Qualitative properties of impulsive semidynamical systems

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Propriedades qualitativas de sistemas semidinâmicos impulsivos

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação - ICMC-USP, como parte dos requisitos para obtenção do título de Doutora em Ciências – Matemática. *EXEMPLAR DE DEFESA*

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*To my mother Gina
(in memoriam).*

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Abstract

The theory of impulsive dynamical systems is an important tool to describe the evolution of systems where the continuous development of a process is interrupted by abrupt changes of state. This phenomenon is called impulse. In many natural phenomena, the real deterministic models are often described by systems which involve impulses.

The aim of this work is to investigate topological properties of impulsive semidynamical systems. We establish necessary and sufficient conditions to obtain uniform and orbital stability via Lyapunov functions. We solve a problem of Jake Hale for impulsive systems where we obtain the existence of a maximal compact invariant set. Also, we obtain results about almost periodic motions and asymptotically almost periodic motions in the context of impulsive systems. Some asymptotic properties for impulsive systems and for their associated discrete systems are investigated.

The new results presented in this text are in the papers [11], [15] and [16].

Keywords: Dynamical systems, impulses, stability, recurrence, dissipativity.

Resumo

A teoria de sistemas dinâmicos com impulsos é apropriada para descrever processos de evolução que sofrem variações de estado de curta duração e que podem ser consideradas instantâneas. Este fenômeno é chamado impulso. Para muitos fenômenos naturais, os modelos determinísticos mais realistas são frequentemente descritos por sistemas que envolvem impulsos.

O objetivo deste trabalho é estudar propriedades topológicas para sistemas semidinâmicos impulsivos. Estabelecemos condições necessárias e suficientes para obtermos estabilidade uniforme e estabilidade orbital utilizando funções do tipo Lyapunov. Resolvemos um problema de Jack Hale para os sistemas impulsivos, onde obtemos a existência de um conjunto invariante compacto maximal. Além disso, obtemos resultados de movimentos quase periódicos e movimentos assintoticamente quase periódicos para sistemas impulsivos. Algumas propriedades assintóticas são estabelecidas para um sistema impulsivo e para seu sistema discreto associado.

Os resultados novos apresentados neste trabalho estão presentes nos artigos [11], [15] e [16].

Palavras Chaves: Sistemas dinâmicos, impulsos, estabilidade, recorrência, dissipatividade.

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Introduction

The dynamics of many mathematical models describe real world phenomena and are subject to abrupt changes of state, whose duration is negligible in comparison with the duration of entire evolution processes. In this way, it is natural to consider that these changes act in form of impulses. It is known, for instance, that many biological phenomena, population dynamics, optimal control model, problems in physics, in medicine, in industrial robotics and frequency modulated systems do exhibit impulsive effects.

The theory of impulsive differential equations is an important tool to describe the evolution of systems where the continuous development of a process is interrupted by abrupt changes of state. An impulsive differential equation is modeled by a system that encompasses a differential equation, which describes the period of continuous variation of state, and additional conditions, which describe the discontinuities of the solutions of the differential equation or of their derivatives at the moments of impulses. The reader may consult [18] to obtain more details.

One of the branches of the theory of impulsive differential equations is the theory of impulsive dynamical systems. An impulsive dynamical system is denoted by $(X, \pi; M, I)$ and it consists of three elements: a continuous dynamical system on a phase space X , a nonempty closed set M in X that is called the impulsive set and a continuous function I defined in M responsible by the discontinuities of the system, called the impulse function. The continuous dynamical system governs the flow until it meets the impulsive set where the flow undergoes a change of state. The impulse function specifies how occurs the change of state and the dynamical system continues the movement after this change. The new flow constructed above governs the impulsive dynamical

system. This construction presents interesting phenomena as oscillation, pulse, beating, dying, non-continuation of process, etc.

As an example, we may cite the billiard-type system which can be modeled by differential systems with impulses acting on the first derivatives of the solutions. Indeed, the positions of the colliding balls do not change at the moments of impact (impulse), but their velocities gain finite increments (the velocity will change according to the position of the ball).

In 1990, Kaul constructed the mathematical foundation of the theory of impulsive dynamical systems at variable times, see [31]. Next, Kaul and Ciesielski published very important results about topological properties on impulsive systems, see [23], [24], [25], [32], [33] and [34]. Thereon a vast literature on this topic has been developed by Bonotto and his collaborators, see [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15] and [16].

In the present work, our main goal is to develop qualitative properties on impulsive semidynamical systems. Qualitative properties of solutions as “asymptotic behavior” and “stability of sets” are very important in the study of trajectories on dynamical systems. Although there are many works of stability on impulsive dynamical systems, some questions concerning attraction and stability still lack answers. For instance, let $(X, \pi; M, I)$ be an impulsive system and $A \subset X$ be a nonempty set. There exist several examples where $A = \bar{A} \setminus M$ is stable “in some sense” but $\bar{A}$ is not stable, see Examples 2.1.4 and 2.1.5. The class of stable sets of the form $\bar{A} \setminus M$, $A \subset X$, is very important in impulsive systems since this class also includes global attractors in the sense of [12]. Thereby, one of the problems of this work is to show necessary and sufficient conditions to obtain results concerning stability, orbital stability and uniform stability for sets in impulsive systems.

An interesting problem in dynamical systems was formulated by Hale in [28]. He studied the theory of dissipative discrete dynamical systems and he proved the existence of the maximal compact invariant set in a class of such systems. This problem is known in the literature as Jack Hale’s problem. In the case of dissipative continuous dynamical systems “without impulses”, the reader may consult results from this theory in [19, 29, 35, 37]. Later, some results were extended to compact dissipative systems with impulses, that is, it was proved that a compact dissipative semidynamical system with impulses admit a maximal compact invariant set called the center of Levinson, see [6]. But this problem has not been investigated for pointwise dissipative systems. In this way, under additional conditions, we prove the existence of a maximal compact invariant set for a pointwise dissipative system with impulses.

The concepts of almost periodic and asymptotically almost periodic functions were introduced by Bohr in [17] and Fréchet in [27], respectively. Later, these concepts were developed in the context of dynamical systems by N. P. Bhatia and G. P. Szego in [1] and by Cheban in [20]. The existence of almost periodic and asymptotically almost periodic solutions is one of the most attractive topics in the qualitative theory of differential equations, see [20] and [30], for instance. On this topic, we consider almost periodic motions in the context of impulsive semidynamical systems. We give sufficient conditions to obtain the existence of asymptotic almost periodic motions in impulsive semidynamical systems. Since almost periodicity of motions is deeply connected with stability, we also investigate this connection on impulsive systems.

The present work is divided in five chapters and it is organized as follows. The main results of this text are presented in the papers [11], [15] and [16].

In the first chapter, we introduce the basis of the theory of semidynamical systems with impulses. In Section 1.1, we present a brief summary of the classical theory of dynamical systems. Section 1.2 deals with the theory of impulsive semidynamical systems, we write ISS for short. In Definition 1.2.3, we exhibit the construction of an ISS and in Proposition 1.2.9 we show that an ISS still satisfying the conditions of identity and semigroup. In Section 1.3, we discuss the continuity of a function which describes the times of meeting impulsive set. In Section 1.4, we establish results of convergence for the impulsive flow. In the last section from this chapter, namely Section 1.5, we present the concepts of invariant sets and positive limit sets and we also include some auxiliary results that will be very useful in the main results.

Chapter 2 is divided in three sections. In Section 2.1, we establish results about uniform stability, orbital stability and attraction. Proposition 2.1.9 shows that a relatively compact set $A \subset X$ is uniformly $\tilde{\pi}$ -stable if and only if $\overline{A}$ is orbitally $\tilde{\pi}$ -stable. The characterization of the positive prolongation set of a relatively compact set is given in Proposition 2.1.11. On locally compact spaces, a relatively compact set $A \subset X$ is uniformly $\tilde{\pi}$ -stable if and only if the positive prolongation set of A coincides with its closure, see Theorem 2.1.13. In Theorem 2.1.14, we present sufficient conditions for a weakly $\tilde{\pi}$ -attractor set to be $\tilde{\pi}$ -attractor. The last result from this section, namely Proposition 2.1.15, exhibits conditions to show that the set $\overline{A} \setminus M$ is contained in the region of attraction of $A \subset X$. In Section 2.2, we present conditions to obtain $\tilde{\pi}$ -stability and orbital $\tilde{\pi}$ -stability for sets of the form $\overline{A} \setminus M$, $A \subset X$. The results are achieved by means of Lyapunov functions. Theorem 2.2.3 concerns sufficient conditions for a set $\overline{A} \setminus M$ to be $\tilde{\pi}$ -stable. In the case of orbital stability, see Theorem 2.2.4, Corollary 2.2.5 and Corollary 2.2.9. A result about insta-

bility is also presented in Theorem 2.2.6. Section 2.3 concerns with the existence of Lyapunov functions under stability conditions as presented in Theorem 2.3.2 and Corollary 2.3.3.

Chapter 3 is divided in four parts. In Section 3.1, we study results about almost periodic motions. In Theorem 3.1.8, we prove the compactness of the closure of an orbit almost $\tilde{\pi}$ -periodic. We show that all almost $\tilde{\pi}$ -periodic points are positively Poisson $\tilde{\pi}$ -stable in impulsive systems, see Theorem 3.1.11. In Section 3.2, we present the concept of the quasi stability of Zhukovskij for impulsive systems and we establish sufficient conditions to obtain Zhukovskij quasi $\tilde{\pi}$ -stable sets, see Theorem 3.2.4. As a consequence of Theorem 3.2.4, we show that a positive limit set of a point $x \in X \setminus M$ without its impulsive points is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to the positive trajectory of x , see Corollary 3.2.5. The Section 3.3 is devoted to present the concepts of asymptotic almost periodic, stationary, periodic, recurrent and Poisson stable motions using time reparametrizations. Some topological properties for these motions are considered. Moreover, the limit set of asymptotic almost $\tilde{\pi}$ -periodic motions coincides with the closure of one an almost $\tilde{\pi}$ -periodic motion, see Theorem 3.3.4. In Theorem 3.3.5, we show conditions for a point to be asymptotically almost $\tilde{\pi}$ -periodic. Section 3.4 presents results concerning on asymptotically $\tilde{\pi}$ -periodic and stationary motions. While Theorem 3.4.2 presents sufficient conditions for a point to be asymptotically $\tilde{\pi}$ -periodic, the Theorem 3.4.4 give us necessary and sufficient conditions for a point to be asymptotically $\tilde{\pi}$ -stationary.

In Chapter 4, we study topological properties on dissipative systems. In Section 4.1, we present the theory of dissipative impulsive systems. In the case of compact dissipative systems, we show the existence of a special attractor namely Center of Levinson. The properties of the center of Levinson are presented in Theorem 4.1.8. Section 4.2 deals with the concept of asymptotic compactness for impulsive systems and its associated discrete system. Theorem 4.2.7 and Theorem 4.2.8 relate the property of asymptotic compactness between an ISS and its associated discrete system. The condition of Ladyzhenskaya is also investigated between an ISS and its associated discrete system, see Theorem 4.2.11 and Theorem 4.2.13. In the Section 4.3, we show that an ISS is $\tilde{\pi}$ -asymptotically compact provided it is μ -condensing, see Proposition 4.3.5. The problem of Jack Hale is studied in Section 4.4. We show sufficient conditions for a pointwise dissipative impulsive system to admit a maximal compact positively invariant set. Finally, in Section 4.5, we present an example where we show an ISS that is pointwise dissipative but does not admit a maximal compact invariant set.

In the last chapter, we study properties of discrete systems in the sense of Kaul. In Section 5.1, we start by presenting the concept of a discrete system in the sense of Kaul. The theory of asymptotic compactness for this class of discrete systems is developed in Section 5.2, see Theorem 5.2.4 and Theorem 5.2.5. Section 5.3 concerns with recurrent motions by time reparametrization. We introduce the notion of almost $\tilde{\pi}$ -periodicity by time reparametrization for an ISS and the notion of almost periodicity for discrete systems in the sense of Kaul. Theorems 5.3.6 and 5.3.8 give us conditions for a point to be almost $\tilde{\pi}$ -periodic by time reparametrization, provided this point is almost periodic in the associated discrete system. Asymptotic properties are obtained in Section 5.4, see Theorem 5.4.3. The last section of this text presents sufficient conditions to obtain Zhukovskij quasi stability via Lyapunov stability, see Theorem 5.5.2.

Impulsive semidynamical systems

In this first chapter, we introduce the basis of the theory of semidynamical systems with impulses effect which consists of three ingredients: a continuous semidynamical system on a phase space X , a nonempty closed set M in X that is called the “impulsive set” and a continuous function I defined in M responsible for the discontinuities of the system, called the “impulse function”. The continuous semidynamical system governs the flow until it meets the impulsive set where the flow undergoes a change of state.

The major of the results from this chapter are presented in [1], [2], [4], [23] and [31]. We also include some technical results from [15].

1.1 Semidynamical systems

The study of impulsive dynamical systems requires a previous knowledge of continuous semidynamical systems. In this way, we present a brief summary of this theory. For more details, the reader may consult [1].

Throughout this work the symbols $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ stand for the set of real, integers and natural numbers, respectively. Let $\mathbb{R}_+(\mathbb{Z}_+)$ be the set of non-negative real (integers) numbers and $\mathbb{T}_+$ be one of the two sets $\mathbb{R}_+$ or $\mathbb{Z}_+$. Also, we denote $\mathbb{N}_0$ by the set of all natural numbers excluding the number zero.

Let X be a metric space with metric ρ . Given $A \subset X$ and $r > 0$, the open r -neighborhood of A is the set defined by

$$B(A, r) = \{x \in X : \rho(x, A) < r\}.$$

The notations ∂A and $\overline{A}$ are used to denote the boundary and the closure of set A , respectively.

The **semi-distance of Hausdorff** from A to B , with A and B bounded subsets from X , is represented by

$$\text{dist}(A, B) = \sup\{\rho(a, B) : a \in A\}.$$

In the sequel, we present the definition of a semidynamical system in the classic sense.

Definition 1.1.1. A **semidynamical system** on X is a triple $(X, \pi, \mathbb{T}_+)$, where π is a map from the product space $X \times \mathbb{T}_+$ into the space X satisfying the following axioms:

- (i) $\pi(x, 0) = x$ for all $x \in X$;
- (ii) $\pi(\pi(x, t), s) = \pi(x, t + s)$ for all $x \in X$ and $t, s \in \mathbb{T}_+$ (semigroup property);
- (iii) $\pi : X \times \mathbb{T}_+ \rightarrow X$ is a continuous function.

If $\mathbb{T}_+ = \mathbb{R}_+$ then the triple $(X, \pi, \mathbb{R}_+)$ will be called a **continuous semidynamical system** and, meanwhile, if $\mathbb{T}_+ = \mathbb{Z}_+$ then we will call the triple $(X, \pi, \mathbb{Z}_+)$ as a **discrete system**.

From now on, we shall denote the system $(X, \pi, \mathbb{R}_+)$ simply by (X, π) and we will call it as a semidynamical system, that is, dropping the word continuous.

Given a discrete system $(X, \pi, \mathbb{Z}_+)$, we shall consider the continuous map

$$g = \pi(\cdot, 1) : X \rightarrow X$$

which satisfies $g^0 = Id$ and $g^n(x) = \pi(x, n)$ for all $x \in X$ and $n \in \mathbb{Z}_+$. Consequently, we write (X, g) instead of $(X, \pi, \mathbb{Z}_+)$.

Let (X, π) be a semidynamical system. The space X is called the **phase space** and the map π is called the **phase function**. The phase map determines two other maps, that is, for a fixed $x \in X$

we define the map $\pi_x : \mathbb{R} \rightarrow X$ by $\pi_x(t) = \pi(x, t)$, $t \in \mathbb{R}_+$, which is called the **motion** through x , and for a fix $t \in \mathbb{R}_+$, we define the map $\pi^t : X \rightarrow X$ by $\pi^t(x) = \pi(x, t)$, $x \in X$, which represents the **transition at time t** .

Remark 1.1.2. If every transition in a semidynamical system (X, π) is a homeomorphism, then we can define the map $\pi^{-t} = (\pi^t)^{-1}$, for $t < 0$, and we obtain a continuous dynamical system $(X, \pi, \mathbb{R})$.

Next, we give examples of classical semidynamical systems.

Example 1.1.3. (Ordinary autonomous differential equations) Consider the autonomous differential equation

$$\frac{dx}{dt} = f(x), \quad (1.1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function. Assume that for every $x \in \mathbb{R}^n$ there is a unique solution $\varphi(t, x)$ of (1.1) defined on $\mathbb{R}_+$ such that $\varphi(0, x) = x$ and

$$\varphi(t, \varphi(s, x)) = \varphi(t + s, x) \quad \text{for all } t, s \in \mathbb{R}_+.$$

The map $\pi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ given by $\pi(x, t) = \varphi(t, x)$, for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, defines a dynamical system on $\mathbb{R}^n$.

Example 1.1.4. (Ordinary non-autonomous differential equations) Consider the non-autonomous differential equation

$$\frac{dx}{dt} = f(t, x), \quad (1.2)$$

where $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous. Assume that for each $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there is a unique solution of (1.2) represented by $\varphi(t, t_0, x_0)$ defined on $\mathbb{R}_+$ such that $\varphi(t_0, t_0, x_0) = x_0$. By uniqueness, we obtain

$$\varphi(t + s_1 + s_2, t, x) = \varphi(t + s_1 + s_2, t + s_1, \varphi(t + s_1, t, x)), \quad t, s_1, s_2 \in \mathbb{R}_+, x \in \mathbb{R}^n.$$

Let $X = \mathbb{R}_+ \times \mathbb{R}^n$. Then the map $\pi : X \times \mathbb{R}_+ \rightarrow X$ given by

$$\pi((t, x), s) = (s + t, \varphi(s + t, t, x)), \quad s \in \mathbb{R}_+, (t, x) \in X,$$

defines a semidynamical system on X .

Example 1.1.5. (Bebutov semidynamical system) Let $C(\mathbb{R}, \mathbb{R})$ be the space of all continuous functions from $\mathbb{R}$ taking values in $\mathbb{R}$ endowed with the topology of uniform convergency on compacts in the following sense: $f_n \xrightarrow{n \rightarrow +\infty} f$ in $C(\mathbb{R}, \mathbb{R})$ if and only if for all compact interval $I \subset \mathbb{R}$, $f_n \xrightarrow{n \rightarrow +\infty} f$ uniformly in I .

Define the mapping $\pi : C(\mathbb{R}, \mathbb{R}) \times \mathbb{R}_+ \rightarrow C(\mathbb{R}, \mathbb{R})$ by $\pi(f, t) = f_t$ with $f \in C(\mathbb{R}, \mathbb{R})$ and $t \in \mathbb{R}_+$, where $f_t(s) = f(t + s)$ for all $s \in \mathbb{R}$. Note that π satisfies the following properties:

- a) $\pi(f, 0) = f$ for all $f \in C(\mathbb{R}, \mathbb{R})$;
- b) $\pi(\pi(f, t), s) = \pi(f, t + s)$ for all $f \in C(\mathbb{R}, \mathbb{R})$ and for all $t, s \in \mathbb{R}_+$.

Moreover, the map π is continuous (see [19, Lemma 1.1]) and consequently, $(C(\mathbb{R}, \mathbb{R}), \pi)$ is a semidynamical system on $C(\mathbb{R}, \mathbb{R})$. The semidynamical system $(C(\mathbb{R}, \mathbb{R}), \pi)$ is called *Bebutov* or *shift* semidynamical system.

Example 1.1.6. Consider a linear differential equation $\dot{x} = Ax$ in the Hilbert space $H = L_2[0, 1]$, with the continuous operator $A : L_2[0, 1] \rightarrow L_2[0, 1]$ defined by

$$(A\varphi)(\tau) = -\tau\varphi(\tau)$$

for all $\tau \in [0, 1]$ and $\varphi \in L_2[0, 1]$. Let $U(t)$ be given by $(U(t)\varphi)(\tau) = e^{-\tau t}\varphi(\tau)$ for all $t \in \mathbb{R}$ and $\varphi \in L_2[0, 1]$. Thus, the dynamical system generated by $\dot{x} = Ax$ is given by $(L_2[0, 1], \pi, \mathbb{R})$, where

$$\pi(\varphi, t) = U(t)\varphi$$

for all $\varphi \in L_2[0, 1]$ and $t \in \mathbb{R}$.

Definition 1.1.7. Let (X, π) be a semidynamical system and $x \in X$. The **positive orbit** of x is given by

$$\pi^+(x) = \{\pi_x(t) : t \in \mathbb{R}_+\}.$$

Given $A \subset X$ and $\Delta \subset \mathbb{R}$, we define

$$\pi^+(A) = \{\pi^+(x) : x \in A\} \quad \text{and} \quad \pi(A, \Delta) = \{\pi(x, t) : x \in A, t \in \Delta\}.$$

Definition 1.1.8. Let (X, π) be a semidynamical system. For $t \geq 0$ and $x \in X$, we write $F(x, t) = \{y \in X : \pi(y, t) = x\}$ and, for $\Delta \subset [0, +\infty)$ and $D \subset X$, we define

$$F(D, \Delta) = \{F(x, t) : x \in D \text{ and } t \in \Delta\}.$$

1.2 Impulsive semidynamical systems

We devote this section to present the definition of impulsive semidynamical systems. The first formal definition appeared in [31] in 1990. The description of this system was improved throughout the years in [23, 24, 25] and posteriorly in [3]. In the sequel, we present the construction of an impulsive system.

Definition 1.2.1. An **impulsive semidynamical system** $(X, \pi; M, I)$, or simply **ISS**, consists of a semidynamical system (X, π) , a closed nonempty subset $M \subset X$ such that for every $x \in M$, there exists $\epsilon_x > 0$ satisfying

$$F(x, (0, \epsilon_x)) \cap M = \emptyset \quad \text{and} \quad \pi(x, (0, \epsilon_x)) \cap M = \emptyset,$$

and a continuous function $I : M \rightarrow X$ whose action we explain below in the description of the impulsive semitrajectory.

Figure 1.1: The trajectory of π through x is, in some sense, transversal to M .

The set M is called the **impulsive set** and the function I is called the **impulse function**. Denote

$$M^+(x) = \left(\bigcup_{t>0} \pi(x, t) \right) \cap M, \quad x \in X.$$

The first result concerns the existence of the smallest number $s > 0$ such that the trajectory $\pi(x, t)$ for $0 < t < s$ does not intercept the set M .

Lemma 1.2.2. [2, Lemma 2.1] *Let $(X, \pi; M, I)$ be an ISS. Assume that $M^+(x) \neq \emptyset$ for some $x \in X$. Then there is a number $s > 0$ such that $\pi(x, t) \notin M$ for $0 < t < s$ and $\pi(x, s) \in M$.*

By means of Lemma 1.2.2 it is possible to define a function ϕ in the following manner:

$$\phi(x) = \begin{cases} s, & \text{if } \pi(x, s) \in M \text{ and } \pi(x, t) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } M^+(x) = \emptyset. \end{cases} \quad (1.3)$$

The number $\phi(x)$, $x \in X$, is the least positive time for which the trajectory of x meets M . Thus for each $x \in X$, we call $\pi(x, \phi(x))$ the *impulsive point* of x .

Definition 1.2.3. The **impulsive semitrajectory** of x in $(X, \pi; M, I)$ is an X -valued function $\tilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such trajectory follows inductively in the following manner: if $M^+(x) = \emptyset$, then $\tilde{\pi}_x(t) = \pi(x, t)$ for all $t \in \mathbb{R}_+$ and $\phi(x) = +\infty$ (this means that the continuous and the impulsive trajectories are the same). However, if $M^+(x) \neq \emptyset$ then $\phi(x) < +\infty$, $\pi(x, \phi(x)) = x_1 \in M$ and $\pi(x, t) \notin M$ for $0 < t < \phi(x)$. In this case, we define $\tilde{\pi}_x$ on $[0, \phi(x)]$ by

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x, t), & 0 \leq t < \phi(x) \\ x_1^+, & t = \phi(x), \end{cases}$$

where $x_1^+ = I(x_1)$. Let us denote x by x_0^+ .

Since $\phi(x) < +\infty$, the process now continues from x_1^+ onwards. If $M^+(x_1^+) = \emptyset$, then we define $\tilde{\pi}_x(t) = \pi(x_1^+, t - \phi(x))$, for $\phi(x) \leq t < +\infty$, and $\phi(x_1^+) = +\infty$. When $M^+(x_1^+) \neq \emptyset$, it follows that $\phi(x_1^+) < +\infty$, $\pi(x_1^+, \phi(x_1^+)) = x_2 \in M$ and $\pi(x_1^+, t - \phi(x)) \notin M$ for $\phi(x) < t < \phi(x) + \phi(x_1^+)$. Then we define $\tilde{\pi}_x$ on $[\phi(x), \phi(x) + \phi(x_1^+)]$ by

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - \phi(x)), & \phi(x) \leq t < \phi(x) + \phi(x_1^+) \\ x_2^+, & t = \phi(x) + \phi(x_1^+), \end{cases}$$

where $x_2^+ = I(x_2)$, and so on. Notice that $\tilde{\pi}_x$ is defined on each interval $[t_n(x), t_{n+1}(x)]$, where $t_0(x) = 0$ and $t_{n+1}(x) = \sum_{i=0}^n \phi(x_i^+)$, $n = 0, 1, 2, \dots$. Hence, $\tilde{\pi}_x$ is defined on $[0, t_{n+1}(x)]$.

The process above ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some $n \in \mathbb{N}$. However, it continues infinitely if $M^+(x_n^+) \neq \emptyset$ for all $n \in \mathbb{N}$, and in this case $\tilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{+\infty} \phi(x_i^+)$.

The Figure 1.2 exhibits a representation of an impulsive trajectory of the point $x \in X$.

Figure 1.2: Impulsive trajectory of $x \in X$.

Next, we present two examples of ISS.

Example 1.2.4. Let $(\mathbb{R}, \pi)$ be a semidynamical system given by

$$\pi(x, t) = x + t, \quad x \in \mathbb{R} \text{ and } t \in \mathbb{R}_+,$$

$M = \{1\}$ and $I : M \rightarrow \mathbb{R}$ be given by $I(1) = 0$. Thus, $(\mathbb{R}, \pi; M, I)$ is an ISS. Figure 1.3 shows the trajectory of $A = [0, 1)$.

Figure 1.3: Impulsive trajectory of $A = [0, 1)$.

Example 1.2.5. Let $(\mathbb{R}^2, \pi; M, I)$ be an ISS, where $(\mathbb{R}^2, \pi)$ is given by

$$\pi((x, y), t) = (x + t, y), \quad (x, y) \in \mathbb{R}^2 \text{ and } t \geq 0,$$

$M = \{(x, y) \in \mathbb{R}^2 : x = 2\}$ and the impulse function $I : M \rightarrow X$ is defined by $I(x, y) = \left(0, \frac{y}{2}\right)$, $(x, y) \in M$. If $p \in \{(x, y) \in \mathbb{R}^2 : x \geq 1\}$ then the impulsive trajectory of p is equal to its continuous trajectory because $M^+(p) = \emptyset$. However, if $q \in \{(x, y) \in \mathbb{R}^2 : x < 1\}$ then $M^+(q_n^+) \neq \emptyset$ for every $n = 0, 1, 2, \dots$, and the trajectory of q suffers an infinite number of impulses. See Figure 1.4.

Figure 1.4: Impulsive trajectory of q .

Definition 1.2.6. The **impulsive positive orbit** of a point $x \in X$ is given by

$$\tilde{\pi}^+(x) = \{\tilde{\pi}(x, t) : t \in [0, T(x))\}.$$

For $A \subset X$ and $\Delta \subset \mathbb{R}_+$, we define

$$\tilde{\pi}^+(A) = \{\tilde{\pi}^+(x) : x \in A\} \quad \text{and} \quad \tilde{\pi}(A, \Delta) = \{\tilde{\pi}(x, t) : x \in A, t \in \Delta\}.$$

Remark 1.2.7. Let $x \in X$. Throughout this text, we shall denote

$$t_0(x) = 0, \quad x_0^+ = x \quad \text{and} \quad t_n(x) = \sum_{i=0}^{n-1} \phi(x_i^+), \quad n = 1, 2, \dots$$

Moreover, we have

$$\tilde{\pi}(x, t_n(x)) = x_n^+ \quad \text{for all} \quad n \in \mathbb{Z}_+.$$

In the next remark, we describe a technical argument that will be often used in this work.

Remark 1.2.8. Let $x \in X$ and $t \in [0, T(x))$ be given. By the construction of an impulsive system, one may obtain $k \in \mathbb{N}$ such that $t_k(x) \leq t < t_{k+1}(x)$. Taking $s = t - t_k(x)$, we get

$$\tilde{\pi}(x, t) = \pi(x_k^+, s).$$

Impulsive systems also satisfy the compatibility conditions as semidynamical systems.

Proposition 1.2.9. [3, Proposition 2.1] *Let $(X, \pi; M, I)$ be an ISS. Then:*

- (i) $\tilde{\pi}(x, 0) = x$ for all $x \in X$;
- (ii) $\tilde{\pi}(\tilde{\pi}(x, t), s) = \tilde{\pi}(x, t + s)$ for all $x \in X$ and $t, s \in [0, T(x))$ such that $t + s \in [0, T(x))$.

Proof. Let $x \in X$. If $\phi(x) = +\infty$ then $\tilde{\pi}_x$ coincides with π_x and the result follows. Now, let us assume that $\phi(x) < +\infty$. Note that item (i) is an immediate consequence of the definition of $\tilde{\pi}_x$. Let $t, s \in [0, T(x))$ be such that $t + s \in [0, T(x))$. Taking $y = \tilde{\pi}(x, t)$, by Remark 1.2.8, there is $k \in \mathbb{N}$ such that $t = t_k(x) + t'$, where $0 \leq t' < \phi(x_k^+)$, and we may write

$$y = \tilde{\pi}(x, t) = \pi(x_k^+, t').$$

Analogously, there is $l \in \mathbb{N}$ such that $s = t_l(y) + s'$, where $0 \leq s' < \phi(y_l^+)$, and we have

$$\tilde{\pi}(y, s) = \pi(y_l^+, s').$$

Since $y = \pi(x_k^+, t')$, we conclude

$$\phi(y) = \phi(x_k^+) - t', \quad y_j^+ = x_{k+j}^+ \quad \text{and} \quad \phi(y_j^+) = \phi(x_{k+j}^+),$$

for all $j \in \mathbb{N}$, that is,

$$t_l(y) = \sum_{j=k}^{k+l-1} \phi(x_j^+) - t'.$$

Then

$$t + s = t_k(x) + t' + t_l(y) + s' = \sum_{j=0}^{k-1} \phi(x_j^+) + t' + \sum_{j=k}^{k+l-1} \phi(x_j^+) + s' - t' = t_{k+l}(x) + s'.$$

Since $0 \leq s' < \phi(y_l^+) = \phi(x_{k+l}^+)$, we conclude that

$$\tilde{\pi}(\tilde{\pi}(x, t), s) = \tilde{\pi}(y, s) = \tilde{\pi}(y_l^+, s') = \pi(x_{k+l}^+, s') = \tilde{\pi}(x, t + s)$$

and the result is proved. □

1.3 Continuity of the function ϕ

This section deals with the continuity of the function ϕ defined in (1.3), which describes the times of meeting the impulsive set M . In [31], the upper semicontinuity of ϕ is proved and it is stated that lower semicontinuity also holds. But, Ciesielski in [23] showed that ϕ need not to be lower semicontinuous in X . In the sequel, we describe the results of semicontinuity of ϕ .

Definition 1.3.1. A function $f : X \rightarrow \mathbb{R}$ is called **upper (lower) semicontinuous** at a point $a \in X$, when for each $\epsilon > 0$ there is $\delta > 0$ such that if $\rho(x, a) < \delta$, for $x \in X$, then

$$f(x) < f(a) + \epsilon \quad (f(x) > f(a) - \epsilon).$$

We say that f is **upper (lower) semicontinuous** when f is upper (lower) semicontinuous at every point of X .

Remark 1.3.2. In terms of sequences, a function $f : X \rightarrow \mathbb{R}$ is upper (lower) semicontinuous at a point $a \in X$, if for each sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ with $x_n \xrightarrow{n \rightarrow +\infty} a$, we have

$$\limsup_{n \rightarrow +\infty} f(x_n) \leq f(a) \quad \left(\liminf_{n \rightarrow +\infty} f(x_n) \geq f(a) \right).$$

Moreover, the function f is **continuous** in X if and only if it is upper and lower semicontinuous.

Firstly, we consider an example of an impulsive system such that its function ϕ is not lower semicontinuous in M .

Example 1.3.3. Let $(\mathbb{R}, \pi; M, I)$ be an ISS on $\mathbb{R}$, where $\pi(x, t) = x + t$ for all $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$, $M = \mathbb{N}$ and $I(n) = n + \frac{1}{2}$ for all $n \in \mathbb{N}$. Then the function ϕ is given by

$$\phi(x) = \begin{cases} -x, & x < 0, \\ 1 - x + E(x), & x \geq 0, \end{cases}$$

where $E(x)$ denotes the integer part of $x \in \mathbb{R}$. Note that ϕ is not lower semicontinuous in M , because for each $n \in \mathbb{N}_0$ we have

$$\liminf_{x \rightarrow n^-} \phi(x) = \liminf_{x \rightarrow n^-} (1 - x + E(x)) = 0 < 1 = \phi(n)$$

and

$$\liminf_{x \rightarrow 0^-} \phi(x) = \liminf_{x \rightarrow 0^-} (-x) = 0 < 1 = \phi(0).$$

However, ϕ is lower semicontinuous in $X \setminus M$ as presented in the next result.

Theorem 1.3.4. [23, Theorem 2.7] *Let $(X, \pi; M, I)$ be an ISS. Then for any $x \in X \setminus M$ the function ϕ is lower semicontinuous at x .*

Proof. Let $x \in X \setminus M$. If $\phi(x) = +\infty$ the result follows. Assume that $\phi(x) = c \in (0, +\infty)$ and let $\{x_n\}_{n \geq 1} \subset X$ be a sequence such that $x_n \xrightarrow{n \rightarrow +\infty} x$ and $\phi(x_n) \xrightarrow{n \rightarrow +\infty} t$. Since M is closed, we have $x_n \notin M$ for n large enough. On the other hand, $\pi(x_n, \phi(x_n)) \in M$ for all $n \in \mathbb{N}$ and by the continuity of π in $X \setminus M$ we conclude that

$$\pi(x_n, \phi(x_n)) \xrightarrow{n \rightarrow +\infty} \pi(x, t) \in M,$$

which means that $t \geq \phi(x)$ and the proof is finished. $\square$

Theorem 1.3.5. [23, Proposition 3.6] *Let $(X, \pi; M, I)$ be an ISS and assume that $x \in M$ is not an initial point. Then ϕ is not lower semicontinuous at x .*

Proof. Take $\epsilon > 0$ and $y \in X$ such that $\pi(y, \epsilon) = x$ and $\pi(y, [0, \epsilon)) \cap M = \emptyset$ (this can be done as the points of M are isolated on the trajectory). Now, let $\{\epsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $0 < \epsilon_n < \epsilon$ for all $n \in \mathbb{N}$ and $\epsilon_n \xrightarrow{n \rightarrow +\infty} 0$. Taking $y_n = \pi(y, \epsilon_n)$ for each $n \in \mathbb{N}$, we obtain $y_n \xrightarrow{n \rightarrow +\infty} y$ and

$$\phi(y_n) = \phi(\pi(y, \epsilon_n)) = \epsilon - \epsilon_n \xrightarrow{n \rightarrow +\infty} 0 < \phi(x).$$

Therefore, ϕ is not lower semicontinuous at x . $\square$

To deal with the upper semicontinuity of ϕ we need to understand the behavior of the impulsive trajectory around M . For that, we make use of “sections” which is a very good tool since it is possible to precisely describe the behavior of a neighborhood of any non-stacionary point.

Definition 1.3.6. A closed set S containing $x \in X$ is called a **section** or **λ -section** through x ($\lambda > 0$), if there is a closed set L such that:

- (i) $F(L, \lambda) = S$;
- (ii) $F(L, [0, 2\lambda])$ is a neighborhood of x ;
- (iii) $F(L, \mu) \cap F(L, \eta) = \emptyset$, for $0 \leq \mu < \eta \leq 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a **tube** (or λ -**tube**) and the set L is called a **bar**. See Figure 1.5.

Figure 1.5: λ -tube $F(L, [0, 2\lambda])$.

In [22], Ciesielski establishes the existence of sections for semidynamical systems and he also presents some properties about sections. The next lemma says that we can reduce the size of a tube.

Lemma 1.3.7. [22, Lemma 1.9] Let (X, π) be a semidynamical system, $\mu \leq \lambda$ and $x \in X$. If S is a λ -section through x , then S is a μ -section through x .

