On Coxeter-Toda lattices

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*To my parents Carlos
and Natalia.*

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Resumo

Esse trabalho visa a investigar a estrutura bi-Hamiltoniana de uma classe de sistemas dinâmicos. Depois de introduzir as ferramentas necessárias, a saber, as noções de variedade de Poisson, de grupo de Poisson-Lie e de network no disco e no anel, introduziremos os sistemas dinâmicos relevantes nessa dissertação, chamados de reticulados de Coxeter-Toda. Esses sistemas dinâmicos, cujo espaço de fase pode ser identificado com um oportuno quociente de uma célula dupla de Coxeter-Bruhat do grupo linear geral, são obtidos por redução do sistema de Toda em GL_n . Na parte final do presente trabalho apresentaremos alguns resultados relacionado à um sistema dinâmico discreto chamado de aplicação do pentagrama, o qual pode ser obtido através uma oportuna discretização do sistema dinâmico de Boussinesq.

Palavras-chave: Variedades de Poisson, colchetes de Poisson de Faybushovich-Gekhtman, networks com pesos, reticulados de Coxeter-Toda, aplicação do pentagrama.

Abstract

This work aims to study the bi-Hamiltonian structure of a class of dynamical systems. After introducing the relevant tools, namely the notions of Poisson manifold, Poisson-Lie group and of network defined in a disc and in an annulus, we will introduce the dynamical systems of interest for this dissertation, i.e., the Coxeter-Toda lattices. These dynamical systems, whose phase space can be identified with a suitable quotient of a Coxeter double Bruhat cell of the general linear group, are obtained by reduction starting from the Toda flow on GL_n . In the final part of the present work will be presented some results concerning a discrete integrable system close to the so called Pentagon map, which is a discretization of the Boussinesq dynamical system.

Keywords: Poisson manifolds, Faybusovich-Gekhtman Poisson brackets, weighted networks, Coxeter-Toda lattices, pentagram map.

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Notation

$\mathbb{Z}$	Integer set numbers.
$\mathbb{R}$	Real set numbers.
$\mathbb{C}$	Complex set numbers.
V^*	Dual vector space of a vector space V .
$\langle \cdot, \cdot \rangle$	Canonical pairing between a vector space V and its dual V^* .
$[\cdot, \cdot]$	Lie bracket on a Lie algebra $\mathfrak{g}$.
$\mathcal{F}(M)$	Algebra of smooth real functions of the smooth manifold M .
$\mathfrak{X}(M)$	The set of smooth vector fields of the smooth manifold M .
$\{\cdot, \cdot\}$	Poisson bracket on the algebra of smooth functions of a manifold.
$\mathcal{X}_f$	Hamiltonian vector field with Hamiltonian function f .
$(\cdot, \cdot)$	Non-degenerate bilinear form in a vector space.
χ	Canonical isomorphism $V \rightarrow V^*$ induced by $(\cdot, \cdot)$ on the vector space V .
$L_{\mathcal{X}}$	Lie derivative with respect to the smooth vector field $\mathcal{X}$.
$[t_1, t_2, t_3, t_4]$	Cross ratio of four points in $\mathbb{RP}^1$.
$\delta_u H$	Variational derivative of the functional H with respect to u .

Introduction

The main goal of this dissertation is the Lie theoretic and combinatorial analysis of a family of integrable Hamiltonian systems. A finite dimensional integrable Hamiltonian system is a dynamical system whose phase space is a finite dimensional Poisson manifold $(M, \{\cdot, \cdot\})$, whose Hamiltonian function belongs to a maximal commutative Poisson sub-algebra of $(\mathcal{F}(M), \{\cdot, \cdot\})$. Under suitable hypothesis, of both topological and of analytical nature, the complete integrability of a Hamiltonian dynamical system implies the existence of the so called action-angle variables, which are coordinates which let to linearize the flow describing the evolution of the system on suitable submanifolds of the phase space.

One of the main features shared by most integrable Hamiltonian systems, is the existence of a bi-Hamiltonian structure, i.e., a pair of compatible Poisson structures defined on the underlying phase space, such that the Hamiltonian vector fields associated to the dynamical systems are Hamiltonian with respect to both Poisson structures. Bi-hamiltonian systems were first defined in [23].

The Hamiltonian and the bi-Hamiltonian formalism can be extended, to some extent, also to dynamical systems with an infinite number of degrees of freedom. The most famous infinite dimensional bi-Hamiltonian system is the so called Korteweg de Vries (KdV) equation which is a hyperbolic, dispersive, non-linear PDE describing the dynamics of waves in shallow water. This equation represents a complete integrable system in the sense that it admits an infinite number of conserved quantities and it turns out to be part of a hierarchy of infinite dimensional Hamiltonian PDE known as the Gelfand-Dickey hierarchy. After a suitable discretization, the KdV equation becomes the Volterra lattice, a finite dimensional integrable Hamiltonian system connected via the Miura map to the one of the main characters of this dissertation, to the open Toda lattice.

This dynamical system is a special case of the famous Fermi-Pasta-Ulam lattice and it describes, in its finite dimensional version, the dynamics of n -particle on a line interacting via a nearest neighbor interaction with a potential $V(r) \sim \exp r$, where r is the distance between two neighbor particles. This system was introduced by Morizako Toda in 1967, see [35] and it was subjected to a deep analysis by Bertram Kostant, [19], who discovered its relevance to the theory of the semi-simple Lie groups and Lie algebras.

In one of its several generalizations, the open Toda lattices can be realized as Hamiltonian systems on a suitable symplectic leaf of an arbitrary simple Lie group G equipped with its standard Poisson-Lie structure, see [29]. The Hamiltonian of these systems are obtained by restriction to the symplectic leaf of an appropriate G -invariant function, see [28]. In [17] it was shown that the symplectic leaves of these kind of Poisson-Lie groups are of the form $S^{u,v}a$, where u, v are elements of the Weyl group W of G of length $l(u)$ and, respectively, $l(v)$, a is an element of the Cartan subgroup H of G and where $S^{u,v}$ is a symplectic leaf of G of dimension $l(u) + l(v) + \text{corank}(uv^{-1} - e)$, contained in the double Bruhat cell $G^{u,v}$. i.e.

$$G^{u,v} = BuB \cap B_-vB_-,$$

where B is a Borel subgroup of G and B_- is its opposite, while in [29] a quite general method to find exact solutions of the Toda flow equations on double Bruhat cells was introduced.

The *Coxeter-Toda lattices* are integrable systems defined on $G^{u,v}/H$ via a reduction process starting from the open Toda system defined on G , when u and v are *Coxeter elements* of the Weyl group W , [15].

The Coxeter-Toda lattice equations for $G = \text{GL}_n$ were studied in detail in [11]. In this case GL_n -invariant function defining the Hamiltonian of the dynamical system are of the form:

$$F_k(X) = \frac{1}{k} \text{Trace}(X^k), \quad k = 1, \dots, n-1.$$

For particular choices of u, v one recovers some of the classical integrable systems: for instance when $v = u^{-1} = s_{n-1} \cdots s_1$, the integrable equations in $G^{u,v}/H$ with Hamiltonian F_1 become the ones describing the open Toda lattice:

$$\dot{a}_j = a_j(b_{j+1} - b_j), \quad j = 1, \dots, n-1, \quad \dot{b}_j = a_j - a_{j-1}, \quad j = 1, \dots, n, \quad a_0 = a_n = 0.$$

On $G^{u,v}/H$, suitable coordinates, denoted by c_i, d_i ($i = 1, \dots, n, c_n = 0$), can be constructed using the so called *factorization parameters* of $G^{u,v}$. More precisely, these coordinates are recovered from the coefficients of the Taylor expansion at infinity of the *Moser map*, the (proper) rational function, defined by

$$m(\lambda, X) = (\lambda I - X)_{11}^{-1}$$

for $X \in G^{u,v}$, see [11]. Using the coordinates c_i, d_i just defined, the standard Poisson-Lie bracket of GL_n assumes the following form:

$$\{c_i, c_{i+1}\} = (\varepsilon_{i+1} - 1)c_i c_{i+1}, \quad \{c_i, d_i\} = -c_i d_i, \quad \{c_i, d_{i+1}\} = c_i d_{i+1}, \quad (1.1)$$

where ε_i ($i = 1, \dots, n$) takes values 0, 1 or 2, depending on the choices of u and v . This Poisson bracket is connected with a *quadratic* Poisson bracket, denoted by $\{\cdot, \cdot\}_1$, which belongs to a family of *compatible* Poisson brackets $\{\cdot, \cdot\}_k$, $k = 0, \dots, n-1$, defined on the space of (proper) rational functions, see [7]. Note that $\{\cdot, \cdot\}_1$ can be recovered using the theory of *weighted networks* on an annulus, see [11].

The main Theorem 4.18 of the present work is an alternative description of the bi-Hamiltonian nature of Coxeter-Toda lattices presenting a non-local Poisson structure depending on $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ and compatible with (1.1). Such Poisson bracket was founded using the Atiyah-Hitchin Poisson bracket and the fact the Moser map is invertible on the quotient $G^{u,v}/H$ when u, v are Coxeter elements. For instance, when $v = u^{-1} = s_{n-1} \dots s_1$ and consequently $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = (2, 0, \dots, 0)$ one has for $i < j$ the non-local Poisson bracket:

$$\begin{aligned} \{c_i, d_i\} &= c_i, & \{c_i, c_j\} &= \frac{2(c_i + 1) \prod_{l=i}^j c_l}{d_j}, & \{c_i, d_j\} &= -(c_i + 1) \prod_{l=i}^{j-1} c_l, \\ \{c_j, d_i\} &= \frac{2d_i \prod_{l=i}^j c_l}{d_j}, & \{d_i, d_j\} &= d_i \prod_{l=i}^{j-1} c_l. \end{aligned} \quad (1.2)$$

which, after the change of variables $a_i = c_i d_i^2, b_i = d_i + c_{i-1} d_{i-1}$, (1.2), becomes the well known linear Poisson structure of the Toda lattice.

The outline of the present dissertation is as follows:

In Chapter 2 we present some elementary background about Poisson manifolds that will be used throughout this dissertation. More precisely, we will present the construction of a family of compatible Poisson brackets due to Faybusovich and Gekhtman and we will sketch the main features of the theory of r -matrix and of Poisson-Lie groups. In spite of its introductory nature, this chapter contains also a more original part, which consists in the re-discovering of a Poisson bracket on the Grassmannians based on a result of [4].

In Chapter 3 we review the concepts and the main properties of the weighted directed networks in a disc and in an annulus and the relevant Poisson geometry of the space of edge weighted networks based on [10, 12]. More precisely, will be presented a 2-parametric family of Poisson brackets on Grassmannians $Gr_k(n)$. The Poisson bracket computed in Chapter 1 belongs to this family.

In Chapter 4 we introduce the Coxeter-Toda flows in GL_n . These flows are complete integrable systems on $G^{u,v}/H$ induced by the Toda flows defined on GL_n . Here, $G^{u,v} \subset GL_n$ are Coxeter double Bruhat cells and H is the subgroup of diagonal matrices. Using the results of weighted networks on Chapter 3 we compute a second Poisson structure for Coxeter-Toda flows compatible with the quadratic one. This lead us to conclude that Coxeter-Toda flows are bi-Hamiltonian systems.

Finally, In Chapter 5 we present a discrete complete integrable system called the pentagram map. The pentagram map T is a particular map of a family of complete integrable maps T_k , $k \geq 2$, which can be described using a particular network on a annulus. The existence of a second Poisson structure for the pentagram map T and consequently for the maps T_k is an

open question. Although we do not have an successful answer in the general case we provide an affirmative answer in the case $k = 2$. Furthermore, we show a relation between the Lax matrix of the T_2 map and the Lax matrix of the relativistic Toda lattice.

Background on Poisson manifolds

In this chapter we collect the results from the theory of Poisson manifolds that we use throughout this dissertation. The material is mostly standard and references are given throughout. The main part of this chapter is the last subsection, where we compute a well known Poisson bracket on Grassmannians based on a result of [4].

2.1 Poisson manifolds

2.1.1 Elementary definitions and results

Let M be a finite dimensional smooth manifold and let $\mathcal{F}(M)$ be the algebra of smooth real valued functions on M .

Definition 2.1. *A Poisson bracket on M is a Lie bracket $\{\cdot, \cdot\}$ on $\mathcal{F}(M)$ which satisfies Leibniz's identity*

$$\{f, gh\} = g\{f, h\} + h\{f, g\} \quad \forall f, g, h \in \mathcal{F}(M). \quad (2.1)$$

*This bracket is called a Poisson bracket. A manifold M equipped with a Poisson bracket is called **Poisson manifold**.*

Example 2.2. (1) Consider $M = \mathbb{R}^{2n}$ with coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$. The canonical Poisson bracket on M is defined by

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right).$$

(2) Let (M, ω) be a symplectic manifold, i.e., ω is a closed non-degenerate 2-form on M . For each $f \in \mathcal{F}(M)$ the Hamiltonian vector field $\mathcal{X}_f$ is defined by $\omega(\mathcal{X}_f, \cdot) = -df$. Then

$$\{f, g\} := \omega(\mathcal{X}_f, \mathcal{X}_g) = \mathcal{X}_f(g)$$

is a Poisson bracket on M .

- (3) Let $(M_i, \{\cdot, \cdot\}_i), i = 1, 2$ be Poisson manifolds. Then $M_1 \times M_2$ is a Poisson manifold. In fact, for any $f, g \in \mathcal{F}(M_1 \times M_2)$ and any $x \in M_1, y \in M_2$,

$$\{f, g\}(x, y) = \{f(\cdot, y), g(\cdot, y)\}_1(x) + \{f(x, \cdot), g(x, \cdot)\}_2(y).$$

is a Poisson bracket on $M_1 \times M_2$ called the *product Poisson bracket*.

Definition 2.3. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and let $H \in \mathcal{F}(M)$. The **Hamiltonian vector field** $\mathcal{X}_H$ corresponding to H , called the Hamiltonian function, is defined by $\mathcal{X}_H f = \{H, f\}$.

The dynamical system

$$\frac{dx(t)}{dt} = \mathcal{X}_H(x(t)),$$

generated by the Hamiltonian vector field $\mathcal{X}_H$, where $x : I \rightarrow M$ is a smooth path on M , is called a *Hamiltonian system*. The flow $\varphi_t : M \rightarrow M$ of the Hamiltonian vector field $\mathcal{X}_H$ is called the *Hamiltonian flow* of the Hamiltonian H . A *Casimir function* is a function that generates the zero Hamiltonian vector field.

We denote by $\text{Ham}(M, \{\cdot, \cdot\})$ the vector space of Hamiltonian vector fields. Let $\Psi : \mathcal{F}(M) \rightarrow \text{Ham}(M, \{\cdot, \cdot\})$ be the map defined by $H \mapsto \mathcal{X}_H$.

Proposition 2.4. $\text{Ham}(M, \{\cdot, \cdot\})$ is a Lie subalgebra of $\mathfrak{X}(M)$ and Ψ is a Lie algebra homomorphism.

Proof. For any $f, g, h \in \mathcal{F}(M)$ we have $[\mathcal{X}_f, \mathcal{X}_g]h = \mathcal{X}_f(\mathcal{X}_g h) - \mathcal{X}_g(\mathcal{X}_f h) = \{f, \{g, h\}\} - \{g, \{f, h\}\} = \{\{f, g\}, h\} = \mathcal{X}_{\{f, g\}}h$. $\square$

Corollary 2.5. Let φ_t be the Hamiltonian flow of $\mathcal{X}_H$, and let ψ_s be the Hamiltonian flow of $\mathcal{X}_F$. If $\{H, F\} = 0$, then the flows φ_t, ψ_s commute.

Proof. Recall that two vector fields commute if and only if their flows commute. $\square$

The next proposition follows from the above definitions.

Proposition 2.6. Let φ_t be the Hamiltonian flow with Hamiltonian $H \in \mathcal{F}(M)$. Then

$$H \circ \varphi_t = H \quad \text{and} \quad \frac{d}{dt}(F \circ \varphi_t) = \{H, F \circ \varphi_t\}. \quad (2.2)$$

In particular, $F \in \mathcal{F}(M)$ is an integral of motion of the flow φ_t i.e., $F \circ \varphi_t = F$, if and only if $\{H, F\} = 0$.

Let (M, ω) be a $2n$ -dimensional symplectic manifold and let $\{\cdot, \cdot\}$ be its respective Poisson bracket.

Definition 2.7. A **complete integrable system** on M (M is called the phase space) is a collection of n functions $f_1, f_2, \dots, f_n \in \mathcal{F}(M)$ satisfying:

- (i) the f_i 's are in involution, i.e., $\{f_i, f_j\} = 0$.
- (ii) $f_1, \dots, f_n$ are functionally independent on an open dense subset of M , i.e., $df_1, \dots, df_n$ are linearly independent for each point of the open dense subset of M .

The next theorem describe the topology of the phase space M of a complete integrable system.

Theorem 2.8 (Arnold-Liouville). *(1) Let $(f_1, \dots, f_n)$ be the set which makes the symplectic manifold $(M, \{\cdot, \cdot\})$ a complete integrable system. Let T be a connected component of a common level set M_c of $c = (c_1, \dots, c_n)$, where c is a regular value. Then T is diffeomorphic to $\mathbb{T}^d \times \mathbb{R}^{n-d}$ with some $0 \leq d \leq n$. In particular, if T is compact, it is necessarily diffeomorphic to $\mathbb{T}^n$.*

- (2) If T is compact, then in some neighborhood $T \times \Omega$ of T , where $\Omega \subset \mathbb{R}^n$ is an open ball, there exist coordinates $(I, \theta) = (I_k, \theta_k)_{k=1}^n$ where $I \in \Omega$ and $\theta \in \mathbb{T}^n$, called **action-angle** coordinates, with the following properties:*

- (i) The actions I_k depend only on f_j 's: $I_k = I_k(f_1, \dots, f_n)$ for $k = 1, \dots, n$.*
- (ii) The Poisson brackets between the coordinate functions are canonical:*

$$\{I_i, I_j\} = \{\theta_i, \theta_j\} = 0, \quad \{I_i, \theta_j\} = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Hence for an arbitrary Hamiltonian function $H = H(f_1, \dots, f_n)$ depending only on f_j 's the Hamilton's equations of motion on M have the form

$$\dot{I}_k = 0, \quad \dot{\theta}_k = \omega_k(I_1, \dots, I_n), \quad k = 1, \dots, n$$

and so the completely integrable systems are solvable by quadratures (a finite number of algebraic operations and integrations of known functions).

The complete integrability notion for Poisson manifolds [20] and mappings [36] of Poisson manifolds on itself with corresponding notion of action-angle coordinates it is also known. Since the Poisson bracket $\{\cdot, \cdot\}$ is a bi-derivation one can define a Poisson structure on a manifold M via a bivector field $\Pi \in \Gamma(\wedge^2 TM)$. More precisely,

$$\Pi(df, dg) = \{f, g\}. \quad (2.3)$$

Associated to Π , there is a bundle map $\tilde{\Pi} : T^*M \rightarrow TM$ given by

$$df \mapsto \tilde{\Pi}(df)g = \Pi(df, dg). \quad (2.4)$$

The unique property of the Poisson bracket which is not clearly encoded in Π is the Jacobi identity.

Definition 2.9. Let $\Pi_i, i = 1, 2$ two bivector fields on a manifold M and let $\{\cdot, \cdot\}_i, i = 1, 2$ be the corresponding brackets defined by (2.3). The *Schouten bracket* of Π_1 and Π_2 is the trivector field, denoted by $[\Pi_1, \Pi_2]$, defined by

$$\begin{aligned} [\Pi_1, \Pi_2](f, g, h) = & \{\{f, g\}_1, h\}_2 + \{\{g, h\}_1, f\}_2 + \{\{h, f\}_1, g\}_2 \\ & + \{\{f, g\}_2, h\}_1 + \{\{g, h\}_2, f\}_1 + \{\{h, f\}_2, g\}_1 \end{aligned} \quad (2.5)$$

where $f, g, h \in \mathcal{F}(M)$.

In other words the bracket $\{\cdot, \cdot\}$ defined by a bivector field Π satisfies the Jacobi identity if and only if $[\Pi, \Pi] = 0$. In this case, Π is called a *Poisson bivector field*.

Definition 2.10. (1) Two Poisson brackets $\{\cdot, \cdot\}_i, i = 1, 2$ on a manifold M are said to be **compatible** if $\alpha\{\cdot, \cdot\}_1 + \beta\{\cdot, \cdot\}_2$ is a Poisson bracket for any $\alpha, \beta \in \mathbb{R}$.

(2) A vector field $X \in \mathfrak{X}(M)$ on a Poisson manifold $(M, \{\cdot, \cdot\})$ is called **Poisson vector field** if $L_X \Pi = 0$ where Π is the bivector field corresponding to $\{\cdot, \cdot\}$.

Remark 2.11. (1) Two Poisson brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are compatible if and only if the corresponding Poisson bivector fields Π_1 and Π_2 satisfy $[\Pi_1, \Pi_2] = 0$.

(2) All Hamiltonian vector fields are Poisson vector fields but the reciprocal is not always true.

Definition 2.12. Let $(M, \{\cdot, \cdot\}_M)$ be a Poisson manifold.

(1) Let $(N, \{\cdot, \cdot\}_N)$ be another Poisson manifold, then a smooth map $\phi : M \rightarrow N$ is called a **Poisson map** if

$$\{f, g\}_N \circ \phi = \{f \circ \phi, g \circ \phi\}_M \quad \forall f, g \in \mathcal{F}(N).$$

(2) An immersed submanifold S of $(M, \{\cdot, \cdot\})$ is a **Poisson submanifold** if it admits a Poisson structure $\{\cdot, \cdot\}_S$ such that the inclusion $\iota : S \rightarrow M$ is a Poisson map.

(3) Let G be a Lie group. A left action $\Psi : G \times M \rightarrow M$ is called **admissible** if the subspace $\mathcal{F}(M)^G$ of G -invariant functions is a Lie subalgebra of $\mathcal{F}(M)$ with respect to the Poisson bracket.

Example 2.13. Consider $(\mathbb{R}^4, (q_1, p_1, q_2, p_2))$, endowed with the canonical Poisson bracket and $(\mathbb{R}^2, (x, y))$ endowed with the Poisson bracket $\{m, n\} = x \left(\frac{\partial m}{\partial x} \frac{\partial n}{\partial y} - \frac{\partial m}{\partial y} \frac{\partial n}{\partial x} \right)$. Then the map $F : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ given by $F(q_1, p_1, q_2, p_2) = (q_1, p_1 q_1 + q_2)$ is a surjective Poisson map.

If M/G is a smooth manifold, an admissible action $\Psi : G \times M \rightarrow M$ gives rise to a unique Poisson structure on M/G , such that the projection map $\pi : M \rightarrow M/G$ is a Poisson map. In fact, for $f, g \in \mathcal{F}(M/G)$ define the Poisson bracket $\{\cdot, \cdot\}_{M/G}$ by

$$\{f, g\}_{M/G} \circ \pi = \{f \circ \pi, g \circ \pi\}.$$

Proposition 2.14. *Let φ_t be the Hamiltonian flow of the Hamiltonian H . Then for each $t \in I \subset \mathbb{R}$ where the flow is defined, the map $\varphi_t : M \rightarrow M$ is a Poisson map.*

Proof. Let $f, g \in \mathcal{F}(M)$ and set

$$f_t = \{f \circ \varphi_t, g \circ \varphi_t\} - \{f, g\} \circ \varphi_t$$

Then

$$\frac{df_t}{dt} = \left\{ \frac{d}{dt}(f \circ \varphi_t), g \circ \varphi_t \right\} + \left\{ f \circ \varphi_t, \frac{d}{dt}(g \circ \varphi_t) \right\} - \frac{d}{dt}(\{f, g\} \circ \varphi_t).$$

From equation (2.2), we have

$$\frac{df_t}{dt} = \{\{H, f \circ \varphi_t\}, g \circ \varphi_t\} + \{f \circ \varphi_t, \{H, g \circ \varphi_t\}\} - \{H, \{f, g\} \circ \varphi_t\}$$

using the Jacobi identity one has

$$\frac{df_t}{dt} = \{H, f_t\},$$

implying that $f_t = f_0 \circ \varphi_t$, and that φ_t is a Poisson map, because $f_0 = 0$. $\square$

Let $(U, x = (x_1, \dots, x_n))$ be a coordinate neighborhood of the n -dimensional Poisson manifold (M, Π) . On U the Poisson bivector field can be written as

$$\Pi = \sum_{i < j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \frac{1}{2} \sum_{i, j=1}^n \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

where $\Pi_{ij} = \Pi(x_i, x_j) = \{x_i, x_j\}$ are called the *structure functions*, and they satisfy:

$$\Pi_{ij} = -\Pi_{ji}, \quad \sum_{l=1}^n (\Pi_{lk} \frac{\partial \Pi_{ij}}{\partial x_l} + \Pi_{li} \frac{\partial \Pi_{jk}}{\partial x_l} + \Pi_{lj} \frac{\partial \Pi_{ki}}{\partial x_l}) = 0$$

which follow from the skew-symmetry and Jacobi identity, respectively. In these local coordinates, the Poisson bracket of $f, g \in \mathcal{F}(M)$ has the form

$$\{f, g\} = \sum_{i, j=1}^n \Pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

The skew-symmetric matrix $(\Pi_{ij})_{i, j=1}^n$ is called the *Poisson matrix* of the Poisson structure Π with respect to the local coordinate system $x = (x_1, \dots, x_n)$. We denote by $\text{Ham}_m(M)$ the vector subspace of $T_m M$ whose tangent vectors are of the form $\mathcal{X}(m)$ where $\mathcal{X} \in \text{Ham}(M, \{\cdot, \cdot\})$.

Definition 2.15. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and let Π the corresponding Poisson bivector field. The rank of the Poisson structure Π at $m \in M$, denoted by $\text{rank}_m \Pi$, is the dimension of $\text{Ham}_m(M)$. The rank of Π , denoted by $\text{rank} \Pi$, is the $\max_{m \in M} \{\text{rank}_m \Pi\}$. The Poisson manifold M is called **regular** if $\text{rank}_m \Pi$ is constant, i.e., does not depend on m . We say that Π has **maximal rank** or is **non-degenerate** at m if $\text{rank}_m \Pi = \dim M$.

Remark 2.16. The $\text{rank}_m \Pi$ is equal to the rank of the Poisson matrix with respect to any local coordinates on a neighborhood of m . Hence $\text{rank}_m \Pi$ is even.

One can see that a submanifold S of a Poisson manifold M is Poisson, if and only if an arbitrary Hamiltonian vector field $\mathcal{X}_H$ on M in the points of S is tangent to S . Thus, Poisson submanifolds are integral manifolds for arbitrary Hamiltonian vector fields. Minimal Poisson submanifolds are the so-called *symplectic leaves* defined as the equivalence classes of the following relation: $x, y \in M$ belong to the same symplectic leaf if there is a piecewise smooth curve in M joining x and y , each segment of which is a trajectory of a locally defined Hamiltonian vector field.

So, the next proposition exposes the global structure of a Poisson manifold.

Proposition 2.17. *A Poisson manifold $(M, \{\cdot, \cdot\})$ is a disjoint union of its symplectic leaves. Each symplectic leaf in M is an injectively immersed Poisson submanifold, and the induced Poisson structure on the leaf is non-degenerate. The dimension of a symplectic leaf is equal to the rank of the Poisson structure in any point of this leaf.*

If some system allows Hamiltonian formulations with respect to two compatible Poisson brackets then the system is called *bi-Hamiltonian*, this notion was discovered by [23]. A bi-hamiltonian system automatically possesses a number of integrals in involution.

Theorem 2.18. *Let M be a manifold equipped with two compatible Poisson brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$, at least one of them, say $\{\cdot, \cdot\}_2$, being non-degenerate. Suppose that there exist two functions $H_1, H_2 \in \mathcal{F}(P)$ such that*

$$\{H_1, \cdot\}_1 = \{H_2, \cdot\}_2 \quad (2.6)$$

holds. Then there exists a sequence of functions $(H_k)_{k=1}^\infty$ starting with H_1, H_2 such that

(i) *For all $k \geq 1$*

$$\{H_k, \cdot\}_1 = \{H_{k+1}, \cdot\}_2$$

(ii) *The functions H_k are in involution with respect to both Poisson brackets:*

$$\{H_k, H_j\}_1 = \{H_k, H_j\}_2 = 0, \quad \forall k, j \geq 1.$$

Proof. See [34], page 19. □

Certainly, only a finite number of the functions H_k are functionally independent. It might happen that the number of independent functions generated by this construction is less than necessary to assure the complete integrability. Nevertheless, in all known examples of bi-Hamiltonian systems turn out to be at the same time completely integrable.

2.1.2 The Faybusovich-Gekhtman Poisson brackets

Denote by Rat_n the space of proper rational functions with complex coefficients of the form

$$\left\{ s(\lambda) = \frac{q(\lambda)}{p(\lambda)} \mid p(\lambda) \text{ is a monic polynomial, } \deg p = n, \deg q < n \right\}.$$

Notice that $s(\lambda) \in \text{Rat}_n$ can be expressed as

$$s(\lambda) = \sum_{i=0}^{\infty} \frac{h^i}{\lambda^{i+1}}$$

after considering its expansion at infinity. Following [7], we give a brief exposition of the family of compatible Faybusovich-Gekhtman Poisson brackets on Rat_n .

For fixed $p(\lambda), q(\lambda)$ and $k = 0, \dots, n-1$, let us denote

$$q^{[k]}(\lambda) = \lambda^k q(\lambda) \pmod{p(\lambda)}$$

and define a skew-symmetric bracket $\{\cdot, \cdot\}_k$ on the coefficients of $p(\lambda), q(\lambda)$ by setting

$$\{p(\lambda), p(\mu)\}_k = \{q(\lambda), q(\mu)\}_k = 0 \quad (2.7)$$

and

$$\{p(\lambda), q(\mu)\}_k = \frac{p(\lambda)q^{[k]}(\mu) - p(\mu)q^{[k]}(\lambda)}{\lambda - \mu} \quad (2.8)$$

This bracket can be re-written in terms of $s(\lambda) \in \text{Rat}_n$ as

$$\{s(\lambda), s(\mu)\}_k = ((\lambda^k s(\lambda))_- - (\mu^k s(\mu))_-) \frac{s(\lambda) - s(\mu)}{\lambda - \mu}, \quad (2.9)$$

where $s(\lambda)_- = s(\lambda) - s(\lambda)_+$ being $s(\lambda)_+$ the polynomial part of the Laurent expansion at infinity of $s(\lambda)$.

In terms of coefficients h_i the brackets $\{\cdot, \cdot\}_k$ take the form, see [7],

$$\{h_i, h_j\}_k = \sum_{r=i}^{j-1} h_{k+r} h_{i+j-1+r}, \quad i < j. \quad (2.10)$$

Remark 2.19. The bracket $\{\cdot, \cdot\}_0$ of the family (2.9) is known in the literature as the Atiyah-Hitchin bracket.

Example 2.20. In this example we compute the brackets $\{\cdot, \cdot\}_k$ in the space Rat_2 . Let $p(\lambda) = \lambda^2 + a_1\lambda + a_0$ and $q(\lambda) = b_1\lambda + b_0$. Then

$$\{p(\lambda), q(\mu)\}_k = \lambda\mu\{a_1, b_1\}_k + \lambda\{a_1, b_0\}_k + \mu\{a_0, b_1\}_k + \{a_0, b_0\}_k, \quad k = 0, 1.$$

On the other hand,

1. If $k = 0$, from the right hand side of (2.8) one obtains

$$\frac{p(\lambda)q(\mu) - p(\mu)q(\lambda)}{\lambda - \mu} = b_1\lambda\mu + b_0\lambda + b_0\mu + a_1b_0 - a_0b_1.$$

Then:

$$\{a_1, b_1\}_0 = b_1, \quad \{a_1, b_0\}_0 = \{a_0, b_1\}_0 = b_0, \quad \{a_0, b_0\}_0 = a_1b_0 - a_0b_1.$$

2. If $k = 1$, then $q^{[1]}(\lambda) = (b_0 - a_1 b_1)\lambda - a_0 b_1$ and

$$\frac{p(\lambda)q^{[1]}(\mu) - p(\mu)q^{[1]}(\lambda)}{\lambda - \mu} = (b_0 - a_1 b_1)\lambda\mu - a_0 b_1(\lambda + \mu) - a_0 b_0.$$

Then:

$$\{a_1, b_1\}_1 = b_0 - a_1 b_1, \quad \{a_1, b_0\}_1 = \{a_0, b_1\}_1 = -a_0 b_1, \quad \{a_0, b_0\}_1 = -a_0 b_0.$$

Proposition 2.21. *The brackets $\{\cdot, \cdot\}_k$ ($k = 0, \dots, n-1$) are compatible Poisson brackets on Rat_n .*

Proof. Consider the dense subset

$$S = \left\{ \frac{p(\lambda)}{q(\lambda)} \in \text{Rat}_n \mid \gcd(p(\lambda), q(\lambda)) = 1, \lambda_i \neq \lambda_j \text{ for } i \neq j \text{ where } p(\lambda_i) = 0, i = 0, \dots, n-1 \right\}$$

of Rat_n . On this subset

$$s(\lambda) = \frac{q(\lambda)}{p(\lambda)} = \sum_{i=0}^{n-1} \frac{r_i}{\lambda - \lambda_i}$$

and the data $\{\lambda_i, q(\lambda_i), i = 0, \dots, n-1\}$ determines $p(\lambda)$ and $q(\lambda)$ completely. In particular,

$$r_i = q(\lambda_i) \prod_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \frac{q(\lambda_i)}{p'(\lambda_i)}.$$

From (2.7), it follows that

$$\{\lambda_i, \lambda_j\}_k = 0. \quad (2.11)$$

Since $\{p(\lambda_i), q(\mu)\}_k = 0$ one has

$$p'(\lambda_i) \{\lambda_i, q(\mu)\}_k + \{p(\lambda), q(\mu)\}_k \big|_{\lambda=\lambda_i} = 0.$$

Using the latter equation and (2.8) one obtains

$$\{\lambda_i, q(\mu)\}_k = \frac{p(\mu)\lambda_i^k q(\lambda_i)}{p'(\lambda_i)(\lambda_i - \mu)} = -\lambda_i^k q(\lambda_i) \prod_{i \neq l} \frac{\mu - \lambda_l}{\lambda_i - \lambda_l},$$

which, using (2.11), implies

$$\{\lambda_i, q(\lambda_j)\}_k = -\delta_{ij} \lambda_i q(\lambda_i).$$

The latter equation and (2.7) implies

$$\{q(\lambda_i), q(\lambda_j)\}_k = 0.$$

Therefore, in coordinates $\lambda_i, q_i = q(\lambda_i)$, any linear combination $\{\cdot, \cdot\}_c = \sum_{k=0}^{n-1} c_k \{\cdot, \cdot\}_k$ has a form

$$\{\lambda_i, \lambda_j\}_c = \{q_i, q_j\}_c = 0, \quad \{\lambda_i, q_j\}_c = -\delta_{ij} c(\lambda_i) q_i \quad (2.12)$$

where $c(\lambda) = \sum_{k=0}^{n-1} c_k \lambda^k$. Hence $\{\cdot, \cdot\}_c$ satisfies Jacobi identity. $\square$

Let

$$\text{Rat}'_n = \left\{ S(\lambda) = \frac{q(\lambda)}{p(\lambda)} \in \text{Rat}_n \mid q(\lambda) \text{ is monic} \right\}$$

be the subset of Rat_n .

Proposition 2.22 ([7]). *The Poisson bracket $\{\cdot, \cdot\}_k$ induced on Rat'_n takes the form*

$$\{S(\lambda), S(\mu)\}_k = ((\lambda^k S(\lambda))_- - (\mu^k S(\mu))_-) \left(\frac{S(\lambda) - S(\mu)}{\lambda - \mu} + S(\lambda)S(\mu) \right) \quad (2.13)$$

One can see that equation (2.13) is equivalent to

$$\begin{aligned} \{p(\lambda), p(\mu)\}_k &= \{q(\lambda), q(\mu)\}_k = 0, \\ \{p(\lambda), q(\mu)\}_k &= \frac{p(\lambda)q^{[k]}(\mu) - p(\mu)q^{[k]}(\lambda)}{\lambda - \mu} - q^{[k]}(\lambda)q(\mu). \end{aligned}$$

In terms of the coefficients H_i of the Laurent expansion at infinity

$$S(\lambda) = \sum_{i=0}^{\infty} \frac{H_i}{\lambda^{i+1}}, \quad H_0 = 1$$

the Poisson bracket (2.13) is equivalent to

$$\{H_i, H_j\}_k = \sum_{r=i}^{j-1} H_{k+r} H_{i+j-1-r} - H_i \sum_{r=0}^{j-1} H_{k+r} H_{j-1+r} + H_j \sum_{r=0}^{i-1} H_{k+r} H_{i-1+r}.$$

2.2 R -matrix and r -matrix

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\mathfrak{g}^*$ its dual vector space. Denote by $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

Definition 2.23. Let $f, g \in \mathcal{F}(\mathfrak{g}^*)$. The bracket

$$\{f, g\}(L) = \langle L, [df(L), dg(L)] \rangle \quad (2.14)$$

is a Poisson bracket called the **Lie-Poisson** bracket of $\mathfrak{g}^*$.

Given $H \in \mathcal{F}(\mathfrak{g}^*)$, the corresponding Hamiltonian vector field $\mathcal{X}_H$ with respect to the Lie-Poisson bracket is

$$\mathcal{X}_H(\xi)(f) = \{H, f\}(\xi) = \langle \xi, [dH(\xi), df(\xi)] \rangle = -\langle \text{ad}_{dH(\xi)}^* \xi, df(\xi) \rangle. \quad (2.15)$$

In other words,

$$\mathcal{X}_H(\xi) = -\text{ad}_{dH(\xi)}^* \xi, \quad \forall \xi \in \mathfrak{g}^*. \quad (2.16)$$

Therefore the Hamilton's equations of motion on $\mathfrak{g}^*$, with respect to the Hamiltonian function H , take the form

$$\frac{d\xi}{dt} = -\text{ad}_{dH(\xi)}^* \xi. \quad (2.17)$$

Proposition 2.24. (1) *The symplectic leaves of the Lie-Poisson structure on $\mathfrak{g}^*$ are the coadjoint orbits of G .*

(2) *The set of Casimir functions of the Lie-Poisson bracket on $\mathfrak{g}^*$ is $\mathcal{F}(\mathfrak{g}^*)^G$.*

Proof. First, recall that for a coadjoint orbit $\mathcal{O}(\xi)$ passing through $\xi \in \mathfrak{g}^*$

$$T_\xi \mathcal{O}(\xi) = \{\text{ad}_X^* \xi \mid X \in \mathfrak{g}\}$$

From (2.15) we have

$$\text{Ham}_\xi(\mathfrak{g}^*) = \{\text{ad}_{df(\xi)}^* \xi \mid f \in \mathcal{F}(\mathfrak{g}^*)\} = \{\text{ad}_X^* \xi \mid X \in \mathfrak{g}\} = T_\xi \mathcal{O}(\xi)$$

This proves the first statement. It follows that the Ad^* -invariant functions are precisely the functions which are constant on every symplectic leaf. From (2.15) and Lemma (A.9), it follows that any $f \in \mathcal{F}(\mathfrak{g}^*)^G$ is a Casimir. Since Casimir functions are constant on symplectic leaves, they are precisely the Ad^* -invariant functions. $\square$

Definition 2.25. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra. A linear operator $R \in \text{End } \mathfrak{g}$ is called a **classical R -matrix** (or simply R -matrix, for short) if the bracket

$$[X, Y]_R = \frac{1}{2} ([RX, Y] + [X, RY]), \quad X, Y \in \mathfrak{g} \quad (2.18)$$

is a Lie bracket. We denote by $\mathfrak{g}_R$ the new Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_R)$ and we will call $[\cdot, \cdot]_R$ an R -bracket.

