The method of exact algebraic restrictions

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O método das restrições algébricas exatas

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *EXEMPLAR DE DEFESA*

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I dedicate this work to my family, relatives and friends. A special feeling of gratitude to my loving parents, Tomas and Bernardina.

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*“But in my opinion,
all things in nature occur mathematically.”
(René Descartes)*

ABSTRACT

BOCANEGRA, L. R. **The method of exact algebraic restrictions.** 2018. 126 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

The aim of this work is to generalize the results given by Domitrz, Janeczko and Zhitomirskii in [10]. In this article they classify in the symplectic manifold $(\mathbb{R}^{2n}, \omega)$ where $\omega = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}$ is the symplectic form given by Darboux's Theorem, all the set which are symplectomorphic to a fixed quasi homogeneous curve N . To do this classification they defined the algebraic restrictions. We develop a new method called the method of exact algebraic restrictions and show that this classification is solved for the non quasi homogeneous case $N = \{f(x_1, x_2) = x_{\geq 3} = 0\}$ in the symplectic manifold $(\mathbb{C}^{2n}, \omega)$, where $f(x_1, x_2) = x_1^4 + x_2^5 + x_1^2 x_2^3$.

Keywords: Symplectic classification, Symplectomorphism, Exact Algebraic Restrictions, Symplectic manifold.

RESUMO

BOCANEGRA, L. R. **O método das restrições algébricas exatas.** 2018. 126 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

Este trabalho tem como objetivo generalizar os resultados feitos por Domitrz, Janeczko e Zhitomirskii em [10]. Neste artigo eles classificaram na variedade simplética $(\mathbb{R}^{2n}, \omega)$ onde $\omega = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}$ é a forma simplética dada pelo Teorema de Darboux, todos os conjuntos que são simplectomorfos a uma curva quase homogênea fixada N . Para fazer a classificação eles definem as restrições algébricas. Nós desenvolvemos um novo método o qual chamamos de método das restrições algébricas exatas e provamos que a classificação é resolvida para o caso não quase homogêneo $N = \{f(x_1, x_2) = x_{\geq 3} = 0\}$ na variedade simplética $(\mathbb{C}^{2n}, \omega)$, onde $f(x_1, x_2) = x_1^4 + x_2^5 + x_1^2 x_2^3$.

Palavras-chave: Classificação simplética, Simpletomorfismos, Restrições Algébricas Exatas, Variedade Simplética.

LIST OF SYMBOLS

$\mathbb{N}$ — the set of natural numbers.

$\mathbb{Z}$ — the set of integers numbers.

$\mathbb{R}$ — the set of real numbers.

$\mathbb{C}$ — the set of complex numbers.

$\mathbb{K}$ — the set of real numbers or complex numbers.

$\Theta(M)$ — the vector space of vector fields on the manifold M .

Θ_n — the vector space of vector fields on the manifold $\mathbb{C}^n$.

$\mathcal{L}_X$ — the Lie derivative along the vector field X .

$j^k f(a)$ — the k -jet of $f : \mathbb{K}^n \rightarrow \mathbb{K}^p$ at $a \in \mathbb{K}^n$.

$\mathcal{O}_n$ — the ring of germ of differential functions over $(\mathbb{C}^n, 0)$.

$\mathcal{M}_n$ — the maximal ideal of $\mathcal{O}_n$.

$\langle \nabla f \rangle$ — the ideal generate by the partial derivatives of $f \in \mathcal{O}_n$.

$\mathcal{R}$ — right equivalence

$\mathcal{C}$ — $\mathcal{C}$ equivalence.

$\mathcal{H}$ — $\mathcal{H}$ equivalence.

$\Omega^k(M)$ — the vector space of all the germs at the origin of smooth k -forms defined on M .

$[\omega]_N$ — the algebraic restriction of ω .

$ind(N)$ — the index of isotropness of N .

$\mu_{symp}(N)$ — the symplectic multiplicity of N .

$\|\omega\|_N$ — the exact algebraic restriction of ω .

$\|\Omega^k(M)\|_N$ — the vector space of the exact algebraic restrictions of k forms to N .

$\overline{\Omega}^k(M)$ — the subspace of $\Omega^k(M)$ containing the germs of closed k -forms.

$\|\overline{\Omega}^k(M)\|_N$ — the vector space of exact algebraic restrictions of closed k -forms to N .

$[\overline{\Omega}^k(M)]_N$ — the vector space of algebraic restrictions of closed k -forms to N .

$G_N(M)$ — the group of local symmetries of N defined on $(M, 0)$.

$\|Symp(\mathbb{C}^{2n})\|_N$ — the set of exact algebraic restrictions of symplectic forms to N .

$T(H)_{\mathcal{K}}$ — the tangent space at the variety H to the $\mathcal{K}$ -orbit.

$T(H)_{\mathcal{K},symp}$ — the tangent space at the variety H to the $\mathcal{K}$ -symplectic orbit.

$Derlog(f)$ — the $\mathcal{O}_2$ -module of vector fields tangents to $f^{-1}(0) \subset \mathbb{C}^2$.

div — the operator from Θ_2 to $\mathcal{O}_2$.

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INTRODUCTION

One of the main purposes in Singularity Theory is to classify germs under an equivalence relation, as examples we have the $\mathcal{A}$, $\mathcal{R}$, $\mathcal{L}$ -equivalence over the $\mathcal{O}_n$ -module of map germs $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^m, 0)$. Another question is given the germ of a subset $(N, 0) \in (\mathbb{C}^n, 0)$, to classify it with respect to the group of diffeomorphisms, in other words, to classify all the germ of subsets $(N_1, 0) \in (\mathbb{C}^n, 0)$ such that there exist a diffeomorphism $\Phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ and $\Phi(N, 0) = (N_1, 0)$.

Now, let (M, ω) be a symplectic manifold, where M is the germ of a manifold of dimension even and ω is the germ of symplectic form defined on M , it is a differential 2-form which is closed and non degenerate. Since we are considering germs, we can assume that $M = (\mathbb{K}^{2n}, 0)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and ω is a symplectic form defined on $(\mathbb{K}^{2n}, 0)$. In this context, a diffeomorphism $\Phi : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ which preserves ω , $\Phi^*\omega = \omega$, is called a symplectomorphism.

Let $(\mathbb{K}^{2n}, \omega)$ be a symplectic manifold. Two curves $f_i : (\mathbb{K}, 0) \rightarrow (\mathbb{K}^{2n}, 0)$, $i = 1, 2$, are called symplectomorphic if there exist a symplectomorphism Φ defined on $(\mathbb{K}^{2n}, 0)$ and a diffeomorphism ψ defined on $(\mathbb{K}, 0)$ satisfying $f_1 = \Phi \circ f_2 \circ \psi$. Consider the following classification problem, which we call symplectic classification of parametrized curves, given a curve f to classify all the curves that are symplectomorphic to f . Arnold in [2] proved that for the curve $A_{2k} = \{(t^2, t^{2k+1}, 0, \dots, 0), t \in \mathbb{C}\} \subset (\mathbb{C}^{2n}, \omega)$, there are $2k + 1$ normal forms for the symplectic classification of parametrized curves. He was interested in this problem because he observed that

" there exist non obvious discrete symplectic invariants of such singularities. This invariants should be expressed in terms of the local algebra's interaction with symplectic structure "

In this work we are interested in the following classification problem, which we call symplectic classification of sets: given a germ of a subset $(N, 0) \subset (\mathbb{K}^n, 0)$, to classify

with respect to the group of symplectomorphism all the germs which are diffeomorphic to $(N, 0)$, in other words, to determine when the subsets $(N_1, 0), ((N_2, 0))$ which are diffeomorphic to $(N, 0)$ are symplectomorphic (there exist a symplectomorphism Φ such that $\Phi(N_1, 0) = (N_2, 0)$).

In [27], Zhitomirskii defined the algebraic restriction of a 1-form to a contact manifold to solve a classification problem analogous to the symplectic classification of sets, but in this case the manifold was a contact manifold and instead of symplectomorphisms he worked with contactomorphisms. In [10] Domitrz, Janeczko and Zhitomirskii, extended the definition of algebraic restrictions of k -differential forms defined on a manifold $(M, 0)$ to a subset N . They used the method of algebraic restrictions to describe the symplectic invariants: index of isotropness and symplectic multiplicity and proved that it is possible to obtain normal forms for the symplectic classification of sets diffeomorphic to $N = \{f(x_1, x_2) = x_{\geq 3} = 0\} \subset (\mathbb{R}^{2n}, \omega)$ when f is a quasi homogeneous function germ. As a particular case, they calculated these normal forms when f is one of the singularities $A - D - E$ of Arnold. Additionally, they used the method of algebraic restrictions to obtain normal forms to the set $S_5 = \{x_1^2 - x_2^2 - x_3^2 = x_2x_3 = x_{\geq 4} = 0\}$ in the symplectic manifold $(\mathbb{R}^{2n}, \omega)$ as well as for regular union singularities.

In several papers Domitrz et al applied this method to obtain several symplectic classification of set [12], [11], [26] and [20]. They also adapted the method of algebraic restrictions to obtain normal forms for the symplectic classification of parametrized curves with semi groups $(4, 5, 6, 7)$, $(4, 5, 6)$ and $(4, 5, 7)$, [19]. In all of these cases, they dealt only with quasi homogeneous case.

On the other hand, in [15], Ishikawa and Janeczko obtained normal forms to the symplectic classification of parametrized simple and unimodal curves, given by Bruce and Gaffney in [5], on the symplectic manifold $(\mathbb{C}^2, dx_1 \wedge dx_2)$. They used Puiseux characteristics to do it. In particular, they proved that the singular curve W_{12} has as normal forms the family of parametrized curves $\{(t^4, t^5 + \lambda t^7), t \in \mathbb{C}, \lambda \in \mathbb{C}/G\}$, where $G = \mathbb{Z}/9\mathbb{Z}$.

Although we can define algebraic restriction of a form to a non quasi homogeneous subset N , the method of algebraic restrictions can not be used to classify subsets diffeomorphic to a non quasi homogeneous N under the action of symplectomorphism preserving N .

In this work we provide another method to deal with the symplectic classification of non quasi homogeneous subsets and we applied this method to obtain the normal forms in this classification for the curve $\{f(x_1, x_2) = x_{\geq 3} = 0\} \subset (\mathbb{C}^{2n}, \omega)$, when $f(x_1, x_2) = W_{12} = x_1^4 + x_2^5 + x_1^2x_2^3$. This work is organized as follows.

In the second chapter we give the basic theory needed in this work such as symplectic manifolds and differential forms on a manifold. We recall the method of algebraic restrictions

and explain why we can not use this method in general.

In the third chapter, we define the exact algebraic restrictions of a form to a subset N , and we show the versions of the theorems in [10]. We prove that the vector space of exact algebraic restriction of 2-closed forms is a finite vector space when N is an isolated complete intersection singularity(ICIS). We find a basis for this vector space when $N = \{f(x_1, x_2) = x_{\geq 3} = 0\} \subset (\mathbb{C}^{2n}, \omega)$, where f has the property of zeros, $\mu(f) - \tau(f) = 1$ and the vector tangents to the singular curve $f^{-1}(0)$ denoted by $Derlog(f)$ satisfy that there exist a $\mathcal{O}_2$ -basis $\{\eta_1, \eta_2\}$ such that $\eta_1(0) = \eta_2(0) = 0$ and $div(\eta_1)(0) = div(\eta_2)(0) = 0$, where $div : \Theta_2 \rightarrow \mathcal{O}_2$, $div(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2}) = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2}$.

In the fourth chapter we proved that $f = W_{12} : x_1^4 + x_2^5 + x_1^2 x_2^3$ has the property of zeros and that for all $g \in \mathcal{M}_2^6$ we have that the 2-form $g dx_1 \wedge dx_2$ has zero exact algebraic restriction to $f^{-1}(0)$.

In the fifth chapter, as an example, we obtain the normal forms of the symplectic classification of sets to the curve $N = \{f(x_1, x_2) = x_{\geq 3} = 0\} \subset (\mathbb{C}^{2n}, \omega)$. As a particular case, when $n = 1$ we have that the normal form to the symplectic classification of sets are the zeros of a family of functions f_c , $c \in \mathbb{C} \setminus \{0\}$.

PRELIMINARY

In this chapter we present basic facts about symplectic geometry and theory of singularities needed in this work and explain the method of algebraic restrictions. We denote by $\mathbb{N}$, $\mathbb{Z}$ the set of natural number and integers numbers respectively and $\mathbb{K}$ either the field of real number $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. In this work, n, m always denote natural numbers and k denotes an positive integer number.

2.1 Symplectic Geometry

The theory of manifolds is well known, the definitions can be found for example in [21] for the complex case or [18] for the real case. The basic definitions and results are the same in both cases. In this work, M always denotes a manifold.

2.1.1 Submanifold

We write manifolds and maps instead of smooth manifolds and smooth maps respectively.

Let M be an m -dimensional manifold and $p \in M$, then there exists a chart or coordinate system (U, ϕ) such that $p \in U$ and $\phi : U \rightarrow \mathbb{K}^m$ is a homeomorphism of U onto an open set in $\mathbb{K}^m$.

Definition 2.1.1. *Let $S \subset M$ be a subset. We say that S is a submanifold if S with the induced topology from M admit an structure of manifold. In this case we denote by $\text{codim} S = \dim M - \dim S$ the codimension of S .*

Proposition 2.1.2. *Let $S \subset M$ be an s -dimensional submanifold of M . Then for all $p \in S$ there exists a coordinate system $(U, \phi = (x_1, x_2, \dots, x_m))$ of M such that $\phi(U \cap S)$ is a subset of*

$$\{(x_1, \dots, x_s, x_{s+1}, \dots, x_m) | x_i = 0, s + 1 \leq i \leq m\}.$$

For $p \in M$, we denote the tangent space to M at p by $T_p M$. We know that $T_p M$ is a $\mathbb{K}$ -vector space whose dimension is $\dim M$. Let $f : M_1 \rightarrow M_2$ be a map, we denote the differential of f in $p \in M_1$ by $d_p f : T_p M_1 \rightarrow T_p M_2$.

2.1.2 Vector fields

We recall that a *section* of a vector bundle $\pi : E \rightarrow M$ is a map $s : M \rightarrow E$ such that $\pi \circ s = id_M$. We say that the section s is smooth when it is a smooth map. Also, given a manifold M the tangent bundle of M is by definition

$$TM := \bigcup_{p \in M} \{p\} \times T_p M.$$

Definition 2.1.3. A vector field on a manifold M is a section of the vector bundle $\pi : TM \rightarrow M$, where $\pi(p, v) = p$ for all $p \in M$, $v \in T_p M$. We denote the $\mathbb{K}$ -vector space of all vector fields on M by $\Theta(M)$. When $M = \mathbb{C}^n$ we denote $\Theta(\mathbb{C}^n)$ by Θ_n .

Let M be an m -dimensional manifold and let X be a vector field on M . Then, given a chart $(U, (x_1, \dots, x_m))$ of M there exist maps $a_i : U \rightarrow \mathbb{K}$, $i = 1, \dots, m$, such that

$$X(p) = \sum_{i=1}^m a_i(p) \frac{\partial}{\partial x_i} \Big|_p, \quad \forall p \in U, \quad (2.1)$$

where $\left\{ \frac{\partial}{\partial x_i} \right\}_{1 \leq i \leq m}$ are the vector fields such that for all $p \in U$, the set $\left\{ \frac{\partial}{\partial x_i} \Big|_p \right\}_{1 \leq i \leq m}$ is a basis for the tangent space $T_p M$.

Proposition 2.1.4. Let X be a vector field on the manifold M . Then X is smooth if and only if for any chart $(U, (x_1, \dots, x_m))$ of M all maps a_i in (2.1) are smooth.

Definition 2.1.5. Let $p \in M$. We say that X has a local flow in p if there exists a smooth map

$$\Phi :]-\varepsilon, \varepsilon[\times V \rightarrow U, \quad \varepsilon > 0$$

where we denote $\Phi_t(x) = \Phi(t, x)$, such that

1. $V \subset U$ are open subsets and contains p ,
2. $\Phi_0 : V \rightarrow U$ is the inclusion map,
3. $\Phi_{t_1+t_2} = \Phi_{t_1} \circ \Phi_{t_2}$, when both expressions make sense,
4. $\frac{d\Phi_t}{dt}(q) = X \circ \Phi_t(q)$ for all $q \in V$.

Proposition 2.1.6. [13] Let M be a manifold, X a smooth vector field on M and $p \in M$. Then there exists a local flow of X in p .

2.1.3 Differential forms

A k -form at $p \in M$ is a function

$$\omega_p : \underbrace{T_p M \times \cdots \times T_p M}_{k\text{-times}} \rightarrow \mathbb{K}$$

which is k -linear, that is, it is linear in each component and $\omega_p(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\omega_p(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$ for all $v_i, v_j \in T_p M$, $i \neq j$, $i, j = 1, \dots, k$.

The set of all the k -forms at p is denoted by $\Lambda^k(T_p M)$. The set

$$\Lambda_*^k(TM) := \bigcup_{p \in M} \{p\} \times \Lambda^k(T_p M)$$

together with the projection map to M is a vector bundle called bundle of k -forms. A smooth section to this vector bundle is called a differential k -form on M . The set of all differential k -forms on M is a $\mathbb{K}$ -vector space and it is denoted by $\Lambda^k(M)$. The differential 0-forms are just the smooth maps on M .

Let $k \in \mathbb{N}$. We denote by S_k the group of permutations of k elements. For each $\sigma \in S_k$ there exists an associated value $\text{sign}(\sigma)$, the sign of a permutation, which is 1 or -1. Let V be a $\mathbb{K}$ -vector space and let $T : V \times \cdots \times V \rightarrow \mathbb{K}$ be a k -linear function, we define the *alternatization* of T as

$$\text{Alt}(T)(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma) T(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Let k, l be natural numbers, we define the *wedge product*

$$\wedge : \Lambda^k(T_p M) \times \Lambda^l(T_p M) \rightarrow \Lambda^{k+l}(T_p M)$$

by $\omega \wedge \alpha := \frac{(k+l)!}{k!l!} \text{Alt}(\omega \otimes \alpha)$, where $\otimes$ denotes the tensorial product (as a reference for tensor product see [17]).

Let M be an m -dimensional manifold, $p \in M$ and let $(U, (x_1, \dots, x_m))$ be a chart of M in p . We denote by $dx_i(p)$ or dx_i (when there is no confusion of the point of M), $i = 1, \dots, m$, the 1-forms at p such that for $j = 1, \dots, m$,

$$dx_i(p) \left(\frac{\partial}{\partial x_j} \Big|_p \right) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Let $\{v_i\}_{1 \leq i \leq k}$ be a subset of $T_p M$, then there exist $\{a_{ij}\}_{1 \leq i \leq k, 1 \leq j \leq m}$ a subset of $\mathbb{C}$ such that $v_i = \sum_{j=1}^m a_{ij} \frac{\partial}{\partial x_j} \Big|_p$, then

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_1, \dots, v_k) = \det \begin{pmatrix} a_{i_1 1} & a_{i_1 2} & \cdots & a_{i_1 k} \\ a_{i_2 1} & a_{i_2 2} & \cdots & a_{i_2 k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k 1} & a_{i_k 2} & \cdots & a_{i_k k} \end{pmatrix}.$$

Proposition 2.1.7. *Let us consider $\omega \in \Lambda^k(M)$, $\eta_1, \eta_2 \in \Lambda^l(M)$ and $\gamma \in \Lambda^n(M)$ then*

1. $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$,
2. $\omega \wedge \eta_1 = (-1)^{kl} \eta_1 \wedge \omega$,
3. $\omega \wedge (\eta \wedge \gamma) = (\omega \wedge \eta) \wedge \gamma$.

Proposition 2.1.8. *Let M be an m -dimensional manifold, $p \in M$ and ω a differential k -form. Then given a chart $(U, (x_1, \dots, x_m))$ of M in p there exist smooth functions $f_I = f_{i_1 i_2 \dots i_k} : U \rightarrow \mathbb{K}$, $1 \leq i_1 < \dots < i_k \leq m$, such that for all $q \in U$ we have*

$$\omega_q = \omega(q) = \sum_I f_I(q) dx_I, \text{ where } dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}. \quad (2.2)$$

Definition 2.1.9. *Let P be a manifold and $f : M \rightarrow P$ a map. We define the pull-back $f^* : \Lambda^k(P) \rightarrow \Lambda^k(M)$ by*

$$(f^*\omega)_p(v_1, \dots, v_k) = \omega_{f(p)}(d_p f(v_1), \dots, d_p f(v_k)), \text{ for } v_1, \dots, v_k \in T_p M.$$

Remark 2.1.10. *In the definition 2.1.9, if $k = 0$, then $f^*g = g \circ f$.*

Proposition 2.1.11. *Let $f : M_1 \rightarrow M_2$ and $g : M_2 \rightarrow M_3$ be maps between manifolds. Suppose $\omega_1, \omega_2 \in \Lambda^k(M_2)$ and $\eta \in \Lambda^l(M_3)$, then*

1. $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$,
2. $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$,
3. $(g \circ f)^*\eta = f^*(g^*\eta)$.

2.1.4 Lie Derivative

Definition 2.1.12. *Let X be a vector field on a manifold M , we define the interior product $i_X : \Lambda^k(M) \rightarrow \Lambda^{k-1}(M)$ by $i_X(\omega)(p, v_1, \dots, v_{k-1}) = \omega(p, X_p, v_1, \dots, v_{k-1}) = \omega_p(X_p, v_1, \dots, v_{k-1})$ where $X_p = X(p)$, $v_1, \dots, v_{k-1} \in T_p M$.*

Definition 2.1.13. *Let M be an m -dimensional manifold and $k \in \mathbb{N}$. An exterior derivative $d : \Lambda^k(M) \rightarrow \Lambda^{k+1}(M)$ is a map satisfying:*

1. d is linear.
2. If $\omega \in \Lambda^k(M)$ and $\alpha \in \Lambda^l(M)$ then

$$d(\omega \wedge \alpha) = (d\omega) \wedge \alpha + (-1)^{kl} \omega \wedge (d\alpha)$$

3. $d \circ d = 0$.

4. For $f \in \Lambda^0(M)$, df is the differential of f given by $df(X) = X(f)$, where $X = \sum_{i=1}^m g_i \frac{\partial}{\partial x_i}$ is a vector field and $X(f) = \sum_{i=1}^m g_i \frac{\partial f}{\partial x_i}$.

Proposition 2.1.14. *Given a manifold M there exists a unique exterior derivative and we denote it by d .*

Proposition 2.1.15. *Given a chart $(U, (x_1, \dots, x_m))$ of the manifold M . If $\omega = \sum_I a_I dx_I$ is a differential form, then $d\omega = \sum_I d(a_I) \wedge dx_I$.*

Proposition 2.1.16. *Let $F : M_1 \rightarrow M_2$ be a map between manifolds. Then $F^* : \Lambda^k(M_2) \rightarrow \Lambda^k(M_1)$ commute with d . In other words, for every $\omega \in \Lambda^k(M_2)$, $F^*(d\omega) = d(F^*\omega)$.*

Definition 2.1.17. *A differential k -form $\omega \in \Lambda^k(M)$ is called closed if $d\omega = 0$ and is called exact if there exists $\alpha \in \Lambda^{k-1}(M)$ such that $d\alpha = \omega$.*

We say that a set $N \subset \mathbb{K}^m$ is called a star domain if there exists $x_0 \in N$ such that for all $x \in N$ the line segment from x_0 to x is in N .

Theorem 2.1.18 (The Poincaré's Lemma). *Let U be an open star domain in $\mathbb{K}^m$, and let k be a positive integer. Then for $\omega \in \Lambda^k(U)$ such that $d\omega = 0$, there exists $\alpha \in \Lambda^{k-1}(U)$ such that $\omega = d\alpha$.*

Definition 2.1.19. *Let X be a vector field on M and ω a differential k -form. The Lie derivative $\mathcal{L}_X : \Lambda^k(M) \rightarrow \Lambda^k(M)$ in $p \in M$ is defined by*

$$(\mathcal{L}_X \omega)_p = (\mathcal{L}_X \omega)(p) = \lim_{t \rightarrow 0} \frac{(\Phi_t^* \omega)_p - \omega_p}{t} = \left. \frac{d}{dt} (\Phi_t^* \omega_p) \right|_{t=0},$$

where Φ is the flow of X in p .

Remark 2.1.20. *The Definition 2.1.19 does not depend of the flow Φ .*

Proposition 2.1.21. *Let X be a vector field on M . Then we have the following properties:*

1. $\mathcal{L}_X f = X(f)$, where f is a function;
2. $\mathcal{L}_X(\omega \wedge \beta) = (\mathcal{L}_X \omega) \wedge \beta + \omega \wedge (\mathcal{L}_X \beta)$, $\forall \omega \in \Lambda^k(M)$ and $\forall \beta \in \Lambda^l(M)$;
3. $\mathcal{L}_X d\omega = d\mathcal{L}_X \omega$, $\forall \omega \in \Lambda^k(M)$;
4. (Cartan's formula) $\mathcal{L}_X \omega = di_X \omega + i_X d\omega$, $\forall \omega \in \Lambda^k(M)$.

Proposition 2.1.22. [25] Let M be a manifold and $I \subset \mathbb{R}$ an open interval containing the zero. Let $\{\omega_t, t \in I\}$ be a family of differential k -forms, $\{X_t, t \in I\}$ a family of vector fields and $\{\Phi_t, t \in I\}$ a family of diffeomorphisms of M such that this last family is smooth with respect to t and $\frac{d\Phi_t}{dt} = X_t \circ \Phi_t$. Then

$$\frac{d}{dt}\Phi_t^*\omega_t = \Phi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}\right) \quad (2.3)$$

2.1.5 Symplectic manifolds

Definition 2.1.23. Let ω be a differential 2-form on M . Then the pair (M, ω) is called symplectic manifold if ω satisfies:

1. It is closed,
2. For each $p \in M$, if $v \in T_pM$ is such that $\omega_p(v, w) = 0$ for all $w \in T_pM$, then $v = 0$.

Proposition 2.1.24. Let (M, ω) be a symplectic manifold, then $\dim M$ is even.

Example 2.1.25. Let us consider the manifold $M = \mathbb{K}^{2n}$ ($n \geq 1$) with coordinate system $(x_1, \dots, x_n, y_1, \dots, y_n)$ and consider the 2-form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. Then the pair (M, ω_0) is a symplectic manifold.

Definition 2.1.26. Let (M, ω) , be a $2n$ -dimensional symplectic manifold and $\phi : M \rightarrow M$ a diffeomorphism. Then ϕ is called a symplectomorphism if $\phi^*\omega = \omega$.

Theorem 2.1.27 (Darboux). Let (M, ω) be a $2n$ -dimensional symplectic manifold and $p \in M$. Then there exists a chart $(U, (x_1, \dots, x_n, y_1, \dots, y_n))$ of M in p such that on U

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

The chart $(U, (x_1, \dots, x_n, y_1, \dots, y_n))$ is called the Darboux's chart.

Proposition 2.1.28. Let ω be a closed 2-form on M and $p \in M$. Then there exist a chart $(U, (x_1, \dots, x_m))$ of M such that $p \in U$ and ω is a symplectic form on U if and only if $\omega(p)(v_1, v_2) = 0$ for all $v_1 \in T_pM$ and $v_2 \in T_pM$ implies $v_2 = 0$.

Proof. Let $(U, x = (x_1, \dots, x_n))$ be a chart on of M such that $x \in U$. Then there exist maps $f_{i,j} : U \rightarrow \mathbb{K}$, $1 \leq i < j \leq 2n$ such that on U

$$\omega = \sum_{1 \leq i < j \leq 2n} f_{ij}(x) dx_i \wedge dx_j.$$

Let us consider $q \in U$ and $V, W \in T_q(U)$ then

$$\omega_q(V, W) = \omega(q, V, W) = \sum_{1 \leq i < j \leq 2n} f_{i,j}(q) dx_i \wedge dx_j(V, W)$$

since $dx_i \wedge dx_j(V, W) = v_i w_j - w_i v_j$ where $V = \sum_{i=1}^2 v_i \frac{\partial}{\partial x_i}$ and $W = \sum_{i=1}^n w_i \frac{\partial}{\partial x_i}$. It is not difficult to prove that if

$$B(x) = \begin{pmatrix} 0 & f_{1,2}(x) & f_{1,3}(x) & \cdots & f_{1,2n-1}(x) & f_{1,2n}(x) \\ -f_{1,2}(x) & 0 & f_{2,3}(x) & \cdots & f_{1,2n-1}(x) & f_{2,2n}(x) \\ -f_{1,3}(x) & -f_{2,3}(x) & 0 & & \cdots & f_{3,2n}(x) \\ \vdots & \vdots & & \ddots & & \vdots \\ -f_{1,2n-1}(x) & -f_{2,2n-1}(x) & & & 0 & f_{2n-1,2n}(x) \\ -f_{1,2n}(x) & -f_{2,2n}(x) & & \cdots & -f_{2n-1,2n}(x) & 0 \end{pmatrix}$$

then $\omega_q(V, W) = V.B(q).W^t$, for all $q \in U$, here we are considering V, W as a vector and W^t denotes the transpose of W .

By definition of a symplectic form, it is enough to prove the "if".

Suppose $\omega(p)(v_1, v_2) = 0$ for all $v_1 \in T_p M$ and $v_2 \in T_p M$ implies $v_2 = 0$. Then the Matrix $B(p)$ is invertible, in fact, if $v_0 \in T_p U = \mathbb{K}^n$ such that $v \in \text{Ker}(B(p))$, then for all $v \in T_p U$ we have $\omega_p(p)(v, v_0^t) = v.B(p).v_0 = 0$, thus $v_0 = 0$. Since the group of invertible matrix is open on the vector space of matrix of order $2n \times 2n$, $B(p)$ is invertible for all $p \in U'$, where $U' \subset U$ is a open subset of M . Let $q \in U'$ and suppose that there exists $V \in T_p U'$ satisfying $\omega_p(V, W) = V.B(p).W^t = 0$ for all $W \in T_p U'$. Let us consider $W_i^t = B(p)^{-1} E_i^t$, for all $i = 1, \dots, 2n$, where $E_i = \frac{\partial}{\partial x_i}$ then $v_i = V.E_i^t = \omega(p, V, W_0) = 0$ which implies $V = 0$. Therefore ω is symplectic on U' , since it already satisfies condition 1 and 2 of the definition. $\square$

Corollary 2.1.29. *Under the same conditions of the above proposition, ω is a symplectic form on some open subset U of M containing p if and only if the matrix B above defined is invertible at p .*

Definition 2.1.30. *Let (M, ω) be a $2n$ -dimensional symplectic manifold. A submanifold $M_1 \subset M$ is called isotropic with respect to ω if the inclusion map $i_{M_1} : M_1 \rightarrow M$ satisfies $i_{M_1}^* \omega = 0$, in other words, ω vanishes on the manifold M_1 .*

Proposition 2.1.31. *With the same conditions of Definition 2.1.30, if the submanifold $M_1 \subset M$ is isotropic with respect to ω then $\dim M_1 \leq n$.*

Definition 2.1.32. *An isotropic submanifold M_1 such that $\dim M_1 = \dim M/2$ is called a Lagrangian submanifold.*

Definition 2.1.33. *Let (M, ω) be a symplectic manifold. We say that $X \in \Theta(M)$ is a Hamiltonian vector field if there exist a function $G : M \rightarrow \mathbb{K}$ such that*

$$i_X \omega = dG.$$

2.2 Singularity theory

The results of this section can be found in [13] , [4].

2.2.1 Germs and Jets

Definition 2.2.1. Let N be a subset of a topological space and $p \in N$. We say that two subsets $A, B \subset N$ are equivalent in $p \in A \cap B$ if there exists an open neighborhood W of p such that $A \cap W = B \cap W$.

Observe that the above relation is an equivalence relation defined on the Power set of N . The equivalence classes are called *germs of sets at p* . When N is a manifold, the equivalence classes are called *germs of manifolds* and it is denoted by (N, p) .

Let P be a manifold. Let us consider the set

$$\{(U, g) | U \text{ is an open neighborhood of } x \in M \text{ and } g : U \rightarrow P \text{ is smooth}\}.$$

On this set we define the equivalence relation: (U, f) and (V, g) are equivalent if and only if there exists an open neighborhood W of x such that $W \subset U \cap V$ and $f|_W = g|_W$.

The equivalence classes are called germs at x of maps from M to P and it is denoted by $f : (M, x) \rightarrow (P, y)$, where $y = f(x)$. Here x and y are called source and target respectively.

Let $f : (M, x) \rightarrow (P, y)$ be the germ of a map (map germ or germ), we define the rank of the germ as the rank of df_x , where f denotes a representative of the germ.

Definition 2.2.2. Let $f : (M_1, x) \rightarrow (M_2, y)$ and $g : (M_2, y) \rightarrow (M_3, z)$ be map germs with representatives $f : U \rightarrow M_2$ and $g : V \rightarrow M_3$ respectively with $f(U) \subset V$. We define the composition of germs denoted by $g \circ f : (M_1, x) \rightarrow (M_3, z)$ as the equivalence class of $g \circ f : U \rightarrow M_3$ in x .

Definition 2.2.3. We say that a map germ $f : (M, x) \rightarrow (P, y)$ is invertible if there exists a germ $g : (P, y) \rightarrow (M, x)$ such that $f \circ g$ is the germ of the identity map in (P, y) and $g \circ f$ is a germ of the identity map in (M, x) . In this case, we say that the germ f is a germ of diffeomorphism.

As a consequence of the Inverse Function Theorem we obtain:

Theorem 2.2.4. Let P be a manifold, $x \in M$ and $y \in P$. A map germ $f : (M, x) \rightarrow (P, y)$ is invertible if and only if $df_x : T_x M \rightarrow T_y P$ is an isomorphism.

We denote by $J^k(n, p)$ the vector space of maps $f : \mathbb{K}^n \rightarrow \mathbb{K}^p$, $f = (f_1, \dots, f_p)$ where each component f_i , $i = 1, \dots, p$, is a polynomial in the variables $x_1, \dots, x_n$, whose

degree is less than or equal to k and with zero constant term. The elements of $J^k(n, p)$ are called k -jets.

Definition 2.2.5. Let $f : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a map and $a \in \mathbb{K}^n$. We define the map $j^k f : \mathbb{K}^n \rightarrow J^k(n, p)$, where $j^k f(a)$ is defined as the Taylor polynomial of $f(x + a) - f(a)$ of order k at the origin. The map $j^k f$ is smooth and $j^k f(a)$ is called k -jet of f at a ,

$$j^k f(a) = df_a(x) + \frac{1}{2}d^2 f_a(x, x) + \cdots + \frac{1}{k!}d^k f_a(x, \dots, x).$$

Remark 2.2.6. Let $f : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a map and $a \in \mathbb{K}^n$. We denote $j^0 f(a) = f(a)$ and we call it as the 0-jet of f in a .

2.2.2 Action of a group

Let G be a group and N a set. An action of G on N is a function $\phi : G \times N \rightarrow N$ denoted by $g \cdot x = \phi(g, x)$ satisfying:

1. $1 \cdot x = x$ for all $x \in N$, where 1 denotes the identity element in G .
2. $(g \cdot h) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and for all $x \in N$.

Definition 2.2.7. Given an action of G on the set N , we can define an equivalence relation on N as follows: if $x, y \in N$, x and y are equivalent if and only if there exists $g \in G$ such that $y = g \cdot x$. The equivalence classes are called orbits. If $x \in N$, then the orbit of x is denoted by $G \cdot x$.

Example 2.2.8. Let $GL(n)$ be the linear group of invertible linear maps on $\mathbb{K}^n$. Consider $N = J^k(n, p)$. An action

$$\phi : GL(n) \times GL(p) \times N \rightarrow N$$

is defined by $\phi((H, K), f) := K \circ f \circ H^{-1}$.

Definition 2.2.9. A Lie group G is a group which is a smooth manifold and the multiplication function $G \times G \rightarrow G$ defined by $(g, h) \mapsto gh$ and inversion function $G \rightarrow G$ defined by $g \mapsto g^{-1}$ are smooth. A Lie group action is a smooth action from a Lie group G acting on a manifold M .

Proposition 2.2.10. Let $\phi : G \times M \rightarrow M$ be a Lie group action. Assume the orbits are submanifolds. Consider $x \in M$, then $\phi_x : G \rightarrow G \cdot x$, defined by $\phi_x(g) = g \cdot x$, is a submersion. Therefore $T_x G \cdot x = d\phi_{x_1}(T_1 G)$.

Example 2.2.11. Let $M = H^d(n, p)$ be the vector subspace of $J^d(n, p)$ whose elements are homogeneous polynomials of degree d in the coordinate system $x_1, \dots, x_n$ of $\mathbb{K}^n$. Consider the group $G = GL(n) \times GL(p)$ and let ϕ be the action defined in Example 2.2.8. Then

$$T_f G \cdot f = \mathbb{K} \left\{ \frac{\partial f}{\partial x_i} x_j \right\}_{i,j=1,\dots,n} + \mathbb{K} \left\{ \underbrace{(0, \dots, f_j, 0, \dots, 0)}_{f_j \text{ in the } i\text{-th place}} \right\}_{i=1,\dots,n, j=1,\dots,p}.$$

2.2.3 The Algebra $\mathcal{O}_n$

Let M be a manifold and $x \in M$. We are interested in germs of functions $f : (M, x) \rightarrow (\mathbb{C}, y)$. Since the germs are described locally we can assume $M = \mathbb{C}^n$ and $x = 0$. Consider

$$\mathcal{O}_n = \{f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, y) / f \text{ is a germ}\}.$$

The set $\mathcal{O}_n$ is a local ring with maximal ideal

$$\mathcal{M}_n = \{f \in \mathcal{O}_n / f(0) = 0\}.$$

Lemma 2.2.12. (*Hadamard's Lemma*) Let $U \subset \mathbb{C}^n$ be a convex neighborhood of the origin, let $f : U \times \mathbb{C}^q \rightarrow \mathbb{C}$ be a smooth function such that $f(0, y) = 0$ for all $y \in \mathbb{C}^q$. Then there exist smooth functions $f_1, \dots, f_n : U \times \mathbb{C}^q \rightarrow \mathbb{C}$ such that $f = x_1 f_1 + \dots + x_n f_n$, where $x_1, \dots, x_n$ are the coordinates in $\mathbb{C}^n$.

Lemma 2.2.13. (*Nakayama's Lemma*) Let R be a commutative ring with unity 1. Let $m \subset R$ be an ideal such that $1 + x$ is invertible for all $x \in m$. Let M be an R -module and let A, B be R -submodules such that A is finitely generated. If $A \subset B + mA$ then $A \subset B$.

Let $f \in \mathcal{O}_n$, we denote by $\langle \nabla f \rangle$ the ideal generated by the partial derivatives of f . Since $\langle \nabla f \rangle$ is an $\mathcal{O}_n$ -submodule, the quotient $\frac{\mathcal{O}_n}{\langle \nabla f \rangle}$ is a vector space. The Milnor number of f is defined as

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \nabla f \rangle}.$$

Proposition 2.2.14. Let f be a germ in $\mathcal{O}_n$ such that $\mu = \mu(f) < \infty$, then $\mathcal{M}_n^\mu \subset \langle \nabla f \rangle$.

Given a germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$ we define the homomorphism of $\mathbb{C}$ -algebras $f^* : \mathcal{O}_p \rightarrow \mathcal{O}_n$ by $f^*(g) = g \circ f$, which is called homomorphism induced by f . Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$ and $g : (\mathbb{C}^p, 0) \rightarrow (\mathbb{C}^k, 0)$ be germs, then $(g \circ f)^* = f^* \circ g^*$.

Proposition 2.2.15. Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ be a germ. Then f^* is an isomorphism if and only if f is germ of a diffeomorphism. Moreover $f^{-1*} = f^{*-1}$.

We denote by D_n the group of germs of diffeomorphisms $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$. We define the action $\phi : D_n \times \mathcal{O}_n \rightarrow \mathcal{O}_n$ given by $\phi(h, f) = f \circ h^{-1}$. We denote the Group D_n with this action by $\mathcal{R}$.

Definition 2.2.16. Let $f, g \in \mathcal{O}_n$. We say that f and g are $\mathcal{R}$ -equivalent if they belong to the same orbit under the action of $\mathcal{R}$.

Proposition 2.2.17. Let $f, g \in \mathcal{O}_n$ be $\mathcal{R}$ -equivalents germs. Then $\langle \nabla f \rangle$ and $\langle \nabla g \rangle$ are isomorphic. In particular, if $\mu(f) < \infty$ then $\mu(f) = \mu(g)$.

Proposition 2.2.17 shows that the Milnor number is an invariant with respect to the $\mathcal{R}$ -equivalence.

Definition 2.2.18. Given a germ $f \in \mathcal{O}_n$, we say that f is k -determined if for every $g \in \mathcal{O}_n$ such that $j^k f(0) = j^k g(0)$ holds then it is $\mathcal{R}$ -equivalent to f . We say that f is finitely determined if f is k -determined for some $k \in \mathbb{N}$.

Proposition 2.2.19. Let $f \in \mathcal{O}_n$ such that $\mathcal{M}_n^k \subset \mathcal{M}_n \langle \nabla f \rangle$ for some $k \in \mathbb{N}$. Then f is k -finitely determined.

As a consequence of Propositions 2.2.14 and 2.2.19 we have

Corollary 2.2.20. Let f be a germ in $\mathcal{O}_n$ such that $\mu = \mu(f) < \infty$. Then f is $(\mu + 1)$ -determined.

2.2.4 $\mathcal{K}$ -symplectic equivalence

We denote by $\mathcal{O}_{n,p}$ the set of germs $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$ and by $\mathcal{M}_n \mathcal{O}_{n,p}$ the set of germs $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$. Clearly $\mathcal{O}_{n,p}$ is an $\mathcal{O}_n$ -module and $\mathcal{M}_n \mathcal{O}_{n,p}$ is an $\mathcal{O}_n$ -submodule of $\mathcal{O}_{n,p}$.

Definition 2.2.21. We say that $f, g \in \mathcal{M}_n \mathcal{O}_{n,p}$ are $\mathcal{C}$ -equivalent if there exist a germ of diffeomorphism $H : (\mathbb{C}^n \times \mathbb{C}^p, 0) \rightarrow (\mathbb{C}^n \times \mathbb{C}^p, 0)$ and a germ $\theta : (\mathbb{C}^n \times \mathbb{C}^p, 0) \rightarrow (\mathbb{C}^p, 0)$ such that $H(x, y) = (x, \theta(x, y))$, $\theta(x, 0) = 0$ for all $x \in (\mathbb{C}^n, 0)$ and $H(x, f(x)) = (x, g(x))$.

Definition 2.2.22. We say that $f, g \in \mathcal{M}_n \mathcal{O}_{n,p}$ are $\mathcal{K}$ -equivalent if there exist a germ of diffeomorphism $H : (\mathbb{C}^n \times \mathbb{C}^p, 0) \rightarrow (\mathbb{C}^n \times \mathbb{C}^p, 0)$, a germ of diffeomorphism $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ and a germ $\theta : (\mathbb{C}^n \times \mathbb{C}^p, 0) \rightarrow (\mathbb{C}^p, 0)$ such that $H(x, y) = (h(x), \theta(x, y))$, $\theta(x, 0) = 0$ for all $x \in (\mathbb{C}^n, 0)$ and $H(x, f(x)) = (h(x), g(h(x)))$.

Definition 2.2.23. Let $f \in \mathcal{O}_{n,p}$. We define the ideal $I(f)$ as the ideal generated by the components of f . That is, if $f = (f_1, \dots, f_n)$ then $I(f) = \langle f_1, \dots, f_n \rangle \subset \mathcal{O}_n$.

Theorem 2.2.24. Let $f, g \in \mathcal{M}_n \mathcal{O}_{n,p}$, then the following statements are equivalent:

1. f and g are $\mathcal{C}$ -equivalent,
2. $I(f) = I(g)$,
3. There exists an invertible matrix $Q = (q_{ij})_{1 \leq i, j \leq p}$ such that $f = Q \cdot g$, where $q_{ij} \in \mathcal{O}_n$ for all $1 \leq i, j \leq p$.

Proposition 2.2.25. Let $f, g \in \mathcal{M}_n \mathcal{O}_{n,p}$. Then f and g are $\mathcal{K}$ -equivalents if and only if there exists a germ of diffeomorphism $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $f \circ h$ and g are $\mathcal{C}$ -equivalent. Hence f and g are $\mathcal{K}$ -equivalent if and only if there exists a germ of diffeomorphism $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $h^*(I(f)) = I(g)$.

From Theorem 2.2.24 and Proposition 2.2.25 we obtain

Corollary 2.2.26. *Consider $f, g \in \mathcal{M}_n \mathcal{O}_{n,p}$. Then f and g are $\mathcal{K}$ -equivalents if and only if there exist an invertible matrix $Q = (q_{ij})_{1 \leq i, j \leq p}$ and a germ of diffeomorphism $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $g = Q \cdot f(h)$.*

Definition 2.2.27. *Let $(\mathbb{C}^{2n}, \omega)$ be a symplectic manifold. The germs $f, g \in \mathcal{M}_{2n} \mathcal{O}_{2n,p}$ are called $\mathcal{K}$ -symplectic equivalents if there exist an invertible matrix $Q = (q_{ij})_{1 \leq i, j \leq p}$ and a germ of diffeomorphism $h : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^{2n}, 0)$ such that $h^* \omega = \omega$ and $g = Q \cdot f(h)$.*

2.3 Algebraic restrictions

The definitions and results of this section can be found in [10]. Let us consider a manifold M and a subset $N \subset M$. We denote by $\Omega^k(M)$ the vector space of all germs at the origin of k -forms defined on the germ of the manifold (M, x) for a fixed $x \in N$.

Remark 2.3.1. *In the notation above, we can assume $x = 0$ and M be a submanifold of $\mathbb{R}^n$ for some n . Even if it is not mentioned, all the germs are considered as germs at the origin, this requires that $0 \in N$.*

Definition 2.3.2. *Two germs of differential k -forms ω_1, ω_2 are equivalent if and only if there exist a germ of a k -form α and a germ of a $k - 1$ -form β vanishing on the germ $(N, 0)$ and such that*

$$\omega_2 - \omega_1 = \alpha + d\beta$$

This relation is an equivalence relation. We denote by $[\omega]_N$ the class of the germ ω under this relation and it is called *the algebraic restriction of ω to N* . We denote by 0 the zero algebraic restriction $[0]_N$.

Proposition 2.3.3. *Let $M = \mathbb{R}^n$ and $N \subset \mathbb{R}^n$ a subset. The exterior derivative d and the exterior product $\wedge$ define an operation on the set of all algebraic restrictions, defined by $d[\omega]_N := [d\omega]_N$ and $[\omega_1]_N \wedge [\omega_2]_N := [\omega_1 \wedge \omega_2]_N$.*

Definition 2.3.4. *Two germs of set $(N_1, 0), (N_2, 0)$ contained in a symplectic manifold $(\mathbb{K}^{2n}, \omega)$ are called symplectomorphic if there exists a symplectomorphism which brings $(N_2, 0)$ to $(N_1, 0)$*

Definition 2.3.5 (The action of the group of diffeomorphism). *Let $(M_1, 0), (M_2, 0)$ be germs of manifolds equidimensional, $\Phi : (M_2, 0) \rightarrow (M_1, 0)$ be a germ of a diffeomorphism and $N_1 \subset M_1$ a subset. It is clear that $[\Phi^* \alpha]_{\Phi^{-1}(N_1)} = 0$ for all $[\alpha]_{N_1} = 0$, where $\alpha \in \Omega^k(M_1)$. Then, the operation induced by the pullback, defined by $\Phi^*([\alpha]_{N_1}) = [\Phi^* \alpha]_{\Phi^{-1}(N_1)}$, is well*

defined. Let $N_2 \subset M_2$ be a subset, $\alpha_j \in \Omega^k(M_j)$ for $j = 1, 2$, then the algebraic restrictions $[\alpha_1]_{N_1}$ and $[\alpha_2]_{N_2}$ are called diffeomorphic if there exists a germ of diffeomorphism $\Phi : (M_2, 0) \rightarrow (M_1, 0)$ such that $\Phi^*[\alpha_1]_{N_1} = [\alpha_2]_{N_2}$. This of course requires $\Phi(N_2, 0) = (N_1, 0)$.