Definition 1.3.8. We say that a tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ is a **TC-tube** if

$$S \subset M \cap F(L, [0, 2\lambda]).$$

A point $x \in M$ satisfies the **Tube Condition** (TC) if there is a TC-tube through x . In particular, if

$$S = M \cap F(L, [0, 2\lambda]),$$

the tube $F(L, [0, 2\lambda])$ given by the section S through $x \in M$ will be called a **STC-tube**. We say that a point $x \in X$ fulfills the **Strong Tube Condition** (STC) if there exists a STC-tube through x .

Example 1.3.9. [23, Example 2.5] Consider the semidynamical system on $\mathbb{R}^2$ given by

$$\pi((x, y), t) = (x + t, y),$$

for $(x, y) \in \mathbb{R}^2$ and $t \in \mathbb{R}_+$. Let $M = \{(x, y) \in \mathbb{R}^2 : x = 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = y, x \geq 0\}$. The point $(0, 0)$ satisfies the tube condition (TC) but it does not satisfy the strong tube condition (STC).

Figure 1.6: $(0, 0)$ satisfies TC but not STC.

As a consequence of Lemma 1.3.7, we have the following result.

Lemma 1.3.10. [23, Lemma 3.3] *Let $x \in X$ be a point satisfying TC (or STC) with a λ -section S . Then for any $\eta < \lambda$ the set S is a η -section with a TC-tube (STC-tube).*

The next theorem concerns the upper semicontinuity of ϕ in X .

Theorem 1.3.11. [23, Theorem 3.4] *Let $(X, \pi; M, I)$ be an ISS such that each point in M satisfies TC. Then ϕ is upper semicontinuous in X .*

Proof. Let $x \in X$. If $\phi(x) = +\infty$ then the result is immediate. Let us suppose that $\phi(x) < +\infty$. Set $\phi(x) = u$, then $\pi(x, u) = y \in M$ and $\pi(x, (0, u)) \cap M = \emptyset$. By Lemma 1.3.10, we may choose $\epsilon < u$ such that $U = F(L, [0, 2\epsilon])$ is a TC-tube through y given by the ϵ -section $F(L, \epsilon)$. Then there is a neighborhood V of x such that $\pi(V, u) \subset U$. Thus, for all $z \in V$, we have $\pi(z, u) \in F(L, [0, 2\epsilon])$. Moreover, for each $z \in V$ there is $\eta_z \in [0, 2\epsilon]$ such that $\pi(z, u + \eta_z) \in L$. Since $\epsilon < u$, it follows that $u - \epsilon + \eta_z > 0$ and

$$\pi(z, u - \epsilon + \eta_z) \in F(L, \epsilon) = S \subset M,$$

that is, $\phi(z) \leq u + \eta_z - \epsilon < u + \epsilon = \phi(x) + \epsilon$. Therefore, ϕ is upper semicontinuous at x . $\square$

The next theorem summarizes the previous results and we conclude the continuity of ϕ in $X \setminus M$.

Theorem 1.3.12. [23, Teorema 3.4] *Let $(X, \pi; M, I)$ be an ISS. Suppose that no initial point of the system (X, π) belongs to the impulsive set M and M satisfies TC. Then the function ϕ is continuous at x if and only if $x \in X \setminus M$.*

In [14], the authors present sufficient conditions to construct impulsive sets in $\mathbb{R}^n$ satisfying STC. In other words, given an impulsive dynamical system generated by the system

$$\begin{cases} \dot{x} = f(x), \\ x(0) = x_0, \\ I: M \rightarrow \mathbb{R}^n, \end{cases} \quad (1.4)$$

with $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$, the authors assume that M is a smooth hypersurface in $\mathbb{R}^n$ such that for each $x \in M$ we have $\langle \vec{n}_x, f(x) \rangle \neq 0$, where $\vec{n}_x$ denotes the normal vector of M at x and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$. Under these conditions, the set M is an impulsive set satisfying STC, see [14, Theorem 9].

1.4 Convergence of the function $\tilde{\pi}$

The continuity of the mapping π is necessary for many results from the classical theory of semidynamical systems without impulse effects. In general, the new phase map $\tilde{\pi}$ of an impulsive semidynamical system is not a continuous function. In this way, we dedicate this section by presenting results about convergence of $\tilde{\pi}$. Most of the results of this section are technical and will be very important for the development of the main results. The reader may consult [12, 6] and [15] for more details.

Let $(X, \pi; M, I)$ be an ISS. Throughout this work, we shall assume the following general conditions:

- (H1) No initial point in (X, π) belongs to the impulsive set M and each element of M satisfies STC, consequently ϕ is continuous on $X \setminus M$;
- (H2) $M \cap I(M) = \emptyset$;
- (H3) For each $x \in X$, the motion $\tilde{\pi}(x, t)$ is defined for every $t \geq 0$.

The above conditions are necessary to obtain qualitative properties for an ISS. The condition (H1) controls how the orbits meet or leave the impulse set M and concerns with the continuity of

the function ϕ in $X \setminus M$. The condition **(H2)** says that an impulsive trajectory can only intercept M when this trajectory start in M and it occurs at $t = 0$. The last condition is important to study the asymptotic behavior of orbits.

In the next remark, we observe how the condition **(H1)** will be used hereafter.

Remark 1.4.1. Let $y \in M$. Since M satisfies STC, see condition **(H1)**, there is a STC-tube $F(L_y, [0, 2\lambda_y])$ through y given by a section S_y . Moreover, since the tube is a neighborhood of y , there is $\eta_y > 0$ such that

$$B(y, \eta_y) \subset F(L_y, [0, 2\lambda_y]).$$

From now on, we shall denote

$$H_1^{(y)} = F(L_y, (\lambda_y, 2\lambda_y]) \cap B(y, \eta_y) \quad \text{and} \quad H_2^{(y)} = F(L_y, [0, \lambda_y]) \cap B(y, \eta_y).$$

Figure 1.7: The set M satisfies STC.

In what follows, we discuss the convergence of $\tilde{\pi}$ on $(X \setminus M) \times \{0\}$.

Lemma 1.4.2. [12, Lemma 3.8] *Let $z \in X \setminus M$ and $\{z_n\}_{n \in \mathbb{N}}$ be a sequence in $X \setminus M$ such that $z_n \xrightarrow{n \rightarrow +\infty} z$. If $\alpha_n \xrightarrow{n \rightarrow +\infty} 0$ and $\alpha_n \geq 0$, for all $n \in \mathbb{N}$, then $\tilde{\pi}(z_n, \alpha_n) \xrightarrow{n \rightarrow +\infty} z$.*

Proof. Since ϕ is continuous on $X \setminus M$, there is $n_0 \in \mathbb{N}$ such that $\frac{\phi(z)}{2} < \phi(z_n) < \frac{3\phi(z)}{2}$ for all $n \geq n_0$. Since $\alpha_n \xrightarrow{n \rightarrow +\infty} 0$ and $\alpha_n \geq 0$, we may assume that $0 \leq \alpha_n < \phi(z_n)$ for all $n \in \mathbb{N}$. Thus,

by the continuity of π , we have

$$\tilde{\pi}(z_n, \alpha_n) = \pi(z_n, \alpha_n) \xrightarrow{n \rightarrow +\infty} \pi(z, 0) = z$$

and the proof is complete. $\square$

The next lemma investigates the convergence of $\tilde{\pi}$ in $(X \setminus M) \times \mathbb{R}_+$.

Lemma 1.4.3. [12, Corollary 3.9] *Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X which converges to $x \in X \setminus M$. Then, given $t \geq 0$, there is a sequence $\{\epsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\epsilon_n \xrightarrow{n \rightarrow +\infty} 0$ and*

$$\tilde{\pi}(x_n, t + \epsilon_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(x, t).$$

Proof. First, let us assume that $\phi(x) = +\infty$. By the continuity of ϕ on $X \setminus M$, given $t \geq 0$, there is a natural number $n_0 \in \mathbb{N}$ such that $\phi(x_n) > t$ for all $n \geq n_0$. Then

$$\tilde{\pi}(x_n, t) = \pi(x_n, t) \xrightarrow{n \rightarrow +\infty} \pi(x, t) = \tilde{\pi}(x, t).$$

In this case, we take the sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ with $\epsilon_n = 0$ for all $n \in \mathbb{N}$.

Now, let us assume that $\phi(x) = +\infty$ and let $t \geq 0$ be fixed. There exists $k \in \mathbb{Z}_+$ and $0 \leq s < \phi(x_k^+)$ such that $t = t_k(x) + s$. By the continuities of π in $X \times \mathbb{R}_+$, ϕ in $X \setminus M$ and I in M , we get

$$\phi((x_n)_j^+) \xrightarrow{n \rightarrow +\infty} \phi((x)_j^+), \quad \text{for all } j \in \mathbb{Z}_+,$$

consequently, $t_k(x_n) \xrightarrow{n \rightarrow +\infty} t_k(x)$. Define the sequence $\{\epsilon_n\}_{n \geq 1} \subset \mathbb{R}_+$ by

$$\epsilon_n = t_k(x_n) - t_k(x) + |t_k(x_n) - t_k(x)|, \quad n \in \mathbb{N}.$$

Then $\epsilon_n \xrightarrow{n \rightarrow +\infty} 0$ as $(t_k(x_n) - t_k(x)) \xrightarrow{n \rightarrow +\infty} 0$. Moreover, $s < \phi((x)_k^+)$ and

$$\phi((x_n)_k^+) \xrightarrow{n \rightarrow +\infty} \phi((x)_k^+).$$

Thus we may assume that $s < \phi((x_n)_k^+)$ for all $n \in \mathbb{N}$. Therefore, by Lemma 1.4.2, we conclude that

$$\begin{aligned} \tilde{\pi}(x_n, t + \epsilon_n) &= \tilde{\pi}(x_n, t_k(x) + s + t_k(x_n) - t_k(x) + |t_k(x_n) - t_k(x)|) \\ &= \tilde{\pi}(\pi((x_n)_k^+, s), |t_k(x_n) - t_k(x)|) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(\pi(x_k^+, s), 0) = \tilde{\pi}(x, t), \end{aligned}$$

and the result is proved. $\square$

We note that if $t \geq 0$ is not a jump instant of a point x , that is, $\tilde{\pi}(x, t) \neq x_j^+$ for every $j = 1, 2, 3, \dots$, then the convergence in Lemma 1.4.3 does not depend on the sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$. Thus, $\tilde{\pi}(x_n, t) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(x, t)$ whenever $t \neq t_k(x)$, for every $k = 1, 2, \dots$. We formalize this fact in the next lemma.

The Lemmas 1.4.4, 1.4.5 and 1.4.6 are original and they are contained on section 2.2 from [15].

Lemma 1.4.4. *Let $\{x_n\}_{n \in \mathbb{N}} \subset X$ be a sequence in X which converges to $x \in X \setminus M$. Given $t \geq 0$ such that $t \neq t_k(x)$, $k = 1, 2, \dots$, and $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ is a sequence with $\lambda_n \xrightarrow{n \rightarrow +\infty} t$ then $\tilde{\pi}(x_n, \lambda_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(x, t)$.*

Proof. If $t = 0$ the result follows by Lemma 1.4.2. Let $k \in \{0, 1, 2, \dots\}$ be such that

$$t_k(x) < t < t_{k+1}(x).$$

Since $\phi((x_n)_j^+) \xrightarrow{n \rightarrow +\infty} \phi(x_j^+)$ for each $j = 0, 1, 2, \dots$, and $\lambda_n \xrightarrow{n \rightarrow +\infty} t$, we may assume that $\lambda_n \in (t_k(x), t_{k+1}(x))$ for all $n \in \mathbb{N}$. Then

$$\tilde{\pi}(x_n, \lambda_n) = \pi((x_n)_k^+, \lambda_n - t_k(x_n)) \xrightarrow{n \rightarrow +\infty} \pi(x_k^+, t - t_k(x)) = \tilde{\pi}(x, t)$$

and we conclude the proof. $\square$

Lemma 1.4.5. *Let $x \in X \setminus M$ and $t \geq 0$. Assume that the sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ is such that $\lambda_n \geq t$, for every $n \in \mathbb{N}$, and $\lambda_n \xrightarrow{n \rightarrow +\infty} t$. If $\{x_n\}_{n \in \mathbb{N}} \subset X$ is a sequence which converges to x then there is a sequence $\{\beta_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\beta_n \xrightarrow{n \rightarrow +\infty} 0$ and*

$$\tilde{\pi}(x_n, \lambda_n + \beta_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(x, t).$$

Proof. If $t \neq t_k(x)$, for every $k = 1, 2, \dots$, then in virtue of Lemma 1.4.2, we get the result. However, if $t = t_k(x)$ for some $k \in \{1, 2, \dots\}$, then since $\lambda_n \geq t$ for every $n \in \mathbb{N}$, we can write $\lambda_n = t + s_n$ with $s_n \geq 0$ and $s_n \xrightarrow{n \rightarrow +\infty} 0$. Now, since $\phi((x_n)_j^+) \xrightarrow{n \rightarrow +\infty} \phi(x_j^+)$ for all $j = 0, 1, 2, \dots$, we have

$$t_k(x_n) \xrightarrow{n \rightarrow +\infty} t_k(x).$$

Define $T_n = t_k(x_n) - t_k(x)$, $n = 1, 2, \dots$. Thus,

$$\lambda_n = t_k(x_n) - T_n + s_n, \quad n = 1, 2, \dots$$

Taking $\beta_n = |T_n|$, $n = 1, 2, \dots$, and using Lemma 1.4.2 we obtain

$$\begin{aligned}\tilde{\pi}(x_n, \lambda_n + \beta_n) &= \tilde{\pi}(x_n, t_k(x_n) - T_n + |T_n| + s_n) = \tilde{\pi}(\tilde{\pi}(x_n, t_k(x_n)), |T_n| - T_n + s_n) \\ &= \tilde{\pi}((x_n)_k^+, |T_n| - T_n + s_n) \xrightarrow{n \rightarrow +\infty} \pi(x_k^+, 0) = \tilde{\pi}(x, t).\end{aligned}$$

□

Lemma 1.4.6. *Let $x \in X \setminus M$ and $t = t_k(x)$, for some $k \in \mathbb{N}$. Suppose $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ is a sequence which converges to t and $\{x_n\}_{n \in \mathbb{N}} \subset X$ is a sequence which converges to x .*

- a) *If $\lambda_n < t$ for all $n \in \mathbb{N}$ then $\tilde{\pi}(x_n, \lambda_n) \xrightarrow{n \rightarrow +\infty} x_k$.*
- b) *If $\lambda_n \geq t$ for all $n \in \mathbb{N}$ then $\{\tilde{\pi}(x_n, \lambda_n)\}_{n \in \mathbb{N}}$ possesses a subsequence which converges in $\overline{\tilde{\pi}^+(x)}$.*

Proof. a) Since $\lambda_n < t$ for every $n \in \mathbb{N}$, then there are a sequence $\{s_n\}_{n \in \mathbb{N}} \subset [0, \phi((x_n)_{k-1}^+))$ with $s_n \xrightarrow{n \rightarrow +\infty} \phi(x_{k-1}^+)$ and $n_0 \in \mathbb{N}$ such that $\lambda_n = t_{k-1}(x_n) + s_n$ for all $n \geq n_0$. Then,

$$\tilde{\pi}(x_n, \lambda_n) = \pi((x_n)_{k-1}^+, s_n) \xrightarrow{n \rightarrow +\infty} \pi(x_{k-1}^+, \phi(x_{k-1}^+)) = x_k.$$

b) Following the proof of Lemma 1.4.3 for $t = t_k(x)$, if $\{s_n - T_n\}_{n \in \mathbb{N}}$ admits a subsequence $\{s_{n_\ell} - T_{n_\ell}\}_{\ell \in \mathbb{N}}$ such that $s_{n_\ell} - T_{n_\ell} \geq 0$ for all $\ell \in \mathbb{N}$ then, using Lemma 1.4.2, we have

$$\tilde{\pi}(x_{n_\ell}, \lambda_{n_\ell}) = \tilde{\pi}(\tilde{\pi}(x_{n_\ell}, t_k(x_{n_\ell})), -T_{n_\ell} + s_{n_\ell}) \xrightarrow{\ell \rightarrow +\infty} x_k^+ \in \overline{\tilde{\pi}^+(x)}.$$

However, if $\{s_n - T_n\}_{n \in \mathbb{N}}$ admits a subsequence $\{s_{m_\ell} - T_{m_\ell}\}_{\ell \in \mathbb{N}}$ such that $s_{m_\ell} - T_{m_\ell} < 0$ for all $\ell \in \mathbb{N}$ then we write $\lambda_{m_\ell} = t_{k-1}(x_{m_\ell}) + \phi((x_{m_\ell})_{k-1}^+) - T_{m_\ell} + s_{m_\ell}$, $\ell = 1, 2, \dots$. Note that $\phi((x_{m_\ell})_{k-1}^+) - T_{m_\ell} + s_{m_\ell} > 0$ for ℓ sufficiently large. Then

$$\begin{aligned}\tilde{\pi}(x_{m_\ell}, \lambda_{m_\ell}) &= \tilde{\pi}(\tilde{\pi}(x_{m_\ell}, t_{k-1}(x_{m_\ell})), \phi((x_{m_\ell})_{k-1}^+) - T_{m_\ell} + s_{m_\ell}) \\ &= \tilde{\pi}((x_{m_\ell})_{k-1}^+, \phi((x_{m_\ell})_{k-1}^+) - T_{m_\ell} + s_{m_\ell}) \xrightarrow{\ell \rightarrow +\infty} \pi(x_{k-1}^+, \phi(x_{k-1}^+)) = x_k \in \overline{\tilde{\pi}^+(x)}.\end{aligned}$$

□

We finish this section by giving conditions to obtain the relative compactness of $\tilde{\pi}(A, [0, \ell])$, for $\ell > 0$ and $A \subset X$. Lemma 1.4.7 will be used in the main results and its proof was done firstly in [6, Lemma 3.6]. However, using the results of this section we present a new proof that is simpler.

Lemma 1.4.7. [6, Lemma 3.6] *Let $A \subset X$ be nonempty and relatively compact. Then the set $\tilde{\pi}(A, [0, \ell])$ is relatively compact in X for each $\ell > 0$.*

Proof. Let $\{y_n\}_{n \in \mathbb{N}} \subset \tilde{\pi}(A, [0, \ell])$, then there are sequences $\{a_n\}_{n \in \mathbb{N}} \subset A$ and $\{s_n\}_{n \in \mathbb{N}} \subset [0, \ell]$ such that $y_n = \tilde{\pi}(a_n, s_n)$, $n \in \mathbb{N}$. Since $\overline{A}$ and $[0, \ell]$ are compact sets in X and in $\mathbb{R}_+$, respectively, we may suppose that

$$a_n \xrightarrow{n \rightarrow +\infty} a \quad \text{and} \quad s_n \xrightarrow{n \rightarrow +\infty} s,$$

with $a \in \overline{A}$ and $s \in [0, \ell]$. We have two cases to consider: $a \notin M$ and $a \in M$.

Case 1: $a \notin M$.

If $s \neq t_k(x)$, for all $k \in \mathbb{Z}_+$, using Lemma 1.4.4 we conclude that

$$y_n = \tilde{\pi}(a_n, s_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(a, s) \in \overline{\tilde{\pi}(A, [0, \ell])}.$$

Now, if $s = t_k(x)$, for some $k \in \mathbb{N}$, the result follows by Lemma 1.4.6.

Case 2: $a \in M$.

Since M satisfies STC, there is a STC-tube $F(L_a, [0, 2\lambda_a])$ through a given by a section S_a . Since the tube is a neighborhood of a , there is $\eta_a > 0$, such that

$$B(a, \eta_a) \subset F(L_a, [0, 2\lambda_a]),$$

and by Remark 1.4.1 we may consider the sets

$$H_1^{(a)} = F(L_a, (\lambda_a, 2\lambda_a]) \cap B(a, \eta_a) \quad \text{and} \quad H_2^{(a)} = F(L_a, [0, \lambda_a]) \cap B(a, \eta_a).$$

In this case, we need to study when $a_n \in H_1^{(a)}$ for infinitely many n and when $a_n \in H_2^{(a)}$ for infinitely many n . We are going to consider without loss of generality the cases: when $\{a_n\}_{n \in \mathbb{N}} \subset H_1^{(a)}$ and when $\{a_n\}_{n \in \mathbb{N}} \subset H_2^{(a)}$.

First, let us assume that $\{a_n\}_{n \in \mathbb{N}} \subset H_2^{(a)}$. If $s < \phi(a)$ then $s_n < \phi(a_n)$ for n large enough. Thus

$$\tilde{\pi}(a_n, s_n) = \pi(a_n, s_n) \xrightarrow{n \rightarrow +\infty} \pi(a, s) = \tilde{\pi}(a, s).$$

But, if $s \geq \phi(a)$ we take $s_0 = \frac{\phi(a)}{2}$. Then

$$\tilde{\pi}(a_n, s_n) = \tilde{\pi}(\tilde{\pi}(a_n, s_0), s_n - s_0)$$

and the result follows by the Case 1.

Finally, let us assume that $\{a_n\}_{n \in \mathbb{N}} \subset H_1^{(a)}$. Then $\phi(a_n) \xrightarrow{n \rightarrow +\infty} 0$. We are going to consider without loss of generality the cases: $\lambda_n < \phi(a_n)$ and $\lambda_n \geq \phi(a_n)$ for $n \in \mathbb{N}$. If $\lambda_n < \phi(a_n)$, for $n \in \mathbb{N}$, then $\lambda_n \xrightarrow{n \rightarrow +\infty} 0$ and consequently $\tilde{\pi}(a_n, \lambda_n) = \pi(a_n, \lambda_n) \xrightarrow{n \rightarrow +\infty} a$. Now, if $\lambda_n \geq \phi(a_n)$, for $n \in \mathbb{N}$, we write

$$\tilde{\pi}(a_n, \lambda_n) = \tilde{\pi}(\tilde{\pi}(a_n, \phi(a_n)), \lambda_n - \phi(a_n)), \quad n \in \mathbb{N}. \quad (1.5)$$

Since $\{a_n\}_{n \in \mathbb{N}} \subset H_1^{(a)}$, we use the continuities of I in M and of π in $X \times \mathbb{R}$ and we obtain

$$\tilde{\pi}(a_n, \phi(a_n)) = I(\pi(a_n, \phi(a_n))) \xrightarrow{n \rightarrow +\infty} I(\pi(a, 0)) = I(a).$$

Note that $I(a) \in X \setminus M$ as we have condition **(H2)**. Now by equality (1.5), Lemma 1.4.5, Lemma 1.4.6 and Case 1, we conclude the result. $\square$

1.5 Invariant sets and positive limit sets

In this section, we define the concepts of invariant sets and positive limit sets in the context of impulsive systems.

Let $(X, \pi; M, I)$ be an ISS. The next definition deals with the concept of invariance in the context of impulsive systems.

Definition 1.5.1. We say that $A \subset X$ is **positively $\tilde{\pi}$ -invariant** if $\tilde{\pi}(A, t) \subset A$ for all $t \in \mathbb{R}_+$. The set A is called **$\tilde{\pi}$ -invariant** if $\tilde{\pi}(A, t) = A$ for all $t \in \mathbb{R}_+$.

Analogously to the continuous case presented in [1, Theorem 1.2], we have the next propositions for impulsive systems.

Proposition 1.5.2. *Let $\{A_i\}_{i \in I}$ be a collection of positively invariant subsets of X . Then their intersection and their union have the same property.*

Proposition 1.5.3. *Let $x \in X$ and $\ell \geq 0$. Then $\{\tilde{\pi}(x, t) : t \geq \ell\}$ is positively $\tilde{\pi}$ -invariant.*

In semidynamical systems without impulsive effects the closure of a positively invariant set is positively invariant too. The next example shows that this property on impulsive systems does not hold in general.

Example 1.5.4. Let $(\mathbb{R}^2, \pi; M, I)$ be the ISS given in Example 1.2.5. The impulsive trajectory of a point $q \in \mathbb{R}^2$ is presented in Figure 1.8.

Figure 1.8: Impulsive trajectory of q .

Note that $A = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x < 2\}$ is positively $\tilde{\pi}$ -invariant, but $\overline{A}$ is not positively $\tilde{\pi}$ -invariant. On the other hand, the set $\overline{A} \setminus M$ is positively $\tilde{\pi}$ -invariant.

Lemma 1.5.5 deals with the closure of a positively $\tilde{\pi}$ -invariant set by excluding the points from M . This is a new result and it is presented in [15, Lema 3.37] .

Lemma 1.5.5. *Let $B \subset X$ be a positively $\tilde{\pi}$ -invariant set. Then $\overline{B} \setminus M$ is positively $\tilde{\pi}$ -invariant.*

Proof. Let $b \in \overline{B} \setminus M$ and $t \geq 0$. Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subset B$ such that $x_n \xrightarrow{n \rightarrow +\infty} b$. By Lemma 1.4.3, there is a sequence $\{\epsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\epsilon_n \xrightarrow{n \rightarrow +\infty} 0$ and

$$\tilde{\pi}(x_n, t + \epsilon_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(b, t).$$

Since $\{\tilde{\pi}(x_n, t + \epsilon_n)\}_{n \in \mathbb{N}} \subset B$ we conclude that $\tilde{\pi}(b, t) \in \overline{B} \setminus M$ as we have condition **(H2)**. $\square$

In what follows, we present the definition of a positive limit set.

Definition 1.5.6. The **positive limit set**, the **positive prolongation limit set** and the **positive prolongation set** of a subset $A \subset X$ are given, respectively, by

$$\tilde{L}^+(A) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \tilde{\pi}(A, \tau)}, \quad \tilde{J}^+(A) = \bigcap_{\epsilon > 0} \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \tilde{\pi}(B(A, \epsilon), \tau)}$$

$$\text{and } \tilde{D}^+(A) = \bigcap_{\epsilon > 0} \overline{\bigcup_{t \geq 0} \tilde{\pi}(B(A, \epsilon), t)}.$$

For each $x \in X$, we set $\tilde{L}^+(x) = \tilde{L}^+(\{x\})$, $\tilde{J}^+(x) = \tilde{J}^+(\{x\})$ and $\tilde{D}^+(x) = \tilde{D}^+(\{x\})$. It is clear that $\tilde{L}^+(A)$, $\tilde{J}^+(A)$ and $\tilde{D}^+(A)$ are closed sets for all $A \subset X$. Moreover, by Lemma 1.5.5, we conclude that $\tilde{L}^+(A) \setminus M$, $\tilde{J}^+(A) \setminus M$ and $\tilde{D}^+(A) \setminus M$ are positively $\tilde{\pi}$ -invariant sets.

We have the following two straightforward results about impulsive limit sets.

Lemma 1.5.7. [6, Lemma 3.27] *Let $A \subset X$. The following statements hold:*

- (i) $y \in \tilde{L}^+(A)$ if and only if there are sequences $\{x_n\}_{n \in \mathbb{N}} \subset A$ and $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \xrightarrow{n \rightarrow +\infty} +\infty$ and $\tilde{\pi}(x_n, t_n) \xrightarrow{n \rightarrow +\infty} y$;
- (ii) $y \in \tilde{J}^+(A)$ if and only if there are sequences $\{x_n\}_{n \in \mathbb{N}} \subset X$ and $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\rho(x_n, A) \xrightarrow{n \rightarrow +\infty} 0$, $t_n \xrightarrow{n \rightarrow +\infty} +\infty$ and $\tilde{\pi}(x_n, t_n) \xrightarrow{n \rightarrow +\infty} y$;
- (iii) $y \in \tilde{D}^+(A)$ if and only if there are sequences $\{x_n\}_{n \in \mathbb{N}} \subset X$ and $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\rho(x_n, A) \xrightarrow{n \rightarrow +\infty} 0$ and $\tilde{\pi}(x_n, t_n) \xrightarrow{n \rightarrow +\infty} y$.

Lemma 1.5.8. [13, Lemma 4.15] *If $\tilde{L}^+(x) \neq \emptyset$ for some $x \in X$ then $\tilde{L}^+(x) \setminus M \neq \emptyset$.*

Next, we present an auxiliary result about the positive prolongation limit set.

Lemma 1.5.9. [6, Lemma 3.31] *Let $x \notin M$ and $y \in \tilde{L}^+(x)$, then $\tilde{J}^+(x) \subset \tilde{J}^+(y)$.*

Analogously to the continuous case, we have the next lemma for impulsive systems.

Lemma 1.5.10. [6, Lemma 3.5] *Let $A \subset X$. In the impulsive semidynamical system $(X, \pi; M, I)$ the following conditions are equivalent:*

- i) *for every sequence $\{x_n\}_{n \in \mathbb{N}} \subset A$ and $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \xrightarrow{n \rightarrow +\infty} +\infty$, the sequence $\{\tilde{\pi}(x_n, t_n)\}_{n \in \mathbb{N}}$ is relatively compact;*
- ii) *$\tilde{L}^+(A)$ is nonempty, compact and $\lim_{t \rightarrow +\infty} \sup_{x \in A} \rho(\tilde{\pi}(x, t), \tilde{L}^+(A)) = 0$;*
- iii) *there exists a nonempty compact subset $K \subset X$ such that*

$$\lim_{t \rightarrow +\infty} \sup_{x \in A} \rho(\tilde{\pi}(x, t), K) = 0.$$

We finalize this chapter with one more property for limit sets.

Lemma 1.5.11. *Let $(X, \pi; M, I)$ be an ISS and $A \subset X$ be such that $\tilde{\pi}^+(A)$ is relatively compact and $\tilde{L}^+(A) \cap M = \emptyset$. Then $\tilde{L}^+(A)$ is nonempty and $\tilde{L}^+(A) \subset \tilde{\pi}(\tilde{L}^+(A), t)$ for all $t \geq 0$.*

Proof. Let $y \in \tilde{L}^+(A)$ and $t \geq 0$. Then there are sequences $\{x_n\}_{n \in \mathbb{N}} \subset A$ and $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \xrightarrow{n \rightarrow +\infty} +\infty$ and

$$\tilde{\pi}(x_n, t_n) \xrightarrow{n \rightarrow +\infty} y.$$

Note that there is $n_0 \in \mathbb{N}$ such that $t_n > t$ for all $n > n_0$. Thus

$$\tilde{\pi}(x_n, t_n) = \tilde{\pi}(\tilde{\pi}(x_n, t_n - t), t),$$

for $n > n_0$. By the compactness of $\overline{\tilde{\pi}^+(A)}$ we may assume that

$$\tilde{\pi}(x_n, t_n - t) \xrightarrow{n \rightarrow +\infty} b,$$

where $b \in \tilde{L}^+(A)$. Since $\tilde{L}^+(A) \cap M = \emptyset$, by Lemma 1.4.3, there is a sequence $\{\epsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ with $\epsilon_n \xrightarrow{n \rightarrow +\infty} 0$ such that

$$\tilde{\pi}(\tilde{\pi}(x_n, t_n - t), t + \epsilon_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(b, t).$$

However, $\tilde{\pi}(\tilde{\pi}(x_n, t_n - t), t + \epsilon_n) = \tilde{\pi}(x_n, t_n + \epsilon_n)$ and by Lemma 1.4.2 we have

$$\tilde{\pi}(x_n, t_n + \epsilon_n) \xrightarrow{n \rightarrow +\infty} y.$$

Therefore, $y = \tilde{\pi}(b, t) \in \tilde{\pi}(\tilde{L}^+(A), t)$ and the lemma is concluded. □

Lyapunov Stability

In this chapter we present a study about attraction and stability of sets in the context of impulsive systems. There are many articles that deal with these concepts for compact or closed sets, see [2], [10] and [5] for instance. However, there exist several examples of impulsive systems where a set A features attraction and stability but this set is not compact neither closed, see Examples 2.1.4 and 2.1.5. Our goal in this chapter is to investigate properties of attraction and stability for relatively compact sets. The results listed in this chapter are new and they are presented in [16].

2.1 Stability and attraction of sets

We start this section by presenting the concepts of regions of attraction. In the sequel, we recall the definitions of stability, uniform stability and orbital stability for impulsive systems.

Definition 2.1.1. The **region of weak attraction** of a set $A \subset X$ is defined by

$$\tilde{P}_w(A) = \{x \in X : \tilde{L}^+(x) \cap \overline{A} \neq \emptyset\}$$

and the **region of attraction** of $A \subset X$ is defined by

$$\tilde{P}(A) = \{x \in X : \tilde{L}^+(x) \neq \emptyset \text{ and } \tilde{L}^+(x) \subset \overline{A}\}.$$

We say that a set $A \subset X$ is **weakly $\tilde{\pi}$ -attractor** if $\tilde{P}_w(A)$ is a neighborhood of A and $\tilde{\pi}$ -attractor if $\tilde{P}(A)$ is a neighborhood of A .

Note that $\tilde{P}(A) \subset \tilde{P}_w(A)$ for all $A \subset X$.

Lemma 2.1.2. *The sets $\tilde{P}(A)$, $\tilde{P}_w(A)$, $\overline{\tilde{P}(A)} \setminus M$ and $\overline{\tilde{P}_w(A)} \setminus M$ are positively $\tilde{\pi}$ -invariant for all $A \subset X$.*

Proof. Let $x \in X$ be such that $\tilde{L}^+(x) \neq \emptyset$ and $t \geq 0$. Note that $\tilde{L}^+(\tilde{\pi}(x, t)) = \tilde{L}^+(x)$. This shows that $\tilde{P}(A)$ and $\tilde{P}_w(A)$ are positively $\tilde{\pi}$ -invariant. Using Lemma 1.5.5, we conclude that the sets $\overline{\tilde{P}(A)} \setminus M$ and $\overline{\tilde{P}_w(A)} \setminus M$ are positively $\tilde{\pi}$ -invariant. $\square$

Definition 2.1.3. A subset $A \subset X$ is said to be:

1. **$\tilde{\pi}$ -stable** if for every $\epsilon > 0$ and $x \in A$ there exists a $\delta = \delta(x, \epsilon) > 0$ such that

$$\tilde{\pi}(B(x, \delta), [0, +\infty)) \subset B(A, \epsilon);$$

2. **uniformly $\tilde{\pi}$ -stable** if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\tilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \epsilon);$$

3. **orbitally $\tilde{\pi}$ -stable** if for every neighborhood U of A there exists a positively $\tilde{\pi}$ -invariant neighborhood V of A with $V \subset U$.

Let $A \subset X$ be a nonempty set and suppose that A is stable in some sense. Is the set $\overline{A}$ stable too? In general, the answer is negative as show the next examples.

Example 2.1.4. Let $(\mathbb{R}, \pi; M, I)$ be the ISS given in Example 1.2.4.

Figure 2.1: $\tilde{\pi}^+(A) = A$ and $\tilde{\pi}^+(\overline{A}) = [0, +\infty)$.

Note that $A = [0, 1)$ is $\tilde{\pi}$ -attractor, orbitally $\tilde{\pi}$ -stable and $\tilde{\pi}$ -stable but it is not uniformly $\tilde{\pi}$ -stable. In the other side, the set $\bar{A}$ is neither $\tilde{\pi}$ -attractor, orbitally $\tilde{\pi}$ -stable nor $\tilde{\pi}$ -stable. See Figure 2.1.

Example 2.1.5. Let $(\mathbb{R}^2, \pi; M, I)$ be the ISS presented in Example 1.2.5. Let $A = [0, 2) \times \{0\}$. The set A is $\tilde{\pi}$ -stable but it is neither orbitally $\tilde{\pi}$ -stable nor uniformly $\tilde{\pi}$ -stable. However, the set $\bar{A}$ is neither $\tilde{\pi}$ -stable, orbitally $\tilde{\pi}$ -stable nor uniformly $\tilde{\pi}$ -stable. Moreover, A is $\tilde{\pi}$ -attractor with region of attraction $\tilde{P}(A) = \{(x, y) \in \mathbb{R}^2 : x < 2\}$, while $\bar{A}$ is not $\tilde{\pi}$ -attractor. See Figure 2.2.

Figure 2.2: Impulsive trajectory of $z \in \mathbb{R}^2$. There is no positively $\tilde{\pi}$ -invariant neighborhood V of A such that $V \subset U_0$, where U_0 is the hatched area.

In many cases, the points of M are responsible to destroy the stability of $\bar{A}$ provided that A is stable in some sense. There are many unstable sets that become stable when we exclude their impulsive points, for instance, as presented in Examples 2.1.4 and 2.1.5, the set $\bar{A}$ is unstable while the set $\bar{A} \setminus M$ is stable. There is still no theory that characterizes the stability of sets of the form $\bar{A} \setminus M$. The class of the sets $\bar{A} \setminus M$, $A \subset X$, is very important in the theory of impulsive dynamical systems. For instance, a global attractor in the sense as presented in [12], belongs to this class. In this way, we shall present results concerning stability of sets of the form $\bar{A} \setminus M$, $A \subset X$. Moreover, we shall study results about uniform stability for relatively compact sets.

When X is locally compact and $A \subset X$ is compact, the concepts of $\tilde{\pi}$ -stability, orbital $\tilde{\pi}$ -stability and uniform $\tilde{\pi}$ -stability of A are equivalent. See the next result.

Theorem 2.1.6. [24, Theorem 4.1] *Let $(X, \pi; M, I)$ be an ISS, X be locally compact and A be a compact subset of X . Then the following conditions are equivalent:*

- (i) A is $\tilde{\pi}$ -stable;
- (ii) A is orbitally $\tilde{\pi}$ -stable;
- (iii) A is uniformly $\tilde{\pi}$ -stable;
- (iv) $\tilde{D}^+(A) = A$.