Given an R -matrix the corresponding Lie-Poisson bracket $\{\cdot, \cdot\}_R$ on $\mathfrak{g}^*$ is

$$\{f, g\}_R(\xi) = \frac{1}{2} \langle \xi, [df(\xi), dg(\xi)]_R \rangle = \frac{1}{2} \langle \xi, [R(df(\xi)), dg(\xi)] + [df(\xi), R(dg(\xi))] \rangle. \quad (2.19)$$

Example 2.26. Typical examples of R -matrices come from a Lie algebra splitting of $\mathfrak{g}$, that is, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where $\mathfrak{g}_+, \mathfrak{g}_-$ are Lie subalgebras. Let $P_\pm : \mathfrak{g} \rightarrow \mathfrak{g}_\pm$ be the corresponding projections operators and, for $X \in \mathfrak{g}$, let $X_\pm := P_\pm(X)$. Then if $R = P_+ - P_-$ then one has that

$$[X, Y]_R := \frac{1}{2} ([RX, Y] + [X, RY]) = [X_+, Y_+] - [X_-, Y_-], \quad X, Y \in \mathfrak{g},$$

satisfies the Jacobi identity. Moreover the splitting of $\mathfrak{g}$ implies a direct sum decomposition of $\mathfrak{g}^*$, namely,

$$\mathfrak{g}^* = \text{Ann}(\mathfrak{g}_-) \oplus \text{Ann}(\mathfrak{g}_+) \cong \mathfrak{g}_+^* \oplus \mathfrak{g}_-^*.$$

Dualizing the projection maps $P_\pm$, one has the linear maps

$$P_\pm^* = \iota_\pm : \mathfrak{g}_\pm^* \rightarrow \mathfrak{g}^*,$$

where $\iota_+ : (\mathfrak{g}_+^*, \{\cdot, \cdot\}_+^*) \rightarrow (\mathfrak{g}^*, \{\cdot, \cdot\}_R)$ a Poisson map, and $\{\cdot, \cdot\}_+^*$ denotes the Lie-Poisson bracket on $\mathfrak{g}_+^*$.

Proposition 2.27. (1) If $f, g \in \mathcal{F}(\mathfrak{g}^*)^G$ then $\{f, g\}_R = 0$.

(2) Given $H \in \mathcal{F}(\mathfrak{g}^*)^G$ then the corresponding Hamiltonian vector field $\mathcal{X}_H$ defined by $\{\cdot, \cdot\}_R$ is

$$\mathcal{X}_H(\xi) = -\text{ad}_{R(dH(\xi))}^* \xi.$$

Proof. Both assertions follow from (A.3) and (2.19). $\square$

The definition of R -matrix can be reformulated as follows.

Proposition 2.28. The bracket (2.18) satisfies the Jacobi identity if and only if

$$[B_R(X, Y), Z] + [B_R(Y, Z), X] + [B_R(Z, X), Y] = 0, \quad (2.20)$$

where $B_R(X, Y) = [RX, RY] - R([RX, Y] + [X, RY])$.

Proof. For $X, Y, Z \in \mathfrak{g}$, after using definition of $[\cdot, \cdot]_R$ and Jacobi identity for $[\cdot, \cdot]$, one can see that

$$4[[X, Y]_R, Z]_R + c.p.(X, Y, Z) = [R([RX, Y] + [X, RY]) - [RX, RY], Z] + c.p.(X, Y, Z)$$

where $c.p.(X, Y, Z)$ means the cyclic permutation sum. $\square$

A sufficient condition for the equation (2.20) to hold is called the **classical Yang-Baxter equation** (CYBE):

$$B_R(X, Y) = [RX, RY] - R([RX, Y] + [X, RY]) = 0, \quad \forall X, Y \in \mathfrak{g}. \quad (2.21)$$

Another sufficient condition is the so called **modified Yang-Baxter equation** (MYBE):

$$B_R(X, Y) = [RX, RY] - R([RX, Y] + [X, RY]) = -c^2[X, Y], \quad (2.22)$$

where $c \neq 0$ is a constant. After rescaling, one can assume $c = 1$. For instance, the linear map $R = P_+ - P_-$ in Example 2.26 satisfies MYBE with $c = 1$. From now on, we refer MYBE to the case $c = 1$.

Let $R \in \text{End}(\mathfrak{g})$ be an R -matrix.

Proposition 2.29. If R satisfies the MYBE for $[\cdot, \cdot]$ then R satisfies the MYBE for $[\cdot, \cdot]_R$.

Proof. Applying the MYBE for the bracket $[\cdot, \cdot]$ to the pairs (RX, Y) and (X, RY) , we have

$$[R^2X, RY] - R([R^2X, Y] + [RX, RY]) = -[RX, Y]$$

and

$$[RX, R^2Y] - R([RX, RY] + [X, R^2Y]) = -[X, RY]$$

respectively. Then, multiplying both equations by $\frac{1}{2}$ and adding them we obtain

$$[RX, RY]_R - R([RX, Y]_R + [X, RY]_R) = -[X, Y]_R,$$

which proves the statement of the Proposition. $\square$

Using the above proposition, one has that if R verifies MYBE then the bracket

$$[X, Y]_{R,R} = \frac{1}{2} ([RX, Y]_R + [X, RY]_R) = \frac{1}{4} ([R^2 X, Y] + 2[RX, RY] + [X, R^2 Y]) \quad (2.23)$$

is a Lie bracket. Assuming $R^2 = I$, (2.23) reduces to

$$[X, Y]_{R,R} = \frac{1}{2} ([X, Y] + [RX, RY]) = R([X, Y]_R) = \frac{1}{2} ([RX, Y]_R + [X, RY]_R)$$

For instance, $R = P_+ - P_-$ as in Example 2.26 satisfies $R^2 = I$, then

$$[X, Y]_{R,R} = \frac{1}{2} ([RX, Y]_R + [X, RY]_R) = [X_+, Y_+] + [X_-, Y_-].$$

One can continue iterating (2.23), the n -th R -bracket will be

$$[X, Y]_{R,R,\dots,R} = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} [R^k X, R^{n-k} Y],$$

and so, one can construct a family of Lie-Poisson brackets on $\mathfrak{g}^*$. In our model example, $R = P_+ - P_-$ when $n = 3$ one has

$$\begin{aligned} [X, Y]_{R,R,R} &= \frac{1}{2} ([RX, Y]_{R,R} + [X, RY]_{R,R}) \\ &= \frac{1}{8} ([R^3 X, Y] + 3[R^2 X, RY] + 3[RX, R^2 Y] + [X, R^3 Y]) = [X, Y]_R. \end{aligned}$$

Let $R \in \text{End}(\mathfrak{g})$ be an R -matrix satisfying the MYBE then the operators

$$R_{\pm} = \frac{1}{2}(R \pm I) : \mathfrak{g}_R \rightarrow \mathfrak{g}, \quad (2.24)$$

are Lie algebra homomorphisms, i.e., $R_{\pm}([X, Y]_R) = [R_{\pm} X, R_{\pm} Y]$. Therefore $\mathfrak{g}_{\pm} = \text{Im } R_{\pm} \subset \mathfrak{g}$ are Lie subalgebras and $\mathfrak{k}_{\pm} = \text{Ker } R_{\mp} \subset \mathfrak{g}_{\pm}$ are ideals.

Let $\theta_R : \mathfrak{g}_+/\mathfrak{k}_- \rightarrow \mathfrak{g}_-/\mathfrak{k}_+$, given by $\theta_R([(R+I)X]) \mapsto [(R-I)X]$. Then the map θ_R is well defined and the following hold:

- (1) θ_R is a Lie algebra isomorphism.
- (2) The map $i_R : \mathfrak{g}_R \rightarrow \mathfrak{g} \oplus \mathfrak{g}$, defined by $i_R = R_+ \oplus R_-$, is a Lie algebra embedding and its image $\tilde{\mathfrak{g}}_R$ in $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ is the subalgebra

$$\tilde{\mathfrak{g}}_R = \{(X, Y) \in \mathfrak{g}_+ \oplus \mathfrak{g}_- \mid \theta_R([X]) = [Y]\}.$$

- (3) Each $X \in \mathfrak{g}$ has a unique decomposition $X = X_+ - X_-$ with $(X_+, X_-) \in \tilde{\mathfrak{g}}_R$ and $RX = X_+ + X_-$.

Let $\Delta_{\mathfrak{g}}$ be the diagonal subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$. Then we have $\mathfrak{g} \oplus \mathfrak{g} = \Delta_{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}_R$, in fact

$$(X, Y) = (Y_+ - X_-, Y_+ - X_-) + (X_+ - Y_+, X_- - Y_-).$$

Let $P_{\Delta_{\mathfrak{g}}}$ and $P_{\tilde{\mathfrak{g}}_R}$ be the projections onto $\Delta_{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_R$, respectively. Then $R_{\mathfrak{g} \oplus \mathfrak{g}} = P_{\Delta_{\mathfrak{g}}} - P_{\tilde{\mathfrak{g}}_R}$ is an R -matrix of $\mathfrak{g} \oplus \mathfrak{g}$.

Remark 2.30. (1) A slightly wider choice of classical R -matrix is obtained by a splitting of $\mathfrak{g}$ in three Lie subalgebras $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$ such that $\mathfrak{g}_0$ normalizes both $\mathfrak{g}_+$ and $\mathfrak{g}_-$, i.e., $[\mathfrak{g}_0, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm$. Let P_+, P_-, P_0 be the projections onto $\mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{g}_0$ and let $R_0 \in \text{End } \mathfrak{g}_0$ satisfying the MYBE then

$$R = P_+ - P_- + R_0 P_0$$

also satisfies the MYBE, see [29].

(2) A linear map $A : \mathfrak{g} \rightarrow \mathfrak{g}$ is called an *intertwining* operator if $A \circ \text{ad } X = \text{ad } X \circ A$. If R is an R -matrix and A is intertwining operator, then $R \circ A$ is also an R -matrix. This gives us of a pencil of Lie brackets $R \circ A$, with R fixed, since the set of intertwining operators is a vector space. Then, the corresponding Lie-Poisson brackets are compatible and have a common set of functions in involution consisting of coadjoint invariants of the Lie algebra $\mathfrak{g}$.

2.2.1 Lie-Poisson bracket on $\mathfrak{g}$

Let $(\cdot, \cdot)$ be an Ad-invariant, non-degenerate bilinear form on $\mathfrak{g}$. Non-degeneracy of $(\cdot, \cdot)$ gives an isomorphism χ between $\mathfrak{g}$ and $\mathfrak{g}^*$ defined by

$$\chi : \mathfrak{g} \rightarrow \mathfrak{g}^* : X \mapsto (X, \cdot).$$

With this at hand, it is defined a one to one correspondence between $\mathcal{F}(\mathfrak{g})$ and $\mathcal{F}(\mathfrak{g}^*)$ which induces a Poisson bracket $\{\cdot, \cdot\}_{\mathfrak{g}}$ on $\mathfrak{g}$. More precisely, for $f, g \in \mathcal{F}(\mathfrak{g})$, the Lie-Poisson bracket $\{f, g\}_{\mathfrak{g}}$ on $\mathfrak{g}$ is

$$\{f, g\}_{\mathfrak{g}}(X) = (X, [\chi^{-1}(df(X)), \chi^{-1}(dg(X))]), \quad X \in \mathfrak{g}. \quad (2.25)$$

Usually, in the literature, the above bracket (2.25) is written as

$$\{f, g\}_{\mathfrak{g}}(X) = (X, [\nabla f(X), \nabla g(X)]), \quad (2.26)$$

where $\nabla f(X) := \chi^{-1}(df(X))$ is the **gradient** of f at X , i.e.,

$$(\nabla f(X), Y) = \langle df(X), Y \rangle = \left. \frac{d}{dt} \right|_{t=0} f(X + tY), \quad \forall Y \in \mathfrak{g}.$$

For $H \in \mathcal{F}(\mathfrak{g})$, using (2.26) and (A.4) one has

$$\{H, f\}_{\mathfrak{g}}(X) = (X, [\nabla H(X), \nabla f(X)]) = ([X, \nabla H(X)], \nabla f(x)) = \langle df(X), [X, \nabla H(X)] \rangle,$$

and the Hamiltonian vector field $\mathcal{X}_H$ with respect to $\{\cdot, \cdot\}_{\mathfrak{g}}$ is given by a **Lax type equation**, i.e.,

$$\mathcal{X}_H(X) = [X, \nabla H(X)]. \quad (2.27)$$

The non-degeneracy of $(\cdot, \cdot)$ implies also $\mathfrak{g} = \mathfrak{g}_+^\perp \oplus \mathfrak{g}_-^\perp$ for $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Moreover, $(\cdot, \cdot)$ defines the isomorphisms

$$\mathfrak{g}_+^\perp \simeq \mathfrak{g}_-^* \quad \text{and} \quad \mathfrak{g}_-^\perp \simeq \mathfrak{g}_+^*,$$

so that the two subspaces $\mathfrak{g}_\pm^\perp$ of $\mathfrak{g}$ carry a linear Poisson structure denoted by $\{\cdot, \cdot\}_{\mathfrak{g}_\pm^\perp}$. For any $H \in \mathcal{F}(\mathfrak{g})$ and $\epsilon \in \mathfrak{g}$, we define $H_\epsilon \in \mathcal{F}(\mathfrak{g}_\perp^\perp)$ and $\tilde{H}_\epsilon \in \mathcal{F}(\mathfrak{g})$ as follows

$$H_\epsilon : \mathfrak{g}_\perp^\perp \rightarrow \mathbb{C} : x \mapsto H(\epsilon + x), \quad \tilde{H}_\epsilon : \mathfrak{g} \rightarrow \mathbb{C} : \xi \mapsto H(\epsilon + \xi).$$

Theorem 2.31 (The Adler-Kostant-Symes theorem on $\mathfrak{g}$). *Under the above assumptions, let $f, g \in \mathcal{F}(\mathfrak{g})^G$ and suppose that $\epsilon \in \mathfrak{g}$ satisfies*

$$[\epsilon, \mathfrak{g}_+] \in \mathfrak{g}_+^\perp, \quad [\epsilon, \mathfrak{g}_-] \in \mathfrak{g}_-^\perp.$$

Then, if $R = P_+ - P_-$ one has

- (1) $\{f, g\}_R = 0$ and $\{f_\epsilon, g_\epsilon\}_{\mathfrak{g}_\pm^\perp} = 0$.
- (2) *The Hamiltonian vector fields $\mathcal{X}_H = \{H, \cdot\}_R$ and $\mathcal{X}_{H_\epsilon} = \{H_\epsilon, \cdot\}_{\mathfrak{g}_\perp^\perp}$ are given respectively by*

$$\begin{aligned} \mathcal{X}_H(X) &= \frac{1}{2} [X, R(\nabla H(X))] = \mp [X, (\nabla H(X))_\mp], \\ \mathcal{X}_{H_\epsilon}(Y) &= \frac{1}{2} [Y, R(\nabla H(Y))] = \mp [Y, (\nabla H(Y))_\mp], \end{aligned} \quad (2.28)$$

where $Y \in \mathfrak{g}_\perp^\perp + \epsilon$.

- (3) *For $x_0 \in \mathfrak{g}$ and for $|t|$ small, let $g_+(t)$ and $g_-(t)$ denote the smooth curves in G_+ and respectively G_- which solve the factorization problem*

$$\exp(-t\nabla H(x_0)) = g_+(t)^{-1}g_-(t), \quad g_\pm(0) = e.$$

Then the integral curve of $\mathcal{X}_H$ which starts at x_0 is given for $|t|$ small by

$$x(t) = \text{Ad}_{g_+(t)} x_0 = \text{Ad}_{g_-(t)} x_0.$$

Proof. See [1], page 90. □

Example 2.32. Let $\mathfrak{g} := \mathfrak{gl}(n)$ be the Lie algebra of $n \times n$ matrices. Consider the splitting

$$\mathfrak{gl}(n) = \mathfrak{g}_+ \oplus \mathfrak{g}_-,$$

where $\mathfrak{g}_+ = \mathfrak{b}_-$ is the Lie subalgebra of lower triangular matrices and $\mathfrak{g}_- = \mathfrak{n}_+$ is the Lie subalgebra of strictly upper triangular matrices. Let $(\cdot, \cdot)$ be the trace form on $\mathfrak{gl}(n)$ defined by:

$$(X, Y) = \text{Trace}(XY).$$

Then $\mathfrak{g}_+^* \simeq \mathfrak{g}_-^\perp = \mathfrak{b}_+$, the Lie subalgebra of upper triangular matrices in $\mathfrak{g}$. For $j \geq i, l \geq k$ the Lie-Poisson bracket of two coordinate linear functions $x_{ij}, x_{kl} \in \mathcal{F}(\mathfrak{b}_+)$ is

$$\begin{aligned} \{x_{ij}, x_{kl}\}(X) &= \text{Trace}(X [\nabla x_{ij}(X), \nabla x_{kl}(X)]) = \text{Trace}(X [E_{ji}, E_{lk}]) \\ &= \text{Trace}(X (\delta_{il}E_{jk} - \delta_{kj}E_{li})) = \delta_{ij}X_{kj} - \delta_{kj}X_{il}. \end{aligned}$$

Let $B_p = \{X \in \mathfrak{b}_+ \mid x_{ij}(X) = 0 \text{ if } j - i > p\}$. Then all Hamiltonian vector fields on $\mathfrak{b}_+$ are tangent to B_p : letting $j - i > p$ and $l \geq k$ we have

$$\mathcal{X}_{x_{kl}}(X)(x_{ij}) = \{x_{ij}, x_{kl}\}(X) = \delta_{kj}X_{il} - \delta_{ij}X_{kl} = 0$$

for any $X \in B_p$, since $j - k > p$ and $l - i > p$. Hence B_p is a Poisson submanifold of $\mathfrak{b}_+$. Taking $p = 1$ and

$$\epsilon = \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ 0 & & & 1 & 0 \end{pmatrix}$$

one has $[\epsilon, \mathfrak{g}_+] \in \mathfrak{g}_+^\perp$ and $[\epsilon, \mathfrak{g}_-] \in \mathfrak{g}_-^\perp$. For $H : Y \mapsto -\frac{1}{2} \text{Trace } Y^2$, we obtain $\nabla H(Y) = -Y$ and from (2.28) the Hamiltonian vector field $\mathcal{X}_{H_\epsilon}$ is given by the Lax equation

$$\dot{Y} = [Y, Y_-] = -[Y, Y_+]$$

where

$$Y = \begin{pmatrix} b_1 & a_1 & \dots & \dots & 0 \\ 1 & b_2 & a_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & b_{n-1} & a_{n-1} \\ 0 & \dots & 0 & 1 & b_n \end{pmatrix},$$

and

$$Y_+ = \begin{pmatrix} 0 & a_1 & \dots & \dots & 0 \\ 0 & 0 & a_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & a_{n-1} \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \quad Y_- = \begin{pmatrix} b_1 & 0 & \dots & \dots & 0 \\ 1 & b_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & b_{n-1} & 0 \\ 0 & \dots & 0 & 1 & b_n \end{pmatrix}$$

Let $\mathfrak{g}$ be a Lie algebra equipped with a non-degenerate Ad-invariant symmetric bilinear form $(\cdot, \cdot)$. Then one has the isomorphism

$$\chi \otimes I_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}^* \otimes \mathfrak{g} \simeq \text{End}(\mathfrak{g})$$

which associates with every element of $\mathfrak{g} \otimes \mathfrak{g}$ a unique linear map $\mathfrak{g} \rightarrow \mathfrak{g}$ and viceversa. Thus, the notion of classical R -matrix can be translated in terms of an element of the tensor product $\mathfrak{g} \otimes \mathfrak{g}$.

Let $\mathfrak{g} \otimes \mathfrak{g}$ be the $\mathfrak{g}$ -module defined by (A.1). For a fixed $r \in \mathfrak{g} \otimes \mathfrak{g}$, consider the 1-coboundary linear map

$$\delta r : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}; X \mapsto [X \otimes 1 + 1 \otimes X, r]$$

Let us suppose that the dual of δr defines a Lie algebra structure $[\cdot, \cdot]_r : \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ on $\mathfrak{g}^*$, i.e., for $\xi, \eta \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$ we have

$$\langle [\xi, \eta]_r, X \rangle = \langle \xi \wedge \eta, \delta r(X) \rangle = \langle \xi \wedge \eta, [X \otimes 1 + 1 \otimes X, r] \rangle.$$

Then $\mathfrak{g}$ inherits a linear Poisson bracket $\{\cdot, \cdot\}_r$ from $[\cdot, \cdot]_r$, since $\mathfrak{g} \cong \mathfrak{g}^{**}$. Explicitly, if $f, g \in \mathcal{F}(\mathfrak{g})$ then

$$\{f, g\}_r(X) = \langle [df(X), dg(X)]_r, X \rangle = \langle df(X) \wedge dg(X), [X \otimes 1 + 1 \otimes X, r] \rangle.$$

Definition 2.33. Let $\mathfrak{g}$ be a Lie algebra. A bivector $r \in \mathfrak{g} \otimes \mathfrak{g}$ is called an *r -matrix* for $\mathfrak{g}$ if $[\cdot, \cdot]_r$ is a Lie bracket on $\mathfrak{g}^*$.

We denote by r^+ and r^- the symmetric and skew-symmetric parts of r , respectively. Then $\delta r^+(X)$ and $\delta r^-(X)$ is a symmetric and skew-symmetric element of $\mathfrak{g} \otimes \mathfrak{g}$, respectively. Therefore, it follows that the skew-symmetry of $[\cdot, \cdot]_r$ is guaranteed if we assume that r to be skew-symmetric, $r = r^-$.

In order to analyze the Jacobi identity for $[\cdot, \cdot]_r$, one introduces the linear map $\underline{r} : \mathfrak{g}^* \rightarrow \mathfrak{g}$ by

$$\langle \eta, \underline{r}(\xi) \rangle = \langle \xi \wedge \eta, r \rangle$$

In terms of $\underline{r}$ the explicit formula for $[\cdot, \cdot]_r$ is given in the following result.

Lemma 2.34. Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be skew-symmetric. For any $\xi, \eta \in \mathfrak{g}^*$,

$$[\xi, \eta]_r = \text{ad}_{\underline{r}(\xi)}^* \eta - \text{ad}_{\underline{r}(\eta)}^* \xi.$$

Proof. See [1]. Lemma 4.40, page 97. □

Proposition 2.35. If $r \in \mathfrak{g} \otimes \mathfrak{g}$ is skew-symmetric and satisfies

$$\langle \zeta, [\underline{r}(\xi), \underline{r}(\eta)] \rangle + \langle \xi, [\underline{r}(\eta), \underline{r}(\zeta)] \rangle + \langle \eta, [\underline{r}(\zeta), \underline{r}(\xi)] \rangle = 0,$$

for all $\zeta, \xi, \eta \in \mathfrak{g}^*$, then r is an r -matrix.

Proof. See [1], Proposition 4.41, page 98. □

For a more general result, without the assumption r to be skew-symmetric, we introduce the following notation

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$$

where for $r = \sum_i a_i \otimes b_i$, we have

$$\begin{aligned} [r_{12}, r_{13}] &= \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j \\ [r_{12}, r_{23}] &= \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j \\ [r_{13}, r_{23}] &= \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j] \end{aligned}$$

Theorem 2.36. Let $\mathfrak{g}$ be a Lie algebra, $r \in \mathfrak{g} \otimes \mathfrak{g}$ is an r -matrix if and only if the following conditions are satisfied:

- i) r^+ is a $\mathfrak{g}$ -invariant element of $\mathfrak{g} \otimes \mathfrak{g}$.
- ii) $[[r, r]]$ is a $\mathfrak{g}$ -invariant element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

Proof. See [21], page 277. □

Example 2.37. Let $\mathfrak{g}$ be a simple Lie algebra with a Cartan-Weyl basis

$$e_\alpha, e_{-\alpha}, h_\alpha, \alpha \in \Phi_+$$

Then the bivector r -matrix r

$$r = \sum_{\alpha \in \Phi_+}^r e_\alpha \wedge e_{-\alpha} \quad (2.29)$$

is an r -matrix.

2.3 Poisson-Lie groups

A Poisson-Lie group is a Lie group equipped with a Poisson bracket which is compatible with the multiplication map, see [5]. A Poisson-Lie group is the non-linear analog of $\mathfrak{g}^*$ equipped with its linear Poisson structure. See the references [18], [22], [32] for more details and topics related.

2.3.1 Lie bialgebras

Definition 2.38. A Lie group G equipped with a Poisson bracket $\{\cdot, \cdot\}$ is called **Poisson-Lie group** if the multiplication map $m : G \times G \rightarrow G : (x, y) \mapsto m(x, y) = xy$ is a Poisson map.

Given $x \in G$ let λ_x, ρ_x be the left and right translation operators on $\mathcal{F}(G)$ defined by

$$\lambda_x \phi(y) = \phi(xy), \quad \rho_x(y) = \phi(yx), \quad \forall \phi, \rho \in \mathcal{F}(G).$$

Then, if $\phi, \psi \in \mathcal{F}(G)$, the multiplication map m is a Poisson map if and only if

$$\{\phi, \psi\}(xy) = \{\lambda_x \phi, \lambda_x \psi\}(y) + \{\rho_y \phi, \rho_y \psi\}(x), \quad \forall x, y \in G. \quad (2.30)$$

Given $\phi \in \mathcal{F}(G)$, the right and left invariant differentials, denoted by $D\phi$ and, respectively, $D'\phi$ are defined by

$$\langle D\phi(x), X \rangle = \left. \frac{d}{dt} \right|_{t=0} \phi(\exp(tX)x), \quad \langle D'\phi(x), X \rangle = \left. \frac{d}{dt} \right|_{t=0} \phi(x \exp(tX)), \quad X \in \mathfrak{g}. \quad (2.31)$$

that is, $D\phi(x) = (R_x^* d\phi)_e$ and $D'\phi(x) = (L_x^* d\phi)_e$.

The Poisson bracket $\{\cdot, \cdot\}$ on G can be written as

$$\{\phi, \psi\}(x) = \langle \eta(x) D\phi(x), D\psi(x) \rangle = \langle \eta'(x) D'\phi(x), D'\psi(x) \rangle \quad (2.32)$$

where $\eta, \eta' : G \rightarrow \text{Hom}(\mathfrak{g}^*, \mathfrak{g})$. Notice that $\text{Hom}(\mathfrak{g}^*, \mathfrak{g}) \simeq \mathfrak{g} \otimes \mathfrak{g}$ and using the fact that $\{\cdot, \cdot\}$ is skew-symmetric we can consider η, η' as maps from G to $\mathfrak{g} \wedge \mathfrak{g}$.

Proposition 2.39. *If G is a Poisson-Lie group, then η and η' verify*

$$\eta(xy) = \text{Ad}_x \eta(y) \text{Ad}_{x^{-1}}^* + \eta(x), \quad \eta'(xy) = \text{Ad}_{y^{-1}} \eta'(x) \text{Ad}_y^* + \eta'(y) \quad (2.33)$$

Proof. Let us prove for η . Notice that

$$D(\lambda_x \phi)(y) = \text{Ad}_{x^{-1}}^* D\phi(xy), \quad D'(\lambda_x \phi)(y) = D'\phi(xy), \quad D(\rho_y \phi)(x) = D\rho(xy).$$

From (2.32) we have

$$\begin{aligned} \{\phi, \psi\}(xy) &= \langle \eta(xy) D\phi(xy), D\psi(xy) \rangle \\ \{\lambda_x \phi, \lambda_x \psi\}(y) &= \langle \eta(y) D(\lambda_x \phi)(y), D(\lambda_x \psi)(y) \rangle \\ &= \langle \eta(y) \text{Ad}_{x^{-1}}^* D\phi(xy), \text{Ad}_{x^{-1}}^* D\psi(xy) \rangle = \langle \text{Ad}_x \eta(y) \text{Ad}_{x^{-1}}^* D\phi(xy), D\psi(xy) \rangle \\ \{\rho_y \phi, \rho_y \psi\}(x) &= \langle \eta(x) D(\rho_y \phi)(x), D(\rho_y \psi)(x) \rangle = \langle \eta(x) D\phi(xy), D\psi(xy) \rangle. \end{aligned}$$

Plugging these equations into (2.30) we obtain the first equality of (2.33). The case for η' can be shown in a similar way. $\square$

It follows from (2.33) that $\eta(e) = \eta'(e) = 0$. Then one can linearize the Poisson bracket $\{\cdot, \cdot\}$ at e to obtain a Lie bracket on $\mathfrak{g}^*$. More precisely, let $\xi_1, \xi_2 \in \mathfrak{g}^*$ and choose $\phi_1, \phi_2 \in \mathcal{F}(G)$ such that $d\phi_i(e) = \xi_i, i = 1, 2$. Define

$$[\xi_1, \xi_2]_* = d\{\phi_1, \phi_2\}(e) \quad (2.34)$$

Proposition 2.40. *Formula (2.34) defines a Lie bracket on $\mathfrak{g}^*$.*

Proof. First, we must prove that (2.34) is well defined, i.e., it does not depend of the choice of $\phi_1, \phi_2 \in \mathcal{F}(G)$ such that $d\phi_i(e) = \xi_i, i = 1, 2$. In fact, from (2.32) and $\eta(e) = 0$ we have

$$d\{\phi_1, \phi_2\}(e) = \langle d\eta(e) D\phi_1(e), D\phi_2(e) \rangle = \langle d\eta(e) \xi_1, \xi_2 \rangle,$$

showing the assertion. Now the Jacobi identity follows from (2.34). $\square$

Let $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ be the map given by

$$\delta(X) = \left. \frac{d}{dt} \right|_{t=0} \eta(\exp tX). \quad (2.35)$$

From (2.33) we have that δ is 1-cocycle of $\mathfrak{g}$ with values on the $\mathfrak{g}$ -module $\mathfrak{g} \otimes \mathfrak{g}$, i.e.,

$$\delta([\xi_1, \xi_2]) = [\xi_1 \otimes 1 + 1 \otimes \xi_1, \delta(\xi_2)] - [\xi_2 \otimes 1 + 1 \otimes \xi_2, \delta(\xi_1)]$$

Definition 2.41. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{g}^*$ be its dual space. One says that $(\mathfrak{g}, \mathfrak{g}^*)$ forms a **Lie bialgebra** if:

- (i) $\mathfrak{g}^*$ is a Lie algebra;

- (ii) the map δ , dual to the commutator $[\cdot, \cdot]_* : \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^*$, is a 1-cocycle on $\mathfrak{g}$ with values on $\mathfrak{g} \otimes \mathfrak{g}$. The map δ is called a **cobacket**.

One says that $\varphi : (\mathfrak{g}, \mathfrak{g}^*) \rightarrow (\mathfrak{h}, \mathfrak{h}^*)$ is a homomorphism of Lie bialgebras if φ is a Lie algebra homomorphism and

$$(\varphi \otimes \varphi) \circ \delta_{\mathfrak{g}} = \delta_{\mathfrak{h}} \circ \varphi.$$

A Lie bialgebra structure $(\mathfrak{g}, \mathfrak{g}^*)$ on $\mathfrak{g}$ give us a Lie-Poisson structure on $\mathfrak{g}$ induced by the Lie algebra structure of $\mathfrak{g}^*$.

Definition 2.42. A **coboundary Lie bialgebra** is a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ such that the cobracket is the 1-cocycle $\delta r := \delta(r)$ where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is an r -matrix.

Example 2.43. (i) Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ be the Lie algebra of the traceless 2×2 complex matrices. Recall that $\mathfrak{g}$ is generated by

$$e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with relations

$$[h, e_{\pm}] = \pm 2e_{\pm}, \quad [e_+, e_-] = h.$$

The linear map

$$\delta(e_{\pm}) = e_{\pm} \wedge h, \quad \delta(h) = 0$$

makes $\mathfrak{g}$ into a Lie bialgebra.

- (ii) Let $\mathfrak{g}$ be a simple complex Lie algebra. Using the notations of Appendix A, the linear map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ acting on the Chevalley generators as

$$\delta(h_i) = 0, \quad \delta(e_{\pm i}) = \frac{1}{2} d_i e_{\pm i} \wedge h_i$$

makes $\mathfrak{g}$ into a Lie bialgebra. This Lie bialgebra on $\mathfrak{g}$ is called the *standard* Lie bialgebra structure on $\mathfrak{g}$.

One can summarize the previous results in the following statement.

Theorem 2.44. *Let G be a Poisson-Lie group then there exists a Lie bialgebra structure on $\mathfrak{g}$, commonly referred as the **tangent Lie bialgebra** of G . Moreover if $\Psi : G_1 \rightarrow G_2$ is a homomorphism of Poisson-Lie groups, then $d\Psi(e)$ is a Lie bialgebra homomorphism of their respective tangent Lie bialgebras.*

The reciprocal is also true,

Theorem 2.45 (Drinfeld). *Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra structure and let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then there exists a unique Poisson structure on G making it into a Poisson-Lie group with tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. Moreover, if $\varphi : (\mathfrak{g}, \mathfrak{g}^*) \rightarrow (\mathfrak{h}, \mathfrak{h}^*)$ is a homomorphism of Lie bialgebras, and H is the Poisson-Lie group with tangent Lie bialgebra $(\mathfrak{h}, \mathfrak{h}^*)$ then there is a unique homomorphism $\Psi : G \rightarrow H$ of Poisson-Lie groups with $d\Psi(e) = \varphi$.*

The definition of a Lie bialgebra is symmetric, i.e., if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, then the pair $(\mathfrak{g}^*, \mathfrak{g})$ is also a Lie bialgebra, called the **dual Lie bialgebra** of $(\mathfrak{g}, \mathfrak{g}^*)$. Then, by Drinfeld's theorem, there exists connected and simply connected Poisson-Lie group G^* with tangent Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$. Such G^* is called the **dual Poisson-Lie group** of G . Importance of the dual Poisson-Lie group G^* in the description of the symplectic leaves of the Poisson-Lie group G and applications to integrable systems can be found in [15, 28].

Definition 2.46. Let G be a Poisson-Lie subgroup and H be a Lie subgroup of G . H is a Poisson-Lie subgroup of G if H is a Poisson submanifold of G such that its multiplication map is Poisson.

Using notation used in the definition above, Poisson-Lie subgroups are characterized as follows.

Proposition 2.47. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of H . H is a Poisson-Lie subgroup of G if and only if the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$ is an ideal in the Lie algebra $\mathfrak{g}^*$.

2.3.2 The Sklyanin bracket

Definition 2.48. A Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is called **factorizable** if the following conditions hold:

- (i) $\mathfrak{g}$ is equipped with an non-degenerate invariant bilinear form $(\cdot, \cdot)$.
- (ii) The Lie bracket on $\mathfrak{g}^* \simeq \mathfrak{g}$ is given by

$$[\xi, \eta]_* = \frac{1}{2}([R\xi, \eta] + [\xi, R\eta]) \quad (2.36)$$

where $R \in \text{End}(\mathfrak{g})$ is a skew-symmetric R -matrix satisfying the MYBE (2.22).

The corresponding connected and simply connected Poisson-Lie group G with factorizable tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ whose existence is guaranteed by Theorem 2.45 is called *factorizable Poisson-Lie group*. The corresponding Poisson bracket is called the *Sklyanin bracket*. The explicit formula formula for the Sklyanin bracket is as follows

$$\{f, g\} = \frac{1}{2} (R(\nabla' f), \nabla' g) + \frac{1}{2} (R(\nabla f), \nabla g) \quad (2.37)$$

where ∇f and $\nabla' f$ are the right and left gradients defined as $\chi^{-1}(Df)$ and $\chi^{-1}(D'f)$, respectively.

The Sklyanin bracket is a particular member of a family of Poisson brackets presented in the following proposition.

Proposition 2.49. Let G be a Lie group with Lie algebra $\mathfrak{g}$ with an invariant bilinear form $(\cdot, \cdot)$. If $R, R' \in \text{End}(\mathfrak{g})$ are two skew-symmetric R -matrices satisfying MYBE then the brackets

$$\{f, g\}_{R, R'} = \frac{1}{2} (R(\nabla' f), \nabla' g) + \frac{1}{2} (R'(\nabla f), \nabla g) \quad (2.38)$$

are Poisson brackets on G .

Proof. See [29], page 202. □

Taking the pair $(R, -R)$ the Poisson bracket (2.38) becomes the Sklyanin bracket.

Let G be a connected, simply connected and simple Lie group with the Lie algebra $\mathfrak{g}$. Let $(\cdot, \cdot)$ be the Killing form. Using the decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, every element $\xi \in \mathfrak{g}$ can be decomposed as

$$\xi = \xi_- + \xi_0 + \xi_+ \quad (2.39)$$

where $\xi_{\pm} \in \mathfrak{n}_{\pm}$ and $\xi_0 \in \mathfrak{h}$. Define $R \in \text{End}(\mathfrak{g})$ by

$$R(\xi) = \xi_+ - \xi_-. \quad (2.40)$$

Then R is an R -matrix satisfying (2.22) and thus (2.37), (2.40) endow G with a Poisson-Lie structure, called *standard* Poisson-Lie structure and denoted by $\{\cdot, \cdot\}_G$

As every multiplicative bracket, the Sklyanin bracket is degenerate at the unit element and its linearization at the unit element defines a structure of a Lie algebra in $\mathfrak{g}^*$. By duality we get a linear Poisson bracket in the tangent space $\mathfrak{g}$; this is precisely the R -bracket seen before. Thus the Sklyanin bracket may be regarded as a nonlinear analog of the R -bracket.

Example 2.50. Take $G = \text{SL}_n$ and $\mathfrak{g} = \mathfrak{sl}_n$ with the trace form

$$(\xi, \eta) = \text{Trace}(\xi\eta).$$

The R -matrix (2.40) is given by

$$R(\xi) = (\text{sgn}(j - i)\xi_{ij})_{i,j=1}^n$$

for $\xi = (\xi_{ij})_{i,j=1}^n \in \mathfrak{sl}_n$. The standard Poisson-Lie structure on coordinate functions x_{ij}, x_{kl} becomes

$$\{x_{ij}, x_{kl}\}_{\text{SL}_n} = \frac{1}{2}(\text{sgn}(k - i) + \text{sgn}(l - j))x_{il}x_{kj} \quad (2.41)$$

This Poisson structure can be extended to a Poisson bracket on $G = \text{GL}_n$, denoted by $\{\cdot, \cdot\}_{\text{GL}_n}$, considering the determinant function as a Casimir with $R = \pi_+ - \pi_- : \mathfrak{gl}_n \rightarrow \mathfrak{gl}_n$ where π_+, π_- are the projections onto the strictly upper and lower parts of an element $A \in \mathfrak{gl}_n$.

In particular, the standard Poisson-Lie structure on

$$\text{SL}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$$

verifies the following relations

$$\begin{aligned} \{a, b\}_{\text{SL}_2} &= \frac{1}{2}ab, & \{a, c\}_{\text{SL}_2} &= \frac{1}{2}ac, & \{a, d\}_{\text{SL}_2} &= bc \\ \{c, d\}_{\text{SL}_2} &= \frac{1}{2}cd, & \{b, d\}_{\text{SL}_2} &= \frac{1}{2}bd, & \{b, c\}_{\text{SL}_2} &= 0. \end{aligned}$$

One has that the image of the embedding $\rho_i : \mathrm{SL}_2 \rightarrow G$ defined in (A.6) is a Poisson-Lie subgroup of G . The restriction $\{\cdot, \cdot\}_{\rho_i(\mathrm{SL}_2)}$ of the Sklyanin bracket (2.37), (2.40) to $\rho_i(\mathrm{SL}_2)$ verifies

$$\{\cdot, \cdot\}_{\rho_i(\mathrm{SL}_2)} = (\alpha_i, \alpha_i) \{\cdot, \cdot\}_{\mathrm{SL}_2}.$$

Thus the standard Poisson-Lie structure is characterized by the condition that for every $i \in [1, l]$ ($l = \mathrm{rank} \mathfrak{g}$) the map

$$\rho_i : (\mathrm{SL}_2, d_i \{\cdot, \cdot\}_{\mathrm{SL}_2}) \rightarrow (G, \{\cdot, \cdot\}_G) \quad (2.42)$$

is Poisson.

2.3.3 A Poisson bracket on Grassmannians

In this subsection, we construct a Poisson bracket on Grassmannians $Gr_k(n)$, $0 < k < n$. The construction is based on a result of [4] for homogeneous spaces M of simple Lie groups using r -matrices.

Recall, the Grassmannian $Gr_k(n)$ is the $k(n-k)$ -dimensional smooth manifold parametrizing the set of k -dimensional subspaces of $\mathbb{R}^n$. Fixing a basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$, we identify $Gr_k(n)$ with the quotient of the matrices in $\mathrm{Mat}_{k \times n}(\mathbb{R})$ of rank k by $\mathrm{GL}_k(\mathbb{R})$. Hence any element $X \in Gr_k(n)$ is represented by a matrix $\bar{X} \in \mathrm{Mat}_{k \times n}$ of rank k . The other representatives of X are $A\bar{X}$ where $A \in \mathrm{GL}_k(\mathbb{R})$. The right action of $G = \mathrm{GL}_n$ on $Gr_k(n)$ given by

$$Gr_k(n) \times \mathrm{GL}_n \rightarrow Gr_k(n) : (X, A) \mapsto \bar{X}A$$

is transitive. Let $X_0 \in Gr_k^\circ(n) = \{A \in Gr_k(n) : x_{[1,k]}(\bar{A}) \neq 0\}$ such that its representative $\bar{X}_0$ is of the form

$$(Id_k \quad 0)$$

One has that the isotropy group G_{X_0} of X_0 is

$$G_{X_0} = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in \mathrm{GL}_k, C \in \mathrm{GL}_{n-k}, B \in \mathrm{Mat}_{k, n-k} \right\}.$$

Then $Gr_k(n) \simeq \mathrm{GL}_n / G_{X_0}$.

