When $\gamma \gg 1$ we can say that the overlap of the Gaussian functions is negligible. In this case, the analytical solution, $|\mathcal{M}| \approx |\mathcal{M}_{\text{non}}|$, is

$$\begin{aligned} |\mathcal{M}_{\text{non}}| &= \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{8\pi\beta} \left| \int_0^\infty d\kappa e^{\frac{1}{2}\kappa^2(1+\delta^2) + i\kappa(-\gamma+\beta)} - \int_0^\infty d\kappa e^{\frac{1}{2}\kappa^2(1+\delta^2) + i\kappa(\gamma+\beta)} \right| \\ &= \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{8\sqrt{2\pi}\beta\sqrt{1+\delta^2}} \left| e^{-\frac{(\beta-\gamma)^2}{2(1+\delta^2)}} \left[1 + \operatorname{erf} \left(i \frac{\beta - \gamma}{\sqrt{2(1+\delta^2)}} \right) \right] \right. \\ &\quad \left. - e^{-\frac{(\gamma+\beta)^2}{2(1+\delta^2)}} \left[1 - \operatorname{erf} \left(i \frac{\beta + \gamma}{\sqrt{2(1+\delta^2)}} \right) \right] \right|. \end{aligned} \quad (4.4.22)$$

Pointlike detectors and Gaussian switching function

This is a special case of the one considered above, where we take the limit $\delta \rightarrow 0$, yielding

$$\mathcal{L}_{ii} = \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{8\pi\beta} \left[2 - \alpha\sqrt{2\pi} e^{\frac{\alpha^2}{2}} \operatorname{erfc} \left(\frac{\alpha}{\sqrt{2}} \right) \right], \quad (4.4.23)$$

$$\mathcal{L}_{AB} = \frac{i\varepsilon^2 e^{-\frac{\alpha^2}{2}} e^{i\alpha\gamma}}{8\sqrt{2\pi}\beta} \left[e^{-\frac{(\beta-\gamma-i\alpha)^2}{2}} \operatorname{erfc} \left(i \frac{\beta - \gamma - i\alpha}{\sqrt{2}} \right) \right]$$

$$-e^{-\frac{(\beta+\gamma+i\alpha)^2}{2}} \operatorname{erfc}\left(-i\frac{\beta+\gamma+i\alpha}{\sqrt{2}}\right)\Big], \quad (4.4.24)$$

$$|\mathcal{M}| = \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{8\pi\beta} \left| \int_0^\infty d\kappa e^{-\frac{1}{2}\kappa^2} \sin(\beta\kappa) [E(\kappa, \gamma) + E(\kappa, -\gamma)] \right|, \quad (4.4.25)$$

$$|\mathcal{M}_{\text{non}}| = \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{8\sqrt{2\pi}\beta} \left| e^{-\frac{(-\gamma+\beta)^2}{2}} \left[1 + \operatorname{erf}\left(i\frac{\beta-\gamma}{\sqrt{2}}\right) \right] - e^{-\frac{(\gamma+\beta)^2}{2}} \left[1 - \operatorname{erf}\left(i\frac{\beta+\gamma}{\sqrt{2}}\right) \right] \right|. \quad (4.4.26)$$

In this case there is another analytical solution for $\mathcal{M}$, which is when both detectors are turned on and off at the same time, that is, $\gamma = 0$. Denoting this case as $\mathcal{M}_{\text{coinc}}$, equation (4.4.19) takes the form $G_2(\boldsymbol{\kappa}) = \pi T^2 e^{\frac{1}{2}(\alpha^2 + \kappa^2)} \operatorname{erfc}(i\kappa/\sqrt{2})$, and

$$\begin{aligned} \mathcal{M}_{\text{coinc}} &= \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{16\pi^2} \int d^3\boldsymbol{\kappa} \frac{e^{-i\boldsymbol{\kappa} \cdot \boldsymbol{\beta} - \kappa^2/2}}{\kappa} \operatorname{erfc}(i\kappa/\sqrt{2}) \\ &= \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{4\pi} \int_0^\infty dt e^{-t^2} \int d\kappa e^{-i\sqrt{2}t\kappa} \kappa \int_{-1}^1 d(\cos\theta) e^{-i\kappa\beta \cos\theta} \\ &= \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{4\pi\sqrt{\pi}\beta} \int_0^\infty dt e^{-t^2} \lim_{\eta \rightarrow 0^+} \left[\int_0^\infty d\kappa e^{-i\kappa(\sqrt{2}t + \beta - i\eta)} - \int_0^\infty d\kappa e^{-i\kappa(\sqrt{2}t - \beta - i\eta)} \right] \\ &= \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{4\pi\sqrt{\pi}\beta} \int_0^\infty dt e^{-t^2} \lim_{\eta \rightarrow 0^+} \left[\frac{1}{\sqrt{2}t + \beta - i\eta} - \frac{1}{\sqrt{2}t - \beta - i\eta} \right]. \end{aligned}$$

To solve the limits above we will use the Sokhotski-Plemelj formula:

$$\lim_{\eta \rightarrow 0^+} \int_a^b \frac{f(x)}{x - i\eta} dx = \mathcal{P} \int_a^b \frac{f(x)}{x} dx + i\pi f(0), \quad (4.4.27)$$

where $\mathcal{P}$ denotes the Cauchy principal value. Placing all the terms of the equation above in the same integral yields

$$\lim_{\eta \rightarrow 0^+} \left(\frac{1}{x - i\eta} \right) = \frac{1}{x} + i\pi\delta(x),$$

where $\delta(x)$ is the Dirac delta. Therefore, we get

$$\mathcal{M}_{\text{coinc}} = \frac{\varepsilon^2 e^{-\frac{\alpha^2}{2}}}{2\pi\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2}}{2t^2 - \beta^2} dt.$$

The result of the integral above can be found in a table of integrals^{¶¶}, resulting in

$$\mathcal{M}_{\text{coinc}} = \frac{\varepsilon^2 e^{-\frac{\alpha^2 + \beta^2}{2}}}{4\sqrt{2\pi}\beta} \left[\operatorname{erfi}\left(\frac{\beta}{\sqrt{2}}\right) - i \right]. \quad (4.4.28)$$

^{¶¶}Equation NI19(13), or 1 from section 3.466, of (38)

Gaussian spatially smeared detectors and sudden switching function

Consider that the detector i is, abruptly, switched on at T_i^{on} and switched off at T_i^{off} . The coupling function is now given by

$$\varepsilon_i(t) = \begin{cases} \varepsilon, & \text{if } T_i^{\text{on}} < t < T_i^{\text{off}}, \\ 0, & \text{otherwise.} \end{cases} \quad (4.4.29)$$

We will still use the dimensionless parameters (see Table 1), but now the time scale of the interaction is given by $T = T_A^{\text{off}} - T_A^{\text{on}} = T_B^{\text{off}} - T_B^{\text{on}}$ and the time delay between interactions takes the form $\gamma = \tau_B^{\text{on}} - \tau_A^{\text{on}} = \tau_B^{\text{off}} - \tau_A^{\text{off}}$. Notice that, because of the form of T , $\tau_i^{\text{off}} - \tau_i^{\text{on}} = 1$.

