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Sets of values of fractional ideals of rings of algebroid curves

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**Conjunto de valores de ideais fracionários de anéis de
curvas algebroides**

Tese apresentada ao Instituto de Ciências
Matemáticas e de Computação – ICMC-USP,
como parte dos requisitos para obtenção do título
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Dedicated to the memory of my brother Walter.

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ABSTRACT

NIÑO DE GUZMÁN, M. E. **Sets of values of fractional ideals of rings of algebroid curves**. 2018. 64 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

The aim of this work is to study rings of algebroid Gorenstein rings. We explore more deeply the symmetry that exists among the sets of values of a fractional ideal and that of its dual and also to express the codimension of a fractional ideal in terms of the maximal points of the value set of the ideal. We apply the formulas we obtained to express the Tjurina number of a complete intersection curve in terms of invariants of its components and the maximal points of the set of values of the Kähler differentials on the curve.

Keywords: Fractional ideal, sets of values, symmetry, Gorenstein rings.

RESUMO

NIÑO DE GUZMÁN, M. E. **Conjunto de valores de ideais fracionários de anéis de curvas algebroides**. 2018. 64 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

O objetivo desse trabalho é o estudo dos anéis de curvas algebroides de Gorenstein. Expolramos mais aprofundadamente a simetria que existe entre os conjuntos de valores de um ideal fracionário e de seu dual e também expressar a codimensão de um ideal fracionário em função dos pontos maximais de seu conjunto de valores. Aplicamos as fórmulas obtidas para relacionar o número de Tjurina de uma curva de interseção completa com certos invariantes de suas componentes e dos pontos maximais do conjunto de valores das diferenciais de Kähler sobre a curva.

Palavras-chave: Ideais fracionários, conjunto de valores, simetria, anéis Gorenstein.

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PREFACE

Semigroup of values of rings of algebroid plane branches were studied by Zariski in [18] and they constitute a complete set of discrete invariants for their topological classification. These semigroups live in $\mathbb{N}$ and they are determined by a finite set of elements, since, as shown by Zariski, they have a conductor, that is, a number c such that $c - 1$ is not in the semigroup, but from c on all natural numbers belong to the semigroup. From the work of Apéry [1], it follows also that this semigroup is symmetric, that is in the interval $[0, c - 1]$ there as many elements in the semigroup as in its complement. Many years later, E. Kunz, in [14], showed that the symmetry of the semigroup of values for branches in a higher dimensional space is equivalent to the property of the branch to be Gorenstein.

For a plane curve singularity with several branches, O. Zariski in [18] characterized its topological type through the semigroups of its branches and their mutual intersection multiplicities. This characterization was shown by R. Waldi in [17] to be equivalent to the knowledge of the semigroup of values of the curve, this time in $\mathbb{N}^r$, where r is the number of branches of the curve. Although not finitely generated, this semigroup, for $r = 2$, was shown by A. Garcia in [9] to be determined in a combinatorial way by a finite set of points that he called *maximal points*. Garcia also showed that these maximal points of the semigroup of a plane curve have a certain symmetry. These results were generalized later for any value of r by F. Delgado in [7]. Delgado discovered several types of maximals among them the relative and absolute maximals and showed that the relative maximals determine the semigroup of values in a combinatorial way and extended for these maximals Garcia's symmetry. Some time later, Delgado, in [8], generalizing the work of Kunz, introduced a concept of symmetry for the semigroup of values in $\mathbb{N}^r$ and showed that this symmetry is equivalent to the property of the ring of the curve to be Gorenstein.

The computation of the codimension of an ideal, using elementary paths in its set of values was also considered by Garcia and Delgado, being generalized for fractional ideals by D'Anna [6] and by Barucci, D'Anna and Fröberg and in [2]. These authors observed that for sets of values of fractional ideals there exist maximal points that determine in the same combinatorial way such sets, as for semigroups of values. What were missing in the literature and we provided was a formula relating the codimension of an ideal with the maximals of its set of values. We also refined this formula for curves in three-space, proving a formula conjectured by M. Hernandez.

On the other hand, D. Pol, in the work [16], showed that the property of the singularity to be Gorenstein is equivalent to the fact that the set of values of the dual of any fractional ideal is determined by the set of values of the ideal. This motivated us to describe the property of a curve singularity to be Gorenstein by means of Apéry sets following the original steps by Apéry. For this, we defined the Apéry set for a set of values of a fractional ideal and characterized Gorenstein rings of algebroid curves by means of some symmetry property between the Apéry set of a fractional ideal and that of its dual. We also showed that for Gorenstein curve singularities one has a symmetry between the absolute maximal of the set of values of a fractional ideal and the relative maximal of the set of values of its dual ideal, generalizing a result by A. Campillo, F. Delgado and K. Kiyek in [5] for semigroups of values. Finally, we apply our formulas for codimensions of ideals to express the Tjurina number of a complete intersection curve in term of the Tjurina numbers of its components and the set of values of the Kähler differentials on the curve, generalizing a work by Bayer, Hefez and Hernandez in [4], where the case of plane curves was treated.

SYMMETRY FOR GORENSTEIN RINGS

In this chapter we will find some new relations between the set of values of a fractional ideal and that of its dual ideal for Gorenstein rings of algebroid curves over an algebraically closed ground field of arbitrary characteristic, generalizing the works of F. Delgado in [7] and [8] and by D. Pohl in [16].

1.1 Algebroid curves

This introductory section is based on the work of Gorenstein [11], Hironaka [13] and Garcia-Lax [10].

Throughout this work we will denote by $\mathbb{Z}$ the set of integers and by $\mathbb{N}$ the set of non-negative integers.

A reduced algebroid curve is given by $C = \text{Spec}(\mathcal{O})$ where $\mathcal{O}$ is a finitely generated local, reduced and complete of pure dimension one k -algebra and k is an algebraically closed field. The ring $\mathcal{O}$ will be called the ring of C .

Let $\wp_1, \dots, \wp_r$ be the minimal primes of $\mathcal{O}$. We will use the notation $I = \{1, \dots, r\}$. The *branches* of C are the integral schemes $C_i = \text{Spec}(\mathcal{O}_i)$, $i \in I$, where $\mathcal{O}_i = \mathcal{O}/\wp_i$. We will denote by $\pi_i: \mathcal{O} \rightarrow \mathcal{O}/\wp_i$ the canonical surjection, which corresponds to an inclusion $C_i \hookrightarrow C$.

Since $\mathcal{O}$ is reduced, we have that $\bigcap_{i=1}^r \wp_i = \sqrt{(0)} = (0)$, so the homomorphism

$$\begin{aligned} \pi: \mathcal{O} &\hookrightarrow \mathcal{O}_1 \times \dots \times \mathcal{O}_r \\ h &\mapsto (\pi_1(h), \dots, \pi_r(h)) \end{aligned}$$

is an injection.

More generally, if $J = \{i_1, \dots, i_s\}$ is any subset of I , we may consider $C_J = \text{Spec}(\mathcal{O}_J)$, where $\mathcal{O}_J = \mathcal{O}/\bigcap_{j=1}^s \wp_{i_j}$. The scheme C_J represents the curve $C_{i_1} \cup \dots \cup C_{i_s}$.

We will consider the natural surjection

$$\pi_J: \mathcal{O} \longrightarrow \mathcal{O}_J.$$

Notice that we have $\mathcal{O}_I = \mathcal{O}$ and $C_I = C$.

We will denote by $\mathcal{K}$ the total ring of fractions of $\mathcal{O}$ and by $\mathcal{K}_J$ the total ring of fractions of the ring $\mathcal{O}_J$, when $J \subset I$.

If $J = \{i\}$, then $\mathcal{O}_{\{i\}}$ is equal to the above defined domain $\mathcal{O}_i$ whose field of fractions will be denoted by $\mathcal{K}_i$. Let $\tilde{\mathcal{O}}$ be the integral closure of $\mathcal{O}$ in $\mathcal{K}$ and $\tilde{\mathcal{O}}_J$ be that of $\mathcal{O}_J$ in $\mathcal{K}_J$. By the splitting principle, one has that $\tilde{\mathcal{O}}_J \simeq \tilde{\mathcal{O}}_{i_1} \times \cdots \times \tilde{\mathcal{O}}_{i_n}$, which in turn is the integral closure of $\mathcal{O}_{i_1} \times \cdots \times \mathcal{O}_{i_n}$ in its total ring of fractions.

We have the following diagram:

$$\begin{array}{ccc} \mathcal{K}_J & \simeq & \mathcal{K}_{i_1} \times \cdots \times \mathcal{K}_{i_n} \\ \uparrow & & \uparrow \\ \tilde{\mathcal{O}}_J & \simeq & \tilde{\mathcal{O}}_{i_1} \times \cdots \times \tilde{\mathcal{O}}_{i_n} \\ \uparrow & & \uparrow \\ \mathcal{O}_J & \hookrightarrow & \mathcal{O}_{i_1} \times \cdots \times \mathcal{O}_{i_n} \end{array}$$

where the up arrows are inclusions.

Since each $\mathcal{O}_i$ is a complete DVR over the field k with valuation v_i , one has that $\tilde{\mathcal{O}}_i \simeq k[[t_i]]$, where $v_i(t_i) = 1$. It follows that $\mathcal{K}_i \simeq k((t_i))$ is a valued field with the extension of the valuation v_i . We also have that $\mathcal{K} \simeq k((t_1)) \times \cdots \times k((t_r))$.

We will denote by φ_i the map $\mathcal{O} \rightarrow \tilde{\mathcal{O}}_i$ and by φ the injective map $\mathcal{O} \rightarrow \tilde{\mathcal{O}}_1 \times \cdots \times \tilde{\mathcal{O}}_r$, $\varphi(h) = (\varphi_1(h), \dots, \varphi_r(h))$.

It is known that $\tilde{\mathcal{O}}_J$ is a finitely generated $\mathcal{O}_J$ -module, hence the set

$$\mathcal{C}_J = \{h \in \mathcal{K}_J; h\tilde{\mathcal{O}}_J \subset \mathcal{O}_J\}$$

has a regular element (ie. a nonzero divisor). It is easily verified that this set $\mathcal{C}_J$ is an ideal of both $\mathcal{O}_J$ and $\tilde{\mathcal{O}}_J$ and, in fact, it contains any simultaneous ideal of these two rings. This ideal is called the *conductor ideal* of $\tilde{\mathcal{O}}_J$ in $\mathcal{O}_J$. When $J = I$, the conductor $\mathcal{C}_I$ will be denoted by $\mathcal{C}$.

An important class of rings $\mathcal{O}$ are the *Gorenstein rings*, for which one has, by definition that, as k -vector spaces,

$$\dim_k \mathcal{O}/\mathcal{C} = \dim_k \tilde{\mathcal{O}}/\mathcal{O}.$$

The next result will characterize the ideals of $\tilde{\mathcal{O}}$.

Proposition 1.1. If $\mathcal{I}$ is an ideal of $\tilde{\mathcal{O}} \simeq k[[t_1]] \times \cdots \times k[[t_r]]$, then $\mathcal{I} = (t_1^{\gamma_1}, \dots, t_r^{\gamma_r})\tilde{\mathcal{O}}$, for some $\gamma_1, \dots, \gamma_r \in \mathbb{N}$.

Proof Let

$$\gamma_i = \min\{v_i(h); h \in \mathcal{I}\}.$$

We will prove that $(t_1^{\gamma_1}, \dots, t_r^{\gamma_r})$ generates $\mathcal{I}$. First of all let us observe that $(t_1^{\gamma_1}, \dots, t_r^{\gamma_r}) \in \mathcal{I}$. Indeed, let $h^i \in \mathcal{I}$ be such that $v_i(\pi_i(h^i)) = \gamma_i$, then if we take $\lambda_1, \dots, \lambda_r \in k$ general enough, then $h = \lambda_1 h^1 + \cdots + \lambda_r h^r \in \mathcal{I}$ and $h = (t_1^{\gamma_1} u_1, \dots, t_r^{\gamma_r} u_r)$, where each u_i is a unit in $k[[t_i]]$, hence $h = u(t_1^{\gamma_1}, \dots, t_r^{\gamma_r})$, where u is a unit in $\tilde{\mathcal{O}}$. This shows that $(t_1^{\gamma_1}, \dots, t_r^{\gamma_r}) \in \mathcal{I}$. Now, any element $g \in \mathcal{I}$ may be written as

$$g = (t_1^{\gamma_1 + \alpha_1} u_1, \dots, t_r^{\gamma_r + \alpha_r} u_r) = (t_1^{\alpha_1} u_1, \dots, t_r^{\alpha_r} u_r)(t_1^{\gamma_1}, \dots, t_r^{\gamma_r}).$$

■

It then follows that there is some $\gamma = (\gamma_1, \dots, \gamma_r) \in \mathbb{N}^r$ such that $\mathcal{C} = \underline{t}^\gamma \tilde{\mathcal{O}}$, where $\underline{t}^\gamma = (t_1^{\gamma_1}, \dots, t_r^{\gamma_r})$.

Example 1.2. (Apéry [1]) If $\text{Spec}(\mathcal{O})$ is a plane branch, then one has that $\mathcal{C} = t^{2\delta} k[[t]]$, where $\delta = \dim_k \frac{\tilde{\mathcal{O}}}{\mathcal{O}} = \dim_k \frac{\mathcal{O}}{\mathcal{C}}$. The number $c = 2\delta$ is the degree of the conductor $\mathcal{C}$.

Example 1.3. (Gorenstein [11]) If $\text{Spec}(\mathcal{O})$ is a reduced plane curve given by an equation $f = f_1 \cdots f_r$, then the conductor of $\tilde{\mathcal{O}}$ in $\mathcal{O}$ is a principal ideal generated by $(t_1^{I_1 + c_1}, \dots, t_r^{I_r + c_r})$, where the c_i denote the degrees of the conductors $\mathcal{C}_i$ of the branches determined by the f_i and $I_i = \sum_{j \neq i} I(f_j, f_i)$.

These two examples will be stated in greater generality later in Theorems 6 and 7.

Let us denote by $Z(A)$ the set of zero divisor of a ring A . One has that $Z(\mathcal{O}) = \cup_i \wp_i$ and $Z(A) = \{(a_1, \dots, a_r) \in A; a_i = 0, \text{ for some } i\}$, if $A = \tilde{\mathcal{O}}$ or $A = \mathcal{K}$. So, one has that $g \in Z(\mathcal{O})$ if and only if $\pi_i(g) = 0$, for some $i = 1, \dots, r$ (if and only if $\varphi_i(g) = 0$).

If J_1 and J_2 are two disjoint subset of I , then we define the intersection multiplicity of C_{J_1} and C_{J_2} as being

$$I(C_{J_1}, C_{J_2}) = \dim_k \frac{\mathcal{O}}{\bigcap_{j \in J_1} \wp_j + \bigcap_{j \in J_2} \wp_j}.$$

In the above situation we also have that $C_{J_1} \cup C_{J_2} = C_{J_1 \cup J_2}$.

1.2 Fractional ideals

An $\mathcal{O}$ -submodule $\mathcal{I}$ of $\mathcal{K}$ will be called a *fractional ideal* of $\mathcal{O}$ if it contains a regular element of $\mathcal{O}$ and there is a regular element d in $\mathcal{O}$ such that $d\mathcal{I} \subset \mathcal{O}$. Such an element d will be called a common denominator for $\mathcal{I}$.

Since $d\mathcal{I}$ is an ideal of $\mathcal{O}$, which is a noetherian ring, one has that $\mathcal{I} \subset \mathcal{K}$ is a nontrivial fractional ideal if and only if it contains a regular element of $\mathcal{O}$ and it is a finitely generated $\mathcal{O}$ -module.

Examples of fractional ideals of $\mathcal{O}$ are $\mathcal{O}$ itself, $\tilde{\mathcal{O}}$, the conductor $\mathcal{C}$ of $\tilde{\mathcal{O}}$ in $\mathcal{O}$, or any ideal of $\mathcal{O}$ or of $\tilde{\mathcal{O}}$ that contains a regular element. Also, if $\mathcal{I}$ is a fractional ideal of $\mathcal{O}$, then for all $\emptyset \neq \mathcal{J} \subset \mathcal{I}$ one has that $\pi_J(\mathcal{I})$ is a fractional ideal of $\mathcal{O}_J$.

Remark 1.4. Any element $h \in \mathcal{K}$ will be written as $h = \frac{h_1}{h_2}$, where $h_1, h_2 \in \mathcal{O}$, $h_2 \notin Z(\mathcal{O})$. If $h = \frac{h_1}{h_2} \in \mathcal{I}$, then $h_1 \in \mathcal{I}$. And, $h \in Z(\mathcal{K})$ if and only if $h_1 \in Z(\mathcal{O})$.

So, we may assume that the regular element contained in a fractional ideal is indeed in $\mathcal{O}$. In the set of fractional ideals of $\mathcal{O}$ there are defined the operations $+$, $\cdot$ and $:$. The first two operations are clear and the last one is defined, for $\mathcal{I} \neq \{0\}$, as follows:

$$(\mathcal{I} : \mathcal{J}) = \{x \in \mathcal{K}; x\mathcal{J} \subset \mathcal{I}\}.$$

This defines a fractional ideal. Indeed, it is an $\mathcal{O}$ -module which contains a regular element: dd' , where d is a common denominator for $\mathcal{I}$ and d' is a common denominator for $\mathcal{J}$. On the other hand, the element bd , where b is a regular element in $\mathcal{J} \cap \mathcal{O}$, is a common denominator for $(\mathcal{I} : \mathcal{J})$.

