On qualitative properties of generalized ODEs

Rogelio Grau Acuña

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Rogelio Grau Acuña

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Advisor: Profe. Dra. Márcia Cristina Anderson Braz Federson

Co-advisor: Profa. Dra. Jaqueline Godoy Mesquita

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Sobre propriedades qualitativas de EDOs generalizadas

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Orientadora: Profa. Dra. Márcia Cristina Anderson
Braz Federson

Coorientadora: Profa.Dra. Jaqueline Godoy Mesquita

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Resumo

Neste trabalho, nosso objetivo é provar resultados sobre prolongamento de soluções, limitação uniforme de soluções, estabilidade uniforme e estabilidade uniforme assintótica (no sentido clássico de Lyapunov) para equações diferenciais em medida e para equações dinâmicas em escalas temporais.

A fim de obter os nossos resultados, empregamos a teoria de EDOs generalizadas, uma vez que estas equações abrangem equações diferenciais em medida e equações dinâmicas em escalas temporais. Portanto, para obter nossos resultados, vamos começar por provar, os resultados que queremos para EDOs generalizadas abstratas. Em seguida, usando a correspondência entre as soluções de EDOs generalizadas e soluções de equações diferenciais em medida (ver [38]), estenderemos os resultados para estas últimas equações. Depois disso, usando a correspondência entre as soluções de equações diferenciais em medida e as soluções de equações dinâmicas em escalas temporais (ver [21]), estenderemos todos os resultados para estas últimas equações.

Finalmente, investigamos EDOs generalizadas autônomas e mostramos que estas equações não aumentam a classe de EDOs autônomas clássicas, mesmo quando consideramos uma classe mais geral de funções nos lados direitos das equações.

Os novos resultados encontrados estão contidos em [16, 17, 18, 19].

Palavras-chaves: Equações diferenciais em medida, equações diferenciais ordinárias generalizadas, equações dinâmicas em escalas temporais, limitação, estabilidade de Lyapunov, prolongamento, integral de kurzweil-Henstock-Stieltjes, funcionais de Lyapunov.

Abstract

In this work, our goal is to prove results on prolongation of solutions, uniform boundedness of solutions, uniform stability as well uniform asymptotic stability (in the classical sense of Lyapunov) for measure differential equations and for dynamic equations on time scales.

In order to get our results, we employ the theory of generalized ODEs, since these equations encompass measure differential equations and dynamic equations on time scales. Therefore, to get our results, we start by proving the expected result for abstract generalized ODEs. Then, using the correspondence between the solutions of these equations and the solutions of measure differential equations (see [38]), we extend all the results to these the latter. After that, using the correspondence between the solutions of measure differential equations and the solutions of dynamic equations on time scales (see [21]), we extend all the results to these last equations.

Finally, we investigate autonomous generalized ODEs and show that these equations do not enlarge the class of classical autonomous ODEs, even when we consider a more general class of functions as right-hand sides.

All the new results presented in this work are contained in papers [16, 17, 18, 19].

Keywords: Measure differential equations, generalized ordinary differential equations, dynamic equations on time scales, boundedness, Lyapunov stability, prolongation, Kurzweil-Henstock-Stieltjes integral, Lyapunov functionals.

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Introduction

The theory of generalized ordinary differential equations (we write generalized ODEs, for short) was introduced by J. Kurzweil in 1957 for Euclidean and Banach space-valued functions with the purpose to generalize certain results on continuous dependence on parameters of the solutions of ordinary differential equations (see [30]).

Since then, these equations have been attracting the attention of many researchers, because they encompass several types of differential equations, such as ordinary and functional differential equations (see [5, 33]), measure differential equations and measure functional differential equations (see [20, 38]), dynamic equations on time scales and functional dynamic equations on time scales (see [20, 42]), differential equations with impulses (see [1, 23]), among others.

We point out that by the correspondences between the solutions of a generalized ODE and the solutions of other types of equations, we are able to translate the results from generalized ODEs to another class of differential equations. The gain in doing so comes from the use of a nonabsolute integral in the integral form of a differential equation. Such integral is the Kurzweil integral whose main feature is to integrate highly oscillating functions. It also copes well with many discontinuities.

In this thesis, our goal is to present results on prolongation of solutions of measure differential equations and of solutions of dynamic equations on time scales, as well as results on boundedness of solutions and Lyapunov-type stability results for these two types of equations. In order to obtain our results, we employ the theory of generalized ODEs, since these equations encompass measure differential equations and dynamic equations on time scale. Therefore, we start by proving our results for abstract generalized ODEs and, then, by the correspondence between the solutions of these equations and the solutions of measure differential equations, we extend our results to this last class of equations. After that, we use the correspondence between the solutions of measure differential equations and the solutions of dynamic equations on time scales to extend our results to these last equations.

Measure differential equations are present in the literature since the decade of 1960 with a work by W. Schmaedeke (see [37]) on control theory. Since then, several works had been developed in measure differential equations on qualitative properties of their solutions, their asymptotic behavior and applications. See [1, 2, 6, 10, 11, 12, 13, 14, 34, 39, 40].

It worths noticing that measure differential equations represent a very important tool to applications, since they describe impulsive systems with discontinuous solutions and, therefore, can be used to studying evolutionary processes such as biological or physical phenomena, optimal control models in economics, among others. See, for instance, [1, 12, 34, 37]

In this work, we consider an integral form of a measure differential equations of type

$$x(t) = x(\tau_0) + \int_{\tau_0}^t f(x(s), s) dg(s), \quad t \geq \tau_0, \quad (0.1)$$

where $\tau_0 \geq t_0$, $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$, $O \subset \mathbb{R}^n$ is an open set, and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ is a nondecreasing function. The integral on the right-hand side of (0.1) is the Kurzweil-Henstock-Stieljes integral. In order to obtain our main results, we assume conditions on indefinite integral instead of conditions on the integrands. This choice enables us to deal with functions which may be highly oscillating or have many discontinuities.

On the other hand, dynamic equations on time scales also play an important role on applications to several fields of knowledge. This is a recent theory, which was introduced by Stefan Hilger in his doctor thesis (see [27]), in order to unify the discrete and continuous analysis and the cases “*in between*”. The main idea of this theory is to prove the results for a type of equation, called *dynamic equation*, where the domain of the unknown function is a time scale, which is an arbitrary closed and nonempty subset of the real numbers. Therefore, depending on the chosen time scale, we obtain a result for a different type of equation. For example, when the time scale is the set of real numbers, we obtain a result for differential equations, and when the time scale is the set of integers, we obtain a result for difference equations. Notice that we can obtain several results for different types of equations depending on the chosen time scale.

Since dynamic equations on time scales have several applications to real-world models, it is very important to understand their properties on unbounded intervals, that is, when the time is sufficiently large, because then one can investigate properties such as asymptotic behavior, stability, boundedness, attractors, bifurcation, among others. Therefore the first step in order to do that is to investigate existence and uniqueness of a maximal solution and prolongation of solutions for these equations.

The present thesis is divided into seven chapters which are organized as follows. In the first chapter, we recall the basic concepts and properties concerning the Kurzweil integral and

its applications to generalized ODEs. The second chapter is devoted to recalling the theory of dynamic equations on time scales, and to presenting the main results and basic concepts of this theory. In the third chapter, we recall the basis of the theory of measure differential equations and its main properties. Also, we investigate the correspondence between the solutions of measure differential equations and the solutions of generalized ODEs. Finally, we present a connection between the solutions of measure differential equations and the solutions of dynamic equations on time scales.

The fourth chapter is devoted to investigating the prolongation of solutions of generalized ODEs, measure differential equations, as well as the solutions of dynamic equations on time scales. We start by investigating the existence and uniqueness of maximal solutions of generalized ODEs defined in a general Banach space. New results are obtained on this subject. Then, using the correspondence between the solutions, we obtain analogous results for measure differential equations and for dynamic equations on time scales. Most of the results presented in this chapter are new.

On the other hand, we are also interested in investigating the boundedness of solutions of measure differential equations and of dynamic equations on time scales. Then we prove new results on boundedness of solutions of generalized ODE which do not require a Lipschitz-type condition with respect to the second variable on the Lyapunov functional, improving the results found in the literature (see [2]). After that, we extend our results to measure differential equations, using the correspondence between the solutions of these equations and the solutions of generalized ODEs, improving the results found in the literature for measure differential equations. Furthermore, we prove a result on boundedness of solutions of dynamic equations on time scales, using the fact that a dynamic equation on time scales is a special case of measure differential equations. Moreover, we introduce new concepts of boundedness of solutions of dynamic equations on time scales, namely, quasi-uniformly ultimately boundedness and uniformly ultimately boundedness. All these result are new and are collected in Chapter 5.

In this work, we also investigate Lyapunov-type stability of the trivial solution of generalized ODEs in Banach spaces. The stability concepts presented here extend the concepts from [44]. Also, we point out that our results on uniform stability and uniform asymptotic stability of the trivial solution of a generalized ODE do not require any Lipschitz-type condition on the Lyapunov functional, improving the results found in the literature (see [3, 22, 24]). Using the correspondences between the solutions, we extend our results to measure differential equations and to dynamic equations on time scales. All these results are original and are contained in Chapter 6.

Finally, we investigate autonomous generalized ODEs and prove that these equations do not enlarge the class of classical autonomous ODEs.

All the new results presented here are contained in the papers [16, 17, 18, 19].

Generalized ODE

In this chapter, we recall the concept of generalized ordinary differential equations (we write generalized ODEs, for short) and present some basic results which play a fundamental role throughout this work.

We divide this chapter into two sections. In the first section, we present the definition of the Kurzweil integral which is the basis to define the concept of a generalized ODE. The second section is devoted to presenting the basic theory of generalized ODEs. The main references for this chapter are [31, 38].

1.1 The Kurzweil integral

We start this section by recalling some basic concepts concerning the Kurzweil integral.

Let $[a, b]$ be an interval of $\mathbb{R}$, $-\infty < a < b < +\infty$. A *tagged division* of $[a, b]$ is a finite collection of point-interval pairs $D = (\tau_i, [s_{i-1}, s_i])$, where $a = s_0 \leq s_1 \leq \dots \leq s_{|D|} = b$ is a division of $[a, b]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \dots, |D|$, where the symbol $|D|$ denotes the number of subintervals in which $[a, b]$ is divided.

A *gauge* on a set $B \subset [a, b]$ is any function $\delta : B \rightarrow (0, \infty)$. Given a gauge δ on $[a, b]$, we say that a tagged division $D = (\tau_i, [s_{i-1}, s_i])$ is δ -*fine* if, for every i , we have

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)).$$

Using these definitions and notations, we are able to present the concept of a Kurzweil integrable function defined on $[a, b] \times [a, b]$ taking values in an abstract space X .

Throughout this section, let us assume that X is a Banach space with norm $\|\cdot\|$.

Definition 1.1. A function $U : [a, b] \times [a, b] \rightarrow X$ is called *Kurzweil integrable* over $[a, b]$, if there is an element $I \in X$ such that given $\varepsilon > 0$, there is a gauge δ on $[a, b]$ such that

$$\left\| \sum_{i=1}^{|D|} [U(\tau_i, s_i) - U(\tau_i, s_{i-1})] - I \right\| < \varepsilon$$

for every δ -fine tagged division $D = (\tau_i, [s_{i-1}, s_i])$ of $[a, b]$. In this case, I is called the Kurzweil integral of U over $[a, b]$ and it is denoted by $\int_a^b DU(\tau, t)$.

We use the notation $S(U, D) = \sum_{i=1}^{|D|} [U(\tau_i, s_i) - U(\tau_i, s_{i-1})]$ for the Riemann-type sum corresponding to the function U and to the tagged division D .

Analogously to the Riemann integral, the Kurzweil integral has the usual properties of linearity, additivity with respect to adjacent intervals and integrability on subintervals.

If the integral $\int_a^b DU(\tau, t)$ exists, then we define

$$\int_b^a DU(\tau, t) = - \int_a^b DU(\tau, t), \text{ if } a < b,$$

and

$$\int_a^b DU(\tau, t) = 0, \text{ if } a = b.$$

In particular, when $U(\tau, t) = f(\tau)t$, where $f : [a, b] \rightarrow \mathbb{R}^n$, then we obtain the usual Kurzweil-Henstock integral of f . More generally, when $U(\tau, t) = f(\tau)g(t)$, we obtain the Kurzweil-Henstock-Stieltjes (or Perron-Stieltjes) integral of $f : [a, b] \rightarrow \mathbb{R}^n$ with respect to a function $g : [a, b] \rightarrow \mathbb{R}$ and we will denote such integral by $\int_a^b f(s)dg(s)$.

Clearly, Definition 1.1 makes sense only if, for a given gauge δ of $[a, b]$, there exists at least one δ -fine division D of $[a, b]$. This is the content of the next result. For more details, see [26, Theorem 4.1].

Lemma 1.2 (Cousin Lemma). *Given a gauge δ of $[a, b]$, there exists a δ -fine tagged division D of $[a, b]$.*

The next result is known as the Saks-Henstock Lemma. A proof of it can be found in [38, Lemma 1.13].

Lemma 1.3 (Saks-Henstock Lemma). *Let $U : [a, b] \times [a, b] \rightarrow X$ be Kurzweil integrable over $[a, b]$. Given $\varepsilon > 0$, let δ be a gauge of $[a, b]$ corresponding to ε in the definition of the*

Kurzweil integral such that

$$\left\| \sum_{i=1}^{|D|} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})] - \int_a^b DU(\tau, t) \right\| < \varepsilon, \quad (1.1)$$

for every δ -fine tagged division $D = (\tau_i, [\alpha_{i-1}, \alpha_i])$ of $[a, b]$. If

$$a \leq \beta_1 \leq \xi_1 \leq \gamma_1 \leq \beta_2 \leq \xi_2 \leq \gamma_2 \leq \dots \leq \beta_m \leq \xi_m \leq \gamma_m \leq b$$

represents a δ -fine tagged partial division $\{(\xi_i, [\beta_i, \gamma_i]) : i = 1, 2, \dots, m\}$, that is,

$$\xi_i \in [\beta_i, \gamma_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)), \quad i = 1, 2, \dots, m,$$

but $\bigcup_{i=1}^m [\beta_i, \gamma_i]$ does not need to equal $[a, b]$, then

$$\left\| \sum_{i=1}^m [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})] - \int_a^b DU(\tau, t) \right\| < \varepsilon.$$

The following result is an immediate consequence of the Saks-Henstock Lemma. See [38].

Corollary 1.4. *Let $U : [a, b] \times [a, b] \rightarrow X$ be Kurzweil integrable over $[a, b]$. Given $\varepsilon > 0$, let δ be a gauge on $[a, b]$ corresponding to ε in the definition of the Kurzweil integral and $[\gamma, v] \subset [a, b]$. Then the following assertions hold.*

(i) *If $(v - \gamma) < \delta(\gamma)$, then*

$$\left\| U(\gamma, v) - U(\gamma, \gamma) - \int_{\gamma}^v DU(\tau, t) \right\| < \varepsilon;$$

(ii) *If $(v - \gamma) < \delta(v)$, then*

$$\left\| U(v, v) - U(v, \gamma) - \int_{\gamma}^v DU(\tau, t) \right\| < \varepsilon.$$

The next result is a generalization of [38, Theorem 1.14] for general Banach space-valued function. We omit its proof here, since it is similar to the finite dimensional case.

Theorem 1.5. *If $U : [a, b] \times [a, b] \rightarrow X$ is a function such that for every $c \in [a, b)$, U is Kurzweil integrable over $[a, c]$ and the limit*

$$\lim_{c \rightarrow b^-} \left[\int_a^c DU(\tau, t) - U(b, c) + U(b, b) \right] = I \in X$$

exists, then the function U is Kurzweil integrable over $[a, b]$ and

$$\int_a^b DU(\tau, t) = I.$$

Similarly, if the function U is Kurzweil integrable over $[c, b]$ for every $c \in (a, b]$ and the limit

$$\lim_{c \rightarrow a^+} \left[\int_c^b DU(\tau, t) + U(a, c) - U(a, a) \right] = I \in X$$

exists, then the function U is Kurzweil integrable over $[a, b]$ and

$$\int_a^b DU(\tau, t) = I.$$

The next result shows that the indefinite Kurzweil integral of $U : [a, b] \times [a, b] \rightarrow X$, the function $s \mapsto \int_a^s DU(\tau, t)$, is not continuous in general. A proof of this fact can be found in [38, Theorem 1.16].

Theorem 1.6 (Hake-type Theorem). *Let $U : [a, b] \times [a, b] \rightarrow X$ be Kurzweil integrable over $[a, b]$ and let $c \in [a, b]$. Then,*

$$\lim_{s \rightarrow c} \left[\int_a^s DU(\tau, t) - U(c, s) + U(c, c) \right] = \int_a^c DU(\tau, t)$$

and

$$\lim_{s \rightarrow c} \left[\int_s^b DU(\tau, t) + U(c, s) - U(c, c) \right] = \int_c^b DU(\tau, t).$$

Remark 1.7. Notice that the indefinite Kurzweil integral is continuous at a point $c \in [a, b]$, if and only if, $U(c, \cdot) : [a, b] \rightarrow X$, given by $U(c, t)$ for $t \in [a, b]$, is continuous at the point $t = c$.

We recall the reader that a function $f : [a, b] \rightarrow X$ is called *regulated*, if the lateral limits

$$f(t^-) = \lim_{s \rightarrow t^-} f(s), \quad t \in (a, b] \quad \text{and} \quad f(t^+) = \lim_{s \rightarrow t^+} f(s), \quad t \in [a, b)$$

exist. On the other hand, given a function $f : [a, b] \rightarrow X$, its variation $\text{var}_{[a,b]} f$ on the interval $[a, b]$ is defined by

$$\text{var}_{[a,b]} f := \sup_{D \in \mathcal{D}[a,b]} \sum_{i=1}^{|D|} |f(t_i) - f(t_{i-1})|,$$

where $\mathcal{D}[a, b]$ denotes the set of all division of $[a, b]$. If $\text{var}_{[a,b]} f < +\infty$, we say that f is of *bounded variation* in the interval $[a, b]$.

The space of all regulated functions $f : [a, b] \rightarrow X$ will be denoted by $G([a, b], X)$ which is a Banach space when endowed with the usual supremum norm

$$\|f\|_\infty = \sup_{s \in [a,b]} \|f(s)\|.$$

It is also known that functions of bounded variation in $[a, b]$ are regulated. For more information, see [25, 29].

The following result, which describes some properties of the indefinite Kurzweil-Henstock-Stieltjes integral, is a special case of Theorem 1.6.

Theorem 1.8. *Let $f : [a, b] \rightarrow \mathbb{R}^n$ and $g : [a, b] \rightarrow \mathbb{R}$ be functions such that g is regulated and $\int_a^b f(s)dg(s)$ exists. Then the functions*

$$h(t) = \int_a^t f(s)dg(s), \quad t \in [a, b] \quad \text{and} \quad k(t) = \int_t^b f(s)dg(s), \quad t \in [a, b],$$

are regulated on $[a, b]$ and satisfy

$$\begin{aligned} h(t^+) &= h(t) + f(t)\Delta^+ g(t), \quad t \in [a, b), \\ h(t^-) &= h(t) - f(t)\Delta^- g(t), \quad t \in (a, b], \\ k(t^+) &= k(t) - f(t)\Delta^+ g(t), \quad t \in [a, b), \\ k(t^-) &= k(t) + f(t)\Delta^- g(t), \quad t \in (a, b], \end{aligned}$$

where

$$\Delta^+ g(t) = g(t^+) - g(t) \quad \text{and} \quad \Delta^- g(t) = g(t) - g(t^-).$$

Also,

$$g(t^+) = \lim_{\xi \rightarrow t^+} g(\xi), \quad t \in [a, b) \quad \text{and} \quad g(t^-) = \lim_{\xi \rightarrow t^-} g(\xi), \quad t \in (a, b].$$

Analogously, one can define $h(t^+)$, $h(t^-)$, $k(t^+)$ and $k(t^-)$.

We finish this section by presenting a result which ensures the existence of the Kurzweil-Henstock-Stieltjes integral $\int_a^b f(s)dg(s)$. This result can be found in [20, Theorem 2.1].

Theorem 1.9. *Let $f : [a, b] \rightarrow \mathbb{R}^n$ be a regulated function on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ be a nondecreasing function. Then, the following conditions hold*

(i) *The integral $\int_a^b f(s)dg(s)$ exists;*

(ii) $\left| \int_a^b f(s)dg(s) \right| \leq \int_a^b |f(s)| dg(s).$

1.2 Generalized ODEs

In this section, we recall the definition of a generalized ODEs and its main properties. The main reference is [38].

Consider a subset $O \subset X$, an interval $[t_0, +\infty) \subset \mathbb{R}$ and an X -valued function $F : \Omega \rightarrow X$ defined for each $(x, t) \in \Omega$, where $\Omega = O \times [t_0, +\infty)$.

Definition 1.10. A function $x : [\alpha, \beta] \rightarrow X$ is called a *solution of the generalized ODE*

$$\frac{dx}{d\tau} = DF(x, t) \tag{1.2}$$

on the interval $[\alpha, \beta] \subset [t_0, +\infty)$, if $(x(t), t) \in \Omega$, for every $t \in [\alpha, \beta]$, and

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t), \tag{1.3}$$

for every $s_1, s_2 \in [\alpha, \beta]$.

The integral on the right-hand side of (1.3) can be understood as the Kurzweil integral of $U(\tau, t) = F(x(\tau), t)$ described in Definition 1.1.

Remark 1.11. It is worth mentioning that the notation in (1.2) is only symbolical. The letter D indicates that (1.2) is a generalized ODE and this concept is defined via its solutions. Even the expression $\frac{dx}{d\tau}$ does not mean that the solution has a derivative.

We can also define a solution of the generalized ODE (1.2) with an initial condition as shows the next definition.

Definition 1.12. The function $x : [\alpha, \beta] \rightarrow X$ is a *solution of the generalized ODE* (1.2) with initial condition $x(\tau_0) = x_0$ on the interval $[\alpha, \beta] \subset [t_0, +\infty)$, if $\tau_0 \in [\alpha, \beta]$, $(x(t), t) \in \Omega$ for every $t \in [\alpha, \beta]$ and

$$x(s) - x_0 = \int_{\tau_0}^s DF(x(\tau), t)$$

for every $s \in [\alpha, \beta]$.

Although a solution x of the generalized ODE (1.2) is defined on a bounded interval $[\alpha, \beta]$ in Definition 1.10, it is possible to extend this definition to a nondegenerated interval I , not necessarily compact. This is the content of the next definition.

Definition 1.13. We say that $x : I \rightarrow X$, where I is a nondegenerated subinterval of $[t_0, +\infty)$, is a *solution* of the generalized ODE (1.2) on I , if $(x(t), t) \in \Omega$ for every $t \in I$ and if the following equality

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t)$$

is satisfied for every $s_1, s_2 \in I$.

In what follows, we present a class of functions $F : \Omega \rightarrow X$ which is be crucial to obtaining important properties concerning the solutions of the generalized ODE (1.2).

Definition 1.14. Given a nondecreasing function $h : [t_0, +\infty) \rightarrow \mathbb{R}$, we say that a function $F : \Omega \rightarrow X$ belongs to the class $\mathcal{F}(\Omega, h)$, if

$$\|F(x, t) - F(x, s)\| \leq |h(t) - h(s)| \quad (1.4)$$

for all $(x, t), (x, s) \in \Omega$, and if

$$\|F(x, t) - F(x, s) - F(y, t) + F(y, s)\| \leq \|x - y\| |h(t) - h(s)| \quad (1.5)$$

for all $(x, t), (x, s), (y, t), (y, s) \in \Omega$.

The following result is essential to our purposes. It can be found in [38, Lemma 3.9].

Lemma 1.15. *Let $F : \Omega \rightarrow X$ satisfy condition (1.4). If $[\alpha, \beta] \subset [t_0, +\infty)$ and $x : [\alpha, \beta] \rightarrow X$ is such that $(x(t), t) \in \Omega$ for every $t \in [\alpha, \beta]$ and if the Kurzweil integral $\int_{\alpha}^{\beta} DF(x(\tau), t)$ exists, then for every $s_2, s_1 \in [\alpha, \beta]$, we have*

$$\left\| \int_{s_1}^{s_2} DF(x(\tau), t) \right\| \leq |h(s_2) - h(s_1)|.$$

The next result describes an important property of the solutions of the generalized ODE (1.2). It can be found in [38, Lemma 3.10].

Lemma 1.16. *Assume $F : \Omega \rightarrow X$ satisfies condition (1.4). If $[a, b] \subset [t_0, +\infty)$ and $x : [a, b] \rightarrow X$ is a solution of the generalized ODE (1.2), then the inequality*

$$\|x(t) - x(s)\| \leq |h(t) - h(s)|$$

holds for all $t, s \in [a, b]$. Also, each point in $[a, b]$ at which the function h is continuous is a continuity point of the solution $x : [a, b] \rightarrow X$.

To the next result, let $\text{var}_{[a,b]} x$ be the variation of a function $x : [a, b] \rightarrow X$. If the right-hand side of the generalized ODE (1.2) satisfies condition (1.4), then the next result ensures that its solution is a function of bounded variation. A proof of it can be found in [38, Lemma 3.11].

Corollary 1.17. *Let $F : \Omega \rightarrow X$ satisfy condition (1.4). If $[\alpha, \beta] \subset [t_0, +\infty)$ and $x : [\alpha, \beta] \rightarrow X$ is a solution of the generalized ODE (1.2), then x is a function of bounded variation (and, therefore, regulated) in $[\alpha, \beta]$ and*

$$\text{var}_{[\alpha, \beta]} x \leq h(\beta) - h(\alpha) < +\infty. \quad (1.6)$$

Now, we present a result which ensures the existence of the integral involved in the definition of the solution of the generalized ODE (1.2). This result is a consequence of [38, Lemma 3.9, Corollary 3.16].

Proposition 1.18. *Let $F : \Omega \rightarrow X$ be an element of the class $\mathcal{F}(\Omega, h)$. If $[\alpha, \beta] \subset [t_0, +\infty)$ and $x : [\alpha, \beta] \rightarrow X$ is regulated (in particular, a function of bounded variation) in $[\alpha, \beta]$ and $(x(s), s) \in \Omega$ for every $s \in [\alpha, \beta]$, then the Kurzweil integral $\int_{\alpha}^{\beta} DF(x(\tau), t)$ exists and the function $s \mapsto \int_{\alpha}^s DF(x(\tau), t)$ is of bounded variation in $[\alpha, \beta]$.*

The next result describes the discontinuities of a solution of the generalized ODE (1.2), provided F satisfies (1.4). For a proof of it, see [38, Lemma 3.12].

Lemma 1.19. Consider $[\alpha, \beta] \subset [t_0, +\infty)$, $x : [\alpha, \beta] \rightarrow X$ is a solution of the generalized ODE (1.2) and $F : \Omega \rightarrow X$ satisfies condition (1.4). Then, we have

$$x(t^+) - x(t) = F(x(t), t^+) - F(x(t), t), \text{ for every } t \in [\alpha, \beta),$$

and

$$x(t) - x(t^-) = F(x(t), t) - F(x(t), t^-), \text{ for every } t \in (\alpha, \beta],$$

where

$$F(x, t^+) = \lim_{s \rightarrow t^+} F(x, s), \text{ for every } t \in [\alpha, \beta),$$

and

$$F(x, t^-) = \lim_{s \rightarrow t^-} F(x, s), \text{ for every } t \in (\alpha, \beta].$$

The next result can be found in [38, Theorem 3.14].

Theorem 1.20. Let $F \in \mathcal{F}(\Omega, h)$. If $x : [a, b] \rightarrow X$, is the uniform limit of a sequence $(x_k)_{k \in \mathbb{N}}$ of step functions $x_k : [a, b] \rightarrow X$ such that $(x(s), s) \in \Omega$ and $(x_k(s), s) \in \Omega$, for every $k \in \mathbb{N}$ and for every $s \in [a, b]$, then the integral $\int_a^b DF(x(\tau), t)$ exists and

$$\int_a^b DF(x(\tau), t) = \lim_{k \rightarrow \infty} \int_a^b DF(x_k(\tau), t).$$

The next result ensures the existence and uniqueness of a local solution for an initial value problem of the generalized ODE (1.2). For a proof of it, see [23, Theorem 2.15].

Theorem 1.21 (Local existence and uniqueness). Let $F : \Omega \rightarrow X$ be an element of the class $\mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous. If $(x_0, \tau_0) \in \Omega$ is such that $x_0 + F(x_0, \tau_0^+) - F(x_0, \tau_0) \in O$, then there exist $\Delta > 0$ and a function $x : [\tau_0, \tau_0 + \Delta] \rightarrow X$ which is the unique solution of the generalized ODE (1.2) for which $x(\tau_0) = x_0$.

Remark 1.22. Notice that the condition $x_0 + F(x_0, \tau_0^+) - F(x_0, \tau_0) \in O$ from Theorem 1.21 is sufficient, but not necessary. The following example illustrates such a claim.

Example 1.23. Let $X = \mathbb{R}$ with norm $|\cdot|$ (absolute value), $B_1 := B(0, 1) \subset \mathbb{R}$, $[0, 1] \subset \mathbb{R}$ and $\Omega := B_1 \times [0, 1]$, where $B(0, 1) = \{x \in \mathbb{R} : |x| < 1\}$. Consider a function $\varphi : [0, 1] \rightarrow \mathbb{R}$ given by

$$\varphi(t) := \begin{cases} 0, & t = 0 \\ t - 1, & 0 < t \leq 1, \end{cases} \quad (1.7)$$

and a function $F : \Omega \rightarrow \mathbb{R}$ given by $F(x, t) := \varphi(t)$, for all $(x, t) \in \Omega$. Notice that the function F is constant with relation to the first variable, that is, $F(x, t) = F(y, t)$ for every

$x, y \in B_1$. Also, define a function $h : [0, 1] \rightarrow \mathbb{R}$ as follows

$$h(t) := \begin{cases} 0, & t = 0 \\ 2t + 1, & 0 < t \leq 1. \end{cases}$$

Note that, by the definition, the function h is left-continuous on $(0, 1]$ and increasing on $[0, 1]$. Consider the generalized ODE given by

$$\begin{cases} \frac{dx}{d\tau} = DF(x, t) = D[\varphi(t)] \\ x(0) = 0. \end{cases} \quad (1.8)$$

Assertion 1. $F \in \mathcal{F}(\Omega, h)$.

i) At first, we will show that $|F(x, t_2) - F(x, t_1)| \leq |h(t_2) - h(t_1)|$, for all $(x, t_2), (x, t_1) \in \Omega$.

– Let $0 < t_1 \leq t_2 \leq 1$ and $x \in B_1$. Then

$$\begin{aligned} |F(x, t_2) - F(x, t_1)| &= |\varphi(t_2) - \varphi(t_1)| \\ &= |t_2 - 1 - (t_1 - 1)| \\ &= |t_2 - t_1| \\ &\leq |2(t_2 - t_1)| \\ &= |2t_2 + 1 - (2t_1 + 1)| \\ &= |h(t_2) - h(t_1)|. \end{aligned}$$

– Let $t_1 = 0, 0 < t_2 \leq 1$ and $x \in B_1$. Then

$$\begin{aligned} |F(x, t_2) - F(x, t_1)| &= |\varphi(t_2) - \varphi(t_1)| \\ &= |t_2 - 1| = 1 - t_2 \\ &< 1 < 2t_2 + 1 \\ &= h(t_2) - h(0) = |h(t_2) - h(t_1)|. \end{aligned}$$

– The case $t_1 = t_2 = 0$ follows trivially. Therefore, we have

$$|F(x, t_2) - F(x, t_1)| \leq |h(t_2) - h(t_1)|, \text{ for every } (x, t_2), (x, t_1) \in \Omega.$$

ii) Also, the following inequality

$$|F(x, t_2) - F(x, t_1) - F(y, t_2) + F(y, t_1)| \leq |h(t_2) - h(t_1)| |x - y|,$$

is satisfied for all $(x, t_2), (x, t_1), (y, t_2), (y, t_1) \in \Omega$. Indeed, notice that

$$\begin{aligned} |F(x, t_2) - F(x, t_1) - F(y, t_2) + F(y, t_1)| &= |\varphi(t_2) - \varphi(t_1) - \varphi(t_2) + \varphi(t_1)| \\ &= 0 \leq |h(t_2) - h(t_1)| |x - y|, \end{aligned}$$

for all $(x, t_2), (x, t_1), (y, t_2), (y, t_1) \in \Omega$.

Assertion 2. For $x_0 = 0$ and $\tau_0 = 0$, we have $0 + F(0, 0^+) - F(0, 0) \notin B_1$.

Indeed, notice that by the definition of F , we have

$$F(0, 0^+) - F(0, 0) = \lim_{t \rightarrow 0^+} F(0, t) - F(0, 0) = \lim_{t \rightarrow 0^+} \varphi(t) = -1,$$

which implies that

$$0 + F(0, 0^+) - F(0, 0) = -1 \notin B_1.$$

Assertion 3. The function φ , given by (1.7), is the unique solution of the generalized ODE (1.8) on $[0, 1]$.

Indeed, we will start by proving that φ is solution of the generalized ODE (1.8) on $[0, 1]$. Let $s \in [0, 1]$, then

$$\varphi(s) - \varphi(0) = \int_0^s D[\varphi(t)] = \int_0^s DF(\varphi(\tau), t).$$

Therefore, φ is solution of (1.8) on $[0, 1]$.

Now, suppose $x : [0, 1] \rightarrow \mathbb{R}$ is also a solution of (1.8) on $[0, 1]$. Then, given $s \in [0, 1]$, we have

$$x(s) = 0 + \int_0^s D \underbrace{F(x(\tau), t)}_{\varphi(t)} = \int_0^s D[\varphi(t)] = \varphi(s) - \underbrace{\varphi(0)}_0 = \varphi(s),$$

that is, $x(s) = \varphi(s)$ for all $s \in [0, 1]$. Notice that the equality $\varphi(s) - \varphi(0) = \int_0^s D[\varphi(t)]$ follows directly from the definition of the Kurzweil integral.

Dynamic equations on time scales

In this chapter, we will present the basic concepts concerning the theory of dynamic equations on time scales. This theory is recent and it was introduced by Stefan Hilger in his doctor thesis (see [27]) in order to unify the discrete and continuous analysis and the cases “in between”.

This chapter is divided into three sections. In the first section, we present the basic definitions and notations about the theory of dynamic equations on time scales. The second section is dedicated to presenting the basic concepts and properties of delta derivatives. The last section is devoted to the study of the Kurzweil-Henstock Δ -integrals.

The main references for this chapter are [7, 8, 27].

2.1 Time scales calculus

In this section, our goal is to present the basic concepts and results concerning the theory of dynamic equations on time scales.

A *time scale* is an arbitrary nonempty and closed subset of the real numbers (with the standard topology of $\mathbb{R}$). Thus

$$\mathbb{R}, \quad [1, 2] \cup \mathbb{N}, \quad \mathbb{Z}, \quad \text{the Cantor set}$$

are examples of time scales. On the other hand, open intervals, half-open intervals, $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{Q}$ are not examples of time scales, since they are not closed sets in $\mathbb{R}$. Here, we denote a time scale by the symbol $\mathbb{T}$.

Let $\mathbb{T}$ be a time scale. We define the *forward jump operator* $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(t) := \inf \{s \in \mathbb{T} : s > t\},$$

while the *backward jump operator* $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is given by

$$\rho(t) := \sup \{s \in \mathbb{T} : s < t\}.$$

In the above definitions, we consider $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. Then, it is not difficult to see that $\sigma(t) = t$, if $\mathbb{T}$ has a maximum t , and $\rho(t) = t$, if $\mathbb{T}$ has a minimum t . Since $\mathbb{T}$ is a closed set, it is clear from the definition of the forward jump operator and backward jump operator, that the values $\sigma(t)$ and $\rho(t)$ belong to $\mathbb{T}$, for all $t \in \mathbb{T}$. Also, we define the *graininess function* $\mu : \mathbb{T} \rightarrow [0, +\infty)$ by

$$\mu(t) := \sigma(t) - t.$$

Definition 2.1. Let $\mathbb{T}$ be a time scale and $t \in \mathbb{T}$. We say that

- (i) t is **right-scattered**, if $\sigma(t) > t$;
- (ii) t is **left-scattered**, if $\rho(t) < t$;
- (iii) t is **isolated**, if it is right-scattered and left-scattered.

Definition 2.2. Let $\mathbb{T}$ be a time scale and $t \in \mathbb{T}$. We say that

- (i) t is **right-dense**, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$;
- (ii) t is **left-dense**, if $\inf \mathbb{T} < t$ and $\rho(t) = t$;
- (iii) t is **dense**, if it is right-dense and left-dense.

In what follows, we present some examples to illustrate the above definitions. They are borrowed from [7].

Example 2.3. Let $\mathbb{T} = \mathbb{Z}$. Then for all $t \in \mathbb{Z}$, we have

$$\sigma(t) = \inf \{s \in \mathbb{Z} : s > t\} = \inf \{t + 1, t + 2, \dots\} = t + 1 > t$$

$$\rho(t) = \sup \{s \in \mathbb{Z} : s < t\} = \sup \{t - 1, t - 2, \dots\} = t - 1 < t;$$

Also, we have

$$\mu(t) = 1.$$

Therefore, we can easily see that every point in $\mathbb{Z}$ is isolated.

Example 2.4. Let $\mathbb{T} = \mathbb{R}$. Then, for each $t \in \mathbb{R}$, we have

$$\sigma(t) = \inf \{s \in \mathbb{R} : s > t\} = \inf(t, +\infty) = t$$

$$\rho(t) = \sup \{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t.$$

Thus, each $t \in \mathbb{R}$ is a right-dense and left-dense point. Hence t is dense. Furthermore, $\mu(t) = 0$, for all $t \in \mathbb{T}$.

Example 2.5. Let $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, with $q > 1$. Then for any $t \in \mathbb{T} \setminus \{0\}$, we have

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\} = \inf \{qt, q^2t, \dots\} = qt > t$$

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\} = \sup \left\{ \frac{t}{q}, \frac{t}{q^2}, \dots \right\} = \frac{t}{q} < t.$$

Thus, each $t \in \mathbb{T} \setminus \{0\}$ is an isolated point. On the other hand, if $t = 0$, then $\sigma(0) = 0$ which implies that 0 is a right-dense point. Also, note that

$$\mu(t) = \begin{cases} (q-1)t, & t \in \mathbb{T} \setminus \{0\}, \\ 0, & t = 0. \end{cases}$$

Example 2.6. Let $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$. Then, for every $t \in \mathbb{T}$, we have

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\} = \inf \{t + nh : n \in \mathbb{N}\} = t + h > t$$

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\} = \sup \{t - nh : n \in \mathbb{N}\} = t - h < t.$$

Hence, every point $t \in \mathbb{T}$ is an isolated point and

$$\mu(t) = \sigma(t) - t = h, \quad \text{for any } t \in \mathbb{T},$$

which implies that μ is constant.

To conclude this section, we define the set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases} \quad (2.1)$$

This definition is important to define the Δ -derivative of a function f , which we will presented in the next section.

2.2 Differentiation

In this section, we present the basic concepts and main properties about the Δ -derivative of a function $f : \mathbb{T} \rightarrow \mathbb{R}$.

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function. We define the so-called delta (or Hilger) derivative of f at a point $t \in \mathbb{T}^\kappa$ as follows.

Definition 2.7. Suppose that $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$. Then, we define $f^\Delta(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a neighborhood U of t (that is, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the *delta (ou Hilger) derivative* of f at t . Moreover, we say that f is delta differentiable on $\mathbb{T}^\kappa$, provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is called the Δ -derivative of f on $\mathbb{T}^\kappa$.