Again, we start by computing $\mathcal{L}_{ij}$. Inserting the coupling function (4.4.29) in equation (4.4.6) results in

$$L_i(\boldsymbol{\kappa}) = \varepsilon \sqrt{T} \frac{e^{-i\boldsymbol{\kappa} \cdot \boldsymbol{\beta}_i - \frac{1}{4}\kappa^2 \delta^2}}{\sqrt{2\kappa(2\pi)^3}} S_1(\kappa, \tau_i), \quad (4.4.30)$$

where

$$\begin{aligned} S_1(\kappa, \tau_i) &= \frac{T}{\varepsilon} \int_{-\infty}^{+\infty} d\tau \varepsilon_i(t) e^{i(\kappa+\alpha)\tau} \\ &= -iT \frac{\left(e^{i\tau_i^{\text{off}}(\kappa+\alpha)} - e^{i\tau_i^{\text{on}}(\kappa+\alpha)} \right)}{\kappa + \alpha}. \end{aligned} \quad (4.4.31)$$

Using this result and equation (4.4.4), for $i = j$, yields to

$$\begin{aligned} \mathcal{L}_{ii} &= \frac{\varepsilon^2}{4\pi^2} \int_0^\infty d\kappa \frac{\kappa e^{-\frac{1}{2}\kappa^2 \delta^2}}{(\kappa + \alpha)^2} \left(2 - e^{i(\tau_i^{\text{off}} - \tau_i^{\text{on}})(\kappa+\alpha)} - e^{-i(\tau_i^{\text{off}} - \tau_i^{\text{on}})(\kappa+\alpha)} \right) \\ &= \frac{\varepsilon^2}{\pi^2} \int_0^\infty d\kappa \frac{\kappa e^{-\frac{1}{2}\kappa^2 \delta^2}}{(\kappa + \alpha)^2} \sin^2 \left[\frac{1}{2}(\kappa + \alpha) \right]. \end{aligned} \quad (4.4.32)$$

For $i \neq j$, we have

$$\begin{aligned} \mathcal{L}_{AB} &= \frac{\varepsilon^2}{8\pi^2} \int_0^\infty d\kappa \frac{\kappa e^{-\frac{1}{2}\kappa^2 \delta^2}}{(\kappa + \alpha)^2} \left[e^{-i\gamma(\kappa+\alpha)} - e^{-i(\gamma-1)(\kappa+\alpha)} \right. \\ &\quad \left. - e^{-i(\gamma+1)(\kappa+\alpha)} + e^{-i\gamma(\kappa+\alpha)} \right] \int_{-1}^1 d(\cos \theta) e^{i\kappa\beta \cos \theta} \\ &= \frac{\varepsilon^2}{4\pi^2 \beta} \int_0^\infty d\kappa \frac{e^{-\frac{1}{2}\kappa^2 \delta^2 - i(\gamma+1)(\kappa+\alpha)}}{(\kappa + \alpha)^2} \sin(\beta\kappa) \left(e^{i(\kappa+\alpha)} - 1 \right)^2. \end{aligned} \quad (4.4.33)$$

Now, let us compute $\mathcal{M}$. Using equations (4.4.29) and (4.4.7), we obtain

$$M(\boldsymbol{\kappa}) = -\varepsilon^2 T e^{i\boldsymbol{\kappa} \cdot (\boldsymbol{\beta}_A - \boldsymbol{\beta}_B)} \frac{e^{-\frac{1}{2}\kappa^2 \delta^2}}{2\kappa(2\pi)^3} S_2(\kappa), \quad (4.4.34)$$

where

$$S_2(\kappa) = \frac{1}{\varepsilon^2} \int_{-\infty}^{+\infty} dt \int_{-\infty}^t dt' e^{i\Omega(t+t')} e^{-i\kappa(t-t')} [\varepsilon_A(t) \varepsilon_B(t') + \varepsilon_B(t) \varepsilon_A(t')]. \quad (4.4.35)$$

To solve equation (4.4.35), we will deal with two distinct cases: the coupling functions do or do not overlap. Let us begin with the non-overlapping case, which we will denote $S_{2\text{non}}(\kappa)$. Without loss of generality, we assume that the detector- A interaction is switched off before the detector- B interaction is switched on. In this case, the integral proportional to $\varepsilon_A(t) \varepsilon_B(t')$ is zero. Therefore, we have

$$\begin{aligned} S_{2\text{non}}(\kappa) &= \frac{T^2}{\kappa^2 - \alpha^2} \left[e^{i(\alpha-\kappa)\tau_B^{\text{off}}} - e^{i(\alpha-\kappa)\tau_B^{\text{on}}} \right] \left[e^{i(\alpha+\kappa)\tau_A^{\text{off}}} - e^{i(\alpha+\kappa)\tau_A^{\text{on}}} \right] \\ &= \frac{T^2}{\kappa^2 - \alpha^2} e^{i\alpha(\tau_B^{\text{on}} + \tau_A^{\text{on}}) - i\kappa\gamma} \left[e^{i(\alpha-\kappa)} - 1 \right] \left[e^{i(\alpha+\kappa)} - 1 \right]. \end{aligned}$$

Inserting the result above in equation (4.4.34) and, then, using equation (4.4.5), we get

$$\begin{aligned} |\mathcal{M}_{\text{non}}| &= \left| \frac{-\varepsilon^2}{2(2\pi)^2} \int_0^\infty d\kappa \frac{e^{i\alpha(\tau_B^{\text{on}} + \tau_A^{\text{on}}) - i\kappa\gamma - \frac{1}{2}\delta^2\kappa^2}}{\kappa^2 - \alpha^2} \left[e^{i(\alpha-\kappa)} - 1 \right] \right. \\ &\quad \times \left. \left[e^{i(\alpha+\kappa)} - 1 \right] \int_{-1}^1 d(\cos\theta) e^{-i\kappa\beta \cos\theta} \right| \\ &= \frac{\varepsilon^2}{4\pi^2\beta} \left| \int_0^\infty d\kappa \frac{e^{-i\kappa\gamma - \frac{1}{2}\delta^2\kappa^2}}{\kappa^2 - \alpha^2} \sin(\kappa\beta) \left[e^{i(\alpha-\kappa)} - 1 \right] \left[e^{i(\alpha+\kappa)} - 1 \right] \right|. \quad (4.4.36) \end{aligned}$$

For the overlapping case, without loss of generality, we assume that the detector- A interaction is switched on simultaneously or before detector- B interaction and detector- B interaction is switched off simultaneously or after detector- A interaction. Under this circumstances, the total interaction time interval $[T_A^{\text{on}}, T_B^{\text{off}}]$ can be divided in one region of no overlap, $[T_A^{\text{on}}, T_B^{\text{on}}] \cup [T_A^{\text{off}}, T_B^{\text{off}}]$, and one region of overlap, $(T_B^{\text{on}}, T_A^{\text{off}})$. Let us begin with the simplest non-overlapping region.

