Pullback dynamics of nonautonomous supercritical wave equations on compact Riemannian manifolds

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Eduardo Henrique Gomes Tavares

**Dinâmica pullback de equações da onda não-autônomas
supercríticas em variedades Riemannianas compactas**

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To my family

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*"Truth is ever to be found in the simplicity,
and not in the multiplicity and confusion of things."
(Isaac Newton).*

ABSTRACT

GOMES TAVARES, E. H. **Pullback dynamics of nonautonomous supercritical wave equations on compact Riemannian manifolds**. 2020. 66 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

This thesis is concerned with large-time dynamics of non-autonomous wave equations defined on compact Riemannian manifolds with boundary. It contains three main contributions. First, we give a detailed proof of well-posedness for the wave equation with supercritical nonlinearities and time-dependent external forces, on the energy space. It is a slight generalization of known results for autonomous problems. However our arguments are different. Thus, the wave problem can be studied as a non-autonomous dynamical system since its finite energy solution flows define a continuous evolution process. Next, we establish the existence of pullback exponential attractors to this non-autonomous system, such that any section have finite fractal dimensions on the natural energy space. Finally, in the case of external force is dependent on a parameter, we study the continuity of pullback attractors with respect to it.

Keywords: Supercritical wave equation, pullback exponential attractor, continuity of attractors.

RESUMO

GOMES TAVARES, E. H. **Dinâmica pullback de equações da onda não-autônomas supercríticas em variedades Riemannianas compactas**. 2020. 66 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2020.

A presente tese é dedicada ao estudo da dinâmica a longo-prazo de equações de ondas definidas em variedades Riemannianas compactas com bordo. Apresentamos três resultados principais. Primeiramente, estudamos a boa colocação do problema com não linearidades supercríticas e forças dependente do tempo. O resultado é uma extensão dos trabalhos anteriores para o caso autônomo. Entretanto a nossa abordagem é diferente. Assim, nosso problema gera um processo de evolução e pode ser estudado como um sistema dinâmico não autônomo. Em seguida, mostramos a existência de atratores pullback exponencial, o qual toda seção possui dimensão fractal finita no espaço de energia. Por fim, no caso em que a força externa depende de um parâmetro, estudamos a continuidade dos atratores em relação ao parâmetro.

Palavras-chave: Equação da onda supercrítica, atrator pullback exponencial, continuidade de atratores.

LIST OF SYMBOLS

$C^k([\tau, t], X)$ — Space of functions $u : [\tau, t] \rightarrow X$ that are continuously differentiable of order k

$L^p(\tau, t; X)$ — Space of functions $u : [\tau, t] \rightarrow X$ such that $\|u(\cdot)\|_X \in L^p(\tau, t)$

$L^\infty(\tau, t; X)$ — Space of functions $u : [\tau, t] \rightarrow X$ such that $\|u(\cdot)\|_X \in L^\infty(\tau, t)$

$X \hookrightarrow Y$ — The inclusion mapping from X to Y is continuous

$\xi_u(t)$ — The pair $(u(t), u_t(t))$

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INTRODUCTION

Let (M, g) be a three-dimensional compact Riemannian manifold with boundary ∂M and metric g . For $\tau \in \mathbb{R}$, we consider the following initial value problem:

$$\begin{cases} u_{tt} - \Delta_g u + \gamma u_t + f(u) = h, & t > \tau, \\ u(\tau) = u_\tau^0, & u_t(\tau) = u_\tau^1 \end{cases} \quad (1.1)$$

with Dirichlet boundary conditions. Here, Δ_g is the Laplace-Beltrami operator on M , γ is a fixed strict positive constant, f is a source with growth rate

$$f(u) \sim |u|^{p-1}u, \quad p \in [3, 5)$$

and h is a time-dependent force.

Semilinear wave equations like (1.1) describes oscillatory process in several areas of mathematical physics, see for instance (BABIN; VISHIK, 1992), (CHEPYZHOV; VISHIK, 2002), (TEMAM, 2012) and references therein.

The global well-posedness of (1.1) with critical Sobolev nonlinearity ($p = 3$) satisfying some Lipschitz condition can be shown using standard tools of semigroup theory, see for instance (CHUESHOV; ELLER; LASIECKA, 2002), (SHOWALTER, 2013). For the reader's convenience, we quote (BALL, 1978) for problems with non-Lipschitz source which the uniqueness of solution is not guaranteed. On the other hand, when the nonlinearity has supercritical Sobolev growth ($3 < p < 5$), the Nemytskii operator $f : H_0^1(M) \rightarrow L^2(M)$ is not, in general, locally Lipschitz. In 90's, Kapitanskii (KAPITANSKII, 1992) and Feireisl (FEIREISL, 1995) used a family of inequalities called Strichartz estimates (STRICHARTZ, 1977) (see also (GINIBRE; VELO, 1985), (SHATAH; STRUWE, 1993)) to overcome this difficult on compact manifolds without boundary and unbounded domains, respectively. Such estimates balance the time and space regularity of solutions of the linear part of (1.1) with suitable weights. More than ten years later, Blair, Smith and Sogge proved the inhomogeneous version of Strichartz estimates

for (1.1) on compact manifolds with boundary (BLAIR; SMITH; SOGGE, 2009). This progress allows some researchers to show the well-posedness of (1.1) in the supercritical case. We refer to (KALANTAROV; SAVOSTIANOV; ZELIK, 2016) for the Faedo-Galerkin method and (CAVALCANTI; AL., 2018) for nonlinear semigroup approach.

Once the problem is well-posed, one can study the dynamics of the solution semigroup (autonomous case) or the solution process (nonautonomous case). One of the first studied objects is the global attractors for semigroups, that is, a compact invariant (under the action of the semigroup) non-empty set which attracts all bounded sets when the time goes to infinity and it is the minimal set with this property. Moreover, in most cases it is possible to prove that the global attractor has finite fractal or Hausdorff dimension. Roughly speaking, one can see the global attractor as a subset in a space of finite dimension, which simplifies the numerical and computational analysis of the trajectories obtained by the action of the flow. Another important property is the rate of attraction of the global attractor. In general, one can prove that it attracts the trajectories exponentially. So, the attractor is less sensible under perturbations (FABRIE *et al.*, 2004). Unfortunately, there are global attractors that has slow rate of attraction, see for instance (KOSTIN, 1998). So, Foias, Sell and Temam (FOIAS; SELL; TEMAM, 1988) constructed a huge smooth set with good properties containing the global attractor, which attracts the trajectories exponentially called inertial manifold. But, the construction of inertial manifolds requires strong assumptions on the system, which reduces the class of equations that the existence of such set can be studied. To overcome this difficult, Eden, Foias, Nicolaenko and Temam (EDEN *et al.*, 1994) proposed the notion of exponential attractor as a intermediate object between the global attractor and the inertial manifold in the following sense: the exponential attractor is not smooth, but it has finite dimension and attract all trajectories exponentially.

Several works generalized the notion of exponential attractors for the nonautonomous scenario, see for instance (EFENDIEV; MIRANVILLE; ZELIK, 2003), (FABRIE; MIRANVILLE, 1998) and (MIRANVILLE, 1998), where they constructed a fixed compact set which forward attracts all the trajectories exponentially, called uniform exponential attractor. On the other hand, Efendiev, Miranville and Zelik (EFENDIEV; MIRANVILLE; ZELIK, 2005) proposed a concept of time dependent exponential attractors. Specifically, the exponential attractor is a family of compact sets indexed on $t \in \mathbb{R}$ which attracts the trajectories exponentially starting in any bounded set of the phase space. In 2010, Langa, Miranville and Real (LANGA; MIRANVILLE; REAL, 2010) introduced the concept of the exponential attractor in the pullback sense. The authors showed an abstract construction of the pullback exponential attractor and also prove that it contains the pullback attractor. Based on the works (CZAJA; EFENDIEV, 2011), (EFENDIEV; MIRANVILLE; ZELIK, 2005) and (LANGA; MIRANVILLE; REAL, 2010), Carvalho and Sonner (CARVALHO; SONNER, 2013) constructed a strong version of the pullback exponential attractor. This result can be applied in asymptotic compact process, and consequently covers a larger class of PDEs than (CZAJA; EFENDIEV, 2011) and (LANGA; MIRANVILLE; REAL, 2010). We also mentioned the recent work (YANG; LI, 2018) for another construction of pullback

attractors based on the quasi-stability inequality (see (CHUESHOV, 2015)).

Dynamics of (1.1) in bounded domains of $\mathbb{R}^3$ or in three dimensional manifolds with boundary is well known on the autonomous scenario, see for instance (ARRIETA; CARVALHO; HALE, 1992), (CHUESHOV; ELLER; LASIECKA, 2002), (CHUESHOV; LASIECKA; TOUNDYKOV, 2008), (FABRIE *et al.*, 2004) and (MA; SEMINARIO-HUERTAS, 2020) for critical Sobolev sources and (KALANTAROV; SAVOSTIANOV; ZELIK, 2016), (JOLY; LAURENT, 2013), (LIU; MENG; SUN, 2017), (LIU; MENG; ZHANG, 2019) and (MENG; LIU, 2019) for supercritical Sobolev nonlinearities. However, pullback dynamics of (1.1) with critical and supercritical sources was explored only recently. With respect to critical nonlinearities, we refer to (SUN; CAO; DUAN, 2006), (CHEPYZHOV; VISHIK, 2002), (CONTI; PATA; TEMAM, 2013), (MA; MARÍN-RUBIO; CHUÑO, 2017) and references therein. For supercritical sources we mentioned (MEI; SUN, 2019) where the authors proved the existence of uniform attractor for (1.1) in a bounded domain of $\mathbb{R}^3$.

Motivated by the above scenario, our main goals in this thesis are the well-posedness and pullback dynamics of (1.1) in a three dimensional compact Riemannian manifold with boundary. Specifically, this work is organized as follows: In Chapter 2, we present the necessary preliminaries for a better understanding of the work. In Chapter 3, we give a detailed proof of the well-posedness of (1.1). To do this, we combine some classical and modern results like Contraction Mapping Theorem, Zorn's Lemma and Strichartz estimates. Finally, in Chapter 4, we use the extra regularity of solutions of (1.1) obtained by Strichartz estimates to prove that the evolution process generated by the weak solutions has a pullback exponential attractor. We also study a parametrized family of problems and, using some results of (CARVALHO; LANGA; ROBINSON, 2012), we prove the upper-semicontinuity of the family of pullback attractors and, as a consequence, we conclude the residual continuity of this family using the results of (HOANG; OLSON; ROBINSON, 2018).

PRELIMINARIES

In this chapter, we list some basic theoretical results that will be used along this thesis. In all results below, we consider vector spaces over the real field $\mathbb{R}$ with the usual sum and multiplication by a real number

2.1 Elements of geometric analysis

In this subsection we give a brief review on Riemannian geometry present in (JOST, 2008) and recall some spectral properties of the Laplace-Beltrami operator present in (CRAIOVEANU; PUTA; RASSIAS, 2013, Chapter 3). To simplify the exposition, we consider (M, g) be a 3-dimensional compact Riemannian with smooth boundary ∂M and metric g . We also consider (V, ψ) be a local chart on M with coordinates (x_1, x_2, x_3) and denote by $T_p M$ the tangent space on M at $p \in M$. So, denoting by $\mathcal{X}(M)$ the set of all vector fields on M , we can represent a element $X \in \mathcal{X}(M)$ in local coordinates as

$$X = \sum_{i=1}^3 a_i \frac{\partial}{\partial x_i}.$$

Then, we can set a norm induced by the metric g as follows

$$|X| = [g(X, X)]^{1/2} = \left(\sum_{i=1}^3 g_{ii} a_i^2 \right)^{1/2}$$

where we set

$$g_{ij} = g \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right), \quad i, j = 1, 2, 3.$$

2.1.1 Differential operators on manifolds

We start give the notion of differentiability on $\mathcal{X}(M)$. A *linear connection* on M is a bilinear mapping $D : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ satisfying

- $D_{fX}Y = f D_X Y$,
- $D_X(fY) = X(f) \cdot Y + f D_X Y$,

for any $X, Y \in \mathcal{X}(M)$ and for any $f \in C^\infty(M) = \bigcap_{k=1}^\infty C^k(M)$, where $D_X Y := D(X, Y)$.

A linear connection D on M can be extended in a natural way to the space of smooth tensors of type $(0, r)$, $r \geq 1$. So, given $f \in C^\infty(M)$, we can define its *covariant differential* as

$$Df := df$$

where df means the usual differential of f . Inductively, we can set the k -th order covariant differential of f by

$$D^k f := D(D^{k-1} f), \quad k \geq 2. \quad (2.1)$$

Since M is a Riemannian manifold, there exists a unique symmetric linear connection $\nabla = \nabla_g$ such that

- $\nabla_X Y - \nabla_Y X = [X, Y]$, for all $X, Y \in \mathcal{X}(M)$,
- $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, for all $X, Y, Z \in \mathcal{X}(M)$,

where $[X, Y]$ is the *Lie bracket* of X and Y . Such linear connection is called the *Levi-Civita connection* associated to g .

The *gradient operator* is the operator $\nabla_g : C^\infty(M) \rightarrow \mathcal{X}(M)$, where ∇_g is the covariant differential associated to the Levi-Civita connection. It satisfies

$$X(f) = g(X, \nabla_g f), \quad \forall X \in \mathcal{X}(M).$$

In the local chart x_1, x_2, x_3 we have

$$\nabla_g f = \sum_{i,j=1}^3 g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} \quad (2.2)$$

where $(g^{ij}) := (g_{ij})^{-1}$. We also set the *divergence operator* $\text{div} : \mathcal{X}(M) \rightarrow C^\infty(M)$ as

$$\text{div}(X) = \text{trace}(\nabla_g X).$$

In the local coordinates x_1, x_2, x_3 ,

$$\text{div}(X) = \frac{1}{[\det(g_{ij})]^{1/2}} \sum_{k=1}^3 \frac{\partial}{\partial x^k} \left([\det(g_{ij})]^{1/2} X^k \right). \quad (2.3)$$

So, we finally define the *Laplace-Beltrami operator* $\Delta_g : C^\infty(M) \rightarrow C^\infty(M)$ as

$$\Delta_g f = -\text{div}(\nabla_g f)$$

By (2.2) and (2.3), the Laplace-Beltrami operator in local coordinates is given by

$$\Delta_g f = -\frac{1}{[\det(g_{ij})]^{1/2}} \sum_{k,l=1}^3 \frac{\partial}{\partial x^k} \left(g^{kl} \frac{\partial f}{\partial x^l} [\det(g_{ij})]^{1/2} \right).$$

Our next aim is define the Lebesgue and Sobolev spaces on M . To do so, we need to introduce the concept of integrable functions on M .

2.1.2 Integrable functions

Let $\zeta_{ij} = g_{ij} \circ \psi^{-1}$ and denote by $g_{\psi(V)} = (\zeta_{ij})_{1 \leq i,j \leq 3}$. Then we have $\det g_{\psi(V)} > 0$ and we set the linear functional

$$\lambda_{V,\psi} : C_0(V) \rightarrow \mathbb{R}, \quad \lambda_{V,\psi}(f) = \int_{\psi(V)} (f|_V \circ \psi^{-1}) [\det g_{\psi(V)}]^{1/2} dx,$$

where $C_0(V)$ is the space of all continuous functions $f : M \rightarrow \mathbb{R}$ such that $\text{supp}(f)$ is compact and $\text{supp}(f) \subset V$. The functional $\lambda_{V,\psi}$ satisfy the following properties:

- If $f \in C_0(V)$ with $f \geq 0$, then $\lambda_{V,\psi}(f) \geq 0$;
- If (V', ψ') is another chart with $V \cap V' \neq \emptyset$, then for any $u \in C_0(V \cap V')$ we have $\lambda_{V,\psi}(f) = \lambda_{V',\psi'}(f)$.

Since the local chart (V, ψ) is arbitrary, there exists a unique linear functional $\lambda_g : C_0(M) \rightarrow \mathbb{R}$ such that (CRAIOVEANU; PUTA; RASSIAS, 2013, Theorem 1.1):

- $\lambda_g \Big|_{C_0(V)} = \lambda_{V,\psi}$;
- If $f \in C_0(M)$ with $f \geq 0$, then $\lambda_g(f) \geq 0$.