When $M_1 = M_2$ and $N_1 = N_2$ this operation defines an action of the group of germs of diffeomorphism on the space of algebraic restrictions to N_1 . The elements of this group are called local symmetries of N_1 .

Proposition 2.3.6. *Let $M \subset \mathbb{R}^n$ be a manifold, $N \subset M$ a subset containing the origin and ω_1, ω_2 two germs of k -forms on $(M, 0)$. Then $[\omega_1]_N = [\omega_2]_N$ if and only if $[\omega_1|_{TM}]_N = [\omega_2|_{TM}]_N$*

Proposition 2.3.7. *Let $M_1, M_2 \subset \mathbb{R}^n$ be two manifolds equal dimensional and $N_1 \subset M_1, N_2 \subset M_2$ two subsets containing the origin. Let ω_1, ω_2 be two germs of k -forms defined on the germs of M_1, M_2 respectively. $[\omega_1]_{N_1}$ is diffeomorphic to $[\omega_2]_{N_2}$ if and only if $[\omega_1|_{TM_1}]_{N_1}$ is diffeomorphic to $[\omega_2|_{TM_2}]_{N_2}$.*

Definition 2.3.8. *The germ $(N, 0) \in (\mathbb{K}^n, 0)$ is called quasi homogeneous if there exist a coordinate system $(x_1, \dots, x_n)$ and positive numbers $\lambda_1, \dots, \lambda_n$ such that if a point $(a_1, \dots, a_n) \in (N, 0)$ then for any $t > 0$ the point $(t^{\lambda_1}a_1, \dots, t^{\lambda_n}a_n)$ also belongs to N .*

Definition 2.3.9. *The germ of a function $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}, 0)$ is called quasi homogeneous if there exist a coordinate system $(x_1, \dots, x_n)$ and positive numbers $\lambda_1, \dots, \lambda_n, d$ such that for all $t > 0$*

$$f(t^{\lambda_1}a_1, \dots, t^{\lambda_n}a_n) = t^d f(a_1, \dots, a_n)$$

as a result we have that if f is a quasi homogeneous function, then $f^{-1}(0)$ is a quasi homogeneous set.

Proposition 2.3.10. *Suppose that the quasi homogeneous analytic real or complex function $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}, 0)$ has an isolated singularity at the origin, then the Milnor number $\mu(f)$ and Tjurina number $\tau(f)$ are equal.*

Proof. Since f is quasi homogeneous, there exist a coordinate system $(x_1, \dots, x_n)$ such that $f(t^{\lambda_1}a_1, \dots, t^{\lambda_n}a_n) = t^d f(a_1, \dots, a_n)$ for some positives numbers $\lambda_1, \dots, \lambda_n, d$. This implies that $f \in \langle \nabla f \rangle$. Therefore $\mu(f) = \tau(f)$. $\square$

We list the main theorems proved in [10] when N is a quasi homogeneous subset of $\mathbb{R}^{2n}$

Theorem 2.3.11 (Theorem A). *(i) Let N be a quasi homogeneous subset of $\mathbb{R}^{2n}$. Let ω_0, ω_1 be germs of symplectic forms on $\mathbb{R}^{2n}$ with the same algebraic restriction to N . Then there exists a germ of diffeomorphism Φ of $(\mathbb{R}^{2n}, 0)$ such that $\Phi(x) = x$ for any $x \in (N, 0)$*

and $\Phi^*\omega_0 = \omega_1$.

(ii) The germs of two quasi homogeneous subsets N_1, N_2 of a fixed symplectic manifold $(\mathbb{R}^{2n}, \omega)$ are symplectomorphic if and only if the algebraic restrictions of the symplectic form ω to N_1 and N_2 are diffeomorphic.

Theorem 2.3.12 (Theorem B). *The germ of a quasi homogeneous set N of a symplectic manifold $(\mathbb{R}^{2n}, \omega)$ is contained in a Lagrangian submanifold if and only if the symplectic form ω has zero algebraic restriction to N .*

Definition 2.3.13. *Given a germ of a differential form ω with zero $(k-1)$ -jet and non-zero k -jet we will say that k is the order of vanishing. If $\omega(0) \neq 0$ then the order of vanishing is 0. If $\omega = 0$ then the order of vanishing is ∞ .*

Definition 2.3.14 (Index of Isotropness). *Let N be a subset of a symplectic manifold $(\mathbb{K}^{2n}, \omega)$. The index of isotropness of N is the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all germs of submanifolds M containing $(N, 0)$. We denote by $\text{ind}(N)$ the index of isotropness of N .*

Theorem 2.3.15 (Theorem C). *The index of isotropness of a quasi homogeneous variety N in a symplectic manifold $(\mathbb{R}^{2n}, \omega)$ is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction $[\omega]_N$.*

Definition 2.3.16. *By a variety in $\mathbb{K}^{2n}$ we mean the zero set of a k -generated ideal having the property of zeros, $k \geq 1$. That is, the functions $g_1, \dots, g_k : \mathbb{K}^{2n} \rightarrow \mathbb{K}$ have the property of zeros if any function g which vanishes on the set $\cap_{i=1}^k g_i^{-1}(0) \neq \emptyset$ belongs to the ideal $\langle g_1, \dots, g_k \rangle$.*

Let $(\mathbb{C}^{2n}, \omega)$ be a symplectic manifold. Denote by $\text{Var}(k, 2n)$ the space of all germs at origin of varieties described by k -generated ideals. We associate to $(N, 0) \in \text{Var}(k, 2n)$ the map germ $H = (h_1, \dots, h_k) : (\mathbb{K}^{2n}, 0) \longrightarrow (\mathbb{K}^k, 0)$ whose k components are generators of the ideal of function germs vanishing on $(N, 0)$. We denote by (N) the orbit of $(N, 0)$ with respect to the group of diffeomorphisms. Suppose $(N_1, 0) \in (N)$ then there exists a diffeomorphism $\phi : (\mathbb{K}^{2n}, 0) \rightarrow (\mathbb{K}^{2n}, 0)$ such that $(N_1, 0) = \phi(N, 0)$. Let $H_1 : (\mathbb{K}^{2n}, 0) \longrightarrow (\mathbb{K}^k, 0)$ whose k components are generators of the ideal of function germs vanishing on $(N_1, 0)$. Since $H \circ \phi^{-1}$ vanishes on $(N_1, 0)$ and $H_1 \circ \phi$ vanishes on $(N, 0)$ there exists Q an invertible matrix $k \times k$ (It follows from the fact $\cap_{j \neq i} h_j^{-1}(0) \not\subset h_i^{-1}(0)$ for all $1 \leq i \leq k$) whose entries are in $\mathcal{O}_{2n}$ such that

$$H_1 = Q \cdot H(\phi^{-1})$$

therefore H and H_1 are $\mathcal{K}$ -equivalent (the $\mathcal{K}$ -equivalence for the real case is defined similarly to the complex case). Thus we can associate to $(N_1, 0) \in (N)$ the map H_1 which is K -equivalent to H .

Denote by $(N)_{\text{symp}}$ the orbit of $(N, 0)$ with respect to the group of symplectomorphisms. Analogously, we can identify to any $(N_1, 0) \in (N)_{\text{symp}}$ a map germ $H_1 : (\mathbb{K}^{2n}, 0) \longrightarrow (\mathbb{K}^k, 0)$ which is $\mathcal{K}$ -symplectic equivalent to H (the $\mathcal{K}$ -symplectic equivalence for the real case is defined similarly to the complex case).

Definition 2.3.17 (Symplectic Multiplicity). *Let N be a variety of a symplectic manifold $(\mathbb{K}^{2n}, \omega)$ and $H : (\mathbb{K}^{2n}, 0) \longrightarrow (\mathbb{K}^k, 0)$ whose k components are generators of the ideal of function germs vanishing on $(N, 0)$. The symplectic multiplicity of N , denoted by $\mu_{\text{symp}}(N)$, is the codimension of the $\mathcal{K}$ -symplectic orbit of H in the $\mathcal{K}$ orbit of H .*

Theorem 2.3.18 (Theorem D). *The symplectic multiplicity of a quasi homogeneous variety in a symplectic manifold $(\mathbb{R}^{2n}, \omega)$ is equal to the codimension of the orbit of the algebraic restriction $[\omega]_N$ with respect to the group of local symmetries of $(N, 0)$ in the space of algebraic restrictions of closed 2-forms.*

With this results it is possible to solve the classification problem described in [10], it is

"To obtain normal forms of curves in a symplectic manifold $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to a fixed quasi homogeneous curve N under the action of the group of symplectomorphism".

This classification was done when N is given by the zeros of the simple germs A_k , D_k , E_6 , E_7 and E_8 in two variables.

Example 2.3.19. *Let $H(x_1, x_2) = x_1^3 - x_2^5$ and $F_0 = \pm 1$, $F_1 = x_2 + bx_1$, $F_2 = x_1 + b_1x_2^2 + b_2x_2^3$, $F_3 = \pm x_2^2 + bx_1x_2$, $F_4 = \pm x_1x_2 + bx_2^3$, $F_5 = x_2^3 + bx_1x_2^2$, $F_6 = x_1x_2^2$, $F_7 = \pm x_1x_2^3$, $F_8 = 0$. Let $\omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$ be a symplectic form in $\mathbb{R}^{2n}$ with coordinate system $(p_1, \cdots, p_n, q_1, \cdots, q_n)$. Then*

(i) *If $n > 1$. Any curve in the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ which is diffeomorphic to the curve $N = \{H(x_1, x_2) = x_{\geq 3} = 0\}$ can be reduced by a symplectomorphism to one and only one of the normal forms*

$$N^i = \left\{ H(p_1, p_2) = q_1 - \int_0^{p_2} F_i(p_1, t) dt = q_{\geq 2} = p_{\geq 3} \right\} \subset (\mathbb{R}^{2n}, \omega_0).$$

(ii) *When $n = 1$, all curves in the symplectic plane $(\mathbb{R}^2, \omega_0)$ which are diffeomorphic to the curve $\{H = 0\}$ are symplectomorphic to one of the curves $p_1^3 \pm q_1^5 = 0$*

An immediate question is whether the same classification can be obtained when the subset N is non quasi homogeneous using the method of algebraic restrictions. The

answer to this question is NO in general. This follows from the fact that in the proof of Theorem A is used the fact that the Relative Cohomology Group associate to N is zero.

$$H^p(N, \mathbb{R}^m) := \frac{\{\omega : \omega \text{ is a closed } p\text{-form and } [\omega]_N = 0\}}{\{d\alpha : \alpha \text{ is a } p-1\text{-form and } [\alpha]_N = 0\}} = 0 \quad (2.4)$$

when $N \subset \mathbb{R}^m$ is quasi homogeneous, see [22] or [8].

Equation 2.4 is not true in general, see the following result from Sebastiani in [24].

Theorem 2.3.20 (Corollary 2). *Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a germ of function such that $(f^{-1}(0), 0) \subset (\mathbb{C}^n, 0)$ is the germ of a singular hypersurface. Denote by H^n the n -th the usual De Rham Cohomology group, $\Omega^p = \Omega^p(\mathbb{C}^n)$ and denote $\Omega_0^p = \frac{\Omega^p}{df \wedge \Omega^{p-1} + f\Omega^p}$. Then for $n > 1$:*

$$\dim_{\mathbb{C}} H^{n-1}(\Omega_0) = \mu(f) - \tau(f),$$

where $\mu(f)$ and $\tau(f)$ are the Milnor and Tjurina numbers of f , respectively.

Theorem 2.3.21. *Let $f \in \mathcal{O}_2$ be a germ of a function which has an isolated singularity at the origin and has the property of zeros. Then $\dim_{\mathbb{C}} H^2(f^{-1}(0), \mathbb{C}^2) = \mu(f) - \tau(f)$.*

Proof. By definition and hypothesis

$$H^2(f^{-1}(0), \mathbb{C}^2) = \frac{f \cdot \Omega^2(\mathbb{C}^2)}{d(f \cdot \Omega^1(\mathbb{C}^2))},$$

and

$$H^1(\Omega_0) = \frac{\{\omega \in \Omega^1(\mathbb{C}^2) : d\omega \in df \wedge \Omega^1(\mathbb{C}^2) + f\Omega^2(\mathbb{C}^2)\}}{d\Omega^0(\mathbb{C}^2) + df \wedge \Omega^0(\mathbb{C}^2) + f\Omega^1(\mathbb{C}^2)}.$$

Since

$$d(d\Omega^0(\mathbb{C}^2) + df \wedge \Omega^0(\mathbb{C}^2) + f\Omega^1(\mathbb{C}^2)) = d(f\Omega^1(\mathbb{C}^2)), \quad (2.5)$$

and for $\omega \in \Omega^1(\mathbb{C}^2)$ such that

$$d\omega = df \wedge \alpha + f\theta \Rightarrow d\omega - f(\theta - d\alpha) \in d(f\Omega^1(\mathbb{C}^2)), \quad (2.6)$$

we have that d induces a well defined linear map

$$d : H^1(\Omega_0) \longrightarrow H^2(f^{-1}(0), \mathbb{C}^2), \quad d\bar{\omega}^1 := \overline{d\omega}^2$$

which is an isomorphism of vector spaces. Here $\bar{\omega}^1$ denotes the class of ω in $H^1(\Omega_1)$ and $\bar{\omega}^2$ denotes the class of ω in $H^2(f^{-1}(0), \mathbb{C}^2)$.

From equation 2.5 it follows that d is well defined, the linearity follows from equation 2.6. If $\overline{d\omega}^2 = 0$ then there exists $\alpha \in \Omega^1(\mathbb{C}^2)$ such that $d\omega = d(f\alpha)$, hence from the Poincaré's Lemma there exists $G(x_1, x_2) \in \mathcal{O}_2$ such that $\omega = f\alpha + dG$. Thus $\bar{\omega}^1 = 0$ and therefore d is injective. If $\theta \in \Omega^2(\mathbb{C}^2)$, by the Poincaré's Lemma there exists a germ of 1-form $\alpha \in \Omega^1(\mathbb{C}^2)$ such that $f\theta = d\alpha$ thus $d\bar{\alpha}^1 = \overline{d\alpha}^2 = \overline{f\theta}^2$ and therefore d is surjective.

□

As a particular case of Theorem [2.3.21](#) when f is a non quasi homogeneous function $H^2(f^{-1}(0), \mathbb{C}^2) \neq 0$.

EXACT ALGEBRAIC RESTRICTIONS

We have seen in the end of the previous chapter that if $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ is a non quasi homogeneous function which has an isolated singularity at the origin and the property of zeros, then $H^2(f^{-1}(0), \mathbb{C}^2) \neq 0$.

Therefore we cannot use the method of algebraic restriction to obtain normal forms to the classification symplectic of set described in [10] in a general case. So, is there some method to do it? The answer to this question is YES. In order to not face the cohomology group we redefine the algebraic restriction to a new algebraic object which we call exact algebraic restriction. Using the exact algebraic restrictions, we prove that the condition of N been a quasi homogeneous set in a symplectic manifold $(\mathbb{C}^{2n}, \omega)$ in the Theorem A can be eliminated. Moreover, we prove a version of Theorem B, C and D to the case when N is a non quasi homogeneous subset. Using this theorems we prove that to obtain normal forms to the symplectic classification of sets is enough to obtain formal norms to the classification of exact algebraic restriction of symplectic forms under the action of the group of diffeomorphism on $(\mathbb{C}^{2n}, 0)$ preserving $(N, 0)$.

Since the set of exact algebraic restrictions of symplectic forms is contained in the vector space of exact algebraic restriction of closed 2-forms, it is natural to ask under what condition on N this vector space is finite dimensional. We prove that this is always true, when N a one-dimensional isolated complete intersection singularity(ICIS). As a particular case for the planar curve $N = \{f(x_1, x_2) = x_{\geq 3} = 0\} \subset \mathbb{C}^{2n}$, where f has an isolated singularity at the origin and the vector tangent to $f^{-1}(0)$ vanishes at zero together with their divergence, we have a way to calculate a basis for the vector space of exact algebraic restriction to N .

Finally, we obtain a brief description of when an exact algebraic restriction is zero in when $N \subset \mathbb{C}^2$. We prove that in general the set of zero exact algebraic restriction is not a module, but it is always a module when N is a quasi homogeneous set.

3.1 The exact algebraic restriction

Definition 3.1.1. Let us consider a manifold M and a subset $N \subset M \subset \mathbb{K}^n$, both containing the origin. Two germs of k -forms $\omega_1, \omega_2 \in \Omega^k(M)$ are equivalent ($\omega_1 \sim_N \omega_2$) if and only if

i) For $k \geq 1$, there exists a germ $\beta \in \Omega^{k-1}(M)$ such that

$$\omega_1 - \omega_2 = d\beta \text{ with } \beta(x) = 0 \text{ for any } x \in (N, 0).$$

ii) For $k = 0$,

$$(\omega_1 - \omega_2)(x) = 0 \text{ for all } x \in (N, 0).$$

This is an equivalence relation and the equivalence classes are called the exact algebraic restriction to N .

Remark 3.1.2. Let $\beta \in \Omega^k(M)$ which vanishes on $(N, 0)$. Let $(x_1, \dots, x_m)$ be a coordinate system of M , then we can write

$$\beta = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

Since $\beta(x) = 0$ for all $x \in (N, 0)$ and there exist vectors $v_{j_1}^x, \dots, v_{j_k}^x \in T_x M$ for all $x \in (N, 0)$ such that

$$dx_{i_1} \wedge \dots \wedge dx_{i_k}(v_{j_1}^x, \dots, v_{j_k}^x) = \begin{cases} 1 & (i_1, \dots, i_k) = (j_1, \dots, j_k), \\ 0 & (i_1, \dots, i_k) \neq (j_1, \dots, j_k). \end{cases}$$

we obtain $f_{i_1, \dots, i_k}(x) = 0$ for all $x \in (N, 0)$, for all $1 \leq i_1 < \dots < i_k \leq n$.

Notation: Let $\omega \in \Omega^k(M)$. We denote the exact algebraic restriction of ω to N by $\|\omega\|_N$, the vector space of all the exact algebraic restrictions by $\|\Omega^k(M)\|_N$, also we write 0 instead of $\|0\|_N$. When N is an specific subset we just write $\|\omega\|$.

If $\omega_1 - \omega_2 = d\beta$, then $d\omega_1 - d\omega_2 = 0$. Therefore the derivative of an exact algebraic restriction to N is well defined by $d\|\omega\|_N := \|d\omega\|_N$. In contrast with the algebraic restriction, we cannot define the operator " $\wedge$ " between exact algebraic restriction, but we can define this operator in a subspace of it.

Remark 3.1.3. From now, whenever we talk about exact algebraic restriction to a set N , we are assuming that the origin is contained in N . Also we assume that $N \subset M$, where M is a manifold.

Proposition 3.1.4. Let $\gamma \in \Omega^l(M)$, $\omega \in \Omega^k(M)$ such that $d\gamma = 0$ and $\|\omega\|_N = 0$. Then $\|\gamma \wedge \omega\|_N = 0$.

Proof. By definition, there exists a germ $\beta \in \Omega^{k-1}(M)$ vanishing on $(N, 0)$ and such that $\omega = d\beta$. Then $\gamma \wedge \omega = \gamma \wedge d\beta = (-1)^{lk-l}d(\gamma \wedge \beta)$ and so $\|\gamma \wedge \omega\|_N = 0$. $\square$

Denote by $\overline{\Omega}^k(M)$ the subspace of $\Omega^k(M)$ containing all germs of closed k -forms on $(M, 0)$. Then we have proved:

Proposition 3.1.5. *The following operators are well defined*

- i) $d : \|\Omega^k(M)\|_N \longrightarrow \|\Omega^{k+1}(M)\|_N$ defined by $d\|\omega\|_N := \|d\omega\|_N$ and
- ii) $\wedge : \|\overline{\Omega}^l(M)\|_N \times \|\Omega^k(M)\|_N \longrightarrow \|\Omega^{k+l}(M)\|_N$ defined by $\|\gamma\|_N \wedge \|\omega\|_N := \|\gamma \wedge \omega\|_N$.

It is natural to ask what is the relation between algebraic restrictions and exact algebraic restriction when N is a quasi homogeneous subset.

Proposition 3.1.6. *Let M be a n dimensional manifold. If N is a quasi homogeneous subset of M , then for $\omega \in \overline{\Omega}^n(M)$ we have $\|\omega\|_N = [\omega]_N$ and therefore $\|\overline{\Omega}^n(M)\|_N = [\overline{\Omega}^n(M)]_N$.*

Proof. In fact, if $\|\omega_1\|_N = \|\omega_2\|_N$ then by definition $[\omega_1]_N = [\omega_2]_N$. Reciprocally, if $[\omega_1]_N = [\omega_2]_N$, where ω_1, ω_2 are two n -forms then $\omega_1 - \omega_2 = \alpha + d\beta$ for some n -form α and $(n-1)$ -form β vanishing both on N . So α is a closed form and since the cohomology group 2.4 is trivial when N is quasi homogeneous there exists a $(n-1)$ -form θ vanishing on N and such that $\omega_1 - \omega_2 = d\theta$. Therefore $\|\omega_1\|_N = \|\omega_2\|_N$. $\square$

Unfortunately, Proposition 3.1.6 is not true for closed k -forms, $k \neq n$.

Example 3.1.7. *Consider $M = \mathbb{C}^n$, $N = \{x_1 - x_2 = x_{\geq 3} = 0\}$ and $k < n$. Let ω_1 be a closed k -form. Then the k -form*

$$\omega_2 = \omega_1 + (x_1 - x_2)dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{k+1}$$

is a non closed k -form, therefore $\|\omega_1\|_N$ is not equal to $[\omega_1]_N$.

Let $(M_1, 0), (M_2, 0)$ be germs of equidimensional manifolds and N_1, N_2 two subsets contained in M_1, M_2 respectively. Analogous to the Definition 2.3.5, we say that two exact algebraic restrictions $\|\alpha_1\|_{N_1}$ and $\|\alpha_2\|_{N_2}$ are diffeomorphic if there exists a germ of diffeomorphism $\Phi : (M_2, 0) \rightarrow (M_1, 0)$ such that $\Phi^*\|\alpha_1\|_{N_1} = \|\alpha_2\|_{N_2}$. This, of course, requires $\Phi(N_2, 0) = (N_1, 0)$. We denote by $G_N(M)$ the group of local symmetries of N defined on $(M, 0)$.

Proposition 3.1.8. *The operator pullback $*$ induces on $G_N(M) \times \|\Omega^k(M)\|_N$ the operator $*$: $G_N(M) \times \|\Omega^k(M)\|_N \longrightarrow \|\Omega^k(M)\|_N$ defined by $g^*\|\omega\|_N := \|g^*\omega\|_N$.*

Proof. Let ω be a germ of a k -form on M such that $\|\omega\|_N = 0$. We shall prove that $g^*\|\omega\|_N = 0$ for all $g \in G_N(M)$.

- i) If $k = 0$: then $\omega(x) = 0$ for all $x \in N$. If $g \in G_N(M)$, then $g(N, 0) = (N, 0)$ and since $g^*\omega = \omega \circ g$ we have $g^*\omega(x) = 0$ for all $x \in N$. Therefore $\|g^*\omega\|_N = 0$
- ii) If $k \geq 1$: then there exists a $(k-1)$ -form β such that $\omega = d\beta$. Since $*$ and d commute we have that for all $g \in G_N(M)$, $g^*\omega = d(g^*\beta)$ and therefore $\|g^*\omega\|_N = 0$. $\square$

Proposition 3.1.9. *Let $N \subset M \subset \mathbb{C}^n$ be a subset contained in a manifold M . Let $\omega_1, \omega_2 \in \Omega^k(\mathbb{C}^n)$, $k \geq 1$, then $\|\omega_1\|_N = \|\omega_2\|_N$ if and only if $d\omega_1 = d\omega_2$ and $\|\omega_1|_{TM}\|_N = \|\omega_2|_{TM}\|_N$.*

Proof. Consider $m = \dim M$. Since we are working with germs at the origin, the case $m = n$ is trivial. Suppose $m < n$, then there exists a germ of diffeomorphism $\phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $\phi(M, 0) = (\mathbb{C}^m \times \{0\}, 0)$. Since $\omega_j|_{TM} = (\phi^*\omega_j)|_{T\mathbb{C}^m \times \{0\}}$, $j = 1, 2$, we can assume that $M = \mathbb{C}^m \times \{0\}$.

- i) Suppose $\|\omega_1\|_N = \|\omega_2\|_N$. Let β be a germ of a $(k-1)$ -form such that $\omega_1 - \omega_2 = d\beta$. Then we have $i_M^*\omega_1 - i_M^*\omega_2 = d(i_M^*\beta)$, where $i_M : (M, 0) \rightarrow (\mathbb{C}^n, 0)$ denotes the inclusion map germ. Therefore $\|\omega_1|_{TM}\|_N = \|\omega_2|_{TM}\|_N$.
- ii) Reciprocally, suppose $d\omega_1 = d\omega_2$ and $\|\omega_1|_{TM} - \omega_2|_{TM}\|_N = 0$. Then by Poincaré's Lemma (Theorem 2.1.18) there exists a germ of a $(k-1)$ -form α such that $\omega_1 - \omega_2 = d\alpha$. If

$$\alpha = \sum_{i_1 < \dots < i_{k-1}} f_{i_1, \dots, i_{k-1}} dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}$$

we can rewrite this expression by

$$\alpha = \sum_{i_1 < \dots < i_{k-1}} \left\{ \hat{f}_{i_1, \dots, i_{k-1}} + (f_{i_1, \dots, i_{k-1}} - \hat{f}_{i_1, \dots, i_{k-1}}) \right\} dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}$$

where $\hat{f}_{i_1, \dots, i_{k-1}} := f_{i_1, \dots, i_{k-1}}(x_1, \dots, x_m, 0, \dots, 0)$. Hence, by Hadamard's Lemma (Lemma 2.2.12) we obtain $f_{i_1, \dots, i_{k-1}} - \hat{f}_{i_1, \dots, i_{k-1}} \in \langle x_{m+1}, \dots, x_n \rangle$, the ideal generated by $x_{k+1}, \dots, x_n$. Thus we can write

$$\alpha = \sum_{i_1 < \dots < i_{k-1}} \hat{f}_{i_1, \dots, i_{k-1}} dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} + \tilde{\alpha}$$

where $\tilde{\alpha}(x) = 0$ for all $x \in (N, 0)$. Moreover

$$\alpha = \sum_{i_1 < \dots < i_{k-1} < m+1} \hat{f}_{i_1, \dots, i_{k-1}} dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} + \hat{\alpha}$$

where $\hat{\alpha}(x) = 0$ for all $x \in (N, 0)$. Therefore

$$\omega_1 - \omega_2 = d \left(\sum_{i_1 < \dots < i_{k-1} < m+1} \hat{f}_{i_1, \dots, i_{k-1}} dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} \right) + d\hat{\alpha}$$

$$i_M^* \omega_1 - i_M^* \omega_2 = di_M^* \left(\sum_{i_1 < \dots < i_{k-1} < m+1} \hat{f}_{i_1, \dots, i_{k-1}} dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} \right) + di_M^* \hat{\alpha}$$

$$i_M^* \omega_1 - i_M^* \omega_2 = d \left(\sum_{i_1 < \dots < i_{k-1} < m+1} \hat{f}_{i_1, \dots, i_{k-1}} dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} \right) + di_M^* \hat{\alpha}$$

So,

$$\omega_1 - \omega_2 = i_M^* \omega_1 - i_M^* \omega_2 + d(\hat{\alpha} - i_M^* \hat{\alpha}) = d(\beta + \hat{\alpha} - i_M^* \hat{\alpha})$$

where β exists by hypothesis. This proves the proposition. $\square$

In the case $k = 0$ we obtain

Proposition 3.1.10. *Let $N \subset M \subset \mathbb{C}^n$ be a subset contained in a manifold M . Let $g_1, g_2 \in \mathcal{O}_n$, then*

$$\|g_1\|_N = \|g_2\|_N \iff \|g_1|_{TM}\|_N = \|g_2|_{TM}\|_N$$

Proof. As in the previous proposition, we can assume $M = \mathbb{C}^m \times \{0\}$. Denote by $i_M : (M, 0) \longrightarrow (\mathbb{C}^n, 0)$ the inclusion map.

First we assume $\|g_1\|_N = \|g_2\|_N$, so by definition $(g_1 - g_2)(x) = 0$ for all $x \in (N, 0)$. Then $i_M^*(g_1 - g_2)(x) = (g_1 - g_2)(x) = 0$ for all $x \in (N, 0)$. Therefore $\|g_1|_{TM}\|_N = \|g_2|_{TM}\|_N$.

Reciprocally, if $\|g_1|_{TM}\|_N = \|g_2|_{TM}\|_N$ then $i_M^*(g_1 - g_2)(x) = 0$ for all $x \in (N, 0)$ and so $(g_1 - g_2)(x) = 0$ for all $x \in (N, 0)$, because $N \subset M$. Therefore $\|g_1\|_N = \|g_2\|_N$. $\square$

Lemma 3.1.11. *Let $M_1, M_2 \subset \mathbb{K}^n$ be two manifolds of equal dimension and $\phi : (M_1, 0) \longrightarrow (M_2, 0)$ a germ of diffeomorphism. Then there exists $\Phi : (\mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^n, 0)$ a germ of diffeomorphism such that $\Phi|_{M_1} = \phi$.*

Proof. Denote by m the dimension of these manifolds and suppose $m \leq n - 1$ (since $m = n$ is a trivial case). Since $(M_j, 0)$, $j = 1, 2$, are germs of manifolds there exist $\psi_j : (\mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^n, 0)$ such that the restrictions $\tilde{\psi}_j := \psi_j|_{\mathbb{K}^m \times \{0\}} : (\mathbb{K}^m \times \{0\}, 0) \longrightarrow (M_j, 0)$, $j = 1, 2$, are diffeomorphisms. Thus $\bar{\phi} := \tilde{\psi}_2^{-1} \circ \phi \circ \tilde{\psi}_1$ is a germ of diffeomorphism on $(\mathbb{K}^m \times \{0\}, 0)$ to itself. Denote by $\tilde{\phi}(x, y) = \bar{\phi}(x, 0) + (0, y)$, where $x \in (\mathbb{K}^m, 0)$, $y \in (\mathbb{K}^{n-m}, 0)$, the differentiable extension of $\bar{\phi}$. So define $\Phi : (\mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^n, 0)$ by $\Phi = \psi_2 \circ \tilde{\phi} \circ \psi_1^{-1}$. It is a germ of diffeomorphism and $\Phi|_{M_1} = \phi$. $\square$

Lemma 3.1.12. *Let M_1, M_2 be as in Lemma 3.1.11. Then there exists $\Phi : (\mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^n, 0)$ diffeomorphism such that $\Phi(M_1, 0) = (M_2, 0)$ and preserving pointwise the set $(M_1 \cap M_2, 0)$.*

Proof. see [10]. □

Lemma 3.1.13. *Let $N \subset \mathbb{K}^n$ be a subset, ω a germ of a closed k -form on $\mathbb{K}^n$ and $\Phi : (\mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^n, 0)$ a diffeomorphism preserving the set $(N, 0)$ pointwise. Then $\Phi^*\|\omega\|_N = \|\omega\|_N$.*

Proof. Since $\Phi = (\Phi_1, \dots, \Phi_n)$ preserves $(N, 0)$ pointwise, we can write $\Phi_i(x) = x_i + \phi_i(x)$, $i = 1, \dots, n$, for $x = (x_1, \dots, x_n)$ and for some map $\phi = (\phi_1, \dots, \phi_n)$ vanishing on $(N, 0)$. Since ω is closed, using Poincaré's Lemma (Theorem 2.1.18) there exists a germ of a $(k-1)$ -form α on $\mathbb{K}^n$ such that $\omega = d\alpha$. So if $\alpha = \sum_{i_1 < \dots < i_{k-1}} f_{i_1, \dots, i_{k-1}} dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}$ then

$$\Phi^*\alpha = \sum_{i_1 < \dots < i_{k-1}} f_{i_1, \dots, i_{k-1}}(x + \phi) d(x_{i_1} + \phi_{i_1}(x)) \wedge \dots \wedge d(x_{i_{k-1}} + \phi_{i_{k-1}}(x))$$

By straightforward calculation it is possible to show that the right term can be written as $\alpha + \tilde{\alpha}$, where $\tilde{\alpha}(x) = 0$ for all $x \in (N, 0)$. Therefore

$$\Phi^*\omega - \omega = d\Phi^*\alpha - d\alpha = d\tilde{\alpha}$$

this proves the lemma. □

Then we can prove a general version of Proposition 3.1.9.

Proposition 3.1.14. *Let $N_1, N_2 \subset \mathbb{C}^n$ be two subsets contained in equidimensional manifolds M_1, M_2 respectively. Let $\omega_1, \omega_2 \in \overline{\Omega}^k(\mathbb{C}^n)$, $k \geq 1$ (when $k = 0$ they are constants functions). Then $\|\omega_1\|_{N_1}$ and $\|\omega_2\|_{N_2}$ are diffeomorphic if and only if $\|\omega_1|_{TM_1}\|_{N_1}$ and $\|\omega_2|_{TM_2}\|_{N_2}$ are diffeomorphic.*

Proof. i) First suppose $\|\omega_1|_{TM_1}\|_{N_1}$ and $\|\omega_2|_{TM_2}\|_{N_2}$ are diffeomorphic. Hence, there exists a diffeomorphism $\Phi : (M_1, 0) \longrightarrow (M_2, 0)$ such that $\Phi(N_1, 0) = (N_2, 0)$ and

$$\|\omega_1|_{TM_1}\|_{N_1} = \Phi^*\|\omega_2|_{TM_2}\|_{N_2} = \|\Phi^*(\omega_2|_{TM_2})\|_{N_1} = \|(\Phi^*\omega_2)|_{TM_1}\|_{N_1}$$

Therefore from Lemma 3.1.11 and Proposition 3.1.9 we obtain that $\|\omega_1\|_{N_1}$ and $\|\omega_2\|_{N_2}$ are diffeomorphic.

ii) Reciprocally, we assume that $\|\omega_1\|_{N_1}$ and $\|\omega_2\|_{N_2}$ are diffeomorphic. Then, there exists a germ of diffeomorphism $\phi : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ such that $\phi(N_1, 0) = (N_2, 0)$ and $\phi^*\|\omega_2\|_{N_2} = \|\phi^*\omega_2\|_{N_1} = \|\omega_1\|_{N_1}$. Consider $M = \phi^{-1}(M_2)$, then $N_1 \subset M$. Thus by Lemma 3.1.12 there exists a diffeomorphism $\bar{\phi}$ from $(\mathbb{C}^n, 0)$ to itself such that $\bar{\phi}(M_1, 0) = (M, 0)$ and preserves pointwise $(N_1, 0)$. By Lemma 3.1.13

$$\|\bar{\phi}^*\phi^*\omega_2\|_{N_1} = \|\bar{\phi}^*\omega_1\|_{N_1} = \|\omega_1\|_{N_1}$$

hence $(\phi \circ \bar{\phi} \circ i_{M_1})^* \|\omega_2\|_{N_2} = \|i_{M_1}^* \bar{\phi}^* \phi^* \omega_2\|_{N_1} = \|i_{M_1}^* \omega_1\|_{N_1}$ where $i_{M_1} : (M_1, 0) \longrightarrow (\mathbb{C}^n, 0)$ is the inclusion map. Therefore $\|\omega_1|_{TM_1}\|_{N_1}$ and $\|\omega_2|_{TM_2}\|_{N_2}$ are diffeomorphic, since $\phi \circ \bar{\phi} \circ i_{M_1} : (M_1, 0) \rightarrow (M_2, 0)$ is a germ of diffeomorphism.

□

The next two lemmas were proved in [10] for the real case, for the sake of completeness we rewrite the same proof for the complex case.

Lemma 3.1.15. *Let $N \subset \mathbb{C}^n$. Let $W \subset T_0(\mathbb{C}^n, 0)$ be the tangent space at 0 of some (and then any) germ of submanifold containing $(N, 0)$ of minimal dimension within such submanifolds. If $\omega \in \bar{\Omega}^k(\mathbb{C}^n)$ is such that $\|\omega\|_N = 0$, then $\omega|_W = 0$.*

Proof. Fix a submanifold M containing N of minimal dimension within such manifolds (then $W = T_0M$). By hypothesis $\|\omega\|_N = 0$, then by Proposition 3.1.9, $\|\omega|_{TM}\|_N = 0$. Thus there exists $\alpha \in \Omega^{k-1}(M)$ vanishing on $(N, 0)$ such that $\omega|_{TM} = d\alpha$. Since $(N, 0)$ is not contained in any hypersurface of $(M, 0)$ then any germ of function vanishing on $(N, 0)$ has zero 1-jet at 0. In fact, if $g \in \mathcal{O}_n$ vanishes on $(N, 0)$ and $j^1g(0) \neq 0$, then dg is a germ of submersion, thus $0 = g(0)$ is a regular value and $g^{-1}(0)$ is a germ hypersurface of $(M, 0)$. Therefore $j^1(0)\alpha = 0$ and $d\alpha(0) = 0$. So $\omega|_{TM}(0) = d\alpha(0) = 0$. □

Lemma 3.1.16. *Let $\omega \in \bar{\Omega}^2(\mathbb{C}^n)$ with $\|\omega\|_N = 0$ and $N \subset \mathbb{C}^n$. Let $M \subset \mathbb{C}^n$ be a submanifold containing N of minimal dimension within such submanifolds. There exists $\theta \in \bar{\Omega}^2(\mathbb{C}^n)$ such that $\theta|_{TM} = \omega|_{TM}$, $\|\theta\|_N = 0$ and $\theta(0) = 0$.*

Proof. Let $\mu = i_M^* \omega = \omega|_{TM}$, where i_M is the inclusion map of $(M, 0)$ in $(\mathbb{C}^2, 0)$. By Lemma 3.1.15 we have $\mu(0) = 0$. Let $\pi : (\mathbb{C}^n, 0) \longrightarrow (M, 0)$ be a submersion germ which is the identity on $(M, 0)$. Let $\theta = \pi^* \mu$. Then θ is a germ of a closed 2-form which vanishes at 0 and whose restriction to TM coincides with that of ω . Since $\|\omega\|_N = 0$ and $\omega|_{TM} = \theta|_{TM}$, by Proposition 3.1.9 we obtain $\|\theta\|_N = 0$. □

Theorem 3.1.17 (Darboux-Givental see [3]). *Let M be a submanifold of $\mathbb{C}^{2n}$ and let ω_1, ω_2 be symplectic forms on $(\mathbb{C}^{2n}, 0)$ with the same restriction to TM . There exists a diffeomorphism Φ of $(\mathbb{C}^{2n}, 0)$ such that $\Phi^* \omega_1 = \omega_2$ and $\Phi|_M = id_M$.*

The following result is the generalization of Theorem A in the quasi homogeneous case proved in [10]

Theorem 3.1.18. (i) *Let $N \subset \mathbb{C}^{2n}$ be a subset. Let ω_1, ω_2 two germs of symplectic forms on $(\mathbb{C}^{2n}, 0)$ with $\|\omega_1\|_N = \|\omega_2\|_N$. Then there exists a germ of diffeomorphism Φ of $(\mathbb{C}^{2n}, 0)$ preserving $(N, 0)$ such that $\Phi^* \omega_1 = \omega_2$.*
(ii) *Two germs of subsets $(N_1, 0), (N_2, 0)$ in the symplectic manifold $(\mathbb{C}^{2n}, \omega)$ are symplectomorphic if and only if $\|\omega\|_{N_1}$ and $\|\omega\|_{N_2}$ are diffeomorphic.*

Proof. i) First we suppose $(\omega_2 - \omega_1)(0) = 0$. The existence of Φ is proved by finding it in a family of germs of diffeomorphisms Φ_t which preserve $(N, 0)$ and bringing $\alpha_t = \omega_1 + t(\omega_2 - \omega_1)$ to ω_1 for $t \in [0, 1]$. We find this family solving the ODE

$$\frac{d\Phi_t}{dt} = X_t \circ \Phi_t, \quad \Phi_0 = id_{\mathbb{C}^{2n}}$$

where X_t is a family of germs of vector fields on $(\mathbb{C}^{2n}, 0)$ vanishing on $(N, 0)$. By the formula 2.3, the requisite $\Phi_t^* \alpha_t = \omega_1$ is equivalent to the condition

$$\mathcal{L}_{X_t} \alpha_t + \frac{\alpha_t}{dt} = 0$$

Since α_t is a closed 2-form we obtain the equation

$$d(i_{X_t}(\omega_1 + t(\omega_2 - \omega_1))) = \omega_1 - \omega_2 \quad (3.1)$$

by hypothesis $\omega_1 - \omega_2 = d\beta$, where β is a 1-form vanishing on $(N, 0)$. Therefore to solve the equation 3.1 it is sufficient to solve the equation

$$i_{X_t}(\omega_1 + t(\omega_2 - \omega_1)) = \beta. \quad (3.2)$$

Since $(\omega_2 - \omega_1)(0) = 0$, we have $\alpha_t(0) = \omega_1(0)$. Therefore α_t is a symplectic form on $(\mathbb{C}^{2n}, 0)$ by Proposition 2.1.28. Hence the equation 3.2 can be solved as a square system of linear equations parametrized

$$X_t(x).B_t(x).\cdot = \beta(x)(\cdot) = V(x).\cdot$$

where B_t is the matrix associated to α_t as in Proposition 2.1.28 and V is a vector which entries are functions which vanish on $(N, 0)$. Therefore it has a unique solution X_t which vanishes on $(N, 0)$. Since Φ_t is the flow of X_t , it preserve $(N, 0)$ see [6].

Now, we assume $(\omega_2 - \omega_1)(0) \neq 0$. We take M as in Lemma 3.1.16. So by the same lemma there exists θ a germ of a closed 2-form such that $\theta|_{TM} = (\omega_2 - \omega_1)|_{TM}$, $\|\theta\|_N = 0$ and $\theta(0) = 0$. Set $\omega_3 = \omega_1 + \theta$. Then ω_3 is a germ of a symplectic form (Lemma 2.1.28); $\omega_3|_{TM} = \omega_2|_{TM}$, $(\omega_3 - \omega_1)(0) = 0$ and $\|\omega_1\|_N = \|\omega_3\|_N$. By Darboux-Givental's Theorem (Theorem 3.1.17) there exists a germ of diffeomorphism Φ_1 of $(\mathbb{C}^{2n}, 0)$ such that $\Phi_1(x) = x$ for all $x \in (M, 0)$ and $\Phi_1^* \omega_2 = \omega_3$. By the first part, there exists a germ of diffeomorphism Φ_2 of $(\mathbb{C}^{2n}, 0)$ such that $\Phi_2(N, 0) = (N, 0)$ and $\Phi_2^* \omega_3 = \omega_1$. Therefore $\Phi := \Phi_1 \circ \Phi_2 : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^{2n}, 0)$ is a germ of diffeomorphism such that $\Phi(N, 0) = (N, 0)$ and $\Phi^* \omega_2 = \omega_1$.

- ii) Suppose $\|\omega_1\|_{N_1}$ and $\|\omega\|_{N_2}$ are diffeomorphic. Then there exists ϕ a diffeomorphism of $(\mathbb{C}^{2n}, 0)$ such that $\phi(N_1, 0) = (N_2, 0)$ and $\|\phi^* \omega\|_{N_1} = \|\omega\|_{N_1}$. Since $\phi^* \omega$ is a germ of a symplectic form, by (i) there exists a germ of diffeomorphism Φ_1 on $(\mathbb{C}^{2n}, 0)$ preserving $(N_1, 0)$ and $\Phi_1^*(\phi^* \omega) = \omega$. Therefore $\Phi = \phi \circ \Phi_1$ is a germ of diffeomorphism satisfying $\Phi^* \omega = \omega$ and $\Phi(N_1, 0) = (N_2, 0)$. The converse is trivial

□

Using Theorem 3.1.18, we can obtain the classification of subsets under symplectomorphisms on the symplectic $(\mathbb{C}^{2n}, \omega)$ which are diffeomorphic to the set N to the Classification of exact algebraic restriction $\|Sym(\mathbb{C}^{2n})\|_N$ with respect to the group $G_N(\mathbb{C}^{2n})$. Here $\|Sym(\mathbb{C}^{2n})\|_N$ denotes the subset of $\|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N$ the set of all exact algebraic restrictions to N which have as a representative one symplectic form on $(\mathbb{C}^{2n}, 0)$.

Proposition 3.1.19. *Let N be a subset of the symplectic manifold $(\mathbb{C}^{2n}, \omega)$. Suppose we find the normal forms of the classification of $\|Sym(\mathbb{C}^{2n})\|_N$ with respect to the group $G_N(\mathbb{C}^{2n})$. Then it is possible to obtain the normal forms of the symplectic classification of set to $(N, 0)$.*

Proof. Let us consider $\{\|\theta_i\|_N\}_{i \in I} \subset \|Sym(\mathbb{C}^{2n})\|_N$ a list of normal forms, where $\theta_i \in \|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N$ for all $i \in I$. By definition, for each i there exists a germ of a symplectic form ω_i on $(\mathbb{C}^{2n}, 0)$ such that $\|\theta_i\|_N = \|\omega_i\|_N$. By the Classical Darboux's Theorem (3.1.17) for each i there exists a germ of diffeomorphism $\phi_i : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^{2n}, 0)$ such that $\phi_i^* \omega_i = \omega$. We denote $(N_i, 0) = \phi_i^{-1}(N, 0)$.

Assume there exist $i, j \in I$ such that $(N_i, 0), (N_j, 0)$ are symplectomorphic for $i \neq j$. Then there exists a germ of symplectomorphism Φ of $(\mathbb{C}^{2n}, 0)$ sending $(N_j, 0)$ to $(N_i, 0)$ and therefore $\psi = \phi_i \circ \Phi \circ \phi_j^{-1}$ belongs to $G_N(\mathbb{C}^{2n})$ and $\psi^* \|\omega_i\|_N = \|\omega_j\|_N$, which is a contradiction.

Now, suppose $(N', 0)$ is diffeomorphic to $(N, 0)$. Then there exists a germ of diffeomorphism $\psi : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^{2n}, 0)$ such that $\psi(N, 0) = (N', 0)$. By hypothesis there exists $\phi \in G_N(\mathbb{C}^{2n})$ such that $\phi^* \|\psi^* \omega\|_N = \|\omega_i\|_N$ for some $i \in I$.

This implies that $\phi^* \psi^* \|\omega\|_{N_0} = \|\omega_i\|_N$ and consequently $\phi_i^* \phi^* \psi^* \|\omega\|_{N_0} = \|\omega\|_{N_i}$. Therefore $\|\omega\|_{N_0}$ and $\|\omega\|_{N_i}$ are diffeomorphic, so by Theorem 3.1.18 $(N_0, 0)$ and $(N_i, 0)$ are symplectomorphic.

Therefore $\{(N_i, 0)\}_{i \in I}$ is a list of normal forms of subsets diffeomorphic to $(N, 0)$. $\square$

Theorem 3.1.20. *A germ of a subset $(N, 0)$ of a symplectic manifold $(\mathbb{C}^{2n}, \omega)$ is contained in a germ of a isotropic submanifold if and only if $\|\omega\|_N = 0$.*

Proof. i) Suppose $(N, 0)$ is contained in a germ of a isotropic submanifold $(L, 0)$, then by definition we have $i_L^* \omega = 0$ (i_L denotes the inclusion map germ of $(L, 0)$), so by Proposition 3.1.9, $\|\omega\|_N = 0$ because ω is closed.

ii) Reciprocally, suppose $\|\omega\|_N = 0$. So we fix a submanifold M and a germ of a closed 2-form θ as in Lemma 3.1.16. Since $\theta(0) = 0$ the germ of 2-form $\omega - \theta$ is symplectic. Then $(M, 0)$ is isotropic with respect to $\omega - \theta$ ($\omega|_{TM} - \theta|_{TM} = 0$) and by Theorem 3.1.18(i) there exists a germ of diffeomorphism Φ sending $\omega - \theta$ to ω preserving

$(N, 0)$. It sends $(M, 0)$ to a germ of submanifold $(\tilde{M}, 0)$ which contains $(N, 0)$ and it is isotropic with respect to ω .