In the interval $[T_A^{\text{on}}, T_B^{\text{on}}] \cup [T_A^{\text{off}}, T_B^{\text{off}}]$, the coupling function of detector A is non zero (equals ε) when $T_A^{\text{on}} \leq t \leq T_B^{\text{on}}$, and, later, when $T_B^{\text{on}} \leq t \leq T_A^{\text{off}}$. The non-overlapping condition implies that the integral proportional to $\varepsilon_A(t) \varepsilon_B(t')$, in equation (4.4.35), is zero. Therefore, we have

$$\begin{aligned} S_2^{[T_A^{\text{on}}, T_B^{\text{on}}]}(\kappa) &= T^2 \int_{\tau_B^{\text{on}}}^{\tau_B^{\text{off}}} d\tau \int_{\tau_A^{\text{on}}}^{\tau_A^{\text{on}}} d\tau' e^{i(\alpha-\kappa)\tau + i(\alpha+\kappa)\tau'} \\ &= \frac{T^2}{\alpha^2 - \kappa^2} e^{i\alpha(\tau_A^{\text{on}} + \tau_B^{\text{on}} - \gamma)} \\ &\quad \times \left[e^{i(1+\gamma)(\alpha-\kappa)} - e^{i\alpha(1+2\gamma) - i\kappa} + e^{2i\gamma\alpha} - e^{i\gamma(\alpha-\kappa)} \right] \quad (4.4.37) \end{aligned}$$

for the first part of the region, and

$$S_2^{[T_A^{\text{off}}, T_B^{\text{off}}]}(\kappa) = T^2 \int_{\tau_A^{\text{off}}}^{\tau_B^{\text{off}}} d\tau \int_{\tau_A^{\text{on}}}^{\tau_A^{\text{off}}} d\tau' e^{i(\alpha-\kappa)\tau + i(\alpha+\kappa)\tau'}$$

$$\begin{aligned}
&= \frac{T^2}{\alpha^2 - \kappa^2} e^{i\alpha(\tau_A^{\text{on}} + \tau_B^{\text{on}} - \gamma)} \\
&\quad \times \left[e^{i\alpha(1+2\gamma) - i\kappa} - e^{i\alpha(2+\gamma) - \gamma\kappa} + e^{2i\alpha} - e^{i\alpha(1+\gamma) + \kappa(\gamma-1)} \right]
\end{aligned} \tag{4.4.38}$$

for the second part of the region.

Let us now deal with the overlapping region $(T_B^{\text{on}}, T_A^{\text{off}})$. In this case, the coupling functions are non zero when $T_B^{\text{on}} \leq t \leq T_A^{\text{off}}$. Now, there are contribution of both terms of the equation (4.4.35), and they both equal ε^2 . Therefore, we have

$$\begin{aligned}
S_2^{(T_B^{\text{on}}, T_A^{\text{off}})}(\kappa) &= 2T^2 \int_{\tau_B^{\text{on}}}^{\tau_A^{\text{off}}} d\tau e^{i(\alpha - \kappa)\tau} \int_{\tau_B^{\text{on}}}^{\tau} d\tau' e^{i(\alpha + \kappa)\tau'} \\
&= \frac{T^2}{\alpha^2 - \kappa^2} e^{i\alpha(\tau_A^{\text{on}} + \tau_B^{\text{on}} - \gamma)} \\
&\quad \times \left[2e^{i\alpha(\gamma+1) + i\kappa(1-\gamma)} - e^{2i\alpha\gamma} - e^{2i\alpha} - \frac{\kappa}{\alpha} (e^{2i\alpha\gamma} - e^{2i\alpha}) \right].
\end{aligned} \tag{4.4.39}$$

The solution of equation (4.4.35) is given by the sum of all contributions (4.4.37)-(4.4.39). Hence, we obtain

$$\begin{aligned}
S_{2\text{over}}(\kappa) &= S_2^{[T_A^{\text{on}}, T_B^{\text{on}}]}(\kappa) + S_2^{[T_A^{\text{off}}, T_B^{\text{off}}]}(\kappa) + S_2^{(T_B^{\text{on}}, T_A^{\text{off}})}(\kappa) \\
&= \frac{T^2}{\alpha^2 - \kappa^2} e^{i\alpha(\tau_A^{\text{on}} + \tau_B^{\text{on}} - \gamma)} \left[e^{i(\gamma+1)(\alpha - \kappa)} (e^{2i\gamma\kappa} - e^{i(\alpha + \kappa)} + 1) \right. \\
&\quad \left. + e^{i\gamma(\alpha - \kappa)} - \frac{\kappa}{\alpha} (e^{2i\alpha\gamma} - e^{2i\alpha}) \right].
\end{aligned} \tag{4.4.40}$$

Inserting in equation (4.4.34), equation (4.4.5) reads

$$|\mathcal{M}_{\text{over}}| = \frac{\varepsilon^2}{4\pi^2\beta} \left| \int_0^\infty d\kappa \sin(\beta\kappa) e^{-\frac{1}{2}\delta^2\kappa^2} S_{2\text{over}}(\kappa) \right|. \tag{4.4.41}$$

Pointlike detectors and sudden switching function

Sudden switching for pointlike detectors leads to divergences (9, 49). Nevertheless, we can consider quasi-pointlike detectors, for which we will use the expression obtained for Gaussian spatially smeared detectors and sudden switching function and take the spatial smearing to be much smaller than the duration of the interaction, i.e., $\sigma/T = \delta \ll 1$.

Entanglement harvesting

With the expressions obtained above, we can quantify, for each case, the entanglement harvested by the detectors after their interaction with the vacuum state of the field. As the final state (4.4.3) takes the same form of the final state (4.1.5), obtained in the first section of this chapter, the negativity will be analog to equation (4.1.10), and it is given by

$$\mathcal{N} = \max \left[0, -\frac{1}{2} \left(\mathcal{L}_{AA} + \mathcal{L}_{BB} - \sqrt{(\mathcal{L}_{AA} - \mathcal{L}_{BB})^2 + 4|\mathcal{M}|^2} \right) \right].$$

As, in every case, $\mathcal{L}_{AA} = \mathcal{L}_{BB}$, the negativity assumes the simple form

$$\mathcal{N} = \max(0, |M| - \mathcal{L}_{ii}). \quad (4.4.42)$$

This expression tells us that the detectors will be entangled if the nonlocal term $\mathcal{M}$ is larger than the local term $\mathcal{L}_{ii}$. The impact of the parameters and case scenarios are not straightforward from equation (4.4.42). Thus, we are going to analyze some plots to compare each scenario.

In the plots, we chose the analytical solutions $|\mathcal{M}_{\text{non}}|$, equations (4.4.22) and (4.4.26), for Gaussian switching cases, and we chose the overlapping solutions $|\mathcal{M}_{\text{over}}|$, equation (4.4.41), for the sudden switching cases. We took $\delta = 1$, for the Gaussian spatial profiles, and $\delta = 0.01$, for the quasi-pointlike case.