By (FOLLAND, 1999, Theorem 7.2) there exists a unique Radon measure μ_g on M such that

$$\lambda_g(f) = \int_M f d\mu_g, \quad \forall f \in C_0(M).$$

The measure μ_g is called *canonical measure* of M and

$$\text{vol}(M) := \int_M d\mu_g$$

is the *volume* of M . Now we are able to define the concept of integrable functions on M . To this end, we recall that a function $\kappa : M \rightarrow \mathbb{R} \cup \{+\infty\}$ is a *lower semicontinuous function*, l.s.f. for short, if $p_n \rightarrow p$ in M implies that $\liminf \kappa(p_n) \geq \kappa(p)$. Given a positive l.s.f. κ on M we set

$$\overline{\lambda_g}(\kappa) := \sup \{ \lambda_g(f), f \in C_0(M) \text{ such that } f \leq \kappa \}.$$

So, for a positive any positive function $h : M \rightarrow \mathbb{R}$ we also define

$$\overline{\lambda}_g(h) := \sup \left\{ \overline{\lambda}_g(\kappa), \kappa \text{ is l.s.f. and } h \leq \kappa \right\}.$$

A function $f : M \rightarrow \mathbb{R}$ is *integrable* on M if there exists a sequence $(f_n) \subset C_0(M)$ such that

$$\lim_{n \rightarrow \infty} \overline{\lambda}_g(|f_n - f|) = 0.$$

Thus, the sequence $(\overline{\lambda}_g(f_n))$ converges, its limit is independent on (f_n) and then we can set

$$\int_M f d\mu_g := \lim_{n \rightarrow \infty} \overline{\lambda}_g(f_n). \quad (2.4)$$

2.1.3 Function spaces on Riemannian manifolds

With the integral operator (2.4), we can define the following inner product in $C^\infty(M)$:

$$(f_1, f_2) := \int_M f_1 f_2 d\mu_g.$$

The completion of C^∞ with the norm $\|\cdot\|_2 := [(\cdot, \cdot)]^{1/2}$ is a Hilbert space denoted by $L^2(M)$. Moreover, for $p \geq 1$ we denote by $L^p(M)$ the space of all (class of) measurable functions $u : M \rightarrow \mathbb{R}$ such that $|u|^p$ is integrable on M . For any $p \geq 1$, the space $L^p(M)$ is a Banach space endowed with the norm

$$\|u\|_p := \left(\int_M |u|^p d\mu_g \right)^{1/p} < +\infty.$$

For $k \in \mathbb{N}$ we also consider the Hilbert spaces $H^k(M)$ defined as the completion of $C^\infty(M)$ with the inner product and norm

$$(u, v)_{H^k} = \sum_{i=0}^k (\nabla_g^i u, \nabla_g^i v), \quad \|u\|_{H^k(M)} = [(u, v)_{H^k}]^{1/2},$$

where we simply denote

$$(\nabla_g^i u, \nabla_g^i v) = \int_M g(\nabla_g^i u, \nabla_g^i v) d\mu_g, \quad i = 1, \dots, k.$$

For any $s > 0$ there exist $k \in \mathbb{N}$ and $\theta \in (0, 1)$ such that $s = k\theta$. So, by interpolation we set

$$H^s(M) = \left[H^k(M), L^2(M) \right]_\theta.$$

We denote by $H_0^s(M)$ the closure of $C_0^\infty(M)$ in $H^s(M)$ and by $H^{-s}(M)$ the dual of $H_0^s(M)$.

Now we see the constructions of Hilbert spaces involving the eigenvalues of Laplace-Beltrami operator with Dirichlet boundary conditions. All the results can be found in (CRAIOVEANU; PUTA; RASSIAS, 2013, Chapter 3).

Let us consider $A : C_0^\infty(M) \rightarrow C_0^\infty(M)$ where $A = -\Delta_g$. Since M is compact (and then closed), then A is a symmetric positive operator in $C_0^\infty(M)$ and consequently, A has a complete orthonormal system of eigenfunctions (e_n) in $L^2(M)$ of class C^∞ with eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = +\infty.$$

Moreover, by the mini-max characterization of (λ_k) (see Theorem 3.2, (CRAIOVEANU; PUTA; RASSIAS, 2013)) we have the Poincaré's inequality

$$\|u\|_2^2 \leq \lambda_1^{-1} \|\nabla_g u\|_2^2, \quad \forall u \in H_0^1(M). \quad (2.5)$$

So, A admits an extension $\tilde{A} : H^2(M) \cap H_0^1(M) \rightarrow L^2(M)$ such that $\tilde{A}$ is a unbounded positive self-adjoint operator satisfying the Green's formula

$$(\tilde{A}u, v) = (\nabla_g u, \nabla_g v), \quad \forall u \in H^2(M) \cap H_0^1(M), \quad \forall v \in H_0^1(M) \quad (2.6)$$

and, for any $n \in \mathbb{N}$, the operator

$$\mathcal{R}_n := (I + n^{-1}\tilde{A})^{-1} : H^{-1}(M) \rightarrow H_0^1(M) \quad (2.7)$$

exists and it is bounded. Here we see $\tilde{A}$ as a H_0^1 -extension of the operator defined in $H^2(M) \cap H_0^1(M)$. Such extension can be obtained by the formula (2.6). Then, we have the following result for compact Riemannian manifolds.

Proposition 2.1.1 (Proposition A.1, (GUO *et al.*, 2014)). *For each $n \in \mathbb{N}$, let $u_n := \mathcal{R}_n u$. The following statements hold.*

- *If $u \in L^2(M)$, then $\|u_n\|_2 \leq \|u\|_2$ and $\lim_{n \rightarrow \infty} u_n = u$ in $L^2(M)$.*
- *If $u \in H_0^1(M)$, then $\|\nabla_g u_n\|_2 \leq \|\nabla_g u\|_2$ and $\lim_{n \rightarrow \infty} u_n = u$ in $H_0^1(M)$* □.

For any $\delta \in \mathbb{R}$ we can define the power $\tilde{A}^{\delta/2}$ which its domain (TEMAM, 2001)

$$H_\delta := D(\tilde{A}^{\delta/2}) = \left\{ u = \sum_{j=1}^{\infty} a_j e_j \in L^2(M), \sum_{j=1}^{\infty} \lambda_j^\delta a_j^2 < +\infty \right\}$$

is a Hilbert space with inner product and norm

$$(u, v)_{H_\delta} = (\tilde{A}^{\delta/2} u, \tilde{A}^{\delta/2} v), \quad \|u\|_{H_\delta} = \|\tilde{A}^{\delta/2} u\|_2.$$

The following lemma give us some conditions to have $H_\delta \hookrightarrow L^p(M)$.

Lemma 2.1.2. *If $\delta < \frac{3}{2}$ and*

$$\delta \geq \frac{3}{2} - \frac{3}{p}, \quad 2 < p < +\infty \quad (2.8)$$

then H_δ is continuously embedded in the space $L^p(M)$. If the inequality for δ in (2.8) is strict, then the embedding is compact.

Proof. The proof is a consequence of Theorem 5.1.5 in (AGRANOVICH, 2015) and Theorem 16.1 in (YAGI, 2009). □

2.2 Vector-valued spaces

In this section X and Y are Banach spaces.

Definition 2.2.1. Let $(S, \mathfrak{M}, \mu)$ be a measure space. A step function $u : S \rightarrow X$ is

- **measurable** if the set $u^{-1}(\{x\}) \in \mathfrak{M}$ for each $x \in X$.
- **integrable** if u is measurable and if $\mu(u^{-1}(x)) < +\infty$ for $x \neq 0$. Then we set the integral

$$\int_S u d\mu := \sum_{x \in B} \mu(u^{-1}(x)).$$

We also say that a function $u : S \rightarrow X$ is

- **measurable** if there is a sequence $u_n : S \rightarrow X$ of measurable step functions for which $u_n(s) \rightarrow u(s)$ in X for almost ever $s \in S$.
- **integrable** if there is a sequence $u_n : S \rightarrow X$ of integrable step functions such that

$$\lim_{n \rightarrow \infty} \int_S \|u_n(s) - u(s)\|_X d\mu = 0$$

and each integrand is integrable. Then $\int_S u_n d\mu$ converges in X to a limit which is the same for any such sequence. This limit is denoted by $\int_S u d\mu$.

Proposition 2.2.2 (Dominated Convergence Theorem). Let $u_n : S \rightarrow X$ be a sequence of integrable functions and $u : S \rightarrow X$ such that

- $u_n(s) \rightarrow u(s)$ for almost $s \in S$ and
- there exists a integrable function $u : S \rightarrow X$ such that $\|u_n(s)\|_X \leq u(s)$ for all $n \in \mathbb{N}$ and for almost ever $s \in S$. Then u is integrable and

$$\lim_{n \rightarrow \infty} \int_S u_n d\mu = \int_S u d\mu \quad \text{in } X.$$

Proof. See (SHOWALTER, 2013), Theorem 1.4, page 103. □

Definition 2.2.3. A function $u : [\tau, t] \rightarrow X$ is **absolutely continuous** if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each sequence of disjoint intervals $(a_n, b_n) \subset [\tau, t]$, $n \in \mathbb{N}$, satisfying $\sum_{n=1}^{\infty} (b_n - a_n) < \delta$ there follows $\sum_{n=1}^{\infty} \|u(b_n) - u(a_n)\|_X < \varepsilon$.

Proposition 2.2.4. Let $u : [\tau, t] \rightarrow X$ be an absolutely continuous function. If X is reflexive, then u is (strong) differentiable almost ever, u' is integrable and

$$u(s) = u(\tau) + \int_{\tau}^s u'(y) dy, \quad s \in [\tau, t].$$

Proof. See (SHOWALTER, 2013), Theorem 1.7, page 105 □

Definition 2.2.5. Let $\tau < t$.

- For each $k \in \mathbb{N} \cup \{0\}$ we denote by $C^k([\tau, t], X)$ the set of all functions $u : [\tau, t] \rightarrow X$ that are continuously differentiable of order k . The function

$$\|u\|_{C^k([\tau, t], X)} := \sum_{j=0}^k \max_{s \in [\tau, t]} \|u^{(j)}(s)\|_X$$

is a norm in $C^k([\tau, t], X)$. In particular, we denote $C^0([\tau, t], X) = C([\tau, t], X)$.

- For $p \in [1, \infty)$ we denote by $L^p(\tau, t; X)$, the set of all measurable functions $u : (\tau, t) \rightarrow X$ such that

$$\|u\|_{L^p(\tau, t; X)} := \left(\int_{\tau}^t \|u(s)\|_X^p ds \right)^{1/p} < \infty. \quad (2.9)$$

The function (2.9) is a norm in $L^p(\tau, t; X)$.

- We denote by $L^\infty(\tau, t; X)$ the set of all essentially bounded measurable functions $u : (\tau, t) \rightarrow X$, that is, the functions such that

$$\|u\|_{L^\infty(\tau, t; X)} := \text{ess sup}_{s \in (\tau, t)} \|u(s)\|_X < \infty. \quad (2.10)$$

The function (2.10) is a norm in $L^\infty(\tau, t; X)$.

Proposition 2.2.6. Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$. Then

- $(C^k([\tau, t], X), \|\cdot\|_{C^k([\tau, t], X)})$ is a Banach space.
- $(L^p(\tau, t; X), \|\cdot\|_{L^p(\tau, t; X)})$ is a Banach space.
- The embedding $C([\tau, t], X) \hookrightarrow L^p(\tau, t; X)$ is continuous and dense.
- If X is separable, then $L^p(\tau, t; X)$ is separable for $p \in [1, \infty)$.
- If $X \hookrightarrow Y$, then $L^p(\tau, t; X) \hookrightarrow L^p(\tau, t; Y)$, for all $1 \leq p \leq \infty$.
- If X is a Hilbert space with inner product $(\cdot, \cdot)_X$, then $L^2(\tau, t; X)$ is a Hilbert with the following inner product

$$(u, v)_{L^2(\tau, t; X)} := \int_{\tau}^t (u(s), v(s))_X ds, \quad u, v \in L^2(\tau, t; X).$$

Proof. See (ZEIDLER, 198), Proposition 23.2. □

Proposition 2.2.7. Let $p \in (1, \infty)$ and $q \in \mathbb{R}$ be the **conjugated exponent** of p , that is, q satisfies

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If X is reflexive, then the dual space of $L^p(\tau, t; X)$ is isometrically isomorph with $L^q(\tau, t; X')$. In particular, $L^p(\tau, t; X)$ is reflexive.

Proof. See (ZEIDLER, 198), Proposition 23.7. □

Proposition 2.2.8 (Hölder's Inequality). Let $p \in (1, \infty)$ and $q \in \mathbb{R}$ the conjugated exponent of p . For any $u \in L^p(\tau, t; X)$ and $v \in L^q(\tau, t; X')$ we have

$$\int_{\tau}^t |\langle v(s), u(s) \rangle_{X', X}| ds \leq \left(\int_{\tau}^t \|v(s)\|_{X'} ds \right)^{1/q} \left(\int_{\tau}^t \|u(s)\|_X ds \right)^{1/p}.$$

Here, $\langle \cdot, \cdot \rangle_{X', X}$ denotes the duality between X' and X .

Proof. See (ZEIDLER, 198), Proposition 23.6. □

Proposition 2.2.9. For any $u \in L^1(\tau, t; X)$ we have

$$\left\| \int_{\tau}^t u(s) ds \right\|_X \leq \int_{\tau}^t \|u(s)\|_X ds$$

Proof. See (YOSIDA, 1980), Corollary 1, page 133. □

Proposition 2.2.10. Let $p \in (1, \infty)$.

- If $u \in L^p(\tau, t; X)$, then

$$\left\langle v, \int_{\tau}^t u(s) ds \right\rangle = \int_{\tau}^t \langle v, u(s) \rangle ds, \quad \forall v \in X'.$$

- If $u \in L^p(\tau, t; X')$, then

$$\left\langle \int_{\tau}^t u(s) ds, v \right\rangle = \int_{\tau}^t \langle u(s), v \rangle ds, \quad \forall v \in X.$$

Proof. See (ZEIDLER, 198), Proposition 23.9. □

Definition 2.2.11. Let $\tau < t$. A linear mapping $T : \mathcal{D}(\tau, t) \rightarrow X$ is a **distribution with values in X** if it is continuous in the sense of $\mathcal{D}(\tau, t)$. The space of all distribution with values in X is denoted by $\mathcal{D}'(\tau, t; X)$.

Remark 2.2.12. Let $p \in [1, \infty]$. Given $u \in L^p(\tau, t; X)$, the mapping $T_u : \mathcal{D}(\tau, t) \rightarrow X$ given by

$$\langle T_u, \varphi \rangle := \int_{\tau}^t u(s) \varphi(s) ds, \quad \forall \varphi \in \mathcal{D}(\tau, t)$$

belongs to $\mathcal{D}'(\tau, t; X)$. Moreover, the mapping

$$T : L^p(\tau, t; X) \rightarrow T(L^p(\tau, t; X)) \subset \mathcal{D}'(\tau, t; X)$$

defined by $Tu = T_u$ is a continuous isomorphism. So, roughly speaking, we have the continuous embedding

$$L^p(\tau, t; X) \hookrightarrow \mathcal{D}'(\tau, t; X).$$

For details and more properties, see for instance (ZEIDLER, 1989), Appendix, page 1047.

Definition 2.2.13. Let $T \in \mathcal{D}'(\tau, t; X)$ and $k \in \mathbb{N} \cup \{0\}$. The k -th **derivative** of T is the distribution with values in X defined by

$$\langle T_t^{(k)}, \varphi \rangle := (-1)^k \langle T, \varphi^{(k)} \rangle, \quad \forall \varphi \in \mathcal{D}(\tau, t).$$

For small orders, we write $T_t^{(0)} = T$, $T_t^{(1)} = T_t$ and $T_t^{(2)} = T_{tt}$.

Theorem 2.2.14. For any $u, v \in L^1(\tau, t; X)$, the following conditions are equivalent:

(i) u is almost ever equal to a primitive of v , that is, there exists $\mathfrak{u} \in X$, such that

$$u(s) = \mathfrak{u} + \int_{\tau}^s v(y) dy$$

almost ever $s \in [\tau, t]$.

(ii) $v = u_t$ almost ever in (τ, t) , that is,

$$\int_{\tau}^t u(s) \varphi'(s) ds = - \int_{\tau}^t v(s) \varphi(s) ds, \quad \forall \varphi \in \mathcal{D}(\tau, t).$$

(iii) For each $\eta \in X'$,

$$\frac{d}{dt} \langle u, \eta \rangle = \langle v, \eta \rangle$$

in the scalar distribution sense on (τ, t) .

In this case, u is almost ever equal to a continuous function from $[\tau, t]$ into X .

Proof. See (TEMAM, 2001), Lemma 1.1, page 250. □

2.3 Existence Results

In this section we present classical existence results in the following sequence:

- Zorn's Lemma: Existence of a maximal element in a partial ordered set.
- Contraction Mapping Theorem: Existence of a unique fixed point in a complete metric space.

- Lax-Milgram Theorem: Existence of solutions for variational equations.
- A result about non-homogeneous Cauchy problems: Existence of classical and generalized solutions for a class of initial value problems.

Definition 2.3.1. A **partially ordered set** is a pair $(X, \preceq)$ where X is a set and $\preceq$ is a partial order relation. In particular, we say that a subset $Y \subset X$ is a **chain** in X if given $z, y \in Y$ we have $y \preceq z$ or $z \preceq y$.

An element $x \in X$ is said to be an **upper bound** for a subset $Y \subset X$ if $y \preceq x$, for all $y \in Y$. We also say that an element $y \in Y \subset X$ is a **maximal element** of Y if there is no $z \in Y$ such that $y \preceq z$ and $z \neq y$.