□

Example 3.1.21. Consider $n \geq 2$. Let $f(x_1, x_2) = x_1^4 + x_2^5 + x_1^2 x_2^3$ and $N = f^{-1}(0) \times \{0\} \subset \mathbb{C}^{2n}$. Consider the germ of a symplectic form

$$\omega = f dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$$

Note that the algebraic restriction $[\omega]_N = 0$. It will be shown later that $\|\omega\|_N \neq 0$. This example proves that Theorem B [10] is not true in a general case.

In [10] they described the symplectic invariants Symplectic multiplicity and the Index of isotropness in terms of the algebraic restriction, Theorems 2.3.15 and 2.3.18 respectively. A natural question arises, could we use exact algebraic restrictions to describe the same invariants? The answer to this question is YES.

Theorem 3.1.22. Let N be a variety in a Symplectic manifold $(\mathbb{C}^{2n}, \omega)$. Then the index of isotropness of N in the Symplectic manifold $(\mathbb{C}^{2n}, \omega)$ is equal to maximal order of vanishing of germs of closed 2-forms representing the exact algebraic restriction $\|\omega\|_N$.

Proof. Let denote the maximal order of vanishing of germs of closed 2-forms representing the exact algebraic restriction $\|\omega\|_N$ by t (even when it is infinite).

- i) First, we assume $\text{ind}(N) > 0$. Let $r \in \mathbb{N}$ such that there exist a germ of submanifold $(M, 0)$ containing $(N, 0)$ and $\omega|_{TM}$ has $(r-1)$ -jet zero. Let $\pi : (\mathbb{C}^{2n}, 0) \rightarrow (M, 0)$ be a germ which is the identity on $(M, 0)$ and let $\tilde{\omega} = \pi^*(\omega|_{TM})$. Then $\tilde{\omega}$ is a germ of a closed form, it has $(r-1)$ -jet zero at origin and $\tilde{\omega}|_{TM} = \omega|_{TM}$. Thus $\|\tilde{\omega}\|_N = \|\omega\|_N$ and $r \leq t$. Then $\text{ind}(N) \geq 1$ and $t \geq 1$.

Reciprocally, let $r \in \mathbb{N}$ such that there exists a germ of a closed 2-form $\tilde{\alpha}$ which has $(r-1)$ -jet zero at origin and $\|\omega\|_N = \|\tilde{\alpha}\|_N$. Since $r \geq 1$, $(\omega - \tilde{\alpha})$ is a germ of a symplectic form and $\|\omega - \tilde{\alpha}\|_N = 0$. Thus, by Theorem 3.1.20, $(N, 0)$ is contained in a germ of manifold $(M, 0)$ such that $i_M^*(\omega - \tilde{\alpha}) = 0$. Since $\tilde{\alpha}$ has $(r-1)$ -jet zero then $\omega|_{TM}$ has $(r-1)$ -jet zero. Thus $r \leq \text{ind}(N)$ and therefore $t \leq \text{ind}(N)$.

- ii) The case $\text{ind}(N) = 0$ follows trivially from the before case, because if $t \geq 1$ then $1 \leq t \leq \text{ind}(N)$. Therefore $\text{ind}(N) = t = 0$.

□

Before prove the analogous to the Theorem D, we have to define the tangent space to one orbit under the action of $G_N(\mathbb{C}^n)$. It natural to try to define the tangent space via

derivatives of curves, it is, let Φ_{tt} be a family smooth with respect to $t \in [0, 1]$ in $G_N(\mathbb{C}^n)$. We would like to define the tangent space to the orbit of $\|\alpha\|_N$ as the elements of the form

$$\lim_{t \rightarrow 0} \frac{\Phi_t^* \|\alpha\|_N - \|\alpha\|_N}{t}$$

but there is a problem with that expression and it is that we can not be sure that taken two representatives of the exact algebraic restriction $\|\alpha\|_N$, the limit above converge, it is, we do not know if given β_1, β_2 both forms vanishing on $(N, 0)$, the expression

$$\lim_{t \rightarrow 0} \frac{\Phi_t^*(\alpha + d\beta_1) - (\alpha + d\beta_2)}{t}$$

converge. Since the exact algebraic restriction is an equivalence class, we can not scape from that problem, unless we take

$$\left\| \lim_{t \rightarrow 0} \frac{\Phi_t^* \alpha - \alpha}{t} \right\|_N.$$

We have to prove that this definition does not depend on the representative of $\|\alpha\|_N$

Lemma 3.1.23. *Let N be a variety, $H = (H_1, \dots, H_l) : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^l, 0)$ a map germ whose l components are the generators of the ideal of functions vanishing on $(N, 0)$ and $\{\Phi_t\}_t \subset G_N(\mathbb{C}^n)$ a family smooth with respect to $t \in [0, 1]$, such that $\Phi_0 = id_{\mathbb{C}^n}$. If $\beta \in \Omega^k(\mathbb{C}^n)$, $k \geq 0$ vanishes on $(N, 0)$. Then*

$$\left. \frac{d(\Phi_t^* d\beta)}{dt} \right|_{t=0} = d\theta$$

where $\theta \in \Omega^k(\mathbb{C}^n)$ vanishes on $(N, 0)$.

Proof. In fact, let $\{X_t\}_t \in \Theta_n$ be the smooth family satisfying

$$\frac{d\Phi_t}{dt} = X_t \circ \Phi_t,$$

Then by equation 2.3,

$$\left. \frac{d(\Phi_t^* d\beta)}{dt} \right|_{t=0} = \Phi_t^* \mathcal{L}_{X_t} d\beta \Big|_{t=0} = \mathcal{L}_{X_0} d\beta = di_{X_0} d\beta.$$

Since $\Phi_t \in G_N(\mathbb{C}^n)$, we have $H \circ \Phi_t(x) = 0$ for all $x \in (N, 0)$, therefore $dH(x).X_0(x) = 0$ for all $x \in (N, 0)$. Thus for $i = 1, \dots, l$ we have

$$\sum_{j=0}^n \frac{\partial H_i}{\partial x_j} u_j \in \langle H_1, \dots, H_l \rangle,$$

where $X_0 = \sum_{j=1}^n u_j \frac{\partial}{\partial x_j}$. Let us write $\beta = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1, i_2, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$ for some $f_{i_1, i_2, \dots, i_k} \in \mathcal{O}_n$. Since β vanishes on $(N, 0)$, we have $f_{i_1, i_2, \dots, i_k}$ vanishes on $(N, 0)$ and therefore $f_{i_1, i_2, \dots, i_k} \in \langle H_1, \dots, H_l \rangle$. Then there exist $\alpha_1, \dots, \alpha_l \in \Omega^k(\mathbb{C}^n)$ such that

$$\beta = \sum_{i=1}^l H_i \alpha_i.$$

thus

$$d\beta = \sum_{i=1}^l d(H_i) \wedge \alpha_i + \sum_{i=1}^l H_i d(\alpha_i)$$

which implies

$$i_{X_0} d\beta = \sum_{i=1}^l (i_{X_0} d(H_i)) \alpha_i - \sum_{i=1}^l d(H_i) \wedge (i_{X_0} \alpha_i) + \sum_{i=1}^l H_i i_{X_0} d(\alpha_i).$$

Since

$$i_{X_0} d(H_i) = i_{X_0} \sum_{j=1}^n \frac{\partial H_i}{\partial x_j} dx_j = \sum_{j=0}^n \frac{\partial H_i}{\partial x_j} u_j$$

we have that there exists $\tilde{\alpha}_1, \dots, \tilde{\alpha}_l \in \Omega^k(\mathbb{C}^n)$ such that

$$i_{X_0} d\beta = \sum_{i=1}^l H_i \tilde{\alpha}_i - \sum_{i=1}^l d(H_i) \wedge (i_{X_0} \alpha_i),$$

thus

$$d(i_{X_0} d\beta) = d\left(\sum_{i=1}^l H_i \tilde{\alpha}_i\right) - d\left(\sum_{i=1}^l d(H_i) \wedge (i_{X_0} \alpha_i)\right).$$

From the equality

$$d\left(\sum_{i=1}^l H_i \wedge (i_{X_0} \alpha_i)\right) = \sum_{i=1}^l d(H_i) \wedge (i_{X_0} \alpha_i) + \sum_{i=1}^l H_i \wedge d(i_{X_0} \alpha_i)$$

we obtain

$$d\left(\sum_{i=1}^l d(H_i) \wedge (i_{X_0} \alpha_i)\right) = -d\left(\sum_{i=1}^l H_i \wedge d(i_{X_0} \alpha_i)\right).$$

Therefore the lemma is proved. $\square$

Definition 3.1.24. Let $N \subset \mathbb{C}^n$ be a variety and $\|\alpha\|_N \in \overline{\Omega}^k(\mathbb{C}^n)$, $k \geq 1$. We define the tangent space to the orbit of $\|\alpha\|_N$ under the action of $G_N(\mathbb{C}^n)$, which is denoted by $T(\|\alpha\|_N)$, as

$$T(\|\alpha\|_N) = \left\{ \left\| \lim_{t \rightarrow 0} \frac{\Phi_t^* \alpha - \alpha}{t} \right\|_N \mid \Phi_t \in G_N(\mathbb{C}^n), t \in [0, 1], \Phi_0 = id_{\mathbb{C}^n} \right\}.$$

Affirmation: The definition of $T(\|\alpha\|_N)$ does not depend on the representative of $\|\alpha\|_N$. Let $\{\Phi_t\}_t$ be a family as in Lemma 3.1.23. Let $\alpha, \theta \in \overline{\Omega}^k(\mathbb{C}^n)$ such that there exists $\beta \in \Omega^{k-1}(\mathbb{C}^n)$ vanishing on $(N, 0)$ and

$$\alpha - \theta = d\beta.$$

then

$$d\left(\frac{d(\Phi_t^* \beta)}{dt} \Big|_{t=0}\right) = \lim_{t \rightarrow 0} \frac{\Phi_t^*(\alpha - \theta) - (\alpha - \theta)}{t} = \lim_{t \rightarrow 0} \frac{\Phi_t^* \alpha - \alpha}{t} - \lim_{t \rightarrow 0} \frac{\Phi_t^* \theta - \theta}{t}$$

Therefore the affirmation follows by the Lemma 3.1.23.

An immediate question arises, Is that tangent space a vector space? To answer this question we should describe this tangent space.

Proposition 3.1.25. *Let $N \subset \mathbb{C}^n$ be a subset such that $H : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^l, 0)$ is a map germ whose l components are the generators of the ideal of functions vanishing on $(N, 0)$ and $\alpha \in \overline{\Omega}^k(\mathbb{C}^n)$. Then*

$$T(\|\alpha\|_N) = \{ \|\mathcal{L}_X \alpha\|_N \mid dH(x) \cdot X(x) = 0, \forall x \in (N, 0), X \in \Theta_n \text{ such that } X(0) = 0 \}$$

Proof. By definition

$$T(\|\alpha\|_N) = \left\{ \left\| \lim_{t \rightarrow 0} \frac{\Phi_t^* \alpha - \alpha}{t} \right\|_N \mid \Phi_t \in G_N(\mathbb{C}^n), \Phi_0 = id_{\mathbb{C}^n} \right\}.$$

We have

$$\left\| \lim_{t \rightarrow 0} \frac{\Phi_t^* \alpha - \alpha}{t} \right\|_N = \left\| \frac{d}{dt} \Phi_t^* \alpha \Big|_{t=0} \right\|_N = \|\mathcal{L}_{X_0} \alpha\|_N$$

where $X_t \in \Theta_n$ such that, $\frac{d\Phi_t}{dt} = X_t \circ \Phi_t$. From this last equation, we have $X_0(0) = 0$, because $\Phi_t(0) = 0$ for all $t \in [0, 1]$.

Since $H(\Phi_t(x)) = 0$ for all $x \in (N, 0)$, we obtain $\frac{dH \circ \Phi_t}{dt}(x) = 0$ for all $x \in (N, 0)$ and thus $dH(x) \cdot X_0(x) = 0$ for all $x \in (N, 0)$.

Reciprocally, if $X \in \Theta_n$ such that $dH(x) \cdot X(x) = 0$ for all $x \in (N, 0)$ and $X(0) = 0$. Let $\Phi_t \in \mathcal{O}_n$ be the flow associated to X which preserve $(N, 0)$. Thus $\left\| \lim_{t \rightarrow 0} \frac{\Phi_t^* \alpha - \alpha}{t} \right\|_N = \|\mathcal{L}_X \alpha\|_N$ and $\|\mathcal{L}_X \alpha\|_N \in T(\|\alpha\|_N)$. This proves the proposition. $\square$

Corollary 3.1.26. *Under the same hypothesis as in the proposition above. The tangent space $T(\|\alpha\|_N)$ is a Linear subspace of $\|\overline{\Omega}^k(\mathbb{C}^n)\|_N$.*

Proof. The fact that $T(\|\alpha\|_N)$ is a linear space is a consequence of the before proposition. The fact that it is contained in $\|\overline{\Omega}^k(\mathbb{C}^n)\|_N$ follows from $\mathcal{L}_X \alpha = di_X \alpha$ for any $X \in \Theta_n$ and $\alpha \in \overline{\Omega}^k(\mathbb{C}^n)$. $\square$

Proposition 3.1.27. *Let $N \subset \mathbb{C}^{2n}$ a subset and $H : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^j, 0)$ a map whose l components are the generators of the ideal vanishing on $(N, 0)$. Let $\alpha, \beta \in \overline{\Omega}^2(\mathbb{C}^{2n})$ such that there exists $\Phi \in G_N(\mathbb{C}^{2n})$ and $\Phi^* \alpha = \beta$. Then $T(\|\alpha\|_N)$ and $T(\|\beta\|_N)$ are isomorphic.*

Proof. Let us denote by ω_0 the symplectic form $dx_1 \wedge dx_2 + \dots + dx_{2n-1} \wedge dx_{2n}$. Let $\theta \in \Omega^1(\mathbb{C}^{2n})$ vanishing on $(N, 0)$. Then by the same arguments used to solve equation 3.2, there exists an unique vector field X_θ vanishing on $(N, 0)$ (which implies $dH(x) \cdot X_\theta(x) = 0$ for all $x \in (N, 0)$) such that $i_{X_\theta} \omega_0 = \theta$. Thus $\mathcal{L}_{X_\theta} \omega_0 = d\theta$.

Let $\gamma \in \Omega^1(\mathbb{C}^{2n})$ vanishing on $(N, 0)$ such that $\Phi^* \alpha = \beta + d\gamma$. Let $X \in \Theta_{2n}$ such that $dH(x) \cdot X(x) = 0$ for all $x \in (N, 0)$ and $X(0) = 0$. Let $\{\Phi_t\}_t \in G_N(\mathbb{C}^{2n})$ be the flow

associate to X . Let $\{X_t\}_t \subset \Theta_{2n}$ be a smooth families with respect to t such that

$$\frac{d(\Phi \circ \Phi_t)}{dt} = X_t \circ (\Phi \circ \Phi_t),$$

thus

$$X_t \circ (\Phi \circ \Phi_t) = \frac{d(\Phi \circ \Phi_t)}{dt} = d\Phi \cdot \frac{d\Phi_t}{dt} = d\Phi \cdot (X \circ \Phi_t) \Rightarrow X_0 \circ \Phi(x) = d\Phi(x) \cdot X(x).$$

In addition to that, since $\Phi_t \in G_N(\mathbb{C}^{2n})$ we have $H \circ \Phi_t(X) = 0$ for all $x \in (N, 0)$, hence $dH \circ \Phi(x) \cdot X_0 \circ \Phi(x) = 0$ for all $x \in (N, 0)$ and thus $dH(x) \cdot X_0(x) = 0$ for all $x \in (N, 0)$. It is obvious that

$$\Phi^* \mathcal{L}_{d\Phi \cdot X} \alpha = \Phi^* \mathcal{L}_{X_0 \circ \Phi} \alpha = \left. \frac{d((\Phi \circ \Phi_t)^* \alpha)}{dt} \right|_{t=0} = \left. \frac{d(\Phi_t^* \Phi^* \alpha)}{dt} \right|_{t=0} = \left. \frac{d(\Phi_t^* (\beta + \gamma))}{dt} \right|_{t=0} = \mathcal{L}_X (\beta + d\gamma).$$

Similarly, it is possible to prove that

$$\Phi^* \mathcal{L}_{d\Phi \cdot X_\theta} \omega_0 = \mathcal{L}_{X_\theta} \Phi^* \omega_0$$

since X_θ vanishes on $(N, 0)$, $\mathcal{L}_{X_\theta} \Phi^* \omega$ is the derivative of a 1-form on $(\mathbb{C}^{2n}, 0)$ which vanishes on $(N, 0)$.

Thus, if $\mathcal{L}_X \alpha + d\theta \in \overline{\Omega}^2(\mathbb{C}^{2n})$ is a representative of $\|\mathcal{L}_X \alpha\|_N \in T(\|\alpha\|_N)$, we have that

$$\|\Phi^* \mathcal{L}_{d\Phi \cdot X} \alpha + \Phi^* \mathcal{L}_{d\Phi \cdot X_\theta} \omega_0\|_N = \|\mathcal{L}_X \beta\|_N.$$

Therefore the map

$$\Psi_1 : T(\|\alpha\|_N) \rightarrow T(\|\beta\|_N), \quad \Psi(\|\mathcal{L}_X \alpha\|_N) = \|\mathcal{L}_X \beta\|_N$$

is a well defined linear map. Analogously we define the linear map

$$\Psi_2 : T(\|\beta\|_N) \rightarrow T(\|\alpha\|_N), \quad \Psi_2(\|\mathcal{L}_X \beta\|_N) = \|\mathcal{L}_X \alpha\|_N$$

which satisfy $\Psi_2 \circ \Psi_1 = id_{T(\|\alpha\|_N)}$ and $\Psi_1 \circ \Psi_2 = id_{T(\|\beta\|_N)}$. Therefore these tangent spaces are isomorphic. $\square$

Theorem 3.1.28. *Let N be a variety in a symplectic manifold $(\mathbb{C}^{2n}, \omega)$. The symplectic multiplicity of N in $(\mathbb{C}^{2n}, \omega)$ is the codimension of the orbit of the exact algebraic restriction $\|\omega\|_N$ with respect to the group $G_N(\mathbb{C}^{2n})$ in the space $\|\overline{\Omega}^2(\mathbb{C}^{2n})\|_N$.*

In order to prove the above result we need some results.

Notation: We denote by $\mathbb{C}^{k \times k}$ to the vector space of matrices of order $k \times k$ with entries belonging to $\mathbb{C}$ and $\mathcal{O}_n^{k \times k}$ to the vector space of matrices of order $k \times k$ with entries belonging to $\mathcal{O}_n$

Proposition 3.1.29. *Let $H(x) = H(x_1, \dots, x_{2n}) : (\mathbb{C}^{2n}, 0) \longrightarrow (\mathbb{C}^k, 0)$ be the map germ whose k components are the generators of the ideal of functions vanishing on $(N, 0)$. Then the tangent space to H under the $\mathcal{K}$ -equivalence, which is denoted by $T(H)_{\mathcal{K}}$, is given by*

$$\{A(x) \cdot H(x) + dH(x) \cdot X(x) \mid A \in \mathcal{O}_{2n}^{k \times k} \text{ and } X \in \Theta_{2n}, X(0) = 0\}$$

and the tangent space to H under the $\mathcal{K}$ -symplectic equivalence denoted by $T(H)_{\mathcal{K}, \text{Symplectic}}$ is given by

$$\{A(x) \cdot H(x) + dH(x) \cdot X(x) \mid A \in \mathcal{O}_{2n}^{k \times k} \text{ and } X \in \Theta_{2n} \text{ is a Hamiltonian vector field, } X(0) = 0\}$$

Proof. The proof of the first part can be found in [4]. Let us prove the second part. Considering the elements of the tangent space as the derivatives of smooth paths

$$T(H)_{\mathcal{K}, \text{Symplectic}} = \left\{ \lim_{t \rightarrow 0} \frac{M_t(x) \cdot H(\Phi_t(x)) - H(x)}{t} \mid \begin{array}{l} M_0(x) = id_{GL(k)}, \quad M_t(x) \in GL(k) \\ \Phi_0 = id_{\mathbb{C}^{2n}}, \quad \Phi_t \text{ is a symplectomorphism} \end{array} \right\}.$$

From the equality

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{M_t(x) \cdot H(\Phi_t(x)) - H(x)}{t} &= \frac{dM_t}{dt}(x) \Big|_{t=0} \cdot H(\Phi_0(x)) + M_0(x) \cdot dH_0(x) \cdot \frac{d\Phi_t}{dt}(x) \Big|_{t=0} \\ &= \frac{dM_t}{dt}(x) \Big|_{t=0} \cdot H(x) + dH_0(x) \cdot \frac{d\Phi_t}{dt}(x) \Big|_{t=0} \end{aligned}$$

We have $\frac{dM_t}{dt}(x) \Big|_{t=0} \in \mathbb{C}^{k \times k}$ for all $x \in (\mathbb{C}^{2n}, 0)$, thus $\frac{dM_t}{dt} \Big|_{t=0} \in \mathcal{O}_{2n}^{k \times k}$. Let us consider X_t be the germ of a vector field associated to Φ_t such that

$$\frac{d\Phi_t}{dt} = X_t \circ \Phi_t \implies \frac{d\Phi_t}{dt} \Big|_{t=0} = X_0$$

as $\omega = \Phi_t^* \omega$ for all t using equation 2.3,

$$0 = \Phi_t^* \mathcal{L}_{X_t} \omega \implies 0 = \mathcal{L}_{X_0} \omega = di_{X_0} \omega.$$

Therefore, by Poincaré's Lemma (Theorem 2.1.18), X_0 is a germ of a Hamiltonian vector field.

Reciprocally, let X be a germ of a Hamiltonian vector field and $A \in \mathcal{O}_{2n}^{k \times k}$. Let us denote by Φ_t the germ of the flow associate to X . Since X is Hamiltonian, $\mathcal{L}_X \omega = 0$, which implies $\frac{d}{dt} \Phi_t^* \omega = 0$ for all $t \in [0, 1]$ and therefore $\Phi_t^* \omega = \omega$. Finally,

$$\lim_{t \rightarrow 0} \frac{(id_{GL(k)} + tA)(x) \cdot H(\Phi_t(x)) - H(x)}{t} = A(x) \cdot H(x) + dH(x) \cdot X(x),$$

thus $A(x) \cdot H(x) + dH(x) \cdot X(x) \in T(H)_{\mathcal{K}, \text{Symplectic}}$. This proves the equality of sets. $\square$

Proof. of Theorem 3.1.28

By hypothesis $H : (\mathbb{C}^{2n}, 0) \longrightarrow (\mathbb{C}^k, 0)$ is a map germ whose k components are the generators of the ideal of functions vanishing on $(N, 0)$. We define the application

$$\Psi : \frac{T(H)_{\mathcal{H}}}{T(H)_{\mathcal{H}, \text{Symp}}} \longrightarrow \frac{\|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N}{T(\|\omega\|_N)}, \quad \Psi(\overline{A \cdot H + dH \cdot X}) = \overline{\mathcal{L}_X \omega}_N$$

where $\overline{A \cdot H + dH \cdot X} \in \frac{T(H)_{\mathcal{H}}}{T(H)_{\mathcal{H}, \text{Symp}}}$ and $\overline{\mathcal{L}_X \omega}_N \in \frac{\|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N}{T(\|\omega\|_N)}$.

i) Ψ is well defined: Consider $A_1 \cdot H + dH \cdot X_1, A_2 \cdot H + dH \cdot X_2 \in T(H)_{\mathcal{H}}$ such that

$$(A_1 - A_2) \cdot H + dH \cdot (X_1 - X_2) \in T(H)_{\mathcal{H}, \text{Symp}}$$

so, there exist a matrix of germ of functions $A \in \mathcal{O}_{2n}^{k \times k}$ and a germ of a Hamiltonian vector field $X \in \Theta_{2n}$ such that

$$dH \cdot (X_1 - X_2 - X) = (A - A_1 + A_2) \cdot H$$

thus $dH(x) \cdot (X_1 - X_2 - X)(x) = 0$ for all $x \in (N, 0)$ and

$$\|\mathcal{L}_{X_1} \omega - \mathcal{L}_{X_2} \omega - \mathcal{L}_X \omega\|_N = \|\mathcal{L}_{(X_1 - X_2 - X)} \omega\|_N \in T(\|\omega\|_N)$$

the fact that Ψ is well defined follows from the fact that $\mathcal{L}_X \omega = 0$.

ii) It follows from its the definition that Ψ is linear.

iii) Ψ is surjective: Let $\beta \in \bar{\Omega}^2(\mathbb{C}^{2n})$ and $\alpha \in \Omega^1(\mathbb{C}^{2n})$ such that $\beta = d\alpha = d(\alpha - \alpha(0))$. Let $X \in \Theta_{2n}$ such that

$$i_X \omega = \alpha - \alpha(0)$$

thus $\mathcal{L}_X \omega = \beta$. Using the associate matrix to ω (Proposition 2.1.28) it is possible to prove that $X(0) = 0$. Therefore $\Psi(\overline{dH \cdot X}) = \overline{\mathcal{L}_X \omega}_N = \overline{\|\beta\|_N}$. Hence Ψ is surjective.

iv) Ψ is injective: Let $\overline{A \cdot H + dH \cdot X} \in \text{Ker}(\Psi)$, then there exists $X_1 \in \Theta_{2n}$ such that $dH(x) \cdot X_1(x) = 0$ for all $x \in (N, 0)$, $X_1(0) = 0$ and

$$\|\mathcal{L}_X \omega\|_N = \|\mathcal{L}_{X_1} \omega\|_N.$$

Hence there exists $\gamma \in \Omega^1(\mathbb{C}^{2n})$ vanishing on $(N, 0)$ such that

$$\mathcal{L}_{X-X_1} \omega = d\gamma \iff i_{X-X_1} \omega = \gamma + dG$$

for some function germ $G \in \mathcal{O}_{2n}$. Let $X_2 \in \Theta_{2n}$ such that $i_{X_2} \omega = \gamma$. Thus $i_{X-X_1-X_2} \omega = dG$ and therefore $X' = X - X_1 - X_2 \in \Theta_{2n}$ is a germ of a Hamiltonian

vector field. Since γ vanishes on $(N, 0)$, X_2 also vanishes on $(N, 0)$ and there exists a $B \in \mathcal{O}_{2n}^{k \times k}$ such that

$$dH \cdot X_1 + dH \cdot X_2 = B \cdot H$$

because the components of H generate the ideal of functions vanishing on N . So

$$\begin{aligned} \overline{A \cdot H + dH \cdot X} &= \overline{A \cdot H + dH \cdot X' + dH \cdot X_1 + dH \cdot X_2} \\ &= \overline{(A + B) \cdot H + dH \cdot X'} = 0 \end{aligned}$$

and therefore Ψ is injective.

Consequently

$$\mu_{\text{Symp}}(N) = \dim \frac{\|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N}{T(\|\omega\|_N)}$$

□

3.2 Singular planar curves

A singular planar curve in $\mathbb{C}^{2n}$ is a curve given in suitable coordinates by

$$N = \{f(x_1, x_2) = x_{\geq 3} = 0\} \subset \mathbb{C}^{2n}$$

where $f = f(x_1, x_2)$ is a germ of function such that:

- (1) Belongs to $\mathcal{M}_n$,
- (2) Has an isolated singularity at the origin,
- (3) Has the properties of zeros.

By Proposition 3.1.19, we just need to classify $\|\text{Symp}(\mathbb{C}^{2n})\|_N$ with respect to the group $G_N(\mathbb{C}^{2n})$, it will imply to work with the tangent space $T(\|\alpha\|_N)$ of $\|\alpha\|_N \in \|\text{Symp}(\mathbb{C}^{2n})\|_N$. Since it is a sub space of $\|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N$ we need condition on N such that it has finite dimension. First, we see the case $n = 2$.

Lemma 3.2.1. *Let $f \in M_n$ having the property of zeros. Let $N = f^{-1}(0)$ and $\omega \in \bar{\Omega}^2(\mathbb{C}^n)$, then $\|\omega\|_N = 0$ if and only if $\omega = d(f\alpha)$ for some $\alpha \in \Omega^1(\mathbb{C}^n)$.*

Proof. The proof follows from the definition of exact algebraic restriction and Remark 3.1.2. □

Proposition 3.2.2. *Let μ be the Milnor number of f . Then $\dim_{\mathbb{C}} \|\Omega^2(\mathbb{C}^2)\|_N = \mu$.*

Proof. By the previous lemma to $n = 2$, we have the isomorphism

$$\phi_n : \|\Omega^2(\mathbb{C}^n)\|_N \rightarrow \frac{\Omega^2(\mathbb{C}^n)}{d(f \cdot \Omega^1(\mathbb{C}^n))}; \quad \phi(\|\omega\|_N) = \bar{\omega}$$

where $\bar{\omega}$ represents the class of ω in $\frac{\Omega^2(\mathbb{C}^2)}{d(f.\Omega^1(\mathbb{C}^2))}$. Therefore the proposition follows from the isomorphism of vector spaces

$$\frac{\Omega^2(\mathbb{C}^2)}{d(f.\Omega^1(\mathbb{C}^2))} / \frac{f.\Omega^2(\mathbb{C}^2) + d(f.\Omega^1(\mathbb{C}^2))}{d(f.\Omega^1(\mathbb{C}^2))} \cong \frac{\Omega^2(\mathbb{C}^2)}{f.\Omega^2(\mathbb{C}^2) + d(f.\Omega^1(\mathbb{C}^2))}$$

and the facts that

$$\dim_{\mathbb{C}} \frac{\Omega^2(\mathbb{C}^2)}{f.\Omega^2(\mathbb{C}^2) + d(f.\Omega^1(\mathbb{C}^2))} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \rangle} = \tau$$

and $\dim_{\mathbb{C}} \frac{f.\Omega^2(\mathbb{C}^2) + d(f.\Omega^1(\mathbb{C}^2))}{d(f.\Omega^1(\mathbb{C}^2))} = \mu - \tau$ by Theorem 2.3.21. $\square$

The general case is proved using a result shown in [14]. The next theorem is the analogous to Proposition 2.16 in [10].

Theorem 3.2.3. *Let $N \subset \mathbb{C}^n$ be an m -dimensional isolated complete intersection singularity with finite Milnor number $\mu(N)$. Then*

$$\dim_{\mathbb{C}} \|\bar{\Omega}^2(\mathbb{C}^n)\|_N = \mu(N).$$

Proof. In fact, since N is an isolated complete intersection singularity of dimension m by [14].

$$\dim_{\mathbb{C}} \frac{[\Omega^1(\mathbb{C}^n)]_N}{d([\Omega^0(\mathbb{C}^n)]_N)} = \mu(N),$$

where $[\]_N$ denotes the algebraic restriction to N .

Let $\|d\alpha\|_N, \|d\beta\|_N \in \|\bar{\Omega}^2(\mathbb{C}^n)\|_N$ be two exact algebraic restrictions with $\|d\alpha\|_N = \|d\beta\|_N$. By definition there exists $\theta \in \Omega^1(\mathbb{C}^n)$ vanishing on $(N, 0)$ such that $d\alpha - d\beta = d\theta$. Hence there exists $G \in \mathcal{O}_n$ such that $\alpha - \beta - \theta = dG$, so $[\alpha]_N - [\beta]_N = [dG]_N = d[G]_N$. Therefore $[\alpha]_N = [\beta]_N$ in $[\Omega^1(\mathbb{C}^n)]_N / d([\Omega^0(\mathbb{C}^n)]_N)$ and we have a well defined linear map

$$\|\bar{\Omega}^2(\mathbb{C}^n)\|_N \longrightarrow \frac{[\Omega^1(\mathbb{C}^n)]_N}{d([\Omega^0(\mathbb{C}^n)]_N)}, \quad \|d\alpha\|_N \mapsto [\alpha]_N$$

This map is surjective. Let $\|d\alpha\|_N$ be an exact algebraic restriction such that $[\alpha]_N = 0$, then there exist $G \in \mathcal{O}_n$ satisfying $[\alpha]_N = d[G]_N$. Hence there exist $\gamma \in \Omega^1(\mathbb{C}^n)$ vanishing on $(N, 0)$ such that

$$\alpha - dG = \gamma,$$

thus $\|d\alpha\|_N = 0$ and this map is injective, so the result follows. $\square$

Consider N the planar curve given by $\{f(x_1, x_2) = x_{\geq 3} = 0\} \subset \mathbb{C}^{2n}$. Then

Theorem 3.2.4. *Let $N' = f^{-1}(0)$, as $\dim_{\mathbb{C}} \|\Omega^2(\mathbb{C}^2)\|_{N'} = \mu$ where μ is the Milnor number of f , then*

- i) There exist functions $f_1, \dots, f_\mu \in \mathcal{O}_2$ such that $f_1(0) \neq 0$, $f_i(0) = 0$ for all $i \geq 2$ and $\{\|f_1 dx_1 \wedge dx_2\|_{N'}, \dots, \|f_\mu dx_1 \wedge dx_2\|_{N'}\}$ is a basis for $\|\Omega^2(\mathbb{C}^2)\|_{N'}$.
- ii) The subset $\{\|f_1 dx_1 \wedge dx_2\|_N, \dots, \|f_\mu dx_1 \wedge dx_2\|_N\} \subset \|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N$ is a basis, where the functions $f_1, \dots, f_\mu$ are given in the item i).
- iii) If $n \geq 2$ then $\|Sym(\mathbb{C}^{2n})\|_N = \|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N$. The manifold $\|Sym(\mathbb{C}^2)\|_{N'}$ consist of exact algebraic restrictions of the form $\{\sum_1^\mu c_i \|f_i dx_1 \wedge dx_2\|_{N'}, c_1 \neq 0\}$.

Proof. i) Since f has an isolated singularity at the origin, its Milnor number is finite. We know by [10], that the space of algebraic restrictions to N' has dimension τ (Tjurina number) and there exists a basis for this space given by $\{[f_1 dx_1 \wedge dx_2]_{N'}, \dots, [f_\tau dx_1 \wedge dx_2]_{N'}\}$, where $f_1(0) \neq 0$ and for $i \geq 2$, $f_i(0) = 0$. This follows from the fact that the map

$$[\Omega^2(\mathbb{C}^2)]_{N'} \rightarrow \frac{\mathcal{O}_2}{\langle f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \rangle}, \quad [g(x_1, x_2) dx_1 \wedge dx_2]_{N'} \rightarrow \bar{g}$$

is an isomorphism of vector spaces.

Suppose that there exist constants c_i for $1 \leq i \leq \tau$ such that

$$c_1 \|f_1 dx_1 \wedge dx_2\|_{N'} + c_2 \|f_1 dx_1 \wedge dx_2\|_{N'} + \dots + c_\tau \|f_\tau dx_1 \wedge dx_2\|_{N'} = 0$$

this implies

$$c_1 [f_1 dx_1 \wedge dx_2]_{N'} + c_2 [f_1 dx_1 \wedge dx_2]_{N'} + \dots + c_\tau [f_\tau dx_1 \wedge dx_2]_{N'} = 0$$

because every zero exact algebraic restriction is a zero algebraic restriction. Therefore $c_i = 0$ for $1 \leq i \leq \tau$. The statement follows from Theorem 3.2.3 and basic results of linear algebra.

- ii) Note that it is enough to prove that $\{\|f_1 dx_1 \wedge dx_2\|_N, \dots, \|f_\mu dx_1 \wedge dx_2\|_N\}$ generate $\|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N$.

Let $\alpha \in \bar{\Omega}^2(\mathbb{C}^{2n})$ and $\beta \in \Omega^1(\mathbb{C}^{2n})$ such that $d\beta = \alpha$. Write $\beta = g_1 dx_1 + g_2 dx_2 + \dots + g_{2n} dx_{2n}$, where $g_i = g_i(x_1, x_2, \dots, x_{2n}) \in \mathcal{O}_{2n}$. By Hadamard's Lemma (Lemma 2.2.12) we can write

$$g_i = g_i(x_1, x_2, 0, \dots, 0) + G_i(x_1, x_2, \dots, x_{2n}), \quad i = 1, 2$$

for some $G_i \in \langle x_3, \dots, x_{2n} \rangle$. So β has the form

$$\beta = g'_1(x_1, x_2) dx_1 + g'_2(x_1, x_2) dx_2 + \theta$$

where $g'_i(x_1, x_2) = g_i(x_1, x_2, 0, \dots, 0)$, $i = 1, 2$, and $\theta \in \Omega^1(\mathbb{C}^{2n})$ vanishes on $(N, 0)$. Hence

$$\alpha = \left(\frac{\partial g'_2}{\partial x_1} - \frac{\partial g'_1}{\partial x_2} \right) dx_1 \wedge dx_2 + d\theta \quad (3.3)$$

Now, by the item i), there exist constants c_i , $1 \leq i \leq \mu$ and a germ of 1-form γ vanishing on $(N', 0)$ such that

$$\left(\frac{\partial g'_2}{\partial x_1} - \frac{\partial g'_1}{\partial x_2} \right) dx_1 \wedge dx_2 = c_1 f_1 dx_1 \wedge dx_2 + \cdots + c_\mu f_\mu dx_1 \wedge dx_2 + d\gamma.$$

Therefore,

$$\alpha = c_1 f_1 dx_1 \wedge dx_2 + \cdots + c_\mu f_\mu dx_1 \wedge dx_2 + d(\gamma + \theta)$$

here we are considering the 2-forms on $(\mathbb{C}^2, 0)$ as 2-forms on $(\mathbb{C}^{2n}, 0)$. Since γ considered as a 2-form on $(\mathbb{C}^2, 0)$ vanishes on $(N', 0)$, γ considered as a 2-form on $(\mathbb{C}^{2n}, 0)$ vanishes on $(N, 0)$. Therefore

$$\|\alpha\|_N = c_1 \|f_1 dx_1 \wedge dx_2\|_N + \cdots + c_\mu \|f_\mu dx_1 \wedge dx_2\|_N$$

and the statement is proved.

- iii) To the first part ($n \geq 2$), it is enough to prove that $\|\overline{\Omega}^2(\mathbb{C}^{2n})\|_N \subset \|Sym(\mathbb{C}^{2n})\|_N$. Let $\alpha \in \overline{\Omega}^2(\mathbb{C}^{2n})$. Similarly to the item ii) we can write

$$\alpha = g(x_1, x_2) dx_1 \wedge dx_2 + d\theta$$

where $\theta \in \Omega^1(\mathbb{C}^{2n})$ vanishes on $(N, 0)$. We prove the inclusion of sets in two cases. First, suppose $g(0) \neq 0$, then

$$\|\alpha\|_N = \|g(x_1, x_2) dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}\|_N$$

since $x_3 dx_4 + \cdots + x_{2n-1} dx_{2n}$ vanishes on $(N, 0)$. By Proposition 2.1.28, $g(x_1, x_2) dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$ is a symplectic on $(\mathbb{C}^{2n}, 0)$, for this reason $\|\alpha\|_N \in \|Sym(\mathbb{C}^{2n})\|_N$.

Now, consider $g(0) = 0$, then

$$\|\alpha\|_N = \|g(x_1, x_2) dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}\|_N$$

since $x_1 dx_3 + x_2 dx_4 + x_5 dx_6 + \cdots + x_{2n-1} dx_{2n}$ vanishes on $(N, 0)$. By Proposition 2.1.28, $g(x_1, x_2) dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$ is a symplectic form, hence $\|\alpha\|_N \in \|Sym(\mathbb{C}^{2n})\|_N$. Therefore $\|\overline{\Omega}^2(\mathbb{C}^{2n})\|_N \subset \|Sym(\mathbb{C}^{2n})\|_N$.

Let suppose $n = 1$. Let α be a germ of a symplectic form on $(\mathbb{C}^2, 0)$, then by item i) and Lemma 3.2.1 there exists a germ of 1-form β such that

$$\alpha = \sum_1^\mu c_i f_i dx_1 \wedge dx_2 + d(f\beta)$$

From the facts that f is singular at the origin, $f_1(0) \neq 0$, $f_i(0) = 0$ for $2 \leq i \leq \mu$ and α is symplectic, we obtain $c_1 \neq 0$.

Reciprocally, if $\theta = \sum_1^\mu c_i f_i dx_1 \wedge dx_2$ with $c_1 \neq 0$, then it is a symplectic form on $(\mathbb{C}^2, 0)$ by Proposition 2.1.28. So $\{\sum_1^\mu c_i \|f_i dx_1 \wedge dx_2\|_N, c_1 \neq 0\} \subset \|Sym(\mathbb{C}^{2n})\|_N$ and therefore the statement is proved.

□

3.3 Vector fields tangent to N'

Since our fixed subset N is included in $(\mathbb{C}^2 \times \{0\}, 0)$, the classification of exact algebraic restrictions in $\| \text{Symp}(\mathbb{C}^{2n}) \|_N$ under the action of the group $G_N(\mathbb{C}^{2n})$ will involve just the classification of $\| \Omega^2(\mathbb{C}^2) \|_{N'}$ under the action of the group $G_{N'}(\mathbb{C}^2)$. As the group $G_{N'}(\mathbb{C}^2)$ consists of germs of diffeomorphisms that preserve $(N', 0)$ we need to know the vector fields tangent to N' . It is well known that the set of vector field tangent to N' is an $\mathcal{O}_2$ submodule of Θ_2 denoted by $\text{Derlog}(N')$. Since $N' = \{f = 0\}$ we denote $\text{Derlog}(N') = \text{Derlog}(f)$.

Definition 3.3.1. *The module of vector fields tangent to N' is*

$$\text{Derlog}(f) = \left\{ X = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} \in \Theta_2 \mid X(f) = v_1 \frac{\partial f}{\partial x_1} + v_2 \frac{\partial f}{\partial x_2} \in \langle f \rangle \right\}$$

Saito proved that in the case of an 1-dimensional isolated singularity, there exists an $\mathcal{O}_2$ -basis for $\text{Derlog}(f)$ [23].

We prove a relation between the $\text{Derlog}(f)$, Milnor and Tjurina numbers of f .

Proposition 3.3.2. *Let $g \in \mathcal{M}_2$ be a germ of an isolated singularity. Then*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_2}{Q} = \mu(g) - \tau(g)$$

where Q denotes the ideal $\left\{ h = \frac{X(g)}{g} \mid X \in \text{Derlog}(g) \right\}$.

Proof. It is clear that $h \in Q$ if and only if $h \cdot g \in \langle \nabla g \rangle$. Therefore Q is the kernel of the map $\mathcal{O}_2 \rightarrow \frac{\langle g, \nabla g \rangle}{\langle \nabla g \rangle}$, $h \rightarrow \overline{g \cdot h}$. Thus the map

$$g \cdot : \frac{\mathcal{O}_2}{Q} \rightarrow \frac{\langle g, \nabla g \rangle}{\langle \nabla g \rangle}$$

is an isomorphism. The proposition follows from the well known result $\dim_{\mathbb{C}} \frac{\langle g, \nabla g \rangle}{\langle \nabla g \rangle} = \mu(g) - \tau(g)$. $\square$

Corollary 3.3.3. *Let $g \in \mathcal{M}_2$ and $\mu(g) - \tau(g) = 1$, then there exists a basis $\{\eta_1, \eta_2\}$ for $\text{Derlog}(g)$ such that $\eta_1(g) = x_1 g$ and $\eta_2(g) = x_2 g$.*

Proof. By Proposition 3.3.2, $Q = \mathcal{M}_2$. Let $\{\eta'_1 = A_1 \frac{\partial}{\partial x_1} + B_1 \frac{\partial}{\partial x_2}, \eta'_2 = A_2 \frac{\partial}{\partial x_1} + B_2 \frac{\partial}{\partial x_2}\} \subset \Theta_2$ which is a basis for $\text{Derlog}(g)$ and $F_1, F_2 \in \mathcal{O}_2$ such that $\eta_1(g) = g F_1$ and $\eta_2(g) = g F_2$. It is clear that $\mathcal{M}_2 = \langle F_1, F_2 \rangle$, thus $F_1(0) = F_2(0) = 0$ and there exist $a_1, a_2, b_1, b_2 \in \mathcal{O}_2$

$$x_1 = a_1 F_1 + a_2 F_2$$

$$x_2 = b_1 F_1 + b_2 F_2$$

comparing the 1-jets at origin, $j^1 F_1(0) = ax_1 + bx_2$ and $j^1 F_2(0) = cx_1 + dx_2$, we obtain

$$\begin{aligned} 1 &= a_1(0)a + a_2(0)c & 0 &= a_1(0)b + a_2(0)d \\ 0 &= b_1(0)a + b_2(0)c & 1 &= b_1(0)b + b_2(0)d \end{aligned}$$

Thus the matrix $\begin{pmatrix} a_1(0) & a_2(0) \\ b_1(0) & b_2(0) \end{pmatrix}$ is invertible, so the determinant of $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ is non zero in a neighborhood of the origin. Therefore

$$\begin{aligned} \eta_1 &= a_1\eta'_1 + a_2\eta'_2 \\ \eta_2 &= b_1\eta'_1 + b_2\eta'_2 \end{aligned}$$

form is a basis for $Derlog(g)$ and $\eta_1(g) = x_1g$, $\eta_2(g) = x_2g$. $\square$

When f is quasi homogeneous, $Derlog(f)$ is well known. We use the software Singular to calculate it when it is not quasi homogeneous.

Definition 3.3.4. We define a operator $div : \Theta_2 \longrightarrow \mathcal{O}_2$ by: $div(X) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$, where $X = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}$.

It follows from this definition that

Proposition 3.3.5. The operator div defined above satisfies

- (i) $div(X_1 + X_2) = div(X_1) + div(X_2)$, $X_1, X_2 \in \Theta_2$.
- (ii) $div(gX) = g \cdot div(X) + X(g)$, $X_1 \in \Theta_2$, $g \in \mathcal{O}_2$.

Corollary 3.3.6. Let $\{\eta_1, \eta_2\}$ be a basis for $Derlog(f)$ such that $div(\eta_j)(0) = 0$ and $\eta_j(0) = 0$, $j = 1, 2$. So, $div(X)(0) = 0$, for all $X \in Derlog(f)$.

