Long-time dynamics of two classes of beam and
plate equations

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equations

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Dinâmica a longo prazo de duas classes de equações de vigas e placas

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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To my parents

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*Uma ocasião,
meu pai pintou a casa toda
de alaranjado brilhante.
Por muito tempo moramos numa casa,
como ele mesmo dizia,
constantemente amanhecendo.
Adélia Prado*

RESUMO

MONTEIRO, R. N.. **Long-time dynamics of two classes of beam and plate equations**. 2016. 125 f. Doctoral dissertation (Doctorate Candidate Program in Mathematics) – Instituto de Ciências Matemáticas e de Computação (ICMC/USP), São Carlos – SP.

Neste trabalho iremos discutir a existência, unicidade, dependência contínua e a dinâmica a longo prazo das soluções de um sistema de equações que modela a vibração de vigas curvas e um modelo de placas termoelásticas. Primeiro consideramos o modelo de Bresse com dissipação não linear e forças externas. Provamos que o sistema de Timoshenko pode ser obtido como limite do sistema de Bresse quando o arco de curvatura ℓ tende para zero e sob algumas hipóteses, mostramos a existência de um atrator global com dimensão fractal finita. Também comparamos o sistema de Bresse com o sistema de Timoshenko no sentido da semicontinuidade de seus atratores quando o parâmetro $\ell \rightarrow 0$. Na segunda parte estudamos o sistema de full Von Karmán. Neste modelo adicionamos efeitos térmicos e condições de fronteira do tipo livre. Mostramos que esse problema, sem dissipação mecânica no deslocamento vertical, também possui um atrator global regular com dimensão infinita.

Palavras-chave: Atrator Exponencial, Atrator Global, Equações diferenciais parciais, Semicontinuidade, Termoelasticidade.

ABSTRACT

MONTEIRO, R. N.. **Long-time dynamics of two classes of beam and plate equations.** 2016. 125 f. Doctoral dissertation (Doctorate Candidate Program in Mathematics) – Instituto de Ciências Matemáticas e de Computação (ICMC/USP), São Carlos – SP.

In this thesis we will discuss the well-posedness and long-time dynamics of curved beam and thermoelastic plates. First, we considered the Bresse system with nonlinear damping and forcing terms. For this model we show the Timoshenko system as a singular limit of the Bresse system as the arch curvature ℓ goes to 0 and under suitable assumptions on the nonlinearity we prove the existence of a smooth global attractor with finite fractal dimension and exponential attractors as well. We also compare the Bresse system with the Timoshenko system, in the sense of upper-semicontinuity of their attractors as $\ell \rightarrow 0$. Second, we study a full von Karman system, this model accounts for vertical and in plane displacements. For this system we add a nonlinear thermal coupling and free boundary conditions. It is shown that the system, without any mechanical dissipation imposed on vertical displacements, admits a global attractor which is also smooth and of finite fractal dimension.

Key-words: Exponential attractors, Global attractor, Partial differential equations, Thermoelasticity, Upper-semicontinuity.

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INTRODUCTION

The analysis of wellposedness and long-time behavior in partial differential equations has been a subject of long lasting interest. Various models with different boundary conditions have been considered, but the physical interest-relevance and the degree of mathematical challenge do depend critically on the specific model and the associated boundary conditions. These aspects create different configurations that require very different mathematical treatments. The overriding desire has been to control long-time behavior of the model with a *minimal* amount of dissipation. By controlling, we mean either stabilize to zero when the sources are absent, or driving solutions asymptotically to a pre-assigned bounded set in the phase space (attractor).

In this thesis we study two different models arising in mathematical elasticity. The first one is the Bresse system, a model that describes vibrations of curved beams. The linear system which models the vibrations is given by

$$\begin{cases} \rho h \varphi_{tt} - Q_x - \ell N = 0, \\ \rho I \psi_{tt} - M_x + Q = 0, \\ \rho h w_{tt} - N_x + \ell Q = 0, \end{cases} \quad (1.1)$$

where $N = Eh(w_x - \ell \varphi)$, $Q = K' Gh(\varphi_x + \psi + \ell w)$ and $M = EI \psi_x$ denote the axial force, shear force and bending moment, respectively. The functions φ , ψ and w represent the vertical, shear angle and longitudinal displacements of elastic materials like flexible beams. The positive constants ρ, h, I, E, G, K' represent the density of the material, cross-sectional area, second moment of area of the cross-section, modulus of elasticity, shear modulus, shear factor, and the parameter ℓ stands for the curvature of the beam. In the context of a circular arch of radius R one has $\ell = R^{-1}$. A description of the model can be found in [38, Chap. 3]. The original derivation of Bresse system was presented in [12]. In order to simplify the notation we adopt the notation $\rho_1 = \rho h$, $\rho_2 = \rho I$, $k = K' Gh$, $k_0 = Eh$, and $b = EI$. Then the Bresse system (1.1) can

be rewritten as

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + \ell w)_x - k_0 \ell (w_x - \ell \varphi) = 0, \\ \rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi + \ell w) = 0, \\ \rho_1 w_{tt} - k_0 (w_x - \ell \varphi)_x + k \ell (\varphi_x + \psi + \ell w) = 0. \end{cases} \quad (1.2)$$

We observe that when the curvature $\ell \rightarrow 0$ the system (1.2) uncouples into the well-known Timoshenko system,

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0 \\ \rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) = 0 \end{cases} \quad (1.3)$$

and an independent wave equation $\rho_1 w_{tt} - k_0 w_{xx} = 0$. Therefore sometimes the Timoshenko system is called Bresse-Timoshenko system. The derivation of the Timoshenko system is presented in [58].

There are several works dedicated to the mathematical analysis of the Bresse system. They are mainly concerned with decay rates of solutions of the linear system. This is done by adding suitable damping effects that can be of thermal, viscous or viscoelastic nature. Here we will present some works related with stability of the Bresse system introducing linear internal dissipation. If we consider frictional damping in each displacement the Bresse system is exponential stable as proved by [3], where they considered full Dirichlet boundary conditions. The same result can be easily adapted to the others boundary conditions. In general the system is not exponential stable and this is achieved if we consider the system partially damped, that is, when we do not consider dissipation in each displacement. In this case a remarkable stability criteria for the Bresse system is the equal wave speeds assumption

$$\frac{\rho_1}{k} = \frac{\rho_2}{b} \text{ and } k = k_0. \quad (1.4)$$

Now it is worth noting that assumption (1.4) is only a mathematical condition. In fact, it never happens from physical point of view since from the theory of elasticity, E and G satisfy

$$G = \frac{E}{2(1+\mu)},$$

with Poisson's ratio $\mu \in (0, 1/2)$. See [14].

Concerning to the model with only one weak damping on the angle displacement (ψ_t), the result in [2] asserts that the system (1.2) is exponentially stable only when the mathematical condition (1.4) holds. When this equality fails, it is shown in [2] that system with mixed boundary condition (Dirichlet-Neumann-Dirichlet) does not have exponential decay rate. Instead, the solution goes to zero polynomially with rate depending on the coefficients. In [23] the authors considered the same problem as in [2] and they determined, when the condition (1.4) does

not hold, the polynomial decay of solution with optimal decay rate to the system with mixed boundary condition. Other interesting results can be found in, for instance, [4, 22, 53, 55, 56, 60].

On the other hand, it is worthy mentioning that all above cited works on Bresse systems were concerned with linear problems. With respect to nonlinear problems, the stability of the Bresse system was studied in [16], with three locally defined nonlinear damping without assuming the equal speeds assumption.

It turns out that long-time dynamics characterized by global attractors was not discussed for Bresse systems. Even for the Timoshenko system there are only a few works in this direction. Motivated by this, in the present work we study the long-time dynamics of the Bresse system. Here we add in the model (1.2) full Dirichlet boundary conditions, nonlinear damping and forcing terms. For this system we first give a meaning of the Bresse-Timoshenko limit. Second, under some assumptions on nonlinear damping and forcing terms, we prove the existence of a smooth global attractor with finite fractal dimension and exponential attractors as well. We also compare the Bresse system with the Timoshenko system, in the sense of upper-semicontinuity of their attractors as $\ell \rightarrow 0$.

The second part of this work is concerned with long-time behavior and theory of global attractors associated with the full Von Karman system. This system models nonlinear oscillations in a plate dynamics which account for both vertical (w) and in plane ($u = (u_1, u_2)$) displacements. The original derivation of the full Von Karman system can be found in [37, 39] and the evolutionary system is given by the following equations

$$\begin{cases} u_{tt} - \operatorname{div}\{\sigma[\varepsilon(u) + f(\nabla w)]\} = 0, \\ w_{tt} + \Delta^2 w - \operatorname{div}\{\sigma[\varepsilon(u) + f(\nabla w)]\nabla w\} = 0. \end{cases} \quad (1.5)$$

Regarding physical parameters in the equations we have that $\sigma[\cdot]$ is a fourth order tensor defined by

$$\sigma[A] = \lambda \operatorname{trace}[A]I + 2\eta A, \quad (1.6)$$

where $\lambda = E\mu/(1 - 2\mu)(1 + \mu)$, $\eta = E/2(1 + \mu)$, the constant $\mu \in (0, \frac{1}{2})$ is the Poisson's modulus and E is the Young's modulus. The strain tensor $\varepsilon(\cdot)$ is given by

$$\varepsilon(u) = \frac{1}{2}[\nabla u + (\nabla u)^\top], \quad (1.7)$$

where ∇u denotes the Jacobian matrix of vector u , and the nonlinearity $f(\cdot)$ is defined by

$$f(s) = \frac{1}{2}[s \otimes s], \quad s \in \mathbb{R}^2, \quad (1.8)$$

which in our case $f(\nabla w)$ is given by

$$f(\nabla w) = \frac{1}{2} \begin{bmatrix} w_x^2 & w_x w_y \\ w_x w_y & w_y^2 \end{bmatrix}.$$

An important remark is that the full Von Karman system reduces to the well know Von Karman *scalar* system. This is done imposing that in-plane accelerations ($u_t = (u_{1,t}, u_{2,t})$) are null. Then, as in [37], the system (1.5) can be decoupled and the in-plane displacement (u) defines an Airy's stress function $v = v(u)$. Then, under this condition, the vertical displacement (w) satisfy

$$\begin{cases} w_{tt} + \Delta^2 w - [w, v] = 0, \\ \Delta^2 v + \frac{E}{2} [w, w] = 0, \end{cases} \quad (1.9)$$

where the bracket is defined by

$$[w, v] = \frac{\partial^2 w}{\partial^2 x} \frac{\partial^2 v}{\partial^2 y} + \frac{\partial^2 w}{\partial^2 y} \frac{\partial^2 v}{\partial^2 x} - 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y}$$

and Airy's stress function v is given by

$$\frac{\partial^2 v}{\partial^2 y} = N_{1,1}, \quad \frac{\partial^2 v}{\partial^2 x} = N_{2,2} \quad \text{and} \quad -\frac{\partial^2 v}{\partial x \partial y} = N_{1,2},$$

where $N = [N_{i,j}]$ denotes the symmetric tensor $\sigma[\varepsilon(u) + f(\nabla w)]$.

For readers' orientation in what follows we shall provide additional literature pertaining to the problem with a particular reference to the model under consideration. Full theory of a long-time behavior (including theory of attractors) has been developed for a thermoelastic von Karman *scalar* equation in the variables (w, θ) , that is the model (1.9) with thermal effects, with either clamped or hinged boundary conditions [20] and without any mechanical dissipation. The analysis in [20] relies on techniques developed in [6, 7] that includes special nonlocal multipliers along with quisi-stability theory developed in [19]. In the case of clamped or hinged boundary conditions one also shows that the estimates for the dimension of the attracting set are independent on rotational inertia ($\gamma \Delta w_{tt}$). Thus the result is also valid in the limit case when $\gamma = 0$.

In the case of *free boundary conditions*, the situation is more complex. In fact, still speaking of a Von Karman *scalar* model, one obtains uniform convergence to an attracting set without any mechanical dissipation for the model that accounts additionally rotational inertia (the term $\gamma \Delta w_{tt}$ is added to the plate equation) [6]. This latter effect provides regularizing effect on the velocity of oscillations. In order to obtain the same result for a limiting case ($\gamma = 0$) - with free boundary conditions - additional boundary mechanical dissipation is needed. Whether the same result can be proved without this additional dissipation is still an open question.

The case of full von Karman model (both vertical and in plane displacements are accounted for) is more complicated - in fact for several reasons. First - the presence of in plane displacements modeled by the elastodynamic system prevents uniform decay rates (in $2d$) unless one adds mechanical dissipation to an already present thermal dissipation. This follows from a well

known argument due to Dafermos [21] and also [31]. Thus, in order to obtain valid results some form of dissipation affecting in plane displacements is necessary. In [10] the considered the full Von Karmam system with thermoelastic damping and an additional dissipation was imposed on rotational components of the velocity u_t . The corresponding result provides uniform stabilization in the case of both hinged and clamped boundary conditions. This stabilization is uniform with respect to rotational inertia parameter $\gamma \geq 0$. The method of obtaining estimates relies on multipliers developed in [6] and also [41]. However, this methods fail when free boundary conditions are imposed. The very first result addressing this problem was presented [42] where full von Karman system, without rotational inertia ($\gamma = 0$) and with thermal effects (with nonlinear coupling) was considered. Uniform stabilization to zero with free boundary conditions and boundary dissipation imposed on the in plane velocities was established. The main ingredients used for the analysis in [42] were partial smoothing obtained due to thermal effects affecting the horizontal displacements. Thermal effects being absent, mechanical dissipation on both in plane and horizontal displacements is imposed, in this case, we refer [41, 49].

The present thesis takes of the analysis in the direction of dynamical systems and theory of attractors. Under supervision of Professor Irena Lasiecka in this second part we studied the problem (1.5) combined with partially dissipative free boundary conditions and thermal effects (with nonlinear coupling) affecting both displacements. This presents new set of challenges due to nonlinear effects which are supercritical and make unable the use of the known tools in the area of attractors. Nevertheless we shall show that this strongly nonlinear dynamics can be reduced asymptotically to a smooth and finite dimensional set.

MATHEMATICAL BACKGROUND

The main purpose of this chapter is to provide the necessary background material to the reader. Here we shall introduce definitions, notations and theoretical results that will be used along this thesis. We begin with the definition of some function spaces and their properties. In a second part we present some useful inequalities. In a third part we collected some results related with the semigroup theory and the long-time behavior of dynamical systems. As a final remark, in the beginning of each section we list some references where the proofs of the results can be found.

2.0.1 Function spaces

The content of this section can found in the references [1, 11, 32, 46, 52, 59]. Here we will define and give pertinent properties on the Lebesgue, Sobolev and intermediate spaces. Consider $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and $1 \leq p < \infty$. We denote by $L^p(\mathcal{O})$ the set of (equivalence classes of) all Lebesgue measurable functions $u: \mathcal{O} \rightarrow \mathbb{R}$ such that $|u|^p: \mathcal{O} \rightarrow \mathbb{R}$ is a Lebesgue integrable function, that is

$$L^p(\mathcal{O}) = \left\{ u: \mathcal{O} \rightarrow \mathbb{R} \text{ a measurable function} \mid \int_{\mathcal{O}} |u(x)|^p dx < \infty \right\}.$$

If $u \in L^p(\mathcal{O})$ we define the L^p -norm of u by

$$\|u\|_{L^p(\mathcal{O})} = \left[\int_{\mathcal{O}} |u(x)|^p dx \right]^{\frac{1}{p}}, \quad 1 \leq p < \infty. \quad (2.1)$$

The set $(L^p(\mathcal{O}), \|\cdot\|_{L^p(\mathcal{O})})$ is a Banach space. The case when $p = 2$, the space $L^2(\mathcal{O})$ have a natural inner-product

$$(u, v)_{L^2(\mathcal{O})} = \int_{\mathcal{O}} u(x)v(x)dx.$$

This inner-product yields the norm given in (2.1) when $p = 2$, that makes

$$(L^2(\mathcal{O}), \|\cdot\|_{L^2(\mathcal{O})})$$

a Hilbert space. For $p = \infty$, the $L^\infty(\mathcal{O})$ is given by the set (equivalence classes) of all almost everywhere bounded functions. For a function u in this set we have that

$$\|u\|_{L^\infty(\mathcal{O})} = \inf \left\{ C \mid |u(x)| \leq C \text{ a.e on } \mathcal{O} \right\},$$

defines a norm in $L^\infty(\mathcal{O})$ which makes it a Banach space.

Let $x \in \mathbb{R}^n$ with coordinates $(x_1, x_2, \dots, x_n)$. A multi-index is an n -tuple

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \alpha_i \geq 0, \alpha_i \text{ integers.}$$

Associated to a multi-index α , we have the following symbols

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \dots + \alpha_n, \\ \alpha! &= \alpha_1! \alpha_2! \dots \alpha_n!, \\ x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, x \in \mathbb{R}^n. \end{aligned}$$

The differentiation operator D^α is defined by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}.$$

For any $m \in \mathbb{Z}_+$, $C^m(\mathcal{O})$ is the space of all the functions $u: \mathcal{O} \rightarrow \mathbb{R}$ for which all the partial derivatives of order m exist and are continuous. $C^0(\mathcal{O})$ is also denoted by $C(\mathcal{O})$. We denote by $C^\infty(\mathcal{O})$ the intersection of all the spaces $C^m(\mathcal{O})$, $m \in \mathbb{Z}_+$. The set $C_0^m(\mathcal{O})$, $m \in \mathbb{Z}_+$ denotes the space of all functions in $C^m(\mathcal{O})$ which have compact support contained in $\mathcal{O}$ and by $\mathcal{D}(\mathcal{O})$ we denote the set of all functions in $C_0^\infty(\mathcal{O})$. These functions are called test functions.

Definition 2.1. A sequence of test functions $\{\varphi_n\}_{n \in \mathbb{N}}$ in $\mathcal{D}(\mathcal{O})$ is said to be convergent to 0 if there exists a fixed compact set $K \subset \mathcal{O}$, such that, $\text{supp}(\varphi_n) \subset K$, for all $n \in \mathbb{N}$ and all its derivatives converge uniformly to zero on K .

Definition 2.2. A linear functional $u: \mathcal{D}(\mathcal{O}) \rightarrow \mathbb{R}$ (or $\mathbb{C}$) is said to be a distribution on $\mathcal{O}$ if whenever $\{\varphi_n\}_{n \in \mathbb{N}}$ converges to 0 in $\mathcal{D}(\mathcal{O})$ we have that $\{\langle u, \varphi_n \rangle\}_{n \in \mathbb{N}}$ converges to 0 in $\mathbb{R}$ (or $\mathbb{C}$). Here the symbol $\langle u, \varphi \rangle$ denotes the action of u on a test function φ in $\mathcal{D}(\mathcal{O})$.

The space of distributions, which is the dual of the space of test functions, is denoted by $\mathcal{D}'(\mathcal{O})$ and for a function $u \in \mathcal{D}'(\mathcal{O})$ its derivative of order α is the distribution defined by

$$\langle D^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle, \text{ for all } \varphi \in \mathcal{D}(\mathcal{O}).$$

Now we give a brief overview on basic results of the theory of Sobolev spaces and their associated trace and dual spaces. We begin with $m \geq 0$ an integer number and $1 \leq p \leq \infty$. By $W^{m,p}(\mathcal{O})$ we denote the set (equivalence classes) of all functions $u \in L^p(\mathcal{O})$ such that all distribution derivatives $D^\alpha u$, $|\alpha| \leq m$, are also in $L^p(\mathcal{O})$, that is

$$W^{m,p}(\mathcal{O}) = \{u \in L^p(\mathcal{O}) \mid D^\alpha u \in L^p(\mathcal{O}) \text{ for all } |\alpha| \leq m\}.$$

In the Sobolev space $W^{m,p}(\mathcal{O})$ we also define the following norms

$$\|u\|_{W^{m,p}(\mathcal{O})} = \left[\sum_{|\alpha| \leq m} \int_{\mathcal{O}} |D^\alpha u(x)|^p dx \right]^{\frac{1}{p}}, \text{ for } 1 \leq p < \infty$$

and

$$\|u\|_{W^{m,\infty}(\mathcal{O})} = \max_{|\alpha| \leq m} \{\|D^\alpha u\|_{L^\infty(\mathcal{O})}\}.$$

For every $1 \leq p \leq \infty$, the space $(W^{m,p}(\mathcal{O}), \|\cdot\|_{W^{m,p}(\mathcal{O})})$ is a Banach space and by $W_0^{m,p}(\mathcal{O})$ we denote the closure of $C_0^\infty(\mathcal{O})$ in $W^{m,p}(\mathcal{O})$. The particular case when $p = 2$, these spaces will be denoted by $H^m(\mathcal{O})$ and $(H^m(\mathcal{O}), \|\cdot\|_{H^m(\mathcal{O})})$ is a Hilbert space with inner-product

$$(u, v)_{H^m(\mathcal{O})} = \sum_{|\alpha| \leq m} \int_{\mathcal{O}} D^\alpha u(x) D^\alpha v(x) dx.$$

We also define the Sobolev space $W^{s,p}(\mathcal{O})$ for positive real s , $s \notin \mathbb{N}$, and $1 \leq p \leq \infty$ by

$$W^{s,p}(\mathcal{O}) = \left\{ u \in W^{m,p}(\mathcal{O}) \mid \|u\|_{W^{s,p}(\mathcal{O})}^p = \|u\|_{W^{m,p}(\mathcal{O})}^p + \sum_{|\alpha|=m} I_{\sigma,p}(\partial^\alpha u) \leq \infty \right\},$$

where $s = m + \sigma$ with $m \in \mathbb{N}$, $0 < \sigma < 1$ and

$$I_{\sigma,p}(u) = \int_{\mathcal{O} \times \mathcal{O}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\sigma p}} dx dy.$$

In particular, if $p = 2$, we use the notation $H^s(\mathcal{O})$ to denote the Hilbert space $W^{s,2}(\mathcal{O})$. A first important property of these spaces is that for every $s_1, s_2 \in \mathbb{R}$ $s_1 > s_2 > 0$ the inclusion $H^{s_1}(\mathcal{O}) \subset H^{s_2}(\mathcal{O})$ is compact. For every real $s > 0$ we define the Sobolev space with a negative order $-s$ as the dual space of $H_0^s(\mathcal{O})$. The space $H_0^{-s}(\mathcal{O})$ is equipped with norm of dual spaces given by

$$\|u\|_{H^{-s}(\mathcal{O})} = \sup \left\{ \langle u, v \rangle \mid v \in H_0^s(\mathcal{O}) \text{ and } \|v\|_{H_0^s(\mathcal{O})} = 1 \right\},$$

where $\langle \cdot, \cdot \rangle$ is the duality between $H^{-s}(\mathcal{O})$ and $H_0^s(\mathcal{O})$.

Next we will collected some extra properties of Sobolev spaces.

Theorem 2.1. [Embeddings Theorem]. Consider $\mathcal{O}$ a bounded subset of $\mathbb{R}^n$ with sufficiently regular boundary $\partial\mathcal{O}$. Then the following inclusions are continuous

$$W^{s,p}(\mathcal{O}) \subset C^\sigma(\overline{\mathcal{O}}), \text{ if } s - \frac{n}{p} > \sigma, \ p \in (0, \infty) \text{ and } s, \sigma \geq 0,$$

(if σ is not an integer the embedding holds also for $\sigma = s - \frac{n}{p}$) and

$$W^{s,p}(\mathcal{O}) \subset W^{s^*,p^*}(\mathcal{O}), \text{ if } s - \frac{n}{p} \geq s^* - \frac{n}{p^*}, \ 1 < p \leq p^* < \infty \text{ and } s^* \geq 0.$$

Remark 2.1. This Theorem implies, when $s^* = 0$ and $p^* = 2$, that $H^s(\mathcal{O}) \subset L^p(\mathcal{O})$, if $s \geq \frac{n}{2} - \frac{n}{p}$ and $p \geq 2$. In the particular case when $n = \dim(\mathcal{O}) = 2$ the result implies the following inclusions $H^s(\mathcal{O}) \subset L^\infty(\mathcal{O})$, if $s > 1$ and $H^s(\mathcal{O}) \subset L^{2/(1-s)}(\mathcal{O})$, if $0 \leq s < 1$. $\square$

Theorem 2.2. [Trace Theorem]. Consider $\mathcal{O}$ a bounded domain of class C^ℓ , $\ell \in \mathbb{N}$. Let $k \in \mathbb{Z}_+$, $s > k + \frac{1}{2}$, $s \leq \ell$ and $\gamma_j : C^\ell(\overline{\mathcal{O}}) \rightarrow C^{\ell-j}(\partial\mathcal{O})$ be trace operator given by $\gamma_j u = \frac{\partial^j u}{\partial \nu^j}$ (where $\frac{\partial^j}{\partial \nu^j}$ denote the normal derivative of order j). Then the operator γ_j can be extended (uniquely) to linear continuous operator $\gamma_j : H^s(\mathcal{O}) \rightarrow H^{s-j-\frac{1}{2}}(\partial\mathcal{O})$, $j = 0, 1, \dots, k$.

Now we will introduce the vector-valued spaces. Let X a Banach space and $[a, b] \subset \mathbb{R}$. We denote by $C^0([a, b]; X)$ the space of continuous functions $u : [a, b] \rightarrow X$ such that $\|u(t)\|_X \in C^0([0, T])$. This space equipped with the norm

$$\|u\|_{C^0([0, T]; X)} = \max_{t \in [a, b]} \{\|u(t)\|_X\},$$

is a Banach space. By $C^m([a, b]; X)$ we denote the space of functions $u : [a, b] \rightarrow X$ such that $\left\| \frac{d^k u}{dt^k}(t) \right\|_X \in C^0([a, b])$, for $0 \leq k \leq m$. The space $C^m([a, b]; X)$ is a Banach space with norm given by

$$\|u\|_{C^m([a, b]; X)} = \|u\|_{C^0([a, b]; X)} + \left\| \frac{du}{dt} \right\|_{C^0([a, b]; X)} + \dots + \left\| \frac{d^m u}{dt^m} \right\|_{C^0([a, b]; X)}.$$

Let $p \in [1, \infty]$. We denote by $L^p(a, b; X)$ the set of measurable functions $u : (0, T) \rightarrow X$ such that $\|u(t)\|_X \in L^p(a, b)$. The space $L^p(a, b; X)$ with the following norm

$$\|u\|_{L^p(a, b; X)} = \left[\int_a^b \|u(t)\|_X^p dt \right]^{\frac{1}{p}}, \text{ for } p \in [1, \infty)$$

and

$$\|u\|_{L^\infty(a, b; X)} = \text{esssup} \left\{ \|u(t)\|_X \mid t \in [a, b] \right\}.$$

The particular case when $p = 2$ and $(X, (\cdot, \cdot)_X)$ a Hilbert space, we have that $L^2(a, b; X)$ is a Hilbert space with inner-product given by

$$(u, v)_{L^2(a, b; X)} = \int_a^b (u(t), v(t))_X dt.$$

Then we have the following properties and an important compactness result:

- If the inclusion $X \subset Y$ is continuous then the inclusion $L^p(a, b; X) \subset L^p(a, b; Y)$, $p \in [1, \infty]$, is also continuous.
- The following inclusions are continuous $L^\infty(a, b; X) \subset L^p(a, b; X) \subset L^1(a, b; X)$, $p \in (1, \infty)$.

Theorem 2.3. [Aubin-Lions-Simon Theorem] Let $X \subset Y \subset Z$ be three Banach spaces with X is compactly embedded in Y and that Y is continuously embedded in Z . For $1 \leq p, q \leq \infty$ we define

$$W = \left\{ u \in L^p(a, b; X) \mid \frac{du}{dt} \in L^q(a, b; Z) \right\},$$

with norm $\|u\|_W = \|u\|_{L^p(a, b; X)} + \left\| \frac{du}{dt} \right\|_{L^q(a, b; Z)}$. Then

- If $p < \infty$ then the embedding of W into $L^p(a, b; Y)$ is compact;
- If $p = \infty$ and $q > 1$ then the embedding of W into $C^0([0, T]; Y)$ is compact.

Finally, by $W^{m,p}(a, b; X)$, $p \in [1, \infty]$, we denote the space

$$W^{m,p}(a, b; X) = \{u \in L^p(a, b; X) \mid D^\alpha u \in L^p(a, b; X), |\alpha| \leq m\},$$

where $D^\alpha u$ is the derivative in the distributional sense. This space with norm defined by

$$\|u\|_{W^{m,p}(a, b; X)} = \left[\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(a, b; X)}^p \right]^{\frac{1}{p}},$$

is a Banach space. The case $p = 2$ and $(X, (\cdot, \cdot)_X)$ a Hilbert space. The set $W^{m,2}(a, b; X)$, now denoted by $H^m(a, b; X)$, is a Hilbert space with natural inner-product

$$(u, v)_{H^m(a, b; X)} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2(a, b; X)}.$$

To end this section we will define and give some properties on intermediate spaces. Here we are following the definitions and notations of [46]. Let us consider $(X, (\cdot, \cdot)_X)$ and $(Y, (\cdot, \cdot)_Y)$ two separable Hilbert spaces, such that the inclusion $X \subset Y$, X dense in Y and X is given by a domain of an operator Λ , which is self-adjoint, positive, unbounded in Y and the norm $\|\cdot\|_X$ is equivalent to the norm of the graph

$$\left[\|u\|_Y^2 + \|\Lambda u\|_Y^2 \right]^{\frac{1}{2}}, \quad u \in D(\Lambda) = X.$$

By $D(S)$ we denote the set of elements u such that the antilinear form

$$v \mapsto (u, v)_X, \quad v \in X$$

is continuous in the topology induced by Y . Then the following inequality defines S as an unbounded operator in Y

$$(Su, v)_Y = (u, v)_X.$$

We have that $D(S)$ is dense in Y , S is a self-adjoint operator and

$$(Su, u)_Y = \|u\|_X^2 \geq C\|u\|_Y^2.$$

where C is a positive constant related with the continuous embedding $X \subset Y$. Then we can define the powers S^θ of S , $\theta \in \mathbb{R}$. We will use

$$\Lambda = S^{\frac{1}{2}}. \tag{2.2}$$

Then Λ is self-adjoint and positive in Y operator, with domain X and

$$(u, v)_X = (\Lambda u, \Lambda v)_Y, \text{ for all } u, v \in X.$$

Now we can define the intermediate space

Definition 2.3. Let X and Y two Hilbert spaces satisfying the the previous properties and Λ defined by (2.2). The intermediate space $[X, Y]_\theta$, $\theta \in [0, 1]$, is defined by

$$[X, Y]_\theta = D(\Lambda^{1-\theta})$$

with norm on $[X, Y]_\theta$ given by

$$\|u\|_{[X, Y]_\theta} = \left[\|u\|_Y^2 + \|\Lambda^{1-\theta} u\|_Y^2 \right]^{\frac{1}{2}}.$$

Next we collected an important result when X, Y denotes the Hilbert spaces $H^s(\mathcal{O})$ with $s \in \mathbb{R}$. First, we need some assumptions about the set $\mathcal{O}$

- The boundary of $\mathcal{O} \subset \mathbb{R}^n$ is a $(n-1)$ dimensional infinitely differentiable variety, $\mathcal{O}$ being locally on one side of $\partial \mathcal{O}$ (that is we consider $\overline{\mathcal{O}}$ a variety with boundary of class C^∞ , the boundary being $\partial \mathcal{O}$).
- $\mathcal{O}$ is bounded.

Theorem 2.4. Assume that $\mathcal{O}$ satisfy the previous properties. Then

$$[H^{s_1}(\mathcal{O}), H^{s_2}(\mathcal{O})]_\theta = H^{(1-\theta)s_1 + \theta s_2}(\mathcal{O}),$$

for all $0 < s_2 < s_1$ and $\theta \in (0, 1)$ and the following interpolation inequality holds

$$\|u\|_{H^{(1-\theta)s_1 + \theta s_2}(\mathcal{O})} \leq C_{\theta, s_1, s_2} \|u\|_{H^{s_1}(\mathcal{O})}^{1-\theta} \|u\|_{H^{s_2}(\mathcal{O})}^\theta. \tag{2.3}$$

Now we give an useful characterization on domains of fractional powers of operator. See [11, Proposition 6.1].

Theorem 2.5. Assume that $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ is a closed maximal accretive operator in the Hilbert space X for which $\mathcal{A}^{-1}$ is bounded in X . Then

$$D(\mathcal{A}^\alpha) = [D(\mathcal{A}), X]_{1-\alpha}, \quad \alpha \in [0, 1]. \quad (2.4)$$

2.0.2 Inequalities

In this section we will summarize some useful inequalities used in this work. The proof of each result can be found in [1, 13].

Theorem 2.6. [Poincaré Inequality] Let $\mathcal{O} \subset \mathbb{R}^n$ a bounded domain and $p \in (1, \infty)$. Then there exists a positive constant $C_{p, |\mathcal{O}|}$ such that

$$\|u\|_{L^p(\mathcal{O})} \leq C_{p, |\mathcal{O}|} \|\nabla u\|_{L^p(\mathcal{O})}, \quad \forall u \in W^{1,p}(\mathcal{O}).$$

Theorem 2.7. [Hölder Inequality] Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\mathcal{O} \subset \mathbb{R}^n$. If $u \in L^p(\mathcal{O})$ and $v \in L^q(\mathcal{O})$ then the product $uv \in L^1(\mathcal{O})$ and the following inequality holds true

$$\int_{\mathcal{O}} |uv| \, dx \leq \|u\|_{L^p(\mathcal{O})} \|v\|_{L^q(\mathcal{O})}.$$

Theorem 2.8. [Friedrichs-type Inequality] Let $\mathcal{O}$ bounded domain for which the Gauss - Green formula holds and $u \in H^1(\mathcal{O})$. Then

$$\int_{\mathcal{O}} |u|^2 \, dx \leq C \left[\int_{\mathcal{O}} |\nabla u|^2 \, dx + \int_{\partial \mathcal{O}} |u|^2 \, dx \right].$$

Theorem 2.9. [Generalized Hölder Inequality] Let $1 \leq p_1, \dots, p_m \leq \infty$ be real numbers such that $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{r} \leq 1$. Let $u_i \in L^{p_i}(\mathcal{O})$, $i = 1, \dots, m$. Then $u = \prod_{i=1}^m u_i \in L^r(\mathcal{O})$ and

$$\|u\|_{L^r(\mathcal{O})} \leq \prod_{i=1}^m \|u_i\|_{L^{p_i}(\mathcal{O})}.$$

Lemma 2.1. [Young Inequality] Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a, b \geq 0.$$

2.0.3 Semigroup and Global Attractors

Here in this section we denote by $(X, S(t))$ a dynamical system where (X, d) is a complete metric space and $\{S(t)\}_{t \geq 0}$ a C_0 -semigroup, that is, for each $t \in [0, \infty)$ the operator

$S(t)$ is bounded, satisfy $S(0) = I$, $S(t+s) = S(t)S(s)$, $\forall t, s \in [0, \infty)$ (semigroup property) and $\lim_{t \rightarrow 0^+} S(t)u = u$, for each $u \in X$ with respect to the norm on X .

The first part of this section is devoted to the results about the well-posedness of Lipschitz perturbations of linear evolution equations. The second part is related with the results about existence and properties of attractors.

2.0.3.1 Abstract Cauchy Problem

Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be the infinitesimal generator of a C_0 -semigroup $\{S(t)\}_{t \geq 0}$ and let $\mathcal{F} : [0, T) \times X \rightarrow X$ an arbitrary function. Let us consider the following inhomogeneous initial value problem

$$\begin{cases} y_t(t) - \mathcal{A}y(t) = \mathcal{F}(t, y(t)), & t > 0 \\ y(0) = y_0, \end{cases} \quad (2.5)$$

In this section we will enunciate some results concerning with the existence of solutions to the problem (2.5). As references we cite [9, 51]. Related with linear evolution equations we refer [13]. First we have some definitions of solution associated with (2.5):

Definition 2.4. A function $y : [0, T) \rightarrow X$ is a classical solution of (2.5) on $[0, T)$ if y is continuous on $[0, T)$, continuously differentiable on $(0, T)$, $y(t) \in D(\mathcal{A})$ for $0 < t < T$ and (2.5) is satisfied on $[0, T)$.

Definition 2.5. A function y which is differentiable almost everywhere on $[0, T]$ such that $y_t \in L^1(0, T; X)$ is called a strong solution of the initial value problem (2.5) if $y(0) = y_0$ and $y_t(t) - \mathcal{A}y(t) = \mathcal{F}(y(t))$ a.e. on $[0, T]$.

Definition 2.6. Let X a Banach space and $\mathcal{A}$ be the infinitesimal generator of C_0 -semigroup $\{S(t)\}_{t \geq 0}$ with $y_0 \in X$. A function $y \in C([0, T]; X)$ is a mild solution to the problem (2.5) if y satisfy the following integral equation

$$y(t) = S(t)y_0 + \int_0^t S(t-s)\mathcal{F}(s, y(s))ds, \quad t \in [0, T].$$

Theorem 2.10. Let $\mathcal{F} : [0, \infty) \times X \rightarrow X$ be continuous in t for $t \geq 0$ and locally Lipschitz continuous in y , uniformly in t on bounded intervals. If $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ is the infinitesimal generator of a C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on X then for every initial data $y_0 \in X$ there is a $T_{\max} \leq \infty$ such that the initial value problem (2.5) has a unique mild solution y on $[0, T_{\max})$. Moreover, if $T_{\max} < \infty$ then $\lim_{t \rightarrow T_{\max}^+} \|y(t)\|_X = +\infty$.

The following result gives hypothesis for the mild solution of (2.5) be a strong solution.

Theorem 2.11. Let $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be the infinitesimal generator of a C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on a reflexive Banach space X . $\mathcal{F} : [0, \infty) \times X \rightarrow X$ is Lipschitz continuous in both variables, $y_0 \in D(\mathcal{A})$ and y is the mild solution of the initial value problem (2.5) then y is the strong solution of this initial value problem.

Next result imply the existence of classical solution to the initial value problem (2.5). First consider $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be the infinitesimal generator of a C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on Banach space X and define $Y = (D(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$, where $\|y\|_{\mathcal{A}} = \|y\|_X + \|\mathcal{A}y\|_X$. Note that the closedness of $\mathcal{A}$ imply that Y is a Banach space and we have the following result

Theorem 2.12. Let $\mathcal{F} : [0, \infty) \times Y \rightarrow Y$ be uniformly Lipschitz in Y and for each $y \in Y$ let $\mathcal{F}(t, y)$ be continuous from $[0, T]$ into Y . If $y_0 \in D(\mathcal{A})$ then the initial value problem (2.5) has a unique classical solution on $[0, T]$.

2.0.3.2 Global Attractors

This section is devoted to the definitions and results on the nonlinear infinite dimensional dynamical systems theory. The main goal of this section is to present the theory pertinent to the existence of global attractors. We also provide some results that allow us to conclude extra properties as well the structure, dimensionality and smoothness of this object. Most of results can be found in classical references such as [8, 18, 27, 34, 57]. We shall follow more closely [19, Chapter 7].

Definition 2.7. Let $(X, S(t))$ a dynamical system.

- The dynamical system $(X, S(t))$ is called **dissipative** if it has an absorbing set, that is, a bounded set $\mathcal{B} \subset X$ that attracts any bounded set B in a finite time $T_B > 0$. In other words,

$$S(t)B \subset \mathcal{B}, \quad t \geq T_B.$$

- $(X, S(t))$ is said to be **asymptotically compact** iff there exists an attracting compact set K , that is, for any bounded set D we have

$$\lim_{t \rightarrow \infty} d_X(S(t)D, K) = 0,$$

where $d_X(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$, for $A, B \subset X$.

- The dynamical system $(X, S(t))$ is said to be **asymptotically smooth** iff for any bounded set $D \subset X$ such that $S(t)D \subset D$ for $t > 0$ there exists a compact set $K \subset \overline{D}$, such that

$$\lim_{t \rightarrow \infty} d_X(S(t)D, K) = 0.$$

• A **global attractor** for the dynamical system $(X, S(t))$ is a closed set $\mathbf{A} \subset X$ that is fully invariant and uniformly attracting, that is,

$$S(t)\mathbf{A} = \mathbf{A} \text{ and } \lim_{t \rightarrow \infty} d_X(S(t)B, \mathbf{A}) = 0,$$

for any bounded set $B \subset X$.

The following result gives an relation between the definition of asymptotically compact and asymptotically smooth dynamical system.

Proposition 2.1. Let X is a Banach space and $(X, S(t))$ a dissipative dynamical system. Then the following assertions are equivalent.

- $(X, S(t))$ is asymptotically compact;
- $(X, S(t))$ is asymptotically smooth.

The existence of a global attractor relies on two properties, dissipativeness and compactness. Next result provides a result on the existence of a compact global attractor.

Theorem 2.13. Any dissipative asymptotically smooth dynamical system $(X, S(t))$ in a Banach space X possesses a unique compact global attractor $\mathbf{A}$. The attractor $\mathbf{A}$ is a connected set and can be described as a set of all bounded full trajectories.

The definition of asymptotically compact (asymptotically smooth) dynamical system is often hard to prove and some compactness (smoothness) criteria are used instead. Here we present a criterion for asymptotic compactness and asymptotic smoothness.

The first is very useful for hyperbolic like systems and involves a function $\Psi : B \times B \rightarrow \mathbb{R}$ such that

$$\liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \Psi(y_n, y_m) = 0, \tag{2.6}$$

for every sequence $\{y_n\} \subset B$, where B is a bounded set of X .

Theorem 2.14. Let $(X, S(t))$ be a dynamical system where X is a Banach space. Assume that for any bounded positively invariant set $B \subset X$ and any $\varepsilon > 0$, there exists a time $T = T_{\varepsilon, B}$ and a function $\Psi_{\varepsilon, B, T} : B \times B \rightarrow \mathbb{R}$ satisfying (2.6) such that

$$\|S(T)y^1 - S(T)y^2\|_X \leq \varepsilon + \Psi_{\varepsilon, B, T}(y^1, y^2), \quad \forall y^1, y^2 \in B.$$

Then $(X, S(t))$ is asymptotically compact.

The second criterion is related with the following definitions.

Definition 2.8. A function $n_X : X \rightarrow [0, \infty)$ is said to be a **compact seminorm** on X if

$$\lim_{m \rightarrow \infty} n_X(x_m) = 0,$$

for any sequence $\{x_m\} \subset X$ such that $x_m \rightharpoonup 0$ in X .

Definition 2.9. Let X, Y be two reflexive Banach spaces with X compactly embedded into Y and put $H = X \times Y$. Consider the dynamical system $(H, S(t))$ given by

$$S(t)y = (u(t), u_t(t)), \quad y = (u(0), u_t(0)) \in H, \quad (2.7)$$

where the functions u have regularity

$$u \in C([0, \infty); X) \cap C^1([0, \infty); Y). \quad (2.8)$$

We say that the dynamical system is **quasi-stable** on a set $B \subset H$, if there exist a compact semi-norm n_X on X and nonnegative scalar functions $a(t)$ and $c(t)$, locally bounded in $[0, \infty)$, and $b(t) \in L^1(0, \infty)$ with $\lim_{t \rightarrow \infty} b(t) = 0$, such that,

$$\|S(t)y^1 - S(t)y^2\|_X^2 \leq a(t)\|y^1 - y^2\|_X^2, \quad (2.9)$$

and

$$\|S(t)y^1 - S(t)y^2\|_X^2 \leq b(t)\|y^1 - y^2\|_X^2 + c(t) \sup_{0 < s < t} [n_X(u^1(s) - u^2(s))]^2, \quad (2.10)$$

for any $y^1, y^2 \in B$, where $S(t)y^i = (u^i(t), u_t^i(t))$, $i = 1, 2$.

The first property of quasi-stable system is the asymptotic compactness.

Theorem 2.15. Let $(H, S(t))$ be a dynamical system given by (2.7) and satisfying (2.8). Suppose that the system is quasi-stable on every bounded positively invariant set B of H . Then $(H, S(t))$ is asymptotically compact.

Quasistability also implies further properties on the global attractors such as finite fractal dimension and improved spatial regularity.

Definition 2.10. The **fractal dimension** of a compact set A is a number defined by

$$\dim_F A = \limsup_{\varepsilon \rightarrow 0} \frac{\ln N_\varepsilon(A)}{\ln(1/\varepsilon)},$$

where $N_\varepsilon(A)$ is the minimal quantity of closed balls of radius 2ε necessary to cover A .

Theorem 2.16. Let $(H, S(t))$ be a dynamical system. Suppose that it has a global attractor and it is quasi-stable on it. Then A has finite fractal dimension.

The next result is about gain of regularity in the t variable.

Theorem 2.17. Let $(H, S(t))$ be a dynamical system with $c(t)$ bounded. Assume in addition that the system has a global attractor $\mathbf{A}$ and it is quasi-stable on $\mathbf{A}$. Then any full trajectory $(u(t), u_t(t))$ in the attractor have additional regularity

$$u_t \in L^\infty(\mathbb{R}, X) \cap C(\mathbb{R}, Y) \text{ and } u_{tt} \in L^\infty(\mathbb{R}, Y).$$

In addition,

$$\|u_t(t)\|_X^2 + \|u_{tt}(t)\|_Y^2 \leq R^2, \quad t \in \mathbb{R}, \quad (2.11)$$

where $R > 0$ depends on $\sup_{t>0} \{c(t)\}$, n_X , and on the compact embedding $X \subset Y$.