Let G be a simple Lie group with Lie algebra $\mathfrak{g}$ and M be a G -homogeneous space. Fix $x_0 \in M$ and let $G_{x_0} \subset G$ be the stabilizer of x_0 with Lie algebra $\mathfrak{g}_{x_0}$. Let $r \in \Lambda^2 \mathfrak{g}$ be an r -matrix and let $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be the Lie algebra homomorphism and consider the bivector field on M

$$r_M = (\rho \otimes \rho)(r) \quad (2.43)$$

Proposition 2.51 ([4]). *If $\mathfrak{n}_+ \subset \mathfrak{g}_{x_0}$ then r_M is a Poisson bivector field.*

We apply the above proposition to our example $M = Gr_k(n)$ with $G = \mathrm{GL}_n$. One verifies the condition $\mathfrak{n}_+ \subset \mathfrak{g}_{x_0}$. To apply the above proposition one considers the map ρ given by

the infinitesimal action defined by the action of GL_n on $Gr_k(n)$ and the r -matrix of $\mathfrak{gl}(n, \mathbb{R})$ given $r = \sum_{i < j}^n E_{ij} \wedge E_{ji}$.

Let $A \in Gr_k^o(n)$ and $\bar{A} = (a_{ij}) \in \text{Mat}_{k,n}$ be a representative of A where $\bar{A}$ has rank k . The local coordinates of $Gr_k^o(n)$ are given by

$$u_{rs} = \frac{1}{\bar{A}_{[1,k]}} \sum_{p=1}^k C_{pr} a_{p,k+s}$$

for $r = 1, \dots, k$ and $s = 1, \dots, n - k$, where $C = (C_{pq})$ is the cofactor matrix of the principal submatrix $k \times k$ of $\bar{A}$. Let $\gamma_{ij}(t) = A \cdot \exp(tE_{ij}) = (b_{rs})$ be the path in $Gr_k^o(n)$ passing through A . One has

(i) If $i = j$ then

$$b_{rs} = \begin{cases} a_{rs} & \text{if } s \neq j \\ a_{ri} \exp t & \text{if } s = j. \end{cases} \quad (2.44)$$

(ii) If $i \neq j$ then

$$b_{rs} = \begin{cases} a_{rs} & \text{if } s \neq j \\ a_{rj} + t a_{ri} & \text{if } s = j. \end{cases} \quad (2.45)$$

We compute the fundamental vector fields $(E_{ij})_M$ in the local coordinates of the open cell $Gr_k^o(n)$ for any $1 \leq i, j \leq n$.

Case: $i = j = k$

In this case, from (2.44), the principal submatrix $k \times k$ of $\gamma_{kk}(t)$ is as follows

$$\begin{pmatrix} a_{11} & \dots & a_{1k} \exp t \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \exp t \end{pmatrix}$$

Its determinant is $\exp t \cdot \bar{A}_{[1,k]}$ and its cofactor matrix $C(t) = (C_{pq}(t))$ is given by

$$C_{pk}(t) = C_{pk}, \quad C_{pq}(t) = \exp t \cdot C_{pq}, \quad p \in [1, k], q \in [1, k-1].$$

Then

$$\begin{aligned} u_{ks}(t) &= \exp(-t) u_{ks}, \quad s \in [1, n-k], \\ u_{rs}(t) &= u_{rs}, \quad r \in [1, k], s \in [1, n-k], \end{aligned}$$

and therefore

$$(E_{kk})_M = - \sum_{s=1}^{n-k} u_{ks} \frac{\partial}{\partial u_{ks}} \quad (2.46)$$

Case: $i = k, j \neq k$

a) $j < k$: In this case, from (2.45), the principal submatrix $k \times k$ of $\gamma_{kj}(t)$ is

$$\begin{pmatrix} a_{11} & \dots & a_{1j} + ta_{1k} & \dots & a_{1k} \\ \vdots & & \vdots & & \vdots \\ a_{k1} & \dots & a_{kj} + ta_{kk} & \dots & a_{kk} \end{pmatrix}$$

Its determinant is $\bar{A}_{[1,k]}$ and its cofactor matrix is given by

$$C_{pr}(t) = C_{pr}, \quad r \neq k, \quad C_{pk}(t) = C_{pk} - tC_{pj}, \quad p \in [1, k].$$

Then

$$\begin{aligned} u_{rs}(t) &= u_{rs}, \quad r \in [1, k-1], s \in [1, n-k], \\ u_{ks}(t) &= u_{ks} - tu_{js}, \quad s \in [1, n-k], \end{aligned}$$

and therefore

$$(E_{kj})_M = - \sum_{s=1}^{n-k} u_{js} \frac{\partial}{\partial u_{ks}} \quad (2.47)$$

b) $j > k$: In this case one can see that

$$\begin{aligned} u_{rs}(t) &= u_{rs}, \quad r \in [1, k], s \neq j-k \\ u_{r,j-k}(t) &= \begin{cases} u_{r,j-k} + t & \text{if } r = k. \\ u_{r,j-k} & \text{if } r \neq k. \end{cases} \end{aligned}$$

Then

$$(E_{kj})_M = \frac{\partial}{\partial u_{k,j-k}} \quad (2.48)$$

Case: $j = k, i \neq k$

a) $i < k$: In this case one obtains

$$\begin{aligned} u_{is}(t) &= u_{is} - tu_{ks}, \quad s \in [1, n-k], \\ u_{rs}(t) &= u_{rs}, \quad r \neq i, s \in [1, n-k], \end{aligned}$$

and therefore

$$(E_{ik})_M = - \sum_{s=1}^{n-k} u_{ks} \frac{\partial}{\partial u_{is}} \quad (2.49)$$

b) $i > k$: After a lengthy but straightforward computation one can see that

$$(E_{ik})_M = - \sum_{r=1}^k \sum_{s=1}^{n-k} u_{ks} u_{r,i-k} \frac{\partial}{\partial u_{rs}} \quad (2.50)$$

Case: $i = j \neq k$

a) $i < k$: In this case one has

$$\begin{aligned} u_{is}(t) &= \exp(-t)u_{is}, \quad s \in [1, n-k], \\ u_{rs}(t) &= u_{rs}, \quad r \neq i, s \in [1, n-k]. \end{aligned}$$

Then

$$(E_{ii})_M = - \sum_{s=1}^{n-k} u_{is} \frac{\partial}{\partial u_{is}} \quad (2.51)$$

b) $i > k$: In this case one has

$$\begin{aligned} u_{r,i-k}(t) &= \exp t \cdot u_{r,i-k}, \quad r \in [1, k], \\ u_{rs}(t) &= u_{rs}, \quad r \in [1, k], s \neq i-k. \end{aligned}$$

Then

$$(E_{ii})_M = \sum_{r=1}^k u_{r,i-k} \frac{\partial}{\partial u_{r,i-k}} \quad (2.52)$$

Case: $i \neq j$: After lengthy but straightforward computation one obtains

a) $i, j > k$:

$$(E_{ij})_M = \sum_{r=1}^k u_{r,i-k} \frac{\partial}{\partial u_{r,j-k}} \quad (2.53)$$

b) $i, j < k$:

$$(E_{ij})_M = \sum_{s=1}^{n-k} u_{js} \frac{\partial}{\partial u_{is}} \quad (2.54)$$

c) $j < k < i$:

$$(E_{ij})_M = - \sum_{r=1}^k \sum_{s=1}^{n-k} u_{js} u_{r,i-k} \frac{\partial}{\partial u_{rs}} \quad (2.55)$$

d) $i < k < j$:

$$(E_{ij})_M = \frac{\partial}{\partial u_{i,j-k}} \quad (2.56)$$

After computation, one can see

$$\sum_{i=1}^{k-1} (E_{ik})_M \wedge (E_{ki})_M + \sum_{j=k+1}^n (E_{kj})_M \wedge (E_{jk})_M = -2 \sum_{i,j=1}^{n-k} \sum_{s=1}^{k-1} u_{ki} u_{sj} \frac{\partial}{\partial u_{kj}} \wedge \frac{\partial}{\partial u_{si}}. \quad (2.57)$$

and

$$\begin{aligned} \sum_{\substack{1 \leq i < j \leq n \\ (i,j) \neq (k,k)}} (E_{ij})_M \wedge (E_{ji})_M &= \sum_{1 \leq i < j \leq n-k} \sum_{r,s=1}^k u_{ri} u_{sj} \frac{\partial}{\partial u_{rj}} \wedge \frac{\partial}{\partial u_{si}} - \sum_{r=1}^{k-1} \sum_{s=1}^k \sum_{i,j=1}^{n-k} u_{ri} u_{sj} \frac{\partial}{\partial u_{rj}} \wedge \frac{\partial}{\partial u_{si}} \\ &+ \sum_{1 \leq r < s < k} \sum_{i,j=1}^{n-k} u_{ri} u_{sj} \frac{\partial}{\partial u_{rj}} \wedge \frac{\partial}{\partial u_{si}}. \end{aligned} \quad (2.58)$$

In the particular case $n = 4, k = 2$ one has

$$\begin{aligned} \sum_{i < j} (E_{ij})_M \wedge (E_{ji})_M &= u_{11}u_{21} \frac{\partial}{\partial u_{11}} \wedge \frac{\partial}{\partial u_{21}} + 2u_{11}u_{22} \frac{\partial}{\partial u_{12}} \wedge \frac{\partial}{\partial u_{21}} + u_{12}u_{22} \frac{\partial}{\partial u_{12}} \wedge \frac{\partial}{\partial u_{22}} \\ &\quad - u_{11}u_{12} \frac{\partial}{\partial u_{11}} \wedge \frac{\partial}{\partial u_{12}} - u_{21}u_{22} \frac{\partial}{\partial u_{21}} \wedge \frac{\partial}{\partial u_{22}}. \end{aligned}$$

After computation, the Poisson bivector field $r_M = \sum_{i < j} (E_{ij})_M \wedge (E_{ji})_M$ on $Gr_k(n)$ has the form

$$r_{Gr_k(n)} = \sum_{r,i,s,j=1} (\operatorname{sgn}(s-r) - \operatorname{sgn}(i-j)) u_{ri} u_{sj} \frac{\partial}{\partial u_{rj}} \wedge \frac{\partial}{\partial u_{si}} \quad (2.59)$$

which induces the following Poisson bracket

$$\{u_{rj}, u_{si}\} = (\operatorname{sgn}(s-r) - \operatorname{sgn}(i-j)) u_{ri} u_{sj}. \quad (2.60)$$

The Poisson structure (2.59) is also defined as the pushforward of the corresponding Poisson bivector field of $\{\cdot, \cdot\}_{GL_n}$ under the projection $GL_n \rightarrow Gr_k(n)$. For details, see [9].

Weighted networks

In this chapter we review the theory of weighted networks on two kind of surfaces: the disc and the annulus. The chapter is mainly based on [10, 12] and [27], where detailed proofs of the statements made here below can be found. A weighted network N in a disc is a planar directed graph G in a disc with two types of boundary vertices of degree 1, internal vertices of degree 3 of two types and each edge of G is equipped with a weight function. The construction of a weighted network in an annulus is similar to the case of a weighted network in a disc with an additional *cut*, i.e., a path joining the connected components of the boundary. The boundary measurement of a directed path joining two boundary vertices in a weighted network on both surfaces are defined. In the annulus case a spectral parameter λ associated to the cut is necessary to count intersections of a path with the cut. The main part of this chapter is devoted to the construction of a 2-parametric family of Poisson brackets on the space of real matrices (resp. space of real rational matrix functions in one variable) and on the Grassmannians based on Poisson structures on the space of edge weights of the network in a disc (resp. in an annulus) via the boundary measurement map. In what follows we occasionally use the term “network” instead of weighted network.

3.1 Networks in a disc

We present the main definitions related to networks in a disc. To each network N in a disc one associates a matrix and a representative of an element of the Grassmannian using the boundary measurements of directed paths between sources and sinks of the network. The space of the edge weights is equipped with a 6-parametric family of Poisson brackets which will induce a 2-parametric family of Poisson brackets on matrices and on Grassmannians via the boundary measurement map. Finally, the 6-parametric family of Poisson brackets on the

space of edge weights induce a 2-parametric family of Poisson brackets on the space of face weights which can be described using the dual weighted network N^* of N .

3.1.1 Boundary measurement of a weighted network

Let $G = (V, E)$ be a directed planar graph inside a disc with vertex set V and edge set E . Among the vertices exactly n of them will be located on the boundary, the ones called *boundary vertices*, and will be labelled counterclockwise as $b_1, \dots, b_n$. One has two types of boundary vertices: the *sources* and the *sinks*. A source (resp. sink) is a vertex with exactly one outgoing (resp. incoming) edge and no incoming (resp. outgoing) edges. We denote the number of sources (resp. sinks) by k (resp. $m := n - k$) and its corresponding set of indices by $I \subset [1, n]$ (resp. $J \subset [1, n]$). The internal vertices, having degree 3, are of two types: white or black. White (resp. black) vertices have exactly one incoming (resp. outgoing) edge.

Let $x_1, \dots, x_d$ be independent variables. A *weighted network* $N = (G, w)$ in a disc is one of the directed planar graphs G described above, who has been assigned an *edge weight* w_e , which is a rational function with integer coefficients in variables $x_1, \dots, x_d$, to each edge $e \in G$. Thus each network N defines a rational map $w : \mathbb{R}^d \rightarrow \mathbb{R}^{|E|}$; the *space of edge weights* $\mathcal{E}_N$ is defined as $w((\mathbb{R}^*)^d) \cap (\mathbb{R}^*)^{|E|}$, i.e., a point in $\mathcal{E}_N$ is a graph G whose edges are weighted by nonzero real numbers obtained after assign nonzero values $x_1, \dots, x_d$ in the expressions for w_e . We shall assume that if the weight of an edge e of G is not specified in the graph then $w_e = 1$.

An example of a weighted network in a disc is shown in Figure 3.1.

Figure 3.1: A weighted network on a disc

A *path* P in N is an alternating sequence $(v_1, e_1, v_2, \dots, e_r, v_{r+1})$ of vertices and edges such that $e_i = (v_i, v_{i+1})$ for any $i \in [1, r]$. A path is called a cycle if $v_{r+1} = v_1$. Given a path P we define its winding index, denoted by $\text{wind}(P)$, as follows: If the path P is simple, i.e., P has no self-intersections, then $\text{wind}(P) = 0$. Suppose that P has at least one self-intersection and let C be the first simple cycle that appears in the path P . Let $\tilde{P}$ be the path after the cycle C of the path P has been deleted and set

$$\text{wind}(P) = \begin{cases} \text{wind}(\tilde{P}) + 1 & \text{if } C \text{ is a counterclockwise cycle;} \\ \text{wind}(\tilde{P}) - 1 & \text{if } C \text{ is a clockwise cycle.} \end{cases}$$

One defines the winding index of P by repeatedly deleting cycles until one gets a path without self-intersections. The weight of path P of a network N , denoted by w_P , is defined as

$$w_P = (-1)^{\text{wind}(P)} \prod_{e \in P} w_e. \quad (3.1)$$

Definition 3.1. Let N be a network in a disc. We define

- (1) Given a source b_i and a sink b_j . The **boundary measurement** $M(i, j)$ is the sum of the weights of all paths starting in b_i and ending at b_j , i.e.,

$$M(i, j) = \sum_{P: b_i \rightarrow b_j} w_P \quad (3.2)$$

- (2) The **boundary measurement matrix** M_N is the $k \times m$ matrix $M_N = (M_{pq})$ where $M_{pq} := M(i_p, j_q)$. The map $M_N : \mathcal{E}_N \rightarrow \text{Mat}_{k,m}$ is called the **boundary measurement map** corresponding to N .

Proposition 3.2. Let N be a network in a disc, then each boundary measurement in N is a rational function in the weights w_e admitting a subtraction-free rational expression.

Proof. The proof is by induction on the number of internal vertices. The technique is based in splitting an internal vertex (white or black). Splitting a black vertex of a network N is explained in Lemma 3.16. Splitting a white vertex can be done in a similar way, see [10]. $\square$

Example 3.3. Consider the network N shown in Figure 3.1. The matrix given by

$$\begin{pmatrix} \frac{w_1 w_3 w_8 w_{11} (w_2 + w_6 w_9 w_{10})}{1 + w_3 w_7 w_{10} w_{11}} & \frac{w_1 w_3 w_4 (w_2 + w_6 w_9 w_{10})}{1 + w_3 w_7 w_{10} w_{11}} \\ \frac{w_3 w_5 w_6 w_8 w_{10} w_{11}}{1 + w_3 w_7 w_{10} w_{11}} & \frac{w_3 w_4 w_5 w_6 w_{10}}{1 + w_3 w_7 w_{10} w_{11}} \end{pmatrix}$$

is the boundary measurement matrix M_N of the network N .

A natural operation on networks in a disc is their *concatenation*. To obtain from two networks in a disc a new one by concatenation: First, we select a segment from the boundary of each disc and we identify these segments via a homeomorphism in such a way that every sink (resp. source) contained in the segment of the first network is glued to a source (resp. sink) of the second network. Then, we erase the common piece of the boundary along which the gluing was performed. Finally, we identify every pair of glued edges in the resulting network with a single edge of the same orientation and with the weight equal to the product of the two weights assigned to the edges that were glued.

To illustrate the concatenation consider the particular case of networks whose sources and sinks do not interlace. We can arrange the boundaries vertices in such a way that the sources (resp. sinks) are located on the left (resp. right) side, see Figure 3.2. To a network N of this type, with k sources and m sinks, we associate the matrix $A_N = M_N W_0$, where M_N is the boundary measurement of N and $W_0 = (\delta_{i, m+1-j})_{i,j=1}^m$. Concatenating two networks N_1 and

Figure 3.2: Concatenation of networks in a disc

N_2 of this kind, one with k sources and m sinks and another with m sources and l sinks, one obtains a network N whose matrix A_N is $A_{N_1}A_{N_2}$, where A_{N_i} is the matrix associated to N_i for $i = 1, 2$.

Example 3.4. Let N_1 and N_2 be the two networks of Figure 3.2 before concatenate and N be the concatenated network. One has

$$A_{N_1} = \begin{pmatrix} w_1^{-1} & w_2 \\ 0 & w_1 \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} w_3^{-1} & 0 \\ w_4 & w_3 \end{pmatrix}$$

and

$$A_N = A_{N_1}A_{N_2} = \begin{pmatrix} w_1^{-1}w_3^{-1} + w_2w_4 & w_2w_3 \\ w_1w_4 & w_1w_3 \end{pmatrix}.$$

One can now extend the boundary measurement map to a map into the Grassmannian $Gr_k(n)$. Let N be a network with sources $b_i, i \in I$ and sinks $b_j, j \in J$. Define the $k \times n$ matrix $\bar{X}_N$ as follows:

- (i) The $k \times k$ submatrix of $\bar{X}_N$ formed by k columns indexed by I is the identity matrix Id_k .
- (ii) for $p \in [1, k]$ and $j = j_q \in J$, the (p, j) -entry of $\bar{X}_N$ is $m_{pj}^J = (-1)^{s(p,j)} M_{pq}$, where $s(p, j) = \#\{i \in I : \min\{i_p, j\} < i < \max\{i_p, j\}\}$. The choice of the sign $(-1)^{s(p,j)}$ is such that the minor $(\bar{X}_N)_{[1,k]}^{I(i_p \rightarrow j)}$ is equal to the entry M_{pq} of the boundary measurement matrix M_N .

The matrix $\bar{X}_N$ is a matrix representative of an element $X_N \in Gr_k^I(n) = \{X \in Gr_k(n) : x_I(\bar{X}) \neq 0\}$ and the corresponding rational map $X_N : \mathcal{E}_N \rightarrow Gr_k(n)$ is called the **Grassmannian boundary measurement map**.

3.1.2 Poisson structures on the space of edges weights

Let G be one of the graphs in a disc described in Subsection 3.1.1. To each internal vertex v of G we assign a 3-dimensional Poisson manifold $((\mathbb{R}^3)_v, \{\cdot, \cdot\}_v)$ with coordinates $x_v^i, i = 1, 2, 3$. To each boundary vertex we assign the 1-dimensional manifold $(\mathbb{R} \setminus 0)_j$ with coordinate x_j^1 . We use the coordinates x_v^i to label the edges involving the vertex v as in Figure 3.3.

Let $\mathcal{R}$ be the $2|E|$ -dimensional manifold which is the cartesian product of all the manifolds associated to the vertices of G . Note that $\mathcal{R}$ equipped with a Poisson bracket $\{\cdot, \cdot\}_{\mathcal{R}}$, which is

defined as the direct sum of all $\{\cdot, \cdot\}_v$, i.e., $\{x_v^i, x_{v'}^j\}_{\mathcal{R}} = 0$ for all $v \neq v'$. The bracket $\{\cdot, \cdot\}_{\mathcal{R}}$ is called **universal** if each $\{\cdot, \cdot\}_v$ depends only on the color of the vertex v . For an edge e ,

Figure 3.3: Labelling of edges involving v

one defines the weight w_e as the product of the corresponding labels of e with respect to its corresponding vertices, i.e., if $e = (u, v)$ then

$$w_e = x_v^i x_u^j \quad (3.3)$$

where x_v^i (resp. x_u^j) is the label of e with respect to v (resp. u). Thus, we have the map $w : (\mathbb{R}^*)^{2|E|} \rightarrow (\mathbb{R}^*)^{|E|}$ called the *weight map*. One requires that the pushforward of $\{\cdot, \cdot\}_{\mathcal{R}}$ to $(\mathbb{R}^*)^{|E|}$ by the weight map to be a well defined Poisson bracket denoted by $\{\cdot, \cdot\}_N$.

Proposition 3.5. *Universal Poisson brackets $\{\cdot, \cdot\}_{\mathcal{R}}$ such that the weight map w is a Poisson map form a 6-parametric family defined by relations*

$$\{x_v^i, x_v^j\}_v = \alpha_{ij} x_v^i x_v^j, \quad 1 \leq i \neq j \leq 3, \quad (3.4)$$

at each white vertex v and

$$\{x_v^i, x_v^j\}_v = \beta_{ij} x_v^i x_v^j, \quad 1 \leq i \neq j \leq 3, \quad (3.5)$$

at each black vertex v .

Proof. Let v be a white vertex and let $e = (v, u)$ and $e' = (v, u')$ be its outgoing edges. Now $w_e = x_v^i x_u^k$ and $w_{e'} = x_v^j x_{u'}^l$ for some $1 \leq i \neq j, k, l \leq 3$. Then

$$\{w_e, w_{e'}\}_N = \{x_v^i x_u^k, x_v^j x_{u'}^l\}_{\mathcal{R}} = x_u^k x_{u'}^l \{x_v^i, x_v^j\}_v$$

Hence the unique possibility to the right hand side be a function of w_e and $w_{e'}$ occurs when $\{x_v^i, x_v^j\}_v = \alpha_{ij} x_v^i x_v^j$. Analogously for the case of black vertices. $\square$

Definition 3.6. A **gauge transformation** on the network N is a transformation $t : (\mathbb{R}^*)^{|E|} \rightarrow (\mathbb{R}^*)^{|E|}$ of the weights of N defined as follows: For each internal vertex v and a Laurent monomial L on the independent variables that define the weights of N , the weights of all edges leaving v are multiplied by L and the weights of all edges entering v are multiplied by L^{-1} .

One uses gauge transformations to decrease the number of parameters of the 6-parametric family in Proposition 3.5 in the following way: For a white vertex v we consider $L = 1/x_v^1$. Then the new coordinates of v are $\bar{x}_v^1 = 1, \bar{x}_v^2 = x_v^1 x_v^2, \bar{x}_v^3 = x_v^1 x_v^3$ satisfying

$$\{\bar{x}_v^2, \bar{x}_v^3\}_v = \alpha \bar{x}_v^2 \bar{x}_v^3, \quad \{\bar{x}_v^1, \bar{x}_v^2\}_v = \{\bar{x}_v^1, \bar{x}_v^3\}_v = 0 \quad (3.6)$$

with

$$\alpha = \alpha_{23} + \alpha_{13} - \alpha_{12} \quad (3.7)$$

Similarly, for each black vertex v after take $L = x_v^1$, we have

$$\{\bar{x}_v^2, \bar{x}_v^3\}_v = \beta \bar{x}_v^2 \bar{x}_v^3, \quad \{\bar{x}_v^1, \bar{x}_v^2\}_v = \{\bar{x}_v^1, \bar{x}_v^3\}_v = 0 \quad (3.8)$$

with

$$\beta = \beta_{23} + \beta_{13} - \beta_{12} \quad (3.9)$$

From now on we consider the 2-parametric family (3.6), (3.8) instead of the parametric family (3.4), (3.5).

Fix an arbitrary disjoint partition $I \cup J = [1, n]$ and let $k = |I|$, $m := n - k = |J|$. Let $\text{Net}_{I,J}$ be the set of all networks in a disc with sources b_i , $i \in I$, sinks b_j , $j \in J$ and edge weights w_e defined in (3.3). For any $i, i' \in I$, $j, j' \in J$ define

$$s_1(i, j, i', j') = \begin{cases} 1 & \text{if } i \prec i' \prec j' \prec j, \\ -1 & \text{if } i' \prec i \prec j \prec j', \\ \frac{1}{2} & \text{if } i = i' \prec j' \prec j \text{ or } i \prec i' \prec j' = j, \\ -\frac{1}{2} & \text{if } i' = i \prec j \prec j' \text{ or } i' \prec i \prec j = j', \\ 0 & \text{otherwise .} \end{cases} \quad (3.10)$$

and

$$s_2(i, j, i', j') = \begin{cases} 1 & \text{if } i' \prec i \prec j' \prec j, \\ -1 & \text{if } i \prec i' \prec j \prec j', \\ \frac{1}{2} & \text{if } i = i' \prec j' \prec j \text{ or } i' \prec i \prec j' = j, \\ -\frac{1}{2} & \text{if } i' = i \prec j \prec j' \text{ or } i \prec i' \prec j = j', \\ 0 & \text{otherwise .} \end{cases} \quad (3.11)$$

where we use $a \prec b$ to denote the cyclical order between a and b in the counterclockwise orientation. Note that s_1 and s_2 are skew-symmetric:

$$s_k(i, j, i', j') = -s_k(i', j', i, j), \quad k = 1, 2.$$

For a picture representation of non-zero values of s_1 and s_2 , see [10].

Define the following 2-parametric bracket on $\text{Mat}_{k,m}$ with respect to the coordinates $M_{pq}, M_{\bar{p}\bar{q}}$, $1 \leq p, \bar{p} \leq k$, $1 \leq q, \bar{q} \leq m$.

$$\{M_{pq}, M_{\bar{p}\bar{q}}\}_{I,J} = (\alpha - \beta)s_1(i_p, j_q, i_{\bar{p}}, j_{\bar{q}})M_{p\bar{q}}M_{\bar{p}q} + (\alpha + \beta)s_2(i_p, j_q, i_{\bar{p}}, j_{\bar{q}})M_{pq}M_{\bar{p}\bar{q}} \quad (3.12)$$

The following proposition can be found in [10].

Proposition 3.7. (1) The bracket $\{\cdot, \cdot\}_{I,J}$ on $\text{Mat}_{k,m}$ in (3.12) is a Poisson bracket.

(2) For any choice of parameters α_{ij}, β_{ij} in (3.4), (3.5) and for any network $N \in \text{Net}_{I,J}$ the map $M_N : ((\mathbb{R}^*)^{|E|}, \{\cdot, \cdot\}_N) \rightarrow (\text{Mat}_{k,m}, \{\cdot, \cdot\}_{I,J})$ is a Poisson map.

Although we do not reproduce a proof of the above proposition, we present an example to verify the assertions.

Example 3.8. Consider a network N with $I = \{1, 2\}$, $J = \{3, 4\}$ with weights w_k , $k = 1, \dots, 4$ as is shown in Figure 3.4, after apply gauge transformations. The non-zero Poisson

Figure 3.4: Network on a disc with $I = \{1, 2\}$

brackets $\{\cdot, \cdot\}_N$ among these weights are:

$$\begin{aligned} \{w_1, w_2\}_N &= \alpha w_1 w_2, & \{w_4, w_3\}_N &= -\alpha w_4 w_3 \\ \{w_3, w_1\}_N &= \beta w_3 w_1, & \{w_2, w_4\}_N &= -\beta w_2 w_4 \end{aligned} \quad (3.13)$$

and its boundary measurement matrix is

$$M_N = \begin{pmatrix} w_2 w_4 & w_1 + w_2 w_3 w_4 \\ w_4 & w_3 w_4 \end{pmatrix}$$

Computing the Poisson bracket $\{\cdot, \cdot\}_N$ in the coordinates M_{ij} we have

$$\begin{aligned} \{M_{11}, M_{12}\}_N &= -\alpha M_{11} M_{12}, & \{M_{11}, M_{21}\}_N &= -\beta M_{11} M_{21}, \\ \{M_{11}, M_{22}\}_N &= -(\alpha + \beta) M_{11} M_{22} & \{M_{12}, M_{21}\}_N &= (\alpha - \beta) M_{11} M_{22}, \\ \{M_{12}, M_{22}\}_N &= -\beta M_{12} M_{22}, & \{M_{21}, M_{22}\}_N &= -\alpha M_{21} M_{22}. \end{aligned}$$

and we can see that it coincides with the Poisson bracket $\{\cdot, \cdot\}_{I,J}$ given by (3.12), i.e., $\{M_{pq}, M_{\bar{p}\bar{q}}\}_{I,J} = \{M_{pq}, M_{\bar{p}\bar{q}}\}_N$. Thus, the map M_N is a Poisson map.

In the particular case, $I = [1, k]$ and $J = [k+1, n]$, we write $\text{Net}_{k,m}$ and $\{\cdot, \cdot\}_{k,m}$ instead of $\text{Net}_{I,J}$ and $\{\cdot, \cdot\}_{I,J}$. The formula (3.12) takes the form

$$\begin{aligned} 2\{M_{ij}, M_{i'j'}\}_{k,m} &= (\alpha - \beta)(\text{sgn}(i' - i) - \text{sgn}(j' - j))M_{ij'}M_{i'j} \\ &\quad + (\alpha + \beta)(\text{sgn}(i' - i) + \text{sgn}(j' - j))M_{ij}M_{i'j'} \end{aligned} \quad (3.14)$$

where M_{ij} is the boundary measurement between b_i and b_{j+k} . For a network $N \in \mathcal{N}_{k,k}$ the Poisson bracket (3.14) written in terms of the entries of A_{ij} of A_N becomes

$$\begin{aligned} 2\{A_{ij}, A_{i'j'}\}_{k,m} &= (\alpha - \beta)(\text{sgn}(i' - i) + \text{sgn}(j' - j))A_{ij'}A_{i'j} \\ &\quad + (\alpha + \beta)(\text{sgn}(i' - i) - \text{sgn}(j' - j))A_{ij}A_{i'j'} \end{aligned} \quad (3.15)$$

Equation (3.15) is the Sklyanin bracket (2.37) on $\text{Mat}_{k,k}$ associated to the R -matrix

$$R_{\alpha,\beta} = (\alpha - \beta)R_0 + (\alpha + \beta)SP_0 \quad (3.16)$$

where

$$S(E_{jj}) = \sum_{i=1}^k \text{sgn}(j-i)E_{ii}, \quad j = 1, \dots, k,$$

and P_0 is the projection onto the diagonal matrices.

The Poisson bracket (3.12) written in terms of u_{pj}^I , which are coordinates functions on the cell $Gr_k^I(n)$ of the Grassmannian $Gr_k(n)$ is:

$$\{u_{pj}^I, u_{p'j'}^I\}_{I,J} = (\alpha - \beta)s_1(i_p, j, i_{p'}, j')u_{pj}^I u_{p'j'}^I + (\alpha + \beta)s_2(i_p, j, i_{p'}, j')u_{pj}^I u_{p'j'}^I \quad (3.17)$$

The next theorem states that the families $\{\cdot, \cdot\}_{I,J}$ on different cells $Gr_k^I(n)$ can be glued together to form the unique 2-parametric family of Poisson brackets on $Gr_k(n)$ that makes all X_N into Poisson maps, see [10].

Theorem 3.9. (i) For any choice of parameters α and β there exists a unique Poisson bracket $\mathcal{P}_{\alpha,\beta}$ on $Gr_k(n)$ such that for any network N with k sources, $n - k$ sinks and weights defined by (3.3), the map $X_N : (\mathbb{R}^*)^{|E|} \rightarrow Gr_k(n)$ is Poisson provided the parameters α_{ij} and β_{ij} defining $\{\cdot, \cdot\}_N$ on $(\mathbb{R}^*)^{|E|}$ satisfy (3.7) and (3.9).

(ii) For any $I \subset [1, n]$, $|I| = k$, and $J = [1, n] \setminus I$, the restriction of $\mathcal{P}_{\alpha,\beta}$ to the cell $Gr_k^I(n)$ coincides with the bracket (3.17) in coordinates u_{pj}^I .

Remark 3.10. The natural right action of GL_n on $Gr_k(n)$ can be interpreted in term of networks via concatenation. To do this one must build an appropriate network on a disc for a generic element of GL_n . Details can be found in [10].

Let $N = (G, w)$ be a network in a disc. The graph G divides the disc into a finite number of connected components called *faces*. A face is *bounded* if its boundary contains only edges of G and *unbounded* otherwise. Given a face f , we define its *face weight* y_f as the Laurent monomial in the edge weights $w_e, e \in E$ given by

$$y_f = \prod_{e \in \partial f} w_e^{\text{sgn } e} \quad (3.18)$$

where $\text{sgn } e$ is 1 if the direction of e is compatible with the counterclockwise orientation of the boundary ∂f and $\text{sgn } e = -1$ otherwise. In particular, the product of face weights over all faces equals 1, since each edge enters the boundaries of exactly two faces in opposite direction. We denote by $\mathcal{F}_N$ the *space of face weights*.

For example, in the network of Figure 3.4 we have 1 bounded face f_0 and its face weight $y_{f_0} = \frac{w_2 w_3 w_4}{w_1}$. The other faces $f_i, i = 1, \dots, 4$ are unbounded with face weights $y_{f_1} = w_1, y_{f_2} = \frac{1}{w_2}, y_{f_3} = \frac{1}{w_3}, y_{f_4} = \frac{1}{w_4}$.

In general, the edge weights can not be recovered from the face weights. Nevertheless, one can restore the path weight w_P of a path P between a source b_i and a sink b_j from face weights, in fact, w_P is a monomial in the face weights. For instance, in our example, the weight w_P of the unique path P from b_2 to b_4 is w_3w_4 which is the $y_{f_0}y_{f_1}y_{f_2}$. More precisely, one can factor the boundary measurement maps $M_N : \mathcal{E}_N \rightarrow \text{Mat}_{k,m}$ and $X_N : \mathcal{E}_N \rightarrow \text{Gr}_k(n)$ through $\mathcal{F}_N$:

$$M_N^{\mathcal{F}} \circ y = M_N, \quad X_N^{\mathcal{F}} \circ y = X_N$$

where $y : \mathcal{E}_N \rightarrow \mathcal{F}_N$ is the map given by (3.18). The 2-parametric family of Poisson brackets on $\mathcal{E}_N$ induced on $\mathcal{F}_N$ by the map y is described via the *directed dual network* $N^* = (G^*, w^*)$ as follows: vertices of G^* are the faces of N , edges of G^* correspond to the edges of N with endpoints of different colors or edges with a boundary endpoint. An edge e^* of G^* corresponding to e is directed in such a way that the white endpoint of e (if it exists) lies to the left of e^* and the black endpoint of e (if it exists) lies to the right of e . The weight of $w^*(e^*)$ equals $\alpha - \beta$ if the endpoints of e are white and black, α if the endpoints of e are white and a boundary vertex and $-\beta$ if the endpoints of e are black and a boundary vertex. Figure 3.5 shows the dual network of the network in Figure 3.4.

Figure 3.5: Dual network

Proposition 3.11 ([10]). *The 2-parametric family $\{\cdot, \cdot\}_{\mathcal{F}_N}$ is given by*

$$\{y_f, y_{f'}\}_{\mathcal{F}_N} = \left(\sum_{e^*: f \rightarrow f'} w^*(e^*) - \sum_{e^*: f' \rightarrow f} w^*(e^*) \right) y_f y_{f'}.$$

3.2 Networks in an annulus

Let $G = (V, E)$ be a directed planar graph drawn inside an annulus with exactly n vertices located on the connected components of the boundary; $n_1 \geq 0$ lying on the outer circle and $n_2 = n - n_1$ on the inner circle, and edge set E . The boundary vertices will be either sources or sinks and they are defined in the same way as in the case of networks on the disc. Also in this case, the internal vertices have degree 3 and are of two types, white or blacks.

A *cut* ρ is an oriented non-selfintersecting curve starting on the inner circle and ending on the outer circle considered up to isotopy relative to fixed endpoints which are distinct of

boundary vertices of G . For an arbitrary oriented curve γ with endpoints not lying on the cut ρ we denote by $\text{ind}(\gamma)$ the algebraic intersection number of γ and ρ . Let $x_1, \dots, x_d$ be independent variables. A network $N = (G, \rho, w)$ is obtained from a graph G equipped with a cut ρ and a map $w : (\mathbb{R}^*)^d \rightarrow (\mathbb{R}^*)^{|E|}$, i.e., a weight function w_e , as in the case of the disc, to each edge $e \in G$. As in the case of networks in a disc the space of edge weights $\mathcal{E}_N$ is the intersection of $w((\mathbb{R}^*)^d)$ with $(\mathbb{R}^*)^{|E|}$.

Let P be a path in N . The weight of P , denoted by w_P , is defined as

$$w_P = w_P(\lambda) = (-1)^{\text{wind}(P)} \lambda^{\text{ind}(P)} \prod_{e \in P} w_e, \quad (3.19)$$

where λ is an auxiliary parameter. If the network N does not have internal vertices of G lying on the cut ρ , the weight (3.19) can be rewrite as

$$w_P = (-1)^{\text{wind}(P)} \prod_{e \in P} \bar{w}_e$$

where $\bar{w}_e = w_e \lambda^{\text{ind}(e)}$ are the modified edge weights.

Remark 3.12. Moving a base point of the cut affects the weights of paths. For example, let $N = (G, \rho, w)$ and $N' = (G, \rho', w)$ be two networks with the same underlying graph, the same weights and suppose that ρ' is obtained from ρ moving a base point in the counterclockwise direction interchanging its position with the first boundary vertex b in the counterclockwise direction with respect to the base point. Then the relation between the weight w_P of a path in N and the weight w'_P in N' is

$$w'_P(\lambda) = ((-1)^{\alpha(P)} \lambda)^\beta(b, P) w_P, \quad (3.20)$$

where

$$\beta(b, P) = \begin{cases} 1 & \text{if } b \text{ is the sink of } P, \\ -1 & \text{if } b \text{ is the source of } P, \\ 0 & \text{otherwise.} \end{cases}$$

and $\alpha(P)$ equals 0 if the endpoints of P lie on the same circle and 1 otherwise.

Given a network N as above, the boundary vertices lying in the outer circle are labelled by $b_1, \dots, b_{n_1}$ in the counterclockwise orientation starting from the first vertex that follows after the base point of the cut and the boundary vertices lying on the inner circle are labelled by $b_{n_1+1}, \dots, b_n$ in the clockwise orientation starting from the first vertex that follows after the base point of the cut. The number of sources lying on the outer (resp. inner) circle is denoted by k_1 (resp. k_2), its corresponding set of indices is denoted by $I_1 \subset [1, n]$ (resp. $I_2 \subset [n_1 + 1, n]$) and the set of the remaining indices $m_1 := n_1 - k_1$ (resp. $m_2 := n_2 - k_2$) is denoted by J_1 (resp. J_2). Denote $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$, their cardinalities of I and J are denoted by $k = k_1 + k_2$ and $m = m_1 + m_2$, respectively.

Given a source $b_i, i \in I$ and a sink $b_j, j \in J$ of N , the *boundary measurement* $M(i, j)$ is defined as the sum of the weights of all paths starting in b_i and ending at b_j , i.e., $M(i, j)$

is given by (3.2) using (3.19). The following result is the analogous of Proposition 3.2 for networks in an annulus.

Proposition 3.13. *If N is a network in an annulus then each boundary measurement in N is a rational function in the parameter λ and the weights $w_e, e \in E$.*

Let $\text{Rat}_{k,m}$ be the space of real rational $k \times m$ matrix functions in one variable. For each network N , one can define a *boundary measurement map* $M_N : \mathcal{E}_N \rightarrow \text{Rat}_{k,m}$ through the $k \times m$ boundary measurement matrix M_N : If $I = \{i_1 < i_2 < \dots < i_k\}$ and $J = \{j_1 < j_2 < \dots < j_m\}$ then $M_N = (M_{pq})$, $p \in [1, k]$, $q \in [1, m]$ where $M_{pq} = M(i_p, j_q)$.

