Surfaces in 4-space from the affine differential geometry viewpoint

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Superfícies em 4-espaço desde o ponto de vista da geometria diferencial afim

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minhas amadas esposa
e filha, com muito amor.

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Abstract

In this thesis, we study locally strictly convex surfaces from the affine differential viewpoint and generalize some tools for locally strictly submanifolds of codimension 2. We introduce a family of affine metrics on a locally strictly convex surface M in affine 4-space. Then, we define the symmetric and antisymmetric equiaffine planes associated with each metric. We show that if M is immersed in a locally strictly convex hyperquadric, then the symmetric and the antisymmetric planes coincide and contain the affine normal to the hyperquadric. In particular, any surface immersed in a locally strictly convex hyperquadric is affine semiumbilical with respect to the symmetric or antisymmetric equiaffine planes. More generally, by using the metric of the transversal vector field on M we introduce the affine normal plane and the families of the affine distance and height functions on M. We show that the singularities of the family of the affine height functions appear at directions on the affine normal plane and the singularities of the family of the affine distance functions appear at points on the affine normal plane and the affine focal points correspond as degenerate singularities of the family of affine distance functions. Moreover we show that if M is immersed in a locally strictly convex hypersurface, then the affine normal plane contains the affine normal vector to the hypersurface. Finally, we conclude that any surface immersed in a locally strictly convex hypersphere is affine semiumbilical.

Resumo

Nesta Tese estudamos as superfícies localmente estritamente convexas desde o ponto de vista da geometria diferencial afim e generalizamos algumas ferramentas para subvariedades localmente estritamente convexas de codimensão 2. Introduzimos uma família de métricas afins sobre uma superfície localmente estritamente convexa M no 4-espaço afim. Então, definimos os planos equiafins simétrico e antissimétrico associados com alguma métrica. Mostramos que se M é imersa em uma hiperquádrica localmente estritamente convexa, então os planos simétrico e antissimétrico são iguais e contêm o campo vetorial normal afim à hiperquádrica. Em particular, qualquer superfície imersa em uma hiperquádrica localmente estritamente convexa é semiumbílica afim com relação ao plano equiafim simétrico ou antissimétrico. Mais geralmente, usando a métrica do campo transversal sobre M introduzimos o plano normal afim e as famílias de funções distância e altura afim sobre M. Provamos que as singularidades da família de funções altura afim aparecem como direções do plano normal afim e as singularidades da família de funções distância afim aparecem como pontos sobre o plano normal afim e os pontos focais correspondem às singularidades degeneradas da família de funções distância afim. Também provamos que se M é uma superfície imersa em uma hipersuperfície localmente estritamente convexa, então o plano normal afim contém o vetor normal afim à hipersuperfície. Finalmente, concluímos que qualquer superfície imersa em uma hiperesfera localmente estritamente convexa é semiumbílica afim.

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CHAPTER 1

Introduction

The Erlanger Program by Sophus Lie and Felix Klein proposes that geometry (projective, affine, etc) is the theory of invariants of a transitive transformation group; more precisely, it studies the properties of spaces that are invariant relative to those symmetries.

The main purpose of this thesis is the study of locally strictly convex surfaces from the affine differential geometry viewpoint. We also generalize some concepts for locally strictly convex submanifolds of codimension 2 in the affine context.

1.1 Why study locally strictly convex surfaces?

Our interest to study affine differential geometry of surfaces $M \subset \mathbb{R}^4$ came from the understanding of the asymptotic configuration of M near an inflection (point where the two second fundamental forms are collinear). There is a conjecture (see [9]) that any locally strictly convex surface homeomorphic to the sphere has at least two inflection points. It is well known that a positive answer to this conjecture should imply a proof of the celebrated Carathéodory conjecture that any convex compact surface $M \subset \mathbb{R}^3$ has at least two umbilic points (see for instance [2]).

Partial proofs for the conjecture in $\mathbb{R}^4$ can be found in [8] for generic surfaces or in [14] for semiumbilical surfaces in the Euclidean sense (i.e., there is a non zero normal vector field whose shape operator is a multiple of the identity). The problem with the result in [14] is that the semiumbilical condition is not affine invariant, although the conjecture in $\mathbb{R}^4$ itself is affine in nature. Surfaces immersed in an Euclidean hypersphere or surfaces given by a product of two plane curves are examples of semiumbilical surfaces. However, surfaces immersed in other strictly convex hyperquadrics (like elliptic paraboloids or hyperboloids of two sheets) are not semiumbilical in general.

The affine differential geometry of hypersurfaces was developed by Blaschke (see [2, 10, 12]).

Concerning submanifolds of codimension 2 there are few results. Nomizu and Vrancken in [13] developed an affine theory for surfaces in $\mathbb{R}^4$. They used the affine metric of Burstin and Mayer [4], which is affine invariant, to construct the affine normal plane.

However, this affine metric and the corresponding affine normal plane present several problems if the surface is locally strictly convex (i.e., at each point $p \in M$ there is a tangent hyperplane with a non-degenerate contact which locally supports M).

The first point is that in order to define the affine metric, we need that the surface is non-degenerate, in particular, M cannot has inflections. But it is well known that any locally strictly convex compact surface M with Euler characteristic $\chi(M) \neq 0$ has at least an inflection, because of the Poincaré-Hopf formula (see [11]).

Another point is that even if M is non-degenerate, the affine metric of Burstin and Mayer is indefinite when M is locally strictly convex. This is the opposite of what you expect, for instance, if M is contained in a locally strictly convex hypersurface N, then the affine metric of N is positive definite.

Finally, if M is contained in a hypersurface N, you expect also some type of compatibility between the affine normals. This is important, for instance, if you want to consider contacts of the surface with affine hyperspheres. However, the affine normal plane (of Nomizu and Vrancken) to M does not contain the affine normal vector to N in general (see Remark 4.17).

1.2 An equiaffine theory

We introduce a new family of affine metrics g_ξ on a locally strictly convex surface $M \subset \mathbb{R}^4$ which are positive definite. Here, ξ is a transversal vector field such that ξ and T_pM span a local support hyperplane with non-degenerate contact at p. We show that when M is immersed in a locally strictly convex hypersurface N, then there is a natural choice of ξ in such a way that g_ξ coincides with the Blaschke metric of N restricted to M.

For each affine metric g_ξ, we define the symmetric and the antisymmetric equiaffine planes, by using analogous arguments to that of Nomizu and Vrancken in [13]. We also obtain algorithms to compute these normal planes. The main result is that if M is immersed in a locally strictly convex hyperquadric N, then the symmetric and the antisymmetric equiaffine planes coincide and contain the affine normal vector to N. As a consequence, any surface contained in a locally strictly convex hyperquadric is affine semiumbilical with respect to the symmetric or antisymmetric equiaffine planes. Another class of surfaces with the same property are those given by a product of two plane curves, hence our definition of affine semiumbilical surface has analogous properties as in the Euclidean case.

1.3 Using singularity theory

With the aim of studying locally strictly convex surfaces $M \subset \mathbb{R}^4$, we introduce the affine normal plane using the metric of transversal vector field g_ξ. This affine normal plane in general does not coincide with the affine normal plane of Nomizu and Vrancken nor with the symmetric and antisymmetric equiaffine planes. Then, by using the metric g_ξ we introduce the families of affine distance and height functions on M.

It is natural to study the singularities of the family of affine distance and height functions. It is known from the study of surfaces in 4-Euclidean space that the singularities of the families of distance and height functions are related to the extrinsic geometry of the surface. In our study, we find analogous results with the Euclidean case, by considering the new affine normal plane and the singularities of the corresponding families of affine distance and height functions on M. We also introduce the affine focal points as the degenerate singularities of the family of affine distance functions and define the affine normal curvature μ_ν in the direction ν. We show that the function μ_ν reaches its extrema at the affine ν-principal directions v and the ν-principal curvatures are given by $\mu_\nu(v)$.

When the surface M is immersed in a locally strictly convex hypersurface, then we show that the affine normal plane contains the affine normal vector field to the hypersurface. In particular, any surface immersed in a locally strictly convex hypersphere is affine semiumbilical (i.e., there is a vector field ν on the affine normal plane such that S_ν is a multiple of the identity). This result generalizes the main result in [16]: if M is immersed in a locally strictly convex hyperquadric N, then the symmetric and the antisymmetric equiaffine planes coincide and contain the affine normal vector to N. We also show that the semiumbilical points on a surface are the points such that the affine normal curvature $\mu_\nu(v)$ is a constant function. Finally we show that the product of two plane curves is an affine semiumbilical surface. In this way, we have generalized the results of the Euclidean case: surfaces immersed in hyperspheres and surfaces given by a product of plane curves are semiumbilical.

CHAPTER 2

Equiaffine structures on surfaces in $\mathbb{R}^4$

In this chapter we introduce an affine theory to study locally strictly convex surfaces in $\mathbb{R}^4$, in other words, we define an affine metric and some equiaffine planes on the surface.

2.1 Basic affine geometry

The affine geometry studies the properties of the affine space (Definition 2.1) that are invariant relative to the group of affine transformations (Definition 2.2). In this section we recall the basic definitions of surfaces in $\mathbb{R}^4$ (see [12, 13]).

Definition 2.1. Denote by A^{n+1} an $(n+1)$-dimensional affine space, by V the real vector space associated with A^{n+1}, and by $\pi : A^{n+1} \times A^{n+1} \to V$ the mapping relating A^{n+1} and V such that

1. for any three points $p, q, r \in A^{n+1}$ we have $\pi(p, q) + \pi(q, r) = \pi(p, r)$,
2. for any $p \in A^{n+1}$ and $v \in V$ there exists a unique $q \in A^{n+1}$ such that $\pi(p, q) = v$.

The mapping π allows to define the affine structure of A^{n+1} from the structure of V. It is very suggestive to write $\overrightarrow{pq} = \pi(p, q) = v$.

Definition 2.2. Let A_1, A_2 be affine spaces and V_1, V_2, resp., their associated vector spaces and $\pi_i : A_i \times A_i \to V_i$. *A mapping* $\alpha : A_1 \to A_2$ *is affine*, if there exists a linear mapping $L_\alpha : V_1 \to V_2$ such that $\pi_2(\alpha(p), \alpha(q)) = L_\alpha(\pi_1(p, q))$, $\forall p, q \in A_1$. The mapping L_α is uniquely determined and is called the linear mapping associated with α. The affine mapping α is injective (surjective) if and only if L_α is injective (surjective). For $A_1 = A_2$, it is defined the determinant of α to be $\det \alpha := \det L_\alpha$.

We denote by $\mathbb{R}^{n+1}$ *the* $(n+1)$*-dimensional affine space associated with* $(n+1)$*-dimensional vector space* $\mathbb{R}^{n+1}$*, with the mapping* $(p,q) \mapsto \overrightarrow{pq} = q - p$*.*

For the affine space $\mathbb{R}^{n+1}$ we have the following affine transformation groups.

- The affine regular group: $\mathfrak{a}(n+1) := \{\alpha : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} / L_\alpha \text{ is regular}\}$.
- The unimodular or equiaffine group : $\mathfrak{s}(n+1) := \{\alpha \in \mathfrak{a}(n+1) / \det \alpha = 1\}$.
- The centroaffine group with center $p \in \mathbb{R}^{n+1}$: $\mathfrak{z}_p(n+1) := \{\alpha \in \mathfrak{a}(n+1) / \alpha(p) = p\}$.

In particular one of the purposes of the affine differential geometry is the study of properties of submanifolds M^m in n-affine space that are invariant under the group of unimodular affine transformations $\mathfrak{s}(n)$. Therefore we recall some concepts as: affine immersion, affine fundamental form and affine induced connection.

Let $(\mathbb{R}^{n+k}, D)$ be the affine $(n+k)$-space with D the usual flat connection on $\mathbb{R}^{n+k}$ and let (M, ∇) a differentiable n-manifold M with ∇ an affine connection.

Definition 2.3. A smooth immersion $f : M \to \mathbb{R}^{n+k}$ is said to be an *affine immersion* if: there is a smooth k-dimensional distribution σ along f: $p \in M \mapsto \sigma_p$, a subspace of $T_{f(p)}(\mathbb{R}^{n+k})$ such that

$$T_{f(p)}(\mathbb{R}^{n+k}) = f_*(T_pM) \oplus \sigma_p \tag{2.1}$$

and for all tangent vector fields X, Y on M,

$$(D_X f_* Y)_p = (f_*(\nabla_X Y))_p + (h(X,Y))_p, \tag{2.2}$$

where $(\nabla_X Y)_p \in T_pM$ and $h(X,Y)_p \in \sigma_p$, for all $p \in M$.

Since the distribution σ: $p \in M \mapsto \sigma_p$, is smooth, each point p has a local basis, namely, a system of k smooth vector fields $\xi_1, \ldots, \xi_k$ on a neighborhood U of p such that

$$\sigma_y = \text{span}\{\xi_1(y), \ldots, \xi_k(y)\}, \quad \forall y \in U.$$

Note that the distribution σ may be regarded as a bundle of transversal k-subspaces.

Definition 2.4. The map $(X,Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto h(X,Y)$ defines for each point $p \in M$ a symmetric bilinear map $T_pM \times T_pM \to \sigma_p$, by taking local extensions of the tangent vectors. It is well known that the definition of $h(X,Y)$ is independent of the local extensions. The map h is called the *affine fundamental form.*

Suppose a smooth manifold M, not provided with any particular affine connection, is immersed into $(\mathbb{R}^{n+k}, D)$. If we take a distribution of transversal subspaces as in (2.1), then we can get a torsion-free affine connection ∇ with (2.2) as defining equation.

Definition 2.5. The map $\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfies the conditions for covariant differentiation, it is called the *induced connection by* σ, i.e. $\nabla = \nabla(\sigma)$.

Remark 2.6. As a result $(M, \nabla) \to (R^{n+k}, D)$ is an affine immersion.

Next we show basic concepts for surfaces in $\mathbb{R}^4$ (see [13]).

Let $\mathbb{R}^4$ be the affine 4-space and D the usual flat connection on $\mathbb{R}^4$. Let $M \subset \mathbb{R}^4$ be an immersed surface and let σ be a transversal plane bundle on M. Then, for all $p \in M$, $\sigma_p \subset T_p\mathbb{R}^4$ is a plane such that

$$T_p\mathbb{R}^4 = T_pM \oplus \sigma_p,$$

and for all tangent vector fields X, Y on M,

$$(D_XY)_p = (\nabla_XY)_p + h(X,Y)_p,$$

where $(\nabla_XY)_p \in T_pM$ and $h(X,Y)_p \in \sigma_p$, for all $p \in M$.

We note that for $p \in M$, there are ξ_1, ξ_2 transversal vector fields defined on some neighborhood U_p such that: $\sigma_q = \mathrm{span}\{\xi_1(q), \xi_2(q)\}$, $\forall q \in U_p$ (see Definition 2.3).

Then for tangent vector fields X, Y on M we have:

$$D_XY = \nabla_XY + h^1(X,Y)\xi_1 + h^2(X,Y)\xi_2, \tag{2.3}$$

$$D_X\xi_1 = -S_1X + \tau_1^1(X)\xi_1 + \tau_1^2(X)\xi_2, \tag{2.4}$$

$$D_X\xi_2 = -S_2X + \tau_2^1(X)\xi_1 + \tau_2^2(X)\xi_2, \tag{2.5}$$

where $\nabla = \nabla(\sigma)$ is a torsion free affine connection, h^1, h^2 are bilinear symmetric forms, S_1, S_2 are $(1,1)$ tensor fields, and τ_i^j are 1-forms on M. We call ∇ the affine connection induced by the transversal plane bundle σ.

For a transversal vector field ξ, we can write

$$D_X\xi = -S_\xi X + \nabla_X^\perp\xi,$$

where $-S_\xi X$ is the tangent component of $D_X\xi$ and $\nabla_X^\perp\xi$ is the σ-component of $D_X\xi$. The operator $-S_\xi$ is linear (in fact, a $(1,1)$-tensor field) and is called shape operator. We also call $\nabla^\perp$ the affine normal connection induced by the transversal plane bundle σ.

Let $\overline{\sigma}$ be another transversal plane bundle and let $\overline{\xi}_1$ and $\overline{\xi}_2$ be other transversal vector fields that locally span $\overline{\sigma}$.

Then we can write

$$\xi_1 = \phi\overline{\xi}_1 + \psi\overline{\xi}_2 + Z_1, \tag{2.6}$$
$$\xi_2 = \rho\overline{\xi}_1 + \beta\overline{\xi}_2 + Z_2, \tag{2.7}$$

where ϕ, ψ, ρ, β are local functions on M satisfying $\phi\beta - \psi\rho \neq 0$, and Z_1 and Z_2 are tangent vector fields on M. Substituting Equations (2.6) and (2.7) into Equation (2.3), we obtain

$$D_XY = (\nabla_XY + h^1(X,Y)Z_1 + h^2(X,Y)Z_2)$$
$$+(\phi h^1(X,Y) + \rho h^2(X,Y))\overline{\xi}_1 + (\psi h^1(X,Y) + \beta h^2(X,Y))\overline{\xi}_2.$$

On the other hand, we can write Equation (2.3) for $\overline{\xi}_1$ and $\overline{\xi}_2$,

$$D_XY = \overline{\nabla}_XY + \overline{h}^1(X,Y)\overline{\xi}_1 + \overline{h}^2(X,Y)\overline{\xi}_2.$$

By comparing we obtain, the following relations

$$\overline{\nabla}_XY = \nabla_XY + h^1(X,Y)Z_1 + h^2(X,Y)Z_2, \tag{2.8}$$
$$\overline{h}^1(X,Y) = \phi h^1(X,Y) + \rho h^2(X,Y), \tag{2.9}$$
$$\overline{h}^2(X,Y) = \psi h^1(X,Y) + \beta h^2(X,Y). \tag{2.10}$$

In the study of the affine differential geometry of surfaces in $\mathbb{R}^4$ it is important to consider the following equations which appear in [13]: the equation of Gauss (2.11), the Equations of Codazzi (2.12), (2.13), (2.14) and (2.15), and the Equations of Ricci (2.16), (2.17), (2.18) and (2.19).

$$R(X,Y)Z = h^1(Y,Z)S_1X + h^2(Y,Z)S_2X - h^1(X,Z)S_1Y - h^2(X,Z)S_2Y. \tag{2.11}$$

$$(\nabla_Xh^1)(Y,Z) + \tau_1^1(X)h^1(Y,Z) + \tau_2^1(X)h^2(Y,Z), \tag{2.12}$$

$$(\nabla_Xh^2)(Y,Z) + \tau_1^2(X)h^1(Y,Z) + \tau_2^2(X)h^2(Y,Z), \tag{2.13}$$

$$(\nabla_XS_1)Y - (\nabla_YS_1)X = -\tau_1^1(Y)S_1X + \tau_1^1(X)S_1Y - \tau_1^2(Y)S_2X + \tau_1^2(X)S_2Y, \tag{2.14}$$

$$(\nabla_XS_2)Y - (\nabla_YS_2)X = -\tau_2^1(Y)S_1X + \tau_2^1(X)S_1Y - \tau_2^2(Y)S_2X + \tau_2^2(X)S_2Y. \tag{2.15}$$

$$h^1(X,S_1Y) - h^1(Y,S_1X) = d\tau_1^1(X,Y) + \tau_1^2(Y)\tau_2^1(X) - \tau_2^1(Y)\tau_1^2(X), \tag{2.16}$$

$$h^2(X, S_2Y) - h^2(Y, S_2X) = d\tau_2^2(X, Y) + \tau_2^1(Y)\tau_1^2(X) - \tau_1^2(Y)\tau_2^1(X), \tag{2.17}$$

$$\begin{aligned} h^2(X, S_1Y) - h^2(Y, S_1X) = d\tau_1^2(X, Y) + \tau_1^1(Y)\tau_1^2(X) - \tau_1^2(Y)\tau_1^1(X) \\ + \tau_1^2(Y)\tau_2^2(X) - \tau_2^2(Y)\tau_1^2(X), \end{aligned} \tag{2.18}$$

$$\begin{aligned} h^1(X, S_2Y) - h^1(Y, S_2X) = d\tau_2^1(X, Y) + \tau_1^1(X)\tau_2^1(Y) - \tau_2^1(X)\tau_1^1(Y) \\ + \tau_2^1(X)\tau_2^2(Y) - \tau_2^2(X)\tau_2^1(Y). \end{aligned} \tag{2.19}$$

2.2 The metric of the transversal vector field

In this section, we introduce a family of affine metrics and the affine normal planes: the antisymmetric and symmetric equiaffine planes. We prove the existence and unicity of these planes. This study is developed on locally strictly convex surfaces in $\mathbb{R}^4$.

Definition 2.7. A submanifold $M \subset \mathbb{R}^m$ has *non-degenerate contact* with a hyperplane H at $p \in M$ if $A : \mathbb{R}^m \to \mathbb{R}$ is any linear function such that $H = \{x : A(x-p) = 0\}$ then $A_{|M} : M \to \mathbb{R}$ has a non-degenerate critical point at $p \in M$.

Note that the hyperplane H in the Definition 2.7 is an affine hyperplane.

Definition 2.8. A hyperplane H is a nonsingular support hyperplane of M if M lies on one side of H , $H \cap M = \{p\}$, and H has non-degenerate contact with M.

Definition 2.9. A submanifold $M \subset \mathbb{R}^m$ is strictly convex if through every point of M there passes a nonsingular support hyperplane, i.e., a hyperplane with non-degenerate contact with respect to which M lies strictly on one side.

Definition 2.10. A submanifold $M \subset \mathbb{R}^m$ is called *locally strictly convex* at p if there is a neighborhood U of p such that $M \cap U$ is strictly convex; and M is locally strictly convex surface if is locally strictly convex in each point $p \in M$.

Let $M \subset \mathbb{R}^4$ be an oriented locally strictly convex surface, let $\mathfrak{u} = \{X_1, X_2\}$ be a positively oriented local tangent frame of a point $p \in M$ and let ξ be a transversal vector field on M.

Definition 2.11. We define the symmetric bilinear form $G_\mathfrak{u}$ on M to be

$$G_\mathfrak{u}(Y, Z) = [X_1, X_2, D_ZY, \xi].$$

We fix ξ such that $G_{\mathfrak{u}}$ is a positive definite symmetric bilinear form, this is possible because M is locally strictly convex and we call such a ξ a *metric field.* (Just consider a transversal vector field ξ such that $[X_1(p), X_2(p), x-p, \xi(p)] = 0$ determines a support hyperplane).

The metric field ξ is defined only locally, but since all our results are local, we can assume without lost of generality that ξ is globally defined and that M is globally oriented.

We define the *metric of the transversal vector field*, denoted by g_ξ, by

$$g_\xi(Y, Z) = \frac{G_{\mathfrak{u}}(Y, Z)}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{\frac{1}{4}}},$$

where $\det_{\mathfrak{u}} G_{\mathfrak{u}} = \det(G_{\mathfrak{u}}(X_i, X_j))$.

Lemma 2.12. *The symmetric bilinear form g_ξ does not depend on the choice of the local tangent frame $\mathfrak{u}$, provided it is positively oriented.*

Proof. Let $\mathfrak{v} = \{Y_1, Y_2\}$ be another local tangent frame on a neighborhood U of $p \in M$, then there exist functions a, b, c and d with $ad - bc > 0$, defined on U such that $Y_1 = aX_1 + bX_2$ and $Y_2 = cX_1 + dX_2$. Note that

$$G_{\mathfrak{v}}(Y, Z) = [Y_1, Y_2, D_Z Y, \xi] = (ad - bc) G_{\mathfrak{u}}(Y, Z).$$

By properties of the determinant, it follows that $\det_{\mathfrak{v}} G_{\mathfrak{v}} = (ad - bc)^2 \det_{\mathfrak{v}} G_{\mathfrak{u}}$. On the other hand, from a simple computation $\det_{\mathfrak{v}} G_{\mathfrak{u}} = (ad - bc)^2 \det_{\mathfrak{u}} G_{\mathfrak{u}}$, therefore

$$\det_{\mathfrak{v}} G_{\mathfrak{v}} = (ad - bc)^4 \det_{\mathfrak{u}} G_{\mathfrak{u}}.$$

Finally,

$$\frac{G_{\mathfrak{v}}(Y, Z)}{(\det_{\mathfrak{v}} G_{\mathfrak{v}})^{1/4}} = \frac{(ad - bc) G_{\mathfrak{u}}(Y, Z)}{((ad - bc)^4 \det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/4}} = \frac{G_{\mathfrak{u}}(Y, Z)}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/4}}.$$

□

Remark 2.13. Let $\xi \in \mathbb{R}^4$ be a metric field, the definition of the metric g_ξ depends only on the equivalence class of ξ in the quotient space $\mathbb{R}^4/T_pM$, which is a 2-dimensional vector space. In fact, if $[\xi] = [\xi']$ then $\xi = \xi' + Z$, with $Z \in T_pM$ therefore $g_\xi = g_{\xi'}$. Thus, we denote $g_{[\xi]} = g_\xi$.

In this way, the family of metrics $\{g_{[\xi]}\}_{[\xi] \in A}$ is parameterized by an open set of $\mathbb{R}^4/T_pM$ given by:

$$A = \{[\xi] \in \mathbb{R}^4/T_pM : \ g_{[\xi]} \text{ is positive definite } \}.$$

This open subset A is not empty whenever M is strictly convex in a neighborhood of p.

It follows from the definition that the family of metrics $\{g_{[\xi]}\}_{[\xi]\in A}$ is an affine invariant of M that does not depend on the chosen transversal plane bundle σ.

Remark 2.14. Although the metric $g_{[\xi]}$ does not depend on the chosen transversal plane bundle σ, once we fix it, we can use it to compute the metric as a second fundamental form. In fact, let $\{\xi_1, \xi_2\}$ be a transversal frame for σ_p and let us denote the second fundamental forms by $h^1(X, Y)$ and $h^2(X, Y)$. For all $(r, s) \in \mathbb{R}^2$ we denote

$$h_{r,s}(X, Y) = rh^1(X, Y) + sh^2(X, Y)$$

and consider the open subset $\tilde{A} \subset \mathbb{R}^2$ given by:

$$\tilde{A} = \{(r, s) \in \mathbb{R}^2 : \ h_{r,s} \text{ is positive definite }\}.$$

Note that, for all $[\xi] \in A$, there is a unique $(r, s) \in \tilde{A}$ such that $g_{[\xi]} = h_{r,s}$. In fact, there is a unique representative $\xi \in \sigma_p$ of $[\xi]$ given by $\xi = b_1\xi_1 + b_2\xi_2$ and therefore $(r, s) = \lambda(b_2, -b_1)$, where

$$\lambda = \frac{[X_1, X_2, \xi_1, \xi_2]}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{\frac{1}{4}}}.$$

Remark 2.15. Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and let π be a support hyperplane with non-degenerate contact at p. Then there is a transversal vector field ξ such that

$$\pi = \ker\{x \mapsto [Y_1(p), Y_2(p), x - p, \xi(p)]\}$$

and g_ξ is positive definite, where $\{Y_1, Y_2\}$ is a tangent local frame on M. We say that ξ determines the support hyperplane π. Note that $\lambda\xi$ determines the same hyperplane π. So, the support hyperplane π determines a family of metrics $(g_{\lambda\xi})_\lambda$.