Finally, we conclude this section with the concepts of generalized exponential attractor and gradient systems.

Definition 2.11. A compact set $\mathbf{A}^{\text{exp}} \subset H$ is called a **generalized exponential attractor** if

- it is positively invariant,
- it attracts exponentially fast the trajectories from any bounded set of initial data,
- it has finite fractal dimension in an extended space $\tilde{H} \supseteq H$.

About the existence of such attractor we have the following theorem.

Theorem 2.18. Let $(H, S(t))$ be a dissipative dynamical system satisfying (2.7)-(2.8) and quasi-stable on some bounded absorbing set $\mathcal{B}$. In addition, suppose there exists an extended space $\tilde{H} \supseteq H$ such that, for each $T > 0$,

$$\|S(t_1)y - S(t_2)y\|_{\tilde{H}} \leq C_{BT}|t_1 - t_2|^\eta, \quad t_1, t_2 \in [0, T], \quad y \in \mathcal{B},$$

where $C_{BT} > 0$ and $\eta \in (0, 1]$ are constants. Then this system has a generalized exponential attractor $\mathbf{A}^{\text{exp}} \subset H$ with finite fractal dimension in $\tilde{H}$.

To obtain information on the structure of the attractors we need the concept of gradient systems.

Definition 2.12. A functional $\Phi : H \rightarrow \mathbb{R}$ is a strict **Lyapunov function** for a system $(H, S(t))$ if,

- the map $t \mapsto \Phi(S(t)z)$ is non-increasing for any $z \in H$,
- if $\Phi(S(t)z) = \Phi(z)$ for all t , then z is a stationary point of $S(t)$.

About the structure of the attractors we know that $\mathbb{M}_+(\mathcal{N}) \subset \mathbf{A}$, where $\mathcal{N}$ is the set of stationary points of $S(t)$ and $\mathbb{M}_+(\mathcal{N})$ is the unstable manifold of $y \in H$ such that there exists a full trajectory $u(t)$ satisfying

$$u(0) = y \text{ and } \lim_{t \rightarrow -\infty} d(u(t), \mathcal{N}) = 0.$$

For gradient systems it is possible to prove that the unstable manifold $\mathbb{M}_+(\mathcal{N})$ coincides with the attractor $\mathbf{A}$. The following result is well-known.

Theorem 2.19. Let $(H, S(t))$ be an asymptotically compact gradient system with the corresponding Lyapunov functional denoted by Φ . Suppose that

$$\Phi(y) \rightarrow \infty \text{ if and only if } \|y\|_H \rightarrow \infty, \quad (2.12)$$

and that the set of stationary points $\mathcal{N}$ is bounded. Then $(H, S(t))$ has a compact global attractor which coincides with the unstable manifold $\mathbb{M}_+(\mathcal{N})$.

SEMILINEAR DISSIPATIVE BRESSE SYSTEM

3.1 Introduction

In this chapter we deal with the semilinear Bresse system given by the following three motion equations,

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + \ell w)_x - k_0 \ell(w_x - \ell \varphi) + g_1(\varphi_t) + f_1(\varphi, \psi, w) = 0 & \text{in } Q, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + \ell w) + g_2(\psi_t) + f_2(\varphi, \psi, w) = 0 & \text{in } Q, \\ \rho_1 w_{tt} - k_0(w_x - \ell \varphi)_x + k\ell(\varphi_x + \psi + \ell w) + g_3(w_t) + f_3(\varphi, \psi, w) = 0 & \text{in } Q, \end{cases} \quad (3.1)$$

where $Q = \Omega \times (0, \infty)$ with $\Omega = (0, L)$, $g_1(\varphi_t), g_2(\psi_t), g_3(w_t)$ denote the nonlinear dissipation (damping terms) acting on Ω and $f_i(\varphi, \psi, w)$, $i = 1, 2, 3$ represent internal forces exerted by some nonlinear elastic foundation.

We complete this system with zero full Dirichlet boundary condition, that is

$$\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = w(0, t) = w(L, t) = 0, \quad t \in (0, \infty), \quad (3.2)$$

and we consider initial condition given by

$$\varphi(0) = \varphi_0, \quad \varphi_t(0) = \varphi_1, \quad \psi(0) = \psi_0, \quad \psi_t(0) = \psi_1, \quad w(0) = w_0, \quad w_t(0) = w_1. \quad (3.3)$$

Remark 3.1. Since our problem has damping terms in all equations of (3.1) we shall not assume the equal speeds assumption (1.4). $\square$

3.2 Goals and Plans of this Chapter

The main features of this chapter are summarized as follows.

- (i) We present a rigorous proof showing that solutions of the Bresse system converge to that of the Timoshenko system as $\ell \rightarrow 0$. This is called singular limit because at $\ell = 0$ the Bresse uncouples. See Theorem 3.2.
- (ii) By considering a nonlinear damping, without growth restriction near zero, we establish the existence of a global attractor. We also show that the system is gradient and therefore the attractors are characterized as unstable manifold of the set of stationary solutions. See Theorem 3.3.
- (iii) By assuming further that damping terms are Lipschitz, we derive a stability inequality and prove that the global attractor is regular and has finite fractal dimension. The same hypotheses allow us to show the existence of exponential attractors as well. See Theorem 3.4.
- (iv) We also compare the attractors of the Bresse system with those of the Timoshenko system. More precisely, we show the upper semicontinuity of attractors of (3.1)-(3.3) as $\ell \rightarrow 0$. In this case we shall assume that the external forces f_1, f_2 are not depending on w . This is reasonable since in the limit $\ell = 0$ we obtain the Timoshenko system, where longitudinal displacement w is neglected. That is, after limit, we get

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + g_1(\varphi_t) + f_1(\varphi, \psi) = 0 & \text{in } \Omega \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + g_2(\psi_t) + f_2(\varphi, \psi) = 0 & \text{in } \Omega \times (0, \infty), \end{cases} \quad (3.4)$$

subjected to Dirichlet boundary conditions

$$\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = 0, \quad t \in (0, \infty), \quad (3.5)$$

and initial conditions

$$\varphi(0) = \varphi_0, \quad \varphi_t(0) = \varphi_1, \quad \psi(0) = \psi_0, \quad \psi_t(0) = \psi_1. \quad (3.6)$$

See Theorem 3.5.

The plan of this chapter is the following. In Section 3.3, we present the notations that will follow in this chapter also give the needed assumptions and some properties on the energy associated with the Bresse system. The Section 3.4 is devoted to the proof of the wellposedness of the model. In Section 3.5 we show the Timoshenko system as limit of Bresse system. In Section 3.6 we show the first global attractor result and the existence of an uniform (independent on ℓ) absorbing set. In Section 3.7 we prove a result about existence of a smooth global attractor with finite fractal dimension and the existence of generalized attractor. Finally, in Section 3.8 we prove the upper-semicontinuity result with respect to the curvature parameter ℓ .

3.3 Preliminaries, Assumptions and Energy Identities

3.3.1 Preliminaries

We begin presenting our notations for the norms of standard $H^s(\Omega)$ (Sobolev) and $L^2(\Omega)$ spaces. We use

$$\|u\|_{\alpha,\Omega} = \|u\|_{H^\alpha(\Omega)}$$

and the case $\alpha = 0$, which corresponds to L^2 spaces,

$$\|u\|_{\Omega} = \|u\|_{L^2(\Omega)}.$$

The corresponding inner-products is denoted by

$$(u, v) = (u, v)_{L^2(\Omega)}.$$

For the Sobolev space $H_0^1(\Omega)$ we have

$$\|u\|_{\Omega} \leq \frac{L}{\pi} \|u_x\|_{\Omega} \text{ and } \|u\|_{H_0^1(\Omega)} = \|u_x\|_{\Omega}.$$

The phase space we consider is that of weak solutions

$$\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$$

equipped with the standard norm

$$\|y\|_{\mathcal{H}}^2 = \|\varphi_x\|_{\Omega}^2 + \|\psi_x\|_{\Omega}^2 + \|w_x\|_{\Omega}^2 + \|\tilde{\varphi}\|_{\Omega}^2 + \|\tilde{\psi}\|_{\Omega}^2 + \|\tilde{w}\|_{\Omega}^2,$$

where $y = (\varphi, \psi, w, \tilde{\varphi}, \tilde{\psi}, \tilde{w})$. For convenience, sometimes we use the (ℓ -dependent) norm,

$$\|y\|_{\mathcal{H}_\ell}^2 = \rho_1 \|\tilde{\varphi}\|_{\Omega}^2 + \rho_2 \|\tilde{\psi}\|_{\Omega}^2 + \rho_1 \|\tilde{w}\|_{\Omega}^2 + b \|\psi_x\|_{\Omega}^2 + k \|\varphi_x + \psi + \ell w\|_{\Omega}^2 + k_0 \|w_x - \ell \varphi\|_{\Omega}^2, \quad (3.7)$$

which is equivalent to the standard norm. Indeed, clearly there exists $\gamma_1 > 0$ such that

$$\|y\|_{\mathcal{H}_\ell}^2 \leq \gamma_1 \|y\|_{\mathcal{H}}^2.$$

Then from the Open Mapping Theorem, there exists $\gamma_2 > 0$ such

$$\|y\|_{\mathcal{H}}^2 \leq \gamma_2 \|y\|_{\mathcal{H}_\ell}^2, \quad (3.8)$$

which shows the equivalence of the norms. In particular there exists $\gamma_3 > 0$ such that

$$\|\varphi_x\|_{\Omega}^2 + \|\psi_x\|_{\Omega}^2 + \|w_x\|_{\Omega}^2 \leq \gamma_3 [b \|\psi_x\|_{\Omega}^2 + k \|\varphi_x + \psi + \ell w\|_{\Omega}^2 + k_0 \|w_x - \ell \varphi\|_{\Omega}^2]. \quad (3.9)$$

Remark 3.2. In the study of continuity of attractors as $\ell \rightarrow 0$ some energy estimates, uniform with respect to ℓ , are used. To this end we need $\gamma_1, \gamma_2, \gamma_3$ independent of ℓ , for ℓ small. It is clear that we can choose such γ_1 if $\ell \leq \ell_0$, for some ℓ_0 . Here, we show a simple argument to obtain γ_3 independently on $\ell \in [0, \ell_0]$ with $\ell_0 < \frac{\pi}{2L}$. Given $\varphi, \psi, w \in H_0^1(\Omega)$,

$$\begin{aligned} & \|\varphi_x\|_\Omega^2 + \|\psi_x\|_\Omega^2 + \|w_x\|_\Omega^2 \\ & \leq \|\psi_x\|_\Omega^2 + 2\|\varphi_x + \psi + \ell w\|_\Omega^2 + 2\|w_x - \ell\varphi\|^2 + 4\|\psi\|_\Omega^2 + 4\ell^2\|w\|_\Omega^2 + 2\ell^2\|\varphi\|_\Omega^2 \\ & \leq \left(1 + \frac{4L^2}{\pi^2}\right) \|\psi_x\|_\Omega^2 + 2\|\varphi_x + \psi + \ell w\|_\Omega^2 + 2\|w_x - \ell\varphi\|^2 + \frac{2\ell^2 L^2}{\pi^2} [2\|w_x\|_\Omega^2 + \|\varphi_x\|_\Omega^2]. \end{aligned}$$

Then, for $\ell \in [0, \ell_0]$,

$$\begin{aligned} & \|\varphi_x\|_\Omega^2 + \|\psi_x\|_\Omega^2 + \|w_x\|_\Omega^2 \\ & \leq \left(1 - \frac{4\ell_0^2 L^2}{\pi^2}\right)^{-1} \left[\left(1 + \frac{4L^2}{\pi^2}\right) \|\psi_x\|_\Omega^2 + 2\|\varphi_x + \psi + \ell w\|_\Omega^2 + 2\|w_x - \ell\varphi\|^2 \right]. \end{aligned}$$

Hence there exists a constant $\gamma_3 > 0$ such that (3.9) holds for all $\ell \in [0, \ell_0]$. In this case, we take $\gamma_2 = \max\{(\min\{\rho_1, \rho_2\})^{-1}, \gamma_3\}$. $\square$

3.3.2 Assumptions

In this section we established the hypotheses on the source terms (f_1, f_2, f_3) and damping functions (g_1, g_2, g_3) .

The assumptions we make on the source terms $f_1, f_2, f_3 : \mathbb{R} \rightarrow \mathbb{R}$ are those of locally Lipschitz and gradient type. In addition we assume there exists a $C^2(\mathbb{R}^3)$ function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that

$$\nabla F = (f_1, f_2, f_3), \tag{3.10}$$

and satisfies the following conditions: There exist $M, m_F \geq 0$ such that

$$F(u, v, w) \geq -M[|u|^2 + |v|^2 + |w|^2] - m_F, \quad \forall u, v, w \in \mathbb{R}, \tag{3.11}$$

where

$$0 \leq M < \frac{\pi^2}{2\gamma_3 L^2}, \tag{3.12}$$

and there exist $p \geq 1$ and $M_f > 0$ such that, for $i = 1, 2, 3$,

$$|\nabla f_i(u, v, w)| \leq M_f [1 + |u|^{p-1} + |v|^{p-1} + |w|^{p-1}], \quad \forall u, v, w \in \mathbb{R}. \tag{3.13}$$

In particular this implies that there exists $M_F > 0$ such that

$$F(u, v, w) \leq M_F [1 + |u|^{p+1} + |v|^{p+1} + |w|^{p+1}], \quad \forall u, v, w \in \mathbb{R}. \tag{3.14}$$

Furthermore, we assume that, for all $u, v, w \in \mathbb{R}$,

$$\nabla F(u, v, w) \cdot (u, v, w) - F(u, v, w) \geq -M[|u|^2 + |v|^2 + |w|^2] - m_F. \tag{3.15}$$

Remark 3.3. A simple example of F satisfying all above assumptions is

$$F(u, v, w) = |u + v|^4 - |u + v|^2 + \alpha_1 |uv|^2 + \alpha_2 |w|^3, \quad \alpha_1, \alpha_2 \geq 0.$$

In this case we have

$$F(u, v, w) \geq \min_{z \in \mathbb{R}} \{z^4 - z^2\} = -\frac{1}{4},$$

and

$$\begin{aligned} f_1(u, v, w) &= \frac{\partial F}{\partial u} = 4(u + v)^3 - 2(u + v) + 2\alpha_1 uv^2, \\ f_2(u, v, w) &= \frac{\partial F}{\partial v} = 4(u + v)^3 - 2(u + v) + 2\alpha_1 u^2 v, \\ f_3(u, v, w) &= \frac{\partial F}{\partial w} = 3\alpha_2 |w|w. \end{aligned}$$

Then conditions (3.11)-(3.13) hold with $m_F = 1/4$ and $p = 3$. In addition,

$$\nabla F(u, v, w) \cdot (u, v, w) - F(u, v, w) \geq 3|u + v|^4 - |u + v|^2 \geq -\frac{1}{16} \geq -m_F,$$

which shows that (3.15) also holds. $\square$

With respect to the damping functions we assume that $g_1, g_2, g_3 \in C^1(\mathbb{R})$,

$$g_i \text{ is increasing and } g_i(0) = 0, \quad (3.16)$$

and there exist constants $m_i, M_i > 0$ such that

$$m_i s^2 \leq g_i(s)s \leq M_i s^2, \quad \forall |s| > 1. \quad (3.17)$$

To establish the regularity and finite dimension of the attractors we assume further that (3.17) holds for all $s \in \mathbb{R}$, that is

$$m_i \leq g'_i(s) \leq M_i, \quad \forall s \in \mathbb{R}. \quad (3.18)$$

Remark 3.4. We observe that conditions (3.16) and (3.17) imply that for any given $\delta > 0$ there exists $C_\delta > 0$ such that

$$C_\delta (g_i(u) - g_i(v))(u - v) + \delta \geq |u - v|^2, \quad \forall u, v \in \mathbb{R}, \quad (3.19)$$

cf. [19, Proposition B.1.2]. On the other hand, assumption (3.18) implies promptly the usual monotonicity property

$$(g_i(u) - g_i(v))(u - v) \geq m_i |u - v|^2, \quad \forall u, v \in \mathbb{R}. \quad (3.20)$$

$\square$

3.3.3 Energy identities

The linear energy of the system, along a solution (φ, ψ, w) , is defined by

$$E_\ell(t) = \frac{1}{2} \|(\varphi(t), \psi(t), w(t), \varphi_t(t), \psi_t(t), w_t(t))\|_{\mathcal{H}_\ell}^2, \quad (3.21)$$

where $\|\cdot\|_{\mathcal{H}_\ell}$ is defined in (3.7). The contribution of the forcing terms implies the following identity

$$\mathcal{E}_\ell(t) = E_\ell(t) + \int_0^L F(\varphi, \psi, w) dx. \quad (3.22)$$

In particular we have that $\mathcal{E}_\ell(\cdot)$ satisfy

$$\frac{d}{dt} \mathcal{E}_\ell(t) = - \int_0^L [g_1(\varphi_t) \varphi_t + g_2(\psi_t) \psi_t + g_3(w_t) w_t] dx, \quad t \geq 0,$$

which holds for weak solutions and by integration in time variable we deduce that

$$\mathcal{E}_\ell(t) + \int_s^t \int_0^L [g_1(\varphi_t) \varphi_t + g_2(\psi_t) \psi_t + g_3(w_t) w_t] dx d\tau = \mathcal{E}_\ell(s), \quad 0 \leq t < s. \quad (3.23)$$

Moreover, we have the following useful energy inequality.

Lemma 3.1. There exists a positive constant M_E such that the following estimate holds true

$$\mathcal{E}_\ell(t) \geq M_E E_\ell(t) - Lm_F, \quad \forall t \geq 0. \quad (3.24)$$

In addition, if $\ell \in (0, \frac{\pi}{2L})$ then M_E is independent of ℓ .

Proof. The combination of the (3.22) and (3.11) imply that

$$\begin{aligned} \mathcal{E}_\ell(t) &\geq E_\ell(t) - M \left[\|\varphi\|_\Omega^2 + \|\psi\|_\Omega^2 + \|w\|_\Omega^2 \right] - Lm_F \\ &\geq \left(1 - \frac{2\beta\gamma_3 L^2}{\pi^2} \right) E_\ell(t) - Lm_F. \end{aligned}$$

Then from (3.12) we take $M_E = 1 - \frac{2\beta\gamma_3 L^2}{\pi^2}$. Finally, if $\ell \in (0, \frac{\pi}{2L})$, then from Remark 3.2 we can take γ_3 independent of ℓ , and then M_E is independent of ℓ . $\square$

3.4 Wellposedness/regularity Result

In this section the existence of global weak and strong solutions to the Bresse system will be established. The proof will be given through nonlinear semigroup theory. To this end we shall rewrite the system (3.1)-(3.3) as an abstract Cauchy problem

$$\begin{cases} y_t(t) - (\mathcal{A}_\ell + \mathcal{B})y(t) = \mathcal{F}(y(t)), & t \geq 0 \\ y(0) = y_0, \end{cases} \quad (3.25)$$

where

$$y(t) = (\varphi(t), \psi(t), w(t), \tilde{\varphi}(t), \tilde{\psi}(t), \tilde{w}(t)) \in \mathcal{H}, \quad \tilde{\varphi} = \varphi_t, \quad \tilde{\psi} = \psi_t, \quad \tilde{w} = w_t$$

and

$$y_0 = (\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1).$$

The operators are given by $\mathcal{A}_\ell : D(\mathcal{A}_\ell) \subset \mathcal{H} \rightarrow \mathcal{H}$,

$$\mathcal{A}_\ell \begin{bmatrix} \varphi \\ \psi \\ w \\ \tilde{\varphi} \\ \tilde{\psi} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} \tilde{\varphi} \\ \tilde{\psi} \\ \tilde{w} \\ k/\rho_1(\varphi_x + \psi + \ell w)_x + k_0\ell/\rho_1(w_x - \ell\varphi) \\ b/\rho_2\psi_{xx} - k/\rho_2(\varphi_x + \psi + \ell w) \\ k_0/\rho_1(w_x - \ell\varphi)_x - k\ell/\rho_1(\varphi_x + \psi + \ell w) \end{bmatrix},$$

with domain

$$D(\mathcal{A}_\ell) = [H^2(0, L) \cap H_0^1(0, L)]^3 \times [H_0^1(0, L)]^3,$$

and $\mathcal{B} : \mathcal{H} \rightarrow \mathcal{H}$,

$$\mathcal{B} \begin{bmatrix} \varphi \\ \psi \\ w \\ \tilde{\varphi} \\ \tilde{\psi} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -g_1(\tilde{\varphi})/\rho_1 \\ -g_2(\tilde{\psi})/\rho_2 \\ -g_3(\tilde{w})/\rho_1 \end{bmatrix},$$

with domain

$$D(\mathcal{B}) = \mathcal{H}.$$

The forcing terms are represented by a nonlinear function $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{H}$ defined by

$$\mathcal{F} \begin{bmatrix} \varphi \\ \psi \\ w \\ \tilde{\varphi} \\ \tilde{\psi} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -f_1(\varphi, \psi, w)/\rho_1 \\ -f_2(\varphi, \psi, w)/\rho_2 \\ -f_3(\varphi, \psi, w)/\rho_1 \end{bmatrix}.$$

Our existence theorem is given in terms of equivalent problem (3.25).

Theorem 3.1 (Well-posedness). Assume that $\ell > 0$ and the hypotheses (3.10)-(3.17) hold. Then for any initial data $y_0 \in \mathcal{H}$ and $T > 0$, problem (3.25) has a unique weak solution

$$y \in C([0, T]; \mathcal{H}), \quad y(0) = y_0,$$

given by

$$y(t) = e^{t(\mathcal{A}_\ell + \mathcal{B})} y_0 + \int_0^t e^{(t-s)(\mathcal{A}_\ell + \mathcal{B})} \mathcal{F}(y(s)) ds, \quad t \in [0, T], \quad (3.26)$$

and depends continuously on the initial data. In particular, if $y_0 \in D(\mathcal{A}_\ell)$ then the solution is strong.

Proof. Under the hypotheses (3.16)-(3.17) it was proved in [16, Theorem 2.2] that $\mathcal{A}_\ell + \mathcal{B}$ is maximal monotone in $\mathcal{H}$. Then from standard theory the Cauchy problem

$$\begin{cases} y_t(t) - (\mathcal{A}_\ell + \mathcal{B})y(t) = \mathcal{F}(y(t)), & t \geq 0 \\ y(0) = y_0, \end{cases} \quad (3.27)$$

has a unique solution. We will show that system (3.25) is a locally Lipschitz perturbation of (3.27). Then from classical results in [9] (see a detailed proof in [17, Theorem 7.2]), we obtain a local solution defined in an interval $[0, t_{\max})$ where, if $t_{\max} < \infty$, then

$$\lim_{t \rightarrow \infty} \|y(t)\|_{\mathcal{H}} = +\infty. \quad (3.28)$$

To show that operator $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{H}$ is locally Lipschitz, let B be a bounded set of $\mathcal{H}$ and $y^1, y^2 \in B$. We can write $y^i = (z^i, z_t^i)$, where $z^j = (\varphi^j, \psi^j, w^j)$, $i = 1, 2$. Then from assumption (3.13) we obtain, for $j = 1, 2, 3$,

$$\begin{aligned} |f_j(z^1) - f_j(z^2)|^2 &= |\nabla f_j(\lambda z^1 + (1 - \lambda)z^2)|^2 |z^1 - z^2|^2 \\ &\leq C(\nabla f) \left[|\varphi^1 - \varphi^2|^2 + |\psi^1 - \psi^2|^2 + |w^1 - w^2|^2 \right], \end{aligned}$$

where

$$C(\nabla f_j) = C \left[|\varphi^1|^{p-1} + |\varphi^2|^{p-1} + |\psi^1|^{p-1} + |\psi^2|^{p-1} + |w^1|^{p-1} + |w^2|^{p-1} \right]$$

Then we infer that, for some $C_B > 0$ the following inequality hold true

$$\int_0^L |f_j(z^1) - f_j(z^2)|^2 dx \leq C_B \|z^1 - z^2\|_{\Omega}^2 \leq C_B \|y^1 - y^2\|_{\mathcal{H}}^2.$$

Summing this estimate on j we obtain

$$\|\mathcal{F}(y^1) - \mathcal{F}(y^2)\|_{\mathcal{H}}^2 \leq 3C_B \|y^1 - y^2\|_{\mathcal{H}}^2,$$

which shows that $\mathcal{F}$ is locally Lipschitz on $\mathcal{H}$.

To see that the solution is global, that is, $t_{\max} = \infty$, let $y(t)$ be a mild solution with initial data $y_0 \in D(\mathcal{A}_\ell + \mathcal{B})$. Then it is indeed a strong solution and so we can use energy inequality (3.24) to conclude that

$$\|y(t)\|_{\mathcal{H}_\ell}^2 \leq \frac{2}{M_E} (\mathcal{E}_\ell(0) + Lm_f), \quad t \geq 0.$$

By density, this inequality holds for mild solutions. Then clearly (3.28) does not hold and therefore $t_{\max} = \infty$.

Finally, using (3.26) we can check that for any initial data $y_0^1, y_0^2 \in \mathcal{H}$, the corresponding solutions y^1, y^2 satisfy

$$\|y^1(t) - y^2(t)\|_{\mathcal{H}}^2 \leq C \|y_0^1(t) - y_0^2(t)\|_{\mathcal{H}}^2, \quad 0 < t < T,$$

which shows the continuous dependence on initial data. $\square$

Remark 3.5. The well-posedness of the Bresse system shows that its solution operator $S_\ell(t)$ is a C^0 -semigroup on $\mathcal{H}$. Then we denote by $(\mathcal{H}, S_\ell(t))$ the dynamical system generated by the problem (3.1)-(3.3). $\square$

3.5 Singular limit

Here we establish the Timoshenko limit of Bresse systems. With respect to the linear system (1.2), if $\ell = 0$, it uncouples into the Timoshenko system (1.3) system plus an independent wave equation in w . Therefore, in order to study the singular limit $\ell \rightarrow 0$ for the nonlinear model, we shall need some compatibility condition. More precisely, we assume that f_1, f_2 do not depend on w , that is,

$$f_1(\varphi, \psi, w) = f_1(\varphi, \psi) \quad \text{and} \quad f_2(\varphi, \psi, w) = f_2(\varphi, \psi). \quad (3.29)$$

Remark 3.6. If the assumption (3.29) holds, then taking $\ell = 0$, the same argument used in the proof of Theorem 3.1 shows that Timoshenko system (3.4)-(3.6) is well-posed in the phase space

$$\mathcal{H}_0 = [H_0^1(0, L)]^2 \times [L^2(0, L)]^2.$$

Its solution operator $S_0(t)$ generates a dynamical system denoted by $(\mathcal{H}_0, S_0(t))$. $\square$

Theorem 3.2. Assume that the hypotheses of Theorem 3.1 and (3.29) hold. Given any sequence $\{\ell_n\}_{n \in \mathbb{N}}$ of positive numbers let (φ^n, ψ^n, w^n) be the corresponding weak solution of the Bresse

system (3.1)-(3.3), with $\ell = \ell_n$, and fixed initial data $(\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1) \in \mathcal{H}$. Then if $\ell_n \rightarrow 0$ as $n \rightarrow \infty$, there exists (φ, ψ, w) such that, for any $T > 0$,

$$\begin{aligned} (\varphi^n, \psi^n, w^n) &\xrightarrow{*} (\varphi, \psi, w) \text{ in } L^\infty(0, T; [H_0^1(\Omega)]^3), \\ (\varphi_t^n, \psi_t^n, w_t^n) &\xrightarrow{*} (\varphi_t, \psi_t, w_t) \text{ in } L^\infty(0, T; [L^2(\Omega)]^3), \end{aligned}$$

and (φ, ψ) is a weak solution of the Timoshenko system (3.4)-(3.6).

Proof. The proof is divided into three steps.

Step 1. A priori estimates: Since $\{\ell_n\}_{n \in \mathbb{N}}$ is uniformly bounded, there exists a positive constant C_0 , such that,

$$\mathcal{E}_{\ell_n}(0) = E_{\ell_n}(0) + \int_0^L F(\varphi_0, \psi_0, w_0) dx \leq C_0, \quad \forall n.$$

Then, because $\mathcal{E}_{\ell_n}(\cdot)$ is decreasing, we get from (3.24),

$$E_{\ell_n}(t) \leq \frac{1}{\beta_0}(C_0 + Lm_F), \quad \forall t \geq 0,$$

From definition of $E_\ell(\cdot)$ in (3.21) we conclude that the following sequences

$$\{\varphi_t^n\}_{n \in \mathbb{N}}, \{\psi_t^n\}_{n \in \mathbb{N}}, \{w_t^n\}_{n \in \mathbb{N}}, \{\varphi_x^n + \psi^n + \ell_n w^n\}_{n \in \mathbb{N}}, \{w_x^n - \ell_n \varphi^n\}_{n \in \mathbb{N}},$$

are bounded in $L^\infty(0, T; L^2(\Omega))$ and $\{\psi^n\}_{n \in \mathbb{N}}$ is bounded in $L^\infty(0, T; H_0^1(\Omega))$. Let us show that $\{\varphi^n\}_{n \in \mathbb{N}}$ and $\{w^n\}_{n \in \mathbb{N}}$ are also bounded in $L^\infty(0, T; H_0^1(\Omega))$. Indeed, from

$$\varphi^n(x, t) = \int_0^t \varphi_t^n(x, s) ds + \varphi_0(x),$$

we infer that $\{\varphi^n\}_{n \in \mathbb{N}}$ is bounded in $L^\infty(0, T; L^2(\Omega))$. Now, using the relation

$$w_x^n = (w_x^n - \ell_n \varphi^n) + \ell_n \varphi^n,$$

we find that $\{w^n\}_{n \in \mathbb{N}}$ is bounded in $L^\infty(0, T; H_0^1(\Omega))$. Similar arguments show that $\{\varphi^n\}_{n \in \mathbb{N}}$ is bounded in $L^\infty(0, T; H_0^1(\Omega))$.

Step 2. Limits: Using a subsequence if necessary, there exist functions $\varphi, \psi, w, \vartheta_1, \vartheta_2$ such that

$$(\varphi^n, \psi^n, w^n) \xrightarrow{*} (\varphi, \psi, w) \text{ in } L^\infty(0, T; H_0^1(\Omega)), \quad (3.30)$$

$$(\varphi_t^n, \psi_t^n, w_t^n) \xrightarrow{*} (\varphi_t, \psi_t, w_t) \text{ in } L^\infty(0, T; L^2(\Omega)), \quad (3.31)$$

$$\{\varphi_x^n + \psi^n + \ell_n w^n\}_{n \in \mathbb{N}} \xrightarrow{*} \vartheta_1 \text{ in } L^\infty(0, T; L^2(\Omega)), \quad (3.32)$$

$$\{w_x^n - \ell_n \varphi^n\}_{n \in \mathbb{N}} \xrightarrow{*} \vartheta_2 \text{ in } L^\infty(0, T; L^2(\Omega)). \quad (3.33)$$

It follows from (3.32)-(3.33) and uniqueness of the weak limit that

$$\vartheta_1 = \varphi_x + \psi \text{ and } \vartheta_2 = w_x.$$

In addition convergences (3.30) and (3.31) imply that

$$(\varphi^n, \psi^n) \rightarrow (\varphi, \psi) \text{ in } L^2(0, T; L^2(\Omega)). \quad (3.34)$$

Now, from the definition of generalized solution for the Bresse system, we know that (φ^n, ψ^n, w^n) satisfies

$$\begin{aligned} & \rho_1 \frac{d}{dt}(\varphi_t^n, \tilde{\varphi}) + \rho_2 \frac{d}{dt}(\psi_t^n, \tilde{\psi}) + \rho_1 \frac{d}{dt}(w_t^n, \tilde{w}) + k((\varphi_x^n + \psi^n + \ell_n w^n), (\tilde{\varphi}_x + \tilde{\psi} + \ell_n \tilde{w})) \\ & + b(\psi_x^n, \tilde{\psi}_x) + k_0((w_x^n - \ell_n \varphi^n), (\tilde{w}_x - \ell_n \tilde{\varphi})) + N_{\varphi, \psi}^n + N_w^n = 0, \end{aligned} \quad (3.35)$$

for all $\tilde{\varphi}, \tilde{\psi}, \tilde{w} \in H_0^1(\Omega)$ and by $N_{\varphi, \psi}^n, N_w^n$ we denote the following nonlinear terms

$$N_{\varphi, \psi}^n = \int_0^L f_1(\varphi^n, \psi^n) \tilde{\varphi} dx + \int_0^L f_2(\varphi^n, \psi^n) \tilde{\psi} dx + \int_0^L g_1(\varphi_t^n) \tilde{\varphi} dx + \int_0^L g_2(\psi_t^n) \tilde{\psi} dx,$$

and

$$N_w^n = \int_0^L f_3(\varphi^n, \psi^n, w^n) \tilde{w} dx + \int_0^L g_3(w_t^n) \tilde{w} dx.$$

Using the assumption (3.31) and applying similar arguments as in [45] we infer that

$$\int_0^L (g_1(\varphi_t^n) - g_1(\varphi_t)) \tilde{\varphi} dx \rightarrow 0$$

and

$$\int_0^L (g_2(\psi_t^n) - g_2(\psi_t)) \tilde{\psi} dx \rightarrow 0.$$

The hypothesis on the sources and the convergence (3.34) imply

$$\int_0^L (f_1(\varphi^n, \psi^n) - f_1(\varphi, \psi)) \tilde{\varphi} dx \rightarrow 0$$

and

$$\int_0^L (f_2(\varphi^n, \psi^n) - f_2(\varphi, \psi)) \tilde{\psi} dx \rightarrow 0.$$

Then, taking test functions with $\tilde{w} = 0$, we see that convergences (3.30)-(3.34) imply that

$$\begin{aligned} & \rho_1 \frac{d}{dt}(\varphi_t, \tilde{\varphi}) + \rho_2 \frac{d}{dt}(\psi_t, \tilde{\psi}) + b(\psi_x, \tilde{\psi}_x) + k((\varphi_x + \psi), (\tilde{\varphi}_x + \tilde{\psi})) \\ & + \int_0^L [f_1(\varphi, \psi) \tilde{\varphi} + f_2(\varphi, \psi) \tilde{\psi}] dx + \int_0^L [g_1(\varphi_t) \tilde{\varphi} + g_2(\psi_t) \tilde{\psi}] dx = 0, \end{aligned} \quad (3.36)$$

for all $\tilde{\varphi}, \tilde{\psi} \in H_0^1(\Omega)$. This means that the limit (φ, ψ) is a weak solution of the Timoshenko system (3.4)-(3.5).

Step 3. Initial conditions: From (3.30)-(3.31) we obtain (cf. [54]),

$$(\varphi^n, \psi^n) \rightarrow (\varphi, \psi) \text{ in } C([0, T], [L^2(\Omega)]^2),$$

and therefore

$$(\varphi(0), \psi(0)) = (\varphi_0, \psi_0). \quad (3.37)$$

It remains to show that $(\varphi_t(0), \psi_t(0)) = (\varphi_1, \psi_1)$. To this end, we multiply (3.35) by a test function

$$\theta \in H^1(0, T), \quad \theta(0) = 1, \quad \theta(T) = 0,$$

and integrate over $[0, T]$. Taking also $\tilde{w} = 0$, we find that

$$\begin{aligned} & \rho_1(\varphi_1, \tilde{\varphi}) - \rho_1 \int_0^T (\varphi_t^n, \tilde{\varphi}) \theta_t dt + \rho_2(\psi_1, \tilde{\psi}) - \rho_2 \int_0^T (\psi_t^n, \tilde{\psi}) \theta_t dt + b \int_0^T (\psi_x^n, \tilde{\psi}_x) \theta dt \\ & + k \int_0^T ((\varphi_x^n + \psi_x^n + \ell_n w^n), (\tilde{\varphi}_x + \tilde{\psi})) \theta dt + k_0 \int_0^T \ell_n^2(\varphi^n, \tilde{\varphi}) \theta dt + \int_0^T N_{\varphi, \psi}^n \theta dt = 0, \end{aligned}$$

for all $\tilde{\varphi}, \tilde{\psi} \in H_0^1(\Omega)$.

Taking the limit $n \rightarrow \infty$, we obtain

$$\begin{aligned} & \rho_1(\varphi_1, \tilde{\varphi}) - \rho_1 \int_0^T (\varphi_t, \tilde{\varphi}) \theta_t dt + \rho_2(\psi_1, \tilde{\psi}) - \rho_2 \int_0^T (\psi_t, \tilde{\psi}) \theta_t dt \\ & + b \int_0^T (\psi_x, \tilde{\psi}_x) \theta dt + k \int_0^T ((\varphi_x + \psi_x), (\tilde{\varphi}_x + \tilde{\psi})) \theta dt + \int_0^T N_{\varphi, \psi}^\infty \theta dt = 0, \end{aligned}$$

for all $\tilde{\varphi}, \tilde{\psi} \in H_0^1(\Omega)$.

On the other hand, multiplying (3.36) by θ and integrating over $[0, T]$, we obtain

$$\begin{aligned} & \rho_1(\varphi_t(0), \tilde{\varphi}) - \rho_1 \int_0^T (\varphi_t, \tilde{\varphi}) \theta_t dt + \rho_2(\psi_t(0), \tilde{\psi}) - \rho_2 \int_0^T (\psi_t, \tilde{\psi}) \theta_t dt \\ & + b \int_0^T (\psi_x, \tilde{\psi}_x) \theta dt + k \int_0^T ((\varphi_x + \psi_x), (\tilde{\varphi}_x + \tilde{\psi})) \theta dt + \int_0^T N_{\varphi, \psi}^\infty \theta dt = 0, \end{aligned}$$

for all $\tilde{\varphi}, \tilde{\psi} \in H_0^1(\Omega)$.

The last two identities imply that

$$(\varphi_t(0), \psi_t(0)) = (\varphi_1, \psi_1). \quad (3.38)$$

Therefore (3.36), (3.37) and (3.38) show that the limit pair (φ, ψ) is a solution of the Timoshenko system (3.4)-(3.6). $\square$

Remark 3.7. We observe that the singular limit holds for the linear problem with $f_i = 0$, $g_i = 0$, $i = 1, 2, 3$. In this case, the energy is conservative and then $E_{\ell_n}(t) = E_{\ell_n}(0) \leq C_0$, for all $t \geq 0$. $\square$

3.6 Global attractors I

In this section we prove a first result on global attractors for Bresse systems. Some definitions and abstract results for global attractors are presented in the Appendix.

Theorem 3.3. Under the hypotheses (3.10)-(3.17), for each $\ell > 0$, the dynamical system $(\mathcal{H}, S_\ell(t))$ generated by the problem (3.1)-(3.3) has a global attractor $\mathbf{A}_\ell$. In addition, it is characterized by

$$\mathbf{A}_\ell = \mathbb{M}_+(\mathcal{N}_\ell),$$

where $\mathbb{M}_+(\mathcal{N}_\ell)$ is the unstable manifold emanating from $\mathcal{N}_\ell$, the set of stationary points of $S_\ell(t)$.

The proof of this theorem is based on Theorem 2.19. We first show that the system is asymptotically compact.

Lemma 3.2. Under the hypotheses Theorem 3.3, given a bounded subset B of $\mathcal{H}$, let $S_\ell(t)y^i = (\varphi^i, \psi^i, w^i, \varphi_t^i, \psi_t^i, w_t^i)$ be, with $i = 1, 2$, two solutions of problem (3.1)-(3.3) with initial data $y^1, y^2 \in B$. Then, for every $\delta > 0$, there exists a constant $C_{\delta, B} > 0$ such that for $T > 0$ sufficiently large, one has

$$E_\ell(T) \leq \delta + \frac{C_{\delta, B}}{T} + C_{\delta, B} \int_0^T \left[\|\tilde{\varphi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)} \right] dt, \quad (3.39)$$

where $\tilde{\varphi} = \varphi^1 - \varphi^2$, $\tilde{\psi} = \psi^1 - \psi^2$, $\tilde{w} = w^1 - w^2$.

Proof. The proof of this lemma is divided in several steps.

Step 1. First energy identity: For $\tilde{u} = u^1 - u^2$ we use the following notation

$$G_i(\tilde{u}) = g_i(u^1) - g_i(u^2) \text{ and } F_i(\tilde{u}) = f_i(u^1) - f_i(u^2), \quad i = 1, 2, 3.$$

Then $(\tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{\varphi}_t, \tilde{\psi}_t, \tilde{w}_t)$ is the solution of the problem

$$\begin{cases} \rho_1 \tilde{\varphi}_{tt} - k(\tilde{\varphi}_x + \tilde{\psi} + \ell \tilde{w})_x - k_0 \ell (\tilde{w}_x - \ell \tilde{\varphi}) = -G_1(\tilde{\varphi}_t) - F_1(\tilde{\varphi}, \tilde{\psi}, \tilde{w}), \\ \rho_2 \tilde{\psi}_{tt} - b \tilde{\psi}_{xx} + k(\tilde{\varphi}_x + \tilde{\psi} + \ell \tilde{w}) = -G_2(\tilde{\psi}_t) - F_2(\tilde{\varphi}, \tilde{\psi}, \tilde{w}), \\ \rho_1 \tilde{w}_{tt} - k_0 (\tilde{w}_x - \ell \tilde{\varphi})_x + k \ell (\tilde{\varphi}_x + \tilde{\psi} + \ell \tilde{w}) = -G_3(\tilde{w}_t) - F_3(\tilde{\varphi}, \tilde{\psi}, \tilde{w}), \end{cases} \quad (3.40)$$

with Dirichlet boundary conditions and initial conditions,

$$(\tilde{\varphi}(0), \tilde{\psi}(0), \tilde{w}(0), \tilde{\varphi}_t(0), \tilde{\psi}_t(0), \tilde{w}_t(0)) = y^1 - y^2.$$

Our objective is to obtain an estimate for $E_\ell(T)$ along the solution $(\tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{\varphi}_t, \tilde{\psi}_t, \tilde{w}_t)$. We begin by multiplying the equations (3.40)₁-(3.40)₃ by $\tilde{\varphi}$, $\tilde{\psi}$ and $\tilde{w}$, respectively, and integrate over

$[0, T] \times [0, L]$. Then adding the kinetic energy, we obtain

$$\begin{aligned} \int_0^T E_\ell(t) dt &= -\frac{1}{2} \int_0^L [\rho_1 \tilde{\phi} \tilde{\phi}_t + \rho_2 \tilde{\psi} \tilde{\psi}_t + \rho_1 \tilde{w} \tilde{w}_t] dx \Big|_0^T + \int_0^T \int_0^L [\rho_1 \tilde{\phi}_t^2 + \rho_2 \tilde{\psi}_t^2 + \rho_1 \tilde{w}_t^2] dx dt \\ &\quad - \frac{1}{2} \int_0^T \int_0^L [G_1(\tilde{\phi}_t) \tilde{\phi} + G_2(\tilde{\psi}_t) \tilde{\psi} + G_3(\tilde{w}_t) \tilde{w}] dx dt \\ &\quad - \frac{1}{2} \int_0^T \int_0^L [F_1(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \tilde{\phi} + F_2(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \tilde{\psi} + F_3(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \tilde{w}] dx dt. \end{aligned} \quad (3.41)$$

We shall estimate the right-hand side of (3.41).