Figures 12, 13, 14 and 15 present four plots for each scenario, and each of them represent one of the cases. The first plots of the first rows (12a, 13a, 14a, 15a) are binary plots that show the parameter region, for fixed time delay $(t_B - t_A)/T$, where it is possible to harvest entanglement. The second plots of the first rows (12b, 13b, 14b, 15b) show the values of negativity $\mathcal{N}$ in the same region. In the latter case, the dependence over the energy gap Ω is clear. The order of magnitude of the negativity decreases as the energy gap increases, although entanglement harvesting is still possible. In the sudden switching cases we chose no time delay $\gamma = 0$, which maximizes entanglement harvesting. In the Gaussian switching cases we chose $\gamma = 3$, because we cannot choose no time delay, as we are plotting the analytical approximation for $\gamma \gg 1$.

The first plots of the second rows (12c, 13c, 14c, 15c) are binary plots that show the parameter region, for fixed energy gap ΩT , where entanglement harvesting is possible. The second plots of the second rows (12d, 13d, 14d, 15d) show values of negativity $\mathcal{N}$ in half of the latter plot regions. We chose $\alpha = 2.5$, a value for which there is entanglement harvesting in all the cases. We can see that the time delay between interactions clearly affects the ability of the detectors to harvest entanglement from the field, as the time delay increases, the entanglement of the final state decreases. In the sudden switching and quasi-pointlike detectors, the order of magnitude of the negativity is greater than in the other cases, which is in accordance with the divergence of the integral as $\delta \rightarrow 0$, as we take a small value for δ . We can also see that the growth of the time delay does not seem to affect the entanglement of the final state. This different result might be associated with numerical errors.

Analyzing all the plots, we see that there is a direct dependence with the distance between the detectors, which gives an actual limit to entanglement harvesting. The greater the distance between the detectors, the smaller the amount of entanglement harvested. The second thing we notice is that the switching manner is extremely relevant. Smooth Gaussian switching are much more efficient for harvesting entanglement. This result can

be associated with the fact that sudden switching enhances the transition probabilities of the detector. By comparing the Gaussian switching cases (figures 12 and 13), we notice that pointlike detectors are able to harvest “more” entanglement - in the sense that the magnitude of the negativity $\mathcal{N}$ is greater - even though it harvests for lower ranges of the parameters.

Finally, we emphasize that the results of this section may not be general. We have already seen in previous sections that the enlargement of energy gap does not permanently enhance the parameter region where entanglement harvesting is possible. However, as the Minkowski case was the most efficient one, we would expect the dependences over spatial smearing to be general - as pointlike detectors suppress the ability to harvest entanglement.

Figure 12 – Negativity for Gaussian switching and Gaussian spatial profile. The binary plots on the left show in the red region where entanglement harvesting is possible, and in gray region where it is not possible. The plots on the right show the values of negativity $\mathcal{N}$. The plots in the first row show different values for the distance between the detectors $(\mathbf{x}_B - \mathbf{x}_A)/T$ and the energy gap ΩT , for a fixed time delay between interaction $(t_B - t_A)/T = 3$. The plots in the second row show different values for the distance between the detectors $(\mathbf{x}_B - \mathbf{x}_A)/T$ and the time delay between interaction $(t_B - t_A)/T$, for a fixed energy gap $\Omega T = 2.5$. In all the plots we chose $\sigma/T = 1$. We can see that the entanglement of the final state decreases as all the parameters increase.

Source: By the author.

Figure 13 – Negativity for Gaussian switching and pointlike detectors. The binary plots on the left show in the red region where entanglement harvesting is possible, and in gray region where it is not possible. The plots on the right show the values of negativity $\mathcal{N}$. The plots in the first row show different values for the distance between the detectors $(\mathbf{x}_B - \mathbf{x}_A)/T$ and the energy gap ΩT , for a fixed time delay between interaction $(t_B - t_A)/T = 3$. The plots in the second row show different values for the distance between the detectors $(\mathbf{x}_B - \mathbf{x}_A)/T$ and the time delay between interaction $(t_B - t_A)/T$, for a fixed energy gap $\Omega T = 2.5$. We can see that the entanglement of the final state decreases as all the parameters increase. Comparing with Figure 12 we see that, even though the parameter region where entanglement harvesting is possible is smaller, the value of negativity is higher for pointlike detectors.

Source: By the author.

Figure 14 – Negativity for sudden switching and Gaussian spatial profile. The binary plots on the left show in the red region where entanglement harvesting is possible, and in gray region where it is not possible. The plots on the right show the values of negativity $\mathcal{N}$. The plots in the first row show different values for the distance between the detectors $(\mathbf{x}_B - \mathbf{x}_A)/T$ and the energy gap ΩT , for a fixed time delay between interaction $(t_B - t_A)/T = 0$. The plots in the second row show different values for the distance between the detectors $(\mathbf{x}_B - \mathbf{x}_A)/T$ and the time delay between interaction $(t_B - t_A)/T$, for a fixed energy gap $\Omega T = 2.5$. In all the plots we chose $\sigma/T = 1$. We can see that the entanglement of the final state decreases as all the parameters increase. Comparing with Figures 12 and 13, we see that sudden switching suppresses the ability of the detectors to harvest entanglement from the field.

Source: By the author.

Figure 15 – Negativity for Gaussian switching and Gaussian spatial profile. The binary plots on the left show in the red region where entanglement harvesting is possible, and in gray region where it is not possible. The plots on the right show the values of negativity $\mathcal{N}$. The plots in the first row show different values for the distance between the detectors $(\mathbf{x}_B - \mathbf{x}_A)/T$ and the energy gap ΩT , for a fixed time delay between interaction $(t_B - t_A)/T = 0$. The plots in the second row show different values for the distance between the detectors $(\mathbf{x}_B - \mathbf{x}_A)/T$ and the time delay between interaction $(t_B - t_A)/T$, for a fixed energy gap $\Omega T = 2.5$. In all the plots we chose $\sigma/T = 0.01$. We can see that, in this case, it is very difficult to harvest entanglement. In contrast to the other cases, the entanglement of the final state does not decrease when the time delay between interactions increases. This different result may be due to numerical errors.

Source: By the author.

5 CONCLUSIONS

This work is centered in the recently discovered phenomenon of entanglement harvesting. In chapter 2 we introduced entanglement in a historical manner. The main motivation of this approach was to show the conceptual importance of this quantum property. Unlike many Quantum Mechanical properties, entanglement was derived from the theory itself, that is, there were no previous experiments made such that entanglement was needed to explain the results. The important thing is that a consequence of a probabilistic physical theory, that physicists with a “classical mind”, like Einstein, were unwilling to accept, was proved right by experiments, defying the deterministic hidden local-variable theories. It is just another evidence that the microscopic world is *not* deterministic.

In chapter 3 we introduced Quantum Field Theory in Curved Spacetime and two of its most famous results, the Unruh effect and the Hawking radiation. This is a step further in comparison to the quantum theory introduced in chapter 2. Now, the states are not fundamental objects, but fields are. Working in general spacetimes also enhances our ability to predict new phenomena. We learned that the concept of particles is observer dependent, and, thus, all the phenomena in the one-particle theory that depend on a precise definition of states should be observer dependent.

In chapter 4 we introduced the phenomenon of entanglement harvesting. We began by determining the conditions for two Unruh-DeWitt detectors to extract entanglement from the field. We found out that the amount of entanglement harvested depends on the transition probabilities of the detectors, decreasing as the probabilities increase.

