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Gradient structure of the cascade system

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Estrutura gradiente de sistemas do tipo cascata

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*This work is dedicated to my parents,
who worked hard so that one day
I could write this thesis.*

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*“There are many paths to the top of the mountain,
but the view is always the same.”
(Chinese Proverb)*

RESUMO

TAKAESSU, C. R. **Estrutura gradiente de sistemas do tipo cascata**. 2021. 116 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

Neste trabalho abordamos a teoria necessária para resolver um dos problemas presentes no artigo ([ARAGÃO-COSTA *et al.*, 2013](#)), relacionado ao sistema do tipo cascata. O presente trabalho está dividido em três partes: a primeira parte dedica-se a estudar resultados e conceitos de sistemas dinâmicos autônomos, como a robustez de semigrupos gradiente, por exemplo. A segunda parte dedica-se a conceitos da teoria de sistemas dinâmicos não autônomos como atratores pullback, dicotomia exponencial e soluções globais hiperbólicas. A terceira parte é a aplicação, relacionada aos sistemas do tipo cascata, das duas primeiras partes. Apesar dos sistemas do tipo cascata serem autônomos, tais sistemas se beneficiam da teoria de atratores não autônomos, em particular dos conceitos de skew-product.

Palavras-chave: Semigrupos, Sistemas dinâmicos, Sistema do tipo cascata.

ABSTRACT

TAKAESSU, C. R. **Gradient structure of the cascade system**. 2021. 116 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2021.

In this work we approach the theory necessary to solve one of the problems present in the article ([ARAGÃO-COSTA *et al.*, 2013](#)), related to the cascade system. The present paper is divided in three parts: the first part is dedicated to study results and concepts from autonomous dynamical systems, such as the robustness of gradient semigroups, for example. The second part is dedicated to the concepts of non-autonomous dynamical systems like pullback attractors, exponential dichotomy and hyperbolic global solutions. The third part is the application, related to the cascade systems, of the first two chapters. Even though the cascade systems are autonomous, it benefits considerably from the theory of non-autonomous attractors, in particular of the skew-product formulation.

Keywords: Semigroups, Dynamical Systems, Cascade system.

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INTRODUCTION

Since the 17th century, the theory of differential equations became a powerful tool to model several phenomena in physics (Navier-Stokes, wave equation, heat equation, Maxwell's Laws), biology (population modeling), economy (Kaldor model), engineering (study of beams), medicine (growth of bacteria and tumors), etc. Therefore, we often want to know what behavior is assumed by the solution of a differential equation over time, that is, we want to identify the asymptotic behavior of the solution.

As in most cases we cannot compute explicitly the solution of the differential equation (we only know about its existence), dynamical systems arise as a tool to deal with this kind of problem. Through the study of dynamical systems, it is possible to understand, under certain conditions, the behavior of the solution and therefore understand how the system will act over time. Briefly, our study of dynamical systems, seeks to identify the presence of a global attractor and study its structure, since the attractor has the most relevant information related to the asymptotic behavior.

In order to comprehend the intuition behind the theory, consider the Cauchy problem

$$\begin{cases} \dot{x} = f(t, x), & t \geq \tau \\ x(\tau) = x_0 \in \mathbb{R}^n \end{cases} \quad (1.1)$$

with $f \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, where $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ denotes the continuously differentiable maps of $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$. If (1.1) has a unique solution $u(\cdot, \tau, x_0)$ in $[\tau, \infty)$, we can define a family of operators continuous operators $S = \{S(t, s) : t, s \in \mathbb{R} \text{ and } t \geq s\}$ given by

$$S(t, s)x = u(t, s, x), \forall t \geq s. \quad (1.2)$$

In fact, we can use the results of existence, uniqueness and continuity with respect to the initial parameters of solutions to show that (1.2) satisfies the definition of a dynamical system (see Definition 23 and Example 2). This means that we can associate the problem (1.1) to a

dynamical system, that is, we can study the properties of the dynamical system instead of trying to solve (1.1). Hence, studying the dynamical system and their attractors we may be able to give information about the behavior of the solutions of (1.1). This shows very briefly why dynamical systems are so powerful and useful in mathematics.

Note that our function f depends on t and because of that we call (1.1) as a non-autonomous problem. In the simpler case that $f(t, x) = g(x)$, for all $t \in \mathbb{R}$ and some $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, we have the autonomous problem

$$\begin{cases} \dot{x} = g(x), & t \geq \tau \\ x(\tau) = x_0 \in \mathbb{R}^n. \end{cases} \quad (1.3)$$

Again, we obtain a dynamical system as in (1.2) and we say that S is a autonomous dynamical system, because it has the extra property $S(t, s) = S(t - s, 0)$, for all $t \geq s$, since f does not depends on t . Hence, we can define a new object called semigroup (Definition 1), given by

$$T(t) = S(t, 0), \forall t \in \mathbb{R}, t \geq 0. \quad (1.4)$$

Therefore, we can associate to the autonomous problem (1.3) the autonomous dynamical systems, or semigroup, $\{T(t) : t \geq 0\}$ and obtain information about its dynamics through it. Since autonomous problems are simpler, our starting point will be the study of semigroups (Chapters 2 and 3).

At times non-autonomous dynamical system can help us understand the asymptotic behavior of the solutions of (1.1). Indeed, in Chapter 4 we will study some important semigroups, namely the driving and the skew-product semigroups, denoted by θ and Π , respectively. Those semigroups are related to a non-autonomous dynamical system $\mathcal{K}$, called cocycle, and it is directly connected to our main problem, the cascade system:

$$\begin{cases} \dot{x}_1 = f_1(x_1) \\ \dot{x}_2 = f_2(x_1, x_2). \end{cases} \quad (1.5)$$

In order to understand the behavior of a solution of (1.1), we need to understand how the vector fields generated by f evolves as $t \rightarrow \infty$. The driving semigroup helps us to understand how these vector fields (one for each instant of time) evolve and, together with the skew-product semigroup and the cocycle, we can determine the asymptotics of the solutions.

We will be also interested in another kind of semigroup or ‘autonomous dynamical system’, that has nice properties, called gradient semigroups. We will prove, through the non-autonomous theory, that if the autonomous problem $\dot{x}_1 = f_1(x_1)$ is gradient, then the cascade system will behave like a gradient system too, as long as

$$\dot{x}_2 = f_2(x_1^*, x_2) \quad (1.6)$$

is gradient for each equilibrium x_1^* of $\dot{x}_1 = f_1(x_1)$ and each equilibria of (1.6) is hyperbolic. To be able to do this, we have to understand some important concepts like the topological structural stability of gradient semigroups (Chapter 3) and the permanence of global hyperbolic solutions (Chapter 5).

SEMIGROUPS AND THEIR ATTRACTORS

2.1 Semigroups and Global Attractors

As we said in the introduction, our starting point will be the study of semigroups or autonomous dynamical systems, since they are a particular case of the general notion of dynamical system (Chapter 4). Although semigroups are a simpler case, it is a very important topic and will have a huge role on our final problem, the cascade system. For a more thorough approach on the topics of this chapter, the reader may consult (CARVALHO; LANGA; ROBINSON, 2013), (BORTOLAN; CARVALHO; LANGA, 2020), (CARVALHO; ROBINSON, 2010), (HALE, 1997) and (TEMAM, 1997). Also, throughout this text, we will use a lot of concepts from metric space theory, qualitative theory of ODE, topology, functional analysis and analysis on euclidian space. There are several books containing these topics, as (LIMA, 2017) for metric space theory, (HALE, 1980) for ODE, (MUNKRES, 1975) for topology, (BREZIS, 2010) for functional analysis and (LIMA, 2015) for analysis on euclidian space.

We will denote by X a metric space with metric d and $\mathcal{C}(X)$ by the set of continuous maps from X into itself. Also, $\mathbb{T}$ will denote either $\mathbb{R}$ or $\mathbb{Z}$, $\mathbb{T}^+$ will denote the set $\{t \in \mathbb{T} : t \geq 0\}$ and $\mathbb{T}^-$ will denote $\{t \in \mathbb{T} : t \leq 0\}$.

Definition 1 (Semigroup). A **semigroup** in X is a family $\mathcal{T} = \{T(t) : t \in \mathbb{T}^+\} \subset \mathcal{C}(X)$ with the properties:

1. $T(0)x = x, \forall x \in X$.
2. $T(t+s) = T(t)T(s), \forall t, s \geq 0$.
3. The map $\mathbb{T}^+ \times X \ni (t, x) \mapsto T(t)x \in X$ is continuous.

If $\mathbb{T} = \mathbb{R}$ we say that $\mathcal{T}$ is a continuous semigroup. If $\mathbb{T} = \mathbb{Z}$, we call $\mathcal{T}$ a discrete semigroup.

Remark 1. If $\mathcal{T}$ is a discrete semigroup, item 3 follows immediatly.

Definition 2 (Invariance). Let A be a subset of X and $\mathcal{T} = \{T(t) : t \in \mathbb{T}^+\}$ be a semigroup in X . We say that A is $\mathcal{T}$ -**invariant** (positively $\mathcal{T}$ -invariant) if $T(t)A = A$ for all $t \in \mathbb{T}^+$ (if $T(t)A \subset A$ for all $t \in \mathbb{T}^+$).

If $A = \{a^*\}$ is an invariant unitary set we say that a^* is an equilibrium for $\mathcal{T}$.

Definition 3 (Hausdorff semidistance). The Hausdorff semidistance $d_H(A, B)$ between two nonempty subsets A and B of X is given by:

$$d_H(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y) = \sup_{x \in A} d(x, B),$$

where $d(x, B)$ is the usual distance between two nonempty sets of X .

Remark 2. The following will be very useful: $d_H(A, B) = 0$ if, and only if $\bar{A} \subset \bar{B}$.

Definition 4. Given a nonempty subset A of the metric space X and $r > 0$, the r -neighborhood of A in X is the set

$$\mathcal{O}_r(A) = \{x \in X : d(x, a) < r \text{ for some } a \in A\}.$$

Definition 5 (Attraction and Absorption). Let A and B be nonempty subsets of X and $\mathcal{T}$ be a semigroup in X . We say that:

1. A $\mathcal{T}$ -**attracts** B if $\lim_{t \rightarrow \infty} d_H(T(t)B, A) = 0$.
2. A $\mathcal{T}$ -**absorbs** B if there exists $t_0 \geq 0$ such that $T(t)B \subset A$, for all $t \geq t_0$.

In particular, if A $\mathcal{T}$ -absorbs B , then A $\mathcal{T}$ -attracts B (the converse is not true in general). However, if A $\mathcal{T}$ -attracts B , then given $r > 0$, the set $\mathcal{O}_r(A)$ $\mathcal{T}$ -absorbs B .

Definition 6 (Global Attractor). A subset $\mathcal{A} \subset X$ is called a **global attractor** for the semigroup $\mathcal{T}$ in X if it satisfies:

1. $\mathcal{A}$ is compact.
2. $\mathcal{A}$ is $\mathcal{T}$ -invariant.
3. $\mathcal{A}$ $\mathcal{T}$ -attracts all bounded subsets of X .

Remark 3. The global attractor for a semigroup $\mathcal{T}$ is unique. In fact, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are global attractors for this semigroup, it follows from the invariance of $\mathcal{A}_1$ and Remark 2 that

$$d_H(\mathcal{A}_1, \mathcal{A}_2) = d_H(T(t)\mathcal{A}_1, \mathcal{A}_2) \xrightarrow{t \rightarrow \infty} 0 \implies \mathcal{A}_1 \subset \mathcal{A}_2.$$

Analogously $\mathcal{A}_2 \subset \mathcal{A}_1$, and the equality follows.

We will introduce the following concept in order to characterize the global attractor.

Definition 7 (Global Solution). Let $\mathcal{T}$ be a semigroup in X . We say that $\phi : \mathbb{T} \rightarrow X$ is a **global solution** of $\mathcal{T}$ if $T(t)\phi(s) = \phi(t+s)$, for all $t \in \mathbb{T}^+$ and $s \in \mathbb{T}$. If $x = \phi(0)$, we say that ϕ is a global solution through x . Also, if $\phi(\mathbb{T})$ is bounded, we say that ϕ is a bounded global solution. Lastly, we say that ϕ is a stationary solution for $\mathcal{T}$ if ϕ is constant.

We can now present a characterization, that will be often used, for global attractors.

Theorem 1. Let $\mathcal{T}$ be a semigroup in X with a global attractor $\mathcal{A}$. Then, given $x \in \mathcal{A}$, there exists a bounded global solution $\phi : \mathbb{T} \rightarrow \mathcal{A}$ such that $\phi(0) = x$. Consequently,

$$\mathcal{A} = \{x \in X : \text{there exists a bounded global solution through } x \text{ of } \mathcal{T}\}.$$

Proof. Let $\mathcal{A}$ be the global attractor for $\mathcal{T}$ and $x \in \mathcal{A}$. We will show that there is a global solution $\phi : \mathbb{T} \rightarrow \mathcal{A}$ through x . Since $x \in \mathcal{A} = T(1)\mathcal{A}$, there exists $x_{-1} \in \mathcal{A}$ such that $T(1)x_{-1} = x$ and we can construct a sequence (not necessarily unique) $\{x_{-n}\}_{n \in \mathbb{N}}$ in $\mathcal{A}$ such that $x_0 = x$ and $T(1)x_{-n-1} = x_{-n}$ for all $n \in \mathbb{N}$. Define

$$\phi(t) = \begin{cases} T(t)x, & t \in \mathbb{T}^+ \\ T(j+t)x_{-j}, & t \in [-j, -j+1) \cap \mathbb{T}, j \in \mathbb{N}^*. \end{cases}$$

It follows from the positive $\mathcal{T}$ -invariance of $\mathcal{A}$ that $\phi : \mathbb{T} \rightarrow \mathcal{A}$ is a well defined global solution of $\mathcal{T}$ through x , and since $\phi(\mathbb{T}) \subset \mathcal{A}$, ϕ is bounded.

Conversely, given a bounded global solution $\phi : \mathbb{T} \rightarrow X$ of $\mathcal{T}$, using that $\phi(\mathbb{T})$ is a bounded $\mathcal{T}$ -invariant set and that $\mathcal{A}$ $\mathcal{T}$ -attracts bounded sets, we obtain $\phi(\mathbb{T}) \subset \mathcal{A}$, which concludes the proof. $\square$

Remark 4. If a global solution through x exists, it does not need to be unique, since $T(t)$ is not necessarily injective for all $t \in \mathbb{T}^+$.

Remark 5. If A is any $\mathcal{T}$ -invariant set of X , then (proceeding as in the proof of Theorem 1) there exists a global solution $\phi : \mathbb{T} \rightarrow A$ through x , for any given $x \in A$.

The following definitions play a fundamental role in semigroup theory and can be interpreted geometrically as the set where solutions accumulate asymptotically.

Definition 8 (ω -limit). Let $\mathcal{T}$ be a semigroup in X .

1. If B is a subset of X , we define for each $t \in \mathbb{T}^*$ the partial positive orbit of B by

$$\gamma_t^+(B) = \bigcup_{\substack{s \in \mathbb{T} \\ s \geq t}} T(s)B.$$

When $t = 0$, we call it simply the positive orbit of B , and denote it simply by $\gamma^+(B)$.

2. The ω -**limit** set of $B \subset X$ is defined by

$$\omega(B) = \bigcap_{t \in \mathbb{T}^+} \overline{\gamma_t^+(B)}.$$

Definition 9 (α -limit and orbit). Let $\mathcal{T}$ be a semigroup in X .

1. When a global solution $\phi: \mathbb{T} \rightarrow X$ through x exists, we define the **entire orbit** of x relative to the global solution ϕ by

$$\gamma_\phi(x) = \{\phi(t) : t \in \mathbb{T}\}.$$

In this case, for each $t \in \mathbb{T}$, we define the partial negative orbit of x relative to ϕ by

$$(\gamma_\phi)_t^-(x) = \{\phi(s) : s \in \mathbb{T} \text{ and } s \leq t\}.$$

2. When a global solution $\phi: \mathbb{T} \rightarrow X$ through x exists, we define the α -**limit** set of x relative to ϕ as

$$\alpha_\phi(x) = \bigcap_{t \in \mathbb{T}^-} \overline{(\gamma_\phi)_t^-(x)}.$$

The following characterization of the ω -limit sets and α -limit sets will be frequently used in the proofs of the results that follow.

Proposition 1. Let $\mathcal{T}$ be a semigroup in X . If $B \subset X$, then:

1. $\omega(B)$ is closed and

$$\omega(B) = \{y \in X : \text{there are sequences } \{t_n\}_{n \in \mathbb{N}} \text{ in } \mathbb{T}^+ \text{ and } \{x_n\}_{n \in \mathbb{N}} \text{ in } B \text{ such that } t_n \xrightarrow{n \rightarrow \infty} \infty \text{ and } y = \lim_{n \rightarrow \infty} T(t_n)x_n\}.$$

2. If $\phi: \mathbb{T} \rightarrow X$ is a global solution of $\mathcal{T}$ through $x \in X$, then $\alpha_\phi(x)$ is closed and

$$\alpha_\phi(x) = \{v \in X : \text{there is a sequence } \{t_n\}_{n \in \mathbb{N}} \text{ in } \mathbb{T}^+ \text{ such that } t_n \xrightarrow{n \rightarrow \infty} \infty \text{ and } v = \lim_{n \rightarrow \infty} \phi(-t_n)\}.$$

Proof. We will prove item 1 and the proof of item 2 will follow analogously. The closedness of the set $\omega(B)$ is straightforward, since it is intersection of closed sets.

Let $y \in \omega(B) = \bigcap_{t \in \mathbb{T}^+} \overline{\gamma_t^+(B)}$. Then $y \in \overline{\gamma_n^+(B)}$ for each $n \in \mathbb{N}$, that is, for each n there exists $y_n \in B$ and $t_n \geq n$ such that $d(T(t_n)y_n, y) < \frac{1}{n}$. Obviously $t_n \rightarrow \infty$ and $T(t_n)y_n \rightarrow y$ as $n \rightarrow \infty$, which concludes one inclusion.

Now, suppose that $y \in X$ is such that there exist sequences $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{T}^+$, $\{x_n\}_{n \in \mathbb{N}} \subset B$ with $t_n \xrightarrow{n \rightarrow \infty} \infty$ and $y = \lim_{n \rightarrow \infty} T(t_n)x_n$. Hence, if $\tau \geq 0$ we have $\{T(t_n)x_n\}_{t_n \geq \tau} \subset \gamma_\tau^+(B)$, and $y \in \overline{\gamma_\tau^+(B)}$. Consequently $y \in \omega(B)$ and the characterization of $\omega(B)$ is proved. □

Proposition 2. Let K be a compact subset of X and $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X such that

$$\lim_{n \rightarrow \infty} d(x_n, K) = 0.$$

Then $\{x_n\}_{n \in \mathbb{N}}$ has a convergent subsequence to a point in K .

Proof. Given $m \in \mathbb{N}$, there are $n_m \in \mathbb{N}$ and $y_{n_m} \in K$ such that $d(x_{n_m}, y_{n_m}) < \frac{1}{m}$. Since K is compact we may assume, passing to a subsequence if necessary, that $y_{n_m} \xrightarrow{m \rightarrow \infty} y_0$, for some $y_0 \in K$. Hence, we obtain

$$d(x_{n_m}, y_0) \leq d(x_{n_m}, y_{n_m}) + d(y_{n_m}, y_0) \xrightarrow{m \rightarrow \infty} 0,$$

that is, $\{x_n\}_{n \in \mathbb{N}}$ has a convergent subsequence to a point in K . $\square$

Lemma 1. Let $\mathcal{T}$ be a semigroup in X and $B \subset X$. Then $\omega(B)$ is positively $\mathcal{T}$ -invariant. If $\omega(B)$ is compact and $\mathcal{T}$ -attracts B , then $\omega(B)$ is $\mathcal{T}$ -invariant and if $\mathbb{T} = \mathbb{R}$, then $\omega(B)$ is connected provided B is connected.

Proof. If $\omega(B) = \emptyset$ then there is nothing to prove. Assume then that $\omega(B) \neq \emptyset$ and fix $t \in \mathbb{T}^+$. From Proposition 1, given $y \in \omega(B)$ there exist sequences $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$ with $t_n \xrightarrow{n \rightarrow \infty} \infty$ and $\{x_n\}_{n \in \mathbb{N}}$ in B such that $y = \lim_{n \rightarrow \infty} T(t_n)x_n$. It follows from the continuity of $T(t)$ that $T(t)y = \lim_{n \rightarrow \infty} T(t+t_n)x_n$ and hence $T(t)y \in \omega(B)$, using once more the characterization given by Proposition 1. Hence $T(t)\omega(B) \subset \omega(B)$, for each $t \in \mathbb{T}^+$, and $\omega(B)$ is positively $\mathcal{T}$ -invariant.

It remains to prove that if $\omega(B)$ is compact and $\mathcal{T}$ -attracts B , then $\omega(B)$ is $\mathcal{T}$ -invariant. For $y \in \omega(B)$ there are sequences $t_n \rightarrow \infty$ and $x_n \in B$ such that $y = \lim_{n \rightarrow \infty} T(t_n)x_n$. For $t \in \mathbb{T}^+$ fixed, we can assume that $t_n \geq t$, for all $n \in \mathbb{N}$. Therefore, $T(t)T(t_n - t)x_n = T(t_n)x_n \rightarrow y$ as $n \rightarrow \infty$. Since $\omega(B)$ attracts B we have $d(T(t_n - t)x_n, \omega(B)) \xrightarrow{n \rightarrow \infty} 0$. It follows from Proposition 2 that $\{T(t_n - t)x_n\}_{n \in \mathbb{N}}$ has a subsequence (which we denote the same) that converges to some $x \in \omega(B)$. Hence, by continuity of $T(t)$, we have $T(t)x = y$. Therefore $\omega(B) \subset T(t)\omega(B)$, for all $t \in \mathbb{T}^+$, and $\omega(B)$ is $\mathcal{T}$ -invariant.

Finally, if $\mathbb{T} = \mathbb{R}$ then $\gamma^+(B)$ is connected for $t \geq 0$, since $[0, \infty) \times X \ni (s, x) \mapsto T(s)x \in X$ is continuous and takes the connected set $[t, \infty) \times B$ into $\gamma^+(B)$. Suppose that $\omega(B)$ is not connected, that is, there exists two nonempty compact sets $\omega_1(B)$, $\omega_2(B)$ and $\rho > 0$ such that $d(\omega_1(B), \omega_2(B)) = 2\rho$ and $\omega(B) = \omega_1(B) \cup \omega_2(B)$. Fix $t_0 \in \mathbb{R}^+$ such that $T(t)B \subset \mathcal{O}_\rho(\omega(B))$, for all $t \geq t_0$ and assume (without loss of generality) that $T(t_0)B \subset \mathcal{O}_\rho(\omega_1(B))$. Note that there exists $t_1 > t_0$ such that $T(t_1)B \subset \mathcal{O}_\rho(\omega_2(B))$, otherwise $\omega_2(B) = \emptyset$. This implies that $\gamma^+(B) \cap \mathcal{O}_\rho(\omega_1(B))$ and $\gamma^+(B) \cap \mathcal{O}_\rho(\omega_2(B))$ are two open (in $\gamma^+(B)$) nonempty sets which are disjoint and $\gamma^+(B) = (\gamma^+(B) \cap \mathcal{O}_\rho(\omega_1(B))) \cup (\gamma^+(B) \cap \mathcal{O}_\rho(\omega_2(B)))$. Thus $\gamma^+(B)$ is not connected, leading to a contradiction and concluding that $\omega(B)$ is connected. $\square$

Remark 6. From Lemma 1 we verify that if $x \in X$, $\omega(x)$ attracts x and if $\omega(x) = \{x^*\}$ then x^* is an equilibrium for $\mathcal{T}$ (see Definition 2).

Lemma 2. Let $\mathcal{T}$ be a semigroup in X , $x \in X$ and assume that there is a global solution $\phi: \mathbb{T} \rightarrow X$ through x of $\mathcal{T}$ such that $\overline{\phi(\mathbb{T}^-)}$ is compact. Then $\alpha_\phi(x)$ is nonempty, compact and $\mathcal{T}$ -invariant. Moreover, if $\mathbb{T} = \mathbb{R}$ then $\alpha_\phi(x)$ is connected.

Proof. From the definition of $\alpha_\phi(x)$ and the compactness of $\overline{\phi(\mathbb{T}^-)}$ it follows that $\alpha_\phi(x)$ is nonempty (using Proposition 1) and compact (because $\alpha_\phi(x)$ is closed). Now, we will show that $\alpha_\phi(x)$ is $\mathcal{T}$ -invariant.

Fix $t \in \mathbb{T}^+$. From Proposition 1, given $y \in \alpha_\phi(x)$, there exists a sequence $t_n \xrightarrow{n \rightarrow \infty} +\infty$ such that $\phi(-t_n) \xrightarrow{n \rightarrow \infty} y$. From the continuity of $T(t)$ we obtain $T(t)\phi(-t_n) = \phi(t - t_n) \xrightarrow{n \rightarrow \infty} T(t)y$ and therefore $T(t)y \in \alpha_\phi(x)$, concluding that $\alpha_\phi(x)$ is positively invariant. To show the other inclusion, let $w \in \alpha_\phi(x)$. Then there exists a sequence $t_n \xrightarrow{n \rightarrow \infty} \infty$ such that $\phi(-t_n) \xrightarrow{n \rightarrow \infty} w$. Since $\{\phi(-t_n - t)\}_{n \in \mathbb{N}}$ is relatively compact, passing to a subsequence if necessary, there exists $z \in X$ such that $\phi(-t_n - t) \xrightarrow{n \rightarrow \infty} z \in \alpha_\phi(x)$. It follows from the continuity of $T(t)$ that $T(t)z = w$ and consequently $\alpha_\phi(x)$ is $\mathcal{T}$ -invariant. This shows that $\alpha_\phi(x)$ is $\mathcal{T}$ -invariant.

Finally, if $\mathbb{T} = \mathbb{R}$, the connectedness follows as in Lemma 1. $\square$

The following definitions has an important role in the characterization of the semigroups that possess global attractors.

Definition 10. Let $\mathcal{T}$ be a semigroup in X . We say that:

1. $\mathcal{T}$ is **bounded dissipative** if there exists a bounded set $B_0 \subset X$ that $\mathcal{T}$ -attracts bounded subsets of X .
2. $\mathcal{T}$ is **eventually bounded** if for each bounded subset B of X exists $t_B > 0$ such that $\gamma_{t_B}^+(B)$ is bounded.

Definition 11. A semigroup $\mathcal{T}$ in X is said to be **asymptotically compact** if, for each nonempty, closed, positively $\mathcal{T}$ -invariant and bounded set $B \subset X$, there exists a compact subset $K \subset B$ that $\mathcal{T}$ -attracts B .

Lemma 3. Let $\mathcal{T}$ be an eventually bounded and asymptotically compact semigroup in X . If $B \subset X$ is a bounded nonempty subset of X , then $\omega(B)$ is nonempty, compact, $\mathcal{T}$ -invariant and $\mathcal{T}$ -attracts B .

Proof. Let $B \subset X$ be a bounded nonempty subset of X . Since $\mathcal{T}$ is eventually bounded, there exists $t_0 \in \mathbb{T}^+$ such that $\gamma_{t_0}^+(B)$ is bounded. From the continuity of $T(t)$ and $T(t)\gamma_{t_0}^+(B) \subset \gamma_{t_0}^+(B)$, we have $T(t)\overline{\gamma_{t_0}^+(B)} \subset \overline{\gamma_{t_0}^+(B)}$, for all $t \in \mathbb{T}^+$. Thus $\overline{\gamma_{t_0}^+(B)}$ is a nonempty closed bounded and positively $\mathcal{T}$ -invariant subset of X , implying (because $\mathcal{T}$ is asymptotically compact) that exists a compact set $J \subset \overline{\gamma_{t_0}^+(B)}$ that $\mathcal{T}$ -attracts $\overline{\gamma_{t_0}^+(B)}$. Thus J is a compact that attracts B and consequently $\omega(B) \neq \emptyset$ and $\omega(B) \subset J$ is a compact set that attracts B (argue by contradiction). Thus $\omega(B)$ is invariant, from Lemma 1, and we conclude the proof.

□

Lemma 4. Let $\mathcal{T}$ be a bounded dissipative and asymptotically compact semigroup in X which contains a bounded subset $B_0 \subset X$ that $\mathcal{T}$ -absorbs all bounded subsets of X . Then, for each nonempty bounded subset B of X we have $\omega(B) \subset \omega(B_0) \subset \overline{B_0}$.

Proof. Let B be a bounded subset of X . We already know, from Lemma 3, that $\omega(B)$ is nonempty, compact, $\mathcal{T}$ -invariant and $\mathcal{T}$ -attracts B . Since B_0 absorbs B , exists $t_0 \geq 0$ such that $\gamma_{t_0}^+(B) \subset B_0$. Given $y = \lim_{n \rightarrow \infty} T(t_n)x_n \in \omega(B)$, with $\{x_n\}_{n \in \mathbb{N}}$ in B , $\{t_n\}_{n \in \mathbb{N}}$ in $\mathbb{T}^+$, $t_n \xrightarrow{n \rightarrow \infty} \infty$ and assuming, without loss of generality, that $t_n \geq t_0$ for all $n \in \mathbb{N}$, we have

$$T(t_n)x_n = T(t_n - t_0)T(t_0)x_n \subset T(t_n - t_0)T(t_0)B \subset T(t_n - t_0)B_0,$$

showing that $y \in \omega(B_0)$. The inclusion $\omega(B_0) \subset \overline{B_0}$ follows from the fact that B_0 $\mathcal{T}$ -absorbs itself. □

The next result characterizes the asymptotical compactness.

Proposition 3. Let $\mathcal{T}$ be a bounded dissipative semigroup in X . Then, $\mathcal{T}$ is asymptotically compact if, and only if for each bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ in X and $\{t_n\}_{n \in \mathbb{N}}$, with $t_n \xrightarrow{n \rightarrow \infty} \infty$, the set $\{T(t_n)x_n\}_{n \in \mathbb{N}}$ is relatively compact.

Proof. Suppose that $\mathcal{T}$ is bounded dissipative. If $\mathcal{T}$ is an asymptotically compact semigroup, $\{x_n\}_{n \in \mathbb{N}}$ is a bounded sequence in X and $\{t_n\}_{n \in \mathbb{N}}$ is a sequence in $\mathbb{T}^+$ such that $t_n \xrightarrow{n \rightarrow \infty} \infty$, then there exists $t_0 > 0$ such that $B = \overline{\gamma_{t_0}^+(\{x_n\}_{n \in \mathbb{N}})}$ is a bounded set. Note that B is nonempty, positively $\mathcal{T}$ -invariant and closed, thus exists a compact subset $K \subset B$ which $\mathcal{T}$ -attracts B . In particular, $\{T(t_n)x_n\}_{n \in \mathbb{N}}$ converges to K as $n \rightarrow \infty$ (because $T(t_n)x_n \in \gamma_{t_0}^+(\{x_n\}_{n \in \mathbb{N}})$ for large t) and by Proposition 2 $\{T(t_n)x_n\}_{n \in \mathbb{N}}$ has a subsequence that converges, that is, $\{T(t_n)x_n\}_{n \in \mathbb{N}}$ is relatively compact.

Now, suppose that for each bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ in X and $\{t_n\}_{n \in \mathbb{N}}$, with $t_n \xrightarrow{n \rightarrow \infty} \infty$, the set $\{T(t_n)x_n\}_{n \in \mathbb{N}}$ is relatively compact. We will show that $\mathcal{T}$ is asymptotically compact. Let $B \subset X$ be nonempty, closed, bounded and positively $\mathcal{T}$ -invariant. Using the hypothesis we can prove that $\omega(B) \subset B$ is nonempty, compact, $\mathcal{T}$ -invariant and $\mathcal{T}$ -attracts B , simply rewriting the proof of Lemma 3. Therefore $\omega(B) \subset B$ is a compact subset that $\mathcal{T}$ -attracts B and $\mathcal{T}$ is asymptotically compact. □

The following theorem characterizes the semigroups that have a global attractor and will be very useful in the future.

Theorem 2. A semigroup $\mathcal{T}$ in X has a global attractor $\mathcal{A}$ if, and only if $\mathcal{T}$ is bounded dissipative and asymptotically compact. In that case, if B_0 is a bounded subset of X that $\mathcal{T}$ -absorbs all bounded subsets of X , then $\mathcal{A} = \omega(B_0)$.

Proof. Assume that $\mathcal{T}$ has a global attractor $\mathcal{A}$. Then $\mathcal{T}$ is bounded dissipative, since $\mathcal{A}$ is a bounded subset of X that $\mathcal{T}$ -attracts all bounded subsets of X . Using the fact that $\mathcal{A}$ is compact with Proposition 2 and Proposition 3, it follows that $\mathcal{T}$ is asymptotically compact.

Now, suppose that $\mathcal{T}$ is bounded dissipative and asymptotically compact and let B_0 be a bounded subset of X that $\mathcal{T}$ -absorbs all bounded subsets of X . Define $\mathcal{A} = \omega(B_0)$, which is nonempty, compact, $\mathcal{T}$ -invariant and $\mathcal{T}$ -attracts B_0 , by Lemma 3.

Given a nonempty bounded subset B of X , Lemmas 3 and 4 show that $\omega(B)$ is nonempty, compact, $\mathcal{T}$ -invariant, $\mathcal{T}$ -attracts B and $\omega(B) \subset \mathcal{A}$. It follows that $\mathcal{A}$ $\mathcal{T}$ -attracts B , and the proof that $\mathcal{A}$ is the global attractor of $\mathcal{T}$ is complete. $\square$

2.2 Gradient semigroups

Now we know the basics concepts from semigroups, we can study a particular case of them, named gradient semigroups. Such type of semigroups are very useful because we know the structure of its attractor and appear in a lot of applications and examples, as we will see throughout this subsection. Also, gradient semigroups will be essential to understand our problem related to the structure of cascade systems.

Definition 12. Let $\mathcal{T}$ be a semigroup in X and Ξ be a $\mathcal{T}$ -invariant set. Given an open neighborhood U of Ξ , we say that Ξ is the **maximal $\mathcal{T}$ -invariant set in U** if for all $B \subset U$ $\mathcal{T}$ -invariant we have $B \subset \Xi$. Also, if Ξ is the maximal $\mathcal{T}$ -invariant set in $\mathcal{O}_\delta(\Xi)$ for some $\delta > 0$, we say that Ξ is an **isolated invariant set**.

Definition 13. Let $\mathcal{T}$ be a semigroup in X . We say that $\Xi = \{\Xi_1, \dots, \Xi_n\}$ is a **disjoint collection of isolated $\mathcal{T}$ -invariant sets** if each $\Xi_i \subset X$ is an isolated $\mathcal{T}$ -invariant set and exists $\delta > 0$ such that $\mathcal{O}_\delta(\Xi_i) \cap \mathcal{O}_\delta(\Xi_j) = \emptyset$, for all $i, j \in \{1, \dots, n\}$ with $i \neq j$.

Definition 14 (Gradient Semigroup). Let $\mathcal{T}$ be a semigroup in X with global attractor and $\Xi = \{\Xi_1, \dots, \Xi_n\}$ be a collection of isolated $\mathcal{T}$ -invariant sets. We say that the semigroup $\mathcal{T}$ is **gradient** relatively to Ξ if there is a continuous function $V : X \rightarrow \mathbb{R}$, called Lyapunov function for $\mathcal{T}$ relatively to Ξ , such that:

1. The map $\mathbb{T}^+ \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is nonincreasing, for each $x \in X$.
2. If $V(T(t)x) = V(x), \forall t \in \mathbb{T}^+$, then $x \in \bigcup_{i=1}^n \Xi_i$.
3. V is constant in $\Xi_i, \forall i = 1, \dots, n$.

Example 1. Let $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, $\delta > 0$ and $C > 0$ be such that $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has finite zeros $x_1^*, \dots, x_n^*$ and

$$\langle \nabla f(x), x \rangle \leq -\delta, \forall \|x\| \geq C. \quad (2.1)$$

Consider the Cauchy problem

$$\begin{cases} \dot{x} = -\nabla f(x), & t > 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \quad (2.2)$$

with solution $x(\cdot, x_0)$. Defining $T(t)x_0 = x(t, x_0)$, for all $t \in \mathbb{R}^+$ and for all $x_0 \in \mathbb{R}^n$, the collection $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ is a gradient semigroup relative to $\mathcal{E} = \{x_1^*, \dots, x_n^*\}$.

To prove that $\mathcal{T}$ is a semigroup with global attractor, see example 2. Now we will show that f is a Lyapunov function for $\mathcal{T}$. Note that f is continuous and satisfies condition 3. To show that f is nonincreasing along solutions, fix $x_0 \in \mathbb{R}^n$ and note that the function $[0, \infty) \ni t \mapsto f(T(t)x_0) = f(x(t, x_0))$ is $\mathcal{C}^1$. Moreover, we have

$$\frac{d}{dt}(f \circ x(\cdot, x_0))(t) = \langle \nabla f(x(t, x_0)), \dot{x}(t, x_0) \rangle = - \|\nabla f(x(t, x_0))\|^2 \leq 0, \quad (2.3)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$. Therefore, it follows from (2.3) that f is nonincreasing along solutions.