Step 2. Boundary terms estimate: Using Hölder's inequality and norm inequality (3.8) there exists a constant $C > 0$, independent of ℓ , such that

$$\int_0^L [\rho_1 \tilde{\phi} \tilde{\phi}_t + \rho_2 \tilde{\psi} \tilde{\psi}_t + \rho_1 \tilde{w} \tilde{w}_t] dx \leq C E_\ell(t), \quad \forall t \geq 0.$$

Then we obtain

$$\int_0^L [\rho_1 \tilde{\phi} \tilde{\phi}_t + \rho_2 \tilde{\psi} \tilde{\psi}_t + \rho_1 \tilde{w} \tilde{w}_t] dx \Big|_0^T \leq C [E_\ell(T) + E_\ell(0)]. \quad (3.42)$$

Step 3. Kinetic energy estimate: Applying (3.19), given $\delta > 0$, we have that there exists $C_\delta > 0$ such that

$$\int_0^T \int_0^L \tilde{\phi}_t^2 dx dt \leq TL\delta + C_\delta \int_0^T \int_0^L G_1(\tilde{\phi}_t) \tilde{\phi}_t dx dt.$$

Same argument holds for $\iint \tilde{\psi}_t^2$ and $\iint \tilde{w}_t^2$ and therefore, given $\delta > 0$, there exists $C_\delta > 0$ such that

$$\begin{aligned} \int_0^T \int_0^L [\rho_1 \tilde{\phi}_t^2 + \rho_2 \tilde{\psi}_t^2 + \rho_1 \tilde{w}_t^2] dx dt \\ \leq TL\delta + C_\delta \int_0^T \int_0^L [G_1(\tilde{\phi}_t) \tilde{\phi}_t + G_2(\tilde{\psi}_t) \tilde{\psi}_t + G_3(\tilde{w}_t) \tilde{w}_t] dx dt. \end{aligned} \quad (3.43)$$

Step 4. Damping terms estimate: Let us consider the integral over $|\tilde{\phi}_t| \leq 1$ and $|\phi_t| > 1$. Then (3.17) implies that

$$\begin{aligned} \int_0^T \int_0^L G_1(\tilde{\phi}_t) \phi dx dt &\leq \int_0^T \int_0^L [|g_1(\phi_t^1)| + |g_1(\phi_t^2)|] |\tilde{\phi}| dx dt \\ &\leq 2 \int_0^T \int_0^L \|g_1\|_{L^\infty(-1,1)} |\tilde{\phi}| dx dt + \int_0^T \int_0^L M_1 [|\phi_t^1| + |\phi_t^2|] |\tilde{\phi}| dx dt \\ &\leq C_B \int_0^T \|\tilde{\phi}\|_{L^{p+1}(\Omega)} dt. \end{aligned}$$

The same argument holds for $\iint G_2(\tilde{\psi}_t) \tilde{\psi}$ and $\iint G_3(\tilde{w}_t) \tilde{w}$. Therefore we obtain the following estimate

$$\begin{aligned} -\frac{1}{2} \int_0^T \int_0^L [G_1(\tilde{\phi}_t) \tilde{\phi} + G_2(\tilde{\psi}_t) \tilde{\psi} + G_3(\tilde{w}_t) \tilde{w}] dx dt \\ \leq C_B \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] dt. \end{aligned} \quad (3.44)$$

Step 5. Forcing terms estimate: Using assumption (3.13) we have

$$\int_0^T \int_0^L F_1(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \tilde{\phi} dx dt \leq C \int_0^T \int_0^L C(\nabla f_1) [|\tilde{\phi}| + |\tilde{\psi}| + |\tilde{w}|] |\tilde{\phi}| dx dt$$

where

$$C(\nabla f_1) = C(1 + |\phi|^p + |\phi^2|^{p-1} + |\psi|^p + |\psi^2|^{p-1} + |w|^p + |w^2|^{p-1}).$$

Using Hölder's inequality with $1/p_1 = \frac{(p-1)}{2(p+1)}$, $1/p_2 = \frac{1}{p+1}$, $1/p_3 = \frac{1}{2}$ and continuous embedding $L^{p+1}(\Omega) \subset L^2(\Omega)$ we have

$$\begin{aligned} \int_0^T \int_0^L C(\nabla f_1) [|\tilde{\phi}| + |\tilde{\psi}| + |\tilde{w}|] |\tilde{\phi}| dx dt \\ \leq C_B \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] \|\tilde{\phi}\|_{\Omega} dt \\ \leq C_B \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] \|\tilde{\phi}\|_{L^{p+1}(\Omega)} dt \\ \leq C_B \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)}^2 + \|\tilde{\psi}\|_{L^{p+1}(\Omega)}^2 + \|\tilde{w}\|_{L^{p+1}(\Omega)}^2] dt \\ \leq C_B \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] dt. \end{aligned}$$

Then we find that

$$\int_0^T \int_0^L F_1(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \tilde{\phi} dx dt \leq C_B \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] dt.$$

A similar argument applied in $\iint F_2 \tilde{\psi}$ and $\iint F_3 \tilde{w}$ imply that

$$\begin{aligned} -\frac{1}{2} \int_0^T \int_0^L [F_1(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \phi_t + F_2(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \psi_t + F_3(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \tilde{w}_t] dx dt \\ \leq C_B \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] dt. \end{aligned} \quad (3.45)$$

Step 6. An energy inequality: We multiply the equations (3.40)₁-(3.40)₃ by $\tilde{\phi}_t$, $\tilde{\psi}_t$ and $\tilde{w}_t$, respectively, and then integrate over $[s, T] \times [0, L]$. Then we find that

$$\begin{aligned} E_\ell(T) = E_\ell(s) - \int_s^T \int_0^L [G_1(\tilde{\phi}_t) \tilde{\phi}_t + G_2(\tilde{\psi}_t) \tilde{\psi}_t + G_3(\tilde{w}_t) \tilde{w}_t] dx dt \\ - \int_0^T \int_0^L [F_1(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \phi_t + F_2(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \psi_t + F_3(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \tilde{w}_t] dx dt. \end{aligned} \quad (3.46)$$

As before we see that

$$\begin{aligned} \int_0^T \int_0^L [F_1(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \phi_t + F_2(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \psi_t + F_3(\tilde{\phi}, \tilde{\psi}, \tilde{w}) \tilde{w}_t] dx dt \\ \leq C_B \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] dt. \end{aligned}$$

Then identity (3.46) gives

$$\begin{aligned} & \int_0^T \int_0^L [G_1(\tilde{\phi}_t)\tilde{\phi}_t + G_2(\tilde{\psi}_t)\tilde{\psi}_t + G_3(\tilde{w}_t)\tilde{w}_t] dx dt \\ & \leq E_\ell(T) + E_\ell(0) + C_B \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] dt. \end{aligned} \quad (3.47)$$

Step 7. Conclusion: Inserting the estimates (3.42)-(3.45) into (3.41) we obtain

$$\begin{aligned} & \int_0^T E_\ell(t) dt \leq C[E_\ell(T) + E_\ell(0)] + \delta TL \\ & \quad + C_\delta \int_0^T \int_0^L [G_1(\tilde{\phi}_t)\tilde{\phi}_t + G_2(\tilde{\psi}_t)\tilde{\psi}_t + G_3(\tilde{w}_t)\tilde{w}_t] dx dt \\ & \quad + C_B \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] dt. \end{aligned}$$

The estimate (3.47) together with the fact that $E(0) \leq C_B$ imply

$$\begin{aligned} & \int_0^T E_\ell(t) dt \leq \delta TL + C_\delta E_\ell(T) + C_{\delta,B} \\ & \quad + C_{\delta,B} \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] dt. \end{aligned} \quad (3.48)$$

Now integrating (3.46) over $[0, T]$ with respect to the variable s and taking into account that $G_1(\phi_t)\phi_t + G_2(\psi_t)\psi_t + G_3(w_t)w_t \geq 0$ we have

$$\begin{aligned} & TE_\ell(T) \leq \delta TL + C_\delta E_\ell(T) + C_{\delta,B} \\ & \quad + C_{\delta,B} \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] dt. \end{aligned} \quad (3.49)$$

Combining (3.48) and (3.49) we find that

$$\begin{aligned} & TE_\ell(T) \leq \delta TL + C_\delta E_\ell(T) + C_{\delta,B} \\ & \quad + C_{\delta,B}(1+T) \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] dt. \end{aligned}$$

Given $\delta > 0$ and taking T sufficiently large (say $T > \max\{1, 2C_\delta\}$) we have that the following estimate holds true

$$E_\ell(T) \leq 2\delta L + \frac{2C_{\delta,B}}{T} + 2C_{\delta,B} \int_0^T [\|\tilde{\phi}\|_{L^{p+1}(\Omega)} + \|\tilde{\psi}\|_{L^{p+1}(\Omega)} + \|\tilde{w}\|_{L^{p+1}(\Omega)}] dt.$$

Then, renaming the constants we see that the inequality (3.39) holds. $\square$

Lemma 3.3. Under the hypotheses of Theorem 3.3 the system $(\mathcal{H}, S_\ell(t))$ is asymptotically compact.

Proof. Let B be a positively invariant bounded set of $\mathcal{H}_\ell$. Given $\varepsilon > 0$, we take δ sufficiently small and T sufficiently large, say

$$\delta < \frac{\varepsilon^2}{8} \quad \text{and} \quad \frac{C_{\delta,B}}{T} < \frac{\varepsilon^2}{8}.$$

Then from (3.39)

$$\|S(T)y^1 - S(T)y^2\|_{\mathcal{H}} \leq \varepsilon + \Phi_{\varepsilon,B,T}(y^1, y^2),$$

where

$$\Phi_{\varepsilon,B,T}(y^1, y^2) = 2\sqrt{C_{\delta,B}} \left(\int_0^T \left[\|\varphi^1 - \varphi^2\|_{L^{p+1}(\Omega)} + \|\psi^1 - \psi^2\|_{L^{p+1}(\Omega)} + \|w^1 - w^2\|_{L^{p+1}(\Omega)} \right] dt \right)^{\frac{1}{2}}.$$

Let us show that condition (2.6) holds. Given $\{y^n\}_{n \in \mathbb{N}}$ in B , by positive invariance, we see that $S_\ell(t)y^n = (\varphi^n, \varphi_t^n, \psi^n, \psi_t^n, w^n, w_t^n)$ is uniformly bounded in $\mathcal{H}$. Then

$$\begin{aligned} (\varphi^n, \psi^n, w^n) &\text{ is bounded in } L^\infty(0, T; [H_0^1(\Omega)]^3), \\ (\varphi_t^n, \psi_t^n, w_t^n) &\text{ is bounded in } L^\infty(0, T; [L^2(\Omega)]^3), \end{aligned}$$

and therefore ([54]),

$$(\varphi^n, \psi^n, w^n) \text{ is pre-compact in } C^0([0, T], [L^{p+1}(\Omega)]^3).$$

It follows that condition (2.6) holds. Then the asymptotic compactness follows from Theorem 2.14. $\square$

One of the assumptions of Theorem 2.19 is that the set of stationary points is bounded.

Lemma 3.4. Under the hypotheses of Theorem 3.3, the set of equilibrium points $\mathcal{N}_\ell$ is bounded in $\mathcal{H}$.

Proof If $y \in \mathcal{N}_\ell$ we know that $y = (\varphi, \psi, w, 0, 0, 0)$ and satisfies

$$\begin{cases} -k(\varphi_x + \psi + \ell w)_x - k_0\ell(w_x - \ell\varphi) + f_1(\varphi, \psi, w) = 0, \\ -b\psi_{xx} + k(\varphi_x + \psi + \ell w) + f_2(\varphi, \psi, w) = 0, \\ -k_0(w_x - \ell\varphi)_x + k\ell(\varphi_x + \psi + \ell w) + f_3(\varphi, \psi, w) = 0. \end{cases} \quad (3.50)$$

Multiplying equations (3.50)₁, (3.50)₂, (3.50)₃ by φ, ψ, w , respectively, and integrating over $[0, L]$, we obtain

$$\begin{aligned} &\int_0^L [b\psi_x^2 + k(\varphi_x + \psi + \ell w)^2 + k_0(w_x + \ell\varphi)^2] dx \\ &= - \int_0^L [f_1(\varphi, \psi, w)\varphi + f_2(\varphi, \psi, w)\psi + f_3(\varphi, \psi, w)w] dx. \end{aligned}$$

Then, using (3.9), (3.15) and (3.11), we get

$$\frac{1}{\gamma_3} [\|\varphi_x\|_\Omega^2 + \|\psi_x\|_\Omega^2 + \|w_x\|_\Omega^2] \leq \frac{2\beta L^2}{\pi^2} [\|\varphi_x\|_\Omega^2 + \|\psi_x\|_\Omega^2 + \|w_x\|_\Omega^2] + 2m_F L,$$

and consequently,

$$\left(1 - \frac{2\beta L^2 \gamma_3}{\pi^2}\right) [\|\varphi_x\|_\Omega^2 + \|\psi_x\|_\Omega^2 + \|w_x\|_\Omega^2] \leq 2m_F L \gamma_3.$$

Therefore the set $\mathcal{N}_\ell$ is bounded in $\mathcal{H}$. $\square$

Proof of Theorem 3.3 (Completion). We already known that the system is asymptotically compact and the set of its stationary points $\mathcal{N}_\ell$ is bounded. To apply Theorem 2.19, it remains to show that the dynamical system $(\mathcal{H}, S_\ell(t))$ is gradient and satisfies condition (2.12). Indeed, we can take the energy functional $\mathcal{E}_\ell$ as a Lyapunov function Φ , since $t \rightarrow \Phi(S_\ell(t)y)$ is a strictly decreasing for any $y \in \mathcal{H}$. Moreover, from (3.22) and (3.13) we see that $\mathcal{E}_\ell(t) \leq \|y(t)\|_{\mathcal{H}_\ell}^2 + C(1 + \|y(t)\|_{\mathcal{H}_\ell}^{p+1})$. Then

$$\mathcal{E}_\ell(t) \rightarrow \infty \text{ implies that } \|y(t)\|_{\mathcal{H}} \rightarrow \infty, \quad t \geq 0.$$

On the other hand, the inequality (3.24) implies that $E_\ell(t) \leq \frac{1}{\beta_0}(\mathcal{E}(t) + Lm_F)$, and then

$$\|y(t)\|_\ell \rightarrow \infty \text{ implies that } \mathcal{E}_\ell(t) \rightarrow \infty, \quad t \geq 0.$$

Therefore all the assumptions of Theorem 2.19 are fulfilled and consequently the system $(\mathcal{H}, S_\ell(t))$ has a global attractor $\mathbf{A}_\ell = \mathbb{M}_+(\mathcal{N}_\ell)$. This ends the proof of Theorem 3.3. $\square$

Remark 3.8. The existence of a global attractor implies that the system has a bounded absorbing set. But in principle it depends on ℓ . For completeness, we shall construct an absorbing set which is uniform bounded for $\ell \in [0, \ell_0]$, with ℓ_0 small. $\square$

Lemma 3.5. Under hypotheses of Theorem 3.3, with $\ell \in (0, \pi/2L)$, the dynamical system $(\mathcal{H}, S_\ell(t))$ has a bounded absorbing set $\mathcal{B}$ independent of ℓ .

Proof. Step 1. First energy inequality: Multiply the equations (3.1)₁ – (3.1)₃ by φ , ψ and w , respectively, and integrate over $[0, L] \times [0, T]$. We obtain

$$\begin{aligned} & \int_0^T [b\|\psi_x\|^2 + k\|\varphi_x + \psi + \ell w\|^2 + k_0\|w_x - \ell\varphi\|^2] dt + \int_0^T \int_0^L \nabla F(\varphi, \psi, w) \cdot (\varphi, \psi, w) dx dt \\ &= - \int_0^L [\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t] dx \Big|_0^T + \int_0^T \int_0^L [\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2] dx dt \\ & \quad - \int_0^T \int_0^L [g_1(\varphi_t) \varphi + g_2(\psi_t) \psi + g_3(w_t) w] dx dt. \end{aligned} \quad (3.51)$$

Inequality (3.15) together with (3.9) and (3.12) results that

$$\begin{aligned}
& \int_0^T \int_0^L \nabla F(\varphi, \psi, w) \cdot (\varphi, \psi, w) dx dt \\
& \geq \int_0^T \int_0^L F(\varphi, \psi, w) dx dt - M \int_0^T \int_0^L [|\varphi|^2 + |\psi|^2 + |w|^2] dx dt - T L m_F \\
& \geq \int_0^T \int_0^L F(\varphi, \psi, w) dx dt - \frac{1}{2} \int_0^T [b \|\psi_x\|_{\Omega}^2 + k \|\varphi_x + \psi + \ell w\|_{\Omega}^2 + k_0 \|w_x - \ell \varphi\|_{\Omega}^2] dt \\
& \quad - T L m_F.
\end{aligned}$$

Inserting this inequality into (3.51) and adding the kinetic energy we find that

$$\begin{aligned}
\int_0^T \mathcal{E}_\ell(t) dt & \leq - \int_0^L [\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t] dx \Big|_0^T \\
& \quad + \frac{3}{2} \int_0^T \int_0^L [\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2] dx dt \\
& \quad - \int_0^T \int_0^L [g_1(\varphi_t) \varphi + g_2(\psi_t) \psi + g_3(w_t) w] dx dt + T L m_F.
\end{aligned} \tag{3.52}$$

In the following we will estimate the terms on the right-hand side of (3.52).

Step 2. Boundary terms estimate: Young's inequality and (3.9) imply that

$$- \int_0^L [\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t] dx \leq C E_\ell(t)$$

for some constant $C > 0$, independent of T and ℓ . Using inequality (3.24) we obtain

$$- \int_0^L [\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t] dx \leq \frac{2C}{M_E} \mathcal{E}_\ell(t) + \frac{2CLm_F}{M_E}.$$

Noting that M_E does not depend on ℓ , there exists $C_1 > 0$, independent of T and ℓ , such that

$$- \int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t) dx \Big|_0^T \leq C_1 [\mathcal{E}_\ell(T) + \mathcal{E}_\ell(0)] + C_1. \tag{3.53}$$

Step 3. Damping terms estimate: Using Young's inequality we have that

$$\begin{aligned}
& - \int_0^T \int_0^L [g_1(\varphi_t) \varphi + g_2(\psi_t) \psi + g_3(w_t) w] dx dt \\
& \leq \frac{1}{2} \int_0^T E_\ell(t) dt + C \int_0^T \int_0^L [g_1(\varphi_t)^2 + g_2(\psi_t)^2 + g_3(w_t)^2] dx dt.
\end{aligned}$$

From assumption (3.17), for $i = 1, 2, 3$, we get that

$$\int_0^T \int_{|u| \leq 1} g_i(u)^2 dx dt \leq \max\{g(-1)^2, g(1)^2\} L T$$

and

$$\int_0^T \int_{|u| > 1} g_i(u)^2 dx dt \leq M_i \int_0^T \int_0^L g_i(u) u dx dt.$$

Then there exists a constant $C_2 > 0$, independent of T and ℓ , such that

$$\begin{aligned} & - \int_0^T \int_0^L [g_1(\varphi_t)\varphi + g_2(\psi_t)\psi + g_3(w_t)w] dx dt \\ & \leq \frac{1}{2} \int_0^T \mathcal{E}_\ell(t) dt + C_2 \int_0^T \int_0^L [g_1(\varphi_t)\varphi_t + g_2(\psi_t)\psi_t + g_3(w_t)w_t] dx dt + C_2 T. \end{aligned} \quad (3.54)$$

Step 4. Estimates for the kinetic energy: Firstly we note that using (3.17) we have

$$\int_0^T \int_{|\varphi_t|>1} \varphi_t^2 dx dt \leq \frac{1}{m_1} \int_0^T \int_0^L g_1(\varphi_t)\varphi_t dx dt$$

and then

$$\begin{aligned} \int_0^T \int_0^L \varphi_t^2 dx dt &= \int_0^T \int_{|\varphi_t|>1} \varphi_t^2 dx dt + \int_0^T \int_{|\varphi_t|\leq 1} \varphi_t^2 dx dt \\ &\leq \frac{1}{m_1} \int_0^T \int_0^L g_1(\varphi_t)\varphi_t dx dt + TL. \end{aligned}$$

Similar estimate holds for $\iint \psi_t^2$ and $\iint w_t^2$. Therefore there exists a positive constant C_3 , independent of T and ℓ , such that

$$\begin{aligned} & \frac{3}{2} \int_0^T \int_0^L [\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_3 w_t^2] dx dt \\ & \leq C_3 \int_0^T \int_0^L [g_1(\varphi_t)\varphi_t + g_2(\psi_t)\psi_t + g_3(w_t)w_t] dx dt + C_3 T. \end{aligned} \quad (3.55)$$

Step 5. A second energy inequality: Inserting estimates (3.53), (3.54), (3.55) into (3.52) we obtain

$$\begin{aligned} \frac{1}{2} \int_0^T \mathcal{E}_\ell(t) dt &\leq C_1 [\mathcal{E}_\ell(T) + \mathcal{E}_\ell(0)] + C_1 + (C_2 + C_3)T \\ &\quad + (C_2 + C_3) \int_0^T \int_0^L [g_1(\varphi_t)\varphi_t + g_2(\psi_t)\psi_t + g_3(w_t)w_t] dx dt. \end{aligned}$$

Using the energy identity (3.23) and noting that $\mathcal{E}_\ell(T) \leq \mathcal{E}_\ell(t)$ in the left-hand side integral,

$$\frac{T}{2} \mathcal{E}_\ell(T) \leq (C_1 - C_2 - C_3) \mathcal{E}_\ell(T) + (C_1 + C_2 + C_3) \mathcal{E}_\ell(0) + C_1 + (C_2 + C_3)T.$$

Taking T sufficiently ($T > 2C_1$) we can write

$$\mathcal{E}_\ell(T) \leq \gamma_T \mathcal{E}_\ell(0) + K_T, \quad (3.56)$$

where

$$\gamma_T = \frac{2(C_1 + C_2 + C_3)}{T - 2(C_1 - C_2 - C_3)} < 1, \quad K_T = \frac{2C_1 + 2(C_2 + C_3)T}{T - 2(C_1 - C_2 - C_3)} > 0.$$

From (3.56) and well-known argument shows that

$$\mathcal{E}_\ell(t) \leq \gamma \mathcal{E}_\ell(0) e^{-\alpha t} + \frac{K_T}{1 - \gamma_T}, \quad \forall t \geq 0,$$

for some $\alpha, \gamma > 0$. For completeness we sketch its proof here. Indeed, the same argument can be repeated for any interval $[mT, (m+1)T]$, $m \in \mathbb{N}$. Then

$$\begin{aligned} \mathcal{E}_\ell(mT) &\leq \gamma_T \mathcal{E}_\ell((m-1)T) + K_T \\ &\leq \gamma_T^m \mathcal{E}_\ell(0) + \left(\sum_{j=0}^{m-1} \gamma_T^j \right) K_T \\ &\leq \gamma_T^m \mathcal{E}_\ell(0) + \frac{K_T}{1 - \gamma_T}. \quad (\text{since } \gamma_T < 1) \end{aligned}$$

Now given $t \geq 0$, there exists $m \in \mathbb{N}$ and $r \in [0, T)$ such that $t = mT + r$. Then

$$\mathcal{E}_\ell(t) \leq \mathcal{E}_\ell(mT) \leq \gamma_T^m \mathcal{E}_\ell(0) + \frac{K_T}{1 - \gamma_T}.$$

It follows that

$$\mathcal{E}_\ell(t) \leq \gamma_T^{\frac{t-r}{T}} \mathcal{E}_\ell(0) + \frac{K_T}{1 - \gamma_T} \leq \gamma_T^{-1} \gamma_T^{\frac{t}{T}} \mathcal{E}_\ell(0) + \frac{K_T}{1 - \gamma_T}.$$

Therefore choosing $\gamma = \gamma_T^{-1}$ and $\alpha = -\ln(\gamma_T)/T$ we obtain (3.56).

Step 6. Conclusion: We observe that combining (3.56) and (3.24) yields

$$\|S(t)y_0\|_{\mathcal{H}_\ell}^2 \leq \frac{2\gamma}{M_E} \mathcal{E}_\ell(0) e^{-\alpha t} + \frac{2Lm_F K_T}{M_E(1 - \gamma_T)},$$

and then clearly any closed ball $\bar{B}_{\mathcal{H}}(0, R_0)$ with $R_0^2 > \frac{2Lm_F K_T}{M_E(1 - \gamma_T)}$ is a bounded absorbing set, not depending on ℓ . $\square$

3.7 Global attractors II

In this section we assume that damping terms satisfy condition (3.18). Then we show that the global attractor obtained in Theorem 3.3 has further properties.

Theorem 3.4. Under the hypotheses of Theorem 3.3, with (3.17) replaced by (3.18), one has:

- (i) The global attractor $\mathbf{A}_\ell$ has finite fractal dimension.
- (ii) Any full trajectory $(\varphi(t), \psi(t), w(t), \varphi_t(t), \psi_t(t), w_t(t))$ inside the attractor $\mathbf{A}_\ell$, has further regularity

$$\|(\varphi, \psi, w)\|_{[H^2(\Omega)]^3} + \|(\varphi_t, \psi_t, w_t)\|_{[H_0^1(\Omega)]^3} + \|(\varphi_{tt}, \psi_{tt}, w_{tt})\|_{[L^2(\Omega)]^3} \leq C_\ell, \quad (3.57)$$

for some $C_\ell > 0$.

(iii) The dynamical system $(\mathcal{H}, S_\ell(t))$ possesses a generalized exponential attractor $\mathbf{A}_\ell^{exp}$, with finite fractal dimension in a extended space $\widetilde{\mathcal{H}}_{-\eta}$, defined as interpolation of

$$\widetilde{\mathcal{H}}_0 := \mathcal{H} \quad \text{and} \quad \widetilde{\mathcal{H}}_{-1} := [L^2(\Omega)]^3 \times [H^{-1}(\Omega)]^3,$$

for any $\eta \in (0, 1]$.

Remark 3.9. As discussed in Remark 3.6, we can prove an analogous result for the Timoshenko system, that is, the dynamical system $(\mathcal{H}_0, S_0(t))$ generated by (3.4)-(3.6) has a regular global attractor $\mathbf{A}_0$ in $\mathcal{H}_0 = [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^2$, with finite fractal dimension. $\square$

The proof of Theorem 3.4 relies on the properties of quasi-stable systems.

3.7.1 Quasistability

Lemma 3.6. In the context of Lemma 3.2, with (3.17) replaced by (3.18), given a bounded invariant set B , there exist constants $\alpha_B, \gamma_B, C_B > 0$, such that

$$E_\ell(t) \leq \gamma_B E_\ell(0) e^{-\alpha_B t} + C_B \sup_{\sigma \in [0, t]} [\|\tilde{\Phi}(\sigma)\|_{L^{2p}(\Omega)}^2 + \|\tilde{\Psi}(\sigma)\|_{L^{2p}(\Omega)}^2 + \|\tilde{W}(\sigma)\|_{L^{2p}(\Omega)}^2]. \quad (3.58)$$

Proof. We begin as in the proof of Lemma 3.2, since (3.18) implies (3.17). Then we only need estimate the right-hand side of (3.41). Here $C > 0$ will represents several constants independent of B or t .

Step 1. First remarks and new forcing estimate: We observe that estimate (3.42) holds unchanged. We also observe that using the continuous embedding $L^{2p}(\Omega) \subset L^{p+1}(\Omega)$ the estimate (3.45) can be changed to

$$\begin{aligned} & -\frac{1}{2} \int_0^T \int_0^L [F_1(\tilde{\Phi}, \tilde{\Psi}, \tilde{W})\tilde{\Phi} + F_2(\tilde{\Phi}, \tilde{\Psi}, \tilde{W})\tilde{\Psi} + F_3(\tilde{\Phi}, \tilde{\Psi}, \tilde{W})\tilde{W}] dx dt \\ & \leq C_B \int_0^T [\|\tilde{\Phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\Psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{W}\|_{L^{2p}(\Omega)}^2] dt. \end{aligned} \quad (3.59)$$

Step 2. Role of assumption (3.18): Now since (3.20) holds we see that estimate (3.43) becomes

$$\int_0^T \int_0^L [\rho_1 \tilde{\Phi}_t^2 + \rho_2 \tilde{\Psi}_t^2 + \rho_3 \tilde{W}_t^2] dx dt \leq C \int_0^T \int_0^L [G_1(\tilde{\Phi}_t)\tilde{\Phi}_t + G_2(\tilde{\Psi}_t)\tilde{\Psi}_t + G_3(\tilde{W}_t)\tilde{W}_t] dx dt.$$

In addition, (3.18) implies that $|g_i(u) - g_i(v)| \leq M_i |u - v|$ for all $u, v \in \mathbb{R}$. Then

$$\int_0^T \int_0^L G_1(\tilde{\Phi}_t)\tilde{\Phi} dx dt \leq \frac{1}{6} \int_0^T \|\tilde{\Phi}_t\|_{\Omega}^2 dt dx + C \int_0^T \|\tilde{\Phi}\|_{\Omega}^2 dt.$$

Applying the same argument to $\iint G_2(\psi_t)\psi$ and $\iint G_3(w_t)w$ we infer that (3.44) becomes

$$\begin{aligned} & -\frac{1}{2} \int_0^T \int_0^L [G_1(\tilde{\phi}_t)\tilde{\phi} + G_2(\tilde{\psi}_t)\tilde{\psi} + G_3(\tilde{w}_t)\tilde{w}] dx dt \\ & \leq \frac{1}{2} \int_0^T E_\ell(t) dt + C \int_0^T [\|\tilde{\phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{w}\|_{L^{2p}(\Omega)}^2] dt. \end{aligned} \quad (3.60)$$

Step 3. First energy inequality: Using the inequalities (3.58)-(3.60) into (3.41) we obtain that

$$\begin{aligned} \int_0^T E_\ell(t) dt & \leq C[E_\ell(T) + E_\ell(0)] + C_B \int_0^T \int_0^L [G_1(\tilde{\phi}_t)\tilde{\phi} + G_2(\tilde{\psi}_t)\tilde{\psi} + G_3(\tilde{w}_t)\tilde{w}] dx dt \\ & \quad + C_B \int_0^T [\|\tilde{\phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{w}\|_{L^{2p}(\Omega)}^2] dt. \end{aligned} \quad (3.61)$$

Step 4. Damping estimate: The energy identity (3.46) implies that

$$\begin{aligned} & \int_0^T \int_0^L [G_1(\tilde{\phi}_t)\tilde{\phi} + G_2(\tilde{\psi}_t)\tilde{\psi} + G_3(\tilde{w}_t)\tilde{w}] dx dt \\ & = E_\ell(0) - E_\ell(T) - \int_0^T \int_0^L [F_1(\tilde{\phi}, \tilde{\psi}, \tilde{w})\tilde{\phi} + F_2(\tilde{\phi}, \tilde{\psi}, \tilde{w})\tilde{\psi} + F_3(\tilde{\phi}, \tilde{\psi}, \tilde{w})\tilde{w}] dx dt. \end{aligned} \quad (3.62)$$

Let us estimate the forcing terms. Note that, for $\varepsilon > 0$,

$$\begin{aligned} \int_0^L F_1(\tilde{\phi}, \tilde{\psi}, \tilde{w})\tilde{\phi} dx & \leq C(\nabla f_1) [\|\tilde{\phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{w}\|_{L^{2p}(\Omega)}^2] \|\tilde{\phi}\|_\Omega \\ & \leq \frac{\varepsilon}{3T} \|\phi_t\|_\Omega^2 + TC_{\varepsilon,B} [\|\tilde{\phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{w}\|_{L^{2p}(\Omega)}^2]. \end{aligned}$$

where

$$C(\nabla f_1) = C[1 + \|\phi^1\|_{2p}^{p-1} + \|\phi^2\|_{2p}^{p-1} + \|\psi^1\|_{2p}^{p-1} + \|\psi^2\|_{2p}^{p-1} + \|w^1\|_{2p}^{p-1} + \|w^2\|_{2p}^{p-1}].$$

Similar estimate holds for $\int F_2(\phi, \psi, w)\psi_t$ and $\int F_3(\phi, \psi, w)w_t$. Then we obtain

$$\begin{aligned} & \int_0^L [F_1(\tilde{\phi}, \tilde{\psi}, \tilde{w})\tilde{\phi} + F_2(\tilde{\phi}, \tilde{\psi}, \tilde{w})\tilde{\psi} + F_3(\tilde{\phi}, \tilde{\psi}, \tilde{w})\tilde{w}] dx \\ & \leq \frac{\varepsilon}{T} (\|\tilde{\phi}_t\|_\Omega^2 + \|\tilde{\psi}_t\|_\Omega^2 + \|\tilde{w}_t\|_\Omega^2) + TC_{\varepsilon,B} [\|\tilde{\phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{w}\|_{L^{2p}(\Omega)}^2] \end{aligned}$$

and integrating this inequality over $[0, T] \times [s, T]$ we find

$$\begin{aligned} & \int_0^T \int_s^T \int_0^L [F_1(\tilde{\phi}, \tilde{\psi}, \tilde{w})\tilde{\phi} + F_2(\tilde{\phi}, \tilde{\psi}, \tilde{w})\tilde{\psi} + F_3(\tilde{\phi}, \tilde{\psi}, \tilde{w})\tilde{w}] dx dt ds \\ & \leq \varepsilon \int_0^T E_\ell(t) dt + C_{\varepsilon,B,T} \int_0^T [\|\tilde{\phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{w}\|_{L^{2p}(\Omega)}^2] dt. \end{aligned} \quad (3.63)$$

This inequality together with (3.62) results that

$$\begin{aligned} & \int_0^T \int_0^L [G_1(\tilde{\phi}_t)\tilde{\phi} + G_2(\tilde{\psi}_t)\tilde{\psi} + G_3(\tilde{w}_t)\tilde{w}] dx dt \\ & \leq E_\ell(0) - E_\ell(T) + \varepsilon \int_0^T E_\ell(t) dt \\ & \quad + C_{\varepsilon,B,T} \int_0^T [\|\tilde{\phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{w}\|_{L^{2p}(\Omega)}^2] dt. \end{aligned} \quad (3.64)$$

Step 5. Second energy inequality: Applying the damping estimate (3.64) in (3.61) we obtain, for ε small enough,

$$\begin{aligned} \int_0^T E_\ell(t) dt &\leq (C - C_B)E_\ell(T) + (C + C_B)E_\ell(0) \\ &\quad + C_B \int_0^T [\|\tilde{\Phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\Psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{W}\|_{L^{2p}(\Omega)}^2] dt. \end{aligned} \quad (3.65)$$

Step 6. Estimating $E(T)$: Integrating the energy identity (3.46) it follows that

$$\begin{aligned} TE_\ell(T) &= \int_0^T E_\ell(t) dt - \int_0^T \int_s^T \int_0^L [G_1(\tilde{\Phi}_t)\tilde{\Phi} + G_2(\tilde{\Psi}_t)\tilde{\Psi} + G_3(\tilde{W}_t)\tilde{W}] dx dt ds \\ &\quad - \int_0^T \int_s^T \int_0^L [F_1(\tilde{\Phi}, \tilde{\Psi}, \tilde{W})\tilde{\Phi} + F_2(\tilde{\Phi}, \tilde{\Psi}, \tilde{W})\tilde{\Psi} + F_3(\tilde{\Phi}, \tilde{\Psi}, \tilde{W})\tilde{W}] dx dt ds. \end{aligned}$$

Taking into account that $G_1(\tilde{\Phi}_t)\tilde{\Phi} + G_2(\tilde{\Psi}_t)\tilde{\Psi} + G_3(\tilde{W}_t)\tilde{W} \geq 0$ and using the estimate (3.63) we obtain

$$TE_\ell(T) \leq 2 \int_0^T E_\ell(t) dt + C_{B,T} \int_0^T [\|\tilde{\Phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\Psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{W}\|_{L^{2p}(\Omega)}^2] dt. \quad (3.66)$$

Step 7. Conclusion: Inserting (3.65) into (3.66) we obtain

$$\begin{aligned} TE_\ell(T) &\leq 2(C - C_B)E_\ell(T) + 2(C + C_B)E_\ell(0) \\ &\quad + C_{B,T} \int_0^T [\|\tilde{\Phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\Psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{W}\|_{L^{2p}(\Omega)}^2] dt. \end{aligned}$$

Taking $T > 4C$ we can write

$$E_\ell(T) \leq \gamma_T E_\ell(0) + C_{B,T} \sup_{\sigma \in [0, T]} [\|\tilde{\Phi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\Psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{W}\|_{L^{2p}(\Omega)}^2]$$

where

$$\gamma_T = \frac{2(C + C_B)}{T - 2(C - C_B)} < 1.$$

Then a standard argument, similar to one employed in Lemma 3.5, shows that there exists $\gamma_{B,T}, \alpha_{B,T}, C_{B,T} > 0$ such that

$$E_\ell(t) \leq \gamma_{B,T} E_\ell(0) e^{-\alpha_{B,T} t} + C_{B,T} \sup_{\sigma \in [0, t]} [\|\tilde{\Phi}(\sigma)\|_{L^{2p}(\Omega)}^2 + \|\tilde{\Psi}(\sigma)\|_{L^{2p}(\Omega)}^2 + \|\tilde{W}(\sigma)\|_{L^{2p}(\Omega)}^2].$$

Since $T > 0$ is a fixed time-step which depends on B , we can simply write γ_B, α_B, C_B , and therefore (3.58) holds. $\square$

Proof of Theorem 3.4 (Fractal dimension). We begin by observing that from the variation of parameters formula (3.26) we obtain inequality (2.9). On the other hand, Lemma 3.6 implies

that for any bounded positively invariant set B , the condition (2.10) is valid with $X = [H_0^1(0, L)]^3$, $Y = [L^2(0, L)]^3$, $b(t) = \gamma_B e^{-\alpha_B t}$, $c(t) = C_B$, and

$$n_X(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) = \sqrt{\|\tilde{\varphi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{\psi}\|_{L^{2p}(\Omega)}^2 + \|\tilde{w}\|_{L^{2p}(\Omega)}^2},$$

which is compact in X . Then, in particular, $(\mathcal{H}, S_\ell(t))$ is quasi-stable on the attractor $\mathbf{A}_\ell$. Therefore this attractor has finite fractal dimension from Theorem 2.16. $\square$

Proof of Theorem 3.4 (Regularity). Since we know that the system is quasi-stable, Theorem 2.17, implies that any full trajectory $(\varphi(t), \psi(t), w(t), \varphi_t(t), \psi_t(t), w_t(t))$ inside the attractor has regularity

$$\varphi_t, \psi_t, w_t \in L^\infty(\mathbb{R}, H_0^1(0, L)) \cap C(\mathbb{R}, L^2(0, L)) \text{ and } \varphi_{tt}, \psi_{tt}, w_{tt} \in L^\infty(\mathbb{R}, L^2(0, L)).$$

Now, by continuity of the nonlinear terms, we have

$$\begin{aligned} \varphi_{xx} &= 1/k [\rho_1 \varphi_{tt} - k(\psi + \ell w)_x - k_0 \ell (w_x - \ell \varphi) + g_1(\varphi_t) + f_1(\varphi, \psi, w)] \in L^\infty(\mathbb{R}; L^2(0, L)), \\ \psi_{xx} &= 1/b [\rho_2 \psi_{tt} + k(\varphi_x + \psi + \ell w) + g_2(\psi_t) + f_2(\varphi, \psi, w)] \in L^\infty(\mathbb{R}; L^2(0, L)), \\ w_{xx} &= 1/k_0 [\rho_1 w_{tt} + k_0 \ell \varphi_x + k \ell (\varphi_x + \psi + \ell w) + g_3(w_t) + f_3(\varphi, \psi, w)] \in L^\infty(\mathbb{R}; L^2(0, L)). \end{aligned}$$

Then, elliptic regularity implies that $\varphi, \psi, w \in L^\infty(\mathbb{R}, H^2(0, L) \cap H_0^1(0, L))$ and therefore estimate (3.57) is verified. $\square$

Proof of Theorem 3.4 (Exponential attractors). Let $\mathcal{B}$ be the bounded absorbing set of $(\mathcal{H}, S_\ell(t))$ given by Lemma 3.5. Then for $T > 0$ and $y = (\varphi, \psi, w, \tilde{\varphi}, \tilde{\psi}, \tilde{w}) \in \mathcal{B}$, there exists $C_{\mathcal{B}} > 0$ such that

$$\|S_\ell(t)y\|_{\mathcal{H}} \leq C_{\mathcal{B}}, \quad 0 \leq t \leq T.$$

Using this in (3.1₁)-(3.1₃) we obtain $(\varphi_{tt}, \psi_{tt}, w_{tt}) \in [H^{-1}(0, L)]^3$. Taking a larger $C_{\mathcal{B}}$ if necessary, we have

$$\left\| \frac{d}{dt} S_\ell(t) \right\|_{\widetilde{\mathcal{H}}_{-1}} \leq C_{\mathcal{B}}, \quad 0 \leq t \leq T.$$

Consequently we obtain

$$\|S_\ell(t_1)y - S_\ell(t_2)y\|_{\widetilde{\mathcal{H}}_{-1}} \leq \int_{t_1}^{t_2} \left\| \frac{d}{dt} S_\ell(t) \right\|_{\widetilde{\mathcal{H}}_{-1}} dt \leq C_{\mathcal{B}} |t_1 - t_2|, \quad 0 \leq t_1 < t_2 \leq T. \quad (3.67)$$

That is, the map $t \mapsto S_\ell(t)y$ is Hölder continuous from $[0, T]$ to $\widetilde{\mathcal{H}}_{-1}$ (with exponent 1). Therefore Theorem 2.18 implies the existence of a generalized exponential attractor $\mathbf{A}_\ell^{exp}$ whose fractal dimension is finite in $\widetilde{\mathcal{H}}_{-1}$.

We can choose smaller extended spaces. Indeed, since $\widetilde{\mathcal{H}}_0 \subset \widetilde{\mathcal{H}}_{-1}$ continuously, given $\eta \in (0, 1)$, the interpolation theorem implies that

$$\|y\|_{\widetilde{\mathcal{H}}_{-\eta}} \leq C \|y\|_{\widetilde{\mathcal{H}}_0}^{1-\eta} \|y\|_{\widetilde{\mathcal{H}}_{-1}}^\eta \leq C_{\mathcal{B}}^{1-\eta} \|y\|_{\widetilde{\mathcal{H}}_{-1}}^\eta.$$

In particular,

$$\|S_\ell(t_1)y - S_\ell(t_2)y\|_{\widetilde{\mathcal{H}}_{-\eta}} \leq C_{\mathcal{B}}^{1-\eta} \|S_\ell(t_1)y - S_\ell(t_2)y\|_{\widetilde{\mathcal{H}}_{-1}}^\eta.$$

Then, combining this with (3.67) we find that

$$\|S_\ell(t_1)y - S_\ell(t_2)y\|_{\widetilde{\mathcal{H}}_{-\eta}} \leq C_{\mathcal{B}} |t_1 - t_2|^\eta, \quad 0 \leq t_1 < t_2 \leq T.$$

This shows that $t \mapsto S_\ell(t)y$ is Hölder continuous in space $\widetilde{\mathcal{H}}_{-\eta}$. Then the existence of a generalized exponential attractor, with finite fractal dimension in $\widetilde{\mathcal{H}}_{-\eta}$, follows from Theorem 2.18. $\square$

3.8 Upper-semicontinuity

Our last result is concerned with the convergence of attractors of the Bresse system $(\mathbf{A}_\ell)$ to that of the Timoshenko system $(\mathbf{A}_0)$.

Theorem 3.5. Under the hypotheses of Theorem 3.4, assume further that f_i satisfy condition (3.29). Then the attractor $\mathbf{A}_\ell$ is upper-semicontinuous with respect to $\ell \rightarrow 0$, in the sense that,

$$\lim_{\ell \rightarrow 0} d^{\mathcal{H}_0}(\mathcal{P}\mathbf{A}_\ell, \mathbf{A}_0) = 0, \quad (3.68)$$

where $d^{\mathcal{H}_0}$ denotes Hausdorff semi-distance, and $\mathcal{P} : \mathcal{H} \rightarrow \mathcal{H}_0$ is the projection map defined by $\mathcal{P}(\varphi, \psi, w, \tilde{\varphi}, \tilde{\psi}, \tilde{w}) = (\varphi, \psi, \tilde{\varphi}, \tilde{\psi})$.