As an example of this construction we have that $\mathcal{C} = (\mathcal{O} : \tilde{\mathcal{O}})$.

Proposition 1.5. Given a fractional ideal $\mathcal{I}$ of $\mathcal{O}$ there exist $\beta \in \mathbb{N}^r$ and $\alpha \in \mathbb{Z}^r$ such that

$$\underline{t}^\beta \tilde{\mathcal{O}} \subset \mathcal{I} \subset \underline{t}^\alpha \tilde{\mathcal{O}}.$$

Proof From the definition of a fractional ideal we know that there exists a regular element d in $\mathcal{O}$ such that $d\mathcal{I} \subset \mathcal{O} \subset \tilde{\mathcal{O}}$. Therefore, as subsets of $\mathcal{K}$, one has that $\mathcal{I} \subset \frac{1}{d}\tilde{\mathcal{O}}$. Since we may write $\frac{1}{d} = \underline{t}^\alpha u$, where $\alpha \in \mathbb{Z}^r$ and u is a unit of $\tilde{\mathcal{O}}$, we get that

$$\mathcal{I} \subset \underline{t}^\alpha u \tilde{\mathcal{O}} = \underline{t}^\alpha \tilde{\mathcal{O}}.$$

On the other hand, we now that there is a regular element of $\mathcal{O}$ in $\mathcal{I}$. Write $a = \underline{t}^\epsilon v$, where $\epsilon \in \mathbb{N}^r$ and v is a unit of $\tilde{\mathcal{O}}$. Let $\underline{t}^\gamma$ be a generator of the conductor ideal of $\tilde{\mathcal{O}}$ in $\mathcal{O}$, then

$$\underline{t}^{\epsilon+\gamma} \tilde{\mathcal{O}} = \underline{t}^\epsilon v \underline{t}^\gamma \tilde{\mathcal{O}} \subset a\mathcal{O} \subset \mathcal{I}.$$

The remaining inclusion we are looking for follows taking $\beta = \epsilon + \gamma$. ■

We define the *dual module* of $\mathcal{I}$ as being the $\mathcal{O}$ -module $\mathcal{I}^\vee = \text{Hom}_{\mathcal{O}}(\mathcal{I}, \mathcal{O})$.

If $\varphi \in \mathcal{I}^\vee$ and $h = \frac{h_1}{h_2} \in \mathcal{I}$, then it is easy to verify that

$$\varphi\left(\frac{h_1}{h_2}\right) = \frac{\varphi(h_1)}{h_2}.$$

Our aim now is to realize $\mathcal{I}^\vee$ as a fractional ideal of $\mathcal{O}$. To do this, we begin with the following result.

Lemma 1.6. Let $\varphi \in \mathcal{I}^\vee$. If $g \in \mathcal{I} \setminus Z(\mathcal{K})$, then $\frac{\varphi(g)}{g}$ is independent of g .

Proof Let $g = \frac{g_1}{g_2}$ and $g' = \frac{g'_1}{g'_2}$ in $\mathcal{I} \setminus Z(\mathcal{K})$, then

$$\begin{aligned} \frac{\varphi(g)}{g} &= \frac{g_2 \varphi(g)}{g_1} = \frac{\varphi(g_2 g)}{g_1} = \frac{\varphi(g_1)}{g_1}, \\ \frac{\varphi(g')}{g'} &= \frac{g'_2 \varphi(g')}{g'_1} = \frac{\varphi(g'_2 g')}{g'_1} = \frac{\varphi(g'_1)}{g'_1}, \end{aligned}$$

which make sense since g_1 and g'_1 are not in $Z(\mathcal{O})$.

But, since $\varphi(g_1 g'_1) = \varphi(g'_1 g_1)$ and φ is an $\mathcal{O}$ -modules homomorphism, it follows that $g_1 \varphi(g'_1) = g'_1 \varphi(g_1)$, which implies our result. ■

For $g \in \mathcal{I} \setminus Z(\mathcal{K})$, the map

$$\begin{aligned} \Phi: \text{Hom}_{\mathcal{O}}(\mathcal{I}, \mathcal{O}) &\rightarrow \mathcal{K} \\ \varphi &\mapsto \frac{\varphi(g)}{g}, \end{aligned}$$

which is independent of g , is a homomorphism of $\mathcal{O}$ -modules.

Given a fractional ideal $\mathcal{I}$ of $\mathcal{O}$, we will use the notation

$$\mathcal{I}^* = (\mathcal{O} : \mathcal{I}) = \{x \in \mathcal{K}; x\mathcal{I} \subset \mathcal{O}\}.$$

Proposition 1.7. The map Φ induces an $\mathcal{O}$ -module isomorphism between $\mathcal{I}^\vee$ and $\mathcal{I}^*$.

Proof We first show that Φ is injective. Let $\varphi \in \text{Hom}_{\mathcal{O}}(\mathcal{I}, \mathcal{O})$ be such that $\frac{\varphi(g)}{g} = 0$, for some $g \in (\mathcal{I} \cap \mathcal{O}) \setminus Z(\mathcal{O})$. Let $h_1 \in \mathcal{I} \cap \mathcal{O}$, then

$$\varphi(h_1) = \frac{g \varphi(h_1)}{g} = h_1 \frac{\varphi(g)}{g} = 0,$$

which implies that $\varphi\left(\frac{h_1}{h_2}\right) = \frac{\varphi(h_1)}{h_2} = 0$, for $\frac{h_1}{h_2} \in \mathcal{I}$, so Φ is injective.

Now we show that $Im(\Phi) \subset \mathcal{I}^*$. Indeed, let $a \in Im(\Phi)$, then $a = \frac{\varphi(g)}{g}$ for some $\varphi \in \mathcal{I}^*$ and some $g \in (\mathcal{I} \cap \mathcal{O}) \setminus Z(\mathcal{O})$. Now, let $h = \frac{h_1}{h_2} \in \mathcal{I}$, then

$$a \frac{h_1}{h_2} = \frac{\varphi(g)}{g} \frac{h_1}{h_2} = \frac{\varphi(gh_1)}{gh_2} = \frac{g}{gh_2} \varphi(h_1) = \frac{\varphi(h_1)}{h_2} = \varphi\left(\frac{h_1}{h_2}\right) \in \mathcal{O}.$$

Finally, let us show that $\mathcal{I}^* \subset Im(\Phi)$. Take a in $\mathcal{I}^*$ and define

$$\begin{aligned} \varphi_a: \mathcal{I} &\rightarrow \mathcal{O} \\ h &\mapsto ah. \end{aligned}$$

It is easy to verify that φ_a is a homomorphism of $\mathcal{O}$ -modules, and

$$a = \frac{ag}{g} = \frac{\varphi_a(g)}{g} \in Im(\Phi).$$

■

We will always view $\mathcal{I}^\vee = Hom_{\mathcal{O}}(\mathcal{I}, \mathcal{O})$, as the fractional ideal $\mathcal{I}^*$, which will be called the dual ideal of the fractional ideal $\mathcal{I}$. It is easy to verify that taking duals is order reversing with respect to inclusions; that is,

$$\mathcal{I} \subset \mathcal{J} \implies \mathcal{I}^* \supset \mathcal{J}^*.$$

As examples of duals of fractional ideals one has that $\mathcal{O}^* = \mathcal{O}$ and $\widetilde{\mathcal{O}}^* = \mathcal{C}$.

One always has an inclusion $\mathcal{I} \subset (\mathcal{I}^*)^*$, which is almost tautological, on the other hand, it is not always true that $(\mathcal{I}^*)^* \subset \mathcal{I}$. But we have the following result.

Theorem 1.8. ([3, Theorem 6.3]) The following statemens are equivalent:

- i) $\mathcal{O}$ is a Gorenstein ring;
- ii) $(\mathcal{I}^*)^* \subset \mathcal{I}$ for all fractional ideal $\mathcal{I}$ of $\mathcal{O}$;
- iii) $\dim_k \frac{\mathcal{J}}{\mathcal{I}} = \dim_k \frac{\mathcal{I}^*}{\mathcal{J}^*}$, for all fractional ideals $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{I} \subset \mathcal{J}$.

If $J \subset I$ we define

$$\delta_J = \dim_k \widetilde{\mathcal{O}_J} / \mathcal{O}_J,$$

and if we put $J' = I \setminus J$, then one defines $I_J = I(C_J, C_{J'})$, we have the following results:

Theorem 1.9. ([10, Corollary 3.10]) Let $\mathcal{O}$ be a Gorenstein ring, $J \subset I$ and $T_1, \dots, T_t$ a partition of I . Then one has

$$\delta_J + \frac{1}{2} I_J = \sum_{j=1}^t \delta_{T_j} + \frac{1}{2} \sum_{j=1}^t I_{T_j}.$$

Theorem 1.10. ([10, Theorem 3.13]) Let $\mathcal{O}$ be a Gorenstein ring, then the conductor $\mathcal{C}$ of $\widetilde{\mathcal{O}}$ in $\mathcal{O}$ is given by

$$\mathcal{C} = (t_1^{I_1+2\delta_1}, \dots, t_r^{I_r+2\delta_r}) \widetilde{\mathcal{O}}.$$

1.3 Value sets of fractional ideals

For any n , we will consider on $\mathbb{Z}^n$ the partial order given by

$$(a_1, \dots, a_n) \leq (b_1, \dots, b_n) \iff a_i \leq b_i, \ i = 1, \dots, n.$$

We will write $(a_1, \dots, a_n) < (b_1, \dots, b_n)$ when $a_i < b_i$, for all $i = 1, \dots, n$.

Consider the value map

$$\begin{aligned} v: \mathcal{K} \setminus Z(\mathcal{K}) &\rightarrow \mathbb{Z}^r \\ h &\mapsto (v_1(\pi_1(h)), \dots, v_r(\pi_r(h))), \end{aligned}$$

where π_i here denotes the projection $\mathcal{K} \rightarrow \mathcal{K}_i$, which is the extension of the previously defined projection map $\pi_i: \mathcal{O} \rightarrow \mathcal{O}_i$. This map v can be extended to a map $\bar{v}: \mathcal{K} \rightarrow \bar{\mathbb{Z}}^r = (\mathbb{Z} \cup \{\infty\})^r$.

If $\mathcal{I}$ is a fractional ideal of $\mathcal{O}$, we define the *value set* of $\mathcal{I}$ as being

$$E = v(\mathcal{I} \setminus Z(\mathcal{K})) \subset \mathbb{Z}^r,$$

and the *extended value set* of $\mathcal{I}$ as being

$$\bar{E} = v(\mathcal{I}) \subset \bar{\mathbb{Z}}^r.$$

We will denote by E^* the value set of $\mathcal{I}^*$. The value set of $\mathcal{O}$ will be denoted by $S(\mathcal{O})$, or simply by S , and it is a subsemigroup of $\mathbb{N}^r$, called the semigroup of values of $\mathcal{O}$. The value sets E of fractional ideals are not necessarily closed under addition, but they are such that $S + E \subset E$. For this reason they are called monomodules over the semigroup S .

From the definition of the ideal $\mathcal{I}^*$, if E is the value set of $\mathcal{I}$ and E^* is the value set of $\mathcal{I}^*$, it follows immediately that

$$E + E^* \subset S(\mathcal{O}). \tag{1.1}$$

Let $I = \{1, \dots, r\}$ and $J \subset I$. If $J = \{i_1, \dots, i_n\}$, then we denote by π_J the projection $\mathcal{K} \rightarrow \mathcal{K}_{i_1} \times \dots \times \mathcal{K}_{i_n}$ and by pr_J the corresponding projection $\mathbb{Z}^r \rightarrow \mathbb{Z}^n$, $(\alpha_1, \dots, \alpha_r) \mapsto (\alpha_{i_1}, \dots, \alpha_{i_n})$.

Let us define

$$E_J = v(\pi_J(\mathcal{I}) \setminus Z(\mathcal{K}_J)) \quad \text{and} \quad \bar{E}_J = v(\pi_J(\mathcal{I})).$$

Caution If $j \in J = \{i_1, \dots, i_t, \dots, i_s\} \subset I$, with $i_t = j$, for $\beta = (\beta_{i_1}, \dots, \beta_{i_s}) \in E_J$, we will define

$$pr_j(\beta) = \beta_{i_t} = \beta_j,$$

instead of $pr_j(\beta) = \beta_{i_j}$, as it would be natural.

From Proposition 2 one has that, given a set of value E , there exist $\beta \in E$ and $\alpha \in \mathbb{Z}^r$ such that

$$\beta + \mathbb{N}^r \subset E \subset \alpha + \mathbb{N}^r.$$

Value sets of fractional ideals have the following fundamental properties:

Property 1.11 (A). If $\alpha = (\alpha_1, \dots, \alpha_r)$ and $\beta = (\beta_1, \dots, \beta_r)$ belong to E , then

$$\min(\alpha, \beta) = (\min(\alpha_1, \beta_1), \dots, \min(\alpha_r, \beta_r)) \in E.$$

Property 1.12 (B). If $\alpha = (\alpha_1, \dots, \alpha_r), \beta = (\beta_1, \dots, \beta_r)$ belong to E , $\alpha \neq \beta$ and $\alpha_i = \beta_i$ for some $i \in \{1, \dots, r\}$, then there exists $\gamma \in E$ such that $\gamma_i > \alpha_i = \beta_i$ and $\gamma_j \geq \min\{\alpha_j, \beta_j\}$ for each $j \neq i$, with equality holding if $\alpha_j \neq \beta_j$.

Since there is an $\alpha \in \mathbb{Z}^r$ such that $E \subset \alpha + \mathbb{N}^r$, then from **Property (A)** there exists a unique $m_E = \min(E)$, that is, for all $(\beta_1, \dots, \beta_r) \in E$, one has $\beta_i \geq m_{Ei}$, $i = 1, \dots, r$.

On the other side, one has the following

Lemma 1.13. If $\beta, \beta' \in E$ are such that $\beta + \mathbb{N}^r \subset E$ and $\beta' + \mathbb{N}^r \subset E$, then $\min(\beta, \beta') + \mathbb{N}^r \subset E$.

Proof Let $\theta = \min(\beta, \beta')$, we want to prove that $\theta + \mathbb{N}^r$ is a subset of E .

Let $a \in \mathbb{N}^r$. Since $\beta + a \in E$ and $\beta' + a \in E$, from **Property (A)**, we have that

$$\min(\beta, \beta') + a = \min(\beta + a, \beta' + a) \in E.$$

Therefore, $\min(\beta, \beta') + \mathbb{N}^r \subset E$. ■

The above lemma guarantees that there is an unique least element $\gamma \in E$ with the property that $\gamma + \mathbb{N}^r \subset E$. This element is what we call the conductor of E and will denote it by $c(E)$.

Observe that one always have

$$c(E_J) \leq c(pr_J(E)), \quad \forall J \subset I.$$

One has the following result:

Lemma 1.14. If $\mathcal{I}$ be a fractional ideal of $\mathcal{O}$ and $J \subset I$, then $pr_J(E) = E_J$.

Proof One has obviously that $pr_J(E) \subset E_J$. On the other hand, let $\alpha_J \in E_J$. Take $h \in \mathcal{I}$ such that $v_J(\pi_J(h)) = \alpha_J$. If $h \notin Z(\mathcal{K})$ we are done. Otherwise, choose any $h' \in \mathcal{I} \setminus Z(\mathcal{K})$

such that $pr_J(v(h')) > \alpha_J$, which exists since E has a conductor. Hence, $v_J(h + h') = \alpha_J$, proving the other inclusion. $\blacksquare$

Let $S = S(\mathcal{O})$. We will use the following notation:

$$\mathfrak{f}(S) = c(S) - (1, \dots, 1),$$

and call $\mathfrak{f}(S)$ the *Frobenius vector* of S or of $\mathcal{O}$.

When $\mathcal{O}$ is Gorenstein, from Theorem 1.10, one has that

$$c(S) = (I_1 + 2\delta_1, \dots, I_r + 2\delta_r),$$

where $I_j = \sum_{i \neq j} I(C_j, C_i)$ and δ_j is the singularity index of C_j .

1.3.1 Maximal points

We now introduce the important notion of a *fiber* of an element $\alpha \in E$ with respect to a subset $J \subset I = \{1, \dots, r\}$ that will play a central role in what follows.

Definition 1.15. Given $\alpha \in \mathbb{Z}^r$ and $\emptyset \neq J \subset I$, we define:

$$F_J(\alpha) = \{\beta \in \mathbb{Z}^r; pr_J(\beta) = pr_J(\alpha) \text{ and } pr_{I \setminus J}(\beta) > pr_{I \setminus J}(\alpha)\},$$

$$\overline{F}_J(\alpha) = \{\beta \in \mathbb{Z}^r; pr_J(\beta) = pr_J(\alpha), \text{ and } pr_{I \setminus J}(\beta) \geq pr_{I \setminus J}(\alpha)\},$$

$$F_J(E, \alpha) = F_J(\alpha) \cap E \quad \text{and} \quad \overline{F}_J(E, \alpha) = \overline{F}_J(\alpha) \cap E,$$

$$F(\alpha) = \bigcup_{i=1}^r F_{\{i\}}(\alpha), \quad F(E, \alpha) = F(\alpha) \cap E.$$

The last set above will be called the *fiber* of α .