It is also possible to extend the concept of Δ -derivative presented in Definition 2.7 for an $\mathbb{R}^n$ -valued functions.

Definition 2.8. Let $t \in \mathbb{T}^\kappa$ and $f : \mathbb{T} \rightarrow \mathbb{R}^n$ be an $\mathbb{R}^n$ -valued function, $f = (f_1, \dots, f_n)$ where each component f_i is a real function defined on $\mathbb{T}$. We say that f is Δ -differentiable at t if each component f_i is Δ -differentiable at t and we define

$$f^\Delta(t) = (f_1^\Delta(t), \dots, f_n^\Delta(t)).$$

In the sequel, we present some examples of the Δ -derivative of a function $f : \mathbb{T} \rightarrow \mathbb{R}$. Such examples are borrowed from [7].

Example 2.9. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be given by $f(t) = c$, for all $t \in \mathbb{T}$, where $c \in \mathbb{R}$ is a constant. Then $f^\Delta(t) = 0$ for any $t \in \mathbb{T}^\kappa$. Indeed, for every $\epsilon > 0$, we have

$$|(f(\sigma(t)) - f(s)) - 0(\sigma(t) - s)| = |c - c| = 0 \leq \epsilon |\sigma(t) - s|, \quad \text{for all } s \in \mathbb{T}^\kappa.$$

Example 2.10. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function given by $f(t) = t^2$, for any $t \in \mathbb{T}$. Then $f^\Delta(t) = t + \sigma(t)$, for all $t \in \mathbb{T}^\kappa$. In fact, given $\epsilon > 0$, let us take $U = (t - \delta, t + \delta)$ with

$0 < \delta \leq \epsilon$. Then, for all $s \in U$, we obtain:

$$\begin{aligned} |(f(\sigma(t)) - f(s)) - (t + \sigma(t))(\sigma(t) - s)| &= |(\sigma(t))^2 - s^2 - t\sigma(t) + ts - (\sigma(t))^2 + s\sigma(t)| \\ &= |\sigma(t)(s - t) - s(s - t)| \\ &= |s - t| |\sigma(t) - s| \\ &< \delta |\sigma(t) - s| \leq \epsilon |\sigma(t) - s|. \end{aligned}$$

The next result presents some important properties of the Δ -derivatives. For a proof of it, see [7, Theorem 1.16].

Theorem 2.11. *Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$. Then the following assertions hold*

(i) *If f is Δ -differentiable at t , then f is continuous at t .*

(ii) *If f is continuous at t and t is right-scattered, then f is Δ -differentiable at t , with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If t is right-dense, then f is Δ -differentiable at t if, and only if, the limit*

$$\lim_{s \rightarrow t} \frac{f(s) - f(t)}{s - t}$$

exists as a finite number. In this case, we have

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If f is Δ -differentiable at t , then*

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

The next example illustrates the above properties. Such example is borrowed from [7].

Example 2.12. Let $h > 0$ and

$$\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}.$$

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function. As it was showed in Example 2.6, any $t \in \mathbb{T}$ is right-scattered. Also, note that f is continuous on $\mathbb{T} = h\mathbb{Z}$ (since each point $t \in \mathbb{T}$ isolated). Thus, by the

statement (ii) from Theorem 2.11, we have

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+h) - f(t)}{h}, \quad \text{for all } t \in \mathbb{T}.$$

The next result describes algebraic properties of the Δ -derivatives. A proof of it can be found in [7, Theorem 1.20].

Theorem 2.13. *Suppose $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are Δ -differentiable at $t \in \mathbb{T}^\kappa$. Then, we have*

(i) $f + g : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t , with

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(ii) For every $\alpha \in \mathbb{R}$, $\alpha f : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t , with

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

(iii) $f \cdot g : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t , with

$$(f \cdot g)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

(iv) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is Δ -differentiable at t and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

Remark 2.14. By Theorem 2.13, it is not difficult to see that if $\mathbb{T} = \mathbb{R}$, we have the usual properties for the Frechet derivative.

2.3 Kurzweil-Henstock delta integrals

In this section, we present the basic concepts and the main properties of the Kurzweil-Henstock Δ -integrals. This section is based on [35].

Let $\mathbb{T}$ be a time scale. For each pair of numbers $a, b \in \mathbb{T}$, $a \leq b$, set $[a, b]_\mathbb{T} = [a, b] \cap \mathbb{T}$. The open and half-open time scales intervals are defined in a similar way.

We say that $\delta = (\delta_L, \delta_R)$ is a Δ -gauge in $[a, b]_\mathbb{T}$, provided $\delta_L(t) > 0$ in $(a, b]_\mathbb{T}$, $\delta_R(t) > 0$ in $[a, b)_\mathbb{T}$, $\delta_L(a) \geq 0$, $\delta_R \geq 0$ and $\delta_R(t) \geq \mu(t)$ for all $t \in [a, b)_\mathbb{T}$.

We say that P is a tagged division of $[a, b]_{\mathbb{T}}$ if

$$P = \{a = s_0 \leq \xi_1 \leq s_1 \leq \xi_2 \leq s_2 \leq \cdots \leq s_{n-1} \leq \xi_n \leq s_n = b\}$$

with $s_{i-1} < s_i$ for all $i = 1, \dots, n$ and $s_i, \xi_i \in \mathbb{T}$. We denoted such a tagged division by $P = \{[s_{i-1}, s_i]_{\mathbb{T}} : \xi_i\}$, where $[s_{i-1}, s_i]_{\mathbb{T}}$ denotes an interval in $\mathbb{T}$ and ξ_i is the associated “tag point” in $[s_{i-1}, s_i]_{\mathbb{T}}$.

If δ is a Δ -gauge of $[a, b]_{\mathbb{T}}$, then a tagged division P is called δ -fine, whenever

$$\xi_i - \delta_L(\xi_i) \leq s_{i-1} < s_i \leq \xi_i + \delta_R(\xi_i),$$

for $i = 1, \dots, n$.

In the sequel, we recall the concept of the Kurzweil-Henstock Δ -integral. This concept was introduced for the first time by A. Peterson and B. Thompson in [35].

Definition 2.15. A function $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is called *Kurzweil-Henstock Δ -integrable* (or KH delta integrable) over $[a, b]_{\mathbb{T}}$, if there is a number I such that given $\epsilon > 0$, there exists a Δ -gauge δ on $[a, b]_{\mathbb{T}}$ such that

$$\left| I - \sum_{i=1}^n f(\xi_i)(s_i - s_{i-1}) \right| < \epsilon$$

for all δ -fine tagged division P of $[a, b]_{\mathbb{T}}$. In this case, I is called the Kurzweil-Henstock Δ -integral of f over $[a, b]_{\mathbb{T}}$ and it will be denoted by $KH \int_a^b f(t) \Delta t$.

Remark 2.16. When it is clear, we write simply $\int_a^b f(t) \Delta t$ instead of $KH \int_a^b f(t) \Delta t$.

Notice that Definition 2.15 makes sense only if, for a given Δ -gauge δ on $[a, b]_{\mathbb{T}}$, there exists at least one δ -fine tagged division of $[a, b]_{\mathbb{T}}$. The next result ensures this existence and it is a generalization of the known Cousin Lemma for a Δ -gauge on a time scale interval. This result can be found [35, Lemma 1.9].

Lemma 2.17. *If δ is a Δ -gauge on $[a, b]_{\mathbb{T}}$, then there exists a δ -fine tagged division P of $[a, b]_{\mathbb{T}}$.*

Finally, we present the last result of this section. It describes a very important property of the Kurzweil-Henstock Δ -integral.

Theorem 2.18. *Let $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a function and $c \in [a, b]$. Then f is Kurzweil-Henstock Δ -integrable over $[a, b]_{\mathbb{T}}$ if, and only if, f is Kurzweil-Henstock Δ -integrable over $[a, c]_{\mathbb{T}}$ and $[c, b]_{\mathbb{T}}$. In this case,*

$$\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t.$$

Also, if $f, g : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ are Kurzweil-Henstock Δ -integrable functions over $[a, b]_{\mathbb{T}}$, then $\alpha f + \beta g$ is a Kurzweil-Henstock Δ -integrable function over $[a, b]_{\mathbb{T}}$ and

$$\int_a^b (\alpha f + \beta g)(t) \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t,$$

for all $\alpha, \beta \in \mathbb{R}$.

Correspondences between equations

In this chapter, we present the correspondence between the solutions of a measure differential equation and the solutions of a generalized ODE. Furthermore, we prove that the dynamic equations on time scales can be viewed as a measure differential equations (we write MDE, for short). These correspondences are fundamental to our purposes.

We divide this chapter into three sections. In the first section, we discuss the main properties of measure differential equations (MDEs). In the second section, we present the correspondence between MDEs and generalized ODEs. In the third section, we recall the reader that dynamic equations on time scales are a special class of measure differential equations. The main references for this chapter are [10, 20, 21, 38, 42].

3.1 Measure Differential Equations

Throughout this section, let $\mathbb{R}^n$ be the n -dimensional Euclidean space with norm $|\cdot|$. Consider an open set $O \subset \mathbb{R}^n$. Also, let $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ be functions.

In this work, we consider the integral form of a measure differential equations of the type

$$x(t) = x(\tau_0) + \int_{\tau_0}^t f(x(s), s) dg(s), \quad t \geq \tau_0, \quad (3.1)$$

where $\tau_0 \geq t_0$, and the integral on the right-hand side is in the sense of Kurzweil-Henstock-Stieltjes taken with respect to $g : [t_0, +\infty) \rightarrow \mathbb{R}$, which is a nondecreasing and left-continuous function.

Since these equations are very important to applications, their qualitative properties have been investigated by different authors (see, for instance, [9, 10, 32, 40, 41]). It is a known fact that these equations encompass integral form ordinary and impulsive differential equations

Remark 3.1. It is not difficult to see that if $I \subset [t_0, +\infty)$ is a nondegenerate interval. Then, the function $x : I \rightarrow \mathbb{R}^n$ is a solution of the measure differential equation (3.1) if, and only if, $(x(t), t) \in O \times I$ for all $t \in I$ and

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} f(x(s), s) dg(s),$$

for every $s_1, s_2 \in I$, where the integral on the right-hand side is in the sense of Kurzweil-Henstock-Stieltjes.

3.2 Measure differential equations and Generalized ODEs

In this section, we present a correspondence inspired in a result found in [38, Theorem 5.17]. Such correspondence describes the relation between the solutions of a measure differential equation and the solutions of a generalized ODE. This correspondence is crucial to prove our main results.

We will show that, under certain assumptions, we can establish a correspondence between the solutions of a integral form of a measure differential equation of type

$$x(t) = x(\tau_0) + \int_{\tau_0}^t f(x(s), s) dg(s), \quad t \geq \tau_0,$$

and the solutions of the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t),$$

where F is given by $F(x, t) = \int_{\tau_0}^t f(x, s) dg(s)$, for $x \in O$ and $t \in [t_0, +\infty)$.

We recall the reader that $G([t_0, +\infty), \mathbb{R}^n)$ denotes the vector space of functions $x : [t_0, +\infty) \rightarrow \mathbb{R}^n$ such that $x|_{[\alpha, \beta]}$ belongs to the space $G([\alpha, \beta], \mathbb{R}^n)$, for all $[\alpha, \beta] \subset [t_0, +\infty)$. We use the symbol $G_0([t_0, +\infty), \mathbb{R}^n)$ to denote the vector space of all functions $x \in G([t_0, +\infty), \mathbb{R}^n)$ such that $\sup_{s \in [t_0, +\infty)} e^{-(s-t_0)} |x(s)| < +\infty$. This space is endowed with the norm

$$\|x\|_{[t_0, +\infty)} = \sup_{s \in [t_0, +\infty)} e^{-(s-t_0)} |x(s)|, \quad x \in G_0([t_0, +\infty), \mathbb{R}^n),$$

and it becomes a Banach space (see [28]).

We will use the notation $x \in G([t_0, +\infty), O)$ for a function $x \in G([t_0, +\infty), \mathbb{R}^n)$ such that $x(s) \in O$, for all $s \in [t_0, +\infty)$. The notation $x \in G_0([t_0, +\infty), O)$ is defined in a similar way.

In what follows, we say that $\gamma : [t_0, +\infty) \rightarrow \mathbb{R}^+$ is locally Kurzweil-Henstock-Stieltjes integrable with respect to g if, and only if, the function $[s_1, s_2] \ni t \mapsto \gamma(t)$ is Kurzweil-Henstock-Stieltjes integrable with respect to g on $[s_1, s_2]$, for every $s_1, s_2 \in [t_0, +\infty)$.

Throughout this section, let us assume the following conditions on the functions $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ and $g : [t_0, +\infty) \rightarrow \mathbb{R}$.

(A1) The function $g : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous on $(t_0, +\infty)$.

(A2) The Kurzweil-Henstock-Stieltjes integral

$$\int_{s_1}^{s_2} f(x(s), s) dg(s)$$

exists, for all $x \in G([t_0, +\infty), O)$ and all $s_1, s_2 \in [t_0, +\infty)$.

(A3) There exists a Kurzweil-Henstock-Stieltjes integrable function $M : [t_0, +\infty) \rightarrow \mathbb{R}^+$ with respect to g such that

$$\left| \int_{s_1}^{s_2} f(x(s), s) dg(s) \right| \leq \int_{s_1}^{s_2} M(s) dg(s),$$

for all $x \in G([t_0, +\infty), O)$ and all $s_1, s_2 \in [t_0, +\infty)$, $s_1 \leq s_2$.

(A4) There exists a Kurzweil-Henstock-Stieltjes integrable function $L : [t_0, +\infty) \rightarrow \mathbb{R}^+$ with respect to g such that

$$\left| \int_{s_1}^{s_2} [f(x(s), s) - f(z(s), s)] dg(s) \right| \leq \|x - z\|_{[t_0, +\infty)} \int_{s_1}^{s_2} L(s) dg(s),$$

for all $x, z \in G_0([t_0, +\infty), O)$ and all $s_1, s_2 \in [t_0, +\infty)$, $s_1 \leq s_2$.

The next result ensures that if the function $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies the conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1), then the function F given by $F(x, t) = \int_{\tau_0}^t f(x, s) dg(s)$, for $(x, t) \in O \times [t_0, +\infty)$ belongs to the class $\mathcal{F}(O \times [t_0, +\infty), h)$, where $h(t) = \int_{\tau_0}^t (M(s) + L(s)) dg(s)$, $t \in [t_0, +\infty)$.

Theorem 3.2. *Assume $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1). Choose an arbitrary $\tau_0 \in [t_0, +\infty)$ and define*

$F : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by

$$F(x, t) = \int_{\tau_0}^t f(x, s) dg(s), \quad (x, t) \in O \times [t_0, +\infty). \quad (3.2)$$

Then $F \in \mathcal{F}(\Omega, h)$, where $\Omega = O \times [t_0, +\infty)$, and $h : [t_0, +\infty) \rightarrow \mathbb{R}$ given by

$$h(t) = \int_{\tau_0}^t (M(s) + L(s)) dg(s), \quad t \in [t_0, +\infty) \quad (3.3)$$

is a nondecreasing function.

Proof. At first, we prove that F is well-defined. Indeed, let $t \in [t_0, +\infty)$ and $x \in O$. Define the following function

$$\begin{aligned} c_x : [t_0, +\infty) &\rightarrow O \\ s &\mapsto c_x(s) = x. \end{aligned} \quad (3.4)$$

Note that $c_x \in G_0([t_0, +\infty), O)$ (in particular, $x \in G([t_0, +\infty), O)$). Thus, condition (A2) implies $\int_{\tau_0}^t f(c_x(s), s) dg(s) = \int_{\tau_0}^t f(x, s) dg(s)$ exists and, therefore, F is well-defined. On the other hand, since M and L are Kurzweil-Henstock-Stieltjes integrable functions, h is well-defined. Also, by the left-continuity of g , h is a left-continuous function.

Now, for an arbitrary $x \in O$ and for $t_0 \leq s_1 \leq s_2 < +\infty$, by condition (A3), we have

$$\begin{aligned} |F(x, s_2) - F(x, s_1)| &= \left| \int_{\tau_0}^{s_2} f(x, s) dg(s) - \int_{\tau_0}^{s_1} f(x, s) dg(s) \right| \\ &= \left| \int_{s_1}^{s_2} f(x, s) dg(s) \right| = \left| \int_{\tau_0}^{s_2} f(c_x(s), s) dg(s) \right| \\ &\leq \int_{s_1}^{s_2} M(s) dg(s) \leq \int_{s_1}^{s_2} (M(s) + L(s)) dg(s) = (h(s_2) - h(s_1)). \end{aligned}$$

Analogously, condition (A4) implies that if $x, y \in O$ and $t_0 \leq s_1 \leq s_2 < +\infty$, then

$$\begin{aligned} &|F(x, s_2) - F(x, s_1) - F(y, s_2) + F(y, s_1)| \\ &= \left| \int_{s_1}^{s_2} [f(x, s) - f(y, s)] dg(s) \right| = \left| \int_{s_1}^{s_2} [f(c_x(s), s) - f(c_y(s), s)] dg(s) \right| \\ &\leq \|c_x - c_y\|_{[t_0, +\infty)} \int_{s_1}^{s_2} L(s) dg(s) \leq |x - y| \int_{s_1}^{s_2} L(s) dg(s) = |x - y| (h(s_2) - h(s_1)). \end{aligned}$$

(Note that $\|c_x - c_y\|_{[t_0, +\infty)} = \sup_{s \in [t_0, +\infty)} e^{-(s-t_0)} \underbrace{|c_x(s) - c_y(s)|}_{|x-y|} \leq |x - y|$). $\square$

Remark 3.3. Since g is left-continuous, by relation (3.3), clearly h is left-continuous.

Remark 3.4. Given a compact interval $[a, b] \subset [t_0, +\infty)$ and a function $x \in G([a, b], O)$, we identify the function x with its constant prolongation to the whole interval $[t_0, +\infty)$, that is, with the function

$$\widehat{x}(t) = \begin{cases} x(a), & t \in [t_0, a] \\ x(t), & t \in [a, b] \\ x(b), & t \in [b, +\infty). \end{cases} \quad (3.5)$$

It is clear that $\widehat{x} \in G_0([t_0, +\infty), O)$ and $\|\widehat{x}\|_{[t_0, +\infty)} \leq \|x\|_\infty$.

The next result brings an important property of the regulated function on $[a, b]$. It can be found in [29].

Theorem 3.5. *Any function in $G([a, b], X)$ (in particular, any function of bounded variation in $[a, b]$) can be written as the uniform limit of step functions.*

Remark 3.6. Notice that by the Theorem 3.5 we can ensure that given $x \in G([a, b], X)$ and $\epsilon > 0$, there exists a step function $\varphi : [a, b] \rightarrow X$ such that $\|\varphi(t) - x(t)\| < \epsilon$ for every $t \in [a, b]$. On the other hand, it is important to point out that the result still valid if we change X by an set $O \subset X$, that is, given $x : [a, b] \rightarrow O$ a step function, there exist a step function $\phi : [a, b] \rightarrow O$ such that $\|\phi(t) - x(t)\| < \epsilon$, for every $t \in [a, b]$. In particular, given $x : [a, b] \rightarrow O$ a step function, there exist a sequence of finite step functions $\phi_k : [a, b] \rightarrow O$, $k = 1, 2, \dots$ such that

$$\|\phi_k - x\|_\infty = \sup_{s \in [a, b]} |\phi_k(s) - x(s)| \xrightarrow{k \rightarrow +\infty} 0.$$

For more details, see [42].

The next result describes the relation between the Kurzweil-Henstock-Stieltjes integral and the Kurzweil integral. A similar version of this result was proved in [38, Proposition 5.12], for the case where f satisfies the usual conditions of Carathéodory. Also, [38, Proposition 5.12] describes the relation between the Lebesgue-Stieltjes integral and the Kurzweil integral.

Theorem 3.7. *Assume $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1). Let $F : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ be defined by (3.2). If $[a, b] \subset [t_0, +\infty)$ and $x : [a, b] \rightarrow O$ is a regulated function (in particular, a function of bounded variation), then both the Kurzweil integral $\int_a^b DF(x(\tau), t)$ and the Kurzweil-Henstock-Stieltjes integral $\int_a^b f(x(s), s) dg(s)$ exist and have the same value.*

Proof. Let $F : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ be defined by (3.2). By Theorem 3.2, $F \in \mathcal{F}(\Omega, h)$, where $\Omega = O \times [t_0, +\infty)$, and h is defined by (3.3). Now, suppose $x : [a, b] \rightarrow O$ is a regulated function. Then by Proposition 1.18, the Kurzweil integral $\int_a^b DF(x(\tau), t)$ exists. Also, according to condition (A2), $\int_a^b f(\widehat{x}(s), s)dg(s)$ exists, where $\widehat{x}$ is defined as in Remark 3.4. On the other hand, $\int_a^b f(\widehat{x}(s), s)dg(s) = \int_a^b f(x(s), s)dg(s)$ and, therefore, the last integral exists.

We will show that $\int_a^b DF(x(\tau), t) = \int_a^b f(x(s), s)dg(s)$.

Assertion 1. For any finite step function $\varphi : [a, b] \rightarrow O$, we have

$$\int_b^a f(\varphi(s), s)dg(s) = \int_a^b DG(\varphi(\tau), t).$$

Indeed, let $\varphi : [a, b] \rightarrow O$ be a finite step function. Then, φ is a regulated function on $[a, b]$ and, therefore, the Kurzweil integral $\int_a^b DF(\varphi(\tau), t)$ exists and the Kurzweil-Henstock-Stieltjes integral $\int_a^b f(\varphi(s), s)dg(s)$ also exists.

Since φ is a step function, there exists a division

$$a = s_0 < s_1 < \dots < s_n = b$$

and vectors $c_1, c_2, \dots, c_n \in \mathbb{R}^n$ such that

$$x(s) = c_k \text{ for every } s \in (s_{k-1}, s_k), \quad k = 1, 2, \dots, n.$$

It is easy to check that if $s_{k-1} < \sigma_1 < \sigma_2 < s_k$, then

$$\int_{\sigma_1}^{\sigma_2} DG(\varphi(\tau), t) = \int_{\sigma_1}^{\sigma_2} DF(c_k, t) = F(c_k, \sigma_2) - F(c_k, \sigma_1).$$

Since

$$\begin{aligned} F(c_k, \sigma_2) - F(c_k, \sigma_1) &= \int_{t_0}^{\sigma_2} f(c_k, s)dg(s) - \int_{t_0}^{\sigma_1} f(c_k, s)dg(s) \\ &= \int_{\sigma_1}^{\sigma_2} f(c_k, s)dg(s) = \int_{\sigma_1}^{\sigma_2} f(\varphi(s), s)dg(s), \end{aligned}$$

we have

$$\int_{\sigma_1}^{\sigma_2} f(\varphi(s), s)dg(s) = F(c_k, \sigma_2) - F(c_k, \sigma_1) = \int_{\sigma_1}^{\sigma_2} DF(\varphi(\tau), t).$$

Let $k \in \{1, 2, \dots, n\}$ and suppose $\sigma \in (s_{k-1}, s_k)$ is given. By Theorem 1.8, we obtain

$$\begin{aligned} \int_{s_{k-1}}^{\sigma} f(\varphi(s), s) dg(s) - f(\varphi(s_{k-1}), s_{k-1}) \Delta^+ g(s_{k-1}) &= \lim_{\xi \rightarrow s_{k-1}^+} \int_{\xi}^{\sigma} f(\varphi(s), s) d(s) \\ &= \lim_{\xi \rightarrow s_{k-1}^+} F(c_k, \sigma) - F(c_k, \xi) \\ &= F(c_k, \sigma) - F(c_k, s_{k-1}^+), \end{aligned}$$

that is,

$$\int_{s_{k-1}}^{\sigma} f(\varphi(s), s) dg(s) = F(c_k, \sigma) - F(c_k, s_{k-1}^+) + f(\varphi(s_{k-1}), s_{k-1}) \Delta^+ g(s_{k-1}). \quad (3.6)$$

On the other hand, according to Theorem 1.6, we get

$$\begin{aligned} \int_{s_{k-1}}^{\sigma} DF(\varphi(\tau), t) &= \lim_{\xi \rightarrow s_{k-1}^+} \left(\int_{\xi}^{\sigma} DF(\varphi(\tau), t) + F(\varphi(s_{k-1}), \xi) - F(\varphi(s_{k-1}), s_{k-1}) \right) \\ &= \lim_{\xi \rightarrow s_{k-1}^+} \left(\int_{\xi}^{\sigma} DF(c_k, t) + \int_{t_0}^{\xi} f(\varphi(s_{k-1}), s) dg(s) - \int_{t_0}^{s_{k-1}} f(\varphi(s_{k-1}), s) dg(s) \right) \\ &= \lim_{\xi \rightarrow s_{k-1}^+} \left(F(c_k, \sigma) - F(c_k, \xi) + \int_{s_{k-1}}^{\xi} f(\varphi(s_{k-1}), s) dg(s) \right) \\ &= F(c_k, \sigma) - F(c_k, s_{k-1}^+) + f(\varphi(s_{k-1}), s_{k-1}) \Delta^+ g(s_{k-1}). \end{aligned} \quad (3.7)$$

Notice that the last equality follows directly by (i) from the Corollary 1.8. Hence, by (3.6) and (3.7),

$$\int_{s_{k-1}}^{\sigma} f(\varphi(s), s) dg(s) = \int_{s_{k-1}}^{\sigma} DF(\varphi(\tau), t),$$

for every $k \in \{1, 2, \dots, n\}$ and for all $\sigma \in (s_{k-1}, s_k)$.

Analogously, we can prove that

$$\int_{\sigma}^{s_k} f(\varphi(s), s) dg(s) = \int_{\sigma}^{s_k} DF(\varphi(\tau), t),$$

for each $k \in \{1, 2, \dots, n\}$ and for each $\sigma \in (s_{k-1}, s_k)$. Therefore, we get

$$\int_a^b f(\varphi(s), s) dg(s) = \sum_{k=1}^n \int_{s_{k-1}}^{s_k} f(\varphi(s), s) dg(s)$$

$$= \sum_{k=1}^n \int_{s_{k-1}}^{s_k} DF(\varphi(\tau), t) = \int_a^b DF(\varphi(\tau), t),$$

that is,

$$\int_a^b f(\varphi(s), s) dg(s) = \int_a^b DF(\varphi(\tau), t).$$

This proves Assertion 1.

Since $x : [a, b] \rightarrow O$ is a regulated function, by Theorem 3.5 (see Remark 3.6), there exists a sequence of finite step functions $\varphi_k : [a, b] \rightarrow O$, $k = 1, 2, \dots$ such that

$$\|\varphi_k - x\|_\infty = \sup_{s \in [a, b]} |\varphi_k(s) - x(s)| \xrightarrow{k \rightarrow +\infty} 0.$$

By Assertion 1,

$$\int_a^b f(\varphi_k(s), s) dg(s) = \int_a^b DF(\varphi_k(\tau), t). \quad (3.8)$$

Now, using Theorem 1.20, we have

$$\lim_{k \rightarrow \infty} \int_a^b DF(\varphi_k(\tau), t) = \int_a^b DF(x(\tau), t). \quad (3.9)$$

Then, by (3.8) and (3.9), we obtain

$$\lim_{k \rightarrow \infty} \int_a^b f(\varphi_k(s), s) dg(s) = \lim_{k \rightarrow \infty} \int_a^b DF(\varphi_k(\tau), t) = \int_a^b DF(x(\tau), t). \quad (3.10)$$

On the other hand, condition (A4) and Remark 3.4 imply

$$\begin{aligned} \left\| \int_a^b [f(\varphi_k(s), s) - f(x(s), s)] dg(s) \right\| &= \left\| \int_a^b [f(\widehat{\varphi}_k(s), s) - f(\widehat{x}(s), s)] dg(s) \right\| \\ &\leq \|\widehat{\varphi}_k - \widehat{x}\|_{[t_0, +\infty)} \int_a^b L(s) dg(s) \\ &\leq \|\varphi_k - x\|_\infty \int_a^b L(s) dg(s) \xrightarrow{k \rightarrow +\infty} 0, \end{aligned}$$

that is,

$$\lim_{k \rightarrow \infty} \int_a^b f(\varphi_k(s), s) dg(s) = \int_a^b f(x(s), s) dg(s). \quad (3.11)$$

Finally, by (3.10) and (3.11),

$$\int_a^b DF(x(\tau), t) = \int_a^b f(x(s), s) dg(s),$$

obtaining the desired result. $\square$

The next result gives us a correspondence between the solutions of the integral form of an MDE of type

$$x(t) = x(\tau_0) + \int_{\tau_0}^t f(x(s), s) dg(s), \quad t \geq \tau_0, \quad (3.12)$$

and the solutions of the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t),$$

where F is given by $F(x, t) = \int_{\tau_0}^t f(x, s) dg(s)$. This result is inspired in [38, Theorem 5.17]. Since this result is crucial to our main results, we repeat its proof here, following the ideas of [38].

Theorem 3.8. *Assume $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1). Then, the function $x : I \rightarrow \mathbb{R}^n$ is a solution of the MDE (3.12) on $I \subset [t_0, +\infty)$ if, and only if, x is a solution of the generalized ODE*

$$\frac{dx}{d\tau} = DF(x, t)$$

on I with the function F is given by (3.2).

Proof. Let $x : I \rightarrow \mathbb{R}^n$ be a solution of the MDE (3.12) on $I \subset [t_0, +\infty)$. Then, given $s_1, s_2 \in I$, we have

$$x(t) - x(s_1) = \int_{s_1}^t f(x(s), s) dg(s), \quad \text{for all } t \in [s_1, s_2]. \quad (3.13)$$

Now, from Theorem 1.8, $[s_1, s_2] \ni t \mapsto \int_{s_1}^t f(x(s), s) dg(s)$ is a regulated function and, therefore, $x|_{[s_1, s_2]}$ is also a regulated function for any $s_1, s_2 \in I$.

Let $F : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ be defined by (3.2). By Theorem 3.7, given $s_1, s_2 \in I$, the integrals $\int_{s_1}^{s_2} DF(x(\tau), t)$ and $\int_{s_1}^{s_2} f(x(s), s) dg(s)$ exist and

$$\int_{s_1}^{s_2} DF(x(\tau), t) = \int_{s_1}^{s_2} f(x(s), s) dg(s).$$

Thus

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} f(x(s), s) dg(s) = \int_{s_1}^{s_2} DF(x(\tau), t),$$

that is,

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t).$$

This shows that x is a solution of the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t).$$

Conversely, let $x : I \rightarrow \mathbb{R}^n$ be a solution of the generalized ODE $\frac{dx}{d\tau} = DF(x, t)$ on I , with the function F given by (3.2).

Let $s_1, s_2 \in I$ be such that $s_1 \leq s_2$. According to the Theorem 3.2, $F \in \mathcal{F}(\Omega, h)$, where h is a nondecreasing function. Thus, by Corollary 1.17, $x|_{[s_1, s_2]}$ is a function of bounded variation in $[s_1, s_2]$. Again, using Theorem 3.7, we have

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t) = \int_{s_1}^{s_2} f(x(s), s) dg(s),$$

that is,

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} f(x(s), s) dg(s).$$

This shows that x is a solution of the MDE (3.12), which completes the proof. $\square$

We finish this section by presenting an example which illustrates the previous theorem.

Example 3.9. Let $\mathbb{R}$ be the 1-dimensional Euclidean space with norm $|\cdot|$ (absolute value). Consider the MDE in its integral form

$$x(t) = x(0) + \int_0^t f(x(s), s) dg(s), \quad t \geq 0 \tag{3.14}$$

where $f : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$ and $g : [0, +\infty) \rightarrow \mathbb{R}$ are given by respectively

$$f(x, t) = \frac{\{t\} \sin x}{4^t + [|t|]},$$

for all $(x, t) \in \mathbb{R} \times [0, +\infty)$ and

$$g(t) = \begin{cases} t, & t \in [0, 1] \\ t + 1, & t \in (1, +\infty), \end{cases}$$

where the symbol $[t]$ denotes the integer part of t , and the symbol $\{t\} := t - [t]$ denotes the fractional part of t .

We will show that f satisfies conditions (A2), (A3), (A4), and g satisfies condition (A1). Indeed, clearly g satisfies condition (A1). Consider an arbitrary $x \in G([0, +\infty), \mathbb{R})$ and let $s_1, s_2 \in [0, +\infty)$ be such that $s_1 \leq s_2$. Notice that $[s_1, s_2] \ni t \mapsto f(x(t), t)$ is a regulated function on $[s_1, s_2]$. Thus by Theorem 1.9 (item (i)), $\int_{s_1}^{s_2} f(x(s), s) dg(s)$ exists and, therefore, f satisfies (A2).

Now, define $M : [0, +\infty) \rightarrow \mathbb{R}$ by $M(t) = \{t\}$, for $t \in [0, +\infty)$. Evidently, M is a locally Kurzweil-Henstock-Stieltjes integrable with respect to g and

$$\begin{aligned} \left| \int_{s_1}^{s_2} f(x(s), s) dg(s) \right| &\stackrel{\text{Theorem 1.9}}{\leq} \int_{s_1}^{s_2} |f(x(s), s)| dg(s) \\ &= \int_{s_1}^{s_2} \left| \frac{\{s\} \sin(x(s))}{4^s + [s]} \right| dg(s) \\ &\leq \int_{s_1}^{s_2} \{s\} dg(s) = \int_{s_1}^{s_2} M(s) dg(s), \end{aligned}$$

for every $x \in G([0, +\infty), \mathbb{R})$ and $s_1, s_2 \in [0, +\infty)$, $s_1 \leq s_2$. Thus, f satisfies condition (A3).

On the other hand, define $L : [0, +\infty) \rightarrow \mathbb{R}$ by $L(t) := M(t)$ for $t \in [0, +\infty)$. Then

$$\begin{aligned} \left| \int_{s_1}^{s_2} [f(x(s), s) - f(y(s), s)] dg(s) \right| &\stackrel{\text{Theorem 1.9}}{\leq} \int_{s_1}^{s_2} |f(x(s), s) - f(y(s), s)| dg(s) \\ &\leq \int_{s_1}^{s_2} \{s\} e^{-s} |\sin(x(s)) - \sin(y(s))| dg(s) \\ &\leq \int_{s_1}^{s_2} L(s) e^{-s} |x(s) - y(s)| dg(s) \\ &\leq \int_{s_1}^{s_2} L(s) \|x - y\|_{[0, +\infty)} dg(s) \\ &= \|x - y\|_{[0, +\infty)} \int_{s_1}^{s_2} L(s) dg(s), \end{aligned}$$

for all $x, y \in G_0([0, +\infty), \mathbb{R})$ and $s_1, s_2 \in [0, +\infty)$, $s_1 \leq s_2$. Hence f and g fulfill all the hypotheses of Theorem 3.8.

To obtain the corresponding generalized ODE, we choose an arbitrary $\tau_0 \in [0, +\infty)$ and define

$$F(x, t) := \int_{\tau_0}^t f(x, s) dg(s), \quad (3.15)$$

for $(x, t) \in \mathbb{R} \times [0, +\infty)$. Notice that

$$F(x, t) = \varphi(t) \sin x,$$

where

$$\varphi(t) = \int_{\tau_0}^t \frac{\{s\}}{4^s + [|s|]} dg(s),$$

for $t \in [0, +\infty)$.

On the other hand, if $x : I \rightarrow \mathbb{R}$, where $0 \in I \subset [0, +\infty)$, is a solution of the MDE (3.14), by Theorem 3.8, x is also a solution of the generalized ODE $\frac{dx}{d\tau} = DF(x, t)$ with the function F given by (3.15) and, therefore,

$$x(\xi) = x(0) + \int_0^\xi DF(x(\tau), t) = x(0) + \int_0^\xi D[\varphi(t) \sin x(\tau)] = x(0) + \int_0^\xi \sin x(s) d\varphi(s),$$

for all $\xi \in I$.

3.3 Dynamic equation on time scales and measure differential equations

In this section, we recall some definitions and concepts concerning dynamic equations on time scales and prove that these equations are a special case of MDEs.

We recall that, given a time scale $\mathbb{T}$, for each pair of numbers $a, b \in \mathbb{T}$, $a \leq b$, we define a closed time scale interval by $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. The open and half-open intervals are defined in a similar way.

Let $\mathbb{T}$ be a time scale. The next notation is borrowed from [42]. Given a real number $t \leq \sup \mathbb{T}$, define

$$t^* = \inf \{s \in \mathbb{T} : s \geq t\}. \quad (3.16)$$

By this definition, $t \leq t^*$, for every real number $t \leq \sup \mathbb{T}$ and $t^* = t$ for any $t \in \mathbb{T}$.

Note that t^* may differ from $\sigma(t)$. The following example illustrates this fact.

Example 3.10. Let $\mathbb{T} = \mathbb{N}$ and $t \in \mathbb{T}$. Then

$$t^* = \inf \{s \in \mathbb{T} : s \geq t\} = \inf \{t, t+1, t+2, \dots\} = t$$

and

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\} = \inf \{t+1, t+2, t+3, \dots\} = t+1.$$

Moreover, $t^* \neq \sigma(t)$.

Coming back to time scales $\mathbb{T}$, since $\mathbb{T}$ is a closed set, we have $t^* \in \mathbb{T}$. Now, define the extension of $\mathbb{T}$ by

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, \infty), & \text{otherwise.} \end{cases} \quad (3.17)$$

Given a function $f : \mathbb{T} \rightarrow \mathbb{R}^n$, we define its extension $f^* : \mathbb{T}^* \rightarrow \mathbb{R}^n$ by

$$f^*(t) = f(t^*), \quad t \in \mathbb{T}^*.$$

Analogously, for each function $f : B \times \mathbb{T} \rightarrow \mathbb{R}^n$, $B \subset \mathbb{R}^n$, we define $f^* : B \times \mathbb{T}^* \rightarrow \mathbb{R}^n$ by

$$f^*(x, t) = f(x, t^*), \quad x \in B, \quad t \in \mathbb{T}^*.$$

Note that, while the function f is defined on $\mathbb{T}$, its extension f^* is defined on the whole interval $(-\infty, \sup \mathbb{T}]$, whenever $\sup \mathbb{T} < +\infty$, and $(-\infty, +\infty)$, whenever $\sup \mathbb{T} = +\infty$.

Let $a, b \in \mathbb{T}$, $a \leq b$, and $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be a function. Notice that if $t \in [a, b]$, then t^* is well-defined and $t^* \in [a, b]_{\mathbb{T}}$, since $a^* = a \leq t \leq b = b^* \leq \sup \mathbb{T}$. Thus, we conclude that $f^* : [a, b] \rightarrow \mathbb{R}^n$ given by $f^*(t) = f(t^*)$, for every $t \in [a, b]$, is well-defined.

On the other hand, given a real number $t_0 \in \mathbb{T}$ and let $f : [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be a function, it is clear that we may have points $t \in [t_0, +\infty)$ such that $t > \sup \mathbb{T}$. Therefore, t^* is not well-defined. Thus, in order to ensure that the function $t \mapsto t^*$ is well-defined, for $t \in [t_0, +\infty)$, it is necessary to require that $\sup \mathbb{T} = +\infty$.

In this work, we consider a dynamic equation on time scales of type

$$x^\Delta(t) = f(x^*, t) \quad (3.18)$$

where $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$, where $O \subset \mathbb{R}^n$ is an open subset. Also, x^Δ denotes the Δ -derivative of x , as defined in Chapter 2. The integral form of the dynamic equation on time scales (3.18) is given by

$$x(t) - x(t_0) = \int_{t_0}^t f(x^*(s), s) \Delta s, \quad t \geq t_0, \quad (3.19)$$

where the integral on the right-hand side is the Kurzweil-Henstock Δ -integral.