Proof. The proof is based on the arguments in [28] and also in [25]. Suppose by contradiction that the statement (3.68) is false. Then there exists $\varepsilon > 0$ and a sequence $\ell_n \rightarrow 0$ such that

$$\sup_{y \in \mathbf{A}_{\ell_n}} \inf_{z \in \mathbf{A}_0} \|\mathcal{P}y - z\|_{\mathcal{H}_0} \geq \varepsilon, \quad \forall n.$$

Since $\mathbf{A}_{\ell_n}$ and $\mathbf{A}_0$ are compact sets, there exist $y_0^n \in \mathbf{A}_{\ell_n}$ such that

$$\inf_{z \in \mathbf{A}_0} \|\mathcal{P}y_0^n - z\|_{\mathcal{H}_0} \geq \varepsilon, \quad \forall n. \quad (3.69)$$

Let $y^n(t)$ the full trajectory in $\mathbf{A}_{\ell_n}$ defined by

$$y^n(t) = (\varphi^n(t), \psi^n(t), w^n(t), \varphi_t^n(t), \psi_t^n(t), w_t^n(t)), \quad y^n(0) = y_0^n.$$

We can assume $\ell_n \in (0, \pi/2L)$, and then the absorbing ball $\mathcal{B} = \overline{B}(0, R_0)$ given by Lemma 3.5 is independent of ℓ_n . Then

$$\|(\varphi^n(t), \psi^n(t), w^n(t))\|_{[H_0^1(0,L)]^3}^2 + \|(\varphi_t^n(t), \psi_t^n(t), w_t^n(t))\|_{[L^2(0,L)]^3}^2 \leq R_0^2, \quad \forall t, n. \quad (3.70)$$

In addition, since (3.9) and (3.24) are now independent of ℓ_n , we see that coefficients in (3.58) do not depend on ℓ_n . Then (2.11) asserts that there is $R_1 > 0$ such that

$$\|(\varphi_t^n(t), \psi_t^n(t), w_t^n(t))\|_{[H_0^1(0,L)]^3}^2 + \|(\varphi_{tt}^n(t), \psi_{tt}^n(t), w_{tt}^n(t))\|_{[L^2(0,L)]^3}^2 \leq R_1^2, \quad \forall t, n. \quad (3.71)$$

As in the proof Theorem 3.4 (regularity), using elliptic regularity combined with (3.70)-(3.71), we obtain $R_2 > 0$ such that

$$\|(\varphi^n(t), \psi^n(t), w^n(t))\|_{[H^2(0,L)]^3}^2 \leq R_2^2, \quad \forall t, n.$$

Consequently,

$$\begin{aligned} \{y^n\} &\text{ is bounded in } L^\infty(\mathbb{R}, [H^2(0,L)]^3 \times [H^1(0,L)]^3), \\ \{y_t^n\} &\text{ is bounded in } L^\infty(\mathbb{R}, \mathcal{H}), \end{aligned}$$

and for every $T > 0$, we have

$$\{y^n\} \text{ is precompact in } C([-T, T], \mathcal{H}).$$

From this, there exists a subsequence $\{y^{n_k}\}$ and $y \in C([-T, T], \mathcal{H})$ such that

$$\lim_{k \rightarrow \infty} \sup_{t \in [-T, T]} \|y^{n_k}(t) - y(t)\|_{\mathcal{H}} = 0.$$

In particular, denoting $\tilde{z} = \mathcal{P}y$, we have

$$\lim_{k \rightarrow \infty} \|\mathcal{P}y_0^{n_k} - \tilde{z}(0)\|_{\mathcal{H}_0} \rightarrow 0. \quad (3.72)$$

Let us show that $\tilde{z}(0) \in \mathbf{A}_0$. Indeed, the same argument used in Theorem 3.2 proves that $\tilde{z} = \mathcal{P}y$ is a solution of the Timoshenko system (3.4)-(3.6) in $[-T, T]$. Since $T > 0$ is arbitrary, it follows from (3.70) that $\tilde{z}(t)$ is a bounded full trajectory for the Timoshenko system and thus $\tilde{z}(0) \in \mathbf{A}_0$. Therefore, (3.72) contradicts (3.69). This completes the proof of the Theorem 3.5. $\square$

THERMOELASTIC FULL VON KARMAN SYSTEM

4.1 Introduction

In this chapter we will study the full Von Karman system combined with thermal effects affecting both displacements and weakly dissipative free boundary conditions. Here $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain with two nonintersecting portions of the boundary Γ_0 and Γ_1 such that $\Gamma_0 \cup \Gamma_1 = \partial\Omega := \Gamma$. The displacement equations are given by the following system

$$\begin{cases} u_{tt} - \operatorname{div}\{\sigma[\varepsilon(u) + f(\nabla w)]\} + \nabla\phi + p_1(u, w) = 0 & \text{in } \Omega \times (0, \infty), \\ w_{tt} + \Delta^2 w - \operatorname{div}\{\sigma[\varepsilon(u) + f(\nabla w)]\nabla w + \phi\nabla w\} + \Delta\theta + p_2(u, w) = 0 & \text{in } \Omega \times (0, \infty), \end{cases} \quad (4.1)$$

with Dirichlet boundary conditions on the portion of the boundary Γ_0 ,

$$u = 0, w = 0 \text{ and } \nabla w = 0 \text{ on } \Gamma_0 \times (0, \infty). \quad (4.2)$$

The boundary conditions on the portion Γ_1 are of *free* type and given by

$$\begin{cases} \sigma[\varepsilon(u) + f(\nabla w)]\nu + \kappa u - \phi\nu + u_t = 0, \\ [\Delta w + (1 - \mu)B_1 w] + \theta = 0, \\ \left[\frac{\partial}{\partial\nu} \Delta w + (1 - \mu)B_2 w \right] - \sigma[\varepsilon(u) + f(\nabla w)]\nu \cdot \nabla w - \phi \frac{\partial w}{\partial\nu} + \frac{\partial\theta}{\partial\nu} = 0, \end{cases} \quad (4.3)$$

where $\sigma[\cdot], \varepsilon(\cdot), f(\cdot)$ are given in (1.6), (1.7), (1.8), respectively and κ is a positive constant. The boundary operators B_1, B_2 are defined by

$$\begin{aligned} B_1 w &= 2\nu_1\nu_2 \frac{\partial^2 w}{\partial x \partial y} - \nu_1^2 \frac{\partial^2 w}{\partial y^2} - \nu_2^2 \frac{\partial^2 w}{\partial x^2}, \\ B_2 w &= \frac{\partial}{\partial\tau} \left[(\nu_1^2 - \nu_2^2) \frac{\partial^2 w}{\partial x \partial y} - \nu_1\nu_2 \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \right) \right] - lw, \end{aligned}$$

and the vectors $\nu = (\nu_1, \nu_2)$ and $\tau = (\tau_1, \tau_2)$ represent normal and tangential directions to the boundary Γ_1 , and l is a positive parameter.

The averaged of thermal stress (ϕ) and thermal moment (θ) are given by the following system of equations

$$\begin{cases} \phi_t - \Delta\phi + \operatorname{div}\{u_t\} - \nabla w \cdot \nabla w_t = 0 & \text{in } \Omega \times (0, \infty), \\ \theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } \Omega \times (0, \infty), \end{cases} \quad (4.4)$$

with Robin boundary conditions

$$\frac{\partial \phi}{\partial \nu} + \lambda_1 \phi = \frac{\partial \theta}{\partial \nu} + \lambda_2 \theta = 0 \quad \text{on } \Gamma \times (0, \infty), \quad (4.5)$$

where $\lambda_1, \lambda_2 > 0$. The system is subject to a nonlinear source term $p(u) = (p_1(u), p_2(u))$ and initial conditions

$$u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, w(\cdot, 0) = w_0, w_t(\cdot, 0) = w_1, \phi(\cdot, 0) = \phi_0, \theta(\cdot, 0) = \theta_0. \quad (4.6)$$

4.2 Goals and Plans of this Chapter

The goal of the chapter is to establish existence of global attractor which captures asymptotic behavior of the dynamics described by the problem (4.1)-(4.6). In addition, we shall prove that such attractor is both finite dimensional and smooth.

This chapter is organized as follows. In Section 4.3, we give some remarks and extra references on the model. In Section 4.4 we presented the notations, assumptions and some energy estimates. In Section 4.5 we formulated our main results. The Section 4.6 contains some additional background pertinent to this problem. The Section 4.7 is devoted to the proof of the wellposedness/regularity result of the model. Finally, in Section 4.8 we show the proof of the global attractor result.

4.3 Comments and Remarks on the Model

The system described by (4.1)-(4.6) involves strongly and nonlinearly coupled thermoelastic plate with thermoelastic waves. Since thermoelastic plates are associated with analytic semigroups [43, 44, 48], we are faced with a combination of parabolic and hyperbolic like dynamics. The nonlinear effects are supercritical level (this means that the nonlinear terms for finite energy solutions are not bounded in a finite energy space). Indeed, finite energy displacements $u \in H^1(\Omega)$, $w \in H^2(\Omega)$ produce nonlinear stresses

$$\operatorname{div}\{f(\nabla w)\} \in H^{-\varepsilon}(\Omega) \quad \text{and} \quad \operatorname{div}\{\sigma[\varepsilon(u) + f(\nabla w)]\nabla w\} \in H^{-1-\varepsilon}(\Omega), \quad \varepsilon > 0.$$

Thus we are dealing with a loss of $1 + \varepsilon$, ε small, derivative. This feature becomes a major difficulty in the study of Hadamard wellposedness (uniqueness and continuous dependence on the initial data) and, above all, in obtaining the needed compactness for the existence of attractors. While parabolic like structure is typically equipped with additional regularity properties, the challenge in the problem is the “transfer” of these beneficial effects to the hyperbolic part of the system. The carriers of propagation in the case of *free* boundary conditions are boundary traces. Thus, at the technical level, we will be concerned with “hidden” trace regularity properties which will play a role of propagators of both regularity and stability.

There is a strong belief and experimental evidence that thermal dissipation should provide substantial damping mechanism for the oscillations so that there may be no need for mechanical dissipation. In fact, such property has been proved in a special case of a scalar linear plate equation with hinged boundary conditions [33] or in one-dimensional configuration such as thermoelastic rods [29]. However, in the case of *free* boundary conditions stabilization results in [35, 36] do require mechanical dissipation imposed on the boundary of the plate. Only several years later it was shown that in the case of linear thermoelastic plates, uniform decay to zero of the energy can be achieved without any mechanical dissipation, regardless of the boundary conditions [6, 7, 43, 44]. The situation is very different when one considers thermoelastic waves. Here no longer one has smoothing property of the dynamics or uniform decay to zero of the energy. The best one can achieve without additional mechanical dissipation is strong stability to zero with a polynomial rate [21, 31]. Thus the problem considered in this work falls into a category of mixed (parabolic-hyperbolic) dynamics with thermal plates and waves which are nonlinear and strongly coupled. Our aim is to show that nonlinear coupling, while making estimates challenging (due to severe unboundedness of nonlinear terms), does provide beneficial mechanism in transferring thermal dissipation onto the entire system thus forging desired long-time behavior. The final result is that the dynamics becomes asymptotically finite dimensional and smooth. While this kind of result is to be expected for the dynamics which is fully analytic with an overall smoothing effect, it is much less expected in hyperbolic type of models without strong mechanical dissipation and with highly unbounded nonlinear effects. Finally we achieved our result through combining the thermal damping mechanism with a boundary frictional damping (u_t) applied only in plane displacements and without any mechanical damping imposed on vertical displacements. The necessity of some mechanical damping imposed on in plane displacements results from well known negative results showing the lack of uniform stability in thermal linear waves whenever the dimension of the domain is greeter than one [21].

4.4 Notations, Assumptions and Energy Relations

The aim of this section is to present the notations, the needed assumptions as well some remarks on the energy.

4.4.1 Notations and Assumptions

Before presenting our assumptions we fix some notations for the norms of standard $H^s(\mathcal{O})$ (Sobolev) and $L^2(\mathcal{O})$ spaces. We use

$$\|u\|_{\alpha,\Omega} = \|u\|_{H^\alpha(\Omega)}, \quad \|u\|_{\alpha,\Gamma} = \|u\|_{H^\alpha(\Gamma)},$$

and the case $\alpha = 0$, which corresponds to L^2 spaces,

$$\|u\|_{\Omega} = \|u\|_{L^2(\Omega)}, \quad \|u\|_{\Gamma} = \|u\|_{L^2(\Gamma)}.$$

The corresponding inner-products are denoted by

$$(u, v)_{\Omega} = (u, v)_{L^2(\Omega)} \quad \text{and} \quad \langle u, v \rangle_{\Gamma} = \langle u, v \rangle_{L^2(\Gamma)}.$$

For $\alpha > 0$, the space $H_0^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, and $H^{-\alpha}(\Omega) = [H_0^\alpha(\Omega)]'$, where the duality is taken with respect to $L^2(\Omega)$ inner-product. Occasionally, by the same symbol, we denote norms and inner-products of n -copies of $L^2(\mathcal{O})$, where $\mathcal{O}$ is either Ω or Γ . The same is applied to $H^\alpha(\mathcal{O})$. We also consider the Sobolev space

$$H_{\Gamma_0}^1(\Omega) = \{v \in H^1(\Omega) \mid \text{trace of } v = 0 \text{ on } \Gamma_0\}, \quad \|v\|_{H_{\Gamma_0}^1(\Omega)} = \|\nabla v\|_{\Omega}.$$

In this chapter the assumptions that we make on the forcing terms are the same as in Chapter 3, that is, p_1, p_2 is locally Lipschitz and gradient type.

More precisely, let us fix the notation

$$u = (u_1, u_2) \quad \text{and} \quad p_1(u, w) = (p_{1,1}(u, w), p_{1,2}(u, w)).$$

Assume there exists a $C^2(\mathbb{R}^3)$ function $P : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that

$$\nabla P(u, w) = (p_{1,1}(u, w), p_{1,2}(u, w), p_2(u, w)), \quad (4.7)$$

satisfying the following conditions: there exist $M, m_P \geq 0$ such that

$$P(u, w) \geq -M(|u_1|^2 + |u_2|^2 + |w|^2) - m_P, \quad \forall u_1, u_2, w \in \mathbb{R}, \quad (4.8)$$

where

$$0 \leq M < M_0, \quad (4.9)$$

where M_0 is a positive constant to be defined in (4.19), depends on the definition tensor $\sigma[\cdot]$ and on the Korn's and Sobolev's inequalities. We also assume there exist $r \geq 1$ and $M_p > 0$ such that, for $i = 1, 2$,

$$|\nabla p_{1,i}(u, w)| \leq M_p(1 + |u_1|^{r-1} + |u_2|^{r-1} + |w|^{r-1}), \quad \forall u_1, u_2, w \in \mathbb{R}, \quad (4.10)$$

$$|\nabla p_2(u, w)| \leq M_P(1 + |u_1|^{r-1} + |u_2|^{r-1} + |w|^{r-1}), \quad \forall u_1, u_2, w \in \mathbb{R}. \quad (4.11)$$

Furthermore, we assume that, for all $u_1, u_2, w \in \mathbb{R}$,

$$\nabla P(v, w) \cdot (v, w) - P(v, w) \geq -M(|u_1|^2 + |u_2|^2 + |w|^2) - m_P. \quad (4.12)$$

We note that (4.10) implies that there exists $M_P > 0$ such that

$$P(u, w) \leq M_P(1 + |u_1|^{r+1} + |u_2|^{r+1} + |w|^{r+1}), \quad \forall u_1, u_2, w \in \mathbb{R}. \quad (4.13)$$

4.4.2 Energy relations

In what follows we make some remarks on the energy of the system. Along a solution $(u, u_t, w, w_t, \phi, \theta)$, the energy of the system is defined by

$$\mathcal{E}(t) = E(t) + \int_{\Omega} P(u, w) \, d\Omega, \quad (4.14)$$

where $E(t) = E_k(t) + E_p(t)$. Here, E_k is the kinetic energy defined by

$$E_k(t) = \frac{1}{2} \int_{\Omega} [|u_t|^2 + |w_t|^2] \, d\Omega,$$

and E_p is the potential energy defined by

$$E_p(t) = \frac{1}{2} \int_{\Omega} [\sigma[N(u, w)]N(u, w) + |\phi|^2 + |\theta|^2] \, d\Omega + \frac{1}{2} a(w, w) + \frac{\kappa}{2} \int_{\Gamma_1} |u|^2 \, d\Gamma_1,$$

where the resultant stress $N(u, w)$ is given by

$$N(u, w) = \varepsilon(u) + f(\nabla w),$$

and

$$a(w, z) = \int_{\Omega} [w_{xx}z_{xx} + w_{yy}z_{yy} + \mu w_{xx}z_{yy} + \mu w_{yy}z_{xx} + 2(1 - \mu)w_{xy}z_{xy}] \, d\Omega + l \int_{\Gamma_1} wz \, d\Gamma_1.$$

It follows that the energy satisfies the identity

$$\mathcal{E}(t) + \int_s^t [\|u_t(\tau)\|_{\Gamma_1}^2 + \|\nabla \phi(\tau)\|_{\Omega}^2 + \|\nabla \theta(\tau)\|_{\Omega}^2 + \lambda_1 \|\phi(\tau)\|_{\Gamma}^2 + \lambda_2 \|\theta(\tau)\|_{\Gamma}^2] \, d\tau = \mathcal{E}(s) \quad (4.15)$$

For strong solutions, the proof of (4.15) is standard and follows from classical energy type arguments. Due to the uniqueness of weak solutions, by using density arguments, it can be extended to finite energy solutions.

Next we establish a relation between $\mathcal{E}(\cdot)$ and $E(\cdot)$. To this end, we note that for $u \in H^1(\Omega)$, the Korn inequality (See [50, Theorem 2.1]) together with Sobolev's embedding lead to

$$\|u\|_{1,\Omega}^2 \leq M_K [\|N(u, v)\|_{\Omega}^2 + \|w\|_{W^{1,4}(\Omega)}^4]. \quad (4.16)$$

Also, the definition of tensor $\sigma[\cdot]$ (1.6) implies that

$$\int_{\Omega} \sigma[N(u, w)]N(u, w) \, d\Omega \geq M_{\sigma} [\|N(u, v)\|_{\Omega}^2 + \|w\|_{W^{1,4}(\Omega)}^4]. \quad (4.17)$$

Then we have the following estimate.

Lemma 4.1. There exists a constant $M_E > 0$ such that

$$\mathcal{E}(t) \geq M_E E(t) - m_P |\Omega|, \quad \forall t \geq 0. \quad (4.18)$$

Proof. As mentioned above, we work firstly with the energy along a strong solution and then use density arguments to extend the result to finite energy solutions. Let us define

$$M_0 = \min \left\{ \frac{M_\sigma}{4M_p M_K}, \frac{M_a}{4M_p} \right\}, \quad (4.19)$$

with $M_p = \max\{M_1, M_2\} > 0$, where M_1, M_2 denote the embedding constants

$$\|u\|_\Omega^2 \leq M_1 \|u\|_{1,\Omega}^2, \quad \|w\|_\Omega^2 \leq M_2 \|w\|_{2,\Omega}^2,$$

and $M_a > 0$ is a constant such that $a(w, w) \geq M_a \|w\|_{2,\Omega}^2$. Then, from inequalities (4.16)-(4.17) and (4.9), we obtain

$$\begin{aligned} \mathcal{E}(t) &= E_k(t) + \frac{1}{2} E_p(t) + \frac{1}{4} \int_\Omega [\sigma[N(u, w)]N(u, w) + |\phi|^2 + |\theta|^2] d\Omega \\ &\quad + \frac{1}{4} a(w, w) + \frac{\kappa}{4} \int_{\Gamma_1} |u|^2 d\Gamma_1 + \int_\Omega P(u, w) d\Omega \\ &\geq CE(t) + \frac{M_\sigma}{4} [\|N(u, v)\|_\Omega^2 + \|w\|_{W^{1,4}(\Omega)}^4] + \frac{M_a}{4} \|w\|_{2,\Omega}^2 - M [\|u\|_\Omega^2 + \|w\|_\Omega^2] - m_P |\Omega| \\ &\geq CE(t) + \frac{M_\sigma}{4} [\|N(u, v)\|_\Omega^2 + \|w\|_{W^{1,4}(\Omega)}^4] + \frac{M_a}{4} \|w\|_{2,\Omega}^2 - MM_p [\|u\|_{1,\Omega}^2 + \|w\|_{2,\Omega}^2] - m_P |\Omega| \\ &\geq CE(t) + \left[\frac{M_\sigma}{4} - MM_p M_K \right] [\|N(u, v)\|_\Omega^2 + \|w\|_{W^{1,4}(\Omega)}^4] + \left[\frac{M_a}{4} - MM_p \right] \|w\|_{2,\Omega}^2 - m_P |\Omega| \\ &\geq CE(t) - m_P |\Omega|. \end{aligned}$$

Therefore (4.18) holds. $\square$

Remark 4.1. We note that the potential energy $E_p(\cdot)$ is topologically equivalent to space $H^2(\Omega) \times [H^1(\Omega)]^2 \times [L^2(\Omega)]^2$ and therefore $E(\cdot)$ is topologically equivalent to the space $\mathcal{H}$. Consequently, since we know that $E(\cdot)$ decays to zero [42], the forcing terms $p_1(\cdot), p_2(\cdot)$ plays an essential role for the existence of a nontrivial global attractor.

4.5 Main Results

Before we state the main results we begin this section introducing some spaces. Then we give the wellposedness result and with additional geometric assumptions on the portion Γ_0 we present the global attractor result.

The analysis of for weak and regular solutions of our system will be done on the (phase) spaces

$$\mathcal{H} = [H^1(\Omega)]^2 \times [L^2(\Omega)]^2 \times H^2(\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2$$

and

$$\mathcal{H}_1 = [H^2(\Omega)]^2 \times [H^1(\Omega)]^2 \times H^4(\Omega) \times H^2(\Omega) \times [H^2(\Omega)]^2,$$

respectively.

The wellposedness/regularity is given by the next Theorem and its proof is given in Section 4.7.

Theorem 4.1. Assume that the source term satisfy (4.7)-(4.12). Then:

(i) **Weak solutions.** For any $T > 0$ and initial data $(u_0, u_1, w_0, w_1, \phi_0, \theta_0) \in \mathcal{H}$, problem (4.1)-(4.6) has a unique weak (finite energy) solution

$$(u, u_t, w, w_t, \phi, \theta) \in C([0, T]; [H^1(\Omega)]^2 \times [L^2(\Omega)]^2 \times H^2(\Omega) \times L^2(\Omega) \times [H^1(\Omega)]^2).$$

Moreover, this solution depends continuously on the initial data.

(ii) **Regular solutions.** Assume that above initial data has further regularity

$$(u_0, u_1, w_0, w_1, \phi_0, \theta_0) \in [H^2(\Omega)]^2 \times [H_{\Gamma_0}^1(\Omega)]^2 \times H^4(\Omega) \times H^2(\Omega) \times [H^2(\Omega)]^2.$$

Then problem (4.1)-(4.6) has a unique regular solution

$$(u, u_t, w, w_t, \phi, \theta) \in C([0, T]; \mathcal{H}_1) \text{ with } (\phi_t, \theta_t) \in C([0, T]; [L^2(\Omega)]^2).$$

Remark 4.2. The above wellposedness result shows that the (weak) solution operator of problem (4.1)-(4.6) is a strongly continuous semigroup $S(t)$ on $\mathcal{H}$, which generates a well-defined nonlinear dynamical system $(\mathcal{H}, S(t))$. $\square$

To establish the existence of attractors we need the following geometric condition on the boundary Γ_0 . There exists $x_0 \in \mathbb{R}^2$ such that

$$(x - x_0) \cdot \nu \leq 0, \quad x \in \Gamma_0. \quad (4.20)$$

Then the global attractor result reads as follows.

Theorem 4.2. Assume that conditions (4.7)-(4.12) and (4.20) are in force. Then the dynamical system $(\mathcal{H}, S(t))$ generated by the problem (4.1)-(4.6) admits a compact global attractor $\mathbf{A} \subset \mathcal{H}$ with finite fractal dimension. Moreover it is bounded in the more regular space $\mathcal{H}_1$.

The proof of Theorem 4.2 will be given in Section 4.8.

Remark 4.3. By assuming additional regularity on the source p one can reiterate the proof of Theorem 4.2 in order to obtain C^∞ dynamics on the attractor $\mathbf{A}$. See for instance [19, 26]. $\square$

4.6 Additional Background

In this section we collected a number of important results such as some tensor identities, a result on trace estimates and analytic semigroup properties.

4.6.0.1 Tensor identities

In order to simplify the verification of some rather long calculations, we provide a few elementary tensor identities. Let us define the vector field

$$h(x) = x - x_0 \text{ with } x_0 \in \mathbb{R}^2.$$

Then we have that

$$\varepsilon(h\nabla u) = \varepsilon(u) + \mathcal{R}, \quad (4.21)$$

where $\mathcal{R}$ is a tensor given by

$$\mathcal{R} = \begin{bmatrix} \sum_{i=1}^2 \frac{\partial^2 u_1}{\partial x_1 \partial x_i} h_i & \frac{1}{2} \sum_{i=1}^2 \left[\frac{\partial^2 u_1}{\partial x_i \partial x_2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_i} \right] h_i \\ \frac{1}{2} \sum_{i=1}^2 \left[\frac{\partial^2 u_1}{\partial x_i \partial x_2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_i} \right] h_i & \sum_{i=1}^2 \frac{\partial^2 u_2}{\partial x_i \partial x_2} h_i \end{bmatrix}.$$

Given two (fourth order) tensors A, B , written as 4-vectors, we define $(A, B) = (A, B)_{\mathbb{R}^4}$. Then, if $A = [a_{i,j}]$ is a symmetric tensor, we can show that

$$(A, \mathcal{R}) = \sum_{i,j,k=1}^2 \left(a_{k,j}, \frac{\partial^2 u_j}{\partial x_k \partial x_i} h_i \right). \quad (4.22)$$

Let $B = [b_{i,j}]$ be another symmetric tensor such that

$$a_{j,i} = \sum_{l=1}^2 c_{j,l} b_{l,i},$$

with constant and symmetric coefficients $c_{j,i}$. Then

$$\begin{aligned} \operatorname{div}\{(A, B)h\} &= (A, B) \operatorname{div}\{h\} + \sum_{i,j,k,l=1}^2 c_{i,l} \frac{\partial}{\partial x_k} [b_{l,j} b_{j,i}] h_k \\ &= (A, B) \operatorname{div}\{h\} + 2 \sum_{i,j,k=1}^2 \left(a_{j,i}, \frac{\partial b_{j,i}}{\partial x_k} h_k \right). \end{aligned}$$

Taking $A = \sigma[\varepsilon(u)]$ and $B = \varepsilon(u)$, we obtain

$$\operatorname{div}\{(\sigma[\varepsilon(u)], \varepsilon(u))_{\Omega} h\} = 2(\sigma[\varepsilon(u)], \varepsilon(u))_{\Omega} + 2 \sum_{i,j,k=1}^2 \left(a_{i,j}, \frac{\partial^2 u_i}{\partial x_k \partial x_j} h_k \right)_{\Omega}. \quad (4.23)$$

Now, taking $A = \sigma[\varepsilon(u)]$ and using identities (4.21) and (4.22), we obtain

$$(\sigma[\varepsilon(u)], \varepsilon(h\nabla u))_{\Omega} = (\sigma[\varepsilon(u)], \varepsilon(u))_{\Omega} + \sum_{i,j,k=1}^2 \left(a_{i,j}, \frac{\partial^2 u_i}{\partial x_k \partial x_j} h_k \right)_{\Omega}. \quad (4.24)$$

4.6.0.2 Trace Regularity for the Linear Elasticity System

Let Ω a n -dimensional bounded domain with smooth boundary Γ and consider the system of linear elasticity given by

$$\begin{cases} u_{tt} - \operatorname{div}\{\sigma[\varepsilon(u)]\} = F & \text{in } \Omega \times (0, \infty), \\ \sigma[\varepsilon(u)]\nu = g & \text{on } \Gamma \times (0, \infty). \end{cases} \quad (4.25)$$

Corresponding to this problem, we have the following trace regularity result. See [30, Theorem 1.2]

Theorem 4.3. Let $u = u(x, t)$ the solution to the system (4.25) and $\alpha \in (0, \frac{T}{2})$. Then the following estimate holds for $\varepsilon \in (0, \frac{1}{2})$

$$\int_{\alpha}^{T-\alpha} \|\nabla u \tau\|_{\Gamma}^2 dt \leq C_{\alpha} \int_0^T \left[\|u_t\|_{\Gamma}^2 + \|F\|_{-\frac{1}{2}, \Omega}^2 + \|\sigma[\varepsilon(u)]\nu\|_{\Gamma}^2 + \|u\|_{\frac{1}{2}+\varepsilon, \Omega}^2 \right] dt.$$

4.6.0.3 Properties of Analytic Semigroups

In this section we will present the definition and some properties of analytic semigroups. In addition we introduce the analyticity result of dynamics associated with thermoelastic plates in the non-rotational case.

Definition 4.1. Define $\Lambda = \{z \in \mathbb{C} \mid \varphi_1 < \arg\{z\} < \varphi_2 \text{ and } \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Lambda$ let $S(z)$ be a bounded linear operator. The family $\{S(z) \mid z \in \Lambda\}$ is an **analytic** semigroup in Λ if

- $z \rightarrow S(z)$ is analytic in Λ ;
- $S(0) = I_X$ and $\lim_{z \rightarrow 0, z \in \Lambda} S(z)x = x$, for every $x \in X$;
- $S(z_1 + z_2) = S(z_1)S(z_2)$, for every $z_1, z_2 \in \Lambda$.

Definition 4.2. A semigroup $\{S(t)\}_{t \geq 0}$ will be called **analytic** if it is analytic in some sector Λ containing the nonnegative real axis.

About operators for which are the infinitesimal generator of an analytic semigroup we have the following result:

Theorem 4.4. Let $-\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be the infinitesimal generator of an analytic semigroup $S(t)$. If $0 \in \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$, then

- $S(t) : X \rightarrow D(\mathcal{A}^{\alpha})$ for every $t > 0$ and $\alpha \geq 0$;
- For every $x \in D(\mathcal{A}^{\alpha})$ we have $S(t)\mathcal{A}^{\alpha}x = \mathcal{A}^{\alpha}S(t)x$;
- For every $t > 0$ the operator $\mathcal{A}^{\alpha}S(t) : X \rightarrow X$ is bounded and the following inequality holds true

$$\|\mathcal{A}^{\alpha}S(t)\|_{\mathcal{L}(X)} \leq C_{\alpha} t^{-\alpha} e^{-\delta t}. \quad (4.26)$$

We end this section with the an important analyticity result. This results can be adapted to our corresponding problem and it's role is essential to obtain the stabilization inequality (2.10).

Let Ω a two-dimensional bounded domain with smooth boundary Γ and consider the thermoelastic plate problem given by the system

$$\begin{cases} w_{tt} + \Delta^2 w + \Delta \theta = 0 & \text{in } \Omega \times (0, \infty), \\ \theta_t - \Delta \theta - \Delta w_t = 0 & \text{in } \Omega \times (0, \infty), \end{cases} \quad (4.27)$$

with boundary conditions

$$\begin{cases} \Delta w + (1 - \mu)B_1 w + \theta = 0 & \text{on } \Gamma \times (0, \infty), \\ \frac{\partial}{\partial \nu} \Delta w + (1 - \mu)B_2 w - w + \frac{\partial \theta}{\partial \nu} = 0 & \text{on } \Gamma \times (0, \infty), \\ \frac{\partial \theta}{\partial \nu} + \lambda \theta = 0, \lambda > 0 & \text{on } \Gamma \times (0, \infty). \end{cases} \quad (4.28)$$

For this system we have the following result. See [43, Theorem 1.3].

Theorem 4.5. The semigroup associated with the problem (4.27)-(4.28) is analytic on the phase space $H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.

4.7 Proof of Wellposedness/Regularity Result

In this section we will demonstrate the wellposedness of weak and regular solutions of our system. We begin with an outline of the proof and finally in Section 4.7.2 we give the prove of the Theorem 4.1.

4.7.1 Outline of the Proof to Theorem 4.1

The proof of the Theorem 4.1 will be given by several steps. Here, just to guide, we will indicate the main ideas of the proof:

- (i) First we consider the problem with general sufficiently regular functions (forcing terms). To this problem we used semigroup theory to obtain the existence of local solutions. To this end we introduce some operators in order to consider the abstract version of the model.
- (ii) Second, we construct an application defined in an appropriated space. We show that this application is a contraction map, then the existence of local solution to the problem is well defined and given by fixed point argument. To accomplish this we used fact that the semigroup associated with the thermoelastic plate is analytic. To extend the solution we used energy estimates.
- (iii) Finally, we prove the existence of regular solutions and again we used the fact that the semigroup associated with the thermoelastic plate is analytic to transfer the regularity properties of parabolic part to hyperbolic part of the system.

4.7.2 Proof of Theorem 4.1

Proof of Theorem 4.1 (i):

Step 1. Existence of weak solution with general and regular functions: First let us consider the following system

$$\begin{cases} u_{tt} - \operatorname{div}\{\sigma[\varepsilon(u)]\} + \nabla\phi + p_1(u, w) - \operatorname{div}\{\mathcal{J}\} = 0 & \text{in } \Omega \times (0, \infty), \\ w_{tt} + \Delta^2 w + \Delta\theta + p_2(u, w) - \operatorname{div}\{\mathcal{J}\} = 0 & \text{in } \Omega \times (0, \infty), \\ \phi_t - \Delta\phi + \operatorname{div}\{u_t\} - \mathcal{K} = 0 & \text{in } \Omega \times (0, \infty), \\ \theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } \Omega \times (0, \infty), \end{cases} \quad (4.29)$$

subjected to the boundary conditions

$$\begin{cases} u = 0, w = 0, \nabla w = 0 & \text{on } \Gamma_0 \times (0, \infty), \\ \sigma[\varepsilon(u)]\nu + \kappa u - \phi\nu + u_t + \mathcal{J}\nu = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ [\Delta w + (1 - \mu)B_1 w] + \theta = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \left[\frac{\partial}{\partial \nu}\Delta w + (1 - \mu)B_2 w\right] + \frac{\partial \theta}{\partial \nu} - \mathcal{J}\nu = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \frac{\partial \phi}{\partial \nu} + \lambda_1 \phi = \frac{\partial \theta}{\partial \nu} + \lambda_2 \theta = 0 & \text{on } \Gamma \times (0, \infty), \end{cases} \quad (4.30)$$

where $\mathcal{J}$, $\mathcal{J}$ and $\mathcal{K}$ are sufficiently regular functions. In order to put the problem (4.29)-(4.30) into an abstract setting we have to consider some operators and spaces.

- The operator associated to the in-plane displacement u : Let $\mathcal{A}_u$ an operator on $L^2(\Omega)$ given by $\mathcal{A}_u v = -\operatorname{div}\{\sigma[\varepsilon(v)]\}$ with domain

$$D(\mathcal{A}_u) = \left\{ v \in [H^2(\Omega)]^2 \left| \begin{array}{l} v = 0 \text{ on } \Gamma_0, \\ \sigma[\varepsilon(v)]\nu + \kappa v = 0 \text{ on } \Gamma_1. \end{array} \right. \right\}.$$

- The Green operator corresponding to the in-plane displacement boundary condition:

$$G_u g = v \Leftrightarrow \begin{cases} \operatorname{div}\{\sigma[\varepsilon(v)]\} = 0 & \text{in } \Omega, \\ \begin{cases} v = 0 \text{ on } \Gamma_0, \\ \sigma[\varepsilon(v)]\nu + \kappa v = g \text{ on } \Gamma_1. \end{cases} \end{cases}$$

Elliptic regularity (e.g. [46]) we can show that $G_u : H^s(\Gamma_1) \rightarrow H^{s+\frac{3}{2}}(\Omega)$ for all $s \in \mathbb{R}$.

- The biharmonic operator: Let $\mathcal{A}_M$ a positive and self-adjoint operator on $L^2(\Omega)$ given by $\mathcal{A}_M v = \Delta^2 v$ with domain

$$D(\mathcal{A}_M) = \left\{ v \in H^4(\Omega) \left| \begin{array}{l} v = 0, \nabla v = 0 \text{ on } \Gamma_0, \\ [\Delta v + (1 - \mu)B_1 v]|_{\Gamma_1} = 0, \\ \left[\frac{\partial}{\partial \nu}(\Delta v) + (1 - \mu)B_2 v\right]|_{\Gamma_1} = 0. \end{array} \right. \right\}.$$

- The Green's operators corresponding to the vertical displacement: Let G_i , $i = 1, 2$, be the operators corresponding to the mechanical boundary conditions defined by

$$G_1 g = v \Leftrightarrow \begin{cases} \Delta^2 v = 0 \text{ in } \Omega, \\ \begin{cases} v = 0, \nabla v = 0 \text{ on } \Gamma_0, \\ [\Delta v + (1 - \mu)B_1 v]|_{\Gamma_1} = g, \\ \left[\frac{\partial}{\partial \nu}(\Delta v) + (1 - \mu)B_2 v \right]|_{\Gamma_1} = 0, \end{cases} \end{cases}$$

and

$$G_2 g = v \Leftrightarrow \begin{cases} \Delta^2 v = 0 \text{ in } \Omega, \\ \begin{cases} v = 0, \nabla v = 0 \text{ on } \Gamma_0, \\ [\Delta v + (1 - \mu)B_1 v]|_{\Gamma_1} = 0, \\ \left[\frac{\partial}{\partial \nu}(\Delta v) + (1 - \mu)B_2 v \right]|_{\Gamma_1} = g. \end{cases} \end{cases}$$

Elliptic regularity (e.g. [46]) gives $G_1 : L^2(\Gamma_1) \rightarrow H^{\frac{5}{2}}(\Omega) \subset H^{\frac{5}{2}-4\epsilon}(\Omega) \equiv D(\mathcal{A}_M^{\frac{5}{8}-\epsilon})$ and $G_2 : L^2(\Gamma_1) \rightarrow H^{\frac{7}{2}}(\Omega) \subset H^{\frac{7}{2}-4\epsilon}(\Omega) \equiv D(\mathcal{A}_M^{\frac{7}{8}-\epsilon})$, $\epsilon > 0$. By application of Green's formula (e.g. [44, 46]) we get, for $v \in D(\mathcal{A}_M)$,

$$\begin{aligned} G_1^* \mathcal{A}_M v &= -\frac{\partial}{\partial \nu}(v|_{\Gamma_1}), \\ G_2^* \mathcal{A}_M v &= -v|_{\Gamma_1}. \end{aligned} \tag{4.31}$$

where $(G_i v, w)_{L^2(\Omega)} = (v, G_i^* w)_{L^2(\Gamma)}$, $i = 1, 2$.

Using the definition of the above operators we rewrite (4.29)-(4.30) as

$$\begin{cases} u_{tt} - \operatorname{div}\{\sigma[\varepsilon(u + G_u(-\phi v + u_t + \mathcal{J} v))]\} + \nabla \phi + p_1(u, w) - \operatorname{div}\{\mathcal{J}\} = 0, \\ w_{tt} + \Delta^2[w + G_1(\theta) + G_2\left(\frac{\partial \theta}{\partial \nu}\right) - G_2(\mathcal{J} v)] + \Delta \theta + p_2(u, w) - \operatorname{div}\{\mathcal{J}\} = 0, \\ \phi_t - \Delta \phi + \operatorname{div}\{u_t\} - \mathcal{K} = 0, \\ \theta_t - \Delta \theta - \Delta w_t = 0, \end{cases}$$

in $\Omega \times (0, \infty)$ and subjected to the boundary conditions

$$\begin{cases} \mathbf{u} = 0, \mathbf{w} = 0, \nabla \mathbf{w} = 0 \text{ on } \Gamma_0 \times (0, \infty), \\ \sigma[\varepsilon(\mathbf{u})] - \kappa \mathbf{u} = 0 \text{ on } \Gamma_1 \times (0, \infty), \\ [\Delta \mathbf{w} + (1 - \mu)B_1 \mathbf{w}] + \theta = 0 \text{ on } \Gamma_1 \times (0, \infty), \\ \left[\frac{\partial}{\partial \nu} \Delta \mathbf{w} + (1 - \mu)B_2 \mathbf{w} \right] + \frac{\partial \theta}{\partial \nu} - \mathcal{J} v = 0 \text{ on } \Gamma_1 \times (0, \infty), \\ \frac{\partial \phi}{\partial \nu} + \lambda_1 \phi = \frac{\partial \theta}{\partial \nu} + \lambda_2 \theta = 0 \text{ on } \Gamma \times (0, \infty), \end{cases}$$

where

$$\begin{aligned} \mathbf{u} &= u + G_u(-\phi v + u_t + \mathcal{J} v), \\ \mathbf{w} &= w + G_1(\theta) + G_2\left(\frac{\partial \theta}{\partial \nu}\right) - G_2(\mathcal{J} v). \end{aligned}$$

Note that $\mathbf{u} \in D(\mathcal{A}_u)$ and $\mathbf{w} \in D(\mathcal{A}_M)$, then the following equalities for u and w holds in $[D(\mathcal{A}_u)]'$ and $[D(\mathcal{A}_M)]'$, respectively [43]

$$\begin{aligned} u_{tt} + \mathcal{A}_u u + \mathcal{A}_u G_u(\phi v) + \mathcal{A}_u G_u(u_t) + \mathcal{A}_u G_u(\mathcal{J} v) + \nabla \phi + p_1(u, w) - \operatorname{div}\{\mathcal{J}\} &= 0, \\ w_{tt} + \mathcal{A}_M w + \mathcal{A}_M G_1(\theta) + \mathcal{A}_M G_2\left(\frac{\partial \theta}{\partial v}\right) - \mathcal{A}_M G_2(\mathcal{J} v) + \Delta \theta + p_2(u, w) - \operatorname{div}\{\mathcal{J}\} &= 0. \end{aligned}$$

This equality implies the following abstract representation for (u, ϕ) problem

$$\begin{bmatrix} u_t \\ u_{tt} \\ \phi_t \end{bmatrix} = \mathcal{A}_0 \begin{bmatrix} u \\ u_t \\ \phi \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathcal{A}_u G_u(\mathcal{J} v) + \operatorname{div}\{\mathcal{J}\} \\ \mathcal{K} \end{bmatrix} - \begin{bmatrix} 0 \\ p_1(u, w) \\ 0 \end{bmatrix}, \quad (4.32)$$

where $\mathcal{A}_0 : [H^1(\Omega)]^2 \times [L^2(\Omega)]^3 \rightarrow [H^1(\Omega)]^2 \times [L^2(\Omega)]^3$ is a linear operator with domain

$$D(\mathcal{A}_0) = \left\{ (u, v, \phi) \in [H^2(\Omega)]^2 \times [H^1(\Omega)]^2 \times H^1(\Omega) \left| \begin{array}{l} u = 0 \text{ on } \Gamma_0, \\ \sigma[\varepsilon(u)] v - \kappa u = \phi v - v \text{ on } \Gamma_1, \\ \frac{\partial \phi}{\partial v} + \lambda_1 \phi = 0 \text{ on } \Gamma. \end{array} \right. \right\}$$

and given by

$$\mathcal{A}_0 \begin{bmatrix} u \\ v \\ \phi \end{bmatrix} = \begin{bmatrix} v \\ -\mathcal{A}_u u - \mathcal{A}_u G_u(\phi v) - \mathcal{A}_u G_u(v) - \nabla \phi \\ \Delta \phi - \operatorname{div}\{v\} \end{bmatrix}.$$

About the problem for (w, θ) we may rewrite

$$\begin{bmatrix} w_t \\ w_{tt} \\ \theta_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{A}_M G_2(\mathcal{J} v) + \operatorname{div}\{\mathcal{J}\} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ p_2(u, w) \\ 0 \end{bmatrix}, \quad (4.33)$$

where $\mathcal{A} : H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \rightarrow H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ with domain

$$D(\mathcal{A}) = \left\{ (w, v, \theta) \in H^4(\Omega) \times [H^2(\Omega)]^2 \left| \begin{array}{l} w = 0, \nabla w = 0 \text{ on } \Gamma_0, \\ [\Delta w + (1 - \mu)B_1 w] + \theta|_{\Gamma_1} = 0, \\ \left[\frac{\partial}{\partial v} \Delta w + (1 - \mu)B_2 w \right] + \frac{\partial \theta}{\partial v} \Big|_{\Gamma_1} = 0, \\ \frac{\partial \theta}{\partial v} + \lambda_2 \theta = 0 \text{ on } \Gamma. \end{array} \right. \right\}$$

and defined by

$$\mathcal{A} \begin{bmatrix} w \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} v \\ -\mathcal{A}_M w - \mathcal{A}_M(G_1 \theta) - \mathcal{A}_M G_2\left(\frac{\partial \theta}{\partial v}\right) - \Delta \theta \\ \Delta \theta + \Delta w \end{bmatrix}. \quad (4.34)$$

We have that the operators $\mathcal{A}_0$ and $\mathcal{A}$ are generators of a contraction semigroup on the spaces $\mathcal{Y}_1 = [H^1(\Omega)]^2 \times [L^2(\Omega)]^3$ and $\mathcal{Y}_2 = H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, respectively (See [6]). To conclude the existence of local solutions we need to take functions $\mathcal{J}, \mathcal{J}, \mathcal{K}$ sufficiently regular, that is, for functions such that the following operators

$$[u, u_t, \phi] \rightarrow [0, -\mathcal{A}_u G_u(\mathcal{J} v) + \operatorname{div}\{\mathcal{J}\}, \mathcal{K}] \in L^1(0, T; \mathcal{Y}_1)$$

and

$$[w, w_t, \theta] \rightarrow [0, \mathcal{A}_M G_2(\mathcal{J} v) + \operatorname{div}\{\mathcal{J}\}, 0] \in L^1(0, T; \mathcal{Y}_2).$$

The fact that the Nemytskii mapping given by the forcing $p_1(u, w)$ and $p_2(u, w)$ is locally Lipschitz on $\mathcal{H}$, the existence of a unique local solution for the problem (4.29)-(4.30) is granted by semigroup theory (e.g. [17, Theorem 7.2]).

Step 2. Lipschitz-type estimate: The next result will play an important role in the next section.

Proposition 4.1. Let y^1, y^2 two solutions of the system (4.29)-(4.30) corresponding to forcing terms $\mathcal{J}_1, \mathcal{J}_1, \mathcal{K}_1$ and $\mathcal{J}_2, \mathcal{J}_2, \mathcal{K}_2$. Then for any $\varepsilon > 0$ and $0 < t \leq T$, the following estimates holds true

$$\|\tilde{y}(t)\|_{\mathcal{Y}} \leq C [\|\tilde{y}(0)\|_{\mathcal{Y}} + \|\tilde{\mathcal{J}}_t\|_{L^1(0, T; L^2(\Omega))} + \|\tilde{\mathcal{J}}\|_{L^{2+\varepsilon}(0, T; L^2(\Omega))} + \|\tilde{\mathcal{K}}\|_{L^2(0, T; H^{-1}(\Omega))}]$$

and

$$\int_0^T [\|\tilde{w}\|_{3, \Omega}^2 + \|\tilde{w}_t\|_{1, \Omega}^2 + \|\tilde{\theta}\|_{1, \Omega}^2] dt \leq C [\|\tilde{y}(0)\|_{\mathcal{Y}}^2 + \|\tilde{\mathcal{J}}\|_{L^2(0, T; L^2(\Omega))}^2],$$

where $\tilde{y} = y^1 - y^2$, $\tilde{\mathcal{J}} = \mathcal{J}^1 - \mathcal{J}^2$, $\tilde{\mathcal{J}} = \mathcal{J}^1 - \mathcal{J}^2$ and $\tilde{\mathcal{K}} = \mathcal{K}^1 - \mathcal{K}^2$.