From now on, we fix a metric field ξ and consider a local orthonormal tangent frame relative to the metric g_ξ on M, that is, $\mathfrak{u} = \{X_1, X_2\}$ is a tangent frame defined on some neighborhood U of $p \in M$ such that

$$g_\xi(X_i, X_j) = \delta_{ij}.$$

Theorem 2.16. *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ a metric field. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local orthonormal tangent frame of g_ξ and let σ be an arbitrary transversal plane bundle. Then there exists a unique local frame $\{\xi_1, \xi_2\}$ of σ, such that*

$$[X_1, X_2, \xi_1, \xi_2] = 1, \qquad h^1(X_1, X_1) = 0, \qquad -\xi_1 \in [\xi], \qquad h^2(X_i, X_j) = \delta_{ij}.$$

Proof. Let p be a point in M and let $\{\nu_1, \nu_2\}$ be any local frame of σ in a neighborhood U of p. We can assume that X_1 and X_2 are defined on U. Now, we write

$$[\xi] = \lambda_3\nu_1 + \lambda_4\nu_2 + T_pM.$$

Using the notation: $h^1(X_1, X_1) = a$, $h^1(X_1, X_2) = b$, $h^1(X_2, X_2) = c$, $h^2(X_1, X_1) = e$, $h^2(X_1, X_2) = f$, $h^2(X_2, X_2) = g$ and $K = [X_1, X_2, \nu_1, \nu_2]$, we compute the bilinear form $G_{\mathfrak{u}}$: $G_{\mathfrak{u}}(X_1, X_1) = (a\lambda_4 - e\lambda_3)K$, $G_{\mathfrak{u}}(X_1, X_2) = (b\lambda_4 - f\lambda_3)K$, $G_{\mathfrak{u}}(X_2, X_2) = (c\lambda_4 - g\lambda_3)K$. By using the change

$$\nu_1 = \alpha\xi_1 + \beta\xi_2, \qquad \nu_2 = \varphi\xi_1 + \psi\xi_2,$$

we obtain the affine fundamental forms from the new frame $\{\xi_1, \xi_2\}$:

$$\overline{h}^1(X_1, X_1) = \alpha a + \varphi e, \quad \overline{h}^1(X_1, X_2) = \alpha b + \varphi f, \quad \overline{h}^1(X_2, X_2) = \alpha c + \varphi g,$$

$$\overline{h}^2(X_1, X_1) = \beta a + \psi e, \quad \overline{h}^2(X_1, X_2) = \beta b + \psi f, \quad \overline{h}^2(X_2, X_2) = \beta c + \psi g.$$

Note that $G_{\mathfrak{u}}(X_1, X_2) = 0$, hence $G_{\mathfrak{u}}(X_1, X_1) \neq 0$ and $a\lambda_4 - e\lambda_3 \neq 0$ and then the following system

$$\begin{aligned} 1 &= \beta a + \psi e, \\ 0 &= \beta\lambda_3 + \psi\lambda_4, \end{aligned}$$

has solution (β, ψ) given by

$$\beta = \frac{\lambda_4}{a\lambda_4 - e\lambda_3}, \qquad \psi = \frac{-\lambda_3}{a\lambda_4 - e\lambda_3}.$$

Now substitute β and ψ in $\overline{h}^2(X_i, X_j)$ and we prove that $\overline{h}^2(X_i, X_j) = \delta_{ij}$. In fact,

$$\begin{aligned} \overline{h}^2(X_1, X_2) &= \beta b + \psi f = (\frac{\lambda_4}{a\lambda_4 - e\lambda_3})b + (\frac{-\lambda_3}{a\lambda_4 - e\lambda_3})f = \frac{G_{\mathfrak{u}}(X_1, X_2)}{(a\lambda_4 - e\lambda_3)K} \\ &= \frac{G_{\mathfrak{u}}(X_1, X_2)}{G_{\mathfrak{u}}(X_1, X_1)} = \frac{G_{\mathfrak{u}}(X_1, X_2)/(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/4}}{G_{\mathfrak{u}}(X_1, X_1)/(det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/4}} = \frac{g_\xi(X_1, X_2)}{g_\xi(X_1, X_1)} = 0. \end{aligned}$$

From the equation $0 = \overline{h}^1(X_1, X_1) = \alpha a + \varphi e$ we can write $\alpha = Re$ and $\varphi = -Ra$, therefore

$$\begin{aligned} [X_1, X_2, \nu_1, \nu_2] &= [X_1, X_2, \xi_1, \xi_2](\alpha\psi - \beta\varphi) = (\alpha\psi - \beta\varphi) \\ &= ((Re)\psi - \beta(-Ra)) = R, \end{aligned}$$

we conclude $R = K$, $\alpha = Ke$ and $\varphi = -Ka$.

It only remains to prove that $[\xi] = -[\xi_1]$. First we note that $G_{\mathfrak{u}}(X_1, X_2) = 0$, because $\{X_1, X_2\}$ is an orthonormal tangent frame relative to g_ξ. Moreover,

$$(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/2} = \frac{\det_{\mathfrak{u}} G_{\mathfrak{u}}}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/2}} = \frac{G_{\mathfrak{u}}(X_1,X_1)}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/4}} \frac{G_{\mathfrak{u}}(X_2,X_2)}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/4}} = 1.$$

It follows that $\lambda_3\alpha + \lambda_4\varphi = \lambda_3 Ke - \lambda_4 Ka = K(\lambda_3 e - \lambda_4 a) = -(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/4} = -1$. Finally, we compute $[\xi]$:

$$\begin{aligned}
[\xi] &= \lambda_3\nu_1 + \lambda_4\nu_2 + T_pM \\
&= \lambda_3(\alpha\xi_1 + \beta\xi_2) + \lambda_4(\varphi\xi_1 + \psi\xi_2) + T_pM \\
&= \underbrace{(\lambda_3\alpha + \lambda_4\varphi)}_{-1}\xi_1 + \underbrace{(\lambda_3\beta + \lambda_4\psi)}_{0}\xi_2 + T_pM.
\end{aligned}$$

□

Lemma 2.17. *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ a metric field. Let $\mathfrak{u} = \{X_1, X_2\}$ and $\mathfrak{v} = \{Y_1, Y_2\}$ be two orthonormal frames and let σ a transversal plane bundle. So we can write*

$$Y_1 = \cos\theta X_1 + \sin\theta X_2, \tag{2.20}$$

$$Y_2 = \epsilon(-\sin\theta X_1 + \cos\theta X_2), \tag{2.21}$$

where $\epsilon = \pm 1$. If we denote by $\{\xi_1, \xi_2\}$ (resp. $\{\bar{\xi}_1, \bar{\xi}_2\}$) the frame of Theorem 2.16 corresponding to $\mathfrak{u}$ (resp. $\mathfrak{v}$), then

$$\begin{aligned}
&\xi_1 = \bar{\xi}_1, \\
&\xi_2 = -(\sin 2\theta h^1(X_1, X_2) + \sin^2\theta h^1(X_2, X_2))\bar{\xi}_1 + \bar{\xi}_2,
\end{aligned}$$

and also

$$\begin{aligned}
2\bar{h}^1(Y_1, Y_2) &= \epsilon(2\cos 2\theta h^1(X_1, X_2) + \sin 2\theta h^1(X_2, X_2)), \\
\bar{h}^1(Y_2, Y_2) &= \cos 2\theta h^1(X_2, X_2) - 2\sin 2\theta h^1(X_1, X_2), \\
4\bar{h}^1(Y_1, Y_2)^2 + \bar{h}^1(Y_2, Y_2) &= 4h^1(X_1, X_2)^2 + h^1(X_2, X_2)^2.
\end{aligned}$$

Proof. From Theorem 2.16, we have $[\xi_1] = -[\xi] = [\bar{\xi}_1]$. Since ξ_1 and $\bar{\xi}_1$ belong to the same transversal plane we conclude that $\xi_1 = \bar{\xi}_1$. We compute now the affine connection in the different frames to compare them. By using the frame $\{\bar{\xi}_1, \bar{\xi}_2\}$, it follows from Theorem

2.16 that

$$D_{Y_1}Y_1 = \nabla_{Y_1}Y_1 + \overline{\xi}_2,$$

and by using the frame $\{\xi_1, \xi_2\}$ and equation,

$$D_{Y_1}Y_1 = \nabla_{Y_1}Y_1 + h^1(Y_1, Y_1)\xi_1 + h^2(Y_1, Y_1)\xi_2.$$

Hence, $\overline{\xi}_2 = h^1(Y_1, Y_1)\xi_1 + h^2(Y_1, Y_1)\xi_2$, and from equation (2.20) we obtain:

$$\overline{\xi}_2 = (\sin 2\theta h^1(X_1, X_2) + \sin^2\theta h^1(X_2, X_2))\xi_1 + \xi_2.$$

Analogously, by comparing $D_{Y_1}Y_2$ (and $D_{Y_2}Y_2$) in the two frames, we obtain:

$$\overline{h}^1(Y_1, Y_2) = \cos 2\theta h^1(X_1, X_2) + \sin\theta\cos\theta h^1(X_2, X_2),$$

$$\overline{h}^1(Y_2, Y_2) = \cos 2\theta h^1(X_2, X_2) - 2\sin 2\theta h^1(X_1, X_2).$$

The last equality follows by direct computation. □

Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and let ξ be a metric field. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local orthonormal tangent frame. If we denote the corresponding transversal vector fields obtained by Theorem 2.16 by ξ_1 and ξ_2, we define the metric $g^\perp_{\mathfrak{u}}$ by setting

$$\begin{aligned} g^\perp_{\mathfrak{u}}(\xi_1, \xi_1) &= 1, \\ g^\perp_{\mathfrak{u}}(\xi_1, \xi_2) &= -\frac{1}{2}h^1(X_2, X_2), \\ g^\perp_{\mathfrak{u}}(\xi_2, \xi_2) &= 4h^1(X_1, X_2)^2 + \frac{5}{4}h^1(X_2, X_2)^2, \end{aligned}$$

and extending it linearly on σ.

Lemma 2.18. *Take $\mathfrak{u} = \{X_1, X_2\}$, $\mathfrak{v} = \{Y_1, Y_2\}$, ξ_1, ξ_2, $\overline{\xi}_1$ and $\overline{\xi}_2$ as in Lemma 2.17 and Theorem 2.16. Then*

$$g^\perp_{\mathfrak{u}}(\xi, \eta) = g^\perp_{\mathfrak{v}}(\xi, \eta).$$

Proof. It is enough to show that equality occurs on the frame $\{\xi_1, \xi_2\}$. We have $g^\perp_{\mathfrak{v}}(\xi_1, \xi_1) = g^\perp_{\mathfrak{v}}(\overline{\xi}_1, \overline{\xi}_1) = 1$, but also

$$\begin{aligned} g^\perp_{\mathfrak{v}}(\xi_1, \xi_2) &= g^\perp_{\mathfrak{v}}(\overline{\xi}_1, h^1(Y_1, Y_1)\overline{\xi}_1 + \overline{\xi}_2) \\ &= h^1(Y_1, Y_1) - \frac{1}{2}\overline{h}^1(Y_2, Y_2) = -\frac{1}{2}h^1(X_2, X_2), \end{aligned}$$

and finally,

$$\begin{aligned} g_{\mathfrak{v}}^{\perp}(\xi_2,\xi_2) &= g_{\mathfrak{v}}^{\perp}(h^1(Y_1,Y_1)\overline{\xi}_1+\overline{\xi}_2, h^1(Y_1,Y_1)\overline{\xi}_1+\overline{\xi}_2) \\ &= h^1(Y_1,Y_1)^2+2h^1(Y_1,Y_1)g_{\mathfrak{v}}^{\perp}(\overline{\xi}_1,\overline{\xi}_2)+g_{\mathfrak{v}}^{\perp}(\overline{\xi}_2,\overline{\xi}_2) \\ &= h^1(Y_1,Y_1)^2-h^1(Y_1,Y_1)\overline{h}^1(Y_2,Y_2)+4\overline{h}^1(Y_1,Y_2)^2+\frac{5}{4}\overline{h}^1(Y_2,Y_2)^2 \\ &= (h^1(Y_1,Y_1)-\frac{1}{2}\overline{h}^1(Y_2,Y_2))^2+4\overline{h}^1(Y_1,Y_2)^2+\overline{h}^1(Y_2,Y_2)^2 \\ &= \frac{1}{4}h^1(X_2,X_2)^2+4h^1(X_1,X2)^2+h^1(X_2,X_2)^2. \end{aligned}$$

□

By Lemma 2.18, $g_{\mathfrak{u}}^{\perp}$ is independent of the choice of the tangent frame $\mathfrak{u}$, we denote it by $g_{\xi}^{\perp}$.

Remark 2.19. Other metrics appear on the transverse plane bundle that does not depend on the tangent frame. For example the metric given by

$$\begin{aligned} g_{\mathfrak{u}}^{\perp}(\xi_1,\xi_1) &= 1, \\ g_{\mathfrak{u}}^{\perp}(\xi_1,\xi_2) &= -\frac{1}{2}h^1(X_2,X_2), \\ g_{\mathfrak{u}}^{\perp}(\xi_2,\xi_2) &= \frac{1}{4}h^1(X_1,X_2)^2, \end{aligned}$$

that is positive definite and the metric given by

$$\begin{aligned} g_{\mathfrak{u}}^{\perp}(\xi_1,\xi_1) &= -1, \\ g_{\mathfrak{u}}^{\perp}(\xi_1,\xi_2) &= \frac{1}{2}h^1(X_2,X_2), \\ g_{\mathfrak{u}}^{\perp}(\xi_2,\xi_2) &= h^1(X_1,X_2)^2, \end{aligned}$$

that is indefinite.

The following lemma gives the relation between the transversal frames of Theorem 2.16.

Lemma 2.20. *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface. Let ξ be a metric field and $\mathfrak{u} = \{X_1, X_2\}$ a local orthonormal tangent frame. Let σ and $\overline{\sigma}$ be two transversal plane bundles. We denote by $\{\xi_1, \xi_2\}$ and $\{\overline{\xi}_1, \overline{\xi}_2\}$ the transversal frames obtained from Theorem 2.16 for σ and $\overline{\sigma}$, respectively. Then there are Z_1 and Z_2 tangent vector fields* *on M such that*

$$\overline{\xi}_1 = \xi_1 + Z_1,$$
$$\overline{\xi}_2 = \xi_2 + Z_2.$$

Proof. We suppose that

$$\xi_1 = \phi\overline{\xi}_1 + \psi\overline{\xi}_2 + Z_1, \qquad \xi_2 = \rho\overline{\xi}_1 + \beta\overline{\xi}_2 + Z_2.$$

Since $[\xi_1] = [\overline{\xi}_1]$ we have $\psi = 0$ and $\phi = 1$. By Theorem 2.16 $[X_1, X_2, \xi_1, \xi_2] = 1$, which implies $\phi\beta - \psi\rho = 1$ and it follows that $\beta = 1$. We denote by $\overline{h}^1$ and $\overline{h}^2$ the affine fundamental forms of the frame $\{\overline{\xi}_1, \overline{\xi}_2\}$. We note that

$$0 = \overline{h}^1(X_1, X_1) = \phi h^1(X_1, X_1) + \rho h^2(X_1, X_1) = \rho.$$

□

2.3 The metric of Burstin and Mayer

Another affine metric that appears in the study of surfaces in $\mathbb{R}^4$ is the affine metric of Burstin and Mayer (see [4, 13]).

Let $M \subset \mathbb{R}^4$ be a surface and let σ be a transversal plane bundle over M. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local tangent frame on a neighborhood U of a point $p \in M$. It is defined the symmetric bilinear form

$$B_{\mathfrak{u}}(Y, Z) = \frac{1}{2}([X_1, X_2, D_Y X_1, D_Z X_2] + [X_1, X_2, D_Z X_1, D_Y X_2]).$$

A surface M is called non-degenerate if $B_{\mathfrak{u}}$ is non-degenerate.

Definition 2.21. It follows from [13, Lemma 3.3] that the symmetric bilinear form, which we denote by g_{BM},

$$g_{BM}(Y, Z) = \frac{B_{\mathfrak{u}}(Y, Z)}{(\det_{\mathfrak{u}} B_{\mathfrak{u}})^{\frac{1}{3}}},$$

does not depend on the choice of tangent frame $\mathfrak{u}$. The symmetric bilinear form g_{BM} is called *the metric of Burstin and Mayer.*

Let M be a surface in $\mathbb{R}^4$ and σ a transversal plane bundle. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local tangent frame on a neighborhood U of a point $p \in M$. Then $\det_{\mathfrak{u}} B_{\mathfrak{u}} = \Delta$, where

$$\Delta = \frac{1}{4}\begin{vmatrix} a & 2b & c & 0 \\ e & 2f & g & 0 \\ 0 & a & 2b & c \\ 0 & e & 2f & g \end{vmatrix}$$

$a = h^1(X_1, X_1)$, $b = h^1(X_1, X_2)$, $c = h^1(X_2, X_2)$, $e = h^2(X_1, X_1)$, $f = h^2(X_1, X_2)$ and $g = h^2(X_2, X_2)$.

Lemma 2.22. The metric g_{BM} of each locally strictly convex surface $M \subset \mathbb{R}^4$ is indefinite.

Proof. In fact, let M be a locally strictly convex surface and $p \in M$. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local tangent frame and let σ be a transversal plane bundle. Since M is locally strictly convex, there is a transversal vector field ξ such that $[X_1(p), X_2(p), x - p, \xi(p)] = 0$ is a support hyperplane in p. We choose a frame $\{\xi_1, \xi_2\}$ on σ and note that

$$\xi = r_1 X_1 + r_2 X_2 + r_3 \xi_1 + r_4 \xi_2$$

for some functions r_1, r_2, r_3 and r_4.
By a simple computation

$$\begin{aligned} [X_1(p), X_2(p), X_{uu}(p), \xi(p)] &= ar_4 - er_3, \\ [X_1(p), X_2(p), X_{uv}(p), \xi(p)] &= br_4 - fr_3, \\ [X_1(p), X_2(p), X_{vv}(p), \xi(p)] &= cr_4 - gr_3. \end{aligned}$$

Now, the matrix

$$A = \begin{pmatrix} ar_4 - er_3 & br_4 - fr_3 \\ br_4 - fr_3 & cr_4 - gr_3 \end{pmatrix}$$

has determinant greater than zero because $[X_1(p), X_2(p), x - p, \xi(p)] = 0$ determines a support hyperplane. Since A is symmetric and $\det[A] > 0$, there is (u_0, v_0) an eigenvector of A, hence we obtain (u_0, v_0) as solution of the equation

$$(af - eb)u^2 + (ag - ec)uv + (bg - fc)v^2 = 0,$$

we conclude that: $-\Delta = \frac{1}{4}(ag - ec)^2 - (af - eb)(bg - fc) > 0$ and $\Delta = \det_{\mathfrak{u}} B_{\mathfrak{u}} < 0$. □

In [13, Theorem 4.1, Lemma 4.1, Remark 4.2] we find results analogous to Theorem 2.16, Lemma 2.17 and Lemma 2.20.

2.4 The equiaffine transversal plane bundles

Nomizu and Vrancken in [13] defined the concept of equiaffine plane as a transversal plane bundle σ such that the affine connection induced by σ, $\nabla = \nabla(\sigma)$ satisfies $\nabla\omega_g = 0$ where ω_g is the metric volume form for the Burstin and Mayer affine metric g:

$$\omega_g(X, Y) = \sqrt{|g(X, X)g(Y, Y) - g(X, Y)^2|},$$

where $\{X, Y\}$ is any positively oriented basis of T_pM. In our case, we consider the same definition, but we use the metric of the transversal vector field g_ξ instead of the Burstin and Mayer affine metric. This definition is based on the compatibility between the volume form and the affine connection.

Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ a metric field and $g = g_\xi$ the metric of the transversal field ξ.

Definition 2.23. We say a transversal plane bundle σ is *equiaffine* if the connection $\nabla = \nabla(\sigma)$ induced by σ satisfies $\nabla\omega_g = 0$.

If $\mathfrak{u} = \{X_1, X_2\}$ is a local orthonormal tangent frame and $\{\xi_1, \xi_2\}$ is the transversal frame given by Theorem 2.16, then $\omega_g = \theta$, where θ is the volume form induced by the determinant:

$$\theta(X, Y) = [X, Y, \xi_1, \xi_2], \ \forall X, Y \in T_pM.$$

This is because $\omega_g(X_i, X_j) = \theta(X_i, X_j)$, $\forall i, j$.

Remark 2.24. By using θ instead of ω_g, we see that σ is an equiaffine plane bundle if and only if

$$B_1 := (\nabla g)(X_1, X_1, X_1) + (\nabla g)(X_1, X_2, X_2) = 0, \tag{2.22}$$

$$B_2 := (\nabla g)(X_2, X_1, X_1) + (\nabla g)(X_2, X_2, X_2) = 0. \tag{2.23}$$

Lemma 2.25. *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ a metric field. Then there exists an equiaffine plane bundle σ defined on a neighborhood of p.*

Proof. Let $\mathfrak{u} = \{X_1, X_2\}$ be an orthonormal tangent frame defined on some neighborhood U of p. Let $\overline{\sigma}$ be a transversal plane bundle defined also on U and $\{\overline{\xi}_1, \overline{\xi}_2\}$ the local basis of $\overline{\sigma}$ obtained by Theorem 2.16. Now we want to construct a new equiaffine plane bundle σ defined on U, with local basis $\{\xi_1, \xi_2\}$ obtained also by Theorem 2.16. By Lemma 2.20, we have

$$\xi_1 = \overline{\xi}_1 - Z_1, \qquad \xi_2 = \overline{\xi}_2 - Z_2,$$

where Z_1 and Z_2 are tangent vector fields. We denote the connection induced by σ (resp. $\overline{\sigma}$) by ∇ (resp. $\overline{\nabla}$). On the other hand, by a simple calculation we obtain

$$\overline{B}_1 = B_1 + 2g(Z_2, X_1) + 2h^1(X_1, X_2)g(Z_1, X_2),$$
$$\overline{B}_2 = B_2 + 2h^1(X_1, X_2)g(Z_1, X_1) + 2h^1(X_2, X_2)g(Z_1, X_2) + 2g(Z_2, X_2).$$

Note that σ is equiaffine if and only if $B_1 = B_2 = 0$. By writing $Z_1 = aX_1 + bX_2$ and $Z_2 = cX_1 + dX_2$, this is equivalent to

$$\overline{B}_1 = 2c + 2bh^1(X_1, X_2),$$
$$\overline{B}_2 = 2d + 2ah^1(X_1, X_2) + 2bh^1(X_2, X_2).$$

The lemma follows since the system above has a solution. For instance, set $a = b = 0$, $c = \frac{\overline{B}_1}{2}$ and $d = \frac{\overline{B}_2}{2}$. □

2.5 The equiaffine normal plane bundles

In this section we define the equiaffine normal plane bundles. Our construction is based on the ideas developed in [13]. Because there are many equiaffine plane bundles (Lemma 2.25), we give conditions to choose some special types among them.

Definition 2.26. Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface, ξ a metric field and $\mathfrak{u} = \{X_1, X_2\}$ an orthonormal tangent frame. We say that an equiaffine plane bundle σ is:

- $\mathfrak{u}$-*symmetric, if*

$$D_1 = (\nabla g)(X_2, X_1, X_1) - (\nabla g)(X_1, X_2, X_1) = 0,$$
$$D_2 = (\nabla g)(X_1, X_2, X_2) - (\nabla g)(X_2, X_1, X_2) = 0,$$

- $\mathfrak{u}$-*antisymmetric, if*

$$C_1 = (\nabla g)(X_2, X_1, X_1) + (\nabla g)(X_1, X_2, X_1) = 0,$$
$$C_2 = (\nabla g)(X_1, X_2, X_2) + (\nabla g)(X_2, X_1, X_2) = 0.$$

Lemma 2.27. *The antisymmetric equiaffine plane bundle does not depend on the tangent frame.*

Proof. Let $\mathfrak{v} = \{Y_1, Y_2\}$ be another tangent frame. We use complex notation $Y_1 + iY_2 = e^{-i\theta}(X_1 + iX_2)$. By extending ∇g complex-linearly

$$\nabla g(X_1+iX_2, X_1+iX_2, X_1+iX_2) = (\nabla g(X_1, X_1, X_1) - \nabla g(X_1, X_2, X_2) - \nabla g(X_2, X_1, X_2) - \nabla g(X_2, X_2, X_1)) + i(\nabla g(X_1, X_1, X_2) + \nabla g(X_1, X_2, X_1) + \nabla g(X_2, X_1, X_1) - \nabla g(X_2, X_2, X_2)),$$

and since σ induces an equiaffine structure on M, we get

$$\nabla g(X_1 + iX_2, X_1 + iX_2, X_1 + iX_2) = -2(\nabla g(X_1, X_2, X_2) + \nabla g(X_2, X_1, X_2)) + 2i(\nabla g(X_2, X_1, X_1) + \nabla g(X_1, X_1, X_2)).$$

Therefore

$$\nabla g(X_1 + iX_2, X_1 + iX_2, X_1 + iX_2) = -2C_2 + 2iC_1.$$

The result on antisymmetry follows since

$$\nabla g(Y_1 + iY_2, Y_1 + iY_2, Y_1 + iY_2) = e^{-3i\theta}\nabla g(X_1 + iX_2, X_1 + iX_2, X_1 + iX_2) = 0.$$

□

From Lemma 2.27, we call σ just *antisymmetric* when it is $\mathfrak{u}$-antisymmetric. The next lemma follows by simple computation.

Lemma 2.28. *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface, ξ a metric field and $\mathfrak{u} = \{X_1, X_2\}$ an orthonormal tangent frame. If we write*

$$\nabla_{X_1}X_1 = a_1X_1 + a_2X_2, \qquad \nabla_{X_2}X_1 = a_5X_1 + a_6X_2,$$
$$\nabla_{X_1}X_2 = a_3X_1 + a_4X_2, \qquad \nabla_{X_2}X_2 = a_7X_1 + a_8X_2.$$

Then

$$B_1 = -2(a_1 + a_4), \qquad B_2 = -2(a_5 + a_8),$$
$$C_1 = -a_2 - a_3 - 2a_5, \qquad C_2 = -a_6 - a_7 - 2a_4,$$
$$D_1 = a_2 + a_3 - 2a_5, \qquad D_2 = a_6 + a_7 - 2a_4.$$

In this section, we say that: A point $p \in M$ is an *inflection* if and only if $h^1(X_1, X_2) = h^1(X_2, X_2) = 0$ (for details see Definition 3.4).