Our first example was entanglement harvesting near a BTZ black hole. We found out that, near the black hole horizon, there is a point of sudden death of entanglement harvesting. This is due to redshift effect and the Hawking radiation. It is important to point out that the results of this phenomenon are compatible with well know phenomena, especially the Hawking effect, as both entanglement harvesting and Hawking radiation are results of QFT in CST, and a disagreement would mean that there is a flaw in this theory. Finally, we would expect these results to be general for any black holes, as they depend on general black hole properties, and the sudden death should also be present in spacetimes in which there is a point where the redshift factor goes to zero.

Following, we compared the phenomenon in different scenarios. It was interesting to find out that the most efficient way to harvest entanglement is using inertial detectors in the vacuum of Minkowski spacetime. More surprisingly, we discovered that, although one detector cannot distinguish between being inertial in a thermal bath in flat spacetime and being in uniform acceleration in the vacuum or being inertial in a de Sitter expanding

universe, two detectors can.

Finally, we analyzed how the detectors' parameters affect entanglement harvesting. In accordance to the previous examples, we saw that larger distances between the detectors decrease entanglement harvesting, and, although it can be enhanced by increasing the detectors' energy gap, the amount of entanglement extracted for large values of these parameters is very small. The most remarkable result is that the entanglement of the final state depends severely on the smoothness of the switching of the detectors' coupling to the field, as sudden switching makes the phenomenon very inefficient. With these results, we expect that entanglement harvesting is sensitive to the harvesting apparatus, giving different results if one considers others than Unruh-deWitt detectors.

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Appendix

APPENDIX A – GENERAL RELATIVITY

In this appendix we will give a brief review of the theory of General Relativity, focusing on elements that were used in the main text. It is important to emphasize that we assume previous knowledge of the reader, and here we simply set notation. We consider natural units, $G = c = 1$, and metric signature $(+, -, -, -)$.

In General Relativity, gravity is represented as curvature in a four-dimensional spacetime. The curvature is codified in the *metric tensor*, $g_{\mu\nu}$, where the indices μ, ν (and any Greek indices) take values from 0 to 3. The metric of the flat spacetime (i.e., with no gravity), is $\eta_{\mu\nu}$ and its matrix representation in inertial Cartesian coordinates is $\text{diag}(1, -1, -1, -1)$. Given the metric $g_{\mu\nu}$, the *Christoffel symbols* are defined as

$$\Gamma_{\alpha\beta}^{\mu} \equiv \frac{1}{2}g^{\gamma\mu}(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}),$$

where the comma indicates partial derivative. These symbols are not tensors, and thus they are not covariant under coordinate transformation. We use the Christoffel symbols to express the *Riemann curvature tensor*, defined as

$$R^{\alpha}_{\beta\mu\nu} \equiv \Gamma_{\beta\mu,\nu}^{\alpha} - \Gamma_{\beta\nu,\mu}^{\alpha} + \Gamma_{\sigma\nu}^{\alpha}\Gamma_{\beta\mu}^{\sigma} - \Gamma_{\sigma\mu}^{\alpha}\Gamma_{\beta\nu}^{\sigma}.$$

Curvature of the spacetime is also partially codified in the *Ricci tensor*, $R_{\alpha\beta} \equiv R^{\mu}_{\alpha\mu\beta}$, and the *Ricci scalar*, $R \equiv g^{\alpha\beta}R_{\alpha\beta} = R^{\alpha}_{\alpha}$. In Minkowski spacetime all these tensors equal zero. The *Einstein's field equations* are:

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \quad (\text{A.0.1})$$

where $G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is the *Einstein tensor*, Λ is the *cosmological constant* and $T_{\alpha\beta}$ is the *energy-momentum tensor* that codifies the content of matter and energy. These equations are well summarized in John Wheeler's words: “*Spacetime tells matter how to move; matter tells spacetime how to curve*”.

The *line element* of a metric is defined as $ds^2 \equiv g_{\mu\nu} dx^{\mu} dx^{\nu}$, where $x = (x^0, x^i)$ are coordinates. We take x^0 to be the timelike coordinate and x^i , $i \in \{1, 2, 3\}$, to be spacelike coordinates. The line element of the Minkowski spacetime is $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$. The line element is the separation between two arbitrarily close events in spacetime. If, for two events, $ds^2 = 0$, we say they are lightlike separated, and this separation represents light trajectories. If $ds^2 > 0$, we say they are timelike separated, and the events are casually connected, i.e., one event can affect the other. If $ds^2 < 0$, we say they are spacelike separated, and there is no causal connection between the events.

A *static* spacetime “does not change in time”, that is, in some appropriate coordinate system the metric does not depend on the coordinate $t = x^0$, and the geometry is unchanged

by time reversal, $t \rightarrow -t$. Its line element is given by

$$ds^2 = g_{00}(x^i) dt^2 - h_{jk}(x^i) dx^j dx^k, \quad (\text{A.0.2})$$

where h_{jk} is the induced metric of the hypersurface Σ_{x^0} , which is orthogonal to $\frac{\partial}{\partial x^0}$. Examples of static metrics are the Minkowski metric and the spheric symmetric vacuum solution of the Einstein equations, called Schwarzschild metric, given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2. \quad (\text{A.0.3})$$

For more information about static spacetimes, see pages 128-130 of (37).

A vector field ξ^μ is a *Killing vector field* if the infinitesimal coordinate transformation $x'^\mu = x^\mu + \varepsilon \xi^\mu$, with $\varepsilon \ll 1$, leaves the metric invariant, that is,

$$g'_{\rho\sigma}(x') = g_{\rho\sigma}(x') = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma}.$$

Killing vector fields define symmetries in the spacetime. For example, in Minkowski spacetime they represent all the possible invariant transformations, i.e., translations, rotations and boosts. One could have defined static spacetime in terms of Killing vector fields, in a coordinate-free way (37). A Killing vector field is said to be timelike if $g^{\mu\nu} \xi_\mu \xi_\nu > 0$. If a Killing vector field is timelike, one can always make a coordinate transformation such that $\xi^\mu = \frac{\partial}{\partial x^0}$. When x^0 has the meaning of time, a timelike Killing vector field indicates time-translation symmetry in the spacetime.