Proof. Let us prove the first inequality. Multiplying the equations by (4.29)₁ and (4.29)₃ by $\tilde{u}_t$ and $\tilde{\phi}$, respectively, and integrating over $\Omega \times [0, t]$ then we find the equality

$$\begin{aligned} E_{\tilde{u}, \tilde{\phi}}(t) + 2 \int_0^t [\|\tilde{u}_t(s)\|_{\Gamma_1}^2 + \|\nabla \tilde{\phi}(s)\|_{\Omega}^2 + \lambda_1 \|\tilde{\phi}(s)\|_{\Gamma}^2] ds \\ = E_{\tilde{u}, \tilde{\phi}}(0) + 2 \int_0^t [(\operatorname{div}\{\tilde{\mathcal{J}}\}, \tilde{u}_t)_{\Omega} - \langle \tilde{\mathcal{J}} v, \tilde{u}_t \rangle_{\Gamma_1} + (\tilde{\mathcal{K}}, \tilde{\phi})_{\Omega}] ds, \end{aligned} \quad (4.35)$$

where

$$E_{\tilde{u}, \tilde{\phi}}(t) = \|\tilde{u}_t(t)\|_{\Omega}^2 + \|\varepsilon(\tilde{u}(t))\|_{\Omega}^2 + \kappa \|\tilde{u}(t)\|_{\Omega}^2 + \|\tilde{\phi}(t)\|_{\Omega}^2. \quad (4.36)$$

We have to estimate all the products on the right-hand side of (4.35). To handle with the first and second product we use Divergence Theorem and integration by parts to obtain

$$\begin{aligned} \int_0^t (\operatorname{div}\{\tilde{\mathcal{J}}\}, \tilde{u}_t)_{\Omega} ds - \int_0^t \langle \tilde{\mathcal{J}} v, \tilde{u}_t \rangle_{\Gamma_1} ds &= - \int_0^t (\tilde{\mathcal{J}}, \varepsilon(\tilde{u}_t))_{\Omega} ds \\ &= - (\tilde{\mathcal{J}}, \varepsilon(\tilde{u}))_{\Omega} \Big|_0^t + \int_0^t (\tilde{\mathcal{J}}_t, \varepsilon(\tilde{u}))_{\Omega} ds. \end{aligned} \quad (4.37)$$

By Hölder's and Young's inequalities we can estimate the inner-products on the right-side hand of (4.37), for $0 < t \leq T$, by

$$\begin{aligned} (\tilde{\mathcal{J}}, \varepsilon(\tilde{u}))_{\Omega} \Big|_0^t &\leq C \int_{\Omega} [\|\tilde{\mathcal{J}}\|_{L^\infty(0,T)} \|\varepsilon(\tilde{u})\|_{L^\infty(0,T)}] d\Omega \\ &\leq C_\delta \|\tilde{\mathcal{J}}\|_{L^\infty(0,T;L^2(\Omega))} + \delta \|\varepsilon(\tilde{u})\|_{L^\infty(0,T;L^2(\Omega))} \end{aligned}$$

and

$$\int_0^t (\tilde{\mathcal{J}}_t, \varepsilon(\tilde{u}))_{\Omega} ds \leq C_\delta \|\tilde{\mathcal{J}}_t\|_{L^1(0,T;L^2(\Omega))} + \delta \|\varepsilon(\tilde{u})\|_{L^\infty(0,T;L^2(\Omega))}.$$

Thus, inserting this inequalities into the right-hand side of (4.37) and using the inequality $\|\tilde{\mathcal{J}}\|_{L^\infty(0,T;L^2(\Omega))} \leq C \|\tilde{\mathcal{J}}_t\|_{L^1(0,T;L^2(\Omega))}$ we obtain

$$\int_0^t (\operatorname{div}\{\tilde{\mathcal{J}}\}, \tilde{u}_t)_{\Omega} ds - \int_0^t \langle \tilde{\mathcal{J}} \mathbf{v}, \tilde{u}_t \rangle_{\Gamma_1} ds \leq C_\delta \|\tilde{\mathcal{J}}_t\|_{L^1(0,T;L^2(\Omega))} + \delta \|\varepsilon(\tilde{u})\|_{L^\infty(0,T;L^2(\Omega))}.$$

The third product can be estimated as

$$\int_0^t (\tilde{\mathcal{K}}, \tilde{\phi})_{\Omega} ds \leq C_\delta \|\tilde{\mathcal{K}}\|_{L^2(0,T;H^{-1}(\Omega))} + \delta \|\tilde{\phi}\|_{L^2(0,T;H_0^1(\Omega))}.$$

Combining this estimates together with (4.35) and taking $\delta > 0$ small enough we find that

$$\begin{aligned} E_{\tilde{u}, \tilde{\phi}}(t) + C \int_0^t [\|\tilde{u}_t(s)\|_{\Gamma_1}^2 + \|\nabla \tilde{\phi}(s)\|_{\Omega}^2 + \lambda_1 \|\tilde{\phi}(s)\|_{\Omega}^2] ds \\ \leq CE_{\tilde{u}, \tilde{\phi}}(0) + C \|\tilde{\mathcal{J}}_t\|_{L^1(0,T;L^2(\Omega))}^2 + C \|\tilde{\mathcal{K}}\|_{L^2(0,T;H^{-1}(\Omega))}^2. \end{aligned} \quad (4.38)$$

To complete the proof we need to estimate the w displacement energy. Here we shall use the fact that the operator $\mathcal{A}$ defined in (4.34) generates an analytic and exponentially stable semigroup on the space $\mathcal{B}_2 = H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, cf. [43]. In Step 1 we found the following representation for the problem (w, θ)

$$\begin{bmatrix} w_t \\ w_{tt} \\ \theta_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{A}_M G_2(\tilde{\mathcal{J}} \mathbf{v}) + \operatorname{div}\{\tilde{\mathcal{J}}\} \\ 0 \end{bmatrix}. \quad (4.39)$$

Rewriting the solution of (4.39) using variation of parameters formula,

$$\begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\theta} \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} \tilde{w}_0 \\ \tilde{w}_1 \\ \tilde{\theta}_0 \end{bmatrix} + \int_0^t \mathcal{A}^{\frac{1}{2}} e^{\mathcal{A}(t-s)} \mathcal{A}^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \mathcal{A}_M G_2(\tilde{\mathcal{J}} \mathbf{v}) + \operatorname{div}\{\tilde{\mathcal{J}}\} \\ 0 \end{bmatrix} ds.$$

Using the following estimate

$$\|\mathcal{A}^\alpha e^{\mathcal{A}t}\|_{\mathcal{L}(\mathcal{B}_2)} \leq C \frac{e^{-\omega t}}{t^\alpha}, \quad 0 \leq \alpha \leq 1, \quad (4.40)$$

we obtain

$$\left\| \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\theta} \end{bmatrix} \right\|_{\mathcal{H}_2} = \left\| \begin{bmatrix} \tilde{w}_0 \\ \tilde{w}_1 \\ \tilde{\theta}_0 \end{bmatrix} \right\|_{\mathcal{H}_2} + \int_0^t \frac{e^{-\omega(t-s)}}{(t-s)^{\frac{1}{2}}} \left\| \mathcal{A}^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \mathcal{A}_M G_2(\tilde{\mathcal{J}} \mathbf{v}) + \operatorname{div}\{\tilde{\mathcal{J}}\} \\ 0 \end{bmatrix} \right\|_{\mathcal{H}_2} ds.$$

Using the characterization given by Theorem 2.5 we infer that, for $\alpha \in (0, 1)$,

$$D(\mathcal{A}^\alpha) = [D(\mathcal{A}), \mathcal{H}_2]_{1-\alpha} \subset H^{2(1+\alpha)}(\Omega) \times H^{2\alpha}(\Omega) \times H^{2\alpha}(\Omega). \quad (4.41)$$

Then this equality with $\alpha = \frac{1}{2}$ and by duality we obtain

$$\left\| \mathcal{A}^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \mathcal{A}_M G_2(\tilde{\mathcal{J}} \mathbf{v}) + \operatorname{div}\{\tilde{\mathcal{J}}\} \\ 0 \end{bmatrix} \right\|_{\mathcal{H}_2} \leq C \|\mathcal{A}_M G_2(\tilde{\mathcal{J}} \mathbf{v}) + \operatorname{div}\{\tilde{\mathcal{J}}\}\|_{-1, \Omega}. \quad (4.42)$$

We recall (4.31) identities to show that

$$(\mathcal{A}_M G_2(\tilde{\mathcal{J}} \mathbf{v}) + \operatorname{div}\{\tilde{\mathcal{J}}\}, \psi)_\Omega = (\tilde{\mathcal{J}}, \nabla \psi)_\Omega.$$

for all $\psi \in H_0^1(\Omega)$. Having established this we obtain that

$$\|\mathcal{A}_M G_2(\tilde{\mathcal{J}} \mathbf{v}) + \operatorname{div}\{\tilde{\mathcal{J}}\}\|_{-1, \Omega} \leq C \|\tilde{\mathcal{J}}\|_\Omega. \quad (4.43)$$

Then Hölder's inequality implies the

$$\left\| \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\theta} \end{bmatrix} \right\|_{\mathcal{H}_2} \leq C \left\| \begin{bmatrix} \tilde{w}_0 \\ \tilde{w}_1 \\ \tilde{\theta}_0 \end{bmatrix} \right\|_{\mathcal{H}_2} + C \|\tilde{\mathcal{J}}\|_{L^p(0, t; L^2(\Omega))}. \quad (4.44)$$

This inequality with $p = 2 + \varepsilon$, ε small, combined with (4.38) implies the first part of the Proposition. Similar arguments implies the second part. $\square$

Step 3. Contraction argument: Now, will prove the existence of local solution to the problem. We are following some ideas from [15]. For this, let $\mathcal{X}$ the space of solutions y to the problem, such that,

$$y = (u, u_t, w, w_t, \phi, \theta) \in C([0, T]; Y), w \in L^2(0, T; H^3(\Omega)) \text{ and } w_t \in L^2(0, T; H_0^1(\Omega)).$$

We have that the pair $(\mathcal{X}, \operatorname{dist}_{\mathcal{X}}(\cdot, \cdot))$ is a metric space, where

$$\operatorname{dist}_{\mathcal{X}}(y^1, y^2) = \|y^1 - y^2\|_{C([0, T]; \mathcal{Y})} + \|w^1 - w^2\|_{L^2(0, T; H^3(\Omega))} + \|w_t^1 - w_t^2\|_{L^2(0, T; H_0^1(\Omega))}.$$

Let $B_{R, T}$ the subset of $\mathcal{X}$ given by

$$B_{R, T} = \{y \in \mathcal{X} \mid \|y\|_{\mathcal{X}} = \operatorname{dist}_{\mathcal{X}}(y, 0) \leq R \text{ and } y(0) = y_0\}.$$

- (i) The pair $(B_{R,T}, \text{dist } \mathcal{X}(\cdot, \cdot))$ is a complete metric space;
- (ii) The map $\mathcal{T} : B_{R,T} \rightarrow \mathcal{X}$ defined by $y^* \mapsto Ty^* = y$, where y is a unique solution with

$$\begin{aligned}\mathcal{I} &= \sigma[f(\nabla w^*)], \\ \mathcal{J} &= \sigma[N(u^*, w^*)] \nabla w^* + \phi^* \nabla w^*, \\ \mathcal{K} &= \nabla w^* \cdot \nabla w_t^*.\end{aligned}\tag{4.45}$$

is well defined, $T(B_{R,T}) \subset B_{R,T}$ and a contraction.

Let us prove that $T(B_{R,T}) \subset B_{R,T}$. Consider $y \in T(B_{R,T})$, then $y = Ty^*$ with $y^* \in B_{R,T}$. We will show that the solution $y \in B_{R,T}$ for T sufficiently small. Proceeding as in the proof of Proposition 4.1 we obtain the following inequalities

$$\|y(t)\|_{\mathcal{Y}} \leq C [\|y(0)\|_{\mathcal{Y}} + \|\mathcal{I}_t\|_{L^1(0,T;L^2(\Omega))} + \|\mathcal{J}\|_{L^{2+\varepsilon}(0,T;L^2(\Omega))} + \|\mathcal{K}\|_{L^2(0,T;H^{-1}(\Omega))}] \tag{4.46}$$

and

$$\int_0^T [\|w\|_{3,\Omega}^2 + \|w_t\|_{1,\Omega}^2] dt \leq C [\|y(0)\|_{\mathcal{Y}}^2 + \|\mathcal{J}\|_{L^2(0,T;L^2(\Omega))}^2], \tag{4.47}$$

where $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ are given in (4.45). Let us estimate the norms on the right-hand side of (4.46) and (4.47). The definition of $\mathcal{I}$ imply that

$$\mathcal{I}_t = \frac{d}{dt} \{ \sigma[f(\nabla w^*)] \} = \frac{1}{2} \sigma[\nabla w_t^* \otimes \nabla w^*] + \frac{1}{2} \sigma[\nabla w^* \otimes \nabla w_t^*],$$

By Sobolev's embedding $H^{1+\varepsilon}(\Omega) \subset L^\infty(\Omega)$, for $0 < \varepsilon < 1$, and interpolation inequality

$$\|w^*\|_{2+\varepsilon,\Omega} \leq C \|w^*\|_{\Omega}^{\frac{1-\varepsilon}{3}} \|w^*\|_{3,\Omega}^{\frac{2+\varepsilon}{3}}, \tag{4.48}$$

we obtain an estimate for $\mathcal{I}_t$ in $L^1(0,T;L^2(\Omega))$

$$\begin{aligned}\|\mathcal{I}_t\|_{L^1(0,T;L^2(\Omega))} &\leq C \int_0^T \|\nabla w_t^*\|_{\Omega} \|\nabla w^*\|_{1+\varepsilon,\Omega} dt \\ &\leq C \left[\int_0^T \|\nabla w_t^*\|_{\Omega}^2 dt \right]^{\frac{1}{2}} \left[\int_0^T \|w^*\|_{2+\varepsilon,\Omega}^2 dt \right]^{\frac{1}{2}} \\ &\leq CR \left[\int_0^T \|w^*\|_{\Omega}^{2[\frac{1-\varepsilon}{3}]} \|w^*\|_{3,\Omega}^{2[\frac{2+\varepsilon}{3}]} dt \right]^{\frac{1}{2}} \\ &\leq CR \sup_{t \in [0,T]} \{ \|w^*\|_{\Omega} \}^{[\frac{1-\varepsilon}{3}]} \left[\int_0^T \|w^*\|_{3,\Omega}^{2[\frac{2+\varepsilon}{3}]} dt \right]^{\frac{1}{2}} \\ &\leq CRR^{[\frac{1-\varepsilon}{3}]} \left[\int_0^T \|\tilde{w}^*\|_{3,\Omega}^2 dt \right]^{\frac{1}{2}[\frac{2+\varepsilon}{3}]} T^{[\frac{1-\varepsilon}{6}]} \\ &\leq CRR^{[\frac{1-\varepsilon}{3}]} R^{[\frac{2+\varepsilon}{3}]} T^{[\frac{1-\varepsilon}{6}]} = C R^2 T^{[\frac{1-\varepsilon}{6}]}. \end{aligned}\tag{4.49}$$

Let us estimate $\mathcal{J}$ term. First note that, by interpolation results we have that if

$$w \in L^2(0, T; H^3(\Omega)) \cup H^1(0, T; H^1(\Omega))$$

then

$$w \in H^{1-\zeta}(0, T; H^{1+2\zeta}(\Omega)), \text{ for } \zeta \in (0, 1).$$

Taking $\zeta = \frac{1}{2} + \frac{1}{2}\varepsilon$ and by continuous embedding $H^\zeta(0, T; H^{2+\varepsilon}(\Omega)) \subset L^{\frac{2}{\varepsilon}}(0, T; H^{2+\varepsilon}(\Omega))$ we have that the following inequality holds true

$$\|w\|_{L^{\frac{2}{\varepsilon}}(0, T; H^{2+\varepsilon}(\Omega))} \leq C \left[\|w\|_{L^2(0, T; H^3(\Omega))} + \|w_t\|_{L^2(0, T; H^1(\Omega))} \right]. \quad (4.50)$$

The definition of $\mathcal{J}$ results that

$$\begin{aligned} \|\mathcal{J}\|_{\Omega} &\leq C \left[\|\sigma[N(u^*, w^*)]\|_{\Omega} + \|\phi^*\|_{\Omega} \right] \|w^*\|_{2+\varepsilon, \Omega} \\ &\leq C \left[\|\varepsilon(u^*)\|_{\Omega} + \|w^*\|_{2, \Omega}^2 + \|\phi^*\|_{\Omega} \right] \|w^*\|_{2+\varepsilon, \Omega}. \end{aligned}$$

Hölder's inequality in time variable (with $p_1 = \frac{2+2\varepsilon_0}{2+\varepsilon_0}$ and $p_2 = \frac{2+2\varepsilon_0}{\varepsilon_0}$) implies, for $\varepsilon_0 > 0$ small, that

$$\begin{aligned} \|\mathcal{J}\|_{L^{2+\varepsilon_0}(0, T; L^2(\Omega))} &\leq C \|\varepsilon(u^*)\|_{C([0, T]; L^2(\Omega))} \|w^*\|_{L^{2+\varepsilon_0}(0, T; H^{2+\varepsilon}(\Omega))} \\ &\quad + C \|w^*\|_{C([0, T]; H^2(\Omega))}^2 \|w^*\|_{L^{2+\varepsilon_0}(0, T; H^{2+\varepsilon}(\Omega))} \\ &\quad + C \|\phi^*\|_{C([0, T]; L^2(\Omega))} \|w^*\|_{L^{2+\varepsilon_0}(0, T; H^{2+\varepsilon}(\Omega))} \\ &\leq C [R + R^2] \|w^*\|_{L^{2+\varepsilon_0}(0, T; H^{2+\varepsilon}(\Omega))} \\ &= C [R + R^2] \left[\int_0^T \|w^*\|_{2+\varepsilon, \Omega}^{2+\varepsilon_0} dt \right]^{\left[\frac{1}{2+\varepsilon_0} \right]} \\ &\leq C [R + R^2] \left[\int_0^T \|w^*\|_{2+\varepsilon, \Omega}^{2+2\varepsilon_0} dt \right]^{\left[\frac{1}{2+2\varepsilon_0} \right]} T^{\left[\frac{\varepsilon_0}{(2+\varepsilon_0)(2+2\varepsilon_0)} \right]} \\ &= C [R + R^2] \|w^*\|_{L^{2+2\varepsilon_0}(0, T; H^{2+\varepsilon}(\Omega))} T^{\left[\frac{\varepsilon_0}{(2+\varepsilon_0)(2+2\varepsilon_0)} \right]} \\ &\leq C [R^2 + R^3] T^{\left[\frac{\varepsilon_0}{(2+\varepsilon_0)(2+2\varepsilon_0)} \right]} \end{aligned} \quad (4.51)$$

and here we used interpolation estimate (4.50) with $\frac{2}{\varepsilon} = 2 + 2\varepsilon_0$.

Finally, let us estimate $\mathcal{K}$. For $\psi \in H_0^1(\Omega)$ we find that

$$\begin{aligned} (\mathcal{K}, \psi)_{\Omega} &\leq C \|\nabla w_t^*\|_{-\varepsilon, \Omega} \|\nabla w^* \psi\|_{\varepsilon, \Omega} \\ &\leq C \|\nabla w_t^*\|_{-\varepsilon, \Omega} \|\nabla w^*\|_{\varepsilon+\varepsilon_0, \Omega} \|\psi\|_{1, \Omega} \end{aligned}$$

and this implies that

$$\|\mathcal{K}\|_{-1, \Omega} \leq C \|\nabla w_t^*\|_{-\varepsilon, \Omega} \|\nabla w^*\|_{\varepsilon+\varepsilon_0, \Omega}$$

and

$$\begin{aligned}
\|\mathcal{K}\|_{L^2(0,T;H^{-1}(\Omega))}^2 &\leq C \int_0^T [\|\nabla w_t^*\|_{-\varepsilon,\Omega}^2 \|\nabla w^*\|_{\varepsilon+\varepsilon_0,\Omega}^2] dt \\
&\leq C \|w^*\|_{C([0,T];H^2(\Omega))}^2 \int_0^T \|w_t^*\|_{1-\varepsilon,\Omega}^2 dt \\
&\leq CR^2 \int_0^T \|w_t^*\|_{1,\Omega}^{2(1-\varepsilon)} \|w_t^*\|_{\Omega}^{2\varepsilon} dt \\
&\leq CR^2 R^{2\varepsilon} \left[\int_0^T \|w_t^*\|_{1,\Omega}^2 dt \right]^{1-\varepsilon} T^\varepsilon \\
&\leq CR^2 R^{2\varepsilon} R^{(2-2\varepsilon)} T^\varepsilon = CR^4 T^\varepsilon.
\end{aligned} \tag{4.52}$$

Combining the estimates (4.49)-(4.52) together with (4.46) and (4.47) we find

$$\begin{aligned}
\|y(t)\|_{\mathcal{Y}} &\leq C\|y(0)\|_{\mathcal{Y}} + CR^2 T^{\frac{1-\varepsilon}{6}} + C[R^2 + R^3] T^{r_0} + CR^2 T^{\frac{\varepsilon}{2}}, \\
\int_0^T [\|w\|_{3,\Omega}^2 + \|w_t\|_{1,\Omega}^2] dt &\leq C\|y(0)\|_{\mathcal{Y}}^2 + C[R^2 + R^3] T^{r_0},
\end{aligned}$$

where $r_0 = \frac{\varepsilon_0}{(2+\varepsilon_0)(2+2\varepsilon_0)}$. These inequalities result that

$$\|y\|_{\mathcal{X}} \leq C\|y(0)\|_{\mathcal{Y}} + CR^2 T^{\frac{1-\varepsilon}{6}} + C[R^2 + R^3] T^{r_0} + CR^2 T^{\frac{\varepsilon}{2}}. \tag{4.53}$$

After taking R and T , such that, $C\|y(0)\|_{\mathcal{Y}} \leq \frac{R}{2}$ and

$$[CRT^{\frac{1-\varepsilon}{6}} + C[R + R^2] T^{r_0} + CRT^{\frac{\varepsilon}{2}}] \leq \frac{1}{2}, \tag{4.54}$$

the estimate (4.53) implies

$$\|y\|_{\mathcal{X}} \leq R.$$

Therefore $\mathcal{T}(B_{R,T}) \subset B_{R,T}$.

Now we show that the map $\mathcal{T}$ is a contraction. For this let $y^{1,*}, y^{2,*} \in B_{R,T}$, $\tilde{y}^* = y^{1,*} - y^{2,*}$ with $\mathcal{T}y^{1,*} = y^1$, $\mathcal{T}y^{2,*} = y^2$, and we shall prove that

$$\|\mathcal{T}y^{1,*} - \mathcal{T}y^{2,*}\|_{\mathcal{X}} \leq C\|y^{1,*} - y^{2,*}\|_{\mathcal{X}} \tag{4.55}$$

with constant $C < 1$. In order to prove (4.55) we will use the estimates given by the Proposition 4.1. As before we need to estimate the terms $\tilde{\mathcal{J}} = \mathcal{J}^1 - \mathcal{J}^2$, $\tilde{\mathcal{J}} = \mathcal{J}^1 - \mathcal{J}^2$ and $\tilde{\mathcal{K}} = \mathcal{K}^1 - \mathcal{K}^2$. Using the definition of $\tilde{\mathcal{J}}$ we find

$$\begin{aligned}
\tilde{\mathcal{J}}_t &= \frac{d}{dt} \{ \sigma[f(\nabla w^{1,*})] - \sigma[f(\nabla w^{2,*})] \} \\
&= \frac{d}{dt} \{ \sigma[f(\nabla \tilde{w}^*)] + \sigma[\nabla \tilde{w}^* \otimes \nabla w^{2,*}] + \sigma[\nabla w^{2,*} \otimes \nabla \tilde{w}^*] \} \\
&= \frac{1}{2} \sigma[\nabla \tilde{w}_t^* \otimes \nabla \tilde{w}^*] + \frac{1}{2} \sigma[\nabla \tilde{w}^* \otimes \nabla \tilde{w}_t^*] + \sigma[\nabla \tilde{w}_t^* \otimes \nabla w^{2,*}] \\
&\quad + \sigma[\nabla \tilde{w}^* \otimes \nabla w_t^{2,*}] + \sigma[\nabla w_t^{2,*} \otimes \nabla \tilde{w}^*] + \sigma[\nabla w^{2,*} \otimes \nabla \tilde{w}_t^*],
\end{aligned} \tag{4.56}$$

where $\tilde{w}^* = w^{1,*} - w^{2,*}$.

We have to estimate all the products in (4.56). Proceeding as in (4.49) we obtain

$$\begin{aligned}
& \|\sigma[\nabla \tilde{w}_t^* \otimes \nabla \tilde{w}^*]\|_{L^1(0,T;L^2(\Omega))} \\
& \leq C \int_0^T \|\nabla \tilde{w}_t^*\|_{\Omega} \|\nabla \tilde{w}^*\|_{1+\varepsilon,\Omega} dt \\
& \leq C \left[\int_0^T \|\tilde{w}_t^*\|_{1,\Omega}^2 dt \right]^{\frac{1}{2}} \left[\int_0^T \|\tilde{w}^*\|_{2+\varepsilon,\Omega}^2 dt \right]^{\frac{1}{2}} \\
& \leq C \|\tilde{y}^*\|_{\mathcal{X}} \left[\int_0^T \|\tilde{w}^*\|_{\Omega}^{2[\frac{1-\varepsilon}{3}]} \|\tilde{w}^*\|_{3,\Omega}^{2[\frac{2+\varepsilon}{3}]} dt \right]^{\frac{1}{2}} \\
& \leq C \|\tilde{y}^*\|_{\mathcal{X}} \sup_{t \in [0,T]} \{\|\tilde{w}^*\|_{\Omega}\}^{\frac{1-\varepsilon}{3}} \left[\int_0^T \|\tilde{w}^*\|_{3,\Omega}^{2[\frac{2+\varepsilon}{3}]} dt \right]^{\frac{1}{2}} \\
& \leq C \|\tilde{y}^*\|_{\mathcal{X}} \sup_{t \in [0,T]} \{\|\tilde{w}^*\|_{\Omega}\}^{\frac{1-\varepsilon}{3}} \left[\int_0^T \|\tilde{w}^*\|_{3,\Omega}^2 dt \right]^{\frac{1}{2}[\frac{2+\varepsilon}{3}]} T^{\frac{1-\varepsilon}{6}} \\
& \leq C \|\tilde{y}^*\|_{\mathcal{X}} \left[\sup_{t \in [0,T]} \{\|\tilde{w}^*\|_{\Omega}\} + \left[\int_0^T \|\tilde{w}^*\|_{3,\Omega}^2 dt \right]^{\frac{1}{2}} \right] T^{\frac{1-\varepsilon}{6}} \\
& \leq C \|\tilde{y}^*\|_{\mathcal{X}} \sup_{i=1,2} \{ \|y^{i,*}\|_{\mathcal{X}} \} T^{\frac{1-\varepsilon}{6}}.
\end{aligned}$$

Applying the same argument to the other terms in (4.56) we find the following estimate for $\tilde{\mathcal{J}}_t$

$$\|\tilde{\mathcal{J}}_t\|_{L^1(0,T;L^2(\Omega))} \leq C \|\tilde{y}^*\|_{\mathcal{X}} \sup_{i=1,2} \{ \|y^{i,*}\|_{\mathcal{X}} \} T^{\frac{1-\varepsilon}{6}}. \quad (4.57)$$

Let us estimate $\tilde{\mathcal{J}}$ term. Note that $\tilde{\mathcal{J}}$ can be rewrite as

$$\begin{aligned}
\tilde{\mathcal{J}} &= \sigma[N(u^{1,*}, w^{1,*})] \nabla w^{1,*} - \sigma[N(u^{2,*}, w^{2,*})] \nabla w^{2,*} + \phi^{1,*} \nabla w^{1,*} - \phi^{2,*} \nabla w^{2,*} \\
&= \sigma[\varepsilon(\tilde{u}^*)] \nabla w^{2,*} + \sigma[f(\nabla w^{1,*}) - f(\nabla w^{2,*})] \nabla w^{2,*} + [\varepsilon(u^{1,*}) + f(\nabla w^{1,*})] \nabla w^* \\
&\quad + \tilde{\phi}^* \nabla w^{2,*} + \phi^{1,*} \nabla \tilde{w}^* \\
&= \sigma[\varepsilon(\tilde{u}^*)] \nabla w^{2,*} + (\sigma[f(\nabla \tilde{w}^*)] + \sigma[\nabla \tilde{w}^* \otimes \nabla w^{2,*}] + \sigma[\nabla w^{2,*} \otimes \nabla \tilde{w}^*]) \nabla w^{2,*} \\
&\quad + [\varepsilon(u^{1,*}) + f(\nabla w^{1,*})] \nabla w^* + \tilde{\phi}^* \nabla w^{2,*} + \phi^{1,*} \nabla \tilde{w}^*.
\end{aligned} \quad (4.58)$$

To estimate $\tilde{\mathcal{J}}$ in $L^{2+\varepsilon_0}(0,T;L^2(\Omega))$ we have to estimate all the products in (4.58). Here we will estimate the first and second terms, the others can estimate in a similar away. From Hölder's inequality and (4.48), with $\frac{2}{\varepsilon} = 2 + 2\varepsilon_0$, we can show that

$$\begin{aligned}
\|\sigma[\varepsilon(\tilde{u}^*)] \nabla w^{2,*}\|_{L^{2+\varepsilon_0}(0,T;L^2(\Omega))} &\leq C \|\varepsilon(\tilde{u}^*)\|_{C([0,T];L^2(\Omega))} \|w^{2,*}\|_{L^{2+\varepsilon_0}(0,T;H^{2+\varepsilon}(\Omega))} \\
&\leq C \|\tilde{y}^*\|_{C([0,T];\mathcal{X})} \|w^{2,*}\|_{L^{2+2\varepsilon_0}(0,T;H^{2+\varepsilon}(\Omega))} T^{r_0} \\
&\leq C \|\tilde{y}^*\|_{\mathcal{X}} \|y^{2,*}\|_{\mathcal{X}} T^{r_0} \\
&\leq C \|\tilde{y}^*\|_{\mathcal{X}} \sup_{i=1,2} \{ \|y^{i,*}\|_{\mathcal{X}} \} T^{r_0},
\end{aligned}$$

$$\begin{aligned}
\|\sigma[f(\nabla \tilde{w}^*)]\nabla w^{2,*}\|_{L^{2+\varepsilon_0}(0,T;L^2(\Omega))} &\leq C\|\nabla \tilde{w}^*\|_{C([0,T];L^2(\Omega))}^2\|w^{2,*}\|_{L^{2+\varepsilon_0}(0,T;H^{2+\varepsilon}(\Omega))} \\
&\leq C\|\tilde{y}^*\|_{\mathcal{X}}^2\|y^{2,*}\|_{\mathcal{X}}T^{r_0} \\
&\leq C\|\tilde{y}^*\|_{\mathcal{X}}^2\sup_{i=1,2}\{\|y^{i,*}\|_{\mathcal{X}}\}T^{r_0}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\|\tilde{\mathcal{J}}\|_{L^{2+\varepsilon_0}(0,T;L^2(\Omega))} &\leq C\|\tilde{y}^*\|_{\mathcal{X}}\sup_{i=1,2}\{\|y^{i,*}\|_{\mathcal{X}}\}T^{r_0} + C\|\tilde{y}^*\|_{\mathcal{X}}^2\sup_{i=1,2}\{\|y^{i,*}\|_{\mathcal{X}}\}T^{r_0} \\
&\leq C\|\tilde{y}^*\|_{\mathcal{X}}\sup_{i=1,2}\{\|y^{i,*}\|_{\mathcal{X}}\}T^{r_0} + C\|\tilde{y}^*\|_{\mathcal{X}}\sup_{i=1,2}\{\|y^{i,*}\|_{\mathcal{X}}^2\}T^{r_0}.
\end{aligned} \tag{4.59}$$

Finally let us estimate $\tilde{\mathcal{K}}$. For $\psi \in H_0^1(\Omega)$ we find that

$$\begin{aligned}
(\tilde{\mathcal{K}}, \psi)_{\Omega} &= (\nabla w^{1,*} \cdot \nabla w_t^{1,*} - \nabla w^{2,*} \cdot \nabla w_t^{2,*}, \psi)_{\Omega} \\
&= (\nabla w^{1,*} \cdot \nabla \tilde{w}_t^* + \nabla \tilde{w}^* \cdot \nabla w_t^{2,*}, \psi)_{\Omega} \\
&\leq C\left[\|\nabla \tilde{w}_t^*\|_{-\varepsilon, \Omega}\|\nabla w^{1,*}\psi\|_{\varepsilon, \Omega} + \|\nabla w_t^{2,*}\|_{-\varepsilon, \Omega}\|\nabla \tilde{w}^*\psi\|_{\varepsilon, \Omega}\right] \\
&\leq C\left[\|\nabla \tilde{w}_t^*\|_{-\varepsilon, \Omega}\|\nabla w^{1,*}\|_{\varepsilon+\varepsilon_0, \Omega} + \|\nabla w_t^{2,*}\|_{-\varepsilon, \Omega}\|\nabla \tilde{w}^*\|_{\varepsilon+\varepsilon_0, \Omega}\right]\|\psi\|_{1, \Omega}.
\end{aligned}$$

After integration on time variable, from this inequality we get that

$$\|\tilde{\mathcal{K}}\|_{L^2(0,T;H^{-1}(\Omega))}^2 \leq C\int_0^T\left[\|\nabla \tilde{w}_t^*\|_{-\varepsilon, \Omega}^2\|\nabla w^{1,*}\|_{\varepsilon+\varepsilon_0, \Omega}^2 + \|\nabla w_t^{2,*}\|_{-\varepsilon, \Omega}^2\|\nabla \tilde{w}^*\|_{\varepsilon+\varepsilon_0, \Omega}^2\right]dt \tag{4.60}$$

To conclude the estimate of $\tilde{\mathcal{K}}$ in $L^2(0,T;H^{-1}(\Omega))$ we need to estimate the two integrals on the right-hand side of (4.60). To deal with this we use Interpolation inequality and then we find the estimates

$$\begin{aligned}
\int_0^T\|\nabla w_t^{2,*}\|_{-\varepsilon, \Omega}^2\|\nabla \tilde{w}^*\|_{\varepsilon+\varepsilon_0, \Omega}^2dt &\leq C\|\tilde{w}^*\|_{C([0,T];H^2(\Omega))}^2\int_0^T\|w_t^{2,*}\|_{1-\varepsilon, \Omega}^2dt \\
&\leq C\|\tilde{y}^*\|_{C([0,T];\mathcal{X})}^2\int_0^T\|w_t^{2,*}\|_{1, \Omega}^{2(1-\varepsilon)}\|w_t^*\|_{\Omega}^{2\varepsilon}dt \\
&\leq C\|\tilde{y}^*\|_{\mathcal{X}}^2\|w_t^{2,*}\|_{C(0,T;L^2(\Omega))}^{2\varepsilon}\left[\int_0^T\|w_t^{2,*}\|_{1, \Omega}^2dt\right]^{1-\varepsilon}T^{\varepsilon} \\
&\leq C\|\tilde{y}^*\|_{\mathcal{X}}^2\left[\|w_t^{2,*}\|_{C(0,T;L^2(\Omega))}^2 + \int_0^T\|w_t^{2,*}\|_{1, \Omega}^2dt\right]T^{\varepsilon} \\
&\leq C\|\tilde{y}^*\|_{\mathcal{X}}^2\|y^{2,*}\|_{\mathcal{X}}^2T^{\varepsilon} \leq C\|\tilde{y}^*\|_{\mathcal{X}}^2\sup_{i=1,2}\{\|y^{i,*}\|_{\mathcal{X}}^2\}T^{\varepsilon},
\end{aligned}$$

$$\int_0^T\|\nabla \tilde{w}_t^*\|_{-\varepsilon, \Omega}^2\|\nabla w^{1,*}\|_{\varepsilon+\varepsilon_0, \Omega}^2dt \leq C\|\tilde{y}^*\|_{\mathcal{X}}^2\sup_{i=1,2}\{\|y^{i,*}\|_{\mathcal{X}}^2\}T^{\varepsilon}.$$

This estimates imply that $\tilde{\mathcal{K}}$ can be estimate as

$$\|\tilde{\mathcal{K}}\|_{L^2(0,T;H^{-1}(\Omega))} \leq C\|\tilde{y}^*\|_{\mathcal{X}}\sup_{i=1,2}\{\|y^{i,*}\|_{\mathcal{X}}\}T^{\frac{\varepsilon}{2}}. \tag{4.61}$$

Combining the estimates (4.37), (4.59) and (4.61) together with Proposition 4.1 we get that

$$\begin{aligned} \|\tilde{y}\|_{C([0,T];\mathcal{Y})} &\leq C\|\tilde{y}^*\|_{\mathcal{X}} \sup_{i=1,2} \{\|y^{i,*}\|_{\mathcal{X}}\} T^{\frac{1-\varepsilon}{6}} + C\|\tilde{y}^*\|_{\mathcal{X}} \sup_{i=1,2} \{\|y^{i,*}\|_{\mathcal{X}}\} T^{r_0} \\ &\quad + C\|\tilde{y}^*\|_{\mathcal{X}}^2 \sup_{i=1,2} \{\|y^{i,*}\|_{\mathcal{X}}\} T^{r_0} + C\|\tilde{y}^*\|_{\mathcal{X}} \sup_{i=1,2} \{\|y^{i,*}\|_{\mathcal{X}}\} T^{\frac{\varepsilon}{2}} \\ &\leq \left[CRT^{\frac{1-\varepsilon}{6}} + CRT^{r_0} + CR^2 T^{r_0} + CRT^{\frac{\varepsilon}{2}} \right] \|\tilde{y}^*\|_{\mathcal{X}}, \end{aligned}$$

$$\begin{aligned} \int_0^T [\|\tilde{w}\|_{3,\Omega}^2 + \|\tilde{w}_t\|_{1,\Omega}^2] dt &\leq C\|\tilde{y}^*\|_{\mathcal{X}} \sup_{i=1,2} \{\|y^{i,*}\|_{\mathcal{X}}\} T^{r_0} + C\|\tilde{y}^*\|_{\mathcal{X}}^2 \sup_{i=1,2} \{\|y^{i,*}\|_{\mathcal{X}}\} T^{r_0} \\ &\leq \left[CRT^{r_0} + CR^2 T^{r_0} \right] \|\tilde{y}^*\|_{\mathcal{X}} \end{aligned}$$

This inequalities and (4.54) implies that

$$\begin{aligned} \|\mathcal{T}y^{1,*} - \mathcal{T}y^{2,*}\|_{\mathcal{X}} &\leq \left[CRT^{\frac{1-\varepsilon}{6}} + CRT^{r_0} + CR^2 T^{r_0} + CRT^{\frac{\varepsilon}{2}} \right] \|y^{1,*} - y^{2,*}\|_{\mathcal{X}} \\ &\leq \frac{1}{2} \|y^{1,*} - y^{2,*}\|_{\mathcal{X}}. \end{aligned}$$

Then (4.55) holds true and therefore the map $\mathcal{T}$ possesses a unique fixed point y and this fixed point is the solution of the problem in $[0, T]$

Step 4. Global Solution: First note that from identity (4.15) and inequality (4.18) we obtain a priori bounds on the space $\mathcal{Y}$. The proof is complete if we show the following inequality

$$\int_0^T [\|w\|_{3,\Omega}^2 + \|w_t\|_{1,\Omega}^2 + \|\theta\|_{1,\Omega}^2] dt \leq C_{T, \|y(0)\|_Y}. \quad (4.62)$$

Here we will use again the fact that the semigroup is analytic. Proceeding as in Step 1 we have that (w, θ) satisfy

$$\begin{bmatrix} w_t \\ w_{tt} \\ \theta_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{A}_M G_2(F(u, w, \phi) \cdot \nu) + \operatorname{div}\{F(u, w, \phi)\} \\ 0 \end{bmatrix},$$

with

$$F(u, w, \phi) = \sigma[\varepsilon(u) + f(\nabla w)] \nabla w + \phi \nabla w \quad (4.63)$$

and the following estimate holds

$$\left\| \mathcal{A}^{\frac{1}{2}} \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} \right\|_{L^2(0,T;\mathcal{Y}_2)} \leq C \left\| \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} \right\|_{\mathcal{Y}_2} + C \left\| \mathcal{A}^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \mathcal{F}(u, w, \phi) \\ 0 \end{bmatrix} \right\|_{L^2(0,T;\mathcal{Y}_2)}, \quad (4.64)$$

where

$$\mathcal{F}(u, w, \phi) = \mathcal{A}_M G_2(F(u, w, \phi) \cdot \nu) + \operatorname{div}\{F(u, w, \phi)\}. \quad (4.65)$$

By duality we find that

$$\left\| \mathcal{A}^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \mathcal{F}(u, w, \phi) \\ 0 \end{bmatrix} \right\|_{\mathcal{H}_2} \leq C \|\mathcal{A}_M G_2(F(u, w, \phi) \cdot \nu) + \operatorname{div}\{F(u, w, \phi)\}\|_{-1, \Omega}. \quad (4.66)$$

Equalities (4.31) gives, for every $\psi \in H^{1+\varepsilon}(\Omega)$ with $0 < \varepsilon < 1$,

$$(\mathcal{A}_M G_2(F(u, w, \phi) \cdot \nu) + \operatorname{div}\{F(u, w, \phi)\}, \psi)_\Omega = (-F(u, w, \phi), \nabla \psi)_\Omega$$

and consequently we have by the definition of F and continuous embedding $H^{1+\varepsilon}(\Omega) \subset L^\infty(\Omega)$ that

$$\begin{aligned} & \|\mathcal{A}_M G_2(F(u, w, \phi) \cdot \nu) + \operatorname{div}\{F(u, w, \phi)\}\|_{-1, \Omega} \\ & \leq C \left[\|\sigma[\varepsilon(u) + f(\nabla w)]\|_\Omega + \|\phi\|_\Omega \right] \|w\|_{2+\varepsilon, \Omega}. \end{aligned}$$

This together with (4.66) will give that

$$\left\| \mathcal{A}^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \mathcal{F}(u, w, \phi) \\ 0 \end{bmatrix} \right\|_{\mathcal{H}_2} \leq C \left[\|\sigma[\varepsilon(u) + f(\nabla w)]\|_\Omega + \|\phi\|_\Omega \right] \|w\|_{2+\varepsilon, \Omega}. \quad (4.67)$$

The characterization (4.41) of $D(\mathcal{A}^{\frac{1}{2}})$, estimate (4.67) and the a priori bound on $C([0, T]; \mathcal{Y})$ imply that

$$\begin{aligned} \int_0^T \left[\|w\|_{3, \Omega}^2 + \|w_t\|_{1, \Omega}^2 + \|\theta\|_{1, \Omega}^2 \right] dt & \leq C \|y(0)\|_{\mathcal{Y}}^2 + \delta \int_0^T \|w\|_{3, \Omega}^2 dt \\ & \quad + C_{\delta, \|y\|_{C([0, T], \mathcal{Y})}} \int_0^T \|w\|_{\Omega}^2 dt. \end{aligned}$$

Taking $\delta > 0$ small enough we obtain that (4.62) holds true. $\square$

Proof of Theorem 4.1 (ii):

Step 1. To proof this part we need to consider the time derivative of the original system. Using the following notation

$$\begin{aligned} \bar{u} &= u_t, \bar{w} = w_t, \bar{\bar{u}} = u_t, \bar{\phi} = \phi_t, \bar{\theta} = \theta_t, f_t(\bar{w}, w) = \frac{d}{dt} f(\nabla w), \\ \mathcal{K}_t &= \frac{d}{dt} (\nabla w \cdot \nabla \bar{w}), F_t = \frac{d}{dt} (\sigma[\varepsilon(u) + f(\nabla w)] \nabla w + \phi \nabla w), \end{aligned} \quad (4.68)$$

where $(u, u_t, w, w_t, \phi, \theta)$ is the weak solution of the system and differentiating in time (in the sense of distributions) the system (4.29)-(4.30) we have that $(\bar{u}, \bar{u}_t, \bar{w}, \bar{w}_t, \bar{\phi}, \bar{\theta})$ solves the problem

$$\begin{cases} \bar{u}_{tt} - \operatorname{div}\{\sigma[\varepsilon(\bar{u})]\} + \nabla \bar{\phi} - \operatorname{div}\{\sigma[f_t(\bar{w}, w)]\} = 0 & \text{in } \Omega \times (0, \infty), \\ \bar{w}_{tt} + \Delta^2 \bar{w} + \Delta \bar{\theta} - \operatorname{div}\{F_t\} = 0 & \text{in } \Omega \times (0, \infty), \\ \bar{\phi}_t - \Delta \bar{\phi} + \operatorname{div}\{\bar{u}_t\} - \mathcal{K}_t = 0 & \text{in } \Omega \times (0, \infty), \\ \bar{\theta}_t - \Delta \bar{\theta} - \Delta \bar{w}_t = 0 & \text{in } \Omega \times (0, \infty), \end{cases} \quad (4.69)$$

and satisfy the boundary condition

$$\begin{cases} \bar{u} = 0, \bar{w} = 0, \nabla \bar{w} = 0 & \text{on } \Gamma_0 \times (0, \infty), \\ \sigma[\varepsilon(\bar{u}) + f_t(\bar{w}, w)]\nu + \kappa \bar{u} - \bar{\phi}\nu + \bar{u}_t = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ [\Delta \bar{w} + (1 - \mu)B_1 \bar{w}] + \bar{\theta} = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \left[\frac{\partial}{\partial \nu} \Delta \bar{w} + (1 - \mu)B_2 \bar{w} \right] - F_t \nu + \frac{\partial \bar{\theta}}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \frac{\partial \bar{\phi}}{\partial \nu} + \lambda_1 \bar{\phi} = \frac{\partial \bar{\theta}}{\partial \nu} + \lambda_2 \bar{\theta} = 0 & \text{on } \Gamma \times (0, \infty). \end{cases} \quad (4.70)$$

Step 2. We begin this Step with an auxiliary result which provides higher regularity for w displacement provided that the initial data are smoother. First, from the proof of the first part of Theorem 4.1 we have the following apriori bound

$$\begin{aligned} \sup_{t \geq 0} \left\{ \|N(u, w)\|_{\Omega} + \|u\|_{1, \Omega} + \|u_t\|_{\Omega} + \|w\|_{2, \Omega} + \|w_t\|_{\Omega} + \|\phi\|_{\Omega} + \|\theta\|_{\Omega} \right\} &\leq C_{E(0)}, \\ \int_0^T \left[\|w\|_{3, \Omega}^2 + \|w_t\|_{1, \Omega}^2 + \|\phi\|_{1, \Omega}^2 + \|\theta\|_{1, \Omega}^2 \right] dt &\leq C_{E(0)}, \end{aligned} \quad (4.71)$$

where the positive constant $C_{E(0)}$ denotes the finite energy norm of the initial data.