Proposition 2.29. *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ be a metric field. Suppose $p \in M$ is not an inflection, then there exists a unique antisymmetric equiaffine plane bundle σ defined on a neighborhood of p.*

Proof. Let $\mathfrak{u} = \{X_1, X_2\}$ be an orthonormal tangent frame on a neighborhood U of p. We consider $\overline{\sigma}$ an equiaffine plane bundle defined on U and $\{\overline{\xi}_1, \overline{\xi}_2\}$ the local basis of $\overline{\sigma}$ obtained by Theorem 2.16. Now, we want to construct a new antisymmetric equiaffine plane bundle σ defined on U. Again by Theorem 2.16 we have $\{\xi_1, \xi_2\}$ a basis of σ, and by Lemma 2.20 we write $\xi_1 = \overline{\xi}_1 - Z_1$, $\xi_2 = \overline{\xi}_2 - Z_2$, where Z_1 and Z_2 are tangent vector fields. We denote by ∇ (resp. $\overline{\nabla}$) the affine connection induced by σ (resp. $\overline{\sigma}$). We compute $\overline{C}_1$ and $\overline{C}_2$

$$\begin{aligned}
\overline{C}_1 &= C_1 + 3h^1(X_1, X_2)g(Z_1, X_1) + g(Z_2, X_2),\\
\overline{C}_2 &= C_2 + 3h^1(X_1, X_2)g(Z_1, X_2) + h^1(X_2, X_2)g(Z_1, X_1) + g(X_1, Z_2).
\end{aligned}$$

Since σ is antisymmetric, $C_1 = C_2 = 0$ and writing $Z_1 = aX_1 + bX_2$ and $Z_2 = cX_1 + dX_2$ we obtain the system

$$\begin{aligned}
0 &= c + b\overline{h}^1(X_1, X_2),\\
0 &= d + a\overline{h}^1(X_1, X_2) + b\overline{h}^1(X_2, X_2),\\
\overline{C}_1 &= 3a\overline{h}^1(X_1, X_2) + d,\\
\overline{C}_2 &= 3b\overline{h}^1(X_1, X_2) + a\overline{h}^1(X_2, X_2) + c.
\end{aligned}$$

The determinant of the linear system above in the variables a, b, c, d is

$$4\overline{h}^1(X_1, X_2)^2 + \overline{h}^1(X_2, X_2)^2 \neq 0,$$

since p is not an inflection. Therefore, the system has a unique solution. □

Proposition 2.30. *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ be a metric field. If $p \in M$ is not an inflection. Let $\mathfrak{u} = \{X_1, X_2\}$ be an orthonormal tangent frame, then there exists a unique $\mathfrak{u}$-symmetric equiaffine plane bundle σ defined on a neighborhood of p.*

Proof. We follow the same arguments as in the proof of Proposition 2.29,

$$\begin{aligned}
\overline{D}_1 &= D_1 + h^1(X_1, X_2)g(Z_1, X_1) - g(Z_2, X_2),\\
\overline{D}_2 &= D_2 + h^1(X_1, X_2)g(Z_1, X_2) - h^1(X_2, X_2)g(Z_1, X_1) - g(X_1, Z_2).
\end{aligned}$$

By writing $Z_1 = aX_1 + bX_2$ and $Z_2 = cX_1 + dX_2$ we obtain again a linear system in a, b, c, d:

$$\begin{aligned} 0 &= c + b\overline{h}^1(X_1, X_2), \\ 0 &= d + a\overline{h}^1(X_1, X_2) + b\overline{h}^1(X_2, X_2), \\ \overline{D}_1 &= a\overline{h}^1(X_1, X_2) - d, \\ \overline{D}_2 &= b\overline{h}^1(X_1, X_2) - a\overline{h}^1(X_2, X_2) - c, \end{aligned}$$

whose determinant is again $4\overline{h}^1(X_1, X_2)^2 + \overline{h}^1(X_2, X_2)^2 \neq 0$. □

Remark 2.31. Lemma 2.25 and Proposition 2.29 provide an algorithm to calculate a basis $\{\xi_1, \xi_2\}$ of the antisymmetric equiaffine plane bundle: Let σ be an arbitrary transversal plane and ν_1, ν_2, $h^1(X_1, X_2)$, $h^1(X_2, X_2)$ are obtained by Theorem 2.16. Denote by ∇ the affine connection induced by σ then: $\xi_1 = \nu_1 - aX_1 - bX_2$, and $\xi_2 = \nu_2 - cX_1 - dX_2$, where:

$$\begin{aligned} a &= \frac{-2(a_2 + a_3 + a_5 - a_8)h^1(X_1, X_2) - (a_4 + a_6 + a_7 - a_1)h^1(X_2, X_2)}{4h^1(X_1, X_2)^2 + h^1(X_2, X_2)^2}, \\ b &= \frac{-2(a_4 + a_6 + a_7 - a_1)h^1(X_1, X_2) + (a_2 + a_3 + a_5 - a_8)h^1(X_2, X_2)}{4h^1(X_1, X_2)^2 + h^1(X_2, X_2)^2}, \\ c &= -(a_1 + a_4 + bh^1(X_1, X_2)), \\ d &= -(a_5 + a_8 + ah^1(X_1, X_2) + bh^1(X_2, X_2)). \end{aligned}$$

Remark 2.32. Analogously, by using the same notation as in Remark 2.31, we obtain from Lemma 2.25 and Proposition 2.30 the algorithm to compute a basis $\{\xi_1, \xi_2\}$ of the $\mathfrak{u}$-symmetric equiaffine plane bundle: $\xi_1 = \nu_1 - aX_1 - bX_2$, $\xi_2 = \nu_2 - cX_1 - dX_2$, where:

$$\begin{aligned} a &= \frac{2(a_2 + a_3 - 3a_5 - a_8)h^1(X_1, X_2) - (a_6 + a_7 - a_1 - 3a_4)h^1(X_2, X_2)}{4h^1(X_1, X_2)^2 + h^1(X_2, X_2)^2}, \\ b &= \frac{2(a_6 + a_7 - a_1 - 3a_4)h^1(X_1, X_2) + (a_2 + a_3 - 3a_5 - a_8)h^1(X_2, X_2)}{4h^1(X_1, X_2)^2 + h^1(X_2, X_2)^2}, \\ c &= -(a_1 + a_4 + bh^1(X_1, X_2)), \\ d &= -(a_5 + a_8 + ah^1(X_1, X_2) + bh^1(X_2, X_2)). \end{aligned}$$

Lemma 2.33. *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ, $\widehat{\xi}$ are a metric fields such that $\widehat{\xi} = \lambda\xi$ for some function $\lambda > 0$. Then $g_{\widehat{\xi}} = \sqrt{\lambda} g_\xi$. Moreover if $\mathfrak{u} = \{X_1, X_2\}$ be a local orthonormal tangent of g_ξ then $\mathfrak{v} = \{\lambda^{-1/4}X_1, \lambda^{-1/4}X_2\}$ is a local tangent* *orthonormal frame of* $g_{\widehat{\xi}}$ *and :*

$$\widehat{\nu}_1 = \lambda\nu_1, \qquad\qquad \widehat{\nu}_2 = \lambda^{-1/2}\nu_2.$$

Where: the frames $\{\nu_1, \nu_2\}$ *and* $\{\widehat{\nu}_1, \widehat{\nu}_2\}$ *are obtained by Theorem 2.16 to the frames* $\mathfrak{u}$ *and* $\mathfrak{v}$ *respectively.*

Proof. The relationship $g_{\widehat{\xi}} = \sqrt{\lambda} g_\xi$ follows straightforward from Definition 2.11 and, by a simple computation the local tangent frame $\mathfrak{v}$ is an orthonormal frame of $g_{\widehat{\xi}}$. By Theorem 2.16 there are tangent vector fields Z_1, Z_2 such that $\nu_1 = -\xi + Z_1$ and $\widehat{\nu}_1 = -\widehat{\xi} + Z_2$ and it follows $\widehat{\nu}_1 - \lambda\nu_1 = -\lambda Z_1 + Z_2$. Therefore $\widehat{\nu}_1 - \lambda\nu_1$ is a tangent vector field and $\widehat{\nu}_1 = \lambda\nu_1$. On the other hand, by Theorem 2.16 and Equation 2.3, we have

$$\nabla_{X_1} X_1 + \nu_2 = D_{X_1} X_1 = \nabla_{X_1} X_1 + \widehat{h}^1(X_1, X_1)\widehat{\nu}_1 + \widehat{h}^2(X_1, X_1)\widehat{\nu}_2.$$

Note that:

$$\widehat{h}^1(X_1, X_1) = \widehat{h}^1(\lambda^{1/4} Y_1, \lambda^{1/4} Y_1) = \lambda^{1/2}\widehat{h}^1(Y_1, Y_1) = 0,$$
$$\widehat{h}^2(X_1, X_1) = \widehat{h}^2(\lambda^{1/4} Y_1, \lambda^{1/4} Y_1) = \lambda^{1/2}\widehat{h}^2(Y_1, Y_1) = \lambda^{1/2},$$

hence $\nu_2 = \lambda^{1/2}\widehat{\nu}_2$. □

Proposition 2.34. *Under the same hypotheses of Lemma 2.33, we have*

$$2C_1 - B_2 = \lambda^{1/4}(2\widehat{C}_1 - \widehat{B}_2), \qquad 2C_2 - B_1 = \lambda^{1/4}(2\widehat{C}_2 - \widehat{B}_1),$$
$$2D_1 - B_2 = \lambda^{1/4}(2\widehat{D}_1 - \widehat{B}_2), \qquad 2D_2 - B_1 = \lambda^{1/4}(2\widehat{D}_2 - \widehat{B}_1).$$

Proof. It is enough to see that,

$$\widehat{B}_1 = \lambda^{-1/4} B_1 - 4X_1(\lambda^{-1/4}), \qquad \widehat{B}_2 = \lambda^{-1/4} B_2 - 4X_2(\lambda^{-1/4}),$$
$$\widehat{C}_1 = \lambda^{-1/4} C_1 - 2X_2(\lambda^{-1/4}), \qquad \widehat{C}_2 = \lambda^{-1/4} C_2 - 2X_1(\lambda^{-1/4}),$$
$$\widehat{D}_1 = \lambda^{-1/4} D_1 - 2X_2(\lambda^{-1/4}), \qquad \widehat{D}_2 = \lambda^{-1/4} D_2 - 2X_1(\lambda^{-1/4}).$$

□

Theorem 2.35. *Let* $M \subset \mathbb{R}^4$ *be a locally strictly convex surface and* ξ, $\widehat{\xi}$ *be two metric fields such that* $\widehat{\xi} = \lambda\xi$ *for some function* $\lambda > 0$. *Let* $\mathfrak{u} = \{X_1, X_2\}$, $\mathfrak{v} = \{Y_1, Y_2\}$ *be two orthonormal tangent frames as in Lemma 2.33. Then the vector fields* ξ_1 *and* ξ_2 *generating the antisymmetric affine normal plane relative to* g_ξ *and the vector fields* $\widehat{\xi}_1$

and $\widehat{\xi}_2$ *generating the antisymmetric affine normal plane relative to* $g_{\widehat{\xi}}$ *are related by:*

$$\widehat{\xi}_1 = \lambda \xi_1 \quad \textit{and} \quad \widehat{\xi}_2 = \frac{1}{\lambda^{1/4}}\xi_2 - \frac{2}{\lambda^{1/2}}(Y_1(\lambda^{1/4})Y_1 + Y_2(\lambda^{1/4})Y_2).$$

Proof. Let σ be an arbitrary transversal plane, we denote by σ (resp. $\widehat{\sigma}$) the transversal plane generated by $\{\xi_1, \xi_2\}$ (resp. $\{\widehat{\xi}_1, \widehat{\xi}_2\}$) and ∇ (resp. $\widehat{\nabla}$) the connection induced by σ (resp. $\widehat{\sigma}$). Using the notation

$$\begin{aligned} \nabla_{X_1}X_1 &= a_1X_1 + a_2X_2, & \nabla_{Y_1}Y_1 &= \widehat{a}_1Y_1 + \widehat{a}_2Y_2, \\ \nabla_{X_1}X_1 &= a_3X_1 + a_4X_2, & \nabla_{Y_1}Y_2 &= \widehat{a}_3Y_1 + \widehat{a}_4Y_2, \\ \nabla_{X_1}X_1 &= a_5X_1 + a_6X_2, & \nabla_{Y_2}Y_1 &= \widehat{a}_5Y_1 + \widehat{a}_6Y_2, \\ \nabla_{X_1}X_1 &= a_7X_1 + a_8X_2, & \nabla_{Y_2}Y_2 &= \widehat{a}_7Y_1 + \widehat{a}_8Y_2, \end{aligned}$$

and by a straightforward computation, we have

$$\begin{aligned} a_1 &= Y_1(\lambda^{1/4}) + \lambda^{1/4}\widehat{a}_1, & a_2 &= \lambda^{1/4}\widehat{a}_2, & a_3 &= \lambda^{1/4}\widehat{a}_3, & a_4 &= Y_1(\lambda^{1/4}) + \lambda^{1/4}\widehat{a}_4, \\ a_5 &= Y_2(\lambda^{1/4}) + \lambda^{1/4}\widehat{a}_5, & a_6 &= \lambda^{1/4}\widehat{a}_6, & a_7 &= \lambda^{1/4}\widehat{a}_7, & a_8 &= Y_2(\lambda^{1/4}) + \lambda^{1/4}\widehat{a}_8. \end{aligned}$$

We need to find, the relation between the antisymmetric affine normal planes, therefore we compute their generators as in Remark 2.31, this is, a, b, c and d (resp. $\widehat{a}, \widehat{b}, \widehat{c}$ and $\widehat{d}$).

$$\begin{aligned} \widehat{a} &= \frac{-2(\widehat{a}_2 + \widehat{a}_3 + \widehat{a}_5 - \widehat{a}_8)\widehat{h}^1(Y_1, Y_2) - (\widehat{a}_4 + \widehat{a}_6 + \widehat{a}_7 - \widehat{a}_1)\widehat{h}^1(Y_2, Y_2)}{4\widehat{h}^1(Y_1, Y_2)^2 + \widehat{h}^1(Y_2, Y_2)^2} \\ &= \frac{-2\lambda^{-1/4}(a_2 + a_3 + a_5 - a_8)\lambda^{-3/2}h^1(X_1, X_2) - \lambda^{-1/4}(a_4 + a_6 + a_7 - a_1)\lambda^{-3/2}h^1(X_2, X_2)}{\lambda^{-3}(h^1(X_1, X_2)^2 + h^1(X_2, X_2)^2)} \\ &= \lambda^{5/4}\frac{-2(a_2 + a_3 + a_5 - a_8)h^1(X_1, X_2) - (a_4 + a_6 + a_7 - a_1)h^1(X_2, X_2)}{4h^1(X_1, X_2)^2 + h^1(X_2, X_2)^2} = \lambda^{5/4}a. \end{aligned}$$

$\widehat{b} = \lambda^{5/4}b$, $\widehat{c} = \lambda^{-1/4}c + 2\lambda^{-1/4}Y_1(\lambda^{1/4})$ and $\widehat{d} = \lambda^{-1/4}d + 2\lambda^{-1/4}Y_2(\lambda^{1/4})$. Finally

$$\begin{aligned} \widehat{\xi}_1 &= \widehat{\nu}_1 - \widehat{a}Y_1 - \widehat{b}Y_2 = \lambda\nu_1 - \lambda^{5/4}a\lambda^{-1/4}X_1 - \lambda^{5/4}b\lambda^{-1/4}X_2 = \lambda(\nu_1 - aX_1 - bX_2) = \lambda\xi_1, \\ \widehat{\xi}_2 &= \widehat{\nu}_2 - \widehat{c}Y_1 - \widehat{d}Y_2 = \frac{1}{\sqrt{\lambda}}(\nu_2 - cX_1 - dX_2) - \frac{2}{\sqrt{\lambda}}(Y_1(\lambda^{1/4})X_1 + Y_2(\lambda^{1/4})X_2) \\ &= \frac{1}{\lambda^{1/2}}\xi_2 - \frac{2}{\lambda^{1/2}}(Y_1(\lambda^{1/4})X_1 + Y_2(\lambda^{1/4})X_2). \end{aligned}$$

□

2.6 The cubic forms

Equations (2.12) and (2.13) are used to define the cubic forms which we denote by C^1 and C^2,

$$C^1(X,Y,Z) = (\nabla_X h^1)(Y,Z) + \tau_1^1(X)h^1(Y,Z) + \tau_2^1(X)h^2(Y,Z),$$
$$C^2(X,Y,Z) = (\nabla_X h^2)(Y,Z) + \tau_1^2(X)h^1(Y,Z) + \tau_2^2(X)h^2(Y,Z).$$

Note that C^1 and C^2 are symmetric in X, Y and Z.

In [1] we find the following problem of classification of submanifolds:

Problem 18.(L. Vrancken) *Define the covariant derivative*

$$(\nabla_X h)(Y,Z) := \nabla_X^{\perp}(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Classify the submanifolds satisfying $\nabla h \equiv 0$.

Note that **Problem 18** is equivalent to classify the submanifolds with null cubic forms C^1 and C^2. In fact,

$$(\nabla_X h)(Y,Z) = C^1(X,Y,Z)\xi_1 + C^2(X,Y,Z)\xi_2.$$

Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface, ξ a metric field and σ a transversal plane bundle. Let $\mathfrak{u} = \{X_1, X_2\}$ an orthonormal tangent frame. We denote by

$$\nabla_{X_1}X_1 = a_1X_1 + a_2X_2, \qquad \nabla_{X_2}X_1 = a_5X_1 + a_6X_2,$$
$$\nabla_{X_1}X_2 = a_3X_1 + a_4X_2, \qquad \nabla_{X_2}X_2 = a_7X_1 + a_8X_2.$$

By Theorem 2.16 there exists a frame $\{\xi_1, \xi_2\} \subset \sigma$ such that $h^1(X_1,X_1) = 0$, $h^2(X_i,X_j) = \delta_{ij}$. We denote by $b = h^1(X_1,X_2)$ and $c = h^1(X_2,X_2)$.

Now we compute the cubic forms:

$$C^1(X_1,X_1,X_1) = -2a_2b + \tau_2^1(X_1),$$
$$C^1(X_1,X_1,X_2) = X_1(b) - a_1b - a_2c - a_4b + \tau_1^1(X_1)b,$$
$$C^1(X_2,X_1,X_1) = -2a_6b + \tau_2^1(X_2),$$
$$C^1(X_1,X_2,X_2) = X_1(c) - 2a_3b - 2a_4c + \tau_1^1(X_1)c + \tau_2^1(X_1),$$
$$C^1(X_2,X_1,X_2) = X_2(b) - a_5b - a_6c - a_8b + \tau_1^1(X_2)b,$$
$$C^1(X_2,X_2,X_2) = X_2(c) - 2a_7b - 2a_8c + \tau_1^1(X_2)c + \tau_2^1(X_2).$$

By symmetry of C^1 we have : $C^1(X_1,X_1,X_2) = C^1(X_2,X_1,X_1)$ and $C^1(X_1,X_2,X_2) = C^1(X_2,X_1,X_2)$. Hence

$$b(a_1+a_4-2a_6)+ca_2 = \tau_1^1(X_1)b-\tau_2^1(X_2)+X_1(b), \tag{2.24}$$
$$b(a_5+a_8-2a_3)+c(a_6-2a_4) = \tau_1^1(X_2)b-\tau_2^1(X_1)+X_2(b)-\tau_1^1(X_1)c-X_1(c). \tag{2.25}$$

Analogously we compute C^2:

$$C^2(X_1,X_1,X_1) = -2a_1+\tau_2^2(X_1), \qquad C^2(X_1,X_2,X_2) = -2a_4+\tau_1^2(X_1)c+\tau_2^2(X_1),$$
$$C^2(X_1,X_1,X_2) = -a_2-a_3+\tau_1^2(X_1)b, \qquad C^2(X_2,X_1,X_2) = -a_6-a_7+\tau_1^2(X_2)b,$$
$$C^2(X_2,X_1,X_1) = -2a_5+\tau_2^2(X_2), \qquad C^2(X_2,X_2,X_2) = -2a_8+\tau_1^2(X_2)c+\tau_2^2(X_2).$$

By symmetry of C^2 : $C^2(X_1,X_1,X_2) = C^2(X_2,X_1,X_1)$ and $C^2(X_1,X_2,X_2) = C^2(X_2,X_1,X_2)$. Hence

$$a_2+a_3-2a_5 = \tau_1^2(X_1)b-\tau_2^2(X_2), \tag{2.26}$$
$$a_6+a_7-2a_4 = \tau_1^2(X_2)b-\tau_1^2(X_1)c-\tau_2^2(X_1). \tag{2.27}$$

Combining Equations (2.24) with (2.27) and (2.25) with (2.26) we obtain:

$$\begin{aligned}(\tau_1^1(X_1)b-\tau_2^2(X_1))+(\tau_1^2(X_2)b-\tau_2^1(X_2))\\ = b(a_1+a_4-2a_6)+ca_2+(a_6+a_7-2a_4)+\tau_1^2(X_1)c-X_1(b),\end{aligned} \tag{2.28}$$

$$\begin{aligned}(\tau_1^1(X_2)b-\tau_2^2(X_2))+(\tau_1^2(X_1)b-\tau_2^1(X_1))\\ = b(a_5+a_8-2a_3)+c(a_6-2a_4)+(a_2+a_3-2a_5)+\tau_1^1(X_1)c+X_1(c)-X_2(b).\end{aligned} \tag{2.29}$$

Definition 2.36. Let $M\subset\mathbb{R}^4$ be a locally strictly convex surface, ξ a metric field and $\mathfrak{u}=\{X_1,X_2\}$ an orthonormal tangent frame with g_ξ. We say that the quadratic form h^1 is $\mathfrak{u}$-symmetric, if

$$(\nabla h^1)(X_2,X_1,X_1) = (\nabla h^1)(X_1,X_2,X_1),$$
$$(\nabla h^1)(X_1,X_2,X_2) = (\nabla h^1)(X_2,X_1,X_2).$$

Note that the quadratic form h^1 is $\mathfrak{u}$-symmetric if and only if $X_1(b) = (a_1+a_4-2a_6)b+a_2c$ and $X_1(c) = X_2(b)-(a_5+a_8-2a_3)b-(a_6-2a_4)c$.

Therefore from Equations (2.28) and (2.29)

$$
\begin{aligned}
(\tau_1^1(X_1)b - \tau_2^2(X_1)) + (\tau_1^2(X_2)b - \tau_2^1(X_2)) &= (a_6 + a_7 - 2a_4) + \tau_1^2(X_1)c, \\
(\tau_1^1(X_2)b - \tau_2^2(X_2)) + (\tau_1^2(X_1)b - \tau_2^1(X_1)) &= (a_2 + a_3 - 2a_5) + \tau_1^1(X_1)c.
\end{aligned}
$$

Also since $C^1(X, Y, Z)$ is symmetric in X, Y and Z and h^1 is $\mathfrak{u}$-symmetric we have:

$$
\begin{aligned}
\tau_1^1(X_1)b &= \tau_2^1(X_2), \\
\tau_1^1(X_1)c + \tau_2^1(X_1) &= \tau_1^1(X_2)b.
\end{aligned}
$$

CHAPTER 3

Asymptotic directions, affine binormals and inflections

In this chapter, we introduce the concepts of asymptotic directions, affine binormals and inflections. These concepts are well known in the case of a surface immersed in Euclidean space (see for instance [11, 15]). We show how to adapt all these definitions to the context of the affine differential geometry.

3.1 Asymptotic directions and affine binormals

Let $M \subset \mathbb{R}^4$ be an immersed surface with a transversal plane bundle σ. We denote by σ^* the conormal, that is, the dual vector bundle of σ. For any $p \in M$ and for any conormal vector $\lambda \in \sigma_p^*$, we define the second fundamental form along λ as:

$$h_\lambda(X, Y) = \lambda(h(X, Y)), \ \forall X, Y \in T_pM.$$

Definition 3.1. We say that a non zero $\lambda \in \sigma_p^*$ is an *affine binormal* at p if h_λ is degenerate, that is, if there is a non zero tangent vector $X \in T_pM$ such that

$$h_\lambda(X, Y) = 0, \ \forall Y \in T_pM.$$

Moreover, in such a case, we say that X is an *asymptotic direction* at p associated with the affine binormal λ.

The concepts of asymptotic directions and affine binormals are related to the so-called generalized eigenvalue problem. Let A, B be two $n \times n$ matrices. A pair $(p, q) \in \mathbb{R}^2 - \{0\}$

is a *generalized eigenvalue* of (A, B) if

$$\det(pA + qB) = 0.$$

Analogously, $x \in \mathbb{R}^n - \{0\}$ is a *generalized eigenvector* associated with the generalized eigenvalue (p, q) if

$$(pA + qB)x = 0.$$

In our case, given a point $p \in M$ we fix $\mathfrak{u} = \{X_1, X_2\}$ any tangent frame of T_pM, $\{\xi_1, \xi_2\}$ any transversal frame of σ_p and $\{\lambda_1, \lambda_2\}$ the corresponding dual frame of σ_p^*. We denote by $A = (h^1(X_i, X_j))$ and $B = (h^2(X_i, X_j))$ the coefficient matrices of the second fundamental forms h^1, h^2 respectively. The proof of the following lemma is straightforward from the definitions.

Lemma 3.2. *With the above notation, $X = u_1X_1 + u_2X_2 \in T_pM$ is an asymptotic direction associated with the affine binormal $\lambda = r\lambda_1 + s\lambda_2 \in \sigma_p^*$ if and only if $u = (u_1, u_2)$ is a generalized eigenvector of (A, B) associated with the generalized eigenvalue (r, s).*

It follows from Lemma 3.2 that the affine binormals are determined by the solutions of the quadratic equation $\det(rA + sB) = 0$, so we can have either 2, 1 or 0 affine binormal directions. When M is locally strictly convex, we always have at least one affine binormal.

Corollary 3.3. *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface with a transversal bundle σ. At any point $p \in M$, either:*

1. *there exist exactly two affine binormal directions and two asymptotic directions (one for each binormal), or*
2. *there exists exactly one affine binormal direction and any tangent direction is asymptotic.*

Proof. We choose any metric field ξ on M and consider $\mathfrak{u} = \{X_1, X_2\}$ an orthonormal tangent frame and $\{\xi_1, \xi_2\}$ the associated transversal frame given by Theorem 2.16. The coefficient matrices of the second fundamental forms are

$$A = \begin{pmatrix} 0 & b \\ b & c \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $b = h^1(X_1, X_2)$ and $c = h^1(X_2, X_2)$. By Lemma 3.2, the asymptotic and affine binormal directions are given in terms of the solutions of the homogeneous linear system:

$$\begin{pmatrix} s & rb \\ rb & rc + s \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The affine binormal directions are given by the roots of the determinant $s^2+crs-b^2r^2=0$. Since $(r,s)\neq(0,0)$, we can assume $r\neq 0$ and normalize to $r=1$, so,

$$s=\frac{-c\pm\sqrt{c^2+4b^2}}{2}.$$

If $(b,c)\neq(0,0)$, we have two distinct solutions and one asymptotic direction (u_1,u_2) for each one of them. Otherwise, if $(b,c)=(0,0)$, then $s=0$ and all the directions (u_1,u_2) are asymptotic. □

Definition 3.4. We say that a point $p\in M$ is an *inflection* if all the tangent directions at p are asymptotic, that is, if there is a non zero $\lambda\in\sigma^*$ such that $h_\lambda=0$.

With the notation of Lemma 3.2, p is an inflection if and only if the matrices A,B are collinear. In the case that M is locally strictly convex, we fix a metric field ξ and take an orthonormal tangent frame $\mathfrak{u}=\{X_1,X_2\}$ and a transversal frame $\{\xi_1,\xi_2\}$ as in Theorem 2.16. Then p is an inflection if and only if $h^1(X_1,X_2)=h^1(X_2,X_2)=0$.

We can also use Lemma 3.2 in order to obtain the differential equation of the asymptotic lines of a surface. By definition, an asymptotic line is an integral curve of the field of asymptotic directions, that is, it is a curve whose tangent at any point is asymptotic.

Theorem 3.5. *With the notation of Lemma 3.2, the differential equation for the asymptotic lines of M is:*

$$\begin{vmatrix} dv^2 & -dvdu & du^2 \\ a & b & c \\ e & f & g \end{vmatrix}=0,$$

where $A=\begin{pmatrix} a & b \\ b & c\end{pmatrix}$ *and* $B=\begin{pmatrix} e & f \\ f & g\end{pmatrix}$.

Proof. We just eliminate (r,s) in the linear system $(rA+sB)u=0$, where $u=(du,dv)$. □

Remark 3.6. If M is locally strictly convex and ξ is a metric field, then we can use one of the transversal metrics $g_\xi^\perp$ defined in Section 2.2 in order to define binormal directions also in σ instead of σ^*. In fact, for each $\nu\in\sigma_p$ we have a well defined second fundamental form along ν:

$$h_\nu(X,Y)=g_\xi^\perp(h(X,Y),\nu),\ \forall X,Y\in T_pM.$$

We say that ν is binormal if h_ν is degenerate.

Remark 3.7. It is not difficult to see that if M is non-degenerate in the sense of Nomizu and Vranken [13], then the asymptotic directions of M at p are exactly the null directions of the Burstin-Mayer affine metric (which is indefinite in the case that M is locally strictly convex).