The reader can be found the following result, called Zorn's maximum principle, and its applications in (ZORN, 1935).

Theorem 2.3.2. If $(X, \preceq)$ is any non-empty partially ordered set in which every chain has an upper bound, then X has a maximal element.

Definition 2.3.3. A mapping $f : X \rightarrow Y$ between two metric spaces X and Y is a **Lipschitz mapping** if there exists $L > 0$ such that

$$d_Y(f(x), f(y)) \leq L d_X(x, y), \quad \forall x, y \in X.$$

In particular, when $L \in (0, 1)$ we say that f is a **contraction**.

Theorem 2.3.4. Let X a complete metric space and $f : X \rightarrow X$ a contraction mapping. Then, there exists a unique $x \in X$ such that $f(x) = x$.

Proof. See (BANACH, 1922), Theorem 6, page 330. □

Definition 2.3.5. Let X be a vector space. A **bilinear form** in X is a mapping $b : X \times X \rightarrow \mathbb{R}$ such that for any $x, y \in X$ the mappings

$$b_1 := b(x, \cdot) : X \rightarrow \mathbb{R} \quad \text{and} \quad b_2 := b(\cdot, y) : X \rightarrow \mathbb{R}$$

are linear functionals in X .

Definition 2.3.6. A bilinear form $b : X \times X \rightarrow \mathbb{R}$ defined in a normed space X is **bounded** if there exists $C > 0$ such that

$$|b(x, y)| \leq C \|x\|_X \|y\|_X, \quad \forall x, y \in X.$$

We also say that b is **coercive** if there exists $C' > 0$ such that

$$b(x, x) \geq C' \|x\|_X^2, \quad \forall x \in X.$$

Theorem 2.3.7. *Let X be a Hilbert space and $b : X \times X \rightarrow \mathbb{R}$ a bounded coercive bilinear form in H . Then, for any $f \in X'$ there exists $z \in X$ such that*

$$b(x, z) = \langle f, x \rangle, \quad \forall x \in X.$$

Proof. See (LAX; MILGRAM, 1955), Theorem 2.1. □

Definition 2.3.8. *Let X be a Hilbert space with inner product $(\cdot, \cdot)_H$. An operator $A : D(A) \subset X \rightarrow X$ is **monotone** if*

$$(Au_1 - Au_2, u_1 - u_2)_H \geq 0, \quad \forall u_1, u_2 \in D(A).$$

*We also say that A is **maximal monotone** if A is monotone and $I + A$ is onto.*

Let X be a Hilbert space, $\tau \in \mathbb{R}$ and $A : D(A) \subset X \rightarrow X$ a maximal monotone operator. Given $\xi \in X$, consider the following initial value problem

$$u_t(t) + A(u(t)) = h(t) + \omega u(t), \quad t > \tau, \quad u(\tau) = \xi. \quad (2.11)$$

Here, $h : [\tau, t] \rightarrow X$ is an absolutely continuous function for almost every $t > \tau$ and $\omega \geq 0$.

Definition 2.3.9. *A function $u : [\tau, t] \rightarrow X$ is called:*

- **classical solution** of (2.11) if u is absolutely continuous, $u(\tau) = \xi$ and u satisfy (2.11) for almost every $t > \tau$.
- **generalized solution** of (2.11) if $u \in C([\tau, t], X)$ for all $t > \tau$ and if there exist sequences u^n of classical solutions of (2.11) and $h^n : [\tau, t] \rightarrow X$ of absolutely continuous functions such that

$$u^n \longrightarrow u \quad \text{in } C([\tau, t], X) \quad \text{and} \quad h^n \longrightarrow h \quad \text{in } L^1(\tau, t; X).$$

Theorem 2.3.10. *Let $A : D(A) \subset X \rightarrow X$ be a maximal monotone operator on a Hilbert space X .*

- If $\xi \in D(A)$ and $h : [\tau, t] \rightarrow X$ is absolutely continuous for all $t > \tau$, then the problem (2.11) has a unique classical solution.
- If $\xi \in \overline{D(A)}$ and $h \in L^1(\tau, t; X)$ for all $t > \tau$, then the problem (2.11) has a unique generalized solution.

Proof. See (SHOWALTER, 2013), Theorem 4.1 and Theorem 4.1A, pages 180 and 183. □

2.4 Abstract theory of pullback dynamics

Let X be a Banach space with norm $\|\cdot\|_X$. We denote by $B_X(z, R)$ the open ball in X with center $z \in X$ and radius $R > 0$ and by

$$\text{diam}_X(A) := \sup_{z, w \in A} \|z - w\|_X$$

the *diameter* of a subset $A \subset X$. We also define the *Hausdorff semidistance* of two non-empty subsets $A, B \subset X$ as the number

$$\text{dist}_X(A, B) := \sup_{x \in A} \inf_{y \in B} \|x - y\|_X,$$

and the *Hausdorff symmetric distance* of A and B is given by

$$\text{dist}_X^{\text{symm}}(A, B) := \text{dist}_X(A, B) + \text{dist}_X(B, A).$$

The *fractal dimension* of a compact set $A \subset X$ is set by

$$\dim_f^X(A) := \limsup_{\varepsilon \rightarrow 0} \frac{\ln(N_\varepsilon(A))}{\ln(1/\varepsilon)},$$

where $N_\varepsilon(A)$ is the *Kolmogorov's ε -entropy* of A , that is, the smallest number of closed balls of radius ε covering A .

A *nonautonomous dynamical system* is a pair $(U(t, \tau), X)$ where $U(t, \tau)$ is an *evolution process* acting on X , that is, for any $t \geq \tau$, $U(t, \tau) : X \rightarrow X$ is a linear operator satisfying the following conditions:

- $U(\tau, \tau) = Id_X$,
- $U(t, s) \circ U(s, \tau) = U(t, \tau)$, $\tau \leq s \leq t$,
- the function $(t, \tau, x) \mapsto U(t, \tau)x$ is continuous, for each $\tau \leq t$ and $x \in X$.

A *pullback absorbing family* for the nonautonomous dynamical system $(U(t, \tau), X)$ is a family $\{D(t)\}_{t \in \mathbb{R}}$ of bounded subsets of X such that, for every bounded set $B \subset X$ and $t \in \mathbb{R}$ there exists $T = T(B, t) > 0$ satisfying

$$U(t, t-r)B \subset D(t), \quad r \geq T. \quad (2.12)$$

A family $\mathcal{A} = \{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ of subsets of D is called *pullback attractor* for $(U(t, \tau), X)$ if:

- $\mathcal{A}(t)$ is a compact set of X , for every $t \in \mathbb{R}$;
- $U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$, for every $t \geq \tau$;

- $\mathcal{A}$ pullback attracts every bounded set of X , that is, for any $t \in \mathbb{R}$ and for any bounded set $B \subset X$,

$$\lim_{r \rightarrow \infty} \text{dist}_X(U(t, t-r)B, \mathcal{A}(t)) = 0;$$

- $\mathcal{A}$ is minimal with respect the families of closed subsets that pullback attract all bounded subsets of X .

For each $t \in \mathbb{R}$, the set $\mathcal{A}(t)$ is called *section* of $\mathcal{A}$.

As the global attractors, the definition of pullback attractor does not give any information of the fractal dimension of its sections and how fast it pullback attracts bounded sets. So, this motivate the next definition.

A family $\mathcal{M} = \{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ of subsets of X is called a *pullback exponential attractor* for $(U(t, \tau), X)$ if:

- $\mathcal{M}(t)$ is a compact set of X , for every $t \in \mathbb{R}$;
- $U(t, \tau)\mathcal{M}(\tau) \subset \mathcal{M}(t)$, for every $t \geq \tau$;
- the fractal dimension of $\mathcal{M}(t)$ is uniformly bounded in X , that is,

$$\sup_{t \in \mathbb{R}} \dim_f^X(\mathcal{M}(t)) < +\infty;$$

- $\mathcal{M}$ exponentially attracts all bounded sets of X uniformly, that is, there exists $\mu > 0$ such that

$$\lim_{s \rightarrow +\infty} e^{\mu s} \text{dist}_X(U(t, t-s)B, \mathcal{M}(t)) = 0$$

for any bounded set $B \subset X$.

Since $\mathcal{M}(t)$ is positively semi-invariant for each $t \in \mathbb{R}$, then $\mathcal{M}$ is not unique.

Now, we see a general existence result created by Carvalho and Sonner ([CARVALHO; SONNER, 2013](#)) that ensures the existence of pullback exponential attractor for an evolution process $U(t, \tau) : X \rightarrow X$ acting in a Banach space X . The construction of such attractor rely on the existence of a pullback absorbing family $\{D(t)\}_{t \in \mathbb{R}}$ for $(U(\tau, t), X)$, a suitable $\tilde{t}_0 > 0$ and a decomposition $U = S + C$ satisfying the assumptions below.

Assumptions for $\{D(t)\}_{t \in \mathbb{R}}$:

(D1) The family $\{D(t)\}_{t \in \mathbb{R}}$ is positively semi-invariant, that is,

$$U(t, \tau)D(\tau) \subset D(t), \quad \forall t \geq \tau.$$

(D2) For any bounded subset $B \subset X$ and $t \in \mathbb{R}$, there exists $T = T(B, t) > 0$ such that

$$U(\tau, \tau - s)B \subset D(\tau), \quad \forall \tau \leq t, \quad \forall s \geq T.$$

(D3) The diameter of $\{D(t)\}_{t \in \mathbb{R}}$ grows at most sub-exponentially in the past, that is, there exists $\nu > 0$ such that

$$\lim_{t \rightarrow -\infty} \text{diam}_X(D(t)) e^{-\nu t} = 0.$$

Assumptions for decomposition $U = S + C$:

(S) There exist a normed space Y and a constant $\kappa > 0$ such that $X \hookrightarrow Y$ is compact for every $t \in \mathbb{R}$,

$$\|S(t + \tilde{t}_0, t)\xi^1 - S(t + \tilde{t}_0, t)\xi^2\|_X \leq \kappa \|\xi^1 - \xi^2\|_Y, \quad \forall \xi^1, \xi^2 \in D(t).$$

(C) There exists a constant $\lambda \in [0, \frac{1}{2})$ such that for every $t \in \mathbb{R}$,

$$\|C(t + \tilde{t}_0, t)\xi^1 - C(t + \tilde{t}_0, t)\xi^2\|_X \leq \lambda \|\xi^1 - \xi^2\|_X, \quad \forall \xi^1, \xi^2 \in D(t).$$

(L) For any $t \in \mathbb{R}$ and $r \in (t, t + \tilde{t}_0)$ there exists a constant $L = L(t, r) > 0$ such that

$$\|U(r, t)\xi^1 - U(r, t)\xi^2\|_X \leq L \|\xi^1 - \xi^2\|_X, \quad \forall \xi^1, \xi^2 \in D(t).$$

The next useful lemma gives us a simple condition to verify the conditions **(D1)** - **(D3)**.

Lemma 2.4.1. *Let $(U(t, \tau), X)$ be nonautonomous dynamical system that admits a constant pullback absorbing family $D \subset X$. Then, it admits a pullback absorbing family $\{D(t)\}_{t \in \mathbb{R}}$ satisfying **(D1)** - **(D3)** and such that $D(t) \subset \overline{D}^X$ for all $t \in \mathbb{R}$.*

Proof. See Lemma 3.3 in (MA; MONTEIRO; PEREIRA, 2019). □

The result of existence of pullback exponential attractors is stated as follows.

Theorem 2.4.2. *Let $(U(t, \tau), X)$ be a nonautonomous dynamical system. Suppose the assumptions **(D1)** - **(D3)**, **(S)**, **(C)** and **(L)** hold. Then, there exists a pullback exponential attractor $\mathcal{M} = \{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ for $(U(t, \tau), X)$.*

Proof. See Theorem 4 in (CARVALHO; SONNER, 2013). □

A direct consequence is the existence of the pullback attractor.

Corollary 2.4.3. *Under the assumption of Theorem 2.4.2, the pullback attractor $\mathcal{A} = \{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ for $(U(t, \tau), X)$ exists, it is contained in $\mathcal{M}$ and the fractal dimension of its section is uniformly bounded.*

Proof. See Corollary 1 in (CARVALHO; SONNER, 2013). $\square$

We end this section present two results about continuity of attractors w.r.t. a parameter $\varepsilon \in [0, 1]$.

Proposition 2.4.4. *For each $\varepsilon \in [0, 1]$, let $(U_\varepsilon(t, \tau), X)$ be a nonautonomous dynamical system. Suppose that:*

(U1) $(U_\varepsilon(t, \tau), X)$ has a pullback attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(t)\}_{t \in \mathbb{R}}$ for all $\varepsilon \in [0, 1]$.

(U2) For any $t \in \mathbb{R}$, any $T \geq 0$, and any bounded set $B \subset X$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{r \in [0, T]} \sup_{\xi \in B} \|U_\varepsilon(t+r, t)\xi - U_0(t+r, t)\xi\|_X = 0.$$

(U3) There exist $\delta > 0$ and $t_0 \in \mathbb{R}$ such that

$$\bigcup_{\varepsilon \in (0, \delta)} \bigcup_{\tau \leq t_0} A_\varepsilon(\tau)$$

is bounded.

Then, $\mathcal{A}_\varepsilon$ is upper semicontinuous as $\varepsilon \rightarrow 0$, that is, for each $t \in \mathbb{R}$,

$$\lim_{\varepsilon \rightarrow 0} \text{dist}_X(\mathcal{A}_\varepsilon(t), \mathcal{A}_0(t)) = 0.$$

Proof. See Propostion 1.20 in (CARVALHO; LANGA; ROBINSON, 2012). $\square$

Proposition 2.4.5. *For each $\varepsilon \in [0, 1]$ let $(U_\varepsilon(t, \tau), X)$ be a nonautonomous dynamical system satisfying (U1). Suppose that:*

(R1) There is a bounded subset $D \subset X$ such that $\mathcal{A}_\varepsilon(t) \subset D$ for every $\varepsilon \in [0, 1]$ and every $t \in \mathbb{R}$.

(R2) For any $\tau \in \mathbb{R}$, any $t \geq \tau$, and any bounded set $B \subset X$,

$$\lim_{\varepsilon \rightarrow \varepsilon_0} \sup_{\xi \in B} \|U_\varepsilon(t, \tau)\xi - U_{\varepsilon_0}(t, \tau)\xi\|_X = 0, \quad \forall \varepsilon_0 \in [0, 1].$$

(R3) For any $\varepsilon_0 \in [0, 1]$ and $t \in \mathbb{R}$, there exist $\delta > 0$ such that

$$\overline{\bigcup_{\varepsilon \in (\varepsilon_0, \delta)} A_\varepsilon(t)}^X$$

is compact.

Then, there exists a residual set¹ I in $[0, 1]$ such that for every $t \in \mathbb{R}$ the function $\varepsilon \mapsto \mathcal{A}_\varepsilon(t)$ is continuous at each $\varepsilon \in I$. In other words,

$$\lim_{\varepsilon \rightarrow \varepsilon_0} \text{dist}_X^{\text{symm}}(\mathcal{A}_\varepsilon(t), \mathcal{A}_{\varepsilon_0}(t)) = 0, \quad \forall \varepsilon_0 \in I.$$

Proof. See Theorem 3.3 in (HOANG; OLSON; ROBINSON, 2018). $\square$

¹ Let Y be a complete metric space and $Z \subset Y$. We say that Z is *residual* in Y if $Y \setminus Z$ is a countable union of nowhere dense sets.

2.5 Inequalities of Gronwall type

Here we present the Gronwall inequalities that we used in this work. In all results below, we consider $\tau \in \mathbb{R}$ and $t \geq \tau$.

The first one is a differential version and gives us a estimate for fractional powers of absolutely continuous functions.

Lemma 2.5.1. *Let $a, b \in L^1(\tau, t)$ with $b(s) \geq 0$ a.e., and let absolutely continuous function $\phi : [\tau, t] \rightarrow \mathbb{R}^+$ satisfy*

$$(1 - \alpha) \frac{d}{ds} \phi(s) \leq a(s) \phi(s) + b(s) [\phi(s)]^\alpha, \quad \text{a.e. } s \in [\tau, t],$$

where $0 \leq \alpha < 1$. Then

$$[\phi(s)]^{1-\alpha} \leq [\phi(\tau)]^{1-\alpha} e^{\int_\tau^s a(r) dr} + \int_\tau^s e^{\int_y^s a(r) dr} b(y) dy, \quad s \in [\tau, t].$$

Proof. See Lemma 4.1 in (SHOWALTER, 2013). □

The next one is a integral version for continuous functions.