Remark 2.1.7. In Theorem 2.1.6, the implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) hold for any metric space X and for any nonempty set $A \subset X$.

On continuous dynamical systems theory, the positive invariance of the closure of a set is preserved provided this set is positively invariant. Nonetheless, we already know that this fact is not true in general for impulsive systems, see Example 1.5.4. Under stability condition, it is well-known that compact $\tilde{\pi}$ -stable sets are positively $\tilde{\pi}$ -invariant sets, see [24, Theorem 2.3]. But we may not assure that the closure of a non-compact positively $\tilde{\pi}$ -invariant set still positively $\tilde{\pi}$ -invariant, even if this set is $\tilde{\pi}$ -stable or orbitally $\tilde{\pi}$ -stable, see Example 2.1.4. Yet, in the case of uniform $\tilde{\pi}$ -stability, we have the following straightforward result.

Lemma 2.1.8. *Let $(X, \pi; M, I)$ be an ISS and $A \subset X$. If $A \subset X$ is uniformly $\tilde{\pi}$ -stable then $\bar{A}$ is positively $\tilde{\pi}$ -invariant.*

The concepts of uniform $\tilde{\pi}$ -stability and orbital $\tilde{\pi}$ -stability are not equivalent in general. However, we may relate the uniform $\tilde{\pi}$ -stability of a relatively compact set with the orbital $\tilde{\pi}$ -stability of its closure. See the next result.

Proposition 2.1.9. *Let $(X, \pi; M, I)$ be an ISS and $A \subset X$ be relatively compact. Then the set A is uniformly $\tilde{\pi}$ -stable if and only if $\bar{A}$ is orbitally $\tilde{\pi}$ -stable.*

Proof. First, let us assume that $\bar{A}$ is orbitally $\tilde{\pi}$ -stable. Given $\epsilon > 0$ there exists a positively $\tilde{\pi}$ -invariant neighborhood V of $\bar{A}$ such that $V \subset B(\bar{A}, \epsilon)$. Since $\bar{A}$ is compact one can obtain $\delta = \delta(\epsilon) > 0$ such that $B(\bar{A}, \delta) \subset V$. Consequently, we obtain

$$\tilde{\pi}(B(A, \delta), [0, +\infty)) \subset \tilde{\pi}(V, [0, +\infty)) \subset V \subset B(\bar{A}, \epsilon)$$

and we conclude that A is uniformly $\tilde{\pi}$ -stable.

Now, let us assume that A is uniformly $\tilde{\pi}$ -stable. Let U be a neighborhood of $\bar{A}$. By compactness there is $\epsilon > 0$ such that $B(\bar{A}, \epsilon) \subset U$. Since A is uniformly $\tilde{\pi}$ -stable there exists $\delta = \delta(\epsilon) > 0$ such that $\tilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \epsilon)$, that is,

$$\tilde{\pi}(B(\bar{A}, \delta), [0, +\infty)) \subset B(\bar{A}, \epsilon).$$

Taking $V = \tilde{\pi}(B(\bar{A}, \delta), [0, +\infty))$, we conclude that $\bar{A}$ is orbitally $\tilde{\pi}$ -stable. $\square$

Remark 2.1.10. According to the Proposition 2.1.9, we obtain the equivalence (ii) $\Leftrightarrow$ (iii) from Theorem 2.1.6 for compact sets without assuming that X is locally compact.

Let $A \subset X$ be given. In general, the positive prolongation set $\tilde{D}^+(A)$ is not equal to the set $\{\tilde{D}^+(a) : a \in A\}$, see [6, Example 3.29]. But, when $A \subset X$ is compact we get the equality $\tilde{D}^+(A) = \{\tilde{D}^+(a) : a \in A\}$, see [6, Proposition 3.30]. In the case of relatively compact sets, we have the following result.

Proposition 2.1.11. [16, Proposition 3.11] *If $A \subset X$ is relatively compact then $\tilde{D}^+(A) = \{\tilde{D}^+(a) : a \in \bar{A}\}$.*

Proof. It is enough to note that $\tilde{D}^+(A) = \tilde{D}^+(\bar{A})$ and $\tilde{D}^+(\bar{A}) = \{\tilde{D}^+(a) : a \in \bar{A}\}$ by [6, Proposition 3.30]. $\square$

In Theorem 2.1.6, the equivalence (iii) $\Leftrightarrow$ (iv) holds for compact sets in locally compact spaces. If we assume that $A \subset X$ is relatively compact and uniformly $\tilde{\pi}$ -stable, then we get the following result without assume that X is locally compact.

Theorem 2.1.12. *Let $A \subset X$ be relatively compact and uniformly $\tilde{\pi}$ -stable. Then $\tilde{D}^+(A) = \bar{A}$.*

Proof. Since $\tilde{D}^+(A)$ is closed we have $\bar{A} \subset \tilde{D}^+(A)$. On the other hand, let $z \in \tilde{D}^+(A)$. By Proposition 2.1.11 there is $a \in \bar{A}$ such that $z \in \tilde{D}^+(a)$. Consequently, there are sequences $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ and $\{a_n\}_{n \in \mathbb{N}} \subset X$ with $a_n \xrightarrow{n \rightarrow +\infty} a$ and

$$\tilde{\pi}(a_n, t_n) \xrightarrow{n \rightarrow +\infty} z.$$

Let $\epsilon > 0$ be arbitrary. By the uniform $\tilde{\pi}$ -stability of A , there exists $\delta = \delta(\epsilon) > 0$ such that

$$\tilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \epsilon).$$

Then, for n sufficiently large, we have $\tilde{\pi}(a_n, t_n) \in B(A, \epsilon)$ which implies in

$$z \in \overline{B(A, \epsilon)}.$$

Since $\epsilon > 0$ is taken arbitrary, we have $z \in \bigcap_{\epsilon > 0} \overline{B(A, \epsilon)} = \overline{A}$. Therefore, $\tilde{D}^+(A) = \overline{A}$. $\square$

Assuming X is locally compact, we obtain the converse of Theorem 2.1.12.

Theorem 2.1.13. *Let $(X, \pi; M, I)$ be an ISS, X be locally compact and $A \subset X$ be relatively compact. Then the set A is uniformly $\tilde{\pi}$ -stable if and only if $\tilde{D}^+(A) = \overline{A}$.*

Proof. Assume that $\tilde{D}^+(A) = \overline{A}$. Since $\tilde{D}^+(A) = \tilde{D}^+(\overline{A})$ it follows by Theorem 2.1.6 that $\overline{A}$ is uniformly $\tilde{\pi}$ -stable. But it is equivalent to A be uniformly $\tilde{\pi}$ -stable. The necessary condition follows by Theorem 2.1.12. $\square$

Theorem 2.1.14 establishes sufficient conditions for a relatively compact set to be $\tilde{\pi}$ -attractor.

Theorem 2.1.14. *Let $(X, \pi; M, I)$ be an ISS and $A \subset X$ be relatively compact. Assume that A is uniformly $\tilde{\pi}$ -stable and weakly $\tilde{\pi}$ -attractor. Then A is $\tilde{\pi}$ -attractor and $\overline{A} \subset \tilde{P}(A)$.*

Proof. Since A is weakly $\tilde{\pi}$ -attractor then there is an open set $\mathcal{O}$ in X such that $A \subset \mathcal{O} \subset \tilde{P}_w(A)$. We claim that $\mathcal{O} \subset \tilde{P}(A)$. Indeed, let $x \in \mathcal{O}$ and take $v \in \tilde{L}^+(x) \cap \overline{A}$.

First, let us assume that $x \notin M$. By Lemma 1.5.9 we have

$$\tilde{L}^+(x) \subset \tilde{J}^+(x) \subset \tilde{J}^+(v) \subset \tilde{D}^+(v) \subset \tilde{D}^+(A),$$

where the last set inclusion follows by Proposition 2.1.11. By Theorem 2.1.12 we have $\tilde{D}^+(A) = \overline{A}$. Therefore, $\tilde{L}^+(x) \subset \overline{A}$ and $x \in \tilde{P}(A)$, that is, $\mathcal{O} \subset \tilde{P}(A)$.

Second, if $x \in M$, we may choose $\eta \in (0, \phi(x))$ such that $y = \tilde{\pi}(x, \eta) = \pi(x, \eta) \in \mathcal{O} \setminus M$. Then $\tilde{L}^+(y) \cap \overline{A} = \tilde{L}^+(x) \cap \overline{A} \neq \emptyset$ and using the proof of the previous case we obtain $\tilde{L}^+(x) = \tilde{L}^+(y) \subset \overline{A}$. Thus $x \in \tilde{P}(A)$, that is, $\mathcal{O} \subset \tilde{P}(A)$. Therefore, A is $\tilde{\pi}$ -attractor.

At last, let us show that $\overline{A} \subset \tilde{P}(A)$. Since A is uniformly $\tilde{\pi}$ -stable it follows that $\overline{A}$ is positively $\tilde{\pi}$ -invariant, see Lemma 2.1.8. Thus $\emptyset \neq \tilde{L}^+(x) \subset \overline{A}$ for all $x \in \overline{A}$ and the proof is completed. $\square$

As shown in Example 2.1.5 the set $\overline{A}$ is not contained in $\tilde{P}(A)$ while $\overline{A} \setminus M \subset \tilde{P}(A)$. In the next result, we present sufficient conditions for a set A to satisfy the property $\overline{A} \setminus M \subset \tilde{P}(A)$.

Proposition 2.1.15. *Let $A \subset X$ be a relatively compact set such that $\bar{A} \setminus M$ is $\tilde{\pi}$ -stable. Then $\bar{A} \setminus M \subset \tilde{P}(A)$.*

Proof. Let $x \in \bar{A} \setminus M$ and $\epsilon > 0$. Then there is $\delta = \delta(x, \epsilon) > 0$ such that $\tilde{\pi}(B(x, \delta), [0, +\infty)) \subset B(\bar{A} \setminus M, \epsilon)$. Thus,

$$\tilde{\pi}^+(x) \subset \bigcap_{\epsilon > 0} B(\bar{A} \setminus M, \epsilon) \subset \bar{A}.$$

Since A is a relatively compact set we obtain $\emptyset \neq \tilde{L}^+(x) \subset \bar{A}$ for all $x \in \bar{A} \setminus M$. Hence, $\bar{A} \setminus M \subset \tilde{P}(A)$. $\square$

2.2 Stability via Lyapunov functions

In this section, we present sufficient conditions to obtain $\tilde{\pi}$ -stability and orbital $\tilde{\pi}$ -stability for sets of the form $\bar{A} \setminus M$, $A \subset X$. The results are achieved by means of functionals which play the role of a Lyapunov functional, that is, of a non-negative scalar function of the state which decrease monotonically along trajectories. We also include a result about instability.

First, we present an auxiliary result.

Lemma 2.2.1. *Let $(X, \pi; M, I)$ be an ISS and $\mathcal{O} \subset X$ be an open set such that $I(\bar{\mathcal{O}} \cap M) \subset \mathcal{O}$. Assume that there exist $x \in \mathcal{O}$ and $t_0 > 0$ such that $\tilde{\pi}(x, t_0) \notin \mathcal{O}$. Then there exists $\tau \in (0, t_0]$ such that $\tilde{\pi}(x, [0, \tau)) \subset \mathcal{O}$ and $\tilde{\pi}(x, \tau) \in \partial\mathcal{O} \setminus M$.*

Proof. Let $\tau = \min\{t > 0 : \tilde{\pi}(x, t) \notin \mathcal{O}\}$. By the openness of $\mathcal{O}$ there is $\epsilon \in (0, \phi(x))$ such that $\tilde{\pi}(x, [0, \epsilon)) \subset \mathcal{O}$ and this fact shows that $\tau > 0$.

We claim that $\tau \neq \sum_{i=0}^n \phi(x_i^+)$ for all $n \in \mathbb{N}_0$. In fact, if $\tau = \sum_{i=0}^n \phi(x_i^+)$ for some $n \in \mathbb{N}_0$ then

$$\tilde{\pi}(x, \tau) = x_{n+1}^+.$$

By the minimality of τ we have $\pi(x_n^+, (0, \phi(x_n^+))) \subset \mathcal{O}$ and, consequently, $x_{n+1} = \pi(x_n^+, \phi(x_n^+)) \in \bar{\mathcal{O}} \cap M$. Using the hypothesis $I(\bar{\mathcal{O}} \cap M) \subset \mathcal{O}$ we get $x_{n+1}^+ = I(x_{n+1}) \in \mathcal{O}$ which contradicts the definition of τ . Hence, $\tau \neq \sum_{i=0}^n \phi(x_i^+)$ for all $n \in \mathbb{N}_0$.

Thus, there exists $k \in \mathbb{N}_0$ such that $\sum_{i=-1}^{k-1} \psi(x_i^+) < \tau < \sum_{i=-1}^k \psi(x_i^+)$, where $\psi(x_{-1}^+) = 0$ and $\psi(x_i^+) = \phi(x_i^+)$ for $i = 0, 1, \dots, k$. Denote $\sum_{i=-1}^{k-1} \psi(x_i^+)$ by η_1 and $\sum_{i=-1}^k \psi(x_i^+)$ by η_2 . Since

$$\tilde{\pi}(x, (\eta_1, \tau)) = \pi(x_k^+, (0, \tau - \eta_1)) \in \mathcal{O}, \quad \tilde{\pi}(x, \tau) = \pi(x_k^+, \tau - \eta_1) \notin \mathcal{O}$$

and π is continuous then it implies that $\tilde{\pi}(x, \tau) = \pi(x_k^+, \tau - \eta_1) \in \partial\mathcal{O}$. Now, note that $\tilde{\pi}(x, t) \notin M$ for all $t > 0$, as $I(M) \cap M = \emptyset$ by condition **(H2)**. Hence, $\tilde{\pi}(x, \tau) \in \partial\mathcal{O} \setminus M$. $\square$

Remark 2.2.2. If $A \subset X$ is such that $\rho(I(x), A) < \rho(x, A)$ for all $x \in M$, then the set $\mathcal{O} = B(A, r)$ satisfies the condition $I(\overline{\mathcal{O}} \cap M) \subset \mathcal{O}$.

Note in the next result that we do not need any hypothesis on the boundedness of A .

Theorem 2.2.3. Let $(X, \pi; M, I)$ be an ISS, $A \subset X$ and $r_0 > 0$ be a number such that $I(\overline{B(A, r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Let $\gamma > 0$ and $\mathcal{U}$ be a neighborhood of A with $B(A, \gamma) \subset \mathcal{U}$. Assume that there are $k > 0$ and a mapping $V : \mathcal{U} \rightarrow \mathbb{R}_+$ continuous on $\mathcal{U} \setminus M$ satisfying the following conditions:

- (i) $V(x) = 0$ for $x \in A$ and for every sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{U}$ such that $V(x_n) \xrightarrow{n \rightarrow +\infty} 0$ implies $\rho(x_n, A) \xrightarrow{n \rightarrow +\infty} 0$;
- (ii) $V(\tilde{\pi}(x, t)) \leq kV(x)$ for all $x \in \mathcal{U} \setminus M$ and $t \geq 0$ such that $\tilde{\pi}(x, [0, t]) \subset \mathcal{U}$.

Then $\overline{A} \setminus M$ is $\tilde{\pi}$ -stable.

Proof. Suppose to the contrary that there are $x \in \overline{A} \setminus M$, $0 < \epsilon < \min\{\gamma, r_0\}$ and sequences $\{x_n\}_{n \in \mathbb{N}} \subset X$ and $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $x_n \xrightarrow{n \rightarrow +\infty} x$ and

$$\tilde{\pi}(x_n, t_n) \notin B(A, \epsilon) \quad \text{for all } n \in \mathbb{N}. \quad (2.1)$$

By the continuity of V in $\mathcal{U} \setminus M$ we conclude that $V(z) = 0$ for all $z \in \overline{A} \setminus M$. Since $x \in \overline{A} \setminus M$, we may assume that $x_n \notin M$ for all $n \in \mathbb{N}$ and, consequently, we have

$$V(x_n) \xrightarrow{n \rightarrow +\infty} V(x) = 0. \quad (2.2)$$

Let $n_0 \in \mathbb{N}$ be such that $x_n \in B(A, \epsilon)$ for all $n \geq n_0$. By Lemma 2.2.1, there exists a sequence $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that

$$\tau_n \leq t_n, \quad \tilde{\pi}(x_n, [0, \tau_n)) \subset B(A, \epsilon) \quad \text{and} \quad \tilde{\pi}(x_n, \tau_n) \in S(A, \epsilon) \quad \text{for all } n \geq n_0. \quad (2.3)$$

Using condition (ii) and (2.2) we obtain

$$V(\tilde{\pi}(x_n, \tau_n)) \xrightarrow{n \rightarrow +\infty} 0.$$

Thus, by the condition (i), we get $\rho(\tilde{\pi}(x_n, \tau_n), A) \xrightarrow{n \rightarrow +\infty} 0$. But it contradicts the fact that $\tilde{\pi}(x_n, \tau_n) \in S(A, \epsilon)$ for all $n \geq n_0$. Therefore, $\bar{A} \setminus M$ is $\tilde{\pi}$ -stable. $\square$

Theorem 2.2.4. *Let $(X, \pi; M, I)$ be an ISS, $A \subset X$ and $r_0 > 0$ be a number such that $I(\overline{B(A, r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Let $\mathcal{U}$ be a neighborhood of $\bar{A}$, $\mathcal{O}$ be an open set in X and $\alpha \in (0, r_0)$ be such that $\bar{A} \setminus M \subset \mathcal{O} \subset \mathcal{U}$ and $\rho(z, A) \geq \alpha$ for all $z \in X \setminus \bar{\mathcal{O}}$. Assume that there is a mapping $V : \mathcal{U} \rightarrow \mathbb{R}_+$ continuous on $\mathcal{U} \setminus M$ satisfying the conditions:*

- (i) $V(x) = 0$ for $x \in A$ and for every sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{U}$ such that $V(x_n) \xrightarrow{n \rightarrow +\infty} 0$ implies $\rho(x_n, A) \xrightarrow{n \rightarrow +\infty} 0$;
- (ii) $V(\tilde{\pi}(x, t)) \leq V(x)$ for all $x \in \mathcal{U} \setminus M$ and $t \geq 0$ such that $\tilde{\pi}(x, [0, t]) \subset \mathcal{U}$.

Then $\bar{A} \setminus M$ is orbitally $\tilde{\pi}$ -stable.

Proof. Let $\mathcal{O}_1 = \{z \in \mathcal{O} : \rho(z, A) < \alpha\}$. Note that $\mathcal{O}_1$ is open and $\bar{A} \setminus M \subset \mathcal{O}_1$. Also, we have $I(\bar{\mathcal{O}}_1 \cap M) \subset \mathcal{O}_1$. In fact, if $x \in \bar{\mathcal{O}}_1 \cap M$ then $x \in \bar{\mathcal{O}} \cap M$ and $\rho(x, A) \leq \alpha$. By hypothesis we get $I(x) \in B(A, \alpha)$. Moreover, $I(x) \in \mathcal{O}$ since $\rho(z, A) \geq \alpha$ for all $z \in X \setminus \bar{\mathcal{O}}$.

Now, note that $\rho(z, A) = \alpha$ for all $z \in \partial\mathcal{O}_1 \setminus M$ and define

$$\mu = \inf\{V(z) : z \in \partial\mathcal{O}_1 \setminus M\}.$$

We assert that $\mu > 0$. Indeed, if there is a sequence $\{v_n\}_{n \in \mathbb{N}} \subset \partial\mathcal{O}_1 \setminus M$ such that $V(v_n) \xrightarrow{n \rightarrow +\infty} 0$ then $\rho(v_n, A) \xrightarrow{n \rightarrow +\infty} 0$ as we have condition (i). But it is a contradiction since $\rho(v_n, A) = \alpha$ for all $n \in \mathbb{N}$. Hence, $\mu > 0$.

Let $K = \{x \in \mathcal{O}_1 \setminus M : V(x) < \mu\}$. By the continuity of V in $\mathcal{U} \setminus M$ it is not difficult to see that $V(z) = 0$ for all $z \in A \cup (\partial A \setminus M)$. This shows that $\bar{A} \setminus M \subset K$.

Now, we claim that K is positively $\tilde{\pi}$ -invariant. In fact, let $x \in K$. First, let us show that $\tilde{\pi}(x, t) \in \mathcal{O}_1$ for all $t \geq 0$. For that, suppose to the contrary that there is $t_* > 0$ such that $\tilde{\pi}(x, t_*) \notin \mathcal{O}_1$.

$\mathcal{O}_1$. By Lemma 2.2.1, there is $\tau \in (0, t_*]$ such that $\tilde{\pi}(x, [0, \tau)) \subset \mathcal{O}_1$ and $\tilde{\pi}(x, \tau) \in \partial\mathcal{O}_1 \setminus M$. Then, using condition (ii) and the definition of μ , we obtain

$$\mu \leq V(\tilde{\pi}(x, \tau)) \leq V(x) < \mu$$

which is a contradiction. In conclusion, using again condition (ii), we have $\tilde{\pi}(x, t) \in K$ for all $x \in K$ and $t \geq 0$. Therefore, K is an open positively $\tilde{\pi}$ -invariant neighborhood of $\bar{A} \setminus M$. $\square$

For relatively compact sets, we have the following consequence from Theorem 2.2.4.

Corollary 2.2.5. *Let $(X, \pi; M, I)$ be an ISS, $A \subset X$ be compact and $r_0 > 0$ be such that $I(\overline{B(A, r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Let $\mathcal{U}$ be a neighborhood of A and assume that there is a mapping $V : \mathcal{U} \rightarrow \mathbb{R}_+$ continuous in $\mathcal{U} \setminus M$ satisfying the conditions:*

- (i) $V(x) = 0$ for $x \in A$ and for every sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{U}$ such that $V(x_n) \xrightarrow{n \rightarrow +\infty} 0$ implies $\rho(x_n, A) \xrightarrow{n \rightarrow +\infty} 0$;
- (ii) $V(\tilde{\pi}(x, t)) \leq V(x)$ for all $x \in \mathcal{U} \setminus M$ and $t \geq 0$ such that $\tilde{\pi}(x, [0, t]) \subset \mathcal{U}$.

Then $A \setminus M$ is orbitally $\tilde{\pi}$ -stable.

In the next result, we characterize the sets whose closure are unstable.

Theorem 2.2.6. *Let $A \subset X$ and $\mathcal{U}$ be a neighborhood of A . Assume that there exists a mapping $V : \mathcal{U} \rightarrow \mathbb{R}_+$ continuous on $\mathcal{U} \setminus M$ satisfying the following conditions:*

- (i) $V(x) = 0$ for $x \in A$;
- (ii) there exist $a \in \bar{A}$ and $s > 0$ such that $V(\tilde{\pi}(a, s)) > 0$.

Then $\bar{A}$ is not $\tilde{\pi}$ -stable. In particular, $\bar{A}$ is neither orbitally $\tilde{\pi}$ -stable nor uniformly $\tilde{\pi}$ -stable.

Proof. Suppose to the contrary that $\bar{A}$ is $\tilde{\pi}$ -stable. Then given $\epsilon > 0$ there is $\delta > 0$ such that $\tilde{\pi}(B(a, \delta), [0, +\infty)) \subset B(A, \epsilon)$. Thus $\tilde{\pi}^+(a) \subset \bigcap_{\epsilon > 0} B(A, \epsilon) = \bar{A}$. The condition (H2) implies $\tilde{\pi}(a, s) \in \bar{A} \setminus M$. Since V is continuous in $\mathcal{U} \setminus M$ and we have condition (i), one can conclude that $V(\tilde{\pi}(a, s)) = 0$. But this contradicts the condition (ii). $\square$

Example 2.2.7. Consider the dynamical system $(L_2[0, 1], \pi, \mathbb{R})$ given in Example 1.1.6, the impulsive set

$$M = \left\{ \psi \in L_2[0, 1] : \int_0^1 |\psi(s)|^2 ds = 1 \right\}$$

and let $I : M \rightarrow L_2[0, 1]$ be an impulse function such that $\|I(\psi)\|_2 \leq \alpha \|\psi\|_2$ for all $\psi \in M$, where $0 < \alpha < 1$. Thus, we have the associate impulsive system $(L_2[0, 1], \pi; M, I)$. Note that $I(M) \cap M = \emptyset$ and each point of M satisfies STC.

Let $r > 2$ and $B_{L_2}(0, r)$ be the open ball in $L_2[0, 1]$ with center 0 and radius r , where $\|\cdot\|_2$ is the usual norm in $L_2[0, 1]$.

- (1) The set $A_1 = \{\psi \in L_2[0, 1] : 1 \leq \|\psi\|_2 \leq 2\}$ is not $\tilde{\pi}$ -stable.

In fact, define the mapping $V : B_{L_2}(0, r) \rightarrow \mathbb{R}_+$ by

$$V(x) = \begin{cases} \inf_{a \in A_1} \|x - a\|_2 & \text{if } x \in B_{L_2}(0, r) \setminus A_1, \\ 0 & \text{if } x \in A_1, \end{cases}$$

which is continuous in $B_{L_2}(0, r) \setminus M$. Let $\varphi \in A_1 \cap M$. Since $\|\pi(\varphi, t)\|_2 < 1$ for all $t > 0$ we have $\phi(\varphi) = +\infty$ and

$$\|\tilde{\pi}(\varphi, t)\|_2 = \|\pi(\varphi, t)\|_2 < 1,$$

for all $t > 0$. Thus, for an arbitrary $s > 0$ we get $\tilde{\pi}(\varphi, s) \notin A_1$ and

$$V(\tilde{\pi}(\varphi, s)) = \inf_{a \in A_1} \|\tilde{\pi}(\varphi, s) - a\|_2 \geq \inf_{a \in A_1} \|a\|_2 - \|\tilde{\pi}(\varphi, s)\|_2 \geq 1 - \|\tilde{\pi}(\varphi, s)\|_2 > 0.$$

By Theorem 2.2.6, we have A_1 is not $\tilde{\pi}$ -stable.

- (2) Let $A_2 = \{\psi \in L_2[0, 1] : \|\psi\|_2 \leq 1\}$. Then $A_2 \setminus M$ is orbitally $\tilde{\pi}$ -stable.

Indeed, consider the mapping $V : B_{L_2}(0, r) \rightarrow \mathbb{R}_+$ defined by

$$V(x) = \begin{cases} \|x\|_2 & \text{if } x \in B_{L_2}(0, r) \setminus A_2, \\ 0 & \text{if } x \in A_2. \end{cases}$$

Note that V is continuous on $B_{L_2}(0, r) \setminus M$ and $I(\overline{B_{L_2}(0, \mu)} \cap M) \subset B_{L_2}(0, \mu)$ for all $\mu > 0$. Let $\mathcal{U} = B_{L_2}(0, r)$ and $\mathcal{O} = B_{L_2}(0, \frac{r+1}{2})$. As presented in [6, Example 3.53], we have $\|\tilde{\pi}(\varphi, t)\|_2 \leq \|\varphi\|_2$ for all $\varphi \in L_2[0, 1]$ and $t \geq 0$. Then $V(\tilde{\pi}(\varphi, t)) \leq V(\varphi)$ for all $\varphi \in B_{L_2}(0, r)$ and $t \geq 0$. By Theorem 2.2.4, the set $A_2 \setminus M$ is orbitally $\tilde{\pi}$ -stable.

Proposition 2.2.8 presents sufficient conditions for a set $A \subset X$ to be $\tilde{\pi}$ -attractor.

Proposition 2.2.8. *Let $(X, \pi; M, I)$ be an ISS and $A \subset X$ be a nonempty subset. Assume that there is a real valued function V defined on a neighborhood $\mathcal{U}$ of $\bar{A}$ and continuous in $\mathcal{U} \setminus M$ satisfying:*

- (i) $V(\tilde{\pi}(x, t)) \leq V(x)$ for all $x \in \mathcal{U} \setminus M$ and $t \geq 0$ such that $\tilde{\pi}(x, [0, t]) \subset \mathcal{U}$.
- (ii) If $\tilde{L}^+(x) \cap (\mathcal{U} \setminus A) \neq \emptyset$ for some $x \in \mathcal{U}$, then V is not constant along trajectories in $\tilde{L}^+(x) \cap (\mathcal{U} \setminus A)$.

If there exist an open relatively compact positively $\tilde{\pi}$ -invariant set K and an open set $\mathcal{O} \subset X$ with $A \subset K \subset \bar{K} \subset \mathcal{O} \subset \mathcal{U}$, then A is $\tilde{\pi}$ -attractor.

Proof. Since K is a relatively compact positively $\tilde{\pi}$ -invariant set we have $\emptyset \neq \tilde{L}^+(x) \subset \bar{K}$ for each $x \in K$. We claim that $K \subset \tilde{P}(A)$. Indeed, let $x \in K$ and $y \in \tilde{L}^+(x)$. Let $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ be a sequence such that $s_n \xrightarrow{n \rightarrow +\infty} +\infty$ and $\tilde{\pi}(x, s_n) \xrightarrow{n \rightarrow +\infty} y$.

Case 1: $y \notin M$.

Suppose to the contrary that $y \notin \bar{A}$. Let $t > 0$ be such that $\tilde{\pi}(y, [0, t]) \cap \bar{A} = \emptyset$. Since $\tilde{L}^+(x) \setminus M$ is positively $\tilde{\pi}$ -invariant we get $\tilde{\pi}(y, [0, t]) \subset \tilde{L}^+(x)$. Then there is a sequence $\{t_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ such that $t_n \leq s_n$, for $n \in \mathbb{N}$, $t_n \xrightarrow{n \rightarrow +\infty} +\infty$ and

$$\tilde{\pi}(x, t_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(y, t).$$

We may assume that $\tilde{\pi}(x, s_n) \notin \bar{A}$ and $\tilde{\pi}(x, t_n) \notin \bar{A}$ for all $n \in \mathbb{N}$. As $I(M) \cap M = \emptyset$ and $\bar{K} \subset \mathcal{O}$ we have $\tilde{\pi}(x, s_n) \in \mathcal{U} \setminus (M \cup A)$ and $\tilde{\pi}(x, t_n) \in \mathcal{U} \setminus (M \cup A)$ for all $n \in \mathbb{N}$. By the condition (i) and the continuity of V , we obtain

$$\begin{aligned} V(y) &= \lim_{n \rightarrow +\infty} V(\tilde{\pi}(x, s_n)) = \lim_{n \rightarrow +\infty} V(\tilde{\pi}(\tilde{\pi}(x, t_n), s_n - t_n)) \leq \\ &\leq \lim_{n \rightarrow +\infty} V(\tilde{\pi}(x, t_n)) = V(\tilde{\pi}(y, t)) \leq V(y), \end{aligned}$$

that is, $V(\tilde{\pi}(y, t)) = V(y)$. But condition (ii) implies that $V(\tilde{\pi}(y, t)) \neq V(y)$, which is a contradiction. Hence, $y \in \bar{A}$ and $\tilde{L}^+(x) \subset \bar{A}$.

Case 2: $y \in M$.

Since M satisfies STC (see condition **(H1)**) there exists a STC-tube $F(L, [0, 2\lambda])$ through y given by a section S . As the tube is a neighborhood of y , there is $\eta > 0$ such that

$$B(y, \eta) \subset F(L, [0, 2\lambda]).$$

Denote H_1 and H_2 by

$$H_1 = F(L, (\lambda, 2\lambda]) \cap B(y, \eta) \quad \text{and} \quad H_2 = F(L, [0, \lambda]) \cap B(y, \eta).$$

In the sequel, we consider only the cases: either $\{\tilde{\pi}(x, s_n)\}_{n \in \mathbb{N}} \subset H_1$ or $\{\tilde{\pi}(x, s_n)\}_{n \in \mathbb{N}} \subset H_2$ (in the other cases we take subsequences).

If $\{\tilde{\pi}(x, s_n)\}_{n \in \mathbb{N}} \subset H_2$ then

$$\tilde{\pi}(x, s_n + \epsilon) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(y, \epsilon) \in \tilde{L}^+(x)$$

for all $\epsilon \in (0, \phi(y))$. Note that $\tilde{\pi}(y, \epsilon) \notin M$ for all $\epsilon \in (0, \phi(y))$ since $I(M) \cap M = \emptyset$. Then by the proof of Case 1, we may conclude that $\tilde{\pi}(y, \epsilon) \in \bar{A}$ for all $\epsilon \in (0, \phi(y))$. Consequently, $y \in \bar{A}$ and $\tilde{L}^+(x) \subset \bar{A}$.

If $\{\tilde{\pi}(x, s_n)\}_{n \in \mathbb{N}} \subset H_1$ then $\phi(\tilde{\pi}(x, s_n)) \xrightarrow{n \rightarrow +\infty} 0$. Assume that $s_n > \lambda > 0$ for all $n \in \mathbb{N}$ and consider the sequence $\{\tilde{\pi}(x, s_n - \frac{\lambda}{2})\}_{n \in \mathbb{N}} \subset K$. By the compactness of $\bar{K}$ we may assume that

$$\tilde{\pi}\left(x, s_n - \frac{\lambda}{2}\right) \xrightarrow{n \rightarrow +\infty} z.$$

Since $\tilde{\pi}\left(\tilde{\pi}\left(x, s_n - \frac{\lambda}{2}\right), \frac{\lambda}{2}\right) = \tilde{\pi}(x, s_n)$, $\phi(\tilde{\pi}(x, s_n)) \xrightarrow{n \rightarrow +\infty} 0$ and $F(L, (\lambda, 2\lambda]) \cap M = \emptyset$, we have $\phi(\tilde{\pi}(x, s_n - \frac{\lambda}{2})) = \frac{\lambda}{2} + \phi(\tilde{\pi}(x, s_n))$, $z \notin M$ and

$$\tilde{\pi}(x, s_n) = \pi\left(\tilde{\pi}\left(x, s_n - \frac{\lambda}{2}\right), \frac{\lambda}{2}\right) \xrightarrow{n \rightarrow +\infty} \pi\left(z, \frac{\lambda}{2}\right) \in \tilde{L}^+(x).$$

By uniqueness, we get $y = \pi\left(z, \frac{\lambda}{2}\right)$. Now, note that $\tilde{\pi}(z, [0, \frac{\lambda}{2})) \subset \tilde{L}^+(x) \setminus M$ as $z \in \tilde{L}^+(x) \setminus M$ and $\tilde{L}^+(x) \setminus M$ is positively $\tilde{\pi}$ -invariant. By the proof of Case 1, we have $\pi(z, [0, \frac{\lambda}{2})) \subset \bar{A}$. Hence, $y \in \bar{A}$ and $\tilde{L}^+(x) \subset \bar{A}$.

In conclusion, we have $\emptyset \neq \tilde{L}^+(x) \subset \bar{A}$ for all $x \in K$, that is, $K \subset \tilde{P}(A)$. Therefore, A is $\tilde{\pi}$ -attractor. $\square$

As a consequence of Corollary 2.2.5 and Proposition 2.2.8, we can state the following result.

Corollary 2.2.9. *Let $(X, \pi; M, I)$ be an ISS, X be locally compact, $A \subset X$ be a relatively compact set and $r_0 > 0$ be such that $I(\overline{B(A, r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Assume that there exists a non-negative real valued function V defined on a neighborhood $\mathcal{U}$ of $\overline{A}$ and continuous on $\mathcal{U} \setminus M$ satisfying:*

- (i) $V(x) = 0$ for $x \in A$ and for every sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{U}$ such that $V(x_n) \xrightarrow{n \rightarrow +\infty} 0$ implies $\rho(x_n, A) \xrightarrow{n \rightarrow +\infty} 0$;
- (ii) $V(\tilde{\pi}(x, t)) \leq V(x)$ for all $x \in \mathcal{U} \setminus M$ and $t \geq 0$ such that $\tilde{\pi}(x, [0, t]) \subset \mathcal{U}$.
- (iii) If $\tilde{L}^+(x) \cap (\mathcal{U} \setminus A) \neq \emptyset$ for some $x \in \mathcal{U}$, then V is not constant along trajectories in $\tilde{L}^+(x) \cap (\mathcal{U} \setminus A)$.

Then $\overline{A} \setminus M$ is orbitally $\tilde{\pi}$ -stable and A is $\tilde{\pi}$ -attractor.