The sets $F_{\{i\}}(\alpha)$ and $\overline{F}_{\{i\}}(\alpha)$ will be denoted simply by $F_i(\alpha)$ and $\overline{F}_i(\alpha)$. Notice that $F_I(\alpha) = \{\alpha\}$.

Definition 1.16. Let $\alpha \in E$. We will say that α is a **maximal** point of E if $F(E, \alpha) = \emptyset$.

This means that there is no element in E with one coordinate equal to the corresponding coordinate of α and the other ones bigger.

By convention, when $r = 1$, the maximal points of E are exactly the gaps of E .

From the fact that E has a minimum m_E and a conductor $\gamma = c(E)$, one has immediately that all maximal elements of E are in the limited region

$$\{(x_1, \dots, x_r) \in \mathbb{Z}^r; m_{Ei} \leq x_i < \gamma_i, \quad i = 1, \dots, r\}.$$

This implies that E has finitely many maximal points.

Definition 1.17. We will say that a maximal point α of E is an **absolute maximal** if $F_J(E, \alpha) = \emptyset$ for every $J \subset I$, $J \neq I$. If a maximal point α of E is such that $F_J(E, \alpha) \neq \emptyset$, for every $J \subset I$ with $\#J \geq 2$, then α will be said to be a **relative maximal** of E .

Notice that when $r = 1$, as we said before, the maximals coincide with the gaps, in the case where $r = 2$ the notions of maximal, relative maximal and absolute maximal coincide. For $r = 3$ we only may have relative maximals or absolute maximals.

We will denote by $M(E)$, $RM(E)$ and $AM(E)$ the sets of maximals, of relative maximals and absolute maximals of the set E , respectively.

The theorem below says that the set of relative maximal of E determines E in a combinatorial sense as follows:

Theorem 1.18 (generation). Let $\alpha \in \mathbb{Z}^r$ be such that $p_J(\alpha) \in E_J$ for all $J \subset I$ with $\#J = r - 1$. Then

$$\alpha \in E \iff \alpha \notin F(\mathbb{Z}^r, \beta), \forall \beta \in RM(E).$$

We will omit the proof since this result is a slight modification of [7, Theorem 1.5] with essentially the same proof.

The following result is the content of [8, Corollary 1.9] and of [8, Corollary 2.7].

Theorem 1.19. Let $\mathcal{O}$ be the ring of an algebroid curve and let $S = S(\mathcal{O})$. One has

- i) $F(S, \mathfrak{f}(S)) = \emptyset$;
- ii) If $\mathcal{O}$ is Gorenstein, then $\mathfrak{f}(S)$ is a relative maximal of S .

Part (i) of the above theorem implies the following result:

Corollary 1.20. Let $\mathcal{O}$ be the ring of an algebroid curve, and let $S = S(\mathcal{O})$. Then

$$\alpha \in S \implies F(S, \mathfrak{f}(S) - \alpha) = \emptyset.$$

Proof Indeed, if $\alpha \in S$ and $F(S, \mathfrak{f}(S) - \alpha) \neq \emptyset$, then if $\theta \in F(S, \mathfrak{f}(S) - \alpha)$, then $\theta + \alpha \in F(S, \mathfrak{f}(S))$, which is a contradiction. ■

The following lemma generalizes [7, Lemma 2.9] with the same proof.

Lemma 1.21. Let $\alpha \in E$ be such that $F_i(E, \alpha) = \emptyset$. Then, there exists $J \subset I$ with $\#J \geq 2$, containing i and a relative maximal element β of E_J , such that $\beta_i = \alpha_i$ and $\beta_j \leq \alpha_j$, for all $j \in J$.

The following two lemmas give us other characterizations of the relative and absolute maximal points.

Lemma 1.22. Given a value set $E \subset \mathbb{Z}^d$ and $\alpha \in \mathbb{Z}^d$ with the following properties:

- i) there is $i \in I$ such that $F_i(E, \alpha) = \emptyset$,
- ii) $F_{i,j}(E, \alpha) \neq \emptyset$ for all $j \in I \setminus \{i\}$.

Then α is a relative maximal of E .

Proof Follows the same steps of [7, Lemma 1.3]. ■

Lemma 1.23. Given a value set $E \subset \mathbb{Z}^d$ and $\alpha \in E$, assume that there exists an index $i \in I$ such that $F_J(E, \alpha) = \emptyset$ for every $J \subset I$ with $i \in J$. Then α is an absolute maximal of E .

Proof We have to prove that $F_K(E, \alpha) = \emptyset$ for all $K \subset I$ with $i \notin K$.

Assume, by reductio ad absurdum, that there exists some $K \subset I$ with $i \notin K$ such that $F_K(E, \alpha) \neq \emptyset$. Let β be an element in $F_K(E, \alpha)$, then $\beta_j = \alpha_j$, $\forall j \in K$ and $\beta_k > \alpha_k$, for all $k \notin K$. Applying the **Property (B)** for the index $k \in K$, there exists $\theta \in E$ such that $\theta_k > \beta_k = \alpha_k$, $\theta_l \geq \min\{\alpha_l, \beta_l\}$, $\forall l \in K \setminus \{k\}$ and $\theta_j = \alpha_j$ for all $j \notin K$. If we take the indices l for which we have equality, then $F_B(E, \alpha) \neq \emptyset$, where $B = (I \setminus K) \cup \{l \in K, \theta_l = \alpha_l\}$ is such that $i \in B$. This contradicts the hypothesis of our lemma. ■

1.3.2 Apéry Sets

Let $\mathcal{I}$ be a fractional ideal of $\mathcal{O}$ and let E be its set of values. For any $\alpha \in E \setminus \{0\}$, we define the *Apéry set* of E with respect to α to be the set

$$A_\alpha(E) = \{\beta \in E; \beta - \alpha \notin E\}.$$

We will denote $A_\alpha(S)$ by A_α .

Lemma 1.24. Let S be the set of values of an algebroid curve, let $\alpha \in S \setminus \{0\}$ and let A_α be its corresponding Apéry set. For $\beta \in \mathbb{Z}^r$ one has that $\beta \notin S$ if only if $\beta = a - \rho\alpha$, for some $a \in A_\alpha$ and some $\rho \in \mathbb{N} \setminus \{0\}$.

Proof

($\Rightarrow$) Since $S \subset \mathbb{N}^r$, we have that $\alpha > 0$. Suppose that $\beta \notin S$ and let ρ be the positive integer such that $a = \beta + \rho\alpha \in S$, but $\beta + (\rho - 1)\alpha \notin S$. This implies $a \in A_\alpha$, giving the result.

($\Leftarrow$) We have that $\beta = a - \rho\alpha$ for $a \in A_\alpha$ and $\rho > 0$. Suppose that $\beta \in E$. Since $\alpha \in S$ and $\rho > 0$, we have that

$$a - \alpha = \beta + (\rho - 1)\alpha \in S,$$

which contradicts the fact that $a \in A_\alpha$. ■

1.4 Symmetry

In what follows we will denote by $\mathcal{O}$ the ring of an algebroid curve and by $S = S(\mathcal{O})$ its semigroup of values. Recall that we defined $\mathfrak{f}(S)$ as being $c(\mathcal{O}) - (1, \dots, 1)$. A fractional ideal of $\mathcal{O}$ will be denoted by $\mathcal{I}$, its dual by $\mathcal{I}^*$, while E and E^* will denote the value sets of $\mathcal{I}$ and $\mathcal{I}^*$, respectively.

The property of the ring $\mathcal{O}$ of an algebroid curve to be Gorenstein is equivalent to the symmetry of the semigroup of values $S(\mathcal{O})$ of $\mathcal{O}$ as pointed out by E. Kunz in [14] in the irreducible case and generalized by F. Delgado de la Mata in [8] as follows:

Theorem 1.25. ([8, Theorem 2.8]) The ring $\mathcal{O}$ is Gorenstein if and only if

$$\forall \alpha \in \mathbb{N}^r, \quad \alpha \in S \iff F(S, \mathfrak{f}(S) - \alpha) = \emptyset. \quad (1.2)$$

A semigroup S satisfying the property (1.2) above, will be called a *symmetric semigroup*.

The above result was generalized by D. Pol in [16] as follows:

Theorem 1.26. ([16, Theorem 2.4]) The ring $\mathcal{O}$ is Gorenstein if and only if, for all fractional ideal $\mathcal{I}$ of $\mathcal{O}$, the following property is verified

$$\forall \beta \in \mathbb{Z}^r, \quad \beta \in E^* \iff F(E, \mathfrak{f}(S) - \beta) = \emptyset. \quad (1.3)$$

Property (1.3) says that for a Gorenstein ring $\mathcal{O}$, the values set E of any fractional ideal, determines the values set E^* of its dual $\mathcal{I}^*$, and conversely.

We will show in this section that the Gorenstein property of $\mathcal{O}$ may be detected in several other ways using the Apéry sets, generalizing the above two results.

Motivated by the classical theory of semigroups of values of irreducible curves, we have the following result:

Theorem 1.27. Let $S = S(\mathcal{O})$ be a semigroup of values in $\mathbb{N}^r$ and let $\alpha \in S \setminus \{0\}$. The following conditions are equivalent:

- i) $\mathcal{O}$ is Gorenstein;
- ii) S is symmetric;
- iii) $\forall a \in \mathbb{N}^r, a \in A_\alpha \implies \emptyset \neq F(S, \mathfrak{f}(S) + \alpha - a) \subseteq A_\alpha$.

Proof $i) \Leftrightarrow ii)$ This is the content of Theorem 1.25.

$ii) \Rightarrow iii)$ Suppose that S is symmetric. Let $a \in A_\alpha$, then $a - \alpha \notin S$ and since S is symmetric, we have $F(S, \mathfrak{f}(S) - (a - \alpha)) \neq \emptyset$. Now we will show that $F(S, \mathfrak{f}(S) - (a - \alpha)) \subseteq A_\alpha$.

Let γ be in $F(S, \mathfrak{f}(S) - (a - \alpha))$. If $\gamma - \alpha$ belonged to S , then we would have $\gamma - \alpha \in F(S, \mathfrak{f}(S) - a)$, then $F(S, \mathfrak{f}(S) - a) \neq \emptyset$. But, since $a \in A_\alpha \subset S$ and S is symmetric, it follows that $F(S, \mathfrak{f}(S) - a) = \emptyset$, which is a contradiction. So, $\gamma - \alpha \notin S$ and consequently $\gamma \in A_\alpha$.

$iii) \Rightarrow ii)$ Assuming (iii), let us prove that S is symmetric, i.e., $\gamma \in S \Leftrightarrow F(S, \mathfrak{f}(S) - \gamma) = \emptyset$.

We know from Corollary 1.20 that the implication $\gamma \in S \Rightarrow F(S, \mathfrak{f}(S) - \gamma) = \emptyset$ is always satisfied.

To prove the other direction, we show that if $\gamma \notin S$ then $F(S, \mathfrak{f}(S) - \gamma) \neq \emptyset$. So, let $\gamma \notin S$, then by Lemma 1.24 we have that $\gamma = a + \rho\alpha$, for some $a \in A_\alpha$ and some negative integer ρ . From our hypothesis we know that $F(S, \mathfrak{f}(S) + \alpha - a) \neq \emptyset$, hence take θ in this set.

Since $\theta \in S$ and $-(\rho + 1)\alpha \in S$, we have that

$$\theta - (\rho + 1)\alpha \in F(\mathbb{N}^\rho, \mathfrak{f}(S) - (a + \rho\alpha)) \cap S = F(S, \mathfrak{f}(S) - \gamma),$$

showing that $F(S, \mathfrak{f}(S) - \gamma) \neq \emptyset$ and therefore S is symmetric. ■

Observe that the condition $F(S, \mathfrak{f}(S) + \alpha - a) \neq \emptyset$ implies that $a - \alpha \notin S$, hence if $a \in S$, then $a \in A_\alpha$. It follows that condition (iii) in the above proposition may be readed as

$$\forall a \in \mathbb{N}^r, a \in A_\alpha \iff a \in S \text{ and } \emptyset \neq F(S, \mathfrak{f}(S) + \alpha - a) \subseteq A_\alpha.$$

Theorem 1.28. Let $\mathcal{O}$ be the ring of an algebroid curve. We denote by $\mathcal{I}$ a fractional ideal of $\mathcal{O}$ and by E its value set. For $\alpha \in E \cap E^*$, the following conditions are equivalent:

- i) $\mathcal{O}$ is Gorenstein;
- ii) $\forall \mathcal{I}, \forall \beta \in \mathbb{Z}^r, \beta \in E^* \iff F(E, \mathfrak{f}(S) - \beta) = \emptyset$.
- iii) $\forall \mathcal{I}, \forall a \in \mathbb{Z}^r, a \in A_\alpha(E) \implies \emptyset \neq F(E^*, \mathfrak{f}(S) + \alpha - a) \subseteq A_\alpha(E^*)$.

Proof $i) \iff ii)$ This is the content of Theorem 1.26.

$ii) \Rightarrow iii)$ Let us assume that (ii) is fulfilled. Let $a \in A_\alpha(E)$, then $a - \alpha \notin E$, so from (ii), we have $F(E^*, \mathfrak{f}(S) + \alpha - a) \neq \emptyset$ (recall that (ii) is equivalent to $\mathcal{O}$ is Gorenstein and, in this situation, $(E^*)^* = E$).

Now, suppose that $\gamma \in F(E^*, \mathfrak{f}(S) + \alpha - a)$, then $\gamma \in E^*$. It remains to show that $\gamma - \alpha \notin E^*$. Suppose that the opposite holds, that is, $\gamma - \alpha \in E^*$, then $\gamma - \alpha \in F(E^*, \mathfrak{f}(S) - a)$, hence $F(E^*, \mathfrak{f}(S) - a) \neq \emptyset$. But, since $a \in E$, from (ii) one gets that $F(E^*, \mathfrak{f}(S) - a) = \emptyset$, a contradiction.

$iii) \Rightarrow i)$ Let us assume that (iii) holds for all fractional ideal $\mathcal{I}$. In particular it holds for $\mathcal{I} = \mathcal{O}$. Now, Theorem 1.27 implies that $\mathcal{O}$ is Gorenstein. ■

COLENGTHS OF FRACTIONAL IDEALS

In this chapter we will give a formula relating the codimension of a fractional ideal with the maximal elements in its set of values.

Let $\mathcal{O}$ be the ring of an algebroid curve and $\mathcal{J} \subset \mathcal{I}$ are two fractional ideals with sets of values D and E , respectively. Since $\mathcal{J} \subset \mathcal{I}$, one has that $D \subset E$, hence $c(E) \leq c(D)$. Our aim in this section is to find a formula for the length $\ell_{\mathcal{O}}(\mathcal{I}/\mathcal{J})$ of $\mathcal{I}/\mathcal{J}$ as $\mathcal{O}$ -modules, called the colength of $\mathcal{J}$ in $\mathcal{I}$, in terms of the value sets D and E .

The motivation comes from the case in which $\mathcal{O}$ is a domain, when one has that

$$\ell_{\mathcal{O}}(\mathcal{I}/\mathcal{J}) = \#(E \setminus D).$$

For $\alpha \in \mathbb{Z}^r$ and $\mathcal{I}$ a fractional ideal of $\mathcal{O}$, with value set E , we define

$$\mathcal{I}(\alpha) = \{h \in \mathcal{I} \setminus Z(\mathcal{K}); v(h) \geq \alpha\}.$$

It is clear that if $m_E = \min E$, then $\mathcal{I}(m_E) = \mathcal{I}$.

One has the following result:

Proposition 2.1. ([2, Proposition 2.7]) Let $\mathcal{J} \subseteq \mathcal{I}$ be two fractional ideals of $\mathcal{O}$, with value sets D and E , respectively, then

$$\ell_{\mathcal{O}}\left(\frac{\mathcal{I}}{\mathcal{J}}\right) = \ell_{\mathcal{O}}\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) - \ell_{\mathcal{O}}\left(\frac{\mathcal{J}}{\mathcal{J}(\gamma)}\right),$$

for sufficiently large $\gamma \in \mathbb{N}^r$ (for instance, if $\gamma \geq c(D)$).

Let $e_i \in \mathbb{Z}^r$ denote the vector with zero entries except the i -th entry which is equal to 1, and recall the definition we gave in Subsection 1.3.1

$$\overline{F}_i(E, \alpha) = \{\beta \in E; \beta_i = \alpha_i \text{ and } \beta_j \geq \alpha_j, \forall j \neq i\},$$

then the following result will give us an effective way to calculate colengths of ideals.

Proposition 2.2. [6, Proposition 2.2] If $\alpha \in \mathbb{Z}^r$, then we have

$$\ell_{\mathcal{O}} \left(\frac{\mathcal{I}(\alpha)}{\mathcal{I}(\alpha + e_i)} \right) = \begin{cases} 1, & \text{if } \overline{F}_i(E, \alpha) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

So, to compute, for instance, $\ell_{\mathcal{O}} \left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)} \right)$, one takes a chain

$$m_E = \alpha^0 \leq \alpha^1 \leq \cdots \leq \alpha^m = \gamma,$$

and observe that

$$\ell_{\mathcal{O}} \left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)} \right) = \ell_{\mathcal{O}} \left(\frac{\mathcal{I}(\alpha^0)}{\mathcal{I}(\gamma)} \right) = \sum_{j=1}^m \ell_{\mathcal{O}} \left(\frac{\mathcal{I}(\alpha^{j-1})}{\mathcal{I}(\alpha^j)} \right).$$

This shows that the colength of an ideal $\mathcal{I}$ is independent from the chain one chooses. So, if we take a saturated chain, that is, a chain such that $\alpha^j - \alpha^{j-1} \in \{e_i; i = 1, 2, \dots, r\}$ for all $j = 1, \dots, m$, then the computation may be done using Proposition 2.2.