The next result describes the properties of the extension f^* , based on the properties of f . This result can be found in [42, Lemma 4].

Lemma 3.11. *If $f : \mathbb{T} \rightarrow \mathbb{R}^n$ is a regulated function, then $f^* : \mathbb{T}^* \rightarrow \mathbb{R}^n$ is also regulated. If f is left-continuous on $\mathbb{T}$, then f^* is left-continuous on $\mathbb{T}^*$. If f is right-continuous on $\mathbb{T}$, then f^* is right-continuous at right-dense points of $\mathbb{T}^*$.*

Lemma 3.12. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$, and $t_0 \in \mathbb{T}$. Let $g : [t_0, +\infty) \rightarrow \mathbb{R}$ be given by $g(t) = t^*$, for all $t \in [t_0, +\infty)$. Then g satisfies the following conditions*

- (i) g is a nondecreasing function;
- (ii) g is left-continuous on $(t_0, +\infty)$.

Proof. Item (i) follows immediately from the definition of the function g .

On the other hand, item (ii) is an immediate consequence of Lemma 3.11 by considering the identity function $i : [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ given by $i(t) = t$. $\square$

The next result describes a correspondence between Kurzweil-Henstock-Stieltjes integrals and Kurzweil-Henstock Δ -integrals. This result is very important to establish a correspondence between dynamic equations on time scales and measure differential equations. It can be found in [21, Theorem 4.2].

Theorem 3.13. *Let $\mathbb{T}$ be a time scale and $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be a function. Define $g(t) = t^*$, for every $t \in [a, b]$. Then, the Kurzweil-Henstock Δ -integral $\int_a^b f(t) \Delta t$ exists if, and only if, the Kurzweil-Henstock-Stieltjes integral $\int_a^b f^*(t) dg(t)$ exists; In this case, both integrals have the same value.*

The following result is a consequence of [21, Lemma 4.4]. We repeat its proof here, following the ideas of [21].

Theorem 3.14. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$, $t_0 \in \mathbb{T}$ and $g(t) = t^*$ for every $t \in [t_0, +\infty)$. If $f : [t_0, +\infty) \rightarrow \mathbb{R}^n$ is such that the Kurzweil-Henstock-Stieltjes integral $\int_c^d f(t) dg(t)$ exists for every $c, d \in [t_0, +\infty)$, then*

$$\int_c^d f(t) dg(t) = \int_{c^*}^{d^*} f(t) dg(t),$$

for every $t_0 \leq c < d < \infty$, where the integral on the right-hand side is in the sense of Kurzweil-Henstock-Stieltjes and c^*, d^* are defined as in (3.16).

Proof. Let $c, d \in [t_0, +\infty)$ be such that $c < d$. We will show that g is constant on $[c, c^*]$ and $[d, d^*]$. Indeed, for any $s \in [c, c^*]$ and $t \in [d, d^*]$, statement (i) from Lemma 3.12 implies that

$$g(c) \leq g(s) \leq g(c^*) \quad \text{and} \quad g(d) \leq g(t) \leq g(d^*).$$

Since $g(c) = g(c^*) = c^*$ and $g(d) = g(d^*) = d^*$, we obtain $g(s) = c^*$, for all $s \in [c, c^*]$, and $g(t) = d^*$, for all $t \in [d, d^*]$.

Now, using the definition of the Kurzweil-Henstock-Stieltjes integral and the fact that g is constant on $[c, c^*]$ and $[d, d^*]$, we obtain $\int_c^{c^*} f(t)dg(t) = 0$ and $\int_d^{d^*} f(t)dg(t) = 0$. Hence

$$\int_c^d f(t)dg(t) = \int_c^{c^*} f(t)dg(t) + \int_{c^*}^d f(t)dg(t) + \int_d^{d^*} f(t)dg(t) = \int_{c^*}^{d^*} f(t)dg(t)$$

and this completes the proof. $\square$

The next result describes the relation between the Kurzweil-Henstock Δ -integral and the Kurzweil-Henstock-Stieltjes integral, where the function f is defined on unbounded time scales interval. Such result can be found in [21, Theorem 4.5].

Theorem 3.15. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$, $t_0 \in \mathbb{T}$ and $f : [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be a function such that the Kurzweil-Henstock Δ -integral*

$$\int_a^b f(s)\Delta s$$

exists for every $a, b \in [t_0, +\infty)_{\mathbb{T}}$ and $a < b$. Choose an arbitrary $a \in [t_0, +\infty)_{\mathbb{T}}$ and define

$$F_1(t) = \int_a^t f(s)\Delta s, \quad t \in [t_0, +\infty)_{\mathbb{T}},$$

$$F_2(t) = \int_a^t f^*(s)dg(s), \quad t \in [t_0, +\infty)_{\mathbb{T}},$$

where $g(s) = s^$, for every $s \in [t_0, +\infty)_{\mathbb{T}}$. Then $F_2 = F_1^*$. In particular, $F_2(t) = F_1(t)$ for all $t \in [t_0, +\infty)_{\mathbb{T}}$.*

Proof. Let $t \in [t_0, +\infty)$. Then, by Theorems 3.13 and 3.14 and the fact $a = a^*$, we have

$$\begin{aligned} F_2(t) &= \int_a^t f^*(s) dg(s) = \int_{a^*}^{t^*} f^*(s) dg(s) \\ &= \int_a^{t^*} f^*(s) dg(s) \\ &= \int_a^{t^*} f(s) \Delta s \\ &= F_1(t^*) = F_1^*(t) \end{aligned}$$

and the proof is finished. $\square$

The next result is essential to prove that dynamic equations on time scales are a special case of MDEs. A proof of it can be found in [20, Theorem 4.2.].

Theorem 3.16. *Let $\mathbb{T}$ be a time scale, $g : [a, b] \rightarrow \mathbb{R}$ be defined by $g(s) = s^*$, for every $s \in [a, b]$, and $[a, b] \subset \mathbb{T}^*$. Consider a pair of functions $f_1, f_2 : [a, b] \rightarrow \mathbb{R}^n$ such that $f_1(t) = f_2(t)$ for every $t \in [a, b] \cap \mathbb{T}$. If the Kurzweil-Henstock-Stieltjes integral $\int_a^b f_1(s) dg(s)$ exists, then the Kurzweil-Henstock-Stieltjes integral $\int_a^b f_2(s) dg(s)$ also exists and both integrals have the same value.*

Proof. Suppose the Kurzweil-Henstock-Stieltjes integral $\int_a^b f_1(s) dg(s)$ exists. Given $\epsilon > 0$, there is a gauge $\tilde{\delta} : [a, b] \rightarrow \mathbb{R}^+$ such that

$$\left\| \sum_{i=1}^{|D|} f_1(\tau_i)(g(t_i) - g(t_{i-1})) - \int_a^b f_1(s) dg(s) \right\| < \epsilon,$$

for every $\tilde{\delta}$ -fine tagged division $D = \{([t_{i-1}, t_i], \tau_i)\}$ of $[a, b]$. Let us define a gauge $\delta : [a, b] \rightarrow \mathbb{R}^+$ by

$$\delta(t) = \begin{cases} \tilde{\delta}(t), & t \in [a, b] \cap \mathbb{T}, \\ \min \left\{ \tilde{\delta}(t), \frac{1}{2} \inf \{|t - s| : s \in \mathbb{T}\} \right\}, & t \in [a, b] \setminus \mathbb{T}. \end{cases}$$

Note that $\delta(t) \leq \tilde{\delta}(t)$ for any $t \in [a, b]$. Then, evidently, every δ -fine tagged division D of $[a, b]$ is also $\tilde{\delta}$ -fine. Consider an arbitrary δ -fine tagged division $D = \{([t_{i-1}, t_i], \tau_i)\}$ of $[a, b]$. For every $i \in \{1, 2, \dots, |D|\}$, there are two possibilities

$$\text{either } [t_{i-1}, t_i] \cap \mathbb{T} = \emptyset \quad \text{or} \quad \tau_i \in \mathbb{T}.$$

In the first case, g is constant on $[t_{i-1}, t_i]$. Consequently, $g(t_{i-1}) = g(t_i)$ and, therefore,

$$f_2(\tau_i)(g(t_i) - g(t_{i-1})) = 0 = f_2(\tau_i)(g(t_i) - g(t_{i-1})).$$

In the second case, $f_1(\tau_i) = f_2(\tau_i)$ and

$$f_1(\tau_i)(g(t_i) - g(t_{i-1})) = f_2(\tau_i)(g(t_i) - g(t_{i-1})).$$

In any case,

$$f_1(\tau_i)(g(t_i) - g(t_{i-1})) = f_2(\tau_i)(g(t_i) - g(t_{i-1})), \quad \text{for } i = 1, 2, \dots, |D|.$$

Thus, we have

$$\left\| \sum_{i=1}^{|D|} f_2(\tau_i)(g(t_i) - g(t_{i-1})) - \int_a^b f_1(s) dg(s) \right\| = \left\| \sum_{i=1}^{|D|} f_1(\tau_i)(g(t_i) - g(t_{i-1})) - \int_a^b f_1(s) dg(s) \right\| < \epsilon.$$

Since $\epsilon > 0$ can be arbitrarily small, we conclude that $\int_a^b f_2(s) dg(s)$ exists and $\int_a^b f_2(s) dg(s) = \int_a^b f_1(s) dg(s)$. $\square$

The next result establishes a correspondence between the solutions of the dynamic equation on time scales (3.18) and the solutions of the integral form of an MDE of type

$$x(t) = x(t_0) + \int_{t_0}^t f^*(x, s) dg(s), \quad t \geq t_0, \quad (3.20)$$

where $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ are given by $g(s) = s^*$. This result is a slightly modified version of [21, Theorem 5.2] (which is the special case when the domain of the solution is a compact interval). The proof from [21] can be carried out without any changes and we reproduce it here.

Theorem 3.17. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $[t_0, +\infty)_{\mathbb{T}}$ be a time scale interval. Let $O \subset \mathbb{R}^n$ be an open subset, $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be a function. Assume that for every $x \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$, the function $t \mapsto f(x(t), t)$ is Kurzweil-Henstock Δ -integrable on $[s_1, s_2]_{\mathbb{T}}$, for every $s_1, s_2 \in [t_0, +\infty)_{\mathbb{T}}$. Define $g : [t_0, +\infty) \rightarrow \mathbb{R}$ by $g(s) = s^*$, for every $s \in [t_0, +\infty)$. Also, let $J \subset [t_0, +\infty)$ be a nondegenerate interval such that $J \cap \mathbb{T}$ is nonempty and for each $t \in J$, we have $t^* \in J \cap \mathbb{T}$. If $x : J \cap \mathbb{T} \rightarrow \mathbb{R}^n$ is a solution of the initial value problem given by*

$$\begin{cases} x^\Delta(t) = f(x^*, t), & t \in J \cap \mathbb{T}, \\ x(s_0) = x_0, \end{cases} \quad (3.21)$$

where $x_0 \in O$ and $s_0 \in J \cap \mathbb{T}$, then $x^* : J \rightarrow \mathbb{R}^n$ is a solution of the initial value problem

$$y(t) = x_0 + \int_{s_0}^t f^*(y(s), s) dg(s) = x_0 + \int_{s_0}^t f(y(s), s^*) dg(s) \quad (3.22)$$

Conversely, if $y : J \rightarrow \mathbb{R}^n$ satisfies the initial value problem (3.22), then it must have the form $y = x^*$, where $x : J \cap \mathbb{T} \rightarrow \mathbb{R}^n$ is a solution of the initial value problem (3.21).

Proof. Assume that $x : J \cap \mathbb{T} \rightarrow \mathbb{R}^n$ satisfies the dynamic equation on time scales (3.21). Then

$$x(t) - x_0 = \int_{s_0}^t f(x^*(s), s) \Delta s, \quad t \in J \cap \mathbb{T}.$$

By Theorem 3.15,

$$x^*(t) - x_0 = \int_{s_0}^t f(x^*(s^*), s^*) dg(s), \quad t \in J.$$

Since $f(x^*(s^*), s^*) = f(x^*(s), s^*)$ for all $s \in [s_0, t] \cap \mathbb{T}$, by Theorem 3.16, we get

$$x^*(t) - x_0 = \int_{s_0}^t f(x^*(s), s^*) dg(s), \quad t \in J,$$

that is, $x^* : J \rightarrow \mathbb{R}^n$ satisfies the initial value problem (3.22), which proves the first part of the theorem.

Conversely, suppose $y : J \rightarrow \mathbb{R}^n$ satisfies the initial value problem (3.22). Then

$$y(t) = x_0 + \int_{s_0}^t f(y(s), s^*) dg(s), \quad t \in J.$$

Note that g is constant on every interval $(\alpha, \beta]$, where $\beta \in \mathbb{T}$ and $\alpha = \sup \{s \in \mathbb{T} : s < \beta\}$. Thus, y has the same property and it follows that $y = x^*$ for some $x : J \cap \mathbb{T} \rightarrow \mathbb{R}^n$. By reversing our previous reasoning of proof of the first part of the theorem, we conclude that x satisfies the dynamic equation on time scale (3.21). $\square$

Corollary 3.18. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $[t_0, +\infty)_{\mathbb{T}}$ be a time scale interval. Let $O \subset \mathbb{R}^n$ be an open subset, $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be a function. Assume that for every $x \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$, the function $t \mapsto f(x(t), t)$ is Kurzweil-Henstock Δ -integrable on $[s_1, s_2]_{\mathbb{T}}$, for every $s_1, s_2 \in [t_0, +\infty)_{\mathbb{T}}$. Define $g : [t_0, +\infty) \rightarrow \mathbb{R}$ by $g(s) = s^*$, for every $s \in [t_0, +\infty)$. Also, let $J \subset [t_0, +\infty)$ be a nondegenerate interval such that $J \cap \mathbb{T}$ is nonempty and for each $t \in J$, we have $t^* \in J \cap \mathbb{T}$. If $x : J \cap \mathbb{T} \rightarrow \mathbb{R}^n$ is a solution of the dynamic equation on time scales given by*

$$x^\Delta(t) = f(x^*, t), \quad t \in J \cap \mathbb{T}. \quad (3.23)$$

then $x^* : J \rightarrow \mathbb{R}^n$ is a solution of the MDE

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} f^*(y, s) dg(s), \quad t_1, t_2 \in J. \quad (3.24)$$

Conversely, if $y : J \rightarrow \mathbb{R}^n$ satisfies the MDE (3.24), then it must have the form $y = x^*$, where $x : J \cap \mathbb{T} \rightarrow \mathbb{R}^n$ is a solution of (3.23).

Proof. Suppose $J \cap \mathbb{T}$ is nonempty and $x : J \cap \mathbb{T} \rightarrow \mathbb{R}^n$ is a solution of the dynamic equation on time scales $x^\Delta(t) = f(x^*, t)$. Since $J \cap \mathbb{T}$ is nonempty, there exists $\tau_0 \in \mathbb{R}$ such that $\tau_0 \in J \cap \mathbb{T}$. Hence, $x : J \cap \mathbb{T} \rightarrow \mathbb{R}^n$ is a solution of the initial value problem

$$\begin{cases} x^\Delta(t) = f(x^*, t), & t \in J \cap \mathbb{T}, \\ x(\tau_0) = z_0, \end{cases} \quad (3.25)$$

where $z_0 := x(\tau_0)$. Thus, by Theorem 3.17, $x^* : J \rightarrow \mathbb{R}^n$ is a solution of

$$y(t) = z_0 + \int_{\tau_0}^t f^*(y(s), s) dg(s), \quad t \in J. \quad (3.26)$$

Thus, by Remark 3.1, $x^* : J \rightarrow \mathbb{R}^n$ satisfies

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} f^*(y(s), s) dg(s),$$

for all $t_1, t_2 \in J$.

The converse assertion can be proved similarly. □

We finish this section by presenting an example which illustrates Theorem 3.16.

Example 3.19. Let $\mathbb{R}$ be the 1-dimensional Euclidean space with norm $|\cdot|$ (absolute value) and let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $0 \in \mathbb{T}$. Consider the dynamic equation on time scales given by

$$\begin{cases} x^\Delta(t) = f(x^*, t), & t \in [0, +\infty)_{\mathbb{T}} \\ x(0) = x_0, \end{cases} \quad (3.27)$$

where $x_0 \in \mathbb{R}$ and $f : \mathbb{R} \times [0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ is defined by $f(x, t) = \frac{5 \cos(2x)}{e^t + t + 1}$, for all $(x, t) \in \mathbb{R} \times [0, +\infty)_{\mathbb{T}}$.

We will show that f satisfies conditions (B1), (B2) and (B3). Indeed, consider an arbitrary $y \in G([0, +\infty)_{\mathbb{T}}, \mathbb{R})$ and let $s_1, s_2 \in [0, +\infty)_{\mathbb{T}}$ with $s_1 \leq s_2$. Notice that the function

$$[s_1, s_2] \ni t \mapsto f(y(t^*), t^*) = \frac{5 \cos(2y(t^*))}{e^{t^*} + t^* + 1} = \frac{5 \cos(2y^*(t))}{e^{g(t)} + g(t) + 1}$$

is regulated in $[s_1, s_2]$, and $g : [0, +\infty) \rightarrow \mathbb{R}$ is given by $g(t) = t^*$. Thus, by Theorem 1.9 (item (i)), $\int_{s_1}^{s_2} f(y(t^*), t^*) dg(t)$ exists and, therefore, according to Theorem 3.13, $\int_{s_1}^{s_2} f(y(t), t) \Delta t$ exists and $\int_{s_1}^{s_2} f(y(t^*), t^*) dg(t) = \int_{s_1}^{s_2} f(y(t), t) \Delta t$.

Now, define $M : [0, +\infty) \rightarrow \mathbb{R}$ by $M(t) = t + 5$, for $t \in [0, +\infty)$. Evidently, M is a Kurzweil-Henstock-Stieltjes integrable function with respect to g and

$$\begin{aligned} \left| \int_{s_1}^{s_2} f(y(s), s) \Delta s \right| &= \int_{s_1}^{s_2} f(y(s^*), s^*) dg(s) = \int_{s_1}^{s_2} \left| \frac{5 \cos(2y^*(s))}{e^{g(s)} + g(s) + 1} \right| dg(s) \\ &\leq \int_{s_1}^{s_2} 5 dg(s) \leq \int_{s_1}^{s_2} M(s) dg(s), \end{aligned}$$

for every $y \in G([0, +\infty), \mathbb{R})$ and $s_1, s_2 \in [0, +\infty)_{\mathbb{T}}$, $s_1 \leq s_2$. Thus f satisfies condition (B2).

On the other hand, define $L : [0, +\infty) \rightarrow \mathbb{R}$ by $L(t) := 10$, for $t \in [0, +\infty)$, then

$$\begin{aligned} \left| \int_{s_1}^{s_2} [f(y(s), s) - f(w(s), s)] \Delta s \right| &= \int_{s_1}^{s_2} [f(y(s^*), s^*) - f(w(s^*), s^*)] dg(s) \\ &\leq \int_{s_1}^{s_2} |f(y(s^*), s^*) - f(w(s^*), s^*)| dg(s) \leq \int_{s_1}^{s_2} 5e^{-g(s)} |\cos(2y^*(s)) - \cos(2w^*(s))| dg(s) \\ &\leq \int_{s_1}^{s_2} 10e^{-g(s)} |y^*(s) - w^*(s)| dg(s) = \int_{s_1}^{s_2} L(s)e^{-s^*} |y(s^*) - w(s^*)| dg(s), \\ &\leq \int_{s_1}^{s_2} L(s) \|y - w\|_{[0, +\infty)_{\mathbb{T}}} dg(s) = \|y - w\|_{[0, +\infty)_{\mathbb{T}}} \int_{s_1}^{s_2} L(s) dg(s), \end{aligned}$$

for all $y, w \in G_0([0, +\infty)_{\mathbb{T}}, \mathbb{R})$ and $s_1, s_2 \in [0, +\infty)$, $s_1 \leq s_2$. Hence, f and g fulfill all the hypotheses Theorem 3.17.

Then the corresponding measure differential equation in the integral form is given by

$$x(t) = x_0 + \int_0^t f^*(x(s), s) dg(s), \quad t \geq 0, \quad (3.28)$$

where $f^*(x, t) = f(x, t^*) = \frac{5 \cos(2x)}{e^{t^*} + t^* + 1}$, for all $(x, t) \in \mathbb{R} \times [0, +\infty)$.

Now, let $J \subset [0, +\infty)$ be a nondegenerate interval such that $J \cap \mathbb{T}$ is nonempty and for each $t \in J$, we have $t^* \in J \cap \mathbb{T}$. Theorem 3.17 says that if $x : J \cap \mathbb{T} \rightarrow \mathbb{R}$ is a solution of the

dynamic equation on time scales (3.27). Then the function $x^* : J \rightarrow \mathbb{R}$ is a solution of the measure differential equation (3.28).

Conversely, every solution $y : J \rightarrow \mathbb{R}$ of the measure differential equation (3.28) has the form $y = x^*$, where $x : J \cap \mathbb{T} \rightarrow \mathbb{R}$ is a solution of the dynamic equation on time sales (3.27).

Prolongation of solutions

In this chapter, we present results on the prolongation of solutions of generalized ODEs, measure differential equations (MDEs) and dynamic equations on time scales.

It is important to mention that, in Section 4.1, we present some new results in the theory of generalized ODEs. Furthermore, all the results presented in Sections 4.2 and 4.3 of this chapter are new and either they follow by using the correspondence between MDEs and generalized ODEs or they follow by the correspondence between MDEs and dynamic equations on time scales (see Chapter 3). These results are contained in [16].

4.1 Prolongation of the solutions of generalized ODEs

In this section, our goal is to prove a result on the prolongation of solutions of an abstract generalized ODE.

Let X be a Banach space. Consider $O \subset X$ an open set, an interval $[t_0, +\infty) \subset \mathbb{R}$ and $\Omega = O \times [t_0, +\infty)$. Consider the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t), \quad (4.1)$$

where $F \in \mathcal{F}(\Omega, h)$ and the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous.

The next result ensures the prolongation of a solution of the generalized ODE (4.1). The proof of the following result is inspired in the proof of [38, Proposition 4.15]. Our result generalizes the result found in [38, Lemma 4.4]. In fact, [38, Lemma 4.4] can be obtained as a consequence of our result (see Corollary 4.2 in the sequel).

Throughout this section, for $t_0 < \beta < \vartheta < +\infty$, define $\Gamma_{\beta, \vartheta}^\infty := \{[\beta, \vartheta], [\beta, \vartheta), [\beta, +\infty)\}$.

Theorem 4.1. *Let $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous. If $x : [\gamma, \beta) \rightarrow X$ and $y : I \rightarrow X$, $I \in \Gamma_{\beta, \vartheta}^\infty$, are solutions of the generalized ODE (4.1) on $[\gamma, \beta)$ and I , respectively, where $t_0 \leq \gamma < \beta < \vartheta < +\infty$, and if the limit $\lim_{t \rightarrow \beta^-} x(t)$ exists and $\lim_{t \rightarrow \beta^-} x(t) = y(\beta)$, then the function $z : [\gamma, \beta) \cup I \rightarrow X$ defined by*

$$z(t) = \begin{cases} x(t), & t \in [\gamma, \beta) \\ y(t), & t \in I, \end{cases}$$

is a solution of the generalized ODE (4.1) on $[\gamma, \beta) \cup I = \begin{cases} [\gamma, \vartheta], & \text{if } I = [\beta, \vartheta], \\ [\gamma, \vartheta), & \text{if } I = [\beta, \vartheta), \\ [\gamma, +\infty), & \text{if } I = [\beta, +\infty). \end{cases}$

Proof. Suppose the limit $\lim_{t \rightarrow \beta^-} x(t)$ exists and

$$\lim_{t \rightarrow \beta^-} x(t) = y(\beta). \quad (4.2)$$

Define $z : [\gamma, \beta) \cup I \rightarrow X$ by

$$z(t) = \begin{cases} x(t), & t \in [\gamma, \beta), \\ y(t), & t \in I. \end{cases}$$

Notice that the function z is well-defined and it is a regulated function, since by Lemma 1.17, x and y are regulated functions. We will show that z is a solution of the generalized ODE (4.1) on $[\gamma, \beta) \cup I$.

Since β is an accumulation point of the set $[\gamma, \beta)$, there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \subset [\gamma, \beta)$ such that $t_n \xrightarrow{n \rightarrow \infty} \beta$. Therefore, by (4.2), we have

$$\lim_{n \rightarrow \infty} x(t_n) = y(\beta). \quad (4.3)$$

Let $s_1, s_2 \in [\gamma, \beta) \cup I$ be such that $s_1 \in [\gamma, \beta)$ and $s_2 \in I$. Then, for $n \in \mathbb{N}$ sufficiently large, we have $t_n \in (s_1, \beta)$ and

$$\begin{aligned} \int_{s_1}^{s_2} DF(z(\tau), t) &= \int_{s_1}^{\beta} DF(z(\tau), t) + \int_{\beta}^{s_2} DF(z(\tau), t) \\ &= \int_{s_1}^{t_n} DF(x(\tau), t) + \int_{t_n}^{\beta} DF(z(\tau), t) + \int_{\beta}^{s_2} DF(y(\tau), t) \\ &= x(t_n) - x(s_1) + \int_{t_n}^{\beta} DF(z(\tau), t) + y(s_2) - y(\beta), \end{aligned}$$

that is,

$$\int_{s_1}^{s_2} DF(z(\tau), t) = x(t_n) - y(\beta) + y(s_2) - x(s_1) + \int_{t_n}^{\beta} DF(z(\tau), t).$$

By the definition of z , we have

$$\int_{s_1}^{s_2} DF(z(\tau), t) = x(t_n) - y(\beta) + z(s_2) - z(s_1) + \int_{t_n}^{\beta} DF(z(\tau), t). \quad (4.4)$$

Thus, by Lemma 1.15, we get

$$\left\| \int_{t_n}^{\beta} DF(z(\tau), t) \right\| \leq |h(\beta) - h(t_n)|, \quad \text{for sufficiently large } n. \quad (4.5)$$

On the other hand, since h is left-continuous and $t_n \xrightarrow{n \rightarrow \infty} \beta$, $t_n < \beta$ for all $n \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} |h(\beta) - h(t_n)| = 0. \quad (4.6)$$

Thus, by (4.5) and (4.6), we obtain

$$\lim_{n \rightarrow \infty} \int_{t_n}^{\beta} DF(z(\tau), t) = 0. \quad (4.7)$$

Finally, applying the limit when $n \rightarrow \infty$ in (4.4) and using (4.3) and (4.7), we have

$$\int_{s_1}^{s_2} DF(z(\tau), t) = z(s_2) - z(s_1),$$

for every $s_1, s_2 \in [\gamma, \beta) \cup I$ such that $s_1 \in [\gamma, \beta)$ and $s_2 \in I$.

The cases where $s_1, s_2 \in [\gamma, \beta)$ or $s_1, s_2 \in I$ follow immediately. Therefore z is a solution of the generalized ODE (4.1) on $[\gamma, \beta) \cup I$. $\square$

The next result is inspired in [38, Lemma 4.4] for the case $I = [\beta, \vartheta]$. Here, we present a different proof of it.

Corollary 4.2. *Let $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous. If $x : [\gamma, \beta] \rightarrow X$ and $y : I \rightarrow X$, $I \in \Gamma_{\beta, \vartheta}^\infty$, are solutions of the generalized ODE (4.1) on $[\gamma, \beta]$ and I respectively, where $t_0 \leq \gamma < \beta < \vartheta < +\infty$ and $x(\beta) = y(\beta)$, then the function $z : [\gamma, \beta] \cup I \rightarrow X$ given by*

$$z(t) = \begin{cases} x(t), & t \in [\gamma, \beta] \\ y(t), & t \in I \end{cases} \quad (4.8)$$

is a solution of the generalized ODE (4.1) on $[\gamma, \beta] \cup I$.

Proof. Suppose

$$x(\beta) = y(\beta). \quad (4.9)$$

Since $F \in \mathcal{F}(\Omega, h)$ and h is a left-continuous function on $[t_0, +\infty)$, by Lemma 1.16, x is left-continuous on $(\gamma, \beta]$. In particular, we have

$$\lim_{t \rightarrow \beta^-} x(t) = x(\beta).$$

Thus, by (4.9), it follows that

$$\lim_{t \rightarrow \beta^-} x(t) = y(\beta).$$

On the other hand, since $x : [\gamma, \beta] \rightarrow X$ is a solution of the generalized ODE (4.1) on $[\gamma, \beta]$, $x|_{[\gamma, \beta)}$ is a solution of the equation (4.1) on $[\gamma, \beta)$. Note that all the hypotheses of Theorem 4.1 are satisfied. Then z defined by (4.8) is a solution of the generalized ODE (4.1) on $[\gamma, \beta] \cup I$ and the proof is complete. $\square$

Now, our goal is to prove that, under certain conditions, the unique solution $x : [\tau_0, \tau_0 + \Delta] \rightarrow X$ of (4.1) satisfying $x(\tau_0) = x_0$, which is ensured by Theorem 1.21, can be extended

to intervals greater than $[\tau_0, \tau_0 + \Delta]$, up to a maximal interval, at least while the graph of the solution does not reach the boundary of Ω .

In the sequel, we recall the concept of maximal solution of a generalized ODE. See [38].

Let $F \in \mathcal{F}(\Omega, h)$ and $(x_0, \tau_0) \in \Omega$ be such that

$$x_0 + F(x_0, \tau_0^+) - F(x_0, \tau_0) \in O,$$

where we recall the reader that $F(x_0, \tau_0^+) = \lim_{t \rightarrow \tau_0^+} F(x_0, t)$. Fix $(x_0, \tau_0) \in \Omega$ and define the following set

$$S_{\tau_0, x_0} := \{x : I_x \subset [t_0, +\infty) \rightarrow X : I_x \text{ is an interval with the left endpoint } \tau_0,$$

$$x \text{ is a solution of the generalized ODE (4.1), } x(\tau_0) = x_0\}.$$

Definition 4.3. Given $x : I_x \rightarrow X$ and $z : I_z \rightarrow X$ in S_{τ_0, x_0} , we say that x is smaller than or equal to z ($x \preceq z$), if and only if, $I_x \subset I_z$ and $z|_{I_x} = x$.

The next result shows that the relation $\preceq$ defines a partial order in S_{τ_0, x_0} .

Proposition 4.4. *The relation $\preceq$ defines a partial order relation in S_{τ_0, x_0} .*

Proof. We must show that $\preceq$ is reflexive, antisymmetric and transitive.

Reflexivity. Let $x : I_x \rightarrow X$ in S_{τ_0, x_0} . Then $I_x \subset I_x$ and $x|_{I_x} \equiv x$. Therefore, $x \preceq x$.

Antisymmetry. Let $x : I_x \rightarrow X$ and $z : I_z \rightarrow X$ belong to S_{τ_0, x_0} , with $x \preceq z$ and $z \preceq x$.

Since $x \preceq z$, we have

$$I_x \subset I_z \text{ and } z|_{I_x} = x. \quad (4.10)$$

On the other hand, since $z \preceq x$, we have

$$I_z \subset I_x \text{ and } x|_{I_z} = z. \quad (4.11)$$

Thus, by (4.10) and (4.11), we obtain $I_x = I_z$ and $x = z$.

Transitivity. Let $x : I_x \rightarrow X$, $z : I_z \rightarrow X$ and $y : I_y \rightarrow X$ belong to S_{τ_0, x_0} such that $x \preceq z$ and $z \preceq y$. Since $x \preceq z$, we have

$$I_x \subset I_z \text{ and } z|_{I_x} = x. \quad (4.12)$$

By $z \preceq y$, we obtain

$$I_z \subset I_y \text{ and } y|_{I_z} = z. \quad (4.13)$$

Thus, by (4.12) and (4.13), $I_x \subset I_y$ and $y|_{I_x} = x$, that is, $x \preceq y$. $\square$

Notice that to be able to investigate the forward maximal solution, it is necessary to require

$$x_0 + F(x_0, \tau_0^+) - F(x_0, \tau_0) \in O.$$

Otherwise, it could happen that $x(t) \notin O$ for $t > \tau_0$, contradicting the definition of a solution of a generalized ODE. Therefore, throughout this chapter, we assume

$$\tilde{x} + F(\tilde{x}, t^+) - F(\tilde{x}, t) \in O, \quad \text{for every } \tilde{x} \in O \text{ and } t \in [t_0, +\infty),$$

that is, we consider

$$\Omega = \Omega_F := \{(x, t) \in \Omega : x + F(x, t^+) - F(x, t) \in O\}. \quad (4.14)$$

This means that there are no points in Ω to which the solution of the generalized ODE (4.1) can escape from the set O .

Definition 4.5. (Prolongation of solution) Let $\tau_0 \geq t_0$ and let $x : I \rightarrow X$, $I \subset [t_0, +\infty)$, be a solution of (4.1) on the interval I , I with left endpoint s_0 . The solution $y : J \rightarrow X$, $J \subset [t_0, +\infty)$, with left endpoint τ_0 , of the generalized ODE (4.1) is called a *prolongation to the right* of x , if $I \subset J$ and $x(t) = y(t)$ for all $t \in I$. If $I \subsetneq J$, then y is called a *proper prolongation* of x to the right.

In the next definition, the concept of a maximal solution of a generalized ODE is presented.

Definition 4.6. (Maximal solution) Let $(x_0, \tau_0) \in \Omega$. We say that $x : J \rightarrow X$ is a *maximal solution* of the generalized ODE

$$\begin{cases} \frac{dx}{d\tau} = DF(x, t), \\ x(\tau_0) = x_0, \end{cases} \quad (4.15)$$

if $x \in S_{\tau_0, x_0}$ and, for every $z : I \rightarrow X$ in S_{τ_0, x_0} such that $x \preceq z$, we have $x = z$. In other words, x is a maximal solution of (4.15), whenever $x \in S_{\tau_0, x_0}$, and there is no proper prolongation of x to the right.

The following result is crucial to prove the existence and uniqueness of a maximal solution of the generalized ODE (4.15), for the case where X is a Banach space and $\Omega = O \times [t_0, +\infty)$, where O being an open subset of X .

Lemma 4.7. Let $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous and $\Omega = \Omega_F$, where Ω_F is given by (4.14). Let $(x_0, \tau_0) \in \Omega$ and consider the generalized ODE (4.15). If $y : J_y \rightarrow X$ and $z : J_z \rightarrow X$ are solutions of the generalized

ODE (4.15), where J_y and J_z are intervals, both with left endpoint τ_0 , then $y(t) = z(t)$ for all $t \in J_y \cap J_z$.

Proof. Let $(x_0, \tau_0) \in \Omega$ be fixed. Suppose $y : J_y \rightarrow X$ and $z : J_z \rightarrow X$ are solutions of the generalized ODE (4.15), where J_y and J_z are intervals with left endpoint τ_0 . Then,

$$y(\tau_0) = z(\tau_0) = x_0. \quad (4.16)$$

We will show that $y(t) = z(t)$ for all $t \in J_y \cap J_z$. Note that $J_y \cap J_z$ is an interval of the form

$$[\tau_0, d], \quad \text{or} \quad [\tau_0, d), \quad d \leq +\infty.$$

Therefore, we will consider the next two cases.

Case 1. $J_y \cap J_z = [\tau_0, d]$. We define

$$\Lambda := \{t \in [\tau_0, d] : y(s) = z(s), \text{ for all } s \in [\tau_0, t]\}.$$

Note that $\Lambda \neq \emptyset$, since $\tau_0 \in \Lambda$. We denote $\lambda = \sup \Lambda$. It is clear that $\lambda \leq d$ and $[\tau_0, \lambda) \subset \Lambda$. Therefore,

$$y(t) = z(t), \quad \text{for all } t \in [\tau_0, \lambda). \quad (4.17)$$

On the other hand, since $F \in \mathcal{F}(\Omega, h)$, h is a left-continuous function. Also, since by the hypotheses, the functions y and z are solutions of the generalized ODE (4.15) on $J_y \cap J_z$, it follows by Lemma 1.16 that y and z are left-continuous on $(\tau_0, \lambda]$. Then we obtain

$$y(\lambda) = \lim_{t \rightarrow \lambda^-} y(t) = \lim_{t \rightarrow \lambda^-} z(t) = z(\lambda),$$

that is,

$$y(\lambda) = z(\lambda). \quad (4.18)$$

Thus, by (4.17) and (4.18), we get

$$\lambda \in \Lambda \quad \text{and} \quad [\tau_0, \lambda] \subset \Lambda. \quad (4.19)$$

Finally, we will show that $\lambda = d$. If $\lambda < d$. Then since $(y(\lambda), \lambda) \in \Omega_F = \Omega$, by Theorem 1.21, there are $\delta > 0$ (we can take, for instance, $\lambda + \delta < d$) and a unique solution $x : [\lambda, \lambda + \delta] \rightarrow X$ of the generalized ODE

$$\begin{cases} \frac{dx}{d\tau} = DF(x, t) \\ x(\lambda) = y(\lambda) = z(\lambda). \end{cases} \quad (4.20)$$

On the other hand, $y|_{[\lambda, \lambda+\delta]}$ and $z|_{[\lambda, \lambda+\delta]}$ are solutions of (4.20). Then, by the uniqueness,

$$x(t) = z(t) = y(t), \quad \text{for all } t \in [\lambda, \lambda + \delta]. \quad (4.21)$$

Thus, by (4.19) and (4.21), $\lambda + \delta \in \Lambda$ which contradicts the definition of λ . Then $d = \lambda$ and $y(t) = z(t)$, for all $t \in J_y \cap J_z$.

Case 2. $J_y \cap J_z = [\tau_0, d)$, with $d \leq +\infty$.

In order to prove Case 2, we will prove two assertions.

Assertion 1. For every $\tau \in (\tau_0, d)$, $y(s) = z(s)$, for all $s \in [\tau_0, \tau]$.