3.2 The height function

Another important fact is that we can characterize the asymptotic directions and affine binormals in terms of the singularities of the height functions. Given the direct sum $\mathbb{R}^4 = T_pM \oplus \sigma_p$, we denote by $p_1 : \mathbb{R}^4 \to T_pM$ and $p_2 : \mathbb{R}^4 \to \sigma_p$ the two associated linear projections. Then, for each $\lambda \in \sigma_p^*$, we define the height function $H_\lambda : M \to \mathbb{R}$ by

$$H_\lambda(x) = \lambda(p_2(x)).$$

Proposition 3.8. *Let $\lambda \in \sigma_p^*$ be a non zero conormal vector of M, then:*

1. *H_λ has always a singularity at p;*
2. *λ is an affine binormal if and only if H_λ has a degenerate singularity at p;*
3. *$X \in T_pM$ is an asymptotic direction associated with λ if and only if X belongs to the kernel of the Hessian of H_λ at p;*
4. *p is an inflection if and only if there exists a non zero $\lambda \in \sigma_p^*$ such that H_λ has a corank 2 singularity at p.*

Proof. The differential of H_λ at p is always zero and we have (1):

$$d(H_\lambda)_p(X) = \lambda(p_2(X)) = 0, \ \forall X \in T_pM.$$

But the Hessian of H_λ at p is precisely the second fundamental form h_λ:

$$d^2(H_\lambda)_p(X,Y) = \lambda(p_2(D_XY)) = \lambda(h(X,Y)) = h_\lambda(X,Y), \ \forall X, Y \in T_pM.$$

Then, (2), (3) and (4) follow directly from the definitions of affine binormal, asymptotic direction and inflection. □

The results of Proposition 3.8 can be easily restated in terms of contacts with hyperplanes.

Definition 3.9. We say that π is an *osculating hyperplane* of M at p if it is tangent to M at p and it has a degenerate contact with M at p. If $H : \mathbb{R}^4 \to \mathbb{R}$ is any linear function such that π is given by the equation $H(x-p) = 0$, then we say that $X \in T_pM$ is a *contact direction* if it belongs to the kernel of the Hessian of $H|_M$.

Corollary 3.10. *A tangent vector $X \in T_pM$ is an asymptotic direction if and only if it is a contact direction of some osculating hyperplane. In particular, the asymptotic directions (and hence the inflections) of M are affine invariant, that is, they do not depend on the choice of the transversal plane bundle σ.*

Proof. If X is an asymptotic direction, then there exists an affine binormal $\lambda \in \sigma_p^*$ associated with X. We define π as the hyperplane passing through p and parallel to $T_pM \oplus \ker \lambda$. We can take $H : \mathbb{R}^4 \to \mathbb{R}$ given by $H(x) = \lambda(p_2(x))$ so that $H(x-p) = 0$ is a defining equation of π and $H|_M = H_\lambda$. By Proposition 3.8, π is an osculating hyperplane and X is a contact direction.

Conversely, assume that π is an osculating hyperplane and X is a contact direction. Let $H : \mathbb{R}^4 \to \mathbb{R}$ be any linear function such that $H(x-p) = 0$ is a defining equation of π. We take now $\lambda = H|_{\sigma_p} \in \sigma_p^*$, then

$$H|_M(x) = H(x) = H(p_1(x) + p_2(x)) = H(p_2(x)) = \lambda(p_2(x)) = H_\lambda(x),$$

for all $x \in M$. Again by Proposition 3.8, λ is an affine binormal with associated asymptotic direction X. □

This gives another proof of the fact that the asymptotic directions are affine invariant.

3.3 A theorem of equivalence

Let $M_i \subset \mathbb{R}^4$ be an immersed surface and σ_i a transversal plane bundle on M_i $(i = 1, 2)$. We consider the affine fundamental form h_i associated with the transversal plane bundles σ_i

$$h_i : TM_i \times TM_i \to \sigma_i.$$

Definition 3.11. We say that a bundle morphism $\widetilde{\phi} : \sigma_1 \to \sigma_2$ preserves the affine fundamental forms if $\widetilde{\phi}$ is an isomorphism of vector bundles such that

$$\widetilde{\phi}(h_1(X, Y)) = h_2(\phi_*(X), \phi_*(Y)),$$

where $\phi : M_1 \to M_2$ is the underlying diffeomorphism.

Theorem 3.12. *Let $M_i \subset \mathbb{R}^4$ be a locally strictly convex surface, σ_i a transversal plane bundle ($i = 1, 2$). Assume $p \in M_1$ is not inflection and let $\mathcal{U}$ be a small enough open neighborhood of p. Then there is a morphism $\widetilde{\phi}$ on $\mathcal{U}$ which preserves the affine fundamental forms if and only if there is a diffeomorphism ϕ on $\mathcal{U}$ such that ϕ_* preserves the asymptotic directions.*

Proof. Let $\widetilde{\phi} : \sigma_1 \to \sigma_2$ be a transformation preserving the affine fundamental forms. Let $\{\eta_1, \eta_2\}$ be a frame on σ_1 and $\{\xi_1, \xi_2\}$ frame on σ_2. Since η_1 and η_2 are linearly independent, $\widetilde{\phi}(\eta_1)$ and $\widetilde{\phi}(\eta_2)$ are linearly independent because $\widetilde{\phi}$ is an isomorphism of bundles. Therefore there are functions α_{ij} defined on $\mathcal{U}$ such that

$$\widetilde{\phi}(\eta_1) = \alpha_{11}\xi_1 + \alpha_{12}\xi_2, \qquad \widetilde{\phi}(\eta_2) = \alpha_{21}\xi_1 + \alpha_{22}\xi_2.$$

Since $\widetilde{\phi}$ is a transformation preserving the affine fundamental forms

$$\begin{aligned} h_2(\phi_*(X), \phi_*(Y)) &= \widetilde{\phi}(h_1(X,Y)) = \widetilde{\phi}(h_1^1(X,Y)\eta_1 + h_1^2(X,Y)\eta_2) \\ &= h_1^1(X,Y)\widetilde{\phi}(\eta_1) + h_1^2(X,Y)\widetilde{\phi}(\eta_2) \\ &= h_1^1(X,Y)(\alpha_{11}\xi_1 + \alpha_{12}\xi_2) + h_1^2(X,Y)(\alpha_{21}\xi_1 + \alpha_{22}\xi_2) \\ &= (\alpha_{11}h_1^1(X,Y) + \alpha_{21}h_1^2(X,Y))\xi_1 + (\alpha_{12}h_1^1(X,Y) + \alpha_{22}h_1^2(X,Y))\xi_2. \end{aligned}$$

By definition of h_2, we obtain

$$\begin{aligned} h_2^1(\phi_*(X), \phi_*(Y)) &= \alpha_{11}h_1^1(X,Y) + \alpha_{21}h_1^2(X,Y), \\ h_2^2(\phi_*(X), \phi_*(Y)) &= \alpha_{12}h_1^1(X,Y) + \alpha_{22}h_1^2(X,Y). \end{aligned}$$

Now we fix a tangent frame $\{X_1, X_2\}$ on M and denote by

$$\begin{aligned} (a_1, b_1, c_1) &= (h_1^1(X_1, X_1), h_1^1(X_1, X_2), h_1^1(X_2, X_2)), \\ (e_1, f_1, g_1) &= (h_1^2(X_1, X_1), h_1^2(X_1, X_2), h_1^2(X_2, X_2)), \\ (a_2, b_2, c_2) &= (h_2^1(\phi_*X_1, \phi_*X_1), h_2^1(\phi_*X_1, \phi_*X_2), h_2^1(\phi_*X_2, \phi_*X_2)), \\ (e_2, f_2, g_2) &= (h_2^2(\phi_*X_1, \phi_*X_1), h_2^2(\phi_*X_1, \phi_*X_2), h_2^2(\phi_*X_2, \phi_*X_2)), \end{aligned}$$

$$\Delta_1 = \begin{vmatrix} dv^2 & -dudv & du^2 \\ a_1 & b_1 & c_1 \\ e_1 & f_1 & g_1 \end{vmatrix} \text{ and } \Delta_2 = \begin{vmatrix} dv^2 & -dudv & du^2 \\ a_2 & b_2 & c_2 \\ e_2 & f_2 & g_2 \end{vmatrix}.$$

By a straightforward computation $\Delta_2 = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\Delta_1$.
If $duX_1 + dvX_2$ is an asymptotic direction then $\Delta_1 = 0$ hence $\Delta_2 = 0$. Therefore

$du\phi_*(X_1) + dv\phi_*(X_2) = \phi_*(duX_1 + dvX_2)$ is an asymptotic direction.
Conversely, we suppose that there is a diffeomorphism $\phi : \mathcal{U} \to \mathcal{V}$ such that ϕ_* preserves the asymptotic directions. We fix the tangent frame $\{X_1, X_2\}$ and using the Δ_1 and Δ_2 as above, then $duX_1 + dvX_2$ and $du\phi_*(X_1) + dv\phi_*(X_2)$ are the asymptotic directions on M_1 and M_2 respectively, or $\Delta_1 = \Delta_2 = 0$. Since p is not inflection we can suppose $a_1f_1 - b_1e_1 \neq 0$. Hence, there are functions β_{ij}, $i, j = 1, 2$ defined on $\mathcal{U}$ such that

$$\begin{aligned} a_2 &= \beta_{11}a_1 + \beta_{21}e_1, & e_2 &= \beta_{12}a_1 + \beta_{22}e_1, \\ b_2 &= \beta_{11}b_1 + \beta_{21}f_1, & f_2 &= \beta_{12}b_1 + \beta_{22}f_1. \end{aligned}$$

Now we consider the following system of linear equations with variables a, b, c, d

$$\begin{aligned} c_2 &= ac_1 + bg_1, \\ g_2 &= cc_1 + dg_1. \end{aligned}$$

Since $\Delta_1 = 0$ and $\Delta_2 = 0$ we have

$$\begin{aligned} 0 &= (a_1f_1 - b_1e_1)du^2 + (a_1g_1 - c_1e_1)dudv + (b_1g_1 - c_1f_1)dv^2, \\ 0 &= (a_2f_2 - b_2e_2)du^2 + (a_2g_2 - c_2e_2)dudv + (b_2g_2 - c_2f_2)dv^2. \end{aligned}$$

Now, by a simple computation $a_2f_2 - b_2e_2 = \det[\beta](a_1f_1 - b_1e_1)$, since $a_1f_1 - b_1e_1 \neq 0$ and $(du : dv)$ is solution of the equations above, it follows that

$$\begin{aligned} a_2g_2 - c_2e_2 &= \det[\beta](a_1g_1 - c_1e_1), \\ b_2g_2 - c_2f_2 &= \det[\beta](b_1g_1 - c_1f_1). \end{aligned}$$

We compute

$$\begin{aligned} a_2g_2 - c_2e_2 = a_1c_1(c\beta_{11} - a\beta_{12}) + e_1g_1(d\beta_{21} - b\beta_{22}) + a_1g_1(\beta_{11}d - b\beta_{12}) \\ + c_1e_1(c\beta_{21} - a\beta_{22}), \end{aligned} \tag{3.1}$$

$$\begin{aligned} b_2g_2 - c_2f_2 = b_1c_1(c\beta_{11} - a\beta_{12}) + f_1g_1(d\beta_{21} - b\beta_{22}) + b_1g_1(\beta_{11}d - b\beta_{12}) \\ + c_1f_1(c\beta_{21} - a\beta_{22}). \end{aligned} \tag{3.2}$$

Now since $(a_2g_2 - c_2e_2)(b_1g_1 - c_1f_1) = (b_2g_2 - c_2f_2)(a_1g_1 - c_1e_1)$, from Equations (3.1) and (3.2) we obtain

$$0 = c_1^2(c\beta_{11} - a\beta_{12}) + g_1^2(d\beta_{21} - b\beta_{22}) + c_1 g_1((\beta_{11}d - \beta_{12}b) + (c\beta_{21} - a\beta_{22})).$$

Therefore we have the following system of equations

$$\begin{aligned} 0 &= c\beta_{11} - a\beta_{12}, \\ 0 &= d\beta_{21} - b\beta_{22}, \\ 0 &= (\beta_{11}d - \beta_{12}b) + (c\beta_{21} - a\beta_{22}). \end{aligned}$$

By solving this system we obtain that: there is a λ such that $a = \beta_{11}\lambda$, $b = \beta_{21}\lambda$, $c = \beta_{12}\lambda$ and $d = \beta_{22}\lambda$. Finally, we replace in Equation (3.1), $\lambda = 1$ and $a = \beta_{11}$, $b = \beta_{21}$, $c = \beta_{12}$ and $d = \beta_{22}$. With this, it is enough to define

$$\begin{aligned} \widetilde{\phi}(\eta_1) &= a_{11}\xi_1 + a_{12}\xi_2, \\ \widetilde{\phi}(\eta_2) &= a_{21}\xi_1 + a_{22}\xi_2. \end{aligned}$$

□

3.4 Curvature ellipse

The curvature ellipse appears in the study of the geometry of surfaces in $\mathbb{R}^4$ as the image of the unit circle defined by the Euclidean metric, by the second fundamental form (see [11]).

We see in [18] the following characterization of the semiumbilic points using the curvature ellipse in the Euclidean case.

Theorem 3.13. *Let M be an immersed surface in $\mathbb{R}^4$ and let $p \in M$. The following are equivalent conditions:*

1. *p is semiumbilic.*
2. *The curvature ellipse at p degenerates to a segment.*
3. *There are two orthogonal asymptotic directions at p.*

In the affine case the curvature ellipse is always degenerate, different from the Euclidean case where it is degenerate just in the semiumbilic points. Other difference with the Euclidean case is that the asymptotic directions are always orthogonal with the metric g_ξ.

Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and let ξ be a metric field. Let σ be a transversal plane bundle. We denote by $\mathbb{S}^1_p$ the set

$$\mathbb{S}^1_p := \{X \in T_pM \ \ / \ \ g_\xi(X,X) = 1\}.$$

We consider the map

$$\begin{aligned} \eta :& \mathbb{S}^1_p \to \sigma_p \\ & X \mapsto h(X,X) \end{aligned}$$

The *curvature ellipse* $\triangle_p$ is the image by η of $\mathbb{S}^1_p$ or $\triangle_p = \eta(\mathbb{S}^1_p)$.

Below we characterize the affine asymptotic directions using the curvature ellipse in the affine case.

Lemma 3.14. *The curvature ellipse degenerates to a segment and the asymptotic directions correspond to the vertices of the segment.*

Proof. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local orthonormal tangent frame. By Theorem 2.16, there exists a frame $\{\xi_1, \xi_2\} \subset \sigma_p$ such that $h^1(X_1, X_1) = 0$ and $h^2(X_i, X_j) = \delta_{ij}$. For $X \in T_pM$ such that $g_\xi(X,X) = 1$ we can write $X = \sin\theta X_1 + \cos\theta X_2$. By a direct computation and definition of h we obtain:

$$h(X,X) = h^1(X,X)\xi_1 + h^2(X,X)\xi_2 = (b\sin(2\theta) + c\cos^2\theta)\xi_1 + \xi_2,$$

where $b = h^1(X_1, X_2)$ and $c = h^1(X_2, X_2)$. Hence, Δ_p is a segment.

By Corollary 3.3 the affine binormal directions are given by the roots of the determinant $s^2 + crs - b^2r^2 = 0$. Since $(r,s) \neq (0,0)$, we can assume $r \neq 0$ and normalize to $r = 1$, so,

$$s = \frac{-c \pm \sqrt{c^2 + 4b^2}}{2}.$$

Now we define : $H(\theta) = b\sin(2\theta) + c\cos^2\theta$ and by derivation $H'(\theta) = 2b\cos(2\theta) - 2c\sin\theta\cos\theta$. Therefore $H'(\theta) = 0$ if and only if $\tan 2\theta = \dfrac{2b}{c}$. Now by trigonometric identities $H'(\theta) = 0$ if and only if $\tan\theta = \frac{-c\pm\sqrt{c^2+4b^2}}{2b}$. Hence θ is a local maximum and minimum of H if and only if X is an asymptotic direction. Therefore the asymptotic directions are the vertices of the segment. □

Next we show that the asymptotic directions are orthogonal with the metric of transversal vector fields.

Proposition 3.15. *Let $\mathfrak{u} = \{X_1, X_2\}$ be a local tangent frame with the metric g. The solutions $(du : dv)$ of the binary differential equation*

$$A_0 du^2 + A_1 du dv + A_2 dv^2 = 0 \tag{3.3}$$

are real and orthogonal with the metric g if and only if $A_2 E - A_1 F + A_0 G = 0$, where $E = g(X_1, X_1)$, $F = g(X_1, X_2)$ and $G = g(X_2, X_2)$.

Proof. We denote by $r_1 = \alpha_1 du + \beta_1 dv$ and $r_2 = \alpha_2 du + \beta_2 dv$ the real solutions of the Equation (3.3), this is $(\alpha_1 du + \beta_1 dv)(\alpha_2 du + \beta_2 dv) = A_0 du^2 + A_1 du dv + A_2 dv^2$, hence $\alpha_1\alpha_2 = A_0$, $A_1 = \alpha_1\beta_2 + \alpha_2\beta_1$ and $A_2 = \beta_1\beta_2$. Then the solutions are orthogonal if $-\beta_1 X_1 + \alpha_1 X_2$ and $-\beta_2 X_1 + \alpha_2 X_2$ are orthogonal, this is,

$$0 = g(-\beta_1 X_1 + \alpha_1 X_2, -\beta_2 X_1 + \alpha_2 X_2) = \beta_1\beta_2 E - (\beta_1\alpha_2 + \beta_2\alpha_1)F + \alpha_1\alpha_2 G.$$

Conversely, we suppose that $A_2 E - A_1 F + A_0 G = 0$ and see that the solutions of Equation (3.3) are real. In fact, since g is a positive definite metric $EG - F^2 > 0$, it follows $EG > F^2 \geq 0$ and $EG > 0$, in particular $E \neq 0$, therefore $A_2 = \frac{A_1 F - A_0 G}{E}$. Now we compute the discriminant Δ of Equation (3.3)

$$\Delta = A_1^2 - 4A_0 A_2 = A_1^2 - 4A_0\left(\frac{A_1 F - A_0 G}{E}\right) = \left(A_1 - 2A_0\frac{F}{E}\right)^2 + 4\frac{A_0^2}{E^2}(EG - F^2) > 0,$$

therefore the solutions are real. The real solution $r_1 = \alpha_1 du + \beta_1 dv$ and $r_2 = \alpha_2 du + \beta_2 dv$ are orthogonal since $A_2 E - A_1 F + A_0 G = 0$. □

Corollary 3.16. *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ be a metric field, then the asymptotic lines are orthogonal with the metric g_ξ.*

Proof. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local orthonormal tangent frame. By Theorems 2.16 and 3.5 we obtain the equation of asymptotic lines

$$-b(du^2) - c(du)(dv) + b(dv^2) = 0,$$

and the result follows by Proposition 3.15. □

CHAPTER 4

Affine σ-semiumbilical surfaces.

Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface with a transversal plane bundle σ. It is common to call a point $p \in M$ semiumbilic if S_ν is a multiple of the identity, for some $\nu \in \sigma_p$. In analogy with the Euclidean case we introduce also the concept of semiumbilic point. We show that the surfaces contained in any locally strictly convex hyperquadrics are σ-semiumbilics.

4.1 The affine hyperspheres

We recall the Blaschke construction of affine for a hypersurface.

Let $\mathbb{R}^{n+1}$ be the affine $(n+1)$-space and D the usual flat connection on $\mathbb{R}^{n+1}$, $N \subset \mathbb{R}^{n+1}$ be an immersed n-manifold and be ξ a transversal vector field. Then, for all $p \in N$, $\xi_p \subset T_p\mathbb{R}^{n+1}$ is a vector such that

$$T_p\mathbb{R}^{n+1} = T_pN \oplus \mathrm{span}(\xi_p),$$

and for all tangent vector fields X, Y on N,

$$D_XY = \nabla_XY + h(X, Y)\xi \qquad \text{(formula of Gauss)},$$

where h is a symmetric bilinear function and

$$D_X\xi = -S_\xi X + \tau(X)\xi \qquad \text{(formula of Weingarten)}.$$

Here, S_ξ is a tensor of type $(1, 1)$, called the affine shape operator, and τ is a 1-form called the transversal connection form.

Now we consider the volume forms ω_h and θ given by

$$\omega_h(X_1,\ldots,X_n) := \sqrt{\det(h(X_i,X_j))},$$
$$\theta(X_1,\ldots,X_n) := [X_1,\ldots,X_n,\xi],$$

where $\{X_1,\ldots,X_n\}$ is any positively oriented basis of T_pN.

Theorem 4.1. *[12, Theorem 3.1] There is, up to sign, a unique transverse vector field ξ, for which the following conditions hold:*

1. $\nabla_X\omega = 0$ *for all tangent vector field X on N,*
2. $\theta(X_1,\ldots,X_n) = \omega_h(X_1,\ldots,X_n)$ *for all tangent vector fields $X_1,\ldots,X_n$ on N.*

Definition 4.2. The unique transverse vector field in Theorem 4.1 is the *affine normal vector field* which we denote by $\mathbf{Y}$. The affine normal $\mathbf{Y}$ is also known as the *Blaschke normal field.*

Definition 4.3. A hypersurface $H \subset \mathbb{R}^{n+1}$ is an *improper affine hypersphere* if the shape operator $S_{\mathbf{Y}}$ is identically 0. If $S_{\mathbf{Y}} = \lambda Id$, where λ is a nonzero constant, then H is a *proper affine hypersphere.* An affine hypersphere is either a proper or improper hypersphere.

Particular cases of affine hyperspheres are the hyperquadrics: elliptic paraboloid, ellipsoid and hyperboloid of two sheets. In general the hyperquadrics are hypersurfaces which can be described by second-order polynomials.

Example 4.4. The *Elliptic Paraboloid* is parameterized by

$$X : (x_1,\ldots,x_n) \to (x_1,\ldots,x_n,\frac{1}{2}((x_1)^2+\ldots+(x_n)^2)). \tag{4.1}$$

We consider the tangent frame $\mathfrak{u} = \{X_1,\ldots,X_n\}$ given by $X_1 = (1,0,\ldots,0,x_1)$, $X_2 = (0,1,\ldots,0,x_2)$, $\ldots$, $X_n = (0,0,\ldots,1,x_n)$ and $X_{n+1} = (0,\ldots,0,1)$. We claim that the quadratic form $h(X_i,X_j) = \delta_{ij}$. In fact, denote by $e_1,\ldots,e_n$ the canonical basis of $\mathbb{R}^n$ and note that: $dX(e_1) = X_1,\ldots,$ $dX(e_n) = X_n$ and then, computing $D_{X_i}X_j$ this is

$$\lim_{t\to 0}\frac{(X_j)(x+te_i) - X_j(x)}{t} = \delta_{ij}X_{n+1}.$$

Since $\theta(X_{n+1}) = [X_1,\ldots,X_n,X_{n+1}] = 1$ and $\omega_h(X_1,\ldots,X_n) = \det(h) = 1$ it follows that X_{n+1} is the affine normal vector field $\mathbf{Y}$.

In [2] we find the computation of affine normal vector bundle for the Ellipsoid and the Hyperboloid of two sheets.

Example 4.5. The *Ellipsoid* is parameterized by

$$M : (x^1)^2 + \ldots + (x^{n+1})^2 = r^2 \tag{4.2}$$

and its affine normal vector field is given by $\mathbf{Y} = -r^{\frac{-(2n+2)}{n+2}} x$.

Example 4.6. The *Hyperboloid of two sheets* is parameterized by

$$M : (x^1)^2 + \ldots + (x^n)^2 - (x^{n+1})^2 = -c^2 \tag{4.3}$$

and its affine normal vector field is given by $\mathbf{Y} = c^{\frac{-(2n+2)}{n+2}} x$.

Example 4.7. The hypersurface $Q(c, n) \subset \mathbb{R}^{n+1}$ (see [10]), is an example of an affine hypersphere which is not a hyperquadric:

$$Q(c, n) : x_{n+1} = \frac{c}{x_1 x_2 \ldots x_n},$$

where $c = constant \neq 0$, $x_1 > 0, x_2 > 0, \ldots, x_n > 0$.

4.2 Surfaces in hypersurfaces

We recall the definition of the Blaschke metric of an immersed hypersurface $N \subset \mathbb{R}^4$. Let $\mathfrak{u}' = \{X_1, X_2, X_3\}$ be a tangent frame defined in some neighborhood U of a point p in N. Now we consider

$$H_{\mathfrak{u}'}(Y, Z) = [X_1, X_2, X_3, D_Z Y], \quad \forall Y, Z \in T_p N.$$

Then $H_{\mathfrak{u}'}$ defines a symmetric bilinear form on N that initially depends on the tangent frame $\mathfrak{u}'$. However, if we suppose that $H_{\mathfrak{u}'}$ is non-degenerate then we can normalize it and the symmetric bilinear form

$$\mathfrak{G}(Y, Z) = \frac{H_{\mathfrak{u}'}(Y, Z)}{(\det_{\mathfrak{u}'} H_{\mathfrak{u}'})^{\frac{1}{5}}}, \quad \forall Y, Z \in T_p N,$$

does not depend on the choice of the tangent frame $\mathfrak{u}'$, where $\det_{\mathfrak{u}'} H_{\mathfrak{u}'} = \det(H_{\mathfrak{u}'}(X_i, X_j))$. The metric $\mathfrak{G}$ is called the *Blaschke metric* of N.

If N is locally strictly convex, then $H_{\mathfrak{u}'}$ is always non-degenerate and positive definite, moreover the tangent hyperplane $T_p N$ is a support hyperplane with a non-degenerate contact. In particular, given any immersed surface $M \subset N$ we have $T_p M \subset T_p N \subset \mathbb{R}^4$, and hence, M is also locally strictly convex. Moreover, we can consider the Blaschke metric $\mathfrak{G}$ restricted to M.

Remark 4.8. We can choose a transversal vector field ξ such that $g_{[\xi]}$ coincides with $\mathfrak{G}$ in T_pM. In fact, let $\mathfrak{u} = \{X_1, X_2\}$ be a frame in T_pM and we choose a tangent vector field $X_3 \in T_pN$ such that $\mathfrak{u}' = \{X_1, X_2, X_3\}$ is a frame in T_pN, then

$$G_{\mathfrak{u}}(Y, Z) = -H_{\mathfrak{u}'}(Y, Z), \quad \forall Y, Z \in T_pM.$$

In particular, we have that $g_{[X_3]} = -\lambda\mathfrak{G}$ where λ is given by

$$\lambda = \frac{(\det_{\mathfrak{u}'} H_{\mathfrak{u}'})^{\frac{1}{5}}}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{\frac{1}{4}}}.$$

Then, it is enough to change the transversal vector field X_3 by $\xi = -X_3/\lambda^2$, so that $g_{[\xi]} = \mathfrak{G}$.

4.3 Surfaces contained in hyperquadrics

An interesting case in this context are the immersed surfaces in affine hyperspheres.