In what follows We will denote $\ell_{\mathcal{O}}$ simply by ℓ .

For the application we have in mind, we will have to compute

$$\ell \left(\frac{\mathcal{I}_1 \times \cdots \times \mathcal{I}_r}{\mathcal{I}} \right),$$

where $\mathcal{I}_i$, $i = 1, \dots, r$, is the projection of $\mathcal{I} \subset \mathcal{K}_1 \times \cdots \times \mathcal{K}_r$ into $\mathcal{K}_i$.

Notice that if E is the value set of $\mathcal{I}$ and $E_1, \dots, E_r$ those of $\mathcal{I}_1, \dots, \mathcal{I}_r$, then it is clear that $E_1 \times \cdots \times E_r$ is the value set of $\mathcal{I}_1 \times \cdots \times \mathcal{I}_r$, which obviously has no maximal points, since no element in it has an empty fiber.

By Proposition 2.1 we have that

$$\ell \left(\frac{\mathcal{I}_1 \times \cdots \times \mathcal{I}_r}{\mathcal{I}} \right) = \ell \left(\frac{\mathcal{I}_1 \times \cdots \times \mathcal{I}_r}{(\mathcal{I}_1 \times \cdots \times \mathcal{I}_r)(\gamma)} \right) - \ell \left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)} \right), \quad (2.1)$$

for some $\gamma \in E$ sufficiently large (for instance, if $\gamma \geq c(E)$).

2.1 Case r=2

This simplest case was studied by D'anna, Barucci and Fröberg in [2] and we reproduce it here because it gives a clue on how to proceed in general.

Let $\alpha^0 = m_E$ and consider the saturated chain in $\mathbb{Z}^2$

$$\alpha^0 \leq \cdots \leq \alpha^m = \gamma = (\gamma_1, \gamma_2)$$

such that

$$\alpha^0 = (\alpha_1^0, \alpha_2^0), \alpha^1 = (\alpha_1^0 + 1, \alpha_2^0), \dots, \alpha^s = (\gamma_1, \alpha_2^0),$$

$$\alpha^{s+1} = (\gamma_1, \alpha_2^0 + 1), \alpha^{s+2} = (\gamma_1, \alpha_2^0 + 2), \dots, \alpha^m = (\gamma_1, \gamma_2),$$

and consider the following sets

$$L_1 = \{\alpha^0, \alpha^1, \dots, \alpha^s\} \text{ and } L_2 = \{\alpha^s, \alpha^{s+1}, \dots, \alpha^m\}.$$

By Proposition 2.2, we have

$$\begin{aligned} \ell \left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)} \right) &= \#L_1 - \#\{\alpha \in L_1; \overline{F}_1(E, \alpha) = \emptyset\} + \\ &\quad \#L_2 - \#\{\alpha \in L_2; \overline{F}_2(E, \alpha) = \emptyset\}. \end{aligned}$$

Now, because of our choice of L_1 , we have that

$$\forall \alpha \in L_1, \overline{F}_1(E, \alpha) = \emptyset \iff pr_1(\alpha) \in M(E_1),$$

hence

$$\#\{\alpha \in L_1; \overline{F}_1(E, \alpha) = \emptyset\} = \#M(E_1).$$

Observe that not all $\alpha \in L_2$ with $\overline{F}_2(E, \alpha) = \emptyset$ are such that $pr_2(\alpha) \in M(E_2)$, hence

$$\#\{\alpha \in L_2; \overline{F}_2(E, \alpha) = \emptyset\} = \#M(E_2) - \xi,$$

where ξ is the number of α in L_2 with $pr_2(\alpha) \in E_2$ and $\overline{F}_2(E, \alpha) = \emptyset$. But such α are in one-to-one correspondence with the maximals of E , hence $\xi = \#M(E)$.

Putting all this together, we get

Proposition 2.3. We have that

$$\ell \left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)} \right) = (\gamma_1 - \alpha_1^0) - \#M(E_1) + (\gamma_2 - \alpha_2^0) - \#M(E_2) - \#M(E). \quad (2.2)$$

On the other hand, by Proposition 2.2, we have that

$$\begin{aligned} \ell \left(\frac{\mathcal{I}_1 \times \mathcal{I}_2}{(\mathcal{I}_1 \times \mathcal{I}_2)(\gamma)} \right) &= \#L_1 - \#\{\alpha \in L_1; \overline{F}_1(E_1 \times E_2, \alpha) = \emptyset\} + \\ &\quad \#L_2 - \#\{\alpha \in L_2; \overline{F}_2(E_1 \times E_2, \alpha) = \emptyset\}. \end{aligned}$$

This gives

$$\ell \left(\frac{\mathcal{I}_1 \times \mathcal{I}_2}{(\mathcal{I}_1 \times \mathcal{I}_2)(\gamma)} \right) = (\gamma_1 - \alpha_1^0) - \#M(E_1) + (\gamma_2 - \alpha_2^0) - \#M(E_2), \quad (2.3)$$

Thus, by (2.1), (2.3) and (2.2) we get

Proposition 2.4. One has

$$\ell \left(\frac{\mathcal{I}_1 \times \mathcal{I}_2}{\mathcal{I}} \right) = \#M(E). \quad (2.4)$$

2.2 Case $r \geq 3$

Let us assume that $\mathcal{I}$ is a fractional ideal of $\mathcal{O}$, where $\mathcal{O}$ has r minimal primes.

Let

$$m_E = \alpha^0 \leq \alpha^1 \leq \dots \leq \alpha^m = \gamma,$$

be the saturated chain in $\mathbb{Z}^r$, given by the union of the following paths (see Figure 1, for $r = 3$):

$$L_1: \alpha^0, \alpha^1 = \alpha^0 + e_1, \dots, \alpha^{s_1} = \alpha^0 + (\gamma_1 - \alpha_1^0)e_1 = (\gamma_1, \dots, \alpha_r^0),$$

...

$$L_r: \alpha^{s_{r-1}} = (\gamma_1, \dots, \gamma_{r-1}, \alpha_r^0), \alpha^{s_{r-1}+1} = \alpha^{s_{r-1}} + e_r, \dots, \alpha^m = \gamma.$$

Figure 1 – The saturated chain for $r = 3$

Let us define $I' = \{1, \dots, r-1\}$. We will need the following result:

Lemma 2.5. For any $\alpha \in L_1 \cup \dots \cup L_{r-1}$, and for $i \in I' = \{1, \dots, r-1\}$, one has

$$\overline{F}_i(E, \alpha) \neq \emptyset \iff \overline{F}_i(E_{I'}, pr_{I'}(\alpha)) \neq \emptyset.$$

Proof ($\Rightarrow$) This is obvious.

($\Leftarrow$) Suppose that

$$(\theta_1, \dots, \theta_{r-1}) \in \overline{F}_i(E_{I'}, pr_{I'}(\alpha)) \neq \emptyset.$$

Since by Lemma 1.14 one has that $pr_{I'}(E) = E_{I'}$, then there exists $\theta = (\theta_1, \dots, \theta_{r-1}, \theta_r) \in E$. Since $\alpha \in L_i$ for some $i = 1, \dots, r-1$, it follows that $\alpha_r = \alpha_r^0$. Then one cannot have $\theta_r < \alpha_r = \alpha_r^0$, because otherwise

$$(\alpha_1^0, \dots, \alpha_{r-1}^0, \theta_r) = \min(\alpha^0, \theta) \in E,$$

which is contradiction, since α^0 is the minimum of E . Hence $\theta_r \geq \alpha_r$, so $\theta \in \overline{F}_i(E, \alpha)$, and the result follows. $\blacksquare$

The above lemma allows us to write

$$\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) = \ell\left(\frac{\pi_{I'}(\mathcal{I})}{\pi_{I'}(\mathcal{I})(pr_{I'}(\gamma))}\right) + (\gamma_r - \alpha_r^0) - \#\{\alpha \in L_r; \overline{F}_r(E, \alpha) = \emptyset\}. \quad (2.5)$$

Hence to get an inductive formula for $\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right)$, we only have to compute

$$\#\{\alpha \in L_r; \overline{F}_r(E, \alpha) = \emptyset\},$$

and for this we will need the following lemma.

Lemma 2.6. Let $\alpha \in \mathbb{Z}^r$, then $\overline{F}_j(E, \alpha) = \emptyset$ if only if there exist some $J \subseteq I$ with $j \in J$ and a relative maximal β of E_J such that $pr_j(\beta) = \alpha_j$ and $pr_i(\beta) < \alpha_i$, for all $i \in J, i \neq j$.

Proof ($\Leftarrow$) (We prove more, since it is enough to assume β is any maximal) Let us assume that there exist $J \subseteq I$, with $j \in J$ and $\beta \in M(E_J)$, such that $\beta_j = pr_j(\beta) = \alpha_j$ and $\beta_i = pr_i(\beta) < \alpha_i$, for all $i \in J, i \neq j$.

Suppose by reductio ad absurdum that $\overline{F}_j(E, \alpha) \neq \emptyset$. Let $\theta \in \overline{F}_j(E, \alpha)$ that is $\theta_j = \alpha_j$ and $\theta_i \geq \alpha_i, \forall i \in J \setminus \{j\}$. Now since

$$pr_j(pr_J(\theta)) = \theta_j = \alpha_j = \beta_j \text{ and } pr_i(pr_J(\theta)) = \theta_j \geq \alpha_i > \beta_i, \forall i \in J, i \neq j,$$

then $pr_J(\theta) \in F_j(E_J, \beta)$, which contradicts the assumption that $\beta \in M(E_J)$.

($\Rightarrow$) Since $\overline{F}_j(E, \alpha) = \emptyset$ implies $F_j(E, \alpha) = \emptyset$, the proof follows the same lines as the proof of [8, Theorem 1.5]. $\blacksquare$

Going back to our main calculation, by Lemma 2.6, if $\alpha \in L_r$ is such that $\overline{F}_r(E, \alpha) = \emptyset$, then there exist a subset J of $I = \{1, \dots, r\}$, with $r \in J$, and $\beta \in RM(E_J)$, with $pr_r(\beta) = \alpha_r$ and $pr_i(\beta) < \alpha_i$ for $i \in J, i \neq r$.

Notice that for $\alpha \in L_r$ one has $\alpha_i = \gamma_i$ for $i \neq r$, so the condition $pr_i(\beta) < \alpha_i$ for $i \in J, i \neq r$ is satisfied, since $\beta \in M(E_J)$. So, we have a bijection between the set $\{\alpha \in L_r; \overline{F}_r(E, \alpha) = \emptyset\}$ and the set $\bigcup_{r \in J \subseteq I} pr_r(RM(E_J))$, where $pr_r(\beta)$ is the last coordinate of β , that is, β_r .

This gives us the formula

$$\#\{\alpha \in L_r; \overline{F}_r(E, \alpha) = \emptyset\} = \#\left(\bigcup_{r \in J \subseteq I} pr_r(RM(E_J))\right).$$

If $J = \{r\}$, we have $pr_r(M(E_r)) = M(E_r)$, and for all $J' \subseteq I$ with $r \in J'$ and $J' \neq \{r\}$, also $M(E_r) \cap pr_r(RM(E_{J'})) = \emptyset$, hence

$$\#\{\alpha \in L_r; \overline{F}_r(E, \alpha) = \emptyset\} = \#M(E_r) + \#\left(\bigcup_{\substack{r \in J \subseteq I \\ J \neq \{r\}}} pr_r(RM(E_J))\right). \quad (2.6)$$

Now, putting together Equations (2.5) and (2.6), we get

Theorem 2.7. For a fractional ideal $\mathcal{I}$ of a ring $\mathcal{O}$ with r minimal primes, with values set E , one has

$$\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) = \ell\left(\frac{\pi_{I'}(\mathcal{I})}{\pi_{I'}(\mathcal{I})(pr_{I'}(\gamma))}\right) + (\gamma_r - \alpha_r^0) - \#M(E_r) - \#\left(\bigcup_{\substack{r \in J \subseteq I \\ J \neq \{r\}}} pr_r(RM(E_J))\right). \quad (2.7)$$

We want now to calculate $\ell\left(\frac{\mathcal{I}_1 \times \cdots \times \mathcal{I}_r}{\mathcal{I}}\right)$. By Propositions 2.2 and 2.1 we have:

$$\ell\left(\frac{\mathcal{I}_1 \times \cdots \times \mathcal{I}_r}{\mathcal{I}}\right) = \ell\left(\frac{\mathcal{I}_1 \times \cdots \times \mathcal{I}_r}{(\mathcal{I}_1 \times \cdots \times \mathcal{I}_r)(\gamma)}\right) - \ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right), \quad (2.8)$$

for some $\gamma \geq c(E)$. Suppose that we know the formula for any fractional ideal in a ring with $r - 1$ minimal primes. Then, as we argued in the case $r = 2$, we have

$$\begin{aligned} \ell\left(\frac{\mathcal{I}_1 \times \cdots \times \mathcal{I}_r}{(\mathcal{I}_1 \times \cdots \times \mathcal{I}_r)(\gamma)}\right) &= \ell\left(\frac{\mathcal{I}_1 \times \cdots \times \mathcal{I}_{r-1}}{(\mathcal{I}_1 \times \cdots \times \mathcal{I}_{r-1})(pr_{I'}(\gamma))}\right) + \\ &(\gamma_r - \alpha_r^0) - \#M(E_r). \end{aligned} \quad (2.9)$$

Putting together Equations (2.7), (2.8) and (2.9), we have

Theorem 2.8. For a fractional ideal $\mathcal{I}$ of a ring $\mathcal{O}$ with r minimal primes, with values set E , one has

$$\ell\left(\frac{\mathcal{I}_1 \times \cdots \times \mathcal{I}_r}{\mathcal{I}}\right) = \ell\left(\frac{\mathcal{I}_1 \times \cdots \times \mathcal{I}_{r-1}}{\pi_{I'}(\mathcal{I})}\right) + \#\left(\bigcup_{\substack{r \in J \subseteq I \\ J \neq \{r\}}} pr_r(RM(E_J))\right). \quad (2.10)$$

2.3 Case r=3

In this subsection, we will give nicer formulas than Formulas (2.7) and (2.10), when $r = 3$. To simplify notations, for any $J \subset I = \{1, 2, 3\}$, we will denote by RM_J , AM_J and M_J the sets $RM(E_J)$, $AM(E_J)$ and $M(E_J)$, respectively. Notice also that if $\#J = 1, 2$, then $RM_J = AM_J = M_J$.

From Formulas (2.2) and (2.5), one has

$$\begin{aligned} \ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) &= (\gamma_1 - \alpha_1^0) - \#M_1 + (\gamma_2 - \alpha_2^0) - \#M_2 - M_{\{1,2\}} + \\ &\quad (\gamma_3 - \alpha_3^0) - \#\{\alpha \in L_3; \bar{F}_3(E, \alpha) = \emptyset\}. \end{aligned}$$

Now, from Lemma 2.6, the points $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in L_3$ such that $\bar{F}_3(E, \alpha) = \emptyset$ come from maximal points β of either E_3 , $E_{\{1,3\}}$, $E_{\{2,3\}}$, or E such that $pr_3(\beta) = \alpha_3$, so we have

$$\#\{\alpha \in L_3; \bar{F}_3(E, \alpha) = \emptyset\} = \#M_3 + \#RM_{\{1,3\}} + \#M_{\{2,3\}} + \#RM - \eta, \quad (2.11)$$

where η is some correcting term which will take into account the eventual multiple counting.

To compute η we will analyze in greater detail the geometry of maximal points.

If $\alpha, \beta \in M$ with $\alpha_3 = \beta_3$, then $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$. If $\alpha_1 < \beta_1$, then necessarily $\beta_2 < \alpha_2$.

We say that two relative (respectively, absolute) maximals α and β of E with $\alpha_3 = \beta_3$ and $\alpha_1 < \beta_1$ are *adjacent*, if there is no $(\theta_1, \theta_2, \alpha_3)$ in RM (respectively, in AM) with $\alpha_1 < \theta_1 < \beta_1$ and $\beta_2 < \theta_2 < \alpha_2$.

We will describe below the geometry of the maximal points of E

Lemma 2.9. If $\alpha \in AM$, then one and only one of the following three conditions is verified:

- i) there exist two adjacent relative maximals β and θ of E such that $pr_{\{1,3\}}(\beta) = pr_{\{1,3\}}(\alpha)$ and $pr_{\{2,3\}}(\theta) = pr_{\{2,3\}}(\alpha)$;
- ii) there exists $\beta \in RM$ such that $pr_{\{1,3\}}(\beta) = pr_{\{1,3\}}(\alpha)$ and $pr_{\{2,3\}}(\alpha) \in M_{\{2,3\}}$, or $pr_{\{2,3\}}(\beta) = pr_{\{2,3\}}(\alpha)$ and $pr_{\{1,3\}}(\alpha) \in M_{\{1,3\}}$;
- iii) $pr_{\{1,3\}}(\alpha) \in M_{\{1,3\}}$ and $pr_{\{2,3\}}(\alpha) \in M_{\{2,3\}}$.