Lemma 2.5.2. *Let $\phi, k \in C([\tau, t])$ and $m \in L^1(\tau, t)$ be positive functions and $\alpha \in (0, 1)$ such that*

$$\phi(s) \leq k(s) + \int_\tau^s m(r) [\phi(r)]^\alpha dr, \quad \forall s \in [\tau, t]. \quad (2.13)$$

Then,

$$[\phi(s)]^{1-\alpha} \leq \max_{\tau \leq r \leq s} \{k(r)\}^{1-\alpha} + (1 - \alpha) \int_\tau^s m(r) dr, \quad \forall s \in [\tau, t].$$

Proof. Let $s_0 \in [\tau, t)$. From (2.13) we have

$$\phi(s) \leq \max_{\tau \leq r \leq s_0} \{k(r)\} + \int_\tau^s m(r) [\phi(r)]^\alpha dr := \psi(s), \quad \forall s \in [\tau, s_0]. \quad (2.14)$$

So, we take the derivative of ψ and use (2.14) to get

$$\psi'(s) \leq m(s) [\psi(s)]^\alpha, \quad \psi(\tau) = \max_{\tau \leq y \leq s_0} \{k(y)\}. \quad (2.15)$$

Solving the differential inequality (2.15), we arrive at

$$[\psi(s_0)]^{1-\alpha} \leq \max_{\tau \leq r \leq s_0} \{k(r)\}^{1-\alpha} + (1 - \alpha) \int_\tau^{s_0} m(r) dr. \quad (2.16)$$

Since $s_0 \in [\tau, t)$ is arbitrary, we conclude the desire inequality from (2.14) and (2.16). □

The last inequality gives us a universal bound for positive continuous functions

Lemma 2.5.3. *Let $C_0 > 0$ and let $\phi \in C([\tau, t))$ be a positive function such that $\phi(\tau) = 0$ and*

$$\phi(s) \leq C_0[\phi(s)]^\sigma + \varepsilon, \quad s \in [\tau, t)$$

for some $\sigma > 1$ and $0 < \varepsilon < \frac{1}{2} \left(\frac{1}{2C_0} \right)^{\frac{1}{\sigma-1}}$. Then

$$\phi(s) \leq 2\varepsilon, \quad s \in [\tau, t).$$

Proof. See Lemma 3.1 in ([SAVOSTIANOV, 2015](#)).

□

WELL-POSEDNESS

This chapter is concerned to study the well-podesness and dynamics of the following equation

$$u_{tt} - \Delta_g u + \gamma u_t + f(u) = h \text{ in } M \times (\tau, +\infty) \quad (3.1)$$

with homogeneous Dirichlet boundary condition and initial data

$$\xi_u(\tau) = \xi. \quad (3.2)$$

Here, $\xi_u(t) := (u(t), u_t(t))$, (M, g) is a compact Riemannian manifold with smooth boundary and $\dim M = 3$, Δ_g is the Laplace-Beltrami operator on M , γ is a positive constant, f is a nonlinear function with sub-quintic growth and h is a time dependent external source. We also denote by ∇_g the connection associated to the Laplace-Beltrami operator

To simplify our calculations let us introduce some notations. For each $\delta \in \mathbb{R}$ we denote by $H_\delta := D((-\Delta_g)^{\delta/2})$ endowed with the inner product and norm

$$(u, v)_{H_\delta} := ((-\Delta_g)^{\delta/2} u, (-\Delta_g)^{\delta/2} v), \quad \|u\|_{H_\delta} = \|(-\Delta_g)^{\delta/2} u\|_2.$$

So, for each $\delta \in \mathbb{R}$, we consider the phase space

$$\mathcal{H}_\delta := H_{\delta+1} \times H_\delta$$

with inner product and norm

$$((u, v), (z, w))_{\mathcal{H}_\delta} = (u, z)_{H_{\delta+1}} + (v, w)_{H_\delta}, \quad \|(u, v)\|_{\mathcal{H}_\delta} = \left(\|u\|_{H_{\delta+1}}^2 + \|v\|_{H_\delta}^2 \right)^{1/2}.$$

In particular we denote $\mathcal{H} := \mathcal{H}_0 = H_0^1(M) \times L^2(M)$. We also have the compact and dense embedding $\mathcal{H}_{\delta'} \hookrightarrow \mathcal{H}_\delta$ for $\delta < \delta'$.

3.1 Some properties of the linear problem

Let us recall some results of the following problem

$$\partial_{tt}v - \Delta_g v + \gamma \partial_t v = \Phi \text{ in } M \times (\tau, +\infty) \quad (3.3)$$

with Dirichlet boundary condition

$$v = 0 \quad \text{on} \quad \partial M \times [\tau, +\infty) \quad (3.4)$$

and initial data

$$\xi_v(\tau) = \xi. \quad (3.5)$$

Firstly, we present the well-posedness of (3.3)-(3.5).

Proposition 3.1.1. *Let $t > \tau$.*

- *For any $\xi \in \mathcal{H}$ and any $\Phi \in L^1(\tau, t; L^2(M))$, the problem (3.3)-(3.5) has a unique generalized solution v such that $\xi_v \in C([\tau, t], \mathcal{H})$.*
- *For any $\xi \in \mathcal{H}_1$ and any $\Phi \in L^1(\tau, t; L^2(M))$ with $\partial_t \Phi \in L^1(\tau, t; L^2(M))$, the problem (3.3)-(3.5) has a unique strong solution v such that $\xi_v \in C^1([\tau, t], \mathcal{H}) \cap C([\tau, t], \mathcal{H}_1)$.*

Proof. Since (3.3)-(3.5) is equivalent to

$$\partial_t v + \mathfrak{A}v = \mathfrak{F}, \quad V(\tau) = \xi \quad (3.6)$$

where $\mathfrak{A} : \mathcal{H}_1 \subset \mathcal{H} \rightarrow \mathcal{H}$ is a maximal monotone operator and $\mathfrak{F} \in L^1(\tau, t; \mathcal{H})$ for all $t > \tau$, we can apply Theorem 2.3.10 to obtain the desire result. $\square$

Proposition 3.1.2. *Let $\xi \in \mathcal{H}$ and $\Phi \in L^1(\tau, t; L^2(M))$. Then the generalized solution v of (3.3)-(3.5) satisfies*

$$\frac{1}{2} \|\xi_v(t)\|_{\mathcal{H}}^2 \leq \frac{1}{2} \|\xi\|_{\mathcal{H}}^2 + \int_{\tau}^t \|\Phi(s)\|_2 \|\xi_v(s)\|_2 ds \quad (3.7)$$

for all $t \geq \tau$. Consequently,

$$\|\xi_v(t)\|_{\mathcal{H}} \leq \|\xi\|_{\mathcal{H}} + \int_{\tau}^t \|\Phi(s)\|_2 ds, \quad t > \tau. \quad (3.8)$$

Proof. Inequality (3.7) follows multiplying (3.3) by $\partial_t v$, integrating the result on $M \times (\tau, t)$ and using Green's formula (2.6). Applying Lemma 2.5.2 in (3.7) we obtain (3.8). $\square$

Proposition 3.1.3. *Let $\xi \in \mathcal{H}$ and $t > \tau$. Then there exist constants $\alpha > 0$ and $\tilde{C} > 0$, depending only on $\gamma > 0$, such that*

$$\|\xi_v(t)\|_{\mathcal{H}} \leq \tilde{C} \|\xi\|_{\mathcal{H}} e^{-\alpha(t-\tau)} + \tilde{C} \int_{\tau}^t e^{-\alpha(t-s)} \|\Phi(s)\|_2 ds. \quad (3.9)$$

Proof. Without loss of generality, we consider v a strong solution of (3.3)-(3.5). So, multiplying (3.3) by $\partial_t v + \varepsilon_0 v$ with

$$\varepsilon_0 := \min \left\{ \frac{\lambda_1 \gamma}{2\lambda_1 + \gamma^2}, \frac{\lambda_1^{1/2}}{4} \right\} > 0,$$

integrating the result on M , and using Green's formula (2.6) we have

$$\frac{d\mathcal{G}}{ds}(s) + (\gamma - \varepsilon_0) \|\partial_t v(s)\|_2^2 + \varepsilon_0 \|\nabla_g v(s)\|_2^2 = (\Phi(s), \partial_t v(s) + \varepsilon_0 v(s)) - \gamma \varepsilon_0 (\partial_t v(s), v(s)), \quad (3.10)$$

where we set

$$\mathcal{G}(s) := \frac{1}{2} \|\xi_v(s)\|_{\mathcal{H}}^2 + \varepsilon_0 (\partial_t v(s), v(s)).$$

From Cauchy-Schwarz and Young inequalities, we estimate the right side of (3.10) to get

$$\frac{d\mathcal{G}}{ds}(s) + \frac{\varepsilon_0}{2} \|\xi_v(s)\|_2^2 \leq C \|\Phi(s)\|_2 \|\xi_v(s)\|_2, \quad (3.11)$$

for some constant $C > 0$ depending only on λ_1 and γ . On the other hand, using again the Cauchy-Schwarz and Young inequalities we have

$$\frac{1}{4} \|\xi_v(s)\|_2^2 \leq \mathcal{G}(s) \leq \frac{3}{4} \|\xi_v(s)\|_2^2. \quad (3.12)$$

So, from (3.11) and (3.12) we deduce

$$\frac{1}{2} \frac{d\mathcal{G}}{ds}(s) + \frac{\varepsilon_0}{3} \mathcal{G}(s) \leq C \|\Phi(s)\|_2 [\mathcal{G}(s)]^{1/2}. \quad (3.13)$$

Therefore, from Lemma 2.5.1 we infer

$$[\mathcal{G}(s)]^{1/2} \leq [\mathcal{G}(\tau)]^{1/2} e^{-\left(\frac{\varepsilon_0}{3}\right)(s-\tau)} + 2C \int_{\tau}^s e^{-\left(\frac{\varepsilon_0}{3}\right)(s-r)} \|\Phi(r)\|_2 dr, \quad \forall s \in [\tau, t].$$

Hence, using (3.12) we obtain the desire result with $\alpha := \frac{\varepsilon_0}{3}$ and $\tilde{C} := \max\{\sqrt{3}, 4C\}$. $\square$

To deal with the semilinear wave equation, we need some extra regularity on the weak solutions. The main tool to obtain such regularity is called Strichartz estimates. Specifically, this family of estimates has two properties: it possible to transfer the time-regularity to space-regularity (and vice-versa) with suitable weights, and also provides a boundness for the solution of wave equations on the regular space. Below, we state the classical version of Strichartz estimates for nonhomogeneous wave equations on three-dimensional manifolds with boundary.

Proposition 3.1.4 (Corollary 1.2, (BLAIR; SMITH; SOGGE, 2009)). *Let M be a Riemannian manifold with boundary and $\dim M = 3$. We say that a triple (p, q, γ) is admissible if $2 < p \leq +\infty$, $2 \leq q < +\infty$,*

$$\frac{1}{p} + \frac{3}{q} = \frac{3}{2} - \gamma \quad \text{and} \quad \frac{3}{p} + \frac{2}{q} \leq 1. \quad (3.14)$$

Suppose that, (p, q, γ) and $(r, s, 1 - \gamma)$ are admissible and consider r' and s' the conjugated exponents of r and s , respectively. Then, for any $\xi \in H^\gamma(M) \times H^{\gamma-1}(M)$ and any $G \in L^1(\tau, t; L^2(M))$, the solution of

$$u_{tt} - \Delta_g u = G \text{ in } M \times (\tau, +\infty)$$

with homogeneous Dirichlet boundary condition and initial data $\xi_u(\tau) = \xi$, has an extra regularity,

$$u \in L^p(\tau, t; L^q(M))$$

and the following estimate holds

$$\|u\|_{L^p(\tau, t; L^q(M))} \leq \mathcal{Q}(t - \tau) (\|\xi\|_{H^\gamma(M) \times H^{\gamma-1}(M)} + \|G\|_{L^{p'}(\tau, t; L^{q'}(M))}) \quad (3.15)$$

for some nondecreasing function $\mathcal{Q}$ depending on M . $\square$

Now we prove a Strichartz estimate for (3.3)-(3.5) which is uniform w.r.t. the time variable.

Proposition 3.1.5. *For any $\xi \in \mathcal{H}$, $\Phi \in L^1(\mathbb{R}; L^2(M))$ and $t > \tau$, the generalized solution v of (3.3)-(3.5) satisfies*

$$\|v\|_{L^4(\tau, t; L^{12}(M))} \leq \tilde{C}_0 \left(\|\xi\|_{\mathcal{H}} + \|\Phi\|_{L^1(\mathbb{R}; L^2(M))} \right). \quad (3.16)$$

for some positive constant $\tilde{C}_0$ depending only of $\tilde{C}$ and α given in Proposition 3.1.3.

Proof. Using the Proposition 3.1.4 with $\frac{3}{12} + \frac{1}{4} = \frac{1}{2}$ and from (3.8), we conclude that $v \in L^4(\tau, t; L^{12}(M))$ and there exists a positive non-decreasing function $\mathcal{Q}$ such that

$$\begin{aligned} \|v\|_{L^4(\tau, t; L^{12}(M))} &\leq \mathcal{Q}(t - \tau) \left(\|\xi\|_{\mathcal{H}} + \|\Phi\|_{L^1(\mathbb{R}; L^2(M))} + \gamma \|\partial_t v\|_{C([\tau, t]; L^2(M))} \right) \\ &\leq \mathcal{Q}_0(t - \tau) \left(\|\xi\|_{\mathcal{H}} + \|\Phi\|_{L^1(\mathbb{R}; L^2(M))} \right), \end{aligned} \quad (3.17)$$

where we set $\mathcal{Q}_0(s) := (1 + \gamma) \mathcal{Q}(s)$.

Now, let us consider $\tau(t) = \max\{\tau, t - 1\}$. From (3.17) we get

$$\|v\|_{L^4(\tau(t), t; L^{12}(M))} \leq \mathcal{Q}_0(1) \left(\|\xi_v(\tau(t))\|_{\mathcal{H}} + \int_{\tau(t)}^t \|\Phi(s)\|_2 ds \right). \quad (3.18)$$

If $\tau(t) = \tau$, then $0 < t - \tau < 1$ and

$$\|\xi_v(\tau(t))\|_{\mathcal{H}} \leq e^\alpha \|\xi\|_{\mathcal{H}} e^{-\alpha(t-\tau)}.$$

If $\tau(t) = t - 1$, then from (3.9) we obtain

$$\|\xi_v(\tau(t))\|_{\mathcal{H}} = \|\xi_v(t - 1)\|_{\mathcal{H}} \leq \tilde{C} e^\alpha \|\xi\|_{\mathcal{H}} e^{-\alpha(t-\tau)} + \tilde{C} e^\alpha \int_{\tau}^t e^{-\alpha(t-s)} \|\Phi(s)\|_2 ds.$$

So, in both cases the following estimate holds

$$\|\xi_v(\tau(t))\|_{\mathcal{H}} \leq \max\{\tilde{C}, 1\} e^\alpha \|\xi\|_{\mathcal{H}} e^{-\alpha(t-\tau)} + \tilde{C} e^\alpha \int_{\tau}^t e^{-\alpha(t-s)} \|\Phi(s)\|_2 ds. \quad (3.19)$$

We also note that $0 < t - \tau(t) \leq 1$ and then

$$\int_{\tau(t)}^t \|\Phi(s)\|_2 ds = \int_{\tau(t)}^t e^{\alpha(t-s)} e^{-\alpha(t-s)} \|\Phi(s)\|_2 ds \leq e^\alpha \int_{\tau}^t e^{-\alpha(t-\tau)} \|\Phi(s)\|_2 ds. \quad (3.20)$$

Plugging the estimates (3.19) and (3.20) in (3.18) we arrive at

$$\|v\|_{L^4(\tau(t),t;L^{12}(M))} \leq \tilde{C}_2 \|\xi\|_{\mathcal{H}} e^{-\alpha(t-\tau)} + \tilde{C}_2 \int_{\tau}^t e^{-\alpha(t-s)} \|\Phi(s)\|_2 ds, \quad (3.21)$$

where $\tilde{C}_2 := \mathcal{Q}_0(1) \max\{\tilde{C}, 1\} e^{\alpha} > 0$.