Example 4.9. Immersed surface in an elliptic paraboloid. We take M as the surface parameterized by

$$X : (u, v) \mapsto (u, v, g(u, v), \frac{1}{2}(u^2 + v^2 + g(u, v)^2)).$$

Note that, M is contained in an elliptic paraboloid H given by

$$H : (x, y, z) \mapsto (x, y, z, \frac{1}{2}(x^2 + y^2 + z^2)).$$

From [10], the Blaschke metric on H is given by

$$\mathfrak{G}(e_i, e_j) = \delta_{ij},$$

where $e_1 = (1, 0, 0, x)$, $e_2 = (0, 1, 0, y)$ and $e_3 = (0, 0, 1, z)$. Therefore, the Blaschke metric on M is given by

$$\mathfrak{G}(X_u, X_u) = 1 + g_u^2, \quad \mathfrak{G}(X_u, X_v) = g_u g_v \;\text{ and }\; \mathfrak{G}(X_v, X_v) = 1 + g_v^2.$$

We can choose ξ such that $g_\xi = \mathfrak{G}$. By a simple computation,

$$\xi = -\sqrt{1+g_u^2+g_v^2}(0,0,1,g).$$

Example 4.10. Immersed surface in a hyperboloid of two sheets. We take M as the surface parameterized by

$$X : (u,v) \mapsto (u,v,g(u,v),\sqrt{1+u^2+v^2+g(u,v)^2}).$$

Then, M is contained in the hyperboloid of two sheets H

$$H : x_1^2+x_2^2+x_3^2-x_4^2=-1.$$

The Blaschke metric is calculated in [10, page 64]. We consider the metric field

$$\xi = \lambda\left(0,0,1,\frac{g}{\sqrt{1+u^2+v^2+g(u,v)^2}}\right),$$

where

$$\lambda = -\sqrt{1+g_u^2+g_v^2+(ug_u+vg_v-g(u,v))^2}.$$

Then the Blaschke metric $\mathfrak{G}$ on M coincides with g_ξ. It is not easy to check this computation by hand, but it is possible to do it with the aid of the software Wolfram Mathematica. Explicitly the metric g_ξ is given by

$$\begin{aligned} g_\xi(X_u,X_u) &= (1+g_u^2) - \frac{(u+gg_u)^2}{1+u^2+v^2+g(u,v)^2}, \\ g_\xi(X_u,X_v) &= g_ug_v - \frac{(u+gg_u)(v+gg_v)}{1+u^2+v^2+g(u,v)^2}, \\ g_\xi(X_v,X_v) &= (1+g_v^2) - \frac{(v+gg_v)^2}{1+u^2+v^2+g(u,v)^2}. \end{aligned}$$

Example 4.11. Immersed surface in $Q(1,3)$. We take M as the surface parameterized by

$$X : (u,v) \mapsto \left(u,v,g(u,v),\frac{1}{uvg(u,v)}\right).$$

Note that, M is contained in $Q(1,3)$:

$$H : (x,y,z) \mapsto \left(x,y,z,\frac{1}{xyz}\right).$$

We consider the tangent frame $\{e_1, e_2, e_3\}$ on H to compute the Blaschke metric, where $e_1 = (x, 0, 0, -\frac{1}{xyz)})$, $e_2 = (0, y, 0, -\frac{1}{xyz})$, $e_3 = (0, 0, z, -\frac{1}{xyz})$ and a transversal field $e_4 = (x, y, z, \frac{-2}{xyz})$. By taking derivatives,

$$D_{e_1}e_1 = (x, 0, 0, x, \frac{1}{xyz}) = -e_1 - 2e_2 - 2e_3 + 2e_4,$$

so $h_{11} = 2$ which is the component e_4 of $D_{e_1}e_1$. Analogously we compute $h_{12} = h_{13} = h_{23} = 1$ and $h_{22} = h_{33} = 2$, therefore $\det(h_{ij}) = 4$. Finally

$$\mathfrak{G}_{ij} = \frac{h_{ij}}{(\det(h_{ij}))^{1/5}},$$

that is, $\mathfrak{G}_{ii} = 2^{3/5}$ and $\mathfrak{G}_{ij} = \frac{1}{2^{2/5}}$ $(i \neq j)$. We restrict $\mathfrak{G}$ to the surface M and obtain

$$\mathfrak{G}(X_u, X_u) = \frac{2^{3/5}(g^2 + ugg_u + u^2g_u^2)}{u^2g^2},$$
$$\mathfrak{G}(X_u, X_v) = \frac{g^2 + vg_v(ug_u + g) + ug_u(vg_v + g)}{2^{2/5}uvg^2},$$
$$\mathfrak{G}(X_v, X_v) = \frac{2^{3/5}(g^2 + vgg_v + v^2g_v^2)}{v^2g^2}.$$

It is enough to consider the metric field

$$\xi = -\frac{\sqrt{2g^2 + 2(vg_v - ug_u)^2 + (g + vg_v + ug_u)^2}}{2^{4/5}g(u, v)} \left(0, 0, g(u, v), \frac{-1}{uvg(u, v)}\right),$$

then we have $g_\xi = \mathfrak{G}$.

4.4 σ-semiumbilicity

Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and σ be an arbitrary transversal plane bundle.

Definition 4.12. A point p in $M \subset \mathbb{R}^4$ is called *σ-semiumbilic* if S_ν is a multiple of the identity, for some non zero $\nu \in \sigma_p$. We say that M is *σ-semiumbilical* if all its points are σ-semiumbilic.

In the case that σ is either the antisymmetric or the symmetric equiaffine plane bundle, then we say that M is either *antisymmetric* or *symmetric* affine semiumbilical, respectively (see Definition 2.26).

We see now that the semiumbilic points are related to the vanishing of the normal curvature tensor. We can consider the curvature tensor of the normal connection, called *normal curvature tensor*

$$R_{\nabla^{\perp}} : T_pM \times T_pM \times \sigma_p \to \sigma_p,$$

given by

$$R_{\nabla^{\perp}}(X,Y)\nu = \nabla^{\perp}_X(\nabla^{\perp}_Y\nu) - \nabla^{\perp}_Y(\nabla^{\perp}_X\nu) - \nabla^{\perp}_{[X,Y]}\nu.$$

Since $\mathbb{R}^4$ has vanishing curvature, we obtain $R_{\nabla^{\perp}}(X,Y)\nu = h(X,S_\nu Y) - h(Y,S_\nu X)$.

We fix a metric field ξ on M and let $\mathfrak{u} = \{X_1, X_2\}$ be an orthonormal tangent frame $\{\xi_1, \xi_2\}$ the corresponding transversal frame given by Theorem 2.16. We consider $X = aX_1 + bX_2$ and $Y = cX_1 + dX_2$. We write

$$S_1X_1 = \lambda_1X_1 + \lambda_2X_2,$$
$$S_1X_2 = \lambda_3X_1 + \lambda_4X_2.$$

We have

$$R_{\nabla^{\perp}}(X,Y)\xi_1 = h(X,S_1Y) - h(Y,S_1X).$$

By using the relations above and the billinearity of h, we prove that

$$R_{\nabla^{\perp}}(X,Y)\xi_1 = (ad-bc)((\lambda_4-\lambda_1)h^1(X_1,X_2) - \lambda_2h^1(X_2,X_2))\xi_1 + (\lambda_3-\lambda_2)\xi_2).$$

Therefore, $R_{\nabla^{\perp}}(X,Y)\xi_1 = 0$ if and only if

$$(\lambda_4-\lambda_1)h^1(X_1,X_2) = \lambda_2h^1(X_2,X_2), \tag{4.4}$$
$$\lambda_3 = \lambda_2. \tag{4.5}$$

Analogously, if we write: $S_2X_1 = \mu_1X_1 + \mu_2X_2$, and $S_2X_2 = \mu_3X_1 + \mu_4X_2$, then, $R_{\nabla^{\perp}}(X,Y)\xi_2 = 0$ if and only if

$$(\mu_4-\mu_1)h^1(X_1,X_2) = \mu_2h^1(X_2,X_2), \tag{4.6}$$
$$\mu_3 = \mu_2. \tag{4.7}$$

Definition 4.13. Let ν be a vector in the transversal plane σ_p, with $p \in M$.

- The *affine ν-principal curvatures* in p are the eigenvalues of the affine shape operator $(-S_\nu)_p$.
- The *affine ν-principal directions* in p are the eigenvectors of the affine shape operator $(-S_\nu)_p$.

Theorem 4.14. *Let $p \in M$, then $R_{\nabla^\perp}(p) \equiv 0$ if and only if the following conditions hold:*

- *the shape operator S_ν is self-adjoint $\forall \nu \in \sigma_p$, and*
- *either p is σ-semiumbilic and all the non trivial ν-principal configurations agree with the asymptotic configuration, or p is an inflection.*

Proof. We suppose that $R_{\nabla^\perp}(p) \equiv 0$. By equations (4.5) and (4.7) it follows that $\lambda_2 = \lambda_3$ and $\mu_2 = \mu_3$, in other words $S_1 = S_{\nu_1}$ and $S_2 = S_{\nu_2}$ are self-adjoint. Now if $\nu = \alpha\nu_1 + \beta\nu_2$, then any $S_\nu = \alpha S_1 + \beta S_2$ is also self-adjoint.

If p is not an inflection, then $(b, c) = (h^1(X_1, X_2), h^1(X_2, X_2)) \neq 0$, hence

$$\begin{vmatrix} \lambda_4 - \lambda_1 & \lambda_2 \\ c & b \end{vmatrix} = 0 \Longleftrightarrow (\lambda_4 - \lambda_1, \lambda_2) = t(c, b),$$

$$\begin{vmatrix} \mu_4 - \mu_1 & \mu_2 \\ c & b \end{vmatrix} = 0 \Longleftrightarrow (\mu_4 - \mu_1, \mu_2) = s(c, b),$$

for some $t, s \in \mathbb{R}$, therefore $\begin{vmatrix} \lambda_4 - \lambda_1 & \lambda_2 \\ \mu_4 - \mu_1 & \mu_2 \end{vmatrix} = 0.$

Moreover, if $\nu = \alpha\nu_1 + \beta\nu_2$, with $(\alpha, \beta) \neq 0$ then,

$$\alpha \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} + \beta \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

if and only if $\alpha\lambda_1 + \beta\mu_1 = \alpha\lambda_4 + \beta\mu_4$ and $\alpha\lambda_2 + \beta\mu_2 = 0$ if and only if

$$\begin{vmatrix} \lambda_4 - \lambda_1 & \lambda_2, \\ \mu_4 - \mu_1 & \mu_2, \end{vmatrix} = 0.$$

On the other hand, the asymptotic configuration is given by

$$\begin{vmatrix} y^2 & -xy & x^2 \\ 0 & b & c \\ 1 & 0 & 1 \end{vmatrix} = -bx^2 - cxy + by^2,$$

and the ν-principal configuration is given by:

$$\begin{vmatrix} y^2 & -xy & x^2 \\ \alpha\lambda_1 + \beta\mu_1 & \alpha\lambda_2 + \beta\mu_2 & \alpha\lambda_4 + \beta\mu_4 \\ 1 & 0 & 1 \end{vmatrix},$$

which is equal to $-(\alpha\lambda_2+\beta\mu_2)x^2-(\alpha(\lambda_4-\lambda_1)+\beta(\mu_4-\mu_1))xy+(\alpha\lambda_2+\beta\mu_2)y^2$. Therefore, the asymptotic configuration and the ν-principal configuration are the same if and only if

$$(\alpha(\lambda_4-\lambda_1)+\beta(\mu_4-\mu_1))b-(\alpha\lambda_2+\beta\mu_2)c=0,$$

or equivalently,

$$b(\lambda_4-\lambda_1)-c\lambda_2=0,$$
$$b(\mu_4-\mu_1)-c\mu_2=0.$$

□

4.5 Hyperquadric surfaces

In the last part of this section we will consider an immersed surface M in a locally strictly convex hyperquadric N. By affine transformation, the locally strictly convex hyperquadrics are equivalent to one of the following normal forms:

- Elliptic paraboloid $x_4=\frac{1}{2}(x_1^2+x_2^2+x_3^2)$.
- Ellipsoid $x_1^2+x_2^2+x_3^2+x_4^2=1$.
- Hyperboloid of two sheets $x_1^2+x_2^2+x_3^2-x_4^2=-1$.

Lemma 4.15. *Let $M\subset N\subset\mathbb{R}^4$ be an immersed surface in a locally strictly convex hypersurface and let $-\xi$ be the metric field such that $g_{-\xi}=\mathfrak{G}$ on M. If $\mathfrak{u}=\{X_1,X_2\}$ is a local orthonormal tangent frame and $\{\xi_1,\xi_2\}$ given by Theorem 2.16 on σ. Then the frame $\{X_1,X_2,\xi_1-\tau_1^2(X_1)X_1-\tau_1^2(X_2)X_2\}$ is orthonormal relative to metric $\mathfrak{G}$ on N and there are functions r_1,r_2,r_3 such that $\xi_2=\mathbf{Y}+r_1X_1+r_2X_2+r_3\xi_1$.*

Proof. We have that $\{X_1,X_2\}$ is orthonormal on N relative to metric $\mathfrak{G}$, now by using orthonormalization, we write $\overline{\xi}_1=\lambda\xi_1-\lambda_1X_1-\lambda_2X_2$ such that the frame $\{X_1,X_2,\overline{\xi}_1\}$ is a local orthonormal frame with the Blaschke metric. Since $\{X_1,X_2,\overline{\xi}_1\}$ is an orthonormal frame with the Blaschke metric $[X_1,X_2,\overline{\xi}_1,\mathbf{Y}]=1$ and it follows that $\lambda[X_1,X_2,\xi_1,\mathbf{Y}]=1$. On the other hand $[X_1,X_2,\xi_1,\xi_2]=1$ and we obtain $[X_1,X_2,\xi_1,\xi_2-\lambda\mathbf{Y}]=0$. Therefore there exist r_1,r_2,r_3 such that

$$\xi_2=\lambda\mathbf{Y}+r_1X_1+r_2X_2+r_3\xi_1.$$

On the other hand, $\mathfrak{G}(\overline{\xi}_1, X_1) = 0$, $\mathfrak{G}(\overline{\xi}_1, X_2) = 0$ and $\mathfrak{G}(\overline{\xi}_1, \overline{\xi}_1) = 1$ and it follows:

$$\begin{aligned} \lambda_1 &= \lambda \mathfrak{G}(X_1, \xi_1), \\ \lambda_2 &= \lambda \mathfrak{G}(X_2, \xi_1), \\ 1 + \lambda_1^2 + \lambda_2^2 &= \lambda^2 \mathfrak{G}(\xi_1, \xi_1). \end{aligned}$$

Now we compute the affine fundamental form h on N. By Theorem 2.16

$$\begin{aligned} D_{X_1} X_1 &= \nabla_{X_1} X_1 + \xi_2 = \nabla_{X_1} X_1 + \xi_2 + \lambda Y + r_1 X_1 + r_2 X_2 + r_3 \xi_1 \\ &= (\nabla_{X_1} X_1 + r_1 X_1 + r_2 X_2 + r_3 \xi_1) + \lambda \mathbf{Y}. \end{aligned}$$

Hence $h(X_1, X_1) = \lambda$, analogously we obtain

$$\begin{aligned} h(X_1, X_2) &= 0, & h(X_1, \overline{\xi}_1) &= \lambda^2 \tau_1^2(X_1) - \lambda_1 \lambda, \\ h(X_2, X_2) &= \lambda, & h(X_2, \overline{\xi}_1) &= \lambda^2 \tau_1^2(X_2) - \lambda_2 \lambda. \end{aligned}$$

The equations $\mathfrak{G}(X_1, \overline{\xi}_1) = 0$, $h(X_1, \overline{\xi}_1) = 0$ imply $\lambda_1 = \lambda \tau_1^2(X_1)$. Analogously $\lambda_2 = \lambda \tau_1^2(X_2)$. Now by a simple computation

$$H = \det(h_{ij}) = \lambda^2 h(\overline{\xi}_1, \overline{\xi}_1).$$

By using the definition of the Blaschke metric

$$\begin{aligned} 1 = \mathfrak{G}(X_1, X_1) &= \frac{\lambda}{\lambda^{2/5} h(\overline{\xi}_1, \overline{\xi}_1)^{1/5}} = \frac{\lambda^{3/5}}{h(\overline{\xi}_1, \overline{\xi}_1)^{1/5}}, \\ 1 = \mathfrak{G}(\overline{\xi}_1, \overline{\xi}_1) &= \frac{h(\overline{\xi}_1, \overline{\xi}_1)}{\lambda^{2/5} h(\overline{\xi}_1, \overline{\xi}_1)^{1/5}} = \frac{h(\overline{\xi}_1, \overline{\xi}_1)^{4/5}}{\lambda^{2/5}}. \end{aligned}$$

We conclude that $\lambda = h(\overline{\xi}_1, \overline{\xi}_1) = 1$ and hence

$$\begin{aligned} \xi_2 &= \mathbf{Y} + r_1 X_1 + r_2 X_2 + r_3 \xi_1, \\ \overline{\xi}_1 &= \xi_1 - \tau_1^2(X_1) X_1 - \tau_1^2(X_2) X_2. \end{aligned}$$

□

Theorem 4.16. *Let $M \subset \mathbb{R}^4$ be a surface immersed in a locally strictly convex hyperquadric N. Then the affine normal to N belongs to both the antisymmetric and symmetric equiaffine plane bundles of M, with respect to the Blaschke metric restricted to M.*

Proof. Let $p \in M$, since $M \subset N$ there is a transversal vector field ν_1 on M, which is tangent to N and such that $g_{[-\nu_1]}$ coincides with the Blaschke metric $\mathfrak{G}$ restricted to surface M. Now we consider a local orthonormal tangent frame $\{X_1, X_2\}$ on M relative to the metric $g_{[-\nu_1]} = \mathfrak{G}$. We fix a local transversal plane bundle σ, by Theorem 2.16 there is a local basis $\{\nu_1, \nu_2\}$ on σ such that: $[X_1, X_2, \nu_1, \nu_2] = 1$, $h^1(X_1, X_1) = 0$, $h^2(X_i, X_j) = \delta_{ij}$. Now we consider the local frame $\{e_1, e_2, e_3, e_4\}$ such that $e_1 = X_1$, $e_2 = X_2$, $e_3 = \nu_1$ and $e_4 = Y$, where Y is the affine normal vector field to N. By Lemma 4.15 we have $[e_1, e_2, e_3, e_4] = 1$. Since ν_2 is a transversal vector field on M and $[X_1, X_2, \nu_1, \nu_2] = 1$, we can conclude $\nu_2 = \lambda_3 e_3 + e_4$, for some λ_3. By Theorem 2.16 we write:

$$\begin{aligned}
D_{e_1} e_1 &= a_1 e_1 + a_2 e_2 + \lambda_3 e_3 + e_4,\\
D_{e_1} e_2 &= a_3 e_1 + a_4 e_2 + h^1(X_1, X_2) e_3,\\
D_{e_1} e_3 &= \beta_1 e_1 + \beta_2 e_2 + (\tau_1^1(X_1) + \lambda_3 \tau_1^2(X_1)) e_3 + \tau_1^2(X_1) e_4,\\
D_{e_2} e_1 &= a_5 e_1 + a_6 e_2 + h^1(X_1, X_2) e_3,\\
D_{e_2} e_2 &= a_7 e_1 + a_8 e_2 + (h^1(X_2, X_2) + \lambda_3) e_3 + e_4,\\
D_{e_2} e_3 &= \beta_3 e_1 + \beta_4 e_2 + (\tau_1^1(X_2) + \lambda_3 \tau_1^2(X_2)) e_3 + \tau_1^2(X_2) e_4.
\end{aligned}$$

We note that:

$$h(e_1, e_1) = 1, h(e_1, e_2) = 0, h(e_1, e_3) = \tau_1^2(X_1), h(e_2, e_2) = 1, h(e_2, e_3) = \tau_1^2(X_2).$$

As e_4 is the affine normal vector field, it follows:

$$\begin{aligned}
a_1 + a_4 + \tau_1^1(X_1) + \lambda_3 \tau_1^2(X_1) = 0,\\
a_5 + a_8 + \tau_1^1(X_2) + \lambda_3 \tau_1^2(X_2) = 0.
\end{aligned}$$

Since N is a hyperquadric, then $C(X, Y, Z) := (\nabla_X h)(Y, Z) \equiv 0$ (see [12]):

$$\begin{aligned}
0 = C(e_1, e_1, e_1) &= e_1(h(e_1, e_1)) - 2h(\nabla_{e_1} e_1, e_1)\\
&= -2h(a_1 e_1 + a_2 e_2 + \lambda_3 e_3, e_1)\\
&= -2a_1 - 2\lambda_3 \tau_1^2(X_1).\\
0 = C(e_1, e_1, e_2) &= e_1(h(e_1, e_2)) - h(\nabla_{e_1} e_1, e_2) - h(e_1, \nabla_{e_1} e_2)\\
&= -h(a_1 e_1 + a_2 e_2 + \lambda_3 e_3, e_2) - h(e_1, a_3 e_1 + a_4 e_2 + h^1(X_1, X_2) e_3)\\
&= -a_2 - \lambda_3 \tau_1^2(X_2) - a_3 - h^1(X_1, X_2) \tau_1^2(X_1).
\end{aligned}$$

$$\begin{aligned}
0 &= C(e_1,e_2,e_2) = e_1(h(e_2,e_2)) - 2h(\nabla_{e_1}e_2,e_2)\\
&= -2h(a_3e_1 + a_4e_2 + h^1(X_1,X_2)e_3,e_2)\\
&= -2a_4 - 2h^1(X_1,X_2)\tau_1^2(X_2).\\
0 &= C(e_2,e_1,e_1) = e_2(h(e_1,e_1)) - 2h(\nabla_{e_2}e_1,e_1)\\
&= -2h(a_5e_1 + a_6e_2 + h^1(X_1,X_2)e_3,e_1)\\
&= -2a_5 - 2h^1(X_1,X_2)\tau_1^2(X_1).\\
0 &= C(e_2,e_1,e_2) = e_2(h(e_1,e_2)) - h(\nabla_{e_2}e_1,e_2) - h(e_1,\nabla_{e_2}e_2)\\
&= -h(a_5e_1 + a_6e_2 + h^1(X_1,X_2)e_3,e_2) - h(e_1,a_7e_1 + a_8e_2 + (h^1(X_2,X_2)+\lambda_3)e_3)\\
&= -a_6 - h^1(X_1,X_2)\tau_1^2(X_2) - a_7 - (h^1(X_2,X_2)+\lambda_3)\tau_1^2(X_1).\\
0 &= C(e_2,e_2,e_2) = e_2(h(e_2,e_2)) - 2h(\nabla_{e_2}e_2,e_2)\\
&= -2h(a_7e_1 + a_8e_2 + (h^1(X_2,X_2)+\lambda_3)e_3,e_2)\\
&= -2a_8 - 2(h^1(X_2,X_2)+\lambda_3)\tau_1^2(X_2).
\end{aligned}$$

In the antisymmetric case we have:

$$\begin{aligned}
2h^1(X_1,X_2)\tau_1^2(X_1) - h^1(X_2,X_2)\tau_1^2(X_2) &= a_8 - a_2 - a_3 - a_5,\\
h^1(X_2,X_2)\tau_1^2(X_1) + 2h^1(X_1,X_2)\tau_1^2(X_2) &= a_1 - a_4 - a_6 - a_7,
\end{aligned}$$

and

$$\begin{aligned}
a_1 + a_4 + \tau_1^2(X_2)h^1(X_1,X_2) &= -\lambda_3\tau_1^2(X_1),\\
a_5 + a_8 + \tau_1^2(X_1)h^1(X_1,X_2) + \tau_1^2(X_2)h^1(X_2,X_2) &= -\lambda_3\tau_1^2(X_2).
\end{aligned}$$

From Remark 2.31, the affine normal plane is generated by the fields $\overline{\nu}_1,\overline{\nu}_2$, where:

$$\begin{aligned}
\overline{\nu}_1 &= \nu_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2,\\
\overline{\nu}_2 &= \nu_2 - \lambda_3\tau_1^2(X_1)X_1 - \lambda_3\tau_1^2(X_2)X_2.
\end{aligned}$$

By substituting $\overline{\nu}_2 = \lambda_3\nu_1 + e_4 - \lambda_3\tau_1^2(X_1)X_1 - \lambda_3\tau_1^2(X_2)X_2$, it follows that

$$\overline{\nu}_2 = \lambda_3\overline{\nu}_1 + e_4.$$

Analogously, in the symmetric case:

$$a_2 + a_3 - 3a_5 - a_8 = a_2 + a_3 - 2a_5 - (a_5 + a_8)$$

$$
\begin{aligned}
&= -\lambda_3\tau_1^2(X_2) - h^1(X_1,X_2)\tau_1^2(X_1) + 2h^1(X_1,X_2)\tau_1^2(X_1) + \tau_1^1(X_2) + \lambda_3\tau_1^2(X_2)\\
&= h^1(X_1,X_2)\tau_1^2(X_1) + \tau_1^1(X_2)\\
&= h^1(X_1,X_2)\tau_1^2(X_1) + h^1(X_1,X_2)\tau_1^2(X_1) + h^1(X_2,X_2)\tau_1^2(X_2)\\
&= 2h^1(X_1,X_2)\tau_1^2(X_1) + h^1(X_2,X_2)\tau_1^2(X_2),
\end{aligned}
$$

and

$$
\begin{aligned}
a_6 + a_7 - 3a_4 - a_1 &= a_6 + a_7 - 2a_4 - (a_1 + a_4)\\
&= h^1(X_1,X_2)\tau_1^2(X_2) - h^1(X_2,X_2)\tau_1^2(X_1) + \tau_1^1(X_1)\\
&= h^1(X_1,X_2)\tau_1^2(X_2) - h^1(X_2,X_2)\tau_1^2(X_1) + h^1(X_1,X_2)\tau_1^2(X_2)\\
&= 2h^1(X_1,X_2)\tau_1^2(X_2) - h^1(X_2,X_2)\tau_1^2(X_1).
\end{aligned}
$$

That is,

$$
\begin{aligned}
2h^1(X_1,X_2)\tau_1^2(X_1) + h^1(X_2,X_2)\tau_1^2(X_2) &= a_2 + a_3 - 3a_5 - a_8,\\
-h^1(X_2,X_2)\tau_1^2(X_1) + 2h^1(X_1,X_2)\tau_1^2(X_2) &= a_6 + a_7 - 3a_4 - a_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&a_1 + a_4 + \tau_1^2(X_2)h^1(X_1,X_2) = -\lambda_3\tau_1^2(X_1),\\
&a_5 + a_8 + \tau_1^2(X_1)h^1(X_1,X_2) + \tau_1^2(X_2)h^1(X_2,X_2) = -\lambda_3\tau_1^2(X_2).
\end{aligned}
$$

From Remark 2.32, the affine normal plane is generated by the fields $\overline{\nu}_1, \overline{\nu}_2$ where:

$$
\begin{aligned}
\overline{\nu}_1 &= \nu_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2,\\
\overline{\nu}_2 &= \nu_2 - \lambda_3\tau_1^2(X_1)X_1 - \lambda_3\tau_1^2(X_2)X_2.
\end{aligned}
$$

By substituting $\overline{\nu}_2 = \lambda_3\nu_1 + e_4 - \lambda_3\tau_1^2(X_1)X_1 - \lambda_3\tau_1^2(X_2)X_2$, it follows

$$\overline{\nu}_2 = \lambda_3\overline{\nu}_1 + e_4.$$

□

Remark 4.17. When we consider the affine metric of Burstin and Mayer [4], then Theorem 4.16 fails. In fact, we suppose that M is parameterized by

$$X(u,v) = (u, v, uv, \frac{u^2+v^2+u^2v^2}{2}).$$

Note that M is contained in the paraboloid $w = \frac{1}{2}(x^2 + y^2 + z^2)$. Now by using *Wolfram Mathematica* we compute the affine normal plane of Nomizu and Vrancken [13] which is generated by

$$\nu_1 = \frac{1}{12(1+u^2)^{2/3}(1+v^2)^{2/3}}\left(u, v, 2uv, 12 + 3v^2 + u^2(13 + 14v^2)\right) \text{ and}$$

$$\nu_2 = \frac{1}{12((1+u^2)(1+v^2))^{1/6}}\left(\frac{5v}{1+v^2}, \frac{5u}{1+u^2}, \frac{-12 - 7v^2 - 7u^2 - 2u^2v^2}{(1+u^2)(1+v^2)}, -14uv\right).$$

We can see that $(0,0,0,1)$ does not belong to the plane generated by ν_1 and ν_2.

The following corollary is deduced from the proof of Theorem 4.16.

Corollary 4.18. *With the same hypothesis as in Theorem 4.16, the antisymmetric and symmetric equiaffine plane bundles of M are equal.*

Let $M \subset \mathbb{R}^4$ be an immersed surface in a locally strictly convex hyperquadric N. We can consider on M the extended Blaschke metric, by writing $\mathfrak{G}(e_i, Y) = 0$ for all $i = 1, 2, 3$. Here $\{e_1, e_2, e_3\}$ is a unimodular frame and Y is the affine normal to N.