Define

$$\Lambda := \{t \in (\tau_0, d) : y(s) = z(s), \text{ for all } s \in [\tau_0, t]\}.$$

Let us prove that $\Lambda \neq \emptyset$. Since $(x_0, \tau_0) \in \Omega = \Omega_F$, Theorem 1.21 ensures the existence of $\eta > 0$ (we can take, for instance, $\tau_0 + \eta < d$) and a unique solution $x : [\tau_0, \tau_0 + \eta] \rightarrow X$ of the generalized ODE

$$\begin{cases} \frac{dx}{d\tau} = DF(x, t) \\ x(\tau_0) = x_0 = y(\tau_0) = z(\tau_0). \end{cases} \quad (4.22)$$

On the other hand, since $y|_{[\tau_0, \tau_0+\eta]}$ and $z|_{[\tau_0, \tau_0+\eta]}$ are solutions of (4.22). Then,

$$x(t) = y(t) = z(t), \quad \text{for all } t \in [\tau_0, \tau_0 + \eta]. \quad (4.23)$$

In particular, $y(t) = z(t)$ for all $t \in [\tau_0, \tau_0 + \eta]$, that is, $\tau_0 + \eta \in \Lambda$. Thus Λ is a nonempty set. Denote $\lambda = \sup \Lambda$. Note that $\lambda \leq d$ and $(\tau_0, \lambda) \subset \Lambda$. Let us prove that $\lambda = d$. Suppose the contrary, that is, $\lambda < d$. We will show that $y(t) = z(t)$, for $t \in [\tau_0, \lambda)$ and $y(\lambda) = z(\lambda)$. Indeed, let $t \in [\tau_0, \lambda)$. If $t = \tau_0$, follows immediately by (4.16). On the other hand, if $t \in (\tau_0, \lambda)$, then $t \in \Lambda$ (since $(\tau_0, \lambda) \subset \Lambda$). Thus, $y(s) = z(s)$, for all $s \in [\tau_0, t]$. Since $y|_{(\tau_0, t]}$ and $z|_{(\tau_0, t]}$ are left-continuous functions, we get

$$y(t) = \lim_{s \rightarrow t^-} y(s) = \lim_{s \rightarrow t^-} z(s) = z(t).$$

Therefore

$$y(t) = z(t), \text{ for all } t \in [\tau_0, \lambda). \quad (4.24)$$

Thus, by (4.24) and by the left continuity of the functions $y|_{(\tau_0, \lambda]}$ and $z|_{(\tau_0, \lambda]}$, we have

$$y(\lambda) = z(\lambda). \quad (4.25)$$

Then, by (4.24) and (4.25), we obtain

$$y(t) = z(t), \quad \text{for all } t \in [\tau_0, \lambda]. \quad (4.26)$$

Since $(y(\lambda), \lambda) \in \Omega_F = \Omega$, Theorem 1.21 implies there are $\delta > 0$ (we can take, for instance, $\lambda + \delta < d$) and a unique solution $x : [\lambda, \lambda + \delta] \rightarrow X$ of the generalized ODE

$$\begin{cases} \frac{dx}{d\tau} = DF(x, t) \\ x(\lambda) = y(\lambda) = z(\lambda). \end{cases} \quad (4.27)$$

On the other hand, since $z|_{[\lambda, \lambda + \delta]}$ and $y|_{[\lambda, \lambda + \delta]}$ are solutions of (4.27), the uniqueness of a solution yields

$$x(t) = z(t) = y(t), \quad \text{for all } t \in [\lambda, \lambda + \delta]. \quad (4.28)$$

Thus, by (4.26) and (4.28), we have $\lambda + \delta \in \Lambda$, which contradicts the definition of λ . Then $d = \lambda$ and $\Lambda = (\tau_0, d)$.

Assertion 2. For all $t \in (\tau_0, d)$, $y(t) = z(t)$.

Indeed, let $t \in (\tau_0, d)$. Then, from Assertion 1, $y(s) = z(s)$, for all $s \in [\tau_0, t)$. Thus, by the left-continuity of the functions $y|_{(\tau_0, \lambda]}$ and $z|_{(\tau_0, \lambda]}$, we have

$$y(t) = \lim_{s \rightarrow t^-} y(s) = \lim_{s \rightarrow t^-} z(s) = z(t),$$

and Assertion 2 follows.

Finally, by (4.16) and Assertion 2, we get

$$y(t) = z(t), \quad \text{for all } t \in [\tau_0, d) = J_y \cap J_z,$$

proving the result. □

The next result gives sufficient conditions to ensure the existence and uniqueness of a maximal solution of the generalized ODE (4.1) with initial condition $x(\tau_0) = x_0$, that is,

$$\begin{cases} \frac{dx}{d\tau} = DF(x, t) \\ x(\tau_0) = x_0. \end{cases}$$

A version of this result was proved in [38, Proposition 4.13], considering the case $X = \mathbb{R}^n$ and $\Omega = O \times (a, b)$, where $O = B(0, c) \subset \mathbb{R}^n$ and $(a, b) \subset \mathbb{R}$. Here, we present a proof for the case where X is a Banach space and $\Omega = O \times [t_0, +\infty)$, where O is an open subset of X .

We recall the reader that $\Omega_F = \{(x, t) \in \Omega : x + F(x, t^+) - F(x, t) \in O\}$.

Theorem 4.8. *Let $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous. If $\Omega = \Omega_F$, then, for every $(x_0, \tau_0) \in \Omega$, there exists a unique maximal solution $x : J \rightarrow X$ of (4.1), where $x(\tau_0) = x_0$ and J is an interval with left endpoint τ_0 .*

Proof. Suppose $\Omega = \Omega_F$. Let $(x_0, \tau_0) \in \Omega$ be fixed. At first, we will show the existence of a maximal solution.

Existence. Consider the set

$$S := \{x : J_x \subset [t_0, +\infty) \rightarrow X : J_x \text{ is an interval with left endpoint } \tau_0 \text{ and}$$

$$x \text{ is a solution of the generalized ODE (4.1), } x(\tau_0) = x_0\}$$

The set S is nonempty by the local uniqueness and existence of a solution given by Theorem 1.21.

Define $J := \bigcup_{y \in S} J_y$ and $x : J \rightarrow X$ by the relation $x(t) = y(t)$, where $y \in S$ and $t \in J_y$.

Note that if y and z belongs to S , then $y(s) = z(s)$, for all $s \in J_y \cap J_z$, by Lemma 4.7. Thus, we conclude that x is well-defined. Note that J is an interval with the left endpoint τ_0 (since J is union connected with a common point) and x is a maximal solution of the generalized ODE (4.1), proving the existence of a maximal solution.

It remains to ensure the uniqueness of a maximal solution.

Uniqueness. Assume that $x_1 : J_1 \rightarrow X$ and $x_2 : J_2 \rightarrow X$ are two maximal solutions of the generalized ODE (4.1) with $x_1(\tau_0) = x_2(\tau_0) = \tilde{x}$ and J_1, J_2 are intervals with left endpoint τ_0 . Thus, by Lemma 4.7, we have to show that $x_1(t) = x_2(t)$, for all $t \in J_1 \cap J_2$.

Define $x_3 : J_3 \rightarrow X$, $J_3 := J_1 \cup J_2$ by

$$x_3(t) = \begin{cases} x_1(t), & t \in J_1 \\ x_2(t), & t \in J_2. \end{cases}$$

It is clear that x_3 is a solution of the generalized ODE (4.1) with initial condition $x_3(\tau_0) = x_0$, $J_1, J_2 \subset J_3$ and $x_3|_{J_1} = x_1$, $x_3|_{J_2} = x_2$. Since the solutions x_1 and x_2 are assumed to be maximal, $J_3 = J_2 = J_1$ and $x_3(t) = x_2(t) = x_1(t)$, for all $t \in J_3$, that is, $x_1 = x_2$. $\square$

The following theorem shows that the maximal interval J of definition of a maximal solution is half-open. This result was proved in [38, Proposition 4.14], for the case $X = \mathbb{R}^n$ and $\Omega = O \times (a, b)$, where $O = B(0, c) \subset \mathbb{R}^n$ and $(a, b) \subset \mathbb{R}$. Here, we generalize the result for the case where X is a Banach space and $\Omega = O \times [t_0, +\infty)$, exhibiting a different proof from the one presented in [38].

Theorem 4.9. *Let $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty)$ is nondecreasing and left-continuous, and $\Omega = \Omega_F$. Suppose $(x_0, \tau_0) \in \Omega$ and $x : J \rightarrow X$ is the maximal solution of the generalized ODE (4.1) with $x(\tau_0) = x_0$ and J is an interval with the left endpoint τ_0 . Then $J = [\tau_0, \omega)$ with $\omega \leq +\infty$.*

Proof. It is clear that the maximal interval J satisfies $J \subset [t_0, +\infty)$. Define $\omega := \sup J$. Then clearly $\omega \leq +\infty$. If $\omega = +\infty$, the result follows immediately. Assume that $\omega < +\infty$. Let us prove that $\omega \notin J$. Suppose the contrary, that is, $\omega \in J$. Then $J = [\tau_0, \omega]$ and by the definition of a solution, that $(x(\omega), \omega) \in \Omega = \Omega_F$. Therefore, by Theorem 1.21, there exists $\eta > 0$ such that, on $[\omega, \omega + \eta]$, there is a unique solution $z : [\omega, \omega + \eta] \rightarrow X$ of the generalized ODE (4.1) such that $z(\omega) = x(\omega)$. Thus, by Corollary 4.2,

$$y(t) := \begin{cases} x(t), & t \in J = [\tau_0, \omega] \\ z(t), & t \in [\omega, \omega + \eta] \end{cases}$$

is a solution of the generalized ODE (4.1) with $y(\tau_0) = x_0$. It is easy to see that z is a proper prolongation of x , which is assumed to be maximal. Therefore we have a contradiction. Hence $\omega \notin J$ and $J = [\tau_0, \omega)$, obtaining the desired result. $\square$

The following result is a generalization of [38, Proposition 4.15], for functions F taking values in a Banach space X . The proof of this result follows the same ideas of [38].

Theorem 4.10. *Let $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous, and $\Omega = \Omega_F$. Suppose $(x_0, \tau_0) \in \Omega$ and $x : [\tau_0, +\infty) \rightarrow X$ is the maximal solution of (4.1) with $x(\tau_0) = x_0$. Then, for every compact set $K \subset \Omega$, there exists $t_K \in [\tau_0, \omega)$ such that $(x(t), t) \notin K$, for all $t \in (t_K, \omega)$.*

Proof. Suppose the contrary. Then, we will obtain the existence of a compact set $K \subset \Omega$ and a sequence $\{t_n\}_{n \in \mathbb{N}} \subset [\tau_0, \omega)$ such that

$$t_n \xrightarrow{n \rightarrow +\infty} \omega \quad \text{and} \quad (x(t_n), t_n) \in K, \quad \text{for all } n \in \mathbb{N}.$$

Let us consider two cases: $\omega = +\infty$ and $\omega < +\infty$.

Case 1. Suppose $\omega = +\infty$. Since K is compact, the sequence $\{(x(t_n), t_n)\}_{n \in \mathbb{N}}$ contains a convergent subsequence which, without loss of generality, we denote again by $\{(x(t_n), t_n)\}_{n \in \mathbb{N}}$. Then there exists $(y, \tau) \in K$ such that

$$\lim_{n \rightarrow \infty} (x(t_n), t_n) = (y, \tau).$$

In particular,

$$t_n \xrightarrow{n \rightarrow +\infty} \tau,$$

which contradicts $t_n \xrightarrow{n \rightarrow +\infty} \omega = +\infty$.

Case 2. Suppose $\omega < +\infty$. Since K is compact, the sequence $\{(x(t_n), t_n)\}_{n \in \mathbb{N}}$ contains a convergent subsequence which, without loss of generality, we denote again by $\{(x(t_n), t_n)\}_{n \in \mathbb{N}}$. Then there exists $(y, \tau) \in K \subset \Omega$ such that

$$\lim_{n \rightarrow \infty} (x(t_n), t_n) = (y, \tau).$$

In particular,

$$t_n \xrightarrow{n \rightarrow +\infty} \tau.$$

By the uniqueness of the limit, $\tau = \omega$. Thus, since $(y, \omega) = (y, \tau) \in \Omega = \Omega_F$, we have

$$y + F(y, \omega^+) - F(y, \omega) \in O.$$

Therefore, by Theorem 1.21, there exists a $\eta > 0$ such that on $[\omega, \omega + \eta]$, there is a unique solution $z : [\omega, \omega + \eta] \rightarrow X$ of the generalized ODE (4.1) satisfying $z(\omega) = y$.

Define $u : [\tau_0, \omega + \eta] \rightarrow X$ by the relation

$$u(t) = \begin{cases} x(t), & t \in [\tau_0, \omega), \\ z(t), & t \in [\omega, \omega + \eta]. \end{cases}$$

Thus, by Theorem 4.1, $u : [\tau_0, \omega + \eta] \rightarrow X$ is a solution of the generalized ODE (4.1) with initial condition $u(\tau_0) = x_0$. But notice that u is clearly a proper prolongation of the solution x , which is assumed to be maximal. This contradiction proves the result. $\square$

The following results are completely new in the theory of generalized ODEs, even for the case $X = \mathbb{R}^n$.

Corollary 4.11. *Let $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous, and $\Omega = \Omega_F$. Let $(x_0, \tau_0) \in \Omega$ and $x : [\tau_0, \omega) \rightarrow X$ be the maximal solution of (4.1) with $x(\tau_0) = x_0$. If $x(t) \in N$ for all $t \in [\tau_0, \omega)$, where N is a compact subset of O , then $\omega = +\infty$.*

Proof. Suppose the contrary, that is, $\omega < +\infty$. Then $K := N \times [\tau_0, \omega]$ is a compact subset of Ω and by the hypotheses, we get

$$(x(t), t) \in K, \quad \text{for all } t \in [\tau_0, \omega). \quad (4.29)$$

Since all the hypotheses from Theorem 4.10 are satisfied, there exists $t_K \in [\tau_0, \omega)$ such that

$$(x(t), t) \notin K, \quad \text{for all } t \in (t_K, \omega),$$

which contradicts (4.29). $\square$

Corollary 4.12. *Let $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous, and $\Omega = \Omega_F$. Let $(x_0, \tau_0) \in \Omega$ and $x : [\tau_0, \omega) \rightarrow X$ be the maximal solution of (4.1) with $x(\tau_0) = x_0$. If $\omega < +\infty$, then the following conditions hold*

(i) *The limit $\lim_{t \rightarrow \omega^-} x(t)$ exists;*

(ii) *$(y, \omega) \in \partial\Omega$ and $(x(t), t) \xrightarrow{t \rightarrow \omega^-} (y, \omega)$, where $y := \lim_{t \rightarrow \omega^-} x(t)$.*

Proof. Suppose $\omega < +\infty$. Let $\epsilon > 0$. Since $\omega \in (\tau_0, +\infty)$ and h is left-continuous on $(\tau_0, +\infty)$, the limit $\lim_{t \rightarrow \omega^-} h(t)$ exists. Thus, by the Cauchy condition, there exists $\delta > 0$ (we can take $\tau_0 < \omega - \delta$) such that

$$|h(t) - h(s)| < \epsilon, \quad \text{for all } t, s \in (\omega - \delta, \omega). \quad (4.30)$$

Then, by (4.30) and Lemma 1.16, the following inequality holds

$$\|x(t) - x(s)\| \leq |h(t) - h(s)| < \epsilon,$$

for every $t, s \in (\omega - \delta, \omega)$. Then again by the Cauchy condition, the limit $\lim_{t \rightarrow \omega^-} x(t)$ exists, that is, there exists $y \in X$ such that

$$y = \lim_{t \rightarrow \omega^-} x(t) \quad (4.31)$$

and it proves item (i).

Now, we will prove (ii). Note that

$$(x(t), t) \xrightarrow{t \rightarrow \omega^-} (y, \omega).$$

Since ω is an accumulation point of the set $[\tau_0, \omega)$, there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \subset [\tau_0, \omega)$ such that $t_n \xrightarrow{n \rightarrow \infty} \omega$. Thus, by (4.31), we have

$$x(t_n) \xrightarrow{n \rightarrow \infty} y.$$

Since $\{(x(t_n), t_n)\}_{n \in \mathbb{N}} \subset \Omega$ and $(x(t_n), t_n) \xrightarrow{n \rightarrow \infty} (y, \omega)$, we obtain

$$(y, \omega) \in \overline{\Omega}. \quad (4.32)$$

Let us prove that $(y, \omega) \notin \Omega$. Suppose the contrary, that is, $(y, \omega) \in \Omega = \Omega_F$. Theorem 1.21 yields that there exists $\Delta > 0$ such that, on $[\omega, \omega + \Delta]$, there is a unique solution $z : [\omega, \omega + \Delta] \rightarrow X$ of the generalized ODE (4.1) with $z(\omega) = y$.

Define $u : [t_0, \omega + \Delta] \rightarrow X$ by

$$u(t) := \begin{cases} x(t), & t \in [\tau_0, \omega) \\ z(t), & t \in [\omega, \omega + \Delta]. \end{cases}$$

Thus, by Theorem 4.1, u is a solution of the generalized ODE (4.1) with initial condition $u(\tau_0) = x(\tau_0) = x_0$, which is clearly a proper prolongation of the solution x . But x is assumed to be maximal, hence we have a contradiction. Therefore we conclude that $(y, \omega) \notin \Omega$, that is, $(y, \omega) \in \Omega^c$, which implies that

$$(y, \omega) \in \overline{\Omega^c}. \quad (4.33)$$

By (4.32) and (4.33), we conclude that

$$(y, \omega) \in \partial\Omega, \quad (4.34)$$

obtaining the desired result. $\square$

Even considering $O = X$, it is possible to ensure that the maximal solution is defined on $[\tau_0, +\infty)$, when $x(\tau_0) = x_0$. This is the content of the next result.

Corollary 4.13. *If $\Omega = X \times [t_0, +\infty)$ and $F \in \mathcal{F}(\Omega, h)$, where $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous, then for every $(x_0, \tau_0) \in \Omega$, there exists a unique maximal solution defined on $[\tau_0, +\infty)$ of the generalized ODE (4.1), with $x(\tau_0) = x_0$.*

Proof. At first, we need to ensure that $\Omega = \Omega_F$. Let $(z_0, s_0) \in \Omega$. Since h is a nondecreasing function, the limit $\lim_{s \rightarrow s_0^+} h(s)$ exists. Therefore, by the Cauchy condition, given $\epsilon > 0$, there exists $\delta > 0$ such that if $s, t \in (s_0, s_0 + \delta)$, then $|h(t) - h(s)| < \epsilon$. Then, since $F \in \mathcal{F}(\Omega, h)$, we have

$$\|F(z_0, t) - F(z_0, s)\| \leq |h(t) - h(s)| < \epsilon,$$

for all $t, s \in (s_0, s_0 + \delta)$, that is, the limit $\lim_{s \rightarrow s_0^+} F(z_0, s)$ exists and we denote it by $F(z_0, s_0^+)$.

Therefore

$$x_0 + F(z_0, s_0^+) - F(z_0, s_0) \in X,$$

that is, $(z_0, s_0) \in \Omega_F$.

Now, let $(x_0, \tau_0) \in \Omega$ and $x : [\tau_0, \omega) \rightarrow X$ be the maximal solution of the generalized ODE (4.1) with $x(\tau_0) = x_0$. Note that the existence of such a solution is guaranteed by Theorems 4.8 and 4.9.

Suppose, by contradiction, $\omega < +\infty$. Thus, by Corollary 4.12, the limit $\lim_{t \rightarrow \omega^-} x(t)$ exists. Let

$$y := \lim_{t \rightarrow \omega^-} x(t) \in X. \quad (4.35)$$

Since ω is an accumulation point of the set $[\tau_0, \omega)$, there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \subset [\tau_0, \omega)$ such that $t_n \xrightarrow{n \rightarrow \infty} \omega$. Therefore, by (4.35), we have

$$x(t_n) \xrightarrow{n \rightarrow \infty} y.$$

It is easy to check that $N := \{x(t_n)\}_{n \in \mathbb{N}} \cup \{y\}$ is a compact subset of X . Thus $N \times [\tau_0, \omega]$ is a compact subset of Ω . By Theorem 4.10, there exists $\hat{t} \in [\tau_0, \omega)$ such that $(x(t), t) \notin N \times [\tau_0, \omega]$, for all $t \in (\hat{t}, \omega)$, which contradicts the fact that $x(t_n) \in N$ for all $n \in \mathbb{N}$ and $t_n \xrightarrow{n \rightarrow \infty} \omega$. Therefore $\omega = +\infty$ and we obtain the desired result. $\square$

4.2 Prolongation of solutions of measure differential equations

In this section, our goal is to investigate the prolongation of solutions of measure differential equations.

Let $\mathbb{R}^n$ be the n -dimensional Euclidean space with norm $\|\cdot\|$, and $O \subset \mathbb{R}^n$ be an open set. Consider the integral form of a measure differential equation (MDE, for short) of type

$$x(t) = x(\tau_0) + \int_{\tau_0}^t f(x(s), s) dg(s), \quad t \geq \tau_0, \quad (4.36)$$

where $\tau_0 \geq t_0$, $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ are functions.

We start by presenting a definition of prolongation to the right of a solution $x : J \rightarrow \mathbb{R}^n$ of the MDE (4.36)

Definition 4.14. (Prolongation to the right) Let $\tau_0 \geq t_0$ and $x : J \rightarrow \mathbb{R}^n$, $J \subset [t_0, +\infty)$, be a solution of the MDE (4.36) on the interval J with left endpoint τ_0 . The solution $y : \hat{J} \rightarrow X$, $\hat{J} \subset [t_0, +\infty)$ with left endpoint τ_0 , of the MDE (4.36) is called a *prolongation to the right* of x , if $J \subset \hat{J}$ and $x(t) = y(t)$ for all $t \in J$. If $J \subsetneq \hat{J}$, then y is called a *proper prolongation* of x to the right.

In what follows, we present a definition of maximal solution of the MDE (4.36).

Definition 4.15. (Maximal solution) Let $(x_0, \tau_0) \in O \times [t_0, +\infty)$. The solution $y : I \rightarrow \mathbb{R}^n$, $I \subset [t_0, +\infty)$ and I with the left endpoint τ_0 , of the MDE

$$y(t) = x_0 + \int_{\tau_0}^t f(y(s), s) dg(s), \quad t \geq \tau_0,$$

is called *maximal*, if there is no proper prolongation of y to the right.

In what follows, we present a result which ensures the existence and uniqueness of a solution of the MDE (4.36).

Theorem 4.16. Suppose $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1). Also, assume that for all $(z_0, s_0) \in O \times [t_0, +\infty)$, we have $z_0 + f(z_0, s_0)\Delta^+g(s_0) \in O$. Then, for every $(x_0, \tau_0) \in O \times [t_0, +\infty)$, there exists a unique maximal solution $x : J \rightarrow \mathbb{R}^n$ of the MDE (4.36) with $x(\tau_0) = x_0$ and J (an interval) with left endpoint τ_0 .

Proof. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)$ (arbitrary, but fixed). Define a function $F : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by

$$F(x, t) = \int_{t_0}^t f(x, s) dg(s), \quad (x, t) \in O \times [t_0, +\infty). \quad (4.37)$$

Then, by Theorem 3.2 and Remark 3.3 there is a nondecreasing and left-continuous function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ such that $F \in \mathcal{F}(\Omega, h)$, where $\Omega = O \times [t_0, +\infty)$ and

$$h(t) = \int_{t_0}^t (L(s) + M(s)) dg(s).$$

On the other hand, for each $(z_0, s_0) \in \Omega$, we have

$$\begin{aligned} z_0 + F(z_0, s_0^+) - F(z_0, s_0) &= z_0 + \lim_{s \rightarrow s_0^+} \int_{t_0}^s f(z_0, \tau) dg(\tau) - \int_{t_0}^{s_0} f(z_0, \tau) dg(\tau) \\ &= z_0 + \lim_{s \rightarrow s_0^+} \int_{s_0}^s f(z_0, \tau) dg(\tau) \\ &= z_0 + f(z_0, s_0)\Delta^+g(s_0) \in O, \end{aligned} \quad (4.38)$$

by Theorem 1.8. Thus

$$z_0 + F(z_0, s_0^+) - F(z_0, s_0) \in O,$$

that is, $\Omega = \Omega_F$. Thus all the hypotheses of Theorem 4.8 are satisfied. Therefore, there exists a unique maximal solution $x : J \rightarrow X$ of the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t), \quad (4.39)$$

with $x(\tau_0) = x_0$, where F is given by (4.37) and J an interval with left endpoint τ_0 . Thus, by Theorem 3.8, $x : J \rightarrow \mathbb{R}^n$ is also a solution of the MDE (4.36) with $x(s_0) = x_0$. Let us prove that x (as a solution of (4.36)) does not admit proper extension to the right. Suppose the contrary, that is, there exists a solution $\hat{x} : \hat{J} \rightarrow \mathbb{R}^n$ of the MDE (4.36) with $\hat{x}(\tau_0) = x_0$, where $\hat{J}$ is an interval with left endpoint τ_0 , such that $\hat{x}$ extends x properly. Thus, by Theorem 3.8, $\hat{x}$ is a solution of the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t),$$

with $\hat{x}(\tau_0) = x_0$, where F is given by (4.37), which contradicts the maximality of x (as a solution of the generalized ODE (4.39)). Therefore x (as a solution of the MDE (4.36)) does not admit a proper extension to the right, that is, x is a maximal solution of the MDE (4.36) with $x(\tau_0) = x_0$. Again by Theorems 4.8 and 3.8, we have the uniqueness of the maximal solution x of the MDE (4.36), obtaining the desired result. $\square$

Remark 4.17. Notice that by the proof of Theorem 4.16, $x : J \rightarrow \mathbb{R}^n$ is the maximal solution of the MDE (4.36) with $x(\tau_0) = x_0$ if, and only if, $x : J \rightarrow \mathbb{R}^n$ is the maximal solution of the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t),$$

with $x(\tau_0) = x_0$, where F is given by (4.37).

Theorem 4.18. *Suppose $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1). Also, assume that for all $(z_0, s_0) \in O \times [t_0, +\infty)$, we have $z_0 + f(z_0, s_0)\Delta^+g(s_0) \in O$. Suppose $(x_0, \tau_0) \in O \times [t_0, +\infty)$ and $x : J \rightarrow \mathbb{R}^n$ is the maximal solution of (4.36) with $x(\tau_0) = x_0$ and J an interval with left endpoint τ_0 . Then $J = [\tau_0, \omega)$, with $\omega \leq +\infty$.*

Proof. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)$ and $x : J \rightarrow \mathbb{R}^n$ be the maximal solution of the MDE (4.36) with $x(\tau_0) = x_0$ and J be an interval with left endpoint τ_0 . Note that the existence of such a solution is guaranteed by Theorem 4.16.

On the other hand, define a function $F : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by

$$F(x, t) = \int_{t_0}^t f(x, s)dg(s), \quad (x, t) \in O \times [t_0, +\infty). \quad (4.40)$$

Then using the same arguments as in the proof of Theorem 4.16, we can prove that

$$\Omega = \Omega_F \quad \text{and} \quad F \in \mathcal{F}(\Omega, h),$$

where $\Omega = O \times [t_0, +\infty)$ and $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is a nondecreasing and left-continuous function. Also, by Remark 4.17, $x : J \rightarrow \mathbb{R}^n$ is the maximal solution of the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t),$$

with $x(\tau_0) = x_0$, where F is given by (4.40). Thus, all the hypotheses of Theorem 4.9 are satisfied. Then $J = [\tau_0, \omega)$, where $\omega \leq +\infty$. This completes the proof. $\square$

Theorem 4.19. *Suppose $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1). Also, assume that for all $(z_0, s_0) \in O \times [t_0, +\infty)$, we have $z_0 + f(z_0, s_0)\Delta^+g(s_0) \in O$. Suppose $(x_0, \tau_0) \in O \times [t_0, +\infty)$ and $x : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ is the maximal solution of the MDE (4.36) with $x(\tau_0) = x_0$. Then, for every compact set $K \subset O \times [t_0, +\infty)$, there exists $t_K \in [\tau_0, \omega)$ such that $(x(t), t) \notin K$, for all $t \in (t_K, \omega)$.*

Proof. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)$ and $x : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ be the maximal solution of the MDE (4.36) with $x(\tau_0) = x_0$. Also, assume that K is a compact subset of $B_c \times [t_0, +\infty)$.

On the other hand, define a function $F : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by

$$F(x, t) = \int_{t_0}^t f(x, s)dg(s), \quad (x, t) \in O \times [t_0, +\infty). \quad (4.41)$$

Similarly as in the proof of Theorem 4.16, we can prove that

$$\Omega = \Omega_F \quad \text{and} \quad F \in \mathcal{F}(\Omega, h),$$

where $\Omega = O \times [t_0, +\infty)$ and $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is a nondecreasing and left-continuous function. Also, by Remark 4.17, $x : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ is the maximal solution of generalized ODE

$$\frac{dx}{d\tau} = DF(x, t),$$

with $x(\tau_0) = x_0$, where F is given by (4.41). Hence all the hypotheses of Theorem 4.10 are satisfied. Then there exist $t_K \in [\tau_0, \omega)$ such that $(x(t), t) \notin K$, for all $t \in (t_K, \omega)$, obtaining the desired result. $\square$

Corollary 4.20. *Suppose $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1). Also, assume that for all $(z_0, s_0) \in O \times [t_0, +\infty)$, we have $z_0 + f(z_0, s_0)\Delta^+g(s_0) \in O$. Suppose $(x_0, \tau_0) \in O \times [t_0, +\infty)$ and $x :$*

$[\tau_0, \omega) \rightarrow \mathbb{R}^n$ is the maximal solution of the MDE (4.36) with $x(\tau_0) = x_0$. If $x(t) \in N$ for all $t \in [\tau_0, \omega)$, where N is a closed subset of O , then $\omega = +\infty$.

Proof. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)$ and $x : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ be the maximal solution of the MDE (4.36) with $x(\tau_0) = x_0$. Moreover, suppose $x(t) \in N$ for all $t \in [\tau_0, \omega)$, where N is a closed subset of O .

Define a function $F : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by

$$F(x, t) = \int_{t_0}^t f(x, s) dg(s), \quad (x, t) \in O \times [t_0, +\infty). \quad (4.42)$$

Then, using the same arguments as in the proof of Theorem 4.16, we can prove that

$$\Omega = \Omega_F \quad \text{and} \quad F \in \mathcal{F}(\Omega, h),$$

where $\Omega = O \times [t_0, +\infty)$ and $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is a nondecreasing and left-continuous function. Also, by Remark 4.17, $x : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ is the maximal solution of generalized ODE

$$\frac{dx}{d\tau} = DF(x, t),$$

with $x(\tau_0) = x_0$, where F is given by (4.42).

On the other hand, note that N is compact (since N is closed and bounded in $\mathbb{R}^n$). Thus, all the hypotheses of Corollary 4.11 are satisfied. Then $\omega = +\infty$ and the proof is complete. $\square$

Corollary 4.21. *Suppose $f : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1). Also, assume that for all $(z_0, s_0) \in O \times [t_0, +\infty)$, we have $z_0 + f(z_0, s_0)\Delta^+g(s_0) \in O$. Suppose $(x_0, \tau_0) \in O \times [t_0, +\infty)$ and $x : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ is the maximal solution of the MDE (4.36) with $x(\tau_0) = x_0$. If $\omega < +\infty$, then the following conditions hold*

(i) *The limit $\lim_{t \rightarrow \omega^-} x(t)$ exists;*

(ii) *$(y, \omega) \in \partial\Omega$ and $(x(t), t) \xrightarrow{t \rightarrow \omega^-} (y, \omega)$, where $y := \lim_{t \rightarrow \omega^-} x(t)$.*

Proof. Suppose $\omega < +\infty$. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)$ and $x : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ be the maximal solution of (4.36) with $x(\tau_0) = x_0$.

Define a function $F : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by

$$F(x, t) = \int_{t_0}^t f(x, s) dg(s), \quad (x, t) \in O \times [t_0, +\infty). \quad (4.43)$$

Proceeding as in the proof of Theorem 4.16, we can prove that

$$\Omega = \Omega_F \quad \text{and} \quad F \in \mathcal{F}(\Omega, h),$$

where $\Omega = O \times [t_0, +\infty)$ and $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is a nondecreasing and left-continuous function.

Also, by Remark 4.17 $x : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ is the maximal solution of generalized ODE

$$\frac{dx}{d\tau} = DF(x, t),$$

with $x(\tau_0) = x_0$, where F given by (4.43). Therefore, all the hypotheses of Corollary 4.12 are satisfied. Then the limit $y := \lim_{t \rightarrow \omega^-} x(t)$ exists, $(y, \omega) \in \partial\Omega$ and $(x(t), t) \xrightarrow{t \rightarrow \omega^-} (y, \omega)$. $\square$

Corollary 4.22. *Suppose $f : \mathbb{R}^n \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4) (with $O = \mathbb{R}^n$), and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1). Then for every $(x_0, \tau_0) \in \mathbb{R}^n \times [t_0, +\infty)$, the maximal solution of the MDE (4.36) with $x(\tau_0) = x_0$, is defined in $[\tau_0, +\infty)$.*

Proof. Let $(x_0, \tau_0) \in \mathbb{R}^n \times [t_0, +\infty)$. Define a function $F : \mathbb{R}^n \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by

$$F(x, t) = \int_{t_0}^t f(x, s) dg(s), \quad (x, t) \in \Omega. \quad (4.44)$$

By Theorem 3.2 and Remark 3.3, there exists a nondecreasing and left-continuous function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ such that $F \in \mathcal{F}(\Omega, h)$, where $\Omega = \mathbb{R}^n \times [t_0, +\infty)$. Thus all the hypotheses of Corollary 4.13 are satisfied. Then there exist a unique maximal solution $x : [\tau_0, +\infty) \rightarrow \mathbb{R}^n$ of the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t),$$

with $x(\tau_0) = x_0$, where F is given by (4.44). Now, the result follows from Remark 4.17. $\square$

4.3 Prolongation of solutions of dynamic equation on time scales

In this section, our goal is to prove results on prolongation of solutions of dynamic equations on time scales.

Let $\mathbb{T}$ be a time scale and consider the dynamic equation on time scales given by

$$x^\Delta(t) = f(x^*, t), \quad (4.45)$$

where $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is a function and $O \subset \mathbb{R}^n$ is a open set.

Definition 4.23. Let $x : I_{\mathbb{T}} \rightarrow \mathbb{R}^n$, $I_{\mathbb{T}} \subset [t_0, +\infty)_{\mathbb{T}}$, be a solution of (4.45) on the interval $I_{\mathbb{T}}$, with $\tau_0 = \min I_{\mathbb{T}}$. The solution $y : J_{\mathbb{T}} \rightarrow \mathbb{R}^n$, with $J_{\mathbb{T}} \subset [t_0, +\infty)_{\mathbb{T}}$ and $\tau_0 = \min J_{\mathbb{T}}$, of the dynamic equation on time scales (4.45) is called a *prolongation* of x to the right, if $I_{\mathbb{T}} \subset J_{\mathbb{T}}$ and $x(t) = y(t)$ for all $t \in I_{\mathbb{T}}$. If $I_{\mathbb{T}} \subsetneq J_{\mathbb{T}}$, then y is called a *proper prolongation* of x to the right.

Definition 4.24. (Maximal solution) Let $(x_0, \tau_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$. The solution $y : I_{\mathbb{T}} \rightarrow \mathbb{R}^n$, with $I_{\mathbb{T}} \subset [t_0, +\infty)_{\mathbb{T}}$ and $\tau_0 = \min I_{\mathbb{T}}$, of the dynamic equation on time scales

$$\begin{cases} y^{\Delta}(t) = f(y^*, t) \\ y(\tau_0) = x_0 \end{cases}$$

is called *maximal*, if there is no proper prolongation of y to the right.

The symbol $G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ denotes the set of all regulated functions $x : [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$. Also, the symbol $G_0([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ denotes the set of all functions $x \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ such that $\sup_{s \in [t_0, +\infty)_{\mathbb{T}}} e^{-(s-t_0)} |x(s)|$ is finite. This space is endowed with the norm

$$\|x\|_{[t_0, +\infty)_{\mathbb{T}}} = \sup_{s \in [t_0, +\infty)_{\mathbb{T}}} e^{-(s-t_0)} |x(s)|, \quad x \in G_0([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n),$$

and it becomes a Banach space.

We use the notation $x \in G([t_0, +\infty)_{\mathbb{T}}, O)$ for a function $x \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ such that $x(s) \in O$, for all $s \in [t_0, +\infty)_{\mathbb{T}}$. The notation $x \in G_0([t_0, +\infty)_{\mathbb{T}}, B_c)$ is defined in an similar way.

Remark 4.25. Note that if $x \in G_0([t_0, +\infty)_{\mathbb{T}}, O)$, then $y := x|_{[t_0, +\infty)_{\mathbb{T}}} \in G_0([t_0, +\infty)_{\mathbb{T}}, O)$ and $\|y\|_{[t_0, +\infty)_{\mathbb{T}}} \leq \|x\|_{[t_0, +\infty)_{\mathbb{T}}}$.

From now on, we assume the following conditions on the function $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$:

(B1) The Kurzweil-Henstock Δ -integral $\int_{t_1}^{t_2} f(y(t), t) \Delta t$ exists, for all $y \in G([t_0, +\infty)_{\mathbb{T}}, O)$ and all $t_1, t_2 \in [t_0, +\infty)_{\mathbb{T}}$.

(B2) There exists a Kurzweil-Henstock Δ -integrable function $M : [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ such that

$$\left| \int_{t_1}^{t_2} f(y(t), t) \Delta t \right| \leq \int_{t_1}^{t_2} M(t) \Delta t,$$

for all $y \in G([t_0, +\infty)_{\mathbb{T}}, O)$ and all $t_1, t_2 \in [t_0, +\infty)_{\mathbb{T}}$, $t_1 \leq t_2$.

(B3) There exists a Kurzweil-Henstock Δ -integrable function $L : [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ such that

$$\left| \int_{t_1}^{t_2} [f(y(t), t) - f(w(t), t)] \Delta t \right| \leq \|y - w\|_{[t_0, +\infty)_{\mathbb{T}}} \int_{t_1}^{t_2} L(t) \Delta t,$$

for all $y, w \in G_0([t_0, +\infty)_{\mathbb{T}}, O)$ and all $t_1, t_2 \in [t_0, +\infty)_{\mathbb{T}}, t_1 \leq t_2$.

The next result describes a relation between the conditions on the Kurzweil-Henstock Δ -integrals and those on Kurzweil-Henstock-Stieltjes integrals. Such result is essential to our purposes. Our proof is inspired in [21], Lemma 5.3.

Theorem 4.26. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$, and let $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be a function. Define $g(t) = t^*$ and $f^*(y, t) = f(y, t^*)$ for every $y \in O$ and $t \in [t_0, +\infty)$.*

1. *If $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies the condition (B1), then the integral $\int_{t_1}^{t_2} f^*(x(t), t) dg(t)$ exists, for all $x \in G([t_0, +\infty), O)$ and for all $t_1, t_2 \in [t_0, +\infty)$.*
2. *If $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies the condition (B2), then $f^* : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies the following condition*

$$\left| \int_{t_1}^{t_2} f^*(x(t), t) dg(t) \right| \leq \int_{t_1}^{t_2} M^*(t) dg(t),$$

for all $t_1, t_2 \in [t_0, +\infty)$, $t_1 \leq t_2$, and for all $x \in G([t_0, +\infty), O)$, where $g(t) = t^$.*

3. *If $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies the condition (B3), then $f^* : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies the condition*

$$\left| \int_{t_1}^{t_2} [f^*(x(t), t) - f^*(z(t), t)] dg(t) \right| \leq \|x - z\|_{[t_0, +\infty)} \int_{t_1}^{t_2} L^*(t) dg(t),$$

for all $t_1, t_2 \in [t_0, +\infty)$, $t_1 \leq t_2$, and for all $x, z \in G([t_0, +\infty), O)$, where $g(t) = t^$.*

Proof. Consider an arbitrary $x \in G([t_0, +\infty), O)$. Let $t_1, t_2 \in [t_0, +\infty)$. Then $t_1^*, t_2^* \in [t_0, +\infty)_{\mathbb{T}}$. According to Remark 4.25, $y := x|_{[t_0, +\infty)_{\mathbb{T}}} \in G([t_0, +\infty)_{\mathbb{T}}, O)$ and, therefore, by hypothesis (B1), the Kurzweil-Henstock Δ -integrals $\int_{s_1}^{s_2} f(y(t), t) \Delta t = \int_{s_1}^{s_2} f(x(t), t) \Delta t$ exists, for every $s_1, s_2 \in [t_0, +\infty)_{\mathbb{T}}$. In particular, $\int_{t_1^*}^{t_2^*} f(x(t), t) \Delta t$ exists. Then, by Theorems 3.13 and 3.14, we have

$$\int_{t_1^*}^{t_2^*} f(x(t), t) \Delta t \stackrel{\text{Theorem 3.13}}{=} \int_{t_1^*}^{t_2^*} f(x(t^*), t^*) dg(t) \stackrel{\text{Theorem 3.16}}{=} \int_{t_1^*}^{t_2^*} f(x(t), t^*) dg(t)$$

$$\stackrel{\text{Theorem 3.14}}{\Downarrow} \int_{t_1}^{t_2} f(x(t), t^*) dg(t) = \int_{t_1}^{t_2} f^*(x(t), t) dg(t),$$

that is, the last integral exists for every $t_1, t_2 \in [t_0, +\infty)$ as well. This proves 1.