It remains to show condition 2. If $x_0 \in \mathbb{R}^n$ is such that $f(T(t)x_0) = f(x_0)$, for all $t \in \mathbb{R}^+$, it follows from (2.3) that

$$0 = \frac{d}{dt}f(x_0) = \frac{d}{dt}(f \circ x(\cdot, x_0))(t) = - \|\nabla f(x(t, x_0))\|^2 = - \|\dot{x}(t, x_0)\|^2.$$

Thus $\dot{x}(t, x_0) = 0$, for all $t \in \mathbb{R}^+$, and $x(t, x_0) = x_0$, for all $t \in \mathbb{R}^+$. Thus $x_0 \in \mathcal{E}$ and $\mathcal{T}$ is a gradient semigroup relative to $\mathcal{E}$.

Remark 7. In Example 1 we have a Cauchy problem

$$\begin{cases} \dot{x} = \phi(x), & t > 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \quad (2.4)$$

such that $\phi \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and its Jacobian is a $n \times n$ matrix given by $J(\phi) = (\partial_i \partial_j \phi)$, with $i, j = 1, \dots, n$. This means that the Jacobian of ϕ is symmetrical, by Schwarz theorem. In fact, every time that a function $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ has symmetrical jacobian there will be a function $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ such that $g = \nabla f$, implying that the Cauchy problem

$$\begin{cases} \dot{x} = g(x), & t > 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \quad (2.5)$$

generates a gradient semigroup, as in Example 1. In particular, if $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, the semigroup associated to (2.5) is gradient. In chapter 6 we will see that there exist gradient semigroups that are not coming from problems like (2.5).

Our next goal is to characterize the attractor for a gradient semigroup. The following Lemmas will help us to find such conditions.

Lemma 5. Let $\mathcal{T}$ be a gradient semigroup relative to the collection of isolated $\mathcal{T}$ -invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. Then, for each $x \in X$ we have $\omega(x) \subset \bigcup_{i=1}^n \Xi_i$. Similarly, if there is a global solution $\phi : \mathbb{T} \rightarrow X$ of $\mathcal{T}$ through x , then $\alpha_\phi(x) \subset \bigcup_{i=1}^n \Xi_i$. Moreover, if $\mathbb{T} = \mathbb{R}$ then $\omega(x) \subset \Xi_j$, for some $j = 1, \dots, n$ and if exists a global solution $\phi : \mathbb{T} \rightarrow X$ through x then $\alpha_\phi(x) \subset \Xi_j$, for some $j = 1, \dots, n$.

Proof. We will proof the statement for $\omega(x)$ and the proof for $\alpha_\phi(x)$ will follow by the same argument. Let $x \in X$ and $y \in \omega(x)$. Then, by Proposition 1, exists a sequence $\{t_n\}_{n \in \mathbb{N}}$, with $t_n \xrightarrow{n \rightarrow \infty} \infty$, such that $y = \lim_{n \rightarrow \infty} T(t_n)x$. Since the Lyapunov function is continuous and nonincreasing through the positive orbit of x , we have $V(y) = \lim_{n \rightarrow \infty} V(T(t_n)x) = \lim_{t \rightarrow \infty} V(T(t)x)$. It follows from the continuity of $\mathcal{T}$ and V that $V(y) = \lim_{n \rightarrow \infty} V(T(t + t_n)x) = \lim_{n \rightarrow \infty} V(T(t)T(t_n)x) = V(T(t)y)$, for each $t \in \mathbb{T}^+$. Thus, from property 2, we conclude that $y \in \bigcup_{i=1}^n \Xi_i$ and $\omega(x) \subset \bigcup_{i=1}^n \Xi_i$.

If $\mathbb{T} = \mathbb{R}$ then $\omega(x)$ is connected, from Lemma 1, and from the fact that $\Xi_1, \dots, \Xi_n$ are disjoint we have $\omega(x) \subset \Xi_j$, for some $j = 1, \dots, n$. $\square$

Lemma 6. Let $\mathcal{T}$ be a semigroup in X that is eventually bounded, asymptotically compact and gradient relative to a bounded $\mathcal{T}$ -invariant set Ξ . Then there exists a bounded positively $\mathcal{T}$ -invariant set U which $\mathcal{T}$ -absorbs every compact subsets of X .

Proof. We already know from Lemmas 3 and 5 that for each $x \in X$, $\omega(x)$ is nonempty, compact, $\mathcal{T}$ -invariant, $\mathcal{T}$ -attracts x and $\omega(x) \subset \Xi$. Hence, given $\varepsilon > 0$, the set $\mathcal{O}_\varepsilon(\Xi)$ $\mathcal{T}$ -absorbs points from x (because $\omega(x) \subset \Xi$ attracts x , for each $x \in X$). Thus, the set $B = \{x \in \mathcal{O}_\varepsilon(\Xi) : \gamma^+(x) \subset \mathcal{O}_\varepsilon(\Xi)\}$ is nonempty and from Lemma 3 $\omega(B)$ is nonempty, compact, $\mathcal{T}$ -invariant and $\mathcal{T}$ -attracts B .

Let V be a neighborhood of $\omega(B)$ and $t_0 \in \mathbb{T}^+$ be such that $\gamma_{t_0}^+(V)$ is bounded (it is easy to prove, by contradiction, that such V exists). Then, for each $x \in X$ exists a neighborhood $\mathcal{O}_x$ of x and $t_x \in \mathbb{T}^+$ such that $T(t)(\mathcal{O}_x) \subset \gamma_{t_0}^+(V)$ for all $t \geq t_x$. This means that $\gamma_{t_0}^+(V)$ $\mathcal{T}$ -absorbs a neighborhood of x , for each $x \in X$ and consequently $\gamma_{t_0}^+(V)$ $\mathcal{T}$ -absorbs compact subsets of X . Defining $U = \gamma_{t_0}^+(V)$ we conclude the proof. $\square$

Lemma 7. Let $\mathcal{T}$ be a semigroup in X . If $\mathcal{T}$ is a gradient semigroup relative to the bounded collection of isolated $\mathcal{T}$ -invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$, eventually bounded and asymptotically compact, then $\mathcal{T}$ is bounded dissipative.

Proof. We know, by Lemma 6, that exists a bounded set U that $\mathcal{T}$ -absorbs all compact subsets of X . Hence, by Lemma 3, $\omega(U)$ is nonempty and attracts U , that is, $\omega(U)$ $\mathcal{T}$ -attracts compact subsets of X .

Let B be a bounded subset of X . Using again Lemma 3 we conclude that $\omega(B)$ is nonempty, compact, $\mathcal{T}$ -invariant and $\mathcal{T}$ -attracts B . Thus $\omega(B) \subset \omega(U)$, since $\omega(B)$ is bounded and $\mathcal{T}$ -invariant. Therefore, $\omega(U)$ $\mathcal{T}$ -attracts B and the proof is complete. $\square$

Definition 15 (Unstable Manifold). Let $\mathcal{T}$ be a semigroup in X . Given $\Xi \subset X$, we define the **unstable manifold** of Ξ , denoted by $W^u(\Xi)$, as the set

$$W^u(\Xi) = \{y \in X : \text{exists a global solution } \phi : \mathbb{T} \rightarrow X \text{ through } y \text{ with } \lim_{t \rightarrow -\infty} d(\phi(t), \Xi) = 0\}.$$

Theorem 3. If $\mathcal{T}$ is a semigroup in X that is gradient relative to a bounded collection of isolated $\mathcal{T}$ -invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$, eventually bounded and asymptotically compact then $\mathcal{T}$ has a global attractor $\mathcal{A}$. Also, $\bigcup_{i=1}^n \Xi_i \subset \mathcal{A}$ and if $\mathbb{T} = \mathbb{R}$ we can characterize the attractor as $\mathcal{A} = \bigcup_{i=1}^n W^u(\Xi_i)$.

Proof. The existence of the attractor is an immediate consequence from Lemma 7 and Theorem 2. To see the inclusion $\Xi \subset \mathcal{A}$ just notice that Ξ is $\mathcal{T}$ -invariant and bounded.

Now we will show that $\mathcal{A} = \bigcup_{i=1}^n W^u(\Xi_i)$. If $x \in \mathcal{A}$, it follows from Theorem 1 and Lemma 2 that $\alpha_\phi(x)$ is a nonempty set and from Lemma 5 we have $\alpha_\phi(x) \subset \Xi_i$, for some $i = 1, \dots, n$. Thus $x \in \bigcup_{i=1}^n W^u(\Xi_i)$ and $\mathcal{A} \subset W^u(\Xi)$. On the other hand, given $x \in \bigcup_{i=1}^n W^u(\Xi_i)$, there exists a global solution $\phi : \mathbb{R} \rightarrow X$ such that $\lim_{t \rightarrow -\infty} d_H(\phi(t), \Xi_i) = 0$, for some $i = 1, \dots, n$. Also, from Lemmas 3 and 5, we have $\lim_{t \rightarrow \infty} d_H(\phi(t), \Xi_j) = 0$, for some $j = 1, \dots, n$. Since $\bigcup_{i=1}^n \Xi_i$ is bounded, it follows that $\phi(\mathbb{R})$ is bounded and $\mathcal{T}$ -invariant, that is, $\phi(\mathbb{R}) \subset \mathcal{A}$. In particular, $x \in \mathcal{A}$ and $\bigcup_{i=1}^n W^u(\Xi_i) \subset \mathcal{A}$. $\square$

Lemma 8. Let $\mathcal{T}$ be a semigroup in X and Ξ be a compact $\mathcal{T}$ -invariant subset of X . Given $\varepsilon > 0$ and $t \in \mathbb{T}^+$, there exists $\delta > 0$ such that

$$T(s)\mathcal{O}_\delta(\Xi) \subset \mathcal{O}_\varepsilon(\Xi), \forall s \in [0, t].$$

Proof. We will prove by contradiction. Suppose that exists $t_0 \in \mathbb{T}^+$ and $\varepsilon_0 > 0$ such that, for each $k \in \mathbb{N}^*$, exists $x_k \in \mathcal{O}_{\frac{1}{k}}(\Xi)$ and $s_k \in [0, t_0]$ satisfying $d_H(T(s_k)x_k, \Xi) \geq \varepsilon_0$. Without loss of generality, we can assume that $s_k \rightarrow s_0$, for some $s_0 \in [0, t_0]$ and $x_k \rightarrow x_0$, for some $x_0 \in \Xi$ (using Proposition 2). It follows, from the continuity of the map $\mathbb{T}^+ \times X \ni (t, x) \mapsto T(t)x$, that $0 = d(T(s_0)x_0, \Xi) \geq \varepsilon_0$, which is a contradiction. $\square$

2.3 Dynamically Gradient Semigroups

In this section our main goal is to study some properties of dynamically gradient semigroups so that later we can establish a relation between gradient and dynamically gradient

semigroups.

Let $\mathcal{T}$ be a semigroup with a global attractor $\mathcal{A}$ which contains a collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. From now on we will denote by $\varepsilon_0(\Xi) = \frac{1}{2} \min_{1 \leq i < j \leq n} d(\Xi_i, \Xi_j)$. If it is clear which is our collection of isolated invariant sets, we will simply write ε_0 . Also, note that $\varepsilon_0 > 0$.

Definition 16 (Chain Recurrent). Let $\mathcal{T}$ be a semigroup in X with a global attractor $\mathcal{A}$ which contains a collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. Let $\Xi^* \in \Xi$ and $0 < \varepsilon < \varepsilon_0$. We say that Ξ^* has an ε -chain if the following conditions hold:

1. A disjoint subcollection $\{\Xi_{l_1}, \Xi_{l_2}, \dots, \Xi_{l_k}\}$ of Ξ , with $\Xi^* = \Xi_{l_1}$
2. A finite collection of points $\{x_1, \dots, x_k\}$ in X with $d(x_i, \Xi_{l_i}) < \varepsilon$, for all $1 \leq i \leq k$.
3. Finites collections of points $\{\tau_1, \dots, \tau_k\}$ and $\{t_1, \dots, t_k\}$ in $\mathbb{T}^+$ with $\tau_i < t_i$, for all $i \leq 1 \leq k$, such that, defining $\Xi_{l_{k+1}} := \Xi^*$, satisfies

$$d(T(t_i)x_i, \Xi_{l_{i+1}}) < \varepsilon \text{ and } d(T(\tau_i)x_i, \bigcup_{j=1}^k \mathcal{O}_{\varepsilon_0}(\Xi_{l_j})) > 0, \text{ for } 1 \leq i \leq k.$$

We say that an invariant set $\Xi^* \in \Xi$ is **chain recurrent** if exists $\delta \in (0, \varepsilon_0)$ such that for each $\varepsilon \in (0, \delta)$ exists an ε -chain at Ξ^* .

We are now able to define dynamically gradient semigroups.

Definition 17 (Dynamically Gradient Semigroup). Let $\mathcal{T}$ be a semigroup with a global attractor $\mathcal{A}$ containing a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. We say that $\mathcal{T}$ is a **dynamically gradient semigroup** relative to Ξ if satisfies:

(G1) If $\xi : \mathbb{T} \rightarrow X$ is a bounded global solution of $\mathcal{T}$, that is, $\xi(\mathbb{T}) \subset \mathcal{A}$, then there exists $1 \leq i, j \leq n$ such that

$$\lim_{t \rightarrow -\infty} d(\xi(t), \Xi_i) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} d(\xi(t), \Xi_j) = 0.$$

(G2) No set of Ξ is chain recurrent.

Note that from (G1) we can characterize the global attractor as $\mathcal{A} = \bigcup_{j=1}^n W^u(\Xi_j)$. In fact, if $x \in \mathcal{A}$, then exists a global solution $\phi : \mathbb{T} \rightarrow X$ through x and from (G1) exists $i \in \{1, \dots, n\}$ such that $\lim_{t \rightarrow -\infty} d(\xi(t), \Xi_i) = 0$. Thus $x \in W^u(\Xi_i) \subset \bigcup_{j=1}^n W^u(\Xi_j)$ and $\mathcal{A} \subset \bigcup_{j=1}^n W^u(\Xi_j)$. On the other hand, if $x \in \bigcup_{j=1}^n W^u(\Xi_j)$, exists a global solution $\phi : \mathbb{T} \rightarrow X$ with $\lim_{t \rightarrow -\infty} d(\xi(t), \Xi_i) = 0$, for some $i \in \{1, \dots, n\}$. Again from (G1), exists $j \in \{1, \dots, n\}$ such that $\lim_{t \rightarrow \infty} d(\xi(t), \Xi_j) = 0$.

Since $\Xi_j \subset \mathcal{A}$ is bounded, it follows that $\phi(\mathbb{T})$ is bounded and $\mathcal{T}$ -invariant and consequently $x \in \phi(\mathbb{T}) \subset \mathcal{A}$, concluding our statement.

This means that gradient and dynamically gradient semigroups has the same characterization for global attractors (see Theorem 3). Other similarities will be revealed as we continue to explore the dynamically gradient semigroup in this section.

The following definitions and results will introduce concepts relative to a family of semigroups, that will be essential to study the structural stability of dynamically gradient semigroups (Chapter 3), that is, if dynamically gradient semigroups are stable under perturbations.

Definition 18. Let $\mathcal{T}_\eta$ be a semigroup in X , for each $\eta \in [0, 1]$. We say that the family of semigroups $\{\mathcal{T}_\eta\}_{\eta \in [0, 1]}$ is:

1. **continuous** at $\eta_0 \in [0, 1]$ if the limit

$$\lim_{\eta \rightarrow \eta_0} d(T_\eta(t)x, T_{\eta_0}(t)x) = 0$$

converges uniformly for (t, x) in compact subsets of $\mathbb{T}^+ \times X$.

2. **collectively asymptotically compact** if given sequences $\{t_n\}_{n \in \mathbb{N}}$ in $\mathbb{T}^+$, with $t_n \rightarrow \infty$, $\{\eta_n\}_{n \in \mathbb{N}}$ in $[0, 1]$ and $\{x_n\}_{n \in \mathbb{N}}$ bounded in X , then $\{T_{\eta_n}(t_n)x_n\}_{n \in \mathbb{N}}$ has a convergent subsequence in X . In particular, if $\eta_n \rightarrow 0$, we say that the family is collectively asymptotically compact at 0.

Lemma 9. Let $\{\mathcal{T}_\eta\}_{\eta \in [0, 1]}$ be a continuous and collectively asymptotically compact at $\eta = 0$ family of semigroups. Let $\{\eta_n\}_{n \in \mathbb{N}}$ in $[0, 1]$ be a sequence satisfying $\eta_n \rightarrow 0$ and consider the sequences $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ and $\{c_n\}_{n \in \mathbb{N}}$ in $\mathbb{T}$ such that $b_n - a_n \rightarrow \infty$, $c_n - b_n \rightarrow \infty$ and set $\mathbb{J}_n := [a_n, c_n] \cap \mathbb{T}$. Suppose that $\xi_n : \mathbb{J}_n \rightarrow X$ is a solution of $\mathcal{T}_{\eta_n}$, for each $n \in \mathbb{N}$, and $\overline{\bigcup_{n \in \mathbb{N}} \xi_n(\mathbb{J}_n)}$ is bounded. Then there exist a subsequence $\{\xi_{n_k}\}_{k \in \mathbb{N}}$ of $\{\xi_n\}_{n \in \mathbb{N}}$ and a bounded global solution $\xi_0 : \mathbb{T} \rightarrow X$ of $\mathcal{T}_0$ such that $\xi_{n_k}(t + b_{n_k}) \rightarrow \xi_0(t)$ uniformly for t in compact subsets of $\mathbb{T}$.

Proof. From the collective asymptotically compactness at $\eta = 0$ there exist a subsequence $\{n_{0,k}\}$ of $\mathbb{N}$ and $x_0 \in X$ such that

$$\xi_{n_{0,k}}(b_n) = T_{\eta_{n_{0,k}}}(b_n - a_n)\xi_{n_{0,k}}(a_n) \rightarrow x_0 \quad \text{as } k \rightarrow \infty.$$

Define $\xi_0 : [0, \infty) \cap \mathbb{T} \rightarrow X$ by $\xi_0(t) = T_0(t)x_0$ for $t \geq 0$. Then

$$\xi_0(t) = T_0(t)x_0 = \lim_{k \rightarrow \infty} T_{\eta_{n_{0,k}}}(t)\xi_{n_{0,k}}(b_n) = \lim_{k \rightarrow \infty} \xi_{n_{0,k}}(t + b_{n_k}),$$

with the convergence being uniform for t in compact subsets of $[0, \infty) \cap \mathbb{T}$. In particular we obtain

$$\xi_0([0, \infty) \cap \mathbb{T}) \subset \overline{\bigcup_{n \in \mathbb{N}} \xi_n(\mathbb{J}_n)}.$$

Now, we just have to repeat the same argument above for all $n \in \mathbb{N}$ and take the diagonal subsequence $\{n_{k,k}\}$. Suppose that we have already constructed a subsequence $\{n_{j,k}\}$ of $\{n_{j-1,k}\}$, for $j = 1, \dots, m-1$, such that $\xi_{n_{j,k}}(\cdot + b_{n_{j,k}}) \rightarrow \xi_0(\cdot)$ as $k \rightarrow \infty$ uniformly on compact subintervals of $[-j, \infty) \cap \mathbb{T}$, where $\xi_0: [-m+1, \infty) \cap \mathbb{T} \rightarrow X$ is a solution of $\mathcal{T}_0$ that lies in $\overline{\cup_{n \in \mathbb{N}} \xi_n(\mathbb{J}_n)}$.

Now we find a subsequence $\{n_{m,k}\}$ of $\{n_{m-1,k}\}$ such that $\xi_{n_{m,k}}(-m + b_{n_{m,k}}) \rightarrow x_m$ for some $x_m \in X$, and extend the definition of $\xi_0(\cdot)$ to $[-m, \infty) \cap \mathbb{T}$ by setting $\xi_0(s) = T_0(s+m)x_m$ for $s \in [-m, -m+1) \cap \mathbb{T}$. Then

$$\begin{aligned} T_0(1)x_m &= \lim_{k \rightarrow \infty} T_{\eta_{n_{m,k}}}(1)\xi_{n_{m,k}}(-m + b_{n_{m,k}}) \\ &= \lim_{k \rightarrow \infty} \xi_{n_{m,k}}(-m + 1 + b_{n_{m,k}}) = \xi_0(-m + 1), \end{aligned}$$

and so $\xi_0: [-m, \infty) \cap \mathbb{T} \rightarrow X$ is a solution of $\mathcal{T}_0$. Clearly $\xi_{n_{m,k}}(\cdot + b_{n_{m,k}}) \rightarrow \xi_0(\cdot)$ uniformly on compact subintervals of $[-m, \infty) \cap \mathbb{T}$, and $\xi_0([-m, \infty) \cap \mathbb{T}) \subset \overline{\cup_{n \in \mathbb{N}} \xi_n(\mathbb{J}_n)}$.

Finally, if we let $n_k = n_{k,k}$, the sequence $\{\xi_{n_k}(\cdot + b_{n_k})\}$ and the corresponding global solution ξ_0 have all the properties that we have stated. $\square$

Lemma 10. Let $\{\mathcal{T}_\eta\}_{\eta \in [0,1]}$ be a continuous and asymptotically compact at $\eta = 0$ family of semigroups such that $\mathcal{A}_\eta$ is a global attractor of $\mathcal{T}_\eta$ with a disjoint collection of isolated invariant sets $\Xi_\eta = \{\Xi_{1,\eta}, \dots, \Xi_{n,\eta}\}$, for each $\eta \in [0, 1]$. Also, suppose that $\mathcal{T}_0$ satisfies (G1). Then, given $0 < \delta < \varepsilon_0(\Xi_0)$ and a bounded set $B \subset X$, there exists $t_0 > 0$ and $\eta_0 > 0$ such that

$$\{T_\eta(t)u_0 : 0 \leq t \leq t_0\} \cap \left(\bigcup_{i=1}^n \mathcal{O}_\delta(\Xi_{i,0}) \right) \neq \emptyset, \forall u_0 \in B, \forall \eta \in [0, \eta_0]. \quad (2.6)$$

In the particular case that $\max_{j=1, \dots, n} d_H(\Xi_{j,\eta}, \Xi_{j,0}) \rightarrow 0$ as $\eta \rightarrow 0$, we can exchange $\Xi_{i,0}$ by $\Xi_{i,\eta}$ in (2.6).

Proof. Suppose that exists $0 < \delta < \varepsilon_0(\Xi_0)$ and a bounded set B such that none $t_0 > 0$ and η_0 satisfies (2.6). Then there exist sequences $\{u_k\}_{k \in \mathbb{N}}$ in B , $\{\eta_k\}_{k \in \mathbb{N}}$ in $[0, 1]$, with $\eta_k \rightarrow 0$, and $\{t_k\}_{k \in \mathbb{N}}$ in $\mathbb{T}^+$ with $t_k \rightarrow \infty$ such that

$$\{T_{\eta_k}(t)u_k : 0 \leq t \leq t_k\} \cap \left(\bigcup_{i=1}^n \mathcal{O}_\delta(\Xi_{i,0}) \right) = \emptyset.$$

Defining $a_n = 0, b_n = \frac{t_n}{2}$ and $c_n = t_n$, it follows from Lemma 9 that exists a bounded global solution $\xi: \mathbb{T} \rightarrow X$ of $\mathcal{T}_0$ such that $T_{\eta_k}(t + \frac{t_k}{2})u_k \rightarrow \xi(t)$ uniformly in compact subsets of $\mathbb{T}$. Thus $\xi(t) \notin \bigcup_{i=1}^n \mathcal{O}_\delta(\Xi_{i,0})$, for all $t \in \mathbb{T}$, which contradicts (G1). $\square$

Lemma 11. Let $\{\mathcal{T}_\eta\}_{\eta \in [0,1]}$ be a family of semigroups that is continuous at $\eta = 0$. Assume that, for each $\eta \in [0, 1]$, Ξ_η is a $\mathcal{T}_\eta$ -invariant set and $d_H(\Xi_\eta, \Xi_0) \rightarrow 0$ as $\eta \rightarrow 0$. Then, given $t \in \mathbb{T}^+$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\eta_0 > 0$ such that

$$\{T_\eta(s)y : 0 \leq s \leq t, y \in \mathcal{O}_\delta(\Xi_\eta)\} \subset \mathcal{O}_\varepsilon(\Xi_\eta), \forall \eta \leq \eta_0.$$

Proof. If this is not the case, there exist $t_0 \in \mathbb{T}^+$, $\varepsilon_0 > 0$ and sequences $\{y_k\}_{k \in \mathbb{N}}$ in $\mathcal{O}_{\frac{1}{k}}(\Xi_{\eta_k})$, $\{\eta_k\}_{k \in \mathbb{N}}$ in $[0, 1]$, with $\eta_k \rightarrow 0$, and $\{s_k\}_{k \in \mathbb{N}} \in [0, t_0]$ such that $d(T_{\eta_k}(s_k)y_k, \Xi_{\eta_k}) \geq \varepsilon_0$. From the compactness of $[0, t_0]$ and $\mathcal{A}$, we can suppose, without loss of generality, that $y_k \rightarrow y_0 \in \Xi_0$ and $s_k \rightarrow s_0 \in [0, t_0]$. It follows from the continuity at $\eta = 0$ that $d(T_0(s_0)y_0, \Xi_0) \geq \varepsilon_0$, which is a contradiction. $\square$

Lemma 12. Let $\{\mathcal{T}_\eta\}_{\eta \in [0, 1]}$ be a continuous and collectively asymptotically compact at $\eta = 0$ family of semigroups in X . Assume that for each $\eta \in [0, 1]$, $\mathcal{T}_\eta$ has a global attractor $\mathcal{A}_\eta$ and a disjoint collection of isolated invariant sets $\Xi_\eta = \{\Xi_{1, \eta}, \dots, \Xi_{n, \eta}\}$ in $\mathcal{A}_\eta$ such that $\max_{1 \leq i \leq n} d_H(\Xi_{i, \eta}, \Xi_{i, 0}) \rightarrow 0$ as $\eta \rightarrow 0$. If $\mathcal{T}_0$ is dynamically gradient and $\delta_0 \in (0, \varepsilon_0(\Xi_0))$, then given $\delta \in (0, \delta_0)$, there exist $\eta_0 > 0$ and $\delta' > 0$ such that, if $\eta \in [0, \eta_0]$, $d_H(z_0, \Xi_{i, \eta}) < \delta'$ and $d_H(T_\eta(t_1)z_0, \Xi_{i, \eta}) \geq \delta$ for some $1 \leq i \leq n$ and $t_1 \in \mathbb{T}^+$, we have $d_H(T_\eta(t)z_0, \Xi_{i, \eta}) > \delta'$, $\forall t \geq t_1$.

Proof. We will argue by contradiction. Assume that, for some $1 \leq i \leq n$, there exist a sequence $\{z_k\}_{k \in \mathbb{N}}$ in X , $\{\eta_k\}_{k \in \mathbb{N}}$ in $[0, 1]$ with $\eta_k \rightarrow 0$, $d_H(z_k, \Xi_{i, \eta_k}) < \frac{1}{k}$ and sequences $\sigma_k < \tau_k$ in $\mathbb{T}^+$ such that $d_H(T_{\eta_k}(\sigma_k)z_k, \Xi_{i, \eta_k}) \geq \delta$ and $d_H(T_{\eta_k}(\tau_k)z_k, \Xi_{i, \eta_k}) < \frac{1}{k}$.

We can assume, without loss of generality, that $\{\sigma_k\}$ is such that $T_{\eta_k}(s)z_k \in \mathcal{O}_\delta(\Xi_{i, \eta_k})$ for all $s < \sigma_k$ and, from Lemma 11, $\sigma_k \rightarrow \infty$. Taking subsequences we can construct a global solution $\xi_0: \mathbb{R} \rightarrow \mathcal{A}$ of $\mathcal{T}_0$ such that $\lim_{t \rightarrow -\infty} d_H(\xi_0(t), \Xi_{i, 0}) = 0$. From the fact that $\mathcal{T}_0$ is dynamically gradient there exists $j \neq i$ such that $\lim_{t \rightarrow \infty} d_H(\xi_0(t), \Xi_{j, 0}) = 0$.

From the fact that $\lim_{t \rightarrow \infty} d_H(\xi_0(t), \Xi_{j, 0}) = 0$ and from the construction of ξ_0 , there exists a subsequence $\{\eta_{k_m}\}$ of $\{\eta_k\}$ and subsequences $\{\tau_{k_m}\}$, $\{t_{k_m}\}$ in $\mathbb{T}^+$, $\tau_{k_m} < \sigma_{k_m}$, such that

$$d_H(T_{\eta_{k_m}}(\tau_{k_m})z_{k_m}, \Xi_{j, \eta_{k_m}}) < \frac{1}{m} \quad \text{and} \quad d_H(T_{\eta_{k_m}}(\sigma_{k_m})z_k, \Xi_{j, \eta_{k_m}}) \geq \delta.$$

Proceeding in the same way, we obtain a solution $\xi_1: \mathbb{R} \rightarrow \mathcal{A}$ such that $\lim_{s \rightarrow -\infty} d_H(\xi_1(s), \Xi_{j, 0}) = 0$. From the fact that $\mathcal{T}_0$ is dynamically gradient there exist $\Xi_{\ell, 0} \in \Xi_0$, $\Xi_{\ell, 0} \notin \{\Xi_{i, 0}, \Xi_{j, 0}\}$, such that $\lim_{s \rightarrow \infty} d_H(\xi_1(s), \Xi_{\ell, 0}) = 0$. Thus, in a finite number of steps, we can construct a chain recurrent set in Ξ_0 , which leads to a contradiction with (G2). $\square$

Definition 19 (Homoclinic Structure). Let $\mathcal{T}$ be a semigroup in X with global attractor $\mathcal{A}$ which contains a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. A **homoclinic structure** in $\mathcal{A}$ is a disjoint subcollection $\{\Xi_{\ell_1}, \dots, \Xi_{\ell_k}\}$ of Ξ and a finite number of bounded global solutions $\xi_i: \mathbb{T} \rightarrow X$, $1 \leq i \leq k$, such that, defining $\Xi_{\ell_{k+1}} := \Xi_{\ell_1}$, satisfies:

$$\lim_{t \rightarrow -\infty} d(\xi_i(t), \Xi_{\ell_i}) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} d(\xi_i(t), \Xi_{\ell_{i+1}}) = 0, \forall 1 \leq i \leq k$$

and

$$\min_{1 \leq i \leq k} \sup_{t \in \mathbb{R}} d\left(\xi_i(t), \bigcup_{i=1}^n \mathcal{O}_\delta(\Xi_i)\right) > 0. \quad (2.7)$$

Remark 8. The condition (2.7) is automatically true when $k \geq 2$. However, if $k = 1$, we have to consider the case where $\phi(\mathbb{T})$ is a subset of Ξ_j , for some $j = 1, \dots, k$. Since we do not want this case to be a homoclinical structure, we add the condition (2.7).

Proposition 4. Suppose that $\mathcal{T}$ is a semigroup in X with a global attractor $\mathcal{A}$ which contains a disjoint collection of invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$ and satisfies (G1). Then, $\mathcal{T}$ is dynamically gradient if, and only if $\mathcal{A}$ does not have a homoclinical structure.

Proof. If $\mathcal{A}$ has a homoclinical structure we can easily prove that exists $\Xi^* \in \Xi$ that is chain recurrent. Therefore, by contraposition, if $\mathcal{T}$ is dynamically gradient then $\mathcal{A}$ does not have a homoclinical structure. It remains to show, again by contraposition, that if exists $\Xi^* \in \Xi$ that is chain recurrent, then $\mathcal{A}$ has homoclinical structure. In fact, if Ξ^* is chain recurrent, there exists a subset $\{\Xi_{k_1}, \dots, \Xi_{k_\ell}\}, \Xi^* = \Xi_{k_1} =: \Xi_{l_{k+1}}$ such that for every $k \in \mathbb{N}^*$ there are $x_1^k, \dots, x_\ell^k$ in X and $t_1^k, \dots, t_\ell^k$ in $\mathbb{T}^+$ satisfying

$$d(x_i^k, \Xi_{k_i}) < \frac{1}{k} \quad \text{and} \quad d(T(t_i^k)x_i^k, \Xi_{k_{i+1}}) < \frac{1}{k}, \quad 1 \leq i \leq \ell,$$

and we may assume that $\overline{\bigcup_{k=1}^\infty \bigcup_{t \in [0, t_i^k]} T(t)x_i^k \cap \Xi_j} \neq \emptyset$ if j is different from i and $i+1$ (adding more isolated invariant sets to the $\frac{1}{k}$ chains if needed).

Choosing $0 < \delta < \varepsilon_0(\Xi)$ and $\tau_i^k > 0$ such that $d(T(t)x_i^k, \Xi_i) < \delta$, for all $0 \leq t < \tau_i^k$ and $d(T(\tau_i^k)x_i^k, \Xi_i) = \delta$, follows from the continuity of $\mathcal{T}$ and from the invariance of Ξ_i that $\tau_i^k \rightarrow \infty$ as $k \rightarrow \infty$. For $t \in [-\tau_i^k, \infty)$ let $\xi_i^k(t) = T(\tau_i^k + t)x_i^k$. Taking subsequences we define the global solutions $\xi_i: \mathbb{T} \rightarrow X$ by $\xi_i(t) = \lim_{k \rightarrow \infty} \xi_i^k(t)$. Since each $\xi_i(t)$ must converge to an isolated invariant set as $t \rightarrow +\infty$ and as $t \rightarrow -\infty$ and since $\xi_i(t) \in \mathcal{O}_\delta(\Xi_i)$ for all $t \leq 0$ we have $\lim_{t \rightarrow -\infty} d(\xi_i(t), \Xi_i) = 0$. Since $\overline{\bigcup_{k=1}^\infty \bigcup_{t \in [0, t_i^k]} T(t)x_i^k \cap \Xi_j} \neq \emptyset$, j different from i and $i+1$, we must have $\lim_{t \rightarrow \infty} d(\xi_i(t), \Xi_{i+1}) = 0$.

The collection of isolated invariant sets $\{\Xi_1, \dots, \Xi_\ell\} \subset \Xi$ and the set of global solutions $\{\xi_i: \mathbb{T} \rightarrow X, 1 \leq i \leq \ell\}$ are such that $\Xi = \Xi_1 = \Xi_{\ell+1}$ and

$$\lim_{t \rightarrow -\infty} d(\xi_i(t), \Xi_i) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} d(\xi_i(t), \Xi_{i+1}) = 0,$$

for $1 \leq i \leq \ell$. Hence $\mathcal{A}$ has a homoclinic structure. □

It follows directly from Proposition 4 the following result.

Corollary 1. If $\mathcal{T}$ is a gradient semigroup in X , then $\mathcal{T}$ is a dynamically gradient semigroup.

Proof. Suppose that $\mathcal{T}$ is a gradient semigroup with respect to Ξ . To verify (G1), let $x \in X$ and $\phi: \mathbb{T} \rightarrow X$ be a solution through x . From Lemma 5, we must have $\phi(t) \rightarrow \omega(x) \subset \Xi_j$ as $t \rightarrow \infty$, and $\phi(t) \rightarrow \alpha_\phi(x) \subset \Xi_i$ as $t \rightarrow -\infty$, for some $i, j \in \{1, \dots, n\}$, and consequently (G1) is

satisfied. To prove (G2) we will argue by contradiction. Suppose that $\mathcal{T}$ does not satisfy (G2), then by Proposition 4, there exists a homoclinical structure. Since the Lyapunov function will imply (because is nonincreasing along solutions) that those solutions (from the homoclinical structure) are constant, we have a contradiction. Therefore all gradient semigroup are dynamically gradient. $\square$

DYNAMICALLY GRADIENT SEMIGROUPS ARE GRADIENT

In this chapter we will prove that gradient and dynamically gradient semigroups are equivalents. Then, we will prove the stability of dynamically gradient structure under perturbation, concluding the robustness of the gradient structure. This chapter is very rich mathematically and it is essential to solve our main problem. There are many books and articles related to this topic, for example (CARVALHO; LANGA; ROBINSON, 2013), (BORTOLAN; CARVALHO; LANGA, 2020), (ARAGÃO-COSTA; CARVALHO; LANGA, 2012), (CARVALHO; LANGA, 2009), (CARVALHO; ROBINSON, 2007), (CONLEY, 1978) and (ARAGÃO-COSTA; CARVALHO, 2013).

3.1 Morse Decomposition

Since we want to prove that dynamically gradient semigroups are gradient, that is, there exists a Lyapunov function for the attractor $\mathcal{A}$, we need to know more about the dynamics in the attractor. With this in mind, we have to introduce some new concepts, like the Morse decomposition, that will allow us to construct a Lyapunov function for $\mathcal{A}$.