Let $N := [t - \tau]$ be the integer part of $t - \tau$. From (3.21) we have

$$\begin{aligned} \|v\|_{L^4(\tau,t;L^{12}(M))} &\leq \left(\sum_{j=0}^{N-1} \|v\|_{L^4(\tau+j,\tau+(j+1);L^{12}(M))} \right) + \|v\|_{L^4(\tau+N,t;L^{12}(M))} \\ &\leq \tilde{C}_2 \sum_{j=0}^{N-1} \left(\|\xi_v(\tau+j)\|_{\mathcal{H}} + \int_{\tau+j}^{\tau+(j+1)} \|\Phi(s)\|_2 ds \right) \\ &\quad + \tilde{C}_2 \left(\|\xi_v(\tau+N)\|_{\mathcal{H}} + \int_{\tau+N}^t \|\Phi(s)\|_2 ds \right) \\ &= \tilde{C}_2 \left[\left(\sum_{j=1}^N \|\xi_v(\tau+j)\|_{\mathcal{H}} \right) + \|\xi\|_{\mathcal{H}} + \|\Phi\|_{L^1(\tau,t;L^2(M))} \right]. \end{aligned} \quad (3.22)$$

On the other hand, from (3.9) we get

$$\begin{aligned} \sum_{j=1}^N \|\xi_v(\tau+j)\|_{\mathcal{H}} &\leq \tilde{C} \sum_{j=1}^N \left(\|\xi\|_{\mathcal{H}} e^{-\alpha j} + \int_{\tau}^{\tau+j} e^{-\alpha(\tau+j-s)} \|\Phi(s)\|_2 ds \right) \\ &= \tilde{C} \|\xi\|_{\mathcal{H}} \sum_{j=1}^N e^{-\alpha j} + \tilde{C} \sum_{j=1}^N \int_{\tau}^{\tau+j} e^{-\alpha(\tau+j-s)} \|\Phi(s)\|_2 ds \\ &\leq \tilde{C} \|\xi\|_{\mathcal{H}} \sum_{j=1}^{+\infty} e^{-\alpha j} + \tilde{C} \sum_{j=1}^N \int_{\tau}^{\tau+j} e^{-\alpha(\tau+j-s)} \|\Phi(s)\|_2 ds \\ &\leq \left(\frac{\tilde{C}}{1-e^{-\alpha}} \right) \|\xi\|_{\mathcal{H}} + \tilde{C} \sum_{j=1}^N \int_{\tau}^{\tau+j} e^{-\alpha(\tau+j-s)} \|\Phi(s)\|_2 ds. \end{aligned} \quad (3.23)$$

Now, we claim that

$$\sum_{j=1}^N \int_{\tau}^{\tau+j} e^{-\alpha(\tau+j-s)} \|\Phi(s)\|_2 ds \leq \left(\frac{1}{1-e^{-\alpha}} \right) \|\Phi\|_{L^1(\mathbb{R};L^2(M))}. \quad (3.24)$$

Indeed,

$$\begin{aligned} \sum_{j=1}^N \int_{\tau}^{\tau+j} e^{-\alpha(\tau+j-s)} \|\Phi(s)\|_2 ds &= \sum_{j=1}^N \sum_{k=0}^{j-1} \int_{\tau+k}^{\tau+k+1} e^{-\alpha(\tau+j-s)} \|\Phi(s)\|_2 ds \\ &\leq \sum_{j=1}^N \sum_{k=0}^{j-1} e^{-\alpha(j-k-1)} \left(\int_{\tau+k}^{\tau+k+1} \|\Phi(s)\|_2 ds \right) \\ &\leq \left(\sum_{m=0}^{+\infty} e^{-\alpha m} \right) \left(\sum_{k=0}^{N-1} \int_{\tau+k}^{\tau+k+1} \|\Phi(s)\|_2 ds \right) \\ &\leq \left(\frac{1}{1-e^{-\alpha}} \right) \|\Phi\|_{L^1(\mathbb{R};L^2(M))} \end{aligned}$$

which proves the claim. Hence, from (3.22), (3.23) and (3.24) we conclude that (3.16) holds with

$$\tilde{C}_0 := \tilde{C}_2 \left(\frac{\tilde{C}}{1 - e^{-\alpha}} + 1 \right) > 0. \quad (3.25)$$

□

3.2 Well-posedness

Now we are in conditions to study the well-posedness of our original problem (3.1)-(3.2). To do this, we impose the following assumptions under the function f :

(F) $f \in C^1(\mathbb{R})$ and there exist $C_f, \rho > 0$, $\beta \in (0, \lambda_1)$ and $p \in [3, 5)$ such that

$$|f'(s)| \leq C_f (1 + |s|^{p-1}), \quad \forall s \in \mathbb{R}, \quad (3.26)$$

and

$$-\rho - \frac{\beta}{2} s^2 \leq F(s) := \int_0^s f(y) dy \leq f(s)s + \frac{\beta}{2} s^2 + \rho, \quad (3.27)$$

where $\lambda_1 > 0$ is the first eigenvalue of Δ_g with Dirichlet boundary condition, which also is the sharp constant of the Poincaré's inequality (2.5).

Lemma 3.2.1. *If (F) holds, then we have the following properties:*

(i) For any $\delta \in [0, \frac{1}{2}]$, the Nemytskii operator

$$f : L^{3(p-1)/(1-\delta)}(M) \cap H_{1-\delta} \rightarrow L^2(M)$$

satisfies

$$\|f(z) - f(w)\|_2 \leq C_0 \Theta_\delta(z, w) \|z - w\|_{H_{1-\delta}} \quad (3.28)$$

for some constant $C_0 > 0$ independent on $z, w \in L^{3(p-1)/(1-\delta)}(M) \cap H_{1-\delta}$, where

$$\Theta_\delta(z, w) := 1 + \|z\|_{3(p-1)/(1-\delta)}^{p-1} + \|w\|_{3(p-1)/(1-\delta)}^{p-1}. \quad (3.29)$$

(ii) For any $z \in H_0^1(M)$ we have

$$\int_M [F(z) - f(z)z] d\mu_g \leq \frac{\beta}{2\lambda_1} \|\nabla_g z\|_2^2 + \rho \operatorname{vol}(M). \quad (3.30)$$

Moreover, there exists $C_1 > 0$ such that

$$-\rho \operatorname{vol}(M) - \frac{\beta}{2\lambda_1} \|\nabla_g z\|_2^2 \leq \int_M F(z) d\mu_g \leq C_1 \left(1 + \|\nabla_g z\|_2^{p+1} \right) + \rho \operatorname{vol}(M). \quad (3.31)$$

Proof. Let $\delta \in [0, 1)$ and $t > \tau$.

Proof of (i): Let us consider $z, w \in L^{3(p-1)/(1-\delta)}(M) \cap H_{1-\delta}$ and

$$r_1 = \frac{3}{2(1-\delta)} \quad \text{and} \quad r_2 = \frac{3}{1+2\delta}.$$

So, from (3.26) and from Hölder inequality $\frac{1}{r_1} + \frac{1}{r_2} = 1$ we get

$$\begin{aligned} \|f(z) - f(w)\|_2^2 &= \int_M \left(\int_0^1 f'(\theta z + (1-\theta)w)(z-w) d\theta \right)^2 d\mu_g \\ &\leq C_f \int_M \left(1 + |z|^{2(p-1)} + |w|^{2(p-1)} \right) |z-w|^2 d\mu_g \\ &\leq C_f \left(1 + \|z\|_{2(p-1)r_1}^{2(p-1)} + \|w\|_{2(p-1)r_1}^{2(p-1)} \right) \|z-w\|_{2r_2}^2. \end{aligned}$$

Now, observe that $1 - \delta < \frac{3}{2}$ and

$$\frac{3}{2} - \frac{3}{2r_2} = 3 \left(\frac{1}{2} - \frac{1+2\delta}{6} \right) = 1 - \delta.$$

From Lemma 2.1.2 we conclude that $H_{1-\delta} \hookrightarrow L^{2r_2}(M)$ and (3.28) holds.

Proof of (ii): To show (3.31), we only need integrate (3.27) over M and use the Poincaré's inequality (2.5).

On the other hand, integrating both sides of (3.27) over M and using again the Poincaré inequality we obtain

$$-\rho \operatorname{vol}(M) - \frac{\beta}{2\lambda_1} \|\nabla_g z\|_2^2 \leq \int_M F(z) d\mu_g \leq \int_M f(z)z d\mu_g + \frac{\beta}{2\lambda_1} \|\nabla_g z\|_2^2 + \rho \operatorname{vol}(M). \quad (3.32)$$

From (3.26) and take into account that $H_0^1(M) \hookrightarrow L^{p+1}(M)$, we deduce

$$\begin{aligned} \int_M f(z)z d\mu_g &= \int_M \int_0^1 f'(\theta z) |z|^2 d\theta d\mu_g \\ &\leq \int_M C_f (1 + |z|^{p-1}) |z|^2 d\mu_g \\ &\leq \tilde{C}_f \left(\|\nabla_g z\|_2^2 + \|\nabla_g z\|_2^{p+1} \right), \end{aligned}$$

for some constant $\tilde{C}_f > 0$. Plugging the last estimate in (3.32) we have

$$-\rho \operatorname{vol}(M) - \frac{\beta}{2\lambda_1} \|\nabla_g z\|_2^2 \leq \int_M F(z) d\mu_g \leq C \left(\|\nabla_g z\|_2^2 + \|\nabla_g z\|_2^{p+1} \right) + \rho \operatorname{vol}(M) \quad (3.33)$$

where $C := \tilde{C}_f + \frac{\beta}{2\lambda_1} > 0$. Now, if $\|\nabla_g z\|_2 \leq 1$, then

$$\|\nabla_g z\|_2^2 + \|\nabla_g z\|_2^{p+1} \leq 1 + \|\nabla_g z\|_2^{p+1}.$$

If $\|\nabla_g z\|_2 > 1$, then

$$\|\nabla_g z\|_2^2 + \|\nabla_g z\|_2^{p+1} \leq 2\|\nabla_g z\|_2^{p+1} \leq 2\left(1 + \|\nabla_g z\|_2^{p+1}\right).$$

So, in both cases we get $\|\nabla_g z\|_2^2 + \|\nabla_g z\|_2^{p+1} \leq 2\left(1 + \|\nabla_g z\|_2^{p+1}\right)$ and from (3.33), we conclude that (3.30) holds with $C_1 := 2C > 0$. $\square$

Now we are in position to state and prove that (3.1)-(3.2) is global well-posed.

Theorem 3.2.2. *Let $h \in L^1(\mathbb{R}; L^2(M))$ and suppose that (F) holds. Then, for any $\xi \in \mathcal{H}$ and for any $\varepsilon \in [0, 1]$ the problem (3.1)-(3.2) has a unique generalized solution u^ε defined in $[\tau, +\infty)$, which depends continuously on the initial data. Moreover, for any $t > \tau$, $u^\varepsilon \in L^4(\tau, t; L^{12}(M))$ and the following estimate holds uniformly in $\varepsilon \in [0, 1]$*

$$\|u\|_{L^4(\tau, t; L^{12}(M))} \leq \hat{\mathcal{Q}}(t - \tau) \left[\tilde{\mathcal{Q}}(\|\xi\|_{\mathcal{H}}) + \tilde{\mathcal{Q}}(\|h\|_{L^1(\mathbb{R}; L^2(M))}) \right] \quad (3.34)$$

for some positive non-decreasing functions $\hat{\mathcal{Q}}$ and $\tilde{\mathcal{Q}}$.

The proof of Theorem 3.2.2 is a consequence of the lemmas below.

Lemma 3.2.3 (Local existence). *Assume that the assumptions of Theorem 3.2.2 are in force. Then, for any $\xi \in \mathcal{H}$, there exists*

$$t_0 = t_0(\|\xi\|_{\mathcal{H}}, \|h\|_{L^1(\mathbb{R}; L^2(M))}) > \tau$$

such that the problem (3.1)-(3.2) has a unique generalized solution u defined in $[\tau, t_0]$.

Proof. For any $t > \tau$, we consider the Banach space

$$\mathcal{C}_t := C([\tau, t]; H_0^1(M)) \cap L^4(\tau, t; L^{12}(M)) \quad (3.35)$$

endowed with the norm

$$\|z\|_{\mathcal{C}_t} := \sup_{s \in [\tau, t]} \|\nabla z(s)\|_2 + \|z\|_{L^4(\tau, t; L^{12}(M))}.$$

Let $z \in \mathcal{C}_t$. Using (3.28) with $\delta = 0$ and applying the Hölder inequality with $\frac{5-p}{4} + \frac{p-1}{4} = 1$ we deduce that $f(z) \in L^1(\tau, t; L^2(M))$ and

$$\|f(z)\|_{L^1(\tau, t; L^2(M))} \leq C_f \left((t - \tau) + (t - \tau)^{\frac{5-p}{4}} \|z\|_{L^4(\tau, t; L^{12}(M))}^{p-1} \right) \|z\|_{C([\tau, t]; H_0^1(M))}, \quad (3.36)$$

for some constant $C_f > 0$. Thanks to Proposition 3.1.1, the problem

$$\begin{cases} \partial_{tt} u - \Delta_g u + \gamma \partial_t u = -f(z) + h(t) \text{ in } M \times (\tau, t), \\ u = 0 \quad \text{on } \partial M \times [\tau, t], \\ \xi_u(\tau) = \xi \end{cases} \quad (3.37)$$

has unique generalized solution u such that $\xi_u \in \mathcal{C}_t \times C([\tau, t]; L^2(M))$. So, we can define $\Xi : \mathcal{C}_t \rightarrow \mathcal{C}_t$ by $\Xi(z) = u$, where u is the unique generalized solution of (3.37).

Now, we set $\mathcal{B}_t := \{z \in \mathcal{C}_t; \|z\|_{\mathcal{C}_t} \leq R\}$ with

$$R := (\tilde{C}_0 + 1) \left[\|\xi\|_{\mathcal{H}} + \|h\|_{L^1(\mathbb{R}; L^2(M))} + 1 \right]$$

where $\tilde{C}_0 > 0$ is given by (3.25). We also choose $t_0 > \tau$ such that

$$(t_0 - \tau)^{\frac{(5-p)}{4}} \leq \min \left\{ 1, \frac{1}{4C_f(\tilde{C}_0 + 1)(1 + R^{p-1})}, \frac{1}{C_f(R + R^p)} \right\}.$$

We claim that $\Xi(\mathcal{B}_{t_0}) \subset \mathcal{B}_{t_0}$ and $\Xi : \mathcal{B}_{t_0} \rightarrow \mathcal{B}_{t_0}$ is a contraction mapping. Indeed, let $z \in \mathcal{B}_{t_0}$. From (3.8), (3.16) and (3.36) we obtain

$$\|\Xi(z)\|_{\mathcal{C}_{t_0}} \leq (\tilde{C}_0 + 1) \left[\|\xi\|_{\mathcal{H}} + \|h\|_{L^1(\mathbb{R}; L^2(M))} + C_f(t_0 - \tau)^{\frac{(5-p)}{4}} (R + R^p) \right] \leq R.$$

Hence, $\Xi(z) \in \mathcal{B}_{t_0}$ and the first assert is proved. To show that Ξ is a contraction mapping in $\mathcal{B}_{t_0}$ we consider $z^i \in \mathcal{B}_{t_0}$ and $u^i = \Xi(z^i)$, $i = 1, 2$. So, $\tilde{u} = u^1 - u^2$ is the generalized solution of

$$\begin{cases} \tilde{u}_{tt} - \Delta_g \tilde{u} + \gamma \tilde{u}_t = f(z^2) - f(z^1) \text{ in } M \times (\tau, t_0), \\ \tilde{u} = 0 \quad \text{on} \quad \partial M \times [\tau, t_0], \\ \xi_{\tilde{u}}(\tau) = 0. \end{cases} \quad (3.38)$$

From (3.8) and (3.16) we infer

$$\|\Xi(z^1) - \Xi(z^2)\|_{\mathcal{C}_{t_0}} \leq (\tilde{C}_0 + 1) \|f(z^2) - f(z^1)\|_{L^1(\tau, t_0; L^2(M))}. \quad (3.39)$$

On the other hand, from (3.28) with $\delta = 0$ and Hölder inequality $\frac{5-p}{4} + \frac{p-1}{4} = 1$ we obtain

$$\|f(z^2) - f(z^1)\|_{L^1(\tau, t_0; L^2(M))} \leq 2C_f(t_0 - \tau)^{\frac{(5-p)}{4}} (1 + R^{p-1}) \|z^1 - z^2\|_{C([\tau, t_0]; H_0^1(M))}. \quad (3.40)$$

Plugging (3.40) in (3.39) and using the definition of t_0 , we conclude that

$$\|\Xi(z^1) - \Xi(z^2)\|_{\mathcal{C}_{t_0}} \leq \lambda_0 \|z^1 - z^2\|_{\mathcal{C}_{t_0}},$$

for some $\lambda_0 \in (0, 1)$ and consequently Ξ is a contraction mapping in $\mathcal{B}_{t_0}$.

Since $\mathcal{B}_{t_0}$ is closed in $\mathcal{C}_{t_0}$, we can apply the Contraction Mapping Theorem to obtain a unique generalized solution u of (3.1)-(3.2) such that $\xi_u \in \mathcal{C}_{t_0} \times C([\tau, t_0]; L^2(M))$. The proof is now complete. $\square$

The next lemma is a criterion to extend the generalized solution to a maximal interval.

Lemma 3.2.4 (Global extension criterion). *Under the assumptions (F), the generalized solution u obtained by the previous lemma can be extended to a maximal interval $[\tau, t_{\max})$, such that either $t_{\max} = +\infty$ or $t_{\max} < +\infty$ and*

$$\lim_{t \rightarrow t_{\max}^-} \|\xi_u(t)\|_{\mathcal{H}} = +\infty. \quad (3.41)$$

Moreover, for any $t \in (\tau, t_{\max})$ we have $u \in L^4(\tau, t; L^{12}(M))$.