Corollary 4.19. *With the same hypothesis as in Theorem 4.16, the antisymmetric (and symmetric) equiaffine plane to M is the orthogonal plane to the tangent plane with respect to the extended Blaschke metric $\mathfrak{G}$.*

Proof. By Theorem 4.16, the antisymmetric equiaffine plane is generated by

$$\begin{aligned}\overline{\nu}_1 &= \nu_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2,\\ \overline{\nu}_2 &= \nu_2 - \lambda_3\tau_1^2(X_1)X_1 - \lambda_3\tau_1^2(X_2)X_2.\end{aligned}$$

Now we consider $\mathfrak{u}' = \{X_1, X_2, \nu_1\}$ the tangent frame on N and see that $\{\overline{\nu}_1, \overline{\nu}_2\}$ are orthogonal to the tangent plane:

$$\begin{aligned}\mathfrak{G}(\overline{\nu}_1, X_1) &= \mathfrak{G}(\nu_1, X_1) - \tau_1^2(X_1)\mathfrak{G}(X_1, X_1)\\ &= \frac{H(\nu_1, X_1)}{(\det_{\mathfrak{u}'} H_{u'})^{1/5}} - \tau_1^2(X_1)\frac{H(X_1, X_1)}{(\det_{\mathfrak{u}'} H_{u'})^{1/5}}\\ &= \frac{\tau_1^2(X_1)}{(\det_{\mathfrak{u}'} H_{u'})^{1/5}} - \tau_1^2(X_1)\frac{1}{(\det_{\mathfrak{u}'} H_{u'})^{1/5}} = 0.\end{aligned}$$

Analogously,

$$\begin{aligned}\mathfrak{G}(\overline{\nu}_1, X_2) &= \mathfrak{G}(\nu_1, X_2) - \tau_1^2(X_2)\mathfrak{G}(X_2, X_2) = 0,\\ \mathfrak{G}(\overline{\nu}_2, X_1) &= \mathfrak{G}(\overline{\nu}_2 - e_4, X_1) = \lambda_3\mathfrak{G}(\overline{\nu}_1, X_1) = 0,\\ \mathfrak{G}(\overline{\nu}_2, X_2) &= \mathfrak{G}(\overline{\nu}_2 - e_4, X_2) = \lambda_3\mathfrak{G}(\overline{\nu}_1, X_2) = 0.\end{aligned}$$

□

We have also the following property.

Proposition 4.20. *Let $M \subset H \subset \mathbb{R}^4$ be an immersed surface in an affine hypersphere. Let Y be the affine normal to H and assume that $Y_p \in \sigma_p$, for all $p \in M$. Then the shape operator S_Y on M is a multiple of the identity.*

Proof. Since H is an affine hypersphere, by definition there is $\lambda \in \mathbb{R}$ such that $D_X Y = -\lambda X$, for all $X \in T_pH$. In particular, this is also true for all $X \in T_pM$, hence $S_Y = \lambda Id$. □

Corollary 4.21. *Any surface $M \subset \mathbb{R}^4$ immersed in a locally strictly convex hyperquadric N is antisymmetric (and symmetric) affine semiumbilical.*

Proof. This result follows from Theorem 4.16 and Proposition 4.20. □

Example 4.22. The product of plane curves is also antisymmetric and symmetric affine semiumbilical with respect to some metric field. We consider the product of two convex plane curves parameterized by affine arc length, as

$$X(u, v) = (\alpha_1(u), \alpha_2(u), \beta_1(v), \beta_2(v))$$

and consider the transversal vector field $\xi = (0, \frac{1}{\alpha_1'(u)}, 0, -\frac{1}{\beta_1'(v)})$.

The metric of the transversal field g_ξ is given by

$$g_\xi(X_u, X_u) = 1, \qquad g_\xi(X_u, X_v) = 0, \qquad g_\xi(X_v, X_v) = 1.$$

Now consider the transversal plane bundle $\sigma = span\{X_{vv}, X_{uu}\}$. By Theorem 2.16 we obtain

$$\begin{aligned}\xi_1 &= (-\alpha_1''(u), -\alpha_2''(u), \beta_1''(v), \beta_2''(v)),\\ \xi_2 &= (\alpha_1''(u), \alpha_2''(u), 0, 0).\end{aligned}$$

Since $\nabla_{X_i} X_j = 0$ for $i = u, v$ and $j = u, v$ then $\nabla g = 0$. Therefore, the plane generated by ξ_1 and ξ_2 is the antisymmetric and the symmetric equiaffine plane. Finally, by a simple calculation we note that the normal curvature tensor $R_{\nabla^\perp} \equiv 0$ and by Theorem 4.14 it follows that the product of curves is also antisymmetric (and symmetric) affine semiumbilical.

Example 4.23. In the case of immersed surfaces in affine hyperspheres $Q(c, n)$, in general the symmetric and antisymmetric equiaffine planes bundle are not equal with respect to the Blaschke metric. Moreover, we have examples of immersed surfaces in $Q(c, n)$ which are semiumbilical and some others which are not:

- The surface parameterized by

$$(u, v) \mapsto (u, v, uv, \frac{1}{u^2v^2})$$

  is symmetric and antisymmetric affine semiumbilical,

- and the surface parameterized by

$$(u, v) \mapsto (u, v, v^2 + u^3, \frac{1}{uv(v^2 + u^3)})$$

  is not symmetric nor antisymmetric affine semiumbilical.

CHAPTER 5

Affine normal plane bundle

In [13] Nomizu and Vrancken studied the surfaces in $\mathbb{R}^4$ using the affine metric of Burstin and Mayer. From this construction it is defined an affine normal plane as an equiaffine plane bundle that satisfies a condition of antisymmetry relative to the metric of Burstin and Mayer [4]. In this section, by using the metric of the transversal vector field we introduce a new affine normal plane. This affine normal plane is related with the singularities of the families of affine distance and height functions as we shall see in this section.

Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ a metric field. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local orthonormal tangent frame of g_ξ and let σ be an arbitrary transversal plane bundle. By Theorem 2.16 there exists a unique local frame $\{\xi_1, \xi_2\}$ of σ such that $[X_1, X_2, \xi_1, \xi_2] = 1$, $h^1(X_1, X_1) = 0$, $-\xi_1 \in [\xi]$, $h^2(X_1, X_1) = 1$, $h^2(X_1, X_2) = 0$, $h^2(X_2, X_2) = 1$.

Definition 5.1. We define the affine normal plane bundle, denoted by $\mathbf{A}$, as the transversal plane bundle generated by $\{\bar{\xi}_1, \bar{\xi}_2\}$ where,

$$\bar{\xi}_1 = \xi_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2,$$
$$\bar{\xi}_2 = \xi_2 - \tau_2^2(X_1)X_1 - \tau_2^2(X_2)X_2.$$

Remark 5.2. It is known that: If $\bar{\xi}_1 = \xi_1 - Z_1$ and $\bar{\xi}_2 = \xi_2 - Z_2$ then

$$\bar{\tau}_1^1(X) = \tau_1^1(X) - h^1(X, Z_1), \qquad \bar{\tau}_2^1(X) = \tau_2^1(X) - h^1(X, Z_2),$$
$$\bar{\tau}_1^2(X) = \tau_1^2(X) - h^2(X, Z_1), \qquad \bar{\tau}_2^2(X) = \tau_2^2(X) - h^2(X, Z_2).$$

By a straightforward computation we obtain the torsions $\overline{\tau}_i^j$ on normal plane bundle

$$\begin{aligned}
\overline{\tau}_1^1(X_1) &= \tau_1^1(X_1) - h^1(X_1, X_2)\tau_1^2(X_2),\\
\overline{\tau}_1^1(X_2) &= \tau_1^1(X_2) - h^1(X_1, X_2)\tau_1^2(X_1) - h^1(X_2, X_2)\tau_1^2(X_2),\\
\overline{\tau}_2^1(X_1) &= \tau_2^1(X_1) - h^1(X_1, X_2)\tau_2^2(X_2),\\
\overline{\tau}_2^1(X_2) &= \tau_2^1(X_2) - h^1(X_1, X_2)\tau_2^2(X_1) - h^1(X_2, X_2)\tau_2^2(X_2),\\
\overline{\tau}_1^2(X_1) &= \overline{\tau}_1^2(X_2) = \overline{\tau}_2^2(X_1) = \overline{\tau}_2^2(X_2) = 0.
\end{aligned}$$

Proposition 5.3. *The affine normal plane bundle* $\mathbf{A}$ *does not depend on the transversal plane bundle* σ*.*

Proof. Let $\hat{\sigma}$ be an other transversal plane bundle. By Theorem 2.16 there exists a frame $\{\nu_1, \nu_2\}$ on $\hat{\sigma}$ and by Lemma 2.20 there are Z_1, Z_2 tangent vector fields on M such that $\nu_1 = \xi_1 - Z_1$ and $\nu_2 = \xi_2 - Z_2$. Now by definition we have

$$\begin{aligned}
D_{X_i}\nu_1 &= -S_{\nu_1}X_i + \hat{\tau}_1^1(X_i)\nu_1 + \hat{\tau}_1^2(X_i)\nu_2,\\
D_{X_i}\nu_2 &= -S_{\nu_2}X_i + \hat{\tau}_2^1(X_i)\nu_1 + \hat{\tau}_2^2(X_i)\nu_2.
\end{aligned}$$

On the other hand, by Remark 5.2 $\hat{\tau}_1^1(X_i) = \tau_1^1(X_i) - h^1(X_i, Z_1)$, $\hat{\tau}_2^1(X_i) = \tau_2^1(X_i) - h^1(X_i, Z_2)$, $\hat{\tau}_1^2(X_i) = \tau_1^2(X_i) - h^2(X_i, Z_1)$ and $\hat{\tau}_2^2(X_i) = \tau_2^2(X_i) - h^2(X_i, Z_2)$. Now we write $Z_1 = aX_1 + bX_2$ and $Z_2 = cX_1 + dX_2$ then

$$\begin{aligned}
\overline{\nu}_1 &= \nu_1 - \hat{\tau}_1^2(X_1)X_1 - \hat{\tau}_1^2(X_2)X_2\\
&= \xi_1 - Z_1 - (\tau_1^2(X_1) - h^2(X_1, Z_1))X_1 - (\tau_1^2(X_2) - h^2(X_2, Z_1))X_2\\
&= \xi_1 - aX_1 - bX_2 - (\tau_1^2(X_1) - a)X_1 - (\tau_1^2(X_2) - b)X_2\\
&= \xi_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2 = \overline{\xi}_1.
\end{aligned}$$

Analogously, $\overline{\nu}_2 = \overline{\xi}_2$. □

Proposition 5.4. *The affine normal plane bundle* $\mathbf{A}$ *does not depend on the local orthonormal tangent frame* $\mathfrak{u}$*.*

Proof. Let $\mathfrak{v} = \{Y_1, Y_2\}$ be another orthonormal tangent frame then

$$\begin{aligned}
Y_1 &= \cos\theta X_1 + \sin\theta X_2,\\
Y_2 &= -\sin\theta X_1 + \cos\theta X_2.
\end{aligned}$$

By Lemma 2.17 we have

$$\xi_1 = \hat{\xi}_1,$$
$$\xi_2 = -h^1(Y_1, Y_1)\hat{\xi}_1 + \hat{\xi}_2.$$

By a computation, $\hat{\tau}_1^2(X) = \tau_1^2(X)$ and $\hat{\tau}_2^2(X) = \tau_2^2(X) + h^1(Y_1, Y_1)\tau_1^2(X)$. Now by linearity of τ_i^j follows

$$\overline{\xi}_1 = \hat{\xi}_1 - \hat{\tau}_1^2(Y_1)Y_1 - \hat{\tau}_1^2(Y_2)Y_2,$$
$$\overline{\xi}_2 = h^1(Y_1, Y_1)\overline{\xi}_1 + \hat{\xi}_2 - \hat{\tau}_2^2(Y_1)Y_1 - \hat{\tau}_2^2(Y_2)Y_2.$$

We conclude $\overline{\xi}_1 = \overline{\hat{\xi}}_1$ and $\overline{\xi}_2 = h^1(Y_1, Y_1)\overline{\hat{\xi}}_1 + \overline{\hat{\xi}}_2$. □

In the following proposition we see that the affine normal plane bundle is not an equiaffine plane bundle.

Proposition 5.5. *The affine normal plane* **A** *is an equiaffine plane bundle if and only if* $\overline{\tau}_1^1 = 0$.

Proof. By a straightforward computation (see Remark 2.24) we have

$$\overline{B}_1 = B_1 - 2\tau_2^2(X_1) - 2h^1(X_1, X_2)\tau_1^2(X_2),$$
$$\overline{B}_2 = B_2 - 2h^1(X_1, X_2)\tau_1^2(X_1) - 2h^1(X_2, X_2)\tau_1^2(X_2) - 2\tau_2^2(X_2).$$

On the other hand, since $[X_1, X_2, \xi_1, \xi_2] = 1$ we obtain:

$$B_1 - 2\tau_2^2(X_1) = 2\tau_1^1(X_1),$$
$$B_2 - 2\tau_2^2(X_2) = 2\tau_1^1(X_2).$$

Therefore

$$\overline{B}_1 = 2(\tau_1^1(X_1) - h^1(X_1, X_2)\tau_1^2(X_2)),$$
$$\overline{B}_2 = 2(\tau_1^1(X_2) - h^1(X_1, X_2)\tau_1^2(X_1) - h^1(X_2, X_2)\tau_1^2(X_2)).$$

By Remark 5.2 we have $\overline{B}_1 = 2\overline{\tau}_1^1(X_1)$ and $\overline{B}_2 = 2\overline{\tau}_1^1(X_2)$. □

Proposition 5.6. *The affine normal plane is a symmetric affine plane bundle, i.e. satisfies* $\overline{D}_1 = \overline{D}_2 = 0$.

Proof. By a straightforward computation (see Definition 2.26) we obtain

$$\overline{D}_1 = D_1 - \tau_1^2(X_1)h^1(X_1, X_2) + \tau_2^2(X_2),$$
$$\overline{D}_2 = D_2 - \tau_1^2(X_2)h^1(X_1, X_2) + \tau_1^2(X_1)h^1(X_2, X_2) + \tau_2^2(X_1).$$

On the other hand, the cubic form C^2 (see Equation 2.13) is:

$$C^2(X_2, X_1, X_1) = \nabla g(X_2, X_1, X_1) + \tau_2^2(X_2),$$
$$C^2(X_1, X_1, X_2) = \nabla g(X_1, X_1, X_2) + \tau_1^2(X_1)h^1(X_1, X_2),$$
$$C^2(X_1, X_2, X_2) = \nabla g(X_1, X_2, X_2) + \tau_2^2(X_1) + \tau_1^2(X_1)h^1(X_2, X_2),$$
$$C^2(X_2, X_1, X_2) = \nabla g(X_2, X_1, X_2) + \tau_1^2(X_2)h^1(X_1, X_2).$$

Therefore

$$C^2(X_2, X_1, X_1) - C^2(X_1, X_1, X_2) = \overline{D}_1,$$
$$C^2(X_1, X_2, X_2) - C^2(X_2, X_1, X_2) = \overline{D}_2.$$

Since the cubic form C^2 is symmetric we conclude $\overline{D}_1 = \overline{D}_2 = 0$. □

Proposition 5.7. *The affine normal plane is not an antisymmetric plane bundle. Furthermore we have:*

$$\overline{C}_1/2 = (\nabla g)(X_1, X_1, X_2) - h^1(X_1, X_2)\tau_1^2(X_1) - \tau_2^2(X_2),$$
$$\overline{C}_2/2 = (\nabla g)(X_1, X_2, X_2) - h^1(X_1, X_2)\tau_1^2(X_2) - h^1(X_2, X_2)\tau_1^2(X_1) - \tau_2^2(X_1).$$

Proof. We use the definition of C_i (see Definition 2.26) and symmetry of the cubic form C^2 (see Equação 2.13). □

5.1 Affine distance functions

In this section we introduce the family of affine distance functions on immersed surface in 4-space similarly as described in [6], where it was defined the family of affine distance functions on hypersurfaces. The affine normal plane bundle is related with the singularities of the family of affine distance and height functions.

Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ a metric field. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local orthonormal tangent frame of g_ξ and let σ be an arbitrary transversal plane bundle. By Theorem 2.16 there exists a unique local frame $\{\xi_1, \xi_2\}$ of σ_p.

Definition 5.8. We define the family of affine distance functions

$$\Delta : \mathbb{R}^4 \times M \to \mathbb{R},$$

as follows: for $x \in \mathbb{R}^4$ and $p \in M$, $\Delta(x,p)$ is given by

$$p - x = z(x,p) + \Delta(x,p)\xi_2$$

where $z(x,p) \in \pi_p = T_pM \oplus \xi_p$.

Lemma 5.9. *The family of affine distance functions is independent of the tangent frame* $\mathfrak{u}$ *and independent of the transversal plane bundle* σ*.*

Proof. Let $\mathfrak{v} = \{Y_1, Y_2\}$ be another orthonormal tangent frame, then

$$\begin{aligned} Y_1 &= \cos\theta X_1 + \sin\theta X_2, \\ Y_2 &= -\sin\theta X_1 + \cos\theta X_2. \end{aligned}$$

By Lemma 2.17 we have $\xi_1 = \overline{\xi}_1$, and $\xi_2 = -h^1(Y_1,Y_1)\overline{\xi}_1 + \overline{\xi}_2$. By expanding

$$p - x = \lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3\overline{\xi}_1 + \overline{\Delta}(x,p)\overline{\xi}_2$$

we obtain

$$\begin{aligned} p - x = (\lambda_1\cos\theta - \lambda_2\sin\theta)X_1 + (\lambda_1\sin\theta + \lambda_2\cos\theta)X_2 + (\lambda_3 \\ + \overline{\Delta}(x,p)h^1(Y_1,Y_1))\xi_1 + \overline{\Delta}(x,p)\xi_2. \end{aligned}$$

On the other hand, let $\overline{\sigma}$ be another transversal plane. By Lemma 2.20 there are tangent vector fields Z_1 and Z_2 such that $\overline{\xi}_1 = \xi_1 - Z_1$, and $\overline{\xi}_2 = \xi_2 - Z_2$. Then

$$\begin{aligned} p - x &= \lambda_1 X_1 + \lambda_2 X_2 + \lambda_1\overline{\xi}_1 + \overline{\Delta}(x,p)\overline{\xi}_2, \\ &= \lambda_1 X_1 + \lambda_2 X_2 - \lambda_3 Z_1 - \overline{\Delta}(x,p)Z_2 + \lambda_3\xi_1 + \overline{\Delta}(x,p)\xi_2. \end{aligned}$$

□

We can write the affine distance function as $\Delta(x,p) = [X_1, X_2, \xi_1, p - x]$.

If $X : U \to M$ is a parametrization of p, the distance function is given by

$$\Delta(x,p) = \frac{[X_u, X_v, \xi_1, p - x]}{[X_u, X_v, \xi_1, \xi_2]},$$

and we can write $\Delta_x(p) = \Delta(x, p)$ in local coordinates as

$$\Delta_x(u, v) = \frac{[X_u, X_v, \xi_1, X(u, v) - x]}{[X_u, X_v, \xi_1, \xi_2]}$$

this is,

$$\Delta = \frac{[X_u, X_v, \xi_1, X - x]}{[X_u, X_v, \xi_1, \xi_2]}.$$

Theorem 5.10. *The affine distance function Δ_x has a singularity if and only if $x - X$ belongs to the affine normal plane $\mathbf{A}$.*

Proof. Since $\mathfrak{u} = \{X_1, X_2\}$ is an orthonormal tangent frame and $\{\xi_1, \xi_2\}$ is the frame on σ obtained by Theorem 2.16 then we can write

$$x - X = r_1X_1 + r_2X_2 + r_3\xi_1 + r_4\xi_2.$$

By properties of derivation and determinant we have:

$$\begin{aligned}
X_1(\Delta) &= [D_{X_1}X_1, X_2, \xi_1, x - X] + [X_1, D_{X_1}X_2, \xi_1, x - X] + [X_1, X_2, D_{X_1}\xi_1, x - X] \\
&= a_1r_4 - r_1 + a_4r_4 + \tau_1^1(X_1)r_4 - \tau_1^2(X_1)r_3 \\
&= -r_1 - \tau_2^2(X_1)r_4 - \tau_1^2(X_1)r_3, \\
X_2(\Delta) &= [D_{X_2}X_1, X_2, \xi_1, x - X] + [X_1, D_{X_2}X_2, \xi_1, x - X] + [X_1, X_2, D_{X_2}\xi_1, x - X] \\
&= a_5r_4 - r_2 + a_8r_4 + \tau_1^1(X_2)r_4 - \tau_1^2(X_2)r_3 \\
&= -r_2 - \tau_2^2(X_2)r_4 - \tau_1^2(X_2)r_3.
\end{aligned}$$

Therefore $X_1(\Delta) = X_2(\Delta) = 0$ if and only if

$$x - X = r_3(\xi_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2) + r_4(\xi_2 - \tau_2^2(X_1)X_1 - \tau_2^2(X_2)X_2).$$

□

5.2 Affine height functions

Analogously to the definition of affine distance functions we can define the family of affine height functions.

Definition 5.11. The family of affine height functions

$$H : \mathbb{R}^4 \times M \to \mathbb{R}$$

is defined as follows: for $x \in \mathbb{R}^4$ and $p \in M$, $H(x, p)$ is given by

$$x = z(x, p) + H(x, p)\xi_2$$

where $z(x, p) \in \pi_p = T_pM \oplus \xi_p$.

Lemma 5.12. *The family of affine height functions is independent of the tangent frame* $\mathfrak{u}$ *and independent of the transversal plane bundle* σ.

Proof. The proof is analogous to that of Lemma 5.9. □

We can write the affine distance function as $H(x, p) = [X_1, X_2, \xi_1, x]$.

If $X : U \to M$ is a parametrization of p, the height function is given by

$$H(x, p) = \frac{[X_u, X_v, \xi_1, x]}{[X_u, X_v, \xi_1, \xi_2]},$$

and we can write $H_x(p) = H(x, p)$ in local coordinates as

$$H_x(u, v) = \frac{[X_u, X_v, \xi_1, x]}{[X_u, X_v, \xi_1, \xi_2]}$$

this is,

$$H = \frac{[X_u, X_v, \xi_1, x]}{[X_u, X_v, \xi_1, \xi_2]}.$$

The next theorem characterizes the singularities of the family of affine height functions.

Theorem 5.13. *The affine height function* H_x *has a singularity if and only if* x *is in the affine normal plane* $\mathbf{A}$

Proof. Analogous to the proof of Theorem 5.10. □

5.3 Affine distance on hypersurfaces

The next results give a relation between the family of affine distance functions on hypersurfaces and the family of affine distance functions on surfaces.

Let $N \subset \mathbb{R}^4$ be a locally strictly convex hypersurface.

It is known that the tangent hyperplane T_pN is a support hyperplane with a non-degenerate hyperplane. In particular, given any immersed surface $M \subset N$ we have $T_pM \subset T_pN \subset \mathbb{R}^4$, and hence, M is locally strictly convex. Moreover, we can consider the Blaschke metric $\mathfrak{G}$ restricted to M. On the other hand, in Remark 4.8 it is shown that there is a metric field ξ such that $\mathfrak{G} = g_\xi$.

Definition 5.14. The family of affine distance functions is given by $\widetilde{\Delta} : \mathbb{R}^4 \times N \to \mathbb{R}$ as follows: given a point $x \in \mathbb{R}^{n+1}$ and a point $p \in N$ it is defined the affine distance from x to p implicitly by,

$$p - x = z(x,p) + \widetilde{\Delta}(x,p)\mathbf{Y}(p)$$

where $z \in T_pN$ and $\mathbf{Y}$ is the affine normal vector field on N.

Theorem 5.15. *Let $M \subset N \subset \mathbb{R}^4$ be an immersed surface in a locally strictly convex hypersurface and let $-\xi$ be the metric field such that $g_{-\xi} = \mathfrak{G}$ on M. Then the affine distance function $\widetilde{\Delta}(x,p)$ coincides with the affine distance function $\Delta(x,p)$ and the affine normal vector field $\mathbf{Y}$ belongs to the affine normal plane $\mathbf{A}$.*

Proof. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local orthonormal tangent frame and σ be a transversal plane bundle. By Theorem 2.16 there is a frame $\{\xi_1, \xi_2\}$ on σ and by Lemma 4.15 $\{X_1, X_2, \overline{\xi}_1\}$ is an orthonormal frame with the Blaschke metric and there are functions r_1, r_2, r_3 such that

$$\begin{aligned}\xi_2 &= \mathbf{Y} + r_1X_1 + r_2X_2 + r_3\xi_1,\\ \overline{\xi}_1 &= \xi_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2.\end{aligned}$$

Now we compute the affine distance

$$\Delta = [X_1, X_2, \xi_1, p - x] = \frac{[X_1, X_2, \xi_1, p - x]}{[X_1, X_2, \xi_1, \xi_2]} = \frac{[X_1, X_2, \xi_1, p - x]}{[X_1, X_2, \xi_1, \mathbf{Y}]} = \widetilde{\Delta}.$$

Now we see that $\mathbf{Y}$ belongs to the affine normal plane $\mathbf{A}$. In fact,

$$D_X\mathbf{Y} = z + (\tau_2^2(X) - r_1h^2(X_1, X) - r_2h^2(X_2, X) - r_3\tau_1^2(X))\mathbf{Y},$$

for some $z \in T_pN$, and since $\mathbf{Y}$ is the affine normal on N it follows

$$\tau_2^2(X) - r_1h^2(X_1, X) - r_2h^2(X_2, X) - r_3\tau_1^2(X) = 0.$$

In particular for $X = X_1, X_2$ we obtain

$$\tau_2^2(X_1) = r_1 + r_3\tau_1^2(X_1)$$
$$\tau_2^2(X_2) = r_2 + r_3\tau_1^2(X_2).$$

Finally

$$\begin{aligned}\mathbf{Y} &= \xi_2 - r_1X_1 - r_2X_2 - r_3\xi_1\\ &= \xi_2 - (\tau_2^2(X_1) - r_3\tau_1^2(X_1))X_1 - (\tau_2^2(X_2) - r_3\tau_1^2(X_2))X_2 - r_3\xi_1\\ &= (\xi_2 - \tau_2^2(X_1)X_1 - \tau_2^2(X_2)X_2) - r_3(\xi_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2).\end{aligned}$$

Therefore

$$\mathbf{Y} = \overline{\xi}_2 - r_3\overline{\xi}_1 \in \mathbf{A}.$$

□

Corollary 5.16. *Let $M \subset N \subset \mathbb{R}^4$ be a surface contained in a locally strictly convex affine hypersphere N. Then M is semiumbilic with respect to the affine normal plane.*

Proof. It follows from Theorem 5.15 and Proposition 4.20. □

Corollary 5.17. *With the same hypothesis as in Theorem 5.15, the affine normal plane is the orthogonal plane to the tangent plane with respect to the extended Blaschke metric.*

Proof. By Theorem 5.15 we have $\mathbf{Y} = \overline{\xi}_2 - r_3\overline{\xi}_1$ where

$$\overline{\xi}_1 = \xi_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2,$$
$$\overline{\xi}_2 = \xi_2 - \tau_2^2(X_1)X_1 - \tau_2^2(X_2)X_2.$$

By Lemma 4.15 the frame $\mathfrak{u}' = \{X_1, X_2, \overline{\xi}_1\}$ is orthonormal relative to the Blaschke metric.

$$\mathfrak{G}(\overline{\xi}_2, X_1) = \mathfrak{G}(\mathbf{Y} + r_3\overline{\xi}_1, X_1) = \mathfrak{G}(\mathbf{Y}, X_1) + r_3\mathfrak{G}(\overline{\xi}_1, X_1) = 0,$$
$$\mathfrak{G}(\overline{\xi}_2, X_2) = \mathfrak{G}(\mathbf{Y} + r_3\overline{\xi}_1, X_2) = \mathfrak{G}(\mathbf{Y}, X_2) + r_3\mathfrak{G}(\overline{\xi}_1, X_2) = 0.$$

□

Now we consider an immersed surface M in a locally strictly convex hyperquadric N. These hyperquadrics are particular cases of affine hyperspheres (elliptic paraboloid, ellipsoid and hyperboloid of two sheets), [10].