The remaining two statements follow from Theorems 3.13 and 3.14. For 2., we have

$$\begin{aligned} \left| \int_{t_1}^{t_2} f^*(x(t), t) dg(t) \right| &\stackrel{\text{Theorem 3.16}}{\Downarrow} \left| \int_{t_1}^{t_2} f(x(t^*), t^*) dg(t) \right| \stackrel{\text{Theorem 3.14}}{\Downarrow} \left| \int_{t_1^*}^{t_2^*} f(x(t^*), t^*) g(t) \right| \\ &\stackrel{\text{Theorem 3.13}}{\Downarrow} \left| \int_{t_1^*}^{t_2^*} f(x(t), t) \Delta t \right| = \left| \int_{t_1^*}^{t_2^*} f(y(t), t) \Delta t \right| \leq \int_{t_1^*}^{t_2^*} M(t) \Delta t \\ &= \int_{t_1^*}^{t_2^*} M^*(t) dg(t) = \int_{t_1}^{t_2} M^*(t) dg(t), \end{aligned}$$

obtaining the desired on concerning the function f^* .

For 3., we have

$$\begin{aligned} \left| \int_{t_1}^{t_2} [f^*(x(t), t) - f^*(z(t), t)] dg(t) \right| &\stackrel{\text{Theorem 3.16}}{\Downarrow} \left| \int_{t_1}^{t_2} [f(x(t^*), t^*) - f(z(t^*), t^*)] dg(t) \right| \\ &\stackrel{\text{Theorem 3.14}}{\Downarrow} \left| \int_{t_1^*}^{t_2^*} [f(x(t^*), t^*) - f(z(t^*), t^*)] dg(t) \right| \stackrel{\text{Theorem 3.13}}{\Downarrow} \left| \int_{t_1^*}^{t_2^*} [f(x(t), t) - f(z(t), t)] \Delta t \right| \\ &= \left| \int_{t_1^*}^{t_2^*} [f(y(t), t) - f(w(t), t)] \Delta t \right| \leq \|y - w\|_{[t_0, +\infty)_{\mathbb{T}}} \int_{t_1^*}^{t_2^*} L(t) \Delta t \\ &= \|y - w\|_{[t_0, +\infty)_{\mathbb{T}}} \int_{t_1^*}^{t_2^*} L^*(t) dg(t) = \|y - w\|_{[t_0, +\infty)_{\mathbb{T}}} \int_{t_1}^{t_2} L^*(t) dg(t) \\ &\stackrel{\text{Remark 4.25}}{\Downarrow} \leq \|x - z\|_{[t_0, +\infty)} \int_{t_1}^{t_2} L^*(t) dg(t), \end{aligned}$$

for every $x, z \in G_0([t_0, +\infty), O)$, where $y := x|_{[t_0, +\infty)_{\mathbb{T}}}$ and $w := z|_{[t_0, +\infty)_{\mathbb{T}}}$. $\square$

The next result ensures the existence and uniqueness of a maximal solution of dynamic equations on time scales.

Theorem 4.27. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$. Suppose $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies conditions (B1), (B2) and (B3). Also, assume that for all $(z_0, s_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$, we have $z_0 + f(z_0, s_0)\mu(s_0) \in O$. Then, for all $(x_0, \tau_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$, there exists a unique maximal solution $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$, $\omega \leq +\infty$ of the dynamic equation on*

time scales given by

$$\begin{cases} x^\Delta(t) = f(x^*, t). \\ x(\tau_0) = x_0 \end{cases} \quad (4.46)$$

Also, if $\omega < +\infty$, we get $\omega \in \mathbb{T}$ and ω is left-dense.

Proof. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$ (arbitrary, but fixed).

Define $f^* : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by $f^*(x, t) = f(x, t^*)$, for all $(x, t) \in O \times [t_0, +\infty)$ and $g(t) = t^*$, for every $t \in [t_0, +\infty)$. Since f satisfies conditions (B1), (B2) and (B3), by Theorem 4.26, f^* satisfies conditions (A2), (A3) and (A4). Also, if $(z_0, s_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$, we have

$$\begin{aligned} z_0 + f^*(z_0, s_0)\Delta^+g(s_0) &= z_0 + f(z_0, s_0^*)(g(s_0^+) - g(s_0)) \\ &= z_0 + f(z_0, s_0^*)(\sigma(s_0^*) - s_0^*) \\ &= z_0 + f(z_0, s_0^*)\mu(s_0^*) \\ &= z_0 + f(z_0, s_0)\mu(s_0) \in O \end{aligned} \quad (4.47)$$

that is,

$$z_0 + f^*(z_0, s_0)\Delta^+g(s_0) \in O,$$

for all $(z_0, s_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$. On the other hand, if $(z_0, s_0) \in O \times [t_0, +\infty)$, but $s_0 \notin \mathbb{T}$, then

$$\begin{aligned} z_0 + f^*(z_0, s_0)\Delta^+g(s_0) &= z_0 + f(z_0, s_0^*)(g(s_0^+) - g(s_0)) \\ &= z_0 + f(z_0, s_0^*)(s_0^* - s_0^*) \\ &= z_0 \in O. \end{aligned} \quad (4.48)$$

It implies that for each $(z_0, s_0) \in O \times [t_0, +\infty)$, we obtain $z_0 + f^*(z_0, s_0)\Delta^+g(s_0) \in O$. Note that, by Lemma 3.12, g is a nondecreasing and left-continuous function on $(t_0, +\infty)$. Then f^* and g fulfill all the hypotheses of Theorems 4.16 and 4.18. Thus, there exists a unique maximal solution $y : [\tau_0, \omega) \rightarrow \mathbb{R}^n$, $\omega \leq +\infty$, of the MDE

$$y(t) = x_0 + \int_{\tau_0}^t f^*(y(s), s)dg(s), \quad (4.49)$$

where $g(s) = s^*$.

Let us consider two cases.

Case 1. Suppose $\omega = +\infty$.

By Theorem 3.17, $y : [\tau_0, +\infty) \rightarrow \mathbb{R}^n$ must have the form

$$y = x^*, \quad (4.50)$$

where $x : [\tau_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is a solution of the dynamic equation on time scales (4.46). Evidently, x is a maximal solution of the dynamic equation on time scales (4.46).

Case 2. Now, assume $\omega < +\infty$.

In order to prove that $\omega \in \mathbb{T}$ and ω is left-dense, we will prove two assertions.

Assertion 1. $\omega \in \mathbb{T}$.

Suppose the contrary, that is, $\omega \notin \mathbb{T}$. We define

$$B := \{s \in \mathbb{T} : s < \omega\}.$$

Note that B is nonempty, because $\tau_0 \in B$. Since $\omega \notin \mathbb{T}$, we have $B = \mathbb{T} \cap (-\infty, \omega]$ and, therefore, B is closed subset of $\mathbb{R}$. We denote $\beta := \sup B$. Since B is a closed, we have $\beta \in B$. Also, notice that $\beta \leq \omega$. But $\omega \notin \mathbb{T}$, hence $\beta < \omega$.

Note that g is constant on $(\beta, \omega]$ and, therefore, $\int_s^t f(y(s), s)dg(s) = 0$, for all $s, t \in (\beta, \omega]$.

Now, let $\sigma \in (\beta, \omega)$ be fixed, and define a function $u : [\tau_0, \omega] \rightarrow \mathbb{R}^n$ by

$$u(t) = \begin{cases} y(t), & t \in [\tau_0, \omega) \\ y(\sigma), & t = \omega. \end{cases} \quad (4.51)$$

Note that u is well-defined and $u|_{[\tau_0, \omega)} = y$. We will show that u is a solution of the MDE (4.49) on $[\tau_0, \omega]$. Indeed, let $s_1, s_2 \in [\tau_0, \omega]$ be such that $s_1 \in [\tau_0, \omega)$ and $s_2 = \omega$. Then

$$\begin{aligned} u(s_2) - u(s_1) &= y(\sigma) - y(s_1) \stackrel{(4.49)}{=} \int_{s_1}^{\sigma} f^*(y(s), s)dg(s) \\ &= \int_{s_1}^{\sigma} f^*(y(s), s)dg(s) + \underbrace{\int_{\sigma}^{\omega} f^*(y(s), s)dg(s)}_{=0} \\ &= \int_{s_1}^{\omega} f^*(y(s), s)dg(s) \stackrel{s_2=\omega}{=} \int_{s_1}^{s_2} f^*(y(s), s)dg(s) \\ &= \int_{s_1}^{s_2} f^*(u(s), s)dg(s), \end{aligned}$$

that is,

$$u(s_2) - u(s_1) = \int_{s_1}^{s_2} f^*(u(s), s)dg(s),$$

for all $s_1, s_2 \in [\tau_0, \omega]$ such that $s_1 \in [\tau_0, \omega)$ and $s_2 = \omega$.

The case $s_1, s_2 \in [\tau_0, \omega)$ follows immediately by the definition of u . Then u is a solution of the MDE (4.49) on $[\tau_0, \omega]$. Note that u is clearly a proper prolongation of $y : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ to the right, which contradicts the fact that y is the maximal solution of the MDE (4.49). Therefore, we conclude that $\omega \in \mathbb{T}$.

Assertion 2. ω is left-dense, that is, $\rho(\omega) = \omega$.

Suppose the contrary, that is, $\rho(\omega) = \sup \{s \in \mathbb{T} : s < \omega\} < \omega$. Thus, g is constant on $(\rho(\omega), \omega]$ and, therefore, using the same arguments as in the proof of Assertion 1 (with $\beta = \rho(\omega)$), we can prove that there exists a proper prolongation of $y : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ to the right, which contradicts the fact that y is the maximal solution of the MDE (4.49). Then, ω is left-dense.

On the other hand, by Theorem 3.17, $y : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ must have the form $y = x^*$, where $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is a solution of the dynamic equation on time scales (4.46). We assert that $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is a maximal solution of the dynamic equation on time scales (4.46). Suppose the contrary, that is, let $\phi : J_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be a proper prolongation of $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ to the right. Then, without loss of generality, we can consider $J_{\mathbb{T}} = [\tau_0, \omega]_{\mathbb{T}}$. Since $\phi : [\tau_0, \omega]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is a solution of the dynamic equation on time scales (4.46) (because ϕ is a proper prolongation of x), by Theorem 3.17, $\phi^* : [\tau_0, \omega] \rightarrow \mathbb{R}^n$ is solution of MDE (4.49). Also, note that $\phi^*|_{[\tau_0, \omega)} = y$. Thus, $\phi^* : [\tau_0, \omega] \rightarrow \mathbb{R}^n$ is a proper prolongation of $y : [\tau_0, \omega) \rightarrow \mathbb{R}^n$, which contradicts the fact that y is the maximal solution of the MDE (4.49). Thus, $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is a maximal solution of the dynamic equation on time scales (4.46).

Finally, we will show the uniqueness of the maximal solution x . Suppose $\psi : L_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is also a maximal solution of the dynamic equation on time scales (4.46).

Statement 1. $x(t) = \psi(t)$, for all $t \in [\tau_0, \omega)_{\mathbb{T}} \cap L_{\mathbb{T}}$.

Indeed, by Theorem 3.17, $\psi^* : L \rightarrow \mathbb{R}^n$ is a solution of the MDE (4.49). But $y : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ is the unique maximal solution of (4.49). Then $y(t) = \psi^*(t)$, for every $t \in [\tau_0, \omega) \cap L$. In particular, since

$$[\tau_0, \omega)_{\mathbb{T}} \cap L_{\mathbb{T}} = [\tau_0, \omega) \cap L \cap \mathbb{T} \subset [\tau_0, \omega) \cap L,$$

we have

$$y(t) = \psi^*(t), \quad \text{for all } t \in [\tau_0, \omega)_{\mathbb{T}} \cap L_{\mathbb{T}},$$

and, therefore, for $t \in [\tau_0, \omega)_{\mathbb{T}} \cap L_{\mathbb{T}}$, we get

$$x(t) = x(t^*) = x^*(t) \stackrel{(4.50)}{=} y(t) = \psi^*(t) = \psi(t^*) = \psi(t),$$

that is, $x(t) = \psi(t)$, for all $t \in [\tau_0, \omega)_{\mathbb{T}} \cap L_{\mathbb{T}}$. This ends the proof of Statement 1.

Now, define $\lambda : E_{\mathbb{T}} \rightarrow \mathbb{R}^n$, $E = [\tau_0, \omega) \cup L$, by

$$\lambda(t) = \begin{cases} x(t), & t \in [\tau_0, \omega)_{\mathbb{T}} \\ \psi(t), & t \in L_{\mathbb{T}}. \end{cases}$$

By Statement 1, λ is well-defined. Notice that λ is a solution of (4.46) and the time scales intervals $[\tau_0, \omega)_{\mathbb{T}}$ and $L_{\mathbb{T}}$ are contained in $E_{\mathbb{T}}$ and $\lambda|_{[\tau_0, \omega)_{\mathbb{T}}} = x$, $\lambda|_{L_{\mathbb{T}}} = \psi$. Since x and ψ are maximal solutions of (4.46), we have $E_{\mathbb{T}} = [\tau_0, \omega)_{\mathbb{T}} = L_{\mathbb{T}}$ and $\lambda(t) = x(t) = \psi(t)$, for all $t \in E_{\mathbb{T}}$, that is, $x(t) = \psi(t)$, for all $t \in E_{\mathbb{T}} = [\tau_0, \omega)_{\mathbb{T}} = L_{\mathbb{T}}$, proving the uniqueness of x . $\square$

Corollary 4.28. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$. Suppose $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies conditions (B1), (B2) and (B3). Also, assume that for all $(z_0, s_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$, we have $z_0 + f(z_0, s_0)\mu(s_0) \in O$. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$ and $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be the maximal solution of the dynamic equation on time scales (4.46) (which is ensured by Theorem 4.27). If each point of $\mathbb{T}$ is left-scattered, then $\omega = +\infty$.*

Proof. Suppose the contrary, that is, $\omega < +\infty$. Thus, by Theorem 4.27, $\omega \in \mathbb{T}$ and ω is left-dense, which contradicts the fact that each point of $\mathbb{T}$ is left-scattered. Then $\omega = +\infty$. $\square$

Remark 4.29. Notice that in the proof of Theorem 4.27, $y = x^* : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ is the maximal solution of the measure differential equation (4.49) if, and only if, $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is the maximal solution of the dynamic equation on time scales (4.46), where $(x_0, \tau_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$.

Theorem 4.30. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$. Suppose $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies conditions (B1), (B2) and (B3). Also, assume that for all $(z_0, s_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$, we have $z_0 + f(z_0, s_0)\mu(s_0) \in O$. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$ and $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be the maximal solution of the dynamic equation on time scales (4.46). Then, for every compact set $K \subset O \times [t_0, +\infty)_{\mathbb{T}}$, there exists $t_K \in [\tau_0, \omega)$ such that $(x(t), t) \notin K$, for $t \in (t_K, \omega) \cap \mathbb{T}$.*

Proof. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$ and $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be the maximal solution of the dynamic equation on time scales (4.46). Also, let $K \subset O \times [t_0, +\infty)_{\mathbb{T}}$ be a compact set. In particular, $K \subset O \times [t_0, +\infty)$.

On the other hand, by Remark 4.29, $x^* : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ is the maximal solution of the following integral form of the MDE

$$y(t) = x_0 + \int_{\tau_0}^t f^*(y(s), s)dg(s), \quad t \geq \tau_0,$$

where $f^* : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ is given by $f^*(x, t) = f(x, t^*)$, for all $(x, t) \in O \times [t_0, +\infty)$ and $g(t) = t^*$, for all $t \in [t_0, +\infty)$.

Since f satisfies conditions (B1), (B2) and (B3), by Theorem 4.26, f^* satisfies conditions (A2), (A3) and (A4). Also, by Lemma 3.12, g is a nondecreasing and left-continuous function on $(t_0, +\infty)$. Note that, using the same arguments as in the proof of Theorem 4.27, we can prove that

$$z_0 + f^*(z_0, s_0)\Delta^+g(s_0) \in O,$$

for all $(z_0, s_0) \in O \times [t_0, +\infty)$. Thus, f^* , g and x^* fulfill all the hypotheses of Theorem 4.19 and, therefore, there exists $t_K \in [\tau_0, \omega)$ such that $(x^*(t), t) \notin K$, for all $t \in (t_K, \omega)$. In particular, since $(t_K, \omega) \cap \mathbb{T} \subset (t_K, \omega)$, we get

$$(x(t), t) = (x(t^*), t) = (x^*(t), t) \notin K, \quad \text{for all } t \in (t_K, \omega) \cap \mathbb{T},$$

that is,

$$(x(t), t) \notin K, \quad \text{for all } t \in (t_K, \omega) \cap \mathbb{T}.$$

This completes the proof of the theorem. $\square$

Remark 4.31. Note that in the proof of the previous theorem, the set $(t_K, \omega) \cap \mathbb{T}$ is nonempty. Indeed, if $\omega < +\infty$, according to Theorem 4.27, $\omega \in \mathbb{T}$ and ω is left-dense and, therefore, $(t_K, \omega) \cap \mathbb{T} \neq \emptyset$. On the other hand, if $\omega = +\infty$, then $(t_K, +\infty) \cap \mathbb{T} \neq \emptyset$, because $\sup \mathbb{T} = +\infty$.

Theorem 4.32. Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$. Suppose $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies conditions (B1), (B2) and (B3). Also, assume that for all $(z_0, s_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$, we have $z_0 + f(z_0, s_0)\mu(s_0) \in O$. Suppose $(x_0, \tau_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$ and $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is the unique maximal solution of (4.45) with $x(\tau_0) = x_0$. If $x(t) \in N$ for all $t \in [\tau_0, \omega)_{\mathbb{T}}$, where N is a closed subset of O , then, $\omega = +\infty$.

Proof. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$ and $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be the unique maximal solution of (4.45) with $x(\tau_0) = x_0$. Note that the existence of such a solution is guaranteed by Theorem 4.27. Also, assume $x(t) \in N$ for any $t \in [\tau_0, \omega)_{\mathbb{T}}$, where N is a closed subset of O .

On the other hand, by Remark 4.29, the function $x^* : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ is the unique maximal solution of

$$y(t) = x_0 + \int_{\tau_0}^t f^*(y(s), s)dg(s), \quad (4.52)$$

where $f^* : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is given by $f^*(x, t) = f(x, t^*)$ for all $(x, t) \in O \times [t_0, +\infty)$ and $g(t) = t^*$ for all $t \in [t_0, +\infty)$.

Since $x(t) \in N$ for all $t \in [\tau_0, +\infty)_{\mathbb{T}}$, for each $s \in [\tau_0, \omega)$, we have

$$x^*(s) = x(s^*) \in N$$

because $s^* \in [\tau_0, \omega)_{\mathbb{T}}$. Thus

$$x^*(s) \in N, \quad \text{for all } s \in [\tau_0, \omega).$$

Now, N is compact (since N is closed and bounded in $\mathbb{R}^n$). Moreover, by Theorem 4.26, f^* satisfies conditions (A2), (A3) and (A4) because f satisfies conditions (B1), (B2) and (B3), and by Lemma 3.12, g is a nondecreasing and left-continuous function on $(\tau_0, +\infty)$. We can prove, by the same arguments as in the proof of Theorem 4.27, that $z_0 + f^*(z_0, \tau_0)\Delta^+g(\tau_0) \in O$, for all $(z_0, \tau_0) \in O \times [t_0, +\infty)$. Thus, $f^* : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$, $g : [t_0, +\infty) \rightarrow \mathbb{R}$ and $x^* : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ fulfill all the hypotheses of Corollary 4.20. Then, $\omega = +\infty$ and the proof is complete. $\square$

Theorem 4.33. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$. Assume that $f : O \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies conditions (B1), (B2) and (B3). Also, assume that for all $(z_0, s_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$, we have $z_0 + f(z_0, s_0)\mu(s_0) \in O$. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$ and $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be the maximal solution of the dynamic equation on time scales (4.46). If $\omega < +\infty$, then the following conditions hold*

(i) *The limit $\lim_{t \rightarrow \omega^-} x(t)$ exists;*

(ii) *$(y, \omega) \in \partial\Omega_{\mathbb{T}}$ and $(x(t), t) \xrightarrow{t \rightarrow \omega^-} (y, \omega)$, where $y := \lim_{t \rightarrow \omega^-} x(t)$ and $\Omega_{\mathbb{T}} = O \times [t_0, +\infty)_{\mathbb{T}}$.*

Proof. Let $(x_0, \tau_0) \in O \times [t_0, +\infty)_{\mathbb{T}}$ and $x : [\tau_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ be the maximal solution of the dynamic equation on time scales (4.46). Also, suppose $\omega < +\infty$.

Define a functions $f^* : O \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by $f^*(x, t) = f(x, t^*)$, for all $(x, t) \in O \times [t_0, +\infty)$ and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ by $g(t) = t^*$, for all $t \in [t_0, +\infty)$. Similarly as in the proof of Theorem 4.27, we can prove that f^* satisfies conditions (A2), (A3), (A4), and g satisfies condition (A1). Also, $z_0 + f^*(z_0, s_0)\Delta^+g(s_0) \in O$, for all $(z_0, s_0) \in O \times [t_0, +\infty)$.

On the other hand, by Remark 4.29, $x^* : [\tau_0, \omega) \rightarrow \mathbb{R}^n$ is the maximal solution of the integral form of a MDE of type

$$y(t) = x_0 + \int_{\tau_0}^t f^*(y(s), s)dg(s), \quad t \geq \tau_0.$$

Since all hypotheses from Theorem 4.21 are satisfied, the limit $\lim_{t \rightarrow \omega^-} x^*(t)$ exists and the following conditions hold

$$(y, \omega) \in \partial\Omega \quad \text{and} \quad (x(t), t) \xrightarrow{t \rightarrow \omega^-} (y, \omega), \quad (4.53)$$

where $y := \lim_{t \rightarrow \omega^-} x(t)$ and $\Omega = O \times [t_0, +\infty)$.

Note that by Theorem 4.27, $\omega \in \mathbb{T}$ and ω is left-dense.

In order to prove item (i), let $\{t_k\}_{k \in \mathbb{N}}$ be a sequence in $[\tau_0, \omega)_{\mathbb{T}}$ such that $t_k \xrightarrow{n \rightarrow +\infty} \omega$. Clearly $t_k = t_k^*$, for all $k \in \mathbb{N}$, and, therefore,

$$x(t_k) = x(t_k^*) = x^*(t_k) \xrightarrow{k \rightarrow +\infty} y,$$

that is, the limit $\lim_{t \rightarrow \omega^-} x(t)$ exists and

$$\lim_{t \rightarrow \omega^-} x(t) = y. \quad (4.54)$$

This proves (i).

Now, we will prove (ii). By (4.53), $(y, \omega) \in \overline{\Omega^c}$. But $\overline{\Omega^c} \subset \overline{\Omega_{\mathbb{T}}^c}$, because $\Omega_{\mathbb{T}} \subset \Omega$. Thus

$$(y, \omega) \in \overline{\Omega_{\mathbb{T}}^c}. \quad (4.55)$$

On the other hand, since ω is left-dense, there exists a sequence $\{s_k\}_{k \in \mathbb{N}}$ in $[\tau_0, \omega)_{\mathbb{T}}$ such that $s_k \xrightarrow{k \rightarrow +\infty} \omega$. Thus, by item (4.54), we have

$$x(s_k) \xrightarrow{k \rightarrow +\infty} y.$$

Since $\{(x(s_k), s_k)\}_{k \in \mathbb{N}} \in \Omega_{\mathbb{T}}$ and $(x(s_k), s_k) \xrightarrow{k \rightarrow +\infty} (y, \omega)$, we obtain

$$(y, \omega) \in \overline{\Omega_{\mathbb{T}}}. \quad (4.56)$$

Finally, item (ii) follows from (4.55) and (4.56), obtaining the desired result. $\square$

Theorem 4.34. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$. Suppose $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies conditions (B1), (B2) and (B3) (with $O = \mathbb{R}^n$). Then, for every $(x_0, \tau_0) \in \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}}$ there exists a unique maximal solution of (4.45) defined in $[\tau_0, +\infty)_{\mathbb{T}}$ such that $x(\tau_0) = x_0$.*

Proof. Let $(x_0, \tau_0) \in \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}}$ (arbitrary, but fixed). Then $(x_0, \tau_0) \in \mathbb{R}^n \times [t_0, +\infty)$.

Define $f^* : \mathbb{R}^n \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by $f^*(x, t) = f(x, t^*)$ for all $(x, t) \in \mathbb{R}^n \times [t_0, +\infty)$. Since f satisfies conditions (B1), (B2) and (B3) (with $O = \mathbb{R}^n$), by Theorem 4.26, f^* satisfies conditions (A2), (A3) and (A4) (with $O = \mathbb{R}^n$), and by Lemma 3.12, g is a nondecreasing and left-continuous function on $(t_0, +\infty)$. Hence f^* and g fulfill all the hypotheses of Corolary 4.22, Then there exists a unique maximal solution $y : [\tau_0, +\infty) \rightarrow \mathbb{R}^n$ of

$$y(t) = x_0 + \int_{\tau_0}^t f^*(y(s), s) dg(s).$$

By Theorem 3.17, y must have the form $y = x^* : [\tau_0, +\infty) \rightarrow \mathbb{R}^n$, where $x : [\tau_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is a solution of (4.45) with $x(\tau_0) = x_0$. The result follows from Remark 4.29. $\square$

Boundedness of solutions

In this chapter, we present results concerning the boundedness of the solutions of generalized ordinary differential equations (generalized ODEs), measure differential equations (MDEs) and dynamic equations on time scales.

It worths mentioning that all the results presented in Section 5.1 generalize the results found in [2], and all the results presented in Sections 5.2 and 5.3 of this chapter improve the results found in the literature (see, for instance, [4, 15, 36]) and the improvements are obtained by using the correspondence between the solutions of the mentioned equations (see Chapter 3). Such results are contained in [17].

5.1 Boundedness of solutions of generalized ODEs

In this section, our goal is to prove some results concerning the boundedness of the solutions of generalized ODEs using Lyapunov functionals.

Throughout this section, consider X a Banach space with norm $\|\cdot\|$ and $\Omega = X \times [t_0, +\infty)$ where $t_0 \geq 0$. Let $F : \Omega \rightarrow X$ be a function defined for every $(x, t) \in \Omega$ and taking values in a Banach space X . Also, suppose $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is left-continuous on $(t_0, +\infty)$. Under these conditions, consider the following generalized ODE

$$\frac{dz}{d\tau} = DF(z, t) \tag{5.1}$$

with the initial condition

$$z(s_0) = z_0, \tag{5.2}$$

where $(z_0, s_0) \in \Omega$.

From now on, we assume that for every $(z_0, s_0) \in \Omega$, there exists a unique (maximal) solution $x : [s_0, +\infty) \rightarrow X$ of the generalized ODE (5.1) with $x(s_0) = z_0$. The existence of such a solution is ensured by Theorem 4.13.

In what follows, for every $(z_0, s_0) \in \Omega$, we denote by $x(s, s_0, z_0)$ the unique maximal solution of the generalized ODE (5.1) with $x(s_0) = z_0$.

Next, we present the concepts of uniform boundedness for generalized ODEs. The basic references for this subject are [2] and [43].

Definition 5.1. We say that the generalized ODE (5.1) is

- **Uniformly bounded:** if for every $\alpha > 0$, there exists $M = M(\alpha) > 0$ such that, for all $s_0 \in [t_0, +\infty)$ and all $z_0 \in X$, with $\|z_0\| < \alpha$, we have

$$\|x(s, s_0, z_0)\| < M, \quad \text{for all } s \geq s_0,$$

where $x(s, s_0, z_0)$ is the maximal solution of (5.1) with $x(s_0) = z_0$.

- **Quasi-uniformly ultimately bounded:** if there exists $B > 0$ such that for all $\alpha > 0$, there exists $T = T(\alpha) > 0$, such that for all $s_0 \in [t_0, +\infty)$ and for all $z_0 \in X$, with $\|z_0\| < \alpha$, we have

$$\|x(s, s_0, z_0)\| < B, \quad \text{for all } s \geq s_0 + T,$$

where $x(s, s_0, z_0)$ is the maximal solution of (5.1) with $x(s_0) = z_0$.

- **Uniformly ultimately bounded:** if it is uniformly bounded and quasi-uniformly ultimately bounded.

The following result can be found in [38], Proposition 10.11. It will be essential to our purposes.

Proposition 5.2. *Suppose $-\infty < a < b < +\infty$ and $f, g : [a, b] \rightarrow \mathbb{R}$ are left-continuous functions on $(a, b]$. If for every $\sigma \in [a, b)$, there exists $\delta = \delta(\sigma) > 0$ such that for all $\eta \in (0, \delta)$, the following inequality holds*

$$f(\sigma + \eta) - f(\sigma) \leq g(\sigma + \eta) - g(\sigma),$$

then

$$f(s) - f(a) \leq g(s) - g(a),$$

for every $s \in [a, b]$.

The next auxiliary result will be crucial to prove our main results.

Lemma 5.3. *Let $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous. Also, suppose $V : [t_0, +\infty) \times X \rightarrow \mathbb{R}$ is such that for each left-continuous function $z : [\alpha, \beta] \rightarrow X$ on $(\alpha, \beta]$, the function $[\alpha, \beta] \ni t \mapsto V(t, z(t))$ is left-continuous on $(\alpha, \beta]$. Moreover, suppose V satisfies the following conditions*

(V1) *For all functions $x, y : [\alpha, \beta] \rightarrow X$, $[\alpha, \beta] \subset [t_0, +\infty)$, of bounded variation in $[\alpha, \beta]$, the following condition*

$$\begin{aligned} & |V(t, x(t)) - V(t, y(t)) - V(s, x(s)) + V(s, y(s))| \\ & \leq (h_1(t) - h_1(s)) \sup_{\xi \in [\alpha, \beta]} \|x(\xi) - y(\xi)\| \end{aligned}$$

holds for every $\alpha \leq s < t \leq \beta$, where $h_1 : [t_0, +\infty) \rightarrow \mathbb{R}$ is a nondecreasing and left-continuous function.

(V2) *There exists a function $\Phi : X \rightarrow \mathbb{R}$ such that for every solution $z : [s_0, +\infty) \rightarrow X$, $s_0 \geq t_0$, of (5.1), we have*

$$V(s, z(s)) - V(t, z(t)) \leq (s - t)\Phi(z(t)),$$

for every $s_0 \leq t < s < +\infty$.

If $\bar{x} : [\gamma, v] \rightarrow X$, $t_0 \leq \gamma < v < \infty$, is left-continuous on $(\gamma, v]$ and of bounded variation in $[\gamma, v]$, then

$$V(v, \bar{x}(v)) - V(\gamma, \bar{x}(\gamma)) \leq (h_1(v) - h_1(\gamma)) \sup_{s \in [\gamma, v]} \left\| \bar{x}(s) - \bar{x}(\gamma) - \int_{\gamma}^s DF(\bar{x}(\tau), t) \right\| + (v - \gamma)K,$$

where $K = \sup \{\Phi(\bar{x}(t)) : t \in [\gamma, v]\}$.

Proof. Let $\bar{x} : [\gamma, v] \rightarrow X$, $[\gamma, v] \subset [t_0, +\infty)$, be a left-continuous function on $(\gamma, v]$ and of bounded variation in $[\gamma, v] \subset [t_0, +\infty)$ and $K := \sup \{\Phi(\bar{x}(t)) : t \in [\gamma, v]\}$. If $K = +\infty$, then clearly the desired inequality is trivially satisfied. Therefore, the statement of the theorem holds.

Now, let us assume that $K < +\infty$. Note that Proposition 1.18 implies the existence of the integral $\int_{\gamma}^v DF(\bar{x}(\tau), t)$.

Take $\sigma \in [\gamma, v]$. Since $(\bar{x}(\sigma), \sigma) \in \Omega = X \times [t_0, +\infty)$, by Theorem 4.13, there exists a unique maximal solution $x : [\sigma, +\infty) \rightarrow X$ of the generalized ODE (5.1) on $[\sigma, +\infty)$, satisfying the initial condition $x(\sigma) = \bar{x}(\sigma)$.

Let $\eta_1 > 0$ be fixed. Then, $x|_{[\sigma, \sigma + \eta_1]}$ is also a solution of the generalized ODE (5.1). Thus, by Corollary 1.17 and Proposition 1.18, the integral $\int_{\sigma}^{\sigma + \eta_1} DF(x(\tau), t)$ exists.

Consider $\eta_2 > 0$ such that $\eta_2 \leq \eta_1$ and $\sigma + \eta_2 \leq v$. Then, the integral $\int_{\sigma}^{\sigma+\eta_2} DF(x(\tau), t)$ exists and the integral $\int_{\sigma}^{\sigma+\eta_2} D[F(\bar{x}(\tau), t) - F(x(\tau), t)]$ also exists by the property of integrability on subintervals of the Kurzweil integral. Therefore, given $\varepsilon > 0$, there exists a gauge δ of the interval $[\sigma, \sigma + \eta_2]$ corresponding to ε in the definition of the last integral. We can assume, without loss of generality, that $\eta_2 < \delta(\sigma)$. By hypothesis (V2), there exists a function $\Phi : X \rightarrow \mathbb{R}$ such that

$$V(s, x(s)) - V(t, x(t)) \leq (s - t)\Phi(x(t)),$$

for every $\sigma \leq t < s < +\infty$. In particular for every $0 < \eta < \eta_2$, we have

$$V(\sigma + \eta, x(\sigma + \eta)) - V(\sigma, x(\sigma)) \leq \eta\Phi(x(\sigma)), \quad (5.3)$$

where $s = \sigma + \eta$ and $t = \sigma$. By Corollary 1.4, we have

$$\left\| F(\bar{x}(\sigma), s) - F(\bar{x}(\sigma), \sigma) - \int_{\sigma}^s DF(\bar{x}(\tau), t) \right\| < \frac{\eta\varepsilon}{2(h_1(\sigma + \eta) - h_1(\sigma))} \quad (5.4)$$

and

$$\left\| F(x(\sigma), s) - F(x(\sigma), \sigma) - \int_{\sigma}^s DF(x(\tau), t) \right\| < \frac{\eta\varepsilon}{2(h_1(\sigma + \eta) - h_1(\sigma))}, \quad (5.5)$$

for every $s \in [\sigma, \sigma + \eta]$. Notice that

$$\begin{aligned} & \sup_{s \in [\sigma, \sigma + \eta]} \left\| \int_{\sigma}^s D[F(\bar{x}(\tau), t) - F(x(\tau), t)] \right\| \\ & - \sup_{s \in [\sigma, \sigma + \eta]} \|F(\bar{x}(\sigma), s) - F(\bar{x}(\sigma), \sigma) - F(x(\sigma), s) + F(x(\sigma), \sigma)\| \\ & \leq \sup_{s \in [\sigma, \sigma + \eta]} \left\| \int_{\sigma}^s D[F(\bar{x}(\tau), t) - F(x(\tau), t)] \right. \\ & \quad \left. - (F(\bar{x}(\sigma), s) - F(\bar{x}(\sigma), \sigma) - F(x(\sigma), s) + F(x(\sigma), \sigma)) \right\| \\ & \leq \sup_{s \in [\sigma, \sigma + \eta]} \left\| F(\bar{x}(\sigma), s) - F(\bar{x}(\sigma), \sigma) - \int_{\sigma}^s DF(\bar{x}(\tau), t) \right\| \\ & \quad + \sup_{s \in [\sigma, \sigma + \eta]} \left\| F(x(\sigma), s) - F(x(\sigma), \sigma) - \int_{\sigma}^s DF(x(\tau), t) \right\|. \end{aligned} \quad (5.6)$$

On the other hand,

$$\sup_{s \in [\sigma, \sigma + \eta]} \|F(\bar{x}(\sigma), s) - F(\bar{x}(\sigma), \sigma) - F(x(\sigma), s) + F(x(\sigma), \sigma)\|$$

$$\leq \|\bar{x}(\sigma) - x(\sigma)\| \sup_{s \in [\sigma, \sigma + \eta]} |h(s) - h(\sigma)| = 0, \quad (5.7)$$

where the first inequality follows from the fact that $F \in \mathcal{F}(\Omega, h)$ and the second equality follows since $x(\sigma) = \bar{x}(\sigma)$. Therefore, by (5.4), (5.5), (5.6) and (5.7), we have

$$\sup_{s \in [\sigma, \sigma + \eta]} \left\| \int_{\sigma}^s D[F(\bar{x}(\tau), t) - F(x(\tau), t)] \right\| \leq \frac{\eta \varepsilon}{(h_1(\sigma + \eta) - h_1(\sigma))}. \quad (5.8)$$

Since $F \in \mathcal{F}(\Omega, h)$ and the function h is nondecreasing, by Corollary 1.17, x is of bounded variation in $[\gamma, v]$ and, therefore, in $[\sigma, \sigma + \eta] \subset [\gamma, v]$. Thus, by hypothesis (V1) and by the relation $\bar{x}(\sigma) = x(\sigma)$, we get

$$\begin{aligned} & V(\sigma + \eta, \bar{x}(\sigma + \eta)) - V(\sigma + \eta, x(\sigma + \eta)) \\ &= V(\sigma + \eta, \bar{x}(\sigma + \eta)) - V(\sigma + \eta, x(\sigma + \eta)) - V(\sigma, \bar{x}(\sigma)) + V(\sigma, x(\sigma)) \\ &\leq |V(\sigma + \eta, \bar{x}(\sigma + \eta)) - V(\sigma + \eta, x(\sigma + \eta)) - V(\sigma, \bar{x}(\sigma)) + V(\sigma, x(\sigma))| \\ &\leq (h_1(\sigma + \eta) - h_1(\sigma)) \sup_{s \in [\sigma, \sigma + \eta]} \|\bar{x}(s) - x(s)\| \\ &= (h_1(\sigma + \eta) - h_1(\sigma)) \sup_{s \in [\sigma, \sigma + \eta]} \|\bar{x}(s) - \bar{x}(\sigma) + x(\sigma) - x(s)\| \\ &= (h_1(\sigma + \eta) - h_1(\sigma)) \sup_{s \in [\sigma, \sigma + \eta]} \|\bar{x}(s) - \bar{x}(\sigma) - \int_{\sigma}^s DF(x(\tau), t)\|, \end{aligned}$$

which implies

$$\begin{aligned} & V(\sigma + \eta, \bar{x}(\sigma + \eta)) - V(\sigma + \eta, x(\sigma + \eta)) \\ &\leq (h_1(\sigma + \eta) - h_1(\sigma)) \sup_{s \in [\sigma, \sigma + \eta]} \|\bar{x}(s) - \bar{x}(\sigma) - \int_{\sigma}^s DF(x(\tau), t)\|. \end{aligned} \quad (5.9)$$

Therefore, by (5.3) and (5.9), we obtain

$$\begin{aligned} & V(\sigma + \eta, \bar{x}(\sigma + \eta)) - V(\sigma, \bar{x}(\sigma)) \\ &= V(\sigma + \eta, \bar{x}(\sigma + \eta)) - V(\sigma + \eta, x(\sigma + \eta)) + V(\sigma + \eta, x(\sigma + \eta)) - V(\sigma, x(\sigma)) \\ &\leq (h_1(\sigma + \eta) - h_1(\sigma)) \sup_{s \in [\sigma, \sigma + \eta]} \left\| \bar{x}(s) - \bar{x}(\sigma) - \int_{\sigma}^s DF(x(\tau), t) \right\| + \eta \Phi(x(\sigma)) \\ &\leq (h_1(\sigma + \eta) - h_1(\sigma)) \sup_{s \in [\sigma, \sigma + \eta]} \left\| \bar{x}(s) - \bar{x}(\sigma) - \int_{\sigma}^s DF(x(\tau), t) \right\| + \eta K \\ &\leq (h_1(\sigma + \eta) - h_1(\sigma)) \sup_{s \in [\sigma, \sigma + \eta]} \left\| \bar{x}(s) - \bar{x}(\sigma) - \int_{\sigma}^s DF(\bar{x}(\tau), t) \right\| \end{aligned}$$

$$\begin{aligned}
& +(h_1(\sigma + \eta) - h_1(\sigma)) \sup_{s \in [\sigma, \sigma + \eta]} \left\| \int_{\sigma}^s D[F(\bar{x}(\tau), t) - F(x(\tau), t)] \right\| + \eta K \\
& \leq (h_1(\sigma + \eta) - h_1(\sigma)) \sup_{s \in [\sigma, \sigma + \eta]} \left\| \bar{x}(s) - \bar{x}(\sigma) - \int_{\sigma}^s DF(\bar{x}(\tau), t) \right\| + \eta \varepsilon + \eta K, \quad (5.10)
\end{aligned}$$

where the last inequality follows from (5.8). Given $s \in [\gamma, v]$, define

$$P(s) := \bar{x}(s) - \int_{\gamma}^s DF(\bar{x}(\tau), t).$$

Since $\bar{x}$ is a function of bounded variation in $[\gamma, v]$ and $(\bar{x}(s), s) \in \Omega$ for every $s \in [\gamma, v]$, then by Proposition 1.18, it follows that the Kurzweil integral $\int_{\gamma}^v DF(\bar{x}(\tau), t)$ exists and the function $s \mapsto \int_{\gamma}^s DF(\bar{x}(\tau), t)$ is of bounded variation in $[\gamma, v]$. Hence, for each $s \in [\gamma, v]$, the Kurzweil integral $\int_{\gamma}^s DF(\bar{x}(\tau), t)$ also, exists by the property of integrability on subintervals of the Kurzweil integral. Then the function P is well-defined and is of bounded variation in $[\gamma, v]$. Moreover, by Lemma 1.15, P is left-continuous on $(\gamma, v]$, since $\bar{x}$ and h are left-continuous on $(\gamma, v]$.