APPENDIX B – EXPLICIT COMPUTATION OF $G_2(\kappa)$ (4.4.19)

In this appendix we compute $G_2(\kappa)$. Let us start by making the coordinate transformations $\tau := \tau_1 + \tau_A$ and $\tau' := \tau_2 + \tau_A$:

$$G_2(\kappa) = T^2 e^{-2i\alpha\tau_A} \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{\tau} d\tau' e^{-i\kappa(\tau-\tau')} e^{i\alpha(\tau+\tau')} \\ \times \left[e^{-(\tau-\gamma)^2} e^{-\tau'^2} + e^{-(\tau'-\gamma)^2} e^{-\tau^2} \right].$$

Splitting the integral in τ' into two integrals, taking 0 as the splitting point, we get

$$G_2(\kappa) = T^2 e^{-2i\alpha\tau_A} \int_{-\infty}^{+\infty} d\tau e^{i(\alpha-\kappa)\tau} e^{-\frac{1}{4}(\alpha+\kappa)^2} \left\{ e^{-(\tau-\gamma)^2} \left[\int_{-\infty}^0 d\tau' e^{(\tau'-i(\alpha+\kappa)/2)^2} \right. \right. \\ \left. \left. + \int_0^{\tau} d\tau' e^{(\tau'-i(\alpha+\kappa)/2)^2} \right] + \left[\int_{-\infty}^0 d\tau' e^{[(\tau'-\gamma)-i(\alpha+\kappa)/2]^2} \right. \right. \\ \left. \left. + \int_0^{\tau} d\tau' e^{[(\tau'-\gamma)-i(\alpha+\kappa)/2]^2} \right] e^{-\tau^2+i(\alpha-\kappa)\tau+i\gamma(\kappa+\alpha)} \right\} \\ = \frac{\sqrt{\pi}}{2} T^2 e^{-2i\alpha\tau_A} \int_{-\infty}^{+\infty} d\tau e^{i(\alpha-\kappa)\tau} e^{-\frac{1}{4}(\alpha+\kappa)^2} \\ \times \left\{ e^{(\tau-\gamma)^2} [1 + \operatorname{erf}(\tau - i(\alpha + \kappa)/2)] \right. \\ \left. + e^{-\tau^2+i(\alpha-\kappa)\tau+i\gamma(\kappa+\alpha)} [1 + \operatorname{erf}(\tau - \gamma - i(\alpha + \kappa)/2)] \right\} \\ = \frac{\sqrt{\pi}}{2} T^2 e^{-2i\alpha\tau_A} (I_1 + I_2 + I_3 + I_4), \quad (\text{B.0.1})$$

where

$$I_1 \equiv \int_{-\infty}^{+\infty} d\tau e^{i(\alpha-\kappa)\tau} e^{-\frac{1}{4}(\alpha+\kappa)^2} e^{-(\tau-\gamma)^2} \\ = \int_{-\infty}^{+\infty} d\tau e^{-[(\tau-\gamma)-i(\alpha-\kappa)/2]^2} e^{-\frac{1}{4}(\alpha+\kappa)^2} e^{-\frac{1}{4}(\alpha-\kappa)^2} e^{i(\alpha-\kappa)\gamma} \\ = \sqrt{\pi} e^{\frac{1}{2}(\kappa^2+\alpha^2)+i(\alpha-\kappa)\gamma}, \quad (\text{B.0.2})$$

$$I_2 \equiv \int_{-\infty}^{+\infty} d\tau e^{i(\alpha-\kappa)\tau} e^{-\frac{1}{4}(\alpha+\kappa)^2} e^{-\tau^2+i(\alpha-\kappa)\tau+i\gamma(\kappa+\alpha)} \\ = \sqrt{\pi} e^{-\frac{1}{2}(\kappa^2+\alpha^2)+i\gamma(\kappa+\alpha)}, \quad (\text{B.0.3})$$

$$I_3 \equiv \int_{-\infty}^{+\infty} d\tau e^{i(\alpha-\kappa)\tau} e^{-\frac{1}{4}(\alpha+\kappa)^2} e^{-(\tau-\gamma)^2} \operatorname{erf}(\tau - i(\alpha + \kappa)/2) \\ = e^{-\gamma^2} \int_{-\infty}^{+\infty} d\tau e^{-((\kappa+\alpha)/2)^2 - i(\kappa-\alpha-2i\gamma)\tau - \tau^2} \operatorname{erf}(\tau - i(\alpha + \kappa)/2), \quad (\text{B.0.4})$$

$$I_4 \equiv \int_{-\infty}^{+\infty} d\tau e^{i(\alpha-\kappa)\tau} e^{-\frac{1}{4}(\alpha+\kappa)^2} e^{-\tau^2+i(\alpha-\kappa)\tau+i\gamma(\kappa+\alpha)} \operatorname{erf}(\tau - \gamma - i(\alpha + \kappa)/2) \\ = e^{-\gamma^2} \int_{-\infty}^{+\infty} d\tau e^{-(-i\gamma+(\kappa+\alpha)/2)^2 - i(\kappa-\alpha)\tau - \tau^2} \operatorname{erf}(\tau - \gamma - i(\alpha + \kappa)/2). \quad (\text{B.0.5})$$

Defining

$$I(a, b) := \int_{-\infty}^{+\infty} dy e^{-a^2 - iby - y^2} \operatorname{erf}(y - ia), \quad (\text{B.0.6})$$

we see that $I_3 = e^{-\gamma^2} I((\kappa + \alpha)/2, \kappa - \alpha - 2i\gamma)$ and $I_4 = e^{-\gamma^2} I(-i\gamma + (\kappa + \alpha)/2, \kappa - \alpha)$. We can find the solution of $I(a, b)$ by solving a differential equation. The partial differentiation of $I(a, b)$ with respect to the parameter a gives

$$\begin{aligned} \frac{\partial}{\partial a} I(a, b) &= -2a \int_{-\infty}^{+\infty} dy e^{-a^2 - iby - y^2} \operatorname{erf}(y - ia) \\ &\quad - i \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dy e^{-[\sqrt{2}y - i(2a-b)/2]^2 - (b-2a)^2/8} \\ &= -2a I(a, b) - i\sqrt{2} e^{-\frac{1}{8}(b-2a)^2}. \end{aligned} \quad (\text{B.0.7})$$

This is a first-order, linear, non-homogeneous, partial differential equation. We will solve it by the variation-of-constants method. The solution of the homogeneous equation, $\frac{\partial}{\partial a} I_H(a, b) + 2a I_H(a, b) = 0$, is $I_H(a, b) = C e^{-a^2}$, where C is a constant. Allowing the constant to be a function of a and b , $C = C(a, b)$, and, then, inserting the homogeneous-equation solution into (B.0.7), we have

$$\frac{\partial}{\partial a} C(a, b) e^{-a^2} = -i\sqrt{2} e^{-\frac{1}{8}(b-2a)^2}.$$

Integrating the above equation, by completing squares and making the coordinate transformation $x = (2a + b)/(2\sqrt{2})$, we obtain

$$C(a, b) = -i\sqrt{\pi} e^{-\frac{b^2}{4}} \operatorname{erfi}\left(\frac{2a + b}{2\sqrt{2}}\right),$$

where

$$\operatorname{erfi}(x) = -i \operatorname{erf}(ix) = \frac{2}{\pi} \int_0^x e^{t^2} dt \quad (\text{B.0.8})$$

is the imaginary error function. Thus, the solution of $I(a, b)$ is

$$I(a, b) = -i\sqrt{\pi} e^{-a^2 - \frac{b^2}{4}} \operatorname{erfi}\left(\frac{2a + b}{2\sqrt{2}}\right). \quad (\text{B.0.9})$$

Inserting the solutions obtained above in equation (B.0.1), yields to

$$G_2(\kappa) = \frac{\pi}{2} T^2 e^{-\frac{1}{2}(\alpha^2 + \kappa^2 - 2i\gamma\alpha - 4i\alpha\tau_A)} [E(\kappa, \gamma) + E(\kappa, -\gamma)],$$

where

$$E(\kappa, \gamma) := e^{i\gamma\kappa} \left[1 - \operatorname{erf}\left(\frac{\gamma + i\kappa}{\sqrt{2}}\right) \right].$$