Proposition 5.18. *Let $M \subset H \subset \mathbb{R}^4$ be an immersed surface in an hyperquadric (elliptic paraboloid, ellipsoid and hyperboloid of two sheets) then the symmetric and antisymmetric equiaffine plane bundles are equal to the affine normal plane bundle.*

Proof. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local orthonormal tangent frame and σ be a transversal plane bundle. By Theorem 6.3 there is a frame $\{\xi_1, \xi_2\}$ on σ. By using orthonormalization and Theorem 5.15, we can write $\overline{\xi}_1 = \xi_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2$, such that the frame $\{X_1, X_2, \overline{\xi}_1\}$ is an orthonormal frame with the Blaschke metric and $\xi_2 = Y + r_1X_1 + r_2X_2 + r_3\xi_1$, for some functions r_1, r_2 and r_3. Now, we denote by ∇ the affine connection induced by σ and we write

$$\begin{aligned}
\nabla_{X_1}X_1 &= a_1X_1 + a_2X_2,\\
\nabla_{X_1}X_2 &= a_3X_1 + a_4X_2,\\
\nabla_{X_2}X_1 &= a_5X_1 + a_6X_2,\\
\nabla_{X_2}X_2 &= a_7X_1 + a_8X_2.
\end{aligned}$$

With this notation, we obtain

$$\begin{aligned}
D_{X_1}X_1 &= (a_1 + r_1 + r_3\tau_1^2(X_1))X_1 + (a_2 + r_2 + r_3\tau_1^2(X_2))X_2 + r_3\overline{\xi}_1 + \mathbf{Y},\\
D_{X_1}X_2 &= (a_3 + \ldots)X_1 + (a_4 + h^1(X_1, X_2)\tau_1^2(X_2))X_2 + h^1(X_1, X_2)\overline{\xi}_1,\\
D_{X_1}\overline{\xi}_1 &= (\ldots)X_1 + (\ldots)X_2 + (\tau_1^1(X_1) - \tau_1^2(X_2)h^1(X_1, X_2))\overline{\xi}_1,
\end{aligned}$$

and

$$\begin{aligned}
D_{X_2}X_1 &= (a_5 + h^1(X_1, X_2)\tau_1^2(X_1))X_1 + (a_6 + \ldots)X_2 + h^1(X_1, X_2)\overline{\xi}_1,\\
D_{X_2}X_2 &= \ldots + (a_8 + h^1(X_2, X_2)\tau_1^2(X_2) + r_2 + r_3\tau_1^2(X_2))X_2 + \ldots + \mathbf{Y},\\
D_{X_2}\overline{\xi}_1 &= \ldots + (\tau_1^1(X_2) - \tau_1^2(X_1)h^1(X_1, X_2) - \tau_1^2(X_2)h^1(X_2, X_2))\overline{\xi}_1.
\end{aligned}$$

Since $\mathbf{Y}$ is parallel

$$\begin{aligned}
0 &= a_1 + r_1 + r_3\tau_1^2(X_1) + a_4 + \tau_1^1(X_1),\\
0 &= a_5 + a_8 + r_2 + r_3\tau_1^2(X_2) + \tau_1^1(X_2).
\end{aligned}$$

As $a_1 + a_4 + \tau_1^1(X_1) + \tau_2^2(X_1) = 0$ and $a_5 + a_8 + \tau_1^1(X_2) + \tau_2^2(X_2) = 0$. We obtain

$$\begin{aligned}
\tau_2^2(X_1) &= r_1 + r_3\tau_1^2(X_1),\\
\tau_2^2(X_2) &= r_2 + r_3\tau_1^2(X_2).
\end{aligned}$$

Since N is a hyperquadric, then

$$C(X,Y,Z) := (\overline{\nabla}_X h)(Y,Z) \equiv 0$$

where $\overline{\nabla}$ is the affine connection induced by $\mathbf{Y}$ on N (see [12]):

$$\begin{aligned}
0 &= C(X_1,X_1,X_1) = X_1(h(X_1,X_1)) - 2h(\overline{\nabla}_{X_1}X_1,X_1)\\
&= -2(a_1 + r_1 + r_3\tau_1^2(X_1)),\\
0 &= C(X_1,X_1,\overline{\xi}_1) = X_1(h(X_1,\overline{\xi}_1)) - h(\overline{\nabla}_{X_1}X_1,X_2) - h(X_1,\overline{\nabla}_{X_1}X_2)\\
&= -(a_2 + r_2 + r_3\tau_1^2(X_2)) - (a_3 + h^1(X_1,X_2)\tau_1^2(X_1)),\\
0 &= C(X_1,X_2,X_2) = X_1(h(X_2,X_2)) - 2h(\nabla_{X_1}X_2,X_2)\\
&= -2(a_4 + h^1(X_1,X_2)\tau_1^2(X_2)),\\
0 &= C(X_2,X_1,X_1) = X_2(h(X_1,X_1)) - 2h(\nabla_{X_2}X_1,X_1)\\
&= -2(a_5 + h^1(X_1,X_2)\tau_1^2(X_1)),\\
0 &= C(X_2,X_1,X_2) = X_2(h(X_1,X_2)) - h(\nabla_{X_2}X_1,X_2) - h(X_1,\nabla_{X_2}X_2)\\
&= -(a_6 + h^1(X_1,X_2)\tau_1^2(X_2)) - (a_7 + h^1(X_2,X_2)\tau_1^2(X_1) + r_1 + r_3\tau_1^2(X_1)),\\
0 &= C(X_2,X_2,X_2) = X_2(h(X_2,X_2)) - 2h(\nabla_{X_2}X_2,X_2)\\
&= -2(a_8 + h^1(X_2,X_2)\tau_1^2(X_2) + r_2 + r_3\tau_1^2(X_2)).
\end{aligned}$$

Now we compute the generators of the antisymmetric transversal plane bundle: note that,

$$\begin{aligned}
a_8 - a_2 - a_3 - a_5 &= -h^1(X_2,X_2)\tau_1^2(X_2) + 2h^1(X_1,X_2)\tau_1^2(X_1),\\
a_1 - a_4 - a_6 - a_7 &= 2h^1(X_1,X_2)\tau_1^2(X_2) + h^1(X_2,X_2)\tau_1^2(X_1),
\end{aligned}$$

$$\begin{aligned}
-(a_1 + a_4 + \tau_1^2(X_1)h^1(X_1,X_2)) &= \tau_2^2(X_1),\\
-(a_5 + a_8 + \tau_1^2(X_1)h^1(X_1,X_2) + \tau_1^2(X_2)h^1(X_2,X_2)) &= \tau_2^2(X_2).
\end{aligned}$$

It follows from Remark 2.31 that the antisymmetric equiaffine plane bundle is generated by:

$$\begin{aligned}
&\xi_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2,\\
&\xi_2 - \tau_2^2(X_1)X_1 - \tau_2^2(X_2)X_2.
\end{aligned}$$

Therefore the antisymmetric equiaffine bundle is equal to the affine normal plane bundle.

□

5.4 Affine focal points

In this section we define the affine normal curvature μ_ν in the direction ν, similarly as Davis in [6]. The affine focal points to surfaces in $\mathbb{R}^4$ appear as the degenerate singularities of the family of affine distance functions, analogous result appears in [5] on the study of hypersurfaces in the affine differential geometry.

Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ a metric field. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local orthonormal tangent frame of g_ξ and let σ be an arbitrary transversal plane bundle. By Theorem 2.16 there exists a unique local frame $\{\xi_1, \xi_2\}$ of σ_p. We denote by P the hyperplane through the point $p \in M$ such that is generated by $v \in T_pM$, ξ_1 and ξ_2. We write C the intersection $M \cap P$ which, close to p is a regular curve and $\nu = \lambda_1\xi_1 + \lambda_2\xi_2 \in \sigma_p$. We consider the restriction of $\Delta : \mathbb{R}^4 \times M \to \mathbb{R}$, for $x \in P$ and $p \in C$,

$$p - x = z(x,p) + \widehat{\Delta}(x,p)\xi_2,$$

this gives a family of functions $\widehat{\Delta} : P \times C \to \mathbb{R}$. Locally this is a three parameter family of functions from $\mathbb{R}$ to $\mathbb{R}$, or $\widehat{\Delta} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$.

Definition 5.19. We define the *affine normal curvature in the direction* ν by

$$\mu_\nu(v) = \frac{\lambda_2}{\widehat{\Delta}(x,p)},$$

where $x \in P$ is such that $\widehat{\Delta}_x : C \to \mathbb{R}$ has a degenerate singularity at $p \in C$.

Definition 5.20. A point $x = p + t\nu$ where $\nu \in \mathbf{A}$ is an *affine focal* of M in p if: $t \neq 0$ and $\frac{1}{t}$ is an affine ν-principal curvature.

Theorem 5.21. *An element of the family of affine distance functions* $\Delta_x : M \to \mathbb{R}$

- *has a critical point at* $p \in M$ *if and only if* $p - x$ *belongs to the affine normal plane.*
- *If* Δ_x *has a critical point at* $p \in M$ *we write* $x = p + t\nu$ *: then* p *is a degenerate critical point if and only if* x *is an affine focal point.*

Proof. Let $\mathfrak{u} = \{X_1, X_2\}$ be an orthonormal tangent frame and σ a transversal plane bundle. By Theorem 2.16 there exist a frame $\{\nu_1, \nu_2\}$ on σ. By Definition 5.1

$$\xi_1 = \nu_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2,$$
$$\xi_2 = \nu_2 - \tau_2^2(X_1)X_1 - \tau_2^2(X_2)X_2,$$

generate the affine normal plane **A**. Now by Lemma 5.9 we can write

$$p - x = z + \lambda_3 \xi_1 + \Delta \xi_2$$

where $z \in T_pM$.

By derivation, for $v \in T_pM$

$$v = (\nabla_v z - \lambda_3 S_1 v - \Delta S_2 v) + (v(\lambda_3) + h^1(v,z) + \lambda_3 \tau_1^1(v) + \Delta \tau_2^1(v))\xi_1 \\ + (v(\Delta) + h^2(v,z) + \lambda_3 \tau_1^2(v) + \Delta \tau_2^2(v))\xi_2.$$

Comparing the transversal and tangential components,

$$\begin{aligned} \nabla_v z &= (\lambda_3 S_1 + \Delta S_2 + Id)v \\ 0 &= v(\lambda_3) + h^1(v,z) + \lambda_3 \tau_1^1(v) + \Delta \tau_2^1(v) \\ 0 &= v(\Delta) + h^2(v,z) + \lambda_3 \tau_1^2(v) + \Delta \tau_2^2(v). \end{aligned}$$

By Remark 5.2 we have $\tau_2^2 \equiv 0$ and $\tau_1^2 \equiv 0$ (on the affine normal plane **A**). Since h^2 is nondegenerate, Δ has a critical point if and only if $z = 0$.

Now we write $x = p + t\nu$ where $\nu \in \mathbf{A}_p$, and consider the Hessian

$$(v,w) \mapsto w(v(\Delta)) = -w(h^2(v,z)).$$

By using properties of affine connection,

$$(\nabla_w h^2)(v,z) = w(h^2(v,z)) - h^2(\nabla_w v, z) - h^2(v, \nabla_w z).$$

If $\Delta_x : M \to \mathbb{R}$ has a critical point then $z = 0$ and the Hessian is given by

$$(v,w) \mapsto -h^2(v, \nabla_w z).$$

On the other hand, note that $-t\nu = \lambda_3 \xi_1 + \Delta \xi_2$ and using linearity of the shape operator we obtain

$$-tS_\nu = S_{-t\nu} = S_{\lambda_3 \xi_1 + \Delta \xi_2} = \lambda_3 S_1 + \Delta S_2.$$

Therefore

$$\begin{aligned}-h^2(v,\nabla_w z) &= -h^2(v,(\lambda_3 S_1+\Delta S_2+Id)w)\\ &= -h^2(v,(-tS_\nu+Id)w)\\ &= h^2(v,(tS_\nu-Id)w).\end{aligned}$$

Since h^2 is nondegenerate, the Hessian is degenerate if and only if

$$\det(S_\nu - \frac{1}{t}Id) = 0.$$

□

Proposition 5.22. *Suppose that $h^2(v,v)\neq 0$ then*

$$\mu_\nu(v) = -\frac{h^2(S_\nu v,v)}{h^2(v,v)}. \tag{5.1}$$

Proof. Let $\alpha : I \to M$ be a curve in M, such that $\alpha(0)=p$ and $v=\alpha'(0)$. We write $\alpha(I)=C$ and consider p a degenerate singularity of the family $\widetilde{\Delta} : P\times C\to\mathbb{R}$. By Theorem 5.21, the point $x\in P$ is such that $v(\widehat{\Delta})(0)=v(v(\widehat{\Delta}))(0)=0$ if and only if $p-x$ belongs to affine normal plane $\mathbf{A}_p$ and $h^2(v,(tS_\nu-Id)v)=0$ where $x=p+t\nu$, therefore

$$\frac{1}{t}=\frac{h^2(S_\nu v,v)}{h^2(v,v)}.$$

On the other hand, if we write $\nu=\lambda_1\xi_1+\lambda_2\xi_2$ then $p-x=-t\lambda_1\xi_1-t\lambda_2\xi_2$, and follows $\widehat{\Delta}=-t\lambda_2$. Finally

$$\mu_\nu(v)=\frac{\lambda_2}{\widehat{\Delta}}=-\frac{1}{t}=-\frac{h^2(S_\nu v,v)}{h^2(v,v)}.$$

□

Proposition 5.23. *Let ν be a transversal vector field on the affine normal plane $\mathbf{A}_p$. The function $v\to\mu_\nu(v)$ has an extrema if and only if v is an affine ν-principal direction of M at p. Furthermore, the value of $\mu_\nu(v)$ in such a direction is the corresponding affine ν-principal curvature.*

Proof. We wish to show that, $(d\mu_\nu)(v)=0$ if and only if v is an affine ν-principal direction. Equivalently we will prove $(w(\mu_\nu))(v)=0$ for all w if and only if v is an affine ν-principal direction. We consider the derivative of Equation (5.1) by w, this gives,

$$(w(\mu_\nu))(v)h^2(v,v)+2\mu_\nu(v)h^2(v,w)=-h^2(S_\nu w,v)-h^2(S_\nu v,w).$$

From the Ricci Equations (2.17) and (2.18) and $\tau_1^2 = \tau_2^2 = 0$ (on the affine normal plane $\mathbf{A}$), we obtain

$$h^2(S_\nu w, v) = h^2(w, S_\nu v).$$

Therefore,

$$(w(\mu_\nu))(v)h^2(v,v) + 2\mu_\nu(v)h^2(v,w) = -2h^2(S_\nu v, w).$$

From Equation (5.1) one has,

$$-h^2(v,v)^2(w(\mu_\nu))(v) = 2h^2(v,v)h^2(S_\nu v, w) - 2h^2(S_\nu v, v)h^2(v,w). \tag{5.2}$$

If v is an affine ν-principal direction, then $(-S_\nu)v = \lambda v$ for some $\lambda \in \mathbb{R}$. In this case the right hand of Equation (5.2) vanishes. Since $h^2(v,v) \neq 0$ follows that $(w(\mu_\nu))(v) = 0$ for all w. Next, if $(w(\mu_\nu))(v) = 0$ for all w then Equation (5.2) becomes

$$h^2(v,v)h^2(S_\nu v, w) - h^2(S_\nu v, v)h^2(v,w) = 0.$$

Rearranging this as follows:

$$0 = h^2(h^2(v,v)S_\nu v, w) - h^2(h^2(S_\nu v, v)v, w), \forall w$$
$$0 = h^2(h^2(v,v)S_\nu v - h^2(S_\nu v, v)v, w), \forall w.$$

Since h^2 is nondegenerate this is true for all w if and only if

$$h^2(v,v)S_\nu v - h^2(S_\nu v, v)v = 0.$$

Finally,

$$(-S_\nu)v = -\frac{h^2(S_\nu v, v)}{h^2(v,v)}v.$$

□

5.5 Affine semiumbilical surfaces

Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface with a transversal plane bundle σ. In Chapter 4, it was introduced the concept of σ-semiumbilical points. In this section we fix the transversal plane bundle $\mathbf{A}$, the affine normal plane, and we define the affine semiumbilical point in analogy with the Euclidean case.

Definition 5.24. A point $p \in M$ is called *affine semiumbilic* if it is semiumbilic with respect to the affine normal plane $\mathbf{A_p}$, this is, there exists a non zero normal vector ν such that the affine shape operator $(S_\nu)_p$ is a multiple of the identity.

Corollary 5.25. *Let ν a transversal vector on the affine normal plane $\mathbf{A}_p$. The affine normal curvature $\mu_\nu(v)$ is a constant function if and only if $(S_\nu)_p$ is a multiple of the identity.*

Proof. We assume that the function $\mu_\nu(v)$ is constant, with value λ on T_pM. From Equation (5.1) we have $\lambda h^2(v,v) = -h^2(S_\nu v, v)$ $\forall v \in T_pM$. It follows $h^2((S_\nu + \lambda Id)v, v) = 0$ $\forall v \in T_pM$. Since h^2 is nondegenerate we obtain $S_\nu = -\lambda Id$. □

Corollary 5.26. *Any immersed surface M in an affine hypersphere N in 4-space is affine semiumbilical.*

Proof. This result follows from Theorem 5.15 and Proposition 4.20. □

In particular, Corollary 5.26 is true for hyperquadrics (elliptic paraboloid, ellipsoid and hyperboloid of two sheets). It was also seen that the product of plane curves is symmetric and antisymmetric affine semiumbilical.

Example 5.27. The product of plane curves is affine semiumbilical. In fact, we consider the product of two convex plane curves parameterized by affine arc length

$$X(u,v) = (\alpha_1(u), \alpha_2(u), \beta_1(v), \beta_2(v))$$

and the transversal vector field

$$\xi = (0, \frac{1}{\alpha_1'(u)}, 0, -\frac{1}{\beta_1'(v)}).$$

The frame $\mathfrak{u} = \{X_u, X_v\}$ is orthonormal with the metric g_ξ. Now we consider the transversal plane bundle σ generated by X_{vv} and X_{uu}. By using Theorem 2.16 we obtain

$$\xi_1 = (-\alpha_1''(u), -\alpha_2''(u), \beta_1''(v), \beta_2''(v)),$$
$$\xi_2 = (\alpha_1''(u), \alpha_2''(u), 0, 0),$$

which in turn generates the symmetric and antisymmetric affine normal plane. On the other hand, by a straightforward computation,

$$
\begin{aligned}
D_{X_u}\xi_1 &= \xi_{1,u} = -\alpha^{'''}(u) = K_\alpha \alpha^{'}(u),\\
D_{X_v}\xi_1 &= \xi_{1,v} = \beta^{'''}(v) = -K_\beta \beta^{'}(v),\\
D_{X_u}\xi_2 &= \xi_{2,u} = \alpha^{'''}(u) = -K_\alpha \alpha^{'}(u),\\
D_{X_v}\xi_2 &= \xi_{2,v} = 0.
\end{aligned}
$$

We obtain $\tau_1^1 = \tau_1^2 = \tau_2^1 = \tau_2^2 = 0$ and therefore the frame $\{\xi_1, \xi_2\}$ also generates the affine normal plane. Finally, since the product of curves is symmetric affine semiumbilical follows that is affine semiumbilical.

CHAPTER 6

Submanifolds of codimension 2

In this chapter we extend some of the concepts introduced to locally strictly convex surfaces in $\mathbb{R}^4$ for locally strictly convex submanifolds M^n in $\mathbb{R}^{n+2}(n > 2)$.

Let $\mathbb{R}^{n+2}$ be the affine $(n+2)$-space and D the usual flat connection on $\mathbb{R}^{n+2}$. Let $M \subset \mathbb{R}^{n+2}$ be an immersed n-submanifold and let σ be a transversal plane bundle on M. Then, for all $p \in M$, $\sigma_p \subset T_p\mathbb{R}^{n+2}$ is a plane such that

$$T_p\mathbb{R}^{n+2} = T_pM \oplus \sigma_p,$$

and for all tangent vector fields X, Y on M,

$$(D_X Y)_p = (\nabla_X Y)_p + h(X,Y)_p,$$

where $(\nabla_X Y)_p \in T_pM$ and $h(X,Y)_p \in \sigma_p$, for all $p \in M$.

We note that for $p \in M$, there are ξ_1, ξ_2 transversal vector fields defined on some neighborhood U_p such that: $\sigma_q = \text{span}\{\xi_1(q), \xi_2(q)\}$, $\forall q \in U_p$.

Then for tangent vector fields X, Y on M we have:

$$\begin{aligned}
D_X Y &= \nabla_X Y + h^1(X,Y)\xi_1 + h^2(X,Y)\xi_2, \\
D_X \xi_1 &= -S_1 X + \tau_1^1(X)\xi_1 + \tau_1^2(X)\xi_2, \\
D_X \xi_2 &= -S_2 X + \tau_2^1(X)\xi_1 + \tau_2^2(X)\xi_2,
\end{aligned}$$

where $\nabla = \nabla(\sigma)$ is a torsion free affine connection, h^1, h^2 are bilinear symmetric forms, S_1, S_2 are $(1,1)$ tensor fields, and τ_i^j are 1-forms on M. We call ∇ the affine connection induced by the transversal plane bundle σ.

6.1 The metric of the transversal vector field

Let $M \subset \mathbb{R}^{n+2}$ be a locally strictly convex n-submanifold, let $\mathfrak{u} = \{X_1, X_2, \ldots, X_n\}$ be a local tangent frame of a point $p \in M$ and let ξ be a transversal vector field on M.

Definition 6.1. We define the symmetric bilinear form $G_{\mathfrak{u}}$ on M to be

$$G_{\mathfrak{u}}(Y, Z) = [X_1, X_2, \ldots, X_n, D_Z Y, \xi].$$

We fix ξ such that $G_{\mathfrak{u}}$ is positive definite this is possible because M is locally strictly convex and we call such a ξ a *metric field.*

The metric field ξ is defined only locally, but since all our results are local, we can assume without lost of generality that ξ is globally defined and that M is globally oriented.

We define the *metric of the transversal vector field*, denoted by g_ξ, by

$$g_\xi(Y, Z) = \frac{G_{\mathfrak{u}}(Y, Z)}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{\frac{1}{(n+2)}}},$$

where $\det_{\mathfrak{u}} G_{\mathfrak{u}} = \det(G_{\mathfrak{u}}(X_i, X_j))$.

Lemma 6.2. *The quadratic form g_ξ does not depend on the choice of the local tangent frame $\mathfrak{u}$, provided it is positively oriented.*

Proof. Let $\mathfrak{v} = \{Y_1, Y_2, \ldots, Y_n\}$ be another local tangent frame on a neighborhood U of $p \in M$, then there exists a matrix $a = (a_{ij})$ with $\det a > 0$, defined on U such that $Y_i = \sum_{j=1}^{n} a_{ij} X_j$, $(i = 1, \ldots, n)$. Note that

$$G_{\mathfrak{v}}(Y, Z) = [Y_1, Y_2, \ldots, Y_n, D_Z Y, \xi] = \det(a) G_{\mathfrak{u}}(Y, Z).$$

By properties of the determinant, it follows that $\det_{\mathfrak{v}} G_{\mathfrak{v}} = \det(a)^n \det_{\mathfrak{v}} G_{\mathfrak{u}}$. On the other hand, from a simple computation $\det_{\mathfrak{v}} G_{\mathfrak{u}} = \det(a)^2 \det_{\mathfrak{u}} G_{\mathfrak{u}}$, therefore

$$\det_{\mathfrak{v}} G_{\mathfrak{v}} = \det(a)^{n+2} \det_{\mathfrak{u}} G_{\mathfrak{u}}.$$

Finally,

$$\frac{G_{\mathfrak{v}}(Y, Z)}{(\det_{\mathfrak{v}} G_{\mathfrak{v}})^{1/(n+2)}} = \frac{\det(a) G_{\mathfrak{u}}(Y, Z)}{(\det(a)^{(n+2)} \det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/(n+2)}} = \frac{G_{\mathfrak{u}}(Y, Z)}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/(n+2)}}.$$

□

From now on, we shall restrict ourselves to orthonormal frames $\{X_1, \ldots, X_n\}$ relative to $g = g_\xi$, this is, $g(X_i, X_j) = \delta_{ij}, \forall i, j$.

Theorem 6.3. *Let $M \subset \mathbb{R}^{n+2}$ be a locally strictly convex n-submanifold and ξ a metric field. Let $\mathfrak{u} = \{X_1, \ldots, X_n\}$ be a local orthonormal tangent frame of g_ξ and let σ be an arbitrary transversal plane bundle. Then there exists a unique local frame $\{\xi_1, \xi_2\}$ of σ, such that: $[X_1, \ldots, X_n, \xi_1, \xi_2] = 1$, $-\xi_1 \in [\xi]$, $h^1(X_1, X_1) = 0$ and $h^2(X_i, X_j) = \delta_{ij}$.*

Proof. Let p be a point in M and let $\{\nu_1, \nu_2\}$ be any local frame of σ in a neighborhood U of p. We can assume that $X_1, \ldots, X_n$ are defined on U. Now, we write

$$[\xi] = \lambda_3 \nu_1 + \lambda_4 \nu_2 + T_p M.$$

Using the notation: $h^1(X_i, X_j) = a_{ij}$, $h^2(X_i, X_j) = b_{ij}$ and $K = [X_1, X_2, \nu_1, \nu_2]$, we compute the bilinear form $G_\mathfrak{u}$:

$$G_\mathfrak{u}(X_i, X_j) = (\lambda_4 a_{ij} - \lambda_3 b_{ij})K. \tag{6.1}$$

By using the change: $\nu_1 = \alpha\xi_1 + \beta\xi_2$, $\nu_2 = \varphi\xi_1 + \psi\xi_2$, we obtain the affine fundamental forms in the new frame $\{\xi_1, \xi_2\}$: $\overline{h}^1(X_i, X_j) = \alpha a_{ij} + \varphi b_{ij}$ and $\overline{h}^2(X_i, X_j) = \beta a_{ij} + \psi b_{ij}$. Since $0 \neq G_\mathfrak{u}(X_1, X_1) = a_{11}\lambda_4 - b_{11}\lambda_3$ the following system:

$$\begin{aligned} 1 &= \beta a_{11} + \psi b_{11} \\ 0 &= \beta\lambda_3 + \psi\lambda_4, \end{aligned}$$

has solution (β, ψ) given by

$$\beta = \frac{\lambda_4}{a_{11}\lambda_4 - b_{11}\lambda_3} \qquad \psi = \frac{-\lambda_3}{a_{11}\lambda_4 - b_{11}\lambda_3}.$$

We substitute β and ψ in $\overline{h}^2(X_i, X_j)$ and we prove that $\overline{h}^2(X_i, X_j) = \delta_{ij}$. In fact,

$$\begin{aligned} \overline{h}^2(X_i, X_j) &= \beta a_{ij} + \psi b_{ij} = \left(\frac{\lambda_4}{a_{11}\lambda_4 - b_{11}\lambda_3}\right)a_{ij} + \left(\frac{-\lambda_3}{a_{11}\lambda_4 - b_{11}\lambda_3}\right)b_{ij} \\ &= \frac{K(\lambda_4 a_{ij} - \lambda_3 b_{ij})}{K(\lambda_4 a_{11} - \lambda_3 b_{11})} = \frac{G_\mathfrak{u}(X_i, X_j)}{G_\mathfrak{u}(X_1, X_1)} = \frac{G_\mathfrak{u}(X_i, X_j)/(det_\mathfrak{u} G_\mathfrak{u})^{\frac{1}{n+2}}}{G_\mathfrak{u}(X_1, X_1)/(det_\mathfrak{u} G_\mathfrak{u})^{\frac{1}{n+2}}} \\ &= \frac{g_\xi(X_i, X_j)}{g_\xi(X_1, X_1)} = g_\xi(X_i, X_j). \end{aligned}$$

From the equation $0 = \overline{h}^1(X_1, X_1) = \alpha a_{11} + \varphi b_{11}$ we can write $\alpha = R b_{11}$ and $\varphi = -R a_{11}$, therefore

$$\begin{aligned}[X_1, X_2, \nu_1, \nu_2] &= [X_1, X_2, \xi_1, \xi_2](\alpha\psi - \beta\varphi) = (\alpha\psi - \beta\varphi)\\ &= ((Rb_{11})\psi - \beta(-Ra_{11})) = R,\end{aligned}$$

we conclude $R = K$, $\alpha = Kb_{11}$ and $\varphi = -Ka_{11}$.