Remark 3.14. The boundary measurement matrix has a block structure

$$M_N = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

where M_1 is a $k_1 \times m_1$ matrix and M_4 is a $k_2 \times m_2$ matrix. Moving a base point of the cut, w_P will change as described in Remark 3.12, and consequently M_N will change in the following way. First, let

$$\Lambda_{\pm} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \pm\lambda^{-1} & 0 & \cdots & 0 \end{pmatrix}$$

then interchanging the base point of the cut with b_1 implies the transformation

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \mapsto \begin{pmatrix} \Lambda_+ M_1 & \Lambda_- M_2 \\ M_3 & M_4 \end{pmatrix},$$

if b_1 is a source, and

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \mapsto \begin{pmatrix} M_1 \Lambda_+^{-1} & M_2 \\ M_3 \Lambda_-^{-1} & M_4 \end{pmatrix},$$

if b_1 is a sink. On the other hand, interchanging the base point of the cut with b_n yields

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \mapsto \begin{pmatrix} M_1 & M_2 \\ \Lambda_+^t M_3 & \Lambda_-^t M_4 \end{pmatrix},$$

if b_n is a source and

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \mapsto \begin{pmatrix} M_1 & M_2 (\Lambda_+^{-1})^t \\ M_3 & M_4 (\Lambda_-^{-1})^t \end{pmatrix},$$

if b_n is a sink. The sizes of $\Lambda_{\pm}$ are $k_1 \times k_1$ in the first case, $m_1 \times m_1$ in the second case, $k_2 \times k_2$ in the third case and $m_2 \times m_2$ in the fourth case.

Similarly to the case of networks in a disc, we can concatenate networks in an annulus. The most interesting and simple case is when sources and sinks are separated, that is, when all sources lie on the outer circle, and all sinks lie on the inner circle. For two networks of this kind, one with k sources and m sinks and the other with m sources and l sinks, we perform the concatenation by gluing the sinks of the former to the sources of the latter in such a

way that the base point of the cut are identified, and the i th sink of the former network is identified with the $(m + 1 - i)$ th source of the latter. Lastly, we erase the common boundary and we do identification of the edges being its weight the product of the weights of the edges identified as in the case of the disc.

Let $N = (G, \rho, w)$ be a network in an annulus. The space of the edge weights $\mathcal{E}_N$ can be identified with $\text{Hom}(\mathbb{Z}^E, \mathbb{R}^*)$ via $w(\sum n_i e_i) = \prod w_{e_i}^{n_i}$, where $\mathbb{Z}^E$ is the $\mathbb{Z}$ -module generated by the edges of G . Consider the $\mathbb{Z}$ -module $\mathbb{Z}^V$ (resp. $\mathbb{Z}$ -module $\mathbb{Z}^{V_0}$) generated by all the vertices (resp. all the internal vertices) of G . An element $\phi \in \text{Hom}(\mathbb{Z}^V, \mathbb{R}^*)$ acts on $\mathcal{E}_N$ in the following way: If $e = (u, v)$ then

$$w_e \mapsto w_e \frac{\phi(v)}{\phi(u)}.$$

Then, the weight of a path between boundary vertices b_i and b_j is multiplied by $\phi(b_j)/\phi(b_i)$. Therefore the gauge group $\mathcal{G}$, which preserves the weights of all paths between boundary vertices, is the subset $\phi \in \text{Hom}(\mathbb{Z}^V, \mathbb{R}^*)$ with $\phi(b) = 1$ for any boundary vertex b and it is identified with $\text{Hom}(\mathbb{Z}^{V_0}, \mathbb{R}^*)$. Thus, the boundary measurement map $M_N : \mathcal{E}_N \rightarrow \text{Rat}_{k,m}$ factors through the quotient $\mathcal{F}_N = \mathcal{E}_N / \mathcal{G}$, i.e., $M_N = M_N^{\mathcal{F}} \circ y$ where $y : \mathcal{E}_N \rightarrow \mathcal{F}_N$ is the projection map and $M_N^{\mathcal{F}}$ is a map $\mathcal{F}_N \rightarrow \text{Rat}_{k,m}$. The space $\mathcal{F}_N$ is called the *space of face and trail weights*. The reason for this name stems from the fact that one can identify $\mathcal{F}_N$ with $H^1(G, \partial G; \mathbb{R}^*) \simeq \mathcal{F}_N^f \oplus \mathcal{F}_N^t$, where the space $\mathcal{F}_N^f$ describes the faces of G as in the case where G is embedded in a disc. The space $\mathcal{F}_N^t$ describes the existence or not of a trail connecting the components of the annulus. A *trail* is a sequence $(v_1, \dots, v_{k+1})$ of vertices such that either (v_i, v_{i+1}) or (v_{i+1}, v_i) is an edge in G for all $i \in [1, k]$ and the endpoints v_1, v_{k+1} are boundary vertices of G . The *trail weight* y_t is defined as

$$y_t = \prod_i^k w(v_i, v_{i+1})$$

where

$$w(v_i, v_{i+1}) = \begin{cases} w_e & \text{if } e = (v_i, v_{i+1}) \in E \\ w_e^{-1} & \text{if } e = (v_{i+1}, v_i) \in E. \end{cases}$$

The functions on $\mathcal{F}_N^t$ are generated by the weight of any connecting trail. In the case of a network in a disc $\mathcal{F}_N^t = 0$, and hence $\mathcal{F}_N$ and $\mathcal{F}_N^f$ coincide.

Fix an arbitrary pair of partitions $I_1 \cup J_1 = [1, n_1]$, $I_1 \cap J_1 = \emptyset$, $I_2 \cup J_2 = [n_1 + 1, n]$, $I_2 \cap J_2 = \emptyset$, and denote $k = |I_1| + |I_2|$ and $m := n - k = |J_1| + |J_2|$. Let $\text{Net}_{I_1, J_1, I_2, J_2}$ stand for the set of all networks with the sources $b_i, i \in I_1$ and sinks $b_j, j \in J_1$, on the outer circle, sources $b_i, i \in I_2$ and sinks $b_j, j \in J_2$ on the inner circle, and edge weights w_e defined by (3.3). We equip the space of edge weights $\mathcal{E}_N = (\mathbb{R}^*)^{|E|}$ with the Poisson bracket $\{\cdot, \cdot\}_N$ obtained as the pushforward of the 6-parametric family $\{\cdot, \cdot\}_{\mathcal{R}}$ discussed in Proposition 3.5. Using the factorization $M_N = M_N^{\mathcal{F}} \circ y$ we have the following proposition.

Proposition 3.15. *The 6-parametric family $\{\cdot, \cdot\}_N$ induces a 2-parametric family of Poisson brackets $\{\cdot, \cdot\}_{\mathcal{F}_N}$ on $\mathcal{F}_N$ with parameters α and β given in (3.7) and (3.9).*

Let $N = (G, w)$ be a network in an annulus and let us fix a boundary vertex b_{i_p} . Assume that the only neighbor of b_{i_p} is a black inner vertex u such that $e_0 = (b_{i_p}, u) \in E$. Denote by u_+ the unique vertex in G such that $e_+ = (u, u_+) \in E$ and by u_- the unique vertex in G such that $e_- = (u_-, u) \in E$. Define

$$\widehat{\mathcal{R}} = (\mathcal{R} \oplus (\mathbb{R}^*)_{i_u} \oplus (\mathbb{R}^*)_{j_u}) - \left((R^*)^3_u \oplus \mathbb{R}^*_{i_p} \right)$$

which corresponds to a network $\widehat{N}$ obtained from the graph G splitting the black vertex u , i.e., by deleting b_{i_p} and the edge (b_{i_p}, u) from G , and adding a new source i_u and a new sink j_u so that either $i_p - 1 \prec i_u \prec j_u \prec i_p + 1$ or $i_p - 1 \prec j_u \prec i_u \prec i_p + 1$ and replacing e_+ and e_- by new edges $\widehat{e}_+ = (b_{i_u}, u_+)$ and $\widehat{e}_- = (u_-, b_{j_u})$, respectively.

Lemma 3.16. *In the above conditions and under the assumption that the cut ρ of N does not intersect the edge e_0 , boundary measurements in the networks N and $\widehat{N}$ are related by*

$$\begin{aligned} M(i_p, j) &= \frac{w_{e_0} w_{e_+} \widehat{M}(i_u, j)}{1 + w_{e_-} w_{e_+} \widehat{M}(i_u, j_u)}, \\ M(i_{\bar{p}}, j) &= \widehat{M}(i_{\bar{p}}, j) \pm \frac{w_{e_-} w_{e_+} \widehat{M}(i_{\bar{p}}, j_u) \widehat{M}(i_u, j)}{1 + w_{e_-} w_{e_+} \widehat{M}(i_u, j_u)}, \quad p \neq \bar{p}; \end{aligned}$$

where in the second formula above, the sign $+$ corresponds to the cases

$$i_p - 1 \prec j_u \prec i_u \prec i_p + 1 \preceq j \prec i_{\bar{p}} \quad \text{or} \quad i_{\bar{p}} \prec j \preceq i_p - 1 \prec i_u \prec j_u \prec i_p + 1,$$

and the sign $-$ corresponds to the cases

$$i_p - 1 \prec i_u \prec j_u \prec i_p + 1 \preceq j \prec i_{\bar{p}} \quad \text{or} \quad i_{\bar{p}} \prec j \preceq i_p - 1 \prec j_u \prec i_u \prec i_p + 1$$

Proof. To prove the first formula, one classifies the paths from b_{i_p} to b_j according to the number of times they traverse e_+ . The total weight of paths passing through e_+ only once is given by $w_{e_0} w_{e_+} \widehat{M}(i_u, j)$. One sees that the total contribution of all paths traversing e_+ exactly twice to $M(i_p, j)$ equals

$$-w_{e_0} w_{e_+} \widehat{M}(i_u, j) w_{e_-} w_{e_+} \widehat{M}(i_u, j_u).$$

In general, the total contribution of all paths traversing e_+ exactly $s + 1$ times to $M(i_p, j)$ equals

$$(-1)^s w_{e_0} w_{e_+} \widehat{M}(i_u, j) (w_{e_-} w_{e_+} \widehat{M}(i_u, j_u))^s.$$

Therefore, one has

$$M(i_p, j) = \frac{w_{e_0} w_{e_+} \widehat{M}(i_u, j)}{1 + w_{e_-} w_{e_+} \widehat{M}(i_u, j_u)}.$$

For the second formula, the paths from $b_{i_{\bar{p}}}$ to b_j in N are also classified according to the number of times they traverse the edge e_+ . The total contribution of the paths not traversing e_+ at all to $M(i_{\bar{p}}, j)$ equals to $\widehat{M}(i_{\bar{p}}, j)$. Each path P traversing e_+ exactly once can be

decomposed as $P = (P_1, e_-, e_+, P_2)$ so that $\widehat{P}_1 = (P_1, \widehat{e}_-)$ is a path from $b_{i_{\bar{p}}}$ to b_{j_u} in $\widehat{N}$ and $\widehat{P}_2 = (\widehat{e}_+, P_2)$ is a path from b_{i_u} to b_j in $\widehat{N}$. Assuming the case

$$i_{\bar{p}} \prec j \preceq i_p - 1 \prec j_u \prec i_u \prec i_p + 1$$

Then one sees that $w_P = -w_{P_1}w_{e_-}w_{e_+}w_{P_2}$ and the contribution of all such paths to $M(i_{\bar{p}}, j)$ equals $-w_{e_-}w_{e_+}\widehat{M}(i_{\bar{p}}, j_u)\widehat{M}(i_u, j)$. Each additional traversing of the edge e_+ results in multiplying the last expression by $-w_{e_-}w_{e_+}\widehat{M}(i_u, j_u)$. Summing up one has the second formula with the $-$ sign. The other cases are treated analogously. $\square$

One can obtain other formulas for the boundary measurement of $\widehat{N}$ if we split a white vertex, i.e., if one assumes that u is white vertex.

Denote by $\{\cdot, \cdot\}_N^1$ the member of the 2-parametric family (3.6), (3.8) corresponding to $\alpha = -\beta = 1$ and define $\sigma_1(i, j, i', j') = \text{sgn}(i' - i) - \text{sgn}(j' - j)$.

Proposition 3.17. *The Poisson brackets $\{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^1$ satisfy the following relations:*

(i) *Let $i_p, i_{\bar{p}} \in [1, n_1]$ and $1 \leq \max\{i_p, i_{\bar{p}}\} < j_{\bar{q}} < j_q \leq n$, then*

$$\{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^1 = \sigma_1(i_p, j_q, i_{\bar{p}}, j_{\bar{q}})M_{p\bar{q}}(s)M_{\bar{p}q}(t) - \frac{2}{t-s}\Phi_{pq}^{\bar{p}\bar{q}}(t, s) \quad (3.21)$$

where

$$\Phi_{pq}^{\bar{p}\bar{q}}(t, s) = \begin{cases} (M_{p\bar{q}}(t) - M_{p\bar{q}}(s))(sM_{\bar{p}q}(t) - tM_{\bar{p}q}(s)) & \text{if } j_{\bar{q}} < j_q \leq n_1, \\ sM_{\bar{p}q}(t)(M_{p\bar{q}}(t) - M_{p\bar{q}}(s)) & \text{if } j_{\bar{q}} \leq n_1 < j_q, \\ s(M_{p\bar{q}}(t)M_{\bar{p}q}(s) - M_{p\bar{q}}(s)M_{\bar{p}q}(t)) & \text{if } n_1 < j_{\bar{q}} < j_q. \end{cases}$$

(ii) *Let $j_q, j_{\bar{q}} \in [n_1 + 1, n]$ and $1 \leq i_p < i_{\bar{p}} < \min\{j_q, j_{\bar{q}}\} \leq n$, then*

$$\{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^1 = \sigma_1(i_p, j_q, i_{\bar{p}}, j_{\bar{q}})M_{p\bar{q}}(t)M_{\bar{p}q}(s) - \frac{2}{t-s}\Psi_{pq}^{\bar{p}\bar{q}}(t, s), \quad (3.22)$$

where

$$\Psi_{pq}^{\bar{p}\bar{q}}(t, s) = \begin{cases} t(M_{p\bar{q}}(t)M_{\bar{p}q}(s) - M_{p\bar{q}}(s)M_{\bar{p}q}(t)) & \text{if } i_p < i_{\bar{p}} \leq n_1, \\ -tM_{p\bar{q}}(t)(M_{p\bar{q}}(t) - M_{p\bar{q}}(s)) & \text{if } i_p \leq n_1 < i_{\bar{p}}, \\ -(tM_{p\bar{q}}(t) - sM_{p\bar{q}}(s))(M_{\bar{p}q}(t) - M_{\bar{p}q}(s)) & \text{if } n_1 < i_p < i_{\bar{p}}. \end{cases}$$

(iii) *Let $1 \leq i_p = i_{\bar{p}} < j_q = j_{\bar{q}} \leq n$, then*

$$\begin{aligned} & \{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^1 \\ &= \begin{cases} -\frac{2}{t-s}(M_{pq}(t) - M_{pq}(s))(sM_{pq}(t) - tM_{pq}(s)) & \text{if } i_p < j_q \leq n_1, \\ 0 & \text{if } i_p \leq n_1 < j_q. \end{cases} \end{aligned} \quad (3.23)$$

(iv) *Let $1 \leq i_p < \min\{i_{\bar{p}}, j_q, j_{\bar{q}}\}$, then*

$$\begin{aligned} & \{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^1 \\ &= \begin{cases} \frac{2t}{t-s}(M_{p\bar{q}}(t) - M_{p\bar{q}}(s))(M_{\bar{p}q}(t) - M_{\bar{p}q}(s)) & \text{if } j_{\bar{q}} < i_{\bar{p}} < j_q \leq n_1, \\ -\frac{2t}{t-s}(M_{p\bar{q}}(t)M_{\bar{p}q}(t) - M_{p\bar{q}}(s)M_{\bar{p}q}(s)) & \text{if } j_q \leq n_1 < i_{\bar{p}} < j_{\bar{q}}, \\ 0 & \text{if } j_{\bar{q}} \leq n_1 < i_{\bar{p}} < j_q. \end{cases} \end{aligned} \quad (3.24)$$

Proof. Relations (3.21)-(3.24) allow to compute $\{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^1$ for any $p, \bar{p} \in [1, k]$ and $q, \bar{q} \in [1, m]$. To achieve this one uses the following techniques: moving a base point of the cut, reversing the direction of the cut and reversing the orientation of boundary circles.

For example, let $1 \leq j_q < i_p < i_{\bar{p}} < j_{\bar{q}} \leq n_1$ which is not covered explicitly by relations (3.21)-(3.24). Consider the network N' obtained from N by moving the base point on the outer circle counterclockwise and interchanging it with j_q . In N' one has $1 \leq i_{p'} < i_{\bar{p}'} < j_{\bar{q}'} < j'_q = n_1$ then using the first case of (3.21) yields

$$\begin{aligned} \{M_{p'q'}(t), M_{\bar{p}'\bar{q}'}(s)\}_{N'}^1 &= 2M_{p'\bar{q}'}M_{\bar{p}'q'}(t) \\ &\quad - \frac{2}{t-s} (M_{p'q'}(t) - M_{p'q'}(s)) (sM_{\bar{p}'q'}(t) - tM_{\bar{p}'q'}(s)). \end{aligned}$$

Equation (3.20) implies that

$$M_{p'q'}(t) = tM_{pq}(t), \quad M_{\bar{p}'\bar{q}'}(t) = M_{\bar{p}\bar{q}}, \quad M_{p'q'}(t) = M_{p\bar{q}}(t), \quad M_{\bar{p}'q'}(t) = tM_{\bar{p}q}(t).$$

Finally, $\{\cdot, \cdot\}_N = \{\cdot, \cdot\}_{N'}$ for any pair of edge weights, so one obtains

$$\{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^1 = 2M_{p\bar{q}}(s)M_{\bar{p}q}(t) - \frac{2s}{t-s}(M_{p\bar{q}}(t) - M_{p\bar{q}}(s))(M_{\bar{p}q}(t) - M_{\bar{p}q}(s)). \quad (3.25)$$

Likewise, for the remaining two techniques one exhausts all possible cases for $\{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^1$. Formulas (3.21)-(3.24) are proved by induction in the number of internal vertices in the network N , after splitting the network. If we split a black vertex one uses the Lemma 3.16. $\square$

Denote by $\{\cdot, \cdot\}_N^2$ the member of the 2-parametric family (3.6), (3.8) corresponding to $\alpha = \beta = 1$ and define $\sigma_2(i, j, i', j') = \text{sgn}(i' - i) + \text{sgn}(j' - j)$.

Proposition 3.18. *The Poisson brackets $\{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^2$ satisfy the following relations:*

(i) *Let $1 \leq \max\{i_p, i_{\bar{p}}\} < j_{\bar{q}} < j_q \leq n$, then*

$$\{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^2 = \sigma_2(i_p, j_q, i_{\bar{p}}, j_{\bar{q}})M_{pq}(t)M_{\bar{p}\bar{q}}(s) - 2\Gamma_{pq}^{\bar{p}\bar{q}}(t, s) \quad (3.26)$$

where

$$\Gamma_{pq}^{\bar{p}\bar{q}}(t, s) = \begin{cases} 0 & \text{if } j_q \leq n_1, \\ sM_{pq}(t)M'_{\bar{p}\bar{q}}(s) & \text{if } j_{\bar{q}} \leq n_1 < j_q, \\ tM'_{pq}(t)M_{\bar{p}\bar{q}}(s) - sM_{pq}(t)M'_{\bar{p}\bar{q}}(s) & \text{if } \max\{i_p, i_{\bar{p}}\} \leq n_1 < j_{\bar{q}}, \\ tM'_{pq}(t)M_{\bar{p}\bar{q}}(s) & \text{if } i_p \leq n_1 < i_{\bar{p}}. \end{cases}$$

and $M'_{pq}, M'_{\bar{p}\bar{q}}$ are derivatives of M_{pq} and $M_{\bar{p}\bar{q}}$.

(ii) *Let $1 \leq i_p < j_{\bar{q}} < i_{\bar{p}} < j_q \leq n$ and either $j_{\bar{q}} \leq n_1 < i_{\bar{q}}$ or $j_q \leq n_1$, then*

$$\{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^2 = 0 \quad (3.27)$$

(iii) Let $1 \leq i_p < j_q < i_{\bar{p}} < j_{\bar{q}} \leq n$ and either $j_q \leq n_1 < i_{\bar{p}}$ or $j_{\bar{q}} \leq n_1$, then

$$\{M_{pq}(t), M_{\bar{p}\bar{q}}(s)\}_N^2 = 0 \quad (3.28)$$

One says that $F \in \text{Rat}_{k,m}$ is represented by a network N if F belongs to the image of the boundary measurement map M_N . The detailed proof of the next proposition is in [12].

Proposition 3.19. *For any $F \in \text{Rat}_{k,m}$ there exists a network $N \in \text{Net}_{I_1, J_1, I_2, J_2}$ such that F is represented by N .*

From Propositions 3.17, 3.18 and 3.19, one has the next result.

Proposition 3.20. *On $\text{Rat}_{k,m}$ there exists a 2-parametric family of Poisson brackets, denoted by $\{\cdot, \cdot\}_{I_1, J_1, I_2, J_2}$, with the following property: for any choice of the parameters α_{ij}, β_{ij} in (3.4), (3.5) this family contains a unique Poisson bracket such that for every network $N \in \text{Net}_{I_1, J_1, I_2, J_2}$ the map $M_N : (\mathbb{R}^*)^{|E|} \rightarrow \text{Rat}_{k,m}$ is Poisson.*

For a network $N \in \text{Net}_{[1,k], \emptyset, \emptyset, [k+1, 2k]}$ with k sources and k sinks the boundary measurement map takes $\mathcal{E}_N$ to $\text{Rat}_{k,k}$. Taking $A_N = M_N W_0$ and for $p < \bar{p}$ and $q < \bar{q}$, the Poisson bracket $\{\cdot, \cdot\}_1^N$ between the entries $a_{pq}(t)$ and $a_{\bar{p}\bar{q}}(s)$ takes the form:

$$\{a_{pq}(t), a_{\bar{p}\bar{q}}(s)\} = 2 \frac{ta_{p\bar{q}}(s)a_{\bar{p}q}(t) - sa_{p\bar{q}}(t)a_{\bar{p}q}(s)}{t-s}. \quad (3.29)$$

$$\{a_{p\bar{q}}(t), a_{\bar{p}q}(s)\} = 2t \frac{a_{pq}(s)a_{\bar{p}\bar{q}}(t) - a_{pq}(t)a_{\bar{p}\bar{q}}(s)}{t-s}. \quad (3.30)$$

$$\{a_{pq}(t), a_{p\bar{q}}(s)\} = \frac{(t+s)a_{p\bar{q}}(s)a_{pq}(t) - 2sa_{p\bar{q}}(t)a_{pq}(s)}{t-s}. \quad (3.31)$$

$$\{a_{pq}(t), a_{\bar{p}\bar{q}}(s)\} = \frac{2ta_{pq}(s)a_{\bar{p}q}(t) - (t+s)a_{pq}(t)a_{\bar{p}\bar{q}}(s)}{t-s}. \quad (3.32)$$

More exactly (3.29) and (3.31) are equivalent to the third case of (3.21), while (3.30) and (3.32) are equivalent to the first case of (3.22). Expressions (3.29)-(3.32) are equivalent to the following Poisson bracket on $\text{Rat}_{k,k}$

$$\{A(t) \otimes A(s)\} = [A(s) \otimes A(t), r(t, s)], \quad (3.33)$$

where $r(t, s)$ is the trigonometric r -matrix given by

$$r(\lambda, \mu) = \frac{\lambda + \mu}{\lambda - \mu} \sum_{i=1}^n E_{ii} \otimes E_{ii} + \frac{2}{\lambda - \mu} \sum_{i < j} (\lambda E_{ij} \otimes E_{ji} + \mu E_{ji} \otimes E_{ij})$$

Summarizing, one has

Proposition 3.21. *For any network $N \in \text{Net}_{[1,k], \emptyset, \emptyset, [k+1, 2k]}$ and any choice of parameters α_{ij}, β_{ij} in (3.4), (3.5) such that $\alpha = -\beta = 1$ in (3.7), (3.9), the map $A_N : (\mathbb{R}^*)^{|E|} \rightarrow \text{Rat}_{k,k}$ is Poisson with respect to the Poisson bracket (3.33).*

Coxeter-Toda lattices

This chapter is the main part of the dissertation. We provide an explicit Poisson bracket to assert the bi-Hamiltonian nature of Coxeter-Toda lattices. Coxeter-Toda lattices are Hamiltonian systems induced on $G^{u,v}/H$ by the restriction of Toda flows on GL_n , where $G^{u,v}$ are double Bruhat cells and $u, v \in S_n$ are Coxeter elements. First, we review briefly the main features of particular Coxeter-Toda lattices, namely, the Toda lattice (relativistic and non-relativistic), presenting the Poisson brackets which define their bi-Hamiltonian structure. Then we present the Coxeter-Toda lattices worked out in detail in [11]. Factorization parameters of a Coxeter double Bruhat cell $G^{u,v}$ induce a coordinate system on $G^{u,v}/H$. In such coordinate system of $G^{u,v}/H$ there is a quadratic Poisson bracket which explains the Hamiltonian nature of Coxeter-Toda lattices, see equation (4.29), induced by the standard Poisson-Lie bracket (2.41) on GL_n . Finally, on the mentioned coordinate system of $G^{u,v}/H$ we present a new non-local Poisson bracket which is compatible with the Poisson bracket (4.29).

4.1 The Toda lattice

The finite Toda lattice is a system of $n \geq 3$ particles on the line with exponential interaction of nearest neighbours, see equation (4.1). There are two kinds of Toda lattice: non-periodic or open, and periodic. In the non-periodic lattice of n particles on the line, the last particle does not interact with the first one while in the periodic lattice the last and first particles interact as other nearest neighbours.

4.1.1 The non-relativistic Toda lattice

The equations of motion of the *finite Toda lattice* are

$$\ddot{q}_k = e^{q_{k+1}-q_k} - e^{q_k-q_{k-1}}, \quad k = 1, \dots, n \quad (4.1)$$

with two types of boundary conditions: *non-periodic* and *periodic*. The non-periodic boundary conditions of (4.1) read: $q_0 = \infty, q_{n+1} = -\infty$, while the periodic condition are obtained taking all the subscripts mod n , i.e., imposing $q_0 = q_n, q_{n+1} = q_1$. After the Flaschka change of variables,

$$b_k = \dot{q}_k, \quad a_k = e^{q_{k+1}-q_k},$$

the equations of motion (4.1) take the form

$$\dot{b}_k = a_k - a_{k-1}, \quad \dot{a}_k = a_k(b_{k+1} - b_k), \quad k = 1, \dots, n \quad (4.2)$$

with non-periodic condition: $a_0 = a_n = 0$, or periodic condition: $a_0 = a_n, b_{n+1} = b_1$.

Denote by $\mathcal{T}$ the phase space $\mathbb{R}^{2n}(a_1, \dots, a_n, b_1, \dots, b_n)$ of the periodic Toda lattice (4.2). In the non-periodic boundary conditions the phase space is $\mathcal{T}_0 = \{(a, b) \in \mathcal{T} | a_n = 0\}$. The next statement asserts that the finite Toda lattice possesses at least two Hamiltonian forms, i.e., is a bi-Hamiltonian system.

Proposition 4.1. (i) *The relations*

$$\{b_k, a_k\}_1 = -a_k, \quad \{a_k, b_{k+1}\}_1 = -a_k \quad (4.3)$$

define a Poisson bracket on $\mathcal{T}$. $\mathcal{T}_0$ is a Poisson submanifold of $\mathcal{T}$. The Toda system (4.2) is a Hamiltonian system on $(\mathcal{T}, \{\cdot, \cdot\}_1)$, on $(\mathcal{T}_0, \{\cdot, \cdot\}_1)$, with the Hamiltonian function

$$H_2(a, b) = \frac{1}{2} \sum_{k=1}^n b_k^2 + \sum_{k=1}^n a_k$$

(ii) *The relations*

$$\begin{aligned} \{b_k, a_k\}_2 &= -b_k a_k, & \{a_k, b_{k+1}\}_2 &= -a_k b_{k+1} \\ \{b_k, b_{k+1}\}_2 &= -a_k, & \{a_k, a_{k+1}\}_2 &= -a_k a_{k+1} \end{aligned} \quad (4.4)$$

define a Poisson bracket on $\mathcal{T}$ compatible with $\{\cdot, \cdot\}_1$. $\mathcal{T}_0$ is a Poisson submanifold of $\mathcal{T}$. The system (4.2) is a Hamiltonian system on $(\mathcal{T}, \{\cdot, \cdot\}_2)$, on $(\mathcal{T}_0, \{\cdot, \cdot\}_2)$, with the Hamiltonian function $H_1 = (a, b) = \sum_{k=1}^n b_k$.

Let

$$T(a, b, \lambda) = \lambda^{-1} \sum_{k=1}^n a_k E_{k,k+1} + \sum_{k=1}^n b_k E_{kk} + \lambda \sum_{k=1}^n E_{k+1,k},$$

where, in the periodic case, we set $E_{n+1,n} = E_{1,n}$, $E_{n,n+1} = E_{n,1}$ while in the non-periodic case we set $E_{n+1,n} = E_{n,n+1} = 0$ and we always put $\lambda = 1$. T is called the “big” Lax matrix of the Toda lattice.

Proposition 4.2. *The equations of motion (4.2) admit the following Lax representations with spectral parameter*

(1)

$$\dot{T} = [T, B] = -[T, A] \quad (4.5)$$

with

$$B(a, b, \lambda) = \sum_{k=1}^n b_k E_{kk} + \lambda \sum_{k=1}^n E_{k+1,k}, \quad A(a, b, \lambda) = \lambda^{-1} \sum_{k=1}^n a_k E_{k,k+1}.$$

(2)

$$\dot{L}_k = M_k L_k - L_k M_{k-1} \quad (4.6)$$

where

$$L_k = \begin{pmatrix} b_k + \lambda & a_{k-1} \\ -1 & 0 \end{pmatrix}, \quad M_k = \begin{pmatrix} -\lambda & -a_k \\ 1 & b_k \end{pmatrix}$$

The matrix $T_n = L_n L_{n-1} \cdots L_1$, called the **monodromy matrix** of (4.2), satisfies

$$T_n = M_n T_n - T_n M_0$$

In the periodic case one has $M_n = M_0$, so that the monodromy matrix remains isospectral.

Remark 4.3. A well known discretization of the famous KdV equation, see (B.7), is the Volterra lattice. This is a dynamical system defined by the system of ODE's:

$$\frac{dr_i}{dt} = r_i(r_{i+1} - r_{i-1}), \quad i = 1, \dots, 2n. \quad (4.7)$$

The non-periodic Toda lattice is related to the Volterra lattice system via the Miura map

$$\begin{aligned} a_k &= r_{2k-1} r_{2k}, & k &= 1, \dots, n-1, \\ b_k &= r_{2k-1} + r_{2k-2}, & k &= 1, \dots, n \end{aligned} \quad (4.8)$$

The pullback of the Poisson bracket (4.4) via the Miura map takes the form

$$\{r_i, r_{i+1}\} = -r_i r_{i+1}. \quad (4.9)$$

4.1.2 The relativistic Toda Lattice

Another dynamical system we will consider is the so called relativistic Toda lattice. The equations of motion of the relativistic Toda lattice with coupling constant $g^2 \in \mathbb{R}$ are

$$\ddot{q}_k = g^2 \dot{q}_k \left[\dot{q}_{k+1} \frac{e^{q_{k+1}-q_k}}{1 + g^2 e^{q_{k+1}-q_k}} - \dot{q}_{k-1} \frac{e^{q_k-q_{k-1}}}{1 + g^2 e^{q_k-q_{k-1}}} \right], \quad 1 \leq k \leq n. \quad (4.10)$$

Together with the above equations, one will consider two kind of boundary conditions; *non-periodic*, $q_0 = \infty$ and $q_{n+1} = -\infty$ or *periodic*, $q_0 = q_n$ and $q_{n+1} = q_1$. Setting $\dot{Q}_n = \dot{q}_n + 1/g$

and letting $g \rightarrow 0$ the system (4.10) reduces to the equation of motion of the nonrelativistic Toda lattice

$$\ddot{Q}_n = e^{Q_{n+1}-Q_n} - e^{Q_n-Q_{n-1}}, \quad k = 1, \dots, n.$$

The equations (4.10) can be put in a Hamiltonian form. Consider the variables p_k canonically conjugated to q_k :

$$e^{p_k} = \frac{\dot{q}_k}{1 + g^2 e^{q_{k+1}-q_k}}$$

which leads to the Hamiltonian system

$$\begin{aligned} \dot{q}_k &= e^{p_k} (1 + g^2 e^{q_{k+1}-q_k}) \\ \dot{p}_k &= g^2 e^{q_{k+1}-q_k+p_k} - g^2 e^{q_k-q_{k-1}+p_{k-1}} \end{aligned}$$

with the Hamiltonian function $H_1 = \sum_{k=1}^n e^{p_k} (1 + g^2 e^{q_{k+1}-q_k})$. If

$$x_k = e^{p_k}, \quad y_k = g^2 e^{q_{k+1}-q_k+p_k},$$

then

$$\{y_k, x_k\}_2 = y_k x_k, \quad \{y_k, x_{k+1}\}_2 = -y_k x_{k+1}, \quad \{y_k, y_{k+1}\}_2 = -y_k y_{k+1}. \quad (4.11)$$

Using this new set of variables, one has

$$H_1 = \sum_{k=1}^n (y_k + x_k), \quad (4.12)$$

and the corresponding Hamiltonian equations become:

$$\dot{x}_k = x_k (y_k - y_{k-1}), \quad \dot{y}_k = y_k (x_{k+1} - x_k + y_{k+1} - y_{k-1}). \quad (4.13)$$

Now consider the $n \times n$ matrices depending on y_k, x_k ($k = 1, \dots, n$) and on the spectral parameter λ ,

$$U = \sum_{k=1}^n x_k E_{kk} + \lambda \sum_{k=1}^n E_{k+1,k}, \quad V = I - \lambda^{-1} \sum_{k=1}^n y_k E_{k,k+1}. \quad (4.14)$$

In the periodic case we set $E_{n+1,n} = E_{1,n}$, $E_{n,n+1} = E_{n,1}$; in the non-periodic case we set $\lambda = 1$ and $E_{n+1,n} = E_{n,n+1} = 0$.

Theorem 4.4 ([33]). *The equations of motion (4.13) can be written in the following matrix differential equations:*

$$\dot{U} = UB - AU, \quad \dot{V} = VB - AV$$

where

$$A = \sum_{k=1}^n (x_k + y_{k-1}) E_{kk} + \lambda \sum_{k=1}^n E_{k+1,k}, \quad B = \sum_{k=1}^n (x_k + y_k) E_{kk} + \lambda \sum_{k=1}^n E_{k+1,k}.$$

which imply that the equations (4.13) can be rewritten as

$$\dot{T} = [T, B], \quad \text{where } T = V^{-1}U. \quad (4.15)$$

The matrix T serves as a Lax matrix for the system (4.13).

Remark 4.5. The equations (4.13) have also two other Hamiltonian representations. The first one with respect to a linear bracket:

$$\{y_k, x_k\}_1 = y_k, \quad \{y_k, x_{k+1}\}_1 = -y_k, \quad \{x_k, x_{k+1}\}_1 = y_k$$

and Hamiltonian $H_2 = \sum_{k=1}^n [\frac{1}{2}(y_k + x_k)^2 + (y_k + x_k)y_{k-1}]$. The second one with respect to the cubic bracket

$$\begin{aligned} \{x_k, y_k\}_3 &= -x_k y_k (x_k + y_k), & \{y_k, x_{k+1}\}_3 &= -y_k x_{k+1} (y_k + x_{k+1}), \\ \{x_k, x_{k+1}\}_3 &= -x_k y_k x_{k+1}, & \{y_k, y_{k+1}\}_3 &= -y_k y_{k+1} (y_k + 2x_{k+1} + y_{k+1}), \\ \{x_k, y_{k+1}\}_3 &= -x_k y_k y_{k+1}, & \{y_k, x_{k+2}\}_3 &= -y_k y_{k+1} x_{k+2}, \\ \{y_k, y_{k+2}\}_3 &= -y_k y_{k+1} y_{k+2}. \end{aligned} \quad (4.16)$$

and Hamiltonian $H_{-1} = \sum_{k=1}^n \log(x_k)$.

Example 4.6. The Lax matrix T takes the for $n = 3$ is

$$T = \frac{1}{y_1 y_2 y_3 - \lambda^3} \begin{pmatrix} -\lambda^3(y_1 + x_1) & -\lambda^2 y_1(y_2 + x_2) & -\lambda(y_1 y_2 x_3 + \lambda^3) \\ -\lambda(y_2 y_3 x_1 + \lambda^3) & -\lambda^3(y_2 + x_2) & -\lambda^2 y_2(y_3 + x_3) \\ -\lambda^2 y_3(y_1 + x_1) & -\lambda(y_1 y_3 x_2 + \lambda^3) & -\lambda^3(y_3 + x_3) \end{pmatrix}$$

and for $n = 5$ we have that T is the scalar $\frac{1}{c - \lambda^5}$ times

$$\begin{pmatrix} -\lambda^5 \sigma_1 & -\lambda^4 y_1 \sigma_2 & -\lambda^3 y_1 y_2 \sigma_3 & -\lambda^2 y_1 y_2 y_3 \sigma_4 & -\lambda(\frac{c x_5}{y_5} + \lambda^5) \\ -\lambda(\frac{c x_1}{y_1} + \lambda^5) & -\lambda^5 \sigma_2 & -\lambda^4 y_2 \sigma_3 & -\lambda^3 y_2 y_3 \sigma_4 & -\lambda^2 y_2 y_3 y_4 \sigma_5 \\ -\lambda^2 y_3 y_4 y_5 \sigma_1 & -\lambda(\frac{c x_2}{y_2} + \lambda^5) & -\lambda^5 \sigma_3 & -\lambda^4 y_3 \sigma_4 & -\lambda^3 y_3 y_4 \sigma_5 \\ -\lambda^3 y_4 y_5 \sigma_1 & -\lambda^2 y_4 y_5 y_1 \sigma_2 & -\lambda(\frac{c x_3}{y_3} + \lambda^5) & -\lambda^5 \sigma_4 & -\lambda^4 y_4 \sigma_5 \\ -\lambda^4 y_5 \sigma_1 & -\lambda^3 y_5 y_1 \sigma_2 & -\lambda^2 y_5 y_1 y_2 \sigma_3 & -\lambda(\frac{c x_4}{y_4} + \lambda^5) & -\lambda^5 \sigma_5 \end{pmatrix}$$

where $c = \prod_{i=1}^5 y_i$.

The following proposition gives the tensor formalism for the linear and quadratic bracket of the relativistic Toda lattice

Proposition 4.7 ([33]). The matrix $T := T(\lambda)$ satisfy

$$\{T(\lambda) \otimes T(\mu)\}_1 = [I \otimes T(\mu), r(\lambda, \mu) + p_0] + [T(\lambda) \otimes I, r(\lambda, \mu) - p_0] \quad (4.17)$$

$$\begin{aligned} \{T(\lambda) \otimes T(\mu)\}_2 &= [T(\lambda) \otimes T(\mu)] a_1(\lambda, \mu) - a_2(\lambda, \mu) [T(\lambda) \otimes T(\mu)] \\ &\quad + [I \otimes T(\mu)] s_1 [T(\lambda) \otimes I] - [T(\lambda) \otimes I] s_2 [I \otimes T(\mu)] \end{aligned} \quad (4.18)$$

where

$$\begin{aligned}
r(\lambda, \mu) &= \frac{1}{2} = \frac{\lambda^n + \mu^n}{\lambda^n - \mu^n} \sum_{i=1}^n E_{ii} \otimes E_{ii} + \sum_{k=1}^{n-1} \frac{\lambda^k \mu^{n-k}}{\lambda^n - \mu^n} \sum_{i-j=k} E_{ij} \otimes E_{ji} \\
p_0 &= \frac{1}{2} \sum_{i=1}^n E_{ii} \otimes E_{ii} \\
p_1 &= \frac{1}{2} \left(\sum_{i>j} E_{ii} \otimes E_{jj} - \sum_{i<j} E_{ii} \otimes E_{jj} \right) \\
a_1(\lambda, \mu) &= r(\lambda, \mu) + p_1, \quad a_2(\lambda, \mu) = r(\lambda, \mu) - p_1, \quad s_1 = p_0 - p_1, \quad s_2 = p_0 + p_1.
\end{aligned}$$

4.2 Coxeter-Toda lattices

Throughout this section $G = \mathrm{GL}_n$, we denote the subgroup of upper (resp. lower) triangular matrices of G by B_+ (resp. B_-). Let $W = S_n$ be the symmetric group on n letters and think of W as a subgroup of G by identifying $w \in S_n$ with the permutation matrix $w = (\delta_{iv(j)})_{i,j=1}^n$.