On the other hand, for $s \in [\gamma, v]$, we have

$$\begin{aligned}
P(s) - P(\sigma) &= \bar{x}(s) - \bar{x}(\sigma) - \int_{\gamma}^s DF(\bar{x}(\tau), t) + \int_{\gamma}^{\sigma} DF(\bar{x}(\tau), t) \\
&= \bar{x}(s) - \bar{x}(\sigma) - \int_{\sigma}^s DF(\bar{x}(\tau), t). \quad (5.11)
\end{aligned}$$

Now, we define the function $f : [\gamma, v] \rightarrow \mathbb{R}$ by

$$f(t) = \begin{cases} (h_1(t) - h_1(\sigma)) \sup_{s \in [\gamma, t]} \|P(s) - P(\sigma)\| + \varepsilon t + Kt, & t \in [\gamma, \sigma] \\ (h_1(t) - h_1(\sigma)) \sup_{s \in [\sigma, t]} \|P(s) - P(\sigma)\| + \varepsilon t + Kt, & t \in [\sigma, v]. \end{cases}$$

Clearly, f is well-defined. Moreover, by the left-continuity of the functions h_1 and P , f is left-continuous on $(\gamma, v]$. Also, since $\bar{x} : [\gamma, v] \rightarrow X$ is left-continuous, it follows from the hypotheses that the function $[\gamma, v] \ni t \mapsto V(t, \bar{x}(t))$ is left-continuous on $(\gamma, v]$.

On the other hand, by (5.10) and (5.11), we have

$$\begin{aligned}
V(\sigma + \eta, \bar{x}(\sigma + \eta)) - V(\sigma, \bar{x}(\sigma)) &\leq (h_1(\sigma + \eta) - h_1(\sigma)) \sup_{s \in [\sigma, \sigma + \eta]} \|P(s) - P(\sigma)\| + \eta \varepsilon + \eta K \\
&= f(\sigma + \eta) - f(\sigma).
\end{aligned}$$

Thus, the functions $[\gamma, v] \ni t \mapsto V(t, \bar{x}(t))$ and $[\gamma, v] \ni t \mapsto f(t)$ satisfy all the hypotheses of Proposition 5.2. Hence

$$\begin{aligned}
V(v, \bar{x}(v)) - V(\gamma, \bar{x}(\gamma)) &\leq f(v) - f(\gamma) \\
&= (h_1(v) - h_1(\sigma)) \sup_{s \in [\sigma, v]} \|P(s) - P(\sigma)\| + \varepsilon v + Kv \\
&\quad - (h_1(\gamma) - h_1(\sigma)) \sup_{s \in [\gamma, \gamma]} \|P(s) - P(\sigma)\| - \varepsilon \gamma - K\gamma \\
&= (h_1(v) - h_1(\sigma)) \sup_{s \in [\sigma, v]} \|P(s) - P(\sigma)\| + \varepsilon v + Kv \\
&\quad + (h_1(\sigma) - h_1(\gamma)) \sup_{s \in [\gamma, \gamma]} \|P(s) - P(\sigma)\| - \varepsilon \gamma - K\gamma \\
&\leq (h_1(v) - h_1(\sigma)) \sup_{s \in [\gamma, v]} \|P(s) - P(\sigma)\| + \varepsilon v + Kv \\
&\quad + (h_1(\sigma) - h_1(\gamma)) \sup_{s \in [\gamma, v]} \|P(s) - P(\sigma)\| - \varepsilon \gamma - K\gamma \\
&= (h_1(v) - h_1(\gamma)) \sup_{s \in [\gamma, v]} \|P(s) - P(\sigma)\| + \varepsilon(v - \gamma) + K(v - \gamma) \\
&= (h_1(v) - h_1(\gamma)) \sup_{s \in [\gamma, v]} \left\| \bar{x}(s) - \bar{x}(\sigma) - \int_{\sigma}^s DF(\bar{x}(\tau), t) \right\| \\
&\quad + K(v - \gamma) + \varepsilon(v - \gamma).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the result follows. $\square$

In the sequel, we present a result which ensures that the generalized ODE (5.1) is uniformly bounded. Our result generalizes the result found in [2].

Theorem 5.4. *Let $V : [t_0, +\infty) \times X \rightarrow \mathbb{R}$ be a function such that, for each left-continuous function $z : [\alpha, \beta] \rightarrow X$ on $(\alpha, \beta]$, the function $[\alpha, \beta] \ni t \mapsto V(t, z(t))$ is left-continuous on $(\alpha, \beta]$. Moreover, suppose V satisfies the following conditions*

- (i) *There are two monotone increasing functions $p, b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $p(0) = b(0) = 0$,*

$$\lim_{s \rightarrow +\infty} b(s) = +\infty \quad (5.12)$$

and

$$b(\|z\|) \leq V(t, z) \leq p(\|z\|), \quad (5.13)$$

for every pair $(t, z) \in [t_0, +\infty) \times X$.

- (ii) *For every solution of type $z : [s_0, +\infty) \rightarrow X$, $s_0 \geq t_0$, of the generalized ODE (5.1), we have*

$$V(s, z(s)) - V(t, z(t)) \leq 0,$$

for every $s_0 \leq t < s < +\infty$

Then the generalized ODE (5.1) is uniformly bounded.

Proof. Let $\alpha > 0$ be fixed. Since $p(\alpha) > 0$, by (5.12), there exists $M = M(\alpha) > 0$ such that

$$p(\alpha) < b(s), \text{ for all } s \geq M.$$

In particular, for $s = M$, we obtain

$$p(\alpha) < b(M). \quad (5.14)$$

Now, let $s_0 \in [t_0, +\infty)$, $z_0 \in X$ and $x(\cdot) = x(\cdot, s_0, z_0) : [s_0, +\infty) \rightarrow X$ be the solution of the generalized ODE (5.1) with initial condition $x(s_0) = z_0$, where $\|z_0\| < \alpha$. We will show that

$$\|x(t)\| < M, \text{ for all } s \geq s_0.$$

Indeed, by hypothesis (ii) and condition (5.13), for each $s \geq s_0$, we have

$$\begin{aligned} V(s, x(s)) &\leq V(s_0, x(s_0)) = V(s_0, z_0) \\ &\leq p(\|z_0\|) \\ &\leq p(\alpha) \\ &< b(M), \end{aligned} \quad (5.15)$$

that is,

$$V(s, x(s)) < b(M), \text{ for all } s \geq s_0. \quad (5.16)$$

Finally, we will show that $\|x(s, s_0, z_0)\| = \|x(s)\| < M$, for all $s \geq s_0$. Suppose the contrary, that is, suppose there exists $\bar{s} \in [s_0, +\infty)$ such that $\|x(\bar{s})\| \geq M$. Then, by hypothesis (5.13) and using the fact that b is an increasing function, we have

$$V(\bar{s}, x(\bar{s})) \geq b(\|x(\bar{s})\|) \geq b(M),$$

which contradicts (5.16). Therefore $\|x(s)\| < M$, for all $s \geq s_0$, and the result follows. $\square$

The next result establishes conditions which guarantee that the generalized ODE (5.1) is uniformly ultimately bounded. Our result generalizes a result found in [2].

Theorem 5.5. *Let $V : [t_0, \infty) \times X \rightarrow \mathbb{R}$ be a function such that for each left-continuous function $z : [\alpha, \beta] \rightarrow X$ on $(\alpha, \beta]$, the function $[\alpha, \beta] \ni t \mapsto V(t, z(t))$ is left-continuous on $(\alpha, \beta]$ and satisfies condition (i) from Theorem 5.4. Moreover, suppose V satisfies the following conditions*

(V1) For every $x, y : [\alpha, \beta] \rightarrow X$, $[\alpha, \beta] \subset [t_0, +\infty)$, of bounded variation in $[\alpha, \beta]$, we have

$$\begin{aligned} & |V(t, x(t)) - V(t, y(t)) - V(s, x(s)) + V(s, y(s))| \\ & \leq (h_1(t) - h_1(s)) \sup_{\xi \in [\alpha, \beta]} \|x(\xi) - y(\xi)\|, \end{aligned}$$

for every $\alpha \leq s < t \leq \beta$, where $h_1 : [t_0, +\infty) \rightarrow \mathbb{R}$ is a nondecreasing and left-continuous function.

(V2) There exists a continuous function $\Phi : X \rightarrow \mathbb{R}$, with $\Phi(0) = 0$ and $\Phi(x) > 0$, $x \neq 0$, such that for every solution $z : [s_0, +\infty) \rightarrow X$, $s_0 \geq t_0$, of (5.1), we have

$$V(s, z(s)) - V(t, z(t)) \leq (s - t) \left(-\Phi(z(t)) \right),$$

for every $s_0 \leq t < s < +\infty$.

Then the generalized ODE (5.1) is uniformly ultimately bounded.

Proof. At first, notice that, by hypothesis (V2),

$$V(s, z(s)) - V(t, z(t)) \leq (s - t) \left(-\Phi(z(t)) \right) \leq 0,$$

for every solution $z : [s_0, +\infty) \rightarrow X$, $s_0 \in [t_0, +\infty)$, of the generalized ODE (5.1), with $s_0 \leq t < s < +\infty$. Hence, all the hypotheses of the Theorem 5.4 are satisfied and, consequently, the generalized ODE (5.1) is uniformly bounded. It remains to show that equation (5.1) is quasi-uniformly ultimately bounded.

By the uniform boundedness of the generalized ODE (5.1), there exists $B = B(t_0 + 1) > 0$ such that, for every $\bar{t} \in [t_0, +\infty)$ and for every $\bar{x} \in X$ with $\|\bar{x}\| < t_0 + 1$, we have

$$\|x(s, \bar{t}, \bar{x})\| < B, \quad \text{for all } s \geq \bar{t}, \quad (5.17)$$

where $x(s, \bar{t}, \bar{x})$ is the maximal solution of the generalized ODE (5.1) with $x(\bar{t}) = x(\bar{t}, \bar{t}, \bar{x}) = \bar{x}$. Without loss of generality, we can take $B \in (t_0 + 1, +\infty)$ (because otherwise we can take $B > B'$ such that $B' \in (t_0 + 1, +\infty)$) and we have

$$\|x(s, \bar{t}, \bar{x})\| < B, \quad \text{for all } s \geq \bar{t}.$$

Let $\alpha > 0$, $s_0 \in [t_0, +\infty)$, $z_0 \in X$ and $x(\cdot) = x(\cdot, s_0, z_0) : [s_0, +\infty) \rightarrow X$ be the solution of the generalized ODE (5.1) with initial condition (5.2), where $\|z_0\| < \alpha$. Since (5.1) is uniformly bounded, there exists a positive number $M_1 = M_1(\alpha)$ (we can take $M_1 >$

$\max\{\alpha, t_0 + 1\}$) such that

$$\|x(s, s_0, z_0)\| < M_1, \quad \text{for all } s \geq s_0.$$

On the other hand, using the same argument as in (5.14) from the proof of Theorem 5.4, there exists $M_2 = M_2(\alpha) > 0$ such that $p(\alpha) < b(M_2)$.

Now, let $M = M(\alpha) := \max\{M_1(\alpha), M_2(\alpha)\}$. Notice that

$$\|x(s, s_0, z_0)\| < M, \quad \text{for all } s \geq s_0. \quad (5.18)$$

and

$$p(\alpha) < b(M). \quad (5.19)$$

Define

$$N := \sup\{-\Phi(z) : t_0 + 1 \leq \|z\| < M\} < 0$$

and

$$T(\alpha) := -\frac{2b(M)}{N} > 0.$$

We want to show that $\|x(s, s_0, z_0)\| < B$, for all $s \geq s_0 + T(\alpha)$. Suppose the contrary, that is, there exists $\bar{s} > s_0 + T(\alpha)$ such that

$$\|x(\bar{s}, s_0, z_0)\| \geq B > t_0 + 1. \quad (5.20)$$

Assertion 1. The following inequality holds

$$\|x(s, s_0, z_0)\| \geq t_0 + 1, \quad \text{for all } s \in [s_0, \bar{s}].$$

Suppose the assertion is false, that is, there exists $\bar{t} \in [s_0, \bar{s}]$ such that

$$\|x(\bar{t}, s_0, z_0)\| < t_0 + 1.$$

On the other hand, by (5.17) (with $\bar{x} = x(\bar{t}, s_0, z_0)$) that

$$\|x(s, \bar{t}, \bar{x})\| < B, \quad \text{for all } s \geq \bar{t}. \quad (5.21)$$

Also, we know that $x(s, \bar{t}, \bar{x})$, $s \in [\bar{t}, +\infty)$, is the unique solution of the initial value problem

$$\begin{cases} \frac{dz}{d\tau} = DF(z, t) \\ z(\bar{t}) = \bar{x} = x(\bar{t}, s_0, z_0). \end{cases} \quad (5.22)$$

But $x(\cdot, s_0, z_0)|_{[\bar{t}, +\infty)}$ is also a solution of the generalized ODE (5.22). Therefore, we have

$$x(s, s_0, z_0) = x(s, \bar{t}, \bar{x}), \quad \text{for all } s \in [\bar{t}, +\infty).$$

In particular, since $\bar{s} \in [\bar{t}, +\infty)$, we obtain

$$x(\bar{s}, s_0, z_0) = x(\bar{s}, \bar{t}, \bar{x}). \quad (5.23)$$

Thus, (5.21) and (5.23) imply that $\|x(\bar{s}, s_0, z_0)\| < B$, which contradicts (5.20). Thus, Assertion 1 follows.

By Assertion 1, we have

$$\|x(s, s_0, z_0)\| \geq t_0 + 1, \quad \text{for all } s \in \left[s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)\right]. \quad (5.24)$$

since $[s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)] \subset [s_0, \bar{s}]$.

However, since $x(\cdot) := x(\cdot, s_0, z_0)|_{[s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)]}$ is a solution of the generalized ODE (5.1) and $F \in \mathcal{F}(\Omega, h)$, where the function h is left-continuous and nondecreasing, then the function $x(\cdot, s_0, z_0)|_{[s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)]}$ is left-continuous on $(s_0 + T(\alpha)/2, s_0 + T(\alpha)]$ and of bounded variation in $I_\alpha := [s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)]$ by Lemma 1.16 and Corollary 1.17. Thus, by Lemma 5.3, it follows that

$$\begin{aligned} V(s_0 + T(\alpha), x(s_0 + T(\alpha))) &\leq V\left(s_0 + \frac{T(\alpha)}{2}, x\left(s_0 + \frac{T(\alpha)}{2}\right)\right) + \\ &+ \left(h_1(s_0 + T(\alpha)) - h_1\left(s_0 + \frac{T(\alpha)}{2}\right)\right) \sup_{s \in I_\alpha} \underbrace{\left\|x(s) - x\left(s_0 + \frac{T(\alpha)}{2}\right) - \int_{s_0 + \frac{T(\alpha)}{2}}^s DF(x(\tau, s_0, z_0), t)\right\|}_{\text{zero}} + \\ &+ \frac{T(\alpha)}{2} \cdot \sup\{-\Phi(x(s)) : s \in [s_0 + T(\alpha)/2, s_0 + T(\alpha)]\} \end{aligned}$$

which implies

$$\begin{aligned}
V(s_0 + T(\alpha), x(s_0 + T(\alpha))) &\leq V\left(s_0 + \frac{T(\alpha)}{2}, x\left(s_0 + \frac{T(\alpha)}{2}\right)\right) \\
&\quad + \frac{T(\alpha)}{2} \cdot \sup\left\{-\Phi(x(s)) : s \in \left[s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)\right]\right\} \\
&\leq V\left(s_0 + \frac{T(\alpha)}{2}, x\left(s_0 + \frac{T(\alpha)}{2}\right)\right) \\
&\quad + \frac{T(\alpha)}{2} \cdot \sup\{-\Phi(z) : t_0 + 1 \leq \|z\| < M\}, \tag{5.25}
\end{aligned}$$

where the last inequality follows from the relation

$$t_0 + 1 \leq \|x(s)\| = \|x(s, s_0, z_0)\| < M$$

for all $s \in [s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)]$. Also, by (5.19) and using the same argument as in (5.15) from the proof of Theorem 5.4, we get

$$V\left(s_0 + \frac{T(\alpha)}{2}, x\left(s_0 + \frac{T(\alpha)}{2}\right)\right) < b(M).$$

Therefore, by (5.25), we have

$$\begin{aligned}
V(s_0 + T(\alpha), x(s_0 + T(\alpha))) &< b(M) + \frac{T(\alpha)}{2} \cdot \sup\{-\Phi(z) : t_0 + 1 \leq \|z\| < M\} \\
&= b(M) + \frac{T(\alpha)}{2} \cdot N \\
&= b(M) - \frac{2b(M)}{2N} \cdot N \\
&= 0,
\end{aligned}$$

which implies that

$$V(s_0 + T(\alpha), x(s_0 + T(\alpha))) < 0. \tag{5.26}$$

On the other hand, by condition (5.13) of the present theorem and by (5.24), we have

$$V(s_0 + T(\alpha), x(t_0 + T(\alpha))) \geq b(\|x(s_0 + T(\alpha))\|) \geq b(t_0 + 1) > 0,$$

which contradicts (5.26). Therefore $\|x(s, s_0, z_0)\| < B$, for all $s \geq s_0 + T(\alpha)$. The generalized ODE (5.1) is quasi-uniformly ultimately bounded and the proof is complete. $\square$

5.2 Boundedness of solutions of measure differential equations

In this section, our goal is to present results concerning boundedness of solutions of measure differential equations.

Let $\mathbb{R}^n$ be the n -dimensional Euclidean space with norm $\|\cdot\|$.

Consider the integral form of a measure differential equation of type

$$x(t) = x(\tau_0) + \int_{\tau_0}^t f(x(s), s) dg(s), \quad t \geq \tau_0, \quad (5.27)$$

where $\tau_0 \geq t_0$, $f : \mathbb{R}^n \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ and $g : [t_0, +\infty) \rightarrow \mathbb{R}$.

From now on, we assume that for every $(z_0, s_0) \in \mathbb{R}^n \times [t_0, +\infty)$, there exists a unique (maximal) solution $x : [s_0, +\infty) \rightarrow \mathbb{R}^n$ of the measure differential equation (5.27) with $x(s_0) = z_0$. The Corollary 4.13, ensures the existence and uniqueness of such a solution. Also, assume f satisfies conditions (A2), (A3) and (A4), and g satisfies condition (A1) defined in Chapter 3.

In what follows, for every $(z_0, s_0) \in \mathbb{R}^n \times [t_0, +\infty)$, we denote by $x(s, s_0, z_0)$ the unique maximal solution of the measure differential equation (5.27) with $x(s_0) = z_0$.

Now, we present some concepts related to uniform boundedness for measure differential equations.

Definition 5.6. We say that the measure differential equation (5.27) is

- **Uniformly bounded:** if for every $\alpha > 0$, there exists $M = M(\alpha) > 0$ such that, for all $s_0 \in [t_0, +\infty)$ and all $z_0 \in \mathbb{R}^n$, where $\|z_0\| < \alpha$, we have

$$\|x(s, s_0, z_0)\| < M, \quad \text{for all } s \geq s_0.$$

- **Quasi-uniformly ultimately bounded:** if there exists $B > 0$ such that for every $\alpha > 0$, there exists $T = T(\alpha) > 0$, such that for all $s_0 \in [t_0, +\infty)$ and all $z_0 \in \mathbb{R}^n$, where $\|z_0\| < \alpha$, we have

$$\|x(s, s_0, z_0)\| < B, \quad \text{for all } s \geq s_0 + T.$$

- **Uniformly ultimately bounded:** if it is uniformly bounded and quasi-uniformly ultimately bounded.

The next result ensures that the measure differential equation (5.27) is uniformly bounded.

Theorem 5.7. Assume $f : \mathbb{R}^n \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4) and the function $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1). Let $U : [t_0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a function such that for each left-continuous function $z : [\alpha, \beta] \rightarrow \mathbb{R}^n$ on $(\alpha, \beta]$, the function $[\alpha, \beta] \ni t \mapsto U(t, z(t))$ is left-continuous on $(\alpha, \beta]$. Moreover, suppose U satisfies the following conditions

(i) There are two monotone increasing functions $p, b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $p(0) = b(0) = 0$,

$$\lim_{s \rightarrow +\infty} b(s) = +\infty$$

and

$$b(\|z\|) \leq U(t, z) \leq p(\|z\|)$$

for every pair $(t, z) \in [t_0, +\infty) \times \mathbb{R}^n$.

(ii) For every solution $z : [s_0, +\infty) \rightarrow \mathbb{R}^n$, $s_0 \geq t_0$, of the measure differential equation (5.27), we have

$$U(s, z(s)) - U(t, z(t)) \leq 0,$$

for every $s_0 \leq t < s < +\infty$.

Then the measure differential equation (5.27) is uniformly bounded.

Proof. Define a function $F : \mathbb{R}^n \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by

$$F(x, t) = \int_{t_0}^t f(x, s) dg(s) \quad (5.28)$$

for all $(x, t) \in \mathbb{R}^n \times [t_0, +\infty)$. Since f satisfies conditions (A2), (A3) and (A4) and g satisfies condition (A1), by Theorem 3.2, $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is given by

$$h(t) = \int_{t_0}^t (M(s) + L(s)) dg(s).$$

By Remark 3.3, $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is left-continuous on $(t_0, +\infty)$. Also, by Theorem 3.8 and the hypotheses of the present theorem, it is not difficult to see that $U : [t_0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies all the hypotheses from Theorem 5.4. Hence the generalized ODE (5.1) is uniformly bounded, where F is given by (5.28).

Again, by Theorem 3.8, it follows that the measure differential equation (5.27) is also uniformly bounded, obtaining the desired result. $\square$

Finally, we present the last result of this section. Such result ensures that the measure differential equation (5.27) is uniformly ultimately bounded.

Theorem 5.8. *Assume $f : \mathbb{R}^n \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4) and the function $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1). Let $U : [t_0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a function such that for each left-continuous function $z : [\alpha, \beta] \rightarrow \mathbb{R}^n$ on $(\alpha, \beta]$, the function $[\alpha, \beta] \ni t \mapsto U(t, z(t))$ is left-continuous on $(\alpha, \beta]$ and satisfies condition (i) from Theorem 5.7. Moreover, suppose U satisfies the following conditions*

(U1) *For every $x, y : [\alpha, \beta] \rightarrow \mathbb{R}^n$, $[\alpha, \beta] \subset [t_0, +\infty)$, of bounded variation in $[\alpha, \beta]$, we have*

$$|U(t, x(t)) - U(t, y(t)) - U(s, x(s)) + U(s, y(s))| \leq \left(\int_s^t K(\tau) du(\tau) \right) \sup_{\xi \in [\alpha, \beta]} \|x(\xi) - y(\xi)\|,$$

for every $\alpha \leq s < t \leq \beta$, where $u : [t_0, +\infty) \rightarrow \mathbb{R}$ is a nondecreasing and left-continuous function and $K : [t_0, +\infty) \rightarrow \mathbb{R}$ is a locally Kurzweil-Henstock-Stieltjes integrable function with respect to u .

(U2) *There exists a continuous function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, with $\phi(0) = 0$ and $\phi(x) > 0$, $x \neq 0$, such that for every solution $z : [s_0, +\infty) \rightarrow \mathbb{R}^n$, $s_0 \geq t_0$, of the measure differential equation (5.27), we have*

$$U(s, z(s)) - U(t, z(t)) \leq (s - t) \left(-\phi(z(t)) \right),$$

for every $s_0 \leq t < s < +\infty$.

Then, the measure differential equation (5.27) is uniformly ultimately bounded.

Proof. Define a function $F : \mathbb{R}^n \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by

$$F(x, t) = \int_{t_0}^t f(x, s) dg(s), \quad (5.29)$$

for all $(x, t) \in \mathbb{R}^n \times [t_0, +\infty)$. Since f satisfies conditions (A2), (A3) and (A4) and g satisfies condition (A1), by Theorem 3.2, $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is given by

$$h(t) = \int_{t_0}^t (M(s) + L(s)) dg(s).$$

Notice that the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is left-continuous by Remark 3.3.

By Theorem 3.8, U satisfies hypothesis (i) from Theorem 5.4. Also, defining the function $h_1 : [t_0, +\infty) \rightarrow \mathbb{R}$ by

$$h_1(t) = \int_{t_0}^t K(\tau) du(\tau),$$

for every $t \in [t_0, +\infty)$, it follows that h_1 is a nondecreasing and left-continuous function. Moreover, by (U1), U satisfies the following condition

$$|U(t, x(t)) - U(t, y(t)) - U(s, x(s)) + U(s, y(s))| \leq (h_1(t) - h_1(s)) \sup_{\xi \in [\alpha, \beta]} \|x(\xi) - y(\xi)\|,$$

for every $\alpha \leq s < t \leq \beta$, and for every $x, y : [\alpha, \beta] \rightarrow \mathbb{R}^n$, $[\alpha, \beta] \subset [t_0, +\infty)$, of bounded variation in $[\alpha, \beta]$.

Also, by Theorem 3.8 and by hypothesis (U2), it is clear that U satisfies the hypothesis (V2) from Theorem 5.5. Therefore, all the hypotheses from Theorem 5.5 are fulfilled and the generalized ODE (5.1) is uniformly ultimately bounded. By Theorem 3.8, the measure differential equation (5.27) is also uniformly ultimately bounded, obtaining the desired result. $\square$

5.3 Boundedness of solutions of dynamic equations on time scales

In this section, our goal is to prove the results concerning boundedness of solutions of dynamic equations on time scales.

Consider the dynamic equation on time scales given by

$$x^\Delta(t) = f(x^*, t), \quad (5.30)$$

where $f : \mathbb{R}^n \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies the following conditions

(B1) The Kurzweil-Henstock Δ -integral $\int_{t_1}^{t_2} f(y(t), t) \Delta t$ exists, for all $y \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ and all $t_1, t_2 \in [t_0, +\infty)_{\mathbb{T}}$.

(B2) There exists a locally Kurzweil-Henstock Δ -integrable function $M : [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ such that

$$\left| \int_{t_1}^{t_2} f(y(t), t) \Delta t \right| \leq \int_{t_1}^{t_2} M(t) \Delta t,$$

for all $y \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ and all $t_1, t_2 \in [t_0, +\infty)_{\mathbb{T}}$, $t_1 \leq t_2$.

(B3) There exists a locally Kurzweil-Henstock Δ -integrable function $L : [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ such that

$$\left| \int_{t_1}^{t_2} [f(y(t), t) - f(w(t), t)] \Delta t \right| \leq \|y - w\|_{[t_0, +\infty)_{\mathbb{T}}} \int_{t_1}^{t_2} L(t) \Delta t,$$

for all $y, w \in G_0([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ and all $t_1, t_2 \in [t_0, +\infty)_{\mathbb{T}}$, $t_1 \leq t_2$.

Moreover, assume that for every $(z_0, s_0) \in \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}}$, there exists a unique (maximal) solution $x : [s_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ of the dynamic equation on time scales (5.30) with $x(s_0) = z_0$. The existence of such a solution is ensured by Theorem 4.34.

In what follows, for every $(z_0, s_0) \in \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}}$, we denote by $x(s, s_0, z_0)$ the unique (maximal) solution of the dynamic equation on time scales (5.30) with $x(s_0) = z_0$ defined on $[s_0, +\infty)_{\mathbb{T}}$.

In the sequel, we introduce the concepts concerning the boundedness of the solutions of the dynamic equation on time scales (5.30).

Definition 5.9. Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$. We say that the dynamic equation on time scales (5.30) is

• **Uniformly bounded:** if for every $\alpha > 0$, there exists $M = M(\alpha) > 0$ such that, for all $s_0 \in [t_0, +\infty)_{\mathbb{T}}$ and all $z_0 \in \mathbb{R}^n$, where $\|z_0\| < \alpha$, we have

$$\|x(s, s_0, z_0)\| < M, \quad \text{for all } s \in [s_0, +\infty)_{\mathbb{T}},$$

where x is the maximal solution of (5.30) with $x(s_0) = z_0$.

• **Quasi-uniformly ultimately bounded:** if there exists a $B > 0$ such that for every $\alpha > 0$, there exists a $T = T(\alpha) > 0$, such that for all $s_0 \in [t_0, +\infty)_{\mathbb{T}}$ and for all $z_0 \in \mathbb{R}^n$, where $\|z_0\| < \alpha$, we have:

$$\|x(s, s_0, z_0)\| < B, \quad \text{for all } s \in [s_0 + T, +\infty) \cap \mathbb{T},$$

where x is the maximal solution of (5.30) with $x(s_0) = z_0$.

• **Uniformly ultimately bounded:** if it is uniformly bounded and quasi-uniformly ultimately bounded.

In what follows, we will prove a result which ensures that the dynamic equation on time scales (5.30) is uniformly bounded.

Theorem 5.10. Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $[t_0, +\infty)_{\mathbb{T}}$ be a time scale interval. Suppose $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies conditions (B1), (B2) and (B3), and $U : [t_0, +\infty)_{\mathbb{T}} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a function such that for each left-continuous function $z : [\alpha, \beta]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ on $(\alpha, \beta]_{\mathbb{T}}$, the function $[\alpha, \beta]_{\mathbb{T}} \ni t \mapsto U(t, z(t))$ is left-continuous on $(\alpha, \beta]_{\mathbb{T}}$. Moreover, suppose the following conditions concerning U are satisfied

(i) There are two monotone increasing functions $p, b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $b(0) = 0$,

$$\lim_{s \rightarrow +\infty} b(s) = +\infty$$

and

$$b(\|z\|) \leq U(t, z) \leq p(\|z\|),$$

for every pair $(t, z) \in [t_0, +\infty)_{\mathbb{T}} \times \mathbb{R}^n$.

(ii) For every solution $z : [s_0, +\infty) \cap \mathbb{T} \rightarrow \mathbb{R}^n$, $s_0 \geq t_0$, of the dynamic equation on time scales (5.30), we have

$$U(s, z(s)) - U(t, z(t)) \leq 0,$$

for every $s, t \in [s_0, +\infty) \cap \mathbb{T}$ with $t \leq s$.

Then, the dynamic equation on time scales (5.30) is uniformly bounded.

Proof. Since f satisfies conditions (B1), (B2) and (B3), by Theorem 4.26, f^* satisfies conditions (A2), (A3) and (A4). Define a function $U^* : [t_0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$U^*(t, x) = U(t^*, x), \quad t \in [t_0, +\infty), \quad x \in \mathbb{R}^n.$$

Then, by hypothesis (i), there are two monotone increasing functions $p, b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $p(0) = b(0) = 0$,

$$\lim_{s \rightarrow +\infty} b(s) = +\infty$$

and

$$b(\|z\|) \leq \underbrace{U^*(t, z)}_{=U(t^*, z)} \leq p(\|z\|)$$

for every pair $(t, z) \in [t_0, +\infty) \times \mathbb{R}^n$, which implies that U^* satisfies the hypothesis (i) from Theorem 5.7.

Now, let $y : [s_0, +\infty) \rightarrow \mathbb{R}^n$, $s_0 \geq t_0$, be a solution of the measure differential equation (5.27). By Corollary 3.18, $y : [s_0, +\infty) \rightarrow \mathbb{R}^n$ must have the form $y = z^*$, where $z : [s_0, +\infty) \cap \mathbb{T} \rightarrow \mathbb{R}^n$ is a solution of the dynamic equation on time scales (5.30). Hence, for each $s_0 \leq t < s < +\infty$, we have

$$\begin{aligned} U^*(s, y(s)) - U^*(t, y(t)) &= U^*(s, z^*(s)) - U^*(t, z^*(t)) \\ &= U(s^*, z^*(s)) - U(t^*, z^*(t)) = U(s^*, z(s^*)) - U(t^*, z(t^*)) \leq 0, \end{aligned}$$

by hypothesis (ii). Therefore all the hypotheses from Theorem 5.7 are satisfied. Then the measure differential equation (5.27) is uniformly bounded. By Corollary 3.18, the dynamic equation on time scales (5.30) is uniformly bounded, obtaining the desired result. $\square$

Finally, we present the last result of our chapter. Our result ensures that the dynamic equation on time scales (5.30) is uniformly ultimately bounded.

Theorem 5.11. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$, $t_0 \in \mathbb{T}$ and $t_0 \geq 0$. Suppose $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies conditions (B1), (B2) and (B3), and $U : [t_0, +\infty)_{\mathbb{T}} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a function such that for each left-continuous function $z : [\alpha, \beta]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ on $(\alpha, \beta]_{\mathbb{T}}$, the function $[\alpha, \beta]_{\mathbb{T}} \ni t \mapsto U(t, z(t))$ is left-continuous on $(\alpha, \beta]_{\mathbb{T}}$ and satisfies condition (i) from Theorem 5.10. Moreover, suppose U satisfies the following conditions*

(I) *For all $x, y, z, w \in \mathbb{R}^n$ and for all $\alpha, \beta \in [t_0, +\infty)_{\mathbb{T}}$ with $\alpha \leq \beta$, we have*

$$|U(\beta, x) - U(\beta, y) - U(\alpha, z) + U(\alpha, w)| \leq \left(\int_{\alpha}^{\beta} K(\tau) \Delta \tau \right) \max \{ \|x - y\|, \|z - w\| \},$$

where $K : [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ is a locally Kurzweil-Henstock Δ -integrable function.

(II) *There exists a continuous function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, with $\phi(0) = 0$ and $\phi(x) > 0$, $x \neq 0$ such that for every solution $z : [s_0, +\infty)_{\mathbb{T}} \cap \mathbb{T} \rightarrow \mathbb{R}^n$, $s_0 \geq t_0$, of the dynamic equation on time scales (5.30), we have*

$$U(s^*, z(s^*)) - U(t^*, z(t^*)) \leq (s - t)^* \left(-\phi(z(t^*)) \right),$$

for every $t, s \in [s_0, +\infty) \cap \mathbb{T}$ with $t \leq s$.

Then, the dynamic equation on time scales (5.30) is uniformly ultimately bounded.

Proof. Since f satisfies conditions (B1), (B2) and (B3), by Theorem 4.26, f^* satisfies conditions (A2), (A3) and (A4), and by Lemma 3.12, g is a nondecreasing and left-continuous function.

Also, define the functional $U^* : [t_0, +\infty)_{\mathbb{T}} \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$U^*(t, x) = U(t^*, x),$$

for every pair $(t, x) \in [t_0, +\infty) \times \mathbb{R}^n$.

Since K is a locally Kurzweil-Henstock Δ -integrable function on $[t_0, +\infty)_{\mathbb{T}}$, it follows from Theorem 3.13 that the function $K^* : [t_0, +\infty) \rightarrow \mathbb{R}$ is locally Kurzweil-Henstock integrable with respect to the nondecreasing function $u : \mathbb{T}^* \rightarrow \mathbb{R}$, given by $u(t) = t^*$ and

$$\int_{\alpha}^{\beta} K(\tau) \Delta \tau = \int_{\alpha}^{\beta} K^*(\tau) du(\tau),$$

for all $\alpha, \beta \in [t_0, +\infty)_{\mathbb{T}}$. Thus, for all $x, y : [v, \gamma] \rightarrow \mathbb{R}^n$, $[v, \gamma] \subset [t_0, +\infty)$, of bounded variation in $[v, \gamma]$ and all $v \leq s < t \leq \gamma$, we have

$$\begin{aligned}
& |U^*(t, x(t)) - U^*(t, y(t)) - U^*(s, x(s)) + U^*(s, y(s))| \\
&= |U(t^*, x(t)) - U(t^*, y(t)) - U(s^*, x(s)) + U(s^*, y(s))| \\
&\leq \left(\int_{s^*}^{t^*} K(\tau) \Delta\tau \right) \max \{ \|x(t) - y(t)\|, \|x(s) - y(s)\| \} \\
&= \left(\int_{s^*}^{t^*} K^*(\tau) du(\tau) \right) \max \{ \|x(t) - y(t)\|, \|x(s) - y(s)\| \} \\
&\leq \left(\int_{s^*}^{t^*} K^*(\tau) du(\tau) \right) \sup_{\xi \in [v, \gamma]} \|x(\xi) - y(\xi)\| \\
&= \left(\int_s^t K^*(\tau) du(\tau) \right) \sup_{\xi \in [v, \gamma]} \|x(\xi) - y(\xi)\|,
\end{aligned}$$

where Theorem 3.14 is used to establish the last equality. Thus, the hypothesis (U1) from Theorem 5.8 is fulfilled.