Proof. By Corollary 2.2.5 the set $\overline{A} \setminus M$ is orbitally $\tilde{\pi}$ -stable. On the other hand, since A is relatively compact and X is locally compact there exists $\alpha \in (0, r_0)$ such that $B(A, \alpha) \subset \mathcal{U}$ with $\overline{B(A, \alpha)}$ compact. Define $\mu = \inf\{V(z) : z \in S(A, \frac{\alpha}{2})\}$. By condition (i) we have $\mu > 0$. Now, consider the set

$$K = \left\{x \in B\left(A, \frac{\alpha}{2}\right) : V(x) < \mu\right\}.$$

Using the same arguments of the proof of Theorem 2.2.4, we conclude that K is positively $\tilde{\pi}$ -invariant. Hence, K is an open relatively compact positively $\tilde{\pi}$ -invariant set with $A \subset K \subset \overline{K} \subset B(A, \alpha) \subset \mathcal{U}$. By Proposition 2.2.8, A is $\tilde{\pi}$ -attractor. $\square$

Example 2.2.10. Consider the impulsive dynamical system

$$\begin{cases} \dot{x} = f(x), \\ x(0) = x_0, \\ I: M \rightarrow \mathbb{R}^n, \end{cases} \quad (2.4)$$

where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, $M \subseteq \mathbb{R}^n$ is an impulsive set and $I: M \rightarrow \mathbb{R}^n$ is an impulse function such that $\|I(x) - I(y)\| \leq \eta\|x - y\|$ for all $x, y \in M$ with $0 < \eta < 1$. We assume that conditions **(H1)**, **(H2)** and **(H3)** hold. Also, we assume that all the solutions of the non-impulsive system $\dot{x} = f(x)$ are defined in the whole real line and give rise to a semigroup π on $\mathbb{R}^n$.

Now, let $V \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ be a function satisfying the following conditions:

- (i) There exists a bounded subset $\mathcal{A} \subset \mathbb{R}^n$ such that $V(x) = 0$ if and only if $x \in \overline{\mathcal{A}}$;
- (ii) $\nabla V(x) \cdot f(x) \leq -\alpha V(x)$ for all $x \in \mathbb{R}^n \setminus \mathcal{A}$, where $\alpha > 0$;
- (iii) $V(I(x)) < V(x)$ for all $x \in M \setminus \overline{\mathcal{A}}$ and $V(I(x)) = V(x)$ for all $x \in M \cap \overline{\mathcal{A}}$.

First, let us prove that $I(\overline{B(\mathcal{A}, r)} \cap M) \subset B(\overline{\mathcal{A}}, r)$ for all $r > 0$. It is enough to assume that $\overline{\mathcal{A}} \cap M \neq \emptyset$. According to the hypotheses (i) and (iii), we have $I(\overline{\mathcal{A}} \cap M) \subset \overline{\mathcal{A}}$. Since I is a Lipschitz function, we have

$$I\left(B\left(\overline{\mathcal{A}} \cap M, \frac{r}{\eta}\right) \cap M\right) \subset B(I(\overline{\mathcal{A}} \cap M), r) \subset B(\overline{\mathcal{A}}, r) = B(\mathcal{A}, r) \quad \text{for all } r > 0.$$

Moreover, it is easy to see that $\overline{B(\mathcal{A}, r)} \cap M \subset \overline{B(\overline{\mathcal{A}} \cap M, r)} \cap M \subset B(\overline{\mathcal{A}} \cap M, \frac{r}{\eta}) \cap M$. Thus, the assertion is proved.

Let $\mu > 0$, $\gamma > 0$ and define $\mathcal{U} = \{x \in B(\mathcal{A}, \gamma) : V(x) < \mu\}$. Note that $\mathcal{U}$ is a neighborhood of $\overline{\mathcal{A}}$. Let $\tilde{\pi}(x_0, \cdot)$ be the impulsive solution of (2.4).

It is not difficult to see that V satisfies the condition (i) from Corollary 2.2.9.

Now, for $x \in \mathcal{U} \setminus \mathcal{A}$ and $s > 0$ such that $\pi(x, [0, s]) \subset \mathcal{U} \setminus \mathcal{A}$, we have

$$\frac{d}{dt}V(\pi(x, t)) = \nabla V(\pi(x, t)) \cdot f(\pi(x, t)) \leq -\alpha V(\pi(x, t)) \quad \text{for } t \in [0, s].$$

Therefore,

$$V(\pi(x, t)) \leq e^{-\alpha t} V(x) < V(x) \quad \text{for all } t \in (0, s].$$

Since $V(I(x)) < V(x)$, $x \in M \setminus \overline{\mathcal{A}}$, we conclude that $V(\tilde{\pi}(x, t)) < V(x)$ for all $x \in \mathcal{U} \setminus \mathcal{A}$ and $t > 0$ such that $\tilde{\pi}(x, [0, t]) \subset \mathcal{U} \setminus \mathcal{A}$. This implies conditions (ii) and (iii) from Corollary 2.2.9. In conclusion, the set $\overline{\mathcal{A}} \setminus M$ is orbitally $\tilde{\pi}$ -stable and $\mathcal{A}$ is $\tilde{\pi}$ -attractor.

2.3 A converse-type result

Our next aim is to show the existence of a Lyapunov function satisfying the conditions (i), (ii) and (iii) of Corollary 2.2.9 provided that $\overline{\mathcal{A}} \setminus M$ is orbitally $\tilde{\pi}$ -stable and $\mathcal{A}$ is $\tilde{\pi}$ -attractor. For that, we present an auxiliary result.

Lemma 2.3.1. *Let $(X, \pi; M, I)$ be an ISS and A be a relatively compact subset of X . Let A be a $\tilde{\pi}$ -attractor and $\bar{A} \setminus M$ be a $\tilde{\pi}$ -stable set. Then the mapping $W : \tilde{P}(A) \rightarrow \mathbb{R}_+$ given by*

$$W(x) = \begin{cases} \sup_{t \in \mathbb{R}_+} \rho(\tilde{\pi}(x, t), A), & x \in \tilde{P}(A) \setminus M, \\ \rho(x, A), & x \in \tilde{P}(A) \cap M, \end{cases}$$

is continuous in $\tilde{P}(A) \setminus M$. Moreover, if $\bar{A}$ is $\tilde{\pi}$ -stable then W is continuous in $\tilde{P}(A) \setminus (M \setminus \bar{A})$.

Proof. Let $x \in \tilde{P}(A) \setminus M$. Then $\emptyset \neq \tilde{L}^+(x) \subset \bar{A}$. By Lemma 1.5.8 there exists $y \in \tilde{L}^+(x) \setminus M \subset \bar{A} \setminus M$. Thus there is a sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \xrightarrow{n \rightarrow +\infty} +\infty$ and $\tilde{\pi}(x, t_n) \xrightarrow{n \rightarrow +\infty} y$. Since $\bar{A} \setminus M$ is $\tilde{\pi}$ -stable, for a given $\epsilon_0 > 0$, there exists $\delta > 0$ such that

$$\tilde{\pi}(B(y, \delta), [0, +\infty)) \subset B(A, \epsilon_0).$$

Hence, there is $n_0 \in \mathbb{N}$ such that $\tilde{\pi}(x, [t_{n_0}, +\infty)) \subset B(A, \epsilon_0)$. Consequently, we have $W(x) \leq \sup\{\rho(\tilde{\pi}(x, [0, t_{n_0}]), A), \epsilon_0\} < +\infty$ and W is well defined.

Let us show the continuity of W in $\tilde{P}(A) \setminus M$. Indeed, let $x \in \tilde{P}(A) \setminus M$ and assume that $\phi(x_k^+) < \infty$ for all $k = 0, 1, 2, \dots$. Then we have

$$W(x) = \sup_{t \in \mathbb{R}_+} \rho(\tilde{\pi}(x, t), A) = \sup_{k \in \mathbb{N}} \sup_{0 \leq t < \phi(x_k^+)} \rho(\pi(x_k^+, t), A). \quad (2.5)$$

Let $\{w_n\}_{n \in \mathbb{N}} \subset \tilde{P}(A)$ be a sequence such that $w_n \xrightarrow{n \rightarrow +\infty} x$. Since M is closed, $I(M) \cap M = \emptyset$, I is continuous in M , ϕ is continuous in $X \setminus M$ and $x \notin M$, one can show that for each $k \in \mathbb{N}$ we have

$$(w_n)_k^+ = I(\pi((w_n)_{k-1}^+, \phi((w_n)_{k-1}^+))) \xrightarrow{n \rightarrow +\infty} I(\pi(x_{k-1}^+, \phi(x_{k-1}^+))) = x_k^+$$

and

$$\sup_{0 \leq t \leq \phi((w_n)_{k-1}^+)} \rho(\pi((w_n)_{k-1}^+, t), A) \xrightarrow{n \rightarrow +\infty} \sup_{0 \leq t \leq \phi(x_{k-1}^+)} \rho(\pi(x_{k-1}^+, t), A).$$

This shows that $W(w_n) \xrightarrow{n \rightarrow +\infty} W(x)$ and W is continuous in $\tilde{P}(A) \setminus M$.

Now, suppose that $\bar{A}$ is $\tilde{\pi}$ -stable. It is enough to prove that W is continuous in $\bar{A} \cap M \cap \tilde{P}(A)$ in order to conclude the continuity of W in $\tilde{P}(A) \setminus (M \setminus \bar{A})$. For $x \in \bar{A} \cap M \cap \tilde{P}(A)$ we have $W(x) = \rho(x, A) = 0$. Since $\bar{A}$ is $\tilde{\pi}$ -stable, given $\epsilon > 0$ there is $\delta = \delta(x, \epsilon) > 0$ such that $\tilde{\pi}(B(x, \delta), [0, +\infty)) \subset B(A, \epsilon)$. If $\{z_n\}_{n \in \mathbb{N}}$ is a sequence in $\tilde{P}(A)$ such that $z_n \xrightarrow{n \rightarrow +\infty} x$, then there is a positive integer $n_0 > 0$ such that $z_n \in B(x, \delta)$ for all $n > n_0$. Consequently,

$\tilde{\pi}(z_n, [0, +\infty)) \subset B(A, \epsilon)$ for all $n > n_0$. Then $W(z_n) < \epsilon$ for all $n > n_0$ and it implies that $W(z_n) \xrightarrow{n \rightarrow +\infty} 0 = W(x)$. Therefore, W is continuous in $\tilde{P}(A) \setminus (M \setminus \bar{A})$. $\square$

Theorem 2.3.2. *Let $(X, \pi; M, I)$ be an ISS, $A \subset X$ be a relatively compact set and $r_0 > 0$ be such that $I(\overline{B(A, r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Assume that $\bar{A} \setminus M$ is $\tilde{\pi}$ -stable and A is $\tilde{\pi}$ -attractor. Then there exists a non-negative real valued function V defined on a neighborhood $\mathcal{U}$ of A and continuous on $\mathcal{U} \setminus M$ satisfying:*

(i) $V(x) = 0$ for $x \in A$ and for every sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{U}$ such that $V(x_n) \xrightarrow{n \rightarrow +\infty} 0$ implies $\rho(x_n, A) \xrightarrow{n \rightarrow +\infty} 0$;

(ii) $V(\tilde{\pi}(x, t)) < V(x)$ for all $x \in \mathcal{U} \setminus (M \cup \bar{A})$ and $t > 0$ such that $\tilde{\pi}(x, [0, t]) \subset \mathcal{U} \setminus (M \cup \bar{A})$.

Proof. Let $\mathcal{U} = \tilde{P}(A)$ which is a positively $\tilde{\pi}$ -invariant neighborhood of A , see Lemma 2.1.2. By Proposition 2.1.15 we have $\bar{A} \setminus M \subset \tilde{P}(A)$. Define the mapping $W : \tilde{P}(A) \rightarrow \mathbb{R}_+$ by

$$W(x) = \begin{cases} \sup_{t \in \mathbb{R}_+} \rho(\tilde{\pi}(x, t), A), & x \in \tilde{P}(A) \setminus M \\ \rho(x, A), & x \in \tilde{P}(A) \cap M. \end{cases}$$

By Lemma 2.3.1 the mapping W is well defined and continuous in $\tilde{P}(A) \setminus M$. Note that $W(x) = 0$ for all $x \in A \setminus M$ since $\bar{A} \setminus M$ is $\tilde{\pi}$ -stable, $W(x) = 0$ for $x \in A \cap M$ and $\rho(x, A) \leq W(x)$ for all $x \in \tilde{P}(A)$. Hence, W satisfies the condition (i).

Now, we assert that $W(\tilde{\pi}(x, t)) \leq W(x)$ for all $x \in \tilde{P}(A) \setminus M$ and $t \geq 0$. Let us assume that $\phi(x_k^+) < \infty$ for all $k = 0, 1, 2, \dots$. If $0 \leq s < \phi(x)$ and $y = \tilde{\pi}(x, s)$, then $x_k^+ = y_k^+$ for each $k = 1, 2, \dots$. Thus

$$W(\tilde{\pi}(x, s)) = \sup_{t \in \mathbb{R}_+} \rho(\tilde{\pi}(x, t + s), A) = \sup_{t \geq s} \rho(\tilde{\pi}(x, t), A) \leq W(x)$$

and

$$W(\tilde{\pi}(x, \phi(x))) = W(x_1^+) = \sup_{t \in \mathbb{R}_+} \rho(\tilde{\pi}(x_1^+, t), A) \leq \sup_{t \in \mathbb{R}_+} \rho(\tilde{\pi}(x, t), A) = W(x).$$

Now, for each $t \geq \phi(x)$, there is $n = n(t) \in \mathbb{N}$ such that $t = t_n(x) + r$ with $0 < r \leq \phi(x_n^+)$. Thus

$$W(\tilde{\pi}(x, t)) = W(\pi(x_n^+, r)) \leq W(x_n^+). \quad (2.6)$$

In this way, we conclude that $W(\tilde{\pi}(x, t)) \leq W(x)$ for all $x \in \tilde{P}(A) \setminus M$ and $t \geq 0$. However, the function W may not be strictly decreasing along to the trajectories in $\tilde{P}(A) \setminus (\bar{A} \cup M)$. Thus, we define the mapping $V : \tilde{P}(A) \rightarrow \mathbb{R}_+$ by

$$V(x) = \begin{cases} \int_0^{+\infty} W(\tilde{\pi}(x, \tau)) \exp(-\tau) d\tau, & x \in \tilde{P}(A) \setminus M, \\ 0, & x \in A \cap M, \\ 1, & x \in (\tilde{P}(A) \setminus A) \cap M. \end{cases}$$

It is not difficult to see that V satisfies the condition (i). It is also clear that $V(\tilde{\pi}(x, t)) \leq V(x)$ for all $x \in \tilde{P}(A) \setminus (\bar{A} \cup M)$ and $t \geq 0$. Suppose to the contrary that $V(\tilde{\pi}(x_0, s)) = V(x_0)$ for some $x_0 \in \tilde{P}(A) \setminus (\bar{A} \cup M)$ and $s > 0$ with $\tilde{\pi}(x_0, [0, s]) \subset \tilde{P}(A) \setminus (\bar{A} \cup M)$. Then

$$\int_0^{+\infty} [W(\tilde{\pi}(x_0, s + \tau)) - W(\tilde{\pi}(x_0, \tau))] \exp(-\tau) d\tau = 0,$$

that is, $W(\tilde{\pi}(x_0, s + \tau)) = W(\tilde{\pi}(x_0, \tau))$ for every $\tau \in [0, +\infty)$. In particular, $W(\tilde{\pi}(x_0, ms)) = W(x_0)$ for all $m \in \mathbb{N}$.

We claim that $W(x_0) = 0$. In fact, given $\epsilon > 0$ there is $t_0 > 0$ such that

$$\tilde{\pi}(x_0, [t_0, +\infty)) \subset B(A, \epsilon)$$

as A is $\tilde{\pi}$ -attractor, $\bar{A} \setminus M$ is $\tilde{\pi}$ -stable and we have Lemma 1.5.8. Thus the sequence $\{\tilde{\pi}(x_0, ns)\}_{n \in \mathbb{N}}$ admits a convergent subsequence. We may assume that

$$\tilde{\pi}(x_0, ns) \xrightarrow{n \rightarrow +\infty} a \in \tilde{L}^+(x_0).$$

Case 1: $a \in \tilde{L}^+(x_0) \setminus M$.

In this case $a \in \bar{A} \setminus M$ as A is $\tilde{\pi}$ -attractor. Using the continuity of W in $\tilde{P}(A) \setminus M$ we get $W(a) = 0$ and

$$W(x_0) = W(\tilde{\pi}(x_0, ns)) \xrightarrow{n \rightarrow +\infty} W(a) = 0.$$

Hence, $W(x_0) = 0$.

Case 2: $a \in \tilde{L}^+(x_0) \cap M$.

By condition **(H1)** the set M satisfies STC. Then there is a STC-tube $F(L, [0, 2\lambda])$ through a given by a section S . Moreover, since the tube is a neighborhood of a , there is $\eta > 0$ such that

$$B(a, \eta) \subset F(L, [0, 2\lambda]).$$

Denote $H_1 = F(L, (\lambda, 2\lambda]) \cap B(a, \eta)$ and $H_2 = F(L, [0, \lambda]) \cap B(a, \eta)$.

Next, we consider just two subcases since the other cases are analogous by taking subsequences.

Subcase 2.1: $\{\tilde{\pi}(x_0, ns)\}_{n \in \mathbb{N}} \subset H_1$.

Then $\phi(\tilde{\pi}(x_0, ns)) \xrightarrow{n \rightarrow +\infty} 0$ and $\tilde{\pi}(\tilde{\pi}(x_0, ns), \phi(\tilde{\pi}(x_0, ns))) \xrightarrow{n \rightarrow +\infty} I(a) \in \tilde{L}^+(x_0) \setminus M \subset \overline{A} \setminus M$. Consequently,

$$W(\tilde{\pi}(\tilde{\pi}(x_0, ns), \phi(\tilde{\pi}(x_0, ns)))) \xrightarrow{n \rightarrow +\infty} W(I(a)).$$

On the other hand, we have $W(\tilde{\pi}(x_0, \phi(\tilde{\pi}(x_0, ns)))) = W(\tilde{\pi}(\tilde{\pi}(x_0, ns), \phi(\tilde{\pi}(x_0, ns))))$ and $W(\tilde{\pi}(x_0, \phi(\tilde{\pi}(x_0, ns)))) \xrightarrow{n \rightarrow +\infty} W(x_0)$. Hence, $W(x_0) = W(I(a)) = 0$ as $W(x) = 0$ for all $x \in \overline{A} \setminus M$.

Subcase 2.2: $\{\tilde{\pi}(x_0, ns)\}_{n \in \mathbb{N}} \subset H_2$.

We may use the proof of Lemma 1.4.3 and obtain a sequence $\{\epsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ with $\epsilon_n \xrightarrow{n \rightarrow +\infty} 0$ such that

$$\tilde{\pi}(\tilde{\pi}(x_0, ns), s + \epsilon_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(a, s).$$

Note that $\tilde{\pi}(a, s) \in \tilde{L}^+(x_0) \setminus M \subset \overline{A} \setminus M$. Since $W(\tilde{\pi}(\tilde{\pi}(x_0, ns), s + \epsilon_n)) = W(\tilde{\pi}(x_0, \epsilon_n))$, $n \in \mathbb{N}$, we conclude that $W(x_0) = W(\tilde{\pi}(a, s)) = 0$.

In conclusion, $W(x_0) = 0$ and it contradicts the fact that $x_0 \notin \overline{A}$. Therefore, V satisfies the condition (ii). $\square$

As a consequence of Theorem 2.3.2, we have a converse type of Corollary 2.2.9.

Corollary 2.3.3. *Let $(X, \pi; M, I)$ be an ISS, $A \subset X$ be a relatively compact set and $r_0 > 0$ be such that $I(\overline{B(A, r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Assume that $\overline{A} \setminus M$ is orbitally $\tilde{\pi}$ -stable and A is $\tilde{\pi}$ -attractor. Then there exists a non-negative real valued function V defined on a neighborhood $\mathcal{U}$ of A and continuous on $\mathcal{U} \setminus M$ satisfying:*

(i) $V(x) = 0$ for $x \in A$ and for every sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{U}$ such that $V(x_n) \xrightarrow{n \rightarrow +\infty} 0$ implies $\rho(x_n, A) \xrightarrow{n \rightarrow +\infty} 0$;

(ii) $V(\tilde{\pi}(x, t)) \leq V(x)$ for all $x \in \mathcal{U} \setminus M$ and $t \geq 0$ such that $\tilde{\pi}(x, [0, t]) \subset \mathcal{U}$.

(iii) If $\tilde{L}^+(x) \cap (\mathcal{U} \setminus A) \neq \emptyset$ for some $x \in \mathcal{U}$, then V is not constant along trajectories in $\tilde{L}^+(x) \cap (\mathcal{U} \setminus A)$.

At last, we present a result that concerns the existence of a relatively compact positively $\tilde{\pi}$ -invariant set.

Proposition 2.3.4. *Let $(X, \pi; M, I)$ be an ISS, X be locally compact, $A \subset X$ be relatively compact and $r_0 > 0$ be such that $I(\overline{B(A, r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Assume that there is a function $V : \mathcal{U} \rightarrow \mathbb{R}_+$, where $\mathcal{U}$ is a neighborhood of $\overline{A}$, continuous in $\mathcal{U} \setminus M$ satisfying the conditions (i) – (iii) of Corollary 2.3.3. Then there is $\alpha_0 > 0$ such that for $0 < \alpha < \alpha_0$ the set $G_\alpha = \{x \in \mathcal{U} \setminus M : V(x) < \alpha\}$ admits a relatively compact positively $\tilde{\pi}$ -invariant subset K_α such that K_α is a neighborhood of $A \cup (\partial A \setminus M)$, $K_\alpha \subset \tilde{P}(A)$ and*

$$K_\alpha \cap \left(\overline{G_\alpha \setminus K_\alpha} \right) = \emptyset.$$

Proof. Let $\epsilon > 0$, $\epsilon < r_0$, be such that $\overline{B(A, \epsilon)} \subset \mathcal{U}$ is compact. Set

$$m(\epsilon) = \inf\{V(x) : x \in S(A, \epsilon)\}$$

which is strictly positive by condition (i). Now let $0 < \alpha < m(\epsilon)$ and define the set

$$K_\alpha = G_\alpha \cap B(A, \epsilon).$$

By the continuity of V in $\mathcal{U} \setminus M$ we have $A \cup (\partial A \setminus M) \subset G_\alpha$. Then K_α is an open relatively compact neighborhood of $A \cup (\partial A \setminus M)$. Using the last part of the proof of Theorem 2.2.4, we conclude that K_α is positively $\tilde{\pi}$ -invariant. Now, by the proof of Proposition 2.2.8, we get $K_\alpha \subset \tilde{P}(A)$. Since $K_\alpha \cap \left(\overline{G_\alpha \setminus K_\alpha} \right) = \emptyset$, the proof is complete. $\square$

Asymptotically almost periodic motions

In this chapter, we consider the theory of asymptotically almost periodic motions in the context of impulsive semidynamical systems. We establish sufficient conditions in order to obtain the existence of asymptotically almost periodic motions. The results of this chapter are new and they are presented in [15].

3.1 Almost periodic motions

A motion is called almost periodic when the orbit of a point from this motion regularly returns close to itself within some fixed interval of time, independent of the point. This theory was extensively investigated in dynamical systems without impulsive effects, see [1], [20], [30] and [38]. In the case of impulsive systems, this theory was developed in [9].

The concept of almost periodic motions is defined as follows.

Definition 3.1.1. A point $x \in X$ is said to be **almost $\tilde{\pi}$ -periodic** if for every $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that for every $\alpha \geq 0$, the interval $[\alpha, \alpha + T]$ contains a number $\tau_\alpha > 0$ such that

$$\rho(\tilde{\pi}(x, t + \tau_\alpha), \tilde{\pi}(x, t)) < \epsilon \quad \text{for all } t \geq 0. \quad (3.1)$$

The set $\{\tau_\alpha : \alpha \geq 0\}$ is called a **family of almost period** of x .

On continuous dynamical systems, every point from the closure of an almost periodic orbit is almost periodic, see [20]. In general, it does not hold for impulsive systems since the impulsive set may “destroy” this property. In the sequel, we discuss some results about this fact.

In [9, Lemma 4.22] it is proved that whether $x \in X$ is almost $\tilde{\pi}$ -periodic, then every point $y \in \tilde{\pi}^+(x)$ is also almost $\tilde{\pi}$ -periodic. However, we have a more general result as presented in the next theorem.

Theorem 3.1.2. *If $x \in X$ is almost $\tilde{\pi}$ -periodic, then every point $y \in \overline{\tilde{\pi}^+(x)} \setminus M$ is almost $\tilde{\pi}$ -periodic. Moreover, if $\{\tau_\alpha : \alpha \geq 0\}$ is a family of almost period of x then $\{\tau_\alpha : \alpha \geq 0\}$ is also a family of almost period for each $y \in \overline{\tilde{\pi}^+(x)} \setminus M$.*

Proof. Let $\epsilon > 0$ be given. Since $x \in X$ is almost $\tilde{\pi}$ -periodic, then there is $T = T(\frac{\epsilon}{3}) > 0$ such that for every $\alpha \geq 0$, the interval $[\alpha, \alpha + T]$ contains a number $\tau_\alpha > 0$ such that

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(x, t + \tau_\alpha)) < \frac{\epsilon}{3} \quad \text{for all } t \geq 0. \quad (3.2)$$

Let $y \in \overline{\tilde{\pi}^+(x)} \setminus M$. Then there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ such that

$$y_n = \tilde{\pi}(x, \lambda_n) \xrightarrow{n \rightarrow +\infty} y.$$

For each $\alpha \geq 0$, we consider $\tau_\alpha \in [\alpha, \alpha + T]$ which satisfies (3.2). Let $t \geq 0$ be fixed and arbitrary. By Lemma 1.4.3 there is a sequence $\{\epsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\epsilon_n \xrightarrow{n \rightarrow +\infty} 0$ and $\tilde{\pi}(y_n, t + \epsilon_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(y, t)$. Since $t + \tau_\alpha + \epsilon_n \geq t + \tau_\alpha$, for every $n \in \mathbb{N}$, and $t + \tau_\alpha + \epsilon_n \xrightarrow{n \rightarrow +\infty} t + \tau_\alpha$, it follows by Lemma 1.4.3 that there exists a sequence $\{\beta_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\beta_n \xrightarrow{n \rightarrow +\infty} 0$ and

$$\tilde{\pi}(y_n, t + \tau_\alpha + \epsilon_n + \beta_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(y, t + \tau_\alpha).$$

Moreover, since $\tilde{\pi}(y, t) \notin M$ it follows by Lemma 1.4.4, that

$$\tilde{\pi}(y_n, t + \epsilon_n + \beta_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(y, t).$$

Thus, there is $n_0 \in \mathbb{N}$ such that

$$\rho(\tilde{\pi}(y_n, t + \epsilon_n + \beta_n), \tilde{\pi}(y, t)) < \frac{\epsilon}{3} \quad \text{and} \quad \rho(\tilde{\pi}(y_n, t + \tau_\alpha + \epsilon_n + \beta_n), \tilde{\pi}(y, t + \tau_\alpha)) < \frac{\epsilon}{3},$$

for all $n \geq n_0$.

Define $\eta_n = \epsilon_n + \beta_n$, $n \in \mathbb{N}$. Then by the above inequalities and (3.2), we have

$$\begin{aligned} \rho(\tilde{\pi}(y, t), \tilde{\pi}(y, t + \tau_\alpha)) &\leq \rho(\tilde{\pi}(y, t), \tilde{\pi}(y_{n_0}, t + \eta_{n_0})) + \rho(\tilde{\pi}(y_{n_0}, t + \eta_{n_0}), \tilde{\pi}(y_{n_0}, t + \tau_\alpha + \eta_{n_0})) \\ &\quad + \rho(\tilde{\pi}(y_{n_0}, t + \tau_\alpha + \eta_{n_0}), \tilde{\pi}(y, t + \tau_\alpha)) < \epsilon. \end{aligned}$$

Since t was taken arbitrary, it shows that $y \in \overline{\tilde{\pi}^+(x)} \setminus M$ is almost $\tilde{\pi}$ -periodic with the same family of almost period $\{\tau_\alpha : \alpha \geq 0\}$. $\square$

Theorem 3.1.2 shows that all the points from $\overline{\tilde{\pi}^+(x)} \setminus M$ are almost $\tilde{\pi}$ -periodic provided that x is almost $\tilde{\pi}$ -periodic. In Theorem 3.1.3, we characterize the points from $\overline{\tilde{\pi}^+(x)} \cap M$.

Theorem 3.1.3. *Let $x \in X$ be almost $\tilde{\pi}$ -periodic, $y \in \overline{\tilde{\pi}^+(x)} \cap M$ and $\{y_n\}_{n \in \mathbb{N}} \subset \tilde{\pi}^+(x)$ be a sequence such that $y_n \xrightarrow{n \rightarrow +\infty} y \in X$. Assuming the notations of Remark 1.4.1 we have:*

- i) if $\{y_n\}_{n \in \mathbb{N}}$ admits a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}} \subset H_2^{(y)}$, then y is almost $\tilde{\pi}$ -periodic. Moreover, y admits the same family of almost period of x ;*
- ii) if $\{y_n\}_{n \in \mathbb{N}}$ admits a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}} \subset H_1^{(y)}$, then $I(y)$ is almost $\tilde{\pi}$ -periodic. Moreover, $I(y)$ admits the same family of almost period of x .*

Proof. Let $y \in \overline{\tilde{\pi}^+(x)} \cap M$. The proof of item *i)* is similar to the proof of Theorem 3.1.2.

Let us show that item *ii)* holds. Suppose without loss of generality that $\{y_n\}_{n \in \mathbb{N}} \subset H_1^{(y)}$. Then $\phi(y_n) \xrightarrow{n \rightarrow +\infty} 0$ and by continuity of I and π , we obtain

$$z_n = \tilde{\pi}(y_n, \phi(y_n)) \xrightarrow{n \rightarrow +\infty} I(y).$$

Since $\{z_n\}_{n \in \mathbb{N}} \subset \tilde{\pi}^+(x) \setminus M$ (see condition **(H2)**), it follows by Theorem 3.1.2 that z_n is almost $\tilde{\pi}$ -periodic, for each $n \in \mathbb{N}$, with the same family of almost period of x . Consequently, given $\epsilon > 0$, there is $T = T(\frac{\epsilon}{3}) > 0$ such that for every $\alpha \geq 0$, we can find $\tau_\alpha \in [\alpha, \alpha + T]$ which satisfies

$$\rho(\tilde{\pi}(z_n, t), \tilde{\pi}(z_n, t + \tau_\alpha)) < \frac{\epsilon}{3} \quad \text{for all } t \geq 0 \quad \text{and} \quad \text{for all } n \in \mathbb{N}. \quad (3.3)$$

Fix $\alpha \geq 0$ and take $\tau_\alpha \in [\alpha, \alpha + T]$. Now let $t \geq 0$ be fixed and arbitrary. Since $I(y) \notin M$ by condition **(H2)**, it follows by Lemma 1.4.3 that there exists a sequence $\{\epsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\epsilon_n \xrightarrow{n \rightarrow +\infty} 0$ and $\tilde{\pi}(z_n, t + \epsilon_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(I(y), t)$. Note that $t + \tau_\alpha + \epsilon_n \geq t + \tau_\alpha$, for every $n \in \mathbb{N}$,

and $t + \tau_\alpha + \epsilon_n \xrightarrow{n \rightarrow +\infty} t + \tau_\alpha$. Thus, by Lemma 1.4.5, there exists a sequence $\{\beta_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\beta_n \xrightarrow{n \rightarrow +\infty} 0$ and

$$\tilde{\pi}(z_n, t + \tau_\alpha + \epsilon_n + \beta_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(I(y), t + \tau_\alpha).$$

Also, since $\tilde{\pi}(I(y), t) \notin M$ it follows by Lemma 1.4.2 that

$$\tilde{\pi}(z_n, t + \epsilon_n + \beta_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(I(y), t).$$

Set $\eta_n = \epsilon_n + \beta_n$, $n \in \mathbb{N}$. We can choose $n_0 \in \mathbb{N}$ such that

$$\rho(\tilde{\pi}(z_n, t + \eta_n), \tilde{\pi}(I(y), t)) < \frac{\epsilon}{3} \quad \text{and} \quad \rho(\tilde{\pi}(z_n, t + \tau_\alpha + \eta_n), \tilde{\pi}(I(y), t + \tau_\alpha)) < \frac{\epsilon}{3},$$

for all $n \geq n_0$. In virtue of the above inequalities and (3.3), we have

$$\begin{aligned} \rho(\tilde{\pi}(I(y), t), \tilde{\pi}(I(y), t + \tau_\alpha)) &\leq \rho(\tilde{\pi}(I(y), t), \tilde{\pi}(z_{n_0}, t + \eta_{n_0})) \\ &\quad + \rho(\tilde{\pi}(z_{n_0}, t + \eta_{n_0}), \tilde{\pi}(z_{n_0}, t + \tau_\alpha + \eta_{n_0})) \\ &\quad + \rho(\tilde{\pi}(z_{n_0}, t + \tau_\alpha + \eta_{n_0}), \tilde{\pi}(I(y), t + \tau_\alpha)) < \epsilon. \end{aligned}$$

Since the above inequality holds for each $\tau_\alpha \in [\alpha, \alpha + T]$ ($\alpha \geq 0$) and $t \geq 0$ was taken arbitrary, then $I(y)$ is almost $\tilde{\pi}$ -periodic. $\square$

Remark 3.1.4. If $x \in X$ is $\tilde{\pi}$ -periodic, then using the same arguments of the proof of Theorem 3.1.2, we can prove that each point $y \in \overline{\tilde{\pi}^+(x)} \setminus M$ is $\tilde{\pi}$ -periodic. If $y \in \overline{\tilde{\pi}^+(x)} \cap M$ then y is $\tilde{\pi}$ -periodic or $I(y)$ is $\tilde{\pi}$ -periodic, its proof is analogous to the proof of Theorem 3.1.3.

Definition 3.1.5 deals with the concept of relatively dense sets. This concept is presented in Definition 3.11, Chapter III, from [1].

Definition 3.1.5. A set $D \subset \mathbb{R}_+$ is said to be **relatively dense** in $\mathbb{R}_+$ if there is a number $L > 0$ such that $D \cap [t, t + L] \neq \emptyset$ for all $t \in \mathbb{R}_+$.

Next, we present a result which relates the concept of almost $\tilde{\pi}$ -periodic motions with relatively dense sets.

Lemma 3.1.6. A point $x \in X$ is almost $\tilde{\pi}$ -periodic if and only if for every $\epsilon > 0$ the set

$$\rho(\epsilon) = \left\{ \tau \in \mathbb{R}_+ : \sup_{t \in \mathbb{R}_+} \rho(\tilde{\pi}(x, t + \tau), \tilde{\pi}(x, t)) < \epsilon \right\}$$

is relatively dense in $\mathbb{R}_+$.

Proof. Let $x \in X$ be almost $\tilde{\pi}$ -periodic and $\epsilon > 0$ be given. Then there is $T = T(\frac{\epsilon}{2}) > 0$ such that for all $\alpha \geq 0$, one can obtain $0 \neq \tau_\alpha \in [\alpha, \alpha + T]$ such that

$$\sup_{t \in \mathbb{R}_+} \rho(\tilde{\pi}(x, t + \tau_\alpha), \tilde{\pi}(x, t)) \leq \frac{\epsilon}{2} < \epsilon.$$

Thus, $\rho(\epsilon) \cap [\alpha, \alpha + T] \neq \emptyset$ and the set $\rho(\epsilon)$ is relatively dense in $\mathbb{R}_+$. The converse is straightforward. $\square$

In what follows, we present the definition of $\tilde{\pi}$ -recurrent points.

Definition 3.1.7. A point $x \in X$ is called $\tilde{\pi}$ -**recurrent** if for every $\epsilon > 0$ there exists a $T = T(\epsilon) > 0$, such that for every $t, s \geq 0$, the interval $[0, T]$ contains a number $\tau > 0$ such that $\rho(\tilde{\pi}(x, t), \tilde{\pi}(x, s + \tau)) < \epsilon$.

In [9, Theorem 4.23], the authors show that if a point $x \in X \setminus M$ is almost $\tilde{\pi}$ -periodic then it is $\tilde{\pi}$ -recurrent provided that $\overline{\tilde{\pi}^+(x)}$ is compact. However, if X is a complete metric space then $\overline{\tilde{\pi}^+(x)}$ is compact provided $x \in X$ is almost $\tilde{\pi}$ -periodic, see Theorem 3.1.8 below. In conclusion, on complete metric spaces any almost $\tilde{\pi}$ -periodic point is also $\tilde{\pi}$ -recurrent, see Corollary 3.1.9.

Theorem 3.1.8. *Let X be a complete metric space. If $x \in X$ is almost $\tilde{\pi}$ -periodic then the set $\overline{\tilde{\pi}^+(x)}$ is compact.*

Proof. Let $\epsilon > 0$ and $x \in X$ be almost $\tilde{\pi}$ -periodic. From Lemma 3.1.6, the set $\rho(\frac{\epsilon}{4})$ is relatively dense in $\mathbb{R}_+$. Thus

$$\rho(\tilde{\pi}(x, t + \tau), \tilde{\pi}(x, t)) < \frac{\epsilon}{4}, \quad (3.4)$$

for all $t \in \mathbb{R}_+$ and for all $\tau \in \rho(\frac{\epsilon}{4})$. Besides, we obtain

$$\rho(\tilde{\pi}(x, t + \tau_1), \tilde{\pi}(x, t + \tau_2)) \leq \rho(\tilde{\pi}(x, t + \tau_1), \tilde{\pi}(x, t)) + \rho(\tilde{\pi}(x, t + \tau_2), \tilde{\pi}(x, t)) < \frac{\epsilon}{2}, \quad (3.5)$$

for all $t \in \mathbb{R}_+$ and for all $\tau_1, \tau_2 \in \rho(\frac{\epsilon}{4})$.