4.2.1 Coxeter double Bruhat cells

The Bruhat decompositions of G with respect to B_+ and B_- are

$$G = \bigcup_{u \in S_n} B_+ u B_+, \quad G = \bigcup_{v \in S_n} B_- v B_-$$

where $B_+ u B_+ = \{X \in \mathrm{GL}_n \mid \exists A, B \in B_+ \text{ such that } X = AuB\}$ and analogous definition for $B_- v B_-$. The double Bruhat cell $G^{u,v}$ for a pair $u, v \in S_n$ is defined as

$$G^{u,v} = B_+ u B_+ \cap B_- v B_-$$

The dimension of a double Bruhat cell $G^{u,v}$ is $l(u) + l(v) + n$, where $l(u)$ is the length of u . A Coxeter element $w \in S_n$ is a product of $n-1$ distinct transpositions.

Definition 4.8. Let $u, v \in S_n$ be two Coxeter elements. The double Bruhat cell $G^{u,v}$ is called a **Coxeter double Bruhat cell**.

Notice that for any Coxeter double Bruhat cell $G^{u,v}$ one has $\dim G^{u,v} = 3n - 2$.

Example 4.9. Let $v = u^{-1} = s_{n-1} \cdots s_1$. Then

$$\begin{aligned}
B_+ u B_+ &= \{(x_{ij}) \in \mathrm{GL}_n \mid x_{ij} = 0 \text{ if } i - j > 1 \text{ and } \prod_{i=1}^{n-1} x_{i+1,i} \neq 0\}, \\
B_- v B_- &= \{(x_{ij}) \in \mathrm{GL}_n \mid x_{ij} = 0 \text{ if } j - i > 1 \text{ and } \prod_{i=1}^{n-1} x_{i,i+1} \neq 0\}.
\end{aligned}$$

Therefore

$$G^{u,v} = \{(x_{ij}) \in \mathrm{GL}_n \mid x_{ij} = 0 \text{ if } |i - j| > 1 \text{ and } \prod_{i=1}^{n-1} x_{i,i+1} x_{i+1,i} \neq 0\}$$

is the set of tridiagonal matrices with nonzero off-diagonal entries.

Recall that G is equipped with the standard Poisson-Lie structure (2.41). Double Bruhat cells are regular Poisson submanifolds of G . Any symplectic leaf of G is of the form $S^{u,v}a$ where $S^{u,v} \subset G^{u,v}$ is a certain distinguished symplectic leaf and a is an element of the Cartan subgroup. Furthermore, the symplectic leaves in $G^{u,v}$ have dimension $l(u) + l(v) + n - \text{rank}(uv^{-1} - I)$.

Conjugation-invariant functions on G form a Poisson commuting family. Each conjugation-invariant function F generates a Hamiltonian flow

$$\frac{dX}{dt} = \left[X, -\frac{1}{2}R(\nabla F(X)) \right] \quad (4.19)$$

where $R(X) = \pi_+(X) - \pi_-(X)$ being $\pi_+(X)$ and $\pi_-(X)$ the strictly upper and lower parts of X , respectively. If we consider $F(X) = F_k(X) = \frac{1}{k} \text{Trace}(X^k)$ ($k = 1, \dots, n-1$) equation (4.19) becomes

$$\frac{dX}{dt} = \left[X, -\frac{1}{2}R(X^k) \right] \quad (4.20)$$

Functions $F_1, \dots, F_{n-1}$ form a maximal family of algebraically independent conjugation-invariant functions of G .

For an element $h \in G$ and any function $f \in G$ we have $\nabla(f \circ C_h)(X) = \text{Ad}_{h^{-1}}(\nabla f)(C_h(X))$. In fact,

$$\begin{aligned} (\text{Ad}_{h^{-1}}(\nabla f(C_h(X))), \xi) &= (\nabla f(C_h(X)), \text{Ad}_h \xi) = \left. \frac{d}{dt} \right|_{t=0} f(e^{t \text{Ad}_h \xi} C_h(X)) \\ &= \left. \frac{d}{dt} \right|_{t=0} f(C_h(e^{t\xi}) C_h(X)) = \left. \frac{d}{dt} \right|_{t=0} (f \circ C_h)(e^{t\xi} X) \\ &= (\nabla(f \circ C_h)(X), \xi). \end{aligned}$$

Furthermore, if $h \in H$ then $R(\text{Ad}_{h^{-1}}(\xi)) = \text{Ad}_{h^{-1}}(R(\xi))$ for any $\xi \in \mathfrak{gl}_n$. Denoting $f_i \circ C_h$ by $\tilde{f}_i$ for $i = 1, 2$ and using the last two identities one has:

$$\begin{aligned} \{\tilde{f}_1, \tilde{f}_2\}(X) &= \frac{1}{2} [(R(\nabla'(f_1 \circ C_h)(X)), \nabla'(f_2 \circ C_h)(X)) - (R(\nabla(f_1 \circ C_h)(X)), \nabla(f_2 \circ C_h)(X))] \\ &= \frac{1}{2} [(R(\nabla' f_1(C_h(X))), \nabla' f_2(C_h(X))) - (R(\nabla f_1(C_h(X))), \nabla f_2(C_h(X)))] \\ &= \{f_1, f_2\} \circ C_h(X). \end{aligned}$$

Therefore, the action of H on G by conjugation is Poisson with respect to the standard Poisson-Lie structure. Since the action of H by conjugation preserves double Bruhat cells, in fact any double Bruhat cell is invariant under the left and right multiplication by elements of $H \subset G$, the standard Poisson-Lie structure induces a Poisson structure on $G^{u,v}/H$ and the Toda hierarchy (4.20) induce a family of commuting Hamiltonian flows on $G^{u,v}/H$.

In the particular case $v = u^{-1} = s_{n-1} \cdots s_1$, see Example 4.9, $G^{u,v}/H$ can be described as the set of tridiagonal matrices of the form

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ 1 & b_2 & a_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & b_{n-1} & a_{n-1} \\ 0 & \cdots & 0 & 1 & b_n \end{pmatrix}, \quad a_1 \cdots a_{n-1} \neq 0, \quad \det L \neq 0. \quad (4.21)$$

The Lax equations (4.20) then become a hierarchy of equations called the **finite non-periodic Toda hierarchy**

$$\frac{d}{dt} L = -[L, \pi_+(L^k)] \quad (4.22)$$

Notice that if $k = 1$, (4.22) is the Lax representation (4.5) in Proposition 4.2.

A Coxeter element $v \in S_n$ can be written in the form

$$v = s_{[i_{k-1}, i_k]} \cdots s_{[i_1, i_2]} s_{[1, i_1]} \quad (4.23)$$

for some subset $I = \{1 = i_0 < i_1 < \cdots < i_k = n\} \subset [1, n]$ where $s_{[p, q]} = s_p s_{p+1} \cdots s_{q-1}$ for $1 \leq p < q \leq n$. Define $L = \{1 = l_0 < l_1 < \cdots < l_{n-k} = n\}$ where $\{l_1 < \cdots < l_{n-k-1}\} = [1, n] \setminus I$. The inverse v^{-1} of a Coxeter element v as in (4.23) is given by

$$v^{-1} = s_{[l_{n-k-1}, l_{n-k}]} \cdots s_{[l_1, l_2]} s_{[1, l_1]}.$$

Let (u, v) be a pair of Coxeter elements and let

$$\begin{aligned} I^\pm &= \{1 = i_0^\pm < i_1^\pm < \cdots < i_{k^\pm}^\pm = n\}, \\ L^\pm &= \{1 = l_0^\pm < l_1^\pm < \cdots < l_{n-k^\pm-1}^\pm < l_{n-k^\pm}^\pm = n\}, \end{aligned} \quad (4.24)$$

be subsets of $[1, n]$ that correspond to v and u^{-1} .

For a set of complex parameters $c_1^-, \dots, c_{n-1}^-; c_1^+, \dots, c_{n-1}^+; d_1, \dots, d_n$ define the matrices $D = \text{diag}(d_1, \dots, d_n)$ and

$$C_j^\pm = \sum_{\alpha=i_{j-1}^\pm}^{i_j^\pm-1} c_\alpha^\pm e_{\alpha, \alpha+1}, \quad j \in [1, k^\pm], \quad \bar{C}_j^\pm = \sum_{\alpha=l_{j-1}^\pm}^{l_j^\pm-1} c_\alpha^\pm e_{\alpha, \alpha+1}, \quad j \in [1, n-k^\pm].$$

Proposition 4.10. *A generic element $X \in G^{u,v}$ and its inverse X^{-1} can be written as*

$$\begin{aligned} X &= (1 - C_1^-)^{-1} \cdots (1 - C_{k^-}^-)^{-1} D (1 - C_{k^+}^+)^{-1} \cdots (1 - C_1^+)^{-1}, \\ X^{-1} &= (1 + \bar{C}_{n-k^+}^+)^{-1} \cdots (1 + \bar{C}_1^+)^{-1} D^{-1} (1 + \bar{C}_1^-)^{-1} \cdots (1 + \bar{C}_{k^-}^-)^{-1}. \end{aligned} \quad (4.25)$$

Proof. See [11]. □

For a fixed pair (u, v) of Coxeter elements and its respective sets $I^\pm$ given by (4.24) set

$$\varepsilon_i^\pm = \begin{cases} 0 & \text{if } i = i_j^\pm \text{ for some } 0 < j \leq k^\pm, \\ 1 & \text{otherwise.} \end{cases}$$

and define

$$\varepsilon_i = \varepsilon_i^+ + \varepsilon_i^-, \quad \varkappa_i = i + 1 - \sum_{\beta=1}^i \varepsilon_\beta.$$

4.2.2 Coxeter-Toda lattices and weighted networks

An important tool in the study of integrable systems like the Toda lattice is the *Weyl function*

$$m(\lambda; X) = ((\lambda I - X)^{-1} e_1, e_1) = \frac{q(\lambda)}{p(\lambda)} \quad (4.26)$$

where $p(\lambda)$ is the characteristic polynomial of X and $q(\lambda)$ is the characteristic polynomial of the $(n-1) \times (n-1)$ submatrix of X formed by deleting the first row and first column. One sees that $m(\lambda; X)$ is invariant under conjugation by H . Then we have a map $m(\lambda, \cdot) : G^{u,v}/H \rightarrow \mathcal{W}_n$, where

$$\mathcal{W}_n = \left\{ m(\lambda) = \frac{q(\lambda)}{p(\lambda)} : \partial p = n, \partial q = n-1, p \text{ and } q \text{ are monic and coprime, } p(0) \neq 0 \right\}$$

The case $v = u^{-1} = s_{n-1} \cdots s_1$ (the tridiagonal case) was worked out by Moser [25]. He has shown that the Weyl map verifies the following properties

- (i) The Moser map $m_{u,v} : G^{u,v}/H \rightarrow \mathcal{W}_n$ is invertible. Namely, the matrix entries of a Jacobi matrix is obtained from its Weyl function $m(\lambda; X)$ via determinantal formulas in terms of Hankel determinants built from the coefficients of the Laurent expansion of $m(\lambda; X)$.
- (ii) The Toda hierarchy defines a completely integrable system on level sets of the determinant in $G^{u,v}/H$.

It was shown in [11] that for a pair $u, v \in S_n$ of any Coxeter elements the two properties above are also satisfied. The integrable equations induced on $G^{u,v}/H$ by Toda flows will be called *Coxeter-Toda lattices*.

In the following we summarize the results of [11] to give a brief explanation of the validity of the properties above. Let $X \in G^{u,v}$ be an element of a Coxeter double Bruhat cell with factorization (4.25) and consider its Weyl function

$$m(\lambda; X) = \frac{q(\lambda)}{p(\lambda)} = \sum_{j=0}^{\infty} \frac{h_j(X)}{\lambda^{j+1}},$$

where $h_j(X) = (X^j)_{11} = (X^j e_1, e_1)$, $j \in \mathbb{Z}$ is the j th moment of X .

For any $l \in \mathbb{Z}$, $i \in \mathbb{N}$ define the Hankel matrices

$$\mathcal{H}_i^{(l)} = (h_{\alpha+\beta+l-i-1})_{\alpha,\beta=1}^i$$

and Hankel determinants

$$\Delta_i^{(l)} = \det \mathcal{H}_i^{(l)}$$

where $\Delta_0^l = 1$ for any $l \in \mathbb{Z}$. The next theorem asserts that one can restore the factorization parameters (4.25) of $X \in G^{u,v}$ from its Weyl function $m(\lambda; X)$ only modulo the diagonal conjugation.

Theorem 4.11. *If X possesses factorization (4.25), then*

$$\begin{aligned} d_i &= \frac{\Delta_i^{(\varkappa_i+1)} \Delta_{i-1}^{(\varkappa_{i-1})}}{\Delta_i^{(\varkappa_i)} \Delta_{i-1}^{(\varkappa_{i-1}+1)}}, \\ c_i^+ c_i^- &= \frac{\Delta_{i-1}^{(\varkappa_{i-1})} \Delta_{i+1}^{(\varkappa_{i+1})}}{(\Delta_i^{(\varkappa_i+1)})^2} \left(\frac{\Delta_{i+1}^{(\varkappa_{i+1}+1)}}{\Delta_{i+1}^{(\varkappa_{i+1})}} \right)^{\varepsilon_{i+1}} \left(\frac{\Delta_{i-1}^{(\varkappa_{i-1}+1)}}{\Delta_{i-1}^{(\varkappa_{i-1})}} \right)^{2-\varepsilon_i} \end{aligned} \quad (4.27)$$

for any $i \in [1, n]$.

Proof. See [11]. □

Now consider the Toda hierarchy (4.20). Equations on X induce an evolution of the corresponding Weyl function $m(\lambda; X)$ which are described in terms of its Laurent coefficients h_i .

Proposition 4.12. *If $X = X(t)$ satisfies (4.20), then the coefficients $h_i(X) = (X^i e_1, e_1)$ of the Laurent expansion of the Weyl function $m(\lambda; X)$ evolve according to*

$$\frac{dh_i(X)}{dt} = h_{i+k}(X) - h_k(X)h_i(X).$$

Proof. See [11]. □

Any $X \in G^{u,v}$ which possesses a factorization (4.25) can be represented via weighted network $N_{u,v}$ in a disc.

Example 4.13. The weighted network $N_{u,v}$ associated to (4.25) when $n = 3$ and $v = u^{-1} = s_{[2,3]}s_{[1,2]} = s_2s_1$ for a generic element $X = E_1^-(c_1^-)E_2^-(c_2^-)DE_2^+(c_2^+)E_1^+(c_1^+) \in G^{u,v}$ is shown in Figure 4.1.

Figure 4.1: Network associated to a generic $X \in G^{s_1s_2, s_2s_1}$.

Let $N_{u,v}$ be the network associated with $X \in G$ and the factorization (4.25). The construction of a network in an annulus $N_{u,v}^\circ$ associated to $N_{u,v}$ is explained in [11]. We present an example of such network $N_{u,v}^\circ$ in an annulus, see Figure 4.2, corresponding to Figure 4.1.

Let $M(\lambda)$ be the boundary measurement for the network $N_{u,v}^\circ$ and consider the space

$$\mathcal{R}_n = \left\{ \frac{Q(\lambda)}{P(\lambda)} : P \text{ monic } \deg P = n, \deg Q < n, P \text{ and } Q \text{ are coprime, } P(0) \neq 0 \right\}.$$

Figure 4.2: Network $N_{s_1 s_2, s_2 s_1}^\circ$ on the annulus.

Proposition 4.14. *Let $X \in G^{u,v}$ with factorization (4.25) then*

(i) $M(\lambda) = \sum_{k=0}^{\infty} (-1)^k H_k \lambda^{-k-1} = -H_0 m(-\lambda, X)$. In particular $H_k = h_k H_0$, where $H_0 = w_{in} w_{out}$.

(ii) *The space of boundary measurements associated with the network $N_{u,v}^\circ$ is dense in $\mathcal{R}_n$.*

Proof. See [11]. □

The network $N_{u,v}^\circ$ in an annulus corresponding to $N_{u,v}$ which is constructed associated to $X \in G^{u,v}$ with factorization (4.25), has $2(n-1)$ bounded faces $f_{0i}, f_{1i}, i \in [1, n-1]$ which are enumerated as follows: each face f_{0i} contains a piece of the cut and each face f_{1i} does not, and the value i is assigned according to the natural bottom to top order inherited from $N_{u,v}$. There are also three unbounded faces: two of them, adjacent to the outer boundary of the annulus denoted by f_{00}, f_{10} using the previous convention for bounded faces. The third unbounded face is adjacent to the inner boundary will be denoted by f_{0n} . The space $\mathcal{F}_{N_{u,v}}$ (resp. $\mathcal{F}_{N_{u,v}^\circ}$) is equipped with the Poisson bracket, called *standard* Poisson bracket, from the 2-parametric family described in Proposition (3.11) (resp. Proposition (3.15)) with $\alpha = -\beta = 1/2$.

The edges of the directed dual network $(N_{u,v}^\circ)^*$, defined in a similar way as for networks in a disc, is described as follows:

- (i) There are two edges directed from f_{0i} to f_{1i} for any $i \in [0, n-1]$.
- (ii) There are $1 - \varepsilon_{i+1}$ edges directed from f_{1i+1} to f_{1i} for any $i \in [1, n-2]$.
- (iii) There are $2 - \varepsilon_{i+1}$ edges directed from f_{1i} to f_{0i+1} for any $i \in [1, n-2]$.
- (iv) There are $1 - \varepsilon_{i+1}$ edges directed from f_{0i+1} to f_{0i} for any $i \in [1, n-2]$.
- (v) There are ε_{i+1} edges directed from f_{1i+1} to f_{0i} for any $i \in [1, n-2]$.

- (vi) There is one edge directed from f_{1n-1} to f_{0n} and one edge directed from f_{0n} to f_{0n-1} .
- (vii) For $|i - j| > 1$ and $a, b \in \{0, 1\}$, vertices f_{ai} and f_{bj} in $(N_{u,v}^\circ)^*$ are not connected by edges.

Next, the face weights y_{st} to every face f_{st} in $N_{u,v}^\circ$ are given by

$$y_{0i} = c_i^{-1}, \quad y_{1i} = \frac{c_i d_i}{d_{i+1}}$$

for $i \in [1, n-1]$ and

$$y_{00} = H_0^{-1}, \quad y_{10} = \frac{H_0}{d_1}, \quad y_{0n} = d_n.$$

Since $N_{u,v}^\circ$ has one source and one sink, both lying on the same connected component of the boundary, and the other connected component of the boundary does not have boundary vertices, one has that $\mathcal{F}_{N_{u,v}^\circ} = \mathcal{F}_{N_{u,v}^\circ}^f$. Therefore, the standard Poisson bracket on $\mathcal{F}_{N_{u,v}^\circ}$ is completely described by Proposition 3.11. From the above description of $(N_{u,v}^\circ)^*$ we have the following Poisson relations in face weights y_{st} :

$$\begin{aligned} \{y_{0i}, y_{1i}\} &= 2y_{0i}y_{1i}, & \{y_{1i}, y_{1i+1}\} &= -(1 - \varepsilon_{i+1})y_{1i}y_{1i+1} \\ \{y_{1i}, y_{0i+1}\} &= (2 - \varepsilon_{i+1})y_{1i}y_{0i+1}, & \{y_{0i}, y_{0i+1}\} &= -(1 - \varepsilon_{i+1})y_{0i}y_{0i+1} \\ \{y_{0i}, y_{1i+1}\} &= -\varepsilon_{i+1}y_{0i}y_{1i+1} \end{aligned}$$

for $i \in [1, n-2]$ and

$$\begin{aligned} \{y_{00}, y_{10}\} &= y_{00}y_{10}, & \{y_{10}, y_{01}\} &= 2y_{10}y_{01}, \\ \{y_{10}, y_{11}\} &= -y_{10}y_{11}, & \{y_{00}, y_{01}\} &= -y_{00}y_{01}, \\ \{y_{0n}, y_{0n-1}\} &= y_{0n}y_{0n-1}, & \{y_{0n}, y_{1n-1}\} &= -y_{0n}y_{1n-1}, \end{aligned}$$

and the rest of the brackets are zero. The next proposition is a particular case of Proposition 3.17.

Proposition 4.15. *The standard Poisson bracket on $\mathcal{F}_{N_{u,v}^\circ}$ induces a Poisson bracket on $\mathcal{R}_n$. This bracket is given by*

$$\{M(\lambda), M(\mu)\} = -(\lambda M(\lambda) - \mu M(\mu)) \frac{M(\lambda) - M(\mu)}{\lambda - \mu} \quad (4.28)$$

Notice that (4.28) is the Poisson bracket $\{\cdot, \cdot\}_1$ of the family of Faybusovich-Gekhtman Poisson brackets.

Coxeter-Toda flows on $G^{u,v}/H$ are induced by the restriction of the Toda hierarchy to $G^{u,v}$. The parameters $c_i = c_i^+ c_i^-$, d_i are coordinates on an open dense set in $G^{u,v}/H$.

Proposition 4.16. *The standard Poisson-Lie structure on GL_n induces the following Poisson brackets for variables c_i, d_i*

$$\{c_i, c_{i+1}\} = (\varepsilon_{i+1} - 1)c_i c_{i+1}, \quad \{c_i, d_i\} = -c_i d_i, \quad \{c_i, d_{i+1}\} = c_i d_{i+1}, \quad (4.29)$$

and the rest of the brackets are zero.

Proof. See [11]. □

Due to their invariance under conjugation by elements of H , Hamiltonians $F_k(X) = \frac{1}{k} \text{Trace}(X^k)$ of the Toda flows, when restricted to a Coxeter double Bruhat cell $G^{u,v}$ can be expressed as functions of c_i, d_i , which, in turn, serve as Hamiltonians for Coxeter-Toda flows on $G^{u,v}/H$. One can write F_k as a function of c_i, d_i as the sum of weights of all paths that start and end at the same level in the network obtained by concatenation of k copies of $N_{u,v}$. For instance, when $k = 1$ consider $I^- \cup I^+ = \{1 = i_1, \dots, i_m = n\}$ then

$$F_1 = F_1(c, d) = d_1 + \sum_{l=1}^{m-1} \sum_{j=i_l+1}^{i_{l+1}} (d_j + c_{j-1}d_{j-1} + \dots + c_{j-1} \dots c_{i_l}d_{i_l})$$

Example 4.17. Let $v = s_{n-1} \dots s_1$, then $I^+ \cup I^- = [1, n]$ and $F_1(c, d) = d_1 + d_2 + c_1d_1 + \dots + d_n + c_{n-1}d_{n-1}$. Furthermore, if $u = v^{-1}$ then $\varepsilon = (2, 0, \dots, 0)$. Then F_1 and (4.29) generate the Hamiltonian equations

$$\begin{aligned} \frac{dd_i}{dt} &= \{d_i, F_1\} = d_i(c_i d_i - c_{i-1} d_{i-1}) \\ \frac{dc_i}{dt} &= \{c_i, F_1\} = c_i(d_{i+1} - d_i + c_{i-1} d_{i-1} - c_i d_i) \end{aligned} \quad (4.30)$$

The change of variables $a_i = c_i d_i^2$, $b_i = d_i + c_{i-1} d_{i-1}$, leads the system (4.30) into the non-periodic Toda lattice (4.2), and the Poisson relations (4.29) become the Poisson relations (4.4).

4.3 A second Poisson bracket for Coxeter-Toda lattices

In this section, we exhibit our main result of this dissertation. Namely, we present a non-local rational Poisson bracket $\{\cdot, \cdot\}$ in c_i, d_i coordinates which is compatible with (4.29). To this end recall the following fundamental facts: First, the boundary measurement $M(\lambda)$ of $N_{u,v}^\circ$ belongs to $\mathcal{R}_n$ and is equal up to a constant and a sign to the Weyl function $m(\lambda)$, item (i) of Proposition 4.14. Furthermore, the Moser map $m(\lambda, \cdot)$ is invertible in $G^{u,v}/H$, see Theorem 4.11. Second, using the standard Poisson structure and the boundary measurement $M(\lambda)$ one recovers $\{\cdot, \cdot\}_1$ on $\mathcal{R}_n$ and $\mathcal{R}_n$ is endowed with the Faybusovich-Gekhtman family of compatible Poisson brackets $\{\cdot, \cdot\}_k$, $k = 0, \dots, n-1$. To achieve our goal we have used the Atiyah-Hitchin bracket $\{\cdot, \cdot\}_0$, and the result is shown in the next theorem.

Theorem 4.18. *The bracket*

$$\{d_i, d_{i+k}\} = \begin{cases} c_i d_i & \text{if } k = 1, \\ d_i \prod_{j=i}^{i+k-1} c_j & \text{if } k > 1, \epsilon_{i+1} = \dots = \epsilon_{i+k-1} = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (4.31)$$

$$\{c_i, d_{i+k}\} = \begin{cases} c_i \delta_{k,0} - c_i(c_i + 1) \delta_{k,1} & \\ -(c_i + 1) \prod_{j=i}^{i+k-1} c_j & \text{if } k > 1, \epsilon_{i+1} = \dots = \epsilon_{i+k-1} = 0, \\ \prod_{j=i}^{i+k-1} c_j & \text{if } k > 1, \epsilon_{i+1} = 2, \epsilon_{i+2} = \dots = \epsilon_{i+k-1} = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (4.32)$$

$$\{c_{i+k}, d_i\} = \begin{cases} (2 - \epsilon_{i+1})c_i c_{i+1} \frac{d_i}{d_{i+1}} & \text{if } k = 1, \\ (2 - \epsilon_{i+k}) \frac{d_i}{d_{i+k}} \prod_{j=i}^{i+k} c_j & \text{if } k > 1, \epsilon_{i+1} = \dots = \epsilon_{i+k-1} = 0, \\ 0 & \text{otherwise .} \end{cases} \quad (4.33)$$

$$\{c_i, c_{i+k}\} = \begin{cases} (2 - \epsilon_{i+1}) \frac{c_{i+1}}{d_{i+1}} c_i c_{i+1} & \text{if } k = 1, \\ (2 - \epsilon_{i+k}) \frac{c_{i+1}}{d_{i+k}} \prod_{j=i}^{i+k} c_j & \text{if } k > 1, \epsilon_{i+1} = \dots = \epsilon_{i+k-1} = 0, \\ (\epsilon_{i+k} - 2) \frac{\prod_{j=i}^{i+k} c_j}{d_{i+k}} & \text{if } k > 1, \epsilon_{i+1} = 2, \epsilon_{i+2} = \dots = \epsilon_{i+k-1} = 0, \\ 0 & \text{otherwise .} \end{cases} \quad (4.34)$$

is a Poisson bracket which is compatible with (4.29).

Proof. The proof follows from the following two lemmas. □

Lemma 4.19. *The bracket given by relations (4.31)–(4.34) is a Poisson bracket.*

Proof. We must prove that relations (4.31)–(4.34) satisfy Jacobi identity. Let

$$\epsilon_{i+1} = \dots = \epsilon_{i+m-1} = 0, \quad (A1)$$

$$\epsilon_{i+1} = 2, \epsilon_{i+2} = 0, \dots = \epsilon_{i+m-1} = 0, \quad (A2)$$

$$\epsilon_{i+m+1} = \dots = \epsilon_{i+m+n-1} = 0, \quad (B1)$$

$$\epsilon_{i+m+1} = 2, \epsilon_{i+m+2} = \dots = \epsilon_{i+m+n-1} = 0, . \quad (B2)$$

Case 1:

$$\{d_i, \{d_{i+m}, d_{i+m+n}\}\} + \{d_{i+m}, \{d_{i+m+n}, d_i\}\} + \{d_{i+m+n}, \{d_i, d_{i+m}\}\}. \quad (4.35)$$

Consider the case $m > 1$ and $n > 1$.

(i) Assume that condition (A1) is false.

- If condition (B1) is false then (4.35) is zero.

- If condition (B1) is true then (4.35) is equal to

$$\{d_i, d_{i+m} \prod_{j=i+m}^{i+m+n-1} c_j\} = \{d_i, d_{i+m}\} \prod_{j=i+m}^{i+m+n-1} c_j + \{d_i, \prod_{j=i+m}^{i+m+n-1} c_j\} d_{i+m} = 0.$$

(ii) Assume that condition (A1) is true

- If condition (B1) is false then (4.35) becomes

$$\{d_{i+m+n}, d_i \prod_{j=i}^{i+m-1} c_j\} = \{d_{i+m+n}, \prod_{j=i}^{i+m-1} c_j\} d_i = 0.$$

The last equality is due that each bracket $\{d_{i+m+n}, c_l\} = 0$ for $l = i, \dots, i+m-1$, since (B1) is false.

- If condition (B1) is true and $\varepsilon_{i+m} \neq 0$. Then (4.35) is equal to

$$\begin{aligned} \{d_i, d_{i+m} \prod_{j=i+m}^{i+m+n-1} c_j\} + \{d_{i+m+n}, d_i \prod_{j=i}^{i+m-1} c_j\} &= \{d_i, d_{i+m}\} \prod_{j=i+m}^{i+m+n-1} c_j \\ &+ \{d_i, \prod_{j=i+m}^{i+m+n-1} c_j\} d_{i+m} + \{d_{i+m+n}, \prod_{j=i}^{i+m-1} c_j\} d_i \quad (4.36) \end{aligned}$$

From $\varepsilon_{i+m} \neq 0$ one has

$$\{d_i, c_l\} = 0, \quad l = i + m + 1, \dots, i + m + n - 1; \quad \{d_{i+m+n}, c_l\} = 0, \quad l = i, \dots, i + m - 2.$$

Then (4.36) is equal to

$$d_i \prod_{j=i}^{i+m+n-1} c_j + (\varepsilon_m - 2) d_i \prod_{j=i}^{i+m+n-1} c_j + \{d_{i+m+n}, c_{i+m-1}\} d_i \prod_{j=i}^{i+m-2} c_j$$

which is zero for ε_m equal to 1 or 2, because

$$\{d_{i+m+n}, c_{i+m-1}\} = \begin{cases} -\prod_{j=i+m-1}^{i+m+n-1} c_j & \text{if } \varepsilon_m = 2 \\ 0 & \text{if } \varepsilon_m = 1. \end{cases}$$

- If condition (B1) is true and $\varepsilon_m = 0$ then (4.35), after simplify, is equal to

$$\begin{aligned} d_i \prod_{j=i}^{i+m+n-1} c_j + d_{i+m} \{d_i, \prod_{j=i+m}^{i+m+n-1} c_j\} - d_i \prod_{j=i+m}^{i+m+n-1} c_j \{d_{i+m}, \prod_{j=i}^{i+m-1} c_j\} \\ - d_i \prod_{j=i}^{i+m-1} c_j \{d_{i+m}, \prod_{j=i+m}^{i+m+n-1} c_j\} + d_i \{d_{i+m+n}, \prod_{j=i}^{i+m-1} c_j\} \end{aligned}$$

This expression is equal to zero, since

$$\begin{aligned} d_{i+m} \{d_i, \prod_{j=i+m}^{i+m+n-1} c_j\} &= -2d_i d_{i+m} \prod_{j=i+m}^{i+m+n-1} c_j \sum_{r=i+m}^{i+m+n-1} \frac{1}{d_r} \prod_{j=i}^{r-1} c_j, \\ d_i \prod_{j=i+m}^{i+m+n-1} c_j \{d_{i+m}, \prod_{j=i}^{i+m-1} c_j\} &= d_i \prod_{j=i}^{i+m+n-1} c_j \left(\sum_{r=i}^{i+m-2} (c_r + 1) \prod_{j=r+1}^{i+m-1} c_j + c_{i+m-1} + 1 \right), \\ d_i \prod_{j=i}^{i+m-1} c_j \{d_{i+m}, \prod_{j=i+m}^{i+m+n-1} c_j\} &= d_i \prod_{j=i}^{i+m+n-1} c_j \left(-2d_{i+m} \sum_{r=i+m+2}^{i+m+n-1} \frac{1}{d_r} \prod_{j=i+m}^{r-1} c_j \right. \\ &\quad \left. - 2c_{i+m} \frac{d_{i+m}}{d_{i+m+1}} - 1 \right), \\ d_i \{d_{i+m+n}, \prod_{j=i}^{i+m-1} c_j\} &= d_i \prod_{j=i}^{i+m-1} c_j \sum_{r=i}^{i+m-1} (c_r + 1) \prod_{j=r+1}^{i+m+n-1} c_j. \end{aligned}$$

In a similar way one proves that (4.35) is equal to zero for $m = 1$, $n > 1$ and $m = n = 1$.

Case 2:

$$\{d_i, \{d_{i+m}, c_{i+m+n}\}\} + \{d_{i+m}, \{c_{i+m+n}, d_i\}\} + \{c_{i+m+n}, \{d_i, d_{i+m}\}\}. \quad (4.37)$$

Consider the case $m > 1$ and $n > 1$.

(i) If condition (A1) is false then one can see that (4.37) is zero.

(ii) Assume that condition(A1) is true.

- If condition (B1) is false then (4.37) is equal to

$$\{c_{i+m+n}, d_i \prod_{j=i}^{i+m-1} c_j\} = \{c_{i+m+n}, \prod_{j=i}^{i+m-1} c_j\} d_i = 0,$$

because $\{c_{i+m+n}, c_l\} = 0$ for $l = i, \dots, i + m - 1$, since (B1) is false.

- If condition (B1) is true and $\varepsilon_{i+m} \neq 0$ then (4.37) is equal to

$$\{d_i, (\varepsilon_{i+m+n} - 2) \frac{d_{i+m}}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\} + \{c_{i+m+n}, d_i \prod_{j=i}^{i+m-1} c_j\}.$$

After simplifying, one gets

$$\frac{(\varepsilon_{i+m+n} - 2)}{d_{i+m+n}} (\varepsilon_{i+m} - 1) \prod_{j=i}^{i+m+n} c_j + d_i \prod_{j=i}^{i+m-2} c_j \{c_{i+m+n}, c_{i+m-1}\},$$

which is zero for $\varepsilon_{i+m} = 1$ or $\varepsilon_{i+m} = 2$.

- If condition (B1) is true and $\varepsilon_m = 0$ then (4.37) is equal to

$$\begin{aligned} & (\varepsilon_{i+m+n} - 2) \{d_i, \frac{d_{i+m}}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\} + (2 - \varepsilon_{i+m+n}) \{d_{i+m}, \frac{d_i}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j\} \\ & + (2 - \varepsilon_{i+m+n}) \frac{d_i}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j \prod_{j=i}^{i+m-1} c_j + d_i \sum_{r=i}^{i+m-1} \prod_{\substack{l=i \\ l \neq r}}^{i+m-1} c_l \{c_{i+m+n}, c_r\} \end{aligned}$$

The last equation is zero, using the following relations

$$\begin{aligned}
\{d_i, \frac{d_{i+m}}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\} &= \frac{d_i}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j - 2 \frac{d_i d_{i+m}}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j \sum_{r=i+m}^{i+m+n} \frac{1}{d_r} \prod_{j=i}^{r-1} c_j \\
&\quad + (\varepsilon_{i+m+n} - 3) \frac{d_i d_{i+m}}{d_{i+m+n}^2} \prod_{j=i+m}^{i+m+n} c_j \prod_{j=i}^{i+m+n-1} c_j \\
\{d_{i+m}, \frac{d_i}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j\} &= - \frac{d_i}{d_{i+m}} \prod_{j=i}^{i+m-1} c_j \prod_{j=i}^{i+m+n} c_j + \frac{d_i}{d_{i+m+n}} \left(\prod_{j=i}^{i+m+n} c_j \sum_{r=i}^{i+m-2} (c_r + 1) \prod_{j=r+1}^{i+m-1} c_r \right. \\
&\quad + c_{i+m-1} \prod_{j=i}^{i+m+n} c_j - 2 \frac{c_{i+m} d_{i+m}}{d_{i+m+1}} \prod_{j=i}^{i+m+n} c_j \\
&\quad \left. + d_{i+m} \prod_{j=i+m}^{i+m+n} c_j \sum_{r=i+m+2}^{i+m+n} \frac{\varepsilon_r - 2}{d_r} \prod_{j=i}^{r-1} c_j \right) - \frac{d_i d_{i+m}}{d_{i+m+n}^2} \prod_{j=i+m}^{i+m+n-1} c_j \prod_{j=i}^{i+m+n} c_j.
\end{aligned}$$

Analogously, equation (4.37) is also equal to zero for $m > 1$, $n = 1$ and $m = n = 1$.

One can verify the next two cases after a straightforward but lengthy computation.

Case 3:

$$\{d_i, \{c_{i+m}, d_{i+m+n}\}\} + \{c_{i+m}, \{d_{i+m+n}, d_i\}\} + \{d_{i+m+n}, \{d_i, c_{i+m}\}\}.$$

Case 4:

$$\{c_i, \{d_{i+m}, d_{i+m+n}\}\} + \{d_{i+m}, \{d_{i+m+n}, c_i\}\} + \{d_{i+m+n}, \{c_i, d_{i+m}\}\}.$$

Case 5:

$$\{d_i, \{c_{i+m}, c_{i+m+n}\}\} + \{c_{i+m}, \{c_{i+m+n}, d_i\}\} + \{c_{i+m+n}, \{d_i, c_{i+m}\}\}. \quad (4.38)$$

Consider the case $m > 1$ and $n > 1$.

(i) If condition (A1) is false then one can see that (4.38) is zero.

(ii) Assume that condition (A1) is true.

- If condition (B2) is true then (4.38), after simplifying, is equal to

$$\frac{(\varepsilon_{i+m+n} - 2)}{d_{i+m+n}} \{d_i, c_{i+m}\} \prod_{j=i+m+1}^{i+m+n} c_j + (\varepsilon_{i+m} - 2) \frac{d_i}{d_{i+m}} \{c_{i+m+n}, c_{i+m}\} \prod_{j=i}^{i+m-1} c_j,$$

which is zero using the corresponding relations of $\{d_i, c_{i+m}\}$ and $\{c_{i+m+n}, c_{i+m}\}$.

- If condition (B1) is true and $\varepsilon_{i+m} \neq 0$ then (4.38) is equal to

$$\frac{(2 - \varepsilon_{i+m+n})}{d_{i+m+n}} \{d_i, (c_{i+m} + 1) \prod_{j=i+m}^{i+m+n} c_j\} + (\varepsilon_{i+m} - 2) d_i \{c_{i+m+n}, \frac{1}{d_{i+m}} \prod_{j=i}^{i+m} c_j\},$$

which is zero for either $\varepsilon_{i+m} = 1$ or $\varepsilon_{i+m} = 2$ since one has

$$\begin{aligned} \{d_i, (c_{i+m} + 1) \prod_{j=i+m}^{i+m+n} c_j\} &= (\varepsilon_{i+m} - 2) \frac{d_i}{d_{i+m}} \prod_{j=i}^{i+m+n} c_j (2c_{i+m} + 1), \\ \{c_{i+m+n}, \frac{1}{d_{i+m}} \prod_{j=i}^{i+m} c_j\} &= (\varepsilon_{i+m+n} - 2) \frac{d_i}{d_{i+m}} \prod_{j=i}^{i+m+n} c_j (2c_{i+m} + 1) \\ &\quad + \{c_{i+m+n}, c_{i+m-1}\} \frac{1}{d_{i+m}} \prod_{\substack{j=i \\ j \neq i+m-1}}^{i+m} c_j. \end{aligned}$$

- If condition (B1) is true and $\varepsilon_{i+m} = 0$ then (4.38) is equal to

$$\begin{aligned} (2 - \varepsilon_{i+m+n}) \{d_i, \frac{(c_{i+m} + 1)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\} &+ (2 - \varepsilon_{i+m+n}) \{c_{i+m}, \frac{d_i}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j\} \\ &- 2 \{c_{i+m+n}, \frac{d_i}{d_{i+m}} \prod_{j=i}^{i+m} c_j\}. \end{aligned}$$

which is zero using the relations

$$\begin{aligned} \{d_i, \frac{(c_{i+m} + 1)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\} &= \frac{1}{d_{i+m+n}} \left(-2 \frac{d_i c_{i+m}}{d_{i+m}} \prod_{j=i}^{i+m+n} c_j + (c_{i+m} + 1) \sum_{r=i+m}^{i+m+n} \prod_{\substack{l=i+m \\ l \neq r}}^{i+m+n} c_l \{d_i, c_r\} \right) \\ &\quad - \frac{d_i}{d_{i+m+n}^2} (c_{i+m} + 1) \prod_{j=i}^{i+m+n-1} c_j \prod_{j=i+m}^{i+m+n} c_j. \\ \{c_{i+m}, \frac{d_i}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j\} &= \frac{2d_i}{d_{i+m} d_{i+m+n}} \prod_{j=i}^{i+m} c_j \prod_{j=i}^{i+m+n} c_j + \frac{d_i}{d_{i+m+n}} \left(\prod_{j=i+m}^{i+m+n} c_j \times \right. \\ &\quad \left. \sum_{r=i}^{i+m-1} \{c_{i+m}, c_r\} \prod_{\substack{l=i \\ l \neq r}}^{i+m-1} c_l + \prod_{j=i}^{i+m} c_j \sum_{r=i+m+1}^{i+m+n} \{c_{i+m}, c_r\} \prod_{\substack{l=i+m+1 \\ l \neq r}}^{i+m+n} c_r \right) \\ &\quad + (c_{i+m} + 1) \frac{d_i}{d_{i+m+n}^2} \prod_{j=i+m}^{i+m+n-1} c_j \prod_{j=i}^{i+m+n} c_j. \\ \{c_{i+m+n}, \frac{d_i}{d_{i+m}} \prod_{j=i}^{i+m} c_j\} &= \frac{(2 - \varepsilon_{i+m+n}) d_i}{d_{i+m} d_{i+m+n}} \left[\prod_{j=i}^{i+m+n} c_j \prod_{j=i}^{i+m} c_j - \prod_{j=i}^{i+m} c_j \sum_{r=i}^{i+m} (c_r + 1) \prod_{j=r+1}^{i+m+n} c_j \right. \\ &\quad \left. - c_{i+m} \prod_{j=i}^{i+m+n} c_j \right]. \end{aligned}$$

For the remaining cases of m and n is also verified that (4.38) is equal to zero.