Proof Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in AM$, then $F(E, \alpha) = \emptyset$. We consider the following sets:

$$R_1 = \{\beta \in \mathbb{Z}^3; \beta_3 = \alpha_3, \beta_1 > \alpha_1, \beta_2 < \alpha_2\}$$

and

$$R_2 = \{\theta \in \mathbb{Z}^3; \theta_3 = \alpha_3, \theta_1 < \alpha_1, \theta_2 > \alpha_2\}.$$

Then there are four possibilities:

$$R_1 \cap E \neq \emptyset \text{ and } R_2 \cap E \neq \emptyset, \quad R_1 \cap E \neq \emptyset \text{ and } R_2 \cap E = \emptyset.$$

$$R_1 \cap E = \emptyset \text{ and } R_2 \cap E \neq \emptyset, \quad R_1 \cap E = \emptyset \text{ and } R_2 \cap E = \emptyset.$$

Suppose $R_1 \cap E \neq \emptyset$ and $R_2 \cap E \neq \emptyset$. Choose $\beta \in R_1 \cap E$ and $\theta \in R_2 \cap E$, such that $\alpha_2 - \beta_2$ and $\alpha_1 - \theta_1$ are as small as possible. Then by **Property (A)**, we have $\min(\alpha, \beta), \min(\alpha, \theta) \in E$. Obviously $pr_{\{1,3\}}(\beta) = pr_{\{1,3\}}(\alpha)$ and $pr_{\{2,3\}}(\theta) = pr_{\{2,3\}}(\alpha)$. Moreover, according to Lemma 1.22, these are relative maximals because $F_3(E, \min(\alpha, \beta)) = F_3(E, \min(\alpha, \theta)) = \emptyset$, $F_{\{1,3\}}(E, \min(\alpha, \beta)) \neq \emptyset$, $F_{\{1,3\}}(E, \min(\alpha, \theta)) \neq \emptyset$, $F_{\{2,3\}}(E, \min(\alpha, \beta)) \neq \emptyset$ and $F_{\{2,3\}}(E, \min(\alpha, \theta)) \neq \emptyset$. It follows that $\min(\alpha, \beta)$ and $\min(\alpha, \theta)$ are adjacent relative maximals.

Suppose $R_1 \cap E \neq \emptyset$ and $R_2 \cap E = \emptyset$. Choose $\beta \in R_1 \cap E$ such that $\alpha_2 - \beta_2$ is as small as possible, then, as we argued above, we have that $\min(\alpha, \beta) \in \text{RM}$ and $pr_{\{1,3\}}(\beta) = pr_{\{1,3\}}(\alpha)$. Moreover, as $R_2 \cap E = \emptyset$, it follows that the projection $pr_{\{2,3\}}(\alpha) \in M_{\{2,3\}}$.

The case $R_1 \cap E = \emptyset$ and $R_2 \cap E \neq \emptyset$ is similar to the above one, giving us the second possibility in (ii).

Suppose $R_1 \cap E = \emptyset$ and $R_2 \cap E = \emptyset$. It is obvious that $pr_{\{1,3\}}(\alpha) \in M_{\{1,3\}}$ and $pr_{\{2,3\}}(\alpha) \in M_{\{2,3\}}$. ■

Given two points $\theta^1, \theta^2 \in \mathbb{Z}^3$ such that $pr_3(\theta^1) = pr_3(\theta^2)$, we will denote by $\mathcal{R}(\theta^1, \theta^2)$ the parallelogram determined by the coplanar points $\theta^1, \theta^2, \min(\theta^1, \theta^2)$ and $\max(\theta^1, \theta^2)$. We have the following result:

Corollary 2.10. Let $\theta^1, \theta^2 \in \text{AM}$, such that $pr_3(\theta^1) = pr_3(\theta^2)$. Then one has $\mathcal{R}(\theta^1, \theta^2) \cap \text{RM} \neq \emptyset$.

Proof Because $\theta^1, \theta^2 \in \text{AM}$, it follows immediately that (iii) of Lemma 2.9 cannot happen, therefore, the existence of the relative maximal is ensured by (i) or (ii). ■

Lemma 2.11. If β and β' are adjacent relative maximals, with $\beta_3 = \beta'_3$, then $\max(\beta, \beta')$ is an absolute maximal of E .

Proof We may suppose that $\beta_1 > \beta'_1$ and $\beta_2 < \beta'_2$. As β and β' are adjacent, we have that $F_{\{1,3\}}(E, \beta) \cap F_{\{2,3\}}(E, \beta') \neq \emptyset$, because otherwise if we take $\alpha^1 \in F_{\{1,3\}}(E, \beta)$, with α^1_2 the greatest possible and $\alpha^2 \in F_{\{2,3\}}(E, \beta')$, with α^2_1 the greatest possible. From Lemma 1.23 it follows that α^1 and α^2 are absolute maximals of E , then by Corollary 2.10 there exists a relative maximal in the region $\mathcal{R}(\alpha^1, \alpha^2)$, this contradicts the fact that β and β' are adjacent relative maximals.

Then, effectively, $F_{\{1,3\}}(E, \beta) \cap F_{\{2,3\}}(E, \beta') = \{\max(\beta, \beta')\}$, which is an absolute maximal. ■

We will use the following notation: $L'_3 = \{\alpha \in L_3; \overline{F}_3(E, \alpha) = \emptyset\}$. Recall that the elements in L_3 are of the form $(\gamma_1, \gamma_2, \alpha_3)$, with $\alpha_3^0 \leq \alpha_3 \leq \gamma_3$.

Lemma 2.12. Let $\alpha \in L'_3$ be such that $\alpha_3 \in (pr_3(M_{\{1,3\}}) \setminus pr_3(M_{\{2,3\}})) \cap pr_3(RM)$, or $\alpha_3 \in (pr_3(M_{\{2,3\}}) \setminus pr_3(M_{\{1,3\}})) \cap pr_3(RM)$, then there are the same number of relative as absolute maximals in E with third coordinate equal to α_3 .

Proof We assume that $\alpha_3 \in (pr_3(M_{\{1,3\}}) \setminus pr_3(M_{\{2,3\}})) \cap pr_3(RM)$, since the other case is analogous.

Since $\alpha_3 \in pr_3(RM)$, we may assume that there are $s (\geq 1)$ relative maximals $\beta^1, \dots, \beta^s$ in E with third coordinate equal to α_3 . We may suppose that $\beta_1^1 < \beta_1^2 < \dots < \beta_1^s$, so the β^i 's are successively adjacent relative maximals, hence, by lemma 2.11, we have that

$$\max(\beta^1, \beta^2), \dots, \max(\beta^{s-1}, \beta^s) \in AM.$$

This shows that there are at least $s-1$ absolute maximals in E with third coordinate α_3 .

Now as $pr_3(\alpha) \in pr_3(M_{\{1,3\}})$, then there is a $(\eta_1^1, \alpha_3) \in M_{\{1,3\}}$ with $\eta_1^1 \leq \alpha_1 (= \gamma_1)$, because $c(E_{\{1,3\}}) \leq pr_{\{1,3\}}(c(E)) = (\gamma_1, \gamma_3)$. Because of our hypothesis, the elements δ in the fiber $F_{\{1,3\}}(E, \beta^s)$ are such that $\beta_1^s < \delta_1 \leq \eta_1^1$. But we must have $\delta_1 = \eta_1^1$, because, otherwise, there would be a point $\eta^1 = (\eta_1^1, \eta_2^1, \alpha_3) \in pr_{\{1,3\}}^{-1}(\eta_1^1, \alpha_3)$, with $\eta_2^1 < \beta_2^s$, and a point $\eta^2 \in F_{\{2,3\}}(E, \beta^s)$ with $\eta_1^2 < \eta_1^1$ and $\eta_2^2 = \beta_2^s$. These η^1 and η^2 are absolute maximals, due to Lemma 1.23, then from Corollary 2.10, there would exist a relative maximal in the region $\mathcal{R}(\eta^1, \eta^2)$, which contradicts the fact that we have s relative maximals. This implies that $(\beta_1^s, \eta_2^1, \alpha_3)$ is an absolute maximal of E .

We have to show that there are no other absolute maximals. If such maximal existed, then one of the three conditions in Lemma 2.9 would be satisfied. Obviously conditions (i) and (iii) cannot be satisfied, but neither condition (ii) can be satisfied, because otherwise $\alpha_3 \in pr_3(M_{\{2,3\}})$, which is a contradiction. ■

Lemma 2.13. Let $\alpha \in L'_3$ such that $\alpha_3 \in (pr_3(M_{\{1,3\}}) \cap pr_3(M_{\{2,3\}})) \setminus pr_3(RM)$, then there exists one and only one absolute maximal of E with third coordinate equal to α_3 .

Proof As $\alpha_3 \in pr_3(M_{\{1,3\}}) \cap pr_3(M_{\{2,3\}})$, then there exist $(\beta_1^1, \alpha_3) \in M_{\{1,3\}}$ and $(\beta_2^2, \alpha_3) \in M_{\{2,3\}}$ such that $\beta_1^1 < \alpha_1 (= \gamma_1)$ and $\beta_2^2 < \alpha_2 (= \gamma_2)$.

So, there are also elements $\delta^1 = (\beta_1^1, \delta_2^1, \alpha_3) \in pr_{\{1,3\}}^{-1}(\beta_1^1, \alpha_3)$ and $\delta^2 = (\delta_1^2, \beta_2^2, \alpha_3) \in pr_{\{2,3\}}^{-1}(\beta_2^2, \alpha_3)$ in E such that $\delta_1^2 \leq \beta_1^1$ and $\delta_2^1 \leq \beta_2^2$, because otherwise (β_1^1, α_3) and/or (β_2^2, α_3) would not be maximals of $E_{\{1,3\}}$ and/or $E_{\{2,3\}}$.

From our hypothesis, at least one of the equalities $\delta_1^2 = \beta_1^1$ or $\delta_2^1 = \beta_2^2$ holds, because otherwise, there would exist absolute maximals $\theta^1 \in pr_{\{1,3\}}^{-1}(\beta_1^1, \alpha_3)$ and $\theta^2 \in pr_{\{2,3\}}^{-1}(\beta_2^2, \alpha_3)$, hence from Corollary 2.10 there would exist a relative maximal in $\mathcal{R}(\theta^1, \theta^2)$, hence with third projection α_3 , a contradiction. Thus we have $(\delta_1^1, \beta_2^2, \alpha_3) \in \text{AM}$, because $\overline{F}_3(E, (\delta_1, \beta_2, \alpha_3)) = \emptyset$.

■

Lemma 2.14. Let $\alpha \in L'_3$ such that $\alpha_3 \in pr_3(M_{\{1,3\}}) \cap pr_3(M_{\{2,3\}}) \cap pr_3(\text{RM})$. If there exist s relative maximals with third coordinate equal to α_3 , then there exist $s + 1$ absolute maximals with third coordinate equal to α_3 .

Proof Following the proof of Lemma 2.12, we have $s - 1$ absolute maximals obtained taking the maximum of each pair of adjacent relative maximals. The conditions $\alpha_3 \in pr_3(M_{\{1,3\}})$ and $\alpha_3 \in pr_3(M_{\{2,3\}})$ give us two extra absolute maximals, and the same argument used there, shows that there are no other.

■

Lemma 2.15. Let $\alpha \in L'_3$ such that $\alpha_3 \in pr_3(\text{RM}) \setminus (pr_3(M_{\{1,3\}}) \cup pr_3(M_{\{2,3\}}))$. If there exists s relative maximals with third coordinate equal to α_3 , then we have $s - 1$ absolute maximals with third coordinate equal to α_3 .

Proof The arguments used in the proofs of the last two lemmas give us the result.

■

Going back to Formula 2.11, we have to calculate η . From Lemma 2.6 we can ensure that $\alpha \in L'_3 = \{\alpha \in L_3; \overline{F}_3(E, \alpha) = \emptyset\}$, in each of the following five cases:

i) $\alpha_3 \in (pr_3(M_{\{1,3\}}) \setminus pr_3(M_{\{2,3\}})) \cap pr_3(\text{RM})$.

If there exist such α , then they are related to a unique element of $M_{\{1,3\}}$ and if there are s_1 relative maximals with third coordinate α_3 , then in our formula α was counted $s_1 + 1$ times. By Lemma 2.12 we know that there exist s_1 absolute maximals of E with third coordinate α_3 . So, we subtract s_1 from our counting to partially correct the formula.

ii) $\alpha_3 \in (pr_3(M_{\{2,3\}}) \setminus pr_3(M_{\{1,3\}})) \cap pr_3(\text{RM})$.

Analogously to (i), α is related to a unique element of $M_{\{2,3\}}$ and if there are s_2 relative maximals with third coordinate α_3 , then α was counted $s_2 + 1$ times in the formula. Again, by Lemma 2.12 we know that there are s_2 absolute maximals of E with third coordinate α_3 . So, we subtract s_2 from our counting to partially correct the formula.

iii) $\alpha_3 \in (pr_3(M_{\{1,3\}}) \cap pr_3(M_{\{2,3\}})) \setminus pr_3(\text{RM})$.

In this case, α is related to a unique elements in $M_{\{1,3\}}$ and in $M_{\{2,3\}}$, so in the formula we are counting α twice. By Lemma 2.13 there is a unique absolute maximal of E with third coordinate α_3 such that its projections $pr_{\{1,3\}}$ and $pr_{\{2,3\}}$ are in $M_{\{1,3\}}$ and $M_{\{1,3\}}$, respectively. So, we correct partially the formula by subtracting 1, which corresponds to this unique absolute maximal.

$$\text{iv) } \alpha_3 \in pr_3(M_{\{1,3\}}) \cap pr_3(M_{\{2,3\}}) \cap pr_3(RM).$$

In this case, α is related to a unique element of $M_{\{1,3\}}$, to a unique element of $M_{\{2,3\}}$ and, let us say, s_3 elements of RM , so in our counting, α was counted $s_3 + 2$ times. By Lemma 2.14 there exist $s_3 + 1$ absolute maximals of E with third coordinate α_3 . In this case, the correcting term is $s_3 + 1$, equal to the number of these absolute maximals.

$$\text{v) } \alpha_3 \in (pr_3(RM) \setminus (pr_3(M_{\{1,3\}}) \cup pr_3(M_{\{2,3\}}))).$$

In this case, α is related with, let us say, s_4 elements of RM with third coordinate equal to α_3 , so we are counting it s_4 times. By Lemma 2.15 there exist $s_4 - 1$ absolute maximals with third coordinate α_3 . This is exactly the correcting term we must apply to our formula.

Observe that the above cases exhaust all absolute maximals of E , implying the following results:

Theorem 2.16. Let $\mathcal{O}$ be the ring of an algebroid curve with three minimal primes and let $\mathcal{I}$ be a fractional ideal of $\mathcal{O}$, with values set E . Then one has

$$\begin{aligned} \ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) &= (\gamma_1 - \alpha_1^0) - \# M_1 + (\gamma_2 - \alpha_2^0) - \\ &\quad \# M_2 - M_{\{1,2\}} + (\gamma_3 - \alpha_3^0) - \# M_3 - \# M_{\{1,3\}} - \# M_{\{2,3\}} - \\ &\quad \# RM + \# AM. \end{aligned}$$

Theorem 2.17. Let $\mathcal{O}$ be the ring of an algebroid curve with three minimal primes and let $\mathcal{I}$ be a fractional ideal of $\mathcal{O}$, with values set E . Then one has

$$\ell\left(\frac{\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3}{\mathcal{I}}\right) = \# M_{\{1,2\}} + \# RM_{\{1,3\}} + \# RM_{\{2,3\}} + \# RM - \# AM.$$

SYMMETRY OF MAXIMALS

The central result in this chapter will establish a correspondence between absolute maximal of the set of values of a fractional ideal of a Gorenstein algebroid curve and the relative maximal of the set of values of its dual ideal, generalizing the symmetry relation among relative maximal and absolute maximal in the semigroup of values of a Gorenstein ring, discovered by F. Delgado in [8].

Proposition 3.1. Let $\mathcal{I}$ be a fractional ideal of $\mathcal{O}$ and $\alpha \in \mathbb{Z}^r$. Then

$$\ell\left(\frac{\mathcal{I}(\alpha)}{\mathcal{I}(\alpha + e_i)}\right) + \ell\left(\frac{\mathcal{I}^*(c(S) - \alpha - e_i)}{\mathcal{I}^*(c(S) - \alpha)}\right) \leq 1 \text{ for every } i \in I.$$

Proof We know by Proposition 2.2 that $\ell\left(\frac{\mathcal{I}(\alpha)}{\mathcal{I}(\alpha + e_i)}\right) \leq 1$ and $\ell\left(\frac{\mathcal{I}^*(c(S) - \alpha - e_i)}{\mathcal{I}^*(c(S) - \alpha)}\right) \leq 1$. Thus, it is sufficient to show that if $\ell\left(\frac{\mathcal{I}(\alpha)}{\mathcal{I}(\alpha + e_i)}\right) = 1$ then $\ell\left(\frac{\mathcal{I}^*(c(S) - \alpha - e_i)}{\mathcal{I}^*(c(S) - \alpha)}\right) = 0$ and vice versa.