Define $\alpha = \inf\{\tau : \tau \in \rho(\frac{\epsilon}{4})\}$. Then there is a sequence $\{\tau_n\}_{n \in \mathbb{N}} \subset \rho(\frac{\epsilon}{4})$ such that $\tau_n \xrightarrow{n \rightarrow +\infty} \alpha$ and $\tau_n \geq \alpha$ for all $n \in \mathbb{N}$. Since $\tilde{\pi}(x, \cdot)$ is continuous from the right, we get

$$\tilde{\pi}(x, t + \tau_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(x, t + \alpha).$$

Then, using (3.5) we obtain

$$\begin{aligned} \rho(\tilde{\pi}(x, t + \alpha), \tilde{\pi}(x, t + \tau)) &\leq \rho(\tilde{\pi}(x, t + \alpha), \tilde{\pi}(x, t + \tau_n)) + \rho(\tilde{\pi}(x, t + \tau_n), \tilde{\pi}(x, t + \tau)) \\ &\leq \rho(\tilde{\pi}(x, t + \alpha), \tilde{\pi}(x, t + \tau_n)) + \frac{\epsilon}{2} \end{aligned} \quad (3.6)$$

for all $t \in \mathbb{R}_+$ and for all $\tau \in \rho(\frac{\epsilon}{4})$. When n approaches to $+\infty$ in (3.6), we obtain

$$\rho(\tilde{\pi}(x, t + \alpha), \tilde{\pi}(x, t + \tau)) \leq \frac{\epsilon}{2} \quad \text{for all } t \in \mathbb{R}_+ \text{ and for all } \tau \in D(\frac{\epsilon}{4}). \quad (3.7)$$

On the other hand, as $\rho(\frac{\epsilon}{4})$ is relatively dense in $\mathbb{R}_+$, there is $L > 0$ such that $\rho(\frac{\epsilon}{4}) \cap [t, t + L] \neq \emptyset$ for all $t \in \mathbb{R}_+$. Let $s > L$, then one can choose $\tau_s \in \rho(\frac{\epsilon}{4}) \cap [s - L, s]$. By (3.7) we have

$$\rho(\tilde{\pi}(x, s), \tilde{\pi}(x, s - \tau_s + \alpha)) = \rho(\tilde{\pi}(x, (s - \tau_s) + \tau_s), \tilde{\pi}(x, (s - \tau_s) + \alpha)) < \epsilon,$$

which implies that $\tilde{\pi}(x, s) \in B(\tilde{\pi}(x, [\alpha, L + \alpha]), \epsilon)$ for all $s > L$. Thus,

$$\tilde{\pi}(x, t) \in B(\tilde{\pi}(x, [0, L + \alpha]), \epsilon) \subset B(Q_\alpha, \epsilon) \quad \text{for all } t \geq 0,$$

where $Q_\alpha = \overline{\tilde{\pi}(x, [0, L + \alpha])}$. By Lemma 1.4.7 the set Q_α is compact. Then $\overline{\tilde{\pi}^+(x)}$ is totally bounded and as X is complete, we conclude that $\overline{\tilde{\pi}^+(x)}$ is compact. $\square$

Corollary 3.1.9. *If X is a complete metric space and $x \in X \setminus M$ is almost $\tilde{\pi}$ -periodic then x is $\tilde{\pi}$ -recurrent.*

Proof. The proof follows by Theorem 3.1.8 and [9, Theorem 4.23]. $\square$

In the next definition, we present the concept of Poisson stability. The reader may consult [4] to obtain more details about this kind of stability.

Definition 3.1.10. A point $x \in X$ is said to be **positively Poisson $\tilde{\pi}$ -stable**, if $x \in \tilde{L}^+(x)$.

The next result says that if $x \in X$ is almost $\tilde{\pi}$ -periodic then the positive orbit of x is dense in its positive limit set. As a consequence of Theorem 3.1.11, we conclude that x is positively Poisson $\tilde{\pi}$ -stable.

Theorem 3.1.11. *If $x \in X$ is almost $\tilde{\pi}$ -periodic, then $\tilde{L}^+(x) = \overline{\tilde{\pi}^+(x)}$. Moreover, x is positively Poisson $\tilde{\pi}$ -stable.*

Proof. It is clear that $\tilde{L}^+(x) \subset \overline{\tilde{\pi}^+(x)}$. Let us show that $\overline{\tilde{\pi}^+(x)} \subset \tilde{L}^+(x)$. Let $\epsilon > 0$ and $p \in \overline{\tilde{\pi}^+(x)}$, then there is a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\tilde{\pi}(x, \lambda_n) \xrightarrow{n \rightarrow +\infty} p$. Take $n_0 \in \mathbb{N}$ such that

$$\rho(\tilde{\pi}(x, \lambda_n), p) < \frac{\epsilon}{2}, \quad n \geq n_0.$$

Since x is almost $\tilde{\pi}$ -periodic, then one can conclude that $\tilde{\pi}(x, \lambda_n)$ is almost $\tilde{\pi}$ -periodic for each $n \in \mathbb{N}$ with the same family of almost period of x , see Theorem 3.1.2. Then there are $T > 0$ and $\tau_n \in [n, n + T]$, $n \in \mathbb{N}$, such that

$$\rho(\tilde{\pi}(x, \lambda_n + \tau_n), \tilde{\pi}(x, \lambda_n)) < \frac{\epsilon}{2} \quad \text{for all } n \in \mathbb{N}.$$

Thus, for all $n \geq n_0$, we get

$$\rho(\tilde{\pi}(x, \lambda_n + \tau_n), p) \leq \rho(\tilde{\pi}(x, \lambda_n + \tau_n), \tilde{\pi}(x, \lambda_n)) + \rho(\tilde{\pi}(x, \lambda_n), p) < \epsilon.$$

As $\lambda_n + \tau_n \xrightarrow{n \rightarrow +\infty} +\infty$, we have $p \in \tilde{L}^+(x)$ and the proof is complete. $\square$

3.2 Zhukovskij quasi $\tilde{\pi}$ -stability

In the classical theory of dynamical systems, almost periodicity of motions is deeply connected with stability, see [1] and [20] for instance. However, due to the discontinuities of impulsive systems, it is very hard to deal with the connection between almost periodic motions and stability in the sense of Lyapunov. On the other hand, there exists a type of stability that is very useful when we are working with impulsive systems. This stability is known as “quasi-stability of Zhukovskij” and it is a type of phase stability where the traces of the orbits are close but with a certain time lag. The quasi-stability of Zhukovskij was initially introduced by Changming Ding in [26]. In this section, we investigate the connection between asymptotically almost period motions and quasi-stability of Zhukovskij.

For this purpose, we introduce the notion of time reparametrization.

Definition 3.2.1. A function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a **time reparametrization** if h is a homeomorphism and $h(0) = 0$.

Definition 3.2.2. A point $x \in X \setminus M$ is called **Zhukovskij quasi $\tilde{\pi}$ -stable** with respect to the set $A \subset X$, if for every $\epsilon > 0$ there is $\delta = \delta(x, \epsilon) > 0$ such that if $\rho(x, y) < \delta$ for $y \in A$, then one can find a time reparametrization h_y such that

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(y, h_y(t))) < \epsilon \quad \text{for all } t \geq 0.$$

A subset $B \subset X \setminus M$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to the set $A \subset X$, if each point $z \in B$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to A .

Example 3.2.3. Let $(\mathbb{R}, \pi; M, I)$ be the ISS given in Example 1.2.4. Let $x \in [0, 1)$. Note that $\tilde{\pi}^+(x) = [0, 1)$. For each $y \in [0, 1)$ we have $\phi(y) = 1 - y$, $y_n^+ = x_n^+$ and $\phi(y_n^+) = \phi(x_n^+) = 1$ for all $n \in \mathbb{N}_0$. Consider the time reparametrization

$$h_y(t) = \begin{cases} \left(\frac{1-y}{1-x}\right)t, & 0 \leq t < 1-x, \\ t + (x-y), & t \geq 1-x. \end{cases}$$

Let $\epsilon > 0$ be given and $y \in \tilde{\pi}^+(x)$ be such that $\rho(x, y) < \frac{\epsilon}{2}$. Suppose that $t < 1-x = \phi(x)$, then

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(y, h_y(t))) = \rho\left(\pi(x, t), \pi\left(y, \frac{(1-y)}{1-x}t\right)\right) = \left|x + t - y - \frac{(1-y)}{1-x}t\right| \leq 2|x-y| < \epsilon.$$

For $t \geq 1-x$, there is $n \in \mathbb{N}$ such that $n + \phi(x) < t < n + 1 + \phi(x)$, hence, we get

$$\begin{aligned} \rho(\tilde{\pi}(x, t), \tilde{\pi}(y, h_y(t))) &= \rho(\tilde{\pi}(x, \phi(x) + t - \phi(x)), \tilde{\pi}(y, t + x - y + (1 - 1))) \\ &= \rho(\tilde{\pi}(x_1^+, t - \phi(x)), \tilde{\pi}(y_1^+, t - \phi(x))) = 0 < \epsilon. \end{aligned}$$

Therefore, $\tilde{\pi}^+(x)$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to $\tilde{\pi}^+(x)$.

The next result shows sufficient conditions for a set to be Zhukovskij quasi $\tilde{\pi}$ -stable.

Theorem 3.2.4. Let $(X, \pi; M, I)$ be an ISS and $x \in X \setminus M$. Suppose that $T = \sup_{k \geq 1} \phi(x_k^+) < +\infty$ and the following assumptions hold:

- i) $\rho(I(p), I(q)) \leq \lambda_1 \rho(p, q)$ for all $p, q \in M$ and $\rho(\pi(p, \phi(p)), \pi(q, \phi(q))) \leq \lambda_2 \rho(p, q)$ for all $p, q \in \overline{\tilde{\pi}^+(x)} \setminus M$, where $\lambda_1, \lambda_2 > 0$ and $\lambda_1 \lambda_2 \leq 1$;
- ii) $|\phi(p_1^+) - \phi(q_1^+)| \leq |\phi(p) - \phi(q)|$ for all $p, q \in \overline{\tilde{\pi}^+(x)} \setminus M$;
- iii) $\overline{\tilde{\pi}^+(x)}$ is compact.

Then every subset A in $\overline{\tilde{\pi}^+(x)} \setminus M$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to $\tilde{\pi}^+(x)$.

Proof. Let $\epsilon > 0$ and $z \in \overline{\tilde{\pi}^+(x)} \setminus M$. Since π is uniformly continuous on $K = \overline{\tilde{\pi}^+(x)} \times [0, T]$, there is $\delta_1 = \delta_1(K, \epsilon) > 0$, $\delta_1 < \epsilon$, such that if $y \in \tilde{\pi}^+(x)$, $t_1, t_2 \in [0, T]$ and $\max\{\rho(y, z), |t_1 - t_2|\} < \delta_1$, then

$$\rho(\pi(z, t_1), \pi(y, t_2)) < \epsilon. \quad (3.8)$$

Now, since ϕ is continuous on $X \setminus M$, then there is $\delta = \delta(z, \delta_1) > 0$, $\delta < \delta_1$, such that if $y \in X$ and $\rho(y, z) < \delta$ then

$$|\phi(y) - \phi(z)| < \delta_1. \quad (3.9)$$

Thus, if $\rho(y, z) < \delta$, it follows by conditions *i*), *ii*) and (3.9) that

$$\rho(z_1^+, y_1^+) \leq \lambda_1 \lambda_2 \rho(z, y) < \delta \implies |\phi(z_1^+) - \phi(y_1^+)| \leq |\phi(z) - \phi(y)| < \delta_1,$$

where $z_1 = \pi(z, \phi(z))$, $y_1 = \pi(y, \phi(y))$, $z_1^+ = I(z_1)$ and $y_1^+ = I(y_1)$. Using the principle of induction, if $\rho(z, y) < \delta$ then

$$\rho(z_n^+, y_n^+) < \delta,$$

and therefore,

$$|\phi(z_n^+) - \phi(y_n^+)| < \delta_1, \quad \text{for all } n = 0, 1, 2, \dots$$

Define the time reparametrization $h_y : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$h_y(t) = t_n(y) + \frac{\phi(y_n^+)}{\phi(z_n^+)}(t - t_n(z)) \quad \text{if } t \in [t_n(z), t_{n+1}(z)), \quad n = 0, 1, 2, \dots$$

Thus, if $t \in [t_n(z), t_{n+1}(z))$, $n = 0, 1, 2, \dots$, we have

$$\rho(\tilde{\pi}(z, t), \tilde{\pi}(y, h_y(t))) = \rho\left(\pi(z_n^+, t - t_n(z)), \pi\left(y_n^+, \frac{\phi(y_n^+)}{\phi(z_n^+)}(t - t_n(z))\right)\right).$$

If $\rho(z, y) < \delta$ then $\rho(z_n^+, y_n^+) < \delta$ for all $n = 0, 1, 2, \dots$, consequently

$$\left|t - t_n(z) - \frac{\phi(y_n^+)}{\phi(z_n^+)}(t - t_n(z))\right| < |\phi(z_n^+) - \phi(y_n^+)| < \delta_1,$$

and by (3.8), we obtain

$$\rho(\tilde{\pi}(z, t), \tilde{\pi}(y, h_y(t))) < \epsilon.$$

Since $t \geq 0$ was taken arbitrary, we conclude that every point $z \in A \subset \overline{\tilde{\pi}^+(x)} \setminus M$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to the set $\tilde{\pi}^+(x)$ and the proof is complete. $\square$

Corollary 3.2.5. *Let $(X, \pi; M, I)$ be an ISS satisfying the conditions *i*) and *ii*) of Theorem 3.2.4, X be a complete metric space and $x \in X \setminus M$ be almost $\tilde{\pi}$ -periodic. If $\phi(x_k^+) < +\infty$ for all $k \in \mathbb{N}$ then $\tilde{L}^+(x) \setminus M$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to $\tilde{\pi}^+(x)$.*

Proof. Since x is almost $\tilde{\pi}$ -periodic, the set $\overline{\tilde{\pi}^+(x)}$ is compact. Thus $\overline{\{x_k^+ : k \in \mathbb{N}\}}$ is compact. By the compactness of $\overline{\{x_k^+ : k \in \mathbb{N}\}}$, condition **(H2)** and since $\{x_k^+\}_{k \in \mathbb{N}} \subset I(M)$ we have $\overline{\{x_k^+ : k \in \mathbb{N}\}} \cap M = \emptyset$ and therefore, $\sup_{k \geq 1} \phi(x_k^+) < +\infty$ because ϕ is uniformly continuous on compact sets $K \subset X \setminus M$. The result follows by Theorem 3.2.4. $\square$

3.3 Asymptotically almost periodic motions

In [20], the author presents a study of asymptotic stability in the sense of Poisson for continuous dynamical systems (in particular, asymptotic almost periodic motions). He shows various results of asymptotic periodicity and asymptotic almost periodicity using Lyapunov stability. In this section, we present some generalizations for these results to the impulsive case.

Definition 3.3.1. A point $x \in X$ is called **asymptotically $\tilde{\pi}$ -stationary** (resp., **asymptotically $\tilde{\pi}$ -periodic, asymptotically almost $\tilde{\pi}$ -periodic, asymptotically $\tilde{\pi}$ -recurrent, asymptotically Poisson $\tilde{\pi}$ -stable**) if there exist a point $p \in X$ stationary (resp., $\tilde{\pi}$ -periodic, almost $\tilde{\pi}$ -periodic, $\tilde{\pi}$ -recurrent, positively Poisson $\tilde{\pi}$ -stable) and a reparametrization h_p such that

$$\lim_{t \rightarrow +\infty} \rho(\tilde{\pi}(x, t), \tilde{\pi}(p, h_p(t))) = 0. \quad (3.10)$$

Remark 3.3.2. If h is a time reparametrization, then the inverse h^{-1} is also a time reparametrization. Indeed, choosing $s = h(t)$, $t \in \mathbb{R}_+$, one can write

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(y, h(t))) = \rho(\tilde{\pi}(x, h^{-1}(s)), \tilde{\pi}(y, s)).$$

The next result says that whether x possesses an asymptotic property then each point from its positive orbit also possesses this property.

Lemma 3.3.3. *If $x \in X$ is asymptotically almost $\tilde{\pi}$ -periodic (resp., asymptotically $\tilde{\pi}$ -stationary, asymptotically $\tilde{\pi}$ -periodic), then every point $y \in \tilde{\pi}^+(x)$ is also asymptotically almost $\tilde{\pi}$ -periodic (resp., asymptotically $\tilde{\pi}$ -stationary, asymptotically $\tilde{\pi}$ -periodic).*

Proof. Suppose that $x \in X$ is asymptotically almost $\tilde{\pi}$ -periodic. Then, there exist a point $p \in X$ almost $\tilde{\pi}$ -periodic and a time reparametrization $h_p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\lim_{t \rightarrow +\infty} \rho(\tilde{\pi}(x, t), \tilde{\pi}(p, h_p(t))) = 0. \quad (3.11)$$

Let $y \in \tilde{\pi}^+(x)$, then $y = \tilde{\pi}(x, s)$ for some $s \in \mathbb{R}_+$. Note that $q = \tilde{\pi}(p, h_p(s))$ is almost $\tilde{\pi}$ -periodic, see Theorem 3.1.2. Consider the function $g_y : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by

$$g_y(t) = h_p(t + s) - h_p(s) \quad \text{for all } t \in \mathbb{R}_+.$$

It is clear that $g_y(0) = 0$ and g_y is a continuous function possessing a continuous inverse function given by $g_y^{-1}(t) = h_p^{-1}(t + h_p(s)) - s$. Then g_y is a time reparametrization. Moreover,

$$\begin{aligned} \rho(\tilde{\pi}(y, t), \tilde{\pi}(q, g_y(t))) &= \rho(\tilde{\pi}(x, t + s), \tilde{\pi}(p, h_p(s) + h_p(t + s) - h_p(s))) \\ &= \rho(\tilde{\pi}(x, t + s), \tilde{\pi}(p, h_p(t + s))) \xrightarrow{t \rightarrow +\infty} 0, \end{aligned}$$

where the convergence follows by (3.11). Therefore, $y \in \tilde{\pi}^+(x)$ is asymptotically almost $\tilde{\pi}$ -periodic. The other cases are analogous. $\square$

Theorem 3.3.4. *Let X be a complete metric space. If $x \in X$ is asymptotically almost $\tilde{\pi}$ -periodic, then:*

- i) $\overline{\tilde{\pi}^+(x)}$ is compact;
- ii) $\tilde{L}^+(x)$ coincides with the closure of an almost $\tilde{\pi}$ -periodic orbit.

Proof. First, we show that i) holds. Let $x \in X$ be asymptotically almost $\tilde{\pi}$ -periodic and $\epsilon > 0$. Then there are a point $p \in X$ almost $\tilde{\pi}$ -periodic, a time reparametrization $h_p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $t_0 > 0$ such that

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(p, h_p(t))) < \frac{\epsilon}{4} \quad \text{for all } t \geq t_0. \quad (3.12)$$

It shows that $\tilde{\pi}(x, [t_0, +\infty)) \subset B(\overline{\tilde{\pi}^+(p)}, \frac{\epsilon}{4})$.

According to Theorem 3.1.8, we have $\overline{\tilde{\pi}^+(p)}$ is compact because X is complete. Thus, there are $p_1, \dots, p_k \in \overline{\tilde{\pi}^+(p)}$ such that

$$\overline{\tilde{\pi}^+(p)} \subset B\left(p_1, \frac{\epsilon}{4}\right) \cup \dots \cup B\left(p_k, \frac{\epsilon}{4}\right). \quad (3.13)$$

Consequently,

$$B\left(\overline{\tilde{\pi}^+(p)}, \frac{\epsilon}{4}\right) \subset B\left(p_1, \frac{\epsilon}{2}\right) \cup \dots \cup B\left(p_k, \frac{\epsilon}{2}\right).$$

Let $h_p(t_0) = \eta$. By Theorem 3.1.11, we can obtain $\lambda_j > \eta$ such that $\rho(\tilde{\pi}(p, \lambda_j), p_j) < \frac{\epsilon}{4}$, for each $j = 1, 2, \dots, k$. Then, using (3.13) we get

$$\overline{\tilde{\pi}^+(p)} \subset B\left(\tilde{\pi}(p, \lambda_1), \frac{\epsilon}{2}\right) \cup \dots \cup B\left(\tilde{\pi}(p, \lambda_k), \frac{\epsilon}{2}\right).$$

For each $j = 1, 2, \dots, k$, let $s_j > t_0$ (because $\lambda_j > \eta$) be such that $h_p(s_j) = \lambda_j$. From (3.12) we have

$$\rho(\tilde{\pi}(x, s_j), \tilde{\pi}(p, \lambda_j)) = \rho(\tilde{\pi}(x, s_j), \tilde{\pi}(p, h_p(s_j))) < \frac{\epsilon}{4}.$$

We claim that $\tilde{\pi}(x, [t_0, +\infty)) \subset B(\tilde{\pi}(x, s_1), \epsilon) \cup \dots \cup B(\tilde{\pi}(x, s_k), \epsilon)$. Indeed, let $a \in \tilde{\pi}(x, [t_0, +\infty))$, then there is $j_0 \in \{1, 2, \dots, k\}$ such that $\rho(a, p_{j_0}) < \frac{\epsilon}{2}$. Thus,

$$\rho(a, \tilde{\pi}(x, s_{j_0})) \leq \rho(a, p_{j_0}) + \rho(p_{j_0}, \tilde{\pi}(p, \lambda_{j_0})) + \rho(\tilde{\pi}(p, \lambda_{j_0}), \tilde{\pi}(x, s_{j_0})) < \epsilon.$$

This shows that $\overline{\tilde{\pi}(x, [t_0, +\infty))}$ is totally bounded and therefore, it is compact since X is complete. By Lemma 1.4.7 the set $\overline{\tilde{\pi}(x, [0, t_0])}$ is compact. Hence, $\overline{\tilde{\pi}^+(x)}$ is compact.

Now, let us show that assertion *ii*) holds. It is enough to show that $\tilde{L}^+(x) = \overline{\tilde{\pi}^+(p)}$, where p is the point almost $\tilde{\pi}$ -periodic found above.

Let $q \in \overline{\tilde{\pi}^+(p)}$. By Theorem 3.1.11 we have $q \in \tilde{L}^+(p)$. Then there is a sequence $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ with $s_n \xrightarrow{n \rightarrow +\infty} +\infty$ such that $\tilde{\pi}(p, s_n) \xrightarrow{n \rightarrow +\infty} q$. Since

$$\rho(\tilde{\pi}(x, h_p^{-1}(s_n)), q) \leq \rho(\tilde{\pi}(x, h_p^{-1}(s_n)), \tilde{\pi}(p, s_n)) + \rho(\tilde{\pi}(p, s_n), q)$$

and x is asymptotically almost $\tilde{\pi}$ -periodic, we have

$$\rho(\tilde{\pi}(x, h_p^{-1}(s_n)), q) \xrightarrow{n \rightarrow +\infty} 0.$$

But $h_p^{-1}(s_n) \rightarrow +\infty$ as $n \rightarrow +\infty$, which implies that $q \in \tilde{L}^+(x)$. Thus $\overline{\tilde{\pi}^+(p)} \subset \tilde{L}^+(x)$.

On the other side, take $q \in \tilde{L}^+(x)$. Then there is a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\lambda_n \xrightarrow{n \rightarrow +\infty} +\infty$ and $\tilde{\pi}(x, \lambda_n) \xrightarrow{n \rightarrow +\infty} q$. By compactness of $\overline{\tilde{\pi}^+(p)}$, the sequence $\{\tilde{\pi}(p, h_p(\lambda_n))\}_{n \in \mathbb{N}}$ admits a convergent subsequence. We may assume without loss of generality that $\lim_{n \rightarrow +\infty} \tilde{\pi}(p, h_p(\lambda_n)) = y \in \overline{\tilde{\pi}^+(p)}$. As

$$\rho(\tilde{\pi}(x, \lambda_n), y) \leq \rho(\tilde{\pi}(x, \lambda_n), \tilde{\pi}(p, h_p(\lambda_n))) + \rho(\tilde{\pi}(p, h_p(\lambda_n)), y)$$

and x is asymptotically almost $\tilde{\pi}$ -periodic, we have

$$\rho(\tilde{\pi}(x, \lambda_n), y) \xrightarrow{n \rightarrow +\infty} 0.$$

Thus, by uniqueness, we have $q = y \in \overline{\tilde{\pi}^+(p)}$ which implies that $\tilde{L}^+(x) \subset \overline{\tilde{\pi}^+(p)}$. Consequently, $\tilde{L}^+(x) = \overline{\tilde{\pi}^+(p)}$ and the assertion *ii*) is proved. $\square$

Next, we exhibit sufficient conditions for a point to be asymptotically almost $\tilde{\pi}$ -periodic.

Theorem 3.3.5. *Let $x \in X \setminus M$ satisfying the following conditions:*

- i) $\overline{\tilde{\pi}^+(x)}$ is compact;*

ii) $\tilde{L}^+(x) \setminus M$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to the set $\tilde{\pi}^+(x)$;

iii) $\tilde{L}^+(x)$ coincides with the closure of an almost $\tilde{\pi}$ -periodic orbit.

Then x is asymptotically almost $\tilde{\pi}$ -periodic.

Proof. Let $\epsilon > 0$ be given. By condition iii), there exists an almost $\tilde{\pi}$ -periodic point $p \in X$ such that $\tilde{L}^+(x) = \overline{\tilde{\pi}^+(p)}$. Thus, there are $T = T(\epsilon) > 0$ and $\tau_n \in [n, n + T]$ such that

$$\rho(\tilde{\pi}(p, t + \tau_n), \tilde{\pi}(p, t)) < \frac{\epsilon}{2} \quad \text{for all } t \geq 0 \quad \text{and} \quad \text{for all } n \in \mathbb{N}. \quad (3.14)$$

According to item i), we may assume that $\tilde{\pi}(x, \tau_n) \xrightarrow{n \rightarrow +\infty} q$. Since $\tau_n \xrightarrow{n \rightarrow +\infty} +\infty$, one can conclude that $q \in \tilde{L}^+(x) = \overline{\tilde{\pi}^+(p)}$.

Case 1: $q \in \overline{\tilde{\pi}^+(p)} \setminus M$.

By Theorem 3.1.2, the point q is almost $\tilde{\pi}$ -periodic with the same family of almost period of p . Then

$$\rho(\tilde{\pi}(q, t + \tau_n), \tilde{\pi}(q, t)) < \frac{\epsilon}{2} \quad \text{for all } t \geq 0 \quad \text{and} \quad \text{for all } n \in \mathbb{N}. \quad (3.15)$$

Since $q \in \tilde{L}^+(x) \setminus M$, it follows by ii) that there is $\delta = \delta(q, \frac{\epsilon}{2}) > 0$ such that if $y \in \tilde{\pi}^+(x)$ and $\rho(q, y) < \delta$, then one can find a time reparametrization h_y such that

$$\rho(\tilde{\pi}(y, h_y(t)), \tilde{\pi}(q, t)) < \frac{\epsilon}{2} \quad \text{for all } t \geq 0.$$

By the convergence $\tilde{\pi}(x, \tau_n) \xrightarrow{n \rightarrow +\infty} q$, we may choose $n_0 \in \mathbb{N}$ for which $\rho(\tilde{\pi}(x, \tau_{n_0}), q) < \delta$. Consequently, we may find a time reparametrization h_0 such that

$$\rho(\tilde{\pi}(x, \tau_{n_0} + h_0(t)), \tilde{\pi}(q, t)) < \frac{\epsilon}{2} \quad \text{for all } t \geq 0. \quad (3.16)$$

Let us define $h_q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$h_q(t) = \begin{cases} h_0(t - \tau_{n_0}) + \tau_{n_0}, & t > \tau_{n_0}, \\ t, & t \in [0, \tau_{n_0}]. \end{cases} \quad (3.17)$$

Note that h_q is a time reparametrization. Using (3.15) and (3.16), we have

$$\begin{aligned} \rho(\tilde{\pi}(x, h_q(t)), \tilde{\pi}(q, t)) &= \rho(\tilde{\pi}(x, h_0(t - \tau_{n_0}) + \tau_{n_0}), \tilde{\pi}(q, t)) \\ &\leq \rho(\tilde{\pi}(x, \tau_{n_0} + h_0(t - \tau_{n_0})), \tilde{\pi}(q, t - \tau_{n_0})) \\ &\quad + \rho(\tilde{\pi}(q, t - \tau_{n_0}), \tilde{\pi}(q, t)) < \epsilon, \end{aligned}$$

for all $t \geq \tau_{n_0}$. Hence, x is asymptotically almost $\tilde{\pi}$ -periodic.

Case 2: $q \in \overline{\tilde{\pi}^+(p)} \cap M$.

Since M satisfies STC, using the notations of Remark 1.4.1, we may write

$$H_1^{(q)} = F(L_q, (\lambda_q, 2\lambda_q]) \cap B(q, \eta_q) \quad \text{and} \quad H_2^{(q)} = F(L_q, [0, \lambda_q]) \cap B(q, \eta_q).$$

Now, we need to consider the cases when $\{\tilde{\pi}(x, \tau_n)\}_{n \in \mathbb{N}}$ admits a subsequence in $H_1^{(q)}$ and when $\{\tilde{\pi}(x, \tau_n)\}_{n \in \mathbb{N}}$ admits a subsequence in $H_2^{(q)}$. For that, we shall consider the cases $\{\tilde{\pi}(x, \tau_n)\}_{n \in \mathbb{N}} \subset H_2^{(q)}$ and $\{\tilde{\pi}(x, \tau_n)\}_{n \in \mathbb{N}} \subset H_1^{(q)}$.

Subcase 2.1: $\{\tilde{\pi}(x, \tau_n)\}_{n \in \mathbb{N}} \subset H_2^{(q)}$.

By Theorem 3.1.3, the point q is almost $\tilde{\pi}$ -periodic with the same family of almost period of p . Then, for the $\epsilon > 0$ given before, we have

$$\rho(\tilde{\pi}(q, t + \tau_n), \tilde{\pi}(q, t)) < \frac{\epsilon}{2} \quad \text{for all } t \geq 0 \quad \text{and} \quad \text{for all } n \in \mathbb{N}. \quad (3.18)$$

Set $\bar{q} = \tilde{\pi}(q, s)$ for some $s \in (0, \phi(q))$. By the tube condition we have $\tilde{\pi}(x, \tau_n + s) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(q, s) = \bar{q}$, that is, $\bar{q} \in \tilde{L}^+(x) \setminus M$. Moreover, defining $t_n = \tau_n + s$, it follows by condition *ii*), there is $n_1 \in \mathbb{N}$ such that one can find a time reparametrization h_1 such that

$$\rho(\tilde{\pi}(x, t_{n_1} + h_1(t)), \tilde{\pi}(\bar{q}, t)) < \frac{\epsilon}{2} \quad \text{for all } t \geq 0. \quad (3.19)$$

Suppose that $t \geq t_{n_1} = \tau_{n_1} + s$. From (3.18), we obtain

$$\rho(\tilde{\pi}(\bar{q}, t - t_{n_1}), \tilde{\pi}(q, t)) = \rho(\tilde{\pi}(q, t - \tau_{n_1}), \tilde{\pi}(q, t)) < \frac{\epsilon}{2}. \quad (3.20)$$

Let $h_q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be given by

$$h_q(t) = \begin{cases} h_1(t - t_{n_1}) + t_{n_1}, & t > t_{n_1}, \\ t, & t \in [0, t_{n_1}]. \end{cases} \quad (3.21)$$

Then, h_q is a time reparametrization. Using (3.19) and (3.20), we get

$$\begin{aligned} \rho(\tilde{\pi}(x, h_q(t)), \tilde{\pi}(q, t)) &= \rho(\tilde{\pi}(x, h_1(t - t_{n_1}) + t_{n_1}), \tilde{\pi}(q, t)) \\ &\leq \rho(\tilde{\pi}(x, t_{n_1} + h_1(t - t_{n_1})), \tilde{\pi}(\bar{q}, t - t_{n_1})) + \\ &\quad + \rho(\tilde{\pi}(\bar{q}, t - t_{n_1}), \tilde{\pi}(q, t)) < \epsilon, \end{aligned}$$

for all $t \geq t_{n_1}$. Hence, x is asymptotically almost $\tilde{\pi}$ -periodic.

Subcase 2.2: $\{\tilde{\pi}(x, \tau_n)\}_{n \in \mathbb{N}} \subset H_1^{(q)}$.

In virtue of Theorem 3.1.3, the point $I(q)$ is almost $\tilde{\pi}$ -periodic with the same family of almost period of p . Then, for the $\epsilon > 0$ given before, we have

$$\rho(\tilde{\pi}(I(q), t + \tau_n), \tilde{\pi}(I(q), t)) < \frac{\epsilon}{2} \quad \text{for all } t \geq 0 \quad \text{and} \quad \text{for all } n \in \mathbb{N}. \quad (3.22)$$

Let $y_n = \tilde{\pi}(x, \tau_n)$ and $z_n = \tilde{\pi}(y_n, \phi(y_n))$, for each $n \in \mathbb{N}$. Note that $y_n \xrightarrow{n \rightarrow +\infty} q$, $\phi(y_n) \xrightarrow{n \rightarrow +\infty} 0$ and

$$z_n = \tilde{\pi}(y_n, \phi(y_n)) = I(\pi(y_n, \phi(y_n))) \xrightarrow{n \rightarrow +\infty} I(\pi(q, 0)) = I(q).$$

Thus $I(q) \in \tilde{L}^+(x) \setminus M$. Moreover, since $\tau_n \xrightarrow{n \rightarrow +\infty} +\infty$ we can choose $n_2 \in \mathbb{N}$ such that $\phi(y_n) < \tau_n$ for every $n \geq n_2$. By condition *ii*), there are $n_3 \in \mathbb{N}$ with $n_3 > n_2$ and a time reparametrization h_3 such that

$$\rho(\tilde{\pi}(z_{n_3}, h_3(t)), \tilde{\pi}(I(q), t)) < \frac{\epsilon}{2} \quad \text{for all } t \geq 0. \quad (3.23)$$

Define $h_{I(q)} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$h_{I(q)}(t) = \begin{cases} h_3(t - \tau_{n_3}) + \tau_{n_3} + \phi(y_{n_3}), & t > \tau_{n_3}, \\ t + \phi(y_{n_3}), & t \in (\phi(y_{n_3}), \tau_{n_3}], \\ 2t, & t \in [0, \phi(y_{n_3})]. \end{cases} \quad (3.24)$$

Then $h_{I(q)}$ is a time reparametrization and using (3.22), (3.23) and (3.24) we have

$$\begin{aligned} \rho(\tilde{\pi}(x, h_{I(q)}(t)), \tilde{\pi}(I(q), t)) &= \rho(\tilde{\pi}(x, h_3(t - \tau_{n_3}) + \tau_{n_3} + \phi(y_{n_3})), \tilde{\pi}(I(q), t)) \\ &= \rho(\tilde{\pi}(z_{n_3}, h_3(t - \tau_{n_3})), \tilde{\pi}(I(q), t)) \\ &\leq \rho(\tilde{\pi}(z_{n_3}, h_3(t - \tau_{n_3})), \tilde{\pi}(I(q), t - \tau_{n_3})) + \\ &\quad + \rho(\tilde{\pi}(I(q), t - \tau_{n_3}), \tilde{\pi}(I(q), t)) < \epsilon, \end{aligned}$$

for all $t > \tau_{n_3}$. Therefore, x is asymptotically almost $\tilde{\pi}$ -periodic and the result is proved. $\square$

In the case of asymptotically $\tilde{\pi}$ -periodic points, we have the following result.

Corollary 3.3.6. *Let $x \in X \setminus M$ satisfying the following conditions:*

- i) $\overline{\tilde{\pi}^+(x)}$ is compact;*

ii) $\tilde{L}^+(x) \setminus M$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to the set $\tilde{\pi}^+(x)$;

iii) $\tilde{L}^+(x)$ coincides with the closure of a $\tilde{\pi}$ -periodic orbit.

Then x is asymptotically $\tilde{\pi}$ -periodic.

Proof. Using the same arguments of the proof of Theorem 3.3.5, we get the result. $\square$

Example 3.3.7. Consider the impulsive system $(\mathbb{R}^2, \pi; M, I)$ presented in Example 1.2.5. Note that the impulse function I satisfies

$$\rho(I(p), I(q)) \leq \frac{1}{2} \rho(p, q), \quad \text{for all } p, q \in M.$$

It is not difficult to see that if $(x, y) \in \mathbb{R}^2$ and $x < 2$ then

$$\phi(x, y) = 2 - x \quad \text{and} \quad \pi((x, y), \phi(x, y)) = (2, y).$$

We claim that the point $(1, 1)$ is asymptotically $\tilde{\pi}$ -periodic. In fact, it is enough to show that the conditions of Corollary 3.3.6 are satisfied.