Likewise, the next two cases are verified after a straightforward but lengthy computation.

Case 6:

$$\{c_i, \{d_{i+m}, c_{i+m+n}\}\} + \{d_{i+m}, \{c_{i+m+n}, c_i\}\} + \{c_{i+m+n}, \{c_i, d_{i+m}\}\}.$$

Case 7:

$$\{c_i, \{c_{i+m}, d_{i+m+n}\}\} + \{c_{i+m}, \{d_{i+m+n}, c_i\}\} + \{d_{i+m+n}, \{c_i, c_{i+m}\}\}.$$

Finally, the last case.

Case 8:

$$\{c_i, \{c_{i+m}, c_{i+m+n}\}\} + \{c_{i+m}, \{c_{i+m+n}, c_i\}\} + \{c_{i+m+n}, \{c_i, c_{i+m}\}\} \quad (4.39)$$

Consider the case $m > 1$ and $n > 1$.

(i) If conditions (A1) and (A2) are false then (4.39) is zero.

(ii) Assume that condition (A2) is true then (4.39) is equal to

$$\{c_i, \{c_{i+m}, c_{i+m+n}\}\} + \{c_{i+m}, \{c_{i+m+n}, c_i\}\} + (\varepsilon_{i+m} - 2)\{c_{i+m+n}, \frac{\prod_{j=i}^{i+m} c_j}{d_{i+m}}\}. \quad (4.40)$$

- If condition (B2) is true then (4.40) is equal to

$$\begin{aligned} & \frac{(\varepsilon_{i+m+n} - 2)}{d_{i+m+n}} \left[\{c_i, c_{i+m}\} \prod_{j=i+m+1}^{i+m+n} c_j + \{c_i, c_{i+m+1}\} \prod_{\substack{j=i+m \\ i+m+1}}^{i+m+n} c_j \right] \\ & + \frac{(\varepsilon_{i+m} - 2)(2 - \varepsilon_{i+m+n})}{d_i d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j. \end{aligned}$$

One sees that the last expression is zero either ε_{i+m} is zero or not zero.

- If condition (B1) is true and $\varepsilon_{i+m} \neq 0$ then (4.40), after simplifying, is equal to

$$\begin{aligned} & \frac{(2 - \varepsilon_{i+m+n})(\varepsilon_{i+m} - 2)}{d_{i+m+n} d_{i+m}} (2c_{i+m+1}) \prod_{j=i}^{i+m+n} c_j + \frac{(\varepsilon_{i+m} - 2)}{d_{i+m}} \left[(\varepsilon_{i+m+n} - 2) \frac{(c_{i+m} + 1)}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j \right. \\ & \left. + \{c_{i+m+n}, c_{i+m-1}\} \prod_{\substack{j=i \\ j \neq i+m-1}}^{i+m} c_j - \frac{(2 - \varepsilon_{i+m+n})}{d_{i+m+n}} c_{i+m} \prod_{j=i}^{i+m+n} c_j \right]. \end{aligned}$$

which is zero for either $\varepsilon_{i+m} = 1$ or 2.

- If condition (B1) is true and $\varepsilon_{i+m} = 0$ then (4.40) is equal to

$$\begin{aligned} & (2 - \varepsilon_{i+m+n}) \{c_i, \frac{(c_{i+m} + 1)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\} + (2 - \varepsilon_{i+m+n}) \{c_{i+m}, \frac{\prod_{j=i}^{i+m+n} c_j}{d_{i+m+n}}\} \\ & + (\varepsilon_{i+m} - 2) \{c_{i+m+n}, \frac{\prod_{j=i}^{i+m} c_j}{d_{i+m}}\}. \end{aligned}$$

The last expression is zero, after a computation, using the bracket relations and the following relations

$$\begin{aligned}
\{c_i, \frac{(c_{i+m}+1)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\} &= \frac{-2c_{i+m}}{d_{i+m}d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j + \frac{(c_{i+m}+1)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j \sum_{r=i+m}^{i+m+n} \frac{(\varepsilon_r-2)}{d_r} \prod_{j=i}^{r-1} c_j \\
&\quad - \frac{(c_{i+m}+1)}{d_{i+m+n}^2} \prod_{j=i+m}^{i+m+n} c_j \prod_{j=i}^{i+m+n-1} c_j, \\
\{c_{i+m}, \frac{\prod_{j=i}^{i+m+n} c_j}{d_{i+m+n}}\} &= \frac{1}{d_{i+m+n}} \left[\{c_{i+m}, \prod_{j=i}^{i+m-1} c_j\} \prod_{j=i+m}^{i+m+n} c_j + \{c_{i+m}, \prod_{j=i+m+1}^{i+m+n} c_j\} \prod_{j=i}^{i+m} c_j \right] \\
&\quad + \frac{(c_{i+m}+1)}{d_{i+m+n}^2} \prod_{j=i}^{i+m+n} c_j \prod_{j=i+m}^{i+m+n-1} c_j, \\
\{c_{i+m+n}, \frac{\prod_{j=i}^{i+m} c_j}{d_{i+m}}\} &= \frac{1}{d_{i+m}} \left[\prod_{j=i+1}^{i+m} c_j \{c_{i+m+n}, c_i\} + \sum_{r=i+1}^{i+m} \prod_{\substack{l=i \\ l \neq r}}^{i+m} c_l \{c_{i+m+n}, c_r\} \right] \\
&\quad - (2 - \varepsilon_{i+m+n}) \frac{c_{i+m}}{d_{i+m+n}d_{i+m}} \prod_{j=i}^{i+m+n} c_j.
\end{aligned}$$

- If conditions (B1) and (B2) are false then (4.40) is zero.

(iii) Assume that condition (A1) is true then (4.39) is equal to

$$\{c_i, \{c_{i+m}, c_{i+m+n}\}\} + \{c_{i+m}, \{c_{i+m+n}, c_i\}\} + (2 - \varepsilon_{i+m}) \{c_{i+m+n}, \frac{(c_i+1)}{d_{i+m}} \prod_{j=i}^{i+m} c_j\} \quad (4.41)$$

- If condition (B2) is true then (4.41) is equal to

$$\begin{aligned}
\frac{(\varepsilon_{i+m+n}-2)}{d_{i+m+n}} &\left[\{c_i, c_{i+m}\} \prod_{j=i+m+1}^{i+m+n} c_j + \{c_i, c_{i+m+1}\} \prod_{\substack{j=i+m \\ i+m+1}}^{i+m+n} c_j \right] \\
&+ \frac{(2 - \varepsilon_{i+m})}{d_{i+m}} (c_i+1) \{c_{i+m+n}, c_{i+m}\} \prod_{j=i}^{i+m-1} c_j.
\end{aligned}$$

One sees that the last expression is zero either ε_{i+m} is zero or not zero.

- If condition (B1) is true and $\varepsilon_{i+m} \neq 0$ then (4.41) is equal to

$$\begin{aligned}
&\frac{(2 - \varepsilon_{i+m+n})(2 - \varepsilon_{i+m})}{d_{i+m+n}d_{i+m}} (c_i+1)(2c_{i+m}+1) \prod_{j=i}^{i+m+n} c_j + (2 - \varepsilon_{i+m}) \frac{(c_i+1)}{d_{i+m}} \left[\frac{(\varepsilon_{i+m+n}-2)}{d_{i+m+n}} \times \right. \\
&\left. (c_{i+m}+1) \prod_{j=i}^{i+m+n} c_j + \{c_{i+m+n}, c_{i+m-1}\} \prod_{\substack{j=i \\ j \neq i+m-1}}^{i+m} c_j - \frac{(2 - \varepsilon_{i+m+n})}{d_{i+m+n}} c_{i+m} \prod_{j=i}^{i+m+n} c_j \right],
\end{aligned}$$

which is zero for either $\varepsilon_{i+m} = 1$ or 2.

- If condition (B1) is true and $\varepsilon_{i+m} = 0$ then (4.41) is equal to

$$(2 - \varepsilon_{i+m+n})\{c_i, \frac{(c_{i+m} + 1)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\} + (\varepsilon_{i+m+n} - 2)\{c_{i+m}, \frac{(c_i + 1)}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j\} \\ + (2 - \varepsilon_{i+m})\{c_{i+m+n}, \frac{(c_i + 1)}{d_{i+m}} \prod_{j=i}^{i+m} c_j\}.$$

The last expression is zero, after a computation, using the bracket relations and the following relations

$$\{c_i, \frac{(c_{i+m} + 1)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\} = \frac{2(c_i + 1)c_{i+m}}{d_{i+m}d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j + \frac{(c_{i+m} + 1)(c_i + 1)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j \times \\ \sum_{r=i+m}^{i+m+n} \frac{(2 - \varepsilon_r)}{d_r} \prod_{j=i}^{r-1} c_j + \frac{(c_{i+m} + 1)(c_i + 1)}{d_{i+m+n}^2} \prod_{j=i}^{i+m+n} c_j \prod_{j=i+m}^{i+m+n-1} c_j, \\ \{c_{i+m}, \frac{(c_i + 1)}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j\} = \frac{-2(c_i + 1)}{d_{i+m}d_{i+m+n}} \prod_{j=i}^{i+m} c_j \prod_{j=i}^{i+m+n} c_j + \frac{(c_i + 1)}{d_{i+m+n}} \sum_{r=i}^{i+m+n} \prod_{\substack{l=i \\ l \neq r}}^{i+m+n} c_l \{c_{i+m}, c_r\} \\ + \frac{(c_i + 1)(c_{i+m} + 1)}{d_{i+m+n}^2} \prod_{j=i}^{i+m+n} c_j \prod_{j=i+m}^{i+m+n-1} c_j, \\ \{c_{i+m+n}, \frac{(c_i + 1)}{d_{i+m}} \prod_{j=i}^{i+m} c_j\} = \frac{(\varepsilon_{i+m+n} - 2)(c_i + 1)}{d_{i+m+n}d_{i+m}} \prod_{j=i}^{i+m+n} c_j \prod_{j=i}^{i+m} c_j + \frac{(c_i + 1)(\varepsilon_{i+m+n} - 2)}{d_{i+m}d_{i+m+n}} \times \\ \prod_{j=i}^{i+m} c_j \sum_{r=i}^{i+m} (c_r + 1) \prod_{j=r+1}^{i+m+n} c_j - \frac{(2 - \varepsilon_{i+m+n})(c_i + 1)c_{i+m}}{d_{i+m}d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j.$$

- If conditions (B1) and (B2) are false then (4.41) is zero.

Similarly (4.39) is equal to zero for the remaining cases of m and n . $\square$

Lemma 4.20. *The Poisson bracket (4.31)–(4.34) is compatible with the Poisson bracket (4.29).*

Proof. We will denote the bracket (4.29) by $\{\cdot, \cdot\}_1$. As in the previous lemma one has to verify 8 cases.

Case 1

$$\{d_i, \{d_{i+m}, d_{i+m+n}\}_1\} + \{d_{i+m}, \{d_{i+m+n}, d_i\}_1\} + \{d_{i+m+n}, \{d_i, d_{i+m}\}_1\} + \\ \{d_i, \{d_{i+m}, d_{i+m+n}\}_1\} + \{d_{i+m}, \{d_{i+m+n}, d_i\}_1\} + \{d_{i+m+n}, \{d_i, d_{i+m}\}_1\}. \quad (4.42)$$

Since $\{d_i, d_j\}_1 = 0$, (4.42) becomes

$$\{d_i, \{d_{i+m}, d_{i+m+n}\}_1\} + \{d_{i+m}, \{d_{i+m+n}, d_i\}_1\} + \{d_{i+m+n}, \{d_i, d_{i+m}\}_1\} \quad (4.43)$$

First, consider the cases $m > 1$ and $n > 1$.

(i) If condition (A1) is false then (4.43) is zero.

(ii) Assume that condition (A1) is true:

- If (B1) is false then (4.43) is equal to

$$\{d_{i+m+n}, d_i \prod_{j=i}^{i+m-1} c_j\}_1 = d_i \{d_{i+m+n}, \prod_{j=i}^{i+m-1} c_j\}_1 = 0$$

since $\{d_i, c_j\}_1 = 0$ whenever $i - j > 1$.

- If (B1) is true and $\varepsilon_{i+m} \neq 0$ then (4.43) is equal to

$$d_{i+m} \{d_i, \prod_{j=i+m}^{i+m+n-1} c_j\}_1 + d_i \{d_{i+m+n}, \prod_{j=i}^{i+m-1} c_j\}_1 = 0.$$

- If (B1) is true and $\varepsilon_{i+m} = 0$ then (4.43) is equal to

$$d_{i+m} \{d_i, \prod_{j=i+m}^{i+m+n-1} c_j\}_1 - d_i \{d_{i+m}, \prod_{j=i}^{i+m+n-1} c_j\}_1 + d_i \{d_{i+m+n}, \prod_{j=i}^{i+m-1} c_j\}_1$$

which it reduces to

$$-d_i \left(\prod_{\substack{j=i \\ j \neq i+m-1}}^{i+m+n-1} c_j \{d_{i+m}, c_{i+m-1}\}_1 + \prod_{\substack{j=i \\ j \neq i+m}}^{i+m+n-1} c_j \{d_{i+m}, c_{i+m}\}_1 \right) = 0$$

Now let $m > 1$ and $n = 1$. Then (4.43) is equal to

$$\{d_{i+m}, \{d_{i+m+1}, d_i\}\}_1 + \{d_{i+m+1}, \{d_i, d_{i+m}\}\}_1 \quad (4.44)$$

(i) If condition (A1) is false then (4.44) is zero.

(ii) Assume that condition (A1) is true:

- If $\varepsilon_{i+m} \neq 0$ then (4.44) is equal to

$$\{d_{i+m+1}, d_i \prod_{j=i}^{i+m-1} c_j\}_1 = 0$$

- If $\varepsilon_{i+m} = 0$ then (4.44) is equal to

$$-d_i \{d_{i+m}, \prod_{j=i}^{i+m} c_j\}_1 = -d_i \left(\{d_{i+m}, c_{i+m}\}_1 \prod_{j=i}^{i+m-1} c_j + \{d_{i+m}, c_{i+m-1}\}_1 \prod_{\substack{j=i \\ j \neq i+m-1}}^{i+m} c_j \right) = 0.$$

Finally the case $m = n = 1$ is also verified.

Case 2

$$\begin{aligned} & \{d_i, \{d_{i+m}, c_{i+m+n}\}_1\} + \{d_{i+m}, \{c_{i+m+n}, d_i\}_1\} + \{c_{i+m+n}, \{d_i, d_{i+m}\}_1\} + \\ & \{d_i, \{d_{i+m}, c_{i+m+n}\}_1\} + \{d_{i+m}, \{c_{i+m+n}, d_i\}_1\} + \{c_{i+m+n}, \{d_i, d_{i+m}\}_1\}. \end{aligned} \quad (4.45)$$

From relations of $\{\cdot, \cdot\}_1$, equation (4.45) becomes

$$\{d_i, \{d_{i+m}, c_{i+m+n}\}_1\} + \{d_{i+m}, \{c_{i+m+n}, d_i\}_1\} + \{c_{i+m+n}, \{d_i, d_{i+m}\}_1\} \quad (4.46)$$

First, assume $m > 1$ and $n > 1$.

(i) If condition (A1) is false then one sees that (4.46) is zero.

(ii) Assume that condition (A1) is true then:

- If condition (B1) is false then (4.46) is zero.

- If condition (B1) is true and $\varepsilon_{i+m} \neq 0$ then (4.46) is equal to

$$\{d_i, (\varepsilon_{i+m+n} - 2) \frac{d_{i+m}}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\}_1 = 0$$

-If condition (B1) is true and $\varepsilon_{i+m} = 0$ then (4.46) is equal to

$$\{d_{i+m}, (2 - \varepsilon_{i+m+n}) \frac{d_i}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j\}_1 = (2 - \varepsilon_{i+m+n}) \frac{d_i}{d_{i+m+n}} \{d_{i+m}, \prod_{j=i}^{i+m+n} c_j\}_1$$

which is zero because

$$\{d_{i+m}, \prod_{j=i}^{i+m+n} c_j\}_1 = \{d_{i+m}, c_{i+m}\}_1 \prod_{\substack{j=i \\ j \neq i+m}}^{i+m+n} c_j + \{d_{i+m}, c_{i+m-1}\}_1 \prod_{\substack{j=i \\ j \neq i+m-1}}^{i+m+n} c_j = 0$$

Now, let $m > 1$ and $n = 1$, then (4.46) becomes

$$\{d_{i+m}, \{c_{i+m+1}, d_i\}_1\} + \{c_{i+m+1}, \{d_i, d_{i+m}\}_1\} \quad (4.47)$$

(i) If condition (A1) is false then (4.47) is zero.

(ii) Assume that condition (A1) is true.

- If $\varepsilon_{i+m} \neq 0$ then (4.47) is equal to

$$\{c_{i+m+1}, d_i \prod_{j=i}^{i+m-1} c_j\}_1 = 0$$

- If $\varepsilon_{i+m} = 0$ then (4.47) is equal to

$$\{d_{i+m}, (2 - \varepsilon_{i+m+1}) \frac{d_i}{d_{i+m+1}} \prod_{j=i}^{i+m+1} c_j\}_1 = (2 - \varepsilon_{i+m+1}) \frac{d_i}{d_{i+m+1}} \{d_{i+m}, \prod_{j=i}^{i+m+1} c_j\}_1$$

which is zero, since

$$\{d_{i+m}, \prod_{j=i}^{i+m+1} c_j\}_1 = \{d_{i+m}, c_{i+m-1}\}_1 \prod_{\substack{j=i \\ j \neq i+m-1}}^{i+m+1} c_j + \{d_{i+m}, c_{i+m}\}_1 \prod_{\substack{j=i \\ j \neq i+m}}^{i+m+1} c_j = 0.$$

Finally the case $m = n = 1$ is also verified.

The next two cases are also verified to be equal to zero.

Case 3:

$$\begin{aligned} & \{d_i, \{c_{i+m}, d_{i+m+n}\}_1\}_1 + \{c_{i+m}, \{d_{i+m+n}, d_i\}_1\}_1 + \{d_{i+m+n}, \{d_i, c_{i+m}\}_1\}_1 + \\ & \{d_i, \{c_{i+m}, d_{i+m+n}\}_1\}_1 + \{c_{i+m}, \{d_{i+m+n}, d_i\}_1\}_1 + \{d_{i+m+n}, \{d_i, c_{i+m}\}_1\}_1. \end{aligned}$$

Case 4:

$$\begin{aligned} & \{c_i, \{d_{i+m}, d_{i+m+n}\}_1\}_1 + \{d_{i+m}, \{d_{i+m+n}, c_i\}_1\}_1 + \{d_{i+m+n}, \{c_i, d_{i+m}\}_1\}_1 + \\ & \{c_i, \{d_{i+m}, d_{i+m+n}\}_1\}_1 + \{d_{i+m}, \{d_{i+m+n}, c_i\}_1\}_1 + \{d_{i+m+n}, \{c_i, d_{i+m}\}_1\}_1. \end{aligned}$$

Case 5:

$$\begin{aligned} & \{d_i, \{c_{i+m}, c_{i+m+n}\}_1\}_1 + \{c_{i+m}, \{c_{i+m+n}, d_i\}_1\}_1 + \{c_{i+m+n}, \{d_i, c_{i+m}\}_1\}_1 + \\ & \{d_i, \{c_{i+m}, c_{i+m+n}\}_1\}_1 + \{c_{i+m}, \{c_{i+m+n}, d_i\}_1\}_1 + \{c_{i+m+n}, \{d_i, c_{i+m}\}_1\}_1. \end{aligned} \quad (4.48)$$

First, consider the case: $m > 1$ and $n > 1$. Thus (4.48) becomes

$$\{d_i, \{c_{i+m}, c_{i+m+n}\}_1\}_1 + \{c_{i+m}, \{c_{i+m+n}, d_i\}_1\}_1 + \{c_{i+m+n}, \{d_i, c_{i+m}\}_1\}_1 \quad (4.49)$$

(i) If condition (A1) is false one sees that (4.49) is zero.

(ii) Assume condition (A1) is true.

- If conditions (B1) and (B2) are false then (4.49) is zero.

- If condition (B1) is true and $\varepsilon_{i+m} \neq 0$ then (4.49) is equal to

$$(2 - \varepsilon_{i+m+n}) \{d_i, \frac{(c_{i+m} + 1)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\}_1 = 0$$

- If condition (B1) is true and $\varepsilon_{i+m} = 0$ then (4.49) is equal to

$$(2 - \varepsilon_{i+m+n}) \frac{d_i}{d_{i+m+n}} \{c_{i+m}, \prod_{j=i}^{i+m+n} c_j\}_1.$$

The last expression is zero, since

$$\begin{aligned} \{c_{i+m}, \prod_{j=i}^{i+m+n} c_j\}_1 &= \{c_{i+m}, c_{i+m-1}\}_1 \prod_{\substack{j=i \\ j \neq i+m-1}}^{i+m+n} c_j + \{c_{i+m}, c_{i+m+1}\}_1 \prod_{\substack{j=i \\ j \neq i+m+1}}^{i+m+n} c_j \\ &= (1 - \varepsilon_{i+m}) c_{i+m} \prod_{j=i}^{i+m+n} c_j + (\varepsilon_{i+m+1} - 1) c_{i+m} \prod_{j=i}^{i+m+n} c_j = 0 \end{aligned}$$

- If condition (B2) is true then (4.49) is equal to

$$\{d_i, \frac{(\varepsilon_{i+m+n} - 2)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\}_1 = 0.$$

Now consider the case $m > 1$ and $n = 1$, then (4.48) becomes

$$(\varepsilon_{i+m+1} - 1)\{d_i, c_{i+m}c_{i+m+1}\} + \{c_{i+m}, \{c_{i+m+1}, d_i\}\}_1 + \{c_{i+m+1}, \{d_i, c_{i+m}\}\}_1. \quad (4.50)$$

(i) If condition (A1) is false the (4.50) is equal to zero.

(ii) Assume that condition (A1) is true.

- If $\varepsilon_{i+m} \neq 0$ then (4.50) is equal to

$$\begin{aligned} & (\varepsilon_{i+m+1} - 1)c_{i+m+1}\{d_i, c_{i+m}\} + \{c_{i+m+1}, (\varepsilon_{i+m} - 2)\frac{d_i}{d_{i+m}} \prod_{j=i}^{i+m} c_j\}_1 \\ &= (\varepsilon_{i+m+1} - 1)(\varepsilon_{i+m} - 2)\frac{d_i}{d_{i+m}} \prod_{j=i}^{i+m+1} c_j + (\varepsilon_{i+m} - 2)\frac{d_i}{d_{i+m}} \{c_{i+m+1}, c_{i+m}\}_1 \prod_{j=i}^{i+m-1} c_j = 0. \end{aligned}$$

- If $\varepsilon_{i+m} = 0$ then (4.50) is equal to

$$\begin{aligned} & (\varepsilon_{i+m+1} - 1)c_{i+m}\{d_i, c_{i+m+1}\} + \{c_{i+m}, (2 - \varepsilon_{i+m+1})\frac{d_i}{d_{i+m+1}} \prod_{j=i}^{i+m+1} c_j\}_1 \\ &= (\varepsilon_{i+m+1} - 1)(\varepsilon_{i+m+1} - 2)c_{i+m}\frac{d_i}{d_{i+m+1}} \prod_{j=i}^{i+m+1} c_j + (2 - \varepsilon_{i+m+1})\frac{d_i}{d_{i+m+1}} \times \\ & \quad \left[\{c_{i+m}, c_{i+m-1}\}_1 \prod_{j=i}^{i+m+1} c_j + \{c_{i+m}, c_{i+m+1}\}_1 \prod_{j=i}^{i+m} c_j - \frac{\{c_{i+m}, d_{i+m+1}\}_1}{d_{i+m+1}} \prod_{j=i}^{i+m+1} c_j \right] \\ &= 0. \end{aligned}$$

Finally, the last case $m = n = 1$ also verify that (4.48) is equal to be zero.

Analogously, the next two cases can be verified to be equal to zero.

Case 6:

$$\begin{aligned} & \{c_i, \{d_{i+m}, c_{i+m+n}\}_1\} + \{d_{i+m}, \{c_{i+m+n}, c_i\}_1\} + \{c_{i+m+n}, \{c_i, d_{i+m}\}_1\} + \\ & \{c_i, \{d_{i+m}, c_{i+m+n}\}\}_1 + \{d_{i+m}, \{c_{i+m+n}, c_i\}\}_1 + \{c_{i+m+n}, \{c_i, d_{i+m}\}\}_1. \end{aligned}$$

Case 7:

$$\begin{aligned} & \{c_i, \{c_{i+m}, d_{i+m+n}\}_1\} + \{c_{i+m}, \{d_{i+m+n}, c_i\}_1\} + \{d_{i+m+n}, \{c_i, c_{i+m}\}_1\} + \\ & \{c_i, \{c_{i+m}, d_{i+m+n}\}\}_1 + \{c_{i+m}, \{d_{i+m+n}, c_i\}\}_1 + \{d_{i+m+n}, \{c_i, c_{i+m}\}\}_1. \end{aligned}$$

Case 8:

$$\begin{aligned} & \{c_i, \{c_{i+m}, c_{i+m+n}\}_1\} + \{c_{i+m}, \{c_{i+m+n}, c_i\}_1\} + \{c_{i+m+n}, \{c_i, c_{i+m}\}_1\} + \\ & \{c_i, \{c_{i+m}, c_{i+m+n}\}\}_1 + \{c_{i+m}, \{c_{i+m+n}, c_i\}\}_1 + \{c_{i+m+n}, \{c_i, c_{i+m}\}\}_1. \end{aligned} \quad (4.51)$$

First, assume $m > 1$ and $n > 1$. Then (4.51) becomes

$$\{c_i, \{c_{i+m}, c_{i+m+n}\}\}_1 + \{c_{i+m}, \{c_{i+m+n}, c_i\}\}_1 + \{c_{i+m+n}, \{c_i, c_{i+m}\}\}_1. \quad (4.52)$$

(i) If conditions (A1) and (A2) are false then (4.52) is zero.

(ii) Assume that condition (A1) is true.

- If conditions (B1) and (B2) are false, follows immediately that (4.52) is zero.

- If condition (B2) is true then (4.52) becomes

$$(\varepsilon_{i+m+n} - 2)\{c_i, \frac{1}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\}_1 + (2 - \varepsilon_{i+m})\{c_{i+m+n}, \frac{(c_i + 1)}{d_{i+m}} \prod_{j=i}^{i+m} c_j\}_1 = 0.$$

- If condition (B1) is true and $\varepsilon_{i+m} \neq 0$ then (4.52) becomes

$$(2 - \varepsilon_{i+m+n})\{c_i, \frac{c_{i+m} + 1}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\}_1 + (2 - \varepsilon_{i+m})\{c_{i+m+n}, \frac{(c_i + 1)}{d_{i+m}} \prod_{j=i}^{i+m} c_j\}_1 = 0.$$

- If condition (B1) is true and $\varepsilon_{i+m} = 0$ then (4.52) becomes

$$(\varepsilon_{i+m+n} - 2)\{c_{i+m}, \frac{(c_i + 1)}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j\}_1 = (\varepsilon_{i+m+n} - 2)\frac{(c_i + 1)}{d_{i+m+n}}\{c_{i+m}, \prod_{j=i}^{i+m+n} c_j\}_1 = 0.$$

The last equality was verified in the Case 5 (ii) of this lemma.

(iii) Assume that condition (A2) is true.

- If conditions (B1) and (B2) are false, follows immediately that (4.52) is zero.

- If condition (B2) is true then (4.52) becomes

$$(\varepsilon_{i+m+n} - 2)\{c_i, \frac{1}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\}_1 + (\varepsilon_{i+m} - 2)\{c_{i+m+n}, \frac{1}{d_{i+m}} \prod_{j=i}^{i+m} c_j\}_1 = 0.$$

- If condition (B1) is true and $\varepsilon_{i+m} \neq 0$ then (4.52) becomes

$$(\varepsilon_{i+m+n} - 2)\{c_i, \frac{(c_i + 1)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\}_1 + (\varepsilon_{i+m} - 2)\{c_{i+m+n}, \frac{1}{d_{i+m}} \prod_{j=i}^{i+m} c_j\}_1 = 0.$$

- If condition (B1) is true and $\varepsilon_{i+m} = 0$ then (4.52) becomes

$$\begin{aligned} (\varepsilon_{i+m+n} - 2)\{c_i, \frac{(c_i + 1)}{d_{i+m+n}} \prod_{j=i+m}^{i+m+n} c_j\}_1 &+ (\varepsilon_{i+m+n} - 2)\{c_{i+m}, \frac{(c_i + 1)}{d_{i+m+n}} \prod_{j=i}^{i+m+n} c_j\}_1 \\ &+ (\varepsilon_{i+m} - 2)\{c_{i+m+n}, \frac{1}{d_{i+m}} \prod_{j=i}^{i+m} c_j\}_1 = 0. \end{aligned}$$

since $m > 1$ and $n > 1$.

Now consider the case $m > 1$ and $n = 1$. Then (4.51) reduces to

$$(\varepsilon_{i+m+1} - 1)\{c_i, c_{i+m}c_{i+m+1}\} + \{c_{i+m}\{c_{i+m+1}, c_i\}\}_1 + \{c_{i+m+1}\{c_i, c_{i+m}\}\}_1 \quad (4.53)$$

(i) If (A1) and (A2) are false then (4.53) is zero.

(ii) If (A1) is true.

- If $\varepsilon_{i+m} \neq 0$ then (4.53) is equal to

$$\begin{aligned} & (\varepsilon_{i+m+1} - 1)\{c_i, c_{i+m}\}c_{i+m+1} + \{c_{i+m+1}, \{c_i, c_{i+m}\}\}_1 \\ &= (\varepsilon_{i+m+1} - 1)(2 - \varepsilon_{i+m})\frac{(c_i + 1)}{d_{i+m}} \prod_{j=i}^{i+m+1} c_j + (2 - \varepsilon_{i+m})\frac{(c_i + 1)}{d_{i+m}} \{c_{i+m+1}, c_{i+m}\}_1 \prod_{j=i}^{i+m-1} c_j = 0. \end{aligned}$$

- If $\varepsilon_{i+m} = 0$ then (4.53), after simplifying, is equal to

$$\begin{aligned} & (\varepsilon_{i+m+1} - 1) \left(2\frac{(c_i + 1)}{d_{i+m}} \prod_{j=i}^{i+m-1} c_j + (2 - \varepsilon_{i+m+1})\frac{(c_i + 1)}{d_{i+m+n}} c_{i+m} \prod_{j=i}^{i+m+1} c_j \right) + (\varepsilon_{i+m+1} - 2) \times \\ & \frac{(c_i + 1)}{d_{i+m+1}} \left[\prod_{\substack{j=i \\ j \neq i+m-1}}^{i+m+1} c_j \{c_{i+m}, c_{i+m-1}\}_1 + \prod_{j=i}^{i+m} c_j \{c_{i+m}, c_{i+m+1}\}_1 - \frac{\prod_{j=i}^{i+m+1} c_j}{d_{i+m+1}} \{c_{i+m}, d_{i+m+1}\}_1 \right] \\ & + (2 - \varepsilon_{i+m})\frac{(c_i + 1)}{d_{i+m}} \prod_{j=i}^{i+m-1} c_j \{c_{i+m+1}, c_{i+m}\}_1 \end{aligned}$$

which is zero.

(iii) If (A2) is true.

- If $\varepsilon_{i+m} \neq 0$ then (4.53) will be equal to

$$(\varepsilon_{i+m+1} - 1)\frac{(\varepsilon_{i+m} - 2)}{d_{i+m}} \prod_{j=i}^{i+m+1} c_j + \frac{(\varepsilon_{i+m} - 2)}{d_{i+m}} \{c_{i+m+1}, c_{i+m}\}_1 \prod_{j=i}^{i+m-1} c_j = 0$$

- If $\varepsilon_{i+m} = 0$ then (4.53) is equal to

$$\begin{aligned} & -2\frac{(\varepsilon_{i+m+1} - 1)}{d_{i+m}} \prod_{j=i}^{i+m+1} c_j + (\varepsilon_{i+m+1} - 1)(\varepsilon_{i+m+1} - 2)\frac{c_{i+m}}{d_{i+m+1}} \prod_{j=i}^{i+m+1} c_j + \frac{(2 - \varepsilon_{i+m+1})}{d_{i+m+1}} \times \\ & \left[\{c_{i+m}, c_{i+m+1}\}_1 \prod_{j=i}^{i+m} c_j + \{c_{i+m}, c_{i+m-1}\}_1 \prod_{\substack{j=i \\ j \neq i+m-1}}^{i+m} c_j - \frac{\prod_{j=i}^{i+m+1} c_j}{d_{i+m+1}} \{c_{i+m}, d_{i+m+1}\}_1 \right] \\ & - \frac{2}{d_{i+m}} \prod_{j=i}^{i+m-1} c_j \{c_{i+m+1}, c_{i+m}\}_1 = 0 \end{aligned}$$

Finally the case $m = n = 1$ can be verified after consider the three possible cases for $\varepsilon_{i+1} = 0, 1$ or 2 . $\square$

Theorem 4.18 and the Poisson bracket (4.29) together with functions F_2 and F_1 , respectively, give us an explicit description of the bi-Hamiltonian representation for Coxeter-Toda lattices.

Remark 4.21. (i) We consider would be interesting to compute explicit formulas of the Poisson brackets on coordinates c_i, d_i , as in Theorem 4.18, corresponding to the remaining compatible Poisson brackets $\{\cdot, \cdot\}_k$ ($k = 2, \dots, n-1$) of the Faybusovich-Gekhtman family.

(ii) General formulas of Poisson brackets like (4.29) were presented for Coxeter-Toda flows in simple Poisson-Lie groups using the factorization parameters (see formula (4.1) of [17]). We believe it is possible to extend the Poisson bracket formulae (4.31)–(4.34) in Theorem 4.18 for general Poisson-Lie groups in terms of factorization parameters.

The formulae (4.31)–(4.34) were deduced after finding all possible brackets relations for particular small values of n . We present two cases which were obtained with the help of the MAPLE software.

Example 4.22. The case $n = 3$ is shown in the table below.

	$u^{-1} = s_1 s_2, v = s_1 s_2$	$u^{-1} = s_2 s_1, v = s_1 s_2$	$u^{-1} = s_2 s_1, v = s_2 s_1$
$\{c_1, c_2\}$	0	$\frac{c_1 c_2 (c_1 + 1)}{d_2}$	$\frac{2c_1 c_2 (c_1 + 1)}{d_2}$
$\{c_1, d_1\}$	c_1	c_1	c_1
$\{c_1, d_2\}$	$-c_1(c_1 + 1)$	$-c_1(c_1 + 1)$	$-c_1(c_1 + 1)$
$\{c_1, d_3\}$	$c_1 c_2$	0	$-c_1 c_2 (c_1 + 1)$
$\{c_2, d_1\}$	0	$\frac{c_1 c_2 d_1}{d_2}$	$\frac{2c_1 c_2 d_1}{d_2}$
$\{c_2, d_2\}$	c_2	c_2	c_2
$\{c_2, d_3\}$	$-c_2(c_2 + 1)$	$-c_2(c_2 + 1)$	$-c_2(c_2 + 1)$
$\{d_1, d_2\}$	$c_1 d_1$	$c_1 d_1$	$c_1 d_1$
$\{d_1, d_3\}$	0	0	$c_1 c_2 d_1$
$\{d_2, d_3\}$	$c_2 d_2$	$c_2 d_2$	$c_2 d_2$
ϵ_1	2	2	2
ϵ_2	2	1	0
ϵ_3	0	0	0

Example 4.23. The case $n = 4$, in the two tables below. Here are listed all possible brackets, for others v and u^{-1} not presented in the table, the bracket coincides with one of this list. More precisely, if the ϵ_i and ϵ'_i associated to different v, u^{-1} and v', u'^{-1} coincide then the brackets relations are the same.