The same Proposition 2.2 says that $\ell\left(\frac{\mathcal{I}(\alpha)}{\mathcal{I}(\alpha + e_i)}\right) = 1$ if and only if $\overline{F}_i(E, \alpha) \neq \emptyset$ and that $\ell\left(\frac{\mathcal{I}^*(c(S) - \alpha - e_i)}{\mathcal{I}^*(c(S) - \alpha)}\right) = 0$ if and only if $\overline{F}_i(E^*, c(S) - \alpha - e_i) = \emptyset$.

Let us assume by reductio ad absurdum that $\overline{F}_i(E, \alpha) \neq \emptyset$ and $\overline{F}_i(E^*, c(S) - \alpha - e_i) \neq \emptyset$. Take θ in the first of these sets and θ' in the second, then $\theta + \theta' \in S$; even more, we have that $\theta + \theta' \in F_i(S, \mathfrak{f}(S))$ because $\theta_i + \theta'_i = \mathfrak{f}_i(S)$ and $\theta_j + \theta'_j > \mathfrak{f}_j(S)$ for all $j \neq i$, hence $\theta + \theta' \in F_i(S, \mathfrak{f}(S))$, which is a contradiction, since $F_i(S, \mathfrak{f}(S)) = \emptyset$.

The proof of the other case is similar. ■

The following theorem generalizes [5, Theorem 3.6].

Theorem 3.2. Let $\mathcal{O}$ be a finitely generated local reduced and complete one-dimensional algebra. Then the following statements are equivalent:

i) For all fractional ideal $\mathcal{I}$ of $\mathcal{O}$, one has

$$\ell\left(\frac{\mathcal{I}(\alpha)}{\mathcal{I}(\alpha + e_i)}\right) + \ell\left(\frac{\mathcal{I}^*(c(S) - \alpha - e_i)}{\mathcal{I}^*(c(S) - \alpha)}\right) = 1, \text{ for every } \alpha \in \mathbb{Z}^r \text{ and } i \in I.$$

ii) $\mathcal{O}$ is a Gorenstein ring.

Proof $i) \Rightarrow ii)$ Assuming (i) and taking $\mathcal{I} = \mathcal{O}$, we will show that S is symmetric, proving that $\mathcal{O}$ is Gorenstein.

Recall that S symmetric means that

$$\alpha \in S \iff F(S, \mathfrak{f}(S) - \alpha) = \emptyset.$$

One direction of the above equivalence is known to be true from Corollary 1.20, namely

$$\alpha \in S \Rightarrow F(S, \mathfrak{f}(S) - \alpha) = \emptyset.$$

To prove the other direction, we show firstly that

$$F(S, \mathfrak{f}(S) - \alpha) = \emptyset \Rightarrow \overline{F}_i(S, c(S) - \alpha - e_i) = \emptyset, \text{ for all } i \in I. \quad (3.1)$$

Indeed, if $F(S, \mathfrak{f}(S) - \alpha) = \emptyset$ and $\overline{F}_i(S, c(S) - \alpha - e_i) \neq \emptyset$, for some $i \in I$, take $\theta \in \overline{F}_i(S, c(S) - \alpha - e_i)$, hence $\theta_i = c_i(S) - \alpha_i - 1 = \mathfrak{f}_i(S) - \alpha$ and $\theta_j \geq c_j(S) - \alpha_j > \mathfrak{f}_j(S) - \alpha_j$, for all $j \neq i$. This implies that $\theta \in F_i(S, \mathfrak{f}(S) - \alpha) \subset F(S, \mathfrak{f}(S) - \alpha) = \emptyset$, a contradiction. So, we proved (3.1).

From (3.1) and Proposition 2.2, we get that $\ell\left(\frac{\mathcal{O}(c(S) - \alpha - e_i)}{\mathcal{O}(c(S) - \alpha)}\right) = 0$, for all $i \in I$. This in view of (i) implies that $\ell\left(\frac{\mathcal{O}(\alpha)}{\mathcal{O}(\alpha + e_i)}\right) = 1$, for all $i \in I$, which in turn, in view of Proposition 2.2 implies that $\overline{F}_i(S, \alpha) \neq \emptyset$ for all $i \in I$, from which we conclude that $\alpha \in S$, as we wanted to show.

$ii) \Rightarrow i)$ We suppose now that $\mathcal{O}$ is Gorenstein and $\mathcal{I}$ is a fractional ideal of $\mathcal{O}$.

From Theorem 1.26 we know that

$$\alpha \notin E \Leftrightarrow F(E^*, \mathfrak{f}(S) - \alpha) \neq \emptyset.$$

To conclude the proof of this part of the theorem, it is enough to show that

$$\forall i \in I, \overline{F}_i(E, \alpha) = \emptyset \implies \overline{F}_i(E^*, \mathfrak{f}(S) - \alpha) \neq \emptyset.$$

Indeed, because the converse of the above implication is always true (cf. Proposition 3.1), item (i) will follow by Proposition 2.2.

Suppose now that $\overline{F}_i(E, \alpha) = \emptyset$. This implies that $\alpha \notin E$, so, in view of Theorem 1.26 we get that $F(E^*, \mathfrak{f}(S) - \alpha) \neq \emptyset$. Hence there exists an element $\theta \in F_i(E^*, \mathfrak{f}(S) - \alpha)$, for some $i \in I$. So, $\theta_i = \mathfrak{f}_i(S) - \alpha_i = c_i(S) - 1 - \alpha_i$ and $\theta_j > \mathfrak{f}_j(S) - \alpha_j = c_j(S) - 1 - \alpha_j$, for all $j \neq i$. This last inequality implies that $\theta_j \geq c_j(S) - \alpha_j$, for all $j \neq i$. Thus $\theta \in \overline{F}_i(E^*, c(S) - \alpha - e_i)$, hence this set is nonempty. ■

In [5, Section 5], the authors define certain integers related to the semigroup of values of an algebroid curve to give another characterization for the property of $\mathcal{O}$ being Gorenstein. Now, let us define these numbers for fractional ideals.

We denote $\ell\left(\frac{\mathcal{I}(\alpha)}{\mathcal{I}(\alpha+e_i)}\right)$ by $\ell_E(\alpha, e_i)$ and put $e_J = \sum_{i \in J} e_i$ for $\emptyset \neq J \subset I$, $e_\emptyset = 0$ and $e = e_I = (1, 1, \dots, 1)$. We will also use the notation $J^c = I \setminus J$.

Let us define the following sentences:

$$p(\alpha, 1) : \forall J \subset I, \text{ with } \#J = 1, \quad \ell_E(\alpha + e_{J^c}, e_i) = 0, \forall i \in J$$

$$\vdots$$

$$p(\alpha, r) : \forall J \subset I, \text{ with } \#J = r, \quad \ell_E(\alpha + e_{J^c}, e_i) = 0, \forall i \in J.$$

Notice that $p(\alpha, n) \Rightarrow p(\alpha, n-1)$.

Indeed, let $K \subset I$ with $\#K = n-1$, now K is included in some subset J of I with $\#J = n$. From the assertion $p(\alpha, n)$ we have that $\ell_E(\alpha + e_{J^c}, e_i) = 0$ for all $i \in J$. Suppose by reductio ad absurdum that $p(\alpha, n-1)$ is false, hence $\ell_E(\alpha + e_{K^c}, e_i) = 1$ for some $i \in K$, that is, $\overline{F}_i(E, \alpha + e_{K^c}) \neq \emptyset$. Take $\theta \in \overline{F}_i(E, \alpha + e_{K^c})$, then $\theta \in \overline{F}_i(E, \alpha + e_{J^c})$, because $\theta_i = \alpha_i$ and $\theta_k \geq \alpha_k$ for $k \in K$, $\theta_j \geq \alpha_j + 1$ for $j \in K^c$. Hence $\overline{F}_i(E, \alpha + e_{J^c}) \neq \emptyset$, which contradicts $\ell_E(\alpha + e_{J^c}, e_i) = 0$ for all $i \in J$.

Thus, we can define

$$p(E, \alpha) = \max\{n; \ell_E(\alpha + e_{J^c}, e_i) = 0, \forall J \subset I, \#J = n, \forall i \in J\}.$$

By the definition of $p(E, \alpha)$ it follows that

- i) $0 \leq p(E, \alpha) \leq r$, for every α ;
- ii) $p(E, \alpha) = \max\{n; F_A(E, \alpha) = \emptyset, \forall A \subset I, \#A \leq n\}$;
- iii) $p(E, \alpha) \geq 1 \iff F(E, \alpha) = \emptyset$.

We also define the sentences:

$$q(\alpha, r+1) : F(E, \alpha) \neq \emptyset$$

$$q(\alpha, r) : \forall J \subset I, \text{ with } \#J = r, \quad \ell_E(\alpha + e_{J^c}, e_i) = 1, \forall i \in J$$

$$\vdots$$

$$q(\alpha, 1) : \forall J \subset I, \text{ with } \#J = 1, \quad \ell_E(\alpha + e_{J^c}, e_i) = 1, \forall i \in J.$$

Notice that $q(\alpha, n-1) \Rightarrow q(\alpha, n)$.

Assume that $q(\alpha, n-1)$ is true. Let $K \subset I$, $\#K = n$ and let J be a subset of K with $\#J = n-1$, we have $\ell_E(\alpha + e_{J^c}, e_i) = 1$ for all $i \in J$, i.e., $\overline{F}_i(E, \alpha + e_{J^c}) \neq \emptyset$. Take $\theta \in \overline{F}_i(E, \alpha + e_{J^c})$ and notice that $\theta \in \overline{F}_i(E, \alpha + e_{K^c})$, because $\theta_i = \alpha_i$ and $\theta_j \geq \alpha_j$ for $j \in K$, $\theta_j \geq \alpha_j + 1$ for $j \in K^c$, hence $\ell_E(\alpha + e_{K^c}, e_i) = 1$, for all $i \in K$.

Thus, we can also define

$$q(E, \alpha) = \min\{n; \ell_E(\alpha + e_{J^c}, e_i) = 1, \forall J \subset I, \#J = n, \forall i \in J\}.$$

By definition $q(E, \alpha)$ has that following properties;

- i) $1 \leq q(E, \alpha) \leq r+1$, for every α ;
- ii) $q(E, \alpha) = \min\{n; F_A(E, \alpha) \neq \emptyset, \forall A \subset I, \#A \geq n\}$
- iii) $q(E, \alpha) \leq r \iff \overline{F}_i(E, \alpha) \neq \emptyset$, for $i = 1, \dots, r$;
- iv) $p(E, \alpha) < q(E, \alpha)$ for every α .

The following theorem generalizes [5, Theorem 5.3].

Theorem 3.3. With notations as above, for any fractional ideal $\mathcal{I}$ of $\mathcal{O}$ with value set E and for any $\alpha \in \mathbb{Z}^r$, we have

$$p(E, \alpha) + q(E^*, f(S) - \alpha) \geq r+1. \quad (3.2)$$

Moreover, $\mathcal{O}$ is a Gorenstein ring if and only if equality holds in (3.2) for every fractional ideal $\mathcal{I}$ of $\mathcal{O}$ and every $\alpha \in \mathbb{Z}^r$.

Proof Take any fractional ideal $\mathcal{I}$ and $\alpha \in \mathbb{Z}^r$, and define $n = r+1 - q(E^*, f(S) - \alpha)$. Take $J \subset I$ with $\#J \leq n$ and any $i \in J$. Put $K = J^c \cup \{i\}$. By Proposition 3.1, one has

$$\ell_E(\alpha + e_{J^c}, e_i) + \ell_{E^*}(c(S) - e - \alpha + e_{K^c}, e_i) \leq 1, \text{ for all } i \in J. \quad (3.3)$$

Since $\#K \geq q(E^*, f(S) - \alpha)$ and $i \notin K^c$, by definition of q one has $\ell_{E^*}(c(S) - e - \alpha - e_{K^c}, e_i) = 1$ and therefore $\ell_E(\alpha + e_{J^c}, e_i) = 0$. This shows that $n \leq p(E, \alpha)$, and therefore (3.2) is proven.

Now, if $\mathcal{O}$ is Gorenstein, to prove that (3.2) holds, we must show that $p(E, \alpha) = n$. Suppose by reductio ad absurdum that $p(E, \alpha) \geq n+1$. Then, we have that

$$\ell_E(\alpha + e_{J^c}, e_i) = 0, \forall i \in J, \forall J \subset I \text{ with } \#J = n+1.$$

Consider a set $K \subset I$ with $\#K = r - n$ and define the set $J = K^c \cup \{i\}$, with $i \in K$. Notice that J has $n + 1$ elements, then

$$\ell_E(\alpha + e_{J^c}, e_i) = 0, \quad \forall i \in J.$$

Since, that $\mathcal{O}$ is Gorenstein, we have

$$\ell_{E^*}(\mathfrak{f}(S) - \alpha + e_{K^c}, e_i) = 0, \quad \forall i \in K,$$

and since, i is any element of K , we have that

$$\ell_{E^*}(\mathfrak{f}(S) - \alpha + e_{K^c}, e_i) = 0, \quad \forall i \in K \text{ and } \forall K \subset I, \text{ with } \#K = r - n.$$

As $n = r + 1 - q(\mathfrak{f}(S) - \alpha)$, we have that $\#K < q(\mathfrak{f}(S) - \alpha)$, this contradicts the definition of $q(\mathfrak{f}(S) - \alpha)$. Therefore, it follows that $p(E, \alpha) = n$.

Now, assume that we have equality in (3.2), for every fractional ideal $\mathcal{I}$ of $\mathcal{O}$ and every $\alpha \in \mathbb{Z}^r$. Let $\alpha \in \mathbb{Z}^r$ and $i \in I$. If $\overline{F}_i(E, \alpha) = \emptyset$, then there exists β with $\beta_i = \alpha_i$ and $\beta_j < \alpha_j$ for every $j \neq i$ such that $F(E, \beta) = \emptyset$. From the condition $F(E, \beta) = \emptyset$ we get that $p(E, \beta) \geq 1$, hence from Equality (3.2) we get that $q(E^*, \mathfrak{f}(S) - \beta) \leq r$. Therefore, $\overline{F}_j(E^*, \mathfrak{f}(S) - \beta) \neq \emptyset$ for every $j \in I$. In particular, $\overline{F}_i(E^*, \mathfrak{f}(S) - \beta) \neq \emptyset$. On the other hand, since $\mathfrak{f}_i(S) - \beta_i = \mathfrak{f}_i(S) - \alpha_i$ and $\mathfrak{f}_j(S) - \beta_j > \mathfrak{f}_j(S) - \alpha_j$, it follows that $\mathfrak{f}(S) - \beta \geq \mathfrak{f}(S) - \alpha$, therefore $\overline{F}_i(E^*, \mathfrak{f}(S) - \alpha) \neq \emptyset$. So, we proved that

$$\ell_E(\alpha, e_i) + \ell_{E^*}(\mathfrak{f}(S) - \alpha - e_i, e_i) = 1, \text{ for every } \alpha \in \mathbb{Z}^r \text{ and all } i \in I.$$

So, by Theorem 3.2 it follows that $\mathcal{O}$ is Gorenstein. ■

Notice that in Theorem 3.3 we have the same result with the inequality

$$p(E^*, \beta) + q(E, \mathfrak{f}(S) - \beta) \geq r + 1.$$

Remark 3.4. In terms of $p(E, \alpha)$ and $q(E, \alpha)$, we may characterize the maximals of E as follows:

- i) α is maximal if and only if $p(E, \alpha) \geq 1$ and $q(E, \alpha) \leq r$;
- ii) α is a relative maximal if and only if $p(E, \alpha) = 1$ and $q(E, \alpha) = 2$; and
- iii) α is an absolute maximal if and only if $p(E, \alpha) = r - 1$ and $q(E, \alpha) = r$.

The following result will generalize [7, Theorem 2.10], and is a consequence of Theorem 3.3 and the previous remark.

Theorem 3.5 (Symmetry of maximals). Suppose that $\mathcal{O}$ is Gorenstein with value set S and $\mathcal{I}$ is a fractional ideal of $\mathcal{O}$ with value set E . Then for $\beta \in E^*$, one has that β is a maximal of E^* if and only if $\mathfrak{f}(S) - \beta \in E$. Moreover, if $\beta \in E^*$ and $\alpha \in E$ are such that $\alpha + \beta = \mathfrak{f}(S)$, then α is an absolute maximal of E if and only if β is a relative maximal of E^* .

Proof Suppose that β is a maximal of E^* , then

$$F(E^*, \mathfrak{f}(S) - (\mathfrak{f}(S) - \beta)) = F(E^*, \beta) = \emptyset,$$

hence from (1.3) it follows that $\mathfrak{f}(S) - \beta \in E$.

Conversely, suppose that $\mathfrak{f}(S) - \beta \in E$, then from (1.3) we have that $F(E^*, \beta) = \emptyset$ and, as by hypothesis, $\beta \in E^*$, it follows that β is a maximal of E^* . This proves the first part of the theorem.

We now prove the second part of the theorem. Suppose that β is a relative maximal of E^* and that $\alpha \in E$ is such that $\alpha + \beta = \mathfrak{f}(S)$. From the first part of the theorem that we have just proved, it follows that α is a maximal of E .