Theorem 3.3.7. Let $f \in \mathcal{M}_2$ and suppose $\mu(f) - \tau(f) = 1$, $\{p_1 dx_1 \wedge dx_2, \dots, p_{\mu-1} dx_1 \wedge dx_2\}$ is a $\mathbb{C}$ -basis for $\frac{\Omega^2(\mathbb{C}^2)}{f \cdot \Omega^2(\mathbb{C}^2) + d(f \cdot \Omega^1(\mathbb{C}^2))}$ and $Derlog(f)$ has a basis $\{\eta_1, \eta_2\}$ as in Corollary 3.3.6. Then $\{\|p_1\|, \dots, \|p_\tau\|, \|f\|\}$ is a $\mathbb{C}$ -basis for $\|\Omega^2(\mathbb{C}^2)\|_{N'}$, where $\|g\|$ denotes $\|g dx_1 \wedge dx_2\|_{N'}$,

Proof. By Proposition 3.2.2, it is enough to prove that the set $\{\|p_1\|, \dots, \|p_\tau\|, \|f\|\}$ is linearly independent. Suppose $a_\mu \|f\| + \sum_{i=1}^{\tau} a_i \|p_i\| = 0$ for some $a_i \in \mathbb{C}$. Since f has the property of zeros, there exists $\alpha \in \Omega^1(\mathbb{C}^2)$ such that

$$a_\mu f dx_1 \wedge dx_2 + \sum_{i=1}^{\mu-1} a_i p_i dx_1 \wedge dx_2 = d(f \cdot \alpha) \Rightarrow \sum_{i=1}^{\mu-1} a_i p_i dx_1 \wedge dx_2 = d(f \cdot \alpha) - a_\mu f dx_1 \wedge dx_2$$

Taking the algebraic restriction to N' we have $a_t = 0$ for $1 \leq t \leq \mu-1$. Hence $a_\mu f dx_1 \wedge dx_2 = d(f \cdot \alpha)$. Writing $\alpha = g_1 dx_1 + g_2 dx_2$, we obtain

$$f \cdot \left(a_\mu - \frac{\partial g_2}{\partial x_1} + \frac{\partial g_1}{\partial x_2} \right) = g_2 \frac{\partial f}{\partial x_1} - g_1 \frac{\partial f}{\partial x_2}.$$

Thus $X = g_2 \frac{\partial}{\partial x_1} - g_1 \frac{\partial}{\partial x_2} \in Derlog(f)$ and $a_\mu = \frac{\partial g_2}{\partial x_1}(0) - \frac{\partial g_1}{\partial x_2}(0)$. Hence $a_\mu = 0$ by Corollary 3.3.6. $\square$

3.4 Zero exact algebraic restriction

Let $f \in \mathcal{O}_2$ be a germ as in the subsection 3.2 and $N' = f^{-1}(0)$. We already know that

$$\dim_{\mathbb{C}} \|\Omega^2(\mathbb{C}^2)\|_{N'} = \mu(f)$$

and we want to describe the set of zero exact algebraic restrictions.

By Lemma 3.2.1, $g(x_1, x_2)dx_1 \wedge dx_2 \in \Omega^2(\mathbb{C}^2)$ is such that $\|g(x_1, x_2)dx_1 \wedge dx_2\|_{N'} = 0$ if and only if there exists $\alpha = a_1dx_1 + a_2dx_2 \in \Omega^1(\mathbb{C}^2)$ such that

$$g(x_1, x_2)dx_1 \wedge dx_2 = d(f \cdot \alpha) = \left(\frac{\partial(fa_2)}{\partial x_1} - \frac{\partial(fa_1)}{\partial x_2} \right) dx_1 \wedge dx_2 \quad (3.4)$$

The equation 3.4 implies that there exists a vector field $X = a_2 \frac{\partial}{\partial x_1} - a_1 \frac{\partial}{\partial x_2}$ such that

$$g = \operatorname{div}(fX) \quad (3.5)$$

Therefore we have proved

Proposition 3.4.1. *The 2-form $g(x_1, x_2)dx_1 \wedge dx_2$ has zero exact algebraic restriction if and only if $g \in \operatorname{div}(f \cdot \Theta_2)$, where $f \cdot \Theta_2$ denotes the multiplication of every element of Θ_2 by f .*

An immediate consequence of Proposition 3.4.1 is

Proposition 3.4.2. *Under the same condition on Proposition 3.4.1, we have the isomorphism*

$$\frac{\mathcal{O}_2}{\operatorname{div}(f \cdot \Theta_2)} \longrightarrow \|\Omega^2(\mathbb{C}^2)\|_{N'}$$

Proof. The proof follows from Propositions 3.2.2 and 3.4.1. □

In the case that $f \in \mathcal{M}_2$ is a quasi homogeneous function germ we have that the algebraic restriction $[\Omega^2(\mathbb{C}^2)]_{N'}$ and exact algebraic restriction $\|\Omega^2(\mathbb{C}^2)\|_{N'}$ are the same, therefore by [10] we have $\operatorname{div}(f \cdot \Theta_2) = \langle \nabla f \rangle$. Thus $\operatorname{div}(f \cdot \Theta_2)$ is an $\mathcal{O}_2$ -module. This is not true in general.

Proposition 3.4.3. *Under the same hypothesis as in Corollary 3.3.6, the vector space $\operatorname{div}(f \cdot \Theta_2)$ is not an $\mathcal{O}_2$ -module.*

Proof. Consider $\eta_1 = \frac{\partial}{\partial x_1}$ and $g = -x_1$. Then

$$g \cdot \operatorname{div}(f\eta_1) = \operatorname{div}(fg\eta_1) - f\eta_1(g) = f + \operatorname{div}(fg\eta_1)$$

Therefore the proof follows from the proof of Theorem 3.3.7 and Proposition 3.4.1. □

Theorem 3.4.4 (Saito's criterion [23]). *Let be $g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ a function such that $g^{-1}(0)$ is an isolated hypersurface. Then $\text{Derlog}(g)$ has a basis if and only if there exist $\{\eta_1, \dots, \eta_n\} \subset \text{Derlog}(g)$ such that $\eta_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_j}$ for all $1 \leq i \leq n$ and $\det(a_{ij})$ is unit multiple of g .*

Example 3.4.5. *Consider $f(x_1, x_2) = x_1^4 + x_2^5 + x_1^2 x_2^3 \in \mathcal{O}_2$. It will be proved in the next chapter that f has the property of zeros. A simple calculation shows that it has an isolated singularity at the origin, $\mu(f) = 12$ and $\tau(f) = 11$ thus f is non quasi homogeneous.*

Using the tool Singular and Saito's criterion we can obtain a basis $\{\eta_1, \eta_2\}$ for $\text{Derlog}(f)$.

$$\begin{aligned}\eta_1 &= \left(-10x_1^2 + 3x_1^2 x_2 + x_2^3\right) \frac{\partial}{\partial x_1} + \left(-8x_1 x_2 + 2x_1 x_2^2\right) \frac{\partial}{\partial x_2} \\ \eta_2 &= \left(-25x_1 x_2 + 6x_1 x_2^2\right) \frac{\partial}{\partial x_1} + \left(2x_1^2 - 20x_2^2 + 6x_2^3\right) \frac{\partial}{\partial x_2}\end{aligned}$$

Since the 1-jet, $j^1 \eta_1(0) = j^1 \eta_2(0) = 0$, we have that $\text{div}(\eta_1)(0) = \text{div}(\eta_2)(0) = 0$. Moreover,

$$\mathbb{C} \left\{ 1, x_1, x_2, x_1^2, x_1 x_2, x_2^2, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1^2 x_2^2, x_1 x_2^3 \right\} dx_1 \wedge dx_2 \cong \frac{\Omega^2(\mathbb{C}^2)}{f \cdot \Omega^2(\mathbb{C}^2) + d(f \cdot \Omega^1(\mathbb{C}^2))}.$$

Therefore according to Theorem 3.3.7,

$$\left\{ \|1\|, \|x_1\|, \|x_2\|, \|x_1^2\|, \|x_1 x_2\|, \|x_2^2\|, \|x_1^2 x_2\|, \|x_1 x_2^2\|, \|x_2^3\|, \|x_1^2 x_2^2\|, \|x_1 x_2^3\|, \|f(x_1, x_2)\| \right\}$$

is a $\mathbb{C}$ -basis for $\|\Omega^2(\mathbb{C}^2)\|_{N'}$, where $N' = f^{-1}(0)$. And by Proposition 3.4.3, the vector space $\text{div}(f \cdot \Theta_2)$ is not an $\mathcal{O}_2$ -module.

Now, consider $n \geq 2$ and the planar curve $N = \{f(x_1, x_2) = x_{\geq 3} = 0\}$ in the symplectic manifold $(\mathbb{C}^{2n}, \omega)$, where

$$\omega = f dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \dots + dx_{2n-1} \wedge dx_{2n}$$

By Theorem 3.2.4, the set

$$\left\{ \begin{array}{l} \|1\|_N, \|x_1\|_N, \|x_2\|_N, \|x_1^2\|_N, \|x_1 x_2\|_N, \|x_2^2\|_N, \|x_1^2 x_2\|_N, \\ \|x_1 x_2^2\|_N, \|x_2^3\|_N, \|x_1^2 x_2^2\|_N, \|x_1 x_2^3\|_N, \|f(x_1, x_2)\|_N \end{array} \right\}$$

is a basis for $\|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N$, in particular $\|f\|_N \neq 0$.

ZERO EXACT ALGEBRAIC RESTRICTION TO N'

Recall that $f(x_1, x_2) = x_1^4 + x_2^5 + x_1^2 x_2^3$ and $N' = f^{-1}(0)$. In this chapter we prove that f has the property of zeros and we calculate the 2-forms which have zero exact algebraic restrictions, In particular we prove that for all $g \in \mathcal{M}_2^6$, the exact algebraic restriction $\|gdx_1 \wedge dx_2\|_{N'} = 0$. To make this calculation more friendly, we use the coordinate system (x, y) instead of (x_1, x_2) .

4.1 f has the property of zeros

We start recalling some basic definitions of commutative algebra. The radical of an ideal J in a commutative ring R , denoted by $\sqrt{J}$, is defined as

$$\sqrt{J} = \{r \in R \mid r^n \in J \text{ for some positive integer } n\}.$$

Let I be an ideal of the commutative ring $\mathcal{O}_n$, since it is a Noetherian ring (every ideal is finitely generated), there exist $g_1, \dots, g_k \in \mathcal{O}_n$ such that $I = \langle g_1, \dots, g_k \rangle$. We denote the set of zeros of J by $V(J)$, this is

$$V(J) = \bigcap_{j=1}^k g_j^{-1}(0).$$

Let $(N, 0) \subset (\mathbb{C}^n, 0)$ be a subset. We define

$$I(N, 0) = \{g \in \mathcal{O}_n : (N, 0) \subset (V(g), 0)\}.$$

Now we enunciate the Nullstellensatz Theorem to the ring $\mathcal{O}_n$, the proof of this theorem could be found for example in [16].

Theorem 4.1.1. *Let J be an ideal in $\mathcal{O}_n$, then $I(V(J), 0) = \sqrt{J}$.*

As a particular case, when J is a prime ideal (if $ab \in J$ then $a \in J$ or $b \in J$), we have $I(V(J), 0) = J$.

Applying Nullstellensatz Theorem to an ideal generated by $g \in \mathcal{O}_n$ we have $I(V(g), 0) = \sqrt{\langle g \rangle}$. If $\langle g \rangle$ is an prime ideal, then $I(V(g), 0) = \langle g \rangle$. It means that if $F \in \mathcal{O}_2$ vanishes on $(g^{-1}(0), 0)$ then $F \in \langle g \rangle$, in other words, g has the property of zeros.

Let us prove that the ideal $\langle f \rangle$ is a prime ideal. Recall that an element a of a commutative ring R with unit 1_R is called invertible if there exists $b \in R$ such that $ab = 1_R$. $a \in R$ is called irreducible if $a = cb$ then either b or c is invertible in R . $a \in R$ is called prime if a divides $bc \in R$ then a divides b or a divides c .

By Commutative Ring Theory, it is well known that every irreducible element of a Unique Factorization Domain R is prime and that the ideal $I = \langle a \rangle \subset R$ is a prime ideal if and only if a is prime.

The ring $\mathcal{O}_n$ is a Unique Factorization Domain, see [16], therefore it remains to show that f is irreducible in $\mathcal{O}_2$, we prove it by contradiction. Recall that $g \in \mathcal{O}_2$ is invertible if and only if $g(0) \neq 0$.

Suppose there exist $f_1, f_2 \in \mathcal{M}_2$ such that $f = f_1 f_2$. Writing

$$\begin{aligned} f_1 &= a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + \dots, \\ f_2 &= b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + \dots, \end{aligned}$$

we have the equation

$$x^4 + y^5 + x^2y^3 = (a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + \dots)(b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + \dots)$$

comparing the monomials we have

$$\begin{array}{ll} a_1b_1 = 0 & x^2 \\ b_1a_2 + a_1b_2 = 0 & xy \\ b_2a_2 = 0 & y^2 \\ a_1b_3 + a_3b_1 = 0 & x^3 \\ a_1b_4 + a_2b_3 + b_1a_4 + b_2a_3 = 0 & x^2y \\ a_1b_5 + a_2b_4 + b_1a_5 + b_2a_4 = 0 & xy^2 \\ a_2b_5 + a_5b_2 = 0 & y^3 \end{array}$$

If $b_1 \neq 0$ then $a_1 = a_2 = a_3 = a_4 = a_5 = 0$. We can consider $b_1 = 1$ and

$$x^4 + y^5 + x^2y^3 = (a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3 \dots)(x + b_2y + \dots)$$

comparing the monomials one more time

$$\begin{array}{ll} a_6 = 1 & x^4 \\ a_7 + a_6b_2 = 0 & x^3y \\ a_8 + a_7b_2 = 0 & x^2y^2 \\ a_9 + a_8b_2 = 0 & xy^3 \\ a_9b_2 = 0 & y^4 \end{array}$$

then $b_2 = a_7 = a_8 = a_9 = 0$. This is a contradiction because in the right side it is not possible to have the monomial y^5 .

Therefore $a_1 = b_1 = 0$ because the case $a_1 \neq 0$ is the same case as above, so

$$x^4 + y^5 + x^2y^3 = (a_2y + a_3x^2 + a_4xy + a_5y^2 + \dots)(b_2y + b_3x^2 + b_4xy + b_5y^2 + \dots)$$

$$\begin{array}{ll} b_2a_2 = 0 & y^2 \\ a_2b_3 + b_2a_3 = 0 & x^2y \\ a_2b_4 + b_2a_4 = 0 & xy^2 \\ a_2b_5 + b_2a_5 = 0 & y^3 \\ a_3b_3 = 1 & x^4 \end{array}$$

then $a_2 = b_2 = 0$ and we can assume $a_3 = b_3 = 1$, so $x^4 + y^5 + x^2y^3 = (x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3 \dots)(x^2 + b_4xy + b_5y^2 + b_6x^3 + b_7x^2y + b_8xy^2 + b_9y^3 \dots)$ and

$$\begin{array}{ll} a_4 + b_4 = 0 & x^3y \\ a_5 + b_5 + a_4b_4 = 0 & x^2y^2 \\ a_4b_5 + a_5b_4 = 0 & xy^3 \\ a_5b_5 = 0 & y^4 \\ a_6 + b_6 = 0 & x^5 \\ a_7 + a_6b_4 + b_7 + a_4b_6 = 0 & x^4y \\ a_8 + b_4a_7 + b_5a_6 + b_8 + a_4b_7 + a_5b_6 = 0 & x^3y^2 \\ a_9 + b_4a_8 + b_5a_7 + b_9 + a_4b_8 + a_5b_7 = 1 & x^2y^3 \\ b_4a_9 + b_5a_8 + a_4b_9 + a_5b_8 = 0 & xy^4 \\ b_5a_9 + a_5b_9 = 1 & y^5 \end{array}$$

if $b_5 = 0$ then $a_5 \neq 0$ from the monomial y^5 , thus $a_4 = b_4 = a_5 = 0$ and this is a contradiction. Analogously the case $b_5 \neq 0$ is not possible. Therefore it is not possible to write $f = f_1f_2$ such that $f_1, f_2 \in \mathcal{M}_2$. Thus f is irreducible and has the property of zeros.

4.2 Zero algebraic restriction to N'

By Example 3.4.5, $\{\eta_1, \eta_2\}$ is a basis for $Derlog(f)$, where

$$\begin{aligned} \eta_1 &= (-10x^2 + 3x^2y + y^3) \frac{\partial}{\partial x} + (-8xy + 2xy^2) \frac{\partial}{\partial y} \\ \eta_2 &= (-25xy + 6xy^2) \frac{\partial}{\partial x} + (2x^2 - 20y^2 + 6y^3) \frac{\partial}{\partial y} \end{aligned}$$

and

$$\{\|1\|, \|x\|, \|y\|, \|x^2\|, \|xy\|, \|y^2\|, \|x^2y\|, \|xy^2\|, \|y^3\|, \|x^2y^2\|, \|xy^3\|, \|f\|\}$$

is a $\mathbb{C}$ -basis for $\|\Omega^2(\mathbb{C}^2)\|_{N'}$, where $\|g\|_{N'}$ denotes the exact algebraic restriction $\|gdx \wedge dy\|_{N'}$, for all $g \in \mathcal{O}_2$.

We want to prove that for $g \in \mathcal{M}_2^6$, $\|g\| = 0$. In order to prove that, we first prove that if $G \in \mathcal{M}_2$, then $\|G \cdot f\| = 0$. We do first for $G = x^i y^j$ and then for a general G .

We want to find $h = \sum_{k \geq 0} \sum_{r=0}^k a_{k,r} x^{k-r} y^r$ and $g = \sum_{k \geq 0} \sum_{r=0}^k b_{k,r} x^{k-r} y^r$ two germs in $\mathcal{O}_2$ such that

$$x^i y^j f dx \wedge dy = d(f\alpha) = \left\{ \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y} \right) f + h \frac{\partial f}{\partial x} - g \frac{\partial f}{\partial y} \right\} dx \wedge dy$$

where, $\alpha = gdx + hdy$, or equivalently

$$x^{i+4} y^j + x^i y^{j+5} + x^{i+2} y^{j+3} = \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y} \right) f + h \frac{\partial f}{\partial x} - g \frac{\partial f}{\partial y} \quad (4.1)$$

$$\begin{aligned} x^i y^j f &= \sum_{k \geq 0} \sum_{r=0}^k (k-r) a_{k,r} (x^{k-r+3} y^r + x^{k-r-1} y^{r+5} + x^{k-r+1} y^{r+3}) + \\ &\quad \sum_{k \geq 0} \sum_{r=0}^k -r b_{k,r} (x^{k-r+4} y^{r-1} + x^{k-r} y^{r+4} + x^{k-r+2} y^{r+2}) + \\ &\quad \sum_{k \geq 0} \sum_{r=0}^k (4a_{k,r} x^{k-r+3} y^r + 2a_{k,r} x^{k-r+1} y^{r+3}) + \\ &\quad \sum_{k \geq 0} \sum_{r=0}^k (-5b_{k,r} x^{k-r} y^{r+4} - 3b_{k,r} x^{k-r+2} y^{r+2}) \\ x^i y^j f &= \sum_{k \geq 0} \sum_{r=0}^k (k-r+4) a_{k,r} x^{k-r+3} y^r + \sum_{k \geq 0} \sum_{r=0}^k (k-r) a_{k,r} x^{k-r-1} y^{r+5} + \\ &\quad \sum_{k \geq 0} \sum_{r=0}^k (k-r+2) a_{k,r} x^{k-r+1} y^{r+3} + \sum_{k \geq 0} \sum_{r=0}^k -r b_{k,r} x^{k-r+4} y^{r-1} + \\ &\quad \sum_{k \geq 0} \sum_{r=0}^k -(r+5) b_{k,r} x^{k-r} y^{r+4} + \sum_{k \geq 0} \sum_{r=0}^k -(r+3) b_{k,r} x^{k-r+2} y^{r+2}, \end{aligned}$$

therefore

$$\begin{aligned} x^{i+4} y^j + x^i y^{j+5} + x^{i+2} y^{j+3} &= \sum_{k \geq 3} \sum_{r=0}^{k-3} (k-r+1) a_{k-3,r} x^{k-r} y^r + \sum_{k \geq 5} \sum_{r=5}^{k+1} (k-r+1) a_{k-4,r-5} x^{k-r} y^r + \\ &\quad \sum_{k \geq 4} \sum_{r=3}^{k-1} (k-r+1) a_{k-4,r-3} x^{k-r} y^r + \sum_{k=4} \sum_{r=0}^{k-4} -(r+1) b_{k-3,r+1} x^{k-r} y^r + \\ &\quad \sum_{k \geq 4} \sum_{r=4}^k -(r+1) b_{k-4,r-4} x^{k-r} y^r + \sum_{k \geq 4} \sum_{r=2}^{k-2} -(r+1) b_{k-4,r-2} x^{k-r} y^r \quad (4.2) \end{aligned}$$

We now use *Maple* and *Mathematica* softwares to find a solution for 4.2 as simple as possible (it means with most $a_{i,j} = 0$ and $b_{i,j} = 0$) and prove, using criteria of comparison (see [7]), that this particular solution is smooth in a neighborhood of the origin. We split the proof in several lemmas.

Lemma 4.2.1. *Let $j \in \mathbb{Z}, j \geq 0$. Then there exists $\alpha \in \Omega^1(\mathbb{C}^2)$ such that $xy^j f dx \wedge dy = d(f\alpha)$.*

Proof. Suppose $\alpha = gdx + hdy$, where

$$g = \sum_{k \geq j+2} b_{k,k-1} x y^{k-1}, \quad h = \sum_{k \geq j+2} \{a_{k,k-2} x^2 y^{k-2} + a_{k,k} y^k\}.$$

Substituting in 4.2, we obtain

$$\begin{aligned} x^5 y^j + x y^{j+5} + x^3 y^{j+3} &= 4a_{j+2,j+2} x^3 y^{j+2} + \sum_{k \geq j} \{6a_{k+2,k} - (k+1)b_{k+2,k+1}\} x^5 y^k + \\ &\quad \sum_{k \geq j+5} \{2a_{k-3,k-3} + 2a_{k-3,k-5} - (k+1)b_{k-3,k-4}\} x y^k + \\ &\quad \sum_{k \geq j+3} \{4a_{k,k} + 4a_{k-1,k-3} - (k+1)b_{k-1,k-2}\} x^3 y^k, \end{aligned}$$

thus we have to solve

$$\begin{aligned} a_{j+2,j+2} &= 0 \\ 6a_{j+2,j} - (j+1)b_{j+2,j+1} &= 1 \\ 2a_{j+2,j+2} + 2a_{j+2,j} - (j+6)b_{j+2,j+1} &= 1 \\ 4a_{j+3,j+3} + 4a_{j+2,j} - (j+4)b_{j+2,j+1} &= 1 \\ 6a_{k+2,k} - (k+1)b_{k+2,k+1} &= 0, \quad k \geq j+1 \\ 2a_{k-3,k-3} + 2a_{k-3,k-5} - (k+1)b_{k-3,k-4} &= 0, \quad k \geq j+6 \\ 4a_{k,k} + 4a_{k-1,k-3} - (k+1)b_{k-1,k-2} &= 0, \quad k \geq j+4, \end{aligned}$$

or equivalently

$$\begin{aligned} a_{j+2,j+2} &= 0 \\ 6a_{j+2,j} - (j+1)b_{j+2,j+1} &= 1 \\ 2a_{j+2,j} - (j+6)b_{j+2,j+1} &= 1 \\ 4a_{j+3,j+3} + 4a_{j+2,j} - (j+4)b_{j+2,j+1} &= 1 \\ 6a_{k+2,k} - (k+1)b_{k+2,k+1} &= 0, \quad k \geq j+1 \\ 2a_{k+2,k+2} + 2a_{k+2,k} - (k+6)b_{k+2,k+1} &= 0, \quad k \geq j+1 \\ 4a_{k+3,k+3} + 4a_{k+2,k} - (k+4)b_{k+2,k+1} &= 0, \quad k \geq j+1. \end{aligned}$$

Solving this system we obtain

$$\begin{aligned} a_{j+2,j+2} &= 0 & a_{j+2,j} &= 5/(4j+34) \\ b_{j+2,j+1} &= -4/(4j+34) & a_{j+3,j+3} &= -1/(8j+68), \end{aligned}$$

and for $k \geq j+1$

$$\begin{aligned} a_{k+2,k} &= \frac{2k+2}{4k+34} a_{k+2,k+2} & b_{k+2,k+1} &= \frac{12}{4k+34} a_{k+2,k+2} \\ a_{k+3,k+3} &= \frac{k+10}{4k+34} a_{k+2,k+2}, \end{aligned}$$

Note that for $k \geq j+1$

$$\begin{aligned} |a_{k+2,k}| &\leq \frac{1}{2} |a_{k+2,k+2}| & |b_{k+2,k+1}| &\leq \frac{1}{2} |a_{k+2,k+2}| \\ |a_{k+3,k+3}| &\leq \frac{1}{2} |a_{k+2,k+2}|, \end{aligned}$$

these last relations implies that for $k \geq j+1$

$$\begin{aligned} |a_{k+2,k}| &\leq \left(\frac{1}{2}\right)^{k-j} |a_{j+3,j+3}| & |b_{k+2,k+1}| &\leq \left(\frac{1}{2}\right)^{k-j} |a_{j+3,j+3}| \\ |a_{k+3,k+3}| &\leq \left(\frac{1}{2}\right)^{k-j} |a_{j+3,j+3}|, \end{aligned}$$

Therefore the 1-form α is smooth on $|y| < 2$. □

Lemma 4.2.2. *Let $j \in \mathbb{Z}, j \geq 0$. Then there exists $\alpha \in \Omega^1(\mathbb{C}^2)$ such that $x^2 y^j f dx \wedge dy = d(f\alpha)$.*

Proof. Suppose $\alpha = gdx + hdy$, where

$$g = \sum_{k \geq j+2} \{b_{k+1,k-1} x^2 y^{k-1} + b_{k+1,k+1} y^{k+1}\}, \quad h = \sum_{k \geq j+2} \{a_{k+1,k-2} x^3 y^{k-2} + a_{k+1,k} x y^k\}.$$

Substituting in 4.2, we have that $x^6 y^j + x^2 y^{j+5} + x^4 y^{j+3}$ is equal to

$$\begin{aligned} & \{5a_{j+3,j+2} - (j+3)b_{j+3,j+3}\} x^4 y^{j+2} + \sum_{k \geq j} \{7a_{k+3,k} - (k+1)b_{k+3,k+1}\} x^6 y^k + \\ & \sum_{k \geq j+3} \{5a_{k,k-3} - (k+1)b_{k,k-2} + 5a_{k+1,k} - (k+1)b_{k+1,k+1}\} x^4 y^k + \\ & \sum_{k \geq j+5} \{3a_{k-2,k-3} + 3a_{k-2,k-5} - (k+1)b_{k-2,k-4} - (k+1)b_{k-2,k-2}\} x^2 y^k + \\ & \sum_{k \geq j+7} \{a_{k-4,k-5} - (k+1)b_{k-4,k-4}\} y^k. \end{aligned}$$

Therefore

$$\begin{aligned} & 5a_{j+3,j+2} - (j+3)b_{j+3,j+3} = 0 \\ & 7a_{j+3,j} - (j+1)b_{j+3,j+1} = 1 \\ & 5a_{j+3,j} - (j+4)b_{j+3,j+1} + 5a_{j+4,j+3} - (j+4)b_{j+4,j+4} = 1 \\ & 3a_{j+3,j+2} + 3a_{j+3,j} - (j+6)b_{j+3,j+1} - (j+6)b_{j+3,j+3} = 1 \\ & a_{j+3,j+2} - (j+8)b_{j+3,j+3} = 0 \\ & 7a_{k+3,k} - (k+1)b_{k+3,k+1} = 0, \quad k \geq j+1 \\ & 5a_{k,k-3} - (k+1)b_{k,k-2} + 5a_{k+1,k} - (k+1)b_{k+1,k+1} = 0, \quad k \geq j+4 \\ & 3a_{k-2,k-3} + 3a_{k-2,k-5} - (k+1)b_{k-2,k-4} - (k+1)b_{k-2,k-2} = 0, \quad k \geq j+6 \\ & a_{k-4,k-5} - (k+1)b_{k-4,k-4} = 0, \quad k \geq j+8, \end{aligned}$$

or equivalently

$$\begin{aligned} & 5a_{j+3,j+2} - (j+3)b_{j+3,j+3} = 0 \\ & 7a_{j+3,j} - (j+1)b_{j+3,j+1} = 1 \\ & 5a_{j+3,j} - (j+4)b_{j+3,j+1} + 5a_{j+4,j+3} - (j+4)b_{j+4,j+4} = 1 \\ & 3a_{j+3,j+2} + 3a_{j+3,j} - (j+6)b_{j+3,j+1} - (j+6)b_{j+3,j+3} = 1 \\ & a_{j+3,j+2} - (j+8)b_{j+3,j+3} = 0 \\ & 7a_{k+3,k} - (k+1)b_{k+3,k+1} = 0, \quad k \geq j+1 \\ & 5a_{k+3,k} - (k+4)b_{k+3,k+1} + 5a_{k+4,k+3} - (k+4)b_{k+4,k+4} = 0, \quad k \geq j+1 \\ & 3a_{k+3,k+2} + 3a_{k+3,k} - (k+6)b_{k+3,k+1} - (k+6)b_{k+3,k+3} = 0, \quad k \geq j+1 \\ & a_{k+3,k+2} - (k+8)b_{k+3,k+3} = 0, \quad k \geq j+1. \end{aligned}$$

Solving this system, we obtain

$$\begin{aligned} a_{j+3,j+2} &= 0 & a_{j+3,j} &= 5/(4j+39) & a_{j+4,j+3} &= -2(j+9)/(4j+39)(4j+41) \\ b_{j+3,j+3} &= 0 & b_{j+3,j+1} &= -4/(4j+39) & b_{j+4,j+4} &= -2/(4j+39)(4j+41), \end{aligned}$$

and for $k \geq j+1$

$$\begin{aligned} a_{k+3,k+2} &= (k+8)b_{k+3,k+3} & a_{k+3,k} &= \frac{(2k+18)(k+1)}{4k+39} b_{k+3,k+3} \\ b_{k+3,k+1} &= \frac{7(2k+18)}{4k+39} b_{k+3,k+3} & b_{k+4,k+4} &= \frac{(2k+18)(2k+23)}{(4k+39)(4k+41)} b_{k+3,k+3} \\ a_{k+4,k+3} &= \frac{(2k+18)(2k+23)(k+9)}{(4k+39)(4k+41)} b_{k+3,k+3}. \end{aligned}$$

Using these relations we obtain that for $k \geq j + 1$

$$\begin{aligned} |a_{k+4,k+1}| &\leq \frac{1}{2}|a_{k+3,k}| & |b_{k+4,k+4}| &\leq \frac{1}{2}|b_{k+3,k+3}| \\ |a_{k+4,k+3}| &\leq \frac{1}{2}|a_{k+3,k+2}| & |b_{k+4,k+2}| &\leq \frac{1}{2}|b_{k+3,k+1}|, \end{aligned}$$

by induction we have that for $k \geq j + 1$

$$\begin{aligned} |a_{k+4,k+1}| &\leq \left(\frac{1}{2}\right)^{k-j} |a_{j+4,j+1}| & |b_{k+4,k+4}| &\leq \left(\frac{1}{2}\right)^{k-j} |b_{j+4,j+4}| \\ |a_{k+4,k+3}| &\leq \left(\frac{1}{2}\right)^{k-j} |a_{j+4,j+3}| & |b_{k+4,k+2}| &\leq \left(\frac{1}{2}\right)^{k-j} |b_{j+4,j+2}|, \end{aligned}$$

therefore the 1-form α is smooth on $|y| < 2$. $\square$

Lemma 4.2.3. *Let $j \in \mathbb{Z}, j \geq 0$. Then there exists $\alpha \in \Omega^1(\mathbb{C}^2)$ such that $x^3 y^j f dx \wedge dy = d(f\alpha)$.*

Proof. Consider $\alpha = gdx + hdy$, where

$$g = \sum_{k \geq j+2} \{b_{k+2,k-1} x^3 y^{k-1} + b_{k+2,k+1} x y^{k+1}\} \quad h = \sum_{k \geq j+2} \{a_{k+2,k-2} x^4 y^{k-2} + a_{k+2,k} x^2 y^k + a_{k+2,k+2} y^{k+2}\}.$$

Then 4.2 is satisfied if and only if

$$\begin{aligned} a_{j+4,j+4} &= 0 \\ 6a_{j+4,j+2} - (j+3)b_{j+4,j+3} &= 0 \\ 8a_{j+4,j} - (j+1)b_{j+4,j+1} &= 1 \\ 6a_{j+5,j+3} + 6a_{j+4,j} - (j+4)b_{j+5,j+4} - (j+4)b_{j+4,j+1} &= 1 \\ 4a_{j+5,j+5} + 4a_{j+4,j+2} + 4a_{j+4,j} - (j+6)b_{j+4,j+1} - (j+6)b_{j+4,j+3} &= 1 \\ 2a_{j+4,j+2} + 2a_{j+4,j+4} - (j+8)b_{j+4,j+3} &= 0 \\ \forall k \geq j+3 \\ 8a_{k+2,k-2} - (k-1)b_{k+2,k-1} &= 0 \\ 6a_{k+3,k+1} + 6a_{k+2,k-2} - (k+2)b_{k+3,k+2} - (k+2)b_{k+2,k-1} &= 0 \\ 4a_{k+3,k+3} + 4a_{k+2,k} + 4a_{k+2,k-2} - (k+4)b_{k+2,k-1} - (k+4)b_{k+2,k+1} &= 0 \\ 2a_{k+2,k} + 2a_{k+2,k+2} - (k+6)b_{k+2,k+1} &= 0, \end{aligned}$$

considering $b_{j+4,j+1} = 0$, for $k \geq j + 2$, $a_{k+3,k-1} = b_{k+2,k-1} = 0$, and solving this system, we obtain

$$\begin{aligned} a_{j+4,j+4} &= 0 & b_{j+4,j+3} &= 0 & a_{j+3,j+2} &= 0 & b_{j+5,j+4} &= \frac{1}{2j+13} \\ a_{j+4,j} &= 1/8 & a_{j+5,j+5} &= 1/8 & a_{j+5,j+3} &= \frac{2j+13}{8(2j+23)}, \end{aligned}$$

and for $k \geq j + 3$

$$\begin{aligned} a_{k+3,k+3} &= \frac{1}{4} \{(k+4)b_{k+2,k+1} - 4a_{k+2,k}\} & a_{k+3,k+1} &= \frac{(k+2)}{2k+19} a_{k+3,k+3} \\ b_{k+3,k+2} &= \frac{6}{2k+19} a_{k+3,k+3}. \end{aligned}$$

From these equations we obtain that for $k \geq j + 3$

$$|a_{k+4,k+4}| \leq \frac{1}{2}|a_{k+3,k+3}| \quad |b_{k+4,k+3}| \leq \frac{1}{2}|a_{k+3,k+3}| \quad |a_{k+4,k+2}| \leq \frac{1}{2}|a_{k+3,k+3}|,$$

similarly to the previous cases, α is smooth on $|y| < 2$. $\square$

Lemma 4.2.4. *Let $j \in \mathbb{Z}, j \geq 0, i = 2n, n \geq 2$. Then there exists $\alpha \in \Omega^1(\mathbb{C}^2)$ such that $x^i y^j f dx \wedge dy = d(f\alpha)$.*

Proof. Consider $\alpha = gdx + hdy$, where

$$g = \sum_{k \geq j+2} \sum_{r=0}^n b_{k,r} x^{2n-2r} y^{k-1+2r} \quad h = \sum_{k \geq j+2} \sum_{r=0}^n a_{k,r} x^{2n+1-2r} y^{k-2+2r}$$

Substituting in 4.2, we have that $x^{2n+4}y^j + x^{2n}y^{j+5} + x^{2n+2}y^{j+3}$ is equal to

$$\begin{aligned} & \{(2n+3)a_{j+2,1} - (j+3)b_{j+2,1}\}x^{2n+2}y^{j+2} + \\ & \sum_{r=0}^{n-2} \{(2n+1-2r)a_{j+2,r+2} - (j+5+2r)b_{j+2,r+2}\}x^{2n-2r}y^{j+4+2r} + \\ & \sum_{k \geq j+2} \{(2n+5)a_{k,0} - (k-1)b_{k,0}\}x^{2n+4}y^{k-2} + \\ & \sum_{k \geq j+2} \{(2n+3)a_{k+1,1} - (k+2)b_{k+1,1} + (2n+3)a_{k,0} - (k+2)b_{k,0}\}x^{2n+2}y^{k+1} \\ & \sum_{k \geq j+2} \{a_{k,n} - (k+4+2n)b_{k,n}\}y^{k+3+2n} + \\ & \sum_{k \geq j+2} \{3a_{k,n-1} + 3a_{k,n} - (k+2+2n)b_{k,n-1} - (k+2+2n)b_{k,n}\}x^2y^{k+1+2n} + \\ & \sum_{k \geq j+2} \sum_{r=0}^{n-2} \{(2n+1-2r)a_{k+1,r+2} - (k+4+2r)b_{k+1,r+2}\}x^{2n-2r}y^{k+3+2r} + \\ & \sum_{k \geq j+2} \sum_{r=0}^{n-2} \{(2n+1-2r)a_{k,r} - (k+4+2r)b_{k,r}\}x^{2n-2r}y^{k+3+2r} + \\ & \sum_{k \geq j+2} \sum_{r=0}^{n-2} \{(2n+1-2r)a_{k,r+1} - (k+4+2r)b_{k,r+1}\}x^{2n-2r}y^{k+3+2r}. \end{aligned}$$

Then $(x^{4+2n}y^j + x^{2n}y^{j+5} + x^{2n+2}y^{j+3})dx \wedge dy = d(f\alpha)$ if and only if

$$\begin{aligned} (2n+3)a_{j+2,1} - (j+3)b_{j+2,1} &= 0 \\ (2n+1-2r)a_{j+2,r+2} - (j+5+2r)b_{j+2,r+2} &= 0, \quad 0 \leq r \leq n-2 \\ (2n+5)a_{j+2,0} - (j+1)b_{j+2,0} &= 1 \\ (2n+5)a_{k,0} - (k-1)b_{k,0} &= 0, \quad k \geq j+3 \\ (2n+3)a_{j+3,1} - (j+4)b_{j+3,1} + (2n+3)a_{j+2,0} - (j+4)b_{j+2,0} &= 1 \\ (2n+3)a_{k+1,1} - (k+2)b_{k+1,1} + (2n+3)a_{k,0} - (k+2)b_{k,0} &= 0, \quad k \geq j+3 \\ a_{k,n} - (k+4+2n)b_{k,n} &= 0, \quad k \geq j+2 \\ 3a_{k,n-1} + 3a_{k,n} - (k+2+2n)b_{k,n-1} - (k+2+2n)b_{k,n} &= 0, \quad k \geq j+2 \\ (2n+1)\{a_{j+2,0} + a_{j+2,1} + a_{j+3,2}\} - (j+6)\{b_{j+2,0} + b_{j+2,1} + b_{j+3,2}\} &= 1 \\ (2n+1-2r)\{a_{j+2,r} + a_{j+2,r+1} + a_{j+3,r+2}\} - (j+6+2r)\{b_{j+2,r} + b_{j+2,r+1} + b_{j+3,r+2}\} &= 0, \quad 1 \leq r \leq n-2 \\ (2n+1-2r)\{a_{k,r} + a_{k,r+1} + a_{k+1,r+2}\} - (k+4+2r)\{b_{k,r} + b_{k,r+1} + b_{k+1,r+2}\} &= 0, \quad k \geq j+3, \quad 0 \leq r \leq n-2, \end{aligned}$$

reordering

$$\begin{aligned}
(2n+3)a_{j+2,1} - (j+3)b_{j+2,1} &= 0 \\
(2n+1-2r)a_{j+2,r+2} - (j+5+2r)b_{j+2,r+2} &= 0, \quad 0 \leq r \leq n-2 \\
(2n+5)a_{j+2,0} - (j+1)b_{j+2,0} &= 1 \\
(2n+3)a_{j+3,1} - (j+4)b_{j+3,1} + (2n+3)a_{j+2,0} - (j+4)b_{j+2,0} &= 1 \\
a_{j+2,n} - (j+6+2n)b_{j+2,n} &= 0 \\
3a_{j+2,n-1} + 3a_{j+2,n} - (j+4+2n)b_{j+2,n-1} - (j+4+2n)b_{j+2,n} &= 0 \\
(2n+1)\{a_{j+2,0} + a_{j+2,1} + a_{j+3,2}\} - (j+6)\{b_{j+2,0} + b_{j+2,1} + b_{j+3,2}\} &= 1 \\
(2n+1-2r)\{a_{j+2,r} + a_{j+2,r+1} + a_{j+3,r+2}\} - (j+6+2r)\{b_{j+2,r} + b_{j+2,r+1} + b_{j+3,r+2}\} &= 0 \\
\text{when } 1 \leq r \leq n-2 \\
\forall k \geq j+3 \\
(2n+5)a_{k,0} - (k-1)b_{k,0} &= 0 \\
(2n+3)a_{k+1,1} - (k+2)b_{k+1,1} + (2n+3)a_{k,0} - (k+2)b_{k,0} &= 0 \\
a_{k,n} - (k+4+2n)b_{k,n} &= 0 \\
3a_{k,n-1} + 3a_{k,n} - (k+2+2n)b_{k,n-1} - (k+2+2n)b_{k,n} &= 0 \\
(2n+1-2r)\{a_{k,r} + a_{k,r+1} + a_{k+1,r+2}\} - (k+4+2r)\{b_{k,r} + b_{k,r+1} + b_{k+1,r+2}\} &= 0, \quad 0 \leq r \leq n-2.
\end{aligned}$$

We find this system in 3 cases: $n = 2$, $n = 3$ and for $n \geq 4$.

i) Case $n = 2$. Assuming $b_{j+2,0} = a_{k,0} = b_{k,0} = 0$, for $k \geq j+3$, we obtain

$$\begin{aligned}
a_{j+2,2} &= 0 & b_{j+2,2} &= 0 & a_{j+3,1} &= \frac{2(633+149j+8j^2)}{9(49+4j)(51+4j)} \\
a_{j+2,1} &= 0 & b_{j+2,1} &= 0 & b_{j+3,1} &= \frac{2(483+40j)}{9(49+4j)(51+4j)} \\
a_{j+2,0} &= 1/9 & a_{j+3,2} &= \frac{4(11+j)}{9(49+4j)} & b_{j+3,2} &= \frac{4}{9(49+4j)},
\end{aligned}$$

and for $k \geq j+3$

$$\begin{aligned}
a_{k+1,1} &= \frac{2(2+k)(10+k)}{(41+4k)(43+4k)} \{(k+4)b_{k,1} - 5a_{k,1}\} & a_{k+1,2} &= \frac{9+k}{41+4k} \{(k+4)b_{k,1} - 5a_{k,1}\} \\
b_{k+1,2} &= \frac{1}{41+4k} \{(k+4)b_{k,1} - 5a_{k,1}\} & b_{k+1,1} &= \frac{14(10+k)}{(41+4k)(43+4k)} \{(k+4)b_{k,1} - 5a_{k,1}\}.
\end{aligned}$$

A simple calculation shows that for all $k \geq j+3$

$$\begin{aligned}
|a_{k+2,1}| &\leq \frac{1}{2}|a_{k+1,1}| & |b_{k+2,1}| &\leq \frac{1}{2}|b_{k+1,1}| \\
|a_{k+2,2}| &\leq \frac{1}{2}|a_{k+1,2}| & |b_{k+2,2}| &\leq \frac{1}{2}|b_{k+1,2}|,
\end{aligned}$$

thus α is smooth on $|y| < 2$.

ii) Case $n = 3$. Taking $b_{j+3,1} = b_{j+2,r} = 0$ for $0 \leq r \leq 3$, we obtain

$$\begin{aligned}
a_{j+2,0} &= 1/11 & a_{j+2,r} &= 0, \quad r \geq 1 & a_{j+3,1} &= 2/99 \\
b_{j+3,3} &= 0 & a_{j+3,2} &= \frac{4(11+j)}{11(59+4j)} & b_{j+3,2} &= \frac{12}{11(59+4j)} \\
a_{j+3,3} &= 0,
\end{aligned}$$

taking $a_{k,0} = b_{k,0} = a_{k+1,1} = b_{k+1,1} = 0$ for all $k \geq j+3$, we obtain that for $k \geq j+3$

$$\begin{aligned} a_{k+1,2} &= \frac{441 + 85k + 4k^2}{(49 + 4k)(51 + 4k)} A_k + \frac{96 + 32k + 2k^2}{(49 + 4k)(51 + 4k)} B_k & a_{k+1,3} &= \frac{11 + k}{49 + 4k} B_k \\ b_{k+1,2} &= \frac{147 + 12k}{(49 + 4k)(51 + 4k)} A_k + \frac{168 + 14k}{(49 + 4k)(51 + 4k)} B_k & b_{k+1,3} &= \frac{1}{49 + 4k} B_k, \end{aligned}$$

where

$$\begin{aligned} A_k &= -7\{a_{k,0} + a_{k,1}\} + (k+4)\{b_{k,0} + b_{k,1}\} \\ B_k &= -5\{a_{k,1} + a_{k,2}\} + (k+6)\{b_{k,1} + b_{k,2}\}. \end{aligned}$$

By the assumptions we have that $A_k = 0$ and $B_k = -5a_{k,2} + (k+6)b_{k,2}$ for all $k \geq j+4$. Therefore we obtain that for $k \geq j+5$

$$\begin{aligned} |a_{k+1,2}| &\leq \frac{1}{2}|a_{k,2}| & |b_{k+1,2}| &\leq \frac{1}{2}|b_{k,2}| \\ |a_{k+1,3}| &\leq \frac{1}{2}|a_{k,3}| & |b_{k+1,3}| &\leq \frac{1}{2}|b_{k,3}|, \end{aligned}$$

hence α is smooth on $|y| < 2$.

iii) Case $n \geq 4$. We take $b_{j+3,r} = b_{j+2,r} = 0$ for $0 \leq r \leq n$,

$$\begin{aligned} a_{j+2,0} &= 1/(2n+5) & a_{j+2,r} &= 0, \quad r \geq 1 \\ a_{j+3,1} &= 2/(2n+5)(2n+3) & a_{j+3,2} &= 4/(2n+1)(2n+5) \\ a_{j+3,r} &= 0, \quad 3 \leq r \leq n & a_{j+3,0} &= 0, \end{aligned}$$

for $k \geq j+3$ and $2 \leq r \leq n-2$

$$\begin{aligned} a_{k+1,0} &= 0 & b_{k+1,0} &= 0 & a_{k+1,1} &= 0 \\ b_{k+1,1} &= 0 & a_{k+1,r} &= -a_{k,r-1} - a_{k,r-2} & b_{k+1,r} &= -b_{k,r-1} - b_{k,r-2}. \end{aligned}$$

Thus we obtain that for $k \geq j+3$

$$\begin{aligned} a_{k+1,n} &= \frac{k+5+2n}{25+4k+8n} B_k \\ b_{k+1,n} &= \frac{1}{25+4k+8n} B_k \\ a_{k+1,n-1} &= \frac{4k^2 + k(37+16n) + 16n^2 + 74n + 75}{(27+4k+8n)(25+4k+8n)} A_k + \frac{2k^2 + k(8+8n) + 8n^2 + 16n - 24}{(25+4k+8n)(27+4k+8n)} B_k \\ b_{k+1,n-1} &= \frac{75+12k+24n}{(27+4k+8n)(25+4k+8n)} A_k + \frac{84+14k+28n}{(27+4k+8n)(25+4k+8n)} B_k, \end{aligned}$$

where

$$\begin{aligned} A_k &= -7\{a_{k,n-2} + a_{k,n-3}\} + (k-2+2n)\{b_{k,n-2} + b_{k,n-3}\} \\ B_k &= -5\{a_{k,n-1} + a_{k,n-2}\} + (k+2n)\{b_{k,n-1} + b_{k,n-2}\}. \end{aligned}$$

A simple calculation shows that for $k \geq j + n + 1$, $0 \leq r \leq n - 2$

$$a_{k,r} = 0 \quad b_{k,r} = 0 \quad A_k = 0 \quad B_k = -5a_{k,n-1} + (k + 2n)b_{k,n-1},$$

and for $k \geq j + n + 2$

$$\begin{aligned} |a_{k+1,n-1}| &\leq \frac{1}{2}|a_{k,n-1}| & |b_{k+1,n-1}| &\leq \frac{1}{2}|b_{k,n-1}| \\ |a_{k+1,n}| &\leq \frac{1}{2}|a_{k,n}| & |b_{k+1,n}| &\leq \frac{1}{2}|b_{k,n}|, \end{aligned}$$

thus α is smooth on $|y| < 2$.

□

Now we consider the case when i is odd.