Now, let $y : [s_0, +\infty) \rightarrow \mathbb{R}^n$, $s_0 \geq t_0$, be a solution of the measure differential equation (5.27). Then By Corollary 3.18, $y : [s_0, +\infty) \rightarrow \mathbb{R}^n$ must have the form $y = z^*$, where $z : [s_0, +\infty) \cap \mathbb{T} \rightarrow \mathbb{R}^n$ is a solution of the dynamic equation on time scales (5.30). Hence

$$\begin{aligned}
& U^*(s, y(s)) - U^*(t, y(t)) = U^*(s, z^*(s)) - U^*(t, z^*(t)) \\
&= U(s^*, z^*(s)) - U(t^*, z^*(t)) = U(s^*, z(s^*)) - U(t^*, z(t^*)) \\
&\leq (s - t)^*(-\phi(z(t^*))) = (s - t)^*(-\phi(y(t))) \\
&\leq (s - t)(-\phi(y(t))),
\end{aligned}$$

for every $s_0 \leq t < s < +\infty$. Thus, condition (U2) from the Theorem 5.8 is fulfilled. Therefore, all the hypotheses from Theorem 5.8 are fulfilled and the measure differential equation (5.27) is uniformly ultimately bounded. By Corollary 3.18, it follows that the dynamic equation on time scales (5.30) is uniformly ultimately bounded, obtaining the desired result. $\square$

Lyapunov stability

In this chapter, our goal is to investigate Lyapunov-type stability results for generalized ODEs and, then, to obtain Lyapunov-type results for measure differential equations. Using relations between the solutions of measure differential equations and the solutions of dynamic equations on time scales, we also present Lyapunov-type results for these last equations.

The results presented in this chapter are new and they are contained in the paper [18].

6.1 Lyapunov stability for generalized ODEs

In this section, we will present Lyapunov-type stability results for generalized ODEs.

Let us assume that X is a Banach space with the norm $\|\cdot\|$ and set $\Omega = B_c \times [t_0, +\infty)$, where $B_c = \{x \in X : \|x\| < c\}$, $c > 0$ and $t_0 \geq 0$.

Consider the generalized ODE given by

$$\frac{dx}{d\tau} = DF(x, t). \quad (6.1)$$

We assume that $F \in \mathcal{F}(\Omega, h)$, where the function h is nondecreasing and left-continuous, and $F(0, t) - F(0, s) = 0$, for $t, s \geq t_0$. Then for every $[a, b] \subset [t_0, +\infty)$, we have

$$\int_a^b DF(0, t) = F(0, b) - F(0, a) = 0,$$

which implies that $x \equiv 0$ is a solution of the generalized ODE (6.1) on $[t_0, +\infty)$.

We also assume that $\Omega = \Omega_F$, that is, $\Omega = \{(x, t) \in \Omega : x + F(x, t^+) - F(x, t) \in B_c\}$. Then, by Theorems 4.8 and 4.9, for every $(x_0, s_0) \in \Omega$, there exists a unique maximal solution $x : [s_0, \omega(s_0, x_0)) \rightarrow X$ of the generalized ODE (6.1) such that $x(s_0) = x_0$. Here, we will denote by $x(t) = x(t, s_0, x_0)$, $t \in [s_0, \omega(s_0, x_0))$, the unique maximal solution of the generalized ODE (6.1) with $x(s_0) = x_0$.

Remark 6.1. For simplicity of notations, when it is clear, we write only ω instead of $\omega(s_0, x_0)$.

The following Lyapunov-type stability concepts of the trivial solution of a generalized ODE extend the concepts presented in [44, Definition 4.1].

Definition 6.2. The trivial solution $x \equiv 0$ of the generalized ODE (6.1) is said to be

- **Stable**, if for every $s_0 \geq t_0$, $\epsilon > 0$, there exists $\delta = \delta(\epsilon, s_0) > 0$ such that if $x_0 \in B_c$ with

$$\|x_0\| < \delta,$$

then

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \epsilon,$$

for all $t \in [s_0, \omega)$, where x is the maximal solution of (6.1) with $x(s_0) = x_0$.

- **Uniformly stable**, if it is stable with δ independent of s_0 .
- **Uniformly asymptotically stable**, if there exists $\delta_0 > 0$ and for every $\epsilon > 0$, there exist $T = T(\epsilon) \geq 0$ such that if $s_0 \geq t_0$ and $x_0 \in B_c$ with

$$\|x_0\| < \delta_0,$$

then

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \epsilon,$$

for all $t \in [s_0, \omega) \cap [s_0 + T, +\infty)$, where x is the maximal solution of (6.1) with $x(s_0) = x_0$.

In the sequel, we present a concept of Lyapunov functional for generalized ODEs.

Definition 6.3. We say that $V : [t_0, +\infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$ is a Lyapunov functional with respect to the generalized ODE (6.1), $0 < \rho < c$, if the following conditions are satisfied:

- (i) $V : [t_0, +\infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$ is left-continuous with respect to the first variable on $(t_0, +\infty)$, for all $x \in \overline{B}_\rho$;

- (ii) There exists a continuous increasing function $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $b(0) = 0$ (we say that such a function is of Hahn class) such that

$$V(t, x) \geq b(\|x\|),$$

for every $(t, x) \in [t_0, +\infty) \times \overline{B}_\rho$;

- (iii) The function $[s_0, \omega) \ni t \mapsto V(t, x(t))$ is nonincreasing along every maximal solution $x(t) = x(t, s_0, x_0)$, $(x_0, s_0) \in B_\rho \times [t_0, +\infty)$, of the generalized ODE (6.1) with $x(s_0) = x_0$.

In what follows, we present a result which ensures that the trivial solution of the generalized ODE (6.1) is uniformly stable. Such result does not require any Lipschitz-type condition concerning the Lyapunov functional, improving, then, the results found in the literature. See, for instance, [3, 22, 24, 38].

Theorem 6.4. *Let $F \in \mathcal{F}(\Omega, h)$, where $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is a left-continuous and nondecreasing function, and let $V : [t_0, +\infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$, $0 < \rho < c$, be a Lyapunov functional. Assume that V satisfies the following condition*

(H1) *There exists a continuous increasing function $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $a(0) = 0$, such that for every solution $x : I \rightarrow B_\rho$, $I \subset [t_0, +\infty)$, of the generalized ODE (6.1), we have*

$$V(t, x(t)) \leq a(\|x(t)\|)$$

for all $t \in I$.

Then the trivial solution $x \equiv 0$ of the generalized ODE (6.1) is uniformly stable.

Proof. Since V is a Lyapunov functional, there exists a function of Hahn class $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$V(t, x) \geq b(\|x\|),$$

for every $(t, x) \in [t_0, +\infty) \times \overline{B}_\rho$.

Let $s_0 \geq t_0$ and $\epsilon > 0$. Since $a(0) = 0$, a is increasing and $a|_{[0, \epsilon]}$ is uniformly continuous, there exists $\delta = \delta(\epsilon)$, $0 < \delta < \epsilon$, such that $a(\delta) < b(\epsilon)$.

Suppose $x_0 \in B_\rho$ and the maximal solution $x(t) = x(t, s_0, x_0)$ of the generalized ODE (6.1) satisfies

$$\|x(s_0)\| = \|x_0\| < \delta.$$

We want to prove that

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \epsilon,$$

for all $t \in [s_0, \omega)$.

Since V is a Lyapunov functional, by (iii) from Definition 6.3, we have

$$V(t, x(t)) \leq V(s_0, x(s_0)), \quad \text{for all } t \in [s_0, \omega).$$

Therefore, for every $t \in [s_0, \omega)$,

$$b(\|x(t)\|) \leq V(t, x(t)) \leq V(s_0, x(s_0)) \leq a(\|x(s_0)\|) < a(\delta) < b(\epsilon).$$

Since b is an increasing function, we obtain

$$\|x(t)\| < \epsilon,$$

for all $t \in [s_0, \omega)$, which completes the proof. $\square$

The next result establishes sufficient conditions so that the trivial solution of the generalized ODE (6.1) is uniformly asymptotically stable.

Theorem 6.5. *Suppose $F \in \mathcal{F}(\Omega, h)$, where $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is left-continuous and non-decreasing, $V : [t_0, +\infty) \times \overline{B_\rho} \rightarrow \mathbb{R}$, $0 < \rho < c$, satisfies conditions (i), (ii) from Definition 6.3 and (H1) from Theorem 6.4. Moreover, suppose there exists a continuous function $\Phi : X \rightarrow \mathbb{R}$ satisfying $\Phi(0) = 0$ and $\Phi(x) > 0$ for $x \neq 0$, such that for every maximal solution $x(t) = x(t, s_0, x_0)$, $(x_0, s_0) \in B_\rho \times [t_0, +\infty)$, of the generalized ODE (6.1),*

$$V(s, x(s)) - V(t, x(t)) \leq (s - t) \left(-\Phi(x(t)) \right), \quad (6.2)$$

for all $t, s \in [s_0, \omega)$, with $t \leq s$. Then the trivial solution $x \equiv 0$ of the generalized ODE (6.1) is uniformly asymptotically stable.

Proof. From the inequality (6.2), it is clear that the function $[s_0, +\infty) \ni t \mapsto V(t, x(t))$ is nonincreasing along every maximal solution $x(t) = x(t, s_0, x_0)$, $(x_0, s_0) \in \Omega$, of the generalized ODE (6.1). Since all the hypotheses from Theorem 6.4 are satisfied, the trivial solution $x \equiv 0$ of (6.1) is uniformly stable.

Let $\delta_0 := \frac{\rho}{2}$ and $\epsilon > 0$. By the uniform stability of the solution $x \equiv 0$ of the generalized ODE (6.1), there exists $\delta = \delta(\epsilon) > 0$ (we can take $\delta < \rho$) such that, if $\tau_0 \geq t_0$ and $y_0 \in B_\rho$ satisfies

$$\|y_0\| < \delta,$$

then

$$\|x(t, \tau_0, y_0)\| < \epsilon, \quad \text{for all } t \in [\tau_0, \omega(\tau_0, y_0)). \quad (6.3)$$

Let

$$N := \sup \{ -\Phi(\vartheta) : \delta(\epsilon) \leq \|\vartheta\| < \rho \} < 0$$

and

$$T(\epsilon) := -\frac{a(\delta_0)}{N} > 0.$$

Suppose $s_0 \geq t_0$ and $x_0 \in B_\rho$ and consider the maximal solution $x(\cdot) = x(\cdot, s_0, x_0) : [s_0, \omega^+(s_0, x_0)) \rightarrow B_\rho$ of (6.1) such that

$$\|x(s_0)\| = \|x_0\| < \delta_0. \quad (6.4)$$

We want to prove that

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \epsilon, \text{ for all } t \in [s_0, \omega(s_0, x_0)) \cap [s_0 + T(\epsilon), +\infty). \quad (6.5)$$

At first, assume that $\omega(s_0, x_0) < +\infty$. Then let us consider two cases.

Case 1. Suppose $T(\epsilon) \geq \omega(s_0, x_0) - s_0$, that is, $\omega(s_0, x_0) \leq s_0 + T(\epsilon)$. Therefore,

$$[s_0, \omega(s_0, x_0)) \cap [s_0 + T(\epsilon), +\infty) = \emptyset.$$

Under these conditions, (6.5) holds trivially.

Case 2. Suppose $T(\epsilon) < \omega(s_0, x_0) - s_0$, that is, $s_0 + T(\epsilon) < \omega(s_0, x_0)$. Therefore,

$$[s_0, \omega(s_0, x_0)) \cap [s_0 + T(\epsilon), +\infty) = [s_0 + T(\epsilon), \omega(s_0, x_0)).$$

Now, let us prove that there exists $\bar{t} \in [s_0, s_0 + T(\epsilon)]$ such that $\|x(\bar{t}, s_0, x_0)\| = \|x(\bar{t})\| < \delta(\epsilon)$. Suppose the contrary, that is, $\|x(s, s_0, x_0)\| = \|x(s)\| \geq \delta(\epsilon)$, for every $s \in [s_0, s_0 + T(\epsilon)]$. Thus,

$$\delta(\epsilon) \leq \|x(s)\| < \rho, \quad \text{for all } s \in [s_0, s_0 + T(\epsilon)]. \quad (6.6)$$

By (H1), (6.2), (6.4) and (6.6), we obtain

$$\begin{aligned} V(s_0 + T(\epsilon), x(s_0 + T(\epsilon))) &\leq V(s_0, x(s_0)) + T(\epsilon) \left(-\Phi(x(s_0)) \right) \\ &\leq V(s_0, x(s_0)) + T(\epsilon) \sup_{s \in [s_0, s_0 + T(\epsilon)]} \{-\Phi(x(s))\} \\ &\leq a(\|x(s_0)\|) + T(\epsilon) \underbrace{\sup \{-\Phi(\vartheta) : \delta(\epsilon) \leq \|\vartheta\| < \rho\}}_N \\ &< a(\delta_0) + T(\epsilon)N = a(\delta_0) + \left(-\frac{a(\delta_0)}{N} \right) N = 0, \end{aligned}$$

that is,

$$V(s_0 + T(\epsilon), x(s_0 + T(\epsilon))) < 0,$$

which contradicts the following inequality

$$V(s_0 + T(\epsilon), x(s_0 + T(\epsilon))) \geq b(\|x(s_0 + T(\epsilon))\|) \geq 0.$$

Then we conclude that there exists $\bar{t} \in [s_0, s_0 + T(\epsilon)]$ such that $\|x(\bar{t}, s_0, x_0)\| = \|x(\bar{t})\| < \delta(\epsilon)$. Therefore, by (6.3) (with $\tau_0 = \bar{t}$ and $y_0 = x(\bar{t}) = x(\bar{t}, s_0, x_0) =: \bar{y}$), we obtain

$$\|x(t, \bar{t}, \bar{y})\| < \epsilon, \quad \text{for all } t \in [\bar{t}, \omega(\bar{t}, \bar{y})]. \quad (6.7)$$

Since

$$x(\cdot, \bar{t}, \bar{y}) : [\bar{t}, \omega(\bar{t}, \bar{y})] \rightarrow B_\rho$$

and the restriction

$$x|_{[\bar{t}, \omega(s_0, x_0))}(\cdot, s_0, x_0) : [\bar{t}, \omega(s_0, x_0)) \rightarrow B_\rho$$

is a solution of the generalized ODE

$$\begin{cases} \frac{dx}{d\tau} = DF(x, t) \\ x(\bar{t}) = \bar{y} = x(\bar{t}, s_0, x_0), \end{cases}$$

by Lemma 4.7, it follows that

$$x(t, \bar{t}, \bar{y}) = x|_{[\bar{t}, \omega(s_0, x_0))}(t, s_0, x_0), \quad \text{for all } t \in [\bar{t}, \omega(\bar{t}, \bar{y})] \cap [\bar{t}, \omega(s_0, x_0)),$$

that is,

$$x(t, \bar{t}, \bar{y}) = x(t, s_0, x_0), \quad \text{for all } t \in [\bar{t}, \omega(\bar{t}, \bar{y})] \cap [\bar{t}, \omega(s_0, x_0)). \quad (6.8)$$

Now, we want to prove that $\omega(\bar{t}, \bar{y}) = \omega(s_0, x_0)$. Without loss of generality, suppose $\omega(\bar{t}, \bar{y}) < \omega(s_0, x_0)$. Thus, by (6.8), we have

$$x(t, \bar{t}, \bar{y}) = x(t, s_0, x_0), \quad \text{for all } t \in [\bar{t}, \omega(\bar{t}, \bar{y})]. \quad (6.9)$$

On the other hand, since $\omega(\bar{t}, \bar{y}) < +\infty$ (because $\omega(\bar{t}, \bar{y}) < \omega(s_0, x_0)$), by Proposition 4.12, the limit $\lim_{t \rightarrow \omega(\bar{t}, \bar{y})^-} x(t, \bar{t}, \bar{y})$ exists. Thus, by (6.9) and using the fact that $x(\cdot, s_0, x_0)$ is a left-continuous function at the point $\omega(\bar{t}, \bar{y})$, we obtain

$$x(\omega(\bar{t}, \bar{y}), s_0, x_0) = \lim_{t \rightarrow \omega(\bar{t}, \bar{y})^-} x(t, s_0, x_0) = \lim_{t \rightarrow \omega(\bar{t}, \bar{y})^-} x(t, \bar{t}, \bar{y}),$$

that is,

$$\lim_{t \rightarrow \omega(\bar{t}, \bar{y})^-} x(t, \bar{t}, \bar{y}) = x(\omega(\bar{t}, \bar{y}), s_0, x_0).$$

Therefore, by Lemma 4.1, the function $z_1 : [\bar{t}, \omega(s_0, x_0)) \rightarrow X$ given by the relation

$$z_1(t) = \begin{cases} x(t, \bar{t}, \bar{y}), & t \in [\bar{t}, \omega(\bar{t}, \bar{y})) \\ x(t, s_0, x_0), & t \in [\omega(\bar{t}, \bar{y}), \omega(s_0, x_0)), \end{cases}$$

is a solution of the generalized ODE (6.1) with $z_1(\bar{t}) = x(\bar{t}, \bar{t}, \bar{y}) = \bar{y}$. Notice that z_1 is a proper right prolongation of $x : [\bar{t}, \omega(\bar{t}, \bar{y})) \rightarrow B_\rho$, which contradicts the fact that $x : [\bar{t}, \omega(\bar{t}, \bar{y})) \rightarrow B_\rho$ is the maximal solution of the generalized ODE (6.1) with $x(\bar{t}) = \bar{y}$.

Analogously, if $\omega(s_0, x_0) < \omega(\bar{t}, \bar{y})$ (note that $\omega(s_0, x_0) < +\infty$ in the case we are considering), then the function

$$z_2(t) = \begin{cases} x(t, s_0, x_0), & t \in [s_0, \omega(s_0, x_0)), \\ x(t, \bar{t}, \bar{y}), & t \in [\omega(s_0, x_0), \omega(\bar{t}, \bar{y})) \end{cases}$$

is a proper right prolongation of $x : [s_0, \omega(s_0, x_0)) \rightarrow B_\rho$, which contradicts the fact that $x : [s_0, \omega(s_0, x_0)) \rightarrow B_\rho$ is the maximal solution of the generalized ODE (6.1) with $x(s_0) = x_0$. Then,

$$\omega(\bar{t}, \bar{y}) = \omega(s_0, x_0). \quad (6.10)$$

Thus, by (6.7), (6.8) and (6.10), we obtain

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \epsilon, \quad \text{for all } t \in [\bar{t}, \omega(s_0, x_0)).$$

In particular,

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \epsilon, \quad \text{for all } t \in \underbrace{[s_0, \omega(s_0, x_0)) \cap [s_0 + T(\epsilon), +\infty)}_{[s_0 + T(\epsilon), \omega(s_0, x_0))},$$

since $[s_0 + T(\epsilon), \omega(s_0, x_0)) \subset [\bar{t}, \omega(s_0, x_0))$ (because $\bar{t} \leq s_0 + T(\epsilon)$). Hence the trivial solution $x \equiv 0$ of the generalized ODE (6.1) is uniformly asymptotically stable.

Now, let us consider $\omega(s_0, x_0) = +\infty$. Then, we have

$$[s_0, \underbrace{\omega(s_0, x_0)}_{+\infty}) \cap [s_0 + T(\epsilon), +\infty) = [s_0 + T(\epsilon), +\infty).$$

We have only to consider the case where $s_0 + T(\epsilon) < +\infty$. Using the same arguments as in the first part of proof of Case 2, there exists $\bar{t} \in [s_0, s_0 + T(\epsilon)]$ such that

$$\|x(\bar{t}, s_0, x_0)\| = \|x(\bar{t})\| < \delta(\epsilon).$$

Then, by (6.3) (with $\tau_0 = \bar{t}$ and $y_0 = x(\bar{t}) = x(\bar{t}, s_0, x_0) =: \bar{y}$), we have

$$\|x(t, \bar{t}, \bar{y})\| < \epsilon, \quad \text{for all } t \in [\bar{t}, \omega(\bar{t}, \bar{y})]. \quad (6.11)$$

Since

$$x(\cdot, \bar{t}, \bar{y}) : [\bar{t}, \omega(\bar{t}, \bar{y})) \rightarrow B_\rho$$

and

$$x|_{[\bar{t}, +\infty)}(\cdot, s_0, x_0) : [\bar{t}, +\infty) \rightarrow B_\rho$$

are solutions of

$$\begin{cases} \frac{dx}{d\tau} = DF(x, t) \\ x(\bar{t}) = \bar{y} = x(\bar{t}, s_0, x_0), \end{cases}$$

by Lemma 4.7, we obtain

$$x(t, \bar{t}, \bar{y}) = x|_{[\bar{t}, +\infty)}(t, s_0, x_0), \quad \text{for all } t \in [\bar{t}, \omega(\bar{t}, \bar{y})) \cap [\bar{t}, +\infty), \quad (6.12)$$

that is,

$$x(t, \bar{t}, \bar{y}) = x(t, s_0, x_0), \quad \text{for all } t \in [\bar{t}, \omega(\bar{t}, \bar{y})) \cap [\bar{t}, +\infty). \quad (6.13)$$

Finally, we will prove that $\omega(\bar{t}, \bar{y}) = +\infty$. Suppose the contrary, that is, $\omega(\bar{t}, \bar{y}) < +\infty$. Then using the same argument as before for z_1 , the function $z_3 : [\bar{t}, +\infty) \rightarrow B_\rho$, given by the relation

$$z_3(t) = \begin{cases} x(t, \bar{t}, \bar{y}), & t \in [\bar{t}, \omega(\bar{t}, \bar{y})) \\ x(t, s_0, x_0), & t \in [\omega(\bar{t}, \bar{y}), +\infty), \end{cases}$$

is a proper right prolongation of $x : [\bar{t}, \omega(\bar{t}, \bar{y})) \rightarrow B_\rho$, which contradicts the fact that $x : [\bar{t}, \omega(\bar{t}, \bar{y})) \rightarrow B_\rho$ is the maximal solution of the generalized ODE (6.1) with $x(\bar{t}) = \bar{y}$. Therefore,

$$\omega(\bar{t}, \bar{y}) = +\infty. \quad (6.14)$$

Thus, by (6.11), (6.13) and (6.14), we get

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \epsilon, \quad \text{for all } t \in [\bar{t}, +\infty).$$

In particular

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \epsilon, \quad \text{for all } t \in \underbrace{[s_0, +\infty) \cap [s_0 + T(\epsilon), +\infty)}_{[s_0 + T(\epsilon), +\infty)},$$

because $[s_0 + T(\epsilon), +\infty) \subset [\bar{t}, +\infty)$ (since $\bar{t} \leq s_0 + T(\epsilon)$). The proof is then completed. $\square$

6.2 Lyapunov stability for measure differential equations

In this section, our goal is to prove stability results for measure differential equations using Lyapunov functionals. Throughout this section, let us consider that $\mathbb{R}^n$ is the n -dimensional Euclidean space with norm $\|\cdot\|$, and set

$$B_c = \{x \in \mathbb{R}^n : \|x\| < c\},$$

with $c > 0$.

We consider the integral form of a measure differential equations of type

$$x(t) = x(\tau_0) + \int_{\tau_0}^t f(x(s), s) dg(s), \quad t \geq \tau_0, \quad (6.15)$$

where $\tau_0 \geq t_0$, $f : B_c \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ and $g : [t_0, +\infty) \rightarrow \mathbb{R}$.

Throughout this section, we assume that the function $f : B_c \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3), (A4) and $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1) (all these conditions presented in Chapter 3).

We also assume that $f(0, t) = 0$, for every $t \in [t_0, +\infty)$. This condition implies that $x \equiv 0$ is a solution of (6.15) on $[t_0, +\infty)$ and the function F , given by $F(x, t) = \int_{t_0}^t f(x, s) dg(s)$, satisfies $F(0, s) - F(0, t) = 0$, for all $t, s \in [t_0, +\infty)$. Notice that, under this condition, it makes sense to prove the stability of the trivial solution of the measure differential equation (6.15).

Moreover, we assume that $x_0 + f(x_0, s_0)\Delta^+g(s_0) \in B_c$, for all $s_0 \in [t_0, +\infty)$ and $x_0 \in B_c$. This condition ensures us that for all $(x_0, s_0) \in B_c \times [t_0, +\infty)$, there exists a unique maximal solution $x : [s_0, \omega) \rightarrow \mathbb{R}^n$ of the measure differential equation (6.15) with $x(s_0) = x_0$. Note that the existence of such a solution is guaranteed by Theorems 4.8 and 4.9. In what follows, for all $(x_0, s_0) \in B_c \times [t_0, +\infty)$, we denote by $x(t, s_0, x_0)$ the unique maximal solution of the measure differential equation (6.15) with $x(s_0) = x_0$ defined on $[s_0, +\infty)$.

In the sequel, we present some concepts of stability of the trivial solution of the measure differential equations (6.15).

Definition 6.6. The trivial solution $x \equiv 0$ of the measure differential equation (6.15) is said to be

- **Stable**, if for every $s_0 \geq t_0$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, s_0) > 0$ such that if $x_0 \in B_c$ with

$$\|x_0\| < \delta,$$

then

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \varepsilon,$$

for all $t \in [s_0, \omega^+(s_0, x_0))$, where x is the maximal solution of (6.15) with $x(s_0) = x_0$.

- **Uniformly stable**, if it is stable and δ is independent of s_0 .
- **Uniformly asymptotically stable**, if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exists $T = T(\varepsilon) \geq 0$ such that if $s_0 \geq t_0$ and $x_0 \in B_c$ with

$$\|x_0\| < \delta_0,$$

then

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \varepsilon,$$

for all $t \in [s_0, \omega^+(s_0, x_0)) \cap [s_0 + T, +\infty)$, where x is the maximal solution of (6.15) with $x(s_0) = x_0$.

In what follows, we introduce a concept of Lyapunov functional with respect to the measure differential equation (6.15).

Definition 6.7. We say that $U : [t_0, +\infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$, $0 < \rho < c$, is a *Lyapunov functional with respect to the measure differential equation* (6.15), if the following conditions are satisfied

- (I) $U(\cdot, x) : [t_0, +\infty) \rightarrow \mathbb{R}$ is left-continuous on $(t_0, +\infty)$ for all $x \in \overline{B}_\rho$;
- (II) There exists a continuous increasing function $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $b(0) = 0$ (we say that such function is of *Hahn class*) such that

$$U(t, x) \geq b(\|x\|),$$

for every $(t, x) \in [t_0, +\infty) \times \overline{B}_\rho$;

- (III) The function $[s_0, \omega) \ni t \mapsto U(t, x(t))$ is a nonincreasing function along every maximal solution $x(t) = x(t, s_0, x_0)$, $(x_0, s_0) \in B_\rho \times [t_0, +\infty)$, of the measure differential equation (6.15).

The next result ensures us that the trivial solution $x \equiv 0$ of the measure differential equation (6.15) is uniformly stable.

Theorem 6.8. Suppose $f : B_c \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1) and $f(0, t) = 0$ for every $t \in [t_0, +\infty)$. Also, let $U : [t_0, +\infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$, $0 < \rho < c$, be a Lyapunov functional with respect to the measure differential equation (6.15). Assume that U satisfies the condition

(H2) There exists a continuous increasing function $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $a(0) = 0$ and for all solutions $x : I \rightarrow B_\rho$, $I \subset [t_0, +\infty)$ of (6.15), we have

$$U(t, x(t)) \leq a(\|x(t)\|)$$

for all $t \in I$.

Then the trivial solution $x \equiv 0$ of the measure differential equation (6.15) is uniformly stable.

Proof. Define a function $F : B_c \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by

$$F(x, t) := \int_{t_0}^t f(x, s) dg(s) \quad (6.16)$$

for all $(x, t) \in B_c \times [t_0, +\infty)$. Then, by Theorem 3.2, it follows that $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is given by

$$h(t) = \int_{t_0}^t (M(s) + L(s)) dg(s),$$

since the function $f : B_c \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4) and the function $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1).

By Theorem 3.8, it is clear that $U : [t_0, +\infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$ is a Lyapunov functional with respect to the generalized ODE (6.1), where F is given by (6.16). Moreover, by condition (H2), it is clear that U also satisfies condition (H1) from Theorem 6.4. Therefore, the trivial solution $x \equiv 0$ of the generalized ODE (6.1) is uniformly stable. By Theorem 3.8, it follows that the trivial solution $x \equiv 0$ of the measure differential equation (6.15) is also uniformly stable, obtaining the desired result. $\square$

In the sequel, we present our last result of this section, which ensures that the trivial solution $x \equiv 0$ of the measure differential equation (6.15) is uniformly asymptotically stable.

Theorem 6.9. Suppose $f : B_c \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), $g : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfies condition (A1) and $f(0, t) = 0$ for every $t \in [t_0, +\infty)$. Suppose $U : [t_0, +\infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$, $0 < \rho < c$, satisfies conditions (I) and (II) from Definition 6.7 and condition (H2) from Theorem 6.8. Moreover, suppose there exists a continuous function $\Phi : X \rightarrow \mathbb{R}$ satisfying $\Phi(0) = 0$ and $\Phi(z) > 0$ for $z \neq 0$, such that for every maximal solution $x(t) = x(t, s_0, x_0)$, $(x_0, s_0) \in B_\rho \times [t_0, +\infty)$, of the measure differential equation (6.15), we have

$$U(s, x(s)) - U(t, x(t)) \leq (s - t) \left(-\Phi(x(t)) \right), \quad (6.17)$$

for all $t, s \in [s_0, \omega)$ with $t \leq s$. Then the trivial solution $x \equiv 0$ of (6.15) is uniformly asymptotically stable.

Proof. Define a function $F : B_c \times [t_0, +\infty) \rightarrow \mathbb{R}^n$ by

$$F(x, t) = \int_{t_0}^t f(x, s) dg(s)$$

for all $(x, t) \in B_c \times [t_0, +\infty)$. Since f satisfies conditions (A2), (A3) and (A4) and g satisfies condition (A1), by Theorem 3.2, $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is given by

$$h(t) = \int_{t_0}^t (M(s) + L(s)) dg(s).$$

By Theorem 3.8, it is easy to check that $U : [t_0, +\infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$ satisfies all the hypotheses of Theorem 6.5. Therefore, by Theorem 6.5, the trivial solution $x \equiv 0$ of the generalized ODE (6.1) is uniformly asymptotically stable. Applying Theorem 3.8 again, the trivial solution $x \equiv 0$ of the measure differential equation (6.15) is uniformly asymptotically stable and we obtain the desired result. $\square$

6.3 Lyapunov stability for dynamic equations on time scales

In this section, our goal is to prove results concerning stability of solutions of dynamic equations on time scales using Lyapunov functionals.

Consider the dynamic equation on time scale given by

$$x^\Delta(t) = f(x^*, t), \quad t \in [t_0, +\infty)_{\mathbb{T}}, \quad (6.18)$$

where $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$. We recall the reader that x^* denotes the extension of x defined in Chapter 3.

Throughout this section, we assume that the function f satisfies conditions (B1), (B2) and (B3) presented in Chapter 4.

From now on, let us assume that $f(0, t) = 0$ for every $t \in [t_0, +\infty)_{\mathbb{T}}$. This condition implies that $x \equiv 0$ is a solution of the dynamic equation on time scales (6.18).

We also assume that for all $(z_0, \tau_0) \in B_c \times [t_0, +\infty)_{\mathbb{T}}$, we have $z_0 + f(z_0, \tau_0)\mu(\tau_0) \in B_c$. Then, by Theorem 4.27, for all $(x_0, s_0) \in B_c \times [t_0, +\infty)_{\mathbb{T}}$, there exists a unique maximal solution $x : [s_0, \omega)_{\mathbb{T}} \rightarrow \mathbb{R}^n$, $\omega \leq +\infty$ of the dynamic equation on time scales (6.18) with $x(s_0) = x_0$. Here, we will denote by $x(t, s_0, x_0)$, $t \in [s_0, \omega(s_0, x_0))_{\mathbb{T}}$, the unique maximal solution of the dynamic equation on time scales (6.18) with $x(s_0) = x_0$.

In the sequel, let us present the concept of stability of the trivial solution of the dynamic equations on time scales (6.18).

Definition 6.10. Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$. The trivial solution $x \equiv 0$ of the dynamic equation on time scales (6.18) is said to be

- (i) **Stable**, if for every $s_0 \in \mathbb{T}$, with $s_0 \geq t_0$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, s_0) > 0$ such that if $x_0 \in B_c$ with

$$\|x_0\| < \delta,$$

then

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \varepsilon,$$

for all $t \in [s_0, \omega(s_0, x_0))_{\mathbb{T}}$, where x is the maximal solution of (6.18) with $x(s_0) = x_0$.

- (ii) **Uniformly stable**, if it is stable with δ independent of s_0 .

- (iii) **Uniformly asymptotically stable**, if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exists $T = T(\varepsilon) \geq 0$ such that if $s_0 \in \mathbb{T}$, $s_0 \geq t_0$ and $x_0 \in B_c$ with

$$\|x_0\| < \delta_0,$$

then

$$\|x(t, s_0, x_0)\| = \|x(t)\| < \varepsilon,$$

for all $t \in [s_0, \omega(s_0, x_0)) \cap [s_0 + T, +\infty) \cap \mathbb{T}$, where x is the maximal solution of (6.18) with $x(s_0) = x_0$.

In what follows, we will present the definition of a Lyapunov functional with respect to the dynamic equation on time scales (6.18).

Definition 6.11. Let $t_0 \in \mathbb{T}$ and $c > 0$. We say that $U : [t_0, +\infty)_{\mathbb{T}} \times \overline{B}_\rho \rightarrow \mathbb{R}$, $0 < \rho < c$, is a *Lyapunov functional with respect to the dynamic equation on time scales* (6.18), if the following conditions are satisfied

- (i) $U(\cdot, x) : [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ is left-continuous on $(t_0, +\infty)_{\mathbb{T}}$ for all $x \in \overline{B}_\rho$;
- (ii) There exists a continuous increasing function $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $b(0) = 0$ (we say that such function is of Hahn class) such that

$$U(t, x) \geq b(\|x\|),$$

for every $(t, x) \in [t_0, +\infty)_{\mathbb{T}} \times \overline{B}_\rho$;

- (iii) The function $[\tau_0, \omega)_{\mathbb{T}} \ni t \mapsto U(t, z(t))$ is nonincreasing along every maximal solution $z(t) = z(t, \tau_0, x_0)$, $(x_0, \tau_0) \in B_\rho \times [t_0, +\infty)$, of the dynamic equation on time scales (6.18).

We recall the reader that the function $g : [t_0, +\infty) \rightarrow \mathbb{R}$ given by $g(t) = t^*$, for all $t \in [t_0, +\infty)$, is nondecreasing and left-continuous on $(t_0, +\infty)$.

The following results ensure us the uniform stability and uniform asymptotically stability of the trivial solution of the dynamic equation on time scales (6.18) via Lyapunov functionals.

Theorem 6.12. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $[t_0, +\infty)_{\mathbb{T}}$ be a time scale interval. Assume that $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies conditions (B1), (B2), (B3) and $f(0, t) = 0$ for every $t \in [t_0, +\infty)_{\mathbb{T}}$. Let $U : [t_0, +\infty)_{\mathbb{T}} \times \overline{B}_\rho \rightarrow \mathbb{R}$, $0 < \rho < c$, be a Lyapunov functional with respect to the dynamic equation on time scales (6.18). Assume that U satisfies the following condition*

(H3) *There exists a continuous increasing function $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $a(0) = 0$ and for all $(t, x) \in [t_0, +\infty)_{\mathbb{T}} \times B_c$, we have*

$$U(t, x) \leq a(\|x\|). \quad (6.19)$$

Then the trivial solution $x \equiv 0$ of the dynamic equation on time scales (6.18) is uniformly stable.

Proof. Since f satisfies conditions (B1), (B2) and (B3), by Theorem 4.26, f^* satisfies conditions (A2), (A3) and (A4). Also, since $f(0, t) = 0$, for every $t \in [t_0, +\infty)_{\mathbb{T}}$, clearly $f^*(0, t) = 0$, for every $t \in [t_0, +\infty)$.

Define

$$U^*(t, x) := U(t^*, x),$$

for every $(t, x) \in [t_0, +\infty) \times \overline{B}_\rho$. We affirm that $U^* : [t_0, +\infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$ is a Lyapunov functional with respect to the measure differential equation

$$y(t) = y(\tau_0) + \int_{\tau_0}^t f^*(y(s), s) dg(s), \quad \text{for } t, \tau_0 \in [t_0, +\infty), \quad (6.20)$$

where $g(t) = t^*$. Indeed, by condition (i) from Definition 6.11, $U(\cdot, x) : [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ is left-continuous on $(t_0, +\infty)_{\mathbb{T}}$ for all $x \in \overline{B}_\rho$. Since $U^*(t, x) = U(t^*, x) = U(g(t), x)$ and $g(t) = t^*$ is a left-continuous function on $(t_0, +\infty)$, we have $U^*(\cdot, x) : [t_0, +\infty) \rightarrow \mathbb{R}$ is left-continuous on $(t_0, +\infty)$ for all $x \in \overline{B}_\rho$, that is, U^* satisfies condition (I) from Definition 6.7.

On the other hand, by condition (ii) from Definition 6.11, there exists a continuous and increasing function $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $b(0) = 0$ such that

$$U(s, x) \geq b(\|x\|),$$

for every $(s, x) \in [t_0, +\infty)_{\mathbb{T}} \times \overline{B_\rho}$. Then, for every $(t, x) \in [t_0, +\infty) \times \overline{B_\rho}$, we obtain

$$U^*(t, x) = U(t^*, x) \geq b(\|x\|),$$

that is, U^* satisfies condition (II) from Definition 6.7.

Finally, we will prove that U^* satisfies condition (III) from Definition 6.7. Indeed, let $(x_0, s_0) \in B_\rho \times [t_0, +\infty)$ and $y = y(\cdot, s_0, x_0) : [s_0, \omega) \rightarrow B_\rho$ be the unique maximal solution of measure differential equation (6.20). Since $y(t) \in \overline{B_\rho} \subset B_c$, for $t \in [s_0, \omega)$, by Corollary 4.20 (with $N = \overline{B_\rho}$), $\omega = +\infty$. Now, since $y : [s_0, +\infty) \rightarrow B_\rho$ is a solution of (6.20), we obtain

$$y(\tau) = y(s_0) + \int_{s_0}^{\tau} f(y(s), s^*) dg(s), \quad \tau \in [s_0, +\infty),$$

where $g(\xi) = \xi^*$.

We consider two cases.

Case 1. Suppose $s_0 \in [t_0, +\infty) \setminus \mathbb{T}$. Then g is constant on $[s_0, s_0^*]$ which implies

$$\xi^* = s_0^*, \quad \text{for all } \xi \in [s_0, s_0^*]. \quad (6.21)$$

Therefore, $\int_{s_0}^{\tau} f(y(s), s^*) dg(s) = 0$, for all $\tau \in [s_0, s_0^*]$, that is, y is constant on $[s_0, s_0^*]$. In particular,

$$y(\tau) = y(s_0^*), \quad \text{for all } \tau \in [s_0, s_0^*]. \quad (6.22)$$

On the other hand, by Corollary 3.18, $y|_{[s_0^*, +\infty)}$ must have the form

$$y|_{[s_0^*, +\infty)} = x^*, \quad (6.23)$$

where $x : [s_0^*, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is a solution of the dynamic equation on time scales (6.18). Notice that x is also the maximal solution of the dynamic equation on time scales (6.18) through $(x(s_0^*), s_0^*)$ and $y(s_0^*) = x^*(s_0^*) = x(s_0^*)$. Hence the following holds true

- If $s_0 \leq t < s \leq s_0^*$, then, by (6.21), (6.22) and the fact $y(s_0^*) = x(s_0^*)$, we get

$$\begin{aligned} U^*(s, y(s)) - U^*(t, y(t)) &= U(s^*, y(s)) - U(t^*, y(t)) \\ &= U(s^*, x(s^*)) - U(t^*, x(t^*)) = 0 \end{aligned}$$

- If $s_0 \leq t \leq s_0^* < s$, then, by (6.23), (6.21) and (6.22), we have

$$\begin{aligned}
 U^*(s, y(s)) - U^*(t, y(t)) &= U(s^*, y(s)) - U(t^*, y(t)) \\
 &= U(s^*, x^*(s)) - U(s_0^*, y(s_0^*)) \\
 &= U(s^*, x(s^*)) - U(s_0^*, x(s_0^*)) \\
 &= U(s^*, x(s^*)) - U(t^*, x(t^*)) \leq 0.
 \end{aligned}$$

- If $s_0^* \leq t < s$, then, by (6.23), we obtain

$$\begin{aligned}
 U^*(s, y(s)) - U^*(t, y(t)) &= U(s^*, y(s)) - U(t^*, y(t)) \\
 &= U(s^*, x^*(s)) - U(t^*, x^*(t)) \\
 &= U(s^*, x(s^*)) - U(t^*, x(t^*)) \leq 0.
 \end{aligned}$$

Thus, $U^*(s, y(s)) - U^*(t, y(t)) \leq 0$ for all $s_0 \leq t < s < +\infty$, that is, the function $t \mapsto U^*(t, y(t))$ is nonincreasing and this completes the proof of condition (III) in the situation of case 1.