Proposition 4.2. Let $p, s > 0$ be such that $(\frac{1}{2} - s)p < 1$. Then for any initial data $y(0) \in D(\mathcal{A}^s)$ we have that

$$w \in L^p(0, T, H^3(\Omega)) \cap W^{1,p}(0, T, H^1(\Omega)).$$

Moreover

$$\nabla w \in L^\infty((0, T) \times \Omega)$$

and

$$\|\nabla w\|_{L^\infty((0, T) \times \Omega)} \leq C_{E(0)} \|(w, w_t, \theta)\|_{D(\mathcal{A}^s)} \leq C_{E(0), E_1},$$

where C_{E_1} denotes the positive constant depending on higher norm of the initial data in $\mathcal{H}_1$.

Proof. As before we have the following equality

$$\mathcal{A}^{\frac{1}{2}} \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} = \mathcal{A}^{\frac{1}{2}} e^{\mathcal{A}t} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} + \int_0^t \mathcal{A} e^{\mathcal{A}(t-s)} \mathcal{A}^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \mathcal{F} \\ 0 \end{bmatrix} ds, \quad (4.72)$$

where $\mathcal{F}(\cdot)$ and $F(\cdot)$ was defined in (4.63) and (4.65), respectively. Using fact that the map

$$f \mapsto \int_0^t \mathcal{A} e^{\mathcal{A}(t-s)} f(s) ds$$

is bounded from $L^p(0, T; \mathcal{B}_2)$ into itself, for $1 < p < \infty$ we find that

$$\begin{aligned} \left\| \mathcal{A}^{\frac{1}{2}} \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} \right\|_{L^p(0, T; \mathcal{B}_2)} &\leq C \left\| \mathcal{A}^{\frac{1}{2}-s} e^{\mathcal{A}t} \mathcal{A}^s \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} \right\|_{L^p(0, T; \mathcal{B}_2)} + C \left\| \mathcal{A}^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \mathcal{F} \\ 0 \end{bmatrix} \right\|_{L^p(0, T; \mathcal{B}_2)} \\ &\leq C \left\| \mathcal{A}^{\frac{1}{2}-s} e^{\mathcal{A}t} \mathcal{A}^s (w_0, w_1, \theta_0) \right\|_{L^p(0, T; \mathcal{B}_2)} + C \|F\|_{L^p(0, T; L^2(\Omega))}. \end{aligned} \quad (4.73)$$

To conclude the proof of the first part of Proposition, we have to estimate the norms on the right-hand side of (4.73). The definition of F , Hölder's inequality and the continuous embedding $H^{1+\varepsilon}(\Omega) \subset L^\infty(\Omega)$ yields the estimate

$$\begin{aligned} &\|F\|_{L^p(0, T; L^2(\Omega))} \\ &\leq C \left[\|\sigma[\varepsilon(u) + f(\nabla w)]\|_{C([0, T]; L^2(\Omega))} + \|\phi\|_{C([0, T]; L^2(\Omega))} \right] \|w\|_{L^p(0, T; H^{2+\varepsilon}(\Omega))}. \end{aligned}$$

Taking in this inequality $p = \frac{2}{\varepsilon}$ we have by the interpolation inequality (4.50) and apriori bounds (4.71) that

$$\begin{aligned} &\|F\|_{L^p(0, T; L^2(\Omega))} \\ &\leq C \left[\|\sigma[\varepsilon(u) + f(\nabla w)]\|_{C(0, T; L^2(\Omega))} + \|\phi\|_{C(0, T; L^2(\Omega))} \right] \|w\|_{L^{\frac{2}{\varepsilon}}(0, T; H^{2+\varepsilon}(\Omega))} \\ &\leq C_{E(0)}. \end{aligned}$$

To estimate the contribution of the initial data (w_0, w_1, θ_0) in (4.73) we shall use the estimate (4.40) and the fact that p and s satisfy $(\frac{1}{2} - s)p < 1$ to obtain

$$\begin{aligned} \left\| \mathcal{A}^{\frac{1}{2}-s} e^{\mathcal{A}t} \mathcal{A}^s (w_0, w_1, \theta_0) \right\|_{L^p(0, T; \mathcal{B}_2)} &\leq \left[\int_0^T \frac{1}{t^{(\frac{1}{2}-s)p}} dt \right]^{\frac{1}{p}} \|\mathcal{A}^s (w_0, w_1, \theta_0)\|_{\mathcal{B}_2} \\ &\leq C \|(w_0, w_1, \theta_0)\|_{D(\mathcal{A}^s)}. \end{aligned}$$

This estimate in (4.73) results the first part of the Proposition. The second part is a consequence of Sobolev's embeddings and interpolation theory

$$w \in L^p(0, T; H^3(\Omega)) \cap W^{1,p}(0, T; L^2(\Omega)) \subset W^{\frac{1}{2}-\varepsilon, p}(0, T; L^2(\Omega)) \subset L^\infty((0, T) \times \Omega)$$

for any $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2} - \frac{1}{p}$. □

Step 3. Estimate for u displacement: Multiplying the equation (4.69)₁ by $\bar{u}_t$, (4.69)₃ by $\bar{\phi}$ and integrating over $\Omega \times [0, t]$ we find that

$$\begin{aligned}
 E_{\bar{u}, \bar{\phi}}(t) &+ 2 \int_0^t \left[\|\bar{u}_t(s)\|_{\Gamma_1}^2 + \|\nabla \bar{\phi}(s)\|_{\Omega}^2 + \lambda_1 \|\bar{\phi}(s)\|_{\Gamma}^2 \right] ds \\
 &= E_{\bar{u}, \bar{\phi}}(0) - 2 \int_0^t (\sigma[f_t], \varepsilon(\bar{u}_t))_{\Omega} ds + 2 \int_0^t (\mathcal{K}_t, \bar{\phi})_{\Omega} ds \\
 &= E_{\bar{u}, \bar{\phi}}(0) - 2(\sigma[f_t], \varepsilon(\bar{u}))_{\Omega}|_0^t + 2 \int_0^t \left(\frac{d}{dt} \sigma[f_t], \varepsilon(\bar{u}) \right)_{\Omega} ds \\
 &\quad + 2 \int_0^t (\mathcal{K}_t, \bar{\phi})_{\Omega} ds.
 \end{aligned} \tag{4.74}$$

where $E_{\bar{u}, \bar{\phi}}(\cdot)$ is given (4.36). Then, using Young's and Hölder's inequalities we find

$$\begin{aligned}
 E_{\bar{u}, \bar{\phi}}(t) &+ \int_0^t \left[\|\bar{u}(s)\|_{\Gamma_1}^2 + \|\nabla \bar{\phi}(s)\|_{\Omega}^2 + \|\bar{\phi}(s)\|_{\Gamma}^2 \right] ds \\
 &\leq C_{E_1} + \delta \int_0^t \left\| \frac{d}{dt} f_t \right\|_{\Omega}^2 ds + C_{\delta} \int_0^t \|\varepsilon(\bar{u})\|_{\Omega}^2 ds + C \|f_t\|_{\Omega}^2 + C \int_0^t \|\mathcal{K}_t\|_{-1, \Omega}^2 ds.
 \end{aligned} \tag{4.75}$$

Let us estimate the integrals on the right-hand side of (4.75). The definition of $f_t(\cdot)$ and the continuous embedding $H^{1+\varepsilon}(\Omega) \subset L^{\infty}(\Omega)$ imply that

$$\begin{aligned}
 \left\| \frac{d}{dt} f_t \right\|_{\Omega} &\leq C \left[\|\nabla \bar{w}_t\|_{\Omega} \|\nabla w\|_{L^{\infty}(\Omega)} + \|\nabla \bar{w}\|_{L^{\infty}(\Omega)} \|\nabla w_t\|_{\Omega} \right] \\
 &\leq C \left[\|\nabla \bar{w}_t\|_{\Omega} \|\nabla w\|_{L^{\infty}(\Omega)} + \|\nabla \bar{w}\|_{1+\varepsilon, \Omega} \|\nabla w_t\|_{\Omega} \right].
 \end{aligned}$$

Integrating this inequality over $[0, t]$ and applying Hölder's inequality in time variable (with exponents $p_1 = \infty$, $p_2 = 2$ for the first product and $p_1 = \frac{2}{\varepsilon}$, $p_2 = \frac{2}{1-\varepsilon}$ for the second) we find that

$$\begin{aligned}
 \int_0^t \left\| \frac{d}{dt} f_t \right\|_{\Omega}^2 ds &\leq C \|\nabla w\|_{L^{\infty}((0, T) \times \Omega)}^2 \int_0^t \|\nabla \bar{w}_t\|_{\Omega}^2 ds \\
 &\quad + C \|\bar{w}\|_{L^{\frac{2}{\varepsilon}}(0, T; H^{2+\varepsilon}(\Omega))}^2 \|\nabla w_t\|_{L^{\frac{2}{1-\varepsilon}}(0, T; L^2(\Omega))}^2.
 \end{aligned}$$

Using the result of Proposition 4.2 with $p = \frac{2}{1-\varepsilon}$ and a priori bounds (4.71) we find that

$$\begin{aligned}
 \|\nabla w\|_{L^{\infty}((0, T) \times \Omega)}^2 &\leq C_{E(0), E_1}, \\
 \|\nabla w_t\|_{L^{\frac{2}{1-\varepsilon}}(0, T; L^2(\Omega))}^2 &\leq C \|w_t\|_{L^2(0, T; H^1(\Omega))}^2 \leq C_{E(0)}
 \end{aligned}$$

and (4.50) implies

$$\|\bar{w}\|_{L^{\frac{2}{\varepsilon}}(0, t; H^{2+\varepsilon}(\Omega))} \leq C \left[\|\bar{w}\|_{L^2(0, t; H^3(\Omega))} + \|\bar{w}_t\|_{L^2(0, t; H^1(\Omega))} \right].$$

This inequalities result that

$$\int_0^t \left\| \frac{d}{dt} f_t \right\|_{\Omega}^2 ds \leq C_{E(0), E_1} \int_0^t \left[\|\bar{w}_t\|_{1, \Omega}^2 + \|\bar{w}_t\|_{3, \Omega}^2 \right] ds \tag{4.76}$$

To estimate the integral of $f_t(\cdot)$ in $L^2(\Omega)$ we use interpolation inequality and a priori bound (4.71)

$$\begin{aligned} \|f_t\|_{\Omega}^2 &\leq C \|\nabla \bar{w}\|_{\Omega} \|\nabla w\|_{L^\infty(\Omega)} \leq C_{E_1, \delta} \|\bar{w}\|_{1, \Omega} \leq C_{E_1, \delta} \|\bar{w}\|_{\Omega} + \delta \|\bar{w}\|_{2, \Omega} \\ &\leq C_{E(0), E_1, \delta} + \delta \|\bar{w}\|_{2, \Omega}. \end{aligned} \quad (4.77)$$

Finally let us estimate the last integral. Using interpolation inequality we find

$$\begin{aligned} \|\mathcal{K}_t\|_{-1, \Omega} &\leq C \left[\|\nabla \bar{w}_t \cdot \nabla w\|_{-1, \Omega} + \|\nabla \bar{w} \cdot \nabla w_t\|_{-1, \Omega} \right] \\ &\leq C \left[\|\bar{w}_t\|_{1-\varepsilon, \Omega} \|w\|_{1+\varepsilon+\varepsilon_0, \Omega} + \|\bar{w}\|_{1+\varepsilon+\varepsilon_0, \Omega} \|w_t\|_{1-\varepsilon, \Omega} \right] \\ &\leq C \|w\|_{2, \Omega} \left[\delta \|\bar{w}_t\|_{1, \Omega} + C_\delta \|\bar{w}_t\|_{\Omega} \right] + C \|w_t\|_{1, \Omega} \left[\delta \|\bar{w}\|_{2, \Omega} + C_\delta \|\bar{w}\|_{\Omega} \right] \end{aligned}$$

and integrating this inequality over $[0, t]$

$$\begin{aligned} \int_0^t \|\mathcal{K}_t\|_{-1, \Omega}^2 ds &\leq \delta \int_0^t \|\bar{w}_t\|_{1, \Omega}^2 ds + C_{E(0), \delta} \int_0^t \|\bar{w}_t\|_{\Omega}^2 ds \\ &\quad + C \left[\delta \|\bar{w}\|_{C(0, T; H^2(\Omega))}^2 + C_{E(0), \delta} \right] \int_0^t \|w_t\|_{1, \Omega}^2 ds \\ &\leq \delta \left[\int_0^t \|\bar{w}_t\|_{1, \Omega}^2 ds + \|\bar{w}\|_{C(0, T; H^2(\Omega))}^2 \right] \\ &\quad + C_{E(0), \delta} \int_0^t \|\bar{w}_t\|_{\Omega}^2 ds + C_{E(0), \delta}. \end{aligned} \quad (4.78)$$

Combining the estimates (4.76)-(4.78) in (4.75) we conclude that

$$\begin{aligned} E_{\bar{u}, \bar{\phi}}(t) + \int_0^t \left[\|\bar{u}(s)\|_{\Gamma_1}^2 + \|\nabla \bar{\phi}(s)\|_{\Omega}^2 + \int_0^t \|\bar{\phi}(s)\|_{\Gamma}^2 ds \right] ds \\ \leq C_{E(0), E_1, \delta} + \delta \int_0^t \|\bar{w}_t\|_{1, \Omega}^2 ds + \delta \int_0^t \|\bar{w}_t\|_{3, \Omega}^2 ds \\ + \delta \|\bar{w}\|_{C([0, T]; H^2(\Omega))}^2 + C_{E(0), \delta} \int_0^t [\|\varepsilon(\bar{u})\|_{\Omega}^2 + \|\bar{w}_t\|_{\Omega}^2] ds. \end{aligned} \quad (4.79)$$

Step 4. Estimate for w displacement: Here we will use again the analyticity of the semigroup. The solution $(\bar{w}, \bar{w}_t, \bar{\theta})$ satisfy

$$\mathcal{A}^{\frac{1}{2}} \begin{bmatrix} \bar{w} \\ \bar{w}_t \\ \bar{\theta} \end{bmatrix} = \mathcal{A}^{\frac{1}{2}} e^{\mathcal{A}t} \begin{bmatrix} \bar{w}_0 \\ \bar{w}_1 \\ \bar{\theta}_0 \end{bmatrix} + \int_0^t \mathcal{A} e^{\mathcal{A}(t-s)} \mathcal{A}^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \mathcal{F}_t \\ 0 \end{bmatrix} ds, \quad (4.80)$$

where $\mathcal{F}(u, w, \phi) = \operatorname{div}\{F_t(u, w, \phi)\} + \mathcal{A}_M G_2(F_t(u, w, \phi) \cdot \nu)$ and $F_t(\cdot)$ is defined in (4.68). Proceeding as before we obtain the following estimate

$$\|\mathcal{A}^{\frac{1}{2}}(\bar{w}, \bar{w}_t, \bar{\theta})\|_{L^2(0, t; \mathcal{H}_2)} \leq C \|(\bar{w}_0, \bar{w}_1, \bar{\theta}_0)\|_{\mathcal{H}_2} + C \|F_t\|_{L^2(0, t; L^2(\Omega))}. \quad (4.81)$$

Let us estimate the norm $\|F_t\|_{L^2(0,t;L^2(\Omega))}$. The definition of F_t and Hölder's inequality imply that

$$\begin{aligned} \|F_t\|_{\Omega} &\leq C\|\nabla w\|_{L^\infty(\Omega)}\|\varepsilon(\bar{u})\|_{\Omega} + C\|\nabla w\|_{L^\infty(\Omega)}\|f_t\|_{\Omega} \\ &\quad + C[\|\nabla \bar{w}\|_{L^\infty(\Omega)}\|N(u, w)\|_{\Omega} + \|\bar{\phi}\nabla w\|_{\Omega} + \|\phi\nabla \bar{w}\|_{\Omega}] \end{aligned} \quad (4.82)$$

We have the following estimates

$$\begin{aligned} \int_0^t \|\nabla w\|_{L^\infty(\Omega)}^2 \|\varepsilon(\bar{u})\|_{0,\Omega}^2 ds &\leq C\|\nabla w\|_{L^\infty((0,t)\times\Omega)}^2 \int_0^t \|\varepsilon(\bar{u})\|_{0,\Omega}^2 ds \\ &\leq C_{E(0),E_1} \int_0^t \|\varepsilon(\bar{u})\|_{0,\Omega}^2 ds, \end{aligned}$$

$$\int_0^t \|\nabla w\|_{L^\infty(\Omega)}^2 \|f_t\|_{0,\Omega}^2 ds \leq C\|\nabla w\|_{L^\infty((0,t)\times\Omega)}^4 \int_0^t \|\nabla \bar{w}\|_{0,\Omega}^2 ds \leq C_{E_0,E_1},$$

$$\begin{aligned} \int_0^t \|\nabla \bar{w}\|_{L^\infty(\Omega)}^2 \|N(u, w)\|_{0,\Omega}^2 ds &\leq C \int_0^t \|\nabla \bar{w}\|_{1+\varepsilon,\Omega}^2 \|N(u, w)\|_{0,\Omega}^2 ds \\ &\leq C \int_0^t \|\bar{w}\|_{\Omega}^{2[\frac{1-\varepsilon}{3}]} \|\bar{w}\|_{3,\Omega}^{2[\frac{2+\varepsilon}{3}]} \|N(u, w)\|_{0,\Omega}^2 ds \\ &\leq \delta \int_0^t \|\bar{w}\|_{3,\Omega}^2 ds + C_{\delta,t,E(0)}, \end{aligned}$$

$$\begin{aligned} \int_0^t \|\bar{\phi}\nabla w\|_{0,\Omega}^2 ds &\leq C \int_0^t \|\bar{\phi}\|_{0,\Omega}^2 \|\nabla w\|_{1+\varepsilon,\Omega}^2 ds \\ &\leq C_{E(0),E_1} \int_0^t \|\bar{\phi}\|_{0,\Omega}^2 ds, \end{aligned}$$

$$\begin{aligned} \int_0^t \|\phi\nabla \bar{w}\|_{0,\Omega}^2 ds &\leq C \int_0^t \|\phi\|_{0,\Omega}^2 \|\nabla \bar{w}\|_{1+\varepsilon,\Omega}^2 ds \\ &\leq C \int_0^t \|\phi\|_{0,\Omega}^2 \|\bar{w}\|_{\Omega}^{2[\frac{1-\varepsilon}{3}]} \|\bar{w}\|_{3,\Omega}^{2[\frac{2+\varepsilon}{3}]} ds \\ &\leq \delta \int_0^t \|\bar{w}\|_{3,\Omega}^2 ds + C_{\delta,t,E(0)}. \end{aligned}$$

Inserting this estimates in (4.82) we find that

$$\begin{aligned} \|F_t\|_{L^2(0,T;L^2(\Omega))} &\leq C_{E_1} \int_0^t \|\varepsilon(\bar{u})\|_{0,\Omega}^2 ds + \delta \int_0^t \|\bar{w}\|_{3,\Omega}^2 ds \\ &\quad + C_{E(0),E_1} \int_0^t \|\bar{\phi}\|_{0,\Omega}^2 ds + C_{\delta,t,E(0)}. \end{aligned}$$

Combining this inequality together with inequalities (4.81)

$$\begin{aligned}
& \int_0^t \left[\|\bar{w}\|_{3,\Omega}^2 + \|\bar{w}_t\|_{1,\Omega}^2 + \|\bar{\theta}\|_{1,\Omega}^2 \right] ds \\
& \leq C_{E(0)} + C_{E_1} \int_0^t \|\varepsilon(\bar{u})\|_{0,\Omega}^2 ds + \delta \int_0^t \|\bar{w}\|_{3,\Omega}^2 ds \\
& \quad + C_{E(0),E_1} \int_0^t \|\bar{\phi}\|_{0,\Omega}^2 ds + C_{\delta,t,E(0)}.
\end{aligned} \tag{4.83}$$

Step 5. Estimate for u and w displacements: Estimates (4.83) (4.79) yields the following inequality

$$\begin{aligned}
& E_{\bar{u},\bar{\phi}} + \int_0^t \left[\|\bar{u}\|_{\Gamma_1}^2 + \|\bar{\phi}\|_{1,\Omega}^2 + \|\bar{w}\|_{3,\Omega}^2 + \|\bar{w}_t\|_{1,\Omega}^2 + \|\bar{\theta}\|_{1,\Omega}^2 \right] ds \\
& \leq C_{E(0),E_1,t,\delta} + \delta \left[\int_0^t \|\bar{w}_t\|_{3,\Omega}^2 ds + \int_0^t \|\bar{w}_t\|_{1,\Omega}^2 ds + \|\bar{w}\|_{C([0,T];H^2(\Omega))}^2 \right] \\
& \quad + C_{E(0),\delta} \int_0^t \|\varepsilon(\bar{u})\|_{\Omega}^2 ds + C_{E(0),\delta} \int_0^t \|\bar{w}_t\|_{\Omega}^2 ds + C_{E(0),E_1} \int_0^t \|\bar{\phi}\|_{\Omega}^2 ds.
\end{aligned}$$

Taking δ small and using Gronwall's inequality we obtain, for $t \leq T$,

$$E_{\bar{u},\bar{\phi}} + \int_0^t \left[\|\bar{u}\|_{\Gamma_1}^2 + \|\bar{\phi}\|_{1,\Omega}^2 + \|\bar{w}\|_{3,\Omega}^2 + \|\bar{w}_t\|_{1,\Omega}^2 + \|\bar{\theta}\|_{1,\Omega}^2 \right] ds \leq C_{E(0),E_1,T}. \tag{4.84}$$

To conclude this Step let us show a pointwise estimate for w . Let us consider the well know estimate

$$\|\mathcal{A}^{\frac{1}{2}}(\bar{w}, \bar{w}_t, \bar{\theta})\|_{\mathcal{H}_2} \leq C\|(\bar{w}_0, \bar{w}_1, \bar{\theta}_0)\|_{\mathcal{H}_2} + C\|F_t\|_{L^2(0,T;L^2(\Omega))}. \tag{4.85}$$

We shall estimate $\|F_t\|_{L^2(0,T;L^2(\Omega))}$. Proceeding as in the beginning of Step 4 and using estimate (4.84) we obtain

$$\|F_t\|_{L^2(0,T;L^2(\Omega))} \leq C_{E(0),E_1,T}.$$

Therefore from (4.85) we obtain that

$$\|\bar{w}\|_{C([0,T];H^2(\Omega))}^2 + \|\bar{w}_t\|_{C([0,T];L^2(\Omega))}^2 + \|\bar{\theta}\|_{C([0,T];L^2(\Omega))}^2 \leq C_{E(0),E_1,T}.$$

Finally, this inequality and the estimate (4.84) imply that

$$\begin{aligned}
& \|u_t\|_{C([0,T];H^1(\Omega))}^2 + \|u_{tt}\|_{C([0,T];L^2(\Omega))}^2 + \|\phi_t\|_{C([0,T];L^2(\Omega))}^2 \\
& + \|w_t\|_{C([0,T];H^2(\Omega))}^2 + \|w_{tt}(t)\|_{C([0,T];L^2(\Omega))}^2 + \|\theta_t\|_{C([0,T];L^2(\Omega))}^2 \leq C_{E(0),E_1,T}.
\end{aligned}$$

To obtain the higher regularity in the space variable we can proceed in analogue way as in the end of the proof of Theorem 4.2. $\square$

4.8 Global Attractors

In this section we prove Theorem 4.2. Accordingly, we must show that $(\mathcal{H}, S(t))$ is quasi-stable with Lyapunov function satisfying (2.12). Then Theorem 2.19 will guarantee the existence of a regular global attractor. The finite fractal dimension and the smoothness of the attractor is a consequence of the Theorem 2.16 and 2.17.

4.8.1 Quasistability

In this subsection we shall prove that our problem is quasi-stable. Accordingly, we must show that the difference of two trajectories satisfies estimate the (2.10). The proof of the inequality (2.10) needs rather extensive background and several energy estimates. This will be established in five subsections. In what follows we use the notations

$$Q = [0, T] \times \Omega, \quad T > 0,$$

and

$$\Sigma_\alpha = [\alpha, T - \alpha] \times \Gamma_1, \quad 0 < \alpha < \frac{T}{2}.$$

4.8.1.1 Comparing two trajectories

Let B be a bounded set of $\mathcal{H}$ and consider two solutions of (4.1)-(4.6),

$$S(t)y_i = (u^i, u_t^i, w^i, w_t^i, \phi^i, \theta^i), \quad i = 1, 2, \quad (4.86)$$

with corresponding initial data $y_i(0) = (u_0^i, u_1^i, w_0^1, w_1^1, \phi_0^i, \theta_0^i) \in B, i = 1, 2$. Then the difference

$$(\tilde{u}, \tilde{w}, \tilde{\phi}, \tilde{\theta}) = (u^1 - u^2, w^1 - w^2, \phi^1 - \phi^2, \theta^1 - \theta^2), \quad (4.87)$$

solves the problem,

$$\tilde{u}_{tt} - \operatorname{div}\{\sigma[\varepsilon(\tilde{u})]\} + \nabla \tilde{\phi} + \mathcal{P}_1(\tilde{u}, \tilde{w}) = \operatorname{div}\{\mathcal{H}_1\} \quad \text{in } \Omega \times (0, \infty), \quad (4.88)$$

$$\tilde{w}_{tt} + \Delta^2 \tilde{w} + \Delta \tilde{\theta} + \mathcal{P}_1(\tilde{u}, \tilde{w}) = \operatorname{div}\{\mathcal{H}_2\} \quad \text{in } \Omega \times (0, \infty), \quad (4.89)$$

where

$$\mathcal{P}_1(\tilde{u}, \tilde{w}) = p_1(u^1, w^1) - p_1(u^2, w^2),$$

$$\mathcal{P}_2(\tilde{u}, \tilde{w}) = p_2(u^1, w^1) - p_2(u^2, w^2),$$

$$\mathcal{H}_1 = \sigma[f(\nabla w^1) - f(\nabla w^2)],$$

$$\mathcal{H}_2 = \sigma[N(u^1, w^1)]\nabla w^1 - \sigma[N(u^2, w^2)]\nabla w^2 + \phi^1 \nabla w^1 - \phi^2 \nabla w^2,$$

with thermal components

$$\tilde{\phi}_t - \Delta \tilde{\phi} + \operatorname{div}\{\tilde{u}_t\} - [\nabla w^1 \cdot \nabla w_t^1 - \nabla w^2 \cdot \nabla w_t^2] = 0 \text{ in } \Omega \times (0, \infty), \quad (4.90)$$

$$\tilde{\theta}_t - \Delta \tilde{\theta} - \Delta \tilde{w}_t = 0 \text{ in } \Omega \times (0, \infty), \quad (4.91)$$

and boundary conditions

$$\tilde{u} = \tilde{w} = \nabla \tilde{w} = 0 \text{ on } \Gamma_0 \times (0, \infty), \quad (4.92)$$

$$\sigma[\varepsilon(\tilde{u})]\nu + \sigma[f(\nabla w^1) - f(\nabla w^2)]\nu + \kappa \tilde{u} - \tilde{\phi}\nu + \tilde{u}_t = 0 \text{ on } \Gamma_1 \times (0, \infty), \quad (4.93)$$

$$[\Delta \tilde{w} + (1 - \mu)B_1 \tilde{w}] + \tilde{\theta} = 0 \text{ on } \Gamma_1 \times (0, \infty), \quad (4.94)$$

$$\left[\frac{\partial}{\partial \nu} \Delta \tilde{w} + (1 - \mu)B_2 \tilde{w} \right] - \mathcal{H}_2 \cdot \nu + \frac{\partial \tilde{\theta}}{\partial \nu} = 0 \text{ on } \Gamma_1 \times (0, \infty), \quad (4.95)$$

$$\frac{\partial \tilde{\phi}}{\partial \nu} + \lambda_1 \tilde{\phi} = \frac{\partial \tilde{\theta}}{\partial \nu} + \lambda_2 \tilde{\theta} = 0 \text{ on } \Gamma \times (0, \infty), \quad (4.96)$$

with corresponding initial data

$$\begin{aligned} \tilde{u}(\cdot, 0) &= u_0^1 - u_0^2, \quad \tilde{u}_t(\cdot, 0) = u_1^1 - u_1^2, \\ \tilde{w}(\cdot, 0) &= w_0^1 - w_0^2, \quad \tilde{w}_t(\cdot, 0) = w_1^1 - w_1^2, \\ \tilde{\phi}(\cdot, 0) &= \phi_0^1 - \phi_0^2, \quad \tilde{\theta}(\cdot, 0) = \theta_0^1 - \theta_0^2. \end{aligned} \quad (4.97)$$

To the system (4.88)-(4.97) we define the linear energy functional

$$\tilde{E}(t) = \frac{1}{2} \int_{\Omega} [|\tilde{u}_t|^2 + |\tilde{w}_t|^2 + \sigma[\varepsilon(\tilde{u})]\varepsilon(\tilde{u}) + |\tilde{\phi}|^2 + |\tilde{\theta}|^2] d\Omega + \frac{1}{2} a(\tilde{w}, \tilde{w}) + \frac{\kappa}{2} \int_{\Gamma_1} |\tilde{u}|^2 d\Gamma_1.$$

Then we have the following energy equality,

$$\tilde{E}(t) + D_s^t(\tilde{u}, \tilde{\phi}, \tilde{\theta}) = \tilde{E}(s) + \int_s^t \sum_{k=1}^5 \mathcal{R}_k(\tau) d\tau, \quad (4.98)$$

where

$$\begin{aligned} \mathcal{R}_1(t) &= - \int_{\Omega} \mathcal{P}_1(\tilde{u}, \tilde{w}) \cdot \tilde{u}_t d\Omega, \\ \mathcal{R}_2(t) &= - \int_{\Omega} \mathcal{P}_2(\tilde{u}, \tilde{w}) \tilde{w}_t d\Omega, \\ \mathcal{R}_3(t) &= \int_{\Omega} \sigma[f(\nabla w^1) - f(\nabla w^2)] \tilde{u}_t d\Omega, \\ \mathcal{R}_4(t) &= \int_{\Gamma_1} \sigma[f(\nabla w^1) - f(\nabla w^2)] \cdot \nu \tilde{u}_t d\Gamma_1, \\ \mathcal{R}_5(t) &= \int_{\Omega} [N(u^1, w^1) \nabla w^1 - N(u^2, w^2) \nabla w^2] \nabla \tilde{w}_t d\Omega, \\ \mathcal{R}_6(t) &= - \int_{\Omega} [\phi^1 \nabla w^1 - \phi^2 \nabla w^2] \nabla \tilde{w}_t d\Omega + \int_{\Omega} [\nabla w^1 \cdot \nabla w_t^1 - \nabla w^2 \cdot \nabla w_t^2] \tilde{\phi} d\Omega, \\ D_s^t(\tilde{u}, \tilde{\phi}, \tilde{\theta}) &= \int_s^t [\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\nabla \tilde{\phi}\|_{\Omega}^2 + \|\nabla \tilde{\theta}\|_{\Omega}^2 + \lambda_1 \|\tilde{\phi}\|_{\Gamma}^2 + \lambda_2 \|\tilde{\theta}\|_{\Gamma}^2] d\tau. \end{aligned}$$

Remark 4.4. We plan to verify condition (2.10) by obtaining an estimate like

$$\tilde{E}(t) \leq C\tilde{E}(0)e^{-\beta t} + C \left[\sup_{\tau \in [0,t]} \|\tilde{u}(\tau)\|_{L^{r+1}(\Omega)}^2 + \sup_{\tau \in [0,t]} \|\tilde{u}(\tau)\|_{1-\varepsilon,\Omega}^2 + \sup_{\tau \in [0,t]} \|\tilde{w}(\tau)\|_{2-\varepsilon,\Omega}^2 \right],$$

for suitable constants $C, \beta, \varepsilon > 0$. This will be achieved in Lemma 4.8.

We end this subsection with some estimates for $f(\nabla w^i)$, $i = 1, 2$.

Lemma 4.2. For every $\varepsilon \in (0, 1)$ the following estimates holds:

- (i) $\int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_{\Omega}^2 dt \leq C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}),$
- (ii) $\int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_{1,\Omega}^2 dt \leq C_B \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt,$
- (iii) $\int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_{\Sigma_1}^2 dt \leq C_B \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt,$

where the lower orders terms are given by

$$\text{l.o.t.}(\tilde{u}, \tilde{w}) = \sup_{t \in [0,T]} \|\tilde{u}(t)\|_{L^{r+1}(\Omega)}^2 + \sup_{t \in [0,T]} \|\tilde{u}(t)\|_{1-\varepsilon,\Omega}^2 + \sup_{t \in [0,T]} \|\tilde{w}(t)\|_{2-\varepsilon,\Omega}^2.$$

Proof. We shall use the identity

$$f(\nabla w^1) - f(\nabla w^2) = f(\nabla \tilde{w}) + \nabla \tilde{w} \otimes \nabla w^2 + \nabla w^2 \otimes \nabla \tilde{w}.$$

To prove (i), the inequality $\|u \otimes v\|_{\Omega} \leq C\|u\|_{\varepsilon,\Omega}\|v\|_{1-\varepsilon,\Omega}$ implies that

$$\begin{aligned} \int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_{\Omega}^2 dt &\leq C \int_0^T \left[\|\tilde{w}\|_{1+\varepsilon,\Omega}^2 \|\tilde{w}\|_{2-\varepsilon,\Omega}^2 + \|w^2\|_{1+\varepsilon,\Omega}^2 \|\tilde{w}\|_{2-\varepsilon,\Omega}^2 \right] dt \\ &\leq C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned}$$

To prove (ii), we will use the inequality $\|u \otimes v\|_{1,\Omega} \leq C\|u\|_{1,\Omega}\|v\|_{1+\varepsilon,\Omega}$, and then

$$\begin{aligned} \int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_{1,\Omega}^2 dt &\leq C \int_0^T \left[\|\tilde{w}\|_{2,\Omega}^2 \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 + \|w^2\|_{2,\Omega}^2 \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 \right] dt \\ &\leq C_B \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt. \end{aligned}$$

The estimate (iii) follows from Trace Theorem and estimate (ii). □

4.8.1.2 A first observability inequality

Here we obtain an observability inequality that reconstructs the integral of the linear energy in terms of the dissipation, lower order terms and also boundary traces, which are not a priori bounded by the energy. The estimate will be obtained by multipliers method applied to all

three components of the system [6, 42]. In order to control these boundary terms, more subtle estimates will be needed which invoke partially regularizing effect of thermoelasticity as well as micro local estimates applied to a hyperbolic component represented by u . This will be done in Subsection 4.8.1.3.

Lemma 4.3. Let $(\tilde{u}, \tilde{u}_t, \tilde{w}, \tilde{w}_t, \tilde{\phi}, \tilde{\theta})$ be a solution of the system (4.88)-(4.97). Then there exists $T > 0$ large enough, such that for any $\varepsilon \in (0, \frac{1}{4})$, the following estimate holds.

$$\begin{aligned} \int_0^T \tilde{E}(t) dt \leq & C[\tilde{E}(0) + \tilde{E}(T)] + C_B \int_0^T [\|\tilde{u}_t\|_{\Gamma_1} + \|\tilde{\phi}\|_{1,\Omega}^2 + \|\tilde{\theta}\|_{1,\Omega}^2] dt + C \int_0^T \|\nabla \tilde{u}\|_{\Gamma_1}^2 dt \\ & + C_B \int_0^T [\|\Delta \tilde{w}\|_{-\frac{1}{2},\Gamma_0}^2 + \|\tilde{w}_t\|_{\frac{1}{2}-\varepsilon,\Gamma_1}^2 + \|\tilde{w}\|_{2+\varepsilon,\Omega}^2] dt + C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned} \quad (4.99)$$

Proof. The proof of this lemma is divided in several steps. The geometric condition (4.20) will be used.

Step 1. Multiplier for the kinetic energy of elastic part: Multiply both sides of equation (4.88) by $h\nabla \tilde{u}$, where $h(x) = x - x_0$, and integrate in time and space

$$\int_0^T (\tilde{u}_{tt} - \operatorname{div}\{\sigma[\varepsilon(\tilde{u})] + \sigma[f(\nabla w^1) - f(\nabla w^2)]\} + \nabla \tilde{\phi} + \mathcal{P}_1(\tilde{u}, \tilde{w}), h\nabla \tilde{u})_{\Omega} dt = 0. \quad (4.100)$$

Integrating by parts in time and divergence formula yields

$$\int_0^T (\tilde{u}_{tt}, h\nabla \tilde{u})_{\Omega} dt = \left[(\tilde{u}_t, h\nabla \tilde{u})_{\Omega} \right]_0^T + \int_Q |\tilde{u}_t|^2 dQ - \frac{1}{2} \int_{\Sigma_1} |\tilde{u}_t|^2 h \cdot \nu d\Sigma_1. \quad (4.101)$$

Using divergence formula and Gauss Theorem in the second term of (4.100) we obtain

$$\int_0^T (\operatorname{div}\{\sigma[\varepsilon(\tilde{u})]\}, h\nabla \tilde{u})_{\Omega} dt = \int_0^T \langle \sigma[\varepsilon(\tilde{u})] \nu, h\nabla \tilde{u} \rangle_{\Gamma} dt - \int_0^T (\sigma[\varepsilon(\tilde{u})], \nabla(h\nabla \tilde{u}))_{\Omega} dt. \quad (4.102)$$

Note that

$$\int_0^T \langle \sigma[\varepsilon(\tilde{u})] \nu, h\nabla \tilde{u} \rangle_{\Gamma} dt = \int_0^T \langle \sigma[\varepsilon(\tilde{u})] \nu, h\nabla \tilde{u} \rangle_{\Gamma_0} dt + \int_0^T \langle \sigma[\varepsilon(\tilde{u})] \nu, h\nabla \tilde{u} \rangle_{\Gamma_1} dt.$$

Then, the identity

$$\langle \sigma[\varepsilon(\tilde{u})] \nu, h\nabla \tilde{u} \rangle_{\Gamma_0} = \langle \sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}) \rangle_{\Gamma_0} h \cdot \nu,$$

together with boundary condition (4.93) imply that

$$\begin{aligned} \int_0^T \langle \sigma[\varepsilon(\tilde{u})] \nu, h\nabla \tilde{u} \rangle_{\Gamma} dt = & - \int_0^T \langle \sigma[f(\nabla w^1) - f(\nabla w^2)] \nu + \kappa \tilde{u} - \tilde{\phi} \nu - \tilde{u}_t, h\nabla \tilde{u} \rangle_{\Gamma_1} dt \\ & + \int_0^T \langle \sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}) \rangle_{\Gamma_0} h \cdot \nu dt. \end{aligned} \quad (4.103)$$

It follows from identity (4.24) that

$$\begin{aligned} \int_0^T (\sigma[\varepsilon(\tilde{u})], \nabla(h\nabla\tilde{u}))_{\Omega} dt &= \int_0^T (\sigma[\varepsilon(\tilde{u})], \varepsilon(h\nabla\tilde{u}))_{\Omega} dt \\ &= \int_0^T (\sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}))_{\Omega} dt + \sum_{i,j,k=1}^2 \int_0^T (a_{i,j}, \frac{\partial^2 \tilde{u}_i}{\partial x_k \partial x_j} h_k)_{\Omega} dt, \end{aligned}$$

which combined with (4.23) and Gauss Theorem implies that

$$\begin{aligned} &\int_0^T (\sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}))_{\Omega} dt + \sum_{i,j,k=1}^2 \int_0^T (a_{i,j}, \frac{\partial^2 \tilde{u}_i}{\partial x_k \partial x_j} h_k)_{\Omega} dt \\ &= \frac{1}{2} \int_0^T [\langle \sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}) \rangle_{\Gamma_0} h \cdot \nu + \langle \sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}) \rangle_{\Gamma_1} h \cdot \nu] dt. \end{aligned} \quad (4.104)$$

Consequently from (4.102), (4.103) and (4.104) we find that

$$\begin{aligned} &\int_0^T (\operatorname{div}\{\sigma[\varepsilon(\tilde{u})]\}, h\nabla\tilde{u})_{\Omega} dt \\ &= \frac{1}{2} \int_0^T [\langle \sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}) \rangle_{\Gamma_0} h \cdot \nu - \langle \sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}) \rangle_{\Gamma_1} h \cdot \nu] dt \\ &\quad - \int_0^T \langle \sigma[f(\nabla w^1) - f(\nabla w^2)] \nu + \kappa \tilde{u} - \tilde{\phi} \nu - \tilde{u}_t, h\nabla\tilde{u} \rangle_{\Gamma_1} dt. \end{aligned} \quad (4.105)$$

Combining (4.101) and (4.105) with (4.100) we obtain

$$\begin{aligned} \int_Q |\tilde{u}_t|^2 dQ &= - \left[(\tilde{u}_t, h\nabla\tilde{u})_{\Omega} \right]_0^T + \frac{1}{2} \int_{\Sigma_1} |\tilde{u}_t|^2 h \cdot \nu d\Sigma_1 \\ &\quad + \frac{1}{2} \int_0^T [\langle \sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}) \rangle_{\Gamma_0} h \cdot \nu - \langle \sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}) \rangle_{\Gamma_1} h \cdot \nu] dt \\ &\quad - \int_0^T \langle \sigma[f(\nabla w^1) - f(\nabla w^2)] \nu + \kappa \tilde{u} - \tilde{\phi} \nu - \tilde{u}_t, h\nabla\tilde{u} \rangle_{\Gamma_1} dt \\ &\quad + \int_0^T (\operatorname{div}\{\sigma[f(\nabla w^1) - f(\nabla w^2)]\} - \nabla \tilde{\phi} - \mathcal{P}(\tilde{u}), h\nabla\tilde{u})_{\Omega} dt. \end{aligned}$$

Let us estimate the nonlinear term $\mathcal{P}_1(\tilde{u}, \tilde{w})$. Using the assumption (4.10) we find that

$$|p_{1,i}(u^1, w^1) - p_{1,i}(u^2, w^2)| \leq C(\nabla p)(|\tilde{u}| + |\tilde{w}|), \quad i = 1, 2, \quad (4.106)$$

where

$$C(\nabla p) = C(1 + |u_1^1|^{r-1} + |u_1^2|^{r-1} + |u_2^1|^{r-1} + |u_2^2|^{r-1} + |w^1|^{r-1} + |w^2|^{r-1}).$$

and using Hölder's inequality with $1/p_1 = \frac{(r-1)}{2(r+1)}, 1/p_2 = \frac{1}{r+1}, 1/p_3 = \frac{1}{2}$ we obtain

$$\int_0^T (\mathcal{P}_1(\tilde{u}, \tilde{w}), h\nabla\tilde{u})_{\Omega} dt \leq C_{\delta, B, T} \text{l.o.t.}(\tilde{u}, \tilde{w}) + \delta \int_Q |\nabla\tilde{u}|^2 dQ. \quad (4.107)$$

The geometric condition (4.20) implies that

$$\frac{1}{2} \int_0^T \langle \sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}) \rangle_{\Gamma_0} h \cdot \nu dt \leq 0.$$

Finally, using these inequalities and Lemma 4.2 we obtain

$$\begin{aligned} \int_Q |\tilde{u}_t|^2 dQ &\leq C[\tilde{E}(0) + \tilde{E}(T)] + C_\delta \int_{\Sigma_1} [|\tilde{u}_t|^2 + |\nabla \tilde{u}|^2] d\Sigma_1 + C_\delta \int_0^T \|\tilde{\phi}\|_{1,\Omega}^2 dt \\ &\quad + \delta \kappa \int_{\Sigma_1} |\tilde{u}|^2 d\Sigma_1 + \delta \int_Q |\nabla \tilde{u}|^2 dQ + C_{B,\delta} \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt \\ &\quad + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned} \quad (4.108)$$