Proof. Let us consider $\mathcal{X}$ the set of all functions u such that exists $t \in (\tau, +\infty]$ satisfying the following property: u is the unique generalized solution of (3.1)-(3.2) in $[\tau, t]$ with

$$\xi_u \in \mathcal{C}_s \times C([\tau, s]; L^2(M)), \quad \forall s \in (\tau, t).$$

From Lemma 3.2.3 we have $\mathcal{X} \neq \emptyset$. We also note that $\mathcal{X}$ is a partially ordered set endowed with the relation

$$u^1 \preceq u^2 \iff t_1 \leq t_2,$$

where $u^1, u^2 \in \mathcal{X}$ and $t_1, t_2 \in (\tau, +\infty]$ are such that t_i satisfies the property of $\mathcal{X}$ for u^i , $i = 1, 2$, respectively. So, from Zorn's Lemma we obtain a maximal time $t_{\max} \in (\tau, +\infty]$ and a unique generalized solution u defined in $[\tau, t_{\max})$, such that $\xi_u \in \mathcal{C}_s \times C([\tau, s]; L^2(M))$ for all $s \in (\tau, t_{\max})$.

Note that if $t_{\max} = +\infty$ there is nothing to prove. So, we suppose by contradiction that $t_{\max} < +\infty$, but (3.41) does not occurs. Thus, there is a sequence $t_n > \tau$ and a constant $K > 0$ such that

$$\lim_{n \rightarrow \infty} t_n = t_{\max} \quad \text{and} \quad \|\xi_u(t_n)\|_{\mathcal{H}} \leq K, \quad \forall n \in \mathbb{N}. \quad (3.42)$$

For each $n \in \mathbb{N}$, consider the problem

$$\begin{cases} \partial_{tt} w - \Delta_g w + \gamma \partial_t w + f(w) = h(t + t_n - \tau) \text{ in } M \times (\tau, +\infty), \\ w = 0 \quad \text{on} \quad \partial M \times [\tau, +\infty), \\ \xi_w(\tau) = \xi_u(t_n). \end{cases} \quad (3.43)$$

From (3.42) and using the same arguments to prove Lemma 3.2.3, we obtain a time $\tilde{t} > \tau$, depending on K and $\|h\|_{L^1(\mathbb{R}; L^2(M))}$, such that (3.43) has a unique generalized solution w in $[\tau, \tilde{t})$ with $\xi_w \in \mathcal{C}_{\tilde{t}} \times C([\tau, \tilde{t}]; L^2(M))$.

On the other hand, the convergence in (3.42) implies that exists $n_0 \in \mathbb{N}$ such that $t_{n_0} + \tilde{t} - \tau > t_{\max} > t_{n_0}$. Now, we set

$$\bar{u}(t) = \begin{cases} u(t), & \tau \leq t \leq t_{n_0}, \\ w(t - t_{n_0} + \tau), & t_{n_0} \leq t \leq t_{n_0} + \tilde{t} - \tau. \end{cases} \quad (3.44)$$

By construction, $\bar{u}$ is the generalized solution of (3.1)-(3.2) in $[\tau, t_{n_0} + \tilde{t} - \tau]$. But, this contradicts the maximality of $t_{\max}$. Hence, if $t_{\max} < +\infty$ then (3.41) holds. $\square$

In what follows, we use the above criterion to show that our solution is global. Since we work only with weak solutions, we use the regularizing sequence $\mathcal{R}_n$ defined by (2.7).

Lemma 3.2.5 (Global existence). *Under the assumptions (F), the problem (3.1)-(3.2) has a unique global generalized solution u such that $u \in L^4(\tau, t; L^{12}(M))$ for all $t > \tau$.*

Proof. Suppose by contradiction that $t_{\max} < +\infty$ and consider u the weak solution of (3.1)-(3.2) on $[\tau, t_{\max})$. Applying the regularizing sequence $\mathcal{R}_n$ in (3.1) we have

$$\partial_{tt} u_n - \Delta_g u_n + \gamma \partial_t u_n + \mathcal{R}_n f(u) = \mathcal{R}_n h \quad (3.45)$$

where we set $u_n := \mathcal{R}_n u$. Composing (3.45) with $\partial_t u_n$, integrating the result on (τ, t) , $t \in (\tau, t_{\max})$ and using Green's formula, we have

$$\begin{aligned} \frac{1}{2} \|\xi_{u_n}(t)\|_{\mathcal{H}}^2 - \frac{1}{2} \|\xi_{u_n}(\tau)\|_{\mathcal{H}}^2 &= -\gamma \int_{\tau}^t \|\partial_t u_n(s)\|_2^2 ds - \int_{\tau}^t (\mathcal{R}_n f(u(s)), \partial_t u_n(t)) ds \\ &\quad + \int_{\tau}^t (\mathcal{R}_n h(s), \partial_t u_n(s)) ds. \end{aligned} \quad (3.46)$$

From Proposition 2.1.1 and from Dominated Convergence Theorem (Proposition 2.2.2), we can take the limit in (3.46) to obtain

$$\begin{aligned} \frac{1}{2} \|\xi_u(t)\|_{\mathcal{H}}^2 - \frac{1}{2} \|\xi_u(\tau)\|_{\mathcal{H}}^2 &= -\gamma \int_{\tau}^t \|\partial_t u(s)\|_2^2 ds - \int_{\tau}^t (f(u(s)), \partial_t u(t)) ds \\ &\quad + \int_{\tau}^t (h(s), \partial_t u(s)) ds. \end{aligned} \quad (3.47)$$

Now we observe that

$$\int_{\tau}^t (f(u(s)), \partial_t u(s)) ds = \int_M F(u(t)) d\mu_g - \int_M F(u(\tau)) d\mu_g. \quad (3.48)$$

This implies that $f(u(\cdot), \partial_t u(\cdot))$ is absolutely continuous and then its time derivative exists almost everywhere in (τ, t) with $\frac{d}{dt} \int_M F(u(t)) d\mu_g = (f(u(t)), \partial_t u(t))$. So, from (3.31), (3.47) and (3.48), together with Cauchy-Schwarz inequality, we arrive at

$$\|\xi_u(t)\|_{\mathcal{H}}^2 \leq C \left(1 + \|\xi_u(\tau)\|_{\mathcal{H}}^{p+1} \right) + \int_{\tau}^t \|h(s)\|_2 \|\xi_u(s)\|_2 ds + 2\rho \operatorname{vol}(M). \quad (3.49)$$

Now we apply Lemma 2.5.2 in (3.49) we arrive at

$$\|\xi_u(t)\|_{\mathcal{H}} \leq C \left(1 + \|\xi\|_{\mathcal{H}}^{p+1} \right)^{1/2} + \|h\|_{L^1(\mathbb{R}; L^2(M))} + (2\rho \operatorname{vol}(M))^{1/2}. \quad (3.50)$$

But, this contradicts (3.41). Hence, $t_{\max} = +\infty$ and the solution u is global. Moreover, from Lemma 3.2.4 the solution $u \in L^4(\tau, t; L^{12}(M))$, for all $t \in (\tau, +\infty)$ and this concludes the proof. $\square$

Remark 3.2.6. *The arguments used in the proof of Lemma 3.2.5 to regularize the weak solution u can be used in the results below. So, without loss of generality, we assume that the solution $\xi_u \in H_0^1(M) \times H_0^1(M)$.*

Next, we prove that (3.34) holds.

Lemma 3.2.7 (Strichartz type estimate). *Let the assumptions (F) be in force. Then, for any $t > \tau$, the global generalized solution u given by Lemma 3.2.5 satisfies (3.34) for some non-decreasing positive functions $\hat{\mathcal{Q}}$ and $\tilde{\mathcal{Q}}$.*

Proof. Let us consider $t \in (\tau, \tau + 1)$. We decompose $u = v + z$ where v is the generalized solution of (3.3)-(3.5) with $\Phi = h$ and z is the generalized solution of

$$\begin{cases} \partial_{tt} z - \Delta_g z + \gamma \partial_t z = -f(v + z), & \text{in } M \times (\tau, t), \\ z = 0 & \text{on } \partial M \times [\tau, t], \\ \xi_z(\tau) = 0. \end{cases} \quad (3.51)$$

So, using (3.16) we infer

$$\|v\|_{L^4(\tau,t;L^{12}(M))} \leq \tilde{C}_0 \left(\|\xi\|_{\mathcal{H}} + \|h\|_{L^1(\mathbb{R};L^2(M))} \right), \quad (3.52)$$

$$\|z\|_{L^4(\tau,t;L^{12}(M))} \leq \tilde{C}_0 \|f(v+z)\|_{L^1(\tau,t;L^2(M))}. \quad (3.53)$$

From (3.49) we also have

$$\|\xi_u(t)\|_{\mathcal{H}} \leq \mathcal{Q}_1(\|\xi\|_{\mathcal{H}}) + \mathcal{Q}_1(\|h\|_{L^1(\mathbb{R};L^2(M))}), \quad (3.54)$$

where we set $\mathcal{Q}_1(s) := \max\{(1+|s|^{p+1})^{1/2}, s + (2\rho \operatorname{vol}(M))^{1/2}\}$. Applying (3.28) in (3.53), using the estimates (3.52) and (3.54) and take into account the embedding

$$L^4(\tau,t;L^{12}(M)) \hookrightarrow L^{p-1}(\tau,t;L^{3(p-1)}(M))$$

we obtain

$$\begin{aligned} \|z\|_{L^4(\tau,t;L^{12}(M))} &\leq (t-\tau)^{\frac{(5-p)}{4}} \left(\mathcal{Q}_2(\|\xi\|_{\mathcal{H}}) + \mathcal{Q}_2(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right) \\ &\quad + C_2 \left(\mathcal{Q}_1(\|\xi\|_{\mathcal{H}}) + \mathcal{Q}_1(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right) \|z\|_{L^4(\tau,t;L^{12}(M))}^{p-1}, \end{aligned} \quad (3.55)$$

for some constant $C_2 > 0$ and for some positive non-decreasing function $\mathcal{Q}_2$. Now, we define

$$C_3 := C_2 \left(\mathcal{Q}_1(\|\xi\|_{\mathcal{H}}) + \mathcal{Q}_1(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right), \quad \tilde{\varepsilon} := \min \left\{ 1, \frac{\tilde{c}}{2} \right\}, \quad \tilde{t}_0 := \tau + \frac{\tilde{\varepsilon}}{\tilde{c}}$$

where

$$\tilde{c} := \left\{ [2^{p-1}C_3]^{\frac{1}{p-2}} \left(\mathcal{Q}_2(\|\xi\|_{\mathcal{H}}) + \mathcal{Q}_2(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right) \right\}^{\frac{4}{5-p}}.$$

It follows from (3.55) that

$$\|z\|_{L^4(\tau,\tilde{t}_0;L^{12}(M))} \leq \varepsilon_0 + C_3 \|z\|_{L^4(\tau,\tilde{t}_0;L^{12}(M))}^{p-1} \quad (3.56)$$

where

$$\varepsilon_0 := (\tilde{t}_0 - \tau)^{\frac{(5-p)}{4}} \left(\mathcal{Q}_2(\|\xi\|_{\mathcal{H}}) + \mathcal{Q}_2(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right) \leq \frac{1}{2} \left(\frac{1}{2C_3} \right)^{\frac{1}{p-2}}.$$

So, we can apply Lemma 2.5.3 in (3.56) to obtain

$$\|z\|_{L^4(\tau,\tilde{t}_0;L^{12}(M))} \leq 2\varepsilon_0 = 2(\tilde{t}_0 - \tau)^{\frac{(5-p)}{4}} \left(\mathcal{Q}_2(\|\xi\|_{\mathcal{H}}) + \mathcal{Q}_2(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right) \quad (3.57)$$

and (3.34) holds for all $t \in (\tau, \tilde{t}_0)$. We also observe that there exists a positive non-decreasing function independent on t , τ , $\|\xi\|_{\mathcal{H}}$ and $\|h\|_{L^1(\mathbb{R};L^2(M))}$, which we also call $\mathcal{Q}_2$, such that (3.57) is valid for any interval with length less or equal than $c_0 := (\tilde{t}_0 - \tau)$. It can be proved using (3.54).

It remains to show (3.34) for all $t > \tilde{t}_0$. Indeed, for $t > \tilde{t}_0$ we set $P := \left\lceil \frac{t-\tau}{\tilde{t}_0-\tau} \right\rceil \in \mathbb{N}$. We divide $(t - \tau)$ in $P + 1$ subintervals and apply (3.57) in each one to deduce

$$\begin{aligned} \|z\|_{L^4(\tau,t;L^{12}(M))} &\leq \left(\sum_{j=0}^{P-1} \|z\|_{L^4(\tau+c_0j,\tau+c_0(j+1);L^{12}(M))} \right) + \|z\|_{L^4(\tau+Pc_0,t;L^{12}(M))} \\ &\leq 2(P+1)(\tilde{t}_0 - \tau)^{\frac{(5-p)}{4}} \left(\mathcal{Q}_2(\|\xi\|_{\mathcal{H}}) + \mathcal{Q}_2(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right) \\ &\leq 4(t - \tau)(\tilde{t}_0 - \tau)^{\frac{(1-p)}{4}} \left(\mathcal{Q}_2(\|\xi\|_{\mathcal{H}}) + \mathcal{Q}_2(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right). \end{aligned} \quad (3.58)$$

From (3.57) and (3.58), there is a positive non-decreasing function $\mathcal{Q}_3$ such that

$$\|z\|_{L^4(\tau,t;L^{12}(M))} \leq (t - \tau) \left(\mathcal{Q}_3(\|\xi\|_{\mathcal{H}}) + \mathcal{Q}_3(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right), \quad \forall t > \tau. \quad (3.59)$$

Hence, from (3.52) and (3.59) we conclude

$$\begin{aligned} \|u\|_{L^4(\tau,t;L^{12}(M))} &\leq \|v\|_{L^4(\tau,t;L^{12}(M))} + \|z\|_{L^4(\tau,t;L^{12}(M))} \\ &\leq \hat{\mathcal{Q}}(t - \tau) \left[\tilde{\mathcal{Q}}(\|\xi\|_{\mathcal{H}}) + \tilde{\mathcal{Q}}(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right] \end{aligned}$$

for some positive non-decreasing functions $\hat{\mathcal{Q}}$ and $\tilde{\mathcal{Q}}$. The proof is now complete. $\square$

Finally, we prove the continuous dependence of the initial data.

Lemma 3.2.8 (Continuous dependence). *For any $\xi^1, \xi^2 \in \mathcal{H}$ and $t > \tau$ we have*

$$\|\xi_{u^1}(t) - \xi_{u^2}(t)\|_{\mathcal{H}} \leq C \|\xi^1 - \xi^2\|_{\mathcal{H}}, \quad (3.60)$$

for some constant $C > 0$, depending on $\|\xi^1\|_{\mathcal{H}}$, $\|\xi^2\|_{\mathcal{H}}$, $\|h\|_{L^1(\mathbb{R};L^2(M))}$ and $t - \tau$, where u^1 and u^2 are the generalized solutions of (3.1)-(3.2) with initial data ξ^1 and ξ^2 , respectively.

Proof. Let $w = u^1 - u^2$ and $\tilde{\xi} = \xi^1 - \xi^2$. So, w satisfies

$$\begin{cases} \partial_{tt} w - \Delta_g w + \gamma \partial_t w = f(u^2) - f(u^1), & \text{in } M \times (\tau, t), \\ w = 0 & \text{on } \partial M \times [\tau, t], \\ \xi_w(\tau) = \tilde{\xi}. \end{cases} \quad (3.61)$$

From (3.7) and (3.28) with $\delta = 0$ we have

$$\frac{1}{2} \|\xi_w(t)\|_{\mathcal{H}}^2 \leq \frac{1}{2} \|\tilde{\xi}\|_{\mathcal{H}}^2 + C_0 \int_{\tau}^t \Theta_0(u^2(s), u^1(s)) \|\xi_w(s)\|_{\mathcal{H}}^2 ds.$$

Applying the Gronwall's inequality, we obtain

$$\|\xi_w(t)\|_{\mathcal{H}} \leq \exp \left\{ \frac{C_0}{2} \|\Theta_0(u^2, u^1)\|_{L^1(\tau,t)} \right\} \|\tilde{\xi}\|_{\mathcal{H}}. \quad (3.62)$$

On the other hand, using the embedding $L^4(\tau, t; L^{12}(M)) \hookrightarrow L^{p-1}(\tau, t; L^{3(p-1)}(M))$ and from the estimate (3.34) we infer

$$\|\Theta_0(u^2, u^1)\|_{L^1(\tau,t)} \leq \mathcal{Q}_4(t - \tau) \left[\mathcal{Q}_5(\|\xi^1\|_{\mathcal{H}}) + \mathcal{Q}_5(\|\xi^2\|_{\mathcal{H}}) + \mathcal{Q}_5(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right], \quad (3.63)$$

For some non-decreasing positive functions $\mathcal{Q}_4$ and $\mathcal{Q}_5$. Hence, from (3.62) and (3.63) we conclude that (3.60) holds with

$$C := \exp \left\{ \frac{C_0}{2} \mathcal{Q}_4(t - \tau) \left(\mathcal{Q}_5(\|\xi^1\|_{\mathcal{H}}) + \mathcal{Q}_5(\|\xi^2\|_{\mathcal{H}}) + \mathcal{Q}_5(\|h\|_{L^1(\mathbb{R};L^2(M))}) \right) \right\}.$$

$\square$

Proof of Theorem 3.2.2: Conclusion. From Lemma 3.2.5, Lemma 3.2.7 and Lemma 3.2.8 we obtain the desire result.