By reductio ad absurdum, suppose that α is not an absolute maximal of E , then there exists $J \subset I$ with $2 \leq \#J \leq r-1$ such that $F_J(E, \alpha) \neq \emptyset$. On the other hand, since β is a relative maximal of E^* , then $F_K(E^*, \beta) \neq \emptyset$, for all $K \subset I$ with $\#K \geq 2$. In particular, take $K = (I \setminus J) \cup \{i\}$, where $i \in J$, and choose $\theta \in F_K(E^*, \beta)$ and $\theta' \in F_J(E, \alpha)$. Notice that because of Equation (1.1) one has $\theta + \theta' \in S$. On the other hand, our choice of K implies that $\theta + \theta' \in F(\mathbb{Z}^r, \mathfrak{f}(S))$, hence $\theta + \theta' \in F(S, \mathfrak{f}(S))$, which is a contradiction, since from Corollary 1.20 we know that $F(S, \mathfrak{f}(S)) = \emptyset$. Thus α is an absolute maximal of E .

It remains to prove that if $\alpha + \beta = \mathfrak{f}(S)$ and α is an absolute maximal of E , then β is a relative maximal of E^* . In fact, suppose that α is an absolute maximal of E , which means that $p(E, \alpha) = r-1$ and $q(E, \alpha) = r$, then by Theorem 3.3 we have that $q(E^*, \beta) = 2$ and $p(E^*, \beta) = 1$, i.e., β is a relative maximal of E^* . ■

KÄHLER DIFFERENTIALS ON COMPLETE INTERSECTIONS

In this Chapter we generalize for complete intersections curve singularities the results by Bayer, Hefez and Hernandez in [4], obtained for plane curve singularities.

For the sake of simplicity, we will assume that $\text{char}(k) = 0$, and that $C = \text{Spec}(\mathcal{O})$ is embedded in $\text{Spec}(k[[X_1, \dots, X_n]])$, where n is the embedding dimension of C .

The above embedding of C is determined by a surjection $k[[X_1, \dots, X_n]] \rightarrow \mathcal{O}$, whose kernel we denote by $\mathcal{I}$. Therefore, one has

$$\mathcal{O} \simeq k[[X_1, \dots, X_n]]/\mathcal{I}.$$

We define the module of Kähler differentials as being the $\mathcal{O}$ -module

$$\Omega(\mathcal{O}) = \frac{\mathcal{O}^{\oplus n}}{\langle f_{X_1}e_1 + \dots + f_{X_n}e_n; f \in \mathcal{I} \rangle},$$

where $e_1, \dots, e_n$ is the canonical basis of $\mathcal{O}^{\oplus n}$ and $f_{X_i} = \frac{\partial f}{\partial X_i}$, $i = 1, \dots, n$. This module does not depend neither on $\mathcal{I}$ nor on the embedding of $\text{Spec}(\mathcal{O})$.

We will denote the image of e_i in $\Omega(\mathcal{O})$ by dx_i and for $g \in \mathcal{O}$ we consider the universal differentiation $dg = g_{X_1}dx_1 + \dots + g_{X_n}dx_n \in \Omega(\mathcal{O})$.

The jacobian ideal $\mathcal{J}$ of $\mathcal{O}$ is the smallest nonzero Fitting ideal of $\Omega(\mathcal{O})$, that is, the nonzero ideal generated by the minors of greatest possible size of the Jacobian matrix

$$Jac(f_1, \dots, f_m) = \begin{bmatrix} f_{1X_1} & \cdots & f_{1X_n} \\ \vdots & & \vdots \\ f_{mX_1} & \cdots & f_{mX_n} \end{bmatrix},$$

where $f_1, \dots, f_m$ are generators of the ideal $\mathcal{I}$.

The surjection $\pi_i: \mathcal{O} \rightarrow \mathcal{O}_i$ induces a surjection $\pi_i^*: \Omega(\mathcal{O}) \rightarrow \Omega(\mathcal{O}_i)$, defined by

$$\pi_j^* \left(\sum_{i=1}^n g_i dx_i \right) = \sum_{i=1}^n \pi_j(g_i) dx_i.$$

This in turn, induces an $\mathcal{O}$ -modules homomorphism

$$\begin{aligned} \pi^*: \Omega(\mathcal{O}) &\rightarrow \Omega(\mathcal{O}_1) \times \cdots \times \Omega(\mathcal{O}_r) \\ \omega &\mapsto (\pi_1^*(\omega), \dots, \pi_r^*(\omega)). \end{aligned}$$

We also have an $\mathcal{O}_i$ -modules homomorphism $\Omega(\mathcal{O}_i) \rightarrow \Omega(\tilde{\mathcal{O}}_i) \simeq k[[t_i]]dt_i$, defined by $dx_j \mapsto x_j(t_i)'dt_i$, $j = 1, \dots, n$, which composed with π_i^* gives rise to an $\mathcal{O}$ -modules homomorphism

$$\begin{aligned} \varphi_i^*: \Omega(\mathcal{O}) &\rightarrow \Omega(\tilde{\mathcal{O}}_i), \\ \sum_{j=1}^n g_j dx_j &\mapsto \left(\sum_{j=1}^n \varphi_i(g_j) \varphi_i(x_j)' \right) dt_i \end{aligned}$$

where the dash means derivative with respect to t_i .

Now, we also define the $\mathcal{O}$ -modules homomorphism

$$\begin{aligned} \varphi^*: \Omega(\mathcal{O}) &\rightarrow \Omega(\tilde{\mathcal{O}}_1) \times \cdots \times \Omega(\tilde{\mathcal{O}}_r). \\ \omega &\mapsto (\varphi_1^*(\omega), \dots, \varphi_r^*(\omega)) \end{aligned}$$

The valuation v_i on $\mathcal{O}_i$ extends to a valuation of $\Omega(\mathcal{O}_i)$ in the following way:

$$v_i \left(\sum_{j=1}^n g_j dx_j \right) = v_i \left(\sum_{j=1}^n \varphi_i(g_j) \varphi_i(x_j)' \right) + 1,$$

where φ_i is the map from $\mathcal{O}$ to $\tilde{\mathcal{O}}_i$; and we extend the valuation v on $\mathcal{O}$ to $\Omega(\mathcal{O})$ as follows:

$$v(\omega) = (v_1(\omega), \dots, v_r(\omega)).$$

We define the *ramification ideal*, as the ideal $\mathcal{R}$ generated in $\tilde{\mathcal{O}}$ by the elements

$$(\varphi_1(x_i)', \dots, \varphi_r(x_i)'), \quad i = 1, \dots, n,$$

By choosing properly the coordinates in $k[[X_1, \dots, X_n]]$, we may assume that the hyperplane X_1 is transversal to all the branches C_i of the curve C , so the maximal ideal $\mathcal{M}$ of $\mathcal{O}$ is a fractional ideal and

$$v(x_1) = (\text{mult}(C_1), \dots, \text{mult}(C_r)) = \min v(\mathcal{M} \setminus Z(\mathcal{O})).$$

Since $\text{char}(k) = 0$, it follows that $\mathcal{R}$ is a fractional ideal of $\mathcal{O}$ and

$$v(x_1) - (1, \dots, 1) = v(\varphi_1(x_1)', \dots, \varphi_r(x_1)') = \min v(\mathcal{R} \setminus Z(\tilde{\mathcal{O}})).$$

We will denote by $\mathcal{T}$ and by $\mathcal{T}_i$ the torsion modules of $\Omega(\mathcal{O})$ and of $\Omega(\mathcal{O}_i)$, respectively. Concerning these modules we have the following result:

Proposition 4.1. Suppose that X is irreducible, that is, $\mathcal{O}$ is a domain, then $\ker(\varphi^*) = \mathcal{T}$.

Proof See [12, Proposition 1, Sect. 7.1]. ■

Proposition 4.2. Suppose that $\mathcal{O}$ is reduced, then $\pi^*(\mathcal{T}) \subset \mathcal{T}_1 \times \cdots \mathcal{T}_r$.

Proof Let $\omega \in \mathcal{T}$, then there exists an element $h \notin Z(\mathcal{O}) = \bigcup_{i=1}^r \wp_i$ such that $h\omega = 0$. Since $h \notin Z(\mathcal{O})$, it follows that $\pi_j(h) \neq 0$, for all $j = 1, \dots, r$, hence a nonzero divisor in $\mathcal{O}_j$. Since $0 = \pi_j^*(h\omega) = \pi_j(h)\pi_j^*(\omega)$, this implies that $\pi_j^*(\omega) \in \mathcal{T}_i$, proving the result. ■

From now on we will assume that the curve C is a complete intersection, this means that $C = \text{Spec } k[[X_1, \dots, X_n]]/\langle f_1, \dots, f_{n-1} \rangle$.

In this situation one has the following theorem:

Theorem 4.3 (Piene [15]). Let $\mathcal{O}$ be the ring of an algebroid complete intersection curve and let $\tilde{\mathcal{O}}$ be its normalization. If $\mathcal{J}$ denotes the Jacobian ideal of $\Omega(\mathcal{O})$ and $\mathcal{R}$ and $\mathcal{C}$, respectively, the ramification and conductor ideals of $\tilde{\mathcal{O}}$ in $\mathcal{O}$, then one has

$$\mathcal{J}\tilde{\mathcal{O}} = \mathcal{C}\mathcal{R}.$$

In this situation, one has that $\mathcal{J} = \langle |M_1|, \dots, |M_n| \rangle$, where $|M_i|$ is the determinant of the matrix M_i obtained by deleting the i -th column of the matrix $Jac(f_1, \dots, f_{n-1})$. Since by Theorem 4.3 one has that $\mathcal{J}$ is a fractional ideal, then by reordering the series $f_1, \dots, f_{n-1}$, we may assume that

$$v(|M_1|) = \inf \mathcal{J}\tilde{\mathcal{O}} = \inf v(\mathcal{R}) + c = v(x_1) + \mathfrak{f}(\Gamma), \quad (4.1)$$

where $e = (1, \dots, 1)$ and $c = (c_1, \dots, c_r)$ is the conductor of the semigroup of values Γ . This implies that $|M_1|$ is not a zero divisor in $\tilde{\mathcal{O}}$.

The relation in $\Omega(\mathcal{O})$ are

$$\sum_{j=1}^n f_{iX_j} dx_j = 0, \quad i = 1, \dots, n-1,$$

which may be written in matricial form as follows

$$M_1 \begin{bmatrix} dx_2 \\ \vdots \\ dx_n \end{bmatrix} = - \begin{bmatrix} f_{1X_1} \\ \vdots \\ f_{n-1X_1} \end{bmatrix} dx_1. \quad (4.2)$$

If we multiply both sides of equation (4.2) by the adjoint matrix M_1^* of M_1 , we get

$$|M_1| dx_i = (-1)^{i-1} |M_i| dx_1. \quad (4.3)$$

From Equations (4.1) and (4.3) one gets that

$$v(|M_i|) = v(x_i) + \mathfrak{f}(\Gamma). \quad (4.4)$$

Theorem 4.4. For a complete intersection $\mathcal{O}$ one has that

$$\pi^{*-1}(\mathcal{T}_1 \times \cdots \times \mathcal{T}_r) = \mathcal{T}.$$

Proof From Proposition 4.2 one has that $\mathcal{T} \subset \pi^{*-1}(\mathcal{T}_1 \times \cdots \times \mathcal{T}_r)$.

For the other inclusion, let $\omega = \sum_{j=1}^n g_j dx_j \in \pi^{*-1}(\mathcal{T}_1 \times \cdots \times \mathcal{T}_r)$. Hence $\pi_i^*(\omega) \in \mathcal{T}_i$, therefore $\phi_i^*(\omega) = 0$, for all $i = 1, \dots, r$.

Consider $|M_1|\omega$, which in view of relations (4.3) may be written as

$$|M_1|\omega = \sum_{j=1}^n g_j |M_1| dx_j = \left(\sum_{j=1}^n (-1)^{i-1} g_j |M_j| \right) dx_1.$$

Applying φ_i^* to both sides of the above expression, one gets, for all $i = 1, \dots, r$,

$$0 = \pi_i(|M_1|)\varphi_i^*(\omega) = \varphi_i^*(|M_1|\omega) = \pi_i^*\left(\sum_{j=1}^n (-1)^{i-1} g_j |M_j|\right)\varphi_i^*(x_1)'. \quad \blacksquare$$

Since $\varphi_i^*(x_1)' \neq 0$, for all i , then $\pi_i^*\left(\sum_{j=1}^n (-1)^{i-1} g_j |M_j|\right) = 0$, for all i , which implies that $\sum_{j=1}^n (-1)^{i-1} g_j |M_j| \in \cap_i \mathfrak{p}_i = (0)$. So, $|M_1|\omega = 0$ and since $|M_1| \notin Z(\mathcal{O})$, we have that $\omega \in \mathcal{T}$. ■

Corollary 4.5. Let $\mathcal{O}$ be the ring of an algebroid complete intersection curve, then one has $\omega = \sum_{j=1}^n g_j dx_j \in \mathcal{T}$ if and only if

$$\frac{df_1 \wedge \cdots \wedge df_{n-1} \wedge \omega}{dx_1 \wedge \cdots \wedge dx_n} = \det \begin{bmatrix} f_{1X_1} & \cdots & f_{1X_n} \\ \vdots & & \vdots \\ f_{n-1X_1} & \cdots & f_{n-1X_n} \\ g_1 & \cdots & g_n \end{bmatrix} = 0.$$

From Proposition 4.1 and Theorem 4.4 we have shown that if $\mathcal{C}$ is a complete intersection, then we have inclusions

$$\Omega(\mathcal{O})/\mathcal{T} \hookrightarrow \Omega(\mathcal{O}_1)/\mathcal{T}_1 \times \cdots \times \Omega(\mathcal{O}_r)/\mathcal{T}_r \hookrightarrow \Omega(\tilde{\mathcal{O}}_1) \times \cdots \times \Omega(\tilde{\mathcal{O}}_r) \simeq \tilde{\mathcal{O}},$$

which allows us to view $\Omega(\mathcal{O})/\mathcal{T}$ and $\Omega(\mathcal{O}_1)/\mathcal{T}_1 \times \cdots \times \Omega(\mathcal{O}_r)/\mathcal{T}_r$ as fractional ideals, one contained in the other.

Now, it is easy to verify that the map

$$\begin{aligned} \mathcal{D}: \mathcal{O} &\rightarrow \mathcal{O} \\ g &\mapsto \frac{df_1 \wedge \cdots \wedge df_{n-1} \wedge dg}{dx_1 \wedge \cdots \wedge dx_n} \end{aligned}$$

is a differential operator on $\mathcal{O}$, that may be written as

$$\mathcal{D}(g) = \det \begin{bmatrix} f_{1X_1} & \cdots & f_{1X_n} \\ \vdots & & \vdots \\ f_{n-1X_1} & \cdots & f_{n-1X_n} \\ g_{X_1} & \cdots & g_{X_n} \end{bmatrix}.$$

This operator and the universal differentiation on $\mathcal{O}$ may be extended in a standard way to the total ring of fractions $\mathcal{K}$ of $\mathcal{O}$.

Proposition 4.6. One has that

$$v(\mathcal{D}(g)) = v(g) + \mathfrak{f}(\Gamma).$$

Proof Since $\varphi_i(f_j)' = 0$, for all $i = 1, \dots, r$ and $j = 1, \dots, n$, one has the equalities

$$\begin{bmatrix} \varphi_i(f_{1X_1}) & \cdots & \varphi_i(f_{1X_n}) \\ \vdots & & \vdots \\ \varphi_i(f_{n-1X_1}) & \cdots & \varphi_i(f_{n-1X_n}) \\ \varphi_i(g_{X_1}) & \cdots & \varphi_i(g_{X_n}) \end{bmatrix} \begin{bmatrix} \varphi_i(x_1)' \\ \vdots \\ \varphi_i(x_{n-1})' \\ \varphi_i(x_n)' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \varphi_i(g)' \end{bmatrix}.$$

Calling M the above square matrix on the left, and multiplying both sides of the above equality by the adjoint matrix M^* of M , one gets that

$$\varphi_i(\mathcal{D}(g))\varphi_i(x_j)' = (-1)^{j+n}|M_j|\varphi_i(g)', \quad i = 1, \dots, r; \quad j = 1, \dots, n.$$

From this we get that

$$v_i(\varphi_i(\mathcal{D}(g))) = c_i + v_i(\varphi_i(g)') = c_i + v_i(g) - 1.$$

which implies the result. ■

Proposition 4.7. One has that $\mathcal{D}(\tilde{\mathcal{O}}) \subset \tilde{\mathcal{O}}$.