At first, we note that $\overline{\tilde{\pi}^+(1, 1)}$ is a compact set. If $p, q \in \overline{\tilde{\pi}^+(1, 1)} \setminus M$ then

$$|\phi(p_1^+) - \phi(q_1^+)| = 0 \leq |\phi(p) - \phi(q)|,$$

$$\rho(\pi(p, \phi(p)), \pi(q, \phi(q))) \leq \rho(p, q)$$

and $\phi((1, 1)_k^+) = 1$ for all $k = 1, 2, \dots$. By Theorem 3.2.4, we can conclude that $\tilde{L}^+(1, 1) \setminus M$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to the set $\tilde{\pi}^+(1, 1)$.

Note that $\tilde{L}^+(1, 1) = [1, 2] \times \{0\}$. Clearly $(1, 0)$ is $\tilde{\pi}$ -periodic and $\overline{\tilde{\pi}^+(1, 0)} = [1, 2] \times \{0\} = \tilde{L}^+(1, 1)$. Hence, the conditions of Corollary 3.3.6 are satisfied and we can conclude that $(1, 1)$ is asymptotically $\tilde{\pi}$ -periodic.

3.4 Asymptotically $\tilde{\pi}$ -periodic and stationary motions

In this section, we present results concerning on asymptotically $\tilde{\pi}$ -periodic and stationary motions.

Theorem 3.4.1. *Let $x \in X \setminus M$ be asymptotically $\tilde{\pi}$ -periodic. Then there is $\tau > 0$ such that $\{\tilde{\pi}(x, n\tau) : n \in \mathbb{N}\}$ is relatively compact in X .*

Proof. Since $x \in X \setminus M$ is asymptotically $\tilde{\pi}$ -periodic, then there exist $p \in X$ $\tilde{\pi}$ -periodic with period $\tau > 0$, and a time reparametrization $h_p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying

$$\lim_{t \rightarrow +\infty} \rho(\tilde{\pi}(x, t), \tilde{\pi}(p, h_p(t))) = 0.$$

In particular,

$$\lim_{n \rightarrow +\infty} \rho(\tilde{\pi}(x, n\tau), \tilde{\pi}(p, h_p(n\tau))) = 0. \quad (3.25)$$

For each $n \in \mathbb{N}$, there are $k_n \in \mathbb{N}$ and $r_n \in [0, \tau)$ such that $h_p(n\tau) = k_n\tau + r_n$. We may assume without loss of the generality that $r_n \xrightarrow{n \rightarrow +\infty} r_0 \in [0, \tau]$. Note that $\tilde{\pi}(p, h_p(n\tau)) = \tilde{\pi}(p, r_n)$, then by Lemmas 1.4.4 and 1.4.6, we may conclude that $\{\tilde{\pi}(p, h_p(n\tau))\}_{n \in \mathbb{N}}$ admits a convergent subsequence in X . Using (3.25), we conclude that the sequence $\{\tilde{\pi}(x, n\tau)\}_{n \in \mathbb{N}}$ possesses a convergent subsequence in X . $\square$

In Theorem 3.4.2, we present sufficient conditions to obtain the converse of Theorem 3.4.1.

Theorem 3.4.2. *Suppose that $(X, \pi; M, I)$ satisfies conditions i) and ii) of Theorem 3.2.4, $x \in X \setminus M$ and $\sup_{k \geq 1} \phi(x_k^+) < +\infty$. If the sequence $\{\tilde{\pi}(x, n\tau)\}_{n \in \mathbb{N}}$ converges in $X \setminus M$, for some $\tau > 0$, then x is asymptotically $\tilde{\pi}$ -periodic.*

Proof. In order to show this result, we will show that the conditions of Corollary 3.3.6 are satisfied. Let $p \in X \setminus M$ be such that $\tilde{\pi}(x, n\tau) \xrightarrow{n \rightarrow +\infty} p$. Firstly, let us prove that $\overline{\tilde{\pi}^+(x)}$ is compact. Indeed, let $\{y_n\}_{n \in \mathbb{N}} \subset \tilde{\pi}^+(x)$ be an arbitrary sequence. Then there is a sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $y_n = \tilde{\pi}(x, t_n)$, for every $n \in \mathbb{N}$.

If $\{t_n\}_{n \in \mathbb{N}}$ admits a convergent subsequence then $\{y_n\}_{n \in \mathbb{N}}$ also admits one. Now, if $t_n \xrightarrow{n \rightarrow +\infty} +\infty$, then for each $n \in \mathbb{N}$ there are $a_n \in \mathbb{N}$ and $b_n \in [0, \tau)$ such that $t_n = a_n\tau + b_n$. We may assume that $b_n \xrightarrow{n \rightarrow +\infty} b_0 \in [0, \tau]$. Since $a_n\tau \xrightarrow{n \rightarrow +\infty} +\infty$, $b_n \xrightarrow{n \rightarrow +\infty} b_0$ and $\tilde{\pi}(x, a_n\tau) \xrightarrow{n \rightarrow +\infty} p$, it follows that the sequence $\{y_n\}_{n \in \mathbb{N}}$ ($y_n = \tilde{\pi}(\tilde{\pi}(x, a_n\tau), b_n)$, $n = 1, 2, \dots$) admits a convergent subsequence in $\overline{\tilde{\pi}^+(x)}$, see Lemmas 1.4.3 and 1.4.6. Thus, $\overline{\tilde{\pi}^+(x)}$ is compact.

According to Theorem 3.2.4, $\tilde{L}^+(x) \setminus M$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to the set $\tilde{\pi}^+(x)$.

Now, we need to prove that $\tilde{L}^+(x)$ is the closure of a $\tilde{\pi}$ -periodic orbit. Recall that $p = \lim_{n \rightarrow +\infty} \tilde{\pi}(x, n\tau)$.

Since $p \notin M$, it follows by Lemma 1.4.3, that there is a sequence $\{\epsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ with $\epsilon_n \xrightarrow{n \rightarrow +\infty} 0$ such that

$$\tilde{\pi}(\tilde{\pi}(x, n\tau), \tau + \epsilon_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(p, \tau).$$

On the other hand, using Lemma 1.4.2, we get

$$\tilde{\pi}(\tilde{\pi}(x, n\tau), \tau + \epsilon_n) = \tilde{\pi}(\tilde{\pi}(x, (n+1)\tau), \epsilon_n) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(p, 0) = p.$$

By the uniqueness of limit, we get $p = \tilde{\pi}(p, \tau)$, that is, p is $\tilde{\pi}$ -periodic.

We claim that $\tilde{L}^+(x) = \overline{\tilde{\pi}^+(p)} = \overline{\tilde{\pi}(p, [0, \tau])}$. Indeed, if $q \in \tilde{L}^+(x)$, then there is a sequence $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\tau_n \xrightarrow{n \rightarrow +\infty} +\infty$ and $\tilde{\pi}(x, \tau_n) \xrightarrow{n \rightarrow +\infty} q$. There are $k_n \in \mathbb{N}$ and $r_n \in [0, \tau)$ such that $\tau_n = k_n\tau + r_n$, $n \in \mathbb{N}$. Using Lemmas 1.4.4 and 1.4.6, we may assume (we take subsequence if necessary) that

$$\tilde{\pi}(x, \tau_n) = \tilde{\pi}(\tilde{\pi}(x, k_n\tau), r_n) \xrightarrow{n \rightarrow +\infty} q \in \overline{\tilde{\pi}(p, [0, \tau])}.$$

On the other side, let $q \in \overline{\tilde{\pi}(p, [0, \tau])}$, then there is a sequence $\{s_n\}_{n \in \mathbb{N}} \subset [0, \tau]$ such that $q = \lim_{n \rightarrow +\infty} \tilde{\pi}(p, s_n)$. We may assume without loss of generality that $s_n \xrightarrow{n \rightarrow +\infty} s_0 \in [0, \tau]$. If $s_0 \neq t_k(p)$, for every $k = 1, 2, \dots$, then by Lemma 1.4.4, we get $q = \tilde{\pi}(p, s_0)$. Let $t_n = n\tau + s_0$, thus $t_n \xrightarrow{n \rightarrow +\infty} +\infty$ and using Lemma 1.4.4 again we have

$$\tilde{\pi}(x, t_n) = \tilde{\pi}(\tilde{\pi}(x, n\tau), s_0) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(p, s_0) = q.$$

Then, $q \in \tilde{L}^+(x)$.

Now, if $s_0 = t_k(p)$ for some $k \in \mathbb{N}$, then looking at the proof of Lemma 1.4.6 and taking in account that $\tilde{\pi}(x, n\tau) \xrightarrow{n \rightarrow +\infty} p$, we get that either

$$\tilde{\pi}(x, t_n) = \tilde{\pi}(\tilde{\pi}(x, n\tau), s_0) \xrightarrow{n \rightarrow +\infty} \tilde{\pi}(p, s_0) = p_k^+ = q$$

or

$$\tilde{\pi}(x, t_n) = \tilde{\pi}(\tilde{\pi}(x, n\tau), s_0) \xrightarrow{n \rightarrow +\infty} p_k = q.$$

In both of cases, we have $q \in \tilde{L}^+(x)$. Hence, $\tilde{L}^+(x) = \overline{\tilde{\pi}(p, [0, \tau])}$. By Corollary 3.3.6 we have x is asymptotically $\tilde{\pi}$ -periodic. $\square$

Corollary 3.4.3. *Suppose that $(X, \pi; M, I)$ satisfies the conditions i) and ii) of Theorem 3.2.4, $x \in X \setminus M$ and $\phi(x_k^+) < +\infty$ for all $k \in \mathbb{N}$. If the sequence $\{\tilde{\pi}(x, n\tau)\}_{n \in \mathbb{N}}$ converges in $X \setminus M$, for some $\tau > 0$, then x is asymptotically $\tilde{\pi}$ -periodic.*

Proof. By the proof of Theorem 3.4.2 we have $\overline{\tilde{\pi}^+(x)}$ compact. Thus $\{x_k^+ : k \in \mathbb{N}\}$ is compact. By the compactness of $\{x_k^+ : k \in \mathbb{N}\}$, condition (H2) and since $\{x_k^+\}_{k \in \mathbb{N}} \subset I(M)$ we have $\overline{\{x_k^+ : k \in \mathbb{N}\}} \cap M = \emptyset$ and therefore, $\sup_{k \geq 1} \phi(x_k^+) < +\infty$ because ϕ is uniformly continuous on compact sets $K \subset X \setminus M$. The result follows by Theorem 3.4.2. $\square$

We finish this section presenting necessary and sufficient conditions for a point to be asymptotically $\tilde{\pi}$ -stationary.

Theorem 3.4.4. *Let $(X, \pi; M, I)$ be an ISS. Then $x \in X$ is asymptotically $\tilde{\pi}$ -stationary if and only if the sequence $\{\tilde{\pi}(x, t_n)\}_{n \in \mathbb{N}}$ converges in X , where $t_n = \sum_{k=1}^n \frac{1}{k}$, $n \in \mathbb{N}$.*

Proof. First, we suppose that x is asymptotically $\tilde{\pi}$ -stationary. Then, there is a point $p \in X$ stationary such that

$$\lim_{t \rightarrow +\infty} \rho(\tilde{\pi}(x, t), p) = 0. \quad (3.26)$$

Since $t_n = \sum_{k=1}^n \frac{1}{k} \xrightarrow{n \rightarrow +\infty} +\infty$, it follows from (3.26) that $\{\tilde{\pi}(x, t_n)\}_{n \in \mathbb{N}}$ converges to $p \in X$.

Conversely, we suppose that $\tilde{\pi}(x, t_n) \xrightarrow{n \rightarrow +\infty} p \in X$, where $t_n = \sum_{k=1}^n \frac{1}{k}$, $n \in \mathbb{N}$. Let $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ be an arbitrary sequence such that $s_n \xrightarrow{n \rightarrow +\infty} +\infty$. There are $n_0 \in \mathbb{N}$ and a sequence $\{m_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ such that

$$t_{m_n} < s_n \leq t_{m_n+1},$$

for all $n \geq n_0$. Note that $0 < s_n - t_{m_n} \leq t_{m_n+1} - t_{m_n} = \frac{1}{m_n + 1}$, $n \geq n_0$.

Case 1: $p \notin M$.

In this case we have

$$\tilde{\pi}(x, s_n) = \tilde{\pi}(\tilde{\pi}(x, t_{m_n}), s_n - t_{m_n}) \xrightarrow{n \rightarrow +\infty} p,$$

since $\tilde{\pi}(x, t_{m_n}) \xrightarrow{n \rightarrow +\infty} p$ and $(s_n - t_{m_n}) \xrightarrow{n \rightarrow +\infty} 0$. Since $\{s_n\}_{n \in \mathbb{N}}$ is arbitrary we have

$$\lim_{t \rightarrow +\infty} \rho(\tilde{\pi}(x, t), p) = 0.$$

If $0 \leq s < \phi(p)$ then by Lemma 1.4.4 we get

$$p = \lim_{t \rightarrow +\infty} \tilde{\pi}(x, t + s) = \pi(p, s) = \tilde{\pi}(p, s).$$

Hence, p is stationary and x is asymptotically $\tilde{\pi}$ -stationary.

Case 2: $p \in M$.

By condition **(H1)** the set M satisfies STC. Using Remark 1.4.1, we may write

$$H_1^{(p)} = F(L_p, (\lambda_p, 2\lambda_p]) \cap B(p, \eta_p) \quad \text{and} \quad H_2^{(p)} = F(L_p, [0, \lambda_p]) \cap B(p, \eta_p).$$

We may assume that $\{\tilde{\pi}(x, t_{m_n})\}_{n \in \mathbb{N}} \subset H_1^{(p)}$. In fact, suppose that there is a subsequence $\{m_{n_k}\}_{k \in \mathbb{N}}$ such that $\tilde{\pi}(x, t_{m_{n_k}}) \in H_2^{(p)}$ for all $k \in \mathbb{N}$. Let $\{r_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+$ be an arbitrary sequence such that $r_k \xrightarrow{k \rightarrow +\infty} +\infty$. As we did before, we may assume that $t_{m_{n_k}} < r_k \leq t_{m_{n_k}+1}$ for all $k \in \mathbb{N}$. Then

$$\tilde{\pi}(x, r_k) = \tilde{\pi}(\tilde{\pi}(x, t_{m_{n_k}}), r_k - t_{m_{n_k}}) \xrightarrow{k \rightarrow +\infty} p \in M.$$

Since $\{r_k\}_{k \in \mathbb{N}}$ is arbitrary we have $\lim_{t \rightarrow +\infty} \rho(\tilde{\pi}(x, t), p) = 0$. This shows that p is stationary because

$$p = \lim_{k \rightarrow +\infty} \tilde{\pi}(x, s + t_{m_{n_k}}) = \pi(p, s)$$

for all $s \in [0, \phi(p))$. But $p \in M$ and it contradicts the definition of an impulsive system.

Then, let us assume $\{\tilde{\pi}(x, t_{m_n})\}_{n \in \mathbb{N}} \subset H_1^{(p)}$. In this case, we may also consider that $t_{m_n} < s_n - \phi(\tilde{\pi}(x, t_{m_n})) \leq t_{m_{n+1}}$ for all $n \geq n_0$ as $\phi(\tilde{\pi}(x, t_{m_n})) \xrightarrow{n \rightarrow +\infty} 0$. Consequently, $0 < s_n - t_{m_n} - \phi(\tilde{\pi}(x, t_{m_n})) \leq t_{m_{n+1}} - t_{m_n} = \frac{1}{m_n + 1}$. Since $\tilde{\pi}(x, t_{m_n} + \phi(\tilde{\pi}(x, t_{m_n}))) \xrightarrow{n \rightarrow +\infty} I(p)$ and $[s_n - t_{m_n} - \phi(\tilde{\pi}(x, t_{m_n}))] \xrightarrow{n \rightarrow +\infty} 0$, we have

$$\tilde{\pi}(x, s_n) = \tilde{\pi}(\tilde{\pi}(x, t_{m_n} + \phi(\tilde{\pi}(x, t_{m_n}))), s_n - t_{m_n} - \phi(\tilde{\pi}(x, t_{m_n}))) \xrightarrow{n \rightarrow +\infty} I(p).$$

Since $\{s_n\}_{n \in \mathbb{N}}$ is an arbitrary sequence we have $\lim_{t \rightarrow +\infty} \rho(\tilde{\pi}(x, t), I(p)) = 0$. Hence, $I(p)$ is stationary and x is asymptotically $\tilde{\pi}$ -stationary. Therefore, the result is proved. $\square$

Dissipative impulsive systems

The theory of dissipative autonomous systems with impulses is a recent theory which started by Bonotto and his collaborators in [6, 7] and, posteriorly, in [12]. In this chapter, we present some characterizations for dissipative systems with impulses in order to study a particular problem of Jack Hale, that is, a problem which deals with the existence of a maximal compact invariant set in discrete dynamical systems. A solution for this problem is known for locally bounded dynamical systems. Following this line of research, we consider the class of impulsive semidynamical systems and we present sufficient conditions to obtain the existence of a maximal compact invariant set for a system in this class. We use the theory of asymptotic compactness to get the main results. The new theory obtained in this chapter was inspired in the results established in [19] for continuous dynamical systems. The results of this chapter are presented in the paper [11].

4.1 Levinson's center

In this section, we summarize the basis of the theory of dissipativity for impulsive systems.

Let (X, ρ) be a metric space. Let $Comp(X)$ and $B(X)$ be the collection of all compact subsets and bounded subsets from X , respectively. As presented in Chapter 1, we shall consider the semi-distance of Hausdorff from A to B , with A and B bounded subsets from X , given by

$$dist(A, B) = \sup\{\rho(a, B) : a \in A\}.$$

Definition 4.1.1. Let $\mathfrak{M}$ be a family of subsets of X . An ISS $(X, \pi; M, I)$ is called **$\mathfrak{M}$ -dissipative** if there exists a bounded set $K \subset X \setminus M$ such that for every $A \in \mathfrak{M}$ we have

$$\lim_{t \rightarrow +\infty} dist(\tilde{\pi}(A, t), K) = 0,$$

that is, given $\epsilon > 0$ and $A \in \mathfrak{M}$ there exists $\ell(\epsilon, A) > 0$ such that $\tilde{\pi}(A, t) \subset B(K, \epsilon)$ for all $t \geq \ell(\epsilon, A)$, see Figure 4.1. In this case, the set K is called an **attractor** for the family $\mathfrak{M}$.

Figure 4.1: Dissipative impulsive system $(X, \pi; M, I)$.

In the sequel, we present some types of dissipativity for impulsive systems.

Definition 4.1.2. An ISS $(X, \pi; M, I)$ is called:

1. **point b -dissipative** if there exists a bounded subset $K \subset X \setminus M$ such that for every $x \in X$

$$\lim_{t \rightarrow +\infty} \rho(\tilde{\pi}(x, t), K) = 0; \tag{4.1}$$

2. **compact b -dissipative** if the convergence in (4.1) takes place uniformly with respect to x on the compact subsets from X ;

3. **locally b -dissipative** if for any point $x \in X$ there exists $\delta_x > 0$ such that the convergence in (4.1) takes place uniformly with respect to $y \in B(x, \delta_x)$;
4. **bounded b -dissipative** if the convergence in (4.1) takes place uniformly with respect to x on every bounded subset from X .

We note that a point (compact)(locally)(bounded) dissipative system is a $\mathfrak{M}$ -dissipative system with $\mathfrak{M} = \{\{x\} : x \in X\}$ ($\mathfrak{M} = \text{Comp}(X)$) ($\mathfrak{M} = \{B(x, \delta_x) : x \in X, \delta_x > 0\}$) ($\mathfrak{M} = B(X)$).

Remark 4.1.3. In Definition 4.1.2, when K is compact, we say that the ISS $(X, \pi; M, I)$ is **k -dissipative**.

As presented in [6], we have the following implications: bounded dissipativity $\Rightarrow$ local dissipativity $\Rightarrow$ compact dissipativity $\Rightarrow$ point dissipativity.

Next, we are going to define the center of Levinson of a compact k -dissipative impulsive system. Before that, we present some auxiliaries results.

Lemma 4.1.4. [6, Lemma 3.7] *Let $(X, \pi; M, I)$ be an ISS and $A \subset X$ be a set such that $\tilde{\pi}^+(A)$ is relatively compact. If $A \cap M = \emptyset$ and $\tilde{L}^+(A) \subset A$, then*

$$\tilde{L}^+(A) = \cap \{\tilde{\pi}(A, t) : t \geq 0\}.$$

Let $(X, \pi; M, I)$ be compact k -dissipative and K be a nonempty compact set such that $K \cap M = \emptyset$ and it is an attractor for all compact subsets of X . Then for every compact $A \subset X$ the equality

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(A, t), K) = 0$$

holds. By Lemma 1.5.10, we have $\tilde{L}^+(A)$ nonempty, compact and

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(A, t), \tilde{L}^+(A)) = 0.$$

We also obtain $\tilde{\pi}^+(A)$ relatively compact and $\tilde{L}^+(A) \subset K$. In particular, by taking $A = K$ and using Lemma 4.1.4, it follows that

$$\tilde{L}^+(K) = \cap \{\tilde{\pi}(K, t) : t \geq 0\}.$$

The next lemma states that $\tilde{L}^+(K)$ is well-defined.

Lemma 4.1.5. [6, Lemma 3.10] *Let $(X, \pi; M, I)$ be compact k -dissipative. The set $\tilde{L}^+(K)$ does not depend on the choice of the set K which attracts all compact subsets of X and $K \cap M = \emptyset$.*

Definition 4.1.6. The set

$$J = \tilde{L}^+(K),$$

where K is a nonempty compact set such that $K \cap M = \emptyset$ and it is an attractor for all compact subsets of X , is called the **center of Levinson** of the compact k -dissipative system $(X, \pi; M, I)$.

Example 4.1.7. [23, Example 5.6] Let $X = \mathbb{R}^2 \times \{0, 1\}$ and consider the dynamical system

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y, \end{cases}$$

on $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$, independently. Now let $M_0 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0\}$, $M_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1/4, z = 1\}$ and consider the impulsive set $M = M_0 \cup M_1$. We define the impulse function I by $I(x, y, 0) = (x, y, 1)$ for $(x, y, 0) \in M_0$ and $I(x, y, 1) = (x, y, 0)$ for $(x, y, 1) \in M_1$. Then $(X, \pi; M, I)$ is compact k -dissipative and $J = \{(0, 0, 0), (0, 0, 1)\}$ is its center of Levinson.

The next theorem concerns the compactness, positive invariance, stability and attraction of the center of Levinson.

Theorem 4.1.8. [6, Theorem 3.1] *Let $(X, \pi; M, I)$ be compact k -dissipative and J be its center of Levinson. Then*

- i) J is a compact positively $\tilde{\pi}$ -invariant set;
- ii) J is $\tilde{\pi}$ -stable;
- iii) J is the attractor of the family of all compacts of X ;
- iv) J is the maximal compact positively $\tilde{\pi}$ -invariant set in $(X, \pi; M, I)$ such that $J \subset \tilde{\pi}(J, t)$ for each $t \geq 0$.

Definition 4.1.9. An ISS $(X, \pi; M, I)$ is called **locally asymptotically $\tilde{\pi}$ -condensing**, if for every point $x \in X$ there are $\delta_x > 0$ and a nonempty compact $K_x \subset X$ such that $K_x \cap M = \emptyset$ and

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B(x, \delta_x), t), K_x) = 0.$$

Let $\Omega = \overline{\cup\{\tilde{L}^+(x) : x \in X\}}$. The next theorem will be very useful to show the existence of a maximal compact invariant set for a compact dissipative system with impulses.

Theorem 4.1.10. [6, Theorem 3.11] *Let $(X, \pi; M, I)$ be point k -dissipative. Then $(X, \pi; M, I)$ is locally k -dissipative if and only if $(X, \pi; M, I)$ is locally asymptotically $\tilde{\pi}$ -condensing and $\tilde{D}^+(\Omega) \cap M = \emptyset$.*

4.2 Asymptotic compactness

We start by presenting the concept of $\tilde{\pi}$ -asymptotic compactness on impulsive semidynamical systems which was established in [7]. The results of this section concern with the characterization of asymptotically compact systems.

Definition 4.2.1. An ISS $(X, \pi; M, I)$ is called $\tilde{\pi}$ -**asymptotically compact**, if for every bounded positively $\tilde{\pi}$ -invariant set $B \subset X$ there exists a nonempty compact $K_B \subset X$ such that $K_B \cap M = \emptyset$ and

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, t), K_B) = 0.$$

As a simple consequence of Definition 4.2.1 we have the following result.

Lemma 4.2.2. *Let $(X, \pi; M, I)$ be $\tilde{\pi}$ -asymptotically compact. If $B \subset X$ is a bounded positively $\tilde{\pi}$ -invariant set, then $\tilde{L}^+(B)$ is nonempty, compact, positively $\tilde{\pi}$ -invariant, $\tilde{L}^+(B) \cap M = \emptyset$ and*

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, t), \tilde{L}^+(B)) = 0. \quad (4.2)$$

The next two results deal with the importance of $\tilde{\pi}$ -asymptotic compactness property in dissipative systems.

Theorem 4.2.3. [7, Theorem 3.9] *Let $(X, \pi; M, I)$ be b -dissipative with respect to a family $\mathfrak{M}$, where $\mathfrak{M}$ is a family of subsets of X , and $\tilde{\pi}$ -asymptotically compact. Then $(X, \pi; M, I)$ is k -dissipative with respect to the family $\mathfrak{M}$.*

Theorem 4.2.4. [6, Theorem 3.11] *Let $(X, \pi; M, I)$ be point k -dissipative. For the impulsive semidynamical system $(X, \pi; M, I)$ to be compact k -dissipative, it is necessary and sufficient that $\tilde{D}^+(\Omega) \cap M = \emptyset$ and $\tilde{\pi}^+(A)$ is relatively compact, for any compact $A \subset X$.*

Next, we construct a discrete dynamical system associated to an ISS.

Definition 4.2.5. Let $(X, \pi; M, I)$ be an ISS and $\lambda > 0$. The system $(X, P_\lambda; M, I)$, where $P_\lambda : X \rightarrow X$ is given by $P_\lambda(x) = \tilde{\pi}(x, \lambda)$, $x \in X$, is called **the discrete system associated** to $(X, \pi; M, I)$ at time λ .

In Definition 4.2.5, we consider $P_\lambda^0 = Id$ (Id is the identity function) and $P_\lambda^n = P_\lambda \circ P_\lambda^{n-1}$ for $n = 1, 2, \dots$. Note that, for all $x \in X$ and for all $n \in \mathbb{Z}_+$, we have

$$P_\lambda^n(x) = \tilde{\pi}(x, n\lambda).$$

We say that $B \subset X$ is P_λ -**invariant** if $P_\lambda(B) \subset B$.

For discrete systems, we have the following definition.

Definition 4.2.6. A discrete system $(X, P_\lambda; M, I)$ is called P_λ -**asymptotically compact**, if for every bounded P_λ -invariant set $B \subset X$, there exists a nonempty compact set $K_B \subset X$ such that $K_B \cap M = \emptyset$ and

$$\lim_{n \rightarrow +\infty} \text{dist}(P_\lambda^n(B), K_B) = 0.$$

In the sequel we relate the concept of asymptotic compactness between an impulsive system and its associated discrete system.

Theorem 4.2.7. Let $(X, \pi; M, I)$ be an ISS. Suppose that for some $\lambda > 0$ the discrete system $(X, P_\lambda; M, I)$ is P_λ -asymptotically compact. Then $(X, \pi; M, I)$ is $\tilde{\pi}$ -asymptotically compact.

Proof. Let $B \subset X$ be a bounded positively $\tilde{\pi}$ -invariant set. Then $P_\lambda(B) = \tilde{\pi}(B, \lambda) \subset B$ and according to the asymptotic compactness of the discrete system $(X, P_\lambda; M, I)$, there exists a compact set $K_B \subset X$ such that $K_B \cap M = \emptyset$ and

$$\lim_{n \rightarrow +\infty} \text{dist}(P_\lambda^n(B), K_B) = \lim_{n \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, n\lambda), K_B) = 0. \quad (4.3)$$

Consider two arbitrary sequences $\{x_n\}_{n \in \mathbb{N}} \subset B$ and $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ with $t_n \xrightarrow{n \rightarrow +\infty} +\infty$. Note that we can write $t_n = k_n\lambda + \tau_n$, where k_n is a positive integer number and $\tau_n \in [0, \lambda)$. By (4.3) we may assume without loss of generality that $\{\tau_n\}_{n \in \mathbb{N}}$ is convergent and that $\{\tilde{\pi}(x_n, k_n\lambda)\}_{n \in \mathbb{N}}$ is convergent with limit in K_B . In virtue of the equality

$$\tilde{\pi}(x_n, t_n) = \tilde{\pi}(\tilde{\pi}(x_n, k_n\lambda), \tau_n),$$

it follows by Lemma 1.4.7 that the sequence $\{\tilde{\pi}(x_n, t_n)\}_{n \in \mathbb{N}}$ is relatively compact and from Lemma 1.5.10 we conclude that $\tilde{L}^+(B)$ is nonempty, compact and

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, t), \tilde{L}^+(B)) = 0.$$

We claim that $\tilde{L}^+(B) \cap M = \emptyset$. Suppose to the contrary that there are $z \in M$, $\{b_n\}_{n \in \mathbb{N}} \subset B$, $\{r_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $r_n \xrightarrow{n \rightarrow +\infty} +\infty$ and

$$\tilde{\pi}(b_n, r_n) \xrightarrow{n \rightarrow +\infty} z.$$

We may write $r_n = u_n \lambda + s_n$, where u_n is a positive integer number and $s_n \in [0, \lambda)$, for $n = 1, 2, \dots$. By (4.3), we get

$$\lim_{n \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, u_n \lambda), K_B) = 0.$$

Since $\tilde{\pi}(B, s_n) \subset B$, we have

$$\tilde{\pi}(\tilde{\pi}(B, s_n), u_n \lambda) \subset \tilde{\pi}(B, u_n \lambda),$$

for all $n = 1, 2, \dots$. Consequently,

$$\lim_{n \rightarrow +\infty} \text{dist}(\tilde{\pi}(b_n, u_n \lambda + s_n), K_B) = 0.$$

Hence $z \in K_B$. Is is a contradiction as $K_B \cap M = \emptyset$. Therefore, $(X, \pi; M, I)$ is $\tilde{\pi}$ -asymptotically compact. $\square$

It is not true that the set $\pi(B, [0, \lambda])$ is always bounded when $B \in B(X)$ and $\lambda > 0$, see [21, Proposition 4.1]. It also occurs for impulsive systems. Thus, if we assume that $\tilde{\pi}(B, [0, \lambda])$ is bounded for all $B \in B(X)$ and for some $\lambda > 0$ then we can show the converse of Theorem 4.2.7.

Theorem 4.2.8. *Let $(X, \pi; M, I)$ be $\tilde{\pi}$ -asymptotically compact. If $\tilde{\pi}(B, [0, \lambda])$ is bounded, for all $B \in B(X)$ and some $\lambda > 0$, then its associated discrete system $(X, P_\lambda; M, I)$ is P_λ -asymptotically compact.*

Proof. Let $B \subset X$ be a bounded P_λ -invariant set. We claim that $\tilde{\pi}^+(B)$ is bounded. In fact, for each $y \in \tilde{\pi}^+(B)$, there are $b \in B$ and $s \in \mathbb{R}_+$ such that $y = \tilde{\pi}(b, s)$. Let $s = k\lambda + \tau$, where $k \in \mathbb{Z}_+$ and $\tau \in [0, \lambda)$. Then $y = \tilde{\pi}(b, k\lambda + \tau) = \tilde{\pi}(P_\lambda^k(b), \tau) \in \tilde{\pi}(B, [0, \lambda])$, that is, $\tilde{\pi}^+(B) \subset \tilde{\pi}(B, [0, \lambda])$ which implies that $\tilde{\pi}^+(B)$ is bounded. Thus $A = \tilde{\pi}^+(B)$ is bounded and positively $\tilde{\pi}$ -invariant.

According to the asymptotic compactness of $(X, \pi; M, I)$, there exists a compact set $K_A \subset X$ such that $K_A \cap M = \emptyset$ and

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(A, t), K_A) = 0.$$

Consequently,

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, t), K_A) = 0.$$

Since $P_\lambda^n(B) = \tilde{\pi}(B, n\lambda)$, for all $n \in \mathbb{N}$, it follows that

$$\lim_{n \rightarrow +\infty} \text{dist}(P_\lambda^n(B), K_A) = 0$$

and the result is proved. $\square$

The following definition deals with the condition of Ladyzhenskaya for impulsive systems. See [7] and [19], for instance.

Definition 4.2.9. An ISS $(X, \pi; M, I)$ satisfies the **condition of Ladyzhenskaya**, if for every bounded set $B \subset X$ there exists a nonempty compact set $K_B \subset X$ such that $K_B \cap M = \emptyset$ and $\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, t), K_B) = 0$.

It is obvious that a system $(X, \pi; M, I)$ is $\tilde{\pi}$ -asymptotically compact provided that it satisfies the condition of Ladyzhenskaya.

Analogously to Definition 4.2.9, we present the condition of Ladyzhenskaya for discrete systems.

Definition 4.2.10. A discrete system $(X, P_\lambda; M, I)$ satisfies the **condition of Ladyzhenskaya**, if for every bounded set $B \subset X$ there exists a nonempty compact set $K_B \subset X$ such that $K_B \cap M = \emptyset$ and $\lim_{n \rightarrow +\infty} \text{dist}(P_\lambda^n(B), K_B) = 0$.

Theorem 4.2.11. *If an ISS $(X, \pi; M, I)$ satisfies the condition of Ladyzhenskaya then for all $\lambda > 0$ its associated discrete system $(X, P_\lambda; M, I)$ also satisfies the condition of Ladyzhenskaya.*

Proof. Let $B \subset X$ be a bounded set. By Definition 4.2.9, there exists a compact set $K_B \subset X$ such that $K_B \cap M = \emptyset$ and

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, t), K_B) = 0. \quad (4.4)$$

Since $P_\lambda^n(B) = \tilde{\pi}(B, n\lambda)$, for all $n \in \mathbb{Z}_+$, we get

$$\lim_{n \rightarrow +\infty} \text{dist}(P_\lambda^n(B), K_B) = 0$$

and we have the result. $\square$

Corollary 4.2.12. *If $(X, \pi; M, I)$ satisfies the condition of Ladyzhenskaya, then its associated discrete system $(X, P_\lambda; M, I)$ is P_λ -asymptotically compact for all $\lambda > 0$.*

The converse of Theorem 4.2.11 holds in the following way.

Theorem 4.2.13. *Let $(X, \pi; M, I)$ be an ISS and $(X, P_\lambda; M, I)$ be its associated discrete system with $\lambda > 0$. If $(X, P_\lambda; M, I)$ satisfies the condition of Ladyzhenskaya and $\tilde{\pi}(B, [0, \lambda]) \in B(X)$ for all $B \in B(X)$, then $(X, \pi; M, I)$ satisfies the condition of Ladyzhenskaya.*

Proof. Let $B \subset X$ be a bounded set. Then $\tilde{\pi}(B, [0, \lambda]) \in B(X)$ and there exists a nonempty compact set $K_B \subset X$ such that $K_B \cap M = \emptyset$ and

$$\lim_{n \rightarrow +\infty} \text{dist}(P_\lambda^n(\tilde{\pi}(B, [0, \lambda])), K_B) = 0.$$

First, we note that

$$P_\lambda^n(\tilde{\pi}(B, [0, \lambda])) = \tilde{\pi}(\tilde{\pi}(B, [0, \lambda]), n\lambda) = \tilde{\pi}(\tilde{\pi}(B, n\lambda), [0, \lambda]) = \tilde{\pi}(P_\lambda^n(B), [0, \lambda]).$$

Second, we claim that $\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, t), K_B) = 0$. Suppose that there are $\epsilon_0 > 0$, sequences $\{b_n\}_{n \in \mathbb{N}} \subset B$ and $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \xrightarrow{n \rightarrow +\infty} +\infty$ and

$$\rho(\tilde{\pi}(b_n, t_n), K_B) \geq \epsilon_0,$$

for all $n = 1, 2, \dots$. There are $r_n \in \mathbb{N}$ and $s_n \in [0, \lambda)$ such that $t_n = \lambda r_n + s_n$, $n = 1, 2, \dots$. Thus

$$\epsilon_0 \leq \rho(\tilde{\pi}(b_n, t_n), K_B) = \rho(\tilde{\pi}(P_\lambda^{r_n}(b_n), s_n), K_B) \xrightarrow{n \rightarrow +\infty} 0,$$

which is a contradiction. □

4.3 Condensing maps and asymptotic compactness

In this section, we study another class of asymptotically compact impulsive semidynamical systems which will be obtained from condensing maps.

Definition 4.3.1. A mapping $\mu : B(X) \rightarrow \mathbb{R}_+$ is a *measure of non-compactness* on X if it satisfies the following properties:

- a) $\mu(A) = 0$ if and only if $A \in B(X)$ is relatively compact;

b) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$, for every $A, B \in B(X)$.