It only remains to prove that $[\xi] = -[\xi_1]$. First we note that $G_{\mathfrak{u}}(X_i, X_j) = 0$ $(i \neq j)$ because $\{X_1, \ldots, X_n\}$ is a orthonormal tangent frame relative to g_ξ. Moreover,

$$\frac{\det_{\mathfrak{u}} G_{\mathfrak{u}}}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{n/(n+2)}} = \frac{G_{\mathfrak{u}}(X_1,X_1)}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/(n+2)}} \cdots \frac{G_{\mathfrak{u}}(X_n,X_n)}{(\det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/(n+2)}} = 1.$$

Therefore $\det_{\mathfrak{u}} G_{\mathfrak{u}} = 1$, it follows that

$$\lambda_3\alpha + \lambda_4\varphi = \lambda_3 K b_{11} - \lambda_4 K a_{11} = K(\lambda_3 b_{11} - \lambda_4 a_{11}) = -G_{\mathfrak{u}}(X_1, X_1) = -1.$$

Finally, we compute $[\xi]$:

$$\begin{aligned}[\xi] &= \lambda_3\nu_1 + \lambda_4\nu_2 + T_pM\\ &= \lambda_3(\alpha\xi_1 + \beta\xi_2) + \lambda_4(\varphi\xi_1 + \psi\xi_2) + T_pM\\ &= \underbrace{(\lambda_3\alpha + \lambda_4\varphi)}_{-1}\xi_1 + \underbrace{(\lambda_3\beta + \lambda_4\psi)}_{0}\xi_2 + T_pM.\end{aligned}$$

□

Lemma 6.4. *Let $M \subset \mathbb{R}^{n+2}$ be a locally strictly convex n-submanifold and ξ a metric field. Let $\mathfrak{u} = \{X_1, \ldots, X_n\}$ and $\mathfrak{v} = \{Y_1, \ldots, Y_n\}$ be two orthonormal frames and let σ be a transversal plane bundle. So we can write for $i = 1, \ldots, n$*

$$Y_i = \sum_{j=1}^{n} a_{ij} X_j \tag{6.2}$$

where $\mathbf{a} = (a_{ij})$ is an orthogonal matrix and $\det(\mathbf{a}) = \pm 1$ depending on $\mathfrak{u}$ and $\mathfrak{v}$ have the same orientation or not. If we denote by $\{\xi_1, \xi_2\}$ (resp. $\{\bar{\xi}_1, \bar{\xi}_2\}$) the frame of Theorem 6.3 corresponding to $\mathfrak{u}$ (resp. $\mathfrak{v}$), then

$$\begin{aligned}\xi_1 &= \bar{\xi}_1,\\ \xi_2 &= \lambda\bar{\xi}_1 + \bar{\xi}_2,\end{aligned}$$

for some function λ.

Proof. From Theorem 6.3, we have $[\xi_1] = -[\xi] = [\overline{\xi}_1]$. Since ξ_1 and $\overline{\xi}_1$ belong to the same transversal plane we conclude that $\xi_1 = \overline{\xi}_1$. We compute now the affine connection in the different references to compare the references. By using the frame $\{\overline{\xi}_1, \overline{\xi}_2\}$, it follows from Theorem 6.3 that

$$D_{Y_1}Y_1 = \nabla_{Y_1}Y_1 + \overline{\xi}_2,$$

and by using the reference $\{\xi_1, \xi_2\}$ we have,

$$D_{Y_1}Y_1 = \nabla_{Y_1}Y_1 + h^1(Y_1, Y_1)\xi_1 + h^2(Y_1, Y_1)\xi_2.$$

Hence, $\overline{\xi}_2 = h^1(Y_1, Y_1)\xi_1 + h^2(Y_1, Y_1)\xi_2$. Now how the frame $\mathfrak{u}$ is orthonormal with g_ξ and $h^2 = g_\xi$

$$h^2(Y_1, Y_1) = h^2(\sum_{j=1}^{n} a_{1j}X_j, \sum_{j=1}^{n} a_{1j}X_j) = \sum_{i}^{n} a_{1i}^2 = 1.$$

Therefore

$$\overline{\xi}_2 = \lambda\xi_1 + \xi_2.$$

□

Lemma 6.5. *Let $M \subset \mathbb{R}^{n+2}$ be a locally strictly convex n-submanifold. Let ξ be a metric field and $\mathfrak{u} = \{X_1, \dots, X_n\}$ a local orthonormal tangent frame. Let σ and $\overline{\sigma}$ be two transversal plane bundles. We denote by $\{\xi_1, \xi_2\}$ and $\{\overline{\xi}_1, \overline{\xi}_2\}$ the transversal frames obtained from Theorem 6.3 for σ and $\overline{\sigma}$, respectively. Then there are Z_1 and Z_2 tangent vector fields on M such that*

$$\overline{\xi}_1 = \xi_1 + Z_1,$$
$$\overline{\xi}_2 = \xi_2 + Z_2.$$

Proof. We suppose that $\xi_1 = \phi\overline{\xi}_1 + \psi\overline{\xi}_2 + Z_1$ and $\xi_2 = \rho\overline{\xi}_1 + \beta\overline{\xi}_2 + Z_2$. Since $[\xi_1] = [\overline{\xi}_1]$ we have $\psi = 0$ and $\phi = 1$. By Theorem 6.3 $[X_1, \dots, X_n, \xi_1, \xi_2] = 1$, which implies $\phi\beta - \psi\rho = 1$ and it follows that $\beta = 1$. We denote by $\overline{h}^1$ and $\overline{h}^2$ the affine fundamental forms of the frame $\{\overline{\xi}_1, \overline{\xi}_2\}$. We note that

$$0 = \overline{h}^1(X_1, X_1) = \phi h^1(X_1, X_1) + \rho h^2(X_1, X_1) = \rho.$$

□

6.2 The equiaffine transversal plane bundles

Let $M \subset \mathbb{R}^{n+2}$ be a locally strictly convex n-submanifold and ξ a metric field and $g = g_\xi$ the metric of the transversal field ξ.

We denote by ω_g the metric volume form for the metric $g = g_\xi$

$$\omega_g(Z_1, Z_2, \dots, Z_n) = \sqrt{|\det(g(Z_i, Z_j))|},$$

where $\{Z_1, \dots, Z_n\}$ is any positively oriented basis of T_pM.

Definition 6.6. We say a transversal plane bundle σ is *equiaffine* if the connection $\nabla = \nabla(\sigma)$ induced by σ satisfies $\nabla\omega_g = 0$.

If $\mathfrak{u} = \{X_1, \dots, X_n\}$ is a local orthonormal tangent frame and $\{\xi_1, \xi_2\}$ is the transversal frame given by Theorem 6.3, then $\omega_g = \theta$, where θ is the volume form induced by the determinant:

$$\theta(Z_1, \dots, Z_n) = [Z_1, \dots, Z_n, \xi_1, \xi_2], \ \forall Z_1, \dots, Z_n \in T_pM.$$

This is because $\omega_g(X_1, \dots, X_n) = \theta(X_1, \dots, X_n)$.

For locally strictly convex surfaces $M \subset \mathbb{R}^4$, by using θ instead of ω_g, it is easy to see that σ is equiaffine if and only if $B_1 = B_2 = 0$, see Equations (2.22) and (2.23).

Lemma 6.7. *Let $M \subset \mathbb{R}^{n+2}$ be a locally strictly convex n-manifold and ξ be a metric field. Let σ be a transversal plane bundle and $\mathfrak{u} = \{X_1, \dots, X_n\}$ a local orthonormal tangent frame. Then the plane bundle σ is equiaffine if and only if $B_j = 0$ for all $j = 1, \dots, n$ where*

$$B_j = (\nabla g)(X_j, X_1, X_1) + (\nabla g)(X_j, X_2, X_2) + \dots + (\nabla g)(X_j, X_n, X_n). \tag{6.3}$$

Proof. First, we claim that

$$B_j = -2\sum_{i=1}^{n} \Gamma_{ji}^{i}, \tag{6.4}$$

where

$$\nabla_{X_i} X_j = \sum_{k}^{n} \Gamma_{ij}^{k} X_k. \tag{6.5}$$

In fact, by definition of B_j and since the frame $\mathfrak{u}$ is orthonormal with g,

$$B_j = \sum_{i=1}^{n} (\nabla g)(X_j, X_i, X_i) = \sum_{i=1}^{n} (X_j(g(X_i, X_i)) - 2g(\nabla_{X_j} X_i, X_i))$$
$$= -2\sum_{i=1}^{n} g(\sum_{k=1}^{n} \Gamma_{ji}^{k} X_k, X_i) = -2\sum_{i=1}^{n} \Gamma_{ji}^{i}.$$

Now, let $\{Y_1, Y_2, \ldots, Y_n\}$ be an arbitrary frame on T_pM. For $k \in 1, \ldots, n$ we compute

$$\nabla\omega_g(X_k, Y_1, \ldots, Y_n) := (\nabla_{X_k}\omega_g)(Y_1, \ldots, Y_n) = (\nabla_{X_k}\theta)(Y_1, \ldots, Y_n).$$

In fact, we write $Y_l = \sum_{r=1}^{n} a_{lr} X_r$, where $\det(\mathfrak{a}) \neq 0$ with $\mathfrak{a} = (a_{ij})$ and hence

$$\nabla\omega_g(X_k, Y_1, \ldots, Y_n) = X_k(\omega_g(Y_1, \ldots, Y_n)) - \sum_{l=1}^{n} \omega_g(Y_1, \ldots, \nabla_{X_k} Y_l, \ldots, Y_n)$$
$$= X_k(\theta(Y_1, \ldots, Y_n)) - \sum_{l=1}^{n} \theta(Y_1, \ldots, \nabla_{X_k}(\sum_{r=1}^{n} a_{lr} X_r), \ldots, Y_n).$$

Also we note that: $\nabla_{X_k}(\sum_{r=1}^{n} a_{lr} X_r) = \sum_{r=1}^{n} \nabla_{X_k}(a_{lr} X_r) = \sum_{r=1}^{n} X_k(a_{lr}) X_r + \sum_{r=1}^{n} a_{lr} \nabla_{X_k} X_r$

$$= \sum_{t=1}^{n} X_k(a_{lt}) X_t + \sum_{r=1}^{n} a_{lr} \sum_{t=1}^{n} \Gamma_{kr}^{t} X_t = \sum_{t=1}^{n} (X_k(a_{lt}) + \sum_{r=1}^{n} a_{lr} \Gamma_{kr}^{t}) X_t.$$

We denote by $\widetilde{a}_{lt} = X_k(a_{lt}) + \sum_{r=1}^{n} a_{lr} \Gamma_{kr}^{t}$ and hence

$$\nabla\omega_g(X_k, Y_1, \ldots, Y_n) = X_k(\theta(Y_1, \ldots, Y_n)) - \sum_{l=1}^{n} \theta(Y_1, \ldots, \sum_{t=1}^{n} \widetilde{a}_{lt} X_t, \ldots, Y_n). \tag{6.6}$$

By definition of θ and since the determinant is multilinear

$$\theta(Y_1, \ldots, \textstyle\sum_{t=1}^{n} \widetilde{a}_{lt} X_t, \ldots, Y_n) = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{a}_{l1} & \widetilde{a}_{l2} & \ldots & \widetilde{a}_{ln} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ X_k(a_{l1}) & X_k(a_{l2}) & \dots & X_k(a_{ln}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{r=1}^n a_{lr}\Gamma_{kr}^1 & \sum_{r=1}^n a_{lr}\Gamma_{kr}^2 & \dots & \sum_{r=1}^n a_{lr}\Gamma_{kr}^n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ X_k(a_{l1}) & X_k(a_{l2}) & \dots & X_k(a_{ln}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \sum_{r=1}^n a_{lr} \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \Gamma_{kr}^1 & \Gamma_{kr}^2 & \dots & \Gamma_{kr}^n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Hence $\nabla\omega_g(X_k, Y_1, \dots, Y_n)$

$$= -\left(\sum_{r=1}^n a_{1r} \begin{vmatrix} \Gamma_{kr}^1 & \Gamma_{kr}^2 & \dots & \Gamma_{kr}^n \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \dots + \sum_{r=1}^n a_{nr} \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \Gamma_{kr}^1 & \Gamma_{kr}^2 & \dots & \Gamma_{kr}^n \end{vmatrix} \right). \tag{6.7}$$

By developing Equation (6.7), we obtain

$$\nabla\omega_g(X_k, Y_1, \dots, Y_n) = -\sum_{i,j=1}^n p_{ij}\Gamma_{ki}^j,$$

where

$$p_{ij} = \begin{vmatrix} a_{11} & \dots & a_{1(j-1)} & a_{1i} & a_{1(j+1)} & \dots & a_{1n} \\ a_{21} & \dots & a_{2(j-1)} & a_{2i} & a_{2(j+1)} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{n(j-1)} & a_{ni} & a_{n(j+1)} & \dots & a_{nn} \end{vmatrix}.$$

Therefore for $k = 1, \dots, n$ we obtain

$$\nabla\omega_g(X_k, Y_1, \dots, Y_n) = \frac{1}{2}\det(\mathfrak{a})B_k.$$

□

Lemma 6.8. *Let $M \subset \mathbb{R}^{n+2}$ be a locally strictly convex n-manifold and ξ a metric field. Then there exists an equiaffine plane bundle σ defined on a neighborhood of $p \in M$.*

Proof. Let $\mathfrak{u} = \{X_1, \dots, X_n\}$ be an orthonormal tangent frame defined on some neighborhood U of p. Let $\overline{\sigma}$ be a transversal plane bundle defined also on U and $\{\overline{\xi}_1, \overline{\xi}_2\}$ the local basis of $\overline{\sigma}$ obtained by Theorem 6.3. Now we want to construct a new equiaffine plane bundle σ defined on U, with local basis $\{\xi_1, \xi_2\}$ obtained also by Theorem 6.3. By Lemma 6.5, we have

$$\xi_1 = \overline{\xi}_1 - Z_1, \qquad \xi_2 = \overline{\xi}_2 - Z_2,$$

where Z_1 and Z_2 are tangent vector fields. We denote the connection induced by σ (resp. $\overline{\sigma}$) by ∇ (resp. $\overline{\nabla}$). On the other hand, by a simple calculation we obtain

$$\overline{B}_k = B_k + 2\sum_{r=1}^{n} h^1(X_k, X_r)g(Z_1, X_r) + 2\sum_{r=1}^{n} h^2(X_k, X_r)g(Z_2, X_r).$$

Note that σ is equiaffine if and only if $B_1 = B_2 = 0$. By writing $Z_1 = \lambda_1 X_1 + \ldots + \lambda_n X_n$ and $Z_2 = \beta_1 X_1 + \ldots + \beta_n X_n$, this is equivalent to

$$\begin{aligned}
\overline{B}_1 &= 2h^1(X_1, X_2)\lambda_2 + \ldots + 2h^1(X_1, X_n)\lambda_n + 2\beta_1,\\
\overline{B}_2 &= 2h^1(X_2, X_1)\lambda_1 + 2h^1(X_2, X_2)\lambda_2 + \ldots + 2h^1(X_2, X_n)\lambda_n + 2\beta_2,\\
\vdots &= \vdots\\
\overline{B}_n &= 2h^1(X_n, X_1)\lambda_1 + 2h^1(X_n, X_2)\lambda_2 + \ldots + 2h^1(X_n, X_n)\lambda_n + 2\beta_n.
\end{aligned}$$

The lemma follows since the system above has a solution. For instance, set $\lambda_1 = \ldots = \lambda_n = 0$, $\beta_i = \frac{\overline{B}_i}{2}$, for $i = 1, \ldots, n$. □

6.3 Affine normal plane bundle

In analogy with Chapter 5, by using the metric of the transversal vector field we introduce an affine normal plane for locally strictly convex submanifolds of codimension 2.

Let $M \subset \mathbb{R}^{n+2}$ be a locally strictly convex n-submanifold and ξ a metric field. Let $\mathfrak{u} = \{X_1, \ldots, X_n\}$ be a local orthonormal tangent frame of $g = g_\xi$ and let σ be an arbitrary transversal plane bundle. By Theorem 6.3 there exists a unique local frame $\{\xi_1, \xi_2\}$ of σ such that $[X_1, \ldots, X_n, \xi_1, \xi_2] = 1$, $h^1(X_1, X_1) = 0$, $-\xi_1 \in [\xi]$ and $h^2(X_i, X_j) = \delta_{ij}$.

Definition 6.9. We define the affine normal plane bundle, denoted by $\mathbf{A}$, as the transversal plane bundle generated by $\{\overline{\xi}_1, \overline{\xi}_2\}$ where,

$$\begin{aligned}
\overline{\xi}_1 &= \xi_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2 - \ldots - \tau_1^2(X_n)X_n,\\
\overline{\xi}_2 &= \xi_2 - \tau_2^2(X_1)X_1 - \tau_2^2(X_2)X_2 - \ldots - \tau_2^2(X_n)X_n.
\end{aligned}$$

Note that: if $\overline{\xi}_1 = \xi_1 - Z_1$ and $\overline{\xi}_2 = \xi_2 - Z_2$ then

$$\overline{\tau}_1^1(X) = \tau_1^1(X) - h^1(X, Z_1), \qquad \overline{\tau}_2^1(X) = \tau_2^1(X) - h^1(X, Z_2),$$
$$\overline{\tau}_1^2(X) = \tau_1^2(X) - h^2(X, Z_1), \qquad \overline{\tau}_2^2(X) = \tau_2^2(X) - h^2(X, Z_2).$$

By a straightforward computation we obtain the torsions $\overline{\tau}_i^j$ on normal plane bundle

$$\overline{\tau}_1^1(X_j) = \tau_1^1(X_j) - \sum_{k=1}^n h^1(X_j, X_k)\tau_1^2(X_k),$$
$$\overline{\tau}_2^1(X_j) = \tau_2^1(X_j) - \sum_{k=1}^n h^1(X_j, X_k)\tau_2^2(X_k),$$
$$\overline{\tau}_1^2(X_j) = 0,$$
$$\overline{\tau}_2^2(X_j) = 0.$$

Proposition 6.10. *The affine normal plane bundle* **A** *does not depend on the transversal plane bundle* σ*.*

Proof. Let $\hat{\sigma}$ be an other transversal plane bundle. By Theorem 6.3 there exists a frame $\{\nu_1, \nu_2\}$ on $\hat{\sigma}$ and by Lemma 6.5 there are Z_1, Z_2 tangent vector fields on M such that $\nu_1 = \xi_1 - Z_1$ and $\nu_2 = \xi_2 - Z_2$. Now by definition we have $D_{X_i}\nu_1 = -S_{\nu_1}X_i + \hat{\tau}_1^1(X_i)\nu_1 + \hat{\tau}_1^2(X_i)\nu_2$ and $D_{X_i}\nu_2 = -S_{\nu_2}X_i + \hat{\tau}_2^1(X_i)\nu_1 + \hat{\tau}_2^2(X_i)\nu_2$. On the other hand,

$$\hat{\tau}_1^1(X_i) = \tau_1^1(X_i) - h^1(X_i, Z_1), \qquad \hat{\tau}_2^1(X_i) = \tau_2^1(X_i) - h^1(X_i, Z_2),$$
$$\hat{\tau}_1^2(X_i) = \tau_1^2(X_i) - h^2(X_i, Z_1), \qquad \hat{\tau}_2^2(X_i) = \tau_2^2(X_i) - h^2(X_i, Z_2).$$

Now we write $Z_1 = a_1X_1 + \ldots + a_nX_n$ and $Z_2 = b_1X_1 + \ldots + b_nX_n$ then

$$\overline{\nu}_1 = \nu_1 - \sum_{k=1}^n \hat{\tau}_1^2(X_k)X_k = \xi_1 - Z_1 - \sum_{k=1}^n (\tau_1^2(X_k) - h^2(X_k, Z_1))X_k$$
$$= \xi_1 - \sum_{k=1}^n \tau_1^2(X_k)X_k = \overline{\xi}_1.$$

Analogously, $\overline{\nu}_2 = \overline{\xi}_2$. □

Proposition 6.11. *The affine normal plane bundle* **A** *does not depend on the local orthonormal tangent frame* $\mathfrak{u}$*.*

Proof. Let $\mathfrak{v} = \{Y_1, \ldots, Y_n\}$ be another orthonormal tangent frame then

$$Y_i = \sum_{j=1}^{n} a_{ij} X_j \tag{6.8}$$

where $\mathbf{a} = (a_{ij})$ is an orthogonal matrix and $\det(\mathbf{a}) = \pm 1$ depending on $\mathfrak{u}$ and $\mathfrak{v}$ have the same orientation or not. By Lemma 6.4 we have: $\xi_1 = \hat{\xi}_1$ and $\xi_2 = \lambda\hat{\xi}_1 + \hat{\xi}_2$ for some function λ. By a computation, $\hat{\tau}_1^2(X) = \tau_1^2(X)$ and $\hat{\tau}_2^2(X) = \tau_2^2(X) - \lambda\tau_1^2(X)$. Now by linearity of τ_i^j follows

$$\begin{aligned}
\hat{\xi}_1 - \sum_{k=1}^{n} \hat{\tau}_1^2(Y_k) Y_k = \xi_1 - \sum_{k=1}^{n} \tau_1^2(Y_k) \sum_{j=1}^{n} a_{kj} X_j \\
\bar{\hat{\xi}}_1 = \xi_1 - \sum_{j=1}^{n} (\sum_{k=1}^{n} \tau_1^2(Y_k) a_{kj}) X_j \\
= \xi_1 - \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \tau_1^2 (\sum_{l=1}^{n} a_{kl} X_l) a_{kj} \right) X_j \\
= \xi_1 - \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \left(\sum_{l=1}^{n} \tau_1^2(X_l) a_{kl} a_{kj} \right) \right) X_j \\
= \xi_1 - \sum_{j=1}^{n} \tau_1^2(X_j) X_j = \bar{\xi}_1.
\end{aligned}$$

Analogously we obtain $\bar{\xi}_2 = -\lambda\bar{\hat{\xi}}_1 + \bar{\hat{\xi}}_2$. □

Propositions 6.10 and 6.11 allow us to define the affine distance and height functions for submanifolds of codimension 2. Next we define these functions and characterize the singularities of them.

6.4 Affine distance functions

Let $M \subset \mathbb{R}^{n+2}$ be a locally strictly convex n-submanifold and ξ a metric field. Let $\mathfrak{u} = \{X_1, \ldots, X_n\}$ be a local orthonormal tangent frame of $g = g_\xi$ and let σ be an arbitrary transversal plane bundle. By Theorem 6.3 there exists a unique local frame $\{\xi_1, \xi_2\}$ of σ_p.

Definition 6.12. We define the family of affine distance functions

$$\Delta : \mathbb{R}^{n+2} \times M \to \mathbb{R},$$

as follows: for $x \in \mathbb{R}^{n+2}$ and $p \in M$, $\Delta(x,p)$ is given by

$$p - x = z(x,p) + \Delta(x,p)\xi_2$$

where $z(x,p) \in \pi_p = T_pM \oplus \xi_p$.

Lemma 6.13. *The family of affine distance functions is independent of the tangent frame* $\mathfrak{u}$ *and independent of the transversal plane bundle* σ.

Proof. The proof is analogous to that of Lemma 5.9. □

Theorem 6.14. *The affine distance function* Δ_x *has a singularity if and only if* $x - X$ *belongs to the affine normal plane* $\mathbf{A}$.

Proof. Since $\mathfrak{u} = \{X_1, \dots, X_n\}$ is an orthonormal tangent frame and $\{\xi_1, \xi_2\}$ is the frame on σ obtained by Theorem 6.3 then we can write

$$x - X = r_1X_1 + \ldots + r_nX_n + \alpha_1\xi_1 + \alpha_2\xi_2.$$

We have

$$\Delta = [X_1, \dots, X_n, \xi_1, x - X].$$

By properties of derivation and determinant we have $X_k(\Delta)$:

$$\begin{aligned}
=&[D_{X_1}X_1, \dots, X_n, \xi_1, x - X] + \ldots + [X_1, \dots, D_{X_1}X_n, \xi_1, x - X] + [X_1, \dots, X_n, D_{X_1}\xi_1, x - X]\\
=&(\alpha_2\Gamma^1_{k1} - h^2(X_k, X_1)r_1) + \ldots + (\alpha_2\Gamma^n_{kn} - h^2(X_k, X_n)r_n) + (\alpha_2\tau^1_1(X_k) - \alpha_1\tau^2_1(X_k))\\
=&\alpha_2(\Gamma^1_{k1} + \ldots \Gamma^n_{kn} + \tau^1_1(X_k)) - \alpha_1\tau^2_1(X_k) - r_k.
\end{aligned}$$

Therefore $X_k(\Delta) = 0$ if and only if

$$r_k = \alpha_2(\Gamma^1_{k1} + \ldots \Gamma^n_{kn} + \tau^1_1(X_k)) - \alpha_1\tau^2_1(X_k).$$

By derivation of $[X_1, \dots, X_n, \xi_1, \xi_2] = 1$ we obtain $\Gamma^1_{k1} + \ldots \Gamma^n_{kn} + \tau^1_1(X_k) = -\tau^2_2(X_k)$. It follows that $X_k(\Delta) = 0$ if and only if $r_k = -\alpha_2\tau^2_2(X_k) - \alpha_1\tau^2_1(X_k)$ Therefore $X_1(\Delta) = \ldots = X_k(\Delta) = 0$ if and only if

$$x - X = \alpha_1\left(\xi_1 - \sum_{k=1}^{n}\tau^2_1(X_k)X_k\right) + \alpha_2\left(\xi_2 - \sum_{k=1}^{n}\tau^2_2(X_k)X_k\right).$$

□

6.5 Affine height functions

Analogously to the definition of affine distance functions we can to define the family of affine height functions.

Definition 6.15. The family of affine height functions

$$H : \mathbb{R}^{n+2} \times M \to \mathbb{R},$$

is defined as follows: for $x \in \mathbb{R}^{n+2}$ and $p \in M$, $H(x, p)$ is given by

$$x = z(x, p) + H(x, p)\xi_2$$

where $z(x, p) \in \pi_p = T_pM \oplus \xi_p$.

Lemma 6.16. *The family of affine height functions is independent of the tangent frame $\mathfrak{u}$ and independent of the transversal plane bundle σ.*

Proof. The proof is analogous to that of Lemma 5.12. □

We can write the affine distance function as $H(x, p) = [X_1, \ldots, X_n, \xi_1, x]$.

Theorem 6.17. *The affine height function H_x has a singularity if and only if x is in the affine normal plane* $\mathbf{A}$.

Proof. Analogous to the proof of Theorem 6.14. □

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