Step 2. Multiplier for the difference of potential and kinetic energies: Multiply both sides of equation (4.88) by $\tilde{u}$ and integrate in time and space

$$\int_0^T (\tilde{u}_{tt} - \operatorname{div}\{\sigma[\varepsilon(\tilde{u})] + \sigma[f(\nabla w^1) - f(\nabla w^2)]\} + \nabla \tilde{\phi} + \mathcal{P}_1(\tilde{u}, \tilde{w}), \tilde{u})_\Omega dt = 0. \quad (4.109)$$

Note that

$$\int_0^T (\tilde{u}_{tt}, \tilde{u})_\Omega dt = \left[(\tilde{u}_t, \tilde{u})_\Omega \right]_0^T - \int_Q |\tilde{u}_t|^2 dQ.$$

Using Gauss Theorem in the second term of (4.109) we find

$$\int_0^T (\operatorname{div}\{\sigma[\varepsilon(\tilde{u})]\}, u)_\Omega dt = \int_0^T \langle \sigma[\varepsilon(\tilde{u})] \nu, \tilde{u} \rangle_\Gamma dt - \int_0^T (\sigma[\varepsilon(\tilde{u})], \varepsilon(\tilde{u}))_\Omega dt.$$

Boundary conditions (4.92) and (4.93) imply that

$$\int_0^T \langle \sigma[\varepsilon(u)] \nu, u \rangle_\Gamma dt = - \int_0^T \langle \sigma[f(\nabla w^1) - f(\nabla w^2)] \nu - \tilde{\phi} \nu + \tilde{u}_t, \tilde{u} \rangle_{\Gamma_1} dt.$$

These identities in (4.109) imply in the following equality

$$\begin{aligned} & - \int_0^T \int_\Omega |\tilde{u}_t|^2 d\Omega dt + \int_0^T \int_\Omega \sigma[\varepsilon(\tilde{u})] \varepsilon(\tilde{u}) d\Omega dt + \kappa \int_{\Sigma_1} |\tilde{u}|^2 d\Sigma_1 \\ &= - \left[(\tilde{u}_t, \tilde{u})_\Omega \right]_0^T + \int_0^T (\operatorname{div}\{\sigma[f(\nabla w^1) - f(\nabla w^2)]\}, \tilde{u})_\Omega dt \\ &\quad - \int_0^T \left[\langle \sigma[f(\nabla w^1) - f(\nabla w^2)] \nu - \tilde{\phi} \nu + \tilde{u}_t, \tilde{u} \rangle_{\Gamma_1} + (\mathcal{P}(\tilde{u}) + \nabla \tilde{\phi}, \tilde{u})_\Omega \right] dt. \end{aligned}$$

Proceeding as in (4.107), we find the following estimate

$$\int_0^T (\mathcal{P}_1(\tilde{u}, \tilde{w}), \tilde{u})_\Omega dt \leq C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}).$$

This inequality and trace theorem imply that

$$\begin{aligned} & - \int_Q |\tilde{u}_t|^2 dQ + \int_Q \sigma[\varepsilon(\tilde{u})] \varepsilon(\tilde{u}) dQ + \kappa \int_{\Sigma_1} |\tilde{u}|^2 d\Sigma_1 \\ & \leq C[\tilde{E}(0) + \tilde{E}(T)] + C_\delta \int_0^T \left[\|f(\nabla w^1) - f(\nabla w^2)\|_{1,\Omega}^2 + \|\tilde{u}_t\|_{\Gamma_1}^2 \right] dt \\ & \quad + C_\delta \int_0^T z \|\tilde{\phi}\|_{1,\Omega}^2 dt + \delta \int_0^T \|\tilde{u}\|_{1,\Omega}^2 dt + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned}$$

Choosing $\delta > 0$ small enough and using Lemma 4.2 we find,

$$\begin{aligned} & - \int_Q |\tilde{u}_t|^2 dQ + \int_Q \sigma[\varepsilon(\tilde{u})] \varepsilon(\tilde{u}) dQ + \kappa \int_{\Sigma_1} |\tilde{u}|^2 d\Sigma_1 \\ & \leq C[\tilde{E}(0) + \tilde{E}(T)] + C \int_0^T \left[\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{\phi}\|_{1,\Omega}^2 \right] dt \\ & \quad + C_B \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned} \quad (4.110)$$

Now multiplying both sides of equation (4.89) by $\tilde{w}$ and integrate in time and space

$$\int_0^T (\tilde{w}_{tt} + \Delta^2 \tilde{w} - \text{div}\{\mathcal{H}_2\} + \Delta \tilde{\theta} + \mathcal{P}_2(\tilde{u}, \tilde{w}), \tilde{w})_\Omega dt = 0. \quad (4.111)$$

As before,

$$\int_0^T (\tilde{w}_{tt}, \tilde{w})_\Omega dt = \left[(\tilde{w}_t, \tilde{w})_\Omega \right]_0^T - \int_Q |\tilde{w}_t|^2 dQ. \quad (4.112)$$

To handle the second term in (4.111) we use the following identity (See [44, Proposition C.12])

$$\begin{aligned} (\Delta^2 \tilde{w}, \psi)_\Omega &= a(\tilde{w}, \psi) + \int_{\Gamma_1} \left[\frac{\partial}{\partial \mathbf{v}} \Delta \tilde{w} + (1 - \mu) B_2 \tilde{w} \right] \psi d\Gamma_1 \\ & \quad - \int_{\Gamma_1} [\Delta \tilde{w} + (1 - \mu) B_1 \tilde{w}] \frac{\partial \psi}{\partial \mathbf{v}} d\Gamma_1 + \int_{\Gamma_0} \left[\frac{\partial}{\partial \mathbf{v}} \Delta \tilde{w} \psi - \Delta \tilde{w} \frac{\partial \psi}{\partial \mathbf{v}} \right] d\Gamma_0, \end{aligned}$$

where $\psi \in H^2(\Omega)$. Taking $\psi = \tilde{w}$ and using boundary conditions (4.92), (4.94) and (4.95) we find

$$\int_0^T (\Delta^2 \tilde{w}, \tilde{w})_\Omega dt = \int_0^T a(\tilde{w}, \tilde{w}) dt + \int_0^T \left\langle \mathcal{H}_2 \cdot \mathbf{v} - \frac{\partial \tilde{\theta}}{\partial \mathbf{v}}, \tilde{w} \right\rangle_{\Gamma_1} dt + \int_{\Sigma_1} \tilde{\theta} \frac{\partial \tilde{w}}{\partial \mathbf{v}} d\Sigma_1. \quad (4.113)$$

Using Gauss Theorem we can rewrite the third term of (4.111) as

$$\int_0^T (\text{div}\{\mathcal{H}_2\}, \tilde{w})_\Omega dt = - \int_0^T (\mathcal{H}_2, \nabla \tilde{w})_\Omega dt + \int_0^T \left\langle \mathcal{H}_2 \cdot \mathbf{v}, \tilde{w} \right\rangle_{\Gamma_1} dt. \quad (4.114)$$

Combining (4.112), (4.113) and (4.114) with (4.111) we obtain

$$- \int_Q |\tilde{w}_t|^2 dQ + \int_0^T a(\tilde{w}, \tilde{w}) dt = - \left[(\tilde{w}_t, \tilde{w})_\Omega \right]_0^T - \int_0^T \left[\left\langle \tilde{\theta}, \frac{\partial \tilde{w}}{\partial \mathbf{v}} \right\rangle_{\Gamma_1} - (\nabla \tilde{\theta}, \nabla \tilde{w})_\Omega \right] dt - \mathcal{R} \quad (4.115)$$

where

$$\begin{aligned} \mathcal{R} = & \int_0^T (\sigma[N(u^1, w^1)] \nabla w^1 - \sigma[N(u^2, w^2)] \nabla w^2, \nabla \tilde{w})_{\Omega} dt \\ & + \int_0^T (\phi^1 \nabla w^1 - \phi^2 \nabla w^2, \nabla \tilde{w})_{\Omega} dt + \int_0^T (\mathcal{P}_2(\tilde{u}, \tilde{w}), \tilde{w})_{\Omega} dt. \end{aligned}$$

Next, we estimate the integrals on the right-hand side of (4.115). Trace Theorem provides

$$\left\langle \theta, \frac{\partial w}{\partial \nu} \right\rangle_{\Gamma_1} \leq \|\theta\|_{\Gamma_1} \left\| \frac{\partial w}{\partial \nu} \right\|_{\Gamma_1} \leq C_{\delta} \|\theta\|_{1,\Omega}^2 + \delta \|w\|_{2,\Omega}^2. \quad (4.116)$$

Let us estimate $\mathcal{R}$. Using the definition of stress $N(\cdot, \cdot)$ we find

$$\begin{aligned} & \int_0^T (\sigma[N(u^1, w^1)] \nabla w^1 - \sigma[N(u^2, w^2)] \nabla w^2, \nabla \tilde{w})_{\Omega} dt \\ &= \int_Q [\sigma[\varepsilon(\tilde{u})] \cdot (\nabla w^2 \otimes \nabla \tilde{w}) + \sigma[f(\nabla w^1) - f(\nabla w^2)] \cdot (\nabla w^2 \otimes \nabla \tilde{w})] dQ \\ &+ \int_Q \sigma[\varepsilon(u^1) + f(\nabla w^1)] \cdot (\nabla \tilde{w} \otimes \nabla \tilde{w}) dQ. \end{aligned}$$

The inequality $\|u \otimes v\|_{\Omega} \leq C \|u\|_{\varepsilon,\Omega} \|v\|_{1-\varepsilon,\Omega}$, which holds for $\varepsilon \in (0, 1)$, implies that

$$\begin{aligned} \int_Q \sigma[\varepsilon(\tilde{u})] \cdot (\nabla w^2 \otimes \nabla \tilde{w}) dQ &\leq \delta \int_0^T \|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 dt + C_{\delta} \int_0^T \|\nabla w^2 \otimes \nabla \tilde{w}\|_{\Omega}^2 dt \\ &\leq \delta \int_0^T \|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 dt + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}), \end{aligned}$$

$$\begin{aligned} \int_Q \sigma[f(\nabla w^1) - f(\nabla w^2)] \cdot (\nabla w^2 \otimes \nabla \tilde{w}) dQ \\ \leq C \int_0^T [\|f(\nabla w^1) - f(\nabla w^2)\|_{\Omega}^2 + \|\nabla w^2 \otimes \nabla \tilde{w}\|_{\Omega}^2] dt \\ \leq C \int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_{\Omega}^2 dt + C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}), \end{aligned}$$

$$\begin{aligned} \int_Q [\sigma[\varepsilon(u^1) + f(\nabla w^1)] \cdot (\nabla \tilde{w} \otimes \nabla \tilde{w})] dQ \\ \leq C \int_0^T [\|\sigma[\varepsilon(u^1) + f(\nabla w^1)]\|_{\Omega} \|\nabla \tilde{w} \otimes \nabla \tilde{w}\|_{\Omega}] dt \\ \leq C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}) \end{aligned}$$

and

$$\begin{aligned} \int_0^T (\phi^1 \nabla w^1 - \phi^2 \nabla w^2, \nabla \tilde{w})_{\Omega} dt &= \int_0^T [(\tilde{\phi} \nabla w^2 - \phi^1 \nabla \tilde{w}, \nabla \tilde{w})_{\Omega}] dt \\ &\leq C \int_0^T \|\tilde{\phi}\|_{\Omega}^2 dt + C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}) \end{aligned}$$

and

$$\int_0^T (\mathcal{P}_2(\tilde{u}, \tilde{w}), \tilde{w})_{\Omega} dt \leq C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}).$$

These estimates and Lemma 4.2 imply that

$$\mathcal{R} \leq \delta \int_0^T \|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 dt + C \int_0^T \|\tilde{\phi}\|_{\Omega}^2 dt + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \quad (4.117)$$

Inserting this and (4.116) into (4.115), we obtain

$$\begin{aligned} - \int_Q |\tilde{w}_t|^2 dQ + \int_0^T a(\tilde{w}, \tilde{w}) dt &\leq C[\tilde{E}(0) + \tilde{E}(T)] + C_B \int_0^T [\|\tilde{\phi}\|_{1,\Omega}^2 + \|\tilde{\theta}\|_{1,\Omega}^2] dt \\ &\quad + \delta \int_0^T \|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 dt + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned}$$

This estimate and (4.110), for $\delta > 0$ small enough, shows that

$$\begin{aligned} \int_0^T [\tilde{E}_p(t) - \tilde{E}_k(t)] dt &\leq C[\tilde{E}(0) + \tilde{E}(T)] + C_B \int_0^T \|\tilde{u}\|_{1,\Gamma_1}^2 dt + C_B \int_0^T \|\tilde{\phi}\|_{1,\Omega}^2 dt \\ &\quad + C_B \int_0^T \|\tilde{\theta}\|_{1,\Omega}^2 dt + C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}) + C_B \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt. \end{aligned} \quad (4.118)$$

Step 3. Multiplier for kinetic energy of the plate equation: Let us consider the following operators

- The Laplace operator: $\mathcal{A}_D : L^2(\Omega) \rightarrow L^2(\Omega)$, where $\mathcal{A}_D = -\Delta$, equipped with Dirichlet boundary condition and domain $D(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega)$.
- The elliptic operator $\mathcal{D}$: $\mathcal{D}h = v \Leftrightarrow \begin{cases} \Delta v = 0 \text{ in } \Omega, \\ v = h \text{ on } \Gamma. \end{cases}$

Classical elliptic regularity [46] provides

$$\|\mathcal{A}_D^{-1}v\|_{2,\Omega} \leq C\|v\|_{\Omega} \quad v \in L^2(\Omega),$$

and

$$\mathcal{D} \in \mathcal{L}(H^s(\Gamma), H^{s+\frac{1}{2}}(\Omega)), \quad s \in \mathbb{R}.$$

For $v \in H^2(\Omega)$ we have that

$$-v + \mathcal{D}(v|_{\Gamma}) \in D(\mathcal{A}_D) \quad \text{and} \quad \mathcal{A}_D^{-1}\Delta v = -v + \mathcal{D}(v|_{\Gamma}). \quad (4.119)$$

Now, multiply both sides of equation (4.89) by $\mathcal{A}_D^{-1}\tilde{\theta}$ and integrate in time and space

$$\int_0^T (\tilde{w}_{tt} + \Delta^2 \tilde{w} - \text{div}\{\mathcal{H}_2\} + \Delta \tilde{\theta} + \mathcal{P}_2(\tilde{u}, \tilde{w}), \mathcal{A}_D^{-1}\tilde{\theta})_{\Omega} dt = 0.$$

Proceeding as before we obtain

$$\begin{aligned} \int_0^T (\tilde{w}_{tt}, \mathcal{A}_D^{-1}\tilde{\theta})_{\Omega} dt + \int_0^T [a(\tilde{w}, \mathcal{A}_D^{-1}\tilde{\theta}) + (\mathcal{N}_2, \nabla(\mathcal{A}_D^{-1}\tilde{\theta}))_{\Omega} + (\mathcal{P}_2(\tilde{u}, \tilde{w}), \mathcal{A}_D^{-1}\tilde{\theta})_{\Omega}] dt \\ = \int_0^T \left[-\langle \tilde{\theta}, \frac{\partial}{\partial \nu}(\mathcal{A}_D^{-1}\tilde{\theta}) \rangle_{\Gamma_1} + \langle \Delta \tilde{w}, \frac{\partial}{\partial \nu}(\mathcal{A}_D^{-1}\tilde{\theta}) \rangle_{\Gamma_0} + (\nabla \tilde{\theta}, \nabla(\mathcal{A}_D^{-1}\tilde{\theta}))_{\Omega} \right] dt. \end{aligned} \quad (4.120)$$

Integrating in time and relation (4.119) imply that

$$\begin{aligned} & \int_0^T (\tilde{w}_{tt}, \mathcal{A}_D^{-1} \tilde{\theta})_{\Omega} dt \\ &= \left[(\tilde{w}_t, \mathcal{A}_D^{-1} \tilde{\theta})_{\Omega} \right]_0^T + \int_0^T \left[\|\tilde{w}_t\|_{\Omega}^2 - \left(\tilde{w}_t, \mathcal{D}(\tilde{w}_t|_{\Gamma}) - \tilde{\theta} + \mathcal{D}(\tilde{\theta}|_{\Gamma}) \right)_{\Omega} \right] dt. \end{aligned} \quad (4.121)$$

For every $\delta, \delta_0 > 0$ there exist constants $C_{\delta}, C_{\delta_0} > 0$ such that

$$\begin{aligned} \int_0^T (\tilde{w}_t, \mathcal{D}(\tilde{w}_t|_{\Gamma}))_{\Omega} dt &\leq \int_0^T \|\tilde{w}_t\|_{-(1-\varepsilon), \Omega} \|\mathcal{D}(\tilde{w}_t|_{\Gamma})\|_{1-\varepsilon, \Omega} dt \\ &\leq \int_0^T \|\tilde{w}_t\|_{-(1-\varepsilon), \Omega} \|\mathcal{D}\|_{\mathcal{L}} \|\tilde{w}_t\|_{\frac{1}{2}-\varepsilon, \Gamma_1} dt \\ &\leq \delta_0 \int_0^T \|\tilde{w}_t\|_{\Omega}^2 dt + C_{\delta_0} \int_0^T \|\tilde{w}_t\|_{\frac{1}{2}-\varepsilon, \Gamma_1}^2 dt, \end{aligned} \quad (4.122)$$

$$\int_0^T (\tilde{w}_t, \tilde{\theta} - \mathcal{D}(\tilde{\theta}|_{\Gamma}))_{\Omega} dt \leq \delta_0 \int_0^T \|\tilde{w}_t\|_{\Omega}^2 dt + C_{\delta_0} \int_0^T \|\tilde{\theta}\|_{1, \Omega}^2 dt, \quad (4.123)$$

$$\int_0^T a(\tilde{w}, \mathcal{A}_D^{-1} \tilde{\theta}) dt \leq \delta \int_0^T \|\tilde{w}\|_{2, \Omega}^2 dt + C_{\delta} \int_0^T \|\tilde{\theta}\|_{\Omega}^2 dt, \quad (4.124)$$

$$\int_0^T (\mathcal{P}_2(\tilde{u}, \tilde{w}), \mathcal{A}_D^{-1} \tilde{\theta})_{\Omega} dt \leq C_{B, T} \text{l.o.t.}(\tilde{u}, \tilde{w}) + C \int_0^T \|\tilde{\theta}\|_{\Omega}^2 dt, \quad (4.125)$$

$$\int_0^T \left[(\nabla \tilde{\theta}, \nabla(\mathcal{A}_D^{-1} \tilde{\theta}))_{\Omega} - \left\langle \tilde{\theta}, \frac{\partial}{\partial \mathbf{v}}(\mathcal{A}_D^{-1} \tilde{\theta}) \right\rangle_{\Gamma_1} \right] dt \leq C \int_0^T \|\tilde{\theta}\|_{1, \Omega}^2 dt, \quad (4.126)$$

$$\begin{aligned} \int_0^T \left\langle \Delta \tilde{w}, \frac{\partial}{\partial \mathbf{v}}(\mathcal{A}_D^{-1} \tilde{\theta}) \right\rangle_{\Gamma_0} dt &\leq \int_0^T \|\Delta \tilde{w}\|_{-\frac{1}{2}, \Gamma_0} \|\mathcal{A}_D^{-1} \tilde{\theta}\|_{2, \Omega} dt \\ &\leq C \int_0^T \left[\|\Delta \tilde{w}\|_{-\frac{1}{2}, \Gamma_0}^2 + \|\tilde{\theta}\|_{\Omega}^2 \right] dt. \end{aligned} \quad (4.127)$$

It remains to estimate the nonlinear terms in (4.120). For this, considering the definition of $\mathcal{H}_2$ we find,

$$\begin{aligned} \int_0^T (\mathcal{H}_2, \nabla(\mathcal{A}_D^{-1} \tilde{\theta}))_{\Omega} dt &= \int_0^T (\sigma[N(u^1, w^1)] \nabla w^1 - \sigma[N(u^2, w^2)] \nabla w^2, \nabla(\mathcal{A}_D^{-1} \tilde{\theta}))_{\Omega} dt \\ &\quad + \int_0^T (\phi^1 \nabla w^1 - \phi^2 \nabla w^2, \nabla(\mathcal{A}_D^{-1} \tilde{\theta})) dt. \end{aligned} \quad (4.128)$$

Let us estimate the two integrals on the right side of (4.128). Proceeding as in (4.117) we obtain

$$\begin{aligned} & \int_0^T \left(\sigma[N(u^1, w^1)] \nabla w^1 - \sigma[N(u^2, w^2)] \nabla w^2, \nabla(\mathcal{A}_D^{-1} \tilde{\theta}) \right)_{\Omega} dt \\ & \leq \delta \int_0^T \|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 dt + C_{B, \delta} \int_0^T \|\tilde{\theta}\|_{\Omega}^2 dt + C_{B, T, \delta} \text{l.o.t.}(\tilde{u}, \tilde{w}) \end{aligned}$$

and

$$\begin{aligned} & \int_0^T \left(\phi^1 \nabla w^1 - \phi^2 \nabla w^2, \nabla (\mathcal{A}_D^{-1} \tilde{\theta}) \right)_{\Omega} dt \\ & \leq C_B \int_0^T \left[\|\tilde{\phi}\|_{1,\Omega}^2 + \|\tilde{\theta}\|_{1,\Omega}^2 \right] dt + C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned}$$

Then we have

$$\begin{aligned} & \int_0^T \left(\mathcal{H}_2, \nabla (\mathcal{A}_D^{-1} \tilde{\theta}) \right)_{\Omega} dt \leq \delta \int_0^T \|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 dt + C_{B,\delta} \int_0^T \left[\|\tilde{\phi}\|_{1,\Omega}^2 + \|\tilde{\theta}\|_{1,\Omega}^2 \right] dt \\ & \quad + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned} \quad (4.129)$$

Therefore the estimates (4.122), (4.123), (4.124), (4.126), (4.127) and (4.129) applied in (4.120), for $\delta_0 > 0$ small enough, yield

$$\begin{aligned} & \int_0^T \|\tilde{w}_t\|_{\Omega}^2 dt \leq C[\tilde{E}(0) + \tilde{E}(T)] + \delta \int_0^T \|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 dt + C_{B,\delta} \int_0^T \left[\|\tilde{\theta}\|_{1,\Omega}^2 + \|\tilde{\phi}\|_{1,\Omega}^2 \right] dt \\ & \quad + C_B \int_0^T \left[\|\Delta \tilde{w}\|_{-\frac{1}{2},\Gamma_0}^2 + \|\tilde{w}_t\|_{\frac{1}{2}-\varepsilon,\Gamma_1}^2 + \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 \right] dt + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned} \quad (4.130)$$

Step 4. Completion of the proof: Combining the inequalities (4.108), (4.118), (4.130) and selecting suitable $\delta > 0$ small we obtain (4.99). This ends the proof of Lemma 4.3

4.8.1.3 Trace regularity and analytic estimates

In order to control boundary terms in (4.99), more subtle estimates are needed, including trace regularity and analytic estimates. They are essential to prove the quasistability inequality. Our result is based on the corresponding trace estimate for the linear model of dynamic elasticity given by the Theorem 4.3. The analytic estimates rely on the analyticity of the generator associated with the linear thermoelastic plate (See theorem 4.5).

Lemma 4.4. Let $(\tilde{u}, \tilde{u}_t, \tilde{w}, \tilde{w}_t, \tilde{\phi}, \tilde{\theta})$ be a regular solution of the system (4.88)-(4.97). Then for any $\varepsilon \in (0, \frac{1}{4})$ and $\alpha \in (0, \frac{T}{2})$ the following trace regularity is valid.

$$\int_{\Sigma_{\alpha}} |\nabla \tilde{u}|^2 d\Sigma_{\alpha} \leq C_{\alpha} \int_0^T \left[\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{\phi}\|_{1,\Omega}^2 \right] dt + C_{\alpha,B} \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt + C_{\alpha,B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}).$$

Proof. The proof is divided into several steps.

Step 1. Trace regularity for the linear model: Consider $\tilde{F} = \tilde{F}(x, y, t)$ given by

$$\tilde{F} = \text{div}\{\sigma[f(\nabla w^1 - f(\nabla w^2))]\} - \nabla \tilde{\phi} - \mathcal{P}_1(\tilde{u}, \tilde{w}). \quad (4.131)$$

Then the solution $\tilde{u} = \tilde{u}(x, y, t)$ satisfies the problem

$$\tilde{u}_{tt} - \text{div}\{\sigma[\varepsilon(\tilde{u})]\} = \tilde{F}.$$

By using the trace regularity stated in Theorem 4.3 we obtain the estimate

$$\int_{\Sigma_\alpha} |\nabla \tilde{u} \tau|^2 d\Sigma_\alpha \leq C_\alpha \int_0^T \left[\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{F}\|_{-\frac{1}{2}, \Omega}^2 + \|\sigma[\varepsilon(\tilde{u})] \nu\|_{\Gamma_1}^2 + \|\tilde{u}\|_{1-\varepsilon, \Omega}^2 \right] dt, \quad (4.132)$$

where we used the inequality $\|\tilde{u}\|_{\frac{1}{2}+\varepsilon, \Omega}^2 \leq C \|\tilde{u}\|_{1-\varepsilon, \Omega}^2$.

Step 2. Estimate for $\tilde{F}$ defined in (4.131): For $\varepsilon \in (0, \frac{1}{2})$, we have

$$\|\tilde{F}(t)\|_{-\frac{1}{2}, \Omega}^2 \leq C_B \|\tilde{w}\|_{2+\varepsilon, \Omega}^2 + C \|\tilde{\phi}\|_{1, \Omega}^2 + C_B \text{l.o.t.}(\tilde{u}, \tilde{w}), \quad \forall t \geq 0. \quad (4.133)$$

To prove this, consider $\psi \in H^{\frac{1}{2}}(\Omega)$. The inequality $\|u \otimes v\|_{1, \Omega} \leq C \|u\|_{1, \Omega} \|v\|_{1+\varepsilon, \Omega}$ shows that

$$\begin{aligned} (\operatorname{div}\{\sigma[f(\nabla w^1) - f(\nabla w^2)]\}, \psi)_\Omega &= (\operatorname{div}\{\sigma[f(\nabla \tilde{w}) + \nabla \tilde{w} \otimes \nabla w^2 + \nabla w^2 \otimes \nabla \tilde{w}], \psi)_\Omega \\ &\leq C \|\tilde{w}\|_{2, \Omega} \|\nabla \tilde{w} \cdot \psi\|_\Omega + C \|\tilde{w}\|_{2, \Omega} \|\nabla w^2 \cdot \psi\|_\Omega \\ &\quad + C \|w^2\|_{2, \Omega} \|\nabla \tilde{w} \cdot \psi\|_\Omega. \end{aligned}$$

Hölder's inequality and Sobolev's embeddings $H^{2-\varepsilon}(\Omega) \subset W^{1,4}(\Omega)$ and $H^{\frac{1}{2}}(\Omega) \subset L^4(\Omega)$ imply that

$$\|\tilde{w}\|_{2, \Omega} \|\nabla \tilde{w} \cdot \psi\|_\Omega \leq C \|\tilde{w}\|_{2, \Omega} \|\tilde{w}\|_{W^{1,4}(\Omega)} \|\psi\|_{L^4(\Omega)} \leq C_B \|\tilde{w}\|_{2-\varepsilon, \Omega} \|\psi\|_{\frac{1}{2}, \Omega},$$

$$\|\tilde{w}\|_{2, \Omega} \|\nabla w^2 \cdot \psi\|_\Omega \leq C \|\tilde{w}\|_{2, \Omega} \|w^2\|_{W^{1,4}(\Omega)} \|\psi\|_{L^4(\Omega)} \leq C_B \|\tilde{w}\|_{2, \Omega} \|\psi\|_{\frac{1}{2}, \Omega}$$

and

$$\|w^2\|_{2, \Omega} \|\nabla \tilde{w} \cdot \psi\|_\Omega \leq C \|w^2\|_{2, \Omega} \|\tilde{w}\|_{W^{1,4}(\Omega)} \|\psi\|_{L^4(\Omega)} \leq C_B \|\tilde{w}\|_{2-\varepsilon, \Omega} \|\psi\|_{\frac{1}{2}, \Omega}.$$

These inequalities and Sobolev's embedding $H^{2+\varepsilon}(\Omega) \subset H^2(\Omega)$ imply that

$$\|\operatorname{div}\{\sigma[f(\nabla w^1) - f(\nabla w^2)]\}\|_{-\frac{1}{2}, \Omega}^2 \leq C_B \|\tilde{w}\|_{2+\varepsilon, \Omega}^2 + C_B \text{l.o.t.}(\tilde{u}, \tilde{w}). \quad (4.134)$$

Using Hölder's inequality and Sobolev's embedding we find that

$$(\nabla \tilde{\phi}, \psi)_\Omega \leq \|\nabla \tilde{\phi}\|_\Omega \|\psi\|_\Omega \leq C \|\tilde{\phi}\|_{1, \Omega} \|\psi\|_{\frac{1}{2}, \Omega} \quad (4.135)$$

and (4.10) leads to

$$(\mathcal{P}_1(\tilde{u}, \tilde{w}), \psi)_\Omega \leq C_B \|\tilde{u}\|_{L^{r+1}(\Omega)} \|\psi\|_\Omega \leq C_B \|\tilde{u}\|_{L^{r+1}(\Omega)} \|\psi\|_{\frac{1}{2}, \Omega}. \quad (4.136)$$

Therefore (4.134) together with (4.135) and (4.136) shows that the estimate (4.133) holds.

Step 3. Estimate for the stress tensor: For $\varepsilon \in (0, \frac{1}{2})$, we have

$$\int_{\Sigma_\alpha} |\sigma[\varepsilon(\tilde{u})]|^2 d\Sigma_\alpha \leq C \int_0^T \left[\|f(\nabla w^1) - f(\nabla w^2)\|_{\Gamma_1}^2 + \|\tilde{u}\|_{1-\varepsilon, \Omega}^2 + \|\tilde{\phi}\|_{1, \Omega}^2 + \|\tilde{u}_t\|_{\Gamma_1}^2 \right] dt \quad (4.137)$$

Indeed, the boundary condition (4.93) implies that

$$\|\sigma[\varepsilon(\tilde{u})]\|_{\Sigma_1}^2 \leq C \int_0^T \left[\|f(\nabla w^1) - f(\nabla w^2)\|_{\Gamma_1}^2 + \|\tilde{u}\|_{\Gamma_1}^2 + \|\tilde{\phi}\|_{\Gamma_1}^2 + \|\tilde{u}_t\|_{\Gamma_1}^2 \right] dt.$$

Then using inequalities $\|\tilde{u}\|_{\Gamma_1}^2 \leq C\|\tilde{u}\|_{1-\varepsilon,\Omega}^2$ and $\|\tilde{\phi}\|_{\Gamma_1}^2 \leq C\|\tilde{\phi}\|_{1,\Omega}^2$, we obtain (4.137).

Step 4. Estimate for $|\nabla \tilde{u}|$: We have

$$|\nabla \tilde{u}| \leq C \left[|\nabla \tilde{u} \tau| + |\sigma[\tilde{u}] \nu| \right], \quad (4.138)$$

where $\nu = (\nu_1, \nu_2)$ and $\tau = (\tau_1, \tau_2) = (-\nu_2, \nu_1)$ denote, respectively, the outward unit normal and the unit tangential vectors, at a point of Γ . To prove this, let us denote $\nabla \tilde{u}$ as a 4-vector $\nabla \tilde{u} = (\tilde{u}_{1,x}, \tilde{u}_{1,y}, \tilde{u}_{2,x}, \tilde{u}_{2,y})$. Then we obtain the algebraic system

$$A(\nabla \tilde{u})^\top = (\nabla \tilde{u} \tau, \sigma[\tilde{u}] \nu)^\top,$$

where

$$A = \begin{bmatrix} \tau_1 & \tau_2 & 0 & 0 \\ 0 & 0 & \tau_1 & \tau_2 \\ (\lambda + 2\eta)\nu_1 & \eta\nu_2 & \eta\nu_2 & \lambda\nu_1 \\ \lambda\nu_2 & \eta\nu_1 & \eta\nu_1 & (\lambda + 2\eta)\nu_2 \end{bmatrix}.$$

Note that $\det(A) = (\lambda + 2\eta)\eta$ is constant over Γ . Then we obtain

$$(\nabla \tilde{u})^\top = A^{-1}(\nabla \tilde{u} \tau, \sigma[\tilde{u}] \nu)^\top,$$

and this implies (4.138).

Step 5. Conclusion: Integrating in time and space the inequality (4.138) we obtain

$$\int_{\Sigma_\alpha} |\nabla \tilde{u}|^2 d\Sigma_\alpha \leq C \int_{\Sigma_\alpha} |\nabla \tilde{u} \tau|^2 d\Sigma_\alpha + C \int_{\Sigma_\alpha} |\sigma[\tilde{u}] \nu|^2 d\Sigma_\alpha.$$

Inequalities (4.132) and (4.133) imply

$$\begin{aligned} \int_{\Sigma_\alpha} |\nabla \tilde{u}|^2 d\Sigma_\alpha &\leq C_\alpha \int_0^T \left[\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{\phi}\|_{1,\Omega}^2 + \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 \right] dt \\ &\quad + C_\alpha \int_{\Sigma_\alpha} |\sigma[\tilde{u}] \nu|^2 d\Sigma_\alpha + C_{\alpha,T} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned}$$

Using (4.137) we find

$$\begin{aligned} \int_{\Sigma_\alpha} |\nabla \tilde{u}|^2 d\Sigma_\alpha &\leq C_\alpha \int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_{\Gamma_1}^2 dt + C_\alpha \int_0^T \|\tilde{u}_t\|_{\Gamma_1}^2 dt \\ &\quad + C_\alpha \int_0^T \left[\|\tilde{\phi}\|_{1,\Omega}^2 + \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 \right] dt + C_{\alpha,T} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned}$$

Finally inequality (ii) of Lemma 4.2 results the proof of Lemma 4.4. $\square$

Next we prove an improved regularity on the vertical displacement $\tilde{w}$. This is done by exploiting the analyticity of the thermoelastic semigroup.

Lemma 4.5. Let $(\tilde{u}, \tilde{u}_t, \tilde{w}, \tilde{w}_t, \tilde{\phi}, \tilde{\theta})$ be a regular solution of the system (4.88)-(4.97). Then for any $\varepsilon \in (0, \frac{1}{2})$,

$$\int_0^T [\|\tilde{w}\|_{3-\varepsilon, \Omega}^2 + \|\tilde{w}_t\|_{1-\varepsilon, \Omega}^2 + \|\tilde{\theta}\|_{1-\varepsilon, \Omega}^2] dt \leq C\tilde{E}(0) + C_B \int_0^T \|\tilde{\phi}\|_{1, \Omega}^2 dt + C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}).$$

Proof. The proof of the lemma is divided into three parts.

Step 1. Abstract setting: Proceeding as in Section 1 of Theorem 4.1 we find that

$$\mathbf{w} := \left[\tilde{w} + G_1(\tilde{\theta}|_{\Gamma_1}) + G_2\left(\frac{\partial \tilde{\theta}}{\partial \mathbf{v}}\right) \right] \in D(\mathcal{A}_M).$$

and

$$\begin{aligned} \tilde{w}_{tt} + \mathcal{A}_M \mathbf{w} - \operatorname{div}\{F(\tilde{u}, \tilde{w}, \tilde{\phi})\} - \mathcal{A}_M G_2(F(\tilde{u}, \tilde{w}, \tilde{\phi}) \cdot \mathbf{v}) + \Delta \tilde{\theta} + \mathcal{P}_2(\tilde{u}, \tilde{w}) &= 0, \\ \tilde{\theta}_t - \Delta \tilde{\theta} - \Delta \tilde{w}_t &= 0, \end{aligned}$$

where

$$F(\tilde{u}, \tilde{w}, \tilde{\phi}) = \sigma[N(u^1, w^1)] \nabla w^1 - \sigma[N(u^2, w^2)] \nabla w^2 + \phi^1 \nabla w^1 - \phi^2 \nabla w^2.$$

Therefore, we can rewrite the problem for $(\tilde{w}, \tilde{\theta})$ in the following form,

$$\begin{bmatrix} \tilde{w}_t \\ \tilde{w}_{tt} \\ \tilde{\theta}_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \operatorname{div}\{F(\tilde{u}, \tilde{w}, \tilde{\phi})\} + \mathcal{A}_M G_2(F(\tilde{u}, \tilde{w}, \tilde{\phi}) \cdot \mathbf{v}) - \mathcal{P}_2(\tilde{u}, \tilde{w}) \\ 0 \end{bmatrix}, \quad (4.139)$$

where

$$\mathcal{A} \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\theta} \end{bmatrix} = \begin{bmatrix} \tilde{w}_t \\ -\mathcal{A}_M \tilde{w} - \mathcal{A}_M(G_1 \tilde{\theta}) - \mathcal{A}_M G_2\left(\frac{\partial \tilde{\theta}}{\partial \mathbf{v}}\right) - \Delta \tilde{\theta} \\ \Delta \tilde{\theta} + \Delta \tilde{w}_t \end{bmatrix}.$$

Then, for $\varepsilon < \frac{1}{2}$, we can rewrite the solution of (4.139) using variation of parameters formula,

$$\mathcal{A}^{\frac{1-\varepsilon}{2}} \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\theta} \end{bmatrix} = \mathcal{A}^{\frac{1-\varepsilon}{2}} e^{\mathcal{A}t} \begin{bmatrix} \tilde{w}_0 \\ \tilde{w}_1 \\ \tilde{\theta}_0 \end{bmatrix} + \int_0^t \mathcal{A} e^{\mathcal{A}(t-s)} \mathcal{A}^{-\frac{1+\varepsilon}{2}} \begin{bmatrix} 0 \\ \mathcal{F}(\tilde{u}, \tilde{w}, \tilde{\phi}) \\ 0 \end{bmatrix} ds, \quad (4.140)$$

where $\mathcal{F}(\tilde{u}, \tilde{w}, \tilde{\phi}) = \mathcal{A}_M G_2(F(\tilde{u}, \tilde{w}, \tilde{\phi}) \cdot \mathbf{v}) + \operatorname{div}\{F(\tilde{u}, \tilde{w}, \tilde{\phi})\} - \mathcal{P}_2(\tilde{u}, \tilde{w})$.

Step 2. Some estimates: Since $\mathcal{A}$ is m-dissipative, invertible and generates an analytic semigroup which is exponentially stable, the following estimates are valid

$$\begin{aligned} \left\| \int_0^t \mathcal{A} e^{(t-s)\mathcal{A}} f(s) ds \right\|_{L^2(0,T;H)} &\leq C \|f\|_{L^2(0,T;H)}, \\ \left\| \int_0^t \mathcal{A}^\alpha e^{(t-s)\mathcal{A}} f(s) ds \right\|_H &\leq C \|f\|_{L^2(0,T;H)}, \quad \alpha \leq \frac{1}{2}, \\ \left\| \mathcal{A}^\alpha e^{t\mathcal{A}} x \right\|_{L^2(0,T;H)} &\leq C \|x\|_H, \quad \alpha \leq \frac{1}{2}. \end{aligned} \quad (4.141)$$

Inserting inequalities (4.141) in (4.140), for $\varepsilon \in (0, \frac{1}{2})$, shows that

$$\left\| \mathcal{A}^{\frac{1-\varepsilon}{2}} \begin{bmatrix} \tilde{w} \\ \tilde{w}_t \\ \tilde{\theta} \end{bmatrix} \right\|_{L^2(0,T;H)} \leq C \left\| \begin{bmatrix} \tilde{w}_0 \\ \tilde{w}_1 \\ \tilde{\theta}_0 \end{bmatrix} \right\|_H + C \left\| \mathcal{A}^{-\frac{1+\varepsilon}{2}} \begin{bmatrix} 0 \\ \mathcal{F}(\tilde{u}, \tilde{w}, \tilde{\phi}) \\ 0 \end{bmatrix} \right\|_{L^2(0,T;H)}. \quad (4.142)$$

Therefore, by duality

$$\left\| \mathcal{A}^{-\frac{1+\varepsilon}{2}} \begin{bmatrix} 0 \\ \mathcal{F}(\tilde{u}, \tilde{w}, \tilde{\phi}) \\ 0 \end{bmatrix} \right\|_H \leq C \|\mathcal{F}(\tilde{u}, \tilde{w}, \tilde{\phi})\|_{-(1+\varepsilon), \Omega}. \quad (4.143)$$

Using equalities (4.31) we find, for every $\psi \in H^{1+\varepsilon}(\Omega)$,

$$\begin{aligned} (\mathcal{F}(\tilde{u}, \tilde{w}, \tilde{\phi}), \psi)_\Omega &= (\operatorname{div}\{F(\tilde{u}, \tilde{w}, \tilde{\phi})\} + \mathcal{A}_M G_2(F(\tilde{u}, \tilde{w}, \tilde{\phi}) \cdot \nu), \psi)_\Omega - (\mathcal{P}_2(\tilde{u}, \tilde{w}), \psi)_\Omega \\ &= -(F(\tilde{u}, \tilde{w}, \tilde{\phi}), \nabla \psi)_\Omega - (\mathcal{P}_2(\tilde{u}, \tilde{w}), \psi)_\Omega. \end{aligned} \quad (4.144)$$

Next, we shall estimate the terms in right-hand side of (4.144). First, using the definition of F we obtain

$$\begin{aligned} (F(\tilde{u}, \tilde{w}, \tilde{\phi}), \nabla \psi)_\Omega &= (\sigma[\varepsilon(\tilde{u})] \nabla w^2, \nabla \psi)_\Omega + (\sigma[f(\nabla w^1) - f(\nabla w^2)] \nabla w^2, \nabla \psi)_\Omega \\ &\quad + (\sigma[\varepsilon(u^1) + f(\nabla w^1)] \nabla \tilde{w}, \nabla \psi)_\Omega + (\tilde{\phi} \nabla w^2 + \phi^1 \nabla \tilde{w}, \nabla \psi)_\Omega. \end{aligned} \quad (4.145)$$

Let us estimate all the inner-products in (4.145). For this we recall the following inequalities (cf. [19])

$$\begin{aligned} (uv, w)_\Omega &\leq C \|uv\|_{\eta, \Omega} \|w\|_{-\eta, \Omega}, \\ \|uv\|_{\eta, \Omega} &\leq C \|u\|_{1, \Omega} \|v\|_{\eta + \eta_0, \Omega}, \\ \|uv\|_{-\eta, \Omega} &\leq C \|u\|_{1-\eta, \Omega} \|v\|_\Omega, \end{aligned}$$

where $\eta < \frac{1}{2}$ and $\eta_0 \in \mathbb{R}$.