	$v = s_1 s_2 s_3$ $u^{-1} = s_1 s_2 s_3$	$v = s_1 s_2 s_3$ $u^{-1} = s_1 s_3 s_2$	$v = s_1 s_3 s_2$ $u^{-1} = s_1 s_3 s_2$	$v = s_1 s_2 s_3$ $u^{-1} = s_2 s_3 s_1$	$v = s_1 s_2 s_3$ $u^{-1} = s_3 s_2 s_1$
$\{c_1, c_2\}$	0	0	0	$\frac{c_1 c_2 (c_1 + 1)}{d_2}$	$\frac{c_1 c_2 (c_1 + 1)}{d_2}$
$\{c_1, c_3\}$	0	$-\frac{c_1 c_2 c_3}{d_3}$	$-\frac{2c_1 c_2 c_3}{d_3}$	0	0
$\{c_1, d_1\}$	c_1	c_1	c_1	c_1	c_1
$\{c_1, d_2\}$	$-c_1(c_1 + 1)$	$-c_1(c_1 + 1)$	$-c_1(c_1 + 1)$	$-c_1(c_1 + 1)$	$-c_1(c_1 + 1)$
$\{c_1, d_3\}$	$c_1 c_2$	$c_1 c_2$	$c_1 c_2$	0	0
$\{c_1, d_4\}$	0	0	$c_1 c_2 c_3$	0	0
$\{c_2, c_3\}$	0	$\frac{c_2 c_3 (c_2 + 1)}{d_3}$	$\frac{2c_2 c_3 (c_2 + 1)}{d_3}$	0	$\frac{c_2 c_3 (c_2 + 1)}{d_3}$
$\{c_2, d_1\}$	0	0	0	$\frac{c_1 c_2 d_1}{d_2}$	$\frac{c_1 c_2 d_1}{d_2}$
$\{c_2, d_2\}$	c_2	c_2	c_2	c_2	c_2
$\{c_2, d_3\}$	$-c_2(c_2 + 1)$	$-c_2(c_2 + 1)$	$-c_2(c_2 + 1)$	$-c_2(c_2 + 1)$	$-c_2(c_2 + 1)$
$\{c_2, d_4\}$	$c_2 c_3$	0	$-c_2 c_3 (c_2 + 1)$	$c_2 c_3$	0
$\{c_3, d_1\}$	0	0	0	0	0
$\{c_3, d_2\}$	0	$\frac{c_2 c_3 d_2}{d_3}$	$\frac{2c_2 c_3 d_2}{d_3}$	0	$\frac{c_2 c_3 d_2}{d_3}$
$\{c_3, d_3\}$	c_3	c_3	c_3	c_3	c_3
$\{c_3, d_4\}$	$-c_3(c_3 + 1)$	$-c_3(c_3 + 1)$	$-c_3(c_3 + 1)$	$-c_3(c_3 + 1)$	$-c_3(c_3 + 1)$
$\{d_1, d_2\}$	$c_1 d_1$	$c_1 d_1$	$c_1 d_1$	$c_1 d_1$	$c_1 d_1$
$\{d_1, d_3\}$	0	0	0	0	0
$\{d_1, d_4\}$	0	0	0	0	0
$\{d_2, d_3\}$	$c_2 d_2$	$c_2 d_2$	$c_2 d_2$	$c_2 d_2$	$c_2 d_2$
$\{d_2, d_4\}$	0	0	$c_2 c_3 d_2$	0	0
$\{d_3, d_4\}$	$c_3 d_3$	$c_3 d_3$	$c_3 d_3$	$c_3 d_3$	$c_3 d_3$
ϵ_1	2	2	2	2	2
ϵ_2	2	2	2	1	1
ϵ_3	2	1	0	2	1
ϵ_4	0	0	0	0	0

	$v = s_1 s_3 s_2$ $u^{-1} = s_3 s_2 s_1$	$v = s_2 s_3 s_1$ $u^{-1} = s_2 s_3 s_1$	$v = s_3 s_2 s_1$ $u^{-1} = s_2 s_3 s_1$	$v = s_3 s_2 s_1$ $u^{-1} = s_3 s_2 s_1$
$\{c_1, c_2\}$	$\frac{c_1 c_2 (c_1 + 1)}{d_2}$	$\frac{2(c_1 + 1) c_1 c_2}{d_2}$	$\frac{2(c_1 + 1) c_1 c_2}{d_2}$	$\frac{2(c_1 + 1) c_1 c_2}{d_2}$
$\{c_1, c_3\}$	0	0	$\frac{(c_1 + 1) c_1 c_2 c_3}{d_3}$	$\frac{2(c_1 + 1) c_1 c_2 c_3}{d_3}$
$\{c_1, d_1\}$	c_1	c_1	c_1	c_1
$\{c_1, d_2\}$	$-c_1(c_1 + 1)$	$-c_1(c_1 + 1)$	$-c_1(c_1 + 1)$	$-c_1(c_1 + 1)$
$\{c_1, d_3\}$	0	$-c_1 c_2(c_1 + 1)$	$-c_1 c_2(c_1 + 1)$	$-c_1 c_2(c_1 + 1)$
$\{c_1, d_4\}$	0	0	0	$-c_1 c_2 c_3(c_1 + 1)$
$\{c_2, c_3\}$	$\frac{2c_2 c_3 (c_2 + 1)}{d_3}$	0	$\frac{c_2 c_3 (c_2 + 1)}{d_3}$	$\frac{2c_2 c_3 (c_2 + 1)}{d_3}$
$\{c_2, d_1\}$	$\frac{c_1 c_2 d_1}{d_2}$	$\frac{2c_1 c_2 d_1}{d_2}$	$\frac{2c_1 c_2 d_1}{d_2}$	$\frac{2c_1 c_2 d_1}{d_2}$
$\{c_2, d_2\}$	c_2	c_2	c_2	c_2
$\{c_2, d_3\}$	$-c_2(c_2 + 1)$	$-c_2(c_2 + 1)$	$-c_2(c_2 + 1)$	$-c_2(c_2 + 1)$
$\{c_2, d_4\}$	$-c_2 c_3(c_2 + 1)$	$c_2 c_3$	0	$-c_2 c_3(c_2 + 1)$
$\{c_3, d_1\}$	0	0	$\frac{c_1 c_2 c_3 d_1}{d_3}$	$\frac{2c_1 c_2 c_3 d_1}{d_3}$
$\{c_3, d_2\}$	$\frac{2c_2 c_3 d_2}{d_3}$	0	$\frac{c_2 c_3 d_2}{d_3}$	$\frac{2c_2 c_3 d_2}{d_3}$
$\{c_3, d_3\}$	c_3	c_3	c_3	c_3
$\{c_3, d_4\}$	$-c_3(c_3 + 1)$	$-c_3(c_3 + 1)$	$-c_3(c_3 + 1)$	$-c_3(c_3 + 1)$
$\{d_1, d_2\}$	$c_1 d_1$	$c_1 d_1$	$c_1 d_1$	$c_1 d_1$
$\{d_1, d_3\}$	0	$c_1 c_2 d_1$	$c_1 c_2 d_1$	$c_1 c_2 d_1$
$\{d_1, d_4\}$	0	0	0	$c_1 c_2 c_3 d_1$
$\{d_2, d_3\}$	$c_2 d_2$	$c_2 d_2$	$c_2 d_2$	$c_2 d_2$
$\{d_2, d_4\}$	$c_2 c_3 d_2$	0	0	$c_2 c_3 d_2$
$\{d_3, d_4\}$	$c_3 d_3$	$c_3 d_3$	$c_3 d_3$	$c_3 d_3$
ϵ_1	2	2	2	2
ϵ_2	1	0	0	0
ϵ_3	0	2	1	0
ϵ_4	0	0	0	0

Although the Poisson bracket (4.31)–(4.34) was deduced experimentally using the Atiyah-Hitchin bracket $\{\cdot, \cdot\}_0$ and item (i) of Proposition 4.14, we were not able to prove the following statement.

Conjecture 4.1. *The boundary measurement map $M(\lambda) = -H_0 m(-\lambda, X) : (G^{u,v}/H, \{\cdot, \cdot\}) \rightarrow (\mathcal{R}_n, \{\cdot, \cdot\}_0)$ is a Poisson map, for any $X \in G^{u,v}$ with factorization (4.25).*

4.3.1 Examples

We show that the Poisson bracket defined in (4.31)–(4.34) includes well known Poisson brackets for particular choice of Coxeter elements $u, v \in S_n$.

Case 1: Let $v = u^{-1} = s_{n-1} \dots s_1$ then $\epsilon = (2, 0, \dots, 0)$.

In this case, the Poisson bracket (4.31)–(4.34), one has for $i < j$ the following brackets:

$$\begin{aligned}
 \{c_i, d_j\} &= c_i, & \{c_i, c_j\} &= \frac{2(c_i + 1) \prod_{l=i}^j c_l}{d_j}, & \{c_i, d_j\} &= -(c_i + 1) \prod_{l=i}^{j-1} c_l, \\
 \{c_j, d_i\} &= \frac{2d_i \prod_{l=i}^j c_l}{d_j}, & \{d_i, d_j\} &= d_i \prod_{l=i}^{j-1} c_l.
 \end{aligned} \tag{4.54}$$

Now, the function F_2 becomes

$$F_2 = \frac{1}{2}(d_1^2 + \sum_{i=1}^{n-1} (2d_i^2 c_i + (d_{i+1} + c_i d_i)^2).$$

After computation, using the bracket (4.54) one has

$$\begin{aligned} \{F_2, d_i\} &= d_i(d_i c_i - d_{i-1} c_{i-1}), \quad i = 1, \dots, n \\ \{F_2, d_i c_i\} &= d_i c_i (d_{i+1} - d_i), \quad i = 1, \dots, n-1. \end{aligned} \quad (4.55)$$

Let

$$r_{2i} = d_i, \quad r_{2i+1} = d_i c_i, \quad i = 1, \dots, n. \quad (4.56)$$

with $c_n = 0, r_1 = r_{2n+1} = 0$, and consider the system of ODE's

$$\frac{dr_i}{dt} = \{F_2, r_i\}, \quad i = 2, \dots, 2n.$$

Then the above system, using (4.55), becomes the Volterra lattice

$$\frac{dr_i}{dt} = r_i(r_{i+1} - r_{i-1}), \quad \text{for } i = 2, \dots, 2n, \quad r_1 = r_{2n+1} = 0.$$

Furthermore, one can see that the function $F_1 = d_1 + \sum_{i=1}^{n-1} (d_{i+1} + c_i d_i)$ is a Casimir for the bracket (4.54).

In [6] a non-local Poisson bracket for the Volterra lattice is presented, obtained as the pullback of (4.3) via the Miura map;

$$\{r_i, r_j\}_{nl}^v = (-1)^{i+j} \frac{\prod_{i \leq 2k+1 \leq j} r_{2k+1}}{\prod_{i < 2k < j} r_{2k}}, \quad (i < j).$$

We recover now this bracket using (4.54) and the change of variables (4.56). Let us assume $i < j$. Then

$$\begin{aligned} \{r_{2i-1}, r_{2j}\} &= \{d_{i-1} c_{i-1}, d_j\} = d_{i-1} c_{i-1} \prod_{l=i-1}^{j-1} c_l - d_{i-1} (c_{i-1} + 1) \prod_{l=i-1}^{j-1} c_l \\ &= -d_{i-1} \prod_{l=i-1}^{j-1} c_l \end{aligned}$$

On the other hand:

$$\begin{aligned} \{r_{2i-1}, r_{2j}\}_{nl}^v &= - \frac{\prod_{2i-1 \leq 2k+1 \leq 2j} r_{2k+1}}{\prod_{2i-1 < 2k < 2j} r_{2k}} = \frac{r_{2i-1} \dots r_{2j-1}}{r_{2i} \dots r_{2(j-1)}} \\ &= - \frac{d_{i-1} c_{i-1} \dots d_{j-1} c_{j-1}}{d_i \dots d_{j-1}} = -d_{i-1} \prod_{l=i-1}^{j-1} c_l. \end{aligned}$$

Also

$$\begin{aligned}\{r_{2i}, r_{2j}\}_{nl}^v &= \frac{\prod_{2i \leq 2k+1 \leq 2j} r_{2k+1}}{\prod_{2i < 2k < 2j} r_{2k}} = \frac{d_i c_i \dots d_{j-1} c_{j-1}}{d_{i+1} \dots d_{j-1}} \\ &= d_i \prod_{l=i}^{j-1} c_l.\end{aligned}$$

coincides with the brackets defined previously above. Finally

$$\begin{aligned}\{r_{2i+1}, r_{2j+1}\} &= d_i d_j \{c_i, c_j\} + d_i c_j \{c_i, d_j\} + c_i d_j \{d_i, c_j\} + c_i c_j \{d_i, d_j\} \\ &= d_i d_j \frac{2(c_i + 1) \prod_{l=i}^j c_l}{d_j} - d_i c_j (c_i + 1) \prod_{l=i}^{j-1} c_l - c_i d_j \frac{2d_i \prod_{l=i}^j c_l}{d_j} + c_i c_j d_i \prod_{l=i}^{j-1} c_l \\ &= d_i \prod_{l=i}^j c_l\end{aligned}$$

which coincides with

$$\begin{aligned}\{r_{2i+1}, r_{2j+1}\}_{nl}^v &= \frac{\prod_{2i+1 \leq 2k+1 \leq 2j+1} r_{2k+1}}{\prod_{2i+1 < 2k < 2j+1} r_{2k}} = \frac{r_{2i+1} \dots r_{2j+1}}{r_{2(i+1)} \dots r_{2j}} \\ &= \frac{d_i c_i \dots d_j c_j}{d_{i+1} \dots d_j} = d_i \prod_{l=i}^j c_l.\end{aligned}$$

Another change of variables:

$$a_i = c_i d_i^2, \quad i = 1, \dots, n-1, \quad b_i = d_i + c_{i-1} d_{i-1}, \quad i = 1, \dots, n.$$

with $b_1 = d_1$ allows us to recover the linear Poisson bracket of the of the non-periodic Toda lattice. In fact, if $i < j$ then:

$$\begin{aligned}\{a_i, b_i\} &= d_i^2 \{c_i, d_i\} + \{d_i^2 c_i, d_{i-1} c_{i-1}\} \\ &= d_i^2 c_i + 2c_i d_i (c_{i-1} d_{i-1}) + d_i^2 (-2 \frac{c_{i-1} c_i d_{i-1}}{d_i}) = d_i^2 c_i = a_i. \\ \{a_i, b_{i+1}\} &= 2d_i c_i \{d_i, d_{i+1}\} + d_i^2 \{c_i, d_{i+1}\} + d_i^2 c_i \{d_i, c_i\} \\ &= 2d_i^2 c_i^2 - d_i^2 (c_i + 1) c_i - d_i^2 c_i^2 = -d_i^2 c_i = -a_i. \\ \{a_i, a_j\} &= c_i c_j \{d_i^2, d_j^2\} + c_i d_j^2 \{d_i^2, c_j\} + d_i^2 c_j \{c_i, d_j^2\} + d_i^2 d_j^2 \{c_i, c_j\} \\ &= 4c_i c_j d_i d_j \{d_i, d_j\} + 2c_i d_i d_j^2 \{d_i, c_j\} + 2d_j d_i^2 c_j \{c_i, d_j\} + d_i^2 d_j^2 \{c_i, c_j\} \\ &= 0. \\ \{b_i, b_j\} &= 0 \\ \{a_i, b_j\} &= 0, \quad \text{if } j > i + 1.\end{aligned} \tag{4.57}$$

Furthermore the function F_2 , written in variables a_i, b_i , becomes

$$F_2 = \sum_{i=1}^{n-1} a_i + \frac{1}{2} \sum_{i=1}^n b_i^2$$

which is the Hamiltonian function for the Hamiltonian interpretation of the Toda system using the bracket (4.57).

Case 2: $u = v = s_{n-1} \cdots s_1$, then $\varepsilon = (2, 1, \dots, 1, 0)$.

In this case, for $i < j$ the Poisson bracket (4.31)–(4.34) becomes

$$\begin{aligned} \{c_i, d_i\} &= c_i, & \{c_i, c_{i+1}\} &= \frac{(c_i + 1)c_i c_{i+1}}{d_{i+1}}, & \{c_i, d_{i+1}\} &= -c_i(c_i + 1), \\ \{c_{i+1}, d_i\} &= \frac{d_i c_i c_{i+1}}{d_{i+1}}, & \{d_i, d_{i+1}\} &= d_i c_i. \end{aligned} \tag{4.58}$$

After the change of variables $\tilde{c}_i = c_i d_i$ the above Poisson bracket relations become

$$\{\tilde{c}_i, d_i\} = \tilde{c}_i = \{d_i, d_{i+1}\} = -\{\tilde{c}_i, d_{i+1}\}$$

which is the linear Poisson bracket for the Hamiltonian representation of relativistic Toda lattice.

Pentagram map

The pentagram map T on the plane, introduced by R. Schwartz [30], constructs a new n -gon $T(P)$ ($n \geq 5$) from an old n -gon P defined as the convex hull of the intersection points of the shortest diagonals from each vertex of P . Since T is invariant under projective transformations the map T can be extended to the projective plane on a wider class of generic n -gons called twisted n -gons on $\mathbb{RP}^2$. In [26] was proved that the pentagram map T is a discrete integrable system in the sense of Liouville-Arnold and it is a discretization of Boussinesq equation. Being the Boussinesq equation an infinite dimensional bi-Hamiltonian system, the existence of a second Poisson structure for the pentagram map was conjectured. Many works have appeared giving generalizations of the pentagram map T to greater dimension such that the continuous limit are related to the Gelfand-Dickey flows, see [16, 24]. The work of [8] offers a generalization of the pentagram map, what they called *higher pentagram maps*, depending on a discrete parameter $k \geq 2$ (the case $k = 3$ is the pentagram map T) from the point of view of quivers and networks on a torus. We will see that the map T_2 admits other invariant Poisson structure.

5.1 Schwartz's pentagram map

5.1.1 Complete integrability

One says that three different points in $\mathbb{RP}^2$ are in *general position* if they are not collinear. A twisted n -gon is a sequence $P = (v_i)_{i \in \mathbb{Z}}$ in $\mathbb{RP}^2$ such that each three consecutive points v_i, v_{i+1}, v_{i+2} are in general position and $v_{i+n} = Mv_i$ for a fixed $M \in PGL(3, \mathbb{R})$. M is called the **monodromy** of P . Two twisted n -gons P_1 and P_2 are equivalent if there exists $g \in PGL(3, \mathbb{R})$ such that $gP_1 = P_2$. The space of twisted n -gons modulo $PGL(3, \mathbb{R})$ will be

Figure 5.1: The pentagram map on a closed pentagon.

denoted by $\mathcal{P}_n$. In case the monodromy M is the identity one has that P is a closed n -gon. The space of closed n -gons modulo $PGL(3, \mathbb{R})$ will be denoted by $\mathcal{C}_n$.

The pentagram map T inputs a twisted n -gon P and constructs a new twisted n -gon $T(P) = (v_i^*)_{i \in \mathbb{Z}}$ such that v_i^* is the intersection point of the shortest diagonals $\overline{v_{i-1}v_{i+1}}$ and $\overline{v_i v_{i+2}}$ of P , see Figure 5.1.

The space $\mathcal{P}_n$ is $2n$ -dimensional. For each vertex v_i of a twisted n -gon P , the functions

$$\begin{aligned} X_i &= [v_{i-2}, v_{i-1}, \overline{v_{i-2}v_{i-1}} \cap \overline{v_i v_{i+1}}, \overline{v_{i-2}v_{i-1}} \cap \overline{v_{i+1}v_{i+2}}], \\ Y_i &= [\overline{v_{i-2}v_{i-1}} \cap \overline{v_{i+1}v_{i+2}}, \overline{v_{i-1}v_i} \cap \overline{v_{i+1}v_{i+2}}, v_{i+1}, v_{i+2}], \end{aligned}$$

are $PGL(3, \mathbb{R})$ -invariant and, in particular, $X_{i+n} = X_i$ and $Y_{i+n} = Y_i$. The functions $(X_i, Y_i)_{i=1}^n$ form a coordinate system generically defined on $\mathcal{P}_n$. The space $\mathcal{C}_n$ has dimension $2n - 8$.

The pentagram map T , using the above coordinates, on generic elements of $\mathcal{P}_n$ takes the form:

$$X_i^* := T^* X_i = X_i \frac{1 - X_{i-1}Y_{i-1}}{1 - X_{i+1}Y_{i+1}}, \quad Y_i^* := T^* Y_i = Y_{i+1} \frac{1 - X_{i+2}Y_{i+2}}{1 - X_i Y_i} \quad (5.1)$$

One can see from (5.1) that the functions

$$O_n = \prod_{i=1}^n X_i, \quad E_n = \prod_{i=1}^n Y_i, \quad (5.2)$$

are invariants under the pentagram map T . Furthermore, when n is even, the additional functions

$$O_{n/2} = \prod_{i \text{ even}} X_i + \prod_{i \text{ odd}} X_i, \quad E_{n/2} = \prod_{i \text{ even}} Y_i + \prod_{i \text{ odd}} Y_i, \quad (5.3)$$

are also invariants under the pentagram map T .

R. Schwartz [31] has founded other additional $2[n/2]$ invariants of T , called the *monodromy invariants*. In [26], considering the Poisson bracket (only listed the non-zero brackets)

$$\{X_i, X_{i+1}\} = -X_i X_{i+1}, \quad \{Y_i, Y_{i+1}\} = Y_i Y_{i+1}, \quad (5.4)$$

the next proposition was proved.

Proposition 5.1. (i) *The Poisson bracket (5.4) is invariant with respect to the pentagram map T , i.e., $\{T^*(f), T^*(g)\} = \{f, g\}$ for any $f, g \in \mathcal{F}(\mathbb{R}^{2n})$.*

(ii) *The monodromy invariants Poisson commute.*

(iii) *The functions (5.2) are Casimirs for the Poisson bracket (5.4). If n is even the functions (5.3) are also Casimirs.*

(iv) *The Poisson bracket (5.4) has corank 2 if n is odd and corank 4 if n is even.*

The complete integrability of the pentagram map T follows from the above proposition and the algebraic independence of the monodromy invariants.

There exists another coordinate system on $\mathcal{P}_n$ which allow see $\mathcal{P}_n$ globally diffeomorphic to $\mathbb{R}^{2n}$ provided n is not divisible by 3. Consider two n -periodic sequences of real number (a_i) and (b_i) and define the following difference equation associated to these sequences:

$$V_{i+3} = a_i V_{i+2} + b_i V_{i+1} + V_i \quad (5.5)$$

where $V_i \in \mathbb{R}^3$. The n -periodicity implies that there exists $M \in SL(3, \mathbb{R})$, called the monodromy matrix, such that $V_{i+n} = M V_i$.

Proposition 5.2 ([26]). *The space $\mathcal{P}_n$ is isomorphic to the space of the equations (5.5) if n is not divisible by 3.*

The relation between the two coordinate systems (X_i, Y_i) and (a_i, b_i) , assuming that n is not divisible by 3, is given by

$$X_i = \frac{a_{i-2}}{b_{i-2}b_{i-1}}, \quad Y_i = -\frac{b_{i-1}}{a_{i-2}a_{i-1}}. \quad (5.6)$$

Formulas for the pentagram map T and the Poisson bracket (5.4) on the coordinate system $(a_i, b_i)_{i=1}^n$ can be found in [26].

5.1.2 Pentagram map and Boussinesq equation

The continuous limit of a twisted n -gon is a smooth parametrized curve $\gamma : \mathbb{R} \rightarrow \mathbb{RP}^2$ with fixed monodromy $M \in PSL(3, \mathbb{R})$ such that:

$$\gamma(x+1) = M(\gamma(x)), \quad \forall x \in \mathbb{R}$$

The assumption that every three consecutive points are in general position corresponds to the assumption that the vectors $\gamma'(x)$ and $\gamma''(x)$ are linearly independent for all $x \in \mathbb{R}$. A curve γ satisfying these conditions is called *non-degenerate*. The space of projectively equivalent non-degenerate curves in $\mathbb{RP}^2$ is denoted by $\mathcal{C}$.

Proposition 5.3 ([26]). *There exists a 1-1 correspondence between $\mathcal{C}$ and the space of linear differential operators in $\mathbb{R}$:*

$$A = \left(\frac{d}{dx} \right)^3 + u(x) \frac{d}{dx} + v(x) \quad (5.7)$$

where u and v are smooth periodic functions.

Notice that the linear differential operator (5.7) can be rewritten as a sum of a skew-symmetric and a (zero-order) symmetric operator:

$$A = \left(\frac{d}{dx}\right)^3 + \frac{1}{2} \left(u(x) \frac{d}{dx} + \frac{d}{dx} u(x) \right) + w(x) \quad (5.8)$$

where $w(x) = v(x) - \frac{u'(x)}{2}$. The pair of functions (u, w) are the continuous analagous of the system of coordinates (a_i, b_i) .

Given a non-degenerate curve $\gamma(x)$, at each point one draws a small chord: $(\gamma(x-\varepsilon), \gamma(x+\varepsilon))$ and obtain a new curve $\gamma_\varepsilon(x)$ which is the envelope of these chords, see Figure 5.2. The

Figure 5.2: Curve γ_ε .

differential operator (5.8) corresponding to γ_ε contains new functions $(u_\varepsilon, w_\varepsilon)$. The following proposition was proved in [26].

Proposition 5.4. *For u_ε and w_ε of the form*

$$u_\varepsilon = u + \varepsilon^2 \tilde{u} + O(\varepsilon^3), \quad w_\varepsilon = w + \varepsilon^2 \tilde{w} + O(\varepsilon^3)$$

one has

$$\tilde{u} = w_x, \quad \tilde{w} = -\frac{u u_x}{3} - \frac{u_{xxx}}{12}.$$

Assuming that $u(x)$ and $w(x)$ depend on an additional parameter t , the time, the evolution equation

$$u_t := \dot{u} = \tilde{u}, \quad w_t := \dot{w} = \tilde{w} \quad (5.9)$$

is the continuous limit of the pentagram map T . Equation (5.9) is equivalent to the following nonlinear partial differential equation:

$$u_{tt} = -\frac{(u^2)_{xx}}{6} - \frac{u_{xxxx}}{12} \quad (5.10)$$

which is nothing else but the classical Boussinesq equation, see Appendix B.

The Boussinesq equation is a particular flow of the Gelfand-Dickey flows. These flows are bi-Hamiltonian systems: the first Hamiltonian operator for the Boussinesq equation is

$$\mathcal{D} = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \quad (5.11)$$

with Hamiltonian functional $H = \int_{S^1} (\frac{w^2}{2} - \frac{u^3}{18} - \frac{uw''}{24})dx$. The respective Poisson bracket is

$$\{G, H\} = \int_{S^1} (\delta_u G (\delta_w H)' + \delta_w G (\delta_u H)') dx. \quad (5.12)$$

The second Hamiltonian operator $\mathcal{E}$ is

$$\mathcal{E} = \begin{pmatrix} 2D_x^3 + D_x u + u D_x & -D_x^4 - D_x^2 u + 2D_x v + v D_x \\ D_x^4 + u D_x^2 + D_x v + 2v D_x & -\frac{2}{3}(D_x^5 + D_x^3 u + u D_x^3 + u D_x u) + D_x^2 v - v D_x^2 \end{pmatrix}, \quad (5.13)$$

with Hamiltonian functional $H_0(u, v) = \int v dx$, i.e., $(u, v)_t^T = \mathcal{E}(\delta_u H_0, \delta_v H_0)^T$. Some of the first integrals of (5.9) are the following:

$$H_1 = \int_{S^1} u dx, \quad H_2 = \int_{S^1} w dx, \quad H_3 = \int_{S^1} u w dx$$

being H_1 and H_2 Casimirs of the Poisson bracket (5.12).

The process of discretization of the Boussinesq equation is as follows, see [26]. For a non-degenerate curve $\gamma(x)$, fix an arbitrary x and set

$$v_i := \gamma(x), \quad v_{i+1} := \gamma(x + \varepsilon), \quad v_{i+2} := \gamma(x + 2\varepsilon), \quad v_{i+3} := \gamma(x + 3\varepsilon), \quad \dots$$

for a small ε . Lifting $\gamma(x)$ and (v_i) to $\mathbb{R}^3$ so that (5.5) is satisfied, we are looking for functions $a(x, \varepsilon)$ and $b(x, \varepsilon)$ in

$$\Gamma(x + 3\varepsilon) = a(x, \varepsilon)\Gamma(x + 2\varepsilon) + b(x, \varepsilon)\Gamma(x + \varepsilon) + \Gamma(x)$$

Writing a and b as a series:

$$a(x, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i a_i(x), \quad b(x, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i b_i(x)$$

the first four terms are

$$\begin{aligned} a_0 &= 3, & a_1 &= 0, & a_2 &= -u(x), & a_3 &= -\frac{7}{4}u_x(x) - \frac{1}{2}w(x), & \dots \\ b_0 &= -3, & b_1 &= 0, & b_2 &= u(x), & b_3 &= \frac{5}{4}u_x(x) - \frac{1}{2}w(x), & \dots \end{aligned} \quad (5.14)$$

The Poisson bracket (5.4) was discovered after the computation (5.12) for the functionals

$$\begin{aligned} F_f(a, b) &= \int_{S^1} f(x) a(x, \varepsilon) dx = 3 \int_{S^1} f dx - \varepsilon^2 \int_{S^1} f u dx - \varepsilon^3 \int_{S^1} f \left(\frac{7}{4}u_x + \frac{w}{2} \right) dx + O(\varepsilon^4) \\ G_f(a, b) &= \int_{S^1} f(x) b(x, \varepsilon) dx = -3 \int_{S^1} f dx + \varepsilon^2 \int_{S^1} f u dx + \varepsilon^3 \int_{S^1} f \left(\frac{5}{4}u_x - \frac{w}{2} \right) dx + O(\varepsilon^4) \end{aligned}$$

From (5.12) and (5.14) one has

$$\{F_f, F_g\} = \varepsilon^5 \int_{S^1} f g' dx + O(\varepsilon^6), \quad \{F_f, G_g\} = O(\varepsilon^6), \quad \{G_f, G_g\} = -\varepsilon^5 \int_{S^1} f g' dx + O(\varepsilon^6).$$

Due the bi-Hamiltonian property of the Boussinesq equation and being the pentagram map a discretization of the Boussinesq equation, it was conjectured in [26] the existence of another

Poisson structure which is invariant under the Pentagon and compatible with (5.4) and allow us to prove the complete integrability of the pentagram map. To our knowledge this question still continues open.

If one tries to find the second Poisson bracket for the pentagram map imitating the procedure explained above one should compute the Poisson brackets among the functional F_f and G_g using the second Hamiltonian operator $\mathcal{E}$. The result of this computation is shown in the next proposition.

Proposition 5.5. *The Poisson bracket between the functionals F_g and G_g associated to $\mathcal{E}$ are given by*

$$\begin{aligned} \{F_f, F_g\} &= \varepsilon^4 \int_{S^1} (2fg''' + uW(f, g)) dx + \varepsilon^5 \int_{S^1} \left(\frac{3}{2}wW(f, g) + 2uW'(f, g) \right) dx + O(\varepsilon^6). \\ \{F_f, G_g\} &= -\varepsilon^4 \int_{S^1} (2fg''' + uW(f, g)) dx - \varepsilon^5 \int_{S^1} \left(2fg^{(iv)} + u(2f''g - (fg')') + \frac{w}{2}(fg)' \right) dx \\ &\quad + O(\varepsilon^6) \\ \{G_f, G_g\} &= \varepsilon^4 \int_{S^1} (2fg''' + uW(f, g)) dx - \varepsilon^5 \int_{S^1} \left(\frac{3}{2}wW(f, g) + uW'(f, g) \right) dx + O(\varepsilon^6). \end{aligned} \quad (5.15)$$

where $W(f, g)$ denotes the Wronskian of f and g , i.e., $W(f, g) = fg' - f'g$.

An appropriate discretization of (5.15) should provide us the second Poisson bracket for the pentagram map T .

5.2 The pentagram map and weighted networks

Glick's work [13] presented a cluster algebra interpretation for any iteration of pentagram map T on Y -coordinates via an appropriate bipartite quiver. Based on Glick's quiver, a family of maps T_k depending on the discrete parameter $k \geq 2$ was defined in [8], being the pentagram recovered in the case $k = 3$. Furthermore, the maps T_k have a geometric interpretation.

For any integer $n \geq 2$ let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$. Fixed an integer k such that $2 \leq k \leq n$, consider the maps $T_k : (\mathbf{x}, \mathbf{y}) \rightarrow (\mathbf{x}^*, \mathbf{y}^*) = (x_1^*, \dots, x_n^*, y_1^*, \dots, y_n^*)$ defined by

$$x_i^* = x_{i-r'-1} \frac{x_{i+r} + y_{i+r}}{x_{i-r'-1} + y_{i-r'-1}}, \quad y_i^* = y_{i-r'} \frac{x_{i+r+1} + y_{i+r+1}}{x_{i-r'} + y_{i-r'}}. \quad (5.16)$$

where $r = \lfloor k/2 \rfloor - 1$ and $r + r' = k - 2$. In the case $k = 3$,

$$x_i^* = x_{i-2} \frac{x_i + y_i}{x_{i-2} + y_{i-2}}, \quad y_i^* = y_{i-1} \frac{x_{i+1} + y_{i+1}}{x_{i-1} + y_{i-1}}$$

after a change of variables $Y_i := x_i$, $X_i := \frac{y_{i-1}}{x_{i-1}x_i}$ and a shift of indices the map T_3 becomes the pentagram map T . The maps T_k can be described using networks on an annulus and Postnikov moves. Let $\mathcal{N}'_{k,n}$ be a network in an annulus described as follows, see Figure 5.4 in the case $k = 3, n = 5$. It has n sources on the outer circle labelled from 1 to n in the counterclockwise order and n sinks on the inner circle labelled from 1 to n in the clockwise

order. Between the i th source and the $(i + k - 1)$ -th sink the local structure of the network $\mathcal{N}'_{k,n}$ is shown in Figure 5.3. Each of these blocks are connected by $\bullet \rightarrow \bullet$ edges as in Figure 5.4. The cut of $\mathcal{N}'_{k,n}$ starts at the segment between the sinks n and 1 on the inner circle, crosses $k - 1$ edges incident to the inner boundary, followed by a single $\bullet \rightarrow \bullet$ edge and ends on the corresponding segment on the outer circle. The variable associated with the cut in $\mathcal{N}'_{k,n}$ will be denoted by z .

Figure 5.3: Local structure between the i -th source and $(i + k - 1)$ -th sink of $\mathcal{N}'_{k,n}$

Figure 5.4: Network $\mathcal{N}'_{3,5}$ on an annulus.

The $n \times n$ boundary measurement matrix $A_{k,n}(z)$ of $\mathcal{N}'_{k,n}(z)$ is given by

$$A_{k,n}(z) = Z(D_x + D_y Z)Z^{k-2}(\text{Id} - Z)^{-1}. \quad (5.17)$$

where $D_x = \text{diag}(x_1, \dots, x_n)$, $D_y = \text{diag}(y_1, \dots, y_n)$ and $Z := Z(z) = -ze_{n1} + \sum_{i=1}^{n-1} e_{i,i+1}$ or equivalently

$$(1+z)a_{ij} = \begin{cases} \sigma_{i+1} & \text{if } j - i > k - 1, \\ x_{i+1} - zy_{i+1} & \text{if } j - i = k - 1, \\ -z\sigma_{i+1} & \text{if } k - n - 1 < j - i < k - 1, \\ -z(x_{i+1} - zy_{i+1}) & \text{if } j - i = k - n - 1, \\ z^2\sigma_{i+1} & \text{if } j - i < k - n - 1. \end{cases}$$

where $\sigma_i = x_i + y_i$.

Example 5.6. For $k = 2$ and $n = 5$ one has

$$A_{2,5}(z) = \frac{1}{1+z} \begin{pmatrix} -z\sigma_2 & x_2 - zy_2 & \sigma_2 & \sigma_2 & \sigma_2 \\ -z\sigma_3 & -z\sigma_3 & x_3 - zy_3 & \sigma_3 & \sigma_3 \\ -z\sigma_4 & -z\sigma_4 & -z\sigma_4 & x_4 - zy_4 & \sigma_4 \\ -z\sigma_5 & -z\sigma_5 & -z\sigma_5 & -z\sigma_5 & x_5 - zy_5 \\ -z(x_1 - zy_1) & -z\sigma_1 & -z\sigma_1 & -z\sigma_1 & -z\sigma_1 \end{pmatrix} \quad (5.18)$$

The matrix $A_{k,n}$ gives the $n \times n$ Lax representation of T_k via re-factorization of rational matrix functions, i.e., we have

$$A_{k,n}(z) = A_1(z)A_2(z) \mapsto A_{k,n}^*(z) = A_2(z)A_1(z)$$

where

$$A_1(z) = ZD_\sigma(I - Z)^{-1}Z^{r'}, \quad A_2(z) = Z^{-r'}(D_x + ZD_y)D_\sigma^{-1}Z^{k-2}. \quad (5.19)$$

and $A_{k,n}^*(z)$ is the image of $A_{k,n}(z)$ under T_k .

Example 5.7. For $n = 5$ and $k = 3$, using formulae (5.19) one obtains

$$A_1(z) = \frac{1}{1+z} \begin{pmatrix} -z\sigma_2 & -z\sigma_2 & \sigma_2 & \sigma_2 & \sigma_2 \\ -z\sigma_3 & -z\sigma_3 & -z\sigma_3 & \sigma_3 & \sigma_3 \\ -z\sigma_4 & -z\sigma_4 & -z\sigma_4 & -z\sigma_4 & \sigma_4 \\ -z\sigma_5 & -z\sigma_5 & -z\sigma_5 & -z\sigma_5 & -z\sigma_5 \\ z^2\sigma_1 & -z\sigma_1 & -z\sigma_1 & -z\sigma_1 & -z\sigma_1 \end{pmatrix}, \quad A_2(z) = \begin{pmatrix} \frac{x_5}{\sigma_5} & \frac{y_1}{\sigma_1} & 0 & 0 & 0 \\ 0 & \frac{x_1}{\sigma_1} & \frac{y_2}{\sigma_2} & 0 & 0 \\ 0 & 0 & \frac{x_2}{\sigma_2} & \frac{y_3}{\sigma_3} & 0 \\ 0 & 0 & 0 & \frac{x_3}{\sigma_3} & \frac{y_4}{\sigma_4} \\ \frac{-zy_5}{\sigma_5} & 0 & 0 & 0 & \frac{x_4}{\sigma_4} \end{pmatrix}$$

Proposition 5.8. [8]

i) For $n \geq 2k - 1$ the Poisson structure given by

$$\begin{aligned} \{x_i, x_{i+l}\}_2 &= -x_i x_{i+l}, 1 \leq l \leq k-2; & \{y_i, y_{i+l}\}_2 &= -y_i y_{i+l}, 1 \leq l \leq k-1; \\ \{y_i, x_{i+l}\}_2 &= -y_i x_{i+l}, 1 \leq l \leq k-1; & \{y_i, x_{i-l}\}_2 &= y_i x_{i-l}, 0 \leq l \leq k-2, \end{aligned} \quad (5.20)$$

where the indices are understood mod n and only non-zero brackets are listed, is invariant under the map T_k .

ii) The bracket (5.20) has rank $2(n-d)$, where $d = \gcd(k-1, n)$. Functions

$$\prod_{i=0}^{\frac{n}{d}-1} x_{s+i(k-1)}, \quad \prod_{i=0}^{\frac{n}{d}-1} y_{s+i(k-1)}, \quad s = 1, \dots, d,$$

are Casimir functions for (5.20).

iii) The map T_k is integrable. The conserved quantities are given from the spectral invariants of $A_{k,n}(z)$

In [8], the map T_k , $k \geq 4$ are interpreted as a transformation of a class of twisted polygons in $\mathbb{RP}^{k-1}$, called *corrugated polygons*.

5.2.1 The map T_2

From (5.16) one sees that the map T_2 on coordinates x_i, y_i ($i = 1, \dots, n$) is given by

$$x_i^* = x_{i-1} \frac{x_i + y_i}{x_{i-1} + y_{i-1}}, \quad y_i^* = y_i \frac{x_{i+1} + y_{i+1}}{x_i + y_i}. \quad (5.21)$$

The map T_2 has the following geometrical interpretation. A *twisted n -gon* S in $\mathbb{RP}^1$ is a sequence of points $(S_i)_{i \in \mathbb{Z}}$ such that $S_{i+n} = MS_i$ for every $i \in \mathbb{Z}$ and a fixed $M \in PGL(2, \mathbb{R})$. Let $\mathcal{S}_n$ be the space of twisted n -gons in $\mathbb{RP}^1$ with the same monodromy. The group $PGL(2, \mathbb{R})$ acts in $\mathcal{S}_n$ in a straightforward way. The formulas [8]

$$x_i = \frac{(S_{i+1} - S_{i+2}^-)(S_i^- - S_{i+1}^-)}{(S_i^- - S_{i+1})(S_{i+1}^- - S_{i+2}^-)}, \quad y_i = \frac{(S_{i+1}^- - S_{i+1})(S_{i+2}^- - S_{i+2})(S_i^- - S_{i+1}^-)}{(S_{i+1}^- - S_{i+2})(S_i^- - S_{i+1})(S_{i+1}^- - S_{i+2}^-)}$$

define a map $\varphi : \mathcal{S}_n \rightarrow \mathbb{R}^{2n}(\mathbf{x}, \mathbf{y})$ which allows to identify the $(\mathbf{x}, \mathbf{y})$ with the moduli space $\mathcal{S}_n/PGL(2, \mathbb{R})$. Let $\Phi : \mathcal{S}_n \rightarrow \mathcal{S}_n : (S^-, S) \mapsto (S, S^+)$ be the transformation, where $S^+ = (S_i^+)_{i \in \mathbb{Z}}$ is the sequence defined as follows: given a quadruple of points $S_{i-1}, S_i^-, S_i, S_{i+1}$, the point S_i^+ is the image $P_i(S_i^-)$ where P_i is the projective involution such that $P_i(S_i) = S_i$ and $P_i(S_{i-1}) = S_{i+1}$, see Figure 5.5. The map induced by Φ on the moduli space $\mathcal{S}_n/PGL(2, \mathbb{R})$ is the map T_2 .

Figure 5.5: Projective transformation P_i in $\mathbb{RP}^1$

By other hand, the quadratic Poisson bracket (5.20) becomes

$$\{y_i, x_{i+1}\}_2 = -y_i x_{i+1}, \quad \{y_i, y_{i+1}\}_2 = -y_i y_{i+1}, \quad \{y_i, x_i\}_2 = y_i x_i.$$

However, there are other Poisson brackets which are invariant under T_2 .

Proposition 5.9. *The linear Poisson bracket*

$$\{x_i, x_{i+1}\}_1 = y_i, \quad \{y_i, x_{i+1}\}_1 = -y_i, \quad \{y_i, x_i\}_1 = y_i \quad (5.22)$$

is invariant under T_2 . The latter bracket has two casimirs $\prod_{i=1}^n y_i$ and $\sum_{i=1}^n (x_i + y_i)$. Also the Poisson bracket (4.16) after take $c_k := y_k, d_k := x_k$ is invariant under T_2 .