Definition 20 (Local Attractor and Repeller). Let $\mathcal{T}$ be a semigroup in a metric space X with a global attractor $\mathcal{A}$. We say that a nonempty subset Ξ of $\mathcal{A}$ is a **local attractor** if there exists an $\varepsilon > 0$ such that $\omega(\mathcal{O}_\varepsilon(\Xi)) = \Xi$. We define the **repeller** Ξ^* associated to a local attractor Ξ as the set

$$\Xi^* = \{x \in \mathcal{A} : \omega(x) \cap \Xi = \emptyset\}.$$

The pair (Ξ, Ξ^*) is called an attractor-repeller pair for $\mathcal{T}$.

Lemma 13. Let $\mathcal{T}$ be a semigroup in X with a global attractor $\mathcal{A}$ and a local attractor $\Xi \subset \mathcal{A}$. Then, there exists $\varepsilon > 0$ such that $\Xi^* \cap \mathcal{O}_\varepsilon(\Xi) = \emptyset$. Consequently, the repeller Ξ^* is closed and $\mathcal{T}$ -invariant.

Proof. By hypothesis, there exists $\varepsilon > 0$ such that $\omega(\mathcal{O}_\varepsilon(\Xi)) = \Xi$, that is, if $x \in \mathcal{O}_\varepsilon(\Xi)$ then $\omega(x) \subset \Xi$. Hence $\omega(x) \cap \Xi \neq \emptyset$, for all $x \in \mathcal{O}_\varepsilon(\Xi)$, that is, $\mathcal{O}_\varepsilon(\Xi) \cap \Xi^* = \emptyset$.

To conclude the $\mathcal{T}$ -invariance of Ξ^* , just notice that $\omega(x) = \omega(T(t)x)$, for all $t \in \mathbb{T}^+$. Thus $x \in \Xi^*$ if, and only if $T(t)x \in \Xi^*$, for all $t \in \mathbb{T}^+$, and we proved the $\mathcal{T}$ -invariance of Ξ^* .

It remains to show that $\overline{\Xi^*} \subset \Xi^*$. Since the closure of a $\mathcal{T}$ -invariant set is also $\mathcal{T}$ -invariant, if $x \in \overline{\Xi^*}$ then $T(t)x \in \overline{\Xi^*}$, for all $t \in \mathbb{T}^+$ and $\omega(x) = \bigcap_{t=0}^{\infty} \gamma_t^+(x) \subset \overline{\Xi^*}$. From $\mathcal{O}_\varepsilon(\Xi) \cap \Xi^* = \emptyset$ and $\omega(x) \subset \overline{\Xi^*}$ we conclude that $\omega(x) \cap \Xi = \emptyset$, that is, $x \in \Xi^*$. $\square$

Lemma 14. Let $\mathcal{T}$ be a semigroup in a metric space X with a global attractor $\mathcal{A}$ which contains a $\mathcal{T}$ -invariant subset Ξ . Then Ξ is a local attractor if, and only if it is compact and $\mathcal{T}$ -attracts $\mathcal{O}_\varepsilon(\Xi)$, for some $\varepsilon > 0$.

Proof. Suppose that Ξ is a local attractor, that is, there exists an $\varepsilon > 0$ such that $\omega(\mathcal{O}_\varepsilon(\Xi)) = \Xi$. Then, by Lemma 3, $\omega(\mathcal{O}_\varepsilon(\Xi)) = \Xi$ is compact, invariant and $\mathcal{T}$ -attracts $\mathcal{O}_\varepsilon(\Xi)$.

Now, assume that Ξ is compact, $\mathcal{T}$ -invariant and $\mathcal{T}$ -attracts $\mathcal{O}_\varepsilon(\Xi)$, for some $\varepsilon > 0$. From the compactness and $\mathcal{T}$ -invariance we have $\omega(\mathcal{O}_\varepsilon(\Xi)) \supset \Xi$ and from the fact that Ξ attracts $\mathcal{O}_\varepsilon(\Xi)$ we conclude that $\omega(\mathcal{O}_\varepsilon(\Xi)) \subset \overline{\Xi} = \Xi$. $\square$

Definition 21 (Morse Decomposition). Let $\mathcal{T}$ be a semigroup in X with a global attractor $\mathcal{A}$ which contains a collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. We say that Ξ is a **Morse decomposition** of $\mathcal{A}$ if given a global solution $\phi : \mathbb{T} \rightarrow \mathcal{A}$, then either $\phi(\mathbb{T}) \subset \Xi_j$, for some $j = 1, \dots, n$, or

$$\lim_{t \rightarrow -\infty} d(\phi(t), \Xi_j) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} d(\phi(t), \Xi_i) = 0 \quad \text{with } i < j.$$

Geometrically, Definition 21 says that bounded global solutions always comes from an invariant set Ξ_j to another invariant set Ξ_i , with $i < j$. This means that Ξ_1 can "receive" bounded global solutions that comes from all other invariant sets Ξ_j , $1 \leq j \leq n$. On the other hand, the invariant set Ξ_n does not "receive" bounded global solutions that comes from other invariant sets Ξ_j , except when the solution is always contained in Ξ_n .

Note that our definition of local attractor requires that Ξ attracts an ε -neighborhood of himself in X (Lemma 14). The following lemmas assure that we can exchange this condition to the attraction in an ε -neighborhood of Ξ in $\mathcal{A}$, that is, Ξ attracts $\mathcal{O}_\varepsilon(\Xi) \cap \mathcal{A}$.

Lemma 15. Let $\mathcal{T}$ be a semigroup in X with a global attractor $\mathcal{A}$. If Ξ is compact, $\mathcal{T}$ -attracts $\mathcal{O}_\varepsilon(\Xi) \cap \mathcal{A}$ for some $\varepsilon > 0$ and is $\mathcal{T}$ -invariant then, given $\delta > 0$, there is a $\delta' > 0$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$.

Proof. Suppose that there exists $0 < \delta < \varepsilon$ such that for all $\delta' > 0$ we have $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \not\subset \mathcal{O}_\delta(\Xi)$. Then, there is $x \in \Xi$, $X \ni x_n \rightarrow x$ (because Ξ is compact) and, from Lemma 8, $t_n \rightarrow \infty$ such that $d(T(t_n)x_n, \Xi) = \delta$ and $T(t)x_n \in \mathcal{O}_\delta(\Xi)$, $t \in [0, t_n]$. Since $\mathcal{T}$ has a global attractor, there is a global solution $\xi : \mathbb{T} \rightarrow X$ of $\mathcal{T}$ such that $\xi_{n_k} : [-t_{n_k}, \infty) \rightarrow X$ given by $\xi_{n_k}(t) = T(t_{n_k} + t)x_{n_k}$ satisfies $\xi_{n_k}(t) \rightarrow \xi(t)$ for each $t \in \mathbb{T}$ and some subsequence $n_k \rightarrow \infty$. Hence $\xi(t) \in \overline{\mathcal{O}_\delta(\Xi)} \cap \mathcal{A}$ that is contained in $\mathcal{O}_\varepsilon(\Xi) \cap \mathcal{A}$ for all $t \leq 0$, and $d(\xi(0), \Xi) = \delta$. Consequently Ξ do not $\mathcal{T}$ -attract $\mathcal{O}_\varepsilon(\Xi) \cap \mathcal{A}$, which is a contradiction. $\square$

Lemma 16. Let $\mathcal{T}$ be a semigroup in X with a global attractor $\mathcal{A}$ and $\mathcal{S}$ be the family $\{S(t) = T(t)|_{\mathcal{A}} : t \in \mathbb{T}^+\}$ ($\mathcal{S}$ is a semigroup in the metric space $\mathcal{A}$). If Ξ is a local attractor for $\mathcal{S}$ and K is a compact subset of $\mathcal{A}$ such that $K \cap \Xi^* = \emptyset$, then Ξ attracts K . Furthermore, Ξ is a local attractor for $\mathcal{T}$ in X .

Proof. Let K be a compact subset of $\mathcal{A}$ such that $K \cap \Xi^* = \emptyset$. If Ξ does not $\mathcal{T}$ -attract K , there is a $\delta > 0$, $t_n \rightarrow \infty$, $x \in K$ and $K \ni x_n \rightarrow x$ such that $d(T(t_n)x_n, \Xi) \geq \delta$. From Lemma 15 there exists $0 < \delta' < \delta$ such that $d(T(t)x_n, \Xi) \geq \delta'$ for all $t \in [0, t_n] \cap \mathbb{T}$. This implies that $d(T(t)x, \Xi) \geq \delta'$ for all $t \in \mathbb{T}^+$ and $\omega(x) \cap \Xi = \emptyset$, which contradicts $x \notin \Xi^*$. Thus Ξ attracts K .

It remains to show that Ξ is a local attractor for $\mathcal{T}$. In fact, from Lemma 15, there exists a $\delta' > 0$ such that $\omega(\mathcal{O}_{\delta''}(\Xi)) \cap \Xi^* = \emptyset$ for all $\delta'' < \delta'$. From the $\mathcal{S}$ -invariance of $\omega(\mathcal{O}_{\delta'}(\Xi))$ and the fact that Ξ can $\mathcal{S}$ -attract compact subsets of $\mathcal{A}$ that are disjoint of Ξ^* , we must have that $\omega(\mathcal{O}_{\delta''}(\Xi)) \subset \Xi$. Since $\omega(\mathcal{O}_{\delta''}(\Xi))$ $\mathcal{T}$ -attracts $\mathcal{O}_{\delta''}(\Xi)$, we conclude the proof. $\square$

Lemma 17. Let $\mathcal{T}$ be a semigroup in X with a global attractor $\mathcal{A}$ and an attractor-repeller pair (Ξ, Ξ^*) . If $\xi : \mathbb{T} \rightarrow X$ is a global solution of $\mathcal{T}$ such that, for some $\delta > 0$ satisfies $\mathcal{O}_\delta(\Xi^*) \cap \Xi = \emptyset$, $\xi(t) \in \mathcal{O}_\delta(\Xi^*)$ for all $t \in \mathbb{T}^-$, then $\lim_{t \rightarrow -\infty} d(\xi(t), \Xi^*) = 0$.

Proof. Firstly note that $\overline{\xi(\mathbb{T})} \cap \Xi^* \neq \emptyset$, because if this is not the case, it follows from Lemma 16 that Ξ attracts the invariant set $\overline{\xi(\mathbb{T})}$, that is, $\xi(\mathbb{T}) \subset \Xi$, which is a contradiction. Hence, we have $\overline{\xi(\mathbb{T})} \cap \Xi^* \neq \emptyset$.

Now, we will argue by contradiction to prove our statement. Suppose that there is a $\delta' > 0$ and a sequence $t_n \rightarrow \infty$ such that $d(\xi(-t_n), \Xi^*) \geq \delta'$ and for some $t \in [-t_n - 1, -t_n]$, $d(\xi(t), \Xi^*) < \delta'$ (this is possible because $\overline{\xi(\mathbb{T})} \cap \Xi^* \neq \emptyset$). Note that $K = \{z \in \mathcal{A} : d(z, \Xi^*) \geq \delta'\}$ is a nonempty compact set with $K \cap \Xi^* = \emptyset$ and, by Lemma 16, Ξ must attract K , which is a contradiction. $\square$

Lemma 18. Let $\mathcal{T}$ be a semigroup in X with a global attractor $\mathcal{A}$ and an attractor-repeller pair (Ξ, Ξ^*) . If $\xi : \mathbb{T} \rightarrow X$ is a global bounded solution for $\mathcal{T}$ through $x \notin \Xi \cup \Xi^*$, then

$$\lim_{t \rightarrow \infty} d(\xi(t), \Xi) = 0 \quad \text{and} \quad \lim_{t \rightarrow -\infty} d(\xi(t), \Xi^*) = 0.$$

Moreover, if $x \in X \setminus \mathcal{A}$ then $\lim_{t \rightarrow \infty} d(T(t)x, \Xi \cup \Xi^*) = 0$.

Proof. If $\xi: \mathbb{T} \rightarrow X$ is a bounded global solution through $x \notin \Xi \cup \Xi^*$ it follows from Lemma 16 that $\lim_{t \rightarrow \infty} d(\xi(t), \Xi) = 0$.

Assume now that $\xi(t)$ does not converges to Ξ^* as $t \rightarrow -\infty$. Note that, as in Lemma 17, we can assume that $\overline{\xi(\mathbb{R})} \cap \Xi^* \neq \emptyset$, otherwise it follows from Lemma 16 that $\overline{\xi(\mathbb{T})}$ is attracted by Ξ and the invariance of $\overline{\xi(\mathbb{T})}$ implies that $\overline{\xi(\mathbb{T})} \subset \Xi$, which is a contradiction with $\xi(0) = x \notin \Xi \cup \Xi^*$.

Since $\overline{\xi(\mathbb{R})} \cap \Xi^* \neq \emptyset$, there exist $\delta > 0$ and sequences $t_n, \tau_n \rightarrow \infty$ such that $d(\xi(-t_n), \Xi^*) = \delta$ and $\xi(-t_n + \tau_n) \rightarrow z \in \Xi^*$ and $d(\xi(-t_n + t), \Xi) \leq \delta$ for al $0 \leq t \leq \tau_n$. From this we obtain a global solution $\zeta: \mathbb{R} \rightarrow \mathcal{A}$ of $\mathcal{T}$ with the property that $d(\zeta(t), \Xi^*) \leq \delta$ for all $t \in \mathbb{T}^+$. Therefore we have $\omega(\zeta(0)) \notin \Xi$, which implies that $\zeta(0) \in \Xi^*$, leading to a contradiction. Hence $\xi(t) \rightarrow \Xi^*$ as $t \rightarrow -\infty$.

Finally, let $x \in X \setminus \mathcal{A}$. If $\overline{\gamma^+(x)} \cap \Xi \neq \emptyset$ we have $\lim_{t \rightarrow \infty} d(T(t)x, \Xi) = 0$. Otherwise, there exists $\delta > 0$ with $\gamma^+(z) \cap \mathcal{O}_\delta(\Xi) = \emptyset$, and in this case we claim that $\lim_{t \rightarrow \infty} d(T(t)x, \Xi^*) = 0$. If the claim is false, there are $\varepsilon > 0$ and a sequence $t_n \rightarrow \infty$ such that $d(T(t_n)x, \Xi^*) \geq \varepsilon$. Considering the sequence of functions $\xi_n: [-t_n, \infty) \cap \mathbb{T} \rightarrow X$ defined by $\xi_n(t) = T(t + t_n)x$ for $t \geq -t_n$, we construct a global solution $\xi: \mathbb{T} \rightarrow \mathcal{A}$ of $\mathcal{T}$ such that $d(\xi(0), \Xi^*) \geq \varepsilon$ and $d(\xi(t), \Xi) \geq \delta$ for all $t \in \mathbb{T}$. Hence $\omega(\xi(0)) \cap \Xi = \emptyset$ and $\xi(0) \notin \Xi^*$ which is a contradiction. $\square$

Lemma 19. Let $\mathcal{T}$ be a dynamically gradient semigroup in X with a global attractor $\mathcal{A}$ which has a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. Then Ξ_i is a local attractor, for some $1 \leq i \leq n$.

Proof. Assume that there are no local attractors in Ξ . Choose $0 < \varepsilon < \varepsilon_0(\Xi)$. It follows from Lemma 12 that, for each suitably small $\delta \in (0, \varepsilon)$ there is a $x_{i,\delta} \in \mathcal{O}_\delta(\Xi_i)$ and a $\tau_{i,\delta} > 0$ such that $d(T(\tau_{i,\delta})x_{i,\delta}, \Xi_i) \geq \varepsilon$ and $d(T(t + \tau_{i,\delta})x_{i,\delta}, \Xi_i) \geq \delta$ for all $t \geq 0$. It follows that $\lim_{t \rightarrow \infty} d(T(t)x_{i,\delta}, \Xi_j) = 0$, for some $j \neq i$. This means that there exists $\Xi_i \in \Xi$ that is chain recurrent, contradicting (G2). $\square$

Theorem 4. Let $\mathcal{T}$ be a dynamically gradient semigroup relative to the disjoint collection os isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. Then, there is a reordering of Ξ that is a Morse decomposition of $\mathcal{A}$.

Proof. We know, from Lemma 19, that there is $i = 1, \dots, n$ such that Ξ_i is a local attractor. Suppose, reordering if necessary, that Ξ_1 is a local attractor. From the invariance of Ξ_i , it follows that $\Xi_i \subset \Xi_1^*$, $i > 1$, and that for any $x \in \mathcal{A} \setminus \{\Xi_1 \cup \Xi_1^*\}$ and global solution $\phi: \mathbb{R} \rightarrow \mathcal{A}$ of $\mathcal{T}$ through x we have

$$\lim_{t \rightarrow -\infty} d(\phi(t), \Xi_1^*) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} d(\phi(t), \Xi_1) = 0.$$

Since $\Xi_{1,0}^* := \Xi_1^*$ is $\mathcal{T}$ -invariant by Lemma 13, we can consider the following semigroup $\mathcal{T}_1 = \{T(t)|_{\Xi_{1,0}^*} : t \in \mathbb{T}^+\}$ which is dynamically gradient relative to the disjoint collection of

isolated invariant sets $\{\Xi_2, \dots, \Xi_n\}$. Making this restriction, we have removed all possible global solutions $\phi: \mathbb{T} \rightarrow \mathcal{A}$ of $\mathcal{T}$ such that $\phi(t) \rightarrow \Xi_1$ as $t \rightarrow \infty$, that is, if $\phi_1: \mathbb{T} \rightarrow \mathcal{A}$ is a global solution of $\mathcal{T}_1$ and $\phi_1(t) \rightarrow \Xi_i$, then $i > 1$.

Proceeding as above we can assume, without loss of generality, that Ξ_2 is a local attractor for the semigroup $\mathcal{T}_1$ in $\Xi_{1,0}^*$. If $\Xi_{2,1}^*$ is the repeller associated to the isolated invariant set Ξ_2 for the semigroup $\mathcal{T}_1$ in $\Xi_{1,0}^*$ we may proceed and consider the restriction $\mathcal{T}_2$ of the semigroup $\mathcal{T}_1$ to $\Xi_{2,1}^*$ and $\mathcal{T}_2$ is a dynamically gradient semigroup in $\Xi_{2,1}^*$ with associated disjoint collection of isolated invariant sets $\{\Xi_3, \dots, \Xi_n\}$. Again, we have removed all the global solutions $\phi: \mathbb{T} \rightarrow X$ of $\mathcal{T}$ such that $\phi(t) \rightarrow \Xi_2$ as $t \rightarrow \infty$ and therefore, if $\Xi_j \xleftarrow{t \rightarrow -\infty} \phi(t) \xrightarrow{t \rightarrow \infty} \Xi_2$ then $j \geq 2$.

Proceeding with this until all isolated invariant sets are exhausted we obtain a reordering of $\{\Xi_1, \dots, \Xi_n\}$ in such a way that Ξ_j is a local attractor for the restriction of $\mathcal{T}$ to $\Xi_{j-1,j-2}^*$, where $\Xi_{0,-1}^* = \mathcal{A}$.

With the construction, if a global solution $\phi: \mathbb{R} \rightarrow \mathcal{A}$ of $\mathcal{T}$ satisfies $\phi(\mathbb{T}) \subset \Xi_j$, for some $j = 1, \dots, n$ or

$$\Xi_\ell \xleftarrow{t \rightarrow -\infty} \xi(t) \xrightarrow{t \rightarrow \infty} \Xi_k, \quad k < l. \quad (3.1)$$

Hence this reordering is a Morse decomposition of $\mathcal{A}$. $\square$

Theorem 5. Let $\mathcal{T}$ be a dynamically gradient semigroup relative to the disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$, which is already ordered in such way that is a Morse decomposition. Then, exists a finite sequence $\emptyset = A_0 \subsetneq A_1 \subsetneq \dots \subsetneq A_n = \mathcal{A}$ such that A_j is a local attractor and $\Xi_j = A_j \cap A_{j-1}^*$, for all $1 \leq j \leq n$. Moreover, we have $\bigcap_{j=1}^n (A_j \cup A_j^*) = \bigcup_{j=1}^n \Xi_j$.

Proof. It is easy to see that $W^u(\Xi_j)$ is $\mathcal{T}$ -invariant and therefore A_j is $\mathcal{T}$ -invariant. To prove the compactness, it is sufficient to show that $W^u(\Xi_j)$ is closed, since $\Xi_j \subset \mathcal{A}$. Indeed, suppose that $x_n \in W^u(\Xi_j)$, $x_n \rightarrow x \in \mathcal{A}$ and $x \notin W^u(\Xi_j)$, that is, $x \in W^u(\Xi_k)$, $k \geq j$. Hence, there exists a bounded global solution ξ through x such that $\lim_{t \rightarrow \infty} d(\xi(t), \Xi_l) = 0$, $l \leq k$ and $\lim_{t \rightarrow -\infty} d(\xi(t), \Xi_k) = 0$. We can assume, without loss of generality, that for each n there exists bounded global solutions ξ_n through x_n and $\lim_{t \rightarrow \infty} \xi_n(t) = \Xi_i$, where $i \leq j$. Given $\varepsilon > 0$, there exists t_0 such that $\xi(t) \in \mathcal{O}_\varepsilon(\Xi_k)$. Defining $\psi_n(t) = \xi_n(t + t_0)$ there exists a global solution ψ (using subsequences) such that $\lim_{t \rightarrow -\infty} d(\psi(t), \Xi_j) = 0$ and $\psi(0) \in \mathcal{O}_\varepsilon(\Xi_k)$. Thus (using the same reasoning as in Proposition 4) we have a global solution that connects Ξ_j to Ξ_k , that contradicts the fact that we already have ordered the sets as a Morse decomposition.

It remains to show that A_j is a local attractor for $\mathcal{T}_1 = \{T(t)|_{\mathcal{A}} : t \in \mathbb{T}^+\}$. Choose $d > 0$ such that $\mathcal{O}_d(\bigcup_{i=1}^j W^u(\Xi_i)) \cap (\bigcup_{i=j+1}^n W^u(\Xi_i)) = \emptyset$ and consider the cases:

Case 1: There exists $0 < \delta' < \delta < d$ such that $\gamma^+(\mathcal{O}_{\delta'}(A_j) \cap \mathcal{A}) \subset \mathcal{O}_\delta(A_j) \cap \mathcal{A}$. In this case $\omega(\mathcal{O}_{\delta'}(A_j) \cap \mathcal{A})$ is $\mathcal{T}_1$ -invariant, attracts $\mathcal{O}_{\delta'}(A_j) \cap \mathcal{A}$ and is a subset of A_j , proving that A_j is a local attractor in $\mathcal{A}$.

Case 2: There is no $0 < \delta' < \delta < d$ such that $\gamma^+(\mathcal{O}_{\delta'}(A_j) \cap \mathcal{A}) \subset \mathcal{O}_{\delta}(A_j) \cap \mathcal{A}$. Then exists $\{x_k\}_{k \in \mathbb{N}}$ in $\mathcal{A} \setminus A_j$, $d(x_k, A_j) \rightarrow 0$ as $k \rightarrow \infty$, global solutions $\xi_k : \mathbb{T} \rightarrow \mathcal{A}$ through x_k and $t_k \rightarrow \infty$ such that $d(\xi_k(t), A_j) \leq \delta$ for $t \in [0, t_k]$ and $d(\xi_k(t_k), A_j) = \delta$. From Lemma 9 there exists $\xi : \mathbb{R} \rightarrow \mathcal{A}$ global solution such that $d(\xi(t), A_j) \leq \delta$ for $t \in \mathbb{T}^-$ and $d(\xi(0), A_j) = \delta$, which is a contradiction.

Hence, it follows by Lemma 14 that A_j is a local attractor.

Now we must prove that $\Xi_j = A_j \cap A_{j-1}^*$, for all $j = 1, \dots, n$. The inclusion $\Xi_j \subset A_j \cap A_{j-1}^*$ is a consequence of $A_j \supset \bigcup_{i=1}^j \Xi_i$ and $A_{j-1}^* \supset \bigcup_{i=j}^n \Xi_i$. However, this inclusion also implies that if $z \in A_j \cap A_{j-1}^*$ and $\phi : \mathbb{T} \rightarrow \mathcal{A}$ is a global solution through z , then

$$\Xi_\ell \xrightarrow{t \rightarrow -\infty} \xi(t) \xrightarrow{t \rightarrow \infty} \Xi_k, \quad (3.2)$$

with $k \leq \ell \leq j$ (because $z \in A_j$) and $j \leq k \leq \ell$ (because $z \in A_{j-1}^*$). From this and the fact that $\mathcal{T}$ is a dynamically gradient semigroup, it follows that $z \in \Xi_j$. Thus $\Xi_j = A_j \cap A_{j-1}^*$.

Finally we will prove that $\bigcap_{j=1}^n (A_j \cup A_j^*) = \bigcup_{j=1}^n \Xi_j$. The inclusion $\bigcup_{j=1}^n \Xi_j \subset \bigcap_{j=1}^n (A_j \cup A_j^*)$ follows directly from the property $A_j \subset A_{j+1}$ and $A_{j+1}^* \subset A_j^*$, $j = 0, \dots, n$. On other hand, if $z \in \bigcap_{j=1}^n (A_j \cup A_j^*)$, there exists $k \in \{1, \dots, n\}$ such that $z \in A_j$, $k \leq j \leq n$ and $z \in A_j^*$, $1 \leq j \leq k-1$ (by the same property of inclusion that we used above). Thus $z \in A_k \cap A_{k-1}^* = \Xi_k$ and we conclude the proof. $\square$

3.2 Robustness and Equivalence Between Gradient and Dynamically Gradient Semigroups

The following results will be used to construct a Lyapunov function for a dynamically gradient semigroup relative to a collection of isolated invariant sets $\{\Xi_1, \dots, \Xi_n\}$.

Lemma 20. Let $\mathcal{T}$ be a semigroup in X with global attractor $\mathcal{A}$. Then, the map $h : X \rightarrow \mathbb{R}$ defined by $h(x) = \sup_{t \in \mathbb{T}^+} d(T(t)x, \mathcal{A})$, for all $x \in X$, is well defined, continuous, nonincreasing along solutions of $\mathcal{T}$ and $h^{-1}(0) = \mathcal{A}$.

Proof. To see that h is well defined we use Theorem 2, which ensures that for each $x \in X$ exists $t_0 \in \mathbb{T}^+$ such that $\gamma_{t_0}^+(x)$ is bounded, and the continuity of $\mathcal{T}$, which guarantee that $\bigcup_{t \in [0, t_0] \cap \mathbb{T}} T(t)x$ is compact, hence $\gamma^+(x)$ is bounded and $h(x) < +\infty$. Clearly h is non increasing along solution, since $\{T(t)T(s)x : t \in \mathbb{T}^+\} \subset \{T(t)x : t \in \mathbb{T}^+\}$ for all $s \in \mathbb{T}^+$ and $x \in X$. Given $x \in X$, it follows from the compactness of $\mathcal{A}$ and $\{x\}$ that $h^{-1}(0) = \mathcal{A}$.

Now we will prove the continuity of h . If $x_0 \in \mathcal{A}$ and $\varepsilon > 0$ it follows from Lemma 14 that there exists $\varepsilon' \in (0, \varepsilon)$ such that $\gamma^+(\mathcal{O}_{\varepsilon'}(\mathcal{A})) \subset \mathcal{O}_{\varepsilon}(\mathcal{A})$. Thus $h(z) < \varepsilon$, for $z \in \mathcal{O}_{\varepsilon'}(x)$ and

$|h(x) - h(z)| < \varepsilon$, that is, h is continuous in x_0 . If $x_0 \in X \setminus \mathcal{A}$ we have $h(x_0) \geq d(x_0, \mathcal{A}) > 0$. Fix $0 < \mu < d(x_0, \mathcal{A})$. From the continuity of the map $X \ni x \mapsto d(x, \mathcal{A}) \in [0, \infty)$, we can choose a bounded neighborhood V of x_0 in X such that $d(z, \mathcal{A}) > \mu$ if $z \in V$. Now, let $\tau > 0$ be such that $\gamma^+(T(\tau)V) \subset \mathcal{O}_\mu(\mathcal{A})$. Hence, $h(z) = \sup_{0 \leq s \leq \tau} d(T(s)z, \mathcal{A})$, for each $z \in V$, and from the continuity of $\mathcal{T}$, it follows that h is continuous in x_0 . This proves the continuity of h in x_0 , for all $x_0 \in X$. $\square$

Proposition 5. Let $\mathcal{T}$ be a semigroup in X with global attractor $\mathcal{A}$ and an attractor-repeller pair (Ξ, Ξ^*) . Then, there exists a continuous function $f : X \rightarrow \mathbb{R}$ that satisfies:

1. f is nonincreasing along solutions of $\mathcal{T}$.
2. $f^{-1}(0) = \Xi$ and $f^{-1}(1) \cap \mathcal{A} = \Xi^*$.
3. Given $z \in X$, if $f(T(t)z) = f(z)$ for all $t \in \mathbb{T}^+$, then $z \in \Xi \cup \Xi^*$.

Proof. First note that Ξ and Ξ^* are disjoint compact subsets of X , thus $d_0 = d(\Xi, \Xi^*) > 0$. With the convention that $d(z, \emptyset) = 1$ for each $z \in X$, define $l : X \rightarrow [0, 1]$ as the Urysohn function associated to (Ξ, Ξ^*) , given by

$$l(z) = \frac{d(z, \Xi)}{d(z, \Xi) + d(z, \Xi^*)}, \forall z \in X.$$

Then l is well defined, $l^{-1}(0) = \Xi$ and $l^{-1}(1) = \Xi^*$. Moreover,

$$|l(z) - l(w)| \leq \frac{2}{d_0} d(z, w) \text{ for } z, w \in X,$$

which implies the (uniform) continuity of l .

Now, define $k : X \rightarrow \mathbb{R}$ by

$$k(z) = \sup_{t \in \mathbb{T}^+} l(T(t)z), \forall z \in X. \quad (3.3)$$

We will show that k is continuous, nonincreasing along solutions of $\mathcal{T}$, $k(X) \subset [0, 1]$ and $k^{-1}(0) = \Xi$.

Clearly $k(X) \subset [0, 1]$, k is nonincreasing (by the same argument in Lemma 20) and $k^{-1}(0) = \Xi$. To prove that $k^{-1}(1) \cap \mathcal{A} = \Xi^*$, just note that if $x \in (k^{-1}(1) \cap \mathcal{A})$ is not in Ξ^* , then $T(t)x \in \Xi^*$, for some $t \in \mathbb{T}^+$ and $x \notin \Xi$. From Theorem 1 and Lemma 18, there exists $\phi : \mathbb{T} \rightarrow \mathcal{A}$ through x such that $\phi(t) \rightarrow \Xi$ as $t \rightarrow \infty$, which is a contradiction with the fact that $T(t)x \in \Xi^*$ due to the $\mathcal{T}$ -invariance of Ξ^* . The other inclusion is trivial.

Before showing the continuity of k , note that if $z \in \mathcal{A}$ and $k(T(t)z) = k(z)$ for all $t \geq 0$ then $z \in \Xi \cup \Xi^*$. Otherwise, if $z \in \mathcal{A} \setminus (\Xi \cup \Xi^*)$, $\omega(z) \subset \Xi$ and from the definition of k and the

fact that $\omega(z)$ $\mathcal{T}$ -attracts z we have $k(z) = \lim_{t \rightarrow \infty} k(T(t)z) = 0$. Since $k^{-1}(0) = \Xi$, z must belong to Ξ , which is a contradiction.

Now we will to prove the continuity of k . Let $z_0 \in X$ and consider the cases:

Case 1: $z_0 \in \Xi^*$.

Since $l(z) \leq k(z) \leq 1$, for all $z \in X$, given $z_0 \in \Xi^*$ and $z \in X$ we have

$$|k(z) - k(z_0)| = 1 - k(z) \leq 1 - l(z) = |l(z) - l(z_0)|.$$

Therefore the continuity of l imply the continuity of k in z_0 , that is, k is continuous in Ξ^* .

Case 2: $z_0 \in \Xi$.

From the continuity of l in Ξ , given $\varepsilon > 0$, there is a $\delta > 0$ such that $l(\mathcal{O}_\delta(\Xi)) \subset [0, \varepsilon]$. Hence Lemma 15 implies that there exists $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$, concluding that $k(\mathcal{O}_{\delta'}(\Xi)) \subset [0, \varepsilon]$. Therefore k is continuous in Ξ .

Case 3: Continuity of k in $X \setminus (\Xi \cup \Xi^*)$.

Given $z_0 \in X \setminus (\Xi \cup \Xi^*)$, from Lemma 18, we have either

$$\lim_{t \rightarrow \infty} d(T(t)z_0, \Xi) = 0 \quad \text{or} \quad \lim_{t \rightarrow \infty} d(T(t)z_0, \Xi^*) = 0.$$

Assume that $\lim_{t \rightarrow \infty} d(T(t)z_0, \Xi^*) = 0$ and note firstly that $k(z_0) = 1$. Now, given $\varepsilon > 0$, from the continuity of l in Ξ^* there is a neighborhood V of Ξ^* in X such that $l(V) \subset (1 - \varepsilon, 1]$. If $t_0 > 0$ is such that $T(t_0)z_0 \in V$, from the continuity of $T(t_0): X \rightarrow X$, let U be a neighbourhood of z_0 such that $T(t_0)U \subset V$, from which it follows that $k(z) > 1 - \varepsilon$ for all $z \in U$ (for $T(t_0)z \in V$ and then $1 - \varepsilon < l(T(t_0)z) \leq k(z)$). This proves the continuity of k in points z_0 of $X \setminus (\Xi \cup \Xi^*)$ for which $\lim_{t \rightarrow \infty} d(T(t)z_0, \Xi^*) = 0$.

Now, if $z_0 \in X \setminus (\Xi \cup \Xi^*)$ and $\lim_{t \rightarrow \infty} d(T(t)z_0, \Xi) = 0$, it holds that $l(z_0) > 0$. Choose $\delta > 0$ such that $l(\mathcal{O}_\delta(\Xi)) \subset [0, \frac{l(z_0)}{2})$ and, from Lemma 15, there is a $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$. From this, there is a $t_0 > 0$ with the property that $T(t_0)z_0 \in \mathcal{O}_{\delta'}(\Xi)$ and therefore $T(t)z_0 \in \mathcal{O}_\delta(\Xi)$ for all $t \geq t_0$. From the continuity of $T(t_0): X \rightarrow X$, there is a neighborhood U_1 of z_0 in X such that $T(t_0)U_1 \subset \mathcal{O}_{\delta'}(\Xi)$. Then, for all $z \in U_1$ we have that $T(t)z \in \mathcal{O}_\delta(\Xi)$ for all $t \geq t_0$. Finally, from the continuity of l , let U_2 be a neighborhood of z_0 in X such that $l(z) > \frac{l(z_0)}{2}$ for all $z \in U_2$ and write $U := U_1 \cap U_2$, so that for all $z \in U$ it holds that $k(z) = \sup_{0 \leq t \leq t_0} l(T(t)z)$. Reasoning as before we obtain the continuity of k in points z_0 of $X \setminus (\Xi \cup \Xi^*)$ for which $\lim_{t \rightarrow \infty} d(T(t)z_0, \Xi) = 0$.

Let h be the function defined in Lemma 20 and define $f: X \rightarrow \mathbb{R}$ by

$$f(z) = k(z) + h(z), \forall z \in X.$$

Note that f is continuous, nonincreasing along solutions of $\mathcal{T}$, $f^{-1}(0) = \Xi$ and $f^{-1}(1) \cap \mathcal{A} = k^{-1}(1) \cap \mathcal{A} = \Xi^*$. It remains to verify if f satisfies item 3

Finally, let $z \in X$ such that $f(T(t)z) = f(z)$ for all $t \in \mathbb{T}^+$. Since h and k are nonincreasing along solutions, then $h(T(t)z) = h(z)$ and $k(T(t)z) = k(z)$ for all $t \in \mathbb{T}^+$. It follows from $d(T(t)z, \mathcal{A}) \rightarrow 0$ as $t \rightarrow \infty$ that $h(z) = 0$, and hence $z \in \mathcal{A}$. Then, the equality $k(T(t)z) = k(z)$, for all $t \in \mathbb{T}^+$, implies that $z \in \Xi \cup \Xi^*$ which concludes the proof. $\square$

Now we are able to prove the equivalence between gradient and dynamically gradient semigroups.

Theorem 6. Let $\mathcal{T}$ be a semigroup in X with global attractor $\mathcal{A}$ and a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. Then $\mathcal{T}$ is a gradient semigroup with respect to Ξ if, and only if it is a dynamically gradient semigroup with respect to Ξ .

Proof. We already know one implication from Corollary 1. Suppose now that $\mathcal{T}$ is a dynamically gradient semigroup with respect to Ξ reordered in such a way that it is a Morse decomposition for $\mathcal{A}$. Let $\emptyset = A_0 \subset A_1 \subset \dots \subset A_n = \mathcal{A}$ be the sequence of local attractors defined in Theorem 5 and $\emptyset = A_n^* \subset A_{n-1}^* \subset \dots \subset A_0^* = \mathcal{A}$ their corresponding repellers such that for each $j = 1, 2, \dots, n$, we have $\Xi_j = A_j \cap A_{j-1}^*$.