Consider $\eta < \varepsilon$. Then

$$\begin{aligned} (\sigma[\varepsilon(\tilde{u})] \nabla w^2, \nabla \psi)_\Omega &\leq C \|\varepsilon(\tilde{u})\|_{-\eta, \Omega} \|\nabla w^2 \cdot \nabla \psi\|_{\eta, \Omega} \\ &\leq C \|\varepsilon(\tilde{u})\|_{-\eta, \Omega} \|\nabla w^2\|_{1, \Omega} \|\nabla \psi\|_{\eta + \eta_0, \Omega}, \end{aligned}$$

and taking $\eta_0 \in \mathbb{R}$ such that $\eta_0 + \eta \leq \varepsilon$, we obtain

$$(\sigma[\varepsilon(\tilde{u})] \nabla w^2, \nabla \psi)_\Omega \leq C \|\tilde{u}\|_{1-\eta, \Omega} \|w^2\|_{2, \Omega} \|\psi\|_{1+\varepsilon, \Omega}.$$

To the others terms, we use Hölder's inequality and Sobolev's embedding, so we find

$$\begin{aligned} (\sigma[f(\nabla w^1) - f(\nabla w^2)] \nabla w^2, \nabla \psi)_\Omega &\leq C \|f(\nabla w^1) - f(\nabla w^2)\|_\Omega \|w^2\|_{2+\varepsilon, \Omega} \|\psi\|_{1+\varepsilon, \Omega} \\ &\leq C_B \|\tilde{w}\|_{2-\varepsilon, \Omega} \|w^2\|_{2+\varepsilon, \Omega} \|\psi\|_{1+\varepsilon, \Omega}, \end{aligned}$$

$$\begin{aligned}
(\sigma[\varepsilon(u^1) + f(\nabla w^1)] \nabla \tilde{w}, \nabla \psi)_{\Omega} &\leq C \|\varepsilon(u^1) + f(\nabla w^1)\|_{\Omega} \|\tilde{w}\|_{2+\varepsilon, \Omega} \|\psi\|_{1+\varepsilon, \Omega} \\
&\leq C_B \|\tilde{w}\|_{\Omega}^{\left[\frac{1-2\varepsilon}{3-\varepsilon}\right]} \|\tilde{w}\|_{3-\varepsilon, \Omega}^{\left[\frac{2+\varepsilon}{3-\varepsilon}\right]} \|\psi\|_{1+\varepsilon, \Omega} \\
&\leq \left[\delta \|\tilde{w}\|_{3-\varepsilon, \Omega} + C_{B, \delta} \|\tilde{w}\|_{\Omega} \right] \|\psi\|_{1+\varepsilon, \Omega}
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{\phi} \nabla w^2 + \phi^1 \nabla \tilde{w}, \nabla \psi)_{\Omega} &\leq C \left[\|\tilde{\phi} \nabla w^2\|_{-\varepsilon, \Omega} + \|\phi^1 \nabla \tilde{w}\|_{-\varepsilon, \Omega} \right] \|\nabla \psi\|_{\varepsilon, \Omega} \\
&\leq C \left[\|\tilde{\phi}\|_{\Omega} \|w^2\|_{2-\varepsilon, \Omega} + \|\phi^1\|_{\Omega} \|\tilde{w}\|_{2-\varepsilon, \Omega} \right] \|\psi\|_{1+\varepsilon, \Omega}.
\end{aligned}$$

Finally, by (4.11) we obtain

$$(\mathcal{P}_2(\tilde{u}, \tilde{w}), \psi)_{\Omega} \leq C_B [\|\tilde{u}\|_{L^{r+1}} + \|\tilde{w}\|_{L^{r+1}}] \|\psi\|_{\Omega}.$$

Step 3. Conclusion: Combining these estimates with (4.144) we obtain

$$\|\mathcal{F}(\tilde{u}, \tilde{w}, \tilde{\phi})\|_{-(1+\varepsilon), \Omega}^2 \leq C_B \|\tilde{\phi}\|_{\Omega}^2 + \delta \|\tilde{w}\|_{3-\varepsilon, \Omega}^2 + C_{B, \delta} \text{l.o.t.}(\tilde{u}, \tilde{w}).$$

This estimate together with (4.142)-(4.143), and the characterization of $D(\mathcal{A}^{\frac{1-\varepsilon}{2}})$, imply that

$$\begin{aligned}
\int_0^T \left[\|\tilde{w}\|_{3-\varepsilon, \Omega}^2 + \|\tilde{w}_t\|_{1-\varepsilon, \Omega}^2 + \|\tilde{\theta}\|_{1-\varepsilon, \Omega}^2 \right] dt &\leq C\tilde{E}(0) + C_B \int_0^T \|\tilde{\phi}\|_{\Omega}^2 dt + \delta \int_0^T \|\tilde{w}\|_{3-\varepsilon, \Omega}^2 dt \\
&\quad + C_{\delta, B, T} \text{l.o.t.}(\tilde{u}, \tilde{w}).
\end{aligned}$$

Taking $\delta > 0$ small enough we obtain the proof of Lemma 4.5. $\square$

Remark 4.5. The estimate in the Lemma 4.5 and the Trace Theorem imply that

$$\int_0^T \left[\|\Delta \tilde{w}\|_{\Gamma_0}^2 + \|\tilde{w}_t\|_{\frac{1}{2}-\varepsilon, \Gamma_1}^2 \right] dt \leq C\tilde{E}(0) + C_B \int_0^T \|\tilde{\phi}\|_{1, \Omega}^2 dt + C_{B, T} \text{l.o.t.}(\tilde{u}, \tilde{w}), \quad (4.146)$$

holds for $\varepsilon \in (0, \frac{1}{2})$.

4.8.1.4 A second observability Inequality

Lemma 4.6. Let $(\tilde{u}, \tilde{u}_t, \tilde{w}, \tilde{w}_t, \tilde{\phi}, \tilde{\theta})$ be a solution of the system (4.88)-(4.97). Then for $\alpha \in (0, \frac{T}{2})$, there exist positive constants $C_{\alpha}, C_{\alpha, B}, C_{\alpha, B, T}$, such that

$$\begin{aligned}
T\tilde{E}(T) + \int_0^T \tilde{E}(t) dt + \int_0^T \left[\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{\phi}\|_{1, \Omega}^2 \right] dt &+ \left[\sup_{t \in [0, T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2 \\
&\leq C\tilde{E}(0) + [C + 2\alpha]\tilde{E}(T) + C_{\alpha, B} D_0^T(\tilde{u}, \tilde{\phi}, \tilde{\theta}) \\
&\quad + C_{\alpha} \int_0^T \left| \sum_{k=1}^6 \mathcal{R}_i(t) \right| dt + \int_0^T \int_s^T \left| \sum_{k=1}^6 \mathcal{R}_i(t) \right| dt ds \\
&\quad + C_{\alpha, B, T} \text{l.o.t.}(\tilde{u}, \tilde{w}).
\end{aligned} \quad (4.147)$$

Proof. The Lemma 4.3 applied to the interval $[\alpha, T - \alpha]$ and estimate in the Lemma 4.4 imply that

$$\begin{aligned} \int_{\alpha}^{T-\alpha} \tilde{E}(t) dt &\leq C[\tilde{E}(\alpha) + \tilde{E}(T - \alpha)] + C_{\alpha, B} \int_{\alpha}^{T-\alpha} [\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{\theta}\|_{1, \Omega}^2 + \|\tilde{\phi}\|_{1, \Omega}^2] dt \\ &\quad + C_{\alpha, B} \int_{\alpha}^{T-\alpha} [\|\Delta \tilde{w}\|_{-\frac{1}{2}, \Gamma_0}^2 + \|\tilde{w}_t\|_{\frac{1}{2}-\varepsilon, \Gamma_1}^2 + \|\tilde{w}\|_{2+\varepsilon, \Omega}^2] dt + C_{\alpha, B, T} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned}$$

Interpolation inequality $\|w\|_{2+\varepsilon, \Omega} \leq C\|w\|_{\Omega}^{\frac{1-2\varepsilon}{3-\varepsilon}} \|w\|_{\frac{2+\varepsilon}{3-\varepsilon}, \Omega}^{\frac{2+\varepsilon}{3-\varepsilon}}$, together with estimates from Lemma 4.5 and (4.146) imply that

$$\int_{\alpha}^{T-\alpha} \tilde{E}(t) dt \leq C[\tilde{E}(0) + \tilde{E}(\alpha) + \tilde{E}(T - \alpha)] + C_{\alpha, B} D_0^T(\tilde{u}, \tilde{\phi}, \tilde{\theta}) + C_{\alpha, B, T} \text{l.o.t.}(\tilde{u}, \tilde{w}) \quad (4.148)$$

We shall extend the integral on the left-hand side to the interval $(0, T)$. To this end, using the energy equality (4.98) we find that

$$\begin{aligned} \int_0^{\alpha} \tilde{E}(t) dt &\leq \alpha \tilde{E}(\alpha) + \int_0^{\alpha} \int_0^t \sum_{k=1}^6 \mathcal{R}_i(s) ds dt, \\ \int_{T-\alpha}^{\alpha} \tilde{E}(t) dt &\leq \alpha \tilde{E}(T - \alpha) + \int_{T-\alpha}^{\alpha} \int_0^t \sum_{k=1}^6 \mathcal{R}_i(s) ds dt, \\ \tilde{E}(\alpha) &\leq \tilde{E}(T) + D_{\alpha}^T(\tilde{u}, \tilde{\phi}, \tilde{\theta}) + \left| \int_{\alpha}^T \sum_{k=1}^6 \mathcal{R}_i(t) dt \right|, \\ \tilde{E}(T - \alpha) &\leq \tilde{E}(T) + D_{T-\alpha}^T(\tilde{u}, \tilde{\phi}, \tilde{\theta}) + \left| \int_{T-\alpha}^T \sum_{k=1}^6 \mathcal{R}_i(t) dt \right|, \end{aligned}$$

and thus

$$\int_0^{\alpha} \tilde{E}(t) dt + \int_{T-\alpha}^{\alpha} \tilde{E}(t) dt \leq 2\alpha \tilde{E}(T) + 2D_0^T(\tilde{u}, \tilde{\phi}, \tilde{\theta}) + 2(\alpha + 1) \int_0^T \left| \sum_{k=1}^6 \mathcal{R}_i(t) \right| dt. \quad (4.149)$$

We also have from energy identity (4.98) that

$$T \tilde{E}(T) \leq \int_0^T \tilde{E}(s) ds + \int_0^T \int_s^T \left| \sum_{k=1}^6 \mathcal{R}_i(t) \right| dt ds. \quad (4.150)$$

Combining (4.149) and (4.150) with (4.148) we obtain the inequality

$$\begin{aligned} T \tilde{E}(T) + \int_0^T \tilde{E}(t) dt &\leq C \tilde{E}(0) + [C + 2\alpha] \tilde{E}(T) + C_{\alpha, B} D_0^T(\tilde{u}, \tilde{\phi}, \tilde{\theta}) \\ &\quad + C_{\alpha} \int_0^T \left| \sum_{k=1}^6 \mathcal{R}_i(t) \right| dt + \int_0^T \int_s^T \left| \sum_{k=1}^6 \mathcal{R}_i(t) \right| dt ds \\ &\quad + C_{\alpha, B, T} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned} \quad (4.151)$$

In order to absorb some terms that will come from the nonlinearities, we shall add $\int_0^T [\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{\phi}\|_{1,\Omega}^2] dt + \left[\sup_{t \in [0,T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2$ in both sides of (4.151). Then using the energy equality (4.98) we obtain

$$\left[\sup_{t \in [0,T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2 \leq \tilde{E}(0) + D_0^T(\tilde{u}, \tilde{\phi}, \tilde{\theta}) + \left| \int_0^T \sum_{k=1}^6 \mathcal{R}_i(t) dt \right|.$$

Therefore we obtain (4.147). This completes the proof of Lemma 4.6. $\square$

4.8.1.5 Estimating $\mathcal{R}_i(t)$

Lemma 4.7. One has

$$\begin{aligned} \max \left\{ \int_0^T \int_s^T \left| \sum_{k=1}^6 \mathcal{R}_i(t) \right| dt ds, \int_0^T \left| \sum_{k=1}^6 \mathcal{R}_i(t) \right| dt \right\} \\ \leq \delta \int_0^T \left[\|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 + \|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{u}_t\|_{\Omega}^2 + \|\tilde{\phi}\|_{1,\Omega}^2 \right] dt \\ + \delta \tilde{E}(T) + \delta \left[\sup_{t \in [0,T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2 + C_{B,T,\delta} \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt \\ + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned} \quad (4.152)$$

Proof. The inequalities (4.10) and (4.11) imply that

$$\int_0^T \int_s^T \mathcal{R}_1(t) dt ds \leq C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}) + \delta \int_0^T \|\tilde{u}_t\|_{\Omega}^2 dt, \quad (4.153)$$

and

$$\int_0^T \int_s^T \mathcal{R}_2(t) dt ds \leq C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}) + \delta \int_0^T \|\tilde{w}_t\|_{\Omega}^2 dt. \quad (4.154)$$

Using estimate (ii) of Lemma 4.2, we find that

$$\int_0^T \int_s^T \mathcal{R}_3(t) dt ds \leq C_{B,T,\delta} \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt + \delta \int_0^T \|\tilde{u}_t\|_{\Omega}^2 dt. \quad (4.155)$$

Analogously, using estimate (iii) of Lemma 4.2, we see that

$$\int_0^T \int_s^T \mathcal{R}_4(t) dt ds \leq C_{B,T,\delta} \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt + \delta \int_0^T \|\tilde{u}_t\|_{\Gamma_1}^2 dt. \quad (4.156)$$

Now, we first note that

$$\begin{aligned} \int_0^T \int_s^T \mathcal{R}_5(t) dt ds &= \int_0^T \int_s^T (\sigma[\varepsilon(\tilde{u})] \nabla w^2, \nabla \tilde{w}_t)_{\Omega} dt ds \\ &+ \int_0^T \int_s^T (\sigma[f(\nabla w^1) - f(\nabla w^2)] \nabla w^2, \nabla \tilde{w}_t)_{\Omega} dt ds \\ &+ \int_0^T \int_s^T (\sigma[\varepsilon(u^1) + f(\nabla w^1)] \nabla \tilde{w}, \nabla w_t)_{\Omega} dt ds. \end{aligned} \quad (4.157)$$

We shaw estimate the three integrals in (4.157). Integrate by parts in time we obtain

$$\begin{aligned} \int_0^T \int_s^T (\sigma[\varepsilon(\tilde{u})] \nabla w^2, \nabla \tilde{w}_t)_{\Omega} dt ds &= \int_0^T (\sigma[\varepsilon(\tilde{u})] \nabla w^2, \nabla \tilde{w})_{\Omega} \Big|_s^T ds \\ &\quad - \int_0^T \int_s^T (\sigma[\varepsilon(\tilde{u}_t)] \nabla w^2, \nabla \tilde{w})_{\Omega} dt ds \\ &\quad - \int_0^T \int_s^T (\sigma[\varepsilon(\tilde{u})] \nabla w_t^2, \nabla \tilde{w})_{\Omega} dt ds. \end{aligned} \quad (4.158)$$

But,

$$\begin{aligned} \int_0^T (\sigma[\varepsilon(\tilde{u})] \nabla w^2, \nabla \tilde{w})_{\Omega} \Big|_s^T ds &= T (\sigma[\varepsilon(\tilde{u}(T))] \nabla w^2(T), \nabla \tilde{w}(T))_{\Omega} \\ &\quad - \int_0^T (\sigma[\varepsilon(\tilde{u})] \nabla w^2, \nabla \tilde{w})_{\Omega} dt. \end{aligned}$$

Using Sobolev's embedding $H^{2-\varepsilon}(\Omega) \subset W^{1,4}(\Omega)$, we can show that

$$\begin{aligned} T (\sigma[\varepsilon(\tilde{u}(T))] \nabla w^2(T), \nabla \tilde{w}(T))_{\Omega} &\leq T \|\sigma[\varepsilon(\tilde{u}(T))]\|_{\Omega} \|\nabla w^2(T)\|_{L^4(\Omega)} \|\nabla \tilde{w}(T)\|_{L^4(\Omega)} \\ &\leq \delta \tilde{E}(T) + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}), \end{aligned}$$

and

$$\begin{aligned} \int_0^T (\sigma[\varepsilon(\tilde{u})] \nabla w^2, \nabla \tilde{w})_{\Omega} dt &\leq \int_0^T \|\sigma[\varepsilon(\tilde{u})]\|_{\Omega} \|\nabla w^2\|_{L^4(\Omega)} \|\nabla \tilde{w}\|_{L^4(\Omega)} dt \\ &\leq \delta \int_0^T \|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 dt + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned}$$

Therefore we conclude that

$$\int_0^T (\sigma[\varepsilon(\tilde{u})] \nabla w^2, \nabla \tilde{w})_{\Omega} \Big|_s^T ds \leq \delta \tilde{E}(T) + \delta \int_0^T \|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 dt + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}).$$

Integration by parts in space variable and the trace theorem imply that

$$\begin{aligned} \int_0^T \int_s^T (\sigma[\varepsilon(\tilde{u}_t)] \nabla w^2, \nabla \tilde{w})_{\Omega} dt ds &\leq C_T \int_0^T \left[\|\nabla w^2 \cdot \nabla \tilde{w}\|_{\Gamma_1} \|\tilde{u}_t\|_{\Gamma_1} + \|\nabla w^2 \cdot \nabla \tilde{w}\|_{1,\Omega} \|\tilde{u}_t\|_{\Omega} \right] dt \\ &\leq C_{T,\delta} \int_0^T \|\nabla w^2 \cdot \nabla \tilde{w}\|_{1,\Omega}^2 dt + \delta \int_0^T \left[\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{u}_t\|_{\Omega}^2 \right] dt \\ &\leq C_{B,T,\delta} \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt + \delta \int_0^T \|\tilde{u}_t\|_{\Gamma_1}^2 dt + \delta \int_0^T \|\tilde{u}_t\|_{\Omega}^2 dt. \end{aligned}$$

Hölder inequality and Sobolev's embedding imply that

$$\begin{aligned}
& \int_0^T \int_s^T (\sigma[\varepsilon(\tilde{u})] \nabla w_t^2, \nabla \tilde{w})_{\Omega} dt ds \\
& \leq C_T \sup_{t \in [0, T]} \|\varepsilon(\tilde{u})\|_{\Omega} \int_0^T \left[\|\nabla w_t^2\|_{\Omega} \|\tilde{w}\|_{2+\varepsilon, \Omega} \right] dt \\
& \leq C_T \sup_{t \in [0, T]} \tilde{E}^{\frac{1}{2}}(t) \left[\int_0^T \|\nabla w_t^2\|_{\Omega}^2 dt \right]^{\frac{1}{2}} \left[\int_0^T \|\tilde{w}\|_{2+\varepsilon, \Omega}^2 dt \right]^{\frac{1}{2}} \\
& \leq \delta \left[\sup_{t \in [0, T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2 + C_{B, T, \delta} \int_0^T \|\tilde{w}\|_{2+\varepsilon, \Omega}^2 dt.
\end{aligned}$$

Inserting these estimates in (4.158) we obtain

$$\begin{aligned}
& \int_0^T \int_s^T (\sigma[\varepsilon(\tilde{u})] \nabla w^2, \nabla \tilde{w}_t)_{\Omega} dt ds \\
& \leq \delta \tilde{E}(T) + \delta \int_0^T \left[\|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 + \|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{u}_t\|_{\Omega}^2 \right] dt \\
& + \delta \left[\sup_{t \in [0, T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2 + C_{B, T, \delta} \int_0^T \|\tilde{w}\|_{2+\varepsilon, \Omega}^2 dt \\
& + C_{B, T, \delta} \text{l.o.t.}(\tilde{u}, \tilde{w}).
\end{aligned} \tag{4.159}$$

Let us estimate the second integral in (4.157). Taking $\tilde{\varepsilon} < 1 - 2\varepsilon$, we have that $H^{2-\varepsilon}(\Omega) \subset H^{1+\varepsilon+\tilde{\varepsilon}}(\Omega)$, and then

$$\begin{aligned}
& \int_0^T \int_s^T (\sigma[f(\nabla w^1) - f(\nabla w^2)] \nabla w^2, \nabla \tilde{w}_t)_{\Omega} dt ds \\
& \leq C_T \int_0^T \|f(\nabla w^1) - f(\nabla w^2) \nabla w^2\|_{\varepsilon, \Omega} \|\nabla \tilde{w}_t\|_{-\varepsilon, \Omega} dt \\
& \leq C_T \int_0^T \|f(\nabla w^1) - f(\nabla w^2)\|_{1, \Omega} \|\nabla w^2\|_{\varepsilon+\tilde{\varepsilon}, \Omega} \|\nabla \tilde{w}_t\|_{-\varepsilon, \Omega} dt \\
& \leq C_{B, T, \delta} \int_0^T \|\tilde{w}\|_{2+\varepsilon, \Omega}^2 dt + \delta \int_0^T \|\tilde{w}_t\|_{1-\varepsilon, \Omega}^2 dt.
\end{aligned} \tag{4.160}$$

To conclude, we have to estimate the third integral in (4.157). Integration by parts in time and space variable and the fact that $\|\nabla \tilde{w}\|^2_{1, \Omega} = \|\nabla \tilde{w} \cdot \nabla \tilde{w}\|_{1, \Omega} \leq C \|\tilde{w}\|_{1, \Omega} \|\tilde{w}\|_{2+\varepsilon, \Omega}$ we obtain

$$\begin{aligned}
& \int_0^T \int_s^T (\sigma[\varepsilon(u^1)] \nabla \tilde{w}, \nabla \tilde{w}_t)_{\Omega} dt ds \\
& \leq C_T \|\varepsilon(u^1(T))\|_{\Omega} \|\nabla \tilde{w}(T)\|^2_{\Omega} + C_T \int_0^T \|\varepsilon(u^1)\|_{\Omega} \|\nabla \tilde{w}\|^2_{\Omega} dt \\
& + C_T \int_0^T \|u_t^1\|_{\Gamma_1} \|\nabla \tilde{w}\|^2_{\Gamma_1} dt + C_T \int_0^T \|u_t^1\|_{\Omega} \|\nabla \tilde{w}\|^2_{1, \Omega} dt \\
& \leq C_{B, T} \int_0^T \|\tilde{w}\|_{2+\varepsilon, \Omega}^2 dt + C_{B, T} \text{l.o.t.}(\tilde{u}, \tilde{w}).
\end{aligned} \tag{4.161}$$

From the fact that $H^{2-\varepsilon}(\Omega) \subset H^{1+\varepsilon+\tilde{\varepsilon}}(\Omega)$ we see that

$$\begin{aligned} \int_0^T \int_s^T (\sigma[f(\nabla w^1)] \nabla \tilde{w}, \nabla \tilde{w}_t)_{\Omega} dt ds &\leq C_T \int_0^T \|\sigma[f(\nabla w^1)] \nabla \tilde{w}\|_{\varepsilon, \Omega} \|\nabla \tilde{w}_t\|_{-\varepsilon, \Omega} dt \\ &\leq C_T \int_0^T \|\sigma[f(\nabla w^1)]\|_{1, \Omega} \|\nabla \tilde{w}\|_{\varepsilon+\tilde{\varepsilon}} \|\tilde{w}_t\|_{1-\varepsilon, \Omega} dt \\ &\leq \delta \int_0^T \|\tilde{w}_t\|_{1-\varepsilon, \Omega}^2 + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned} \quad (4.162)$$

The estimates (4.159), (4.160), (4.161) and (4.162) into (4.157) imply that

$$\begin{aligned} \int_0^T \int_s^T \mathcal{R}_5(t) dt ds &\leq \delta \tilde{E}(T) + \delta \int_0^T \left[\|\sigma[\varepsilon(\tilde{u})]\|_{\Omega}^2 + \|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{u}_t\|_{\Omega}^2 + \|\tilde{w}_t\|_{1-\varepsilon, \Omega}^2 \right] dt \\ &\quad + \delta \left[\sup_{t \in [0, T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2 + C_{B,T,\delta} \int_0^T \|\tilde{w}\|_{2+\varepsilon, \Omega}^2 dt + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned} \quad (4.163)$$

Finally we estimate $\int_0^T \int_s^T \mathcal{R}_5(t) dt ds$. Taking $\tilde{\varepsilon} < 1 - 2\varepsilon$, see that

$$\begin{aligned} \int_0^T \int_s^T \mathcal{R}_6(t) dt ds &= \int_0^T \int_s^T \left[-(\phi^1 \nabla \tilde{w}, \nabla \tilde{w}_t)_{\Omega} + (\nabla \tilde{w} \cdot \nabla w_t^2, \tilde{\phi})_{\Omega} + (\tilde{\phi} \nabla \tilde{w}, \nabla \tilde{w}_t)_{\Omega} \right] dt ds \\ &\leq C_T \int_0^T \left[\|\phi^1 \nabla \tilde{w}\|_{\varepsilon, \Omega} \|\nabla \tilde{w}_t\|_{-\varepsilon, \Omega} + \|\nabla \tilde{w}\|_{L^4(\Omega)} \|\nabla w_t^2\|_{\Omega} \|\tilde{\phi}\|_{L^4(\Omega)} \right] dt \\ &\quad + C_T \int_0^T \left[\|\tilde{\phi} \nabla \tilde{w}\|_{\varepsilon, \Omega} \|\nabla \tilde{w}_t\|_{-\varepsilon, \Omega} \right] dt \\ &\leq C_T \int_0^T \left[\|\phi^1\|_{1, \Omega} \|\nabla \tilde{w}\|_{\varepsilon+\tilde{\varepsilon}, \Omega} \|\tilde{w}_t\|_{1-\varepsilon, \Omega} + \|\tilde{w}\|_{W^{1,4}(\Omega)} \|w_t^2\|_{1, \Omega} \|\tilde{\phi}\|_{1, \Omega} \right] dt \\ &\quad + \int_0^T \left[\|\tilde{\phi}\|_{1, \Omega} \|\nabla \tilde{w}\|_{\varepsilon+\tilde{\varepsilon}, \Omega} \|\nabla \tilde{w}_t\|_{-\varepsilon, \Omega} \right] dt \\ &\leq \delta \int_0^T \left[\|\tilde{\phi}\|_{1, \Omega}^2 + \|\tilde{w}_t\|_{1-\varepsilon, \Omega}^2 \right] dt + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}). \end{aligned} \quad (4.164)$$

Combining (4.153), (4.154), (4.155), (4.156), (4.163) and (4.164), and observing that $\|\tilde{w}_t\|_{1-\varepsilon, \Omega}^2$ was estimated in the Lemma 4.5, we conclude that $\int_0^T \int_s^T |\sum \mathcal{R}_i(t)| dt ds$ satisfies estimate (4.152). Analogous argument shows that $\int_0^T |\sum \mathcal{R}_i(t)| dt$ also satisfies estimate (4.152). $\square$

4.8.1.6 Quasistability inequality

Lemma 4.8. (Stabilizability estimate) Under hypotheses of Theorem 4.2. Let B is a bounded set of $\mathcal{H}$. Then in the context of (4.86)-(4.87), for $\varepsilon \in (0, \frac{1}{4})$, there exist constants $\beta > 0$ and $C_1, C_2 > 0$, depending only on B , such that

$$\tilde{E}(t) \leq C_1 \tilde{E}(0) e^{-\beta t} + C_2 \sup_{\tau \in [0, t]} \left\{ \|\tilde{u}(\tau)\|_{L^{r+1}(\Omega)}^2 + \|\tilde{u}(\tau)\|_{1-\varepsilon, \Omega}^2 + \|\tilde{w}(\tau)\|_{2-\varepsilon, \Omega}^2 \right\}. \quad (4.165)$$

Proof. Inserting estimate (4.152) into (4.147) we obtain that

$$\begin{aligned}
T\tilde{E}(T) + \int_0^T \tilde{E}(t)dt + \int_0^T [\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{\phi}\|_{1,\Omega}^2]dt + \left[\sup_{t \in [0,T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2 \\
\leq C\tilde{E}(0) + [C_\delta + 2\alpha]\tilde{E}(T) + C_{B,\alpha}D_0^T(\tilde{u}, \tilde{\phi}, \tilde{\theta}) + C_{\alpha,B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}) \\
+ \delta \int_0^T [\|\sigma[\varepsilon(\tilde{u})]\|_\Omega^2 + \|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{u}_t\|_\Omega^2 + \|\tilde{\phi}\|_{1,\Omega}^2 + \|\tilde{w}_t\|_{1-\varepsilon,\Omega}^2]dt \\
+ \delta \left[\sup_{t \in [0,T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2 + C_{B,T,\delta} \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt.
\end{aligned} \tag{4.166}$$

Interpolation inequality $\|\tilde{w}\|_{2+\varepsilon,\Omega} \leq C\|\tilde{w}\|_\Omega^{\frac{1-2\varepsilon}{3-\varepsilon}} \|\tilde{w}\|_{3-\varepsilon,\Omega}^{\frac{2+\varepsilon}{3-\varepsilon}}$ implies that

$$C_{B,T,\delta} \int_0^T \|\tilde{w}\|_{2+\varepsilon,\Omega}^2 dt \leq \delta \int_0^T \|\tilde{w}\|_{3-\varepsilon,\Omega}^2 dt + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}).$$

This together with (4.166) and estimates from Lemma 4.5) and (4.146) imply that

$$\begin{aligned}
T\tilde{E}(T) + \int_0^T \tilde{E}(t)dt + \int_0^T [\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{\phi}\|_{1,\Omega}^2]dt + \left[\sup_{t \in [0,T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2 \\
\leq C\tilde{E}(0) + [C_\delta + 2\alpha]\tilde{E}(T) + C_{B,\alpha}D_0^T(\tilde{u}, \tilde{\phi}, \tilde{\theta}) + C_{\alpha,B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}) \\
+ \delta \int_0^T [\|\sigma[\varepsilon(\tilde{u})]\|_\Omega^2 + \|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{u}_t\|_\Omega^2 + \|\tilde{\phi}\|_{1,\Omega}^2]dt + \delta \left[\sup_{t \in [0,T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2.
\end{aligned}$$

Now let $\delta > 0$ be small enough. For $T > 4C_\delta = T_0$ and $\alpha = C < \frac{T}{2}$ we have that

$$\begin{aligned}
T\tilde{E}(T) + \int_0^T \tilde{E}(t)dt + \int_0^T [\|\tilde{u}_t\|_{\Gamma_1}^2 + \|\tilde{\phi}\|_{1,\Omega}^2]dt + \left[\sup_{t \in [0,T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2 \\
\leq C\tilde{E}(0) + C_B D_0^T(\tilde{u}, \tilde{\phi}, \tilde{\theta}) + C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}).
\end{aligned} \tag{4.167}$$

Next, we estimate the damping term $D_0^T(\tilde{u}, \tilde{\phi}, \tilde{\theta})$. Energy equality (4.98) and estimate (4.152) imply that

$$\begin{aligned}
D_0^T(\tilde{u}, \tilde{\phi}, \tilde{\theta}) &\leq \tilde{E}(0) - \tilde{E}(T) + \int_\alpha^T \left| \sum_{k=1}^6 \mathcal{R}_i(t) \right| dt \\
&\leq \tilde{E}(0) - \tilde{E}(T) + \delta \tilde{E}(T) + \delta \int_0^T [\|\sigma[\varepsilon(\tilde{u})]\|_\Omega^2 + \|\tilde{u}_t\|_{\Gamma_1}^2] dt \\
&\quad + \delta \int_0^T \|\tilde{u}_t\|_\Omega^2 dt + \delta \int_0^T \|\tilde{\phi}\|_{1,\Omega}^2 dt + \delta \left[\sup_{t \in [0,T]} \tilde{E}^{\frac{1}{2}}(t) \right]^2 \\
&\quad + C_{B,T,\delta} \text{l.o.t.}(\tilde{u}, \tilde{w}).
\end{aligned}$$

This inequality together with (4.167) give, for δ small enough, the inequality

$$\tilde{E}(T) \leq C_B \tilde{E}(0) - C_B \tilde{E}(T) + C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}),$$

and therefore

$$\tilde{E}(T) \leq \frac{C_B}{1+C_B} \tilde{E}(0) + C_{B,T} \text{l.o.t.}(\tilde{u}, \tilde{w}).$$

Repeating this argument on the interval $I_m = [mT, (m+1)T]$, $m \in \mathbb{N}$, we obtain

$$\tilde{E}((m+1)T) \leq \frac{C_B}{1+C_B} \tilde{E}(mT) + C_{B,T} \text{l.o.t.}_m(\tilde{u}, \tilde{w}),$$

for fixed $T > T_0$, where

$$\text{l.o.t.}_m(\tilde{u}, \tilde{w}) = \sup_{t \in I_m} \|\tilde{u}\|_{L^{r+1}(\Omega)}^2 + \sup_{t \in I_m} \|\tilde{u}\|_{1-\varepsilon, \Omega}^2 + \sup_{t \in I_m} \|\tilde{w}\|_{2-\varepsilon, \Omega}^2.$$

Denoting $\gamma_B = \frac{C_B}{1+C_B} < 1$, we can show, by induction, that

$$\tilde{E}(nT) \leq \gamma_B^n \tilde{E}(0) + C_{B,T} \sum_{k=1}^n \gamma_B^{n-k} \text{l.o.t.}_{k-1}(\tilde{u}, \tilde{w}), \quad \forall n \in \mathbb{N}. \quad (4.168)$$

Using the energy equality we can prove that

$$\tilde{E}(t) \leq C_{B,T}^1 \tilde{E}(nT) e^{\omega T} \text{ for all } nT \leq t \leq (n+1)T, \quad (4.169)$$

where the constant ω depends on B . Let $\beta = \frac{1}{T} \ln \frac{1}{\gamma_B}$, then, for $t = nT + m$ with $m < T$, we have

$$\gamma_B^n \leq \exp(-\beta t) \gamma_B^{-1}$$

and for $k \leq n$

$$\gamma_B^{n-k} = \exp(-\beta(n-k)T) \leq 1.$$

These facts combined with (4.168) and (4.169) imply that

$$\begin{aligned} \tilde{E}(t) &\leq C_{B,T}^1 \tilde{E}(t-m) e^{\omega T} \\ &= C_{B,T}^1 \tilde{E}(nT) e^{\omega T} \\ &\leq C_{B,T}^1 \gamma_B^n \tilde{E}(0) e^{\omega T} + C_{B,T}^1 C_{B,T} \sum_{k=1}^n \gamma_B^{n-k} \text{l.o.t.}_{k-1}(\tilde{u}, \tilde{w}) e^{\omega T} \\ &\leq C_{B,T}^1 \gamma_B^{\frac{t}{T}-1} e^{\omega T} \tilde{E}(0) + C_{B,T}^1 C_{B,T} \sup_{\tau \in [0, t]} \left\{ \|\tilde{u}(\tau)\|_{L^{r+1}(\Omega)}^2 + \|\tilde{u}(\tau)\|_{1-\varepsilon, \Omega}^2 + \|\tilde{w}(\tau)\|_{2-\varepsilon, \Omega}^2 \right\} e^{\omega T} \\ &\leq C_1 \tilde{E}(0) e^{-\beta t} + C_2 \sup_{\tau \in [0, t]} \left\{ \|\tilde{u}(\tau)\|_{L^{r+1}(\Omega)}^2 + \|\tilde{u}(\tau)\|_{1-\varepsilon, \Omega}^2 + \|\tilde{w}(\tau)\|_{2-\varepsilon, \Omega}^2 \right\}, \end{aligned}$$

where $C_1 = C_{B,T}^1 \gamma_B^{-1} e^{\omega T}$ and $C_2 = C_{B,T}^1 C_{B,T} e^{\omega T}$. This ends the proof. $\square$

Lemma 4.9. Under hypotheses of Theorem 4.2, the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on every bounded forward invariant set.

Proof. By using an isomorphism, we can reorder the components of a trajectory (solution) as $(u, w, u_t, w_t, \phi, \theta)$. That is, we can assume $S(t) : \mathcal{H} \rightarrow \mathcal{H}$, with $\mathcal{H} = X \times Y \times Z$, where

$$X = [H^1(\Omega)]^2 \times H^2(\Omega), \quad Y = [L^2(\Omega)]^3, \quad Z = [L^2(\Omega)]^2.$$

Then conditions (2.7), (2.8) and (2.9) are clearly satisfied. Let show that (2.10) also holds. To this end, we consider a X -seminorm defined by,

$$n_X(u, w)^2 = \|u\|_{L^{r+1}(\Omega)}^2 + \|u\|_{1-\varepsilon, \Omega}^2 + \|w\|_{2-\varepsilon, \Omega}^2.$$

This is compact on X since the embedding $[H^1(\Omega)]^2 \subset [L^{r+1}(\Omega)]^2$, $[H^1(\Omega)]^2 \subset [H^{1-\varepsilon}(\Omega)]^2$ and $H^2(\Omega) \subset H^{2-\varepsilon}(\Omega)$ are compact. Therefore, given a bounded forward set $B \in \mathcal{H}$, using estimate (4.165) from Lemma 4.8, we can write

$$\|S(t)y^1 - S(t)y^2\|_{\mathcal{H}}^2 \leq b(t)\|y^1 - y^2\|_{\mathcal{H}}^2 + c(t) \sup_{\tau \in [0, t]} \left[n_X(u^1(\tau) - u^2(\tau), w^1(\tau) - w^2(\tau)) \right]^2,$$

where

$$b(t) = C_1 e^{-\beta t} \quad \text{and} \quad c(t) = C_2.$$

This proves that our system is quasi-stable on B . □

4.8.2 Gradient Systems and Proof of Theorem 4.2

The proof of Theorem 4.2 will follows from Theorem 2.19. We observe that system $(\mathcal{H}, S(t))$ is gradient by taking the energy functional $\mathcal{E}$ as a Lyapunov function Φ , since the function $t \rightarrow \Phi(S(t)y)$ is strictly decreasing for any $y \in \mathcal{H}$.

Lemma 4.10. Under the hypotheses of Theorem 4.2, the set of equilibrium points $\mathcal{N}$ is bounded in $\mathcal{H}$.

Proof. If $y \in \mathcal{N}$, we have that $y = (u, 0, w, 0, 0, 0)$ and satisfies the stationary problem

$$\begin{cases} -\operatorname{div}\{\sigma[\varepsilon(u) + f(\nabla w)]\} + p_1(u, w) = 0 \text{ in } \Omega, \\ \Delta^2 w - \operatorname{div}\{\sigma[\varepsilon(u) + f(\nabla w)]\nabla w\} + p_2(u, w) = 0 \text{ in } \Omega, \end{cases} \quad (4.170)$$

with boundary condition

$$\begin{cases} u = 0, w = 0, \nabla w = 0 \text{ on } \Gamma_0, \\ \sigma[\varepsilon(u) + f(\nabla w)]\nu + \kappa u = 0 \text{ on } \Gamma_1, \\ [\Delta w + (1 - \mu)B_1 w] = 0 \text{ on } \Gamma_1, \\ \left[\frac{\partial}{\partial \nu} \Delta w + (1 - \mu)B_2 w \right] - \sigma[\varepsilon(u) + f(\nabla w)]\nu \cdot \nabla w = 0 \text{ on } \Gamma_1. \end{cases}$$

Multiplying (4.170)₁, (4.170)₂ by u, w , respectively, and integrating over Ω , we obtain

$$\frac{1}{2} \int_{\Omega} \sigma[N(u, w)]N(u, w) d\Omega + \frac{1}{2} a(w, w) + \frac{\kappa}{2} \int_{\Gamma_1} |u|^2 d\Gamma_1 = - \int_{\Omega} \nabla P(u, w) \cdot (u, w) d\Omega.$$

Using inequalities (4.12), (4.16) and (4.17) we find that

$$\begin{aligned} & - \int_{\Omega} \nabla P(u, w) \cdot (u, w) d\Omega \\ & \leq 2M [\|u\|_{\Omega}^2 + \|w\|_{\Omega}^2] + 2m_P |\Omega| \\ & \leq 2MM_p [\|u\|_{1,\Omega}^2 + \|w\|_{2,\Omega}^2] + 2m_P |\Omega| \\ & \leq 2MM_p M_K [\|N(u, w)\|_{\Omega}^2 + \|w\|_{W^{1,4}(\Omega)}^4] + 2 \frac{MM_p}{M_a} a(w, w) + 2m_P |\Omega| \\ & \leq 2 \frac{MM_p M_K}{M_{\sigma}} \int_{\Omega} \sigma[N(u, w)]N(u, w) d\Omega + 2 \frac{MM_p}{M_a} a(w, w) + 2m_P |\Omega| \\ & < \frac{1}{2} \int_{\Omega} \sigma[N(u, w)]N(u, w) d\Omega + \frac{1}{2} a(w, w) + 2m_P |\Omega|. \end{aligned}$$

This shows that $\mathcal{N}$ is bounded in $\mathcal{H}$. □

Proof of Theorem 4.2. From Lemmas 4.9 and 4.10 we know that $(\mathcal{H}, S(t))$ is an asymptotically compact gradient system with bounded set of stationary points. To apply Theorem 2.19 it remains to show condition (2.12). To see this, from (4.13) and (4.14) we have that

$$\mathcal{E}(t) \leq \|y(t)\|_{\mathcal{H}}^2 + C(1 + \|y(t)\|_{\mathcal{H}}^{r+1}).$$

Then

$$\mathcal{E}(t) \rightarrow \infty \text{ implies that } \|y(t)\|_{\mathcal{H}} \rightarrow \infty, \quad t \geq 0.$$

On the other hand, the inequality (4.18) implies that $E(t) \leq \frac{1}{M_E} (\mathcal{E}(t) + m_P |\Omega|)$, and then

$$\|y(t)\|_{\mathcal{H}} \rightarrow \infty \text{ implies that } \mathcal{E}(t) \rightarrow \infty, \quad t \geq 0.$$

Then condition (2.12) is satisfied. Therefore system $(\mathcal{H}, S(t))$ has a global attractor $\mathbf{A}$.

From Theorem 2.16, $\mathbf{A}$ has finite fractal dimension and from Theorem 2.17 the attractor has further “time” regularity,

$$\left\| \frac{d}{dt} S(t)y_0 \right\|_{\mathcal{H}} \leq C, \quad \forall t \in \mathbb{R}, \quad \forall y_0 \in \mathbf{A}.$$

This improved regularity applied in the equations (4.4)₁- (4.4)₂ and elliptic regularity imply that $\phi, \theta \in H^2(\Omega)$. Note that $\sigma[f(\nabla w)] \in H^{1-\varepsilon}(\Omega)$, for $\varepsilon > 0$ small, since $\sigma[f(\nabla w)]$ is a product of $L^2(\Omega)$ and $H^1(\Omega)$ functions. It follows from this fact that $\text{div}\{\sigma[f(\nabla w)]\} \in H^{-\varepsilon}(\Omega)$ and applying Trace Theorem we have that $\sigma[f(\nabla w)]v \in H^{\frac{1}{2}-\varepsilon}(\Gamma_1)$. By equations (4.1)₁, (4.2) and

(4.3) we find that u satisfies the following problem

$$\begin{cases} \operatorname{div}\{\sigma[\varepsilon(u)]\} = -\nabla\phi - p_1(u, w) - u_{tt} - \operatorname{div}\{\sigma[f(\nabla w)]\} \in [H^{-\varepsilon}(\Omega)]^2, \\ u = 0 \in [H^{\frac{1}{2}-\varepsilon}(\Gamma_0)]^2, \\ \sigma[\varepsilon(u)]\nu + \kappa u = \phi\nu - u_t - \sigma[f(\nabla w)]\nu \in [H^{\frac{1}{2}-\varepsilon}(\Gamma_1)]^2. \end{cases}$$

Then, elliptic regularity implies that $u \in [H^{2-\varepsilon}(\Omega)]^2$.

Now, we obtain via the fact $u \in [H^{2-\varepsilon}(\Omega)]^2$ that $\sigma[\varepsilon(u)]\nabla w \in [H^{1-\varepsilon}(\Omega)]^2$ and this combined with $\sigma[f(\nabla w)]\nabla w, \phi\nabla w \in [H^{1-\varepsilon}(\Omega)]^2$ we infer that $\operatorname{div}\{\sigma[\varepsilon(u) + f(\nabla w)]\nabla w + \phi\nabla w\} \in H^{-\varepsilon}(\Omega)$. Therefore by elliptic regularity the solution w of the following system is in $H^{4-\varepsilon}(\Omega)$

$$\begin{cases} \Delta^2 w = -w_{tt} + \operatorname{div}\{\sigma[\varepsilon(u) + f(\nabla w)]\nabla w + \phi\nabla w\} - p_2(u, w) \in H^{-\varepsilon}(\Omega) \\ w = \nabla w = 0 \in H^{\frac{1}{2}-\varepsilon}(\Gamma_0), \\ [\Delta w + (1-\mu)B_1 w] = -\theta \in H^{\frac{1}{2}-\varepsilon}(\Gamma_1), \\ \left[\frac{\partial}{\partial \nu}\Delta w + (1-\mu)B_2 w\right] = \sigma[\varepsilon(u) + f(\nabla w)]\nu \cdot \nabla w - \phi \frac{\partial w}{\partial \nu} + \frac{\partial \theta}{\partial \nu} \in H^{\frac{1}{2}-\varepsilon}(\Gamma_1). \end{cases}$$

The fact that $w \in H^{4-\varepsilon}(\Omega)$ implies that $\sigma[f(\nabla w)] \in H^1(\Omega)$ and $\sigma[f(\nabla w)]\nu \in H^{\frac{1}{2}}(\Gamma_1)$. Thus we conclude that u satisfy

$$\begin{cases} \operatorname{div}\{\sigma[\varepsilon(u)]\} = -\nabla\phi - p_1(u, w) - u_{tt} - \operatorname{div}\{\sigma[f(\nabla w)]\} \in [L^2(\Omega)]^2, \\ u = 0 \in [H^{\frac{1}{2}}(\Gamma_0)]^2, \\ \sigma[\varepsilon(u)]\nu + \kappa u = \phi\nu - u_t - \sigma[f(\nabla w)]\nu \in [H^{\frac{1}{2}}(\Gamma_1)]^2 \end{cases}$$

and by elliptic regularity we find that $u \in [H^2(\Omega)]^2$. This regularity results that $\sigma[\varepsilon(u) + f(\nabla w)]\nabla w + \phi\nabla w \in H^1(\Omega)$ and $\operatorname{div}\{\sigma[\varepsilon(u) + f(\nabla w)]\nabla w + \phi\nabla w\} \in L^2(\Omega)$. Then w satisfy

$$\begin{cases} \Delta^2 w = -w_{tt} + \operatorname{div}\{\sigma[\varepsilon(u) + f(\nabla w)]\nabla w + \phi\nabla w\} - p_2(u, w) \in L^2(\Omega) \\ w = \nabla w = 0 \in H^{\frac{1}{2}}(\Gamma_0), \\ [\Delta w + (1-\mu)B_1 w] = -\theta \in H^{\frac{1}{2}}(\Gamma_1), \\ \left[\frac{\partial}{\partial \nu}\Delta w + (1-\mu)B_2 w\right] = \sigma[\varepsilon(u) + f(\nabla w)]\nu \cdot \nabla w - \phi \frac{\partial w}{\partial \nu} + \frac{\partial \theta}{\partial \nu} \in H^{\frac{1}{2}}(\Gamma_1). \end{cases}$$

Again, by elliptic regularity we obtain $w \in H^4(\Omega)$. Therefore we conclude that

$$\|u(t)\|_{2,\Omega} + \|w(t)\|_{4,\Omega} + \|\theta(t)\|_{2,\Omega} + \|\phi(t)\|_{2,\Omega} \leq C, \quad t \in \mathbb{R}.$$

This ends the proof of Theorem 4.2. □

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