Proof. The proof is an elementary computation but lengthy. We present the case of the linear Poisson bracket $\{\cdot, \cdot\}_1$. The case of the cubic Poisson bracket $\{\cdot, \cdot\}_3$ is analogous. Recall

when $k = 2$ we have $r = r' = 0$. One has

$$\begin{aligned} \{x_i^*, y_j^*\}_1 &= (\{x_{i-1}, y_j\}_1 y_j y_{i-1} - \{x_{i-1}, x_j\}_1 y_j y_{i-1} - \{y_{i-1}, y_j\}_1 x_{i-1} x_j + \{y_{i-1}, x_j\}_1 x_{i-1} y_j) \\ &\quad \cdot \frac{\sigma_i \sigma_{j+1}}{\sigma_{i-1}^2 \sigma_j^2} + x_{i-1} y_j \left(\frac{\{x_i, x_{j+1}\}_1 + \{x_i, y_{j+1}\}_1 + \{y_i, x_{j+1}\}_1 + \{y_i, y_{j+1}\}_1}{\sigma_{i-1} \sigma_j} \right. \\ &\quad \left. - \frac{\sigma_i (\{y_{i-1}, x_{j+1}\}_1 + \{y_{i-1}, y_{j+1}\}_1)}{\sigma_{i-1}^2 \sigma_j} - \frac{\sigma_{j+1} (\{x_i, x_j\}_1 + \{y_i, x_j\}_1)}{\sigma_{i-1} \sigma_j^2} \right) \\ &\quad + x_{i-1} x_j \frac{\sigma_{j+1}}{\sigma_j^2} \frac{\{x_i, y_j\}_1 + \{y_i, y_j\}_1}{\sigma_{i-1}} + y_{i-1} y_j \sigma_i \frac{\{x_{i-1}, x_{j+1}\}_1 + \{x_{i-1}, y_{j+1}\}_1}{\sigma_{i-1}^2 \sigma_j} \end{aligned}$$

(1) If $j = i$ we obtain

$$\begin{aligned} \{x_i^*, y_i^*\}_1 &= (-y_{i-1}^2 y_i - y_{i-1} x_{i-1} y_i) \frac{\sigma_{i+1}}{\sigma_{i-1}^2 \sigma_i} + x_{i-1} y_i \left(-\frac{\sigma_{i+1}}{\sigma_{i-1} \sigma_i^2} y_i \right) + x_{i-1} x_i \frac{\sigma_{i+1}}{\sigma_i^2 \sigma_{i-1}} (-y_i) \\ &= -y_{i-1} y_i \frac{\sigma_{i+1}}{\sigma_{i-1} \sigma_i} - x_{i-1} y_i \frac{\sigma_{i+1}}{\sigma_{i-1} \sigma_i} = -y_i \frac{\sigma_{i+1}}{\sigma_i} = -y_i^* \end{aligned}$$

(2) If $i = j + 1$ we obtain

$$\begin{aligned} \{x_{j+1}^*, y_j^*\}_1 &= \frac{\sigma_{j+1}}{\sigma_j^2} [x_j (\{x_{j+1}, y_j\}_1 + \{y_{j+1}, y_j\}_1) + y_j (\{x_j, x_{j+1}\}_1 + \{x_j, y_{j+1}\}_1)] \\ &= \frac{\sigma_{j+1}}{\sigma_j^2} [x_j y_j + y_j y_j] = y_j \frac{\sigma_{j+1}}{\sigma_j} = y_j^*. \end{aligned}$$

We also have

$$\begin{aligned} \{y_i^*, y_j^*\} &= (\{y_i, y_j\} x_i x_j - \{y_i, x_j\} y_j x_i - \{x_i, y_j\} y_i x_j + \{x_i, x_j\} y_i y_j) \frac{\sigma_{i+1} \sigma_{j+1}}{\sigma_i^2 \sigma_j^2} + y_i y_j \cdot \\ &\quad \left(\frac{\{x_{i+1}, x_{j+1}\} + \{x_{i+1}, y_{j+1}\} + \{y_{i+1}, x_{j+1}\} + \{y_{i+1}, y_{j+1}\}}{\sigma_i \sigma_j} - \frac{\sigma_{i+1}}{\sigma_i^2 \sigma_j} \right. \\ &\quad \left. (\{x_i, x_{j+1}\} + \{x_i, y_{j+1}\}) - \frac{\sigma_{j+1}}{\sigma_i \sigma_j^2} (\{x_{i+1}, x_j\} + \{y_{i+1}, x_j\}) \right) \\ &\quad + y_i x_j \frac{\sigma_{j+1}}{\sigma_j^2 \sigma_i} (\{x_{i+1}, y_j\} + \{y_{i+1}, y_j\}) + y_j x_i \frac{\sigma_{i+1}}{\sigma_i^2 \sigma_j} (\{y_i, x_{j+1}\} + \{y_i, y_{j+1}\}) \end{aligned}$$

For $j = i + 1$

$$\begin{aligned} \{y_i^*, y_{i+1}^*\} &= y_i y_{i+1} \frac{\sigma_{i+2}}{\sigma_i \sigma_{i+1}} - y_i y_{i+1}^2 \frac{\sigma_{i+2}}{\sigma_i \sigma_{i+1}^2} - y_{i+1} y_i x_{i+1} \frac{\sigma_{i+2}}{\sigma_{i+1}^2 \sigma_i} \\ &= 0. \end{aligned}$$

Finally,

$$\begin{aligned} \{x_i^*, x_j^*\} &= (\{x_{i-1}, x_{j-1}\} y_{i-1} y_{j-1} - \{x_{i-1}, y_{j-1}\} x_{j-1} y_{i-1} - \{y_{i-1}, x_{j-1}\} x_{i-1} y_{j-1} \\ &\quad + \{y_{i-1}, y_{j-1}\} x_{i-1} x_{j-1}) \cdot \frac{\sigma_i \sigma_j}{\sigma_{i-1}^2 \sigma_{j-1}^2} + x_{i-1} x_{j-1} \left(\frac{\{x_i, x_j\} + \{x_i, y_j\} + \{y_i, x_j\} + \{y_i, y_j\}}{\sigma_{i-1} \sigma_{j-1}} \right. \\ &\quad \left. - \frac{\sigma_i}{\sigma_{i-1}^2 \sigma_{j-1}} (\{y_{i-1}, x_j\} + \{y_{i-1}, y_j\}) - \frac{\sigma_j}{\sigma_{i-1} \sigma_{j-1}^2} (\{x_i, y_{j-1}\} + \{y_i, y_{j-1}\}) \right) + \\ &\quad x_{i-1} y_{j-1} \frac{\sigma_j}{\sigma_{j-1}^2 \sigma_{i-1}} (\{x_i, x_{j-1}\} + \{y_i, x_{j-1}\}) + x_{j-1} y_{i-1} \frac{\sigma_i}{\sigma_{i-1}^2 \sigma_{j-1}} (\{x_{i-1}, x_j\} + \{x_{i-1}, y_j\}) \end{aligned} \quad (5.23)$$

For $j = i + 1$ we have

$$\begin{aligned} \{x_i^*, x_{i+1}^*\}_1 &= (y_{i-1}^2 y_i + y_{i-1} x_{i-1} y_i) \frac{\sigma_i \sigma_{i+1}}{\sigma_{i-1}^2 \sigma_i^2} + x_{i-1} x_i \frac{\sigma_{i+1}}{\sigma_{i-1} \sigma_i^2} y_i + x_{i-1} y_i \frac{\sigma_{i+1}}{\sigma_i^2 \sigma_{i-1}} y_i \\ &= y_i \frac{\sigma_{i+1}}{\sigma_i} = y_i^* \end{aligned}$$

□

In the following we present the relation between the Lax matrix $A_{2,n}(z)$ of the map T_2 and the Lax matrix T (4.15) of the relativistic Toda lattice.

Example 5.10. Let $n = 5$, $c = \prod_{i=1}^5 y_i$ and $\Lambda := \Lambda(\lambda) = \text{diag}(1, \lambda^{-1}, \lambda^{-2}, \lambda^{-3}, \lambda^{-4})$. The matrix $\Lambda T \Lambda^{-1}$ has the form:

$$\frac{1}{c - \lambda^5} \begin{pmatrix} -\lambda^5 \sigma_1 & -\lambda^5 y_1 \sigma_2 & -\lambda^5 y_1 y_2 \sigma_3 & -\lambda^5 y_1 y_2 y_3 \sigma_4 & -\lambda^5 (\frac{cx_5}{y_5} + \lambda^5) \\ -(\frac{cx_1}{y_1} + \lambda^5) & -\lambda^5 \sigma_2 & -\lambda^5 y_2 \sigma_3 & -\lambda^5 y_2 y_3 \sigma_4 & -\lambda^5 y_2 y_3 y_4 \sigma_5 \\ -y_3 y_4 y_5 \sigma_1 & -(\frac{cx_2}{y_2} + \lambda^5) & -\lambda^5 \sigma_3 & -\lambda^5 y_3 \sigma_4 & -\lambda^3 y_3 y_4 \sigma_5 \\ -y_4 y_5 \sigma_1 & -y_4 y_5 y_1 \sigma_2 & -(\frac{cx_3}{y_3} + \lambda^5) & -\lambda^5 \sigma_4 & -\lambda^5 y_4 \sigma_5 \\ -y_5 \sigma_1 & y_5 y_1 \sigma_2 & -y_5 y_1 y_2 \sigma_3 & -(\frac{cx_4}{y_4} + \lambda^5) & -\lambda^5 \sigma_5 \end{pmatrix}.$$

Now consider the matrix $D = \text{diag}(1, c_1, c_1 c_2, c_1 c_2 c_3, c_1 c_2 c_3 c_4)$, using the notation $\sigma_i = x_i + y_i$ the matrix $\tilde{T} = D \Lambda T \Lambda^{-1} D^{-1} = (D \Lambda) T (D \Lambda)^{-1}$ has the form:

$$\tilde{T} = \frac{-1}{c - \lambda^5} \begin{pmatrix} \lambda^5 \sigma_1 & \lambda^5 \sigma_2 & \lambda^5 \sigma_3 & \lambda^5 \sigma_4 & \lambda^5 (x_5 + \frac{y_5}{c} \lambda^5) \\ (cx_1 + \lambda^5 y_1) & \lambda^5 \sigma_2 & \lambda^5 \sigma_3 & \lambda^5 \sigma_4 & \lambda^5 \sigma_5 \\ c \sigma_1 & (cx_2 + \lambda^5 y_2) & \lambda^5 \sigma_3 & \lambda^5 \sigma_4 & \lambda^5 \sigma_5 \\ c \sigma_1 & c \sigma_2 & (cx_3 + \lambda^5 y_3) & \lambda^5 \sigma_4 & \lambda^5 \sigma_5 \\ c \sigma_1 & c \sigma_2 & c \sigma_3 & (cx_4 + \lambda^5 y_4) & \lambda^5 \sigma_5 \end{pmatrix}$$

Notice that the matrix $\tilde{T}$ has the similar form of matrix $A_{2,5}(z)$ in equation (5.18) after a shift of indices and take the transpose of $\tilde{T}$ whenever $c = -1$ and $z = \lambda^5$.

In fact, for our purpose of a more precise relation between the Lax matrix T and $A_{2,n}(z)$ we will make minor changes on the definition of the Lax matrix T keeping the same notations as in the Subsection 4.1.2.

Let $U = \sum_{x=1}^n x_{i+1} + \lambda \sum_{i=1}^n E_{i,i+1}$ and $V = I - \lambda^{-1} \sum_{i=1}^n E_{i,i+1}$. Notice that the new U and V are the transpose of the old ones (4.14) with a shift of the indices of the entries. The equations of motion of the relativistic Toda lattice are equivalent to

$$\dot{U} = BU - UA \quad \dot{V} = BV - VA$$

with $A = \sum_{i=1}^n (x_{i+1} + y_i) E_{ii} + \lambda \sum_{i=1}^n E_{i,i+1}$ and $B = \sum_{i=1}^n (x_{i+1} + y_{i+1}) E_{ii} + \lambda \sum_{i=1}^n E_{i+1,i}$. Consequently, the Lax matrix $T = UV^{-1}$ satisfies $\dot{T} = [B, T]$ and satisfies the analogous formulas (4.17) and (4.18), more precisely

$$\{T(\lambda) \otimes T(\mu)\}_1 = [r(\lambda, \mu)^t + p_0, I \otimes T(\mu)] + [r(\lambda, \mu)^t - p_0, T(\lambda) \otimes I] \quad (5.24)$$

$$\begin{aligned} \{T(\lambda) \otimes T(\mu)\}_2 &= a_1(\lambda, \mu)^t [T(\lambda) \otimes T(\mu)] - [T(\lambda) \otimes T(\mu)] a_2(\lambda, \mu)^t \\ &\quad + [T(\lambda) \otimes I] s_1^t [I \otimes T(\mu)] - [I \otimes T(\mu)] s_2^t [T(\lambda) \otimes I] \end{aligned} \quad (5.25)$$

where $r(\lambda, \mu)$, $a_i(\lambda, \mu)$, s_i , $i = 1, 2$ are the same as in the Subsection 4.1.2. In our above running example, $n = 5$, this new $T := T(z)$ has the form:

$$\frac{1}{c - z^5} \begin{pmatrix} -z^5 \sigma_2 & -z(\frac{cx_2}{y_2} + z^5) & -z^2 y_1 y_4 y_5 \sigma_2 & -z^3 y_1 y_5 \sigma_2 & -z^4 y_1 \sigma_2 \\ -z^4 y_2 \sigma_3 & -z^5 \sigma_3 & -z(\frac{cx_3}{y_3} + z^5) & -z^2 y_1 y_2 y_5 \sigma_3 & -z^3 y_1 y_2 \sigma_3 \\ -z^3 y_2 y_3 \sigma_4 & -z^4 y_3 \sigma_4 & -z^5 \sigma_4 & -z(\frac{cx_4}{y_4} + z^5) & -z^2 y_1 y_2 y_3 \sigma_4 \\ -z^2 y_2 y_3 y_4 \sigma_5 & -z^3 y_3 y_4 \sigma_5 & -z^4 y_4 \sigma_5 & -z^5 \sigma_5 & -z(\frac{cx_5}{y_5} + z^5) \\ -z(\frac{cx_1}{y_1} + z^5) & -z^2 y_3 y_4 y_5 \sigma_1 & -z^3 y_4 y_5 \sigma_1 & -z^4 y_5 \sigma_1 & -z^5 \sigma_1 \end{pmatrix}$$

Now, let $\Gamma := \Gamma(z) = \text{diag}(1, y_2 z^{-1}, y_2 y_3 z^{-2}, \dots, y_2 \dots y_n z^{-(n-1)})$ and consider the matrix $\tilde{T} = \Gamma^{-1} T \Gamma$. One has that $\tilde{T} = -A_{2,n}(z^n)$ whenever $c = -1$. In fact in our example, $n = 5$, one has that $\tilde{T}$ is

$$\frac{1}{c - z^5} \begin{pmatrix} -z^5 \sigma_2 & -(x_2 c + y_2 z^5) & -\sigma_2 c & -\sigma_2 c & -\sigma_2 c \\ -z^5 \sigma_3 & -z^5 \sigma_3 & -(x_3 c + y_3 z^5) & -\sigma_3 c & -\sigma_3 c \\ -z^5 \sigma_4 & -z^5 \sigma_4 & -z^5 \sigma_4 & -(x_4 c + y_4 z^5) & -\sigma_4 c \\ -z^5 \sigma_5 & -z^5 \sigma_5 & -z^5 \sigma_5 & -z^5 \sigma_5 & -(x_5 c + y_5 z^5) \\ -z^5(x_1 + \frac{y_1}{c} z^5) & -z^5 \sigma_1 & -z^5 \sigma_1 & -z^5 \sigma_1 & -z^5 \sigma_1 \end{pmatrix}$$

which is $-A_{2,5}(z^5)$ after put $c = -1$, compare with (5.18).

Let $\tilde{A}_{2,n}(z) = \Gamma A_{2,n}(z^n) \Gamma^{-1}$, from the above considerations, equation (4.18) and the fact that c is a Casimir for $\{\cdot, \cdot\}_2$ the next result follows.

Proposition 5.11. *One has*

$$\begin{aligned} \{\tilde{A}_{2,n}(\lambda) \otimes \tilde{A}_{2,n}(\mu)\}_2 &= a_1(\lambda, \mu)^t [\tilde{A}_{2,n}(\lambda) \otimes \tilde{A}_{2,n}(\mu)] - [\tilde{A}_{2,n}(\lambda) \otimes \tilde{A}_{2,n}(\mu)] a_2(\lambda, \mu)^t \\ &\quad + [\tilde{A}_{2,n}(\lambda) \otimes I] s_1^t [I \otimes \tilde{A}_{2,n}(\mu)] - [I \otimes \tilde{A}_{2,n}(\mu)] s_2^t [\tilde{A}_{2,n}(\lambda) \otimes I] \end{aligned} \quad (5.26)$$

where $a_1(\lambda, \mu)$, $a_2(\lambda, \mu)$, s_1 and s_2 is the notation used in Proposition 4.7.

Remark 5.12. In the literature we found that the second Poisson structure for the reduced Gelfand-Dickey flows is also known as W_n -algebras. In the case of the Boussinesq equation, we have the W_3 algebra. Belov and Chaltikian [2] have constructed a discrete W_3 algebra based on Feigin's construction of lattice screening charges. The same discrete W_3 algebra was recovered in [14], based on q -deformations of W_n algebras. Being the pentagram map a discretization of the Boussinesq equation, it would be interesting find some connections with these works.

On simple Lie algebras

In this Appendix we collect the essential background material of Lie theory used throughout the dissertation.

A.1 Some aspects of Lie Algebras

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra with Lie bracket $[\cdot, \cdot]$. Recall that $\mathfrak{g} := T_e G$ is identified with the space of left invariant vector fields on G , i.e., vector fields $\mathcal{X} \in \mathfrak{X}(G)$ such that $(L_g)_* \mathcal{X}_h = \mathcal{X}_{gh}$. The isomorphism map is $\mathfrak{g} \ni \alpha \mapsto \mathcal{X}_\alpha$ where

$$\mathcal{X}_\alpha(g)(f) := ((L_g)_* \alpha)f = \left. \frac{d}{dt} \right|_{t=0} f(g \exp(t\alpha))$$

and $f \in \mathcal{F}(G)$.

Definition A.1. We say that $\mathfrak{g}$ acts on V or V is a $\mathfrak{g}$ -module if V is the vector space of a representation of $\mathfrak{g}$, i.e., there exist a Lie algebra homomorphism $\mathfrak{g} \rightarrow \text{End}(V)$. The notation $X.u$ indicates the action of $X \in \mathfrak{g}$ on $u \in V$.

Example A.2. $\mathfrak{g}$ act on itself by the adjoint representation: $\mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ given by $X \mapsto \text{ad}_X Y = [X, Y]$. More generally $\mathfrak{g}$ acts on $\mathfrak{g}^{\otimes p} := \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (p -times) as follows: $X \mapsto \text{ad}_X(Y_1 \otimes \cdots \otimes Y_p) = \sum_{i=1}^p Y_1 \otimes \cdots \otimes \text{ad}_X Y_i \otimes \cdots \otimes Y_p$ which extends linearly to any $u \in \mathfrak{g}^{\otimes p}$. In particular for $p = 2$ and $u \in \mathfrak{g}^{\otimes 2}$ we also use the notation

$$[X \otimes 1 + 1 \otimes X, u] := X.u = \text{ad}_X u. \quad (\text{A.1})$$

Also $\mathfrak{g}$ acts on $\bigwedge^p \mathfrak{g}$: $X \mapsto \text{ad}_X(Y_1 \wedge \cdots \wedge Y_p) = \sum_{i=1}^p Y_1 \wedge \cdots \wedge \text{ad}_X Y_i \wedge \cdots \wedge Y_p$

Definition A.3. Let V a $\mathfrak{g}$ -module. For $k \geq 0$ the vector space $C^k(\mathfrak{g}, V) := \text{Hom}(\bigwedge^k \mathfrak{g} \rightarrow V)$ is called the space of k -**cochains** on $\mathfrak{g}$ with values in V . The coboundary of a k -cochain $u \in C^k(\mathfrak{g}, V)$ is the $(k+1)$ -cochain, $\delta u \in C^{k+1}(\mathfrak{g}, V)$ defined by

$$\delta u(x_0, \dots, x_k) = \sum_{i=0}^k (-1)^i x_i \cdot u(x_0, \dots, \hat{x}_i, \dots, x_k) + \sum_{i < j} (-1)^{i+j} u([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k)$$

where $x_0, \dots, x_k \in \mathfrak{g}$ and $\hat{x}_i$ indicates that the element x_i is omitted.

Remark A.4. (1) $C^0(\mathfrak{g}, V) = V$, i.e., a 0-cochain on $\mathfrak{g}$ with values in V is simply an element of V . We say that a 0-cochain u is **$\mathfrak{g}$ -invariant** if $X.u = 0$ for all $X \in \mathfrak{g}$.

$C^1(\mathfrak{g}, V) := \text{Hom}(\mathfrak{g}, V)$, i.e., a 1-cochain on $\mathfrak{g}$ with values on V is a linear map of $\mathfrak{g}$ into V . For instance, the coboundary of $u \in C^1(\mathfrak{g}, V)$, is the 2-cochain given by

$$\delta u(X, Y) = X.u(Y) - Y.u(X) - u([X, Y]) \quad (\text{A.2})$$

(2) For any k -cochain $u \in C^k(\mathfrak{g}, V)$ we have $\delta(\delta u) = 0$. The cochain complex $(C^*(\mathfrak{g}, V), \delta)$ is called the Chevalley-Eilenberg complex.

(3) An element of the subspace $Z^k(\mathfrak{g}, V) = \{u \in C^k(\mathfrak{g}, V) \mid \delta u = 0\}$ is called a k -*cocycle*. An element of the subspace $B^k(\mathfrak{g}, V) = \{u \in C^k(\mathfrak{g}, V) \mid u = \delta v \text{ for some } v \in C^{k-1}(\mathfrak{g}, V)\}$ is called a k -*coboundary*. The quotient $H^k(\mathfrak{g}, V) := Z^k(\mathfrak{g}, V)/B^k(\mathfrak{g}, V)$ is called the k -th cohomology space of $\mathfrak{g}$ with values in V .

Definition A.5. Let $\Psi : G \times M \rightarrow M$ be a left action. The fundamental vector field $X_M \in \mathfrak{X}(M)$ at $m \in M$ associated to Ψ and $X \in \mathfrak{g}$ is given by $X_M(m) = d(\Psi_m)_e(X)$ where $\Psi_m : G \rightarrow M$ is the induced map from Ψ , equivalently

$$X_M(m)(f) = \left. \frac{d}{dt} \right|_{t=0} f(\Psi(\exp tX, m)).$$

Remark A.6. The vector space generated by $\{X_M(m), X \in \mathfrak{g}\}$ is nothing else than $T_m \mathcal{O}(m)$, where $\mathcal{O}(m)$ is the orbit space through $m \in M$.

Example A.7. (1) Take $M = \mathfrak{g}$, and $\Psi = \text{Ad}$ the *adjoint action* or *adjoint representation* of G on $\mathfrak{g}$ defined by

$$\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g}) : g \mapsto \text{Ad}_g := d(C_g)_e, \quad C_g : G \rightarrow G : h \mapsto ghg^{-1}.$$

The fundamental vector field $X_{\mathfrak{g}}$ of the adjoint action at $X \in \mathfrak{g}$ is ad_X .

(2) The coadjoint action Ad^* of G on $\mathfrak{g}^*$ is defined by duality

$$\langle \text{Ad}_g^* \xi, X \rangle = \langle \xi, \text{Ad}_{g^{-1}} X \rangle, \quad \xi \in \mathfrak{g}^*.$$

Its corresponding fundamental vector field $X_{\mathfrak{g}^*}$ at $X \in \mathfrak{g}$ is ad_X^* :

$$\langle \text{ad}_X^* \xi, Y \rangle = \langle \xi, -\text{ad}_X Y \rangle = \langle \xi, [Y, X] \rangle, \quad Y \in \mathfrak{g}, \xi \in \mathfrak{g}^*.$$

Definition A.8. A function $f \in \mathcal{F}(\mathfrak{g})$ ($f \in \mathcal{F}(\mathfrak{g}^*)$) is called Ad-invariant (Ad*-invariant) if $f(\text{Ad}_g X) = f(X)$ for all $g \in G$ and $X \in \mathfrak{g}$ ($f(\text{Ad}_g^* \xi) = f(\xi)$ for all $g \in G$ and $\xi \in \mathfrak{g}^*$). We use the notation $\mathcal{F}(\mathfrak{g})^G$ ($\mathcal{F}(\mathfrak{g}^*)^G$) to the algebra of Ad-invariant functions on $\mathfrak{g}$ (Ad*-invariant functions on $\mathfrak{g}^*$).

Lemma A.9. If $f \in \mathcal{F}(\mathfrak{g}^*)^G$ then $\text{ad}_{df(\xi)}^* \xi = 0$ for all $\xi \in \mathfrak{g}^*$, i.e., for all $\xi \in \mathfrak{g}^*$, $X \in \mathfrak{g}$ we have

$$\langle \xi, [df(\xi), X] \rangle = 0. \quad (\text{A.3})$$

Moreover, for any $g \in G$ and $\xi \in \mathfrak{g}^*$ the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g}^* & \xrightarrow{df} & \mathfrak{g} \\ \text{Ad}_g^* \downarrow & & \downarrow \text{Ad}_g \\ \mathfrak{g}^* & \xrightarrow{df} & \mathfrak{g} \end{array}$$

Proof. In fact, if $f \in \mathcal{F}(\mathfrak{g}^*)^G$

$$\langle \text{ad}_{df(\xi)}^* \xi, X \rangle = -\langle \text{ad}_X^* \xi, df(\xi) \rangle = -\left. \frac{d}{dt} \right|_{t=0} f(\text{Ad}_{\exp tX}^* \xi) = -\left. \frac{d}{dt} \right|_{t=0} f(\xi) = 0,$$

proving the first assertion. For the reciprocal, assuming $\text{ad}_{df(\xi)}^* \xi = 0$ we have The commutativity property of the diagram follows from

$$\langle df(\xi), \eta \rangle = \langle df(\text{Ad}_g^* \xi), \text{Ad}_g^* \eta \rangle = \langle \text{Ad}_{g^{-1}}(df(\text{Ad}_g^* \xi)), \eta \rangle$$

where $\xi \in \mathfrak{g}^*$. □

Last, let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. A bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ is called Ad-invariant, if for all $g \in G$ and for all $X, Y \in \mathfrak{g}$,

$$(\text{Ad}_g X, \text{Ad}_g Y) = (X, Y)$$

If $(\cdot, \cdot)$ is Ad-invariant, then $(\cdot, \cdot)$ is ad-invariant, i.e.,

$$([X, Y], Z) = (X, [Y, Z]) \quad \forall X, Y, Z \in \mathfrak{g}. \quad (\text{A.4})$$

A.2 Simple Lie algebras

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank l with Cartan subalgebra $\mathfrak{h}$. For $\alpha \in \mathfrak{h}^*$, let

$$\mathfrak{g}_\alpha = \{\xi \in \mathfrak{g} \mid [h, \xi] = \alpha(h)\xi \text{ for any } h \in \mathfrak{h}\}.$$

A nonzero α such that $\mathfrak{g}_\alpha \neq 0$ is called a *root* of $\mathfrak{g}$ and a collection Φ of all roots is called the root system of $\mathfrak{g}$. The following verify:

- (i) For any $\alpha \in \Phi$, $\dim \mathfrak{g}_\alpha = 1$,

(ii) $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$ and

(iii) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$, where the right hand side is zero if $\alpha + \beta$ is not a root.

The Killing form on $\mathfrak{g}$:

$$(\xi, \eta) = \text{Trace}(\text{ad}_\xi \text{ad}_\eta)$$

is non-degenerate on $\mathfrak{g}$, and so is its restriction to $\mathfrak{h}$, also denoted by $(\cdot, \cdot)$. Thus $\mathfrak{h}^*$ and $\mathfrak{h}$ can be identified via $(\cdot, \cdot)$:

$$\mathfrak{h}^* \ni \alpha \mapsto h_\alpha \in \mathfrak{h} : \alpha(h) = (h_\alpha, h)$$

for all $h \in \mathfrak{h}$. The real vector space $\mathfrak{h}_0^*$ spanned by $\alpha \in \Phi$ is a real Euclidean vector space with inner product $(\cdot, \cdot)$. In particular, a reflection with respect the hyperplane normal to $\alpha \in \mathfrak{h}_0^*$ can be defined by

$$s_\alpha \beta = \beta - \langle \beta, \alpha \rangle \alpha, \quad \langle \beta, \alpha \rangle = 2 \frac{(\beta, \alpha)}{(\alpha, \alpha)}.$$

The root system Φ has the following properties:

- (i) Φ contains finitely nonzero vectors that span $\mathfrak{h}_0^*$;
- (ii) for any $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$;
- (iii) $s_\alpha(\Phi) \subset \Phi$, and
- (iv) for any $\alpha, \beta \in \Phi$, the number $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

From the second property we have a polarization of Φ , i.e., $\Phi = \Phi_+ \cup \Phi_-$. Elements of Φ_+ (resp. Φ_-) are called positive (resp. negative) roots. Elements of $\Pi = \{\alpha_1, \dots, \alpha_l\} \subset \Phi_+$ such that can not be represented as a positive linear combination of other roots are called *simple roots*.

The Cartan matrix of $\mathfrak{g}$ is defined by

$$A = (a_{ij})_{i,j=1}^l = (\langle \alpha_j, \alpha_i \rangle)_{i,j=1}^l$$

and is characterized by the following properties:

- (i) A is an integral matrix with 2's on the diagonal and non-positive off-diagonal entries.
- (ii) DA is symmetric, where $D = \text{diag}(d_1, \dots, d_l)$ is the diagonal matrix with entries $d_i = (\alpha_i, \alpha_i)$, and
- (iii) DA is positive definite. This implies $a_{ij}a_{ji} < 3$.

The Lie algebra $\mathfrak{g}$ has a presentation with the so called *Chevalley generators* $\{h_i, e_{\pm i}\}_{i=1}^r$ and relations called *Serre relations*:

$$[h_i, h_j] = 0, \quad [h_i, e_{\pm j}] = \pm a_{ij} e_{\pm j}, \quad [e_i, e_{-j}] = \delta_{ij} h_i, \quad \text{ad}_{e_{\pm i}}^{1-a_{ij}} e_{\pm j} = 0. \quad (\text{A.5})$$

In terms of simple roots and Chevalley generators, the entries of the Cartan matrix are

$$a_{ij} = \alpha_j(h_i), i, j \in [1, l]$$

From relations (A.5) one can construct an embedding $\rho_i : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{g}$:

$$\rho_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h_i, \quad \rho_i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e_i, \quad \rho_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = e_{-i}$$

which can be integrated to an embedding, also denoted by ρ_i , of the group $\mathrm{SL}_2(\mathbb{C})$ to G . In particular,

$$\rho_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp(te_i), \quad \rho_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp(te_{-i}) \quad (\text{A.6})$$

for any $t \in \mathbb{C}$. Then we have one-parameter subgroups

$$x_i^\pm(t) = \exp(te_{\pm i}) \quad (\text{A.7})$$

generated by $e_{\pm i}$.

The *Borel* (resp. *opposite Borel*) subalgebra $\mathfrak{b}_+$ (resp. $\mathfrak{b}_-$) of $\mathfrak{g}$ relative to $\mathfrak{h}$ is spanned by $\mathfrak{h}$ and the vectors e_α , $\alpha \in \Phi_+$ (resp. e_α , $\alpha \in \Phi_-$). Let $\mathfrak{n}_\pm = [\mathfrak{b}_\pm, \mathfrak{b}_\pm]$ be the corresponding maximal nilpotent subalgebras of $\mathfrak{g}$. The connected subgroups that correspond to $\mathfrak{h}, \mathfrak{b}_\pm, \mathfrak{n}_\pm$ will be denoted by $H, B_\pm, N_\pm$.

The *weight lattice* P is the set

$$\{\gamma \in \mathfrak{h}^* \mid \gamma(h_i) \in \mathbb{Z} \text{ for } i \in [1, l]\}.$$

The basis of *fundamental weights* $\omega_1, \dots, \omega_l$ in P is defined by $\omega_j(h_i) = \langle \omega_j, \alpha_i \rangle = \delta_{ij}$, $i, j \in [1, l]$. Every weight P defines a multiplicative character $a \mapsto a^\gamma$ of H as follows:

$$a^\gamma = e^{\gamma(h)}, \quad \text{with } a = \exp(h) \text{ for some } h \in \mathfrak{h}.$$

Simple roots, fundamental weights and entries of the Cartan matrix are related via

$$\alpha_i = \sum_{j=1}^l a_{ji} \omega_j. \quad (\text{A.8})$$

The Weyl group W is defined as the quotient $W = N_G(H)/H$. For each of the embeddings $\rho_i : \mathrm{SL}_2 \rightarrow G$, $i \in [1, l]$ put

$$\bar{s}_i = \rho_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in N_G(H)$$

and $s_i = \bar{s}_i H \in W$. All s_i are of order 2 and together they generate W . One identifies s_i with the elementary reflection s_{α_i} , $\alpha_i \in \Pi$ in $\mathfrak{h}_0^*$. Consequently, W is identified with the Coxeter group generated by reflections $s_1, \dots, s_l$.

The Weyl group acts on the weight lattice P : For any $w \in W$ and $\gamma \in P$ the action $\gamma \mapsto w\gamma$ is defined expanding w as the product of elementary reflections and applying the reflections one by one to γ .

A *reduced decomposition* of an element $w \in W$ is a representation of w as a product $w = s_{i_1} \cdots s_{i_l}$ of the smallest possible length. A reduced decomposition is not unique, but the number l of reflections in the product depends only on w and is called the length of w and denoted by $l(w)$.

The Gelfand-Dickey hierarchy

The associative algebra of *differential operators* of order n with the leading term ∂^n allow us to construct a family of non-linear partial differential equations known as Gelfand-Dickey hierarchy. We present this hierarchy and for a more detailed treatment about the subject see [3].

A *differential operator* is a finite sum

$$L = \sum_{i=0}^n u_i \partial^i, \quad \partial = \frac{d}{dx}$$

where the coefficients u_i are differential functions on x . The differential operator L has order n provided its leading coefficient is not identically zero: $u_n \neq 0$. A differential operator of order 0 is of the form $L = u_0$ and is referred to as a *multiplication operator*.

The (noncommutative) multiplication of differential operators is completely described by

$$\partial^i \cdot \partial^j = \partial^{i+j} \tag{B.1}$$

valid for $i, j \geq 0$, and the general Leibniz rule

$$\partial^n \cdot u = \sum_{k=0}^n \binom{n}{k} u^{(k)} \partial^{n-k}, \quad u^{(k)} = \partial^k u \tag{B.2}$$

The rules (B.1) and (B.2) allow us, by linearity, extend the product for any two differential operators. Therefore the space of all differential operators form a noncommutative associative algebra with unity, the identity operator being the multiplication operator by the constant function 1.

Definition B.1. Given a differential operator $L = \sum_{i=0}^n u_i \partial^i$ its formal adjoint is defined by

$$L^* = \sum_{i=0}^n (-1)^i \partial^i \cdot u_i$$

A differential operator is skew-adjoint if $L^* = -L$.

Example B.2. Let $L = \partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x$. In this case we have $u_0 = \frac{1}{3}u_x, u_1 = \frac{2}{3}u, u_2 = 0, u_3 = 1$. Then

$$\begin{aligned} L^* &= \frac{1}{3}u_x - \partial \cdot \frac{2}{3}u - \partial^3 \\ &= \frac{1}{3}u_x - \frac{2}{3}(u_x + u\partial) - \partial^3 = -\frac{1}{3}u_x - \frac{2}{3}u\partial - \partial^3 = -L \end{aligned}$$

In this case L is skew-adjoint

Definition B.3. A (formal) *pseudo-differential operator* is a formal infinite series

$$L = \sum_{i=-\infty}^n u_i \partial^i$$

whose coefficients are differentiable functions. We say that L has order n provided $u_n \neq 0$.

By convention, the zero pseudo-differential operator is said to have order $-\infty$.

Just as we multiply ordinary differential operators, we can define a compatible multiplication process for pseudo-differential operators. We treat ∂^{-1} as the inverse of ∂ , so that $\partial^{-1} \cdot \partial = 1 = \partial \cdot \partial^{-1}$; more generally, we continue to allow the product rule (B.1), for arbitrary integers i, j .

Now if we multiply (B.2) on both the left and the right by the operator ∂^{-1} and rearrange terms, we have

$$\partial^{-1} \cdot Q = Q\partial^{-1} - \partial^{-1} \cdot Q'\partial^{-1} \quad (\text{B.3})$$

Applying the identity (B.3) to the second term on the right hand side (with Q' replacing Q), we deduce

$$\partial^{-1} \cdot Q = Q\partial^{-1} - Q'\partial^{-2} + \partial^{-1} \cdot Q''\partial^{-2}$$

Continuing applying (B.3) we get

$$\partial^{-1} \cdot Q = \sum_{i=0}^{\infty} (-1)^i Q^{(i)} \partial^{-i-1} \quad (\text{B.4})$$

More generally

$$\partial^k Q = \sum_{i=0}^{\infty} (-1)^i \binom{|k| + i - 1}{i} Q^{(i)} \partial^{k-i}, \quad (\text{B.5})$$

for $k < 0$.

Example B.4. Consider

$$(\partial + u^2\partial^{-1}) \cdot (\partial^2 + u\partial^{-1}) = \partial^3 + u^2\partial + \partial \cdot u\partial^{-1} + u^2\partial^{-1} \cdot u\partial^{-1}$$

we have

$$\partial \cdot u\partial^{-1} = (u' + u\partial)\partial^{-1} = u'\partial^{-1} + u$$

and

$$u^2(\partial^{-1} \cdot u)\partial^{-1} = u^2(u\partial^{-1} - u'\partial^{-2} + u''\partial^{-3} - \dots)\partial^{-1} = u^3\partial^{-2} - u^2u'\partial^{-3} + u^2u''\partial^{-4} - \dots$$

then

$$(\partial + u^2\partial^{-1})(\partial^2 + u\partial^{-1}) = \partial^3 + u^2\partial + u + u'\partial^{-1} + u^3\partial^{-2} - u^2u'\partial^{-3} + u^2u''\partial^{-4} - \dots$$

The commutator of two pseudo-differential operators is defined as usual, $[L_1, L_2] = L_1 \cdot L_2 - L_2 \cdot L_1$ and obey the order relation

$$\text{order } [L_1, L_2] \leq \text{order } L_1 + \text{order } L_2 - 1$$

Proposition B.5.

(1) Every non-zero pseudo-differential operator has an inverse, in fact, if $\text{order } L = n$ then its inverse L^{-1} has order $-n$. Consequently, the space of pseudo-differential operators is a noncommutative field.

(2) Every pseudo-differential operator of order $n > 0$ has an n -th root.

The result allow us consider fractional powers $L^{p/n}$ of L for all $p \in \mathbb{Z}$. Furthermore, we have $[L^{p/n}, L^{q/n}] = 0$.

Example B.6. Consider the first order differential operator $L = \partial + u$ then

$$\begin{aligned} L^{-1} &= \partial^{-1} - u\partial^{-2} + (u' + u^2)\partial^{-3} - (u'' + 3uu' + u^3)\partial^{-4} + \dots \\ L^{1/2} &= \partial + \frac{1}{2}u\partial^{-1} - \frac{1}{4}u'\partial^{-2} + \frac{1}{8}(u'' - u^2)\partial^{-3} + \dots \end{aligned}$$

For a pseudo-differential operator L we denote by L_+ its differential part of L and $L_- := L - L_+$.

Let

$$L = \partial^n + u_{n-2}\partial^{n-2} + \dots + u_0,$$

we omit the term ∂^{n-1} , and let $P_m = L_+^{m/n} := (L^{m/n})_+$

Proposition B.7. The commutator $[P_m, L]$ has order $\leq n - 2$.

Let all the coefficients u_i depend on an additional parameter t_m and let

$$\partial_m L = \frac{\partial u_{n-2}}{\partial t_m} \partial^{n-2} + \dots + \frac{\partial u_0}{\partial t_m}$$

Definition B.8. The set of equations

$$\partial_m L = [P_m, L] \tag{B.6}$$

with a fixed n and various m is called the n th **Gelfand-Dickey hierarchy**. The case $n = 2$ is also called the KdV hierarchy.

Example B.9. (i) Case $n = 2, m = 3$. Let $L = \partial^2 + u$, then $P_3 = L_+^{3/2} = \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u_x$. The equation (B.6) is equivalent to the *KdV equation*:

$$4u_t = u_{xxx} + 6uu_x, \quad t = t_3. \quad (\text{B.7})$$

(ii) Case $n = 3, m = 2$. Let $L = \partial^3 + u\partial + v$, then $P_2 = L_+^{2/3} = \partial^2 + \frac{2}{3}u$. Letting $t = t_2$,

$$\begin{aligned} \partial_t L = [P_2, L] &\Leftrightarrow u_t = -u_{xx} + 2v_x, \quad v_t = v_{xx} - \frac{2}{3}u_{xxx} - \frac{2}{3}uu_x \\ &\Leftrightarrow u_{tt} = -\frac{1}{3}u_{xxxx} - \frac{4}{3}(uu_x)_x \end{aligned} \quad (\text{B.8})$$

Equation (B.8) is called *Boussinesq equation*.

Remark B.10. There exists a Hamiltonian formulation for the Gelfand-Dickey hierarchy such that these non-linear partial differential equations can be considered as dynamical systems with an infinite number of degrees of freedom, see [3].

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