Let h be the function defined in Lemma 20 and k_j be the function constructed in (3.3) for the attractor-repeller pair (A_j, A_j^*) , for each $j = 1, \dots, n$. Define the function $V: X \rightarrow \mathbb{R}$ by

$$V(z) = h(z) + \sum_{j=1}^n k_j(z), \forall z \in X.$$

We will verify that V is a Lyapunov function for the dynamically gradient semigroup relative to Ξ . Note that V is continuous and nonincreasing along solutions.

If $z \in X$ is such that $V(T(t)z) = V(z)$ for all $t \in \mathbb{T}^+$ then, arguing as in the proof of item 3 of Proposition 5, we have $z \in A_j \cup A_j^*$, for each $j = 0, 1, \dots, n$. Hence, from Theorem 5, we have

$$z \in \bigcap_{j=0}^n (A_j \cup A_j^*) = \bigcup_{i=1}^n \Xi_i.$$

This means that V is a Lyapunov function for the dynamically gradient semigroup relative to Ξ and consequently, $\mathcal{T}$ is a gradient semigroup. $\square$

Finally, we will prove the robustness of gradient semigroups, that is, gradient semigroups are stable under perturbation. Since we know that gradient and dynamically gradient are equivalent, by Theorem 6, it is sufficient to show that dynamically gradient semigroups are stable under perturbations.

Theorem 7. Let $\{\mathcal{T}_\eta\}_{\eta \in [0,1]}$ be a family of semigroups in X that is continuous and collectively asymptotically compact at $\eta = 0$. Suppose that

1. $\mathcal{T}_\eta$ has a global attractor $\mathcal{A}_\eta$, for each $\eta \in [0, 1]$ and that $\bigcup_{\eta \in [0,1]} \mathcal{A}_\eta$ is bounded.
2. For each $\eta \in [0, 1]$, the attractor $\mathcal{A}_\eta$ contains a disjoint collection of isolated invariant sets $\Xi_\eta = \{\Xi_{1,\eta}, \dots, \Xi_{n,\eta}\}$ such that $\lim_{\eta \rightarrow 0} d_H(\Xi_{i,\eta}, \Xi_{i,0}) = 0$, for all $1 \leq i \leq n$.
3. There exist $\delta > 0$ and $\eta_0 \in (0, 1]$ such that $\Xi_{i,\eta}$ is the maximal invariant set in $\mathcal{O}_\delta(\Xi_{i,\eta})$, for all $1 \leq i \leq n$ and $0 \leq \eta \leq \eta_0$.
4. $\mathcal{T}_0$ is a dynamically gradient semigroup with global attractor $\mathcal{A}_0$ and disjoint collection of isolated invariant sets $\Xi_0 = \{\Xi_{1,0}, \dots, \Xi_{n,0}\}$.

Then there exists an $\eta_1 > 0$ such that, for all $\eta \in (0, \eta_1]$, $\mathcal{T}_\eta$ is a dynamically gradient semigroup with respect to Ξ_η .

Proof. Note that itens 2 and 3 implies that there exists $\delta > 0$ such that, for sufficiently small η , if a solution ξ_η satisfies $d(\xi_\eta(t), \Xi_{i,0}) \leq \delta$ for all $t \geq t_0$ and for some $t_0 > 0$, then $\xi_\eta(t) \rightarrow \Xi_{i,\eta}$ as $t \rightarrow \infty$.

Now we will prove that for suitably small η , $\mathcal{T}_\eta$ satisfies (G1). We will argue by contradiction. Assume that there is a sequence $\eta_k \rightarrow 0$ and corresponding global solutions ξ_k of $\mathcal{T}_{\eta_k}$ in $\mathcal{A}_{\eta_k}$ such that

$$\limsup_{t \rightarrow \infty} d(\xi_k(t), \bigcup_{i=1}^n \Xi_{i,\eta_k}) > \delta. \quad (3.4)$$

From Lemma 9, there is a subsequence (still denoted by $\{\xi_k\}$) and a global solution $\xi_0: \mathbb{T} \rightarrow X$ of $\mathcal{T}_0$ such that $\xi_k(t) \rightarrow \xi_0(t)$ uniformly in compact subsets of $\mathbb{T}$. By item 3, $\xi_0(t) \rightarrow \Xi_{i,0}$, for some $1 \leq i \leq n$. It follows that, for each $\ell \in \mathbb{N}^*$, there are $t_\ell > 0$ and $k_\ell \in \mathbb{N}$ such that $d(\xi_k(t_\ell), \Xi_{i,0}) < \frac{1}{\ell}$, for each $k \geq k_\ell$. From (3.4), there exists $t'_\ell > t_\ell$ such that $d(\xi_{k_\ell}(t), \Xi_{i,0}) < \delta$ for all $t \in [t_\ell, t'_\ell)$ and $d(\xi_{k_\ell}(t_\ell), \Xi_{i,0}) = \delta$. From the continuity of $\mathcal{T}_0$ and the $\mathcal{T}_0$ -invariance of $\Xi_{i,0}$ we have $t_{\ell'} - t_\ell \xrightarrow{\ell \rightarrow \infty} \infty$. From Lemma 9, taking subsequences if necessary, there is a global solution $\xi_1: \mathbb{R} \rightarrow X$ of $\mathcal{T}_0$ such that $\xi_1(t) = \lim_{\ell \rightarrow \infty} \xi_{k_\ell}(t + t_\ell)$ uniformly in compact subsets of $\mathbb{T}$. Then $d(\xi_1(t), \Xi_{i,0}) \leq \delta$ for all $t \leq 0$ and, consequently, $\xi_1(t) \rightarrow \Xi_{i,0}$ as $t \rightarrow -\infty$. Again by item 3, $\xi_1(t) \rightarrow \Xi_{j,0} \in \Xi_0$ as $t \rightarrow \infty$ with $i \neq j$. From the fact that $\xi_{k_\ell} \rightarrow \xi_1(t)$ uniformly in compact subsets of $\mathbb{R}$ we have, for each $m \in \mathbb{N}$, a time $t_m > 0$ and $k_m \in \mathbb{N}$ such that $d(\xi_k(t_m), \Xi_j) < \frac{1}{m}$ for all $k \geq k_m$. Again, from (3.4), there exists $t'_m > t_m$ such that $d(\xi_{k_m}(t), \Xi_j) < \delta$ for all $t \in [t_m, t'_m)$ and $d(\xi_{k_m}(t'_m), \Xi_j) = \delta$. Proceeding exactly as before we obtain a global solution $\xi_2: \mathbb{R} \rightarrow X$ of $\mathcal{T}_0$ such that $\xi_2(t) \rightarrow \Xi_j$ as $t \rightarrow -\infty$ and $\xi_2(t) \rightarrow \Xi_{r,0} \in \Xi_0$ as $t \rightarrow \infty$ with $r \notin \{i, j\}$. In a finite number of steps we construct a homoclinic structure which contradicts $\mathcal{T}_0$ be a dynamically gradient semigroup. Therefore there exists $\eta_1 > 0$ such that, all global solution ξ_η in $\mathcal{A}_\eta$ with $\eta \leq \eta_1$, must satisfy $\lim_{t \rightarrow \infty} d(\xi_\eta(t), \Xi_{i,0}) = 0$. The proof that there is a $\eta_2 > 0$ such that for all

global solution ξ_η of $\mathcal{T}_\eta$ in $\mathcal{A}_\eta$ with $\eta \leq \eta_2$ we have $\lim_{t \rightarrow -\infty} d(\xi_\eta(t), \Xi_{j,0}) = 0$ is analogue. This proves that for suitably small η , $\mathcal{T}_\eta$ satisfies (G1).

It remains to prove that if η , $\mathcal{T}_\eta$ is suitably small, then satisfies (G2). If this is not the case, there is a collection $\Xi_{1,0}, \dots, \Xi_{p,0}$ in Ξ_0 , a sequence $\eta_k \rightarrow 0$, global solutions ξ_k of $\mathcal{T}_{\eta_k}$ in $\mathcal{A}_{\eta_k}$, and times $t_1^k, \dots, t_p^k$ such that

$$d(\xi_k(0), \Xi_{i,0}) < \frac{1}{k}, \quad d(\xi_k(t_i^k), \Xi_{i+1,0}) < \frac{1}{k}, \quad 1 \leq i \leq p,$$

where $\Xi_{1,0} = \Xi_{p+1,0}$. One can choose a fixed sequence $\Xi_{1,0}, \dots, \Xi_{p,0}$ since there are only a finite number of possible sequences from Ξ_0 . Proceeding as in the proof of (G1) we construct a homoclinic structure for $\mathcal{T}_0$ and arrive at a contradiction. This concludes the proof. $\square$

NON-AUTONOMOUS DYNAMICAL SYSTEMS AND THEIR ATTRACTORS

In this chapter we will introduce the evolution process, which extends the notion of semigroups. For this more general concept, we present the notions of ω -limit set, global attractors, dynamically gradient semigroups, etc. As we mentioned in Chapter 1 (and we will prove in this chapter), we can associate an autonomous system to a semigroup and a non-autonomous system to an evolution process. Geometrically, the autonomous problem (1.3), for which the function g does not depend on time, has a vector field driving the solution which is the same on every time t . Thus, given a point x in the phase space, we can study its evolution just knowing the elapsed time, without knowing the initial time (since the field does not change). But in the non-autonomous case (1.1), the function f depends on t , which means that the vector field changes with time. Consequently, to study the evolution of a point x we need to know, besides the elapsed time, the initial time of the evolution. This shows how difficult it is to understand the dynamics of a non-autonomous problem and why the autonomous problem is a very particular case.

For more details about this chapter the reader may consult (BORTOLAN; CARVALHO; LANGA, 2020), (CARVALHO; LANGA; ROBINSON, 2013) and (CARABALLO; RIVERO, 2010).

4.1 Dynamical Systems and Pullback Attractors

In this section we start to studying the non-autonomous theory, which has great importance to solve our main problem: the cascade system. Also, we will introduce the skew-product semigroup, which is an important tool to understand the dynamics of a non-autonomous problem.

4.1.1 Basic Concepts of non-autonomous Theory

Definition 22 (Evolution Process). A family $\mathcal{S} = \{S(t, s) : t, s \in \mathbb{T} \text{ and } t \geq s\} \subset \mathcal{C}(X)$ is called an **evolution process** in X if satisfies:

1. $S(t, t)x = x, \forall t \in \mathbb{T}, \forall x \in X$.
2. $S(t, s) = S(t, r)S(r, s), \forall t, r, s \in \mathbb{T}, t \geq r \geq s$.
3. The map $\mathcal{P} \times X \ni (t, s, x) \mapsto S(t, s)x \in X$ is continuous, where $\mathcal{P} = \{(t, s) \in \mathbb{T}^2 : t \geq s\}$.

As in semigroup theory, we say that $\mathcal{S}$ is a continuous evolution process if $\mathbb{T} = \mathbb{R}$ and is a discrete evolution process if $\mathbb{T} = \mathbb{Z}$.

Definition 23 (Autonomous and non-autonomous Dynamical System). We say that an evolution process $\mathcal{S} = \{S(t, s) : (t, s) \in \mathcal{P}\}$ is **autonomous** if its evolution depends only on the elapsed time $t - s$, that is, if $S(t, s) = S(t - s, 0)$ for all $(t, s) \in \mathcal{P}$. If an evolution process is not autonomous it is said to be a **non-autonomous dynamical system** or simply a dynamical system.

Remark 9. If $\mathcal{S}$ is an autonomous dynamical system in X , we can obtain a semigroup $\mathcal{T}$ in X defined by $T(t) = S(t, 0)$, for all $t \in \mathbb{T}^+$. On other hand, if $\mathcal{T}$ is a semigroup in X , we can define an autonomous dynamical system in X , given by $S(t, s) = T(t - s)$, for all $(t, s) \in \mathcal{P}$. This means that semigroups are a special case of dynamical system and we can apply the semigroup theory to autonomous dynamical system, which justifies the reason to start this thesis at semigroup theory.

In order to understand the next example, it is worth remembering the following theorem from ODE theory:

Theorem 8. Let $f \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and consider the Cauchy problem

$$\begin{cases} \dot{x} = f(t, x), & t > s \\ x(s) = x_s \in \mathbb{R}^n. \end{cases} \quad (4.1)$$

For each $x_s \in \mathbb{R}^n$ and $s \in \mathbb{R}$, there exist $\tau > s$ and continuously differentiable function $[s, \tau) \ni t \mapsto \xi(t) \in \mathbb{R}^n$ such that $\xi(s) = x_s$ and $\dot{\xi}(t) = f(t, \xi(t))$ for all $t \in (s, \tau)$. Moreover, ξ satisfies:

1. If there is a $\sigma > s$ and continuously differentiable function $\eta : [s, \sigma) \rightarrow \mathbb{R}^n$ such that $\eta(s) = x_s$ and $\dot{\eta} = f(t, \eta(t))$ for all $t \in (s, \sigma)$, then $\xi(t) = \eta(t)$ for all $t \in [0, \min\{\sigma, \tau\})$.
2. There is a $\tau(s, x_s) > s$ and a solution $[s, \tau_{s, x_s}) \ni t \mapsto x(t, s, x_s) \in \mathbb{R}^n$ of (4.1) such that either $\tau(s, x_s) = \infty$ or $\limsup_{t \rightarrow \tau(s, x_s)} \|x(t, s, x_s)\| = \infty$.

3. If $E = \{(t, s, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n : s \leq t \leq \tau(s, x)\}$, the function $E \ni (t, s, x_s) \mapsto x(t, s, x_s) \in \mathbb{R}^n$ is continuous.

Example 2. Let $f \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $\delta > 0$ and $C > 0$ be such that

$$\langle f(t, x), x \rangle < -\delta, \forall \|x\| \geq C, \forall t \in \mathbb{R}, \quad (4.2)$$

and consider the Cauchy problem:

$$\begin{cases} \dot{x} = f(t, x), & t > s \\ x(s) = x_s \in \mathbb{R}^n. \end{cases} \quad (4.3)$$

Note that from (4.2) we have

$$\begin{aligned} (\|x\|^2)' &= 2\langle x, \dot{x} \rangle \\ &= 2\langle x, f(t, x) \rangle < -2\delta, \forall \|x\| \geq C, \forall t \in \mathbb{R}, \end{aligned} \quad (4.4)$$

which guarantees, from conditions 1 and 2 of Theorem 8, that problem (4.3) has a unique solution $x(\cdot, s, x_s)$ in $[s, \infty)$. Thus, we can define operators $S(t, s)$ in $\mathbb{R}^n$ as follows

$$S(t, s)x_0 = x(t, s, x_0), \forall t \geq s, \forall x_0 \in \mathbb{R}^n. \quad (4.5)$$

We will show that the family $\mathcal{S} = \{S(t, s) : t \geq s\}$ is a dynamical system. From condition 3 of Theorem 8, follows that $S(t, s)$ is continuous, for all $t \geq s$, and S satisfies condition 3 of Definition 22. Condition 1 of Definition 22 is clearly satisfied. To show condition 2 of Definition 22, note that given $s \leq \sigma$ both $x(\cdot, \sigma, x(\sigma, s, x_s))$ and $x(\cdot, s, x_s)$ are solutions of

$$\begin{cases} \dot{x} = f(t, x), & t > s \\ x(\sigma) = u(\sigma, s, x_s) \in \mathbb{R}^n. \end{cases} \quad (4.6)$$

Thus $x(\cdot, \sigma, x(\sigma, s, x_s)) = x(\cdot, s, x_s)$ and $S(t, \sigma)x(\sigma, s, x_s) = S(t, s)x_s$, for all $t \geq \sigma \geq s$, by (4.5). Hence

$$S(t, \sigma)S(\sigma, s)x_s = S(t, s)x_s, \forall s \leq t, \forall x_s \in \mathbb{R}^n, \quad (4.7)$$

showing that $\mathcal{S}$ is an evolution process. Moreover, if $f(t, x) = g(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, then both $t \mapsto x(t + \tau, \tau, x_s)$ and $t \mapsto x(t + \sigma, \sigma, x_s)$ are solution of

$$\begin{cases} \dot{x} = g(x), & t > 0 \\ x(0) = x_s \in \mathbb{R}^n. \end{cases} \quad (4.8)$$

Therefore, $\mathcal{T} = \{T(t) : t \geq 0\}$ is a semigroup, where $T(t)x_s = x(t, 0, x_s)$, for all $t \geq 0$ and $x_s \in \mathbb{R}^n$. In fact, from (4.7) we have

$$\begin{aligned} T(t)T(s)x_s &= x(t, 0, x(s, 0, x_s)) = x(t + s, s, x(s, 0, x_s)) = S(t + s, s)S(s, 0)x_s \\ &= S(t + s, 0)x_s = T(t + s)x_s, \forall t, s \geq 0, \forall x_s \in \mathbb{R}^n. \end{aligned} \quad (4.9)$$

It follows from (4.9) that the family $\mathcal{T}$ is a semigroup in $\mathbb{R}^n$.

Proceeding in the autonomous case, we can prove that $\mathcal{T}$ has a global attractor. It is sufficient to show that there is a compact set K that attracts bounded sets of $\mathbb{R}^n$ ($\mathcal{T}$ is asymptotically compact and bounded dissipative). We will show that $K = \{x \in \mathbb{R}^n : \|x\| \leq C\}$ is the compact that attracts bounded subsets of $\mathbb{R}^n$. Let $B \subset \mathbb{R}^n$, be a bounded subset, $x_s \in \bar{B}$ and $\varepsilon > 0$. Note that, from (4.4), if $x_s \in \mathcal{O}_\varepsilon(K)$, then $T(t)x_s \in \mathcal{O}_\varepsilon(K)$, for all $t \in \mathbb{R}$. We know, from (4.4), that

$$\|T(t)x_s\|^2 < -t\delta + R, \forall t \in \mathbb{R}, \text{ whenever } \|T(t)x_s\| \geq C,$$

for some $R > 0$. Hence if $x_s \notin K$, there exists $t_{x_s} > 0$ such that $T(t)x_s \in \mathcal{O}_\varepsilon(K)$, for all $t \geq t_s$.

Since $\mathcal{O}_\varepsilon(K)$ is open, there exists $r_{x_s} > 0$ such that $T(t)B(x_s, r_{x_s}) \subset \mathcal{O}_\varepsilon(K)$, for all $t \geq s$. It follows from the compactness of K that there exists $x_1, \dots, x_n \in \bar{B}$ and $t_0 > 0$ such that $\bar{B} \subset \bigcup_{i=1}^n B(x_i, r_{x_i})$ and for each $i = 1, \dots, n$ and

$$T(t)B(x_i, r_{x_i}) \subset \mathcal{O}_\varepsilon(K), t \geq t_0.$$

This means that $T(t)B \subset K$, for all $t \geq t_0$ and $\mathcal{T}$ has a global attractor.

The following definitions and results generalizes concepts from semigroup theory, in the sense that those concepts extend the concepts of semigroup theory, if the dynamical system is autonomous.

Definition 24 (Invariance). Let $\mathcal{S}$ be an evolution process in X . A family of sets $\{B(t)\}_{t \in \mathbb{T}}$ in X is said to be $\mathcal{S}$ -invariant if $S(t, s)B(s) = B(t)$ for all $(t, s) \in \mathcal{P}$.

Definition 25 (Pullback Attraction). Let $\mathcal{S}$ be an evolution process in X . We say that a set B_0 $\mathcal{S}$ -pullback attracts a set $B \subset X$, at time t if

$$\lim_{s \rightarrow -\infty} d_H(S(t, s)B, B_0) = 0.$$

Definition 26 (Pullback Attractor). Let $\mathcal{S}$ be an evolution process in X . A family $\{A(t)\}_{t \in \mathbb{T}}$ is called a **pullback attractor** for $\mathcal{S}$ if

1. $\bigcup_{t \in \mathbb{T}} A(t)$ is bounded.
2. $A(t)$ is compact in X for each $t \in \mathbb{T}$.
3. $\{A(t)\}_{t \in \mathbb{T}}$ is $\mathcal{S}$ -invariant and $\mathcal{S}$ -pullback attracts bounded subsets of X at time t , for each $t \in \mathbb{T}$.

Remark 10. Frequently, in the literature, condition 1 is not required in the definition of pullback attractor. In that case a condition of minimality is required to ensure uniqueness.

Remark 11. If the pullback attractor exists, then it is unique. Indeed, suppose that $\{\tilde{A}(t)\}_{t \in \mathbb{T}}$ and $\{A(t)\}_{t \in \mathbb{T}}$ are pullback attractors for $\mathcal{S}$, since $B = \cup_{t \in \mathbb{T}} \tilde{A}(t)$ is bounded and $\{A(t)\}_{t \in \mathbb{T}}$ pullback attracts bounded subsets of X at time t , the $\mathcal{S}$ -invariance property of pullback attractors implies that

$$d_H(\tilde{A}(t), A(t)) = d_H(S(t, s)\tilde{A}(s), A(t)) \leq d_H(S(t, s)B, A(t)) \xrightarrow{s \rightarrow -\infty} 0.$$

This implies that $\tilde{A}(t) \subset A(t)$, for each $t \in \mathbb{T}$. The other inclusion follows similarly.

The next result show that pullback attractors extend the definition of global attractor, for the non-autonomous case.

Proposition 6. If $\mathcal{S}$ is an autonomous evolution process associated to the semigroup $\mathcal{T}$, then $\mathcal{S}$ has a pullback attractor $\{A(t)\}_{t \in \mathbb{R}}$ if, and only if $\mathcal{T}$ has a global attractor $\mathcal{A}$. Moreover, in both cases we have $A(t) = \mathcal{A}, \forall t \in \mathbb{R}$.

Proof. Suppose that $\mathcal{S}$ has a pullback attractor $\{A(t)\}_{t \in \mathbb{R}}$. For each $t \in \mathbb{R}$ and $B \subset X$ bounded, we have

$$\lim_{s \rightarrow -\infty} d_H(S(t, s)B, A(t)) = 0 \Rightarrow \lim_{s \rightarrow -\infty} d_H(T(t - s)B, A(t)) = 0 \Rightarrow \lim_{s \rightarrow \infty} d_H(T(s)B, A(t)) = 0.$$

Since $A(t)$ is compact and attract bounded sets, it follows that $\mathcal{T}$ is bounded dissipative and asymptotically compact, that is, $\mathcal{T}$ has a global attractor $\mathcal{A} = \omega(A(t))$. Note that the family $\{\omega(A(t))\}_{t \in \mathbb{R}}$ also pullback attracts bounded sets, it is bounded because $\bigcup_{t \in \mathbb{R}} \omega(A(t)) \subset \bigcup_{t \in \mathbb{R}} A(t)$, and $\omega(A(t))$ is compact, for each $t \in \mathbb{R}$, implying that $A(t) = \omega(A(t)) = \mathcal{A}$, for all $t \in \mathbb{R}$.

On other hand, if $\mathcal{T}$ has a global attractor $\mathcal{A}$, define $A(t) = \mathcal{A}$, for each $t \in \mathbb{R}$. Thus, the family $\{A(t)\}_{t \in \mathbb{R}}$ is $\mathcal{S}$ -invariant, $A(t)$ is compact for each $t \in \mathbb{R}$, $\bigcup_{t \in \mathbb{R}} A(t)$ is bounded and for each bounded set $B \subset X$ and $t \in \mathbb{R}$ we have

$$\lim_{s \rightarrow -\infty} d_H(S(t, s)B, A(t)) = \lim_{s \rightarrow -\infty} d_H(T(t - s)B, \mathcal{A}) = \lim_{s \rightarrow \infty} d_H(T(s)B, \mathcal{A}) = 0.$$

This concludes the proof. $\square$

Definition 27 (Global Solution). Let $\mathcal{S}$ be an evolution process in X . We say that a function $\xi: \mathbb{T} \rightarrow X$ is a **global solution for $\mathcal{S}$** if is continuous and $S(t, s)\xi(s) = \xi(t)$ for all $(t, s) \in \mathcal{P}$, that is, if the family $\{\xi(t)\}_{t \in \mathbb{R}}$ is $\mathcal{S}$ -invariant. If $\xi(\mathbb{T})$ is bounded in X , we say that ξ is a **bounded global solution**.

Theorem 9. Let $\mathcal{S}$ be an evolution process in X . If $\mathcal{S}$ has a pullback attractor $\{A(t)\}_{t \in \mathbb{R}}$ then

$$A(t) = \{\xi(t): \xi \text{ is a bounded global solution of } \mathcal{S}\}, \forall t \in \mathbb{T}.$$

Proof. From the $\mathcal{S}$ -invariance of $\{A(t)\}_{t \in \mathbb{R}}$, proceeding as in Theorem 1, for each $x \in A(t_0)$, we can construct a global solution ξ such that $\xi(t_0) = x$ and $\xi(t) \in A(t)$ for all $t \in \mathbb{T}$.

On the other hand, if ξ is a bounded global solution of $\mathcal{S}$ then, for each $n \in \mathbb{N}^*$, there exists $t_n \leq t$ such that $d_H(S(t, s)B, A(t)) \leq \frac{1}{n}$ for all $s \leq t_n$. In particular

$$d_H(S(t, t_n)\xi(t_n), A(t)) = d_H(\xi(t), A(t)) \leq \frac{1}{n} \quad \forall n \in \mathbb{N}^*.$$

Therefore $\xi(t) \in A(t)$ and the proof is complete. $\square$

Definition 28 (ω -limit). Let $\mathcal{S}$ an evolution process in X . Given a subset $B \subset X$ and $t \in \mathbb{T}$, we define the **pullback ω -limit set** of B at time t as the set

$$\omega(B, t) = \bigcap_{\sigma \leq t} \overline{\bigcup_{s \leq \sigma} S(t, s)B}.$$

Remark 12. Note that $\omega(B, t)$ is closed for each $t \in \mathbb{T}$. Moreover, proceeding as in Proposition 1, we have the characterization

$$\begin{aligned} \omega(B, t) = \{y \in X : \text{there are sequences } \{s_k\}_{k \in \mathbb{N}} \text{ in } (-\infty, t], s_k \rightarrow -\infty, \\ \text{and } \{x_k\}_{k \in \mathbb{N}} \text{ in } B, \text{ such that } y = \lim_{k \rightarrow \infty} S(t, s_k)x_k\}. \end{aligned} \quad (4.10)$$

Theorem 10. Let $\mathcal{S}$ an evolution process in X and $K \subset X$ compact such that K $\mathcal{S}$ -pullback attracts bounded subsets of X at time t , for each $t \in \mathbb{T}$. Then, $\mathcal{S}$ has pullback attractor, given by $\{A(t) = \omega(K, t)\}_{t \in \mathbb{T}}$, and $\overline{\bigcup_{t \in \mathbb{T}} A(t)}$ is compact.

Proof. It follows from Proposition 2 and Remark 12 that $\omega(K, t)$ is nonempty and contained in K , for all $t \in \mathbb{T}$. Thus $\overline{\bigcup_{t \in \mathbb{R}} \omega(K, t)}$ is a closed subset of K , that is, $\overline{\bigcup_{t \in \mathbb{R}} \omega(K, t)}$ is compact.

Now we will prove the $\mathcal{S}$ -invariance of $\{\omega(K, t)\}_{t \in \mathbb{T}}$. Let $(t, s) \in \mathcal{P}$. We will only prove the inclusion $S(t, s)\omega(K, s) \subset \omega(K, t)$, since the other one is trivial. If $x \in \omega(K, t)$, there are sequences $\{s_k\}_{k \in \mathbb{N}}$ in $(-\infty, t]$, with $s_k \rightarrow -\infty$, and $\{x_k\}_{k \in \mathbb{N}}$ in K such that $S(t, s_k)x_k \rightarrow x$. We can assume $s_k \leq s$ for all $k \in \mathbb{N}$. Since K $\mathcal{S}$ -pullback attracts itself at time s , $\{S(s, s_k)x_k\}_{k \in \mathbb{N}}$ has a convergent subsequence (which we denote the same) with limit, say, y . Clearly, $y \in \omega(K, s)$ and from the continuity of $S(t, s)$ we have $S(t, s)y = x$. This completes the proof that $\{\omega(K, t)\}_{t \in \mathbb{T}}$ is $\mathcal{S}$ -invariant.

Finally, we will prove that $\omega(K, t)$ $\mathcal{S}$ -pullback attracts bounded subsets of X at time t . In fact, if this is not the case, there is a bounded subset $B \subset X$ that is not $\mathcal{S}$ -pullback attracted by $\omega(K, t)$ at time t then there would exist $\varepsilon > 0$ and sequences $\{s_k\}_{k \in \mathbb{N}}$ in $(-\infty, t]$, with $s_k \rightarrow -\infty$, and $\{x_k\}_{k \in \mathbb{N}}$ in B such that $d_H(S(t, s_k)x_k, \omega(K, t)) \geq \varepsilon$ for all k . Since K is compact and $\mathcal{S}$ -pullback attracts B , $\{S(t, s_k)x_k\}_{k \in \mathbb{N}}$ must have a convergent subsequence (Proposition 2) with limit in $\omega(K, t)$ which is a contradiction. Hence $\omega(K, t)$ $\mathcal{S}$ -pullback attracts bounded subsets of X , and the proof is complete. $\square$

4.1.2 Skew-Product

Now we are going to introduce some modern concepts that help us to understand the asymptotical behavior of the solutions of a non-autonomous equation

$$\begin{cases} \dot{x} = f(t, x), & t > s \\ x(s) = x_s \in \mathbb{R}^n. \end{cases} \quad (4.11)$$

To understand such behavior, we have to study how the vector field f evolves on time. To do this, consider $\mathcal{D}$ the set of continuous functions $f: \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for each bounded subset $B \subset \mathbb{R}^n$ there exist a constant $L_f(B) > 0$ and an increasing continuous function $w_{f,B}: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $w_{f,B}(0) = 0$ such that

$$\|f(t_1, x_1) - f(t_2, x_2)\| \leq w_{f,B}(|t_1 - t_2|) + L_f(B)\|x_1 - x_2\|, \forall t_1, t_2 \in \mathbb{R}^+, \forall x_1, x_2 \in B. \quad (4.12)$$

In $\mathcal{D}$ we consider the metric ρ of the uniform convergence in compact subsets of $\mathbb{R}^+ \times \mathbb{R}^n$. Suppose, in addition, that there exist $M > 0$ and $\delta > 0$ such that

$$\langle f(t, x), x \rangle \leq -\delta, \forall t \in \mathbb{R}^+, \forall \|x\| > M. \quad (4.13)$$

Conditions (4.12) and (4.13) guarantees that (4.11) has a solution in $[s, \infty)$. Since we want to understand the behavior of the field f in (4.11), define Σ the closure (in the metric ρ) of the set $\{f(\cdot + s) : s \geq 0\}$ and θ the semigroup in Σ given by $\theta t(\sigma) = \sigma(t + \cdot)$. It follows from (4.12) and Arzela-Áscoli that Σ is a compact set and consequently θ has and global attractor, since θ is bounded dissipative and asymptotically compact.

For each $\sigma \in \Sigma$ consider the problem

$$\begin{cases} \dot{x} = \sigma(t, x), & t > 0 \\ x(0) = x_0 \in X, \end{cases} \quad (4.14)$$

whose solution at time t is $\mathcal{K}(t, \sigma)x_0$. Then we have a family of maps $\mathcal{K} = \{\mathcal{K}(t, \sigma) : t \geq 0, \sigma \in \Sigma\}$ that satisfies (by the existence and uniqueness of solutions)

1. $\mathcal{K}(0, \sigma)x = x$, for all $x \in X$ and $\sigma \in \Sigma$.
2. $\mathcal{K}(t + s, \sigma) = \mathcal{K}(t, \theta(s)\sigma)\mathcal{K}(s, \sigma)$, for all $t, s \in \mathbb{T}^+$ and $\sigma \in \Sigma$.
3. The map $\mathbb{T}^+ \times \Sigma \times X \ni (t, \sigma, x) \mapsto \mathcal{K}(t, \sigma)x \in X$ is continuous.

Note that if there exists a set $\mathcal{A}$, that we will call uniform attractor, such that for each $B \subset X$ bounded satisfies

$$\limsup_{t \rightarrow \infty} \sup_{\sigma \in \Sigma} \sup_{b \in B} d_H(\mathcal{K}(t, \sigma)b, \mathcal{A}) = 0,$$

then $\mathcal{A}$ will be the set where all solutions of (4.11) approach when t goes to $+\infty$.

Now that we know the geometry intuition, we will introduce the abstract concepts.

Definition 29. Let X and Σ be metric spaces (which we will denote both metrics, that is different in general, by d) and $\Theta = \{\Theta(t) : t \in \mathbb{T}^+\} \subset \mathcal{C}(\Sigma)$ be a semigroup in Σ . We say that Θ is a **driving semigroup** and $\mathcal{K} = \{\mathcal{K}(t, \sigma) \in \mathcal{C}(X) : t \in \mathbb{T}^+, \sigma \in \Sigma\}$ is a **cocycle** relative to Θ if satisfies

1. $\mathcal{K}(0, \sigma)x = x$, for all $x \in X$ and $\sigma \in \Sigma$.
2. $\mathcal{K}(t+s, \sigma) = \mathcal{K}(t, \Theta(s)\sigma)\mathcal{K}(s, \sigma)$, for all $t, s \in \mathbb{T}^+$ and $\sigma \in \Sigma$.
3. The map $\mathbb{T}^+ \times \Sigma \times X \ni (t, \sigma, x) \mapsto \mathcal{K}(t, \sigma)x \in X$ is continuous.

Property 2 is called cocycle property. Consider the metric space $\mathbb{X} = X \times \Sigma$, with the product metric. We define the **skew-product semigroup** $\Pi = \{\Pi(t) : t \in \mathbb{T}^+\} \subset \mathcal{C}(\mathbb{X})$ by

$$\Pi(t)(x, \sigma) = (\mathcal{K}(t, \sigma)x, \Theta(t)\sigma) \quad \text{for } t \in \mathbb{T}^+ \text{ and } (x, \sigma) \in \mathbb{X}.$$

Note that the skew-product is a semigroup indeed. It is easy to see that Π satisfies properties 1 and 3 of Definition 1. Using the semigroup property of Θ and the cocycle property of $\mathcal{K}$, we have

$$\begin{aligned} \Pi(t+s)(x, \sigma) &= (\mathcal{K}(t+s, \sigma)x, \Theta(t+s)\sigma) \\ &= (\mathcal{K}(t, \Theta(s)\sigma)\mathcal{K}(s, \sigma)x, \Theta(t)\Theta(s)\sigma) = \Pi(t)\Pi(s)(x, \sigma), \end{aligned}$$

which implies that Π is actually a semigroup.

Example 3. Let X and Σ be metric spaces, Θ be a semigroup in Σ and $\mathcal{K}$ be a cocycle relative to Θ . If $\eta : \mathbb{T} \rightarrow \Sigma$ is a global solution for Θ , the family $\mathcal{S}_\eta = \{S_\eta(t, s) : t \geq s\}$ defined by

$$S_\eta(t, s) = \mathcal{K}(t-s, \eta(s)), \forall (t, s) \in \mathcal{P},$$

is a dynamical system in X . It is clear that S_η satisfies conditions 1 and 3. To show condition 2, we use the cocycle property:

$$\begin{aligned} S_\eta(t, s)S_\eta(s, r) &= \mathcal{K}(t-s, \eta(s))\mathcal{K}(s-r, \eta(r)) = \mathcal{K}(t-s, \Theta(s-r)\eta(r))\mathcal{K}(s-r, \eta(r)) \\ &= \mathcal{K}(t-r, \eta(r)) = S_\eta(t, r), \forall r, s, t \in \mathbb{T}, r \leq s \leq t. \end{aligned}$$

Remark 13. When Σ is a unitary set $\Sigma = \{\sigma_0\}$, we have $\Theta(t)\sigma_0 = \sigma_0$ for all $t \in \mathbb{T}^+$, and defining

$$T(t)x = \mathcal{K}(t, \sigma_0)x, \text{ for all } t \in \mathbb{T}^+ \text{ and } x \in X, \quad (4.15)$$

the skew-product semigroup Π is given by $\Pi(t)(x, \sigma_0) = (T(t)x, \sigma_0)$, $t \in \mathbb{T}^+$. Therefore, the family $\mathcal{T} = \{T(t) : t \in \mathbb{T}^+\}$ defines a semigroup in X . This is the situation that arises when we consider an autonomous differential equation. For a non-autonomous differential equation, Σ will not be a unitary set.

As in chapter 2, we want to ensure conditions to the existence of a global attractor $\mathbb{A}$ of Π . If Π is bounded dissipative, there are bounded subsets B_0 in X and S_0 in Σ such that, for bounded subsets $B \subset X$ and $S \subset \Sigma$ there exists $t_0 = t_0(B, S) \in \mathbb{T}^+$ such that $\Pi(t)(B \times S) \subset B_0 \times S_0$ for all $t \geq t_0$. This means that Π is bounded dissipative if, and only if, for any bounded subset $S \subset \Sigma$ there exists $t_0 = t_0(S) \in \mathbb{T}^+$ such that $\Theta(t)S \subset S_0$ for all $t \geq t_0$, and also that for bounded subsets $B \subset X$ and $S \subset \Sigma$ there exists $t_0 = t_0(B, S) \in \mathbb{T}^+$ such that $\mathcal{K}(t, \sigma)B \subset B_0$, for all $t \geq t_0$ and $\sigma \in S$.