For example, the measure of Kuratowski given by $\alpha(B) = \inf\{\epsilon > 0 : B \text{ admits a finite } \epsilon\text{-covering}\}$ for $B \in B(X)$, is a measure of non-compactness on X .

Definition 4.3.2. A mapping $P : X \rightarrow X$ is called μ -condensing with respect to a measure of the non-compactness μ , if $\mu(P(A)) < \mu(A)$ for all $A \in B(X)$ such that $\mu(A) > 0$.

A continuous mapping $P : X \rightarrow X$ on a finite-dimensional space X is always a compact mapping and, consequently, it is μ -condensing with respect to the Kuratowski measure.

Next, we give conditions for a discrete system to be P_λ -asymptotic compact.

Lemma 4.3.3. Let $P_\lambda : X \rightarrow X$ be a continuous μ -condensing mapping and $\lambda > 0$. Suppose that $\tilde{L}^+(B) \cap M = \emptyset$ for all $B \in B(X)$. Then the discrete system $(X, P_\lambda; M, I)$ is P_λ -asymptotic compact.

Proof. Let $B \subset X$ be a bounded P_λ -invariant set. Now consider the set

$$C = \{ \{P^{k_n}(b_n)\}_{n \in \mathbb{N}} : \{b_n\}_{n \in \mathbb{N}} \subset B, \{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N} \text{ and } k_n \xrightarrow{n \rightarrow +\infty} +\infty \}.$$

Set $\eta := \sup \{ \lambda(h) : h \in C \}$. By the proof of Lemma 1.23 in [19], we have $\eta = 0$, that is, $\{P^{k_n}(b_n)\}_{n \in \mathbb{N}}$ is relatively compact for any sequences $\{b_n\}_{n \in \mathbb{N}} \subset B$ and $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ such that $k_n \xrightarrow{n \rightarrow +\infty} +\infty$.

Consider two sequences $\{x_n\}_{n \in \mathbb{N}} \subset B$ and $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ with $t_n \xrightarrow{n \rightarrow +\infty} +\infty$. We can write $t_n = k_n \lambda + \tau_n$, where $k_n \in \mathbb{N}$ and $\tau_n \in [0, \lambda)$, and we may assume without loss of generality that $\{\tau_n\}_{n \in \mathbb{N}}$ and $\{\tilde{\pi}(x_n, k_n \lambda)\}_{n \in \mathbb{N}} = \{P_\lambda^{k_n}(x_n)\}_{n \in \mathbb{N}}$ are convergent since $\eta = 0$. Now, since

$$\tilde{\pi}(x_n, t_n) = \tilde{\pi}(\tilde{\pi}(x_n, k_n \lambda), \tau_n),$$

that is, $\{\tilde{\pi}(x_n, t_n)\}_{n \in \mathbb{N}}$ is relatively compact, it follows by Lemma 1.5.10 that $\tilde{L}^+(B)$ is nonempty, compact and

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, t), \tilde{L}^+(B)) = 0.$$

Since $P_\lambda^n(B) = \tilde{\pi}(B, n\lambda)$, for all $n \in \mathbb{Z}_+$, we get $\lim_{n \rightarrow +\infty} \text{dist}(P_\lambda^n(B), \tilde{L}^+(B)) = 0$ and the result holds. $\square$

Definition 4.3.4. An ISS $(X, \pi; M, I)$ is called μ -condensing, if $\mu(\tilde{\pi}(B, t)) < \mu(B)$ and $\tilde{\pi}(B, t) \in B(X)$ for all $t > 0$ and $B \in B(X)$ with $\mu(B) > 0$.

In the next proposition, we show that an impulsive system is $\tilde{\pi}$ -asymptotically compact provided that it is μ -condensing.

Proposition 4.3.5. *Suppose that $(X, \pi; M, I)$ is μ -condensing and $\tilde{L}^+(B) \cap M = \emptyset$ for all $B \in B(X)$. Then $(X, \pi; M, I)$ is $\tilde{\pi}$ -asymptotically compact.*

Proof. Since $(X, \pi; M, I)$ is μ -condensing it follows that the mapping P_λ is μ -condensing, where $(X, P_\lambda; M, I)$ is its associated discrete system at time $\lambda > 0$. According to the Lemma 4.3.3 the system $(X, P_\lambda; M, I)$ is P_λ -asymptotic compact. By Theorem 4.2.7, $(X, \pi; M, I)$ is $\tilde{\pi}$ -asymptotically compact. $\square$

4.4 On the problem of Jack Hale

In this section we present the main results from [11]. We devote our study to the problem of Jack Hale. As was mentioned in [19], the problem of Jack Hale was formulated in [28] in the following way:

“Let a discrete dynamical system (X, P) be pointwise b -dissipative and the mapping $P : X \rightarrow X$ be a μ -contraction. Is there a maximal compact invariant set in the system (X, P) ?”

Cheban considers the problem of Jack Hale in the following manner:

“Let (X, P) be a pointwise b -dissipative discrete dynamical system. What are the conditions which guarantee the existence of a maximal compact invariant set in (X, P) ?”

In [19], Cheban established conditions for a pointwise b -dissipative dynamical system to admit a maximal compact invariant set.

Our intention is to present conditions for a pointwise b -dissipative system with impulses to admit a maximal compact positively invariant set. This result is already known for compact k -dissipative systems where the center of Levinson is the maximal compact positively invariant set in $(X, \pi; M, I)$, see [6] for more details.

In order to obtain the results, we shall assume the conditions **(H1)**, **(H2)** and **(H3)** presented in Section 1.4 .

Definition 4.4.1. An ISS $(X, \pi; M, I)$ is called **locally bounded**, if for every $x \in X$ there are $\delta_x > 0$ and $\ell_x > 0$ such that the set $\cup\{\tilde{\pi}(B(x, \delta_x), t) : t \geq \ell_x\}$ is bounded.

Remark 4.4.2. From Definition 4.1.9 it follows that a locally bounded and $\tilde{\pi}$ -asymptotically compact impulsive system is also a locally asymptotically $\tilde{\pi}$ -condensing impulsive system.

Theorems 4.4.3 and 4.4.4 give us sufficient conditions for a pointwise b -dissipative system to be locally k -dissipative.

Theorem 4.4.3. *Let $(X, \pi; M, I)$ be a pointwise b -dissipative, locally bounded and $\tilde{\pi}$ -asymptotically compact impulsive system and $\tilde{D}^+(\Omega) \cap M = \emptyset$. Then $(X, \pi; M, I)$ is locally k -dissipative.*

Proof. Since $(X, \pi; M, I)$ is $\tilde{\pi}$ -asymptotically compact it follows by Theorem 4.2.3 that $(X, \pi; M, I)$ is pointwise k -dissipative. From Remark 4.4.2, the system $(X, \pi; M, I)$ is also locally asymptotically $\tilde{\pi}$ -condensing. According to Theorem 4.1.10, the system $(X, \pi; M, I)$ is locally k -dissipative. $\square$

Theorem 4.4.4. *Let $(X, \pi; M, I)$ be pointwise b -dissipative and $\tilde{\pi}$ -asymptotically compact. Assume that $\tilde{D}^+(\Omega) \cap M = \emptyset$. If for every $p \in \Omega$, there exist $\delta_p > 0$ and $\ell_p > 0$ such that $\tilde{\pi}(B(p, \delta_p), [\ell_p, +\infty)) \in B(X)$, then $(X, \pi; M, I)$ is locally k -dissipative.*

Proof. It is enough to show that $(X, \pi; M, I)$ is locally bounded and then the proof follows by Theorem 4.4.3. By Theorem 4.2.3 the system $(X, \pi; M, I)$ is point k -dissipative, and consequently, the set Ω is nonempty, compact and

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(x, t), \Omega) = 0 \quad \text{for all } x \in X. \quad (4.5)$$

Note that $\cup\{B(p, \delta_p) : p \in \Omega, \delta_p > 0\}$ is an open cover of Ω , thus we can obtain a finite sub-covering $\{B(p_i, \delta_{p_i}) : p_i \in \Omega, i = 1, 2, \dots, m\}$ of Ω . Choose $\ell_0 := \max\{\ell_{p_i} : i = 1, 2, \dots, m\}$, where each $\ell_{p_i} > 0$ is given by hypothesis.

By Lemma 1.9 in [19] there exists $\gamma > 0$ such that

$$B(\Omega, \gamma) \subset \cup\{B(p_i, \delta_{p_i}) : i = 1, 2, \dots, m\}.$$

Let $x \in X$. Taking into account equation (4.5), there exists $\ell_x > 0$ such that $\tilde{\pi}(x, t) \in B(\Omega, \gamma)$ for all $t \geq \ell_x$. Choose $t_x > \ell_x$ with $t_x \neq \sum_{i=0}^k \phi(x_i^+)$, for all $k = 0, 1, 2, \dots$, such that $\tilde{\pi}(x, t_x) \in B(\Omega, \gamma)$. Since $B(\Omega, \gamma)$ is open, there is $\epsilon > 0$ such that $B(\tilde{\pi}(x, t_x), \epsilon) \subset B(\Omega, \gamma)$. Then

$$\begin{aligned} \tilde{\pi}(B(\tilde{\pi}(x, t_x), \epsilon), [\ell_0, +\infty)) &\subset \tilde{\pi}(B(\Omega, \gamma), [\ell_0, +\infty)) \\ &\subset \bigcup_{i=1}^m \tilde{\pi}(B(p_i, \delta_{p_i}), [\ell_0, +\infty)) \in B(X). \end{aligned} \quad (4.6)$$

Case 1: $x \in X \setminus M$.

By continuity of mapping π , ϕ and I , there is $\delta_x > 0$ such that

$$\tilde{\pi}(B(x, \delta_x), t_x) \subset B(\tilde{\pi}(x, t_x), \epsilon) \subset B(\Omega, \gamma).$$

Hence, by (4.6), we conclude that $\cup\{\tilde{\pi}(B(x, \delta_x), t) : t \geq t_x + \ell_0\}$ is bounded.

Case 2: $x \in M$.

In this case, there exists a *STC*-tube $F(L_x, [0, 2\lambda_x])$ through x given by a section S_x , because M satisfies *STC*. Since the tube is a neighborhood of x , there is $\eta_x > 0$ such that

$$B(x, \eta_x) \subset F(L_x, [0, 2\lambda_x]).$$

By Remark 1.4.1, we denote

$$H_1^{(x)} = F(L_x, (\lambda_x, 2\lambda_x]) \cap B(x, \eta_x) \quad \text{and} \quad H_2^{(x)} = F(L_x, [0, \lambda_x]) \cap B(x, \eta_x).$$

Take $0 < \eta_1 < \eta$ such that $\tilde{\pi}(B(x, \eta_1) \cap H_2^{(x)}, t_x) \subset B(\tilde{\pi}(x, t_x), \epsilon) \subset B(\Omega, \gamma)$. Hence, by (4.6), we get $\cup\{\tilde{\pi}(B(x, \eta_1) \cap H_2, t) : t \geq t_x + \ell_0\} \in B(X)$.

On the other hand, since $I(x) \in X \setminus M$, we may obtain $t_{I(x)} > 0$ and $\nu_{I(x)} > 0$ such that $t_{I(x)} \neq \sum_{i=0}^k \phi(I(x)_i^+)$, for all $k = 0, 1, \dots$, and $\tilde{\pi}(B(I(x), \nu_{I(x)}), t_{I(x)}) \subset B(\Omega, \gamma)$. Take $\eta_2 > 0$, $\eta_2 < \eta$, such that $\tilde{\pi}(B(x, \eta_2) \cap H_1^{(x)}, t_{I(x)}) \subset B(\Omega, \gamma)$. Then

$$\cup\{\tilde{\pi}(B(x, \eta_2) \cap H_1^{(x)}, t) : t \geq \ell_0 + t_{I(x)}\}$$

is bounded.

Taking $\delta_x = \min\{\eta_1, \eta_2\}$ we have the set $\cup\{\tilde{\pi}(B(x, \delta_x), t) : t \geq t_x + t_{I(x)} + \ell_0\}$ is bounded, again by (4.6). Therefore, $(X, \pi; M, I)$ is locally bounded and, by Theorem 4.4.3, $(X, \pi; M, I)$ is locally k -dissipative. $\square$

Definition 4.4.5. An ISS $(X, \pi; M, I)$ is called **C -bounded**, if $\tilde{\pi}^+(K) \in B(X)$ whenever $K \in \text{Comp}(X)$.

Theorem 4.4.6. Let $(X, \pi; M, I)$ be a pointwise b -dissipative, C -bounded and $\tilde{\pi}$ -asymptotically compact impulsive system. If $\tilde{D}^+(\Omega) \cap M = \emptyset$ then $(X, \pi; M, I)$ is compact k -dissipative.

Proof. By Theorem 4.2.3, $(X, \pi; M, I)$ is pointwise k -dissipative. Since $(X, \pi; M, I)$ is C -bounded, the set $\tilde{\pi}^+(K)$ is positively $\tilde{\pi}$ -invariant and bounded for all $K \in \text{Comp}(X)$. Consequently, by the asymptotic compactness we have $\tilde{\pi}^+(K)$ relatively compact. Using Theorem 4.2.4 and Theorem 4.1.8 we get the result. $\square$

By Theorem 4.4.3, Theorem 4.4.4 and Theorem 4.4.6, we can give an answer to the problem of Jack Hale for the case of impulsive systems, since these theorems establish conditions for a pointwise b -dissipative to be compact k -dissipative and, by Theorem 4.1.8, we conclude that the Levinson center is the maximal compact positively $\tilde{\pi}$ -invariant set in $(X, \pi; M, I)$.

4.5 Bebutov semidynamical system with impulses

This section exhibits an example of a pointwise k -dissipative ISS which is not locally bounded and does not admit a maximal compact invariant set.

Consider the Bebutov semidynamical system given in Example 1.1.5 and the function

$$\psi(t) = \begin{cases} \exp[(t-1)^2 - 1] + 1, & t \in (0, 2), \\ 0, & t \in (-\infty, 0] \cup [2, +\infty). \end{cases}$$

Then ψ satisfies the following conditions:

- 1) $\psi(0) = 0$ and $\psi(1) = 1$;
- 2) $\text{supp}(\psi) = [0, 2]$;
- 3) $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$;
- 4) ψ is monotone increasing in $[0, 1]$ and decreasing in $[1, 2]$;
- 5) $t\psi(t^{-1}) \rightarrow 0$ as $t \rightarrow +\infty$.

Let $X = \{\psi_{a,h} : a > 0, h \in \mathbb{R}\} \cup \{\theta\}$, where $\psi_{a,h}(t) = a\psi\left(\frac{t}{a} + h\right)$ and $\theta(t) = 0$ for all $t \in \mathbb{R}$.

Remark 4.5.1. Due to Ascoli's Theorem, see [36, Theorem 47.1], X is not compact since the set $\{a\psi(h) : a > 0, h \in \mathbb{R}\}$ is unbounded in $\mathbb{R}$.

According to [19, Example 1.9], we can state the following lemma.

Lemma 4.5.2. *The set X is closed in $C(\mathbb{R}, \mathbb{R})$ and invariant with respect to $(C(\mathbb{R}, \mathbb{R}), \pi)$.*

Thus, (X, π) is a continuous dynamical system, where $\pi(f, t) = f_t$ for all $f \in X$ and for all $t \geq 0$. Now, define $M = \{\varphi_{c_1}, \dots, \varphi_{c_k}\}$ with $0 < c_1 < \dots < c_k$, where

$$\varphi_{c_j}(t) = c_j \psi\left(\frac{t}{c_j} + j\right), \quad t \in \mathbb{R},$$

and let $I : M \rightarrow X$ be an impulse function given by $I(\varphi_{c_j}) = \psi_{(c_j+\alpha, j+\alpha)}$, $j = 1, 2, \dots, k$, where $\alpha > 0$ is such that $I(M) \cap M = \emptyset$. Hence, we have the ISS $(X, \pi; M, I)$ associated to (X, π) . Recall that $\phi : X \rightarrow (0, +\infty]$ is a function defined by

$$\phi(f) = \begin{cases} s, & \text{if } \pi(f, s) \in M \text{ and } \pi(f, t) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } M^+(f) = \emptyset, \end{cases}$$

where $M^+(f) = \left(\bigcup_{t>0} \pi(f, t)\right) \cap M$.

Proposition 4.5.3. *Let $(X, \pi; M, I)$ be the Bebutov semidynamical system with impulses. For each $\varphi \in X$, there are $f \in X$ and a constant $L \geq 0$ such that*

$$\tilde{\pi}(\varphi, t) = \pi(f, t - L) \quad \text{for all } t \geq L. \quad (4.7)$$

Proof. Let $\varphi \in X$. If $\varphi = \theta$, then (4.7) holds with $f = \theta$ and $L = 0$. Now, suppose that there are $a > 0$ and $h \in \mathbb{R}$ such that $\varphi = \psi_{a,h}$. We have three cases:

Case 1: $a \neq c_j$ for every $j = 1, 2, \dots, k$.

In this case, $\phi(\varphi) = +\infty$. In fact, if $\phi(\varphi) = t$ for some $0 < t < +\infty$, then there is $j \in \{1, 2, \dots, k\}$ such that

$$\pi(\varphi, t) = \varphi_{c_j}, \quad (4.8)$$

that is,

$$a\psi\left(\frac{t+s}{a} + h\right) = c_j\psi\left(\frac{s}{c_j} + j\right) \quad \text{for all } s \in \mathbb{R}.$$

In particular, if $c_j > a$, then by choosing $s = (1 - j)c_j$, we obtain

$$\psi\left(\frac{t + (1 - j)c_j}{a} + h\right) = \frac{c_j}{a}\psi(1) = \frac{c_j}{a} > 1$$

and it is a contradiction. Now, if $c_j < a$, we can choose $s = (1 - h)a - t$ and we obtain the same contradiction. Thus, when $a \neq c_j$, for every $j = 1, 2, \dots, k$, equation (4.7) holds with $f = \varphi$ and $L = 0$.

Case 2: $a = c_j$ and $h \geq j$ for some $j \in \{1, 2, \dots, k\}$.

In this case, we claim that $\phi(\varphi) = +\infty$. In fact, if $\phi(\varphi) = t$ for some $0 < t < +\infty$, we have

$$\pi(\varphi, t) = \varphi_{c_j},$$

that is,

$$\psi\left(\frac{t+s}{c_j} + h\right) = \psi\left(\frac{s}{c_j} + j\right) \quad \text{for all } s \in \mathbb{R}.$$

Taking $s = (1 - j)c_j$, we have

$$\psi\left(\frac{t}{c_j} + 1 - j + h\right) = \psi(1) = 1.$$

From the definition of ψ , we have $\psi(s) = 1$ if and only if $s = 1$. Then, we have

$$\frac{t}{c_j} + 1 - j + h = 1$$

and consequently $t = c_j(j - h) \leq 0$, which is a contradiction. Therefore $\phi(\varphi) = +\infty$ and we have (4.7) satisfied for $f = \varphi$ and $L = 0$.

Case 3: $a = c_j$ and $h < j$ for some $j \in \{1, 2, \dots, k\}$.

Here, we have $\phi(\varphi) = (j - h)c_j$. Indeed,

$$\pi(\varphi, (j - h)c_j) = \varphi_{c_j}$$

and using the same ideas as above we get $\pi(\varphi, t) \notin M$ for all $t < (j - h)c_j$.

Thus, we define

$$\tilde{\pi}(\varphi, t) = \begin{cases} \pi(\varphi, t), & 0 \leq t < \phi(\varphi) \\ \psi_{(c_j+\alpha, j+\alpha)}, & t = \phi(\varphi), \end{cases}$$

where $I(\pi(\varphi, \phi(\varphi))) = I(\varphi_{c_j}) = \psi_{(c_j+\alpha, j+\alpha)}$. We denote $\varphi_1 = \pi(\varphi, \phi(\varphi))$ and $\varphi_1^+ = I(\varphi_1)$.

If $c_j + \alpha \notin \{c_i : i = 1, 2, \dots, k\}$ or if $c_j + \alpha = c_\ell$ for some $\ell \in \{1, 2, \dots, k\}$ and $j + \alpha > \ell$, then by the previous ideas we get $\phi((\varphi_1^+)) = +\infty$. Consequently,

$$\tilde{\pi}(\varphi, t) = \begin{cases} \pi(\varphi, t), & 0 \leq t < \phi(\varphi) \\ \pi(\psi_{(c_j+\alpha, j+\alpha)}, t - \phi(\varphi)), & t \geq \phi(\varphi). \end{cases}$$

In this case, we choose $f = \varphi_1^+ = \psi_{(c_j+\alpha, j+\alpha)}$ and $L = \phi(\varphi)$ and we obtain (4.7).

However, if $c_j + \alpha = c_{j_1}$ for some $j_1 \in \{1, 2, \dots, k\}$ and $j + \alpha < j_1$, then we obtain $\phi(\varphi_1^+) = c_{j_1}(j_1 - j - \alpha)$ because $\pi(\varphi_1^+, t) \neq \varphi_{j_1}$ for $0 < t < (j_1 - j - \alpha)c_{j_1}$ and

$$\pi(\varphi_1^+, (j_1 - j - \alpha)c_{j_1})(s) = c_{j_1} \psi\left(\frac{s}{c_{j_1}} + j_1\right) = \varphi_{c_{j_1}}(s)$$

for all $s \in \mathbb{R}$. Thus, we define

$$\tilde{\pi}(\varphi, t) = \begin{cases} \pi(\varphi, t), & 0 \leq t < \phi(\varphi) \\ \pi(\psi_{(c_j+\alpha, j+\alpha)}, t - \phi(\varphi)), & \phi(\varphi) \leq t < \phi(\varphi) + \phi(\varphi_1^+), \\ \psi_{(c_{j_1}+\alpha, j_1+\alpha)}, & t = \phi(\varphi) + \phi((\varphi)_1^+). \end{cases}$$

Now, we set $\varphi_2 = \pi(\varphi_1^+, \phi(\varphi_1^+))$ and $\varphi_2^+ = I(\varphi_2)$.

Since $M = \{\varphi_{c_1}, \dots, \varphi_{c_k}\}$, $0 < c_1 < \dots < c_k$, there is $0 < \ell < k$ such that

$$\tilde{\pi}(\varphi, t) = \begin{cases} \pi(\varphi, t), & 0 \leq t < r_0, \\ \pi(\psi_{(c_j+\alpha, j+\alpha)}, t - r_0), & r_0 \leq t < r_1, \\ \pi(\psi_{(c_{j_1}+\alpha, j_1+\alpha)}, t - r_1), & r_1 \leq t < r_2, \\ \vdots & \\ \pi(\psi_{(c_{j_\ell}+\alpha, j_\ell+\alpha)}, t - r_\ell), & t \geq r_\ell, \end{cases}$$

where $r_0 = \phi(\varphi)$ and $r_m = \phi(\varphi) + \phi((\varphi)_1^+) + \dots + \phi(\varphi_m^+)$, $m = 1, 2, \dots, \ell$. Thus $f = \psi_{(c_{j_\ell}+\alpha, j_\ell+\alpha)}$ and $L = r_\ell$, so we obtain (4.7). $\square$

Proposition 4.5.4. *Let $(X, \pi; M, I)$ be the Bebutov semidynamical system with impulses. Then $\tilde{L}^+(\varphi) = \{\theta\}$ for all $\varphi \in X$. Furthermore, $(X, \pi; M, I)$ is point k -dissipative with attractor $\Omega := \bigcup_{\varphi \in X} \tilde{L}^+(\varphi) = \{\theta\}$.*

Proof. Let $\varphi \in X$, by Proposition 4.5.3 there are $f \in X$ and $L \geq 0$ such that

$$\tilde{\pi}(\varphi, t) = \pi(f, t - L) \quad \text{for all } t \geq L.$$

Thus $\tilde{L}^+(\varphi) = L^+(f)$. According to Example 1.9 in [19], $L^+(g) = \{\theta\}$ for all $g \in X$, then $\tilde{L}^+(\varphi) = \{\theta\}$ for every $\varphi \in X$ and we get the result. $\square$

In the next result we conclude that $(X, \pi; M, I)$ is not compact dissipative.

Proposition 4.5.5. *Let $(X, \pi; M, I)$ be the Bebutov semidynamical system with impulses. Then $(X, \pi; M, I)$ is not compact dissipative.*

Proof. Let $A = \{\psi_{a,h} : h \in \mathbb{R}, a > 0 \text{ and } a \neq c_j \text{ for all } j = 1, \dots, k\}$. By Remark 4.5.1 the set $\{a\psi(h) : h \in \mathbb{R}, a > 0 \text{ and } a \neq c_j \text{ for all } j = 1, \dots, k\}$ is unbounded, consequently $\bar{A}$ is not compact. Now we show that $\bar{A} \subset \tilde{D}^+(\Omega)$. Let $\varphi \in A$ and $t_n \xrightarrow{n \rightarrow +\infty} +\infty$. Consider the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ given by $\varphi_n(s) = \varphi(s - t_n)$, $s \in \mathbb{R}$. Then $\{\varphi_n\}_{n \in \mathbb{N}} \subset A$ and $\varphi_n \xrightarrow{n \rightarrow +\infty} \theta$ in the uniform convergence topology. Note that $\tilde{\pi}(\varphi_n, t_n) = \pi(\varphi_n, t_n)$ because $\varphi_n \in A$ for all $n = 1, 2, \dots$. Hence,

$$\tilde{\pi}(\varphi_n, t_n)(s) = \pi(\varphi_n, t_n)(s) = \varphi(t_n + s - t_n) = \varphi(s) \quad \text{for all } s \in \mathbb{R},$$

that is, $\tilde{\pi}(\varphi_n, t_n) = \varphi$ which implies that $\bar{A} \subset \tilde{D}^+(\Omega)$. Therefore, $\tilde{D}^+(\Omega)$ is not compact and according to [6, Theorem 3.10] the system $(X, \pi; M, I)$ is not compact dissipative. $\square$

Proposition 4.5.6. *Let $(X, \pi; M, I)$ be the Bebutov semidynamical system with impulses. Then $(X, \pi; M, I)$ is not locally bounded.*

Proof. Let $\delta > 0$ be given arbitrary. As in the proof of Proposition 4.5.5, we have

$$\{\psi_{a,h} : h \in \mathbb{R}, a > 0 \text{ and } a \neq c_j \text{ for all } j = 1, \dots, k\} \subset \tilde{\pi}^+(B(\theta, \delta)).$$

By Remark 4.5.1, $\tilde{\pi}^+(B(\theta, \delta))$ is not bounded. $\square$

Finally, we exhibit the result that shows that $(X, \pi; M, I)$ does not admit a maximal compact positively $\tilde{\pi}$ -invariant set.

Proposition 4.5.7. *Let $(X, \pi; M, I)$ be the Bebutov semidynamical system with impulses. Then $(X, \pi; M, I)$ does not admit a maximal compact positively $\tilde{\pi}$ -invariant set.*

Proof. Suppose to the contrary that there is a maximal compact positively $\tilde{\pi}$ -invariant set I . Since X is not compact, we can choose $\varphi \in X \setminus I$. We claim that $\tilde{\pi}^+(\varphi)$ is relatively compact. In fact, by Proposition 4.5.3, there are $f \in X$ and $L \geq 0$ such that

$$\tilde{\pi}^+(\tilde{\pi}(\varphi, t)) \subset \pi^+(f)$$

for all $t \geq L$. According to [19, Example 1.9], the set $\pi^+(f)$ is relatively compact. Then $\overline{\tilde{\pi}^+(\tilde{\pi}(\varphi, t))}$ is compact and positively $\tilde{\pi}$ -invariant. Note that the set $I' = I \cup \overline{\tilde{\pi}^+(\tilde{\pi}(\varphi, t))}$ is compact, positively $\tilde{\pi}$ -invariant and $I \subset I'$. This contradicts the maximality of I . $\square$

We obtain the existence of a point k -dissipative system that does not admit a maximal compact positively invariant set. Therefore, our conditions in Theorem 4.4.4 and Theorem 4.4.6 are essential to solve the problem of Jack Hale.

Discrete systems in the sense of Kaul

In this chapter, we present the discrete system naturally associated with a given ISS and we study the relationship between the dynamics of these two systems. This type of discrete system was introduced by Kaul in [32].

5.1 Discrete systems

Using the ideas by Kaul, see [32], we may define a discrete system associated to an ISS $(X, \pi; M, I)$ as follows below.

Let $(X, \pi; M, I)$ be an ISS. A point $x \in X$ has **infinite impulsive trajectory** if $\phi(x_n^+) < +\infty$ for all $n = 0, 1, 2, \dots$. The set of all the points that have infinite impulsive trajectories is denoted by

$$H = \{x \in I(M) : \phi(x_n^+) < +\infty \text{ for all } n = 0, 1, 2, \dots\}. \quad (5.1)$$

Now, define the mapping $g : H \rightarrow H$ by

$$g(x) = \tilde{\pi}(x, \phi(x)) = I(\pi(x, \phi(x))) = I(x_1) = x_1^+, \quad (5.2)$$

for every $x \in H$. Note that g is a continuous function on H because I is continuous in M and π is continuous in $X \times \mathbb{R}_+$. The iterates of the function g are given by

$$g^0(x) = x$$

and

$$g^{k+1}(x) = g(g^k(x)) = x_{k+1}^+,$$

for each $k = 0, 1, 2, \dots$, and $x \in H$.

Definition 5.1.1. Let $(X, \pi; M, I)$ be an ISS and let H and g be defined as in (5.1) and (5.2), respectively. The pair (H, g) is called the **discrete dynamical system associated** to the system $(X, \pi; M, I)$ in the sense of Kaul.

Definition 5.1.2. Let $(X, \pi; M, I)$ be an ISS and (H, g) be the discrete dynamical system associated to the system $(X, \pi; M, I)$ in the sense of Kaul. The **discrete orbit** of a point $x \in X$ is given by

$$g^+(x) = \{g^n(x) : n = 0, 1, 2, \dots\}.$$

Given $A \subset X$, we set

$$g^+(A) = \{g^+(x) : x \in A\}.$$

5.2 Asymptotic compactness

In section 4.2, we presented the notion of asymptotically compact systems and we related this concept between an ISS and its associated discrete system in the sense of Definition 4.2.6. Now, we intend to investigate the notion of asymptotic compactness between an ISS and its discrete system in sense of Kaul.

In the sequel, we consider a discrete system (H, g) associated to a given ISS $(X, \pi; M, I)$.

Definition 5.2.1. We say that $B \subset H$ is **invariant under g** if $g(B) \subset B$.

Definition 5.2.2. A discrete system (H, g) is called **asymptotically compact**, if for every bounded invariant set $B \subset H$ under g there exists a nonempty compact set $K_B \subset H$ such that

$$\lim_{n \rightarrow +\infty} \text{dist}(g^n(B), K_B) = 0.$$

In general, if (H, g) is asymptotically compact it does not mean that $(X, \pi; M, I)$ is asymptotically compact, see the next example.

Example 5.2.3. Consider the ISS $(\mathbb{R}^2, \pi; M, I)$ presented in Example 1.2.5. Now, note that the set $A = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x < 2\}$ is positively $\tilde{\pi}$ -invariant and bounded. But there is no compact set K , $K \cap M = \emptyset$, which attracts A . Therefore, $(\mathbb{R}^2, \pi; M, I)$ is not $\tilde{\pi}$ -asymptotically compact.

Set $H = \{(x, y) \in \mathbb{R}^2 : x = 0\}$. Then $\phi((x, y)_k^+) < +\infty$ for all $(x, y) \in H$ and $k = 0, 1, 2, \dots$. Consequently, (H, g) is the discrete system associated to $(\mathbb{R}^2, \pi; M, I)$ which is clearly asymptotically compact.

Figure 5.1: (H, g) is asymptotically compact.

However, under a special condition, we can show that $(X, \pi; M, I)$ is asymptotically compact provided (H, g) is asymptotically compact, see the next theorem. Moreover, we shall consider a collection of bounded sets in X given by $\tilde{B}(X) = \{B \in B(X) : \phi(x) < +\infty, \text{ for all } x \in B\}$.

Theorem 5.2.4. Let $(X, \pi; M, I)$ be an ISS and (H, g) be its associated discrete system in the sense of Kaul. Suppose that:

- i) (X, π) is asymptotically compact;
- ii) (H, g) is asymptotically compact;
- iii) for every bounded positively $\tilde{\pi}$ -invariant set $B \subset X$, we have $\tilde{L}^+(B) \cap M = \emptyset$;

iv) $\phi(x) < \lambda < +\infty$ for all $x \in I(M)$ and $\phi(B) = \{\phi(b) : b \in B\} \in B(X)$ for all $B \in \tilde{B}(X)$.

Then $(X, \pi; M, I)$ is $\tilde{\pi}$ -asymptotically compact.

Proof. Let $B \subset X$ be a bounded positively $\tilde{\pi}$ -invariant set. Let $B = B_1 \cup B_2$ where $B_1 = \{x \in B : \phi(x) = +\infty\}$ and $B_2 = \{x \in B : \phi(x) < +\infty\}$.

It is clear that B_1 is bounded and positively invariant ($\pi(B_1, t) \subset B_1$ for all $t \geq 0$), then using item i) from hypothesis, there is $K_1 \subset \overline{B_1}$ such that

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B_1, t), K_1) = \lim_{t \rightarrow +\infty} \text{dist}(\pi(B_1, t), K_1) = 0.$$

By Lemma 1.5.10, the set $\tilde{L}^+(B_1) = L^+(B_1) \subset X \setminus M$ is compact, nonempty and

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B_1, t), \tilde{L}^+(B_1)) = 0. \quad (5.3)$$

Now, we define $\tilde{B}_2 = \bigcup_{b \in B_2} \tilde{\pi}(b, \phi(b))$. Then $\tilde{B}_2 \subset H$ is bounded and invariant under g . Consequently, by condition ii), there is a compact set $K_2 \subset H$ such that

$$\lim_{n \rightarrow +\infty} \text{dist}(g^n(\tilde{B}_2), K_2) = 0. \quad (5.4)$$

Let $\{y_n\}_{n \in \mathbb{N}} \subset B_2$ and $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ be arbitrary sequences such that $t_n \xrightarrow{n \rightarrow +\infty} +\infty$. Define $w_n = \tilde{\pi}(y_n, \phi(y_n))$, $n = 1, 2, \dots$. For each $n \in \mathbb{N}_0$, there are $z_n \in \mathbb{Z}_+$ and $s_n \in [0, \phi((w_n)_{z_n+1}^+))$ such that

$$t_n = \sum_{k=0}^{z_n} \phi((w_n)_k^+) + s_n.$$

Then

$$\tilde{\pi}(y_n, \phi(y_n) + t_n) = \tilde{\pi}\left(y_n, \phi(y_n) + \sum_{k=0}^{z_n} \phi((w_n)_k^+) + s_n\right) = \tilde{\pi}(g^{z_n+1}(w_n), s_n),$$

$n = 1, 2, \dots$. Note that $z_n \xrightarrow{n \rightarrow +\infty} +\infty$ since $t_n \xrightarrow{n \rightarrow +\infty} +\infty$ and we have condition iv). In virtue of (5.4), the sequence $\{g^{z_n+1}(w_n)\}_{n \in \mathbb{N}}$ is relatively compact in H . As $\phi((w_n)_k^+) < \lambda$ for all $k = 0, 1, 2, \dots$, we may assume without loss of generality that $\{s_n\}_{n \in \mathbb{N}}$ is convergent. Consequently, by Lemma 1.4.7, the sequence $\{\tilde{\pi}(w_n, t_n)\}_{n \in \mathbb{N}}$ is relatively compact. Thus, by Lemma 1.5.10, the set $\tilde{L}^+(\tilde{B}_2)$ is nonempty, compact, $\tilde{L}^+(\tilde{B}_2) \cap M = \emptyset$ and

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(\tilde{B}_2, t), \tilde{L}^+(\tilde{B}_2)) = 0,$$

that is, given $\epsilon > 0$ there is $L > 0$ such that

$$\tilde{\pi}(\tilde{B}_2, t) \subset B(\tilde{L}^+(\tilde{B}_2), \epsilon) \quad \text{for all } t > L. \quad (5.5)$$

We can note that $B_2 \in \tilde{B}(X)$ and by *iv*) there is $\eta > 0$ such that $\phi(x) \leq \eta$ for all $x \in B_2$. Now, taking $t > L + \eta$, it follows by using (5.5) that

$$\tilde{\pi}(B_2, t) = \bigcup_{b \in B_2} \tilde{\pi}(b, t) = \bigcup_{b \in B_2} \tilde{\pi}(\tilde{\pi}(b, \phi(b)), t - \phi(b)) \subset \bigcup_{b \in B_2} \tilde{\pi}(\tilde{B}_2, t - \phi(b)) \subset B(\tilde{L}^+(\tilde{B}_2), \epsilon),$$

and consequently,

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B_2, t), \tilde{L}^+(\tilde{B}_2)) = 0.$$

Define $K = \tilde{L}^+(B_1) \cup \tilde{L}^+(B_2)$. Then K is nonempty, compact, $K \cap M = \emptyset$ and

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, t), K) = 0$$

which concludes the theorem. $\square$

Following the ideas of the proof of Theorem 4.2.8 we establish conditions to show that (H, g) is $\tilde{\pi}$ -asymptotically compact provided that $(X, \pi; M, I)$ is $\tilde{\pi}$ -asymptotically compact.