Proof If $\frac{g}{h} \in \tilde{\mathcal{O}}$, with $g, h \in \mathcal{O}$, then $v(g) \geq v(h)$. Since

$$\mathcal{D}\left(\frac{g}{h}\right) = \frac{h\mathcal{D}(g) - g\mathcal{D}(h)}{h^2},$$

and

$$v\left(\frac{\mathcal{D}(g)}{h}\right) = v\left(\frac{g\mathcal{D}(h)}{h^2}\right) = v(g) - v(h) + \mathfrak{f}(\Gamma),$$

one has that

$$v\left(\mathcal{D}\left(\frac{g}{h}\right)\right) \geq v(g) - v(h) + \mathfrak{f}(\Gamma) \geq \mathfrak{f}(\Gamma) \geq \underline{0}.$$
■

Using the determinantal definition of $\mathcal{D}(u)$ and the relations (4.3) one gets that

$$|M_1|du = (-1)^n \mathcal{D}(u)dx_1. \quad (4.5)$$

So, one has that $\mathcal{D}(u)$ is a zero divisor if and only if $|M_1|du$ is torsion, which means that du is torsion, since in characteristic zero, no exact non-zero differential is torsion, it follows that the differential

$$\omega = \frac{\varphi^*(du)}{\varphi(\mathcal{D}(u))} = \frac{(-1)^n}{\varphi(|M_1|)}\varphi^*(dx_1) \in \Omega(\mathcal{K}),$$

is independent from $u \neq 0$ and $v(\omega) = -c + 1$. We consider the $\mathcal{O}$ -modules homomorphism “multiplication by ω ”:

$$\begin{aligned} m_\omega: \mathcal{O} &\rightarrow \Omega(\mathcal{K}) \\ h &\mapsto \varphi(h)\omega \end{aligned}$$

Recalling that $\mathcal{J}$ and $\mathcal{C}$ are respectively the jacobian and conductor ideals of $\mathcal{O}$, we have the theorem:

Theorem 4.8. The following assertions are true

- i) m_ω is injective;
- ii) $m_\omega(\mathcal{C}) = \Omega(\tilde{\mathcal{O}})$;
- iii) $m_\omega(\mathcal{J}) = \varphi^*(\Omega(\mathcal{O}))$.

Proof (i) This is so, because the homomorphism $\varphi: \mathcal{O} \rightarrow \tilde{\mathcal{O}}$ is injective, $\omega \neq 0$ and $\Omega(\mathcal{K}) \simeq k((t_1))dt_1 \times \cdots \times k((t_r))dt_r$ is free.

(ii) The elements of $\Omega(\tilde{\mathcal{O}})$ are those elements $\rho \in \Omega(\mathcal{K})$ for which $v(\rho) \geq e$.

Since $v(\omega) = \mathfrak{f}(\Gamma)$, it follows that for $h \in \mathcal{C}$, $m_\omega(h) = \omega\varphi(h)$ is such that $v(m_\omega(h)) = v(\omega) + v(h) \geq e$, hence $m_\omega(h) \in \Omega(\tilde{\mathcal{O}})$.

Conversely, let $\rho = (\xi_1 dt_1, \dots, \xi_r dt_r) \in \Omega(\tilde{\mathcal{O}})$, hence $v_i(\xi_i) \geq 0$. We want to find $h = (h_1, \dots, h_r) \in \mathcal{C}$ such that $m_\omega(h) = \rho$. The elements

$$h_i = \frac{\xi_i \varphi_i(|M_1|)}{\varphi_i(x_1)'},$$

will do the job.

iii) Observe that any element in $\mathcal{J}$ may be written in the form

$$h = \sum_{i=1}^n (-1)^{i-1} g_i |M_i|, \quad \text{for } h \in \mathcal{O}$$

and any element of $\varphi^*(\Omega(\mathcal{O}))$ is of the form

$$\sum_{i=1}^n \varphi^*(g_i dx_i), \quad \text{for } g_i \in \mathcal{O}.$$

The result follows by observing that

$$\begin{aligned}
 m_\omega(h) &= (-1)^n \sum_{i=1}^n \frac{(-1)^{i-1} \varpi_i(g_i | M_i|)}{|M_1|} \varphi_i^*(dx_1) \\
 &= (-1)^n \sum_{i=1}^n \varphi(g_i) \varphi^*(dx_i) \\
 &= (-1)^n \varphi^*(\sum_{i=1}^n g_i dx_i).
 \end{aligned}$$

■

Corollary 4.9 (cf. D. Pol [16]). One has the following $\mathcal{O}$ -module isomorphisms

$$\mathcal{J} \simeq \varphi^*(\Omega(\mathcal{O})) \simeq \Omega(\mathcal{O})/\mathcal{T},$$

and

$$v(\mathcal{J} \setminus Z(\mathcal{O})) = v(\Omega(\mathcal{O}) \setminus \mathcal{T}) + \mathfrak{f}(\Gamma).$$

Proof This is just Theorem 4.8 (iii) and a computation of values. ■

Let us define the Tjurina number of $\mathcal{O}$ as being

$$\tau(\mathcal{O}) = \dim_k \frac{\mathcal{O}}{\mathcal{J}}.$$

Corollary 4.10. Let $\mathcal{O}$ be the ring of a complete intersection curve, then one has

$$\tau(\mathcal{O}) = \ell_{\mathcal{O}} \left(\frac{\mathcal{O}}{\mathcal{C}} \right) + \ell_{\mathcal{O}} \left(\frac{\Omega(\tilde{\mathcal{O}})}{\varphi^*(\Omega(\mathcal{O}))} \right).$$

Proof By computing values, it is easy to check that $\mathcal{J} \subset \mathcal{C}$, so

$$\tau(\mathcal{O}) = \ell_{\mathcal{O}} \left(\frac{\mathcal{O}}{\mathcal{J}} \right) = \ell_{\mathcal{O}} \left(\frac{\mathcal{O}}{\mathcal{C}} \right) + \ell_{\mathcal{O}} \left(\frac{\mathcal{C}}{\mathcal{J}} \right).$$

Now, from Theorem 4.8 one gets

$$\ell_{\mathcal{O}} \left(\frac{\mathcal{C}}{\mathcal{J}} \right) = \ell_{\mathcal{O}} \left(\frac{m_\omega(\mathcal{C})}{m_\omega(\mathcal{J})} \right) = \ell_{\mathcal{O}} \left(\frac{\Omega(\tilde{\mathcal{O}})}{\varphi^*(\Omega(\mathcal{O}))} \right),$$

which proves the result. ■

Since $\mathcal{O}$ is Gorenstein, it follows that

$$\ell_{\mathcal{O}} \left(\frac{\mathcal{O}}{\mathcal{C}} \right) = \ell_{\mathcal{O}} \left(\frac{\tilde{\mathcal{O}}}{\mathcal{O}} \right) = \delta.$$

On the other hand, from the chain of monomorphisms

$$\varphi^*(\Omega(\mathcal{O})) = \frac{\Omega(\mathcal{O})}{\mathcal{T}} \longrightarrow \bigoplus_{i=1}^r \frac{\Omega_i}{\mathcal{T}_i} \longrightarrow \Omega(\tilde{\mathcal{O}}),$$

we get that

$$\ell_{\mathcal{O}} \left(\frac{\Omega(\tilde{\mathcal{O}})}{\varphi^*(\Omega(\mathcal{O}))} \right) = \ell_{\mathcal{O}} \left(\frac{\Omega(\tilde{\mathcal{O}})}{\oplus_{i=1}^r \frac{\Omega_i}{\mathcal{T}_i}} \right) + \ell_{\mathcal{O}} \left(\frac{\oplus_{i=1}^r \frac{\Omega_i}{\mathcal{T}_i}}{\frac{\Omega(\mathcal{O})}{\mathcal{T}}} \right).$$

Since

$$\ell_{\mathcal{O}} \left(\frac{\Omega(\tilde{\mathcal{O}})}{\oplus_{i=1}^r \frac{\Omega_i}{\mathcal{T}_i}} \right) = \sum_{i=1}^r \ell_{\mathcal{O}_i} \left(\frac{\Omega(\tilde{\mathcal{O}}_i)}{\frac{\Omega_i}{\mathcal{T}_i}} \right),$$

and

$$\begin{aligned} \ell_{\mathcal{O}_i} \left(\frac{\Omega(\tilde{\mathcal{O}}_i)}{\frac{\Omega_i}{\mathcal{T}_i}} \right) &= \# \left(v(\Omega(\tilde{\mathcal{O}}_i)) \setminus v\left(\frac{\Omega_i}{\mathcal{T}_i}\right) \right) = \# \mathbb{N}^* \setminus \Lambda_i \\ &= \# \mathbb{N} \setminus \Gamma_i - \# \Lambda_i \setminus \Gamma_i \\ &= \delta_i - r_i, \end{aligned}$$

where Γ_i and Λ_i are the semigroup of values of the branch $\mathcal{O}_i$ and the set of values of the Kähler differentials on the branch $\mathcal{O}_i$, respectively. The integer $r_i = \# \Lambda_i \setminus \Gamma_i$ is equal to the number of nonexact linearly independent differentials modulo the exact differentials of the branch $\mathcal{O}_i$. This number r_i may be made more explicit when $\mathcal{O}_i$ is also a complete intersection, because, according Corollary 51, in this case, one has

$$r_i = c_i - \tau_i,$$

where c_i is the conductor of the semigroup of values of $\mathcal{O}_i$ and τ_i is its Tjurina number.

So, to compute $\tau(\mathcal{O})$, it remains only to compute $\ell_{\mathcal{O}} \left(\frac{\oplus_{i=1}^r \frac{\Omega_i}{\mathcal{T}_i}}{\frac{\Omega(\mathcal{O})}{\mathcal{T}}} \right)$, which may be done with Theorem 2.7, in general, and with Proposition 2.4 for $r = 2$ and Theorem 2.16 for $r = 3$.

EXAMPLE

Consider the plane curve C given by

$$f(x, y) = (y^3 - x^7)(y^3 - 3x^5y - x^7 - x^8)(y^4 - 2x^5y^2 - x^7y - x^9 + x^{10})$$

and let $\mathcal{O} = \frac{k[[x, y]]}{\langle f \rangle}$ the ring of the curve.

Let's calculate the Tjurina number of $\mathcal{O}$, given by the following formula

$$\tau(\mathcal{O}) = \dim_k \frac{\mathcal{O}}{\mathcal{J}} \quad (5.1)$$

where $\mathcal{J} = \langle f_x, f_y \rangle$ is the Jacobian ideal of $\mathcal{O}$, since, $\mathcal{O}$ is Gorenstein we have

$$\tau(\mathcal{O}) = \ell_{\mathcal{O}} \left(\frac{\tilde{\mathcal{O}}}{\mathcal{O}} \right) + \ell_{\mathcal{O}} \left(\frac{\tilde{\mathcal{O}}}{\frac{\Omega}{\mathcal{T}}} \right)$$

and

$$\ell_{\mathcal{O}} \left(\frac{\tilde{\mathcal{O}}}{\mathcal{O}} \right) = \sum_{i=1}^3 \delta_i + \sum_{1 \leq i, j \leq 3} I_{i,j}$$

where $\ell_{\mathcal{O}_i} \left(\frac{\tilde{\mathcal{O}}_i}{\mathcal{O}_i} \right) = \delta_i$ and $I_{i,j}$ is the intersection multiplicity of C_i and C_j with $1 \leq i, j \leq 3$.

Let C_1 , C_2 and C_3 be the branches of C given by $f_1(x, y) = y^3 - x^7$, $f_2(x, y) = y^3 - 3x^5y - x^7 - x^8$ and $f_3(x, y) = y^4 - 2x^5y^2 - 4x^7y - x^9 + x^{10}$ respectively.

On the other hand,

$$\begin{aligned} \ell_{\mathcal{O}} \left(\frac{\tilde{\mathcal{O}}}{\frac{\Omega}{\mathcal{T}}} \right) &= \ell_{\mathcal{O}} \left(\frac{\tilde{\mathcal{O}}}{\bigoplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}} \right) + \ell_{\mathcal{O}} \left(\frac{\bigoplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}}{\frac{\Omega}{\mathcal{T}}} \right) \\ &= \ell_{\mathcal{O}} \left(\frac{\bigoplus_{i=1}^3 \tilde{\mathcal{O}}_i}{\bigoplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}} \right) + \ell_{\mathcal{O}} \left(\frac{\bigoplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}}{\frac{\Omega}{\mathcal{T}}} \right) \\ &= \sum_{i=1}^3 (\tau_i - \delta_i) + \ell_{\mathcal{O}} \left(\frac{\bigoplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}}{\frac{\Omega}{\mathcal{T}}} \right) \end{aligned}$$

Substituting in 5.1 we have

$$\begin{aligned}\tau(\mathcal{O}) &= \sum_{i=1}^3 \delta_i + \sum_{1 \leq i < j \leq 3} I_{i,j} + \sum_{i=1}^3 (\tau_i - \delta_i) + \ell_{\mathcal{O}} \left(\frac{\oplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}}{\frac{\Omega}{\mathcal{T}}} \right) \\ &= \sum_{i=1}^3 \tau_i + \sum_{1 \leq i < j \leq 3} I_{i,j} + \ell_{\mathcal{O}} \left(\frac{\oplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}}{\frac{\Omega}{\mathcal{T}}} \right).\end{aligned}$$

So to calculate $\tau(\mathcal{O})$, just calculate

$$\ell_{\mathcal{O}} \left(\frac{\oplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}}{\frac{\Omega}{\mathcal{T}}} \right) = \ell_{\mathcal{O}} \left(\frac{\oplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}(\alpha^0)}{\oplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}(c(\Lambda))} \right) - \ell_{\mathcal{O}} \left(\frac{\frac{\Omega}{\mathcal{T}}(\alpha^0)}{\frac{\Omega}{\mathcal{T}}(c(\Lambda))} \right)$$

By Theorem 2.17 we have

$$\ell_{\mathcal{O}} \left(\frac{\oplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}}{\frac{\Omega}{\mathcal{T}}} \right) = \# M_{\{1,2\}} + \# RM_{\{1,3\}} + \# RM_{\{2,3\}} + \# RM - \# AM, \quad (5.2)$$

where AM and RM are the set of absolute maximals and relative maximals of the set Λ , $RM_{\{1,2\}}$, $RM_{\{1,3\}}$ and $RM_{\{2,3\}}$ are the set of maximals of the set $\Lambda_{\{1,2\}}$, $\Lambda_{\{1,3\}}$ and $\Lambda_{\{2,3\}}$, Λ_1 , Λ_2 and Λ_3 respectively.

Now, we will compute the terms in 5.2 by using a script in Maple developed by M. E. Hernandez.

The figure below shows the elements of $\Lambda_{\{1,2\}}$ with conductor $c(\Lambda_{\{1,2\}}) = (20, 20)$, where the set maximals points $M_{\{1,2\}} = \{(3, 3), (6, 6), (7, 7), (9, 9), (10, 10), (12, 12), (13, 13), (15, 15), (16, 16), (19, 19)\}$ of $\Lambda_{\{1,2\}}$ are marked with circles. Notice that $\# M_{\{1,2\}} = 10$.

The figure below shows the elements of $\Lambda_{\{1,3\}}$ with conductor $c(\Lambda_{\{1,3\}}) = (22, 29)$, where the set maximal points $M_{\{1,3\}} = \{(3, 4), (6, 8), (7, 9), (9, 12), (10, 14), (12, 16), (13, 19), (14, 18), (15, 20), (17, 23), (18, 24), (21, 28)\}$ of $\Lambda_{\{1,3\}}$ are marked with circles. Notice that $\# M_{\{1,3\}} = 12$.

The figure below shows the elements of $\Lambda_{\{2,3\}}$ with conductor $c(\Lambda_{\{2,3\}}) = (22, 29)$, where the set maximal points $M_{\{2,3\}} = \{(3, 4), (6, 8), (7, 9), (9, 12), (10, 14), (11, 13), (12, 16), (13, 19), (14, 18), (15, 20), (17, 23), (18, 24), (21, 28)\}$ of $\Lambda_{\{2,3\}}$ are marked with circles. Notice that $\# M_{\{2,3\}} = 13$.

We have that, $\# \text{RM} = 14$, where $\text{RM} = \{(14, 14, 17), (17, 17, 21), (18, 18, 22), (20, 20, 25), (21, 21, 26), (22, 22, 27), (23, 23, 29), (24, 24, 30), (25, 25, 32), (26, 26, 33), (27, 27, 34), (29, 29, 37), (30, 30, 38), (33, 33, 42)\}$ is the set of relative maximals of Λ .

We also have, $\# \text{AM} = 12$, where $\text{AM} = \{(3, 3, 4), (6, 6, 8), (7, 7, 9), (9, 9, 12), (10, 10, 14), (12, 12, 16), (13, 13, 19), (14, 14, 18), (15, 15, 20), (17, 17, 23), (18, 18, 24), (21, 21, 28)\}$ is the set of absolute maximals of Λ .

Substituting in 5.2 we have

$$\ell_{\mathcal{O}} \left(\frac{\oplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}}{\frac{\Omega}{\mathcal{T}}} \right) = 37$$

Moreover, we have that $\tau_1 = 12$, $\tau_2 = 11$, $\tau_3 = 21$, $I_{1,2} = 22$, $I_{1,3} = 27$ and $I_{2,3} = 27$. Therefore,

$$\begin{aligned} \tau(\mathcal{O}) &= \sum_{i=1}^3 \tau_i + \sum_{1 \leq i < j \leq 3} I_{i,j} + \ell_{\mathcal{O}} \left(\frac{\oplus_{i=1}^3 \frac{\Omega_i}{\mathcal{T}_i}}{\frac{\Omega}{\mathcal{T}}} \right) \\ \tau(\mathcal{O}) &= 12 + 11 + 21 + 22 + 27 + 27 + 37 \\ \tau(\mathcal{O}) &= 157. \end{aligned}$$

This same result is obtained using the routine implemented in Singular to calculate the Tjurina number.

```
LIB "sing.lib";
ring r=0,(x,y),ds;      //local ring
poly f=(y3-x7)*(y3-3*x5y-x7-x8)*(y4-2*x5y2-x7y-x9+x10);
tjurina(f);
//-> 157                //Tjurina number at 0.
```


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