Moreover, from Proposition 3, Π is asymptotically compact (and bounded dissipative) if, and only if, given a sequence $\{t_n\}_{n \in \mathbb{N}}$ in $\mathbb{T}^+$ such that $t_n \rightarrow \infty$, and bounded sequences $\{x_n\}_{n \in \mathbb{N}}$ in X and $\{\sigma_n\}_{n \in \mathbb{N}}$ in Σ we have

1. $\{\Theta(t_n)\sigma_n\}$ has a convergent subsequence in Σ .
2. $\{\mathcal{K}(t_n, \sigma_n)x_n\}$ has a convergent subsequence in X .

Thus, the following definitions are expected.

Definition 30. Let X and Σ be metric spaces, Θ be a semigroup in Σ and $\mathcal{K}$ be a cocycle relative to Θ . We say that $\mathcal{K}$ bounded dissipative if there is a bounded subset B_0 of X such that, for bounded subsets $B \subset X$ and $S \subset \Sigma$, there exists $t_0 = t_0(B, S) \in \mathbb{T}^+$ such that $\mathcal{K}(t, \sigma)B \subset B_0$, for all $t \geq t_0$ and $\sigma \in S$.

Definition 31. Let X and Σ be metric spaces, Θ be a semigroup in Σ and $\mathcal{K}$ be a cocycle relative to Θ . We say that $\mathcal{K}$ is asymptotically compact if given a sequence $\{t_n\}$ in $\mathbb{R}^+$ with $t_n \xrightarrow{n \rightarrow \infty} \infty$, and bounded sequences $\{x_n\}$ in X and $\{\sigma_n\}$ in Σ , the sequence $\{\mathcal{K}(t_n, \sigma_n)x_n\}$ has a convergent subsequence in X .

Theorem 11. Let X and Σ be metric spaces, Θ be a semigroup in Σ , $\mathcal{K}$ be a cocycle relative to Θ and Π be the associated skew-product semigroup in $\mathbb{X} = X \times \Sigma$. Then Π has a global attractor $\mathbb{A}$ in $\mathbb{X}$ if and only if

1. Θ has a global attractor $\mathcal{A}$ in Σ .
2. $\mathcal{K}$ is bounded dissipative.
3. $\mathcal{K}$ is asymptotically compact.

It follows directly from Theorems 10 and 11 the following:

Corollary 2. Let X and Σ be metric spaces and consider $\mathbb{X} = X \times \Sigma$ with the product metric. Suppose that Θ is a driving semigroup in Σ and $\mathcal{K}$ is a cocycle relative to Θ . If the associated skew-product semigroup Π has a global attractor $\mathbb{A}$ in $\mathbb{X}$, then for any bounded global solution $\eta: \mathbb{R} \rightarrow \Sigma$ of Θ , the evolution process $\mathcal{S}_\eta = \{S_\eta(t, s) = \mathcal{K}(t - s, \eta(s)): t \geq s\}$ has a pullback attractor $\{A_\eta(t)\}_{t \in \mathbb{R}}$ and $\overline{\cup_{t \in \mathbb{T}} A_\eta(t)} \subset \pi_X(\mathbb{A})$, where π_X is the projection in X .

Now we can characterize the global attractor of the skew-product semigroup.

Theorem 12. Let (X, d_X) , (Σ, d_Σ) be metric spaces and consider $\mathbb{X} = X \times \Sigma$ with the product metric. Consider also a semigroup Θ in Σ and a cocycle $\mathcal{K}$ relative to Θ . If the associated skew-product semigroup Π has a global attractor $\mathbb{A}$ in $\mathbb{X}$, then the driving semigroup Θ has a global attractor $\mathcal{S}$ in Σ . If $\eta: \mathbb{R} \rightarrow \mathcal{S}$ is a bounded global solution for Θ then the evolution process $\mathcal{S}_\eta = \{S_\eta(t, s): t \geq s\}$, given by $S_\eta(t, s) = \mathcal{K}(t - s, \eta(s))$, possesses a pullback attractor $\{A_\eta(t)\}_{t \in \mathbb{R}}$ with the property that $A_\eta(t) = \{x \in X: (x, \eta(t)) \in \mathbb{A}\}$. Moreover,

$$\mathbb{A} = \bigcup_{\eta} \bigcup_{t \in \mathbb{R}} A_\eta(t) \times \{\eta(t)\},$$

where the first union is taken over all bounded global solutions η of Θ .

Proof. Note that a global bounded solution for Π is a continuous function $(x(\cdot), \eta(\cdot)): \mathbb{R} \rightarrow \mathbb{A}$ that satisfies

$$(x(t+s), \eta(t+s)) = \Pi(t)(x(s), \eta(s)) = (\mathcal{K}(t, \eta(s))x(s), \Theta(t)\eta(s)),$$

for all $t \geq 0$ and $s \in \mathbb{R}$. Thus $x(t+s) = \mathcal{K}(t, \eta(s))x(s) = S_\eta(t+s, s)x(s)$ and $\Theta(t)\eta(s) = \eta(t+s)$, for all $t \geq 0$ and $s \in \mathbb{R}$. It follows that $x: \mathbb{R} \rightarrow X$ is a bounded global solution for $\mathcal{S}_\eta$ and η is a bounded global solution for Θ . Hence, any point of $\mathbb{A}$ is of the form $(x(t), \eta(t))$ where $x(t) \in A_\eta(t)$ and $\eta: \mathbb{R} \rightarrow \Sigma$ is a bounded global solution for Θ and therefore

$$\mathbb{A} \subset \bigcup_{\eta} \bigcup_{t \in \mathbb{R}} A_\eta(t) \times \{\eta(t)\},$$

where the first union is taken over all global bounded solutions η of Θ .

On the other hand, if $\eta: \mathbb{R} \rightarrow \Sigma$ is a bounded global solution for Θ and $x_0 \in A_\eta(t_0)$, there exists global solution $x: \mathbb{R} \rightarrow X$ of $\mathcal{S}_\eta$ such that $x(t_0) = x_0$ and $x(t) \in A_\eta(t)$ for all $t \in \mathbb{R}$. Since $\bigcup_{t \in \mathbb{R}} A_\eta(t)$ is bounded, we have $x: \mathbb{R} \rightarrow X$ a bounded global solution of $\mathcal{S}_\eta$, and consequently $(x(\cdot), \eta(\cdot)): \mathbb{R} \rightarrow \mathbb{X}$ is a bounded global solution of Π , showing that $(x(t), \eta(t)) \in \mathbb{A}$ for all $t \in \mathbb{R}$. Then, in particular, $(x_0, \eta(t_0)) \in \mathbb{A}$ and

$$\bigcup_{\eta} \bigcup_{t \in \mathbb{R}} A_\eta(t) \times \{\eta(t)\} \subset \mathbb{A},$$

where the first union is taken over all bounded global solutions η for Θ , which completes the proof. $\square$

The following definition is fundamental to understand the asymptotic behavior of solutions from (4.11).

Definition 32. Let (X, d_X) , (Σ, d_Σ) be metric spaces and $\mathbb{X} = X \times \Sigma$ with the product metric. Consider Θ a semigroup in Σ and $\mathcal{K}$ a cocycle relative to Θ . The **uniform attractor** for the

cocycle $\mathcal{K}$ is the minimal closed subset $\mathcal{A}$ of X such that for each bounded subset B of X and each bounded subset Γ of Σ we have

$$\limsup_{t \rightarrow \infty} \sup_{\sigma \in \Gamma} d(\mathcal{K}(t, \sigma)B, \mathcal{A}) = 0,$$

where the minimality condition means that if C is any closed subset of X such that

$$\limsup_{t \rightarrow \infty} \sup_{\sigma \in \Gamma} d(\mathcal{K}(t, \sigma)B, C) = 0,$$

for each bounded subset B of X and each bounded subset Γ of Σ , then we have $\mathcal{A} \subset C$.

From Theorem 11 the following result holds:

Theorem 13. Let (X, d_X) , (Σ, d_Σ) be metric spaces and $\mathbb{X} = X \times \Sigma$ with the product metric. Consider Θ a semigroup in Σ and $\mathcal{K}$ be a cocycle relative to Θ . If Θ has a global attractor $\mathcal{S}$ in Σ and $\mathcal{K}$ is bounded dissipative and asymptotically compact, then the cocycle $\mathcal{K}$ has a compact uniform attractor $\mathcal{A}$, and if $\mathbb{A}$ is the global attractor for the associated skew-product semigroup Π then $\mathcal{A} = \pi_X(\mathbb{A})$. Furthermore

$$\mathcal{A} = \bigcup_{\eta} A_{\eta}(0),$$

where the union is taken over all bounded global solutions η for Θ .

Proof. It follows from Theorem 11 and the fact that translations of global solutions of Θ is also a global solution. $\square$

From Theorem 13 we can conclude that all solutions of (4.11) will converge to the uniform attractor $\mathcal{A}$ when $t \rightarrow \infty$.

4.2 Dynamically Gradient Evolution Process

As in chapter 3, we have to introduce some new concepts, very similar to the concepts of dynamically semigroup theory, that enable us to define a dynamically gradient dynamical system and prove their structure stability.

Definition 33 (Isolated $\mathcal{S}$ -invariant Family). Let $\mathcal{S}$ be a dynamical system in X with a pullback attractor $\{A(t)\}_{t \in \mathbb{T}}$. A family $\hat{E} = \{E(t)\}_{t \in \mathbb{R}}$, with $E(t) \subset A(t)$ for each $t \in \mathbb{R}$, is called an **isolated $\mathcal{S}$ -invariant family** if $\{E(t)\}_{t \in \mathbb{R}}$ is $\mathcal{S}$ -invariant and if there exists $r > 0$ such that if $\xi: \mathbb{R} \rightarrow X$ is a global solution of $\mathcal{S}$ with $\xi(t) \in \mathcal{O}_r(E(t))$ for all $t \in \mathbb{T}$, then $\xi(t) \in E(t)$ for all $t \in \mathbb{T}$.

Definition 34 (Disjoint Collection of $\mathcal{S}$ -invariant Families). Let $\mathcal{S}$ be a dynamical system in X with a pullback attractor $\{A(t)\}_{t \in \mathbb{T}}$. A collection $\mathfrak{E} = \{\hat{E}_1, \dots, \hat{E}_n\}$ is called a **disjoint collection of isolated $\mathcal{S}$ -invariant families** if $\hat{E}_i$ is an isolated $\mathcal{S}$ -invariant family for $1 \leq i \leq n$, and there exists $\delta > 0$ such that $\mathcal{O}_\delta(E_j(t)) \cap \mathcal{O}_\delta(E_k(t)) = \emptyset$ for each $1 \leq j < k \leq n$ and all $t \in \mathbb{T}$.

Definition 35 (Homoclinical Structure). Let $\mathcal{S}$ be an evolution process in X with a pullback attractor $\hat{A}$ and a disjoint collection of isolated $\mathcal{S}$ -invariant families $\mathfrak{E} = \{\hat{E}_1, \dots, \hat{E}_n\}$. A **homoclinic structure** in $\hat{A}$ is a subcollection $\{\hat{E}_{i_1}, \dots, \hat{E}_{i_k}\}$ of $\mathfrak{E}$ together with global solutions $\{\xi_1, \dots, \xi_k\}$ of $\mathcal{S}$ such that

$$\lim_{t \rightarrow -\infty} d(\xi_j(t), E_{i_j}(t)) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} d(\xi_j(t), E_{i_{j+1}}(t)) = 0,$$

for $1 \leq j \leq k$, with $\hat{E}_{i_{k+1}} = \hat{E}_{i_1}$.

Definition 36 (Dynamically Gradient Dynamical System). Let $\mathcal{S}$ be a dynamical system in X with a pullback attractor $\hat{A}$ which contains a disjoint collection of isolated $\mathcal{S}$ -invariant families $\mathfrak{E} = \{\hat{E}_1, \dots, \hat{E}_n\}$. We say that $\mathcal{S}$ is a **dynamically gradient dynamical system** relative to $\mathfrak{E}$ if satisfies:

(DGDS1) For every bounded global solution $\xi: \mathbb{T} \rightarrow X$ of $\mathcal{S}$ there exist $1 \leq i, j \leq n$ such that

$$\lim_{t \rightarrow -\infty} d(\xi(t), E_i(t)) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} d(\xi(t), E_j(t)) = 0.$$

(DGDS2) There are no homoclinic structures in $\hat{A}$.

Now that we have defined a dynamically gradient dynamical system, we must show that, as in the autonomous case, they are stable under perturbations. To this end, we have to formalize the idea of a family $\{\mathcal{S}_\eta\}_{\eta \in [0,1]}$ of dynamical systems being close to another dynamical system $\mathcal{S}$.

Definition 37. The family $\{\mathcal{S}_\eta\}_{\eta \in [0,1]}$ of evolution processes is said to be:

1. **collectively asymptotically compact** at $\eta = \eta_0 \in [0, 1]$ if given sequences $\eta_n \rightarrow \eta_0$, $\{x_n\}_{n \in \mathbb{N}}$ bounded in X , $t_n \rightarrow \infty$ and $\{\tau_n\}_{n \in \mathbb{N}}$ in $\mathbb{T}$, then the sequence $\{S_{\eta_n}(t_n + \tau_n, \tau_n)x_n\}_{n \in \mathbb{N}}$ is precompact.
2. **continuous** at $\eta = \eta_0$ if for each compact subset K of X and $T > 0$ we have

$$\sup_{x \in K} \sup_{\tau \in [0, T]} d(S_\eta(t, t - \tau)x, S_{\eta_0}(t, t - \tau)x) \xrightarrow{\eta \rightarrow \eta_0} 0, \forall t \in \mathbb{T}.$$

Lemma 21. Let $\{\mathcal{S}_\eta\}_{\eta \in [0,1]}$ be a family of dynamical system in X , which is continuous at $\eta = 0$. Let $\eta_n \rightarrow 0$, $a_n, b_n \rightarrow \infty$, and set $\mathbb{J}_n = [-a_n, b_n] \cap \mathbb{T}$. Let $\xi_n: \mathbb{J}_n \rightarrow X$ be a solution of $\mathcal{S}_{\eta_n}$ for each $n \in \mathbb{N}$, and suppose that:

1. For some $\tau_0 \in \mathbb{T}$, there exists $n_0 = n_0(t) \in \mathbb{N}$ such that $\bigcup_{n \geq n_0(t)} \xi_n(t)$ is precompact, for each $t \in (-\infty, \tau_0] \in \mathbb{T}$.
2. $\bigcup_{n \in \mathbb{N}} \xi_n(\mathbb{J}_n)$ is bounded.

Then there exist a subsequence $\{n_k\}_{k \in \mathbb{N}}$ of $\mathbb{N}$ and a bounded global solution $\xi_0: \mathbb{T} \rightarrow X$ of $\mathcal{S}_0$ such that $\xi_{n_k} \rightarrow \xi_0$ uniformly on compact subintervals of $\mathbb{T}$.

Proof. This proof will be very similar to the proof of Lemma 9. We can suppose, without loss of generality that $\tau_0 = 0$. As a first step, we find a subsequence $n_{0,k}$ of $\mathbb{N}$ such that $\xi_{n_{0,k}}(0) \xrightarrow{k \rightarrow \infty} x_0$ for some $x_0 \in X$. Now we define $\xi_0: [0, \infty) \cap \mathbb{T} \rightarrow X$ by $\xi_0(t) = S_0(t, 0)x_0$, for $t \in \mathbb{T}^+$. Then

$$\xi_0(t) = S_0(t, 0)x_0 = \lim_{k \rightarrow \infty} S_{\eta_{n_{0,k}}}(t, 0)\xi_{n_{0,k}}(0) = \lim_{k \rightarrow \infty} \xi_{n_{0,k}}(t),$$

with the convergence being uniform in compact subsets of $\mathbb{R}$ and, in particular, $\xi_0([0, \infty) \cap \mathbb{T}) \subset \overline{\bigcup_{n \in \mathbb{N}} \xi_n(\mathbb{J}_n)}$.

Suppose that we have already constructed a subsequence $\{n_{j,k}\}_{k \in \mathbb{N}}$ of $\{n_{j-1,k}\}_{k \in \mathbb{N}}$ for $0 \leq j \leq m-1$ such that $\xi_{n_{m-1,k}}(t) \xrightarrow{k \rightarrow \infty} \xi_0(t)$, uniformly for t on compact subintervals of the set $[-m+1, \infty) \cap \mathbb{T}$, where $\xi_0: [-m+1, \infty) \cap \mathbb{T} \rightarrow X$ is a solution of $\mathcal{S}_0$ that lies in $\overline{\bigcup_{n \in \mathbb{N}} \xi_n(\mathbb{J}_n)}$.

Now find a subsequence $n_{m,k}$ such that $\xi_{n_{m,k}}(-m) \xrightarrow{k \rightarrow \infty} x_m$ for some $x_m \in X$, and extend the definition of ξ_0 to $[-m, \infty) \cap \mathbb{T}$ by setting $\xi_0(s) = S_0(s, -m)x_m$ for $s \in [-m, -(m-1)) \cap \mathbb{T}$. Then

$$\begin{aligned} S_0(-m+1, -m)x_m &= \lim_{k \rightarrow \infty} S_{\eta_{n_{m,k}}}(-m+1, -m)\xi_{n_{m,k}}(-m) \\ &= \lim_{k \rightarrow \infty} \xi_{n_{m,k}}(-m+1) = \xi_0(-m+1), \end{aligned}$$

and so $\xi_0: [-m, \infty) \cap \mathbb{T}$ is a solution of $\mathcal{S}_0$. Clearly $\xi_{n_{m,k}}(t) \rightarrow \xi_0(t)$ uniformly for t in compact subintervals of $[-m, \infty) \cap \mathbb{T}$, and $\xi_0([-m, \infty) \cap \mathbb{T}) \subset \overline{\bigcup_{n \in \mathbb{N}} \xi_n(\mathbb{J}_n)}$.

Finally, if we let $n_k = n_{k,k}$, the sequence ξ_{n_k} satisfies the proper properties. $\square$

A direct consequence of the Lemma 21 is the following corollary.

Corollary 3. Let $\{\mathcal{S}_\eta\}_{\eta \in [0,1]}$ be a family of dynamical systems continuous at $\eta = 0$. Suppose that $\mathcal{S}_\eta$ has a pullback attractor $\hat{A}_\eta$, for each $\eta \in [0, 1]$, and S_0 is an autonomous dynamical system, that is, $S_0(t, s) = T(t - s)$, $t \geq s$, where $\mathcal{T} = \{T(t): t \in \mathbb{T}^+\}$ is a semigroup with a global attractor $\mathcal{A}$. Also, suppose that $\overline{\bigcup_{\eta \in [0,1]} \bigcup_{t \in \mathbb{T}} A_\eta(t)}$ is compact.

If $\eta_k \rightarrow 0$ and, for each $k \in \mathbb{N}$, ξ_k is a global solution of $\mathcal{S}_{\eta_k}$ in $\hat{A}_{\eta_k}$, then there exists a subsequence $\{\xi_{k_j}\}_{j \in \mathbb{N}}$ of $\{\xi_k\}_{k \in \mathbb{N}}$ that converges uniformly in bounded intervals of $\mathbb{T}$ to a global solution ξ of $\mathcal{T}$ with $\xi(\mathbb{T}) \subset \mathcal{A}$.

Now we have the necessary tools to prove our main result.

Theorem 14. Let $\{\mathcal{S}_\eta\}_{\eta \in [0,1]}$ be a family of dynamical systems which is continuous at $\eta = 0$. Suppose that

1. $\mathcal{S}_\eta$ has a pullback attractor $\hat{A}_\eta$, for each $\eta \in [0, 1]$, and $\overline{\bigcup_{\eta \in [0, 1]} \bigcup_{t \in \mathbb{T}} A_\eta(t)}$ is compact.
2. $\mathcal{S}_0(t, s) = T(t - s)$, for all $(t, s) \in \mathbb{P}$, where $\mathcal{T} = \{T(t) : t \in \mathbb{T}^+\}$ is a dynamically gradient semigroup relative to the disjoint collection of equilibria $\mathbf{E}_0 = \{e_1^*, \dots, e_n^*\}$ with global attractor $\mathcal{A}$.
3. For each $\eta \in [0, 1]$, $\mathbf{E}_\eta = \{\{\xi_{1,\eta}(t)\}_{t \in \mathbb{R}}, \dots, \{\xi_{n,\eta}(t)\}_{t \in \mathbb{R}}\}$ is a disjoint collection of (unitary) isolated invariant families for $\mathcal{S}_\eta$ and $\lim_{\eta \rightarrow 0} \max_{1 \leq i \leq n} d(\xi_{i,\eta}^*(\mathbb{T}), e_i^*) = 0$.
4. There exist $\delta > 0$ and $r > 0$ such that, if $\eta \in [0, r]$, ξ_η is a global solution of $\mathcal{S}_\eta$ in $\hat{A}_\eta$ and $\tau_0 \in \mathbb{T}$ satisfies $d(\xi_\eta(t), \xi_{i,\eta}^*(\mathbb{T})) \leq \delta$ for all $t \leq \tau_0$ ($t \geq \tau_0$), then $d(\xi_\eta(t), \xi_{i,\eta}^*(t)) \rightarrow 0$ as $t \rightarrow -\infty$ ($t \rightarrow \infty$).

Then there exists an $\varepsilon > 0$ such that $\mathcal{S}_\eta$ is a dynamically gradient evolution process, for all $\eta \in [0, \varepsilon]$.

Proof. The idea of this proof is essentially the same as in Theorem 7. Note that, from items 3 and 4, $d(\xi_{i,\eta}^*(\mathbb{T}), e_i^*) \xrightarrow{\eta \rightarrow 0} 0$ and there is a $\delta > 0$ such that, for all η suitably close to 0, if a solution ξ_η satisfies $d(\xi_\eta(t), e_i^*) \leq \delta$ for all $t \geq \tau_0$, for some $\tau_0 \in \mathbb{T}$, then $d(\xi_\eta(t), \xi_{i,\eta}^*(t)) \xrightarrow{t \rightarrow \infty} 0$.

Now we will prove that for η suitably close to 0, $\mathcal{S}_\eta$ satisfies (DGDS1). We will argue by contradiction. Suppose that there exists a sequence $\eta_k \rightarrow 0$ and global solutions ξ_k of $\mathcal{S}_{\eta_k}$ in $\hat{A}_{\eta_k}$ such that

$$\limsup_{t \rightarrow \infty} d(\xi_k(t), \xi_{i,\eta_k}^*(\mathbb{T})) > \delta \quad \text{for } 1 \leq i \leq n.$$

Thus, for large values of k , we have

$$\limsup_{t \rightarrow \infty} d(\xi_k(t), e_i^*) > \delta \quad \text{for } 1 \leq i \leq n. \quad (4.16)$$

Taking subsequences, if necessary, we may assume that $\xi_k(t) \rightarrow \xi_0(t)$ uniformly for t in compact subsets of $\mathbb{T}$. From item 2, $d(\xi_0(t), e_i^*) \xrightarrow{t \rightarrow \infty} 0$ for some $1 \leq i \leq n$. Hence, given $\ell > \frac{1}{\delta}$ there is a $t_\ell > 0$ and $k_\ell \in \mathbb{N}$ such that $d(\xi_k(t_\ell), e_i^*) < \frac{1}{\ell}$, for each $k \geq k_\ell$. From (4.16), there exists $t'_\ell > t_\ell$ such that $d(\xi_{k_\ell}(t), e_i^*) < \delta$ for all $t \in [t_\ell, t'_\ell] \cap \mathbb{T}$ and $d(\xi_{k_\ell}(t'_\ell), e_i^*) = \delta$. Taking subsequences, if necessary, let $\xi_1(t) = \lim_{\ell \rightarrow \infty} \xi_{k_\ell}(t + t'_\ell)$, which is a global solution of $\mathcal{T}$ in $\mathcal{A}$. Since $t'_\ell - t_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$, we have $d(\xi_1(t), e_i^*) \leq \delta$, for all $t \in \mathbb{T}^-$, which implies $d(\xi_1(t), e_i^*) \xrightarrow{t \rightarrow -\infty} 0$.

Again from item 2, we know that $d(\xi_1(t), e_j^*) \xrightarrow{t \rightarrow \infty} 0$ for some $j \neq i$. Since $\xi_{k_\ell}(t + t'_\ell)$ converges to $\xi_1(t)$ uniformly in compact subsets of $\mathbb{T}$, for each $m \in \mathbb{N}$ there exists $t_m \in \mathbb{T}^+$ and $k_m \in \mathbb{N}$ such that $d(\xi_k(t_m), e_j^*) < \frac{1}{m}$ for all $k \geq k_m$. Once again, from (4.16), there exists $t'_m > t_m$ such that $d(\xi_{k_m}(t), e_j^*) < \delta$ for all $t \in [t_m, t'_m]$ and $d(\xi_{k_m}(t'_m), e_j^*) = \delta$. Proceeding exactly as before we obtain a global solution ξ_2 of $\mathcal{T}$ in $\mathcal{A}$ such that $d(\xi_2(t), e_j^*) \xrightarrow{t \rightarrow \infty} 0$ and $d(\xi_2(t), e_r^*) \xrightarrow{t \rightarrow \infty} 0$ with $r \notin \{i, j\}$. In a finite number of steps we arrive at a contradiction with the fact that $\mathcal{T}$ is

dynamically gradient relatively to $\mathbf{E}_0$. This proves that there is an $\varepsilon > 0$ such that, for all global solution ξ_η of $\mathcal{S}_\eta$ in $\hat{A}_\eta$ with $\eta \in [0, \varepsilon)$, we have $\lim_{t \rightarrow \infty} d(\xi_\eta(t), \xi_{i,\eta}^*(\mathbb{R})) = 0$. Similarly, we can prove the case $t \rightarrow -\infty$, which proves (DGDS1) for η suitably small. Also, the proof of (DGDS2) follows the same reasoning and we complete the proof. $\square$

DICHOTOMY AND GLOBAL HYPERBOLIC SOLUTIONS

To ensure that conditions 3 and 4 of Theorem 14 in Chapter 4 hold we will need the notion of hyperbolicity in the non-autonomous framework. That notion is called exponential dichotomy and it will be essential to define global hyperbolic solutions, to prove that they form isolated invariant sets and that these are robust under perturbations. Those concepts have a strong geometric interpretation which is very useful to understand the dynamic of the process. For a deeper approach, you can consult (HENRY, 2013), (CARVALHO; LANGA; ROBINSON, 2013), (BORTOLAN; CARVALHO; LANGA, 2020) and (CARVALHO; LANGA, 2007).

5.1 Dichotomy

In this section we want to define exponential dichotomy and prove that this property is stable under perturbation. To do that, we will first prove these results on the discrete case and then use them to prove the robustness on the continuous case.

From now on X will denote a Banach space, and not just a metric space as before.

5.1.1 Discrete Dichotomy

Definition 38. We say that a discrete evolution process $\mathcal{S} = \{T_{n,m} : n, m \in \mathbb{Z}, n \geq m\}$ is a **$\mathcal{C}^1$ -discrete evolution process** if $\mathcal{S} \subset \mathcal{C}^1(X)$, where $\mathcal{C}^1$ denotes the set of continuously differentiable functions. Similarly, if $\mathcal{S} \subset \mathcal{L}(X)$, where $\mathcal{L}(X)$ denotes the set of continuous linear operators in X , we say that it is a **linear discrete evolution process**.

Every discrete evolution process determines a sequence $\{T_n\}_{n \in \mathbb{Z}}$ in $\mathcal{C}(X)$, defined by $T_n = T_{n+1,n}$. Conversely, given a sequence of operators $\{T_n\}_{n \in \mathbb{Z}}$ in $\mathcal{C}(X)$, we can define a discrete

evolution process $\mathcal{S} = \{T_{n,m} : n \geq m\}$ in the following way:

- (a) $T_{m,m} = I$ for all $m \in \mathbb{Z}$, where I denotes the identity in X ;
- (b) $T_{n,m} = T_{n-1} \cdots T_m$, for all $n > m \in \mathbb{Z}$.

Clearly, if $T_n = T$ for all $n \in \mathbb{Z}$, the discrete evolution process associated to the sequence $\{T_n\}_{n \in \mathbb{Z}}$ is $\{T^{n-m} : n \geq m\}$ which coincides with the discrete semigroup $\mathcal{T} = \{T^n : n \in \mathbb{N}\}$. This means that discrete evolution process generalizes the discrete semigroups, as we expected.

Lemma 22 (Discrete Variations of Constants). Suppose that $\{T_n\}_{n \in \mathbb{Z}}$ is a sequence in $\mathcal{L}(X)$. If $\{f_n\}_{n \in \mathbb{Z}}$ is a sequence in X and $x_{n+1} = T_n x_n + f_n$ for $n > m$, then

$$x_n = T_{n,m} x_m + \sum_{k=m}^{n-1} T_{n,k+1} f_k. \quad (5.1)$$

Proof. The proof proceeds by induction on m . Note that (5.1) is valid for $m = n - 1$. Now assume that (5.1) holds for $m = r$, i.e.

$$x_n = T_{n,r} x_r + \sum_{k=r}^{n-1} T_{n,k+1} f_k.$$

Since

$$x_r = T_{r-1} x_{r-1} + f_{r-1},$$

it follows that

$$x_n = T_{n,r} T_{r-1} x_{r-1} + T_{n,r} f_{r-1} + \sum_{k=r}^{n-1} T_{n,k+1} f_k = T_{n,r-1} x_{r-1} + \sum_{k=r-1}^{n-1} T_{n,k+1} f_k.$$

Thus (5.1) holds for $m = r - 1$ and we conclude the proof. $\square$

Definition 39 (Exponential Dichotomy). We say that the discrete linear process $\{T_{n,m} : n \geq m\}$ corresponding to the sequence $\{T_n\}_{n \in \mathbb{Z}}$ has a **discrete dichotomy** with constant $M \geq 1$ and exponent $\omega > 0$ if there are projections $\{P_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ such that:

- (i) $T_n P_n = P_{n+1} T_n, \forall n \in \mathbb{Z}$.
- (ii) For each $m, n \in \mathbb{Z}$, $m > n$ $T_{m,n} : R(P_n) \rightarrow R(P_m)$ is an isomorphism with inverse denoted by $T_{n,m} : R(P_m) \rightarrow R(P_n)$.
- (iii) The following inequalities holds:

$$\|T_{n,m}(I - P_m)\|_{\mathcal{L}(X)} \leq M e^{-\omega(n-m)}, \quad n \geq m, \quad (5.2)$$

$$\|T_{n,m} P_m\|_{\mathcal{L}(X)} \leq M e^{\omega(n-m)}, \quad n < m. \quad (5.3)$$

We often say that $\{T_n\}_{n \in \mathbb{Z}}$ has a discrete dichotomy instead of saying that the discrete linear process $\{T_{n,m} : n \geq m\}$ corresponding to the sequence $\{T_n\}_{n \in \mathbb{Z}}$ has a discrete dichotomy.

Geometrically, the presence of an exponential dichotomy on a discrete evolution process means that, for each instant m , we can split X in two parts, the unstable (where solutions approaches when time retreat) and the stable (where solutions approaches when times evolves). The unstable part is the set $P_m(X)$, since every point in $P_m(X)$ evolves to 0 when $n \rightarrow -\infty$. Similarly, the stable set is given by $(I - P_m)X$, because any point in $(I - P_m)X$ evolves to 0 when $n \rightarrow \infty$.

Remark 14. In the autonomous case, the sequence of projections $\{P_n\}_{n \in \mathbb{Z}}$ is constant, since $\{T_n\}_{n \in \mathbb{Z}}$ is constant.

We will denote by ℓ^∞ the Banach space $\ell^\infty(\mathbb{Z}, X)$ of bounded bi-infinite sequences $\underline{x} = \{x_n\}_{n \in \mathbb{Z}}$ with the norm

$$\|\underline{x}\|_\infty := \|\underline{x}\|_{\ell^\infty(\mathbb{Z}, X)} = \sup\{\|x_n\|_X : n \in \mathbb{Z}\}.$$

We define the Green function $G_{n,m}$ by

$$G_{n,m} = \begin{cases} T_{n,m}(I - P_m), & n \geq m, \\ -T_{n,m}P_m, & n < m. \end{cases} \quad (5.4)$$

Note that from (5.2) and (5.3) we have

$$\|G_{n,m}\|_{\mathcal{L}(X)} \leq Me^{-\omega|n-m|}, \forall n, m \in \mathbb{Z}. \quad (5.5)$$

Theorem 15. If $\{T_n\}_{n \in \mathbb{Z}}$ has a discrete dichotomy and $\{f_n\}_{n \in \mathbb{Z}}$ is a bounded sequence in X , then there is a unique bounded solution $\{x_n\}_{n \in \mathbb{Z}}$ of

$$x_{n+1} = T_n x_n + f_n, \quad (5.6)$$

and this solution is given by

$$x_n = \sum_{k=-\infty}^{\infty} G_{n,k+1} f_k, \quad (5.7)$$

Proof. To see that any bounded solution of (5.6) must be given by (5.7), we use Lemma 22 to write

$$x_n = T_{n,m} x_m + \sum_{k=m}^{n-1} T_{n,k+1} f_k.$$

Since $x_n = (I - P_n)x_n + P_n x_n$ and $\{T_n\}_{n \in \mathbb{Z}}$ has an exponential dichotomy, we will consider $(I - P_n)x_n$ and $P_n x_n$ separately. For $m \leq n$ we have

$$(I - P_n)x_n = T_{n,m}(I - P_m)x_m + \sum_{k=m}^{n-1} T_{n,k+1}(I - P_{k+1})f_k.$$

and from (5.2)

$$\|T_{n,m}(I - P_m)x_m\|_X \rightarrow 0 \text{ as } m \rightarrow -\infty.$$

Since $\{x_n\}$ is bounded it follows that

$$\sum_{k=-\infty}^{n-1} T_{n,k+1}(I - P_{k+1})f_k$$

is convergent and

$$(I - P_n)x_n = \sum_{k=-\infty}^{n-1} T_{n,k+1}(I - P_{k+1})f_k. \quad (5.8)$$

On the other hand, for $r \geq n$

$$P_r x_r = T_{r,n} P_n x_n + \sum_{k=n}^{r-1} T_{r,k+1} P_{k+1} y_k,$$

so we can write

$$T_{n,r} P_r x_r = P_n x_n + \sum_{k=n}^{r-1} T_{n,k+1} P_{k+1} y_k, \quad r > n.$$

Thus

$$\|T_{n,r} P_r x_r\|_X \leq M e^{-(r-n)\omega} \|x_r\|_X \quad n < r.$$

From the fact that $\{x_n\}$ is bounded we conclude that

$$\sum_{k=n}^{\infty} T_{n,k+1} P_{k+1} f_k$$

is convergent and

$$P_n x_n = - \sum_{k=n}^{\infty} T_{n,k+1} P_{k+1} f_k. \quad (5.9)$$

Combining (5.8) and (5.9) yields

$$x_n = P_n x_n + (I - P_n)x_n = \sum_{-\infty}^{\infty} G_{n,k+1} y_k.$$

It remains to check if (5.7) really does give a bounded solution of (5.6). To show that $\{x_n\}$ is bounded, we can write

$$\begin{aligned} \|x_n\|_X &\leq \sum_{-\infty}^{n-1} \|T_{n,k+1}(I - P_{k+1})f_k\|_X + \sum_n^{\infty} \|T_{n,k+1}P_{k+1}f_k\|_X \\ &\leq M \left(\sum_{-\infty}^{n-1} e^{-(n-k-1)\omega} + \sum_n^{\infty} e^{-(k+1-n)\omega} \right) \sup_{n \in \mathbb{Z}} \|f_n\|_X = M \frac{1 + e^{-\omega}}{1 - e^{-\omega}} \sup_{n \in \mathbb{Z}} \|f_n\|_X. \end{aligned}$$

To see that $\{x_n\}$ satisfies (5.6), note that

$$\begin{aligned} x_{n+1} &= \sum_{k=-\infty}^{\infty} G_{n+1,k+1} f_k = \sum_{-\infty}^n T_{n+1,k+1}(I - P_{k+1})f_k - \sum_{n+1}^{\infty} T_{n+1,k+1}P_{k+1}f_k \\ &= \sum_{-\infty}^{n-1} T_{n+1,k+1}(I - P_{k+1})f_k - \sum_n^{\infty} T_{n+1,k+1}P_{k+1}f_k + f_n \\ &= T_n \left(\sum_{-\infty}^{n-1} T_{n,k+1}(I - P_{k+1})f_k - \sum_n^{\infty} T_{n,k+1}P_{k+1}f_k \right) + f_n \\ &= T_n x_n + f_n. \end{aligned}$$

□

Corollary 4. If $\{T_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ has an exponential dichotomy, then the associate projections $\{Q_n\}_{n \in \mathbb{Z}}$ are uniquely determined.