APPENDIX C – EXPLICIT COMPUTATION OF ρ_{AB} (4.4.3)

In this appendix we calculate the density matrix (4.4.3). The time evolution operator is given by the Dyson series. Denoting $\mathcal{U}^{(n)}$, $n \in \mathbb{N}$, for each term of the sum (see equation (2.1.7), where this notation is explicit), the final state is

$$\begin{aligned}\rho &= \mathcal{U} \rho_0 \mathcal{U}^\dagger = (\mathcal{U}^{(0)} + \mathcal{U}^{(1)} + \mathcal{U}^{(2)} + \dots) \rho_0 (\mathcal{U}^{(0)\dagger} + \mathcal{U}^{(1)\dagger} + \mathcal{U}^{(2)\dagger} + \dots) \\ &= \rho_0 + \mathcal{U}^{(0)} \rho_0 \mathcal{U}^{(1)\dagger} + \mathcal{U}^{(1)} \rho_0 \mathcal{U}^{(0)\dagger} + \mathcal{U}^{(1)} \rho_0 \mathcal{U}^{(1)\dagger} + \mathcal{U}^{(0)} \rho_0 \mathcal{U}^{(2)\dagger} + \dots \\ &= \rho_0 + \rho_0^{(0,1)} + \rho_0^{(1,0)} + \rho_0^{(1,1)} + \rho_0^{(0,2)} + \dots,\end{aligned}$$

where $\rho_0^{(i,j)} \equiv \mathcal{U}^{(i)} \rho_0 \mathcal{U}^{(j)\dagger}$. The reduced final state, ρ_{AB} , of the detectors, regardless the field state, is given by the partial trace over the degrees of freedom of the field: $\rho_{AB} = \text{Tr}_\phi(\rho) = \sum_n \int d^3\mathbf{k} \langle n_{\mathbf{k}} | \rho | n_{\mathbf{k}} \rangle$. The partial trace over $\rho^{(i,j)}$ is going to be zero whenever the parity of i and j are different, that is, when $i + j = 2m + 1$, $m \in \mathbb{N}$, because, in this case, $\rho^{(i,j)} \propto |k\rangle\langle\ell|$, with $k \neq \ell$. Thus, the final state of the detectors is given by

$$\rho_{AB} = \rho_{AB,0} + \rho_{AB}^{(0,2)} + \rho_{AB}^{(2,0)} + \rho_{AB}^{(1,1)} + \mathcal{O}(\varepsilon^4), \quad (\text{C.0.1})$$

where $\rho_{AB}^{(i,j)} \equiv \text{Tr}_\phi[\rho^{(i,j)}]$ and $\varepsilon \ll 1$ denotes the coupling strength of the interaction ($\varepsilon_i(t) \propto \varepsilon$).

Let us compute each term of equation (C.0.1) separately:

$$\begin{aligned}\rho_{AB}^{(2,0)} &= \sum_n \int d^3\mathbf{k} \langle n_{\mathbf{k}} | \mathcal{U}^{(2)} \rho_{AB,0} | n_{\mathbf{k}} \rangle \\ &= \sum_n \int d^3\mathbf{k} \langle n_{\mathbf{k}} | \mathcal{U}^{(2)} | 0 \rangle | \downarrow_A \downarrow_B \rangle \langle \downarrow_B \downarrow_A | \langle 0 | n_{\mathbf{k}} \rangle \\ &= \langle 0 | \mathcal{U}^{(2)} | 0 \rangle | \downarrow_A \downarrow_B \rangle \langle \downarrow_B \downarrow_A | \\ &= - \int_{-\infty}^{\infty} dt \int_{-\infty}^t dt' \sum_{i \in \{A,B\}} \varepsilon_i(t) \left[e^{+i\Omega t} \sigma_i^+ + e^{-i\Omega t} \sigma_i^- \right] \int \frac{d^3\mathbf{k} \tilde{F}(\mathbf{k})}{\sqrt{2|\mathbf{k}|}} \\ &\quad \times \langle 0 | \left[a_{\mathbf{k}}^\dagger e^{i(|\mathbf{k}|t - \mathbf{k} \cdot \mathbf{x}_i)} + a_{\mathbf{k}} e^{-i(|\mathbf{k}|t - \mathbf{k} \cdot \mathbf{x}_i)} \right] \sum_{j \in \{A,B\}} \varepsilon_j(t') \left[e^{+i\Omega t'} \sigma_j^+ + e^{-i\Omega t'} \sigma_j^- \right] \\ &\quad \times \int \frac{d^3\mathbf{k}' \tilde{F}(\mathbf{k}')}{\sqrt{2|\mathbf{k}'|}} \left[a_{\mathbf{k}'}^\dagger e^{i(|\mathbf{k}'|t' - \mathbf{k}' \cdot \mathbf{x}_j)} + a_{\mathbf{k}'} e^{-i(|\mathbf{k}'|t' - \mathbf{k}' \cdot \mathbf{x}_j)} \right] | 0 \rangle | \downarrow_A \downarrow_B \rangle \langle \downarrow_B \downarrow_A | \\ &= - \int_{-\infty}^{\infty} dt \int_{-\infty}^t dt' \sum_{i \in \{A,B\}} \varepsilon_i(t) \left[e^{+i\Omega t} \sigma_i^+ + e^{-i\Omega t} \sigma_i^- \right] \int \frac{d^3\mathbf{k} \tilde{F}(\mathbf{k})}{\sqrt{2|\mathbf{k}|}} \\ &\quad \times e^{-i(|\mathbf{k}|t - \mathbf{k} \cdot \mathbf{x}_i)} \sum_{j \in \{A,B\}} \varepsilon_j(t') \left[e^{+i\Omega t'} \sigma_j^+ + e^{-i\Omega t'} \sigma_j^- \right] \int \frac{d^3\mathbf{k}' \tilde{F}(\mathbf{k}')}{\sqrt{2|\mathbf{k}'|}} \\ &\quad \times e^{i(|\mathbf{k}'|t' - \mathbf{k}' \cdot \mathbf{x}_j)} | \downarrow_A \downarrow_B \rangle \langle \downarrow_B \downarrow_A | \langle 1_{\mathbf{k}} | 1_{\mathbf{k}'} \rangle \\ &= - \int_{-\infty}^{\infty} dt \int_{-\infty}^t dt' \sum_{i \in \{A,B\}} \varepsilon_i(t) \left[e^{+i\Omega t} \sigma_i^+ + e^{-i\Omega t} \sigma_i^- \right] \int \frac{d^3\mathbf{k} [\tilde{F}(\mathbf{k})]^2}{2|\mathbf{k}|}\end{aligned}$$