Case 2. Now, assume that $s_0 \in \mathbb{T}$. Then $s_0 = s_0^* \in \mathbb{T}$ and, therefore, $[s_0, +\infty)_{\mathbb{T}} = [s_0^*, +\infty)_{\mathbb{T}}$. By Corollary 3.18, $y : [s_0, +\infty) \rightarrow \mathbb{R}^n$ must have the form $y = x^*$, where $x : [s_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is a solution (more precisely, the maximal solution) of the dynamic equation on time scales (6.18) through $(y(s_0), s_0)$. Hence

$$\begin{aligned}
 U^*(s, y(s)) - U^*(t, y(t)) &= U^*(s, x^*(s)) - U^*(t, x^*(t)) \\
 &= U(s^*, x^*(s)) - U(t^*, x^*(t)) \\
 &= U(s^*, x(s^*)) - U(t^*, x(t^*)) \leq 0,
 \end{aligned}$$

for every $s_0 \leq t < s < +\infty$ and, therefore, the function $t \mapsto U^*(t, y(t))$ is nonincreasing. This completes the proof of condition (III) in case 2.

Conditions (I), (II) and (III) proved previously imply U^* is a Lyapunov functional with respect to the measure differential equation (3.20).

Now, by hypothesis (H3), there exists a continuous and increasing function $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $a(0) = 0$ which satisfies condition (6.19). Then, for all solutions $y : I \rightarrow B_\rho$, $I \subset [t_0, +\infty)$ of the measure differential equation (6.20), we have

$$U^*(t, y(t)) = U(t^*, y(t)) \leq a(\|y(t)\|)$$

for all $t \in I$. Therefore all the conditions of Theorem 6.8 are satisfied and, hence, the trivial solution $x^* \equiv 0$ of the measure differential equation (6.20) is uniformly stable. By Corollary

3.18, the trivial solution $x \equiv 0$ of the dynamic equation on time scales (6.18) is uniformly stable and we obtain the desired result. $\square$

Theorem 6.13. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $[t_0, +\infty)_{\mathbb{T}}$ be a time scale interval. Assume that $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ satisfies conditions (B1), (B2), (B3) and $f(0, t) = 0$ for every $t \in [t_0, +\infty)_{\mathbb{T}}$. Suppose $U : [t_0, +\infty)_{\mathbb{T}} \times \overline{B}_\rho \rightarrow \mathbb{R}$, $0 < \rho < c$, satisfies conditions (i) and (ii) from Definition 6.11 and condition (H3) from Theorem 6.12. Moreover, suppose there exists a continuous function $\phi : X \rightarrow \mathbb{R}$ satisfying $\phi(0) = 0$ and $\phi(v) > 0$ for $v \neq 0$ such that for every maximal solution $x(t) = x(t, s_0, x_0)$, $(x_0, s_0) \in B_\rho \times [t_0, +\infty)_{\mathbb{T}}$, of the dynamic equation on time scales (6.18),*

$$U(s^*, x(s^*)) - U(t^*, x(t^*)) \leq (s - t)^*(-\phi(x(t^*))) \quad (6.24)$$

for all $t, s \in [s_0, \omega)_{\mathbb{T}}$ with $t \leq s$. Then the trivial solution $x \equiv 0$ of the dynamic equation on time scales (6.18) is uniformly asymptotically stable.

Proof. Since f satisfies conditions (B1), (B2) and (B3), by Theorem 4.26, f^* satisfies conditions (A2), (A3) and (A4). Furthermore, $f(0, t) = 0$, for every $t \in [t_0, +\infty)_{\mathbb{T}}$, hence $f^*(0, t) = 0$, for every $t \in [t_0, +\infty)_{\mathbb{T}}$.

Define

$$U^*(t, x) := U(t^*, x),$$

for all $(t, x) \in [t_0, +\infty) \times \overline{B}_\rho$. Then $U^* : [t_0, +\infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$ satisfies conditions (I) and (II) from Definition 6.7, using the same arguments as in the proof of Theorem 6.12.

By hypothesis (H3) from Theorem 6.12, U^* satisfies the hypothesis (H2) from Theorem 6.8, using the same arguments as in the proof of Theorem 6.12.

Finally, we will prove that U^* satisfies the hypothesis (6.17). Indeed, let $(x_0, s_0) \in B_\rho \times [t_0, +\infty)$ and $y = y(\cdot, s_0, x_0) : [s_0, \omega) \rightarrow B_\rho$ be the unique maximal solution of the measure differential equation (6.20). Since $y(t) \in \overline{B}_\rho \subset B_c$, for all $t \in [s_0, \omega)$, by Corollary 4.20 (with $N = \overline{B}_\rho$), we have $\omega = +\infty$.

We consider two cases.

Case 1. Suppose $s_0 \in [t_0, +\infty) \setminus \mathbb{T}$. Using the same argument as in Case 1 from the proof of Theorem 6.12, we can prove that $y|_{[s_0^*, +\infty)}$ must have the form

$$y|_{[s_0^*, +\infty)} = x^*,$$

where $x : [s_0^*, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is the maximal solution of the dynamic equation on time scales (6.18) through $(x(s_0^*), s_0^*)$. Hence

- If $s_0 \leq t < s \leq s_0^*$, we get

$$\begin{aligned} U^*(s, y(s)) - U^*(t, y(t)) &= U(s^*, y(s)) - U(t^*, y(t)) \\ &= U(s^*, x(s^*)) - U(t^*, x(t^*)) = 0 \end{aligned} \quad (6.25)$$

On the other hand, by (6.24),

$$U(s^*, x(s^*)) - U(t^*, x(t^*)) \leq (s - t)^*(-\phi(x(t^*))) \leq 0. \quad (6.26)$$

Then, by (6.25) and (6.26), we have $(-\phi(x(t^*))) = 0$ and, therefore, $x(t^*) = 0$, that is, $y(t) = 0$ (because $x(t^*) \stackrel{(6.21)}{=} x(s_0^*) = y(s_0^*) \stackrel{(6.22)}{=} y(t)$). Thus, we obtain

$$U^*(s, y(s)) - U^*(t, y(t)) = (s - t)(-\phi(y(t))) = 0.$$

- If $s_0 \leq t \leq s_0^* < s$, we have

$$\begin{aligned} U^*(s, y(s)) - U^*(t, y(t)) &= U(s^*, x(s^*)) - U(t^*, x(t^*)) \\ &\leq (s - t)^*(-\phi(x(t^*))) \\ &= (s - t)^*(-\phi(x^*(t))) \\ &= (s - t)^*(-\phi(y(t))) \leq (s - t)(-\phi(y(t))). \end{aligned}$$

- If $s_0^* \leq t < s$, we obtain

$$\begin{aligned} U^*(s, y(s)) - U^*(t, y(t)) &= U(s^*, x(s^*)) - U(t^*, x(t^*)) \\ &\leq (s - t)^*(-\phi(x(t^*))) \\ &= (s - t)^*(-\phi(x^*(t))) \\ &= (s - t)^*(-\phi(y(t))) \leq (s - t)(-\phi(y(t))). \end{aligned}$$

Thus, $U^*(s, y(s)) - U^*(t, y(t)) \leq (s - t)(-\phi(y(t)))$ for all $s_0 \leq t < s < +\infty$.

Case 2. Now, assume that $s_0 \in \mathbb{T}$. Then $s_0 = s_0^* \in \mathbb{T}$ and, therefore, $[s_0, +\infty)_{\mathbb{T}} = [s_0^*, +\infty)_{\mathbb{T}}$. By Corollary 3.18, $y : [s_0, +\infty) \rightarrow \mathbb{R}^n$ must have the form $y = x^*$, where $x : [s_0, +\infty)_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is a solution (more precisely, the maximal solution) of the dynamic equation on time scales

(6.18) through $(x(s_0), s_0)$. Hence

$$\begin{aligned}
 U^*(s, y(s)) - U^*(t, y(t)) &= U^*(s, x^*(s)) - U^*(t, x^*(t)) \\
 &= U(s^*, x^*(s)) - U(t^*, x^*(t)) \\
 &= U(s^*, x(s^*)) - U(t^*, x(t^*)) \\
 &\leq (s - t)^*(-\phi(x(t^*))) \\
 &= (s - t)^*(-\phi(y(t))) \leq (s - t)(-\phi(y(t))).
 \end{aligned}$$

Thus $U^*(s, y(s)) - U^*(t, y(t)) \leq (s - t)(-\phi(y(t)))$, for all $s_0 \leq t < s < +\infty$.

Therefore all the hypotheses of Theorem 6.9 are satisfied. Then the trivial solution $x^* \equiv 0$ of the measure differential equation (6.20) is uniformly asymptotically stable. By Corollary 3.18 the trivial solution $x \equiv 0$ of the dynamic equation on time scales (6.18) is uniformly asymptotically stable, obtaining the desired result. $\square$

Remarks on autonomous generalized ODEs

In this section, our goal is to prove that autonomous generalized ODEs do not enlarge the class of classical autonomous ODEs. More precisely, we will prove that if a function $H : \Omega \rightarrow X$ belongs to the class $\mathcal{F}(\Omega, h, \omega)$ (or to one of the class $\tilde{\mathcal{F}}(\Omega, h, \omega)$, $\tilde{\mathcal{F}}(\Omega_\infty, h, \omega)$ or $\tilde{\mathcal{F}}(\Omega, h, \omega, E)$ defined in this chapter) and $H(x, t) = F(x)t$, for all $(x, t) \in \Omega$, then F is a continuous function and this fact implies that the classes of autonomous generalized ODEs $\frac{dx}{d\tau} = D[F(x)t]$ and of classic autonomous ODE $\frac{dx}{dt} = F(x(t))$ coincide.

The results presented here are new and are contained in [19].

7.1 Autonomous generalized ODEs

We start this section by recalling the concept of an autonomous generalized ODE introduced in [38] and used in [44].

Definition 7.1. Let X be a Banach space and $O \subset X$ be open. An *autonomous generalized ODE* is a generalized ODE of the form

$$\frac{dx}{d\tau} = D[H(x, t)], \quad (7.1)$$

where $H : \Omega \rightarrow X$, $\Omega = O \times [\alpha, \beta]$, is given by

$$H(x, t) = F(x)t$$

with $F : O \rightarrow X$ and $t \in [\alpha, \beta]$.

At this moment, the reader may find awkward the name “autonomous” given after an equation of type

$$\frac{dx}{d\tau} = D[F(x)t], \quad (7.2)$$

with t “appearing” on its right-hand side. Recall that a solution of the generalized ODE (7.2) is any function $x : [\sigma, \eta] \rightarrow X$, $[\sigma, \eta] \subset [\alpha, \beta]$, such that $x(t) \in O$ for all $t \in [\sigma, \eta]$ and

$$x(v) - x(\gamma) = \int_{\gamma}^v D[F(x(\tau))t] = (K) \int_{\gamma}^v F(x(t))dt \quad (7.3)$$

for all $\gamma, v \in [\sigma, \eta]$, where the last integral is precisely the Kurzweil integral (the prefix (K) will be used to distinguish this integral from others). Thus, the integral form of (7.2) is given by

$$x(t) = x(\sigma) + (K) \int_{\sigma}^t F(x(s))ds. \quad (7.4)$$

The next result ensures that all solutions of the autonomous generalized ODE (7.2) are continuous.

Lemma 7.2. *If $x : [\sigma, \eta] \rightarrow X$, $[\sigma, \eta] \subset [\alpha, \beta]$, is a solution of the autonomous generalized ordinary differential equation (7.2) on $[\sigma, \eta]$, then x is a continuous function on $[\sigma, \eta]$.*

Proof. Suppose $x : [\sigma, \eta] \rightarrow X$, $[\sigma, \eta] \subset [\alpha, \beta]$, is a solution of the autonomous generalized ODE (7.2) on $[\sigma, \eta]$. Let $a \in [\sigma, \eta]$. We have

$$x(s) = x(a) + \int_a^s D[F(x(\tau))t], \quad s \in [\sigma, \eta].$$

Now, define $U : [\sigma, \eta] \times [\sigma, \eta] \rightarrow X$ by $U(\tau, t) := F(x(\tau))t$, for every $(\tau, t) \in [\sigma, \eta] \times [\sigma, \eta]$. Notice that the function

$$\begin{aligned} U(a, \cdot) : [\sigma, \eta] &\rightarrow X \\ t &\mapsto U(a, t) = F(x(a))t \end{aligned}$$

is continuous at the point $t = a$. Thus, by Remark 1.7, the function $[\sigma, \eta] \ni s \mapsto \int_a^s D[F(x(\tau))t]$ is continuous at the point $s = a$ and, therefore, x is continuous at a . $\square$

In the sequel, we recall an important class of right-hand sides of generalized ODEs. The reader may want to consult [38]. Assume that $h : [\alpha, \beta] \rightarrow \mathbb{R}$ is a nondecreasing function and that $\omega : [0, +\infty) \rightarrow \mathbb{R}$ is a continuous and increasing function with $\omega(0) = 0$.

Definition 7.3. We say that a function $H : \Omega \rightarrow X$ belongs to the class $\mathcal{F}(\Omega, h, \omega)$, if

$$\|H(x, s_2) - H(x, s_1)\| \leq |h(s_2) - h(s_1)| \quad (7.5)$$

for all $(x, s_2), (x, s_1) \in \Omega$ and

$$\|H(x, s_2) - H(x, s_1) - H(y, s_2) + H(y, s_1)\| \leq \omega(\|x - y\|)|h(s_2) - h(s_1)| \quad (7.6)$$

for all $(x, s_2), (x, s_1), (y, s_2), (y, s_1) \in \Omega$.

Remark 7.4. When the function $\omega : [0, +\infty) \rightarrow \mathbb{R}$ is the identity, we will denote this class simply by $\mathcal{F}(\Omega, h)$.

We recall the reader that if $F : O \rightarrow X$ is a continuous function and $x_0 \in O$, then $x : I \rightarrow X$ is a solution of the autonomous ODE given by

$$\frac{dx}{dt} = F(x(t)). \quad (7.7)$$

on I with $x(t_0) = x_0$ if, and only if, x is a continuous function on I , $x(t) \in O$ for every $t \in I$ and

$$x(t) = x(t_0) + (R) \int_{t_0}^t F(x(s))ds,$$

where the last integral is the Riemann integral on Banach spaces.

The following result ensures us that if $F : O \rightarrow X$ is a continuous function, then the autonomous generalized ODE (7.2) and the autonomous ODE (7.7) coincide. Thus all the theory in the literature concerning autonomous ODE remains true also for autonomous generalized ODEs.

Proposition 7.5. *Let $F : O \rightarrow X$ be a function. If F is a continuous function, then the autonomous generalized ODE (7.2) and the autonomous ODE (7.7) coincide.*

Proof. Suppose $x : [\sigma, \eta] \rightarrow X$ satisfies the autonomous generalized ODE (7.2). Then, by Lemma 7.2, x is a continuous function on $[\sigma, \eta]$. Thus, the function $[\sigma, \eta] \ni \xi \mapsto F(x(\xi))$ is a continuous function on $[\sigma, \eta]$ and, therefore, the Kurzweil integral $\int_{\alpha}^{\xi} D[F(x(\tau))]t = (K) \int_{\alpha}^{\xi} F(x(s))ds$ and the Riemann integral $(R) \int_{\alpha}^{\xi} F(x(s))ds$ coincide (i.e are equal). Then

$$x(\xi) = x(\sigma) + (K) \int_{\sigma}^{\xi} F(x(s))ds = x(\sigma) + (R) \int_{\sigma}^{\xi} F(x(s))ds, \quad \xi \in [\sigma, \eta], \quad (7.8)$$

that is, x satisfies the autonomous ODE (7.7).

Analogously, if $x : [\sigma, \eta] \rightarrow X$ satisfies the autonomous ODE (7.7), by the equality (7.8), x satisfies the autonomous generalized ODE (7.2). $\square$

Proposition 7.6. *Let $H \in \mathcal{F}(\Omega, h, \omega)$ be such that*

$$H(x, t) = F(x)t, \quad (7.9)$$

for all $(x, t) \in \Omega$, where $F : O \rightarrow X$. Then

$$\|F(x) - F(y)\| \leq C\omega(\|x - y\|), \quad (7.10)$$

for every $x, y \in O$, with $C = \frac{|h(\beta) - h(\alpha)|}{\beta - \alpha}$. In particular, F is a continuous function.

Proof. let $x, y \in O$. Then, by condition (7.6) from Definition 7.3, we have

$$\begin{aligned} \|F(x)(\beta - \alpha) - F(y)(\beta - \alpha)\| &= \|H(x, \beta) - H(x, \alpha) - H(y, \beta) + H(y, \alpha)\| \\ &\leq (h(\beta) - h(\alpha))\omega(\|x - y\|), \end{aligned}$$

that is,

$$\|F(x) - F(y)\| \leq C\omega(\|x - y\|),$$

where $C = \frac{h(\beta) - h(\alpha)}{\beta - \alpha}$. Since $C\omega(\|x - y\|) \rightarrow 0$ as $x \rightarrow y$, we get $\|F(x) - F(y)\| \rightarrow 0$ as $x \rightarrow y$. Thus F is a continuous function. The result follows now from Proposition 7.5. $\square$

Corollary 7.7. *Let $H \in \mathcal{F}(\Omega, h, \omega)$ be such that*

$$H(x, t) = F(x)t, \quad (7.11)$$

for all $(x, t) \in \Omega$, where $F : O \rightarrow X$. Then the autonomous generalized ODE (7.2) and the autonomous ODE (7.7) coincide.

Proof. The result follows from Propositions 7.5 and 7.6. $\square$

7.2 Correspondence between $\tilde{\mathcal{F}}(\Omega, h, \omega)$ and $\mathcal{F}(\Omega, h, \omega)$

In this section, we prove that if $H \in \tilde{\mathcal{F}}(\Omega, h, \omega)$ with $H(x, t) = F(x)t$, then the autonomous generalized ODE (7.2) and the autonomous ODE (7.7) are coincide. Also, we will prove that the class $\tilde{\mathcal{F}}(\Omega, h, \omega)$ and the class $\mathcal{F}(\Omega, h, \omega)$ coincide.

The next definition is a slightly modified version of [44, Definition 1.7] (which is concerned with the special case $\omega(t) = t$)

Definition 7.8. A function $H : \Omega \rightarrow X$ belongs to the class $\tilde{\mathcal{F}}(\Omega, h, \omega)$, if the integral $\int_{\alpha}^{\beta} DH(z(\tau), t)$ exists for all $z \in G([\alpha, \beta], O)$ and the following conditions

$$\left\| \int_{s_1}^{s_2} DH(z(\tau), t) \right\| \leq |h(s_2) - h(s_1)| \quad (7.12)$$

and

$$\left\| \int_{s_1}^{s_2} D[H(z(\tau), t) - H(x(\tau), t)] \right\| \leq |h(s_2) - h(s_1)| \omega(\|z - w\|_{\infty}) \quad (7.13)$$

hold, for all $z, x \in G([\alpha, \beta], O)$ and all $s_2, s_1 \in [\alpha, \beta]$.

Note that Proposition 1.18 ensures us the existence of the integrals in (7.12) and (7.13).

The following statement is a slightly modified version of [44, Proposition 1.8] (which is concerned with the special case $\omega(t) = t$). The proof from [44] can be carried over without any changes and we reproduce it here.

Proposition 7.9. $\mathcal{F}(\Omega, h, \omega) \subset \tilde{\mathcal{F}}(\Omega, h, \omega)$.

Proof. Suppose $H \in \mathcal{F}(\Omega, h, \omega)$ and let $z, w \in G([\alpha, \beta], O)$ be given. By Proposition 1.18 and Lemma 1.15, we have (7.12). Thus it remains to prove that (7.13) also holds.

By the existence of the integrals $\int_{\alpha}^{\beta} DH(z(\tau), s)$ and $\int_{\alpha}^{\beta} DH(w(\tau), s)$, and by the integrability on subintervals, given $s_1, s_2 \in [\alpha, \beta]$, with $s_1 < s_2$, and given $\epsilon > 0$, there is a gauge δ of $[s_1, s_2]$ such that, for every δ -fine tagged division $D = (\tau_i, [t_{i-1}, t_i])$ of $[s_1, s_2]$, we have

$$\left\| \int_{s_1}^{s_2} DH(z(\tau), t) - \sum_{i=1}^{|D|} [H(z(\tau_i), t_i) - H(z(\tau_i), t_{i-1})] \right\| < \epsilon$$

and

$$\left\| \int_{s_1}^{s_2} DH(w(\tau), t) - \sum_{i=1}^{|D|} [H(w(\tau_i), t_i) - H(w(\tau_i), t_{i-1})] \right\| < \epsilon.$$

Thus,

$$\begin{aligned}
& \left\| \int_{s_1}^{s_2} [DH(z(\tau), t) - DH(w(\tau), t)] \right\| \\
& \leq \left\| \int_{s_1}^{s_2} DH(z(\tau), t) - \sum_{i=1}^{|D|} [H(z(\tau_i), t_i) - H(z(\tau_i), t_{i-1})] \right\| \\
& + \left\| \int_{s_1}^{s_2} DH(w(\tau), t) - \sum_{i=1}^{|D|} [H(w(\tau_i), t_i) - H(w(\tau_i), t_{i-1})] \right\| \\
& + \sum_{i=1}^{|D|} \|H(z(\tau_i), t_i) - H(z(\tau_i), t_{i-1}) - H(w(\tau_i), t_i) + H(w(\tau_i), t_{i-1})\| \\
& < 2\epsilon + \sum_{i=1}^{|D|} \omega(\|z(\tau_i) - w(\tau_i)\|)[h(t_i) - h(t_{i-1})] \leq 2\epsilon + \omega(\|z - w\|_\infty)[h(s_2) - h(s_1)].
\end{aligned}$$

Hence, since $\epsilon > 0$ is arbitrarily small, we obtain

$$\left\| \int_{s_1}^{s_2} [DH(z(\tau), t) - DH(w(\tau), t)] \right\| \leq \omega(\|z - w\|_\infty)[h(s_2) - h(s_1)]$$

which completes the proof. $\square$

Proposition 7.10. *Let $H \in \tilde{\mathcal{F}}(\Omega, h, \omega)$ be such that*

$$H(x, t) = F(x)t, \quad (7.14)$$

for all $(x, t) \in \Omega$, where $F : O \rightarrow X$. Then

$$\|F(b) - F(d)\| \leq K\omega(\|b - d\|), \quad (7.15)$$

for every $d, b \in O$, with $K = \frac{|h(\beta) - h(\alpha)|}{\beta - \alpha}$. In particular, F is a continuous function.

Proof. Let $d, b \in O$. Define the following functions

$$\begin{aligned}
\tilde{z} : [\alpha, \beta] &\rightarrow O \\
s &\mapsto \tilde{z}(s) = d,
\end{aligned} \quad (7.16)$$

and

$$\begin{aligned}
\tilde{w} : [\alpha, \beta] &\rightarrow O \\
s &\mapsto \tilde{w}(s) = b.
\end{aligned} \quad (7.17)$$

Notice that $\tilde{z}, \tilde{w} \in G([\alpha, \beta], O)$. Thus, by condition (7.13) from Definition 7.8, we have

$$\left\| \int_{\alpha}^{\beta} D[H(\tilde{z}(\tau), t) - H(\tilde{w}(\tau), t)] \right\| \leq |h(\beta) - h(\alpha)| \underbrace{\omega(\|\tilde{z} - \tilde{w}\|_{\infty})}_{=\omega(\|b-d\|)}. \quad (7.18)$$

We assert that

$$\int_{\alpha}^{\beta} DH(\tilde{z}(\tau), t) = F(d)(\beta - \alpha) \quad \text{and} \quad \int_{\alpha}^{\beta} DH(\tilde{w}(\tau), t) = F(b)(\beta - \alpha). \quad (7.19)$$

We will only prove the first equality holds. The second equality follows analogously. Let $U(\tau, t) = H(\tilde{z}(\tau), t)$, for all $(\tau, t) \in [\alpha, \beta] \times [\alpha, \beta]$. Notice that

$$U(\tau, t) = H(\tilde{z}(\tau), t) \stackrel{(7.16)}{\downarrow} H(d, t) \stackrel{(7.14)}{\downarrow} F(d)t,$$

that is,

$$U(\tau, t) = F(d)t, \quad \text{for every } (\tau, t) \in [\alpha, \beta] \times [\alpha, \beta]. \quad (7.20)$$

Let $D = (\tau_k, [t_{k-1}, t_k])$ be an arbitrary tagged division of the interval $[\alpha, \beta]$. Then,

$$\begin{aligned} S(U, D) &= \sum_{k=1}^{|D|} [U(\tau_k, t_k) - U(\tau_k, t_{k-1})] \stackrel{(7.20)}{\downarrow} \underbrace{\sum_{k=1}^{|D|} [F(d)t_k - F(d)t_{k-1}]}_{\text{telescopic sum}} \\ &= F(d)t_{|D|} - F(d)t_0 = F(d)(t_{|D|} - t_0) = F(d)(\beta - \alpha). \end{aligned}$$

Hence, we obtain

$$\int_{\alpha}^{\beta} DH(\tilde{z}(\tau), t) = F(d)(\beta - \alpha),$$

which completes the proof of the assertion.

By (7.18) and our (7.19), we obtain

$$\|F(b)(\beta - \alpha) - F(d)(\beta - \alpha)\| \leq |h(\beta) - h(\alpha)| \omega(\|b - d\|),$$

that is,

$$\|F(b) - F(d)\| \leq K\omega(\|b - d\|),$$

where $K = \frac{|h(\beta) - h(\alpha)|}{\beta - \alpha}$. This completes the proof of the proposition. $\square$

As an immediate consequence of Proposition 7.10, we have the following result.

Corollary 7.11. *Let $H \in \tilde{\mathcal{F}}(\Omega, h, \omega)$ be such that*

$$H(x, t) = F(x)t, \quad (7.21)$$

for all $(x, t) \in \Omega$, where $F : O \rightarrow X$. Then the autonomous generalized ODE (7.2) and the autonomous ODE (7.7) coincide.

Proof. The proof is a consequence of Propositions 7.5 and 7.10. □

Now, let us prove the inclusion $\tilde{\mathcal{F}}(\Omega, h, \omega) \subset \mathcal{F}(\Omega, h, \omega)$.

Proposition 7.12. $\tilde{\mathcal{F}}(\Omega, h, \omega) \subset \mathcal{F}(\Omega, h, \omega)$.

Proof. Suppose $H \in \tilde{\mathcal{F}}(\Omega, h, \omega)$. Let $x, y \in O$ and $s_2, s_1 \in [\alpha, \beta]$ be given. Define the following functions

$$\begin{aligned} \tilde{z} : [\alpha, \beta] &\rightarrow O \\ s &\mapsto \tilde{z}(s) = x, \end{aligned} \quad (7.22)$$

and

$$\begin{aligned} \tilde{w} : [\alpha, \beta] &\rightarrow O \\ s &\mapsto \tilde{w}(s) = y. \end{aligned} \quad (7.23)$$

Using the same arguments as in the proof of Theorem 7.10, we obtain

$$\int_{s_1}^{s_2} DH(\tilde{z}(\tau), t) = H(x, s_2) - H(x, s_1) \quad \text{and} \quad \int_{s_1}^{s_2} DH(\tilde{w}(\tau), t) = H(y, s_2) - H(y, s_1).$$

Thus, since $H \in \tilde{\mathcal{F}}(\Omega, h, \omega)$, we get

$$\|H(x, s_2) - H(x, s_1)\| = \left\| \int_{s_1}^{s_2} DH(\tilde{z}(\tau), t) \right\| \leq |h(s_2) - h(s_1)|$$

and

$$\begin{aligned} \|H(y, s_2) - H(y, s_1) - H(x, s_2) + H(x, s_1)\| &= \left\| \int_{s_1}^{s_2} DH(\tilde{w}(\tau), t) - \int_{s_1}^{s_2} DH(\tilde{z}(\tau), t) \right\| \\ &\leq |h(s_2) - h(s_1)| \omega(\|\tilde{w} - \tilde{z}\|_\infty) \\ &= |h(s_2) - h(s_1)| \omega(\|y - x\|), \end{aligned}$$

that is, $H \in \mathcal{F}(\Omega, h, \omega)$, obtaining the desired result. $\square$

Corollary 7.13. *The class $\tilde{\mathcal{F}}(\Omega, h, \omega)$ and the class $\mathcal{F}(\Omega, h, \omega)$ coincide.*

Proof. The result is a consequence of Propositions 7.9 and 7.12. $\square$

7.3 The classes $\tilde{\mathcal{F}}(\Omega_\infty, h, \omega)$ and $\tilde{\mathcal{F}}(\Omega_\infty, h, \omega, E)$

In this section, let us show that if $H \in \tilde{\mathcal{F}}(\Omega_\infty, h, \omega)$ with $H(x, t) = F(x)t$, then the autonomous generalized ODE (7.2) and autonomous ODE (7.7) are coincide. Moreover, we will prove a surprising consequence when $H \in \tilde{\mathcal{F}}(\Omega_\infty, h, \omega, E)$.

Let us assume that $\Omega_\infty = O \times [t_0, +\infty)$, where $O \subset \mathbb{R}^n$ is open and $t_0 \in \mathbb{R}$.

We recall the reader that $G([t_0, +\infty), \mathbb{R}^n)$ is the vector space of functions $\varphi : [t_0, +\infty) \rightarrow \mathbb{R}^n$ such that $\varphi|_{[u, v]}$ belongs to the space $G([u, v], \mathbb{R}^n)$ for all $[u, v] \subset [t_0, +\infty)$. For every compact interval $[u, v] \subset [t_0, +\infty)$, we can define a seminorm on $G([t_0, +\infty), \mathbb{R}^n)$

$$\|\cdot\|_{\infty, [u, v]} : G([t_0, +\infty), \mathbb{R}^n) \rightarrow \mathbb{R}$$

by

$$\|\psi\|_{\infty, [u, v]} := \sup_{t \in [u, v]} |\psi|_{[u, v]}(t) = \sup_{t \in [u, v]} |\psi(t)|,$$

for all $\psi \in G([t_0, +\infty), \mathbb{R}^n)$. The topology induced by the family of seminorms $\{\|\cdot\|_{\infty, K}\}_{K \in \Gamma}$ on $G([t_0, +\infty), \mathbb{R}^n)$, where $\Gamma := \{[u, v] : [u, v] \subset [t_0, +\infty)\}$ is called the *topology of locally uniform convergence* on $G([t_0, +\infty), \mathbb{R}^n)$. We will use the notation $x \in G([t_0, +\infty), O)$ for a function $x \in G([t_0, +\infty), \mathbb{R}^n)$ such that $x(s) \in O$, for all $s \in [t_0, +\infty)$.

Assume that $h : [t_0, +\infty) \rightarrow \mathbb{R}$ is a nondecreasing function defined on $[t_0, +\infty)$ and $\omega : [0, +\infty) \rightarrow \mathbb{R}$ is a continuous and increasing function with $\omega(0) = 0$.

Definition 7.14. A function $H : \Omega_\infty \rightarrow X$ belongs to the class $\tilde{\mathcal{F}}(\Omega_\infty, h, \omega)$, if the integral $\int_u^v DH(z(\tau), t)$ exists for every $z \in G([t_0, +\infty), O)$ and, for all $[u, v] \subset [t_0, +\infty)$, we have

$$\left\| \int_{s_1}^{s_2} DH(z(\tau), t) \right\| \leq |h(s_2) - h(s_1)| \quad (7.24)$$

and

$$\left\| \int_{s_1}^{s_2} D[H(z(\tau), t) - H(w(\tau), t)] \right\| \leq |h(s_2) - h(s_1)| \omega(\|z - w\|_{\infty, [s_1, s_2]}), \quad (7.25)$$

for all $z, w \in G([t_0, +\infty), O)$ and for all $s_2, s_1 \in [t_0, +\infty)$.

Remark 7.15. When the function ω is the identity, we write $\tilde{\mathcal{F}}(\Omega_\infty, h)$ instead of $\tilde{\mathcal{F}}(\Omega_\infty, h, \omega)$.

In the sequel, we present a surprising result which follows from the definition of the class $\tilde{\mathcal{F}}(\Omega_\infty, h, \omega)$.

Proposition 7.16. *Let $H \in \tilde{\mathcal{F}}(\Omega_\infty, h)$ be such that*

$$H(x, t) = F(x)t, \quad (7.26)$$

for all $(x, t) \in \Omega_\infty$, where $F : O \rightarrow X$. Then

$$\|F(b) - F(d)\| \leq K\omega(\|b - d\|), \quad (7.27)$$

for all $d, b \in O$, where $K = |h(t_0 + 1) - h(t_0)|$. In particular, F is a continuous function.

Proof. Let $d, b \in O$. Define the following functions

$$\begin{aligned} \tilde{z} : [t_0, +\infty) &\rightarrow O \\ s &\mapsto \tilde{z}(s) = d, \end{aligned} \quad (7.28)$$

and

$$\begin{aligned} \tilde{w} : [t_0, +\infty) &\rightarrow O \\ s &\mapsto \tilde{w}(s) = b. \end{aligned} \quad (7.29)$$

Notice that $\tilde{z}, \tilde{w} \in G([t_0, +\infty), O)$. Thus, by condition (7.25) of Definition 7.14, we have

$$\left\| \int_{t_0}^{t_0+1} D[H(\tilde{z}(\tau), t) - H(\tilde{w}(\tau), t)] \right\| \leq |h(t_0 + 1) - h(t_0)| \underbrace{\omega(\|\tilde{z} - \tilde{w}\|_{\infty, [t_0, t_0+1]})}_{=\omega(\|b-d\|)}. \quad (7.30)$$

Now, using the same arguments as in the proof of Proposition 7.10, we have

$$\int_{t_0}^{t_0+1} DH(\tilde{z}(\tau), t) = F(d)(t_0+1-t_0) = F(d) \quad \text{and} \quad \int_{t_0}^{t_0+1} DH(\tilde{w}(\tau), t) = F(b)(t_0+1-t_0) = F(b).$$

Then, (7.30) implies

$$\|F(b) - F(d)\| \leq K\omega(\|b - d\|),$$

where $K = |h(t_0 + 1) - h(t_0)|$ and we complete the proof. $\square$

The next result is an immediate consequence of Propositions 7.5 and 7.16.

Corollary 7.17. *Let $H \in \tilde{\mathcal{F}}(\Omega_\infty, h, \omega)$ be such that*

$$H(x, t) = F(x)t, \quad (7.31)$$

for all $(x, t) \in \Omega$, where $F : O \rightarrow X$. Then the autonomous generalized ODE (7.2) and the autonomous ODE (7.7) coincide.

Note that the main idea behind the proofs of Propositions 7.10, 7.12 and 7.16 is to define constant functions $\tilde{z}$ and $\tilde{w}$. But, now, let us define a class which does not contain such constant functions and let us show that, even in this class, our result remains true.

Consider the following set

$$E := \{\varphi \in G([t_0, +\infty), O) : \varphi \text{ is constant}\}.$$

Definition 7.18. Let $h : [t_0, +\infty) \rightarrow \mathbb{R}$ be a nondecreasing function defined on $[t_0, +\infty)$ and $\Omega_\infty = O \times [t_0, +\infty)$. A function $H : \Omega_\infty \rightarrow X$ belongs to the class $\tilde{\mathcal{F}}(\Omega_\infty, h, \omega, E)$, if the integral $\int_{\alpha}^{\beta} DH(z(\tau), t)$ exists for every $z \in G([t_0, +\infty), O) \setminus E$ and for every interval $[\alpha, \beta] \subset [t_0, +\infty)$, we have

$$\left\| \int_{s_1}^{s_2} DH(z(\tau), t) \right\| \leq |h(s_2) - h(s_1)| \quad (7.32)$$

and

$$\left\| \int_{s_1}^{s_2} D[H(z(\tau), t) - H(w(\tau), t)] \right\| \leq |h(s_2) - h(s_1)| \omega(\|z - w\|_{\infty, [s_1, s_2]}), \quad (7.33)$$

for all $z, w \in G([t_0, +\infty), O) \setminus E$ and all $s_2, s_1 \in [t_0, +\infty)$.

Proposition 7.19. *Let $H \in \tilde{\mathcal{F}}(\Omega_\infty, h, E)$ be such that*

$$H(x, t) = F(x)t, \quad (7.34)$$

for all $(x, t) \in \Omega_\infty$, where $F : O \rightarrow X$. Then

$$\|F(b) - F(d)\| \leq K\omega(\|b - d\|), \quad (7.35)$$

for every $d, b \in O$, with $K = |h(t_0 + 1) - h(t_0)|$. In particular, F is a continuous function.

Proof. Let $d, b \in O$. If $d = b$, then (7.35) holds trivially. We suppose $d \neq b$.

Define functions $\tilde{z}, \tilde{w} : [t_0, +\infty) \rightarrow O$ by

$$\tilde{z}(t) = \begin{cases} d, & t \in [t_0, t_0 + 1] \\ b, & t \in (t_0 + 1, +\infty) \end{cases}$$

and

$$\tilde{w}(t) = \begin{cases} b, & t \in [t_0, t_0 + 1] \\ d, & t \in (t_0 + 1, +\infty). \end{cases}$$

Notice that, $\tilde{z}, \tilde{w} \in G([t_0, +\infty), O) \setminus E$. Therefore, by condition (7.33) of Definition 7.18, we obtain

$$\left\| \int_{t_0}^{t_0+1} D[H(\tilde{z}(\tau), t) - H(\tilde{w}(\tau), t)] \right\| \leq |h(t_0 + 1) - h(t_0)| \underbrace{\omega(\|\tilde{z} - \tilde{w}\|_{\infty, [t_0, t_0+1]})}_{=\omega(\|b-d\|)}. \quad (7.36)$$

Now, using the same arguments as in the proof of Proposition 7.10, we have

$$\int_{t_0}^{t_0+1} DH(\tilde{z}(\tau), t) = F(d)(t_0+1-t_0) = F(d) \quad \text{and} \quad \int_{t_0}^{t_0+1} DH(\tilde{w}(\tau), t) = F(b)(t_0+1-t_0) = F(b).$$

Therefore, (7.36) can be transformed into

$$\|F(b) - F(d)\| \leq K\omega(\|b - d\|),$$

with $K = |h(t_0 + 1) - h(t_0)|$ and the proof is complete. $\square$

As an immediate consequence of Proposition 7.19, we have the following result.

Corollary 7.20. *Let $H \in \tilde{\mathcal{F}}(\Omega_\infty, h, \omega, E)$ be such that*

$$H(x, t) = F(x)t, \quad (7.37)$$

for all $(x, t) \in \Omega$, where $F : O \rightarrow X$. Then the autonomous generalized ODE (7.2) and the autonomous ODE (7.7) coincide.

Proof. The proof is a consequence of Propositions 7.5 and 7.19. $\square$

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