Proof. Assume that $\{Q_n^{(i)}\}_{n \in \mathbb{Z}}$, $i = 1, 2$ are two families of projections associated with the exponential dichotomy of $\{T_n\}_{n \in \mathbb{Z}}$. Let $\{f_n\}_{n \in \mathbb{Z}}$ with $f_n = 0$ for all $n \neq m-1$ and $f_{m-1} = x$. Thus, if $\{x_n\}_{n \in \mathbb{Z}}$ is the unique bounded solution of $x_{n+1} = T_n x_n + f_n$, from Theorem 15 we have

$$x_n = \sum_{k=-\infty}^{\infty} G_{n,k+1}^{(i)} f_k = G_{n,m}^{(i)} x, \quad i = 1, 2,$$

where

$$G_{n,m}^{(i)} = \begin{cases} T_{n,m}(I - Q_m^{(i)}), & n \geq m, \\ -T_{n,m}Q_m^{(i)}, & n < m, \end{cases}$$

for $i = 1, 2$. In particular, $x_m = G_{m,m}^{(i)} x = (I - Q_m^{(i)})x$ and $Q_m^{(1)}x = Q_m^{(2)}x$ for each $x \in X$. $\square$

The converse of Theorem 15 also holds, as we will see in the next theorem. Thus, we have a characterization for discrete linear process with exponential dichotomy.

Theorem 16. If $\{T_n\}_{n \in \mathbb{Z}}$ is a sequence in $\mathcal{L}(X)$, then $\{T_n\}_{n \in \mathbb{Z}}$ has a discrete dichotomy if, and only if, for each bounded sequence $\{f_n\}_{n \in \mathbb{Z}} \in X$ there is a unique bounded solution $\{x_n\}_{n \in \mathbb{Z}}$ of

$$x_{n+1} = T_n x_n + f_n, \quad \forall n \in \mathbb{Z}. \quad (5.10)$$

Proof. We already have one implication by Theorem 15. It remains to show that if for each bounded sequence $\underline{f} \in \ell^\infty$ there exists a unique bounded solution $\underline{x}$ of $x_{n+1} = T_n x_n + f_n$ then $\{T_n\}_{n \in \mathbb{Z}}$ has an exponential dichotomy.

Step 1. We define a map $L : \ell^\infty \rightarrow \ell^\infty$ by setting $L\underline{f} = \underline{x}$, where $\underline{x}$ is the unique bounded solution of (5.10); the existence of such an $\underline{x}$ is our hypothesis. We want to ensure that L is bounded.

From the Closed Graph Theorem, to show that L is bounded, it suffices to show that it is closed, i.e. if $\{\underline{f}^k\} \in \ell^\infty$, $\underline{x}^k = L\underline{f}^k$, and

$$\|\underline{f}^k - \underline{f}\|_\infty \rightarrow 0 \quad \text{and} \quad \|\underline{x}^k - \underline{x}\|_\infty \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty,$$

then $\underline{x} = L\underline{f}$. We first use (5.10) to write

$$f_n^k - f_n = x_{n+1}^k - T_n x_n^k - f_n,$$

and so

$$\|\underline{f}^k - \underline{f}\|_\infty = \|\{x_{n+1}^k\} - \{T_n x_n^k\} - \underline{f}\|_\infty \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty;$$

then, since $\|\{x_{n+1}^k\} - \{x_{n+1}\}\|_\infty \rightarrow 0$, it follows that

$$\|\{x_{n+1}\} - \{T_n x_n^k\} - \underline{f}\|_\infty \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.$$

Since $\|T_n x_n^k - T_n x_n\|_X \rightarrow 0$ as $k \rightarrow \infty$ for each $n \in \mathbb{Z}$, it follows that $\{T_n x_n\} \in \ell^\infty$ and

$$x_{n+1} - T_n x_n = f_n,$$

i.e. $L\underline{f} = \underline{x}$. This proves that L is closed, and therefore L is bounded.

Step 2. In this step we define the "projections" (we will prove later that is indeed a projection).

Fix $y \in X$ and $m \in \mathbb{Z}$; let $\underline{f} = \{f_n\}_{n \in \mathbb{Z}}$ with $f_{m-1} = y$ and $f_n = 0$ for $n \neq m-1$, and define

$$\{G_{n,m}y\}_{n \in \mathbb{Z}} := L\underline{f}.$$

Defined in this way (over all $y \in X$ and $m \in \mathbb{Z}$), the resulting operators $G_{n,m}$ are elements of $\mathcal{L}(X)$ and $\|G_{n,m}\|_{\mathcal{L}(X)} \leq \|L\|_{\mathcal{L}(\ell^\infty)}$ for all $m, n \in \mathbb{Z}$. It follows from the linearity of L that if $A \subset \mathbb{Z}$ is a finite set and $\underline{f} = \{f_n\}_{n \in \mathbb{Z}}$ with $f_n = 0$ if $n \in \mathbb{Z} \setminus A$, then

$$L\underline{f} = \left\{ \sum_{k \in A} G_{n,k+1} f_k \right\}_{n \in \mathbb{Z}}.$$

Defining $G_{m,m} = I - P_m$, we can prove by induction that the following equality holds:

$$G_{n,m} = T_{n,m}(I - P_m), \quad n \geq m, \quad (5.11)$$

$$T_{m,n}G_{n,m} = -P_m, \quad n < m. \quad (5.12)$$

Step 3a. We will show that if $x_{n+1} = T_n x_n$, $n \geq m$, is a bounded sequence, then $P_m x_m = 0$. Setting $x_n = 0$ for $n < m$ we have

$$\begin{aligned} x_m &= T_{m-1}x_{m-1} + x_m \\ x_{n+1} &= T_n x_n, \quad n \neq m-1, \end{aligned}$$

and so $\underline{x}$ is a bounded solution of $x_{n+1} = T_n x_n + f_n$, where $f_n = 0$ if $n \neq m-1$ and $f_{m-1} = x_m$. Hence $x_n = G_{n,m}x_m$ and in particular $x_m = G_{m,m}x_m = (I - P_m)x_m$, that is, $P_m x_m = 0$.

Step 3b. If $\{x_n\}_{n \leq m}$ is a bounded sequence such that $x_{n+1} = T_n x_n$, $n < m$, then $T_m x_m \in R(P_{m+1})$. Setting $x_n = 0$ for $n > m$, we obtain that $\{x_n\}_{n \in \mathbb{Z}}$ is a bounded solution of

$$x_{n+1} = T_n x_n + f_n,$$

where $f_n = 0$ for $n \neq m$ and $f_m = -T_m x_m$. Hence

$$x_n = -G_{n,m+1}T_m x_m \quad \text{for all } n \in \mathbb{Z}.$$

In particular,

$$0 = x_{m+1} = -G_{m+1,m+1}T_m x_m = -(I - P_{m+1})T_m x_m,$$

and so $T_m x_m \in R(P_{m+1})$.

Step 4. Show that P_m is a projection. Given $m \in \mathbb{Z}$ and $x \in X$, let $x_n = G_{n,m}x$. Then $x_{n+1} = T_n x_n$ for $n \geq m$. Since $\{x_n\}_{n \geq m}$ is a bounded sequence, it follows from Step 3a that

$$0 = P_m G_{m,m}x = P_m(I - P_m)x.$$

Thus $P_m^2 x = P_m x$ and P_m is a projection.

Step 5. The equality $P_{m+1}T_mx = T_mP_mx$ holds for every x with $P_mx = 0$.

Given $m \in \mathbb{Z}$ and $x \in X$, define $x_n = G_{n,m}x$. Note that, since $P_mx = 0$, $x_m = G_{m,m}x = x$. Then $x_{n+1} = T_nx_n$ for $n \geq m+1$, and $\{x_n\}_{n \in \mathbb{Z}}$ is bounded, so we can use Step 3a to deduce that $P_{m+1}x_{m+1} = 0$. Whence

$$P_{m+1}T_mx = P_{m+1}T_mx_m = P_{m+1}x_{m+1} = 0 = T_mP_mx \text{ for all } x \in X \text{ with } P_mx = 0.$$

Step 6. We will show that

$$T_m|_{R(P_m)} : R(P_m) \rightarrow R(P_{m+1})$$

is an isomorphism.

For $x \in R(P_m)$ define $y_n = G_{n,m}x$; then $\{y_n\}$ is bounded and $y_{n+1} = T_ny_n$ for every $n < m-1$. In particular, $T_{m-2}y_{m-2} = y_{m-1}$, and so it follows from Step 3b that $y_{m-1} = T_{m-2}y_{m-2} \in R(P_{m-1})$. Furthermore, $y_m = G_{m,m}x = (I - P_m)x = 0$ since $x \in R(P_m)$, and $0 = y_m = T_{m-1}y_{m-1} + x$, proving that $x = T_{m-1}(-y_{m-1})$. This implies that

$$R(P_m) \subseteq T_{m-1}(R(P_{m-1})). \quad (5.13)$$

Now consider $\{\bar{y}_n\}$, defined by $\bar{y}_n = y_n$ for $n \leq m-1$ and $\bar{y}_{n+1} = T_n\bar{y}_n$ for all $n \geq m-1$, so that $\bar{y}_{n+1} = T_n\bar{y}_n$ for every $n \in \mathbb{Z}$. Note that, in particular,

$$\bar{y}_m = T_{m-1}\bar{y}_{m-1} = T_{m-1}y_{m-1} = -x.$$

From its definition $\{\bar{y}_n\}_{n \leq m}$ is bounded, and consequently it follows, using Step 3b, that $T_m\bar{y}_m = -T_mx \in R(P_{m+1})$. This shows that $T_m(R(P_m)) \subseteq R(P_{m+1})$, and, combining this with (5.13), we obtain

$$R(P_{m+1}) = T_m(R(P_m)).$$

Furthermore, if $T_mx = 0$, then

$$\bar{y}_{m+1} = T_m\bar{y}_m = -T_mx = 0,$$

and $\bar{y}_n = 0$ for $n > m$; that is, $\{\bar{y}_n\}_{n \geq m}$ is bounded and $\bar{y}_{m+1} = T_m\bar{y}_m$. From this it follows from Step 3a that $P_m\bar{y}_m = 0$, and hence $P_mx = x = 0$. This shows that $T_m|_{R(P_m)}$ is injective. Since we have already shown that $T_m|_{R(P_m)} : R(P_m) \rightarrow R(P_{m+1})$ is surjective, it follows from the Open Mapping Theorem that

$$T_m|_{R(P_m)} : R(P_m) \rightarrow R(P_{m+1})$$

is an isomorphism.

Now we can apply $T_{n,m}$ to both sides of (5.12) and obtain

$$G_{n,m} = \begin{cases} T_{n,m}(I - P_m), & n \geq m, \\ -T_{n,m}P_m, & n < m; \end{cases}$$

we have recovered (5.4).

Step 7. We will conclude that $T_mP_m = P_{m+1}T_m$. From step 5 it remains to show that if $P_mx = x$, then $T_mP_mx = P_{m+1}T_mx$. In fact, since $T_m(I - P_m)x = 0$, Step 6 implies that $T_mx \in R(P_{m+1})$ (because $x \in R(P_m)$). Thus $P_{m+1}T_mx = T_mx$ and $T_mP_mx = P_{m+1}T_mx$.

Step 8. We will show (5.2). Fix $x \in X$. If $T_{n,m}(I - P_m)x = 0$, then the estimate is trivial, so we assume that $T_{n,m}(I - P_m)x \neq 0$ and consequently,

$$\phi_k := \|T_{k,m}(I - P_m)x\|_X > 0 \quad \text{for all} \quad m \leq k \leq n.$$

Defining $v_k = T_{k,m}(I - P_m)x\phi_k^{-1}$, we have $\|v_k\|_X = 1$ and

$$\sum_{k=m}^n G_{n,k}v_k = \sum_{k=m}^n T_{n,k}(I - P_k)T_{k,m}(I - P_m)x\phi_k^{-1} = T_{n,m}(I - P_m)x \sum_{k=m}^n \phi_k^{-1}.$$

Hence

$$\left\| \sum_{k=m}^n G_{n,k}v_k \right\|_X = \phi_n \sum_{k=m}^n \phi_k^{-1}.$$

Taking $\underline{f} = \{f_n\}_{n \in \mathbb{Z}}$ with $f_j = v_{j+1}$ for $m-1 \leq j \leq n-1$, and $f_j = 0$ otherwise, it follows that $\|\underline{f}\|_\infty = 1$ and

$$\left\| \sum_{k=m}^n G_{n,k}v_k \right\|_X = \left\| \sum_{k=-\infty}^{\infty} G_{n,k+1}f_k \right\|_X \leq \|L\underline{f}\|_\infty \leq \|L\|_{\mathcal{L}(\ell^\infty)} \|\underline{f}\|_\infty = \|L\|_{\mathcal{L}(\ell^\infty)}.$$

Thus

$$1 < \phi_n \sum_{k=m}^n \phi_k^{-1} \leq \|L\|.$$

To relieve the notation, we will denote $\|L\|_{\mathcal{L}(\ell^\infty)}$ by $\|L\|$.

Since

$$\sum_{k=m}^{n-1} \phi_k^{-1} = -\phi_n^{-1} + \sum_{k=m}^n \phi_k^{-1} \leq (1 - \|L\|^{-1}) \left(\sum_{k=m}^n \phi_k^{-1} \right),$$

it follows

$$\begin{aligned} \|T_{n,m}(I - P_m)x\|_X &= \phi_n \leq \|L\| \left(\sum_{k=m}^n \phi_k^{-1} \right)^{-1} \\ &\leq \|L\| (1 - \|L\|^{-1}) \left(\sum_{k=m}^{n-1} \phi_k^{-1} \right)^{-1} \\ &\leq \|L\| (1 - \|L\|^{-1})^{n-m} \phi_m \\ &= \|L\| (1 - \|L\|^{-1})^{n-m} \|(I - P_m)x\|_X \\ &= \|L\| (1 - \|L\|^{-1})^{n-m} \|G_{m,m}x\|_X \\ &\leq \|L\|^2 (1 - \|L\|^{-1})^{n-m} \|x\|_X. \end{aligned}$$

Step 9. Finally, we prove (5.3). Fix $x \in X$. If $T_{n,m}P_mx \neq 0$ for some $n < m$, then we set $\rho_k = \|T_{k,m}P_mx\|$ for $n \leq k \leq m$, define $v_k = T_{k,m}P_mx\rho_k^{-1}$ and observe that

$$\sum_{k=n+1}^m G_{n,k}v_k = \sum_{k=n+1}^m T_{n,k}P_kT_{k,m}P_mx\rho_k^{-1} = T_{n,m}P_mx \sum_{k=m}^n \rho_k^{-1}.$$

Proceeding as in step 8 we obtain

$$\|T_{n,m}P_mx\|_X \leq (1 + \|L\|)^2 \left(\frac{\|L\|}{1 + \|L\|} \right)^{m-n} \|x\|_X \quad \text{for all} \quad n < m.$$

Taking

$$M = (1 + \|L\|)^2 > \|L\|^2 \quad \text{and} \quad e^{-\omega} = \frac{\|L\|}{1 + \|L\|} \geq 1 - \|L\|^{-1},$$

we obtain (5.2) and (5.3). Thus $\{T_n\}_{n \in \mathbb{Z}}$ has a discrete dichotomy with constant M and exponent ω . $\square$

Lemma 23. Suppose that $a \geq 0$, $b \geq 0$, $0 < r < \rho \leq 1$, and

$$b < \frac{\rho - r}{1 + (r/\rho)}.$$

If $\{g_n : n \in \mathbb{Z}\}$ is a non-negative sequence with $\sup \{g_n \rho^{-|n|} : n \in \mathbb{Z}\} < \infty$ and

$$g_n \leq a \rho^{|n|} + b \sum_{-\infty}^{\infty} r^{|n-k-1|} g_k \quad \text{for all } n \in \mathbb{Z},$$

then

$$g_n \leq a \rho^{|n|} \left(1 - b \frac{1 + (r/\rho)}{\rho - r} \right)^{-1} \quad \text{for all } n \in \mathbb{Z}.$$

Proof. Let $f_n = g_n \rho^{-|n|}$; then

$$f_n \leq a + b \sum_{-\infty}^{\infty} r^{|n-k-1|} \rho^{|k|-|n|} f_k \leq a + b \sum_{-\infty}^{\infty} r^{|n-k-1|} \rho^{|k|-|n|} \sup \{f_k : k \in \mathbb{Z}\}.$$

Hence,

$$\sup \{f_k : k \in \mathbb{Z}\} \leq a + b \frac{1 + (r/\rho)}{\rho - r} \sup \{f_k : k \in \mathbb{Z}\}.$$

Consequently,

$$y_n = g_n \rho^{-|n|} \leq \sup \{f_k : k \in \mathbb{Z}\} \leq a \left(1 - b \frac{1 + (r/\rho)}{\rho - r} \right)^{-1} \quad \text{for all } n \in \mathbb{Z},$$

and the result follows. $\square$

The following theorem shows that exponential dichotomy is stable under perturbations, that is, if T is a discrete evolution process with exponential dichotomy and S is "close" to T , then T also has an exponential dichotomy.

Theorem 17. Suppose that the family $\{T_n\}_{n \in \mathbb{Z}} \in \mathcal{L}(X)$ has a discrete dichotomy with constant M and exponent $\omega > 0$. Given any $M_1 > M$ and $0 < \omega_1 < \omega$, there exists an $\varepsilon > 0$ such that any family $\{S_n\}_{n \in \mathbb{Z}} \in \mathcal{L}(X)$ with

$$\sup_{n \in \mathbb{Z}} \|T_n - S_n\|_{\mathcal{L}(X)} < \varepsilon$$

has a discrete dichotomy with constant M_1 and exponent ω_1 .

Proof. We know, from Theorem 16, that $\{S_n\}_{n \in \mathbb{Z}}$ has a discrete dichotomy if and only if, for each $\underline{f} \in \ell^\infty$, the system

$$x_{n+1} = S_n x_n + f_n, \quad n \in \mathbb{Z},$$

has a unique bounded solution. But this holds if and only if

$$x_{n+1} = T_n x_n + (S_n - T_n)x_n + f_n, \quad n \in \mathbb{Z},$$

has a unique bounded solution. Once again, this holds if and only if

$$x_n = \sum_{k=-\infty}^{\infty} G_{n,k+1} [(S_k - T_k)x_k + f_k], \quad n \in \mathbb{Z}, \quad (5.14)$$

has a unique bounded solution.

To see that (5.14) have an unique bounded solution, define $G : \ell^\infty \rightarrow \ell^\infty$ by

$$G\underline{f} = \left\{ \sum_{k=-\infty}^{\infty} G_{n,k+1} f_k \right\}_{n \in \mathbb{Z}}$$

and $H : \ell^\infty \rightarrow \ell^\infty$ by

$$H\underline{x} = \left\{ \sum_{k=-\infty}^{\infty} G_{n,k+1} (S_k - T_k)x_k \right\}_{n \in \mathbb{Z}}.$$

Then, we have

$$(I - H)\underline{x} = G\underline{f}.$$

Therefore, to prove that $\{S_n\}_{n \in \mathbb{Z}}$ has a discrete dichotomy it is sufficient to show that $\|H\|_{\mathcal{L}(\ell^\infty)} < 1$, since $(I - H)$ is bijective for $\|H\|_{\mathcal{L}(\ell^\infty)} < 1$ (with inverse given by the Neumann series). Using (5.5), we obtain

$$\|H\|_{\mathcal{L}(\ell^\infty)} \leq \sup_{n \in \mathbb{Z}} \sum_{k=-\infty}^{\infty} \|G_{n,k+1} (S_k - T_k)\|_{\mathcal{L}(X)} \leq \varepsilon M \frac{1 + e^{-\omega}}{1 - e^{-\omega}},$$

which implies that $\|H\|_{\mathcal{L}(\ell^\infty)} < 1$ if

$$\varepsilon M \frac{1 + e^{-\omega}}{1 - e^{-\omega}} < 1. \quad (5.15)$$

This proves that $\{S_n\}_{n \in \mathbb{Z}}$ has an exponential dichotomy, with constant $\bar{M}$ and exponent $\bar{\omega}$, if

$$\sup_{n \in \mathbb{Z}} \|T_n - S_n\|_{\mathcal{L}(X)} < \varepsilon,$$

for every ε satisfying (5.15). It remains to show that, given $M_1 > M$ and $\omega_1 < \omega$, it is possible to choose ε to ensure that we can take $\bar{M} = M_1$ and $\bar{\omega} = \omega_1$. Let $\{S_{n,m} : (n,m) \in \mathcal{P}\}$ be the discrete linear process associated to $\{S_n\}_{n \in \mathbb{Z}}$ and let

$$\tilde{G}_{n,m} = \begin{cases} S_{n,m}(I - \tilde{P}_m), & n \geq m, \\ -S_{n,m}\tilde{P}_m, & n < m. \end{cases}$$

Then

$$\tilde{G}_{n,m} = G_{n,m} + \sum_{k=-\infty}^{\infty} G_{n,k+1} (S_k - T_k) \tilde{G}_{k,m} \quad (5.16)$$

since, if $f_{m-1} = x$ and $f_n = 0$ otherwise, then

$$x_n = \sum_{k=-\infty}^{\infty} \tilde{G}_{n,k+1} f_k = \tilde{G}_{n,m} x$$

and

$$x_n = \sum_{-\infty}^{\infty} G_{n,k+1} [(S_k - T_k)x_k + f_k] = \sum_{-\infty}^{\infty} G_{n,k+1} (S_k - T_k) \tilde{G}_{k,m} x + G_{n,m} x.$$

This shows that the sequence $\{G_{n,m}\}_{n \in \mathbb{Z}} \subset \ell^\infty(\mathbb{Z}, \mathcal{L}(X))$ is a fixed point of the mapping $\Phi : \ell^\infty(\mathbb{Z}, \mathcal{L}(X)) \rightarrow \ell^\infty(\mathbb{Z}, \mathcal{L}(X))$ given by

$$\Phi(\{R_n\}_{n \in \mathbb{Z}}) = \{G_{n,m}\}_{n \in \mathbb{Z}} + \left\{ \sum_{-\infty}^{\infty} G_{n,k+1} (S_k - T_k) R_k \right\}_{n \in \mathbb{Z}}.$$

Now let

$$B_{\omega_1} = \left\{ \underline{R} = \{R_n\}_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathcal{L}(X)) : e^{|n|\omega_1} \|R_n\|_{\mathcal{L}(X)} < \infty \right\};$$

equipped with the norm

$$\|\underline{R}\|_{B_{\omega_1}} = \sup \{e^{|n|\omega_1} \|R_n\|_{\mathcal{L}(X)} : n \in \mathbb{Z}\}$$

B_{ω_1} is a Banach space. We show that if ε is sufficiently small, then $\Phi : B_{\omega_1} \rightarrow B_{\omega_1}$ is well defined and is a contraction.

If $\underline{R} = \{R_n\}_{n \in \mathbb{Z}} \in B_{\omega_1}$, then

$$\Phi \underline{R} = \{(\Phi \underline{R})_n\}_{n \in \mathbb{Z}} = \{G_{n,m}\}_{n \in \mathbb{Z}} + \left\{ \sum_{-\infty}^{\infty} G_{n,k+1} (S_k - T_k) R_k \right\}_{n \in \mathbb{Z}}$$

and, using the bound on $\|G_{n,m}\|_{\mathcal{L}(X)}$ in (5.5),

$$\begin{aligned} e^{|n|\omega_1} \|(\Phi \underline{R})_n\|_{\mathcal{L}(X)} &\leq M e^{|n|\omega_1} e^{-|n-m|\omega} + M \varepsilon \sum_{-\infty}^{\infty} e^{-|n-k-1|\omega} e^{|k|\omega_1} \|R_k\|_{\mathcal{L}(X)} e^{(|n|-|k|)\omega_1} \\ &\leq M e^{|m|\omega} + M \varepsilon \sum_{-\infty}^{\infty} e^{-|n-k-1|\omega} e^{(|n|-|k|)\omega_1} \|\underline{R}\|_{B_{\omega_1}} \\ &\leq M e^{|m|\omega} + M \varepsilon \frac{1 + e^{\omega_1 - \omega}}{e^{-\omega_1} - e^{-\omega}} \|\underline{R}\|_{B_{\omega_1}}, \end{aligned} \quad (5.17)$$

which shows that $\Phi \underline{R} \in B_{\omega_1}$. Furthermore, if $\underline{R}^{(i)} \in B_{\omega_1}$, $i = 1, 2$, then

$$\Phi \underline{R}^{(1)} - \Phi \underline{R}^{(2)} = \left\{ \sum_{-\infty}^{\infty} G_{n,k+1} (S_k - T_k) (R_k^{(1)} - R_k^{(2)}) \right\}_{n \in \mathbb{Z}},$$

and hence, using exactly the same algebra that leads to (5.17) except without the first term,

$$\|\Phi \underline{R}^{(1)} - \Phi \underline{R}^{(2)}\|_{B_{\omega_1}} \leq M \varepsilon \frac{1 + e^{\omega_1 - \omega}}{e^{-\omega_1} - e^{-\omega}} \|\underline{R}^{(1)} - \underline{R}^{(2)}\|_{B_{\omega_1}}.$$

It follows that Φ is a contraction on B_{ω_1} if we choose $\varepsilon > 0$ such that

$$M \varepsilon \frac{1 + e^{\omega_1 - \omega}}{e^{-\omega_1} - e^{-\omega}} < 1. \quad (5.18)$$

Since (5.18) also ensures (5.15), which guaranteed the uniqueness of bounded solutions of (5.14), the fixed point of Φ must coincide with $\{\tilde{G}_{n,m}\}$, and so $\{\tilde{G}_{n,m}\} \in B_{\omega_1}$.

We can now apply Lemma 23 to

$$\|\tilde{G}_{n,m}\|_{\mathcal{L}(X)} \leq M e^{-|n-m|\omega_1} + M\varepsilon \sum_{-\infty}^{\infty} e^{-|n-k-1|\omega} \|\tilde{G}_{k,m}\|_{\mathcal{L}(X)}$$

to obtain

$$\|\tilde{G}_{n,m}\|_{\mathcal{L}(X)} \leq M \left(1 - M\varepsilon \frac{1 + e^{\omega_1 - \omega}}{e^{-\omega_1} - e^{-\omega}} \right)^{-1} e^{-|n-m|\omega_1} \leq M_1 e^{-|n-m|\omega_1}.$$

For the last inequality to hold we require

$$M \left(1 - M\varepsilon \frac{1 + e^{\omega_1 - \omega}}{e^{-\omega_1} - e^{-\omega}} \right)^{-1} \leq M_1. \quad (5.19)$$

If ε satisfies this condition, then it certainly satisfies (5.18) since (5.19) implies that

$$M\varepsilon \frac{1 + e^{\omega_1 - \omega}}{e^{-\omega_1} - e^{-\omega}} \leq \frac{M_1 - M}{M_1} < 1.$$

So under (5.19) the family $\{S_n\}_{n \in \mathbb{Z}}$ has a discrete dichotomy with constant M_1 and exponent ω_1 , as required. $\square$

Now that we proved the stability of evolution process with exponential dichotomy, we will stabilish a relation between projections of two evolution process that are close.

Theorem 18. Suppose that $\{T_n\}_{n \in \mathbb{Z}}$ and $\{S_n\}_{n \in \mathbb{Z}}$ both have a discrete dichotomy with projections $\{P_n\}_{n \in \mathbb{Z}}$ and $\{\tilde{P}_n\}_{n \in \mathbb{Z}}$, respectively, with the same constant M and exponent ω . If $\|T_n - S_n\|_{\mathcal{L}(X)} \leq \varepsilon$ for all $n \in \mathbb{Z}$, then

$$\|P_n - \tilde{P}_n\|_{\mathcal{L}(X)} \leq \frac{2M^2}{1 - e^{-\omega}} \varepsilon.$$

Proof. Given $z \in X$, define $x_n = G_{n,m}z$ (associated to T_n) and $\tilde{x}_n = \tilde{G}_{n,m}z$ (associated to S_n). Note that

$$x_{n+1} - S_n x_n = \begin{cases} T_n x_n - S_n x_n, & n \neq m-1, \\ T_n x_n - S_n x_n + z, & n = m-1, \end{cases}$$

and

$$\tilde{x}_{n+1} - S_n \tilde{x}_n = \begin{cases} 0, & n \neq m-1, \\ z, & n = m-1. \end{cases}$$

Then $z_n = x_n - \tilde{x}_n$ satisfies $z_{n+1} = S_n z_n + y_n$, where $y_n = (T_n - S_n)x_n$. Since $\{T_n - S_n\}_{n \in \mathbb{Z}}$ is bounded (by hypothesis) it follows that $\{y_n\}_{n \in \mathbb{Z}}$ is bounded, and from Theorem 15

$$z_n = \sum_{k=-\infty}^{\infty} \tilde{G}_{n,k+1} (T_n - S_n) G_{k,m} z, \quad \forall n \in \mathbb{N}.$$

Whence

$$\|z_n\| \leq \sum_{k=-\infty}^{\infty} M^2 e^{-\omega|n-k-1|} e^{-\omega|k-m|} \|T_n - S_n\|_{\mathcal{L}(X)} \|z\|_X \leq \frac{2M^2}{1 - e^{-\omega}} \varepsilon \|z\|, \quad \forall n \in \mathbb{N}.$$

In particular, for $n = m$ we have

$$\|z_m\| = \|\tilde{P}_m - P_m\| \leq \frac{2M^2}{1 - e^{-\omega}}.$$

$\square$

5.1.2 Continuous Dichotomy

In this subsection we define exponential dichotomy for continuous evolution process and connect these concept with the discrete exponential dichotomy. We will also prove some results in the continuous case that are very similar to the discrete case (like the stability of the exponential dichotomy under perturbations).

Definition 40. We say that a continuous process $(\mathbb{T} = \mathbb{R}) \{T(t, s) : (t, s) \in \mathcal{P}\} \subset \mathcal{L}(X)$ has an **exponential dichotomy** with projection $P(\cdot) = \{P(t) : t \in \mathbb{R}\} \subset \mathcal{L}(X)$, exponent $\omega > 0$, and constant $M \geq 1$ if

$$(i) \quad P(t)T(t, \tau) = T(t, \tau)P(\tau), \forall t \geq \tau.$$

(ii) The restriction of $T(t, \tau)$ to $R(P(\tau))$ is an isomorphism from $R(P(\tau))$ onto $R(P(t))$, which inverse is denoted by $T(\tau, t) : R(P(t)) \rightarrow R(P(\tau))$.

(iii) $P(\cdot)$ satisfies

$$\|T(t, \tau)(I - P(\tau))\|_{\mathcal{L}(X)} \leq Me^{-\omega(t-\tau)}, \quad t \geq \tau \quad (5.20)$$

$$\|T(t, \tau)P(\tau)\|_{\mathcal{L}(X)} \leq Me^{\omega(t-\tau)}, \quad t \leq \tau. \quad (5.21)$$

Theorem 19. Let $\{T(t, s) : (t, s) \in \mathcal{P}\}$ be a continuous process that has an exponential dichotomy with projection $P(\cdot)$, constant $M > 1$, and exponent $\omega > 0$. Then, for each $h > 0$ and $t_0 \in \mathbb{R}$, the discrete evolution process

$$\{T_n\}_{n \in \mathbb{Z}} := \{T(t_0 + (n+1)h, t_0 + nh)\}_{n \in \mathbb{Z}} \quad (5.22)$$

has a discrete dichotomy with projections $\{P_n = P(t_0 + nh) : n \in \mathbb{Z}\}$, constant M and exponent ωh .

Proof. For each $n \in \mathbb{Z}$ define

$$T_n = T(t_0 + (n+1)h, t_0 + nh) \quad \text{and} \quad P_n = P(t_0 + nh).$$

It is clear that $T_n P_n = P_{n+1} T_{n+1}$ and $T_n|_{R(P_n)} : R(P_n) \rightarrow R(P_{n+1})$ is an isomorphism. Moreover, for all $n \geq m$

$$\begin{aligned} \|T_{n,m}(I - P_m)x\|_X &= \|T_{n-1} \cdots T_m(I - P_m)x\| \\ &= \|T(t_0 + nh, t_0 + (n-1)h) \cdots T(t_0 + (m+1)h, t_0 + mh)(I - P(t_0 + mh))x\|_X \\ &= \|T(t_0 + nh, t_0 + mh)(I - P(t_0 + mh))x\|_X \leq Me^{-\omega h(n-m)} \|x\|_X. \end{aligned}$$

Also, for $n < m$

$$\begin{aligned} \|T_{n,m}P_mx\|_X &= \|(T_{m,n})^{-1}P_mx\| \\ &= \|(T(t_0 + mh, t_0 + nh))^{-1}P(t_0 + mh)x\|_X \\ &= \|T(t_0 + nh, t_0 + mh)P(t_0 + mh)x\|_X \leq Me^{\omega h(n-m)} \|x\|_X. \end{aligned}$$

Hence $\{T_n\}_{n \in \mathbb{Z}}$ has a discrete dichotomy with constant M and exponent ωh . $\square$

Corollary 5. If the linear continuous process $\{T(t, s) : (t, s) \in \mathcal{P}\}$ has an exponential dichotomy, then the projection $P(\cdot)$ is uniquely determined.

Proof. It follows from Corollary 4 and the arbitrariness of t_0 and h , given in Theorem 19. $\square$

Theorem 20. Let $\{T(t, s) : (t, s) \in \mathcal{P}\}$ be a continuous such that

$$L_h := \sup_{0 \leq t - \tau \leq h} \|T(t, \tau)\|_{\mathcal{L}(X)} < \infty$$

for some $h > 0$. Assume that there are $M > 1$ and $\omega > 0$ such that for each $t_0 \in \mathbb{R}$, the discrete process $\{T(t_0 + (n+1)h, t_0 + nh)\}_{n \in \mathbb{Z}}$ has a discrete dichotomy with constant M and exponent ωh . Then the process $T(\cdot, \cdot)$ has an exponential dichotomy with constant KM and exponent ω , where

$$K = \sup_{0 \leq t - \tau \leq h} \left\{ e^{\omega(t-\tau)} \|T(t, \tau)\|_{\mathcal{L}(X)} \right\} < \infty.$$

Proof. For each $t_0 \in \mathbb{R}$ let $\{P_n(t_0)\}_{n \in \mathbb{Z}}$ be the family of projections associated to the discrete dichotomy of the process $\{T(t_0 + (n+1)h, t_0 + nh)\}_{n \in \mathbb{Z}}$, and define $P(t_0) = P_0(t_0)$.

For each $t \in \mathbb{R}$ and $n \in \mathbb{Z}$ let

$$T_n(t) = T(t + (n+1)h, t + nh).$$

Note that

$$\{T_n(t + kh)\}_{n \in \mathbb{Z}} = \{T_{n+k}(t)\}_{n \in \mathbb{Z}}.$$

Hence, from the uniqueness of the projections, we have $\{P_n(t + kh)\}_{n \in \mathbb{Z}} = \{P_{n+k}(t)\}_{n \in \mathbb{Z}}$. In particular, $P_k(t) = P_0(t + kh)$.

First we show a weak version of part (i) of the definition for time increments that are a multiple of h (we prove the general case later). For now, we will show that for any $n > k$

$$P(t + nh)T(t + nh, t + kh) = T(t + nh, t + kh)P(t + kh). \quad (5.23)$$

Indeed, writing

$$\begin{aligned} & T(t + kh + (n-k)h, t + kh) \\ &= T(t + kh + (n-k)h, t + kh + (n-k-1)h) \cdots T(t + kh + h, t + kh), \end{aligned}$$

we have

$$\begin{aligned} T(t + nh, t + kh)P(t + kh) &= T(t + kh + (n-k)h, t + kh)P_0(t + kh) \\ &= P_{n-k}(t + kh)(T(t + kh + (n-k)h, t + kh)) \\ &= P(t + nh)T(t + nh, t + kh), \end{aligned}$$

which proves (5.23).