Lemma 4.2.5. *Let $j \in \mathbb{Z}$, $j \geq 0$, $i = 2n + 1$, $n \geq 2$. Then there exists $\alpha \in \Omega^1(\mathbb{C}^2)$ such that $x^i y^j f dx \wedge dy = d(f\alpha)$.*

Proof. Consider $\alpha = gdx + hdy$, where

$$g = \sum_{k \geq j+2} \sum_{r=0}^n b_{k,r} x^{2n+1-2r} y^{k-1+2r} \quad h = \sum_{k \geq j+2} \sum_{r=0}^{n+1} a_{k,r} x^{2n+2-2r} y^{k-2+2r}.$$

Substituting in 4.2, we have that $x^{2n+5}y^j + x^{2n+1}y^{j+5} + x^{2n+3}y^{j+3}$ is equal to

$$\begin{aligned} &4a_{j+2,n+1}x^3y^{j+2+2n} + \{(2n+6)a_{j+2,0} - (j+1)b_{j+2,0}\}x^{2n+5}y^j + \\ &\{(2n+6)a_{j+3,0} - (j+2)b_{j+3,0}\}x^{2n+5}y^{j+1} + \{(2n+4)a_{j+2,1} - (j+3)b_{j+2,1}\}x^{2n+3}y^{j+2} + \\ &\sum_{k \geq j+2} \{2a_{k,n+1} + 2a_{k,n} - (k+4+2n)b_{k,n}\}xy^{k+3+2n} \\ &\sum_{k \geq j+2} \{4a_{k,n} + 4a_{k,n-1} + 4a_{k+1,n+1} - (k+2+2n)b_{k,n} - (k+2+2n)b_{k,n-1}\}x^3y^{k+1+2n} \\ &\sum_{k \geq j+2} \{(2n+6)a_{k+2,0} - (k+1)b_{k+2,0}\}x^{2n+5}y^k + \\ &\sum_{k \geq j+2} \{(2n+4)a_{k,0} + (2n+4)a_{k+1,1} - (k+2)b_{k,0} - (k+2)b_{k+1,1}\}x^{2n+3}y^{k+1} + \\ &\sum_{r=0}^{n-2} \{(2n+2-2r)a_{j+2,r+2} - (j+5+2r)b_{j+2,r+2}\}x^{2n+1-2r}y^{j+4+2r} + \\ &\sum_{k \geq j+2} \sum_{r=0}^{n-2} \{(2n+2-2r)a_{k+1,r+2} - (k+4+2r)b_{k+1,r+2}\}x^{2n+1-2r}y^{k+3+2r} + \\ &\sum_{k \geq j+2} \sum_{r=0}^{n-2} \{(2n+2-2r)a_{k,r} - (k+4+2r)b_{k,r}\}x^{2n+1-2r}y^{k+3+2r} + \\ &\sum_{k \geq j+2} \sum_{r=0}^{n-2} \{(2n+2-2r)a_{k,r+1} - (k+4+2r)b_{k,r+1}\}x^{2n+1-2r}y^{k+3+2r}, \end{aligned}$$

thus $(x^{5+2n}y^j + x^{2n+1}y^{j+5} + x^{2n+3}y^{j+3})dx \wedge dy = d(f\alpha)$ if and only if

$$\begin{aligned}
a_{j+2,n+1} &= 0 \\
(2n+6)a_{j+2,0} - (j+1)b_{j+2,0} &= 1 \\
(2n+6)a_{j+3,0} - (j+2)b_{j+3,0} &= 0 \\
(2n+4)a_{j+2,1} - (j+3)b_{j+2,1} &= 0 \\
2a_{k,n+1} + 2a_{k,n} - (k+4+2n)b_{k,n} &= 0, \quad k \geq j+2 \\
4a_{k,n} + 4a_{k,n-1} + 4a_{k+1,n+1} - (k+2+2n)b_{k,n} - (k+2+2n)b_{k,n-1} &= 0, \quad k \geq j+2 \\
(2n+6)a_{k+2,0} - (k+1)b_{k+2,0} &= 0, \quad k \geq j+2 \\
(2n+4)a_{j+2,0} + (2n+4)a_{j+3,1} - (j+4)b_{j+2,0} - (j+4)b_{j+3,1} &= 1 \\
(2n+4)a_{k,0} + (2n+4)a_{k+1,1} - (k+2)b_{k,0} - (k+2)b_{k+1,1} &= 0, \quad k \geq j+3 \\
(2n+2-2r)a_{j+2,r+2} - (j+5+2r)b_{j+2,r+2} &= 0, \quad 0 \leq r \leq n-2 \\
(2n+2)\{a_{j+3,2} + a_{j+2,0} + a_{j+2,1}\} - (j+6)\{b_{j+3,2} + b_{j+2,0} + b_{j+2,1}\} &= 1 \\
(2n+2-2r)\{a_{j+3,r+2} + a_{j+2,r} + a_{j+2,r+1}\} - \\
- (j+6+2r)\{b_{j+3,r+2} + b_{j+2,r} + b_{j+2,r+1}\} &= 0, \quad 1 \leq r \leq n-2 \\
(2n+2-2r)\{a_{k+1,r+2} + a_{k,r} + a_{k,r+1}\} - \\
- (k+4+2r)\{b_{k+1,r+2} + b_{k,r} + b_{k,r+1}\} &= 0, \quad k \geq j+3, \quad 0 \leq r \leq n-2.
\end{aligned}$$

Reordering

$$\begin{aligned}
a_{j+2,n+1} &= 0 \\
(2n+6)a_{j+2,0} - (j+1)b_{j+2,0} &= 1 \\
(2n+6)a_{j+3,0} - (j+2)b_{j+3,0} &= 0 \\
(2n+4)a_{j+2,1} - (j+3)b_{j+2,1} &= 0 \\
2a_{j+2,n+1} + 2a_{j+2,n} - (j+6+2n)b_{j+2,n} &= 0 \\
2a_{j+3,n+1} + 2a_{j+3,n} - (j+7+2n)b_{j+3,n} &= 0 \\
4a_{j+2,n} + 4a_{j+2,n-1} + 4a_{j+3,n+1} - (j+4+2n)b_{j+2,n} - (j+4+2n)b_{j+2,n-1} &= 0 \\
(2n+4)a_{j+2,0} + (2n+4)a_{j+3,1} - (j+4)b_{j+2,0} - (j+4)b_{j+3,1} &= 1 \\
(2n+2-2r)a_{j+2,r+2} - (j+5+2r)b_{j+2,r+2} &= 0, \quad 0 \leq r \leq n-2 \\
(2n+2)\{a_{j+3,2} + a_{j+2,0} + a_{j+2,1}\} - (j+6)\{b_{j+3,2} + b_{j+2,0} + b_{j+2,1}\} &= 1 \\
(2n+2-2r)\{a_{j+3,r+2} + a_{j+2,r} + a_{j+2,r+1}\} - \\
- (j+6+2r)\{b_{j+3,r+2} + b_{j+2,r} + b_{j+2,r+1}\} &= 0, \quad 1 \leq r \leq n-2 \\
\forall k \geq j+3 \\
(2n+6)a_{k+1,0} - kb_{k+1,0} &= 0 \\
2a_{k,n+1} + 2a_{k,n} - (k+4+2n)b_{k,n} &= 0 \\
4a_{k,n} + 4a_{k,n-1} + 4a_{k+1,n+1} - (k+2+2n)b_{k,n} - (k+2+2n)b_{k,n-1} &= 0 \\
(2n+4)a_{k,0} + (2n+4)a_{k+1,1} - (k+2)b_{k,0} - (k+2)b_{k+1,1} &= 0 \\
(2n+2-2r)\{a_{k+1,r+2} + a_{k,r} + a_{k,r+1}\} - (k+4+2r)\{b_{k+1,r+2} + b_{k,r} + b_{k,r+1}\} &= 0, \quad 0 \leq r \leq n-2.
\end{aligned}$$

As in the even case, we find solutions to this system in 3 cases: $n = 2$, $n = 3$ and $n \geq 4$.

i) Case $n = 2$. Assuming for all $k \geq j+3$, $a_{k,0} = b_{k,0} = a_{k,1} = b_{k,1} = a_{j+2,1} = b_{j+2,1} = 0$,

$$\begin{aligned}
a_{j+2,2} = 0 & & b_{j+2,2} = 0 & & a_{j+2,0} = \frac{3}{2(16+j)} & & a_{j+3,2} = \frac{11+j}{2(16+j)(27+2j)} \\
a_{j+2,3} = 0 & & a_{j+3,3} = 0 & & b_{j+2,0} = \frac{-1}{16+j} & & b_{j+3,2} = \frac{1}{(16+j)(27+2j)}
\end{aligned}$$

we obtain that for $k \geq j+3$

$$\begin{aligned}
a_{k+1,2} &= -\frac{4+k}{4(23+2k)}\{4a_{k,2} - (k+6)b_{k,2}\} & b_{k+1,2} &= -\frac{3}{2(23+2k)}\{4a_{k,2} - (k+6)b_{k,2}\} \\
a_{k+1,3} &= -\frac{1}{4}\{4a_{k,2} - (k+6)b_{k,2}\}.
\end{aligned}$$

Thus we obtain that for $k \geq j+4$

$$|a_{k+1,2}| \leq \frac{1}{2}|a_{k,2}| \quad |b_{k+1,2}| \leq \frac{1}{2}|b_{k,2}| \quad |a_{k+1,3}| \leq \frac{1}{2}|a_{k,2}|,$$

therefore α is smooth on $|y| < 2$.

- ii) Case $n = 3$. We assume that for all $k \geq j+3$, $a_{j+3,4} = a_{j+2,1} = b_{j+2,1} = a_{k,0} = b_{k,0} = a_{k,1} = b_{k,1} = 0$,

$$\begin{aligned}
a_{j+2,2} = 0 & & b_{j+2,2} = 0 & & a_{j+2,3} = 0 & & b_{j+2,3} = 0 \\
a_{j+3,3} = 0 & & b_{j+3,3} = 0 & & a_{j+2,0} = \frac{3}{2(19+j)} & & b_{j+2,0} = -\frac{1}{19+j} \\
a_{j+3,2} = \frac{14+j}{7(19+j)} & & b_{j+3,2} = \frac{1}{19+j}.
\end{aligned}$$

Thus for all $k \geq j+3$ we obtain

$$\begin{aligned}
a_{k+1,3} &= -\frac{6+k}{4(27+2k)}A_k + \frac{22+2k}{4(27+2k)}B_k & a_{k+1,2} &= \frac{k+4}{8}b_{k+1,2} \\
b_{k+1,3} &= -\frac{3}{2(27+2k)}A_k + \frac{1}{(27+2k)}B_k & a_{k+1,4} &= -\frac{1}{4}A_k,
\end{aligned}$$

where

$$A_k = 4\{a_{k,3} + a_{k,2}\} - (k+8)\{b_{k,3} + b_{k,2}\} \quad B_k = -6a_{k,2} + (k+6)b_{k,2}.$$

Considering $a_{k,2} = b_{k,2} = 0$ for all $k \geq j+4$ we obtain $B_k = 0$ and $A_k = 4a_{k,3} - (k+8)b_{k,3}$. Thus for $k \geq j+5$ we have

$$|a_{k+2,3}| \leq \frac{1}{2}|a_{k+1,3}| \quad |b_{k+2,3}| \leq \frac{1}{2}|b_{k+1,3}| \quad |b_{k+2,4}| \leq \frac{1}{2}|b_{k+1,4}|.$$

Consequently, α is smooth on $|y| < 2$.

- iii) Case $n \geq 4$. We take

$$\begin{aligned}
a_{j+2,0} &= \frac{3}{2(10+j+3n)} & b_{j+2,0} &= -\frac{1}{10+j+3n} \\
a_{j+2,r} &= b_{j+2,r} = 0, \quad r \geq 1 & a_{j+3,2} &= \frac{6n+17}{2(10+j+3n)(2n+2)} \\
b_{j+3,2} &= 0 & a_{k,0} &= b_{k,0} = a_{k,1} = b_{k,1} = 0, \quad k \geq j+3 \\
a_{j+3,r} &= b_{j+3,r} = 0, \quad r \geq 3,
\end{aligned}$$

for $k \geq j + 3$ and $0 \leq r \leq n - 3$

$$a_{k+1,r+2} = -a_{k,r} - a_{k,r+1}$$

$$b_{k+1,r} = -b_{k,r} - b_{k,r-1}.$$

Therefore for all $k \geq j + 3$ we obtain

$$\begin{aligned} a_{k+1,n} &= -\frac{k+2n}{4(15+2k+4n)}A_k + \frac{10+2k+4n}{4(15+2k+4n)}B_k & a_{k+1,n+1} &= -\frac{1}{4}A_k \\ b_{k+1,n} &= -\frac{3}{2(15+2k+4n)}A_k + \frac{1}{15+2k+4n}B_k, \end{aligned}$$

where

$$\begin{aligned} A_k &= 4\{a_{k,n} + a_{k,n-1}\} - (k+2+2n)\{b_{k,n} + b_{k,n-1}\} \\ B_k &= -6\{a_{k,n-1} + a_{k,n-2}\} + (k+2n)\{b_{k,n-1} + b_{k,n-2}\}. \end{aligned}$$

A simple calculation shows that for all $k \geq j + n + 1$ and $0 \leq r \leq n - 1$

$$a_{k,r} = 0 \quad b_{k,r} = 0 \quad A_k = 4a_{k,n} - (k+2+2n)b_{k,n} \quad B_k = 0,$$

and for all $k \geq j + n + 2$

$$|a_{k+1,n}| \leq \frac{1}{2}|a_{k,n}| \quad |b_{k+1,n}| \leq \frac{1}{2}|b_{k,n}| \quad |a_{k+1,n+1}| \leq \frac{1}{2}|a_{k,n+1}|,$$

thus α is smooth on $|y| < 2$.

□

Lemma 4.2.6. *Let $j \in \mathbb{N}$, $j \geq 2$. Then there exists $\alpha \in \Omega^1(\mathbb{C}^2)$ such that $y^j f dx \wedge dy = d(f\alpha)$.*

Proof. Suppose $\alpha = gdx + hdy$, where

$$g = \sum_{k \geq j} \{b_{k+1,k-1}x^2y^{k-1} + b_{k+1,k+1}y^{k+1}\} \quad h = \sum_{k \geq j} \{a_{k+1,k-2}x^3y^{k-2} + a_{k+1,k}xy^k\}.$$

Substituting in 4.2, we have that $x^4y^j + y^{j+5} + x^2y^{j+3}$ is equal to

$$\begin{aligned} &\{5a_{j+1,j} - (j+1)b_{j+1,j+1}\}x^4y^j + \\ &\sum_{k \geq j-2} \{7a_{k+3,k} - (k+1)b_{k+3,k+1}\}x^6y^k + \\ &\sum_{k \geq j+1} \{5a_{k,k-3} - (k+1)b_{k,k-2} + 5a_{k+1,k} - (k+1)b_{k+1,k+1}\}x^4y^k + \\ &\sum_{k \geq j+3} \{3a_{k-2,k-3} + 3a_{k-2,k-5} - (k+1)b_{k-2,k-4} - (k+1)b_{k-2,k-2}\}x^2y^k + \\ &\sum_{k \geq j+5} \{a_{k-4,k-5} - (k+1)b_{k-4,k-4}\}y^k, \end{aligned}$$

thus $(x^4y^j + y^{j+5} + x^2y^{j+3})dx \wedge dy = d(f\alpha)$ if and only if

$$\begin{aligned}
5a_{j+1,j} - (j+1)b_{j+1,j+1} &= 1 \\
3a_{j+1,j} + 3a_{j+1,j-2} - (j+4)b_{j+1,j-1} - (j+4)b_{j+1,j+1} &= 1 \\
a_{j+1,j} - (j+6)b_{j+1,j+1} &= 1 \\
7a_{j+1,j-2} - (j-1)b_{j+1,j-1} &= 0 \\
7a_{k+3,k} - (k+1)b_{k+3,k+1} &= 0, \quad k \geq j-1 \\
5a_{k,k-3} - (k+1)b_{k,k-2} + 5a_{k+1,k} - (k+1)b_{k+1,k+1} &= 0, \quad k \geq j+1 \\
3a_{k-2,k-3} + 3a_{k-2,k-5} - (k+1)b_{k-2,k-4} - (k+1)b_{k-2,k-2} &= 0, \quad k \geq j+4 \\
a_{k-4,k-5} - (k+1)b_{k-4,k-4} &= 0, \quad k \geq j+6,
\end{aligned}$$

or equivalently

$$\begin{aligned}
5a_{j+1,j} - (j+1)b_{j+1,j+1} &= 1 \\
a_{j+1,j} - (j+6)b_{j+1,j+1} &= 1 \\
7a_{j+1,j-2} - (j-1)b_{j+1,j-1} &= 0 \\
3a_{j+1,j} + 3a_{j+1,j-2} - (j+4)b_{j+1,j-1} - (j+4)b_{j+1,j+1} &= 1, \\
\forall k \geq j-1 \\
7a_{k+3,k} - (k+1)b_{k+3,k+1} &= 0 \\
a_{k+3,k+2} - (k+8)b_{k+3,k+3} &= 0 \\
5a_{k+2,k-1} - (k+3)b_{k+2,k} + 5a_{k+3,k+2} - (k+3)b_{k+3,k+3} &= 0 \\
3a_{k+3,k+2} + 3a_{k+3,k} - (k+6)b_{k+3,k+1} - (k+6)b_{k+3,k+3} &= 0.
\end{aligned}$$

Solving this system, we obtain

$$\begin{aligned}
a_{j+1,j} &= \frac{5}{29+4j} & b_{j+1,j+1} &= -\frac{4}{29+4j} \\
a_{j+1,j-2} &= \frac{2(j-1)}{(29+4j)(31+4j)} & b_{j+1,j-1} &= \frac{14}{(29+4j)(31+4j)},
\end{aligned}$$

and for $k \geq j-1$

$$\begin{aligned}
a_{k+3,k+2} &= \frac{8+k}{39+4k} \{(k+3)b_{k+2,k} - 5a_{k+2,k-1}\} \\
a_{k+3,k} &= \frac{2(7+8k+k^2)}{(39+4k)^2} \{(k+3)b_{k+2,k} - 5a_{k+2,k-1}\} \\
b_{k+3,k+1} &= \frac{14(7+k)}{(39+4k)^2} \{(k+3)b_{k+2,k} - 5a_{k+2,k-1}\} \\
b_{k+3,k+3} &= \frac{1}{39+4k} \{(k+3)b_{k+2,k} - 5a_{k+2,k-1}\}.
\end{aligned}$$

Thus for all $k \geq j$

$$\begin{aligned}
|a_{k+3,k+2}| &\leq \frac{1}{2}|a_{k+2,k+1}| & |b_{k+3,k+3}| &\leq \frac{1}{2}|b_{k+2,k+2}| \\
|a_{k+3,k}| &\leq \frac{1}{2}|a_{k+2,k-1}| & |b_{k+3,k+1}| &\leq \frac{1}{2}|b_{k+2,k}|,
\end{aligned}$$

and therefore the 1-form α is smooth on $|y| < 2$. □

Lemma 4.2.7. *There exists $\alpha \in \Omega^1(\mathbb{C}^2)$ such that $yfdx \wedge dy = d(f\alpha)$.*

Proof. Suppose $\alpha = gdx + hdy$, where

$$g = \sum_{k \geq 2} \{b_{k,k-2}x^2y^{k-2} + b_{k,k}y^k\} \quad h = a_{2,1}xy + \sum_{k \geq 3} \{a_{k,k-3}x^3y^{k-3} + a_{k,k-1}xy^{k-1}\},$$

Substituting in 4.2, we have that $x^4y + y^6 + x^2y^4$ is equal to

$$\begin{aligned} & \{5a_{2,1} - 2b_{2,2}\}x^4y + \{a_{2,1} - 7b_{2,2}\}y^6 + \{3a_{2,1} - 5b_{2,2} - 5b_{2,0}\}x^2y^4 + \\ & \{5a_{3,2} - 3b_{3,3} - 3b_{2,0}\}x^4y^2 + \sum_{k \geq 0} \{7a_{k+3,k} - (k+1)b_{k+3,k+1}\}x^6y^k + \\ & \sum_{k \geq 3} \{5a_{k,k-3} - (k+1)b_{k,k-2} + 5a_{k+1,k} - (k+1)b_{k+1,k+1}\}x^4y^k + \\ & \sum_{k \geq 5} \{3a_{k-2,k-3} + 3a_{k-2,k-5} - (k+1)b_{k-2,k-4} - (k+1)b_{k-2,k-2}\}x^2y^k + \\ & \sum_{k \geq 7} \{a_{k-4,k-5} - (k+1)b_{k-4,k-4}\}y^k, \end{aligned}$$

thus $(x^4y + y^6 + x^2y^4)dx \wedge dy = y^j f dx \wedge dy = d(f\alpha)$ if and only if

$$\begin{aligned} 5a_{2,1} - 2b_{2,2} &= 1 \\ a_{2,1} - 7b_{2,2} &= 1 \\ 3a_{2,1} - 5b_{2,2} - 5b_{2,0} &= 1 \\ 5a_{3,2} - 3b_{3,3} - 3b_{2,0} &= 0 \\ 7a_{k+3,k} - (k+1)b_{k+3,k+1} &= 0, \quad k \geq 0 \\ 5a_{k,k-3} - (k+1)b_{k,k-2} + 5a_{k+1,k} - (k+1)b_{k+1,k+1} &= 0, \quad k \geq 3 \\ 3a_{k-2,k-3} + 3a_{k-2,k-5} - (k+1)b_{k-2,k-4} - (k+1)b_{k-2,k-2} &= 0, \quad k \geq 5 \\ a_{k-4,k-5} - (k+1)b_{k-4,k-4} &= 0, \quad k \geq 7, \end{aligned}$$

or equivalently

$$a_{2,1} = 5/33 \quad b_{2,2} = -4/33 \quad b_{2,0} = 2/165 \quad 5a_{3,2} - 3b_{3,3} = 3b_{2,0},$$

and for $k \geq 0$

$$\begin{aligned} 7a_{k+3,k} - (k+1)b_{k+3,k+1} &= 0 \\ a_{k+3,k+2} - (k+8)b_{k+3,k+3} &= 0 \\ 5a_{k+3,k} - (k+4)b_{k+3,k+1} + 5a_{k+4,k+3} - (k+4)b_{k+4,k+4} &= 0 \\ 3a_{k+3,k+2} + 3a_{k+3,k} - (k+6)b_{k+3,k+1} - (k+6)b_{k+3,k+3} &= 0. \end{aligned}$$

Thus

$$a_{3,0} = 12/26455 \quad b_{3,1} = 84/26455 \quad a_{3,2} = 16/2035 \quad b_{3,3} = 2/2035,$$

and for $k \geq 0$

$$\begin{aligned} 7a_{k+4,k+1} - (k+2)b_{k+4,k+2} &= 0 \\ 5a_{k+4,k+3} - (k+4)b_{k+4,k+4} - 5a_{k+3,k} + (k+4)b_{k+3,k+1} &= 0 \\ a_{k+4,k+3} - (k+9)b_{k+4,k+4} &= 0 \\ 3a_{k+4,k+3} + 3a_{k+4,k+1} - (k+6)b_{k+4,k+2} - (k+6)b_{k+4,k+4} &= 0. \end{aligned}$$

Therefore for $k \geq 0$

$$\begin{aligned} a_{k+4,k+1} &= \frac{(k+2)(2k+21)}{4(4k^2+77k+369)} \{-5a_{k+3,k} + (k+4)b_{k+3,k+1}\} \\ a_{k+4,k+3} &= \frac{k+9}{4k+41} \{-5a_{k+3,k} + (k+4)b_{k+3,k+1}\} \\ b_{k+4,k+4} &= \frac{1}{4k+41} \{-5a_{k+3,k} + (k+4)b_{k+3,k+1}\} \\ b_{k+4,k+2} &= \frac{7(2k+21)}{4(4k^2+77k+369)} \{-5a_{k+3,k} + (k+4)b_{k+3,k+1}\}, \end{aligned}$$

hence for $k \geq 0$

$$\begin{aligned} |a_{k+4,k+1}| &\leq \frac{1}{2}|a_{k+3,k}| & |b_{k+4,k+4}| &\leq \frac{1}{2}|b_{k+3,k+3}| \\ |a_{k+4,k+3}| &\leq \frac{1}{2}|a_{k+3,k+2}| & |b_{k+4,k+2}| &\leq \frac{1}{2}|b_{k+3,k+1}|. \end{aligned}$$

So, α is smooth on $|y| < 2$. □

In conclusion, we have proved the next proposition:

Proposition 4.2.8. *Let i, j be non negatives integers such that $i + j \geq 1$. Then there exists $\alpha \in \Omega^1(\mathbb{C}^2)$ which is smooth on $|y| < 2$, such that*

$$x^i y^j f dx \wedge dy = d(f\alpha). \quad (4.3)$$

We have proved the existence of $\alpha = gdx + hdy$ satisfying equation 4.3 where $g = g(x, y)$ and $h = h(x, y)$ are smooth on $|y| < 2$. In order to prove the general case, we want to bound the norm of g and h in a neighborhood around the origin for x and y . To do it we use the previous calculations. Let $j \geq 0$ be an integer.

a) For $xy^j f dx \wedge dy$:

$$\begin{aligned} g &= \sum_{k \geq j+2} b_{k,k-1} xy^{k-1} & h &= \sum_{k \geq j+2} \{a_{k,k-2} x^2 y^{k-2} + a_{k,k} y^k\} \\ a_{j+2,j+2} &= 0 & a_{j+2,j} &= 5/(4j+34) \\ b_{j+2,j+1} &= -4/(4j+34) & a_{j+3,j+3} &= -1/(8j+68) \\ k &\geq j+1 \\ a_{k+2,k} &= \frac{2k+2}{4k+34} a_{k+2,k+2} & b_{k+2,k+1} &= \frac{12}{4k+34} a_{k+2,k+2} \\ a_{k+3,k+3} &= \frac{k+10}{4k+34} a_{k+2,k+2}. \end{aligned}$$

Note that for $k \geq j+1$

$$\begin{aligned} |a_{k+3,k+1}| &\leq \frac{1}{2}|a_{k+2,k}| & |b_{k+3,k+2}| &\leq \frac{1}{2}|b_{k+2,k+1}| \\ |a_{k+3,k+3}| &\leq \frac{1}{2}|a_{k+2,k+2}|, \end{aligned}$$

and $|a_{j+3,j+1}| \leq \frac{1}{2}|a_{j+2,j}|$, $|b_{j+3,j+2}| \leq \frac{1}{2}|b_{j+2,j+1}|$. Thus by induction we have that for all $k \geq j$

$$\begin{aligned} |a_{k+3,k+1}| &\leq \left(\frac{1}{2}\right)^{k-j} |a_{k+2,k}| & |b_{k+3,k+2}| &\leq \left(\frac{1}{2}\right)^{k-j} |b_{k+2,k+1}| \\ |b_{k+4,k+4}| &\leq \left(\frac{1}{2}\right)^{k+1-j} |b_{k+3,k+3}|. \end{aligned}$$

Hence for $|y| < 2$ we have

$$\begin{aligned} |g| &\leq |x||b_{j+2,j+1}| \sum_{k \geq j} \left(\frac{1}{2}\right)^{k-j} |y|^{k+1} \\ &\leq |2b_{j+2,j+1}| \frac{|x||y|^{j+1}}{2-|y|} = \frac{4}{2j+17} \frac{|x||y|^{j+1}}{2-|y|} \\ |h| &\leq |a_{j+2,j}||x|^2 \sum_{k \geq j} \left(\frac{1}{2}\right)^{k-j} |y|^k + |a_{j+3,j+3}| \sum_{k \geq j+3} \left(\frac{1}{2}\right)^{k-j-3} |y|^k \\ &\leq |2a_{j+2,j}| \frac{|x|^2|y|^j}{2-|y|} + |2a_{j+3,j+3}| \frac{|y|^{j+3}}{2-|y|} \\ &\leq \frac{5}{2j+17} \frac{|x|^2|y|^j}{2-|y|} + \frac{1}{4j+34} \frac{|y|^{j+3}}{2-|y|}. \end{aligned}$$

Therefore for $R < 1$, $|x| < R$ and $|y| < R$ then

$$\begin{aligned} |g(x,y)| = |g| &\leq \frac{4}{2j+17} R^{j+2} \leq \frac{4}{2j+17} R^{j+1} \leq R^{j+1} \\ |h(x,y)| = |h| &\leq \left(\frac{5}{2j+17} + \frac{1}{4j+34}\right) R^{j+3} \leq R^{j+1}. \end{aligned}$$

b) For $x^2 y^j f dx \wedge dy$

$$\begin{aligned} g &= \sum_{k \geq j+2} \{b_{k+1,k-1} x^2 y^{k-1} + b_{k+1,k+1} y^{k+1}\} \\ h &= \sum_{k \geq j+2} \{a_{k+1,k-2} x^3 y^{k-2} + a_{k+1,k} x y^k\} \end{aligned}$$

$$\begin{aligned} a_{j+3,j+2} &= 0 & b_{j+3,j+3} &= 0 \\ a_{j+3,j} &= 5/(4j+39) & b_{j+3,j+1} &= -4/(4j+39) \\ a_{j+4,j+3} &= -2(j+9)/(4j+39)(4j+41) & b_{j+4,j+4} &= -2/(4j+39)(4j+41), \end{aligned}$$

for $k \geq j+1$

$$\begin{aligned} a_{k+3,k+2} &= (k+8)b_{k+3,k+3} \\ a_{k+3,k} &= \frac{(2k+18)(k+1)}{4k+39} b_{k+3,k+3} \\ b_{k+3,k+1} &= \frac{7(2k+18)}{4k+39} b_{k+3,k+3} \\ b_{k+4,k+4} &= \frac{(2k+18)(2k+23)}{(4k+39)(4k+41)} b_{k+3,k+3} \\ a_{k+4,k+3} &= \frac{(2k+18)(2k+23)(k+9)}{(4k+39)(4k+41)} b_{k+3,k+3} \end{aligned}$$

$$\begin{aligned}
|a_{k+4,k+1}| &\leq \frac{1}{2}|a_{k+3,k}| & |b_{k+4,k+4}| &\leq \frac{1}{2}|b_{k+3,k+3}| \\
|a_{k+4,k+3}| &\leq \frac{1}{2}|a_{k+3,k+2}| & |b_{k+4,k+2}| &\leq \frac{1}{2}|b_{k+3,k+1}| \\
|a_{j+4,j+1}| &\leq \frac{1}{2}|a_{j+3,j}| & |b_{j+4,j+2}| &\leq \frac{1}{2}|b_{j+3,k+1}|.
\end{aligned}$$

Thus for $|y| < 2$ we have

$$\begin{aligned}
|g| &\leq |2b_{j+3,j+1}| \frac{|x|^2|y|^{j+1}}{2-|y|} + |2b_{j+4,j+4}| \frac{|y|^{j+4}}{2-|y|} \\
&\leq \frac{8}{4j+39} \frac{|x|^2|y|^{j+1}}{2-|y|} + \frac{4}{(4j+39)(4j+41)} \frac{|y|^{j+4}}{2-|y|} \\
|h| &\leq |2a_{j+3,j}| \frac{|x|^3|y|^j}{2-|y|} + |2a_{j+4,j+3}| \frac{|x||y|^{j+3}}{2-|y|} \\
&\leq \frac{10}{4j+39} \frac{|x|^3|y|^j}{2-|y|} + \frac{4(j+9)}{(4j+39)(4j+41)} \frac{|x||y|^{j+3}}{2-|y|}.
\end{aligned}$$

Hence if $R < 1$, $|x| < R$ and $|y| < R$ we have

$$\begin{aligned}
|g| &\leq \left(\frac{8}{4j+39} + \frac{4}{(4j+39)(4j+41)} \right) R^{j+3} \leq R^{j+2} \\
|h| &\leq \left(\frac{10}{4j+39} + \frac{4(j+9)}{(4j+39)(4j+41)} \right) R^{j+3} \leq R^{j+2}.
\end{aligned}$$

c) For $x^3y^j f dx \wedge dy$

$$\begin{aligned}
g &= \sum_{k \geq j+2} \{b_{k+2,k-1}x^3y^{k-1} + b_{k+2,k+1}xy^{k+1}\} \\
h &= \sum_{k \geq j+2} \{a_{k+2,k-2}x^4y^{k-2} + a_{k+2,k}x^2y^k + a_{k+2,k+2}y^{k+2}\}
\end{aligned}$$

$$\begin{aligned}
a_{j+4,j+4} &= 0 & b_{j+4,j+3} &= 0 & a_{j+4,j+2} &= 0 & b_{j+5,j+4} &= \frac{1}{2j+13} \\
a_{j+4,j} &= 1/8 & a_{j+5,j+5} &= 1/8 & a_{j+5,j+3} &= \frac{2j+13}{8(2j+23)} & b_{j+4,j+1} &= 0
\end{aligned}$$

$$\begin{aligned}
k &\geq j+2 & a_{k+3,k-1} &= b_{k+2,k-1} = 0 \\
k &\geq j+3 & a_{k+3,k+3} &= \frac{1}{4}\{(k+4)b_{k+2,k+1} - 4a_{k+2,k}\} \\
& & a_{k+3,k+1} &= \frac{(k+2)}{2k+19}a_{k+3,k+3} \\
& & b_{k+3,k+2} &= \frac{6}{2k+19}a_{k+3,k+3}.
\end{aligned}$$

Then for $k \geq j+3$

$$\begin{aligned}
|a_{k+4,k+4}| &\leq \frac{1}{2}|a_{k+3,k+3}| & |b_{k+4,k+3}| &\leq \frac{1}{2}|b_{k+3,k+2}| \\
|a_{k+4,k+2}| &\leq \frac{1}{2}|a_{k+3,k+1}|
\end{aligned}$$

and

$$\begin{aligned} |a_{j+6,j+6}| &\leq \frac{1}{2}|a_{j+5,j+5}| & |b_{j+6,j+5}| &\leq \frac{1}{2}|b_{j+5,j+4}| \\ |a_{j+6,j+4}| &\leq \frac{1}{2}|a_{j+5,j+3}|. \end{aligned}$$

Then, for $|y| < 2$ we have

$$\begin{aligned} |g| &\leq |2b_{j+5,j+4}| \frac{|x||y|^{j+4}}{2-|y|} = \frac{2}{2j+13} \frac{|x||y|^{j+4}}{2-|y|} \\ |h| &\leq |a_{j+4,j}||x|^4|y|^j + |2a_{j+5,j+3}| \frac{|x|^2|y|^{j+3}}{2-|y|} + |2a_{j+5,j+5}| \frac{|y|^{j+5}}{2-|y|} \\ |h| &\leq \frac{1}{8}|x|^4|y|^j + \frac{2j+13}{4(2j+23)} \frac{|x|^2|y|^{j+3}}{2-|y|} + \frac{1}{4} \frac{|y|^{j+5}}{2-|y|}, \end{aligned}$$

therefore if $R < 1$, $|x| < R$ and $|y| < R$ we obtain

$$\begin{aligned} |g| &\leq \frac{2}{2j+13} R^{j+5} \leq R^{j+3} \\ |h| &\leq \left(\frac{1}{8} + \frac{2j+13}{4(2j+23)} + \frac{1}{4}\right) R^{j+4} \leq R^{j+3}. \end{aligned}$$

d) For $x^4 y^j f dx \wedge dy$

$$g = \sum_{k \geq j+2} \sum_{t=0}^2 b_{k,t} x^{4-2t} y^{k-1+2t} \quad h = \sum_{k \geq j+2} \sum_{t=0}^2 a_{k,t} x^{5-2t} y^{k-2+2t}$$

$$\begin{aligned} a_{j+2,2} &= 0 & b_{j+2,2} &= 0 \\ a_{j+2,1} &= 0 & b_{j+2,1} &= 0 \\ a_{j+2,0} &= 1/9 & b_{j+2,0} &= 0 \\ a_{j+3,1} &= \frac{2(633 + 149j + 8j^2)}{9(49 + 4j)(51 + 4j)} & b_{j+3,1} &= \frac{2(483 + 40j)}{9(49 + 4j)(51 + 4j)} \\ a_{j+3,2} &= \frac{4(11 + j)}{9(49 + 4j)} & b_{j+3,2} &= \frac{4}{9(49 + 4j)} \end{aligned}$$

$$k \geq j+3 \quad a_{k,0} = b_{k,0} = 0$$

$$\begin{aligned} a_{k+1,1} &= \frac{2(2+k)(10+k)}{(41+4k)(43+4k)} \{(k+4)b_{k,1} - 5a_{k,1}\} \\ b_{k+1,1} &= \frac{14(10+k)}{(41+4k)(43+4k)} \{(k+4)b_{k,1} - 5a_{k,1}\} \\ a_{k+1,2} &= \frac{9+k}{41+4k} \{(k+4)b_{k,1} - 5a_{k,1}\} \\ b_{k+1,2} &= \frac{1}{41+4k} \{(k+4)b_{k,1} - 5a_{k,1}\} \end{aligned}$$

$$\begin{aligned} |a_{k+2,1}| &\leq \frac{1}{2}|a_{k+1,1}| & |b_{k+2,1}| &\leq \frac{1}{2}|b_{k+1,1}| \\ |a_{k+2,2}| &\leq \frac{1}{2}|a_{k+1,2}| & |b_{k+2,2}| &\leq \frac{1}{2}|b_{k+1,2}| \\ |a_{j+4,1}| &\leq \frac{1}{2}|a_{j+3,1}| & |b_{j+4,1}| &\leq \frac{1}{2}|b_{j+3,1}| \\ |a_{j+4,2}| &\leq \frac{1}{2}|a_{j+3,2}| & |b_{j+4,2}| &\leq \frac{1}{2}|b_{j+3,2}|. \end{aligned}$$

Thus for $|y| < 2$ we have

$$|g| \leq \frac{4(40j+483)}{9(4j+49)(4j+51)} \frac{|x|^2|y|^{j+4}}{2-|y|} + \frac{8}{9(4j+49)} \frac{|y|^{j+6}}{2-|y|}$$

$$|h| \leq \frac{1}{9}|x|^5|y|^j + \frac{4(8j^2+149j+633)}{9(4j+49)(4j+51)} \frac{|x|^3|y|^{j+3}}{2-|y|} + \frac{8(11+j)}{9(49+4j)} \frac{|x||y|^{j+5}}{2-|y|}.$$

So, if $R < 1$, $|x| < R$ and $|y| < R$ then

$$|g| \leq \left(\frac{4(40j+483)}{9(4j+49)(4j+51)} + \frac{8}{9(4j+49)} \right) R^{j+6} \leq R^{j+4}$$

$$|h| \leq \left(\frac{1}{9} + \frac{4(8j^2+149j+633)}{9(4j+49)(4j+51)} + \frac{8(11+j)}{9(49+4j)} \right) R^{j+5} \leq R^{j+4}.$$

e) For $x^5 y^j dx \wedge dy$

$$g = \sum_{k \geq j+2} \sum_{t=0}^2 b_{k,t} x^{5-2t} y^{k-1+2t} \quad h = \sum_{k \geq j+2} \sum_{t=0}^3 a_{k,t} x^{6-2t} y^{k-2+2t}$$

$$\begin{aligned} a_{j+2,1} &= 0 & b_{j+2,1} &= 0 \\ a_{j+2,2} &= 0 & b_{j+2,2} &= 0 \\ a_{j+2,3} &= 0 & a_{j+3,3} &= 0 \\ a_{j+2,0} &= \frac{3}{2(16+j)} & b_{j+2,0} &= \frac{-1}{16+j} \\ a_{j+3,2} &= \frac{11+j}{2(16+j)(27+2j)} & b_{j+3,2} &= \frac{1}{(16+j)(27+2j)} \end{aligned}$$

$$\begin{aligned} k \geq j+3 & \quad a_{k,0} = b_{k,0} = a_{k,1} = b_{k,1} = 0 \\ a_{k+1,2} &= -\frac{4+k}{4(23+2k)} \{4a_{k,2} - (k+6)b_{k,2}\} \\ b_{k+1,2} &= -\frac{3}{2(23+2k)} \{4a_{k,2} - (k+6)b_{k,2}\} \\ a_{k+1,3} &= -\frac{1}{4} \{4a_{k,2} - (k+6)b_{k,2}\} \\ |a_{k+2,2}| &\leq \frac{1}{2} |a_{k+1,2}| & |b_{k+2,2}| &\leq \frac{1}{2} |b_{k+1,2}| \\ |a_{k+2,3}| &\leq \frac{1}{2} |a_{k+1,2}| \\ |a_{j+4,2}| &\leq \frac{1}{2} |a_{j+3,2}| & |b_{j+4,2}| &\leq \frac{1}{2} |2b_{j+3,2}|. \end{aligned}$$

Thus for $|y| < 2$ we have

$$|g| \leq \frac{1}{j+16} |x|^5 |y|^{j+1} + \frac{4}{(16+j)(2j+27)} \frac{|x||y|^{j+6}}{2-|y|}$$

$$|h| \leq \frac{2}{2(j+16)} |x|^6 |y|^j + \frac{11+j}{(16+j)(27+2j)} \frac{|x|^2 |y|^{j+5}}{2-|y|} + \frac{2j+26}{4(16+j)(27+2j)} \frac{|y|^{j+8}}{2-|y|}.$$

So, if $R < 1$, $|x| < R$ and $|y| < R$ then

$$|g| \leq \left(\frac{1}{j+16} + \frac{4}{(16+j)(2j+27)} \right) R^{j+6} \leq R^{j+5}$$

$$|h| \leq \left(\frac{2}{2(j+16)} + \frac{11+j}{(16+j)(27+2j)} + \frac{2j+26}{4(16+j)(27+2j)} \right) R^{j+6} \leq R^{j+5}.$$

f) For $x^6 y^j f dx \wedge dy$

$$g = \sum_{k \geq j+2} \sum_{r=0}^3 b_{k,r} x^{6-2r} y^{k-1+2r} \quad h = \sum_{k \geq j+2} \sum_{r=0}^3 a_{k,r} x^{7-2r} y^{k-2+2r}$$

$$b_{j+2,r} = 0, \quad 0 \leq r \leq 3$$

$$a_{j+2,0} = 1/11$$

$$a_{j+3,1} = 2/99$$

$$b_{j+3,3} = 0$$

$$a_{j+3,2} = \frac{4(11+j)}{11(59+4j)}$$

$$a_{j+2,r} = 0, \quad r \geq 1$$

$$a_{j+3,3} = 0$$

$$b_{j+3,1} = 0$$

$$b_{j+3,2} = \frac{12}{11(59+4j)}$$

$$k \geq j+3$$

$$a_{k,0} = b_{k,0} = a_{k+1,1} = b_{k+1,1} = 0$$

$$a_{k+1,2} = \frac{441 + 85k + 4k^2}{(49+4k)(51+4k)} A_k + \frac{96 + 32k + 2k^2}{(49+4k)(51+4k)} B_k$$

$$b_{k+1,2} = \frac{147 + 12k}{(49+4k)(51+4k)} A_k + \frac{168 + 14k}{(49+4k)(51+4k)} B_k$$

$$a_{k+1,3} = \frac{11+k}{49+4k} B_k$$

$$b_{k+1,3} = \frac{1}{49+4k} B_k,$$

where for $k \geq j+3$

$$A_k = -7a_{k,1} + (k+4)b_{k,1}$$

$$B_k = -5\{a_{k,1} + a_{k,2}\} + (k+6)\{b_{k,1} + b_{k,2}\}.$$

Since $a_{k,0} = b_{k,0} = a_{k+1,1} = b_{k+1,1} = 0$ for all $k \geq j+3$ we have that for all $k \geq j+4$

$$A_k = 0$$

$$B_k = -5a_{k,2} + (k+6)b_{k,2},$$

so for all $k \geq j+5$

$$|a_{k+1,2}| \leq \frac{1}{2}|a_{k,2}|$$

$$|b_{k+1,2}| \leq \frac{1}{2}|b_{k,2}|$$

$$|a_{k+1,3}| \leq \frac{1}{2}|a_{k,3}|$$

$$|b_{k+1,3}| \leq \frac{1}{2}|b_{k,3}|.$$

It is not difficult to prove that $|A_{j+3}| = 7|a_{j+3,1}| = 14/99$, $|B_{j+3}| \leq | -5(a_{j+3,1} + a_{j+3,2}) + (j+9)b_{j+3,2}| \leq 4$ and for all $k \geq j+3$

$$|a_{k+1,2}| + |a_{k+1,3}| + |b_{k+1,2}| + |b_{k+1,3}| \leq |A_k| + |B_k|,$$

So for $|y| < 2$ we have

$$\begin{aligned} |g| &\leq \sum_{k \geq j+3} \sum_{r=0}^3 |b_{k,r}| |x|^{6-2r} |y|^{k-1+2r} + \sum_{k \geq j+5} \sum_{r=0}^3 |b_{k,r}| |x|^{6-2r} |y|^{k-1+2r} \\ &\leq \sum_{k \geq j+3} \sum_{r=1}^3 |b_{k,r}| |x|^{6-2r} |y|^{k-1+2r} + \sum_{r=2}^3 |2b_{j+5,r}| |x|^{6-2r} \frac{|y|^{j+4+2r}}{2-|y|} \\ |h| &\leq \sum_{k \geq j+2} \sum_{r=0}^3 |a_{k,r}| |x|^{7-2r} |y|^{k-2+2r} + \sum_{k \geq j+5} \sum_{r=0}^3 |a_{k,r}| |x|^{7-2r} |y|^{k-2+2r} \\ &\leq \sum_{k \geq j+2} \sum_{r=0}^3 |a_{k,r}| |x|^{7-2r} |y|^{k-2+2r} + \sum_{r=2}^3 |2a_{j+5,r}| |x|^{7-2r} \frac{|y|^{j+3+2r}}{2-|y|}. \end{aligned}$$

Then if $R < 1$, $|x| < R$ and $|y| < R$ we obtain

$$\begin{aligned}
|g| &\leq \sum_{k \geq j+3}^{j+4} \sum_{r=1}^3 |b_{k,r}| R^{k+5} + \sum_{r=2}^3 |2b_{j+5,r}| R^{j+10} \\
&\leq \frac{12R^{j+8}}{11(59+4j)} + (|b_{j+4,2}| + |b_{j+4,3}|) R^{j+9} + 2(|A_{j+4}| + |B_{j+4}|) R^{j+10} \\
&\leq \frac{12R^{j+8}}{11(59+4j)} + (|A_{j+3}| + |B_{j+3}|) R^{j+9} + 2(|-5a_{j+4,2} + (k+6)b_{j+4,2}|) R^{j+10} \\
&\leq \left(\frac{12}{11(59+4j)} + 3|A_{j+3}| + 3|B_{j+3}| \right) R^{j+8} \\
&\leq \left(\frac{12}{11(59+4j)} + \frac{14}{33} + 4 \right) R^{j+8} \\
&\leq 5R^{j+6} \\
|h| &\leq \sum_{k \geq j+2}^{j+4} \sum_{r=0}^3 |a_{k,r}| R^{k+5} + \sum_{r=2}^3 |2a_{j+5,r}| R^{j+10} \\
&\leq \frac{R^{j+7}}{11} + (|a_{j+3,1}| + |a_{j+3,2}|) R^{j+8} + (|A_{j+3}| + |B_{j+3}|) R^{j+9} + \sum_{t=2}^3 |2a_{j+5,t}| R^{j+10} \\
&\leq \left(\frac{1}{11} + |a_{j+3,1}| + |a_{j+3,2}| + \frac{14}{33} + 4 \right) R^{j+6} \\
&\leq 5R^{j+6}.
\end{aligned}$$

g) For $x^7 y^j f dx \wedge dy$

$$g = \sum_{k \geq j+2} \sum_{r=0}^3 b_{k,r} x^{7-2r} y^{k-1+2r} \quad h = \sum_{k \geq j+2} \sum_{r=0}^4 a_{k,r} x^{8-2r} y^{k-2+2r}$$

$$\begin{array}{ll}
a_{j+2,1} = 0 & b_{j+2,1} = 0 \\
a_{j+2,2} = 0 & b_{j+2,2} = 0 \\
a_{j+2,3} = 0 & b_{j+2,3} = 0 \\
a_{j+3,3} = 0 & b_{j+3,3} = 0 \\
a_{j+2,0} = \frac{3}{2(19+j)} & b_{j+2,0} = -\frac{1}{19+j} \\
a_{j+3,2} = \frac{14+j}{7(19+j)} & b_{j+3,2} = \frac{1}{19+j} \\
a_{j+3,4} = 0 &
\end{array}$$

$$\begin{aligned}
k \geq j+3 \quad & a_{k,0} = b_{k,0} = a_{k,1} = b_{k,1} = a_{k+1,2} = b_{k+1,2} = 0 \\
& a_{k+1,3} = -\frac{6+k}{4(27+2k)} A_k + \frac{22+2k}{4(27+2k)} B_k \\
& b_{k+1,3} = -\frac{3}{2(27+2k)} A_k + \frac{1}{(27+2k)} B_k \\
& a_{k+1,4} = -\frac{1}{4} A_k,
\end{aligned}$$

where

$$\begin{aligned}
A_k &= 4\{a_{k,3} + a_{k,2}\} - (k+8)\{b_{k,3} + b_{k,2}\} \\
B_k &= -6a_{k,2} + (k+6)b_{k,2}.
\end{aligned}$$