Theorem 5.2.5. *Suppose that $(X, \pi; M, I)$ is $\tilde{\pi}$ -asymptotically compact and (H, g) is its associated discrete system, where H is a closed set. Assume that there are $\xi_1 > 0$ and $\xi_2 > 0$ such that*

$$i) \quad \xi_1 \leq \phi(x) < \xi_2, \text{ for all } x \in H;$$

$$ii) \quad \tilde{\pi}(B, [0, \xi_2]) \text{ is bounded, for all } B \subset H \text{ such that } B \text{ is a bounded invariant set under } g.$$

Then (H, g) is asymptotically compact.

Proof. Let $B \subset H$ be a bounded invariant set under g . For each $y \in \tilde{\pi}^+(B)$, there are $b \in B$ and $s \in \mathbb{R}_+$ such that $y = \tilde{\pi}(b, s)$. Note that there is $k \in \mathbb{Z}_+$ such that $y = \pi(b_k^+, \tau_k) = \tilde{\pi}(b_k^+, \tau)$ with $0 \leq \tau < \phi(b_k^+) < \xi_2$. Since B is invariant under g , we have $b_k^+ \in B$. Then

$$y \in \tilde{\pi}(B, [0, \xi_2]).$$

Thus $\tilde{\pi}^+(B) \subset \tilde{\pi}(B, [0, \xi_2])$ which implies that it is bounded. By hypothesis, there is a compact set $K_B \subset X \setminus M$ such that

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, t), K_B) = 0. \quad (5.6)$$

We have $K_B \cap H \neq \emptyset$. In fact, if we suppose that it is false, then there is $\epsilon_0 > 0$ such that $\rho(H, K_B) > \epsilon_0$. On the other hand, given $b \in B$, we have $s_n = \sum_{j=0}^n \phi(b_j^+) \xrightarrow{n \rightarrow +\infty} +\infty$, by condition *i*). Then $\lim_{n \rightarrow +\infty} \rho(g^{n+1}(b), K_B) = \lim_{n \rightarrow +\infty} \rho(\tilde{\pi}(b, s_n), K_B) = 0$ and we obtain a contradiction.

We claim that $\lim_{n \rightarrow +\infty} \text{dist}(g^n(B), K_B \cap H) = 0$. In fact, suppose to the contrary that there are $\epsilon_0 > 0$, sequences $\{b_k\}_{k \in \mathbb{N}} \subset B$ and $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}_0$ such that $n_k \xrightarrow{k \rightarrow +\infty} +\infty$ and

$$\rho(g^{n_k}(b_k), K_B \cap H) \geq \epsilon_0 \quad \text{for all } k = 0, 1, 2, \dots \quad (5.7)$$

Define $t_k = \sum_{j=0}^{n_k-1} \phi((b_k)_j^+)$, $k = 1, 2, \dots$. By condition *i*), we have $t_k \xrightarrow{k \rightarrow +\infty} +\infty$. Then using (5.6) we get

$$\lim_{k \rightarrow +\infty} \rho(g^{n_k}(b_k), K_B) = \lim_{k \rightarrow +\infty} \rho(\tilde{\pi}(b_k, t_k), K_B) = 0$$

which contradicts (5.7) since H is a closed set. Therefore, the result is proved. $\square$

Next, we define the concept of Ladyzhenskaya condition for discrete systems.

Definition 5.2.6. A discrete system (H, g) satisfies the **condition of Ladyzhenskaya**, if for every bounded set $B \subset H$ there exists a nonempty compact set $K_B \subset H$ such that

$$\lim_{n \rightarrow +\infty} \text{dist}(g^n(B), K_B) = 0.$$

Theorem 5.2.7. Suppose that $(X, \pi; M, I)$ satisfies the condition of Ladyzhenskaya, H is a closed set and there is $\xi > 0$ such that $\phi(x) \geq \xi$ for all $x \in H$. Then (H, g) satisfies the condition of Ladyzhenskaya.

Proof. Let $B \subset X$ be a bounded set. By Definition 4.2.9, there exists a compact set $K_B \subset X$ with $K_B \cap M = \emptyset$ such that

$$\lim_{t \rightarrow +\infty} \text{dist}(\tilde{\pi}(B, t), K_B) = 0.$$

Again, we have $K_B \cap H \neq \emptyset$ as we showed in the proof of Theorem 5.2.5. Then we conclude that $\lim_{n \rightarrow +\infty} \text{dist}(g^n(B), K_B \cap H) = 0$ and the result follows. $\square$

5.3 Recurrent motions by time reparametrization

Let $(X, \pi; M, I)$ be an ISS and (H, g) be its associated discrete dynamical system in the sense of Kaul as presented in Definition 5.1.1.

Our aim is to investigate recursive properties between an ISS and its associated discrete system. In the sequel, we define some concepts of recursiveness.

Definition 5.3.1. A point $x \in X$ is called **almost $\tilde{\pi}$ -periodic by time reparametrization**, if given $\epsilon > 0$ there is a number $T = T(\epsilon) > 0$ such that for every $\alpha \geq 0$, the interval $[\alpha, \alpha + T]$ contains a number $\tau_\alpha > 0$ and one can obtain a time reparametrization h_α such that

$$\rho(\tilde{\pi}(x, h_\alpha(t) + \tau_\alpha), \tilde{\pi}(x, t)) < \epsilon \quad \text{for all } t \geq 0. \quad (5.8)$$

Definition 5.3.2. Let $\sigma > 0$. A **time σ -reparametrization** is a time reparametrization $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $|h(t) - t| < \sigma$ for every $t \in \mathbb{R}_+$. If $0 \leq h(t) - t < \sigma$ for all $t \in \mathbb{R}_+$, we say that h is a **positive time σ -reparametrization**.

Definition 5.3.3. A point $x \in X$ is called **almost $\tilde{\pi}$ -periodic by time σ -reparametrization**, if given $\epsilon > 0$ there is a number $T = T(\epsilon) > 0$ such that for every $\alpha \geq 0$, the interval $[\alpha, \alpha + T]$ contains a number $\tau_\alpha > 0$ and one can obtain a time σ -reparametrization h_α such that (5.8) holds. A point $x \in X$ is called **almost $\tilde{\pi}$ -periodic by a positive time σ -reparametrization**, if the reparametrization h_α is a positive time σ -reparametrization.

The next lemma shows that every point from a positive orbit $\tilde{\pi}^+(x)$ is almost $\tilde{\pi}$ -periodic by time σ -reparametrization provided that x has this positive property.

Lemma 5.3.4. *Let $\sigma > 0$ and $x \in X$ be almost $\tilde{\pi}$ -periodic by a positive time σ -reparametrization. Then every point $y \in \tilde{\pi}^+(x)$ is almost $\tilde{\pi}$ -periodic by time σ -reparametrization.*

Proof. Let $\epsilon > 0$ be given. Since $x \in X$ is almost $\tilde{\pi}$ -periodic by a positive time σ -reparametrization, then there is $T = T(\epsilon) > 0$ such that for every $\alpha \geq 0$, the interval $[\alpha, \alpha + T]$ contains a number $\tau_\alpha > 0$ and one can find a time σ -reparametrization h_α such that

$$\rho(\tilde{\pi}(x, h_\alpha(t) + \tau_\alpha), \tilde{\pi}(x, t)) < \epsilon \quad \text{for all } t \geq 0.$$

Take $y \in \tilde{\pi}^+(x)$, then $y = \tilde{\pi}(x, s)$ for some $s \geq 0$. Let $T_s = T + \sigma$. For each $\alpha \geq 0$ consider the number $\tau_\alpha > 0$ and the function $h_\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ chosen above and define

$$\tau_\alpha^s = \tau_\alpha + h_\alpha(s) - s$$

and

$$H_\alpha(t) = h_\alpha(t + s) - h_\alpha(s) \quad \text{for all } t \geq 0.$$

Then $\tau_\alpha^s \in [\alpha, \alpha + T_s]$, $H_\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a time reparametrization and for all $t \geq 0$ we have

$$-\sigma < H_\alpha(t) - t = h_\alpha(t + s) - h_\alpha(s) - t = (h_\alpha(t + s) - (t + s)) - (h_\alpha(s) - s) < \sigma.$$

Thus H_α is a time σ -reparametrization and

$$\begin{aligned} \rho(\tilde{\pi}(y, t), \tilde{\pi}(y, H_\alpha(t) + \tau_\alpha^s)) &= \rho(\tilde{\pi}(x, s + t), \tilde{\pi}(x, s + h_\alpha(t + s) - h_\alpha(s) + \tau_\alpha + h_\alpha(s) - s)) \\ &= \rho(\tilde{\pi}(x, s + t), \tilde{\pi}(x, h_\alpha(t + s) + \tau_\alpha)) < \epsilon \end{aligned}$$

for all $t \geq 0$. The proof is complete. $\square$

The concept of almost periodicity for discrete systems is defined in the next definition. This concept was introduced in [32]. Let us consider $g : H \rightarrow H$ as defined in (5.2).

Definition 5.3.5. A point $x \in H$ is said to be **almost g -periodic** if given $\epsilon > 0$ there is $N_1 > 0$ such that for each $n_1 \in \mathbb{Z}_+$, the interval $[n_1, n_1 + N_1]$ contains a number $m_{n_1} \in \mathbb{Z}_+$ such that

$$\rho(g^n(x), g^{n+m_{n_1}}(x)) = \rho(x_n^+, x_{n+m_{n_1}}^+) < \epsilon \quad \text{for all } n \in \mathbb{Z}_+.$$

In the next result, we present sufficient conditions for a point to be almost $\tilde{\pi}$ -periodic by time reparametrization provided this point is almost g -periodic in its associated discrete system (H, g) .

Theorem 5.3.6. Let $(X, \pi; M, I)$ be an ISS and (H, g) be its associated discrete system in the sense of Kaul. If $x \in H$ is a point almost g -periodic, $\overline{g^+(x)} \cap M = \emptyset$ and $\overline{g^+(x)}$ is compact, then x is almost $\tilde{\pi}$ -periodic by time reparametrization.

Proof. Let $\epsilon > 0$ be given. Since $\overline{g^+(x)}$ is compact and $\overline{g^+(x)} \cap M = \emptyset$ then we have $T = \sup_{k \geq 0} \phi(x_k^+) < +\infty$ because ϕ is uniformly continuous on the compact set $\overline{g^+(x)}$. The mapping π is uniformly continuous on $\overline{g^+(x)} \times [0, T]$, then there is $\delta \in (0, \epsilon)$ such that if $y, z \in \overline{g^+(x)}$ and $t_1, t_2 \in [0, T]$ satisfying $\max\{\rho(y, z), |t_1 - t_2|\} < \delta$, then

$$\rho(\pi(y, t_1), \pi(z, t_2)) < \epsilon. \tag{5.9}$$

By the uniform continuity of ϕ on $\overline{g^+(x)}$, one can obtain $\delta_1 \in (0, \delta)$ such that if $y, z \in \overline{g^+(x)}$ with $\rho(y, z) < \delta_1$, then $|\phi(y) - \phi(z)| < \delta$.

By hypothesis, $x \in H$ is almost g -periodic. Then, for $\delta_1 > 0$ chosen above, there is $N_0 \in \mathbb{Z}_+$ such that for each $m \in \mathbb{Z}_+$, the interval $[m, m + N_0]$ contains a number $n_m \in \mathbb{Z}_+$ such that

$$\rho(g^n(x), g^{n+n_m}(x)) = \rho(x_n^+, x_{n+n_m}^+) < \delta_1 \quad \text{for all } n \in \mathbb{Z}_+.$$

Let $T_1 = (N_0 + 1)T$. We claim that T_1 satisfies Definition 5.3.5. Indeed, given $\alpha \geq 0$, there is $k \in \mathbb{Z}_+$ such that $t_k(x) \leq \alpha < t_{k+1}(x)$. Thus there is $n_k \in [k + 1, k + 1 + N_0] \cap \mathbb{Z}_+$ such that

$$\rho(x_n^+, x_{n+n_k}^+) < \delta_1 \quad \text{for all } n \in \mathbb{Z}_+.$$

Let $\tau_\alpha = t_{n_k}(x)$. Then $\tau_\alpha \in [\alpha, \alpha + T_1]$ because

$$\alpha < t_{k+1}(x) \leq t_{n_k}(x) = \tau_\alpha < t_{k+1+N_0}(x) = t_k(x) + \sum_{i=k}^{k+N_0} \phi(x_i^+) \leq \alpha + (N_0 + 1)T.$$

Now, we define the time reparametrization $h_\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$h_\alpha(t) = t_n(x_{n_k}^+) + \frac{\phi(x_{n+n_k}^+)}{\phi(x_n^+)}(t - t_n(x)), \quad t \in [t_n(x), t_{n+1}(x)), \quad n = 0, 1, 2, \dots$$

If $t = t_n(x)$, $n \in \mathbb{Z}_+$, we have

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(x, h_\alpha(t) + \tau_\alpha)) = \rho(x_n^+, \tilde{\pi}(x, t_n(x_{n_k}^+) + t_{n_k}(x))) = \rho(x_n^+, x_{n+n_k}^+) < \delta_1 < \epsilon.$$

If $t \in (t_n(x), t_{n+1}(x))$, $n \in \mathbb{Z}_+$, we have

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(x, h_\alpha(t) + \tau_\alpha)) = \rho\left(\pi(x_n^+, t - t_n(x)), \pi\left(x_{n+n_k}^+, \frac{\phi(x_{n+n_k}^+)}{\phi(x_n^+)}(t - t_n(x))\right)\right).$$

Since $\rho(x_n^+, x_{n+n_k}^+) < \delta_1$ and

$$\left|t - t_n(x) - \frac{\phi(x_{n+n_k}^+)}{\phi(x_n^+)}(t - t_n(x))\right| < |\phi(x_n^+) - \phi(x_{n+n_k}^+)| < \delta,$$

we conclude by (5.9) that

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(x, h_\alpha(t) + \tau_\alpha)) < \epsilon.$$

Hence, $\rho(\tilde{\pi}(x, t), \tilde{\pi}(x, h_\alpha(t) + \tau_\alpha)) < \epsilon$ for all $t \geq 0$. □

Definition 5.3.7. A point $x \in H$ is called **strongly almost g -periodic** if given $\epsilon > 0$ there is $N_1 > 0$ such that for each $n_1 \in \mathbb{Z}_+$, the interval $[n_1, n_1 + N_1]$ contains a number $m_{n_1} \in \mathbb{Z}_+$ such that

$$\rho(x_n^+, x_{n+m_{n_1}}^+) < \epsilon \quad \text{and} \quad |t_k(x_n^+) - t_k(x_{n+m_{n_1}}^+)| < \epsilon, \quad \text{for all } n, k \in \mathbb{Z}_+.$$

Theorem 5.3.8. Let $(X, \pi; M, I)$ be an ISS and (H, g) be its associated discrete system in the sense of Kaul. If $x \in H$ is a point strongly almost g -periodic, $\overline{g^+(x)} \cap M = \emptyset$ and $\overline{g^+(x)}$ is compact, then for every $\epsilon > 0$, the point x is almost $\tilde{\pi}$ -periodic by time ϵ -reparametrization.

Proof. Let $\epsilon > 0$ be given. Using the proof of Theorem 5.3.6, we need just to note that if $t \in [t_n(x), t_{n+1}(x))$, $n = 0, 1, 2, \dots$, then

$$\begin{aligned} |h_\alpha(t) - t| &= \left| t_n(x_{n_k}^+) + \frac{\phi(x_{n+n_k}^+)}{\phi(x_n^+)}(t - t_n(x)) - t \right| \\ &\leq \max\{|t_n(x_{n_k}^+) - t_n(x)|, |t_{n+1}(x_{n_k}^+) - t_{n+1}(x)|\} < \epsilon, \end{aligned}$$

where we have used the second condition of Definition 5.3.7.

Therefore, x is almost $\tilde{\pi}$ -periodic by time ϵ -reparametrization. $\square$

5.4 Asymptotically motions by time reparametrization

The concept of asymptotic almost g -periodicity for discrete systems is presented in the next definition.

Definition 5.4.1. A point $x \in H$ is said to be **asymptotically almost g -periodic** if there is a point $p \in H$ almost g -periodic such that

$$\lim_{n \rightarrow +\infty} \rho(g^n(x), g^n(p)) = 0. \quad (5.10)$$

Definition 5.4.2. A point $x \in X$ is called **asymptotically almost $\tilde{\pi}$ -periodic by time reparametrization**, if there are a point p almost $\tilde{\pi}$ -periodic by time reparametrization and a reparametrization h_p such that

$$\lim_{t \rightarrow +\infty} \rho(\tilde{\pi}(x, t), \tilde{\pi}(p, h_p(t))) = 0.$$

Next, we present sufficient conditions for a point in H to be asymptotically almost $\tilde{\pi}$ -periodic by time reparametrization.

Theorem 5.4.3. Let $(X, \pi; M, I)$ be an ISS and (H, g) be its associated discrete system in the sense of Kaul. Suppose that X is a complete metric space. If $x \in H$ is a point asymptotically almost g -periodic and $\{x_n^+\}_{n \in \mathbb{N}}$ is convergent in H , then x is asymptotically almost $\tilde{\pi}$ -periodic by time reparametrization.

Proof. By hypothesis, there exists a point $p \in H$ almost g -periodic such that

$$\lim_{n \rightarrow +\infty} \rho(g^n(x), g^n(p)) = 0. \quad (5.11)$$

Since $\{x_n^+\}_{n \in \mathbb{N}}$ converges in H then we have $\overline{g^+(x)}$ compact and $\overline{g^+(x)} \cap M = \emptyset$ ($\overline{g^+(x)} \subset H \subset I(M)$) and we have condition **(H2)**. Now, using (5.11), we obtain that $\overline{g^+(p)}$ is compact and $\overline{g^+(p)} \cap M = \emptyset$. Hence, by Theorem 5.3.6 the point p is almost $\tilde{\pi}$ -periodic by time reparametrization.

Let $\epsilon > 0$ be given. By the uniform continuity of π on $(\overline{g^+(x)} \cup \overline{g^+(p)}) \times [0, T]$, there is $\delta \in (0, \epsilon)$ such that if $y, z \in \overline{g^+(x)} \cup \overline{g^+(p)}$ and $t_1, t_2 \in [0, T]$ satisfying $\max\{\rho(y, z), |t_1 - t_2|\} < \delta$, then

$$\rho(\pi(y, t_1), \pi(z, t_2)) < \epsilon. \quad (5.12)$$

Let $\lim_{n \rightarrow +\infty} x_n^+ = \lim_{n \rightarrow +\infty} p_n^+ = z \in H$. By the continuity of ϕ at z , there is $\delta_1 \in (0, \delta)$ such that if $\rho(y, z) < \delta_1$ then $|\phi(y) - \phi(z)| < \frac{\delta}{2}$. Let $n_1 \in \mathbb{N}$ be such that $\rho(x_n^+, z) < \delta_1$ and $\rho(p_n^+, z) < \delta_1$ for all $n \geq n_1$.

By (5.11), there is $n_2 \in \mathbb{N}$, $n_2 \geq n_1$, such that

$$\rho(x_n^+, p_n^+) = \rho(g^n(x), g^n(p)) < \delta_1 \quad \text{for all } n \geq n_2.$$

Now, we define a time reparametrization $h_p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$h_p(t) = t_n(p) + \frac{\phi(p_n^+)}{\phi(x_n^+)}(t - t_n(x)), \quad t \in [t_n(x), t_{n+1}(x)), \quad n = 0, 1, 2, \dots$$

If $t = t_n(x)$ for $n \geq n_2$, we have

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(p, h_p(t))) = \rho(x_n^+, p_n^+) < \delta_1 < \epsilon.$$

If $t \in (t_n(x), t_{n+1}(x))$, for $n \geq n_2$, we have

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(p, h_p(t))) = \rho\left(\pi(x_n^+, t - t_n(x)), \pi\left(p_n^+, \frac{\phi(p_n^+)}{\phi(x_n^+)}(t - t_n(x))\right)\right).$$

Since $\rho(x_n^+, p_n^+) < \delta_1$ and

$$\left|t - t_n(x) - \frac{\phi(p_n^+)}{\phi(x_n^+)}(t - t_n(x))\right| < |\phi(x_n^+) - \phi(p_n^+)| \leq |\phi(x_n^+) - \phi(z)| + |\phi(z) - \phi(p_n^+)| < \delta,$$

for $n \geq n_2$, we can conclude by (5.12) that

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(p, h_p(t))) < \epsilon \quad \text{for all } t \geq t_{n_2}(x).$$

Therefore, x is asymptotically almost $\tilde{\pi}$ -periodic by time reparametrization. □

5.5 Lyapunov stability and Zhukovskij quasi stability

In this last section, we present sufficient conditions to obtain Zhukovskij quasi stability via Lyapunov stability. The concept of Lyapunov stability for discrete systems in the sense of Kaul was introduced in [33] as presented below.

Definition 5.5.1. A point $x \in H$ is called **Lyapunov g -stable with respect to a set $P \subset H$** if $x \in \overline{P}$ and given $\epsilon > 0$ there exists $\delta > 0$ such that if $\rho(x, p) < \delta$ with $p \in P$, then

$$\rho(g^n(x), g^n(p)) = \rho(x_n^+, p_n^+) < \epsilon \quad \text{for all } n \in \mathbb{Z}_+.$$

A subset $A \subset H$ is called **Lyapunov g -stable with respect to $P \subset H$** if $A \subset \overline{P}$ and each point $x \in A$ is Lyapunov g -stable with respect to $P \subset H$.

If $A \subset X$ is a set such that $\phi(a) < +\infty$ for every $a \in A$, then we may define

$$\tilde{A} = \bigcup_{x \in A} \tilde{\pi}(x, \phi(x)).$$

Note that $\tilde{A} \subset I(M)$. If $\phi(a_j^+) < +\infty$ for all $j = 0, 1, 2, \dots$, and for all $a \in A$, then $\tilde{A} \subset H$.

Theorem 5.5.2. Let $(X, \pi; M, I)$ be an ISS and (H, g) be its associated discrete system in the sense of Kaul, where H is a closed set. Let $A, B \subset X \setminus M$, $A \subset \overline{B}$ and B be a relatively compact positively $\tilde{\pi}$ -invariant set. Suppose that $\phi(b) < +\infty$ for all $b \in \overline{B} \setminus M$. If $\tilde{A}$ is Lyapunov g -stable with respect to $\overline{B}$, then any set $\mathcal{O} \subset \overline{A} \setminus M$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to $\overline{B} \setminus M$.

Proof. Let $\mathcal{O} \subset \overline{A} \setminus M$, $x \in \mathcal{O}$ be fixed and note that $\phi(x) < +\infty$ by hypothesis. Since $\overline{B}$ is compact and H is closed then we have $\overline{\tilde{B}}$ compact and $\overline{\tilde{B}} \cap M = \emptyset$. Note that $\tilde{B} \subset H$ because B is positively $\tilde{\pi}$ -invariant. The continuity of ϕ on the compact set $\overline{\tilde{B}} \cup \{x\}$ implies that

$$T = \sup_{a \in \overline{\tilde{B}} \cup \{x\}} \phi(a) < +\infty.$$

Let $\epsilon > 0$ be given. Using the uniform continuity of π in $\overline{B} \times [0, T]$, one can obtain $\delta_1 \in (0, \epsilon)$ such that for each $y, z \in \overline{B}$ and $t_1, t_2 \in [0, T]$ satisfying $\max\{\rho(y, z), |t_1 - t_2|\} < \delta_1$ we have

$$\rho(\pi(y, t_1), \pi(z, t_2)) < \epsilon. \quad (5.13)$$

Note that ϕ is uniformly continuous on the compact $\overline{\widetilde{B}}$. Thus there is $\delta_2 \in (0, \delta_1)$ such that if $y, z \in \overline{\widetilde{B}}$ and $\rho(y, z) < \delta_2$ then

$$|\phi(y) - \phi(z)| < \delta_1. \quad (5.14)$$

By the Lyapunov g -stability of $x_1^+ \in \overline{\widetilde{A}}$ with respect to $\overline{\widetilde{B}}$, there is $\delta_3 \in (0, \delta_2)$ such that if $p \in \overline{\widetilde{B}}$ with $\rho(x_1^+, p) < \delta_3$ then

$$\rho(g^n(x_1^+), g^n(p)) < \delta_2 \quad \text{for all } n \in \mathbb{Z}_+. \quad (5.15)$$

Since π is continuous in $X \times \mathbb{R}_+$, I is continuous in M and ϕ is continuous in $X \setminus M$ one can find $\delta_4 \in (0, \delta_3)$ such that if $y \in \overline{B} \setminus M$ with $\rho(x, y) < \delta_4$ then

$$\rho(x_1^+, y_1^+) = \rho(I(\pi(x, \phi(x))), I(\pi(y, \phi(y)))) < \delta_3.$$

Consequently, by (5.15) we have

$$\rho(x_n^+, y_n^+) < \delta_2 \quad \text{for all } n \in \mathbb{Z}_+. \quad (5.16)$$

Since B is positively $\tilde{\pi}$ -invariant then we have $\overline{B} \setminus M$ positively $\tilde{\pi}$ -invariant, see Lemma 1.5.5. Thus, $x_n^+, y_n^+ \in \overline{\widetilde{B}}$ for all $n \in \mathbb{N}$, where $y \in \overline{B} \setminus M$. Then if $y \in \overline{B} \setminus M$ with $\rho(x, y) < \delta_4$ it follows by (5.16) and (5.14) that

$$|\phi(x_n^+) - \phi(y_n^+)| < \delta_1 \quad \text{for every } n \in \mathbb{Z}_+. \quad (5.17)$$

For $y \in \overline{B} \setminus M$ such that $\rho(x, y) < \delta_4$ we define the time reparametrization $h_y : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$h_y(t) = t_n(y) + \frac{\phi(y_n^+)}{\phi(x_n^+)}(t - t_n(x)), \quad t \in [t_n(x), t_{n+1}(x)), \quad n = 0, 1, 2, \dots$$

If $t = t_n(x)$, $n \in \mathbb{N}$, we have

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(y, h_y(t))) = \rho(x_n^+, y_n^+) < \delta_2 < \epsilon.$$

If $t \in (t_n(x), t_{n+1}(x))$, $n \in \mathbb{N}$, we have

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(y, h_y(t))) = \rho\left(\pi(x_n^+, t - t_n(x)), \pi\left(y_n^+, \frac{\phi(y_n^+)}{\phi(x_n^+)}(t - t_n(x))\right)\right).$$

Since $x_n^+, y_n^+ \in \overline{\widetilde{B}}$, $\rho(x_n^+, y_n^+) < \delta_2 < \delta_1$ and

$$\left| t - t_n(x) - \frac{\phi(y_n^+)}{\phi(x_n^+)}(t - t_n(x)) \right| < |\phi(x_n^+) - \phi(y_n^+)| < \delta_1,$$

then we conclude by (5.13) that

$$\rho(\tilde{\pi}(x, t), \tilde{\pi}(y, h_y(t))) < \epsilon.$$

Thus, every point $x \in \mathcal{O} \subset \overline{A} \setminus M$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to the set $\overline{B} \setminus M$ and the proof is complete. $\square$

Corollary 5.5.3. *Let $(X, \pi; M, I)$ be an ISS and (H, g) be its associated discrete system in the sense of Kaul, where H is a closed set. Let $x \in X \setminus M$, $\overline{\tilde{\pi}^+(x)}$ be compact, $\phi(x_n^+) < +\infty$ for all $n \in \mathbb{Z}_+$ and $\{x_n^+\}_{n \in \mathbb{N}}$ be convergent in H . If $\overline{g^+(x_1^+)}$ is Lyapunov g -stable with respect to itself, then any set $\mathcal{O} \subset \overline{\tilde{\pi}^+(x)} \setminus M$ is Zhukovskij quasi $\tilde{\pi}$ -stable with respect to $\overline{\tilde{\pi}^+(x)} \setminus M$.*

Proof. It is enough to note that $\phi(y) < +\infty$ for all $y \in \overline{\tilde{\pi}^+(x)} \setminus M$. In fact, since

$$\overline{\tilde{\pi}^+(x)} \setminus M = \pi^+(x) \cup (\tilde{L}^+(x) \setminus M)$$

and $\phi(y) < +\infty$ for all $y \in \tilde{\pi}^+(x)$, we need to show that $\phi(y) < +\infty$ for all $y \in \tilde{L}^+(x) \setminus M$. Let $y \in \tilde{L}^+(x) \setminus M$ then there is a sequence $\{s_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ such that $s_n \xrightarrow{n \rightarrow +\infty} +\infty$ and

$$\tilde{\pi}(x, s_n) \xrightarrow{n \rightarrow +\infty} y.$$

For each $n \in \mathbb{N}$ there is $k_n \in \mathbb{Z}_+$ such that $t_{k_n}(x) \leq s_n < t_{k_n+1}(x)$. Then

$$\tilde{\pi}(x, s_n) = \pi(x_{k_n}^+, s_n - t_{k_n}) \quad \text{and} \quad \phi(\tilde{\pi}(x, s_n)) = \phi(x_{k_n}^+) - (s_n - t_{k_n}(x)).$$

Thus

$$\phi(\tilde{\pi}(x, s_n)) \leq \phi(x_{k_n}^+), \tag{5.18}$$

for all $n \in \mathbb{N}$. Since $\{x_n^+\}_{n \in \mathbb{N}}$ is convergent in H we may write $\lim_{n \rightarrow +\infty} x_n^+ = z \in H$. Then, using the continuity of ϕ in $X \setminus M$ as $n \rightarrow +\infty$ in (5.18), we get

$$\phi(y) \leq \phi(z).$$

Now, since $z \in H$ we have $\phi(z) < +\infty$. Hence, $\phi(y) < +\infty$ for all $y \in \tilde{L}^+(x) \setminus M$. The result follows by Theorem 5.5.2. $\square$

Bibliography

- [1] Bhatia, N. P. and Szegő, G. P.: Stability theory of dynamical systems, Grundlehren Math. Wiss., Band 161, Springer-Verlag, New York, 1970; reprint of the 1970 original in: Classics Math., Springer-Verlag, Berlin, 2002.
- [2] Bonotto, E. M. and Federson, M.: Topological conjugation and asymptotic stability in impulsive semidynamical systems, J. Math. Anal. Appl., 326 (2007), 869-881.
- [3] Bonotto, E. M.: Flows of characteristic 0^+ in impulsive semidynamical system, J. Math. Anal. Appl., 332 (1) (2007), 81-96.
- [4] Bonotto, E. M. and Federson, M.: Poisson stability for impulsive semidynamical system, Nonlinear Analysis: Theory, Methods & Applications, 71 (12)(2009), 6148-6156.
- [5] Bonotto, E. M. and Jr. Grulha, N. G.: Lyapunov stability of closed sets in impulsive semidynamical systems, Electronic Journal of Differential Equations, 2010 (78) (2010), 1-18.
- [6] Bonotto, E. M. and Demuner, D. P.: Autonomous dissipative semidynamical systems with impulses, Topological Methods in Nonlinear Analysis, 41 (1) (2013), 1-38.
- [7] Bonotto, E. M. and Demuner, D. P.: Attractors of impulsive dissipative semidynamical systems, Bull. Sci. Math., 137 (2013), 617-642.
- [8] Bonotto, E. M. and Jimenez, M. Z.: Weak almost periodic motions, minimality and stability in impulsive semidynamical systems, J. Diff. Equations, 256 (2014), 1683-1701.

- [9] Bonotto, E. M. and Jimenez, M. Z.: On impulsive semidynamical systems: minimal, recurrent and almost periodic motions, *Topological Methods in Nonlinear Analysis*, 44 (2014), 121-141.
- [10] Bonotto, E. M. and Ferreira, J. C.: Uniform attractors of discontinuous semidynamical systems, *Collectanea Mathematica*, 65 (1) (2014), 47-59.
- [11] Bonotto, E. M., Gimenes, L. P. and Souto, G. M.: On Jack Hale's problem for impulsive systems, *J. Diff. Equations*, (2) (2015), 642-665.
- [12] Bonotto, E. M., Bortolan, M. C., Carvalho, A. N. and Czaja, R.: Global attractors for impulsive dynamical systems - a precompact approach, *J. Diff. Equations*, (2015), 2602-2625.
- [13] Bonotto, E. M. and Ferreira, J. C.: Dissipativity in impulsive systems via Lyapunov functions, *Math. Nachr.*, 289 (2016), 213-231.
- [14] Bonotto, E. M., Bortolan, M. C., Caraballo, T. and Collegari, R.: Impulsive surfaces on dynamical systems, *Acta Mathematica Hungarica*, (2016), 209-216.
- [15] Bonotto, E. M., Gimenes, L. P. and Souto, G. M.: Asymptotic almost periodic motions, *Topological Methods in Nonlinear Analysis*, (2016). In press.
- [16] Bonotto, E. M. and Souto G. M.: On the Lyapunov stability theory for impulsive dynamical systems. Submitted.
- [17] Bohr, H.: Zur theorie der fastperiodischen funktionen. II: Zusammenhang der fastperiodischen funktionen mit funktionen von unendlich vielen variablen; gleichmssige approximation durch trigonometrische summen, *Acta Math.*, 46 (1925), 101-214.
- [18] Bainov, B. B., Lakshmikantham, V. and Simeonov, P. S.: *Theory of Impulsive Differential Equations*, Series in Modern Applied Mathematics, vol. 6, World Scientific, Singapore, (1989).
- [19] Cheban, D. N.: *Global attractors of non-autonomous dissipative dynamical systems*, Interdiscip. Math. Sci., vol. 1, World Scientific Publishing, Hackensack, NJ, (2004).
- [20] Cheban, D. N.: *Asymptotically Almost Periodic Solutions of Differential Equations*, Hindawi, Publishing Corporation, (2009).

- [21] Cholewa, J. W. and Hale, J. K.: Some counterexamples in dissipative systems, *Dynamics of Continuous, Discrete and Impulsive Systems*, 7 (2000), 159-176.
- [22] Ciesielski, K.: Sections in semidynamical systems, *Bull. Polish Acad. Sci. Math.*, 40 (1992), 297-307.
- [23] Ciesielski, K.: On semicontinuity in impulsive dynamical systems, *Bull. Polish Acad. Sci. Math.*, 52 (2004), 71-80.
- [24] Ciesielski, K.: On stability in impulsive dynamical systems, *Bull. Polish Acad. Sci. Math.*, 52 (2004), 81-91.
- [25] Ciesielski, K.: On time reparametrization and isomorphisms of impulsive dynamical system, *Ann. Polon. Math.*, 84 (2004), 1-25.
- [26] Ding, C.: Lyapunov quasi-stable trajectories, *Fundamenta Mathematicae*, 220 (2013), 139-154.
- [27] Fréchet, M.: Les fonctions asymptotiquement presque-periodiques continues, *Comptes Rendus de l'Académie des Sciences*, vol. 213, (1941), 520-522.
- [28] Hale, J. K.: Some Recent Results on Dissipative Processes, *Lecture Notes in Mathematics*, 799 (1980), 152-172.
- [29] Hale, J. K.: Asymptotic behavior of dissipative systems. *Mathematical Surveys and Monographs*, 25. American Mathematical Society, Providence, RI, (1988).
- [30] Hino, Y., Naito, T., Nguyen V. M. and Shin, J. S.: Almost periodic solutions of differential equations in Banach spaces, *Stability and Control: Theory, Methods and Applications*, Taylor and Francis, London, (2002).
- [31] Kaul, S. K.: On impulsive semidynamical systems, *J. Math. Anal. Appl.*, 150 (1990), 120-128.
- [32] Kaul, S. K.: On impulsive semidynamical systems II, *Recursive Properties, Nonlinear Analysis, Methods and Applications*, 16 (7/8) (1991), 635-645.
- [33] Kaul, S. K.: On impulsive semidynamical systems III, *Lyapunov stability, Recent Trends in Differential Equations*, World Sci. Ser. Appl. Anal., 1. Publishing, River Edge, NJ, (1992), 335-345.

-
- [34] Kaul, S. K.: Stability and asymptotic stability in impulsive semidynamical systems, *J. Applied Math. and Stochastic Analysis*, 7(4) (1994), 509-523.
 - [35] Ladyzhenskaya, O.: *Attractors for semigroups and evolution equations*, *Lezioni Lincee*. Cambridge University Press, Cambridge, (1991).
 - [36] Munkres, J. R.: *Topology*, second edition, Prentice Hall, (2000).
 - [37] Robinson, J. C.: *Infinite-dimensional dynamical systems. An introduction to dissipative parabolic PDEs and the theory of global attractors*, *Cambridge Texts in Applied Mathematics*, Cambridge University Press, Cambridge, (2001).
 - [38] Saperstone, S. H.: *Semidynamical Systems in Infinite Dimensional Spaces*, Springer, New York, (1981).
 - [39] Zhukovskij N. E.: On solidity of motion, in *Collection of Works*, Leningrad Press, vol 1., (1948).