PULLBACK DYNAMICS

Under the above notations and results, we study now the dynamics of the nonautonomous dynamical system generated by the weak solutions of (3.1)-(3.2). Indeed, Theorem 3.2.2 ensures that for each $\varepsilon \in [0, 1]$, the problem (3.1)-(3.2) is well-posed. Consequently, we can set a family (indexed in $\varepsilon \in [0, 1]$) of evolution process $U_\varepsilon(t, \tau) : \mathcal{H} \rightarrow \mathcal{H}$ give by

$$U_\varepsilon(t, \tau)\xi := \xi_{u^\varepsilon}(t) = (u^\varepsilon(t), \partial_t u^\varepsilon(t)), \quad \tau \in \mathbb{R}, \quad t \geq \tau$$

where u^ε is the weak solution of (3.1)-(3.2) with initial data $\xi \in \mathcal{H}$.

A direct consequence of (3.60) is that, for each $\tau \in \mathbb{R}$, $t \geq \tau$ and $\varepsilon \in [0, 1]$ the evolution process $U_\varepsilon(t, \tau)$ is locally Lipschitz in $\mathcal{H}$, that is, if $\xi^1, \xi^2 \in B$ for some bounded set $B \subset \mathcal{H}$, then

$$\|U_\varepsilon(t, \tau)\xi^1 - U_\varepsilon(t, \tau)\xi^2\|_{\mathcal{H}} \leq C\|\xi^1 - \xi^2\|_{\mathcal{H}} \quad (4.1)$$

for some constant $C = C(B, \|h\|_{L^1(\mathbb{R}, L^2(M))}, t - \tau) > 0$

The main result of this section is present as follows.

Theorem 4.0.1. *Suppose the assumptions (F) hold and assume that $h \in L^1(\mathbb{R}; L^2(M))$. Then, we have:*

- (i) *for each $\varepsilon \in [0, 1]$, the dynamical system $(U_\varepsilon(t, \tau), \mathcal{H})$ posses a pullback exponential attractor $\mathcal{M}_\varepsilon = \{\mathcal{M}_\varepsilon(t)\}_{t \in \mathbb{R}}$;*
- (ii) *for each $\varepsilon \in [0, 1]$, the dynamical system $(U_\varepsilon(t, \tau), \mathcal{H})$ posses a unique global pullback attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(t)\}_{t \in \mathbb{R}}$;*
- (iii) *the family $\mathcal{A}_\varepsilon$ is upper-semicontinuous in $\varepsilon = 0$, that is, for each $t \in \mathbb{R}$,*

$$\lim_{\varepsilon \rightarrow 0} \text{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon(t), \mathcal{A}_0) = 0,$$

where $\mathcal{A}_0$ is the global attractor of the autonomous dynamical system $(U_0(t, \tau), \mathcal{H})$;

(iv) there exists a residual set $I \subset [0, 1]$ such that for every $t \in \mathbb{R}$, the mapping $\varepsilon \mapsto \mathcal{A}_\varepsilon(t)$ is continuous at each $\varepsilon \in I$, that is,

$$\lim_{\varepsilon \rightarrow \varepsilon_0} \text{dist}_{\mathcal{H}}^{\text{symm}}(\mathcal{A}_\varepsilon(t), \mathcal{A}_{\varepsilon_0}(t)) = 0, \quad \forall \varepsilon_0 \in I.$$

To prove Theorem 4.0.1, we prepare some auxiliary estimates that allow us to verify the conditions of Theorem 2.4.2, Proposition 2.4.4 and Proposition 2.4.5.

4.1 Technical estimates

We introduce the total energy associated to the weak solution u^ε as follows

$$E_\varepsilon(t) = \frac{1}{2} \|\xi_{u^\varepsilon}(t)\|_{\mathcal{H}}^2 + \int_M F(u^\varepsilon(t)) d\mu_g, \quad t > \tau,$$

which satisfies the following identity on the distributional sense

$$\frac{d}{dt} E_\varepsilon(t) = -\gamma \|\partial_t u^\varepsilon(t)\|_2^2 + (h(t), \partial_t u^\varepsilon(t)), \quad t > \tau. \quad (4.2)$$

From (3.31) we can see that

$$-\rho \text{vol}(M) + \beta_0 \|\xi_{u^\varepsilon}(t)\|_{\mathcal{H}}^2 \leq E_\varepsilon(t) \leq C(1 + \|\xi_{u^\varepsilon}(t)\|_{\mathcal{H}}^{p+1}) + \rho \text{vol}(M), \quad (4.3)$$

for some constant $C > 0$ and $\beta_0 := \frac{1}{2} \left(1 - \frac{\beta}{\lambda_1}\right) > 0$.

The first proposition give us a uniform estimate (w.r.t. $\varepsilon \in [0, 1]$) for $U_\varepsilon(t, \tau)\xi$ on the phase space $\mathcal{H}$.

Proposition 4.1.1. *For any $\varepsilon \in [0, 1]$ and for any $\xi \in \mathcal{H}$, there exist positive constants C and ω , independent on ε and ξ , such that*

$$\|U_\varepsilon(t, \tau)\xi\|_{\mathcal{H}} \leq C \left(1 + \|\xi\|_{\mathcal{H}}^{(p+1)/2}\right) e^{-\omega(t-\tau)} + C \int_\tau^t e^{-\omega(t-s)} \|h(s)\|_2 ds + C\rho \text{vol}(M). \quad (4.4)$$

Proof. Given $\tilde{\eta} > 0$, let us consider the functional $E_\varepsilon^{\tilde{\eta}} : [\tau, +\infty) \rightarrow \mathbb{R}$ defined by

$$E_\varepsilon^{\tilde{\eta}}(t) = E_\varepsilon(t) + \tilde{\eta} (\partial_t u^\varepsilon(t), u^\varepsilon(t)). \quad (4.5)$$

Using (4.3) and Cauchy-Schwarz inequality, we have

$$E_\varepsilon(t) - \rho \text{vol}(M) \leq 2E_\varepsilon^{\tilde{\eta}}(t) \leq 3E_\varepsilon(t) + \rho \text{vol}(M), \quad \forall t \geq \tau \quad (4.6)$$

for some $\tilde{\eta} > 0$ small enough. Taking the distributional derivative of $E_\varepsilon^{\tilde{\eta}}$ and using (4.2) we obtain

$$\begin{aligned} \frac{dE_\varepsilon^{\tilde{\eta}}}{dt}(t) + (1 - \tilde{\eta}) \|\partial_t u^\varepsilon(t)\|_2^2 + \tilde{\eta} \|\nabla_g u^\varepsilon(t)\|_2^2 &= -\tilde{\eta} (\gamma \partial_t u^\varepsilon(t) + f(u^\varepsilon(t)), u^\varepsilon(t)) \\ &\quad + (\varepsilon h(t), \partial_t u^\varepsilon(t) + \tilde{\eta} u^\varepsilon(t)). \end{aligned} \quad (4.7)$$

Adding $\eta E_\varepsilon(t)$ on both sides of (4.7) we get

$$\frac{dE_\varepsilon^{\tilde{\eta}}}{dt}(t) + \tilde{\eta} E_\varepsilon(t) + \left(1 - \frac{3\tilde{\eta}}{2}\right) \|\partial_t u^\varepsilon(t)\|_2^2 + \frac{\tilde{\eta}}{2} \|\nabla_g u^\varepsilon(t)\|_2^2 = G_\varepsilon^{\tilde{\eta}}(t) \quad (4.8)$$

where,

$$G_\varepsilon^{\tilde{\eta}}(t) = \tilde{\eta} \int_M [F(u^\varepsilon(t)) - f(u^\varepsilon(t))u^\varepsilon(t)] d\mu_g + (\varepsilon h(t), \partial_t u^\varepsilon(t) + \tilde{\eta} u^\varepsilon(t)) - \gamma \tilde{\eta} (\partial_t u^\varepsilon(t), u^\varepsilon(t)).$$

In what follows, we will estimate $G_\varepsilon^{\tilde{\eta}}(t)$. Indeed, from (3.27), Cauchy-Schwarz inequality and Young inequality, we deduce

$$\begin{aligned} G_\varepsilon^{\tilde{\eta}}(t) &\leq \frac{\tilde{\eta}\beta}{2} \|u^\varepsilon(t)\|_2^2 + \tilde{\eta}\rho \operatorname{vol}(M) + C_{\tilde{\eta}} \|h(t)\|_2 \|\xi_{u^\varepsilon}(t)\|_{\mathcal{H}} + \gamma \tilde{\eta} \|\partial_t u^\varepsilon(t)\|_2 \|u^\varepsilon(t)\|_2 \\ &\leq \tilde{\eta} \left(\frac{1-\beta_0}{2} \right) \|\nabla_g u^\varepsilon(t)\|_2^2 + \tilde{\eta}\rho \operatorname{vol}(M) + C_{\tilde{\eta}} \|h(t)\|_2 \|\xi_{u^\varepsilon}(t)\|_{\mathcal{H}} + \tilde{\eta} C \|\partial_t u^\varepsilon(t)\|_2^2 \end{aligned}$$

for some positive constants $C, C_{\tilde{\eta}}$ independent on the initial data. So, plugging the above estimate in (4.8) we infer

$$\frac{dE_\varepsilon^{\tilde{\eta}}}{dt}(t) + \tilde{\eta} E_\varepsilon(t) + [1 - \tilde{\eta} C] \|\partial_t u^\varepsilon(t)\|_2^2 + \frac{\tilde{\eta}\beta_0}{2} \|\nabla_g u^\varepsilon(t)\|_2^2 \leq C_{\tilde{\eta}} \|h(t)\|_2 \|\xi_{u^\varepsilon}(t)\|_{\mathcal{H}} + \tilde{\eta}\rho \operatorname{vol}(M). \quad (4.9)$$

Now, we choose $\tilde{\eta} > 0$ satisfying (4.6) and such that $\tilde{\eta} < \frac{1}{C}$. Then, from (4.6) and (4.9) we obtain

$$\frac{dE_\varepsilon^{\tilde{\eta}}}{dt}(t) + \frac{3\tilde{\eta}}{2} E_\varepsilon^{\tilde{\eta}}(t) \leq C \|h(t)\|_2 \|\xi_{u^\varepsilon}(t)\|_{\mathcal{H}} + C\rho \operatorname{vol}(M). \quad (4.10)$$

So, from (4.6) and (4.10) we arrive at

$$\phi(t) \leq k(t) + \int_\tau^t m(s) [\phi(s)]^{1/2} ds, \quad (4.11)$$

where

$$\phi(t) := \|\xi_{u^\varepsilon}(t)\|_{\mathcal{H}}^2 e^{\frac{3\tilde{\eta}}{2}t}, \quad k(t) := C(1 + \|\xi\|_{\mathcal{H}}^{p+1}) e^{\frac{3\tilde{\eta}}{2}\tau} + C\rho \operatorname{vol}(M) e^{\frac{3\tilde{\eta}}{2}t}, \quad m(t) := C \|h(t)\|_2 e^{\frac{3\tilde{\eta}}{4}t}$$

and C is a positive constant. Hence, applying Lemma 2.5.2 in (4.11) we conclude that (4.4) holds with $\omega = \frac{3\tilde{\eta}}{4} > 0$. $\square$

Two immediate consequences of Proposition 4.1.1 are the existence of a uniformly bounded (w.r.t. $\varepsilon \in [0, 1]$) pullback absorbing set for $(U_\varepsilon(t, \tau), \mathcal{H})$ and the exponential stability (3.1)-(3.2) for a suitable f and $h \equiv 0$.

Corollary 4.1.2. *Under the assumptions (F), there exist a bounded set $D \subset \mathcal{H}$ satisfying the following: for any bounded set $B \subset \mathcal{H}$ and for any $t \in \mathbb{R}$, there exists $T = T(B, t) > 0$ such that*

$$\bigcup_{\varepsilon \in [0, 1]} U_\varepsilon(t, t-r)B \subset D, \quad t \leq T. \quad (4.12)$$

Proof. Let B be a bounded subset of $\mathcal{H}$, $t \in \mathbb{R}$, $T > 0$ and consider $\xi \in B$. From (4.4), there exists $C_B > 0$ such that

$$\sup_{\varepsilon \in [0,1]} \|U_\varepsilon(t, t-T)\xi\|_{\mathcal{H}} \leq C_B e^{-\omega T} + C\|h\|_{L^1(\mathbb{R}; L^2(M))} + C\rho \operatorname{vol}(M).$$

Thus, there exists $T = T(B, t) > 0$ such that

$$\sup_{\varepsilon \in [0,1]} \|U_\varepsilon(t, t-r)\xi\|_{\mathcal{H}} \leq R, \quad \forall r \geq T,$$

where $R := 2C\|h\|_{L^1(\mathbb{R}; L^2(M))} + 2C\rho \operatorname{vol}(M) > 0$. Hence, $D := B_{\mathcal{H}}(0, R)$ satisfy (4.12), as desired. $\square$

Corollary 4.1.3. *If $f \equiv 0$ (or $\rho = 0$) and $h \equiv 0$ then for any $\xi \in \mathcal{H}$, $\tau \in \mathbb{R}$, $t \geq \tau$ and $\varepsilon \in [0, 1]$ we have*

$$\|U_\varepsilon(t, \tau)\xi\|_{\mathcal{H}} \leq \tilde{\mathcal{Q}}(\|\xi\|_{\mathcal{H}}) e^{-\omega(t-\tau)} \quad (4.13)$$

for some non-decreasing function $\tilde{\mathcal{Q}}$ and for some constant $\omega > 0$, both independent on the ξ and $\varepsilon \in [0, 1]$.

Proof. Repeat the process to obtain (4.4). $\square$

To deal with the next result we split the process $U_\varepsilon(t, \tau) = S_\varepsilon(t, \tau) + C_\varepsilon(t, \tau)$ for each $\varepsilon \in [0, 1]$. To do so, for each $\tau \in \mathbb{R}$ and $t \geq \tau$ let us consider the operator

$$C_\varepsilon(t, \tau) : \mathcal{H} \rightarrow \mathcal{H}, \quad C_\varepsilon(t, \tau)\xi := \xi_{v^\varepsilon}(t) = (v^\varepsilon(t), \partial_t v^\varepsilon(t)) \quad (4.14)$$

where v^ε the weak solution of (3.3)-(3.5) with $\Phi = \varepsilon h$ and initial data $\xi \in \mathcal{H}$. Then we set

$$S_\varepsilon(t, \tau) : \mathcal{H} \rightarrow \mathcal{H}, \quad S_\varepsilon(t, \tau)\xi := U_\varepsilon(t, \tau)\xi - C_\varepsilon(t, \tau)\xi = \xi_{z^\varepsilon}(t) = (z^\varepsilon(t), \partial_t z^\varepsilon(t)), \quad (4.15)$$

where we observe that z^ε is the solution of the following problem

$$\begin{cases} \partial_{tt} z^\varepsilon - \Delta_g z^\varepsilon + \gamma \partial_t z^\varepsilon = -f(u^\varepsilon) \text{ in } M \times (\tau, +\infty), \\ z^\varepsilon = 0 \quad \text{on} \quad \partial M \times [\tau, +\infty), \\ \xi_{z^\varepsilon}(\tau) = 0. \end{cases} \quad (4.16)$$

Let us also introduce some notations to simplify the proof of the next proposition. For $\xi^1, \xi^2 \in \mathcal{H}$ and $\varepsilon \in [0, 1]$ we denote

$$U_\varepsilon(t, \tau)\xi^i = (u_i^\varepsilon(t), \partial_t u_i^\varepsilon(t)), \quad C_\varepsilon(t, \tau)\xi^i = (v_i^\varepsilon(t), \partial_t v_i^\varepsilon(t)), \quad S_\varepsilon(t, \tau)\xi^i = (z_i^\varepsilon(t), \partial_t z_i^\varepsilon(t)),$$

for $i = 1, 2$. Then, we set

$$\xi := \xi^1 - \xi^2, \quad u^\varepsilon = u_1^\varepsilon - u_2^\varepsilon, \quad v^\varepsilon = v_1^\varepsilon - v_2^\varepsilon, \quad z^\varepsilon = z_1^\varepsilon - z_2^\varepsilon.$$

Proposition 4.1.4. *Suppose the assumption (F) holds. Then, for any $\xi^1, \xi^2 \in \mathcal{H}$, $\tau \in \mathbb{R}$ and $t \geq \tau$, we have:*