Since $a_{k,2} = b_{k,2} = 0$ for all $k \geq j+4$, we have $B_k = 0$ and $A_k = 4a_{k,3} - (k+8)b_{k,3}$. Therefore for $k \geq j+4$

$$\begin{aligned} |a_{k+2,3}| &\leq \frac{1}{2}|a_{k+1,3}| & |b_{k+2,3}| &\leq \frac{1}{2}|b_{k+1,3}| \\ |a_{k+2,4}| &\leq \frac{1}{2}|a_{k+1,4}|, \end{aligned}$$

and for $k \geq j+3$

$$|a_{k+1,3}| + |b_{k+1,3}| + |a_{k+1,4}| \leq |A_k| + |B_k|.$$

Thus

$$\begin{aligned} |g| &\leq \sum_{k \geq j+2}^{j+4} \sum_{r=0}^3 |b_{k,r}| |x|^{7-2r} |y|^{k-1+2r} + \sum_{k \geq j+5} \sum_{r=2}^3 |b_{k,r}| |x|^{7-2r} |y|^{k-1+2r} \\ &\leq \sum_{k \geq j+2}^{j+4} \sum_{r=0}^3 |b_{k,r}| |x|^{7-2r} |y|^{k-1+2r} + \sum_{r=2}^3 |2b_{j+5,r}| |x|^{7-2r} \frac{|y|^{j+4+2r}}{2-|y|} \\ |h| &\leq \sum_{k \geq j+2}^{j+4} \sum_{r=0}^4 |a_{k,r}| |x|^{8-2r} |y|^{k-2+2r} + \sum_{k \geq j+5} \sum_{r=2}^4 |a_{k,r}| |x|^{8-2r} |y|^{k-2+2r} \\ &\leq \sum_{k \geq j+2}^{j+4} \sum_{r=0}^4 |a_{k,r}| |x|^{8-2r} |y|^{k-2+2r} + \sum_{r=2}^4 |2a_{j+5,r}| |x|^{8-2r} \frac{|y|^{j+3+2r}}{2-|y|} \end{aligned}$$

Thus, if $R < 1$, $|x| < R$ and $|y| < R$ then

$$\begin{aligned} |g| &\leq \sum_{k \geq j+2}^{j+4} \sum_{r=0}^3 |b_{k,r}| |x|^{7-2r} |y|^{k-1+2r} + \sum_{r=3}^3 2|b_{j+5,r}| R^{j+11} \\ &\leq \frac{R^{j+8}}{19+j} + \frac{R^{j+9}}{19+j} + |b_{j+4,4}| R^{j+10} + 2|A_{j+4}| R^{j+11} \\ &\leq \frac{R^{j+8}}{19+j} + \frac{R^{j+9}}{19+j} + |b_{j+4,4}| R^{j+10} + 2|A_{j+3} + B_{j+3}| R^{j+11} \\ &\leq \left(\frac{2}{19+j} + 3|A_{j+3}| + 3|B_{j+3}| \right) R^{j+11} \\ &\leq 3R^{j+7} \\ |h| &\leq \sum_{k \geq j+2}^{j+4} \sum_{r=0}^4 |a_{k,r}| |x|^{8-2r} |y|^{k-2+2r} + \sum_{r=3}^4 |2a_{j+5,r}| R^{j+11} \\ &\leq \frac{3R^{j+8}}{2(j+19)} + \frac{(14+j)R^{j+9}}{7(j+19)} + (|a_{j+4,3}| + |a_{j+4,4}|) R^{j+10} + 2|A_{j+4}| R^{j+11} \\ &\leq \left(\frac{3}{2(j+19)} + \frac{14+j}{7(j+19)} + 3(|A_{j+3}| + |B_{j+3}|) \right) R^{j+7} \\ &\leq 3R^{j+7}. \end{aligned}$$

h) For $x^{2n}y^j f dx \wedge dy$, when $n \geq 4$

$$g = \sum_{k \geq j+2} \sum_{r=0}^n b_{k,r} x^{2n-2r} y^{k-1+2r} \quad h = \sum_{k \geq j+2} \sum_{r=0}^n a_{k,r} x^{2n+1-2r} y^{k-2+2r}$$

$$b_{j+3,r} = b_{j+2,r} = 0, \quad 0 \leq r \leq n$$

$$a_{j+2,r} = 0, \quad r \geq 1$$

$$a_{j+2,0} = 1/(2n+5)$$

$$a_{j+3,r} = 0, \quad 3 \leq r \leq n$$

$$a_{j+3,1} = 2/(2n+5)(2n+3)$$

$$a_{j+3,2} = 4/(2n+1)(2n+5),$$

for $k \geq j + 3$ and $2 \leq r \leq n - 2$

$$\begin{aligned} a_{k+1,0} &= 0 & b_{k+1,0} &= 0 \\ a_{k+1,1} &= 0 & b_{k+1,1} &= 0 \\ a_{k+1,r} &= -a_{k,r-1} - a_{k,r-2} & b_{k+1,r} &= -b_{k,r-1} - b_{k,r-2}, \end{aligned}$$

for $k \geq j + 3$

$$\begin{aligned} a_{k+1,n-1} &= \frac{4k^2 + k(37 + 16n) + 16n^2 + 74n + 75}{(27 + 4k + 8n)(25 + 4k + 8n)} A_k + \\ &\quad \frac{2k^2 + k(8 + 8n) + 8n^2 + 16n - 24}{(25 + 4k + 8n)(27 + 4k + 8n)} B_k \\ b_{k+1,n-1} &= \frac{75 + 12k + 24n}{(27 + 4k + 8n)(25 + 4k + 8n)} A_k + \\ &\quad \frac{84 + 14k + 28n}{(27 + 4k + 8n)(25 + 4k + 8n)} B_k \\ a_{k+1,n} &= \frac{k + 5 + 2n}{25 + 4k + 8n} B_k \\ b_{k+1,n} &= \frac{1}{25 + 4k + 8n} B_k, \end{aligned}$$

where

$$\begin{aligned} A_k &= -7\{a_{k,n-2} + a_{k,n-3}\} + (k - 2 + 2n)\{b_{k,n-2} + b_{k,n-3}\} \\ B_k &= -5\{a_{k,n-1} + a_{k,n-2}\} + (k + 2n)\{b_{k,n-1} + b_{k,n-2}\}. \end{aligned}$$

Since $a_{k,0} = b_{k,0} = a_{k,1} = b_{k,2} = 0$ for all $k \geq j + 4$, we obtain $a_{k,2} = b_{k,2} = 0$ for all $k \geq j + 5$, continuing by induction we have that for all $k \geq j + 1 + n$ and $0 \leq r \leq n - 2$

$$\begin{aligned} a_{k,r} &= 0 & b_{k,r} &= 0 \\ A_k &= 0 & B_k &= -5a_{k,n-1} + (k + 2n)b_{k,n-1}. \end{aligned}$$

In addition, since $b_{j+3,r} = 0$ for all $0 \leq r \leq n - 2$, we have that for $0 \leq r \leq n - 2$, $b_{j+3,r} = 0$, continuing by induction we have that for all $j + 4 \leq k \leq j + n$, $0 \leq r \leq n - 2$

$$\begin{aligned} b_{k,r} &= 0 & A_k &= -7\{a_{k,n-2} + a_{k,n-3}\} \\ B_k &= -5\{a_{k,n-1} + a_{k,n-2}\} + (k + 2n)b_{k,n-1}. \end{aligned}$$

By induction we can prove that there is a relation between $a_{k,r}$ for $k \geq j + 3$, $a_{j+3,1}$ and $a_{j+3,2}$. In the next table we express this relation. We represent $a_{j+3,1}$ by 1, 0 and $a_{j+3,2}$ by 0, 1. For example $a_{j+4,2} = -a_{j+3,1}$ which is represented in the table by

$-1, 0$ or $a_{j+4,3} = -a_{j+3,1} - a_{j+3,2}$ which is represented by $-1, -1$.

$k =$	$j+3$	$j+4$	$j+5$	$j+6$	$j+7$	$j+8$	$j+9$	$j+10$
$a_{k,0}$	0	0	0	0	0	0	0	0
$a_{k,1}$	1, 0	0	0	0	0	0	0	0
$a_{k,2}$	0, 1	-1, 0	0	0	0	0	0	0
$a_{k,3}$	0	-1, -1	1, 0	0	0	0	0	0
$a_{k,4}$	0	0, -1	2, 1	-1, 0	0	0	0	0
$a_{k,5}$	0	0	1, 2	-3, -1	1, 0	0	0	0
$a_{k,6}$	0	0	0, 1	-3, -3	4, 1	-1, 0	0	0
$a_{k,7}$	0	0	0	-1, -3	6, 4	-5, -1	1, 0	0
$a_{k,6}$	0	0	0	0, -1	4, 6	-10, -5	6, 1	-1, 0

Since $|a_{j+3,1}| \leq |a_{j+3,2}|$, we can deduce that for $0 \leq k \leq n-3$

$$\sum_{r=0}^{n-2} |a_{j+3+k,r}| \leq 2^{k+1} a_{j+3,2}.$$

A simple computing shows that for all $k \geq j+3$

$$|a_{k+1,n-1}| + |a_{k+1,n}| + |b_{k+1,n-1}| + |b_{k+1,n}| \leq |A_k| + |B_k|,$$

and for $1 \leq k \leq n-3$, $|A_{j+3+k}| \leq 7(2^{k+1})a_{j+3,2}$ and

$$\begin{aligned} |B_{j+3+k}| &\leq |A_{j+2+k}| + |B_{j+2+k}| + 5(2^{k+1})a_{j+3,2} \\ &\leq |B_{j+2+k}| + 7(2^k + 2^{k+1})a_{j+3,2}. \end{aligned}$$

Since $B_{j+3} = 0$, we have that for $1 \leq k \leq n-3$

$$|B_{j+3+k}| \leq 7(2^{k+3})a_{j+3,2}.$$

In addition, we have that for $k \geq j+1+n$

$$\begin{aligned} |a_{k+2,n-1}| &\leq \frac{1}{2}|a_{k+1,n-1}| & |b_{k+2,n-1}| &\leq \frac{1}{2}|b_{k+1,n-1}| \\ |a_{k+2,n}| &\leq \frac{1}{2}|a_{k+1,n}| & |b_{k+2,n}| &\leq \frac{1}{2}|b_{k+1,n}|. \end{aligned}$$

Therefore if $|y| < 2$

$$\begin{aligned} |g| &\leq \sum_{k \geq j+4}^{j+n+1} \sum_{r=n-1}^n |b_{k,r}| |x|^{2n-2r} |y|^{k-1+2r} + \sum_{k \geq j+2+n} \sum_{r=n-1}^n |b_{k,r}| |x|^{2n-2r} |y|^{k-1+2r} \\ &\leq \sum_{k \geq j+4}^{j+n+1} \sum_{r=n-1}^n |b_{k,r}| |x|^{2n-2r} |y|^{k-1+2r} + \sum_{r=n-1}^n 2|b_{j+n+2,r}| |x|^{2n-2r} \frac{|y|^{j+n+1+2r}}{2-|y|} \\ |h| &\leq \sum_{k \geq j+2}^{j+n+1} \sum_{r=0}^n |a_{k,r}| |x|^{2n+1-2r} |y|^{k-2+2r} + \sum_{k \geq j+2+n} \sum_{r=n-1}^n |a_{k,r}| |x|^{2n+1-2r} |y|^{k-2+2r} \\ &\leq \sum_{k \geq j+2}^{j+n+1} \sum_{r=0}^n |a_{k,r}| |x|^{2n+1-2r} |y|^{k-2+2r} + \sum_{r=n-1}^n 2|a_{j+n+2,r}| |x|^{2n-2r} \frac{|y|^{j+n+2r}}{2-|y|}. \end{aligned}$$

Let $R < 1$, if $|x| < R$ and $|y| < R$ then

$$\begin{aligned}
|g| &\leq \sum_{k \geq j+4}^{j+n+1} \sum_{r=n-1}^n |b_{k,t}| R^{2n+k-1} + 2(|b_{j+n+2,n-1}| + |b_{j+n+2,n}|) R^{j+3n+1} \\
&\leq \sum_{k \geq j+4}^{j+n+1} (|A_{k-1}| + |B_{k-1}|) R^{2n+k-1} + 2|B_{j+n+1}| R^{j+3n+1} \\
&\leq \sum_{k \geq j+5}^{j+n+1} 7(2^{k-j-3} + 2^{k-1-j}) a_{j+3,2} R^{2n+k-1} + 2(|A_{j+n}| + |B_{j+n}|) R^{j+3n+1} \\
&\leq 7a_{j+3,2} R^{2n+j+4} \sum_{k \geq j+5}^{j+n+1} (2^{k-j-3} + 2^{k-1-j}) + 14(2^{n-2} + 2^n) a_{j+3,2} R^{j+3n+1} \\
&\leq 7a_{j+3,2} (2^n + 2^{n+2}) R^{j+2n} \\
&\leq \frac{35 \cdot 2^{n+2}}{(2n+1)(2n+5)} R^{j+2n} \\
|h| &\leq \frac{1}{2n+5} R^{j+2n+1} + \sum_{k \geq j+3}^{j+n+1} \sum_{r=0}^n |a_{k,r}| R^{2n+k-1} + 2(|a_{j+n+2,n-1}| + |a_{j+n+2,n}|) R^{j+3n+1} \\
&\leq \frac{1}{2n+5} R^{j+2n+1} + \sum_{k \geq j+3}^{j+n+1} \sum_{r=0}^{n-2} |a_{k,r}| R^{2n+k-1} + \sum_{k \geq j+3}^{j+n+1} \sum_{r=n-1}^n |a_{k,r}| R^{2n+k-1} + \\
&\quad + 2(|a_{j+n+2,n-1}| + |a_{j+n+2,n}|) R^{j+3n+1} \\
&\leq \frac{1}{2n+5} R^{j+2n+1} + \sum_{k \geq j+3}^{j+n+1} 2^{k+1-j-3} a_{j+3,2} R^{2n+k-1} + \sum_{k \geq j+4}^{j+n+1} (|A_{k-1}| + |B_{k-1}|) R^{2n+k-1} + \\
&\quad + 2(|B_{k+n+1}|) R^{j+3n+1} \\
&\leq \frac{1}{2n+5} R^{j+2n+1} + 2^n a_{j+3,2} R^{2n+j+2} + \sum_{k \geq j+5}^{j+n+1} 7(2^{k-j-3} + 2^{k-1-j}) a_{j+3,2} R^{2n+k-1} + \\
&\quad + 2(|A_{k+n+1}| + |B_{k+n+1}|) R^{j+3n+1} \\
&\leq \left(\frac{1}{2n+5} + 35(2^n) a_{j+3,2} \right) R^{j+2n} \\
&\leq \left(\frac{1}{2n+5} + \frac{35 \cdot 2^{n+2}}{(2n+1)(2n+5)} \right) R^{j+2n}.
\end{aligned}$$

i) For $x^{2n+1} y^j f dx \wedge dy$, when $n \geq 4$

$$\begin{aligned}
g &= \sum_{k \geq j+2} \sum_{r=0}^n b_{k,r} x^{2n+1-2r} y^{k-1+2r} & h &= \sum_{k \geq j+2} \sum_{r=0}^{n+1} a_{k,r} x^{2n+2-2r} y^{k-2+2r} \\
a_{j+2,0} &= \frac{3}{2(10+j+3n)} & b_{j+2,0} &= -\frac{1}{10+j+3n} \\
a_{j+2,r} &= b_{j+2,r} = 0, \quad r \geq 1 & a_{j+3,2} &= \frac{2j+6n+17}{2(10+j+3n)(2n+2)} \\
b_{j+3,2} &= 0 & a_{j+3,r} &= b_{j+3,r} = 0, \quad r \geq 3 \\
a_{k,0} &= b_{k,0} = a_{k,1} = b_{k,1} = 0, \quad k \geq j+3,
\end{aligned}$$

and for $k \geq j+3$ and $0 \leq r \leq n-3$

$$a_{k+1,r+2} = -a_{k,r} - a_{k,r+1} \qquad b_{k+1,r+2} = -b_{k,r} - b_{k,r-1}$$

$$\begin{aligned}
a_{k+1,n} &= -\frac{k+2n}{4(15+2k+4n)}A_k + \frac{10+2k+4n}{4(15+2k+4n)}B_k \\
b_{k+1,n} &= -\frac{3}{2(15+2k+4n)}A_k + \frac{1}{15+2k+4n}B_k \\
a_{k+1,n+1} &= -\frac{1}{4}A_k,
\end{aligned}$$

where

$$\begin{aligned}
A_k &= 4\{a_{k,n} + a_{k,n-1}\} - (k+2+2n)\{b_{k,n} + b_{k,n-1}\} \\
B_k &= -6\{a_{k,n-1} + a_{k,n-2}\} + (k+2n)\{b_{k,n-1} + b_{k,n-2}\}.
\end{aligned}$$

Similarly to the even case, we have that for $k \geq j+n+1$, $0 \leq t \leq n-1$

$$\begin{aligned}
a_{k,r} &= 0 & b_{k,r} &= 0 \\
A_k &= 4a_{k,n} - (k+2+2n)b_{k,n} & B_k &= 0,
\end{aligned}$$

and for all $j+4 \leq k \leq j+n$, $0 \leq t \leq n-1$

$$\begin{aligned}
b_{k,r} &= 0 & A_k &= 4\{a_{k,n} + a_{k,n-1}\} - (k+2+2n)b_{k,n} \\
B_k &= -6\{a_{k,n-1} + a_{k,n-2}\}.
\end{aligned}$$

Also, similar to the case even, we have a table in which is expressed the relation between $a_{k,r}$ for $k \geq j+3$ and $a_{j+3,2}$. Here we represent $a_{j+3,2}$ by 1.

$k =$	$j+3$	$j+4$	$j+5$	$j+6$	$j+7$	$j+8$	$j+9$	$j+10$
$a_{k,0}$	0	0	0	0	0	0	0	0
$a_{k,1}$	0	0	0	0	0	0	0	0
$a_{k,2}$	1	0	0	0	0	0	0	0
$a_{k,3}$	0	-1	0	0	0	0	0	0
$a_{k,4}$	0	-1	1	0	0	0	0	0
$a_{k,5}$	0	0	2	-1	0	0	0	0
$a_{k,6}$	0	0	1	-3	1	0	0	0
$a_{k,7}$	0	0	0	-3	4	-1	0	0
$a_{k,6}$	0	0	0	-1	6	-5	1	0

By induction we obtain that for $0 \leq k \leq n-3$

$$\sum_{t=0}^{n-1} |a_{j+3+k,t}| \leq 2^k a_{j+3,2}.$$

A simple computing shows that for all $k \geq j+3$

$$|a_{k+1,n}| + |a_{k+1,n+1}| + |b_{k+1,n}| \leq |A_k| + |B_k|,$$

and for $1 \leq k \leq n-3$, $|B_{j+3+k}| \leq 6(2^k)a_{j+3,2}$ and

$$\begin{aligned}
|A_{j+3+k}| &\leq |A_{j+2+k}| + |B_{j+2+k}| + 4(2^k)a_{j+3,2} \\
&\leq |A_{j+2+k}| + 6(2^{k-1} + 2^k)a_{j+3,2}.
\end{aligned}$$

Since $A_{j+3} = 0$, we have for $1 \leq k \leq n-3$

$$|A_{j+3+k}| \leq 6(2^{k+2})a_{j+3,2},$$

we know that for all $k \geq j + 1 + n$

$$\begin{aligned} |a_{k+2,n}| &\leq \frac{1}{2}|a_{k+1,n}| & |b_{k+2,n}| &\leq \frac{1}{2}|b_{k+1,n}| \\ |a_{k+2,n+1}| &\leq \frac{1}{2}|a_{k+1,n+1}|. \end{aligned}$$

Therefore for $|y| < 2$

$$\begin{aligned} |g| &\leq |b_{j+2,0}| |x|^{2n+1} |y|^{j+1} + \sum_{k \geq j+4}^{j+n+1} |b_{k,n}| |x| |y|^{k-1+2n} + \sum_{k \geq j+2+n} |b_{k,n}| |x| |y|^{k-1+2n} \\ &\leq |b_{j+2,0}| |x|^{2n+1} |y|^{j+1} + \sum_{k \geq j+4}^{j+n+1} |b_{k,n}| |x| |y|^{k-1+2n} + 2|b_{j+n+2,n}| \frac{|x| |y|^{j+3n+1}}{2 - |y|} \\ |h| &\leq |a_{j+2,0}| |x|^{2n+2} |y|^j + \sum_{k \geq j+2}^{j+n+1} \sum_{r=0}^{n+1} |a_{k,r}| |x|^{2n+2-2r} |y|^{k-2+2r} + \sum_{k \geq j+2+n}^{n+1} \sum_{r=n} |a_{k,r}| |x|^{2n+2-2r} |y|^{k-2+2r} \\ &\leq |a_{j+2,0}| |x|^{2n+2} |y|^j + \sum_{k \geq j+2}^{j+n+1} \sum_{r=0}^{n+1} |a_{k,r}| |x|^{2n+2-2r} |y|^{k-2+2r} + 2|a_{j+n+2,n}| \frac{|x|^2 |y|^{j+3n}}{2 - |y|} + \\ &\quad + 2|a_{j+n+2,n+1}| \frac{|y|^{j+3n+2}}{2 - |y|}. \end{aligned}$$

Then if $R < 1$, $|x| < R$ and $|y| < R$

$$\begin{aligned} |g| &\leq \frac{1}{10+j+2n} R^{2n+j+2} + \sum_{k \geq j+5}^{j+n+1} (|A_{k-1}| + |B_{k-1}|) R^{k+2n} + 2|b_{j+n+2,n}| R^{j+3n+2} \\ &\leq \frac{1}{10+j+2n} R^{2n+j+1} + \sum_{k \geq j+5}^{j+n+1} 6(2^{k-j-2} + 2^{k-1-j-3}) a_{j+3,2} R^{k+2n+1} + \\ &\quad + 2(|A_{j+n+1}|) R^{j+3n+2} \\ &\leq \frac{1}{10+j+2n} R^{2n+j+1} + 6(2^n + 2^{n-2}) a_{j+3,2} R^{j+2n+1} + 2(|A_{j+n}| + |B_{j+n}|) R^{j+2n+1} \\ &\leq \left(\frac{1}{10+j+2n} + 6(2^{n+1} + 2^{n-1}) a_{j+3,2} \right) R^{j+2n+1} \\ &\leq \left(\frac{1}{10+j+2n} + 30 \cdot 2^{n-1} \frac{2j+6n+17}{2(10+j+3n)(2n+2)} \right) R^{j+2n+1} \\ |h| &\leq \frac{3}{2(10+j+3n)} R^{2n+j+1} + \sum_{k=j+3}^{j+n+1} \sum_{r=0}^{n-1} |a_{k,r}| R^{2n+k} + \sum_{k=j+4}^{j+n+1} \sum_{r=n} |a_{k,r}| R^{2n+k} + \\ &\quad + 2(|a_{j+n+2,n}| + |a_{j+n+2,n+1}|) R^{j+3n+2} \\ &\leq \frac{3}{2(10+j+3n)} R^{2n+j+1} + \sum_{k=j+3}^{j+n+1} 2^{k-j-3} a_{j+3,2} R^{2n+k} + \sum_{k=j+5}^{j+n+1} (|A_{k-1}| + |B_{k-1}|) R^{2n+k} \\ &\quad + 2(|A_{j+n+1}|) R^{j+3n+2} \\ &\leq \left(\frac{3}{2(10+j+2n)} + 30 \cdot 2^{n-1} \frac{2j+6n+17}{2(10+j+3n)(2n+2)} \right) R^{j+2n+1} \end{aligned}$$

j) For $y f dx \wedge dy$

$$g = \sum_{k \geq 2} \{b_{k,k-2} x^2 y^{k-2} + b_{k,k} y^k\} \quad h = a_{2,1} xy + \sum_{k \geq 3} \{a_{k,k-3} x^3 y^{k-3} + a_{k,k-1} xy^{k-1}\}$$

$$\begin{aligned}
a_{2,1} &= 5/33 & b_{2,2} &= -4/33 & b_{2,0} &= 2/165 \\
a_{3,0} &= 12/26455 & b_{3,1} &= 84/26455 & a_{3,2} &= 16/2035 \\
b_{3,3} &= 2/2035,
\end{aligned}$$

and for $k \geq 0$

$$\begin{aligned}
a_{k+4,k+1} &= \frac{(2+k)(21+2k)}{4(9+k)(41+4k)} \{-5a_{k+3,k} + (k+4)b_{k+3,k+1}\} \\
b_{k+4,k+2} &= \frac{7(21+2k)}{4(9+k)(41+4k)} \{-5a_{k+3,k} + (k+4)b_{k+3,k+1}\} \\
a_{k+4,k+3} &= \frac{9+k}{41+4k} \{-5a_{k+3,k} + (k+4)b_{k+3,k+1}\} \\
b_{k+4,k+4} &= \frac{1}{41+4k} \{-5a_{k+3,k} + (k+4)b_{k+3,k+1}\}.
\end{aligned}$$

Therefore for $k \geq 0$

$$\begin{aligned}
|a_{k+4,k+1}| &\leq \frac{1}{2}|a_{k+3,k}| & |b_{k+4,k+2}| &\leq \frac{1}{2}|b_{k+3,k+1}| \\
|a_{k+4,k+1}| &\leq \frac{1}{2}|a_{k+3,k}| & |b_{k+4,k+2}| &\leq \frac{1}{2}|b_{k+3,k+1}|.
\end{aligned}$$

Since $|b_{3,1}| \leq \frac{1}{2}|b_{2,0}|$, $|b_{3,3}| \leq \frac{1}{2}|b_{2,2}|$ and $|a_{3,2}| \leq \frac{1}{2}|a_{2,1}|$,

$$\begin{aligned}
|g| &\leq \frac{4}{165} \frac{|x|^2}{2-|y|} + \frac{8}{33} \frac{|y|^2}{2-|y|} \\
|h| &\leq \frac{10}{3} \frac{|x||y|}{2-|y|} + \frac{24}{26455} \frac{|x|^3}{2-|y|}.
\end{aligned}$$

So, if $R < 1$, $|x| < R$ and $|y| < R$ we have

$$\begin{aligned}
|g| &\leq \left(\frac{4}{165} + \frac{8}{33}\right)r^2 \leq R \\
|h| &\leq \left(\frac{10}{3} + \frac{24}{26455}\right)r^3 \leq R.
\end{aligned}$$

k) For $y^j f dx \wedge dy$, $j \geq 2$

$$g = \sum_{k \geq j} \{b_{k+1,k-1}x^2y^{k-1} + b_{k+1,k+1}y^{k+1}\} \quad h = \sum_{k \geq j} \{a_{k+1,k-2}x^3y^{k-2} + a_{k+1,k}xy^k\}$$

$$\begin{aligned}
a_{j+1,j} &= \frac{5}{29+4j} & b_{j+1,j+1} &= -\frac{4}{29+4j} \\
a_{j+1,j-2} &= \frac{2(j-1)}{(29+4j)(31+4j)} & b_{j+1,j-1} &= \frac{14}{(29+4j)(31+4j)},
\end{aligned}$$

and for $k \geq j-1$

$$\begin{aligned}
a_{k+3,k+2} &= \frac{8+k}{39+4k} \{(k+3)b_{k+2,k} - 5a_{k+2,k-1}\} \\
a_{k+3,k} &= \frac{2(7+8k+k^2)}{(39+4k)^2} \{(k+3)b_{k+2,k} - 5a_{k+2,k-1}\} \\
b_{k+3,k+1} &= \frac{14(7+k)}{(39+4k)^2} \{(k+3)b_{k+2,k} - 5a_{k+2,k-1}\} \\
b_{k+3,k+3} &= \frac{1}{39+4k} \{(k+3)b_{k+2,k} - 5a_{k+2,k-1}\}.
\end{aligned}$$

Then for all $k \geq j - 2$ we have

$$\begin{aligned} |a_{k+4,k+3}| &\leq \frac{1}{2} |a_{k+3,k+2}| & |b_{k+4,k+4}| &\leq \frac{1}{2} |b_{k+3,k+3}| \\ |a_{k+4,k+1}| &\leq \frac{1}{2} |a_{k+3,k}| & |b_{k+4,k+2}| &\leq \frac{1}{2} |b_{k+3,k+1}|. \end{aligned}$$

Thus for $|y| < 2$

$$\begin{aligned} |g| &\leq \frac{28}{(29+4j)(31+4j)} \frac{|x|^2 |y|^{j-1}}{2-|y|} + \frac{8}{29+4j} \frac{|y|^{j+1}}{2-|y|} \\ |h| &\leq \frac{4(j-1)}{(29+4j)(31+4j)} \frac{|x|^3 |y|^{j-2}}{2-|y|} + \frac{10}{29+4j} \frac{|x| |y|^j}{2-|y|} \end{aligned}$$

Then for $R < 1$, $|x| < R$ and $|y| < R$ we have

$$\begin{aligned} |g| &\leq \left(\frac{28}{(29+4j)(31+4j)} + \frac{8}{29+4j} \right) R^{j+1} \leq R^j \\ |h| &\leq \left(\frac{4(j-1)}{(29+4j)(31+4j)} + \frac{10}{29+4j} \right) R^{j+1} \leq R^j. \end{aligned}$$

Let us prove the general case.

Theorem 4.2.9. *Let $G \in \mathcal{M}_2$, then there exists $\alpha \in \Omega^1(\mathbb{C}^2)$ such that*

$$Gf dx \wedge dy = d(f\alpha).$$

Proof. Let $r < 1/2$ such that G is smooth on $D = \{(x, y) \in \mathbb{C}^2 \mid |x| \leq 2r, |y| \leq 2r\}$. By the compactness of D there exists $M > 0$ such that $|G(x, y)| \leq M$ for all $(x, y) \in D$. Then by the expansion in series of Taylor of G on D ,

$$G(x, y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j, \quad a_{i,j} = \frac{G^{(i,j)}(0)}{i!j!} = \frac{1}{i!j!} \frac{\partial^{i+j} G}{\partial x^i \partial y^j}(0).$$

By Cauchy inequality $|G^{(i,j)}(0)| \leq \frac{M i! j!}{(2r)^{i+j}}$, see [1]. Let $\alpha_{i,j} = g_{i,j}(x, y) dx + h_{i,j}(x, y) dy$ the smooth 1-form given by the lemmas such that

$$x^i y^j f dx \wedge dy = d(f \alpha_{i,j})$$

Let $\alpha = \sum_{i,j} a_{i,j} \alpha_{i,j} = \sum_{i,j} a_{i,j} g_{i,j} dx + \sum_{i,j} a_{i,j} h_{i,j} dy$. So $d(f\alpha) = fG dx \wedge dy$. Then, in order to get the result, it is enough prove that α is smooth.

i) Denote by $g = \sum_{i,j \geq 0} a_{i,j} g_{i,j}(x, y)$. Then for all $(x, y) \in D$

$$\begin{aligned}
|g| &\leq \sum_{j \geq 1} |a_{0,j}| |g_{0,j}| + \sum_{j \geq 0} |a_{1,j}| |g_{1,j}| + \sum_{j \geq 0} |a_{2,j}| |g_{2,j}| + \sum_{j \geq 0} |a_{3,j}| |g_{3,j}| + \\
&\quad \sum_{j \geq 0} |a_{4,j}| |g_{4,j}| + \sum_{j \geq 0} |a_{5,j}| |g_{5,j}| + \sum_{j \geq 0} |a_{6,j}| |g_{6,j}| + \sum_{j \geq 0} |a_{7,j}| |g_{7,j}| + \\
&\quad + \sum_{n \geq 4} \sum_{j \geq 0} |a_{2n,j}| |g_{2n,j}| + \sum_{n \geq 4} \sum_{j \geq 0} |a_{2n+1,j}| |g_{2n+1,j}| \\
|g| &\leq \sum_{j \geq 1} |a_{0,j}| r^j + \sum_{j \geq 0} |a_{1,j}| r^{j+1} + \sum_{j \geq 0} |a_{2,j}| r^{j+2} + \sum_{j \geq 0} |a_{3,j}| r^{j+3} + \\
&\quad \sum_{j \geq 0} |a_{4,j}| r^{j+4} + \sum_{j \geq 0} |a_{5,j}| r^{j+5} + \sum_{j \geq 0} 5|a_{6,j}| r^{j+6} + \sum_{j \geq 0} 3|a_{7,j}| r^{j+7} + \\
&\quad + \sum_{n \geq 4} \sum_{j \geq 0} |a_{2n,j}| \frac{35 \cdot 2^{n+2}}{(2n+1)(2n+5)} r^{j+2n} + \\
&\quad + \sum_{n \geq 4} \sum_{j \geq 0} |a_{2n+1,j}| \left(\frac{1}{10+j+2n} + 30 \cdot 2^{n-1} \frac{2j+6n+17}{2(10+j+3n)(2n+2)} \right) r^{j+2n+1} \\
|g| &\leq \sum_{j \geq 1} |a_{0,j}| r^j + \sum_{j \geq 0} |a_{1,j}| r^{j+1} + \sum_{j \geq 0} |a_{2,j}| r^{j+2} + \sum_{j \geq 0} |a_{3,j}| r^{j+3} + \\
&\quad \sum_{j \geq 0} |a_{4,j}| r^{j+4} + \sum_{j \geq 0} |a_{5,j}| r^{j+5} + \sum_{j \geq 0} 5|a_{6,j}| r^{j+6} + \sum_{j \geq 0} 3|a_{7,j}| r^{j+7} + \\
&\quad + \sum_{n \geq 4} \sum_{j \geq 0} |a_{2n+1,j}| r^{j+2n+1} + \sum_{n \geq 4} \sum_{j \geq 0} |a_{2n,j}| \frac{35 \cdot 2^n}{n^2} r^{j+2n} + \\
&\quad + \sum_{n \geq 4} \sum_{j \geq 0} 15|2^n a_{2n+1,j}| r^{j+2n+1}.
\end{aligned}$$

Note that the sum of the nine first terms is finite because $\sum_{i,j \leq 0} |a_{i,j}| r^{i+j} < \infty$ and

$$\begin{aligned}
\sum_{n \geq 4} \sum_{j \geq 0} |a_{2n,j}| \frac{2^n}{n^2} r^{j+2n} &\leq \sum_{n \geq 4} \sum_{j \geq 0} \frac{M}{2^{2n+j} r^{2n+j}} \frac{2^n}{n^2} r^{j+2n} \leq \sum_{n \geq 4} \sum_{j \geq 0} \frac{M}{2^{n+j}} \leq M/4 \\
\sum_{n \geq 4} \sum_{j \geq 0} |2^n a_{2n+1,j}| r^{j+2n+1} &\leq \sum_{n \geq 4} \sum_{j \geq 0} 2^n \frac{M}{2^{2n+1+j}} \leq M/8
\end{aligned}$$

It shows that g converges uniformly on D and since D is compact, we have that g converges uniformly on any compact subset of D . Therefore g is smooth on D , see [1]

ii) Denote by $h = \sum_{i,j \geq 0} a_{i,j} h_{i,j}(x, y)$. Then for all $(x, y) \in D$

$$\begin{aligned}
|h| &\leq \sum_{j \geq 1} |a_{0,j}| |h_{0,j}| + \sum_{j \geq 0} |a_{1,j}| |h_{1,j}| + \sum_{j \geq 0} |a_{2,j}| |h_{2,j}| + \sum_{j \geq 0} |a_{3,j}| |h_{3,j}| + \\
&\quad \sum_{j \geq 0} |a_{4,j}| |h_{4,j}| + \sum_{j \geq 0} |a_{5,j}| |h_{5,j}| + \sum_{j \geq 0} 5|a_{6,j}| |h_{6,j}| + \sum_{j \geq 0} 3|a_{7,j}| |h_{7,j}| + \\
&\quad + \sum_{n \geq 4} \sum_{j \geq 0} |a_{2n,j}| |h_{2n,j}| + \sum_{n \geq 4} \sum_{j \geq 0} |a_{2n+1,j}| |h_{2n+1,j}|
\end{aligned}$$

$$\begin{aligned}
|h| \leq & \sum_{j \geq 1} |a_{0,j}| r^j + \sum_{j \geq 0} |a_{1,j}| r^{j+1} + \sum_{j \geq 0} |a_{2,j}| r^{j+2} + \sum_{j \geq 0} |a_{3,j}| r^{j+3} + \\
& \sum_{j \geq 0} |a_{4,j}| r^{j+4} + \sum_{j \geq 0} |a_{5,j}| r^{j+5} + \sum_{j \geq 0} 5|a_{6,j}| r^{j+6} + \sum_{j \geq 0} 3|a_{7,j}| r^{j+7} + \\
& + \sum_{n \geq 4} \sum_{j \geq 0} |a_{2n,j}| \left(\frac{1}{2n+5} + \frac{35 \cdot 2^{n+2}}{(2n+1)(2n+5)} \right) r^{j+2n} + \\
& + \sum_{n \geq 4} \sum_{j \geq 0} |a_{2n+1,j}| \left(\frac{3}{2(10+j+2n)} + 30 \cdot 2^{n-1} \frac{2j+6n+17}{2(10+j+3n)(2n+2)} \right) r^{j+2n+1}.
\end{aligned}$$

Analogous to i) we have that h is smooth on D .

Therefore $\alpha = gdx + hdy$ is smooth on D and holds $d(f\alpha) = Gf dx \wedge dy$. $\square$

Remark 4.2.10. Let us denote by $\text{Derlog}(f).(f) = \{X(f) | X \in \text{Derlog}(f)\}$ the ideal of $\mathcal{O}_2$. Then the result of Theorem 4.2.9 can be written

$$\text{Derlog}(f).(f) \subset \text{div}(f \cdot \Theta_2) \quad (4.4)$$

If $\tilde{f} \in \mathcal{M}_2$ is a quasi homogeneous germ, by Propositions 3.1.6 and 3.4.1 and the fact $[h(x,y)dx \wedge dy]_{\{\tilde{f}^{-1}(0)\}} = 0$ if and only if $h \in \langle \nabla \tilde{f} \rangle$ we have that (4.4) is true to $\tilde{f}$ instead of f . What happens in general? For which $g \in \mathcal{M}_2$, (4.4) is true? We do not know the answer.

Corollary 4.2.11. Let $g \in \mathcal{M}_2^6$, then $\|g\|_{N'} = 0$.

Proof. Since $\frac{\partial f}{\partial x} = 4x^3 + 2xy^3$ and $\frac{\partial f}{\partial y} = 5y^4 + 3x^2y^2$, it is not difficult to prove

$$\begin{aligned}
x^3 & \in Jf + \mathcal{M}_2^4 \\
x^4, x^3y & \in Jf + \mathcal{M}_2^5 \\
x^5, x^4y, x^3y^2, xy^4 & \in Jf + \mathcal{M}_2^6 \\
x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6 & \in Jf + \mathcal{M}_2^7.
\end{aligned}$$

Therefore $\mathcal{M}_2^6 \subset Jf + \mathcal{M}_2^7$ and by Nakayama's Lemma, $\mathcal{M}_2^6 \subset Jf$. Let $g \in \mathcal{M}_2^6$, then there exist $g_1, g_2 \in \mathcal{O}_2$ such that

$$g = g_1 \frac{\partial f}{\partial x} + g_2 \frac{\partial f}{\partial y}.$$

Thus

$$\|g\|_{N'} = \|d(fg_1dy - fg_2dx)\|_{N'} = \left\| f \left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} \right) \right\|_{N'}$$

Since $g \in \mathcal{M}_2^6$ we obtain that $g_1, g_2 \in \mathcal{M}_2^2$, hence by Theorem 4.2.9, $\left\| f \left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} \right) \right\|_{N'} = 0$. Therefore $\|g\|_{N'} = 0$. $\square$

CLASSIFICATION

Recall that $f(x_1, x_2) = x_1^4 + x_2^5 + x_1^2 x_2^3 \in \mathcal{M}_2$. We want to obtain normal forms of curves in a symplectic manifold $(\mathbb{C}^{2n}, \omega)$ which are diffeomorphic to the fixed non quasi homogeneous curve $N = \{f(x_1, x_2) = x_{\geq 3} = 0\}$, under the action of the group of symplectomorphism. By Proposition 3.1.19, it is enough to obtain normal forms of exact algebraic restriction in $\|Sym(\mathbb{C}^{2n})\|_N$ with respect to the group $G_N(\mathbb{C}^{2n})$. In order to do that, we will obtain the normal forms of exact algebraic restriction in $\|\Omega^2(\mathbb{C}^2)\|_{N'}$ with respect to the group $G_{N'}(\mathbb{C}^2)$, where $N' = f^{-1}(0)$.

5.1 Classification of exact algebraic restriction to N'

As it is common in classification, we will work with the tangent space to the orbit of each element of $\|\Omega^2(\mathbb{C}^2)\|_{N'}$ under the action $G_{N'}(\mathbb{C}^2)$, where $N' = f^{-1}(0)$, $f(x, y) = x^4 + y^5 + x^2 y^3$.

Note that by Lemma 3.1.23 it is possible to define the Lie derivative along a vector field X satisfying $df.X(x) = 0$ for all $x \in (N', 0)$, $X(0) = 0$ by $\mathcal{L}_X \|gdx \wedge dy\|_{N'} := \|\mathcal{L}_X gdx \wedge dy\|_{N'}$.

Proposition 5.1.1. *Let $a = \|gdx \wedge dy\|_{N'}$ be an exact algebraic restriction in $\|\Omega^2(\mathbb{C}^2)\|_{N'}$. Then $T(a) = \{\mathcal{L}_X a \mid X \in Derlog(f)\} \subset \|\Omega^2(\mathbb{C}^2)\|_{N'}$.*

Proof. In fact, by Proposition 3.1.25

$$T(a) = \{\mathcal{L}_X a \mid df(x) \cdot X(x) = 0, \forall x \in (N', 0), X \in \Theta_2, X(0) = 0\}$$

By the property of zeros of f , $df(x) \cdot X(x) = 0$ for all $x \in (N', 0)$ if and only if $X \in Derlog(f)$. The result follows from the fact that any $X \in Derlog(f)$ satisfies $X(0) = 0$. $\square$

Lemma 5.1.2. *If $X \in \Theta_2$, then for all $g, h \in \mathcal{O}_2$ we have*

- (i) $\mathcal{L}_{hX}gdx \wedge dy = \mathcal{L}_Xhgdx \wedge dy$,
- (ii) $\mathcal{L}_X(gdx \wedge dy) = \text{div}(gX)dx \wedge dy$.

Proof. The proof of this lemma is a straightforward calculation. $\square$

Let us denote $\theta_1 = \|1\|$, $\theta_2 = \|y\|$, $\theta_3 = \|x\|$, $\theta_4 = \|y^2\|$, $\theta_5 = \|xy\|$, $\theta_6 = \|x^2\|$, $\theta_7 = \|y^3\|$, $\theta_8 = \|xy^2\|$, $\theta_9 = \|x^2y\|$, $\theta_{10} = \|xy^3\|$, $\theta_{11} = \|x^2y^2\|$, $\theta_{12} = \|f\|$. Recall that we denote $\|gdx \wedge dy\|_{N'}$ by $\|g\|$, $g \in \mathcal{O}_2$.

Lemma 5.1.3. *We have that $\|x^2y^3\| = -\frac{29}{2}\|f\|$, $\|x^4\| = 7\|f\|$, $\|y^5\| = \frac{17}{2}\|f\|$ and $\|x^3y\| = \|x^4y\| = \|xy^4\| = \|x^3y^2\| = \|x^5\| = 0$.*

Proof. (i) Consider $\alpha = \frac{x}{4}dy - \frac{y}{5}dx$, so $d(f\alpha) = \left(\frac{29}{20}f + \frac{1}{10}x^2y^3\right)dx \wedge dy$. Then $\frac{29}{20}\|f\| + \frac{1}{10}\|x^2y^3\| = 0$.

(ii) Consider $\alpha = \frac{x}{2}dy + \frac{y}{3}dx$, so $d(f\alpha) = \left(2x^4 - \frac{5}{3}y^5 + \frac{1}{6}f\right)dx \wedge dy$. Hence $\frac{5}{3}\|y^5\| - 2\|x^4\| = \frac{1}{6}\|f\|$, and since $\|x^4\| + \|y^5\| + \|x^2y^3\| = \|f\|$ applying (i) we obtain $\|x^4\| = 7\|f\|$ and $\|y^5\| = \frac{17}{2}\|f\|$.

(iii) $\frac{\partial f}{\partial x} = 4x^3 + 2xy^3$ and $\frac{\partial f}{\partial y} = 5y^4 + 3x^2y^2$. Then $x^3y \in Jf + \langle xy^4 \rangle$, but $xy^4 \in Jf + \mathcal{M}_2^6 \subset Jf$. Thus $x^3y \in Jf$. Let g_1, g_2 be smooth functions such that $x^3y = g_1\frac{\partial f}{\partial x} + g_2\frac{\partial f}{\partial y}$. A simple calculation shows that $g_1 = 1/4y + \tilde{g}_1$ and $g_2 = -1/10x + \tilde{g}_2$, where $\tilde{g}_1, \tilde{g}_2 \in \mathcal{M}_2^2$. Therefore

$$\|x^3y\| = \|d(fg_1dy - fg_2dx)\| - \left\| \left(\frac{\partial \tilde{g}_1}{\partial x} + \frac{\partial \tilde{g}_2}{\partial y} \right) f \right\|$$

This $\|x^3y\| = 0$ by Theorem 4.2.9.

(iv) Applying Corollary 4.2.11 we have. $0 = \|d(xyfdy)\| = \|yf\| + \|xy\frac{\partial f}{\partial x}\| \Rightarrow \|2x^4y + x^2y^4\| = 0 \Rightarrow \|x^4y\| = 0$
 $0 = \|d(yfdy)\| = \left\| y\frac{\partial f}{\partial x} \right\| \Rightarrow \|2x^3y + xy^4\| = 0 \Rightarrow \|xy^4\| = 0$
 $0 = \|d(y^2fdy)\| = \left\| y^2\frac{\partial f}{\partial x} \right\| \Rightarrow \|2x^3y^2 + xy^5\| = 0 \Rightarrow \|x^3y^2\| = 0$
 $0 = \|xf\| \Rightarrow \|x^5\| = 0$ $\square$

Since $\{\eta_1, \eta_2\}$ is a basis for $\text{Derlog}(f)$, where

$$\begin{aligned} \eta_1 &= \left(-10x^2 + 3x^2y + y^3\right) \frac{\partial}{\partial x} + \left(-8xy + 2xy^2\right) \frac{\partial}{\partial y} \\ \eta_2 &= \left(-25xy + 6xy^2\right) \frac{\partial}{\partial x} + \left(2x^2 - 20y^2 + 6y^3\right) \frac{\partial}{\partial y}, \end{aligned}$$

$\|g\| = 0$ for all $g \in \mathcal{M}_2^6$, Lemma 5.1.2(i) and Proposition 5.1.1 to calculate the tangent space of any exact algebraic restriction in $\|\Omega^2(\mathbb{C}^2)\|_{N'}$ is enough to consider the vector

field in the form $x^i y^j \eta_k$, $k = 1, 2$ and $i + j \leq 5$. Also, by Lemma 5.1.2 (i) it is enough to calculate $\mathcal{L}_{\eta_k} \theta_m$, $k = 1, 2$ and $1 \leq m \leq 12$.