Now we will show that

$$\|T(t, s)(I - P(s))\|_{\mathcal{L}(X)} \leq KMe^{-\omega(t-s)}, \quad t \geq s.$$

Given $t \geq s$ choose $n \in \mathbb{N}$ such that $s + nh \leq t < s + (n+1)h$. Then, if $n \in \mathbb{N}$,

$$\begin{aligned} \|T(t, s)(I - P(s))\|_{\mathcal{L}(X)} &\leq \|T(t, s + nh)\|_{\mathcal{L}(X)} \|T(s + nh, s)(I - P(s))\|_{\mathcal{L}(X)} \\ &\leq Me^{-\omega nh} \|T(t, s + nh)\|_{\mathcal{L}(X)} \\ &= Me^{\omega(t-s-nh)} \|T(t, s + nh)\|_{\mathcal{L}(X)} e^{-\omega(t-s)} \\ &\leq KMe^{-\omega(t-s)}. \end{aligned}$$

To prove (5.21), assume that $z \in R(P(s))$ and $t \leq s$. Define

$$T(t, s)z = T(t, s + nh)(T(s, s + nh)|_{R(P(s+nh))})^{-1}z,$$

where $n \in \mathbb{Z}$ is chosen such that $s + (n+1)h > t \geq s + nh$. Then, for $t \leq s$

$$\begin{aligned} \|T(t, s)P(s)\|_{\mathcal{L}(X)} &\leq \|T(t, s + nh)\|_{\mathcal{L}(X)} \|(T(s, s + nh)|_{R(P(s+nh))})^{-1}P(s + nh)\|_{\mathcal{L}(X)} \\ &\leq Me^{\omega nh} \|T(t, s + nh)\|_{\mathcal{L}(X)} \\ &= Me^{-\omega(t-s-nh)} \|T(t, s + nh)\|_{\mathcal{L}(X)} e^{\omega(t-s)} \\ &\leq KMe^{\omega(t-s)}. \end{aligned}$$

We now characterise $\text{Ker}(P(t_0))$ and $R(P(t_0))$. First we show that

$$\text{Ker}(P(t_0)) = \{z \in X : T(t, t_0)z \in X \text{ is bounded for } t \geq t_0\}. \quad (5.24)$$

If $z \in \text{Ker}(P(t_0))$, then $P(t_0)z = 0$ and

$$\|T(t, t_0)z\|_X = \|T(t, t_0)(I - P(t_0))z\|_X \leq KMe^{-\omega(t-t_0)},$$

i.e. $T(t, t_0)z$ is bounded for $t \geq t_0$. On the other hand, if $z \notin \text{Ker}(P(t_0))$, then

$$T(t, t_0)z = T(t, t_0)(I - P(t_0))z + T(t, t_0)P(t_0)z.$$

While $T(t, t_0)(I - P(t_0))z$ is bounded for $t \geq t_0$,

$$\begin{aligned} \|P(t_0)z\|_X &= \|T(t_0, t_0 + nh)T(t_0 + nh, t_0)P(t_0)z\|_X \\ &\leq Me^{-\omega nh} \|T(t_0 + nh, t_0)P(t_0)z\|_X, \end{aligned}$$

which shows that $\|T(t_0 + nh, t_0)P(t_0)z\|_X \geq M^{-1}e^{\omega nh} \|P(t_0)z\|_X$. Thus the function $T(t, t_0)P(t_0)z$ is unbounded on $[t_0, \infty)$. This proves (5.24), from which it follows immediately that

$$T(t, t_0)\text{Ker}(P(t_0)) \subseteq \text{Ker}(P(t)) \quad \text{for all } t \geq t_0. \quad (5.25)$$

To see that $T(t, t_0)|_{R(P(t_0))} : R(P(t_0)) \rightarrow X$ is injective for $t \geq t_0$, take $z \in R(P(t_0))$, with $T(t, t_0)z = 0$. For $n \in \mathbb{N}$ such that $t_0 + nh \geq t$, $T(t_0 + nh, t_0)z = 0$, and consequently $z = 0$. Proceeding exactly in the same manner we can also show that $T(s, t)|_{T(t, t_0)R(P(t_0))} : T(t, t_0)R(P(t_0)) \rightarrow X$ is injective.

The next step is to characterise $R(P(t_0))$; we prove that

$$\begin{aligned} R(P(t_0)) &= \{z \in X : \text{there exists a bounded function } y \in C((-\infty, t_0], X) \\ &\text{with } y(t_0) = z \text{ and } T(s, t)y(t) = y(s) \text{ for all } t_0 \geq s \geq t\}. \end{aligned} \quad (5.26)$$

Given $z \in R(P(t_0))$, if we define $y(t)$ for $t \leq t_0$ by $y(t) = T(t, t_0 + nh)T(t_0 + nh, t_0)z$, where $n \in \mathbb{Z}$ is chosen such that $t_0 + nh \leq t \leq t_0 + (n+1)h$, then $y : (-\infty, t_0] \rightarrow X$ is continuous and bounded and $T(s, t)y(t) = y(s)$ for all $t_0 \geq s \geq t$. Furthermore,

$$y(t_0 + nh) = T(t_0 + nh, t_0)x \in R(P(t_0 + nh))$$

for every negative integer n .

So to prove equality in (5.26), assume that $z \notin R(P(t_0))$ and that there is a function $y : (-\infty, t_0] \rightarrow X$ such that $y(t_0) = z$ and $T(s, t)y(t) = y(s)$ for all $t_0 \geq s \geq t$. Clearly $T(t_0, t_0 + nh)y(t_0 + nh) = z$ and

$$\begin{aligned} P(t_0)z + (I - P(t_0))z &= T(t_0, t_0 + nh)P(t_0 + nh)y(t_0 + nh) \\ &\quad + T(t_0, t_0 + nh)(I - P(t_0 + nh))y(t_0 + nh). \end{aligned}$$

Consequently,

$$T(t_0, t_0 + nh)P(t_0 + nh)y(t_0 + nh) = P(t_0)z$$

and

$$T(t_0, t_0 + nh)(I - P(t_0 + nh))y(t_0 + nh) = (I - P(t_0))z.$$

Hence $P(t_0 + nh)y(t_0 + nh) = T(t_0 + nh, t_0)P(t_0)z$ remains bounded as $n \rightarrow -\infty$, whereas

$$\|(I - P(t_0))z\|_X \leq Me^{\omega nh} \|(I - P(t_0 + nh))y(t_0 + nh)\|_X,$$

and hence

$$\|(I - P(t_0 + nh))y(t_0 + nh)\|_X \geq M^{-1}e^{-\omega nh} \|(I - P(t_0))z\|_X \rightarrow \infty \quad \text{as } n \rightarrow -\infty,$$

which shows that $y : (-\infty, t_0] \rightarrow X$ is not bounded and completes the proof of (5.26).

It remains to prove that $T(t, t_0)R(P(t_0)) = R(P(t))$ for all $t \geq 0$. If $z \in R(P(t_0))$, then let $y : (-\infty, t_0] \rightarrow X$ be such that $y(t_0) = z$ and $T(s, r)y(r) = y(s)$ for all $t_0 \geq s \geq r$. Defining $\bar{y} : (-\infty, t] \rightarrow X$ by $\bar{y}(r) = y(r)$, $r \leq t_0$ and $\bar{y}(r) = T(r, t_0)z$ for $r \in [t_0, t]$, it follows that $\bar{y}(t) = T(t, t_0)z$ and $T(s, r)\bar{y}(r) = \bar{y}(s)$ for all $t \geq s \geq r$. The characterisation of $R(P(t))$ in (5.26) now guarantees that $T(t, t_0)z \in R(P(t))$. On the other hand, if $z \in R(P(t))$ for $t \geq t_0 \geq t + nh$, then $w = T(t_0, t + nh)T(t + nh, t)z \in R(P(t_0))$ and $T(t, t_0)w = z$.

It follows that

$$T(t, t_0)|_{R(P(t_0))} : R(P(t_0)) \rightarrow R(P(t))$$

is an isomorphism, which completes the proof of part (ii) of the definition.

Now, we can finally prove the commutative property of projections. Observe that

$$P(t)T(t, s)x = P(t)T(t, s)(P(s)x + (I - P(s))x) = P(t)T(t, s)P(s)x.$$

Since $(I - P(s))x \in \text{Ker}(P(s))$, (5.25) implies that $T(t, s)(I - P(s))x \in \text{Ker}(P(t))$, and hence $P(t)T(t, s)(I - P(s))x = 0$. Since $P(s)x \in R(P(s))$, $T(t, s)P(s)x \in R(P(t))$, i.e. $P(t)T(t, s)P(s)x = T(t, s)P(s)x$, and thus $P(t)T(t, s) = T(t, s)P(s)$ for all $t \geq s$, as required. $\square$

The following results are the continuous version of Theorems 17 and 18.

Theorem 21. Assume that the linear continuous process $\{T(t, s) : (t, s) \in \mathcal{P}\}$ has an exponential dichotomy with projection $P(\cdot) = \{P(t) : t \in \mathbb{R}\}$, constant M , and exponent ω and that

$$L := \sup_{0 \leq t-s \leq 1} \|T(t, s)\|_{\mathcal{L}(X)} < \infty. \quad (5.27)$$

If $\omega_1 < \omega$ and $M_1 > M$, there exists an $\varepsilon > 0$ such that any process satisfying

$$\sup_{0 \leq t-s \leq 1} \|T(t, s) - S(t, s)\|_{\mathcal{L}(X)} \leq \varepsilon \quad (5.28)$$

has an exponential dichotomy with projection $\tilde{P}(\cdot)$, constant M_1 , and exponent ω_1 . Furthermore,

$$\sup_{t \in \mathbb{R}} \|P(t) - \tilde{P}(t)\|_{\mathcal{L}(X)} \leq \frac{2M_1^2}{1 - e^{-\omega_1}} \varepsilon, \forall t \in \mathbb{R}.$$

Proof. Let $h > 0$ be such that $Me^{-\omega h} \leq e^{-\omega_1 h}$. For each $t_0 \in \mathbb{R}$ set $t_n = t_0 + nh$ for $n \in \mathbb{Z}$. It follows from Theorem 19 that $\{T(t_{n+1}, t_n)\}_{n \in \mathbb{Z}}$ has a discrete dichotomy with constant M and exponent ωh .

To begin, we show that $S(t, s)$ is close to $T(t, s)$ for $0 \leq t - s \leq h$. Choose $\sigma_k \in [s, t]$ such that $\sigma_k = s + k$ with $k \in \mathbb{N}$ and $\sigma_k + 1 > t$. Then

$$\begin{aligned} & \|T(t, s) - S(t, s)\|_{\mathcal{L}(X)} \\ &= \|[T(t, \sigma_k) - S(t, \sigma_k)]S(\sigma_k, s) - T(t, \sigma_k)[S(\sigma_k, s) - T(\sigma_k, s)]\|_{\mathcal{L}(X)}, \end{aligned}$$

and, combining (5.27) and (5.28),

$$\|S(\sigma_k, s)\|_{\mathcal{L}(X)} = \|S(\sigma_k, \sigma_{k-1}) \cdots S(s+1, s)\|_{\mathcal{L}(X)} \leq (L + \varepsilon)^k.$$

Thus,

$$\|T(t, s) - S(t, s)\|_{\mathcal{L}(X)} \leq (L + \varepsilon)^k \varepsilon + L \|S(\sigma_k, s) - T(\sigma_k, s)\|_{\mathcal{L}(X)}.$$

Similarly,

$$\begin{aligned} & \|S(\sigma_k, s) - T(\sigma_k, s)\|_{\mathcal{L}(X)} \\ &= \|[T(\sigma_k, \sigma_{k-1}) - S(\sigma_k, \sigma_{k-1})]S(\sigma_{k-1}, s) \\ &\quad - T(\sigma_k, \sigma_{k-1})[S(\sigma_{k-1}, s) - T(\sigma_{k-1}, s)]\|_{\mathcal{L}(X)} \\ &\leq (L + \varepsilon)^{k-1} \varepsilon + L \|S(\sigma_{k-1}, s) - T(\sigma_{k-1}, s)\|_{\mathcal{L}(X)}. \end{aligned}$$

Hence, for all $0 \leq t - s \leq h$

$$\|T(t, s) - S(t, s)\|_{\mathcal{L}(X)} \leq [(L + \varepsilon)^k + L(L + \varepsilon)^{k-1} + \cdots + L^k] \varepsilon \leq c_h \varepsilon.$$

This means that,

$$\|T(t_{n+1}, t_n) - S(t_{n+1}, t_n)\|_{\mathcal{L}(X)} \leq c_h \varepsilon \quad \text{for all } n \in \mathbb{Z},$$

which implies, from Theorem 17, that the process $\{S(t_{n+1}, t_n)\}_{n \in \mathbb{Z}}$ has a discrete dichotomy with constant M_1 and exponent $\omega_1 h$, for suitably small ε , uniformly for $t_0 \in \mathbb{R}$. It then follows from Theorem 20 that $S(\cdot, \cdot)$ has an exponential dichotomy with constant KM_1 and exponent ω_1 .

It remains to show that the constant can be taken as M_1 instead of KM_1 for suitably chosen ε . If $\tilde{G}(t, s)$ is the Green's function for $S(\cdot, \cdot)$, then

$$\|\tilde{G}(t, s)\|_{\mathcal{L}(X)} \leq M_1 e^{-\omega_1 |t-s|},$$

whenever $t = s + nh$ for some $n \in \mathbb{Z}$ (this follows directly from the discrete dichotomy of $\{T(s + nh, s)\}_{n \in \mathbb{Z}}$). If $G(t, s)$ denotes the Green's function for $T(\cdot, \cdot)$ and $t_n = t_0 + nh$, then from (5.16)

$$\tilde{G}(t_n, t_m) - G(t_n, t_m) = \sum_{-\infty}^{\infty} G(t_n, t_{k+1}) (S(t_{k+1}, t_k) - T(t_{k+1}, t_k)) \tilde{G}(t_k, t_m),$$

and hence

$$\|\tilde{G}(t_n, t_m) - G(t_n, t_m)\|_{\mathcal{L}(X)} \leq c\varepsilon,$$

using the bound on $G(\cdot, \cdot)$ and $\tilde{G}(\cdot, \cdot)$ that follow from the exponential dichotomies of $T(\cdot, \cdot)$ and $S(\cdot, \cdot)$.

If $t \geq s$, then choose r such that $0 \leq t - r < h$ and $\frac{r-s}{h} \in \mathbb{N}$, then

$$\|\tilde{G}(t, s) - G(t, s)\|_{\mathcal{L}(X)} = \|(S(t, r) - T(t, r))\tilde{G}(r, s) + T(t, r)(\tilde{G}(r, s) - G(r, s))\|_{\mathcal{L}(X)},$$

and hence

$$\|\tilde{G}(t, s) - G(t, s)\|_{\mathcal{L}(X)} \leq cM_1\varepsilon + c\varepsilon \sup\{\|T(t, r)\|, 0 \leq t - r \leq h\}.$$

On the other hand, if $s \geq t$, then choose r such that $0 \leq r - s \leq h$ and $\frac{r-t}{h} \in \mathbb{N}$; then

$$\begin{aligned} \|\tilde{G}(t, s) - G(t, s)\|_{\mathcal{L}(X)} &= \|(\tilde{G}(t, r) - G(t, r))T(r, s)P(s) - \tilde{G}(t, r)(T(r, s) - S(r, s))P(s)\|_{\mathcal{L}(X)} \\ &\leq c \sup_{0 \leq r-s \leq h} \|T(r, s)P(s)\|_{\mathcal{L}(X)} \varepsilon + cM_1 M \varepsilon =: R\varepsilon. \end{aligned}$$

So for ε suitably small and for $0 \leq |t - s| \leq h$ we have

$$\begin{aligned} e^{\omega_1 |t-s|} \|\tilde{G}(t, s)\|_{\mathcal{L}(X)} &\leq e^{\omega_1 |t-s|} \|G(t, s)\|_{\mathcal{L}(X)} + e^{\omega_1 |t-s|} \|\tilde{G}(t, s) - G(t, s)\|_{\mathcal{L}(X)} \\ &\leq M e^{-(\omega - \omega_1)|t-s|} + e^{\omega h} R\varepsilon \leq M_1. \end{aligned} \tag{5.29}$$

In particular, if $|t - s| = h$, then

$$e^{\omega_1 h} \|\tilde{G}(t, s)\|_{\mathcal{L}(X)} \leq M e^{-(\omega - \omega_1)h} + e^{\omega_1 h} R\varepsilon \leq 1$$

from the choice of h and for suitably small ε . Hence

$$\|\tilde{G}(t, s)\|_{\mathcal{L}(X)} \leq e^{-\omega_1 h}. \tag{5.30}$$

Now, if t and s are any real numbers, r is between t , and s and satisfies

$$|t - r| < h \quad \text{and} \quad \frac{|r - s|}{h} = m + 1 \in \mathbb{Z},$$

then

$$\tilde{G}(t, s) = \tilde{G}(t, r) \tilde{G}(r, r \pm h) \cdots \tilde{G}(r \pm mh, s)$$

and, from (5.29) and (5.30),

$$\|\tilde{G}(t, s)\|_{\mathcal{L}(X)} \leq \|\tilde{G}(t, r)\|_{\mathcal{L}(X)} e^{-\omega_1 |h(m+1)|} \leq M_1 e^{-\omega_1 |t-r|} e^{-\omega_1 |r-s|} = M_1 e^{-\omega_1 |t-s|}.$$

It follows that $S(\cdot, \cdot)$ has an exponential dichotomy with constant M_1 and exponent ω_1 , as required.

The estimate on the difference of the projections follows immediately from Theorem 18. $\square$

5.2 Global Hyperbolic Solution

Lemma 24 (Gronwall's Lemma). Let $X \in L^1((0, T), \mathbb{R}^+)$, $a \in L^\infty((0, T), \mathbb{R})$ increasing and $b \in L^\infty((0, T), \mathbb{R}^+)$ such that

$$X(t) \leq a(t) + \int_0^t b(s)X(s)ds, \text{ for a.e. } t \in (0, T). \quad (5.31)$$

Then

$$X(t) \leq a(t) e^{\int_0^t b(s)ds}.$$

Proof. Define Y as

$$Y(t) = \int_0^t b(s)X(s)ds, \forall t \in (0, T).$$

From (5.31) we have

$$\frac{dY}{dt} = b(t)X(t) \leq a(t)b(t) + b(t) \int_0^t b(s)X(s)ds = a(t)b(t) + b(t)Y(t).$$

Thus

$$\begin{aligned} \frac{d}{dt} \left(Y(t) e^{-\int_0^t b(s)ds} \right) &= \frac{d}{dt} Y(t) e^{-\int_0^t b(s)ds} - b(t) Y(t) e^{-\int_0^t b(s)ds} \\ &\leq (a(t)b(t) + b(t)Y(t)) e^{-\int_0^t b(s)ds} - b(t)Y(t) e^{-\int_0^t b(s)ds} \\ &= a(t)b(t) e^{-\int_0^t b(s)ds}. \end{aligned}$$

Since $a(\cdot)$ is increasing, we can integrate and obtain

$$\begin{aligned} Y(t)e^{-\int_0^t b(s)ds} &\leq \int_0^t a(r)b(r)e^{-\int_0^r b(s)ds} dr \\ &\leq a(t) \int_0^t b(r)e^{-\int_0^r b(s)ds} dr \\ &\leq a(t) \left(1 - e^{-\int_0^t b(s)ds} \right). \end{aligned}$$

Hence $Y(t) + a(t) \leq a(t)e^{-\int_0^t b(s)ds}$ and from (5.31) we have $X(t) \leq a(t)e^{-\int_0^t b(s)ds}$. $\square$

The proof of the following Lemma can be found in section 7.1 of (HENRY, 2013).

Lemma 25 (Singular Gronwall). Let $X \in L^1((0, \infty), \mathbb{R}^+)$, $\alpha \in (0, 1)$, $\gamma \in [0, 1)$ and $a, b \in [0, \infty)$ such that

$$X(t) \leq at^{-\gamma} + b \int_0^t (t-s)^{1-\alpha} X(s) ds \text{ for a.e. } t \in (0, \infty).$$

Then, for each $K > (2b\Gamma(\alpha))^{\frac{1}{2}}$, where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx,$$

there exists M such that

$$X(t) \leq \frac{Mat^{-\alpha}e^{Kt}}{1-\alpha}, \text{ for a.e. } t \in (0, \infty).$$

If $f : \mathbb{R} \times X \rightarrow X$ is continuous on the first variable, locally Lipschitz continuous on the second variable and $\{T(t, \tau) : (t, \tau) \in \mathcal{P}\}$ is a linear continuous process satisfying (5.27), then (proceeding as in Picard Theorem) the integral equation

$$y(t, \tau, x) = T(t, \tau)x + \int_\tau^t T(t, s)f(s, y(s, \tau, x))ds \quad (5.32)$$

has an unique local solution, that is, for each $(\tau, x) \in \mathbb{R} \times X$ there exists $\sigma = \sigma(\tau, x) > 0$ and and unique continuous function $y(\cdot, \tau, x) : [\tau, \tau + \sigma) \rightarrow X$ satisfying (5.32) for all $t \in [\tau, \tau + \sigma)$. Assuming that $\sigma(\tau, x) = +\infty$ for each $(\tau, x) \in \mathbb{R} \times X$ and defining $S(t, \tau)x = y(t, \tau, x)$, $t \in [\tau, \infty)$, then $\mathcal{S} = \{S(t, \tau) : (t, \tau) \in \mathcal{P}\}$ is an evolution process. In this case, we say that $\mathcal{S}$ is the semilinear evolution process obtained by the perturbation of the linear evolution process $\{T(t, \tau) : (t, \tau) \in \mathcal{P}\}$ by the nonlinear function f , that is,

$$S(t, \tau)x = T(t, \tau)x + \int_\tau^t T(t, s)f(s, S(s, \tau)x)ds \quad (5.33)$$

Definition 41 (Hyperbolic Global Solution). Let $\xi : \mathbb{R} \rightarrow X$ be a global solution of the semilinear evolution process $\mathcal{S} = \{S(t, s) : (t, s) \in \mathcal{P}\}$, as in (5.33), and consider the associated linear process L_ξ that is the solution of

$$L_\xi(t, \tau) = T(t, \tau) + \int_\tau^t T(t, s) D_x f(s, \xi(s)) L_\xi(s, \tau) ds. \quad (5.34)$$

We say that $\xi : \mathbb{R} \rightarrow X$ is a **hyperbolic global solution** of $\mathcal{S}$ if L_ξ has an exponential dichotomy.

Remark 15.

If $\mathcal{S}$ is an evolution process which is related to the Cauchy problem

$$\begin{cases} \dot{y} = f(t, y), & t > \tau \\ y(\tau) = y_0 \in \mathbb{R}^m, \end{cases}$$

then, from the variation of constants formula, ξ is a hyperbolic global solution of $\mathcal{S}$ if the evolution process associated to

$$\begin{cases} \dot{y} = D_y f(t, \xi(t))y, & t > \tau \\ y(\tau) = y_0 \in \mathbb{R}^m, \end{cases} \quad (5.35)$$

has an exponential dichotomy.

Lemma 26. Let $\xi : \mathbb{R} \rightarrow X$ be a bounded hyperbolic global solution of the semilinear process S , that is, $L_\xi(\cdot, \cdot)$ has an exponential dichotomy with projection $P(\cdot)$, constant M , and exponent ω . Then, if $\phi : \mathbb{R} \rightarrow X$ is a bounded global solution of S , we have

$$\phi(t) = \int_{-\infty}^{\infty} G_\xi(t, s) [f(s, \phi(s)) - D_x f(s, \xi(s))\phi(s)] ds,$$

where

$$G_\xi(t, s) = \begin{cases} L_\xi(t, s)(I - P(s)), & t \geq s \\ -L_\xi(t, s)P(s), & t \leq s. \end{cases} \quad (5.36)$$

Proof. Firstly, we will show that if $\phi : [\tau, \tau + \sigma] \rightarrow X$ is any solution of

$$\phi(t) = T(t, \tau)\phi(\tau) + \int_\tau^t T(t, s)g(s, \phi(s)) ds,$$

then

$$\phi(t) = L_\xi(t, \tau)\phi(\tau) + \int_\tau^t L_\xi(t, s)[g(s, \phi(s)) - D_x f(s, \xi(s))\phi(s)] ds. \quad (5.37)$$

In fact, if

$$\psi(t) := L_\xi(t, \tau)\phi(\tau) + \int_\tau^t L_\xi(t, s)[g(s, \phi(s)) - D_x f(s, \xi(s))\phi(s)] ds,$$

then,

$$\begin{aligned}
\psi(t) - \phi(t) &= \int_{\tau}^t T(t,s) D_x f(s, \xi(s)) L_{\xi}(s, \tau) \phi(\tau) ds - \int_{\tau}^t T(t,s) D_x f(s, \xi(s)) \phi(s) ds \\
&\quad + \int_{\tau}^t \int_s^t T(t, \theta) D_x f(\theta, \xi(\theta)) L_{\xi}(\theta, s) d\theta [g(s, \phi(s)) - D_x f(s, \xi(s)) \phi(s)] ds \\
&= \int_{\tau}^t T(t,s) D_x f(s, \xi(s)) L_{\xi}(s, \tau) \phi(\tau) ds - \int_{\tau}^t T(t,s) D_x f(s, \xi(s)) \phi(s) ds \\
&\quad + \int_{\tau}^t T(t, \theta) D_x f(\theta, \xi(\theta)) \int_{\tau}^{\theta} L_{\xi}(\theta, s) [g(s, \phi(s)) - D_x f(s, \xi(s)) \phi(s)] ds d\theta \\
&= \int_{\tau}^t T(t,s) D_x f(s, \xi(s)) [\psi(s) - \phi(s)] ds,
\end{aligned}$$

and, by Lemma 25, we have $\phi(t) = \psi(t)$ for all $t \in [\tau, \tau + \sigma]$.

Note that, applying $P(t)$ to (5.37), we have

$$P(t)\phi(t) = L_{\xi}(t, \tau)P(\tau)\phi(\tau) + \int_{\tau}^t L_{\xi}(t, s)P(s)[f(s, \phi(s)) - D_x f(s, \xi(s))\phi(s)] ds,$$

and consequently

$$L_{\xi}(\tau, t)P(t)\phi(t) = P(\tau)\phi(\tau) + \int_{\tau}^t L_{\xi}(\tau, s)P(s)[f(s, \phi(s)) - D_x f(s, \xi(s))\phi(s)] ds.$$

Letting $t \rightarrow \infty$ we obtain

$$P(t)\phi(t) = - \int_t^{\infty} L_{\xi}(t, s)P(s)[f(s, \phi(s)) - D_x f(s, \xi(s))\phi(s)] ds, \quad t \in \mathbb{R}.$$

Similarly,

$$\begin{aligned}
(I - P(t))\phi(t) &= L_{\xi}(t, \tau)(I - P(\tau))\phi(\tau) \\
&\quad + \int_{\tau}^t L_{\xi}(t, s)(I - P(s))[f(s, \phi(s)) - D_x f(s, \xi(s))\phi(s)] ds,
\end{aligned}$$

and sending $\tau \rightarrow -\infty$ we obtain

$$(I - P(t))\phi(t) = \int_{-\infty}^t L_{\xi}(t, s)(I - P(s))[f(s, \phi(s)) - D_x f(s, \xi(s))\phi(s)] ds.$$

Using this inequalities, we conclude the proof. $\square$

Let $T(\cdot, \cdot)$ be a linear process and $f, g : \mathbb{R} \times X \rightarrow X$ two continuously differentiable functions such that the corresponding semilinear processes

$$S_f(t, \tau)x = T(t, \tau)x + \int_{\tau}^t T(t, s)f(s, S_f(s, \tau)x) ds, \quad \forall t \geq \tau, \forall x \in X$$

and

$$S_g(t, \tau)x = T(t, \tau)x + \int_{\tau}^t T(t, s)g(s, S_g(s, \tau)x) ds, \quad \forall t \geq \tau, \forall x \in X \quad (5.38)$$

are well defined.

The following result plays an important role in our theory and is directly connected with our main problem, the cascade system (Chapter 6).

Lemma 27. Assume that $\xi : \mathbb{R} \rightarrow X$ is a bounded hyperbolic global solution for $S_f(\cdot, \cdot)$, where $\sup_{t \in \mathbb{R}} \|\xi(t)\|_X \leq B < \infty$ and the associated exponential dichotomy has constant M and exponent ω such that

$$\sup_{\|x\|_X \leq M} \|f(t, x)\|_X + \|D_x f(t, x)\|_{\mathcal{L}(X)} < \infty.$$

If $\varepsilon > 0$ satisfies

$$\sup_{\|x\| \leq \varepsilon} \sup_{t \in \mathbb{R}} \|f(t, \xi(t) + x) - f(t, \xi(t)) - D_x f(t, \xi(t))x\|_Y < \frac{\varepsilon}{4M\omega^{-1}} \quad (5.39)$$

and

$$\sup_{\|x\| \leq B} \sup_{t \in \mathbb{R}} \|f(t, x) - g(t, x)\|_X + \|D_x f(t, x) - D_x g(t, x)\|_{\mathcal{L}(X)} < \frac{\varepsilon}{4M\omega^{-1}}, \quad (5.40)$$

then there is a unique global solution $\eta : \mathbb{R} \rightarrow X$ of $S_g(\cdot, \cdot)$ such that

$$\sup_{t \in \mathbb{R}} \|\xi(t) - \eta(t)\|_X < \varepsilon.$$

Moreover, this solution is hyperbolic.

Proof. We know from Lemma 26 that

$$\xi(t) = \int_{-\infty}^{\infty} G_{\xi}(t, s) [f(s, \xi(s)) - D_x f(s, \xi(s))\xi(s)] ds,$$

where G_{ξ} is defined in (5.36).

Now suppose that $y : \mathbb{R} \rightarrow X$ is a bounded global solution of (5.38). Then, using (5.37),

$$y(t) = L_{\xi}(t, \tau)y(\tau) + \int_{\tau}^t L_{\xi}(t, s) [g(s, y(s)) - D_x f(s, \xi(s))y(s)] ds$$

and

$$\xi(t) = L_{\xi}(t, \tau)\xi(\tau) + \int_{\tau}^t L_{\xi}(t, s) [f(s, \xi(s)) - D_x f(s, \xi(s))\xi(s)] ds.$$

Defining $\phi(t) = y(t) - \xi(t)$, $t \in \mathbb{R}$, it follows that

$$\phi(t) = L_{\xi}(t, \tau)\phi(\tau) + \int_{\tau}^t L_{\xi}(t, s) \tilde{g}(s, (\phi(s))) ds, \quad (5.41)$$

where $\tilde{g}(t, \phi) = g(t, \phi(t) + \xi(t)) - f(t, \xi(t)) - D_x f(t, \xi(t))\phi(t)$.

Applying $I - P(t)$ to (5.41) and taking the limit as $\tau \rightarrow -\infty$ gives

$$(I - P(t))\phi(t) = \int_{-\infty}^t L_{\xi}(t, s) (I - P(s)) \tilde{g}(s, (\phi(s))) ds.$$

Applying $P(t)$ to (5.41) yields, for $t \geq \tau$,

$$P(t)\phi(t) = L_{\xi}(t, \tau)P(\tau)\phi(\tau) + \int_{\tau}^t L_{\xi}(t, s) P(s) \tilde{g}(s, (\phi(s))) ds,$$

and consequently

$$L_\xi(\tau, t)P(t)\phi(t) = P(\tau)\phi(\tau) + \int_\tau^t L_\xi(\tau, s)P(s)\tilde{g}(s, (\phi(s))) ds.$$

Taking $t \rightarrow \infty$ we conclude

$$P(\tau)\phi(\tau) = - \int_\tau^\infty L_\xi(\tau, s)P(s)\tilde{g}(s, (\phi(s))) ds.$$

This implies that a bounded global solution of (5.41) exists in a small neighbourhood of $x = 0$ if and only if

$$\begin{aligned} \mathcal{T}(\phi)(t) &= - \int_t^\infty L_\xi(t, s)P(s)\tilde{g}(s, (\phi(s))) ds + \int_{-\infty}^t L_\xi(t, s)(I - P(s))\tilde{g}(s, (\phi(s))) ds \\ &= \int_{-\infty}^\infty G_\xi(t, s)\tilde{g}(s, (\phi(s))) ds \end{aligned}$$

has a fixed point in the set

$$B_\varepsilon := \{\phi : \mathbb{R} \rightarrow X : \phi \text{ is continuous and } \sup_{t \in \mathbb{R}} \|\phi(t)\|_X \leq \varepsilon\}$$

for ε sufficiently small.

Using the exponential dichotomy of $L_\xi(\cdot, \cdot)$ and the Contraction Mapping Theorem we can show that there is a unique global bounded solution of (5.41) in B_ε . To see that $\mathcal{T}$ is well defined, we can use (5.39) to conclude that

$$\begin{aligned} \|\mathcal{T}(\phi)(t)\|_X &\leq M \int_{-\infty}^\infty e^{-\omega|t-s|} \|\tilde{g}(s, (\phi(s)))\|_X ds \\ &\leq 2M\omega^{-1} \sup_{t \in \mathbb{R}} \|g(t, y(t)) - f(t, y(t))\|_X \\ &\quad + 2M\omega^{-1} \sup_{\|x\| \leq \varepsilon} \sup_{t \in \mathbb{R}} \|f(t, \xi(t) + x) - f(t, \xi(t)) - D_x f(t, \xi(t))x\|_X \\ &\leq \varepsilon. \end{aligned}$$

Using (5.40), we can show that, for ε sufficiently small,

$$\|\mathcal{T}(\phi_1)(t) - \mathcal{T}(\phi_2)(t)\|_X \leq \frac{1}{2} \sup_{t \in \mathbb{R}} \|\phi_1(t) - \phi_2(t)\|_X,$$

that is, $\mathcal{T}$ is a contraction and there is a unique global solution $\phi : \mathbb{R} \rightarrow X$ of (5.41) in B_ε . Now define $\eta(\cdot) = \phi(\cdot) + \xi(\cdot)$.

Since $\eta : \mathbb{R} \rightarrow X$ is uniformly close to $\xi : \mathbb{R} \rightarrow X$, it follows from (5.40) and from Theorem 21 that, for suitably small ε ,

$$\begin{aligned} L_\eta(t, \tau) &= T(t, \tau) + \int_\tau^t T(t, s)D_x g(s, \eta(s))L_\eta(s, \tau) ds \\ &= L_\xi(t, \tau) + \int_\tau^t L_\xi(t, s)[D_x g(s, \eta(s)) - D_x f(s, \xi(s))][L_\eta(s, \tau) - L_\xi(s, \tau)] ds \\ &\quad + \int_\tau^t L_\xi(t, s)[D_x g(s, \eta(s)) - D_x f(s, \xi(s))]L_\xi(s, \tau) ds \end{aligned}$$

has an exponential dichotomy, and consequently η is hyperbolic. □

5.3 Unstable and Stable Manifolds

Throughout this section we will continue to assume that X is a Banach space and $S(\cdot, \cdot)$ is given by

$$S(t, \tau)x = T(t, \tau)x + \int_{\tau}^t T(t, s)f(s, S(s, \tau)x) ds, \quad (5.42)$$

where $T(\cdot, \cdot)$ is a linear process and $f : \mathbb{R} \times X \rightarrow X$ is continuously differentiable. Also, $\xi : \mathbb{R} \rightarrow X$ will be a hyperbolic global solution for $S(\cdot, \cdot)$, denoting the corresponding linear process by $L_{\xi}(\cdot, \cdot)$ as in (5.34), and let $P(\cdot)$, M , and ω be respectively the projection, constant, and exponent of the exponential dichotomy of L_{ξ} . We will define $Q(t) := I - P(t)$ for every $t \in \mathbb{R}$.

Definition 42 (Unstable and Stable Manifolds). The unstable manifold of $\xi(\cdot)$ is the time-dependent family $W^u(\xi)(\cdot)$ whose sections are

$$W^u(\xi)(\tau) = \left\{ x \in X : \text{there is a global solution } \eta : \mathbb{R} \rightarrow X \text{ of (5.42)} \right. \\ \left. \text{with } \eta(\tau) = x \text{ and } \lim_{t \rightarrow -\infty} \|\eta(t) - \xi(t)\|_X = 0 \right\},$$

for each $\tau \in \mathbb{R}$. Similarly, the stable manifold of $\xi(\cdot)$, denoted by $W^s(\xi)(\cdot)$, has sections

$$W^s(\xi)(\tau) = \{x \in X : \lim_{t \rightarrow +\infty} \|S(t, \tau)x - \xi(t)\|_X = 0\},$$

for each $\tau \in \mathbb{R}$.

In this section, our main goal is to show that the unstable and stable manifolds of $\xi(\cdot)$ can be given in terms of time-dependent Lipschitz maps $\Sigma^u(t, \cdot) : P(t)X \rightarrow Q(t)X$ and $\Sigma^s(t, \cdot) : Q(t)X \rightarrow P(t)X$, so that

$$W^u(\xi)(t) = \{\xi(t) + p + \Sigma^u(t, p) : p \in P(t)X\}$$

and

$$W^s(\xi)(t) = \{\xi(t) + q + \Sigma^s(t, q) : q \in Q(t)X\}.$$

Geometrically, we will show that the unstable manifold is locally a graph. This information will be useful to fulfill condition 4 from Theorem 14, which is an essential tool to solve our main problem.