- *there exists $\delta_0 \in (0, \frac{1}{2}]$ such that*

$$\|S_\varepsilon(t, \tau)\xi^1 - S_\varepsilon(t, \tau)\xi^2\|_{\mathcal{H}} \leq C_0 \|\xi^1 - \xi^2\|_{\mathcal{H}_{\delta_0}} \quad (4.17)$$

for some $C_0 > 0$ depending on ξ^1, ξ^2, h and $t - \tau$;

- *there exist constants $C > 0$ and $\tilde{\omega} > 0$ such that*

$$\|C_\varepsilon(t, \tau)\xi^1 - C_\varepsilon(t, \tau)\xi^2\|_{\mathcal{H}} \leq C \|\xi^1 - \xi^2\|_{\mathcal{H}} e^{-\tilde{\omega}(t-\tau)}. \quad (4.18)$$

Proof. We start noting that z^ε satisfies

$$\begin{cases} \partial_{tt} z^\varepsilon - \Delta_g z^\varepsilon + \gamma \partial_t z^\varepsilon = f(u_2^\varepsilon) - f(u_1^\varepsilon) \text{ in } M \times (\tau, +\infty), \\ z^\varepsilon = 0 \quad \text{on} \quad \partial M \times [\tau, +\infty), \\ \xi_{z^\varepsilon}(\tau) = 0. \end{cases} \quad (4.19)$$

Composing (4.19) with $\partial_t z^\varepsilon$, integrating the result on (τ, t) and taking into account that $\xi_{z^\varepsilon}(\tau) = 0$, we have

$$\frac{1}{2} \|\xi_{z^\varepsilon}(t)\|_{\mathcal{H}}^2 = - \int_\tau^t \|\partial_t z^\varepsilon(s)\|_2^2 ds + \int_\tau^t (f(u_2^\varepsilon(s)) - f(u_1^\varepsilon(s)), \partial_t z^\varepsilon(s)) ds. \quad (4.20)$$

Now, using Cauchy-Schwarz inequality in the last term of (4.20) and applying (3.28) with $\delta_0 := \frac{5-p}{4} \in (0, 1/2]$ in the result, we get

$$\int_\tau^t (f(u_2^\varepsilon(s)) - f(u_1^\varepsilon(s)), \partial_t z^\varepsilon(s)) ds \leq \tilde{C}_f \int_\tau^t \Theta_{\delta_0}(u_1^\varepsilon(s), u_2^\varepsilon(s)) \|z^\varepsilon(s)\|_{H_{1-\delta_0}} \|\xi_{z^\varepsilon}(s)\|_{\mathcal{H}} ds,$$

for some constant $\tilde{C}_f > 0$ and Θ_{δ_0} is defined by (3.29). Plugging the last estimate in (4.20) and using Lemma 2.5.2 we arrive at

$$\|\xi_{z^\varepsilon}(t)\|_{\mathcal{H}} \leq \tilde{C}_f \int_\tau^t \Theta_{\delta_0}(u_1^\varepsilon(s), u_2^\varepsilon(s)) \|u^\varepsilon(s)\|_{H_{1-\delta_0}} ds. \quad (4.21)$$

On the other hand, let us consider $A = -\Delta_g$. We observe that the function $\tilde{u}^\varepsilon := A^{-\delta_0/2} u^\varepsilon$ fulfils the equation

$$\begin{cases} \partial_{tt} \tilde{u}^\varepsilon - \Delta_g \tilde{u}^\varepsilon + \gamma \partial_t \tilde{u}^\varepsilon = \Psi(u_1^\varepsilon, u_2^\varepsilon) \text{ in } M \times (\tau, +\infty), \\ \tilde{u}^\varepsilon = 0 \quad \text{on} \quad \partial M \times [\tau, +\infty), \\ \xi_{\tilde{u}^\varepsilon}(\tau) = \tilde{\xi} \end{cases} \quad (4.22)$$

where we set

$$\tilde{\xi} := (A^{-\delta_0/2} u^\varepsilon(\tau), A^{-\delta_0/2} \partial_t u^\varepsilon(\tau)), \quad \Psi(u_1^\varepsilon, u_2^\varepsilon) := A^{-\delta_0/2} (f(u_2^\varepsilon) - f(u_1^\varepsilon)).$$

So, composing (4.22) with $\partial_t \tilde{u}^\varepsilon$ and integrating the result on (τ, s) , $s \in (\tau, t)$, we get

$$\frac{1}{2} \|\xi_{u^\varepsilon}(s)\|_{\mathcal{H}_{\delta_0}}^2 \leq \frac{1}{2} \|\xi\|_{\mathcal{H}_{\delta_0}}^2 + \int_\tau^s (\Psi(u_1^\varepsilon(r), u_2^\varepsilon(r)), \partial_t \tilde{u}^\varepsilon(r)) dr. \quad (4.23)$$

Applying the Cauchy-Schwarz inequality, taking into account that $L^2(M) \hookrightarrow H_{-\delta_0}$ and using (3.28) again we infer

$$\begin{aligned} \left| \int_\tau^s (\Psi(u_1^\varepsilon(r), u_2^\varepsilon(r)), \partial_t \tilde{u}^\varepsilon(r)) dr \right| &\leq \int_\tau^s \|f(u_2^\varepsilon(r)) - f(u_1^\varepsilon(r))\|_{H_{-\delta_0}} \|\partial_t \tilde{u}^\varepsilon(r)\|_2 dr \\ &\leq C_f \int_\tau^s \Theta_{\delta_0}(u_1^\varepsilon(r), u_2^\varepsilon(r)) \|u(r)\|_{H_{1-\delta_0}} \|\partial_t \tilde{u}^\varepsilon(r)\|_2 dr \\ &\leq C_f \int_\tau^s \Theta_{\delta_0}(u_1^\varepsilon(r), u_2^\varepsilon(r)) \|\xi_{u^\varepsilon}(r)\|_{\mathcal{H}_{\delta_0}}^2 dr, \end{aligned}$$

Plugging the last estimate in (4.23) we deduce

$$\frac{1}{2} \|\xi_{u^\varepsilon}(s)\|_{\mathcal{H}_{\delta_0}}^2 \leq \frac{1}{2} \|\xi\|_{\mathcal{H}_{\delta_0}}^2 + C_f \int_\tau^s \Theta_{\delta_0}(u_1^\varepsilon(r), u_2^\varepsilon(r)) \|\xi_{u^\varepsilon}(r)\|_{\mathcal{H}_{\delta_0}}^2 dr. \quad (4.24)$$

Since $\frac{3(p-1)}{1-\delta_0} = 12$, we can use the same process to get (3.63) to conclude

$$\|\Theta_{\delta_0}(u_1^\varepsilon, u_2^\varepsilon)\|_{L^1(s, \tau)} \leq \mathcal{Q}_6(s - \tau) \left[\mathcal{Q}_7(\|\xi^1\|_{\mathcal{H}}) + \mathcal{Q}_7(\|\xi^2\|_{\mathcal{H}}) + \mathcal{Q}_7(\|h\|_{L^1(\mathbb{R}; L^2(M))}) \right] \quad (4.25)$$

for some positive non-decreasing functions $\mathcal{Q}_6$ and $\mathcal{Q}_7$. So, from (4.24), (4.25) and by Gronwall inequality we obtain

$$\|\xi_{u^\varepsilon}(s)\|_{\mathcal{H}_{\delta_0}} \leq C \|\xi\|_{\mathcal{H}_{\delta_0}}, \quad s \in (\tau, t), \quad (4.26)$$

for some positive constant C depending on $t - \tau$, $\|\xi^1\|_{\mathcal{H}}$, $\|\xi^2\|_{\mathcal{H}}$ and $\|h\|_{L^1(\mathbb{R}; L^2(M))}$. Combining the estimates (4.21) and (4.26) we get

$$\|\xi_{u^\varepsilon}(t)\|_{\mathcal{H}} \leq C \|\Theta_{\delta_0}(u_1^\varepsilon, u_2^\varepsilon)\|_{L^1(t, \tau)} \|\xi\|_{\mathcal{H}_{\delta_0}}, \quad (4.27)$$

for some $C > 0$ depending on $t - \tau$, $\|\xi^1\|_{\mathcal{H}}$, $\|\xi^2\|_{\mathcal{H}}$ and $\|h\|_{L^1(\mathbb{R}; L^2(M))}$. Hence, using again (4.25), we conclude from (4.27) that (4.17) holds.

Noting that v^ε is the weak solution of (3.3)-(3.5) with $\Phi \equiv 0$ and initial data $\xi \in \mathcal{H}$, the estimate (4.18) follows from Corollary 4.1.3. The proof is complete. $\square$

Now, we show that the mapping $\varepsilon \mapsto U_\varepsilon(t, \tau)$ is Lipschitz from $[0, 1]$ to $\mathcal{H}$, for all $\tau \in \mathbb{R}$ and $t \geq \tau$.

Proposition 4.1.5. *Under the assumption (F), there exists a positive constant $C_0 > 0$, depending on $\|\xi\|_{\mathcal{H}}$, $\|h\|_{L^1(\mathbb{R}; L^2(M))}$ and $t - \tau$ such that*

$$\|U_{\varepsilon_1}(t, \tau)\xi - U_{\varepsilon_2}(t, \tau)\xi\|_{\mathcal{H}} \leq C_0 |\varepsilon_1 - \varepsilon_2| \quad (4.28)$$

for all $\varepsilon_1, \varepsilon_2 \in [0, 1]$.

Proof. Let $\xi \in \mathcal{H}$ and

$$U_{\varepsilon_i}(t, \tau)\xi = \xi_{u^{\varepsilon_i}}(t), \quad i = 1, 2.$$

Note that $w^\varepsilon = u^{\varepsilon_1} - u^{\varepsilon_2}$ satisfies

$$\begin{cases} \partial_{tt} w^\varepsilon - \Delta_g w^\varepsilon + \gamma \partial_t w^\varepsilon = f(u^{\varepsilon_2}) - f(u^{\varepsilon_1}) + (\varepsilon_1 - \varepsilon_2)h \text{ in } M \times (\tau, +\infty), \\ w^\varepsilon = 0 \quad \text{on } \partial M \times [\tau, +\infty), \\ \xi_{w^\varepsilon}(\tau) = 0 \end{cases} \quad (4.29)$$

So, composing (4.29) with $\partial_t w^\varepsilon$ we get

$$\frac{1}{2} \frac{d}{dt} \|\xi_{w^\varepsilon}(t)\|_{\mathcal{H}}^2 \leq (f(u^{\varepsilon_2}(t)) - f(u^{\varepsilon_1}(t))), \partial_t w^\varepsilon(t) + (\varepsilon_1 - \varepsilon_2)(h(t), \partial_t w^\varepsilon(t)), \quad t \geq \tau.$$

From Cauchy-Schwarz inequality and using (3.28) with $\delta = 0$ we infer

$$\frac{1}{2} \frac{d}{dt} \phi(t) \leq a(t)\phi(t) + b(t)[\phi(t)]^{1/2}, \quad t \geq \tau$$

where,

$$\phi(t) := \|\xi_{w^\varepsilon}(t)\|_{\mathcal{H}}^2, \quad a(t) := C_f \Theta_0(u^{\varepsilon_2}(t), u^{\varepsilon_1}(t)), \quad b(t) := |\varepsilon_1 - \varepsilon_2| \|h(t)\|_2,$$

for some constant $C_f > 0$. Hence, from Lemma (Gronwall-Showalter), using (3.63) and taking into account that $\xi_{w^\varepsilon}(\tau) = 0$, we conclude that (4.28) holds. $\square$

4.2 Proof of Theorem 4.0.1

From Corollary 4.1.2, for each $\varepsilon \in [0, 1]$ the nonautonomous dynamical system $(U_\varepsilon(t, \tau), \mathcal{H})$ has a constant pullback absorbing family $D \subset \mathcal{H}$ which is uniform w.r.t. $\varepsilon \in [0, 1]$ and also is closed in $\mathcal{H}$. So, we are in position to apply Lemma 2.4.1 to obtain a pullback absorbing family $\{D(t)\}_{t \in \mathbb{R}}$ for $(U_\varepsilon(t, \tau), \mathcal{H})$ satisfying the conditions **(D1)** - **(D3)** and such that $D(t) \subset D$ for all $t \in \mathbb{R}$. It remains to verify the assumptions **(S)**, **(C)** and **(L)**. To this end, let us fixed $t \in \mathbb{R}$ and take $\xi^1, \xi^2 \in D(t)$. We also consider the decomposition

$$U_\varepsilon = S_\varepsilon + C_\varepsilon$$

where S_ε and C_ε are defined in (4.15) and (4.14), respectively.

From (4.18), we have

$$\|C_\varepsilon(t + \tilde{t}, t)\xi^1 - C_\varepsilon(t + \tilde{t}, t)\xi^2\|_{\mathcal{H}} \leq C\|\xi^1 - \xi^2\|_{\mathcal{H}} e^{-\tilde{\omega}\tilde{t}}, \quad \forall \tilde{t} > 0,$$

for some $C > 0$. So, there exists $\tilde{t}_0 > 0$ such that **(C)** holds for some $\lambda \in (0, \frac{1}{2})$. In the same way, from (4.17) we infer

$$\|S_\varepsilon(t + \tilde{t}_0, t)\xi^1 - S_\varepsilon(t + \tilde{t}_0, t)\xi^2\|_{\mathcal{H}} \leq C_0\|\xi^1 - \xi^2\|_{\mathcal{H}_{\delta_0}}$$

for some $C_0 > 0$ depending on $\|h\|_{L^1(\mathbb{R}; L^2(M))}$ and $\tilde{t}_0$. Since $\delta_0 \in (0, \frac{1}{2}]$, the embedding $\mathcal{H} \hookrightarrow \mathcal{H}_{\delta_0}$ is compact and then, the condition **(S)** holds. We also observe that for any $r \in (t, t + \tilde{t}_0)$, the inequality (4.1) implies that

$$\|U_\varepsilon(r, t)\xi^1 - U_\varepsilon(r, t)\xi^2\|_{\mathcal{H}} \leq L\|\xi^1 - \xi^2\|_{\mathcal{H}},$$

for some $L > 0$ depending on $\|h\|_{L^1(\mathbb{R}; L^2(M))}$ and $r - t > 0$. So, the condition **(L)** is fulfilled.

Hence, for each $\varepsilon \in [0, 1]$, Theorem 2.4.2 ensures that $(U_\varepsilon(t, \tau), \mathcal{H})$ has a pullback exponential attractor $\mathcal{M}_\varepsilon = \{\mathcal{M}_\varepsilon(t)\}_{t \in \mathbb{R}}$. Moreover, from Corollary 2.4.3, $(U_\varepsilon(t, \tau), \mathcal{H})$ posses a unique pullback attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(t)\}_{t \in \mathbb{R}}$ such that $\mathcal{A}_\varepsilon$ is contained in $\mathcal{M}_\varepsilon$ and the fractal dimension of $\mathcal{A}_\varepsilon(t)$ is uniformly bounded w.r.t. $t \in \mathbb{R}$. This proves **(i)** and **(ii)** and also condition **(U1)** of Proposition 2.4.4.

Finally, we prove **(iii)** and **(iv)**. Let $\tau \in \mathbb{R}$, $t \geq \tau$, $\varepsilon \in [0, 1]$ and $B \subset \mathcal{H}$ be a bounded set. Since $\mathcal{A}_\varepsilon(\tau)$ is invariant, $\mathcal{A}_\varepsilon(\tau) \subset D$ and consequently

$$\bigcup_{\varepsilon \in [0, 1]} \bigcup_{\tau \in \mathbb{R}} \mathcal{A}_\varepsilon(\tau) \subset D.$$

This shows the conditions **(U3)** of Proposition 2.4.4 and **(R1)** of Proposition 2.4.5. We also note that (4.28) implies

$$\sup_{s \in [\tau, t]} \sup_{\xi \in B} \|U_\varepsilon(s, \tau)\xi - U_{\varepsilon_0}(s, \tau)\xi\|_{\mathcal{H}} \leq C_0|\varepsilon - \varepsilon_0|, \quad \forall \varepsilon \in [0, 1]$$

for some constant $C_0 > 0$ depending on B , $\|h\|_{L^1(\mathbb{R}; L^2(M))}$ and $t - \tau$. This implies that conditions **(U2)** of Proposition 2.4.4 and **(R2)** of Proposition 2.4.5 hold. Thus, all conditions of Proposition 2.4.4 are valid, and then we obtain the upper semicontinuity of $\mathcal{A}_\varepsilon$ when $\varepsilon \rightarrow 0$. This proves **(iii)**. With property **(iii)** in hands, we can apply (CARVALHO; LANGA; ROBINSON, 2012, Lemma 3.3) to see that condition **(R3)** of Proposition 2.4.5 is satisfied. Hence, **(iv)** holds and the proof of Theorem 4.0.1 is now complete. $\square$

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