By Lemma 5.1.2 (ii) we obtain

$$\begin{aligned} \mathcal{L}_{\eta_1} x^i y^j dx \wedge dy &= \text{div}(x^i y^j \eta_1) dx \wedge dy \\ &= \{x^i y^j \text{div}(\eta_1) + \eta_1(x^i y^j)\} dx \wedge dy \\ &= x^i y^j (-28x + 10xy) + i(-10x^2 + 3x^2 y + y^3) x^{i-1} y^j + \\ &\quad + j(-8xy + 2xy^2) x^i y^{j-1} \end{aligned}$$

$$\mathcal{L}_{\eta_1} x^i y^j dx \wedge dy = (10 + 3i + 2j) x^{i+1} y^{j+1} - (28 + 10i + 8j) x^{i+1} y^j + i x^{i-1} y^{j+3} \quad (5.1)$$

Analogously

$$\begin{aligned} \mathcal{L}_{\eta_2} x^i y^j dx \wedge dy &= \text{div}(x^i y^j \eta_2) dx \wedge dy \\ &= \{x^i y^j \text{div}(\eta_2) + \eta_2(x^i y^j)\} dx \wedge dy \\ &= x^i y^j (-65y + 24y^2) + i(-25xy + 6xy^2) x^{i-1} y^j + \\ &\quad + j(2x^2 - 20y^2 + 6y^3) x^i y^{j-1} \end{aligned}$$

$$\mathcal{L}_{\eta_2} x^i y^j dx \wedge dy = (24 + 6i + 6j) x^i y^{j+2} - (65 + 25i + 20j) x^i y^{j+1} + 2j x^{i+2} y^{j-1} \quad (5.2)$$

Using equations 5.1, 5.2 and Lemma 5.1.3 we obtain:

1. $\mathcal{L}_{\eta_1} \theta_1 = -28\theta_3 + 10\theta_5$,
2. $\mathcal{L}_{\eta_2} \theta_1 = -65\theta_2 + 24\theta_4$,
3. $\mathcal{L}_{\eta_1} \theta_2 = -36\theta_5 + 12\theta_8$.
4. $\mathcal{L}_{\eta_2} \theta_2 = 2\theta_6 - 85\theta_4 + 30\theta_7$,
5. $\mathcal{L}_{\eta_1} \theta_3 = -38\theta_6 + 13\theta_9 + \theta_7$,
6. $\mathcal{L}_{\eta_2} \theta_3 = -90\theta_5 + 30\theta_8$,
7. $\mathcal{L}_{\eta_1} \theta_6 = -48\|x^3\| + 16\|x^3 y\| + 2\|xy^3\| = 26\|xy^3\| = 26\theta_{10}$ since

$$0 = [d(fdy)] = \left\| \frac{\partial f}{\partial x} \right\| = \|4x^3 + 2xy^3\|, \quad (5.3)$$

8. $\mathcal{L}_{\eta_2} \theta_6 = -115\theta_9 + 36\theta_{11}$
9. $\mathcal{L}_{\eta_1} \theta_5 = -46\|x^2 y\| + 15\|x^2 y^2\| + \|y^4\| = -46\theta_9 + \frac{72}{5}\theta_{11}$, because

$$0 = \|d(-fdx)\| = \left\| \frac{\partial f}{\partial y} \right\| = \|5y^4 + 3x^2 y^2\|, \quad (5.4)$$

10. $\mathcal{L}_{\eta_2}\theta_5 = 2\|x^3\| - 110\|xy^2\| + 36\|xy^3\| = -110\theta_8 + 35\theta_{10}$, by 5.3.
11. $\mathcal{L}_{\eta_1}\theta_4 = -44\theta_8 + 14\theta_{10}$
12. $\mathcal{L}_{\eta_2}\theta_4 = 4\|x^2y\| - 105\|y^3\| + 36\|y^4\| = 4\theta_9 - 105\theta_7 - \frac{108}{5}\theta_{11}$, by 5.4.
13. $\mathcal{L}_{\eta_1}\theta_9 = -56\|x^3y\| + 18\|x^3y^2\| + 2\|xy^4\| = 0$,
14. $\mathcal{L}_{\eta_2}\theta_9 = 2\|x^4\| - 135\|x^2y^2\| + 42\|x^2y^3\| = -135\theta_{11} - 595\theta_{12}$,
15. $\mathcal{L}_{\eta_1}\theta_8 = -54\|x^2y^2\| + 17\|x^2y^3\| + \|y^5\| = -54\theta_{11} - 238\theta_{12}$,
16. $\mathcal{L}_{\eta_2}\theta_8 = 4\|x^3y\| - 130\|xy^3\| + 42\|xy^4\| = -130\theta_{10}$,
17. $\mathcal{L}_{\eta_1}\theta_7 = -52\|xy^3\| + 16\|xy^4\| = -52\theta_{10}$,
18. $\mathcal{L}_{\eta_2}\theta_7 = 6\|x^2y^2\| - 125\|y^4\| + 42\|y^5\| = 81\theta_{11} - 357\theta_{12}$,
19. $\mathcal{L}_{\eta_1}\theta_{11} = -64\|x^3y^2\| + 20\|x^3y^3\| + 2\|xy^5\| = 0$,
20. $\mathcal{L}_{\eta_2}\theta_{11} = 4\|x^4y\| - 155\|x^2y^3\| + 48\|x^2y^4\| = \frac{2295}{2}\theta_{12}$,
21. $\mathcal{L}_{\eta_1}\theta_{10} = -62\|x^2y^3\| + 19\|x^2y^4\| + \|y^6\| = 899\theta_{12}$,
22. $\mathcal{L}_{\eta_2}\theta_{10} = 6\|x^3y^2\| - 150\|xy^4\| + 48\|xy^5\| = 0$,
23. $\mathcal{L}_{\eta_1}\theta_{12} = -72\|x^3y^3\| + 22\|x^3y^4\| + 2\|xy^6\| = 0$,
24. $\mathcal{L}_{\eta_2}\theta_{12} = 6\|x^4y^2\| - 175\|x^2y^4\| + 54\|x^2y^5\| = 0$.

Summarizing

$\mathcal{L}_{\eta_i}\theta_j$	η_1	η_2
θ_1	$-28\theta_3 + 10\theta_5$	$-65\theta_2 + 24\theta_4$
θ_2	$-36\theta_5 + 12\theta_8$	$2\theta_6 - 85\theta_4 + 30\theta_7$
θ_3	$-38\theta_6 + 13\theta_9 + \theta_7$	$-90\theta_5 + 30\theta_8$
θ_4	$-44\theta_8 + 14\theta_{10}$	$4\theta_9 - 105\theta_7 - \frac{108}{5}\theta_{11}$
θ_5	$-46\theta_9 + \frac{72}{5}\theta_{11}$	$-110\theta_8 + 35\theta_{10}$
θ_6	$26\theta_{10}$	$-115\theta_9 + 36\theta_{11}$
θ_7	$-52\theta_{10}$	$81\theta_{11} - 357\theta_{12}$
θ_8	$-54\theta_{11} - 238\theta_{12}$	$-130\theta_{10}$
θ_9	0	$-135\theta_{11} - 595\theta_{12}$
θ_{10}	$899\theta_{12}$	0
θ_{11}	0	$\frac{2295}{2}\theta_{12}$
θ_{12}	0	0

Now consider the vector fields $X_1 = \eta_1$, $X_2 = \eta_2$, $X_3 = y\eta_1$, $X_4 = x\eta_1$, $X_5 = y\eta_2$, $X_6 = x\eta_2$, $X_7 = y^2\eta_1$, $X_8 = xy\eta_1$, $X_9 = x^2\eta_1$, $X_{10} = y^2\eta_2$, $X_{11} = xy\eta_2$, $X_{12} = x^2\eta_2$, $X_{13} = y^3\eta_1$, $X_{14} = xy^2\eta_1$, $X_{15} = y^3\eta_2$, $X_{16} = xy^2\eta_2$, $X_{17} = x^2y\eta_2$, $X_{18} = xy^3\eta_1$, $X_{19} = x^2y^2\eta_2$. So by Lemma 5.1.2 we obtain the next tables

$\mathcal{L}_{X_i} \theta_j$	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9
X_3	$\mathcal{L}_{\eta_1} \theta_2$	$\mathcal{L}_{\eta_1} \theta_4$	$\mathcal{L}_{\eta_1} \theta_5$	$\mathcal{L}_{\eta_1} \theta_7$	$\mathcal{L}_{\eta_1} \theta_8$	$\mathcal{L}_{\eta_1} \theta_9$	$-\frac{3}{5} \mathcal{L}_{\eta_1} \theta_{11}$	$\mathcal{L}_{\eta_1} \theta_{10}$	$\mathcal{L}_{\eta_1} \theta_{11}$
X_4	$\mathcal{L}_{\eta_1} \theta_3$	$\mathcal{L}_{\eta_1} \theta_5$	$\mathcal{L}_{\eta_1} \theta_6$	$\mathcal{L}_{\eta_1} \theta_8$	$\mathcal{L}_{\eta_1} \theta_9$	$-\frac{1}{2} \mathcal{L}_{\eta_1} \theta_{10}$	$\mathcal{L}_{\eta_1} \theta_{10}$	$\mathcal{L}_{\eta_1} \theta_{11}$	0
X_5	$\mathcal{L}_{\eta_2} \theta_2$	$\mathcal{L}_{\eta_2} \theta_4$	$\mathcal{L}_{\eta_2} \theta_5$	$\mathcal{L}_{\eta_2} \theta_7$	$\mathcal{L}_{\eta_2} \theta_8$	$\mathcal{L}_{\eta_2} \theta_9$	$-\frac{3}{5} \mathcal{L}_{\eta_2} \theta_{11}$	$\mathcal{L}_{\eta_2} \theta_{10}$	$\mathcal{L}_{\eta_2} \theta_{11}$
X_6	$\mathcal{L}_{\eta_2} \theta_3$	$\mathcal{L}_{\eta_2} \theta_5$	$\mathcal{L}_{\eta_2} \theta_6$	$\mathcal{L}_{\eta_2} \theta_8$	$\mathcal{L}_{\eta_2} \theta_9$	$-\frac{1}{2} \mathcal{L}_{\eta_2} \theta_{10}$	$\mathcal{L}_{\eta_2} \theta_{10}$	$\mathcal{L}_{\eta_2} \theta_{11}$	0
X_7	$\mathcal{L}_{\eta_1} \theta_4$	$\mathcal{L}_{\eta_1} \theta_7$	$\mathcal{L}_{\eta_1} \theta_8$	$-\frac{3}{5} \mathcal{L}_{\eta_1} \theta_{11}$	$\mathcal{L}_{\eta_1} \theta_{10}$	$\mathcal{L}_{\eta_1} \theta_{11}$	0	0	0
X_8	$\mathcal{L}_{\eta_1} \theta_5$	$\mathcal{L}_{\eta_1} \theta_8$	$\mathcal{L}_{\eta_1} \theta_9$	$\mathcal{L}_{\eta_1} \theta_{10}$	$\mathcal{L}_{\eta_1} \theta_{11}$	0	0	0	0
X_9	$\mathcal{L}_{\eta_1} \theta_6$	$\mathcal{L}_{\eta_1} \theta_9$	$-\frac{1}{2} \mathcal{L}_{\eta_1} \theta_{10}$	$\mathcal{L}_{\eta_1} \theta_{11}$	0	0	0	0	0
X_{10}	$\mathcal{L}_{\eta_2} \theta_4$	$\mathcal{L}_{\eta_2} \theta_7$	$\mathcal{L}_{\eta_2} \theta_8$	$-\frac{3}{5} \mathcal{L}_{\eta_2} \theta_{11}$	$\mathcal{L}_{\eta_2} \theta_{10}$	$\mathcal{L}_{\eta_2} \theta_{11}$	0	0	0
X_{11}	$\mathcal{L}_{\eta_2} \theta_5$	$\mathcal{L}_{\eta_2} \theta_8$	$\mathcal{L}_{\eta_2} \theta_9$	$\mathcal{L}_{\eta_2} \theta_{10}$	$\mathcal{L}_{\eta_2} \theta_{11}$	0	0	0	0
X_{12}	$\mathcal{L}_{\eta_2} \theta_6$	$\mathcal{L}_{\eta_2} \theta_9$	$-\frac{1}{2} \mathcal{L}_{\eta_2} \theta_{10}$	$\mathcal{L}_{\eta_2} \theta_{11}$	0	0	0	0	0
X_{13}	$\mathcal{L}_{\eta_1} \theta_7$	$-\frac{3}{5} \mathcal{L}_{\eta_1} \theta_{11}$	$\mathcal{L}_{\eta_1} \theta_{10}$	0	0	0	0	0	0
X_{14}	$\mathcal{L}_{\eta_1} \theta_8$	$\mathcal{L}_{\eta_1} \theta_{10}$	$\mathcal{L}_{\eta_1} \theta_{11}$	0	0	0	0	0	0
X_{15}	$\mathcal{L}_{\eta_2} \theta_7$	$-\frac{3}{5} \mathcal{L}_{\eta_2} \theta_{11}$	$\mathcal{L}_{\eta_2} \theta_{10}$	0	0	0	0	0	0
X_{16}	$\mathcal{L}_{\eta_2} \theta_8$	$\mathcal{L}_{\eta_2} \theta_{10}$	$\mathcal{L}_{\eta_2} \theta_{11}$	0	0	0	0	0	0
X_{17}	$\mathcal{L}_{\eta_2} \theta_9$	$\mathcal{L}_{\eta_2} \theta_{11}$	0	0	0	0	0	0	0
X_{18}	$\mathcal{L}_{\eta_1} \theta_{10}$	0	0	0	0	0	0	0	0
X_{19}	$\mathcal{L}_{\eta_2} \theta_{11}$	0	0	0	0	0	0	0	0

Consider the exact algebraic restriction $\gamma = \sum_{l=1}^{12} c_l \theta_l$. We want to find the orbits of γ under the action of the group $G_{N'}(\mathbb{C}^2)$. To do it, we have to calculate the tangent space to γ and using the homotopy method look for those who are diffeomorphic to γ considering some conditions on c_t , $1 \leq l \leq 12$. Consider the vector field tangent to N' , $V = \sum_{r=1}^{19} d_r X_r$, then we have that

$$\mathcal{L}_V \gamma = \sum_{l,r} c_l d_r \mathcal{L}_{X_r} \theta_l.$$

Therefore every element of $T(\gamma)$ has the form $\sum_{l=2}^{12} A_l \theta_l$, where

$$A_2 = -65d_2c_1,$$

$$A_3 = -28d_1c_1,$$

$$A_4 = (24c_1 - 85c_2)d_2 - 85c_1d_5,$$

$$A_5 = (10c_1 - 36c_2)d_1 - 90c_3d_2 - 36c_1d_3 - 90c_1d_6,$$

$$A_6 = -38d_1c_3 + 2c_2d_2 - 38d_4c_1 + 2d_5c_1,$$

$$A_7 = c_3d_1 + (30c_2 - 105c_4)d_2 + c_1d_4 + (30c_1 - 105c_2)d_5 - 105c_1d_{10},$$

$$A_8 = (12c_2 - 44c_4)d_1 + (30c_3 - 110c_5)d_2 + (12c_1 - 44c_2)d_3 + (30c_1 - 110c_3)d_6 \\ - 110c_3d_5 - 44c_1d_7 - 110c_1d_{11},$$

$$A_9 = (13c_3 - 46c_5)d_1 + (4c_4 - 115c_6)d_2 + (13c_1 - 46c_2)d_4 + 4c_2d_5 - 115c_3d_6 \\ - 46c_3d_3 - 46c_1d_8 - 115c_1d_{12} + 4c_1d_{10},$$

$$A_{10} = (26c_6 + 14c_4 - 52c_7)d_1 + (35c_5 - 130c_8)d_2 + 26c_3d_4 + (14c_2 - 52c_4)d_3 \\ + (35c_2 - 130c_4)d_6 + (35c_3 - 130c_5)d_5 + 26c_1d_9 + (14c_1 - 52c_2)d_7 \\ + (35c_1 - 130c_2)d_{11} - 130c_3d_{10} - 52c_1d_{13} - 130c_1d_{16},$$

$$A_{11} = \left(\frac{72}{5}c_5 - 54c_8\right)d_1 + \left(36c_6 - \frac{108}{5}c_4 - 135c_9 + 81c_7\right)d_2 + \left(\frac{72}{5}c_2 - 54c_4\right)d_4 \\ + \left(\frac{72}{5}c_3 - 54c_5\right)d_3 + (36c_3 - 135c_5)d_6 + \left(-\frac{108}{5}c_2 - 135c_6 + 81c_4\right)d_5$$

$$\begin{aligned}
& + \left(\frac{72}{5}c_1 - 54c_2\right)d_8 - 54c_3d_7 + (36c_1 - 135c_2)d_{12} - 135c_2d_{11} \\
& + \left(-\frac{108}{5}c_1 + 81c_2\right)d_{10} - 54c_1d_{14} - 135c_1d_{17} + 81c_1d_{15}, \\
A_{12} = & (-238c_8 + 899c_10)d_1 + (-595c_9 - 357c_7 + \frac{2295}{2}c_{11})d_2 \\
& + \left(-\frac{899}{2}c_6 - 238c_4 + 899c_7\right)d_4 + (-238c_5 + 899c_8)d_3 + \left(-595c_5 + \frac{2295}{2}c_8\right)d_6 \\
& + (-595c_6 - 357c_4 + \frac{2295}{2}c_9 - \frac{1377}{2}c_7)d_5 - \frac{899}{2}c_3d_9 + (-238c_2 + 899c_4)d_8 \\
& + (-238c_3 + 899c_5)d_7 + (-595c_2 + \frac{2295}{2}c_4)d_{12} + \left(-595c_3 + \frac{2295}{2}c_5\right)d_{11} \\
& + (-357c_2 + \frac{2295}{2}c_6 - \frac{1377}{2}c_4)d_{10} + (-238c_1 + 899c_2)d_{14} + 899c_3d_{13} \\
& + (-595c_1 + \frac{2295}{2}c_2)d_{17} + \frac{2295}{2}c_3d_{16} + \left(-357c_1 - \frac{1377}{2}c_2\right)d_{15} + 899c_1d_{18} \\
& + \frac{2295}{2}c_1d_{19}.
\end{aligned}$$

Therefore, we have the system $A.d = (A_2, \dots, A_{12})^T$, where $d = (d_1, d_2, \dots, d_{19})^T$ denotes

the matrix $\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{19} \end{pmatrix}$, A is the matrix given in the Figure 1 and $A_i \in T(\gamma)$, $i = 2, \dots, 12$,

where

$$\begin{aligned}
M_1 &= 36c_6 - \frac{108}{5}c_4 - 135c_9 + 81c_7 \\
M_2 &= -\frac{108}{5}c_2 - 135c_6 + 81c_4 \\
M_3 &= 26c_6 + 14c_4 - 52c_7 \\
M_4 &= -595c_9 - 357c_7 + \frac{2295}{2}c_{11} \\
M_5 &= -\frac{899}{2}c_6 - 238c_4 + 899c_7 \\
M_6 &= -595c_6 - 357c_4 + \frac{2295}{2}c_9 - \frac{1377}{2}c_7 \\
M_7 &= -357c_2 + \frac{2295}{2}c_6 - \frac{1377}{2}c_4
\end{aligned}$$

Now we classify the exact algebraic restriction classification to N' under the action of the group $G_{N'}(\mathbb{C}^2)$.

Lemma 5.1.4. *Let $\gamma_0 = \sum_{i=1}^{12} m_i \theta_i$, $\gamma_1 = \sum_{i=1}^{12} n_i \theta_i$ be two exact algebraic restrictions to N' . Denote $\gamma_t = \gamma_0 + t(\gamma_1 - \gamma_0)$, $t \in [0, 1]$. So, if the system*

$$A_t d_t = (n_2 - m_2, \dots, n_{12} - m_{12})^T, \quad t \in [0, 1]$$

has smooth solution $d_t = (d_1(t), d_2(t), \dots, d_{19}(t))^T$ where A_t is obtained from the matrix A changing c_i by $m_i + t(n_i - m_i)$ for all $1 \leq i \leq 12$, then γ_0 and γ_1 are diffeomorphic.

$$\begin{pmatrix}
0 & -65c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-28c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 24c_1 - 85c_2 & 0 & 0 & -85c_1 & 0 & 0 & 0 & 0 & 0 \\
10c_1 - 36c_2 & -90c_3 & -36c_1 & 0 & 0 & -90c_1 & 0 & 0 & 0 & 0 \\
-38c_3 & 2c_2 & 0 & -38c_1 & 2c_1 & 0 & 0 & 0 & 0 & 0 \\
c_3 & 30c_2 - 105c_4 & 0 & c_1 & 30c_1 - 105c_2 & 0 & 0 & 0 & 0 & 0 \\
12c_2 - 44c_4 & 30c_3 - 110c_5 & 12c_1 - 44c_2 & 0 & -110c_3 & 30c_1 - 110c_2 & -44c_1 & 0 & 0 & 0 \\
13c_3 - 46c_5 & 4c_4 - 115c_6 & -46c_3 & 13c_1 + 46c_2 & 4c_2 & -115c_3 & -115c_1 & -46c_1 & 0 & 0 \\
M_3 & 35c_5 - 130c_8 & 14c_2 - 52c_4 & 26c_3 & 35c_3 - 130c_5 & 35c_2 - 130c_4 & 14c_1 - 52c_2 & 0 & 26c_1 & 0 \\
\frac{72}{5}c_5 - 54c_8 & M_1 & \frac{72}{5}c_3 - 54c_5 & \frac{72}{5}c_2 - 54c_4 & M_2 & 36c_3 - 135c_5 & -54c_3 & \frac{72}{5}c_1 - 54c_2 & 0 & -\frac{899}{2}c_3 \\
-238c_8 + 899c_{10} & M_4 & -238c_5 + 899c_8 & M_5 & M_6 & -595c_5 + \frac{2295}{2}c_8 & -238c_3 + 899c_5 & -238c_2 + 899c_4 & 0 & -\frac{899}{2}c_3
\end{pmatrix}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-105c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -110c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-130c_3 & 35c_1 - 130c_2 & 0 & -52c_1 & 0 & 0 & 0 & 0 & -130c_1 & 0 \\
-\frac{108}{5}c_1 + 81c_2 & -135c_3 & 36c_1 - 135c_2 & 0 & -54c_1 & 81c_1 & 0 & -135c_1 & 0 & 0 \\
M_7 & -595c_3 + \frac{2295}{2}c_5 & -595c_2 + \frac{2295}{2}c_4 & 899c_3 & -238c_1 + 899c_2 & -357c_1 - \frac{1377}{2}c_2 & \frac{2295}{2}c_3 - 595c_1 + \frac{2295}{2}c_2 & 899c_1 & \frac{2295}{2}c_1 & 0
\end{pmatrix}$$

Figure 1 – Matrix A

Proof. We use the homotopy method. Suppose there exists a family of germs of diffeomorphisms preserving N' Φ_t , $t \in [0, 1]$ such that

$$\Phi_t^* \gamma_t = \gamma_0, \Phi_0 = id_{\mathbb{C}^2} \quad (5.5)$$

Let X_t be the vector field such that $\frac{d\Phi_t}{dt} = X_t \circ \Phi_t$. Differentiating (5.5) we obtain

$$\mathcal{L}_{X_t} \gamma_t = \gamma_1 - \gamma_0$$

and this equation is equivalent to the system

$$A_t d_t = (n_2 - m_2, \dots, n_{12} - m_{12})^T.$$

By hypothesis there exists a smooth solution d_t , so consider vector field $X_t = \sum_{r=1}^{19} d_r(t) X_r \in \text{Derlog}(f)$. Hence the flow of this vector field gives the family of diffeomorphisms Φ_t holding (5.5). These diffeomorphisms preserve N' since $X_t \in \text{Derlog}(f)$. Therefore $\Phi^* \gamma_1 = \gamma_0$. $\square$

Now, we can obtain the normal forms for the classification of exact algebraic restriction to N' . Let us denote by $\gamma_1 \sim \gamma_2$ to indicate that the exact algebraic restrictions to N' , γ_1 and γ_2 , are diffeomorphic.

Lemma 5.1.5. *If $c_1 \neq 0$, $\gamma \sim c_1 \theta_1$.*

Proof. Let $\gamma_0 = c_1 \theta_1$, $\gamma_1 = \gamma$ and $\gamma_t = c_1 \theta_1 + t \sum_{l=2}^{12} c_l \theta_l$. According to Lemma 5.1.4, it is enough to solve the system

$$A_t d_t = (c_2, c_3, \dots, c_{12})^T$$

where A_t comes from matrix A changing c_i by tc_i for $2 \leq i \leq 12$ and maintaining c_1 . Taking

$$d_5(t) = d_7(t) = d_{10}(t) = d_{11}(t) = d_{13}(t) = d_{14}(t) = d_{15}(t) = d_{19}(t) = 0,$$

we reduce the system $A_t d_t = (c_2, c_3, \dots, c_{12})^T$ to the system $\tilde{A}_t \tilde{d}_t = (c_2, c_3, \dots, c_{12})^T$, where $\tilde{d}(t)$ is the vector $d(t)$ without $d_l(t)$, $l = 5, 7, 10, \dots, 15, 19$, and $\tilde{A}$ is a square matrix whose determinant is a multiple non zero of c_1^{11} . Hence, there exist unique $d_l(t)$, $l \neq 5, 7, 10, \dots, 15, 19$ such that $\tilde{A}_t \cdot \tilde{d}_t = (c_2, c_3, \dots, c_{12})^T$. They are smooth because the determinant of $\tilde{A}_t$ does not depend on t . $\square$

Lemma 5.1.6. *If $c_1 = 0$, $c_2 \neq 0$ and $\tilde{c}_6 = -38c_3 \left(-\frac{c_5}{36c_2} + \frac{c_3c_4}{34c_2^2} \right) - \frac{2c_4}{85}$, then $\gamma \sim c_2 \theta_2 + c_3 \theta_3 + (c_6 - \tilde{c}_6) \theta_6$.*

Proof. Let $\gamma_0 = c_2 \theta_2 + c_3 \theta_3 + (c_6 - \tilde{c}_6) \theta_6$, $\gamma_1 = \gamma$ and $\gamma_t = c_2 \theta_2 + c_3 \theta_3 + (c_6 - \tilde{c}_6) \theta_6 + t \tilde{c}_6 \theta_6 + t \sum_{l=4, \neq 6}^{12} c_l \theta_l$. By Lemma 5.1.4 it is enough to solve the system

$$A_t d_t = (0, 0, c_4, c_5, \tilde{c}_6, c_7, \dots, c_{12})^T,$$

where A_t comes from matrix A by replacing c_l by tc_l for $7 \leq l \leq 12, l \neq 6$, c_6 by $c_6 - \tilde{c}_6 + t\tilde{c}_6$ and maintaining c_2, c_3 . Take

$$d_6(t) = d_7(t) = d_9(t) = d_{10}(t) = d_{12}(t) = d_{13}(t) = d_{\geq 15}(t) = 0$$

$$d_1(t) = -\frac{c_5}{36c_2} + \frac{90c_3}{36c_2} \frac{c_4}{85c_2} \quad d_2(t) = -\frac{c_4}{85c_2}.$$

Then $-85c_2d_2(t) = c_4$, $-36c_2d_1(t) - 90c_3d_2(t) = c_5$ and by hypothesis

$$-38c_3d_1(t) + 2c_2d_2(t) = \tilde{c}_6$$

we reduce our system to the system $\tilde{A}_t \tilde{d}_t = \begin{pmatrix} c_7 - a_{6,1}(t)d_1(t) - a_{6,2}(t)d_2(t) \\ c_8 - a_{7,1}(t)d_1(t) - a_{7,2}(t)d_2(t) \\ \vdots \\ c_{12} - a_{11,1}(t)d_1(t) - a_{11,2}(t)d_2(t) \end{pmatrix}$, where

$A_t = (a_{i,j}(t))$, $\tilde{d}(t) = (d_3(t), d_4(t), d_5(t), d_8(t), d_{11}(t), d_{14}(t))^T$ and

$$\tilde{A}_t = \begin{pmatrix} 0 & 0 & -105c_2 & 0 & 0 & 0 \\ -46c_3 & 46c_2 & 4c_2 & 0 & 0 & 0 \\ -44c_2 & 0 & -110c_3 & 0 & 0 & 0 \\ 14c_2 - 52tc_4 & 26c_3 & 35c_3 - 130tc_5 & 0 & -130c_2 & 0 \\ \frac{72}{5}c_3 - 54tc_5 & \frac{72}{5}c_2 - 54tc_4 & \tilde{M}_2 & -54c_2 & -135c_3 & 0 \\ -238tc_5 + 899tc_8 & \tilde{M}_5 & \tilde{M}_6 & * & ** & 899c_2 \end{pmatrix}$$

$$\tilde{M}_2 = -\frac{108}{5}c_2 - 135(c_6 - \tilde{c}_6 + t\tilde{c}_6) + 81tc_4$$

$$\tilde{M}_5 = -\frac{899}{2}(c_6 - \tilde{c}_6 + t\tilde{c}_6) - 238tc_4 + 899tc_7$$

$$\tilde{M}_6 = -595(c_6 - \tilde{c}_6 + t\tilde{c}_6) - 357tc_4 + \frac{2295}{2}tc_9 - \frac{1377}{2}tc_7$$

$$* = -238c_2 + 899tc_4$$

$$** = -595c_3 + \frac{2295t}{2}c_5$$

Since the determinant of $\tilde{A}_t$ is a multiple non zero of c_3^6 which does not depend on t , there exists an unique smooth solution $\tilde{d}_t$. $\square$

Lemma 5.1.7. *If $c_1 = c_2 = 0$, $c_3 \neq 0$ and $\tilde{c}_7 = -\frac{c_6}{38} + \frac{7c_4c_5}{6c_3}$, then $\gamma \sim c_3\theta_3 + c_4\theta_4 + (c_7 - \tilde{c}_7)\theta_7$.*

Proof. Let $\gamma_0 = c_3\theta_3 + c_4\theta_4 + (c_7 - \tilde{c}_7)\theta_7$, $\gamma_1 = \gamma$ and $\gamma_t = c_3\theta_3 + c_4\theta_4 + (c_7 - \tilde{c}_7)\theta_7 + t\tilde{c}_7\theta_7 + t(\sum_{l=5, \neq 7}^{12} c_l\theta_l)$. Hence, we have to solve the system

$$A_t d_t = (0, \dots, 0, c_5, c_6, \tilde{c}_7, c_8, \dots, c_{12})^T, \quad A_t = (a_{i,j}(t)),$$

where A_t comes from matrix A by replacing c_l by tc_l for $5 \leq l \leq 12, l \neq 7$, c_7 by $c_7 - \tilde{c}_7 + t\tilde{c}_7$ and maintaining c_3, c_4 . Take

$$d_1(t) = -\frac{c_6}{38c_3}, \quad d_2(t) = -\frac{c_5}{90c_3}, \quad d_l(t) = 0, \quad l \neq 3, 5, 7, 9, 10.$$

By hypothesis we have that $c_3d_1(t) - 105c_4d_2(t) = \tilde{c}_7$, therefore since $-38c_3d_1(t) = c_6$ and $-90c_3d_2(t) = c_5$ we reduce the first system to the system

$$\tilde{A}_t \tilde{d}_t = \begin{pmatrix} c_8 - a_{7,1}(t)b_1(t) - a_{7,2}(t)b_2(t) \\ c_9 - a_{8,1}(t)b_1(t) - a_{8,2}(t)b_2(t) \\ c_{10} - a_{9,1}(t)b_1(t) - a_{9,2}(t)b_2(t) \\ c_{11} - a_{10,1}(t)b_1(t) - a_{10,2}(t)b_2(t) \\ c_{12} - a_{11,1}(t)b_1(t) - a_{11,2}(t)b_2(t) \end{pmatrix},$$

where $\tilde{d}_t = (d_3(t), d_5(t), d_7(t), d_9(t), d_{10}(t))^T$ and

$$\tilde{A}_t = \begin{pmatrix} 0 & -110c_3 & 0 & 0 & 0 \\ -46c_3 & 0 & 0 & 0 & 0 \\ -52c_4 & 35c_3 - 130tc_5 & 0 & 0 & -130c_3 \\ \frac{72}{5}c_3 - 54tc_5 & M_2 & -54c_3 & 0 & 0 \\ -238tc_5 + 899tc_8 & M_6 & -238c_3 + 899tc_5 & \frac{2295}{2}c_3 & M_7 \end{pmatrix}$$

$$M_2 = -135tc_6 + 81c_4$$

$$M_6 = -595tc_6 - 357c_4 + \frac{2295}{2}tc_9 - \frac{1377}{2}(c_7 - \tilde{c}_7 + t\tilde{c}_7)$$

$$M_7 = \frac{2295}{2}tc_6 - \frac{1377}{2}c_4$$

Since the determinant is a multiple non zero of c_3^5 and does not depend on t , there exists a smooth solution d_t . $\square$

Lemma 5.1.8. *If $c_1 = c_2 = c_3 = 0$, $c_4 \neq 0$ and*

$$\tilde{c}_9 = -46c_5 \left(-\frac{c_8}{44c_4} + \frac{c_5c_7}{42c_4^2} \right) - (4c_4 - 115c_6) \frac{c_7}{105c_4},$$

then $\gamma \sim \sum_{l=4}^6 c_l \theta_l + (c_9 - \tilde{c}_9) \theta_9$.

Proof. Let $\gamma_0 = \sum_{l=4}^6 c_l \theta_l + (c_9 - \tilde{c}_9) \theta_9$, $\gamma_1 = \gamma$ and $\gamma_t = \sum_{l=4}^6 c_l \theta_l + (c_9 - \tilde{c}_9) \theta_9 + t\tilde{c}_9 \theta_9 + t(\sum_{l=7, \neq 9}^{12} c_l \theta_l)$. So, we have to solve the system

$$A_t \cdot d_t = (0, \dots, 0, c_7, c_8, \tilde{c}_9, c_{10}, c_{11}, c_{12})^T, \quad A_t = (a_{i,j}(t)),$$

where A_t comes from matrix A by replacing c_l by tc_l for $7 \leq l \leq 12, l \neq 9$; c_9 by $c_9 - \tilde{c}_9 + t\tilde{c}_9$ and maintaining c_4, c_5, c_6 . Take

$$d_1(t) = -\frac{c_8}{44c_4} + \frac{c_5}{2c_4} \frac{c_7}{21c_4}, \quad d_2(t) = -\frac{c_7}{105c_4}, \quad d_l(t) = 0, \quad l \neq 4, 6, 8.$$

Then $-44c_4d_1(t) - 11c_5d_2(t) = c_8$, $-105c_4d_2(t) = c_7$ and $-46c_5d_1(t) + (4c_4 - 115c_6)d_2(t) = \tilde{c}_9$. Thus, we reduce the first system to the system

$$\tilde{A}_t \tilde{d}_t = \begin{pmatrix} c_{10} - a_{9,1}(t)d_1(t) - a_{9,2}(t)d_2(t) \\ c_{11} - a_{10,1}(t)d_1(t) - a_{10,2}(t)d_2(t) \\ c_{12} - a_{11,1}(t)d_1(t) - a_{11,2}(t)d_2(t) \end{pmatrix},$$

where $\tilde{d}(t) = (d_4(t), d_6(t), d_8(t))$ and

$$\tilde{A}_t = \begin{pmatrix} 0 & -130c_4 & 0 \\ -54c_4 & -135c_5 & 0 \\ -\frac{899}{2}c_6 - 238c_4 + 899tc_7 & -595c_5 + \frac{2295}{2}tc_8 & 899c_4 \end{pmatrix}.$$

Since the determinant of $\tilde{A}$ is a multiple non zero of c_4^3 and it does not depend on t , there exists a smooth solution d_t . $\square$

Lemma 5.1.9. *If $c_1 = c_2 = c_3 = c_4 = 0$, $c_5 \neq 0$, then $\gamma \sim c_5\theta_5 + c_6\theta_6 + c_7\theta_7$.*

Proof. Let $\gamma_0 = \sum_{l=5}^7 c_l\theta_l$, $\gamma_1 = \gamma$ and $\gamma_t = c_5\theta_5 + c_6\theta_6 + c_7\theta_7 + t(\sum_{l=8}^{12} c_l\theta_l)$. So, we have to solve the system

$$A_t \cdot d_t = (0, \dots, 0, c_8, \dots, c_{12})_{11}^T, \quad A_t = (a_{i,j}(t)),$$

where A_t comes from matrix A by replacing c_l by tc_l for $8 \leq l \leq 12$ and maintaining c_5, c_6, c_7 . Take

$$d_l(t) = 0, \quad l \neq 1, 2, 5, 6, 7$$

Thus, we reduce the first system to the system $\tilde{A}_t \tilde{d}_t = (c_8, \dots, c_{12})^T$, where $\tilde{d}(t) = (d_1(t), d_2(t), d_5(t), d_6(t), d_7(t))^T$ and

$$\tilde{A}_t = \begin{pmatrix} 0 & -110c_5 & 0 & 0 & 0 \\ -46c_5 & -115c_6 & 0 & 0 & 0 \\ 26c_6 - 52c_7 & 35c_5 - 130tc_8 & -130c_5 & 0 & 0 \\ \frac{72}{5}c_5 - 54tc_8 & M_1 & -135c_6 & 135c_5 & 0 \\ -238tc_8 + 899tc_{10} & M_4 & M_6 & -595c_5 + \frac{2295}{2}tc_8 & 899c_5 \end{pmatrix}.$$

Since determinant of $\tilde{A}$ is a multiple non zero of c_5^5 and it does not depend on t , there exists a smooth solution $d(t)$. $\square$

Lemma 5.1.10. *If $c_1 = c_2 = c_3 = c_4 = c_5 = 0$, $c_6 \neq 0$, $c_6 = 2c_7$ and $\tilde{c}_{10} = \frac{26c_8c_9}{23c_6}$, then $\gamma \sim \sum_{l=6}^8 c_l\theta_l + (c_{10} - \tilde{c}_{10})\theta_{10}$.*

Proof. Let $\gamma_0 = \sum_{l=6}^8 c_l\theta_l + (c_{10} - \tilde{c}_{10})\theta_{10}$, $\gamma_1 = \gamma$ and $\gamma_t = \sum_{l=6}^8 c_l\theta_l + (c_{10} - \tilde{c}_{10})\theta_{10} + t\tilde{c}_{10}\theta_{10} + t\sum_{l=9, \neq 10}^{12} c_l\theta_l$. So, we have to solve the system

$$A_t \cdot d_t = (0, \dots, 0, c_9, \tilde{c}_{10}, c_{11}, c_{12})^T, \quad A_t = (a_{i,j}(t)),$$

where A_t comes from matrix A by replacing c_l by tc_l for $9 \leq l \leq 12, l \neq 10$; c_{10} by $c_{10} - \tilde{c}_{10} + t\tilde{c}_{10}$ and maintaining c_6, c_7, c_8 . Take

$$d_l(t) = 0, \quad l \neq 2, 5, 10, \quad d_2(t) = -\frac{c_9}{115c_6}.$$

We have $-115c_6d_2(t) = c_9$, $-130c_8d_2(t) = \tilde{c}_{10}$. So, we reduce the first system to $\tilde{A}_t\tilde{d}_t = \begin{pmatrix} c_{11} - a_{10,2}(t)d_2(t) \\ c_{12} - a_{11,2}(t)d_2(t) \end{pmatrix}$, where $\tilde{d}_t = \begin{pmatrix} d_5(t) \\ d_{10}(t) \end{pmatrix}$ and

$$\tilde{A} = \begin{pmatrix} -135c_6 & 0 \\ -595c_6 + \frac{2295}{2}tc_9 - \frac{1377}{2}c_7 & \frac{2295}{2}c_6 \end{pmatrix}.$$

Since the determinant of $\tilde{A}$ is a multiple non zero of c_6^2 and it does not depend on t , there exists a smooth solution $d(t)$. $\square$

Lemma 5.1.11. *If $c_1 = c_2 = c_3 = c_4 = c_5 = 0$, $c_6 \neq 0$ and $c_6 \neq 2c_7$, then $\gamma \sim c_6\theta_6 + c_7\theta_7 + c_8\theta_8$.*

Proof. Let $\gamma_0 = \sum_{l=6}^8 c_l\theta_l$, $\gamma_1 = \gamma$ and $\gamma_t = c_6\theta_6 + c_7\theta_7 + c_8\theta_8 + t(\sum_{l=9}^{12} c_l\theta_l)$. So, we have to solve the system

$$A_t.d_t = (0, \dots, 0, c_9, \dots, c_{12})_{11}^T, \quad A_t = (a_{i,j}(t)),$$

where A_t comes from matrix A by replacing c_l by tc_l for $9 \leq l \leq 12$ and maintaining c_6, c_7, c_8 . By hypothesis it is possible to prove the existence of p_1 , such that

$$(26c_6 - 52c_7)p_1 + 130c_8 \frac{c_9}{115c_6} = c_{10}.$$

Taking

$$d_1(t) = p_1, \quad d_2(t) = -\frac{c_9}{115c_6}d_1(t) = 0, \quad l \neq 5, 10,$$

thus $-115c_5p_2(t) = c_9$ and we reduce the first system to $\tilde{A}_t\tilde{d}(t) = \begin{pmatrix} c_{10} - a_{9,1}p_1(t) - a_{9,2}p_2(t) \\ c_{12} - a_{11,1}p_1(t) - a_{11,2}p_2(t) \end{pmatrix}$, where $\tilde{d}_t = \begin{pmatrix} d_5(t) \\ d_{10}(t) \end{pmatrix}$ and $\tilde{A}_t = \begin{pmatrix} -135c_6 & 0 \\ -595c_6 + \frac{2295}{2}tc_9 - \frac{1377}{2}c_7 & \frac{2295}{2}c_6 \end{pmatrix}$.

Since determinant of $\tilde{A}$ is a multiple non zero of c_6^2 and it does not depend on t , there exists a smooth solution d_t . $\square$

Lemma 5.1.12. *If $c_1 = \dots = c_6 = 0$, $c_7 \neq 0$, $(54c_8)(130c_8) + (52c_7)(81c_7 - 135c_9) = 0$ and $\tilde{c}_{11} = \frac{27c_8c_{10}}{26c_7}$, then $\gamma \sim \sum_{l=7}^9 c_l\theta_l + (c_{11} - \tilde{c}_{11})\theta_{11}$.*

Proof. Let $\gamma_t = \sum_{l=7}^9 c_l\theta_l + (c_{11} - \tilde{c}_{11})\theta_{11} + t\tilde{c}_{11}\theta_{11} + tc_{10}\theta_{10} + tc_{12}\theta_{12}$ and define $p_1(t) = -\frac{c_{10}}{52c_7}$; $d_4(t) = \frac{1}{899c_7}(c_{12} - (238c_8 + 899tc_{10})p_1(t))$; $d_l(t) = 0$, $l \neq 1, 4$. Then

$$52p_1(t) = c_{10},$$

$$-54p_1(t) = \tilde{c}_{11},$$

$$(238c_8 + 899tc_{10})p_1(t) + 899c_7p_4(t) = c_{12}.$$

$\square$

Lemma 5.1.13. *If $c_1 = \dots = c_6 = 0$, $c_7 \neq 0$ and $(54c_8)(130c_8) + (52c_7)(81c_7 - 135c_9) \neq 0$, then $\gamma \sim c_7\theta_7 + c_8\theta_8 + c_9\theta_9$.*

Proof. Let $\gamma_t = c_7\theta_7 + c_8\theta_8 + c_9\theta_9 + t(\sum_{l=10}^{12} c_l\theta_l)$ and define $d_3(t) = d_{l \geq 5}(t) = 0$. Then we have to solve the system $\tilde{A}_t \tilde{d}_t = \begin{pmatrix} c_{10} \\ c_{11} \\ c_{12} \end{pmatrix}$, where $\tilde{d}(t) = \begin{pmatrix} d_1(t) \\ d_2(t) \\ d_4(t) \end{pmatrix}$ and

$$\tilde{A}_t = \begin{pmatrix} -52c_7 & -130c_8 & 0 \\ -54c_8 & -135c_9 + 81c_7 & 0 \\ -238c_8 + 899tc_{10} & -595c_9 - 357c_7 + \frac{2295}{2}tc_{11} & 899c_7 \end{pmatrix} = (\tilde{a}_{i,j}(t)).$$

By hypothesis it is possible to prove that there exist $b_1, b_2 \in \mathbb{C}$ such that

$$(c_{10}, c_{11}) = b_1(-52c_7, -54c_8) + b_2(-130c_8, 81c_7 - 135c_9),$$

thus $d_1(t) = b_1$, $d_2(t) = b_2$ and $d_3 = \frac{1}{899c_7}(c_{12} - \tilde{a}_{3,1}b_1 - \tilde{a}_{3,2}b_2)$, is a solution of the system. $\square$

Lemma 5.1.14. *If $c_1 = \dots = c_7 = 0$ and $c_8 \neq 0$, then $\gamma \sim c_8\theta_8 + c_9\theta_9$.*

Proof. Let $\gamma_t = c_8\theta_8 + c_9\theta_9 + t(\sum_{l=10}^{12} c_l\theta_l)$ and define $d_{l \geq 4}(t) = 0$. Then we have to solve the system $\tilde{A}_t \tilde{d}_t = \begin{pmatrix} c_{10} \\ c_{11} \\ c_{12} \end{pmatrix}$, where $\tilde{d}(t) = \begin{pmatrix} d_1(t) \\ d_2(t) \\ d_4(t) \end{pmatrix}$ and

$$\tilde{A}_t = \begin{pmatrix} 0 & -130c_8 & 0 \\ -54c_8 & -135c_9 & 0 \\ -238c_8 + 899tc_{10} & -595c_9 + \frac{2295}{2}tc_{11} & 899c_8 \end{pmatrix}.$$

Since the determinant of $\tilde{A}$ is a multiple non zero of c_8^3 there exists a smooth solution $\tilde{d}_t$. $\square$

Lemma 5.1.15. *If $c_1 = \dots = c_8 = 0$ and $c_9 \neq 0$, then $\gamma \sim c_9\theta_9 + c_{10}\theta_{10}$.*

Proof. Let us consider $d_1(t) = d_3(t) = d_4(t) = d_{l \geq 6}(t) = 0$. Then we have to solve the system $\tilde{A}\tilde{d}(t) = \tilde{c} = \begin{pmatrix} c_{10} \\ c_{12} \end{pmatrix}$, where $\tilde{d}(t) = \begin{pmatrix} d_2(t) \\ d_5(t) \end{pmatrix}$ and

$$\tilde{A} = \begin{pmatrix} -157c_9 & 0 \\ -595c_9 + \frac{2295}{2}tc_{11} & \frac{2295}{2}c_9 \end{pmatrix}.$$

Since the determinant of $\tilde{A}$ is a multiple non zero of c_9^2 there exists a smooth solution $\tilde{d}_t$. $\square$

Lemma 5.1.16. *If $c_1 = \dots = c_9 = 0$, $c_{10} \neq 0$ or $c_{11} \neq 0$ then $\gamma \sim c_{10}\theta_{10} + c_{11}\theta_{11}$.*

Proof. Take $d_{l \geq 3}(t) = 0$. If $c_{10} \neq 0$, define $d_1(t) = \frac{c_{12}}{899c_{10}}$, $d_2(t) = 0$ and if $c_{11} \neq 0$, define $d_2(t) = \frac{2c_{12}}{2295c_{11}}$, $d_1(t) = 0$. $\square$

Lemma 5.1.17. *Let $\gamma = c_{12}\theta_{12}$. Then the orbit of γ has just two elements γ and $-\gamma$.*

Proof. Since $T(\gamma) = 0$, we have that the orbit of γ is a discrete subset. Note that $\phi(x, y) = (-x, y)$ preserve N' , therefore $\phi^*\gamma = -\gamma$ belongs to the orbit of γ . It is sufficient to prove that any diffeomorphism preserving N' has linear part (x, y) or $(-x, y)$.