$$\begin{aligned}
& \times e^{-i(|\mathbf{k}|t - \mathbf{k} \cdot \mathbf{x}_i)} e^{i(|\mathbf{k}| + \Omega)t'} \left[\varepsilon_A(t') e^{-i\mathbf{k} \cdot \mathbf{x}_A} |\uparrow_A \downarrow_B\rangle + \varepsilon_B(t') e^{-i\mathbf{k} \cdot \mathbf{x}_B} |\downarrow_A \uparrow_B\rangle \right] \langle \downarrow_B \downarrow_A | \\
& = - \int \frac{d^3 \mathbf{k} [\tilde{F}(\mathbf{k})]^2}{2|\mathbf{k}|} \int_{-\infty}^{\infty} dt \int_{-\infty}^t dt' e^{-i|\mathbf{k}|(t-t')} \left[\varepsilon_A(t) \varepsilon_A(t') e^{-i\Omega(t-t')} |\downarrow_A \downarrow_B\rangle \right. \\
& \quad + \left(\varepsilon_A(t) \varepsilon_B(t') e^{-i\Omega(t+t') + \mathbf{k} \cdot (\mathbf{x}_A - \mathbf{x}_B)} + \underbrace{\varepsilon_B(t) \varepsilon_A(t') e^{-i\Omega(t+t') - \mathbf{k} \cdot (\mathbf{x}_A - \mathbf{x}_B)}}_{\star} \right) \\
& \quad \left. \times |\uparrow_A \uparrow_B\rangle + \varepsilon_B(t) \varepsilon_B(t') e^{-i\Omega(t-t')} |\downarrow_A \downarrow_B\rangle \right] \langle \downarrow_B \downarrow_A | \\
& = \mathcal{M} |\uparrow_A \uparrow_B\rangle \langle \downarrow_B \downarrow_A| - \frac{1}{2} (\mathcal{L}_{AA} + \mathcal{L}_{BB}) |\downarrow_A \downarrow_B\rangle \langle \downarrow_B \downarrow_A|,
\end{aligned}$$

where $\mathcal{M}$ and $\mathcal{L}_{ij}$ are defined in equations (4.4.4)-(4.4.7), and the coordinate transformation $\mathbf{k} \rightarrow -\mathbf{k}$ was made in the $\star$ term.

$$\rho_{AB}^{(0,2)} = \rho_{AB}^{(2,0)\dagger} = \mathcal{M}^* |\downarrow_B \downarrow_A\rangle \langle \uparrow_A \uparrow_B| - \frac{1}{2} (\mathcal{L}_{AA} + \mathcal{L}_{BB}) |\downarrow_A \downarrow_B\rangle \langle \downarrow_B \downarrow_A|.$$

$$\begin{aligned}
\rho_{AB}^{(1,1)} &= \sum_n \int d^3 \tilde{\mathbf{k}} \langle n_{\tilde{\mathbf{k}}} | \mathcal{U}^{(1)} \rho_{AB,0} \mathcal{U}^{(1)\dagger} | n_{\tilde{\mathbf{k}}} \rangle \\
&= \sum_n \int d^3 \tilde{\mathbf{k}} \langle n_{\tilde{\mathbf{k}}} | (-i) \int_{-\infty}^{\infty} dt \sum_{i \in \{A,B\}} \varepsilon_i(t) \left[e^{+i\Omega t} \sigma_i^+ + e^{-i\Omega t} \sigma_i^- \right] \int \frac{d^3 \mathbf{k} \tilde{F}(\mathbf{k})}{\sqrt{2|\mathbf{k}|}} \\
& \quad \times \left[a_{\mathbf{k}}^\dagger e^{i(|\mathbf{k}|t - \mathbf{k} \cdot \mathbf{x}_i)} + a_{\mathbf{k}} e^{-i(|\mathbf{k}|t - \mathbf{k} \cdot \mathbf{x}_i)} \right] |0 \downarrow_A \downarrow_B\rangle \langle \downarrow_B \downarrow_A 0| \\
& \quad \times i \int_{-\infty}^{\infty} dt' \sum_{j \in \{A,B\}} \varepsilon_j(t') \left[e^{+i\Omega t'} \sigma_j^+ + e^{-i\Omega t'} \sigma_j^- \right] \\
& \quad \times \int \frac{d^3 \mathbf{k}' \tilde{F}(\mathbf{k}')}{\sqrt{2|\mathbf{k}'|}} \left[a_{\mathbf{k}'}^\dagger e^{i(|\mathbf{k}'|t' - \mathbf{k}' \cdot \mathbf{x}_j)} + a_{\mathbf{k}'} e^{-i(|\mathbf{k}'|t' - \mathbf{k}' \cdot \mathbf{x}_j)} \right] |n_{\tilde{\mathbf{k}}}\rangle \\
&= \sum_n \int d^3 \tilde{\mathbf{k}} \int_{-\infty}^{\infty} dt \sum_{i \in \{A,B\}} \varepsilon_i(t) \left[e^{+i\Omega t} \sigma_i^+ + e^{-i\Omega t} \sigma_i^- \right] |\downarrow_A \downarrow_B\rangle \int \frac{d^3 \mathbf{k} \tilde{F}(\mathbf{k})}{\sqrt{2|\mathbf{k}|}} \\
& \quad \times e^{i(|\mathbf{k}|t - \mathbf{k} \cdot \mathbf{x}_i)} \langle n_{\tilde{\mathbf{k}}} | 1_{\mathbf{k}} \rangle \int_{-\infty}^{\infty} dt' \sum_{j \in \{A,B\}} \varepsilon_j(t') \langle \downarrow_B \downarrow_A | \left[e^{+i\Omega t'} \sigma_j^+ + e^{-i\Omega t'} \sigma_j^- \right] \\
& \quad \times \int \frac{d^3 \mathbf{k}' \tilde{F}(\mathbf{k}')}{\sqrt{2|\mathbf{k}'|}} e^{-i(|\mathbf{k}'|t' - \mathbf{k}' \cdot \mathbf{x}_j)} \langle 1_{\mathbf{k}'} | n_{\tilde{\mathbf{k}}} \rangle \\
&= \int \frac{d^3 \mathbf{k} [\tilde{F}(\mathbf{k})]^2}{2|\mathbf{k}|} \int_{-\infty}^{\infty} dt \sum_{i \in \{A,B\}} e^{i(\Omega + |\mathbf{k}|)t} [\varepsilon_A(t) e^{-i\mathbf{k} \cdot \mathbf{x}_A} |\uparrow_A \downarrow_B\rangle \\
& \quad + \varepsilon_B(t) e^{-i\mathbf{k} \cdot \mathbf{x}_B} |\downarrow_A \uparrow_B\rangle] e^{-i(\Omega + |\mathbf{k}|)t'} [\varepsilon_A(t') e^{i\mathbf{k} \cdot \mathbf{x}_A} \langle \uparrow_A \downarrow_B| + \varepsilon_B(t') e^{i\mathbf{k} \cdot \mathbf{x}_B} \langle \downarrow_A \uparrow_B|] \\
&= \mathcal{L}_{AA} |\uparrow_A \downarrow_B\rangle \langle \downarrow_B \uparrow_A| + \mathcal{L}_{AB} |\uparrow_A \downarrow_B\rangle \langle \uparrow_B \downarrow_A| + \mathcal{L}_{BA} |\downarrow_A \uparrow_B\rangle \langle \downarrow_B \uparrow_A| \\
& \quad + \mathcal{L}_{BB} |\downarrow_A \uparrow_B\rangle \langle \uparrow_B \downarrow_A|.
\end{aligned}$$