Claim: Let ϕ be a diffeomorphism of $(\mathbb{C}^2, 0)$ preserving $(N', 0)$, then the linear part of ϕ is (x, y) or $(-x, y)$.

Proof of the claim

By the property of zeros of f there exists $H \in \mathcal{O}_2$ such that

$$f \circ \phi(x, y) = f(x, y)H(x, y). \quad (5.6)$$

Write $\phi = (\phi_1, \phi_2)$ and

$$\begin{aligned} \phi_1(x, y) &= a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + \dots \\ \phi_2(x, y) &= b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + \dots \\ H(x, y) &= r_0 + r_1x + r_2y + r_3x^2 + r_4xy + r_5y^2 + \dots \end{aligned}$$

Replacing in 5.6 and comparing the jets we obtain:

i) 5-jet: $a_2 = 0$, $a_1^4 = r_0$.

ii) 6-jet:

$$\begin{aligned} 4a_1^3a_3 + b_1^5 + a_1^2b_1^3 &= r_1, \\ 4a_1^3a_4 + 5b_1^4b_2 + 3a_1^2b_1^2 &= r_2, \\ 4a_1^3a_5 + 10b_1^3b_2^2 + 3a_1^2b_1b_2^2 &= 0, \\ 10b_1^2b_2^3 + a_1^2b_2^3 &= r_0, \\ 5b_1b_2^4 &= 0, \\ b_2^5 &= r_0. \end{aligned}$$

Then $b_1 = 0$ and $a_1^4 = a_1^2b_2^3 = b_2^5 = r_0$. Therefore $r_0 = b_2 = a_1^2 = 1$. $\square$

Summarizing we obtain the Table 1. In this table we also show the codimension of the manifold defined by the coefficients c_i on $\mathbb{C}^{12}$ and the dimension of the tangent space. We recall that $\theta_1 = \|1\|$, $\theta_2 = \|y\|$, $\theta_3 = \|x\|$, $\theta_4 = \|y^2\|$, $\theta_5 = \|xy\|$, $\theta_6 = \|x^2\|$, $\theta_7 = \|y^3\|$, $\theta_8 = \|xy^2\|$, $\theta_9 = \|x^2y\|$, $\theta_{10} = \|xy^3\|$, $\theta_{11} = \|x^2y^2\|$, $\theta_{12} = \|f\|$.

$\gamma = \sum_{l=1}^{12} c_l \theta_l$	$\sim$	$cod(c_i)$	$\dim T(\gamma)$
$c_1 \neq 0$	$c_1 \theta_1$	0	11
$c_1 = 0, c_2 \neq 0,$ $\tilde{c}_6 = -38c_3 \left(-\frac{c_5}{36c_2} + \frac{90c_3}{36c_2} \frac{c_4}{85c_2} \right) - \frac{2c_4}{85}$	$\sum_{l=2}^3 c_l \theta_l + (c_6 - \tilde{c}_6) \theta_6$	1	8
$c_1 = c_2 = 0, c_3 \neq 0,$ $\tilde{c}_7 = -\frac{c_6}{38} + \frac{7c_4 c_5}{6c_3}$	$\sum_{l=3}^4 c_l \theta_l + (c_7 - \tilde{c}_7) \theta_7$	2	7
$c_{\leq 3} = 0, c_4 \neq 0,$ $\tilde{c}_9 = -46c_5 \left(-\frac{c_8}{44c_4} + \frac{c_5}{2c_4} \frac{c_7}{21c_4} \right) -$ $-(4c_4 - 115c_6) \frac{c_7}{105c_4}$	$\sum_{l=4}^6 c_l \theta_l + (c_9 - \tilde{c}_9) \theta_9$	3	5
$c_{\leq 3} = c_4 = 0, c_5 \neq 0$	$\sum_{l=5}^7 c_l \theta_l$	4	5
$c_{\leq 3} = c_4 = c_5 = 0, c_6 \neq 0, c_6 \neq 2c_7$	$\sum_{l=6}^8 c_l \theta_l$	5	4
$c_{\leq 3} = c_4 = c_5 = 0, c_6 \neq 0, c_6 = 2c_7$ $\tilde{c}_{10} = \frac{26c_8 c_9}{23c_6}$	$\sum_{l=6}^8 c_l \theta_l + (c_{10} - \tilde{c}_{10}) \theta_{10}$	6	3
$c_{\leq 6} = 0, c_7 \neq 0,$ $(54c_8)(130c_8) + (52c_7)(81c_7 - 135c_9) \neq 0$	$\sum_{l=7}^9 c_l \theta_l$	6	3
$c_{\leq 6} = 0, c_7 \neq 0,$ $(54c_8)(130c_8) + (52c_7)(81c_7 - 135c_9) = 0$ $\tilde{c}_{11} = \frac{54c_8 c_{10}}{52c_7}$	$\sum_{l=7}^9 c_l \theta_l + (c_{11} - \tilde{c}_{11}) \theta_{11}$	7	2
$c_{\leq 6} = c_7 = 0, c_8 \neq 0$	$c_8 \theta_8 + c_9 \theta_9$	7	3
$c_{\leq 6} = c_7 = c_8 = 0, c_9 \neq 0$	$c_9 \theta_9 + c_{10} \theta_{10}$	8	2
$c_{\leq 9} = 0, c_{10} \neq 0$	$c_{10} \theta_{10} + c_{11} \theta_{11}$	9	1
$c_{\leq 10} = 0, c_{11} \neq 0$	$c_{11} \theta_{11}$	10	1
$c_{\leq 11} = 0, c_{12} \neq 0$	$c_{12} \theta_{12}$	11	0
$c_1 = \dots = c_{12} = 0$	0	12	0

Table 1

Therefore we have the normal forms

$$F_1 = c_1 \theta_1, \quad c_1 \neq 0,$$

$$F_2 = \sum_{l=2}^3 c_l \theta_l + \tilde{c}_6 \theta_6, \quad c_2 \neq 0,$$

$$F_3 = \sum_{l=3}^4 c_l \theta_l + \tilde{c}_7 \theta_7, \quad c_3 \neq 0,$$

$$F_4 = \sum_{l=4}^6 c_l \theta_l + \tilde{c}_9 \theta_9, \quad c_4 \neq 0,$$

$$F_5 = \sum_{l=5}^7 c_l \theta_l, \quad c_5 \neq 0,$$

$$F_6 = \sum_{l=6}^8 c_l \theta_l, \quad c_6 \neq 0, \quad c_6 \neq 2c_7,$$

$$F_7 = \sum_{l=6}^8 c_l \theta_l + \tilde{c}_{10} \theta_{10}, \quad c_6 \neq 0, \quad c_6 = 2c_7,$$

$$F_8 = \sum_{l=7}^9 c_l \theta_l, \quad c_7 \neq 0, \quad (54c_8)(130c_8) \neq -(52c_7)(81c_7 - 135c_9),$$

$$F_9 = \sum_{l=7}^9 c_l \theta_l + \tilde{c}_{11} \theta_{11}, \quad c_7 \neq 0, \quad (54c_8)(130c_8) = -(52c_7)(81c_7 - 135c_9),$$

$$F_{10} = c_8 \theta_8 + c_9 \theta_9, \quad c_8 \neq 0,$$

$$F_{11} = c_9 \theta_9 + c_{10} \theta_{10}, \quad c_9 \neq 0,$$

$$\begin{aligned}
F_{12} &= c_{10}\theta_{10} + c_{11}\theta_{11}, \quad c_{10} \neq 0, \\
F_{13} &= c_{11}\theta_{11}, \quad c_{11} \neq 0, \\
F_{14} &= c_{12}\theta_{12}, \quad c_{12} \neq 0, \\
F_{15} &= 0.
\end{aligned}$$

Consider $n \geq 2$, the symplectic form $\omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$ on $\mathbb{C}^{2n}$ with coordinate system $(p_1, q_1, \dots, p_n, q_n)$ and $N = \{f(x_1, x_2) = x_{\geq 3} = 0\}$ a subset of $(\mathbb{C}^{2n}, 0)$ with coordinate system $(x_1, x_2, \dots, x_{2n})$, where $f(x_1, x_2) = x_1^4 + x_2^5 + x_1^2 x_2^3$. Denote by f_i the germ of function such that

$$F_i = \|f_i dx_1 \wedge dx_2\|_{N'}.$$

So the symplectic form $\omega_i = f_i(x_1, x_2)dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + \sum_{i=2}^n dx_{2i-1} \wedge dx_{2i}$, satisfies that $i_{\mathbb{C}^2}^* \|\omega_i\|_N = F_i$, where $i_{\mathbb{C}^2} : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^{2n}, 0)$ is the map inclusion $i_{\mathbb{C}^2}(x_1, x_2) = (x_1, x_2, 0, \dots, 0)$ and it is sent to ω_0 by the diffeomorphism.

$$\begin{aligned}
x_1 &= p_1, \quad x_2 = p_2, \quad x_3 = q_1 - \int_0^{p_2} f_i(p_1, t)dt, \quad x_4 = q_2, \\
x_5 &= p_3, \quad x_6 = q_3, \quad \dots, \quad x_{2n-1} = p_n, \quad x_{2n} = q_n.
\end{aligned} \tag{5.7}$$

Theorem 5.1.18. *For $n \geq 2$, any curve in the symplectic manifold $(\mathbb{C}^{2n}, \omega_0)$ which is diffeomorphic to $N = \{f(x_1, x_2) = x_{\geq 3} = 0\}$ is symplectomorphic to a curve of the form*

$$N_i = \left\{ f(p_1, p_2) = q_1 - \int_0^{p_2} f_i(p_1, t)dt = q_{\geq 2} = p_{\geq 3} = 0 \right\} \subset (\mathbb{C}^{2n}, \omega_0). \tag{5.8}$$

Proof. We have that for $1 \leq i \leq 15$, is diffeomorphic to the symplectic form ω_i is sent to ω_0 via the diffeomorphism 5.7. Therefore the set N is diffeomorphic via 5.7 to N_i in 5.8, thus the exact algebraic restriction $\|\omega_i\|_N$ is diffeomorphic to the exact algebraic restriction $\|\omega_0\|_{N_i}$.

Let $\tilde{N}$ be a curve in $(\mathbb{C}^{2n}, \omega_0)$ which is diffeomorphic to N , then there exists a diffeomorphism $\varphi : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^{2n}, 0)$ such that $\varphi(N, 0) = (\tilde{N}, 0)$. Let $M = \varphi(\mathbb{C}^2 \times \{0\}, 0)$, so

$$\varphi^* \|\omega_0|_{TM}\|_{\tilde{N}} = \|(\varphi^* \omega_0)|_{\mathbb{C}^2 \times \{0\}}\|_N.$$

Moreover there exists i such that $\|(\varphi^* \omega_0)|_{\mathbb{C}^2 \times \{0\}}\|_N$ is diffeomorphic to $\|\omega_i|_{\mathbb{C}^2 \times \{0\}}\|_N$, in fact, there exists i such that $i_{\mathbb{C}^2}^* \|(\varphi^* \omega_0)|_{\mathbb{C}^2 \times \{0\}}\|_N = \|(\varphi \circ i_{\mathbb{C}^2})^* \omega_0\|_{N'}$ is diffeomorphic to $F_i = \|f_i dx_1 \wedge dx_2\|_{N'}$. Let denote by $\phi_i : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ the diffeomorphism such that $\phi_i^* \|(\varphi \circ i_{\mathbb{C}^2})^* \omega_0\|_{N'} = \|f_i dx_1 \wedge dx_2\|_{N'}$, then the diffeomorphism $\Phi : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^{2n}, 0)$

defined by $\Phi(x_1, x_2, x_3, \dots, x_{2n}) = (\phi(x_1, x_2), x_3, \dots, x_{2n})$ preserves N and

$$\begin{aligned} \Phi^* \left\| (\varphi^* \omega_0)|_{\mathbb{C}^2 \times \{0\}} \right\|_N &= \left\| ((\varphi \circ \Phi)^* \omega_0)|_{\mathbb{C}^2 \times \{0\}} \right\|_N \\ &= \left\| (\varphi \circ i_{\mathbb{C}^2} \circ \phi)^* \omega_0 \right\|_N \\ &= \left\| f_i dx_1 \wedge dx_2 \right\|_N \\ &= \left\| \omega_i|_{\mathbb{C}^2 \times \{0\}} \right\|_N. \end{aligned}$$

Here we are using the fact that any form $\alpha \in \Omega^k(\mathbb{C}^2)$ which vanishes on N' vanishes on N considered as a form on $\Omega^k(\mathbb{C}^{2n})$.

So $\|\omega_0|_{TM}\|_{\tilde{N}}$ is diffeomorphic to $\|\omega_i|_{\mathbb{C}^2 \times \{0\}}\|_N$ and by Proposition 3.1.14 $\|\omega_0\|_{\tilde{N}}$ is diffeomorphic to $\|\omega_i\|_N$. Therefore $\|\omega\|_{\tilde{N}}$ is diffeomorphic to $\|\omega_0\|_{N_i}$ and by Theorem 3.1.18 ii) $\tilde{N}$ is symplectomorphic to N_i . $\square$

Therefore, to obtain the formal norms to the classification of N under the action of symplectomorphisms, it is enough to prove that the normal forms in Table 1 are distinct. In order to do that, we calculate the symplectic invariants of N_i .

5.2 Calculation of the invariants

Replacing f, F_i we have the normal forms

$$\begin{aligned} N_1 &= \{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - c_1 p_2 = q_{\geq 2} = p_{\geq 3} = 0\} \\ N_2 &= \left\{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \left(\frac{c_2}{2} p_2^2 + c_3 p_1 p_2 + \tilde{c}_6 p_1^2 p_2\right) = q_{\geq 2} = p_{\geq 3} = 0\right\} \\ N_3 &= \left\{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \left(c_3 p_1 p_2 + \frac{c_4}{3} p_2^3 + \frac{\tilde{c}_7}{4} p_2^4\right) = q_{\geq 2} = p_{\geq 3} = 0\right\} \\ N_4 &= \left\{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \left(\frac{c_4}{3} p_2^3 + \frac{c_5}{2} p_1 p_2^2 + c_6 p_1^2 p_2 + \frac{\tilde{c}_9}{2} p_1^2 p_2^2\right) = q_{\geq 2} = p_{\geq 3} = 0\right\} \\ N_5 &= \left\{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \left(\frac{c_5}{2} p_1 p_2^2 + c_6 p_1^2 p_2 + \frac{c_7}{4} p_2^4\right) = q_{\geq 2} = p_{\geq 3} = 0\right\} \\ N_6 &= \left\{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \left(c_6 p_1^2 p_2 + \frac{c_7}{4} p_2^4 + \frac{c_8}{3} p_1 p_2^3\right) = q_{\geq 2} = p_{\geq 3} = 0\right\} \\ N_7 &= \left\{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \left(c_6 p_1^2 p_2 + \frac{c_7}{4} p_2^4 + \frac{c_8}{3} p_1 p_2^3 + \frac{\tilde{c}_{10}}{4} p_1 p_2^4\right) = q_{\geq 2} = p_{\geq 3} = 0\right\} \\ N_8 &= \left\{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \left(\frac{c_7}{4} p_2^4 + \frac{c_8}{3} p_1 p_2^3 + \frac{c_9}{2} p_1^2 p_2^2\right) = q_{\geq 2} = p_{\geq 3} = 0\right\} \\ N_9 &= \left\{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \left(\frac{c_7}{4} p_2^4 + \frac{c_8}{3} p_1 p_2^3 + \frac{c_9}{2} p_1^2 p_2^2 + \frac{\tilde{c}_{11}}{3} p_1^2 p_2^3\right) = q_{\geq 2} = p_{\geq 3} = 0\right\} \\ N_{10} &= \left\{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \left(\frac{c_8}{3} p_1 p_2^3 + \frac{c_9}{2} p_1^2 p_2^2\right) = q_{\geq 2} = p_{\geq 3} = 0\right\} \\ N_{11} &= \left\{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \left(\frac{c_9}{2} p_1^2 p_2^2 + \frac{c_{10}}{4} p_1 p_2^4\right) = q_{\geq 2} = p_{\geq 3} = 0\right\} \\ N_{12} &= \left\{p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \left(\frac{c_{10}}{4} p_1 p_2^4 + \frac{c_{11}}{3} p_1^2 p_2^3\right) = q_{\geq 2} = p_{\geq 3} = 0\right\} \end{aligned}$$

$$\begin{aligned}
N_{13} &= \left\{ p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - \frac{c_{11}}{3} p_1^2 p_2^3 = q_{\geq 2} = p_{\geq 3} = 0 \right\} \\
N_{14} &= \left\{ p_1^4 + p_2^5 + p_1^2 p_2^3 = q_1 - c_{12} \left(p_1^4 p_2 + \frac{1}{6} p_2^6 + \frac{1}{4} p_1^2 p_2^4 \right) = q_{\geq 2} = p_{\geq 3} = 0 \right\} \\
N_{15} &= \{ p_1^4 + p_2^5 + p_1^2 p_2^3 = q_{\geq 1} = p_{\geq 3} = 0 \}
\end{aligned}$$

Let us consider ω be a symplectic form on $\mathbb{C}^{2n}$, we denote by f_ω the function such that

$$i_{\mathbb{C}^2}^* \omega = f_\omega dx_1 \wedge dx_2,$$

where $i_{\mathbb{C}^2} : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^{2n}, 0)$ is the map inclusion.

Theorem 5.2.1. *Let $N = \{g(x_1, x_2) = x_{\geq 3} = 0\}$ be the germ of a planar curve on the symplectic manifold $(\mathbb{C}^{2n}, \omega)$, where g has the property of zeros. The index of isotropness of N is the maximum $p \in \mathbb{N}$ such that $f_\omega \in \text{div}(g \cdot \Theta_2) + \mathcal{M}_2^p$. If $f_\omega \in \text{div}(g \cdot \Theta_2)$, then $p = \infty$. If $f_\omega(0) \neq 0$, then $p = 0$.*

Proof. If $f_\omega \in \text{div}(f \cdot \Theta_2)$ or $f_\omega(0) \neq 0$, then the theorem is true.

Let $p < \infty$ be the index of isotropness of N and p_0 the maximum integer positive such that $f_\omega \in \text{div}(f \cdot \Theta_2) + \mathcal{M}_2^{p_0}$. By Theorem 3.1.22, then the index of isotropness of N is equal to maximal order of vanishing of germs of closed 2-forms representing the exact algebraic restriction $\|\omega\|_N$.

Let θ be a closed 2-form such that the order of vanishing is p and $\|\omega\|_N = \|\theta\|_N$. Then $\|(f_\omega - f_\theta)dx_1 \wedge dx_2\|_{g^{-1}(0)} = 0$ and therefore $f_\omega - f_\theta \in \text{div}(g \cdot \Theta_2)$, by Proposition 3.4.1.

Since f_θ has zero $(p-1)$ -jet, $f_\theta \in \mathcal{M}_2^p$. Hence $f_\omega \in \text{div}(g \cdot \Theta_2) + \mathcal{M}_2^p$ and $p \leq p_0$.

Reciprocally, if $f_\omega \in \text{div}(g \cdot \Theta_2) + \mathcal{M}_2^{p_0}$, then there exists $h \in \mathcal{M}_2^{p_0}$ and $\theta \in \Theta_2$ such that

$$f_\omega = \text{div}(g \cdot \theta) + h.$$

Then $\|\omega\|_{T\mathbb{C}^2 \times \{0\}} = \|h(x_1, x_2)dx_1 \wedge dx_2\|_N$, because g vanishes on N . Hence $\|\omega\|_N = \|h(x_1, x_2)dx_1 \wedge dx_2\|_N$ and therefore $p_0 \leq p$, because $h(x_1, x_2)dx_1 \wedge dx_2$ is a closed 2 form. $\square$

Now we apply Theorem 5.2.1 to calculate the index of isotropness $\text{ind}(N_i)$ for each subset N_i , this invariant is calculated for $N = \{f(x_1, x_2) = x_{\geq 3} = 0\}$ as a subset of the symplectic manifold $(\mathbb{C}^{2n}, \omega_i)$, where

$$\omega_i = f_i(x_1, x_2)dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + \sum_{i=2}^n dx_{2i-1} \wedge dx_{2i}.$$

- 1) N_1 : $f_\omega = c_1$, with $c_1 \neq 0$. So $\text{ind}(N_1) = 0$.
- 2) N_2 : $f_\omega = c_2x_2 + c_3x_1 + \tilde{c}_6x_1^2$, with $c_2 \neq 0$. Obviously $f_\omega \in \mathcal{M}_2 \subset \text{div}(f \cdot \Theta_2) + \mathcal{M}_2$. If $f_\omega \in \text{div}(f \cdot \Theta_2) + \mathcal{M}_2^2$, there exist $\theta \in \Theta_2$ and $g \in \mathcal{M}_2^2$ such that

$$f_\omega = \text{div}(f \cdot \theta) + g,$$

since $f = x_1^4 + x_2^5 + x_1^2x_2^3$, we have $\text{div}(f \cdot \theta) = f \cdot \text{div}(\theta) + \theta(f) \in \mathcal{M}_2^3$. So $f_\omega \in \mathcal{M}_2^2$, which is a contradiction, because $c_2 \neq 0$. Therefore $\text{ind}(N_2) = 1$.

- 3) N_3 : $f_\omega = c_3x_1 + c_4x_2^2 + \tilde{c}_7x_2^3$, with $c_3 \neq 0$ and similarly to item 2), $\text{ind}(N_3) = 1$.
- 4) N_4 : $f_\omega = c_4x_2^2 + c_5x_1x_2 + c_6x_1^2 + \tilde{c}_9x_1^2x_2$, with $c_4 \neq 0$. Similarly to item 2), $\text{ind}(N_4) = 2$.
- 5) N_5 : $f_\omega = c_5x_1x_2 + c_6x_1^2 + c_7x_2^3$, with $c_5 \neq 0$. Similarly to item 2), $\text{ind}(N_5) = 2$.
- 6) N_6 : $f_\omega = c_6x_1^2 + c_7x_2^3 + c_8x_1x_2^2$, with $c_6 \neq 0$ and $c_6 \neq 2c_7$. Similarly to item 2), $\text{ind}(N_6) = 2$.
- 7) N_7 : $f_\omega = c_6x_1^2 + \frac{c_6}{2}x_2^3 + c_8x_1x_2^2 + \tilde{c}_{10}x_1x_2^3$, with $c_6 \neq 0$. Similarly to item 2), $\text{ind}(N_7) = 2$.
- 8) N_8 : $f_\omega = c_7x_2^3 + c_8x_1x_2^2 + c_9x_1^2x_2$, with $c_7 \neq 0$ and $(54c_8)(130c_8) \neq -(52c_7)(81c_7 - 135c_9)$. It is obvious that

$$f_\omega \in \mathcal{M}_2^3 \subset \text{div}(f \cdot \Theta_2) + \mathcal{M}_2^3$$

Suppose $f_\omega \in \text{div}(f \cdot \Theta_2) + \mathcal{M}_2^4$. Then, there exists $\theta \in \Theta_2$ and $g \in \mathcal{M}_2^4$ such that

$$f_\omega = \text{div}(f \cdot \theta) + g = f \cdot \text{div}(\theta) + \theta(f) \quad (5.9)$$

since $\frac{\partial f}{\partial x_2}, f \in \mathcal{M}_2^4$, if we compare the 3-jets in the equation 5.9, we obtain

$$c_7x_2^3 + c_8x_1x_2^2 + c_9x_1^2x_2 = 4\theta_1(0)x_1^3 \quad (5.10)$$

where $\theta = \theta_1 \frac{\partial}{\partial x_1} + \theta_2 \frac{\partial}{\partial x_2}$, $\theta_1, \theta_2 \in \mathcal{O}_2$ are functions on the variables x_1, x_2 . The equation 5.10 give us a contradiction, because $c_7 \neq 0$. Therefore $\text{ind}(N_8) = 3$.

- 9) N_9 : $f_\omega = c_7x_2^3 + c_8x_1x_2^2 + c_9x_1^2x_2 + \tilde{c}_{11}x_1^2x_2^2$, with $c_7 \neq 0$ and $(54c_8)(130c_8) = -(52c_7)(81c_7 - 135c_9)$. Similarly to item 8) $\text{ind}(N_9) = 3$.
- 10) N_{10} : $f_\omega = c_8x_1x_2^2 + c_9x_1^2x_2$, with $c_8 \neq 0$. Similarly to item 8) $\text{ind}(N_{10}) = 3$.
- 11) N_{11} : $f_\omega = c_9x_1^2x_2 + c_{10}x_1^3$, with $c_9 \neq 0$. Similarly to item 8) $\text{ind}(N_{11}) = 3$.
- 12) N_{12} : $f_\omega = c_{10}x_1x_2^3 + c_{11}x_1^2x_2^2$, with $c_{10} \neq 0$. We have

$$f_\omega \in \mathcal{M}_2^4 \subset \text{div}(f \cdot \Theta_2) + \mathcal{M}_2^4$$

Suppose $f_\omega \in \text{div}(f \cdot \Theta_2) + \mathcal{M}_2^5$. Then, there exists $\theta \in \Theta_2$ and $g \in \mathcal{M}_2^5$ such that

$$f_\omega = \text{div}(f \cdot \theta) + g = f \text{div}(\theta) + \theta(f) \quad (5.11)$$

writing $\theta = \theta_1 \frac{\partial}{\partial x_1} + \theta_2 \frac{\partial}{\partial x_2}$, $\theta_1 = a_0 + a_1 x_1 + a_2 x_2 + \dots$ and $\theta_2 = b_0 + b_1 x_1 + b_2 x_2 + \dots$ and comparing the jets of the equation 5.11 we have

$$\begin{aligned} 3 - jet : \quad & 0 = 4a_0 x_1^3 \implies a_0 = 0 \\ 4 - jet : \quad & c_{10} x_1 x_2^3 + c_{11} x_1^2 x_2^2 = (a_1 + b_2) x_1^4 + b_0 (5x_2^4 + 3x_1^2 x_2^2) + 4(a_1 x_1 + a_2 x_2) x_1^3. \end{aligned} \quad (5.12)$$

This last equation give us a contradiction, because $c_{10} \neq 0$. Therefore $\text{ind}(N_{12}) = 3$.

13) $N_{13}: f_\omega = c_{11} x_1^2 x_2^2$, with $c_{11} \neq 0$. We have

$$f_\omega \in \mathcal{M}_2^4 \subset \text{div}(f \cdot \Theta_2) + \mathcal{M}_2^4$$

If we suppose $f_\omega \in \text{div}(f \cdot \Theta_2) + \mathcal{M}_2^5$, then we obtain a similar equation to 5.12

$$4 - jet : \quad c_{11} x_1^2 x_2^2 = (a_1 + b_2) x_1^4 + b_0 (5x_2^4 + 3x_1^2 x_2^2) + 4(a_1 x_1 + a_2 x_2) x_1^3$$

from this last equation, we obtain $b_0 = \frac{c_{11}}{3}$ and $b_0 = 0$ which is a contradiction. So $\text{ind}(N_{13}) = 4$

14) $N_{14}: f_\omega = c_{12} f = c_{12} x_1^4 + c_{12} x_2^5 + c_{12} x_1^2 x_2^3$, with $c_{12} \neq 0$. Consider the vector field $\theta_0 = x_2 \frac{\partial}{\partial x_2}$, then

$$\text{div}(f \cdot \theta_0) = f + x_2 \frac{\partial f}{\partial x_2},$$

since $\frac{\partial f}{\partial y} \in \mathcal{M}_2^4$, we obtain

$$f_\omega \in \text{div}(f \cdot \Theta_2) + \mathcal{M}_2^5.$$

Suppose $f_\omega \in \text{div}(f \cdot \Theta_2) + \mathcal{M}_2^6$. Then, there exists $\theta \in \Theta_2$ and $g \in \mathcal{M}_2^6$ such that

$$f_\omega = \text{div}(f \cdot \theta) + g = f \text{div}(\theta) + \theta(f) \quad (5.13)$$

writing $\theta = \theta_1 \frac{\partial}{\partial x_1} + \theta_2 \frac{\partial}{\partial x_2}$, $\theta_1 = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2 + \dots$ and $\theta_2 = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1^2 + b_4 x_1 x_2 + b_5 x_2^2 + \dots$ and comparing the jets of the equation 5.13 we have

$$\begin{aligned} 3 - jet : \quad & 0 = 4a_0 x_1^3 \implies a_0 = 0 \\ 4 - jet : \quad & c_{12} x_1^4 = (a_1 + b_2) x_1^4 + b_0 (5x_2^4 + 3x_1^2 x_2^2) + 4(a_1 x_1 + a_2 x_2) x_1^3 \\ & \implies 5a_1 + b_2 = c_{12}, b_0 = a_2 = 0 \\ 5 - jet : \quad & \end{aligned}$$

$$\begin{aligned}
c_{12}x_2^5 + c_{12}x_1^2x_2^3 &= (a_1 + b_2)(x_2^5 + x_1^2x_2^3) + \{(2a_3 + b_4)x_1 + (a_4 + 2b_5)x_2\}x_1^4 + \\
&\quad + 4(a_3x_1^2 + a_4x_1x_2 + a_5x_2^2)x_1^3 + 2(a_1x_1 + a_2x_2)x_1x_2^3 + \\
&\quad + (b_1x_1 + b_2x_2)(5x_2^4 + 3x_1^2x_2^2)
\end{aligned}$$

$$\begin{aligned}
c_{12}x_2^5 + c_{12}x_1^2x_2^3 &= (6a_3 + b_4)x_1^5 + (5a_4 + 2b_5)x_1^4x_2 + (4a_5 + 3b_2)x_1^3x_2^2 + \\
&\quad (3a_1 + 4b_2)x_1^2x_2^3 + (2a_2 + 5b_1)x_1x_2^4 + (5b_2)x_2^5.
\end{aligned}$$

Therefore

$$\begin{aligned}
5a_1 + b_2 - c_{12} &= 0 \\
3a_1 + 4b_2 - c_{12} &= 0 \\
5b_2 - c_{12} &= 0
\end{aligned}$$

which implies $a_1 = b_2 = c_{12} = 0$. It is a contradiction as $c_{12} \neq 0$, consequently $\text{ind}(N_{14} = 5)$

15) N_{15} : $f_\omega = 0$ and $\text{ind}(N_{15}) = \infty$.

Theorem 5.2.2. *Let $N = \{g(x_1, x_2) = x_{\geq 3} = 0\}$ be the germ of a planar curve on the symplectic manifold $(\mathbb{C}^{2n}, \omega)$, where g has the property of zeros. The symplectic multiplicity of N is the codimension of the tangent space to the orbit of $\|f_\omega dx_1 \wedge dx_2\|_{\{g=0\}}$ with respect to the action of $G_{\{g=0\}}(\mathbb{C}^{2n})$ on $\|\Omega^2(\mathbb{C}^2)\|_{\{g=0\}}$.*

Proof. By Theorem 3.1.28

$$\mu_{\text{Simp}}(N) = \dim \frac{\|\overline{\Omega}^2(\mathbb{C}^{2n})\|_N}{T(\|\omega\|_N)}$$

By Proposition 3.1.9, $\|\overline{\Omega}^2(\mathbb{C}^{2n})\|_N = \|\mathcal{O}_2 dx_1 \wedge dx_2\|_N$ and $\|\omega\|_N = \|f_\omega dx_1 \wedge dx_2\|_N$.

Let $\phi = (\phi_1, \dots, \phi_{2n}) \in G_N(\mathbb{C}^{2n})$, then by Nakayama's Lemma, for $3 \leq j \leq 2n$

$$\phi_j \in \langle x_3, \dots, x_{2n} \rangle$$

so $\tilde{\phi} = (\phi_1, \phi_2)|_{\mathbb{C}^2 \times \{0\}}$ is a diffeomorphism which preserves $\{g = 0\}$. Reciprocally, if $\varphi = (\varphi_1, \varphi_2)$ is a local symmetry of $\{g = 0\}$, then $\hat{\varphi} = (\varphi_1, \varphi_2, x_3, \dots, x_{2n})$ is a local symmetry of N .

If $\beta \in T(\|\omega\|_N)$, then there exists a family $\varphi_t \in G_N(\mathbb{C}^{2n})$ such that

$$\begin{aligned}
i_{\mathbb{C}^2 \times \{0\}}^* \beta &= \beta = \lim_{t \rightarrow 0} \left\| \frac{\varphi_t^* \omega - \omega}{t} \right\|_N \\
&= \lim_{t \rightarrow 0} \left\| \frac{\varphi_t^* f_\omega dx_1 \wedge dx_2 - f_\omega dx_1 \wedge dx_2}{t} \right\|_N \\
&= \lim_{t \rightarrow 0} \left\| \frac{\tilde{\varphi}_t^* f_\omega dx_1 \wedge dx_2 - f_\omega dx_1 \wedge dx_2}{t} \right\|_N \\
&= \|\gamma\|_N
\end{aligned} \tag{5.14}$$

N^i	ind	μ_{Symp}	N^i	ind	μ_{Symp}
N^1	0	1	N^9	3	10
N^2	1	4	N^{10}	3	9
N^3	1	5	N^{11}	3	10
N^4	2	7	N^{12}	4	11
N^5	2	7	N^{13}	4	11
N^6	2	8	N^{14}	5	12
N^7	2	9	N^{15}	∞	12
N^8	3	9			

Table 2 – Index of isotropness and symplectic multiplicity of N_i

where $\tilde{\varphi}_t \in G_{\{g=0\}}(\mathbb{C}^2)$ obtained from ϕ_t as before and $\gamma \in \Omega^2(\mathbb{C}^2)$ such that $\|\gamma\|_{\{g=0\}} \in T(\|f_\omega dx_1 \wedge dx_2\|_{\{g=0\}})$ and $i_{\mathbb{C}^2 \times \{0\}}$ denotes the map inclusion of $\mathbb{C}^2 \times \{0\} \subset \mathbb{C}^{2n}$.

Analogously, if $\|\gamma\|_{\{g=0\}} \in T(\|f_\omega dx_1 \wedge dx_2\|_{\{g=0\}})$ there exists a closed 2-form $\beta \in T(\|\omega\|_N)$ such that

$$\|\gamma\|_N = \beta = i_{\mathbb{C}^2 \times \{0\}}^* \beta \quad (5.15)$$

Therefore the linear maps

$$\|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N \xrightarrow{i_{\mathbb{C}^2}} \|\Omega^2(\mathbb{C}^2)\|_{\{g=0\}} \xrightarrow{\pi} \frac{\|\Omega^2(\mathbb{C}^2)\|_{\{g=0\}}}{T(\|f_\omega dx_1 \wedge dx_2\|_{\{g=0\}})}$$

where $i_{\mathbb{C}^2}(\beta) := i_{\mathbb{C}^2}^* \beta$ for all $\beta \in \|\bar{\Omega}^2(\mathbb{C}^{2n})\|_N$, $i_{\mathbb{C}^2} : \mathbb{C}^2 \rightarrow \mathbb{C}^{2n}$ denotes the map inclusion and π denotes the projection map. By 5.14 and 5.15, $\text{Ker}(\pi \circ i_{\mathbb{C}^2}) = T(\|\omega\|_N)$, hence $\pi \circ i_{\mathbb{C}^2}$ is an isomorphism. $\square$

From the table 2 we can not distinguish all the sets N_i , so we need one more invariant. We use the invariant *Lagrangian tangency order* $Lt(h)$ of a parametrized curve h , this invariant was introduced in [2] and formally defined in [8].

If $H_1 = \dots = H_n = 0$ define a smooth submanifold L in the symplectic manifold (M, ω) then the tangency order of a curve $h : \mathbb{C} \rightarrow M$ to L is the minimum of the orders of vanishing at 0 of the functions $H_1 \circ h, \dots, H_n \circ h$. We denote the tangency order of h to L by $t(h, L)$.

Definition 5.2.3. *The Lagrangian tangency order $Lt(h)$ of a curve h is the maximum of $t(f, L)$ over all smooth Lagrangian submanifolds L of the symplectic manifold.*

Lemma 5.2.4. *Consider the symplectic form $\omega_0 = dp_1 \wedge dq_1 + \dots + dp_{2n} \wedge dq_{2n}$ with the coordinate system $(p_1, q_1, \dots, p_n, q_n)$. Let $N \subset \mathbb{C}^3 \times \{0\}$ be a subset in the symplectic manifold $(\mathbb{C}^{2n}, \omega_0)$. Suppose that there exists a non zero smooth function $F(p_1, p_2)$ with $F(p_1, 0) = 0$ for all p_1 , such that N is contained in the set*

$$\{(p_1, F(p_1, p_2), p_2, 0, \dots, 0), p_1, p_2 \in \mathbb{C}\}.$$

Additionally, suppose that N has an smooth parametrization $h(t) = (p_1(t), q_1(t), p_2(t), 0, \dots, 0)$, where $p_1(t) = \sum_{i=i_0} a_i t^i$, $p_2(t) = \sum_{j=j_0} a_j t^j$ such that i_0, j_0 are co-primes. Then $Lt(h)$ is equal to $\max\{i_0, \partial F(p_1(t), p_2(t))\}$, where $\partial F(p_1(t), p_2(t))$ denotes the the first $j \in \mathbb{N}$ such that the monomial t^j appears in the Taylor's expansion of $F(p_1(t), p_2(t))$.

Proof. By Hadamard's Lemma $\partial F(p_1(t), p_2(t))$ is a multiple of j_0 so $i_0 \neq \partial F(p_1(t), p_2(t))$. Let L be a Lagrangian submanifold in the symplectic manifold $(\mathbb{C}^{2n}, \omega_0)$. Then there exist disjoint subsets $J, K \subset \{1, \dots, n\}$, $J \cup K = \{1, \dots, n\}$ and a smooth function $S(p_J, q_K)$ [3] such that

$$L = \left\{ q_j = -\frac{\partial S}{\partial p_j}(p_J, q_K), p_k = \frac{\partial S}{\partial q_k}(p_J, q_K) | j \in J, k \in K \right\}.$$

Since we are interested in the maximum of $t(h, L)$, we can assume

$$\left\{ \frac{\partial S}{\partial p_j}(0) = \frac{\partial S}{\partial q_k}(0) = 0 : j \in J, k \in K \right\},$$

then $t(h, L)$ is the minimum of the orders of vanishing at 0 of

$$\left\{ q_j(t) = -\frac{\partial S}{\partial p_j}(p_J(t), q_K(t)), p_k(t) = \frac{\partial S}{\partial q_k}(p_J(t), q_K(t)) | j \in J, k \in K \right\}.$$

Assume $t_0 < \min \partial F(p_1(t), p_2(t))$, we consider 4 cases:

1) $1, 2 \in J$, so for $J_0 = J \cap \{3, \dots, n\}$ and $K_0 = K \cap \{3, \dots, n\}$ we can write

$$\begin{aligned} F(p_1(t), p_2(t)) &= -\frac{\partial S}{\partial p_1}(p_1(t), p_2(t), 0), & 0 &= -\frac{\partial S}{\partial p_j}(p_1(t), p_2(t), 0), j \in J_0 \\ 0 &= -\frac{\partial S}{\partial p_2}(p_1(t), p_2(t), 0), & 0 &= \frac{\partial S}{\partial q_k}(p_1(t), p_2(t), 0), k \in K_0. \end{aligned}$$

Since the order of vanishing of $\frac{\partial S}{\partial p_2}(p_1(t), p_2(t), 0)$, $\frac{\partial S}{\partial p_j}(p_1(t), p_2(t), 0), j \in J_0$, and $\frac{\partial S}{\partial q_k}(p_1(t), p_2(t), 0), k \in K_0$, depend only on the monomial in the variables p_1, p_2 , we have $t(h, L) \geq \min\{i_0, j_0\}$. From $F(p_1(t), p_2(t)) = -\frac{\partial S}{\partial p_1}(p_1(t), p_2(t), 0)$ we deduced $t(h, L) \leq \min \partial F(p_1(t), p_2(t))$. If $S(p_J, q_K)$ is zero, then $L = \{q_j = p_k = 0 : j \in J, k \in K\}$ and $t(h, L) = \min \partial F(p_1(t), p_2(t))$.

2) $1 \in J$ and $2 \in K$, so for $J_0 = J \cap \{3, \dots, n\}$ and $K_0 = K \cap \{3, \dots, n\}$ we can write

$$\begin{aligned} F(p_1(t), p_2(t)) &= -\frac{\partial S}{\partial p_1}(p_1(t), 0), & 0 &= -\frac{\partial S}{\partial p_j}(p_1(t), 0), j \in J_0 \\ p_2(t) &= \frac{\partial S}{\partial q_2}(p_1(t), 0), & 0 &= \frac{\partial S}{\partial q_k}(p_1(t), 0), k \in K_0, \end{aligned}$$

then $\min\{i_0, j_0\} \leq t(h, L) \leq \max\{i_0, j_0\}$.

3) Analogously to the case 1), when $1, 2 \in K$,

$$\begin{aligned} p_1(t) &= \frac{\partial S}{\partial q_1}(F(p_1(t), p_2(t)), 0), & 0 &= -\frac{\partial S}{\partial p_j}(p_1(t), p_2(t), 0), j \in J_0 \\ 0 &= -\frac{\partial S}{\partial p_2}(F(p_1(t), p_2(t)), 0), & 0 &= \frac{\partial S}{\partial q_k}(p_1(t), p_2(t), 0), k \in K_0, \end{aligned}$$

N^i	ind	μ_{Symp}	Lt
N^1	0	1	5
N^2	1	4	8
N^3	1	5	9
N^4	2	7	12
N^5	2	7	13
N^6	2	8	14
N^7	2	9	14
N^8	3	9	16

N^i	ind	μ_{Symp}	Lt
N^9	3	10	16
N^{10}	3	9	17
N^{11}	3	10	18
N^{12}	4	11	21
N^{13}	4	11	22
N^{14}	5	12	24
N^{15}	∞	12	∞

Table 3 – Comparison between the index of isotropness, symplectic multiplicity and Lagrangian tangency order

so $t(h, L) = i_0$.

4) $2 \in J$ and $1 \in K$, so for $J_0 = J \cap \{3, \dots, n\}$ and $K_0 = K \cap \{3, \dots, n\}$ we can write

$$p_1(t) = \frac{\partial S}{\partial q_1}(F(p_1(t), p_2(t)), 0), \quad 0 = -\frac{\partial S}{\partial p_j}(F(p_1(t), p_2(t)), 0), j \in J_0$$

$$p_2(t) = \frac{\partial S}{\partial q_2}(F(p_1(t), p_2(t)), 0), \quad 0 = \frac{\partial S}{\partial q_k}(F(p_1(t), p_2(t)), 0), k \in K_0,$$

then $\min\{i_0, j_0\} \leq t(h, L) \leq \max\{i_0, j_0\}$.

Therefore $Lt(h) = \max\{i_0, \min \partial F(p_1(t), p_2(t))\}$. The case $\min \partial F(p_1(t), p_2(t)) < i_0$ is analogous. $\square$

Consider a parametrization of $f(x_1, x_2) = x_1^4 + x_2^5 + x_1^2 x_2^3 = 0$ given by $h(t) = (t^5 + t^5 h_1(t), -t^4)$, $t \in \mathbb{C}$ and h_1 fulfill $(1 + h_1(t))^2 = \frac{t^2 + \sqrt{4+t^4}}{2}$. Hence the parametrization of N_i is given by $h_i(t) = (p_1(t), q_{1,i}(t), p_2(t), 0 \dots, 0)$ where $h(t) = (p_1(t), p_t(t))$ and $q_{1,i}(t) = \int_0^{p_2(t)} f_i(p_1(t), r) dr$. In table 3 we present the Lagrangian tangency order of h_i .

5.3 Final results

As summary of the previous sections, we have:

Theorem 5.3.1. *Let F_i be one of the exact algebraic restriction given in the Table 1.*

(i) *Any exact algebraic restriction $a \in \|\Omega^2(\mathbb{C}^2)\|_{N'}$ is diffeomorphic to one of the normal forms F_i .*

(ii) *The normal forms F_i , are disjoint.*

(iii) *The parameters $\tilde{c}_i$ in the normal forms are moduli.*

Proof. It remains to prove (iii).

1) From Lemma 5.1.6, we have

$$T(F_2) = \mathbb{C}\{\theta_{12}, \theta_{11}, \theta_{10}, \theta_9, \theta_8, \theta_7, -36c_2\theta_5 - 38c_3\theta_6, -85c_2\theta_4 - 90c_3\theta_5 + 2c_2\theta_6\}$$

by the Table 1 $\dim T(F_2) = 8$, therefore $\theta_6 \notin T(F_2)$.

2) From Lemma 5.1.7, we have

$$T(F_3) = \mathbb{C}\{\theta_{12}, \theta_{11}, \theta_{10}, \theta_9, \theta_8, -38c_3\theta_6 + c_3\theta_7, -90c_3\theta_5 - 105c_4\theta_7\}$$

by the Table 1 $\dim T(F_3) = 7$, therefore $\theta_7 \notin T(F_3)$.

3) From Lemma 5.1.8, we have

$$T(F_4) = \mathbb{C}\{\theta_{12}, \theta_{11}, \theta_{10}, -44c_4\theta_8 - 46c_5\theta_9, -105c_4\theta_7 - 110c_5\theta_8 - 115c_6\theta_9\}$$

by the Table 1 $\dim T(F_4) = 5$, therefore $\theta_9 \notin T(F_4)$.

4) From Lemma 5.1.10, we have

$$T(F_7) = \mathbb{C}\{\theta_{12}, \theta_{11}, -115c_6\theta_9 - 130c_8\theta_{10}\}$$

by the Table 1 $\dim T(F_7) = 3$, therefore $\theta_{10} \notin T(F_7)$.

5) From Lemma 5.1.12, we have

$$T(F_9) = \mathbb{C}\{\theta_{12}, -54c_7\theta_{10} - 54c_8\theta_{11}\}$$

by the Table 1 $\dim T(F_9) = 2$, therefore $\theta_{11} \notin T(F_9)$. □

As a consequence of Theorem 5.3.1, we have that the normal forms N_i in the Theorem 5.1.18 are distinct. It just remain to obtain the normal forms to the case $n = 1$.

Theorem 5.3.2. *Any curve in the symplectic space $(\mathbb{C}^2, \omega_0 = dp_1 \wedge dq_1)$ which is diffeomorphic to the curve $N' = \{f(x_1, x_2) = 0\}$ can be reduced by a symplectomorphism to the normal form $C : \left\{p_1^4 + \frac{q_1^5}{c_1^5} + p_2^3 \frac{q_1^3}{c_1^3} = 0 \mid c_1 \in \mathbb{C}, c_1 \neq 0\right\}$.*

Proof. Let $\tilde{N}$ be a curve in $(\mathbb{C}^2, \omega_0)$ which is diffeomorphic to N' . Then there exists a germ of diffeomorphism $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ such that $\varphi(N') = (\tilde{N})$. Then $\|\omega_0\|_{\tilde{N}}$ is diffeomorphic to $\|\varphi^*\omega_0\|_{N'}$. By the Theorem 3.2.4 and Table 1 $\|\varphi^*\omega_0\|_{N'}$ is diffeomorphic to $\|c_1 dx_1 \wedge dx_2\|_{N'}$ where $c_1 \neq 0$. Consider the diffeomorphism $x_1 = p_1$ and $c_1 x_2 = q_1$ then $\|c_1 dx_1 \wedge dx_2\|_{N'}$ is diffeomorphic to $\|\omega_0\|_C$. Therefore $\|\omega_0\|_{\tilde{N}}$ is diffeomorphic to $\|\omega_0\|_C$ and by Theorem 3.1.18 ii), $\tilde{N}$ is symplectomorphic to C . □

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