We know that $y(t)$ is a solution of (5.42) if and only if $y(t) = x(t) + \xi(t)$, where $x(t)$ satisfies

$$x(t) = L_{\xi}(t, \tau)x(\tau) + \int_{\tau}^t L_{\xi}(t, s)[f(s, x(s) + \xi(s)) - f(s, \xi(s)) - D_x f(s, \xi(s))x(s)] ds. \quad (5.43)$$

This way, if $x(t)$ is a solution of (5.43) and

$$p(t) = P(t)x(t) \quad \text{and} \quad q(t) = Q(t)x(t),$$

then

$$p(t) = L_{\xi}(t, \tau)p(\tau) + \int_{\tau}^t L_{\xi}(t, s)P(s)F(s, p(s) + q(s)) ds \quad (5.44)$$

and

$$q(t) = L_\xi(t, \tau)q(\tau) + \int_\tau^t L_\xi(t, s)Q(s)F(s, p(s) + q(s))ds, \quad (5.45)$$

where

$$F(t, x) = f(t, x + \xi(t)) - f(t, \xi(t)) - D_x f(t, \xi(t))x.$$

Note that at $(t, 0)$ the function F is zero with zero derivatives (with respect to x). It follows from the continuous differentiability of F , uniform with respect to t , that given $\rho > 0$ there exists $\delta > 0$ such that if $\|x\|_X, \|\tilde{x}\|_X < \delta$, then

$$\sup_{t \in \mathbb{R}} \|F(t, x)\|_X \leq \rho \quad \text{and} \quad \sup_{t \in \mathbb{R}} \|F(t, x) - F(t, \tilde{x})\|_X \leq \rho \|x - \tilde{x}\|_X. \quad (5.46)$$

Even though the result will be local, we can assume that (5.46) holds for all x , because we can extend F in such a way that these inequalities continue to hold for all $x \in X$. Indeed, given a W -valued function g on the ball of radius δ contained in $V \times Z$, where V, Z, W are Banach spaces, define $g_\delta : V \times Z \rightarrow W$ by

$$g_\delta(x) = \begin{cases} g(x) & \|x\|_X \leq \delta \\ g(\delta x / \|x\|_X) & \|x\|_X > \delta; \end{cases} \quad (5.47)$$

this extension is globally Lipschitz, and its Lipschitz constant is the Lipschitz constant for g restricted to the ball of radius δ .

Under the assumption that (5.46) holds for all $x \in X$ for some fixed $\rho > 0$ sufficiently small (which we make precise below), we will show that there exists a bounded and Lipschitz continuous function $\Sigma^u(t, \cdot) : P(t)X \rightarrow Q(t)X$ such that

$$W^u(0)(t) = \{p + \Sigma^u(t, p) : p \in P(t)X\},$$

where $W^u(0)(\cdot)$ is the unstable manifold of the equilibrium solution 0 of (5.43).

To find Σ^u , we will use the Contraction Mapping Theorem on the set

$$\text{Lip}^u(D, L) := \left\{ \Sigma : \mathbb{R} \times X \rightarrow X : \Sigma(t, \cdot) : P(t)X \rightarrow Q(t)X, \right. \\ \left. \sup_{p \in P(t)X} \sup_{t \in \mathbb{R}} \|\Sigma(t, p)\|_X \leq D, \text{ and } \|\Sigma(t, p) - \Sigma(t, \tilde{p})\|_X \leq L\|p - \tilde{p}\|_X \right\}$$

which is a complete metric space equipped with the metric

$$\|\Sigma(\cdot, \cdot) - \tilde{\Sigma}(\cdot, \cdot)\|_\infty := \sup_{t \in \mathbb{R}} \sup_{p \in P(t)X} \|\Sigma(t, p) - \tilde{\Sigma}(t, p)\|_X.$$

Theorem 22. Fix $D > 0, L > 0$, and ϑ with $0 < \vartheta < 1$, and then choose $\rho > 0$ such that

$$\frac{\rho M}{\omega} \leq D, \quad \frac{\rho M}{\omega}(1 + L) \leq \vartheta < 1, \quad (5.48)$$

and

$$\frac{\rho M^2(1+L)}{\omega - \rho M(1+L)} \leq L, \quad \frac{2\omega\rho M + \rho^2 M^3(1+L)}{2\omega - \rho M(1+L)} < (1 - \vartheta)\omega. \quad (5.49)$$

If (5.46) holds for all $x \in X$, then there exists a function $\Sigma^u(t, \cdot) \in \text{Lip}^u(D, L)$ such that the unstable manifold $W^u(0)(\cdot)$ of zero for (5.43) is given by

$$W^u(0)(t) = \{p + \Sigma^u(t, p) : p \in P(t)X\}.$$

Furthermore, if $x(t) = p(t) + q(t)$, $t \geq t_0$, is a solution of (5.44), there are constants $M \geq 1$ and $\gamma > 0$ such that

$$\|q(t) - \Sigma^u(t, p(t))\|_X \leq M e^{-\gamma(t-t_0)} \|q(t_0) - \Sigma^u(t, p(t_0))\|_X \quad \text{for all } t \geq t_0, \quad (5.50)$$

and if $x(t) = p(t) + q(t)$, $t \in \mathbb{R}$, is a global solution of (5.44) with the property that $q(\tau) = \Sigma^u(\tau, p(\tau))$, then

$$\|p(t)\|_X \leq M \|p(\tau)\|_X e^{(\omega - \rho M(1+L))(t-\tau)}, \quad t \leq \tau.$$

Proof. Fix $\tau \in \mathbb{R}$; for arbitrary $\zeta \in P(\tau)X$ and $\Sigma \in \text{Lip}^u(D, L)$ denote by $p(t) = \psi(t, \tau, \zeta, \Sigma)$, defined for $t < \tau$, the solution of

$$p(t) = L_\xi(t, \tau)\zeta + \int_\tau^t L_\xi(t, s)P(s)F(s, p(s) + \Sigma(s, p(s)))ds. \quad (5.51)$$

Next we define the mapping $\mathcal{T}$ by setting, for $\Sigma \in \text{Lip}^u(D, L)$ and for each choice of $\tau \in \mathbb{R}$ and $\zeta \in P(\tau)X$,

$$\mathcal{T}(\Sigma)(\tau, \zeta) = \int_{-\infty}^\tau L_\xi(\tau, s)Q(s)F(s, p(s) + \Sigma(s, p(s)))ds.$$

We will show that, for $\rho > 0$ satisfying (5.48), the map $\mathcal{T}$ takes $\text{Lip}^u(D, L)$ into itself, is a strict contraction, and, hence, possesses a unique fixed point in $\text{Lip}^u(D, L)$.

First note that since $L_\xi(\cdot, \cdot)$ has an exponential dichotomy with constant M and exponent ω , it follows from (5.46) that

$$\|\mathcal{T}(\Sigma)(\tau, P(\tau)\zeta)\|_X \leq \int_{-\infty}^\tau \rho M e^{-\omega(\tau-s)} ds = \frac{\rho M}{\omega},$$

and from (5.48) we have

$$\sup_{(\tau, x) \in \mathbb{R} \times X} \|\mathcal{T}(\Sigma)(\tau, P(\tau)x)\|_\infty \leq D. \quad (5.52)$$

We now estimate the difference between two solutions of (5.51) corresponding to different choices of Σ and ζ . To this end, suppose that $\Sigma, \tilde{\Sigma} \in \text{Lip}^u(D, L)$, $\tau \in \mathbb{R}$, and $\zeta, \tilde{\zeta} \in P(\tau)X$; set $p(t) = \psi(t, \tau, \zeta, \Sigma)$ and $\tilde{p}(t) = \psi(t, \tau, \tilde{\zeta}, \tilde{\Sigma})$. Then

$$\begin{aligned} p(t) - \tilde{p}(t) &= L_\xi(t, \tau)P(\tau)(\zeta - \tilde{\zeta}) \\ &\quad + \int_\tau^t L_\xi(t, s)P(s)[F(s, p(s) + \Sigma(s, p(s))) - F(s, \tilde{p}(s) + \tilde{\Sigma}(s, \tilde{p}(s)))]ds, \end{aligned}$$

and with (5.46) and the Lipschitz property of Σ and $\tilde{\Sigma}$ we obtain

$$\begin{aligned} \|p(t) - \tilde{p}(t)\|_X &\leq M e^{\omega(t-\tau)} \|\zeta - \tilde{\zeta}\|_X \\ &\quad + M \int_t^\tau e^{\omega(t-s)} \|F(s, p(s) + \Sigma(s, p(s))) - F(s, \tilde{p}(s) + \tilde{\Sigma}(s, \tilde{p}(s)))\|_X ds \\ &\leq M e^{\omega(t-\tau)} \|\zeta - \tilde{\zeta}\|_X + \rho M \|\Sigma - \tilde{\Sigma}\|_\infty \int_t^\tau e^{\omega(t-s)} ds \\ &\quad + \rho M (1+L) \int_t^\tau e^{\omega(t-s)} \|p(s) - \tilde{p}(s)\|_X ds. \end{aligned}$$

Let $\phi(t) = e^{-\omega(t-\tau)} \|p(t) - \tilde{p}(t)\|_X$. Then

$$\phi(t) \leq M \|\zeta - \tilde{\zeta}\|_X + \rho M \int_t^\tau e^{\omega(\tau-s)} ds \|\Sigma - \tilde{\Sigma}\|_\infty + \rho M (1+L) \int_t^\tau \phi(s) ds.$$

By Gronwall's Lemma (Lemma 24)

$$\begin{aligned} \|p(t) - \tilde{p}(t)\|_X &\leq [M \|\zeta - \tilde{\zeta}\|_X e^{\omega(t-\tau)} + \rho M \int_t^\tau e^{\omega(t-s)} ds \|\Sigma - \tilde{\Sigma}\|_\infty] e^{-\rho M (1+L)(t-\tau)} \\ &\leq [M \|\zeta - \tilde{\zeta}\|_X + \rho M \omega^{-1} \|\Sigma - \tilde{\Sigma}\|_\infty] e^{-\rho M (1+L)(t-\tau)}. \end{aligned} \quad (5.53)$$

It is now straightforward to show that $\mathcal{T}$ maps $\text{Lip}^u(D, L)$ into itself and is a contraction since

$$\begin{aligned} \|\mathcal{T}(\Sigma)(\tau, \zeta) - \mathcal{T}(\tilde{\Sigma})(\tau, \tilde{\zeta})\|_X &\leq M \int_{-\infty}^\tau e^{-\omega(\tau-s)} \|F(s, p(s) + \Sigma(s, p(s))) - F(s, \tilde{p}(s) + \tilde{\Sigma}(s, \tilde{p}(s)))\|_X ds \\ &\leq \rho M \int_{-\infty}^\tau e^{-\omega(\tau-s)} [(1+L) \|p(s) - \tilde{p}(s)\|_X + \|\Sigma - \tilde{\Sigma}\|_\infty] ds. \end{aligned}$$

Using the estimate for $\|p - \tilde{p}\|_X$ from (5.53) with $\Sigma = \tilde{\Sigma}$ we obtain

$$\begin{aligned} \|\mathcal{T}(\Sigma)(\tau, \zeta) - \mathcal{T}(\Sigma)(\tau, \tilde{\zeta})\|_X &\leq \frac{\rho M^2 (1+L)}{\omega - \rho M (1+L)} \|\zeta - \tilde{\zeta}\|_X \\ &\leq L \|\zeta - \tilde{\zeta}\|, \end{aligned}$$

thanks to our conditions on ρ in (5.49). It follows from this and (5.52) that $\mathcal{T}$ maps $\text{Lip}^u(D, L)$ into itself. Similarly, using (5.53) with $\zeta = \tilde{\zeta}$,

$$\begin{aligned} \|\mathcal{T}(\Sigma)(\tau, \zeta) - \mathcal{T}(\tilde{\Sigma})(\tau, \zeta)\|_X &\leq \frac{\rho M}{\omega} \left[1 + \frac{\rho M (1+L)}{\omega - \rho M (1+L)} \right] \|\Sigma - \tilde{\Sigma}\|_\infty \\ &\leq \vartheta \|\Sigma - \tilde{\Sigma}\|_\infty, \end{aligned}$$

once more thanks to our conditions on ρ (5.49). Since $\vartheta < 1$, this shows that $\mathcal{T}$ is a contraction on $\text{Lip}^u(D, L)$ and so has a unique fixed point $\Sigma^u \in \text{Lip}^u(D, L)$.

It remains to prove that the exponential attraction in (5.50),

$$\|q(t) - \Sigma^u(t, p(t))\|_X \leq M e^{-\gamma(t-t_0)} \|q(t_0) - \Sigma^u(t, p(t_0))\|_X \quad \text{for all } t \geq t_0,$$

holds. To bound $\delta(t) := \|q(t) - \Sigma^u(t, p(t))\|_X$, we will need some auxiliary estimates. Recall that we have a coupled system for $p(t)$ and $q(t)$,

$$\begin{aligned} q(t) &= L_\xi(t, \tau)q(\tau) + \int_\tau^t L_\xi(t, s)Q(s)F(s, p(s) + q(s))ds \\ p(t) &= L_\xi(t, \tau)p(\tau) + \int_\tau^t L_\xi(t, s)P(s)F(s, p(s) + q(s))ds, \end{aligned}$$

and that since Σ^u is a fixed point of $\mathcal{T}$, $\Sigma^u(t, p(t))$ satisfies

$$\Sigma^u(t, p(t)) = \int_{-\infty}^t L_\xi(t, s)Q(s)F(s, \tilde{p}(s; t) + \Sigma^u(s, \tilde{p}(s; t)))ds,$$

where, for $s \leq t$,

$$\tilde{p}(s; t) = L_\xi(s, t)p(t) + \int_t^s L_\xi(s, r)P(r)F(r, \tilde{p}(r; t) + \Sigma^u(r, \tilde{p}(r; t)))dr.$$

For our first estimate, note that

$$\begin{aligned} &\|\tilde{p}(s; t) - p(s)\|_X \\ &= \left\| \int_t^s L_\xi(s, r)P(r)[F(r, \tilde{p}(r; t) + \Sigma^u(r, \tilde{p}(r; t))) - F(r, p(r) + q(r))]dr \right\|_X \\ &\leq \rho M \int_s^t e^{\omega(s-r)} [(1+L)\|\tilde{p}(r; t) - p(r)\|_X + \delta(r)]dr; \end{aligned}$$

setting $\psi(s) = e^{-\omega s}\|\tilde{p}(s; t) - p(s)\|_X$

$$\psi(s) \leq \rho M(1+L) \int_s^t \psi(r)dr + \rho M \int_s^t e^{-\omega r} \delta(r)dr,$$

and we can use Gronwall's Lemma to obtain

$$\|\tilde{p}(s; t) - p(s)\|_X \leq \rho M e^{\rho M(1+L)(t-s)} \int_s^t e^{-\omega(r-s)} \delta(r)dr, \quad s \leq t. \quad (5.54)$$

Secondly, if $s \leq t_0 \leq t$, then

$$\begin{aligned} &\|\tilde{p}(s; t) - \tilde{p}(s; t_0)\|_X = \|L_\xi(s, t_0)P(t_0)[\tilde{p}(t_0; t) - p(t_0)]\|_X \\ &+ \left\| \int_{t_0}^s L_\xi(s, r)P(r)[F(r, \tilde{p}(r; t) + \Sigma^u(r, \tilde{p}(r; t))) - F(r, \tilde{p}(r; t_0) + \Sigma^u(r, \tilde{p}(r; t_0)))]dr \right\|_X \\ &\leq \rho M^2 e^{\omega(s-t_0)} e^{\rho M(1+L)(t-t_0)} \int_{t_0}^t e^{-\omega(r-t_0)} \delta(r)dr \\ &+ \rho M(1+L) \int_s^{t_0} e^{\omega(s-r)} \|\tilde{p}(r; t) - \tilde{p}(r; t_0)\|_X dr, \end{aligned}$$

and using Gronwall's Lemma once more we obtain

$$\|\tilde{p}(s; t) - \tilde{p}(s; t_0)\|_X \leq \rho M^2 e^{\omega(s-t_0)} e^{\rho M(1+L)(t-s)} \int_{t_0}^t e^{-\omega(r-t_0)} \delta(r)dr. \quad (5.55)$$

We use these two estimates to bound $\delta(t)$. Since

$$\begin{aligned}
L_\xi(t, \tau) \Sigma^u(\tau, p(\tau)) &= L_\xi(t, \tau) \int_{-\infty}^{\tau} L_\xi(\tau, s) Q(s) F(s, \tilde{p}(s; \tau) + \Sigma^u(s, \tilde{p}(s; \tau))) ds \\
&= \int_{-\infty}^{\tau} L_\xi(t, s) Q(s) F(s, \tilde{p}(s; \tau) + \Sigma^u(s, \tilde{p}(s; \tau))) ds \\
&= \Sigma^u(t, p(t)) - \int_{\tau}^t L_\xi(t, s) Q(s) F(s, \tilde{p}(s; t) + \Sigma^u(\tilde{p}(s; t))) ds \\
&\quad - \int_{-\infty}^{\tau} L_\xi(t, s) Q(s) [F(s, \tilde{p}(s; t) + \Sigma^u(s, \tilde{p}(s; t))) - F(s, \tilde{p}(s; \tau) + \Sigma^u(s, \tilde{p}(s; \tau)))] ds,
\end{aligned}$$

we can write

$$\begin{aligned}
q(t) - \Sigma^u(t, p(t)) &= L_\xi(t, \tau) [q(\tau) - \Sigma^u(\tau, p(\tau))] \\
&\quad + \int_{\tau}^t L_\xi(t, s) Q(s) [F(s, p(s) + q(s)) - F(s, \tilde{p}(s; t) + \Sigma^u(\tilde{p}(s; t)))] ds \\
&\quad + \int_{-\infty}^{\tau} L_\xi(t, s) Q(s) [F(s, \tilde{p}(s; t) + \Sigma^u(s, \tilde{p}(s; t))) - F(s, \tilde{p}(s; \tau) + \Sigma^u(s, \tilde{p}(s; \tau)))] ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\delta(t) &\leq M e^{-\omega(t-\tau)} \delta(\tau) \\
&\quad + \rho M \int_{\tau}^t e^{-\omega(t-s)} [\|p(s) - \tilde{p}(s; t)\|_X + \|q(s) - \Sigma^u(\tilde{p}(s; t))\|_X] ds \\
&\quad + \rho M(1+L) \int_{-\infty}^{\tau} e^{-\omega(t-s)} \|\tilde{p}(s; t) - \tilde{p}(s; \tau)\|_X ds,
\end{aligned}$$

and using (5.54) and (5.55) we obtain

$$\begin{aligned}
\delta(t) &\leq M e^{-\omega(t-\tau)} \delta(\tau) + \rho M \int_{\tau}^t e^{-\omega(t-s)} \delta(s) ds \\
&\quad + \rho^2 M^2 (1+L) \int_{\tau}^t e^{-(\omega-\rho M(1+L))(t-s)} \int_s^t e^{-\omega(r-s)} \delta(r) dr ds \\
&\quad + \rho^2 M^3 (1+L) \int_{-\infty}^{\tau} e^{-(\omega-\rho M(1+L))(t-s)} \int_{\tau}^t e^{-\omega(r-s)} \delta(r) dr ds.
\end{aligned}$$

Now, note that if we change the order of integration in the third term on the right-hand side and adjust the limits accordingly, then we obtain

$$\begin{aligned}
&\int_{\tau}^t \int_{\tau}^r e^{-(\omega-\rho M(1+L))(t-s)} e^{-\omega(r-s)} \delta(r) ds dr \\
&= \int_{\tau}^t e^{-(\omega-\rho M(1+L))(t-r)} \left(\int_{\tau}^r e^{-(2\omega-\rho M(1+L))(r-s)} ds \right) \delta(r) dr \\
&\leq \frac{1}{2\omega - \rho M(1+L)} \int_{\tau}^t e^{-(\omega-\rho M(1+L))(t-r)} \delta(r) dr.
\end{aligned}$$

The integral in the fourth term is

$$\begin{aligned}
& \int_{-\infty}^{\tau} \int_{\tau}^t e^{-(\omega - \rho M(1+L))(t-s)} e^{-\omega(r-s)} \delta(r) dr ds \\
&= e^{-(\omega - \rho M(1+L))(t-\tau)} \left(\int_{-\infty}^{\tau} e^{-(2\omega - \rho M(1+L))(\tau-s)} ds \right) \int_{\tau}^t e^{-\omega(r-\tau)} \delta(r) dr \\
&= \frac{e^{-(\omega - \rho M(1+L))(t-\tau)}}{2\omega - \rho M(1+L)} \int_{\tau}^t e^{-\omega(r-\tau)} \delta(r) dr \\
&= \frac{1}{2\omega - \rho M(1+L)} \int_{\tau}^t e^{-(\omega - \rho M(1+L))(t-r)} e^{-(2\omega - \rho M(1+L))(r-\tau)} \delta(r) dr \\
&\leq \frac{1}{2\omega - \rho M(1+L)} \int_{\tau}^t e^{-(\omega - \rho M(1+L))(t-r)} \delta(r) dr.
\end{aligned}$$

Thus

$$\begin{aligned}
\delta(t) &\leq M e^{-(\omega - \rho M(1+L))(t-\tau)} \delta(\tau) \\
&\quad + \frac{2\omega \rho M + \rho^2 M^3(1+L)}{2\omega - \rho M(1+L)} \int_{\tau}^t e^{-(\omega - \rho M(1+L))(r-t)} \delta(r) dr;
\end{aligned}$$

with $\psi(t) := e^{(\omega - \rho M(1+L))t} \delta(t)$ this yields

$$\psi(t) \leq M \psi(\tau) + \kappa \int_{\tau}^t \psi(r) dr, \quad \kappa := \frac{2\omega \rho M + \rho^2 M^3(1+L)}{2\omega - \rho M(1+L)},$$

whence, via Gronwall's Lemma, $\psi(t) \leq M e^{\kappa(t-\tau)} \psi(\tau)$, which gives

$$\delta(t) \leq M e^{-(\omega - \rho M(1+L) - \kappa)(t-\tau)} \delta(\tau),$$

i.e. (5.50) with $\gamma = \omega - \rho M(1+L) - \kappa$, which is positive due to the condition on ρ in (5.49).

As a consequence of this attraction property, note that if $x(t) = p(t) + q(t)$, $t \in \mathbb{R}$, is a global solution of (5.44) that is bounded as $t \rightarrow -\infty$, then letting $t_0 \rightarrow -\infty$ in (5.50) implies that $q(t) = \Sigma^u(t, p(t))$ for each $t \in \mathbb{R}$, i.e. any globally defined solution that is bounded in the past must lie on the graph of Σ^u [this also ensures that $\Sigma^u(t, 0) = 0$ since 0 is a stationary solution of (5.44)]. It follows that

$$W^u(0)(t) \subseteq \{p + \Sigma^u(t, p) : p \in P(t)X\}.$$

To show that this is in fact an equality, consider a point $p \in P(\tau)X$ and the solution $p^*(t)$ of the initial value problem

$$p^*(t) = L_{\xi}(t, \tau) p + \int_{\tau}^t L_{\xi}(t, s) P(s) F(s, p^*(s) + \Sigma^u(s, p^*(s))) ds.$$

This defines a curve in X , $p^*(t) + \Sigma^u(t, p^*(t))$, $t \in \mathbb{R}$. Using the fact that Σ^u is a fixed point of $\mathcal{T}$, one can check that

$$\Sigma^u(t, q^*(t)) = \int_{-\infty}^t L_{\xi}(t, s) Q(s) F(s, p^*(s) + \Sigma^u(s, p^*(s))) ds \quad \text{for all } t \in \mathbb{R}.$$

Thus $q^*(t) = \Sigma^u(t, p^*(t))$ solves

$$q^*(t) = L_\xi(t, \tau)q^*(\tau) + \int_\tau^t L_\xi(t, s)Q(s)F(s, p^*(s) + \Sigma^u(s, p^*(s)))ds, \quad t \geq \tau \in \mathbb{R},$$

and we conclude that $(p^*(t) + \Sigma^u(t, p^*(t)))$, $t \in \mathbb{R}$, is a solution of (5.44), passing through $p + \Sigma^u(\tau, p)$ at time τ , with $\Sigma^u(t, p^*(t)) \rightarrow 0$ as $t \rightarrow -\infty$; in other words, $p + \Sigma^u(\tau, p) \in W^u(0)(t)$.

Since $\Sigma^u(t, 0) = 0$, the reasoning that led to (5.53) can be used now to ensure that

$$\|p^*(t)\|_X \leq M\|p\|_X e^{(\omega - \rho M(1+L))(t-\tau)},$$

and the proof is complete. $\square$

We now prove the existence of a stable manifold $W^s(0)(\cdot)$. Once more, we assume that F satisfies

$$\sup_{t \in \mathbb{R}} \|F(t, x)\|_X \leq \rho \quad \text{and} \quad \sup_{t \in \mathbb{R}} \|F(t, x) - F(t, \tilde{x})\|_X \leq \rho \|x - \tilde{x}\|_X \quad (5.56)$$

for all $x \in X$.

We will show that there is a bounded Lipschitz function

$$\Sigma^s(t, \cdot) : (I - P(t))X \rightarrow P(t)X$$

such that

$$W^s(0)(t) = \{q + \Sigma^s(q) : q \in Q(t)X\}.$$

As before, to determine Σ^s we define the set

$$\text{Lip}^s(D, L) := \left\{ \Sigma : \mathbb{R} \times X \rightarrow X : \Sigma(t, \cdot) : Q(t)X \rightarrow P(t)X, \right. \\ \left. \sup_{t \in \mathbb{R}} \sup_{q \in Q(t)X} \|\Sigma(t, q)\|_X \leq D, \text{ and } \|\Sigma(t, q) - \Sigma(t, \tilde{q})\|_X \leq L\|q - \tilde{q}\|_X \right\}$$

which is a complete metric space equipped with the metric

$$\|\Sigma(\cdot, \cdot) - \tilde{\Sigma}(\cdot, \cdot)\|_\infty := \sup_{t \in \mathbb{R}} \sup_{q \in Q(t)X} \|\Sigma(t, q) - \tilde{\Sigma}(t, q)\|_X.$$

Theorem 23. Suppose that the assumptions of Theorem 22 are satisfied, namely F satisfies (5.56), with ρ such that (5.48) and (5.49) both hold. Then there exists a $\Sigma^s \in \text{Lip}^s(D, L)$ such that the stable manifold $W^s(0)(\cdot)$ has sections

$$W^s(0)(t) = \{q + \Sigma^s(t, q) : q \in Q(t)X\}.$$

Furthermore, if $p(t) + q(t)$, $t \in \mathbb{R}$, is a global solution of (5.44), there are constants $M \geq 1$ and $\gamma > 0$ such that

$$\|p(t) - \Sigma^s(t, q(t))\|_X \leq M e^{\gamma(t-t_0)} \|p(t_0) - \Sigma^s(t, q(t_0))\|_X, \quad t \leq t_0, \quad (5.57)$$

and if $p(t) + q(t)$, $t \geq \tau$, is a solution of (5.44) with $p(\tau) = \Sigma^s(\tau, q(\tau))$, then

$$\|q(t)\|_X \leq M\|q(\tau)\|_X e^{-(\omega - \rho M(1+L))(t-\tau)}.$$

As in the unstable manifold, we obtain that the local stable manifold of any given hyperbolic equilibrium can be obtained in the form of a graph.

Proof. For $\tau \in \mathbb{R}$ and arbitrary $q \in Q(\tau)X$, given $\Sigma \in \text{Lip}^s(D, L)$, denote by $q(t)$ the solution $\psi(t, \tau, \zeta, \Sigma)$, defined for $t > \tau$, of

$$q(t) = L_\xi(t, \tau)\zeta + \int_\tau^t L_\xi(t, s)QF(s, q(s) + \Sigma^s(s, q(s)))ds.$$

Next we define, for $\Sigma \in \text{Lip}^s(D, L)$,

$$\mathcal{T}(\Sigma)(\tau, \zeta) = - \int_\tau^\infty L_\xi(\tau, s)P(s)F(s, q(s) + \Sigma(s, q(s)))ds \quad (5.58)$$

for each $\tau \in \mathbb{R}$ and $\zeta \in Q(\tau)X$. Following the ideas from Theorem 22, we can prove that if $\rho > 0$ satisfies (5.48), then the map $\mathcal{T}$ takes $\text{Lip}^s(D, L)$ into itself and is a strict contraction and, consequently, has a unique fixed point in $\text{Lip}^s(D, L)$.

Indeed, by (5.46), one has

$$\|\mathcal{T}(\Sigma)(\tau, \cdot)\|_X \leq \int_\tau^{+\infty} \rho M e^{\omega(\tau-s)} ds = \frac{\rho M}{\omega}, \quad (5.59)$$

and from (5.48) it follows that $\|\mathcal{T}(\Sigma)\|_\infty \leq D$.

Now, suppose that $\Sigma, \tilde{\Sigma} \in \text{Lip}^s(D, L)$, $\zeta, \tilde{\zeta} \in Q(\tau)X$, and let $q(t) = \psi(t, \tau, \zeta, \Sigma)$ and $\tilde{q}(t) = \psi(t, \tau, \tilde{\zeta}, \tilde{\Sigma})$. Then

$$\begin{aligned} q(t) - \tilde{q}(t) &= L_\xi(t, \tau)(\zeta - \tilde{\zeta}) \\ &+ \int_\tau^t L_\xi(t, s)Q(s)[F(s, q(s) + \Sigma(s, q(s))) - F(s, \tilde{q}(s) + \tilde{\Sigma}(s, \tilde{q}(s)))]ds, \end{aligned}$$

from which one can show that

$$\begin{aligned} \|q(t) - \tilde{q}(t)\|_X &\leq [M\|\zeta - \tilde{\zeta}\|_X e^{-\omega(t-\tau)} \\ &+ \rho M \int_\tau^t e^{-\omega(t-s)} ds \|\Sigma - \tilde{\Sigma}\|_\infty] e^{\rho M(1+L)(t-\tau)} \\ &\leq [M\|\zeta - \tilde{\zeta}\|_X + \rho M \omega^{-1} \|\Sigma - \tilde{\Sigma}\|_\infty] e^{\rho M(1+L)(t-\tau)}. \end{aligned} \quad (5.60)$$

Following the arguments in the proof of Theorem 22 leads to

$$\begin{aligned} &\|\mathcal{T}(\Sigma)(\tau, \zeta) - \mathcal{T}(\tilde{\Sigma})(\tau, \tilde{\zeta})\|_X \\ &\leq \frac{\rho M}{\omega} \left[1 + \frac{\rho M(1+L)}{\omega - \rho M(1+L)} \right] \|\Sigma - \tilde{\Sigma}\|_\infty + \frac{\rho M^2(1+L)}{\omega - \rho M(1+L)} \|\zeta - \tilde{\zeta}\|_X. \end{aligned} \quad (5.61)$$

It follows from (5.49) and (5.61) that $I_\Sigma \leq \vartheta$, $I_\zeta \leq L$ and

$$\|\mathcal{T}(\Sigma)(\tau, \zeta) - \mathcal{T}(\tilde{\Sigma})(\tau, \tilde{\zeta})\|_X \leq L\|\zeta - \tilde{\zeta}\|_X + \vartheta\|\Sigma - \tilde{\Sigma}\|. \quad (5.62)$$

This inequality with $\Sigma = \tilde{\Sigma}$, along with (5.59), implies that $\mathcal{T}$ takes $\text{Lip}^s(D, L)$ into itself; and (5.62) with $\zeta = \tilde{\zeta}$ shows that $\mathcal{T}$ is a contraction. It follows that there exists a unique fixed point $\Sigma^s = \mathcal{T}(\Sigma^s)$ in $\text{Lip}^s(D, L)$.

Now, if $p(t) + q(t)$, $t \in \mathbb{R}$, is a global solution of (5.44) that is bounded as $t \rightarrow +\infty$, then it can be proved that there are constants $M \geq 1$ and $\gamma > 0$ such that (5.57) holds. Letting $t_0 \rightarrow +\infty$ in (5.57) it follows that $p(t) = \Sigma^s(t, q(t))$ for each $t \in \mathbb{R}$, i.e. any point in $W^s(0)(t)$ must be contained in the graph of $\Sigma^s(t, \cdot)$. This also ensures that $\Sigma^s(t, 0) = 0$ since 0 is a stationary solution of (5.44).

It remains to prove that the graph of Σ^s is the stable manifold of zero. Consider a fixed $\tau \in \mathbb{R}$, and $q \in Q(\tau)X$, and the solution $q^*(t)$ of the initial value problem

$$q^*(t) = L_\xi(t, \tau)q + \int_\tau^t L_\xi(t, s)Q(s)F(s, q^*(s) + \Sigma^s(s, q^*(s)))ds.$$

This defines a curve $q^*(t) + \Sigma^s(t, q^*(t))$ in X for $t \geq \tau$. Using the fact that Σ^s is a fixed point of the operator $\mathcal{T}$, defined in (5.58), we have

$$\Sigma^s(t, p^*(t)) = \int_{+\infty}^t L_\xi(t, s)P(s)F(s, q^*(s) + \Sigma^s(s, q^*(s)))ds,$$

$t \geq \tau$. Thus $\Sigma^s(t, q^*(t))$ solves

$$p(t) = L_\xi(t, \tau)p(\tau) + \int_\tau^t L_\xi(t, s)PF(s, q^*(s) + \Sigma^s(s, q^*(s)))ds, \quad t \geq \tau,$$

and we conclude that $q^*(t) + \Sigma^s(t, q^*(t))$, $t \geq \tau$, is a solution of (5.44), passing through $q + \Sigma^s(\tau, q)$ at time τ , with $\Sigma^s(t, q^*(t)) \rightarrow 0$ as $t \rightarrow +\infty$. Since $\Sigma^s(t, 0) = 0$, we can argue as in (5.60) to ensure that

$$\|q^*(t)\|_X \leq M\|q^*(\tau)\|_X e^{-(\omega - \rho M(1+L))(t-\tau)}.$$

Hence $q^*(t) \rightarrow 0$ as $t \rightarrow +\infty$, and so $q + \Sigma^s(\tau, q) \in W^s(0)(\tau)$. $\square$

The following remark is included on the previously proofs but it is worth being highlighted.

Remark 16. Let S be an evolution process, ξ be a global hyperbolic solution and η be a global solution. Then, given $\delta_1 > 0$ there exists $0 < \delta_2 < \delta_1$ such that any solution $x(t) = p(t) + q(t)$ in the unstable (stable) manifold with $\|p(t_0)\| + \|q(t_0)\| < \delta_2$ satisfies

$$\|p(t)\| + \|q(t)\| < \delta_1, \forall t \leq t_0 \quad (t \geq t_0).$$

As a consequence, fixing $\delta > 0$ as in (5.47) and taking $0 < \rho < \delta$ satisfying the hypothesis from Theorems 22 and 23, there exists $\delta' > 0$ that satisfies the condition above, that is, if $\|p(t_0)\| + \|q(t_0)\| < \delta'$ then

$$\|p(t)\| + \|q(t)\| < \rho, \forall t \leq t_0 \quad (t \geq t_0).$$

This implies that local unstable and stable manifolds are graphs. Hence, if there exists $t_0 \in \mathbb{R}$ such that

$$\|\xi(t) - \eta(t)\| < \delta', \quad t \leq t_0 \quad (t \geq t_0),$$

then $\|\xi(t) - \eta(t)\| = 0$ ($\lim_{t \rightarrow \infty} \|\xi(t) - \eta(t)\| = 0$).

Such property is very similar to condition 4 from Theorem 14, except for the uniformity get from δ . Our next goal is achieve such uniformity. To do this, we will prove that unstable and stable manifolds behaves continuously under perturbations.

For $i = 1, 2$, consider the problems

$$S_i(t, \tau)x = T(t, \tau)x + \int_{\tau}^t T(t, s)f_i(s, S_i(s, \tau)x) ds \quad (5.63)$$

and assume that $\xi_i : \mathbb{R} \rightarrow X$ is a global solution for S_i . As before, we associate to ξ_i the linear process L_i , given by

$$L_i(t, \tau) = T(t, \tau) + \int_{\tau}^t T(t, s)D_x f_i(s, \xi_i(s))L_i(s, \tau) ds.$$

Suppose that L_1 has an exponential dichotomy with projection $P_1(\cdot)$, constant $M_1 > 0$, and exponent $\omega_1 > 0$. Given $M_2 > M_1$ and $0 < \omega_2 < \omega_1$, if

$$\sup_{t \in \mathbb{R}, x \in X} \|f_1(t, x) - f_2(t, x)\|_X + \|D_x f_1(t, x) - D_x f_2(t, x)\|_{\mathcal{L}(X)} =: \eta,$$

it follows from Theorem 21 that, for suitably small $\eta > 0$, the linear process L_2 also has an exponential dichotomy, with projection $P_2(\cdot)$, constant M_2 and exponent ω_2 .

We already know that $y_i(t)$ is a solution of (5.63) if, and only if, $y_i(t) = x_i(t) + \xi_i(t)$, where

$$\begin{aligned} x_i(t) &= L_i(t, \tau)x(\tau) \\ &+ \int_{\tau}^t L_i(t, s)[f_i(s, x_i(s) + \xi_i(s)) - f_i(s, \xi_i(s)) - D_x f_i(s, \xi_i(s))x_i(s)] ds. \end{aligned} \quad (5.64)$$

Once more we can decompose a solution $x_i(t)$ of (5.64) as $x_i(t) = p_i(t) + q_i(t)$, where $p_i(t) = P_i(t)x_i(t)$, $q_i(t) = Q_i(t)x_i(t)$ and $Q_i(t) = I - P_i(t)$. Thus

$$\begin{aligned} p_i(t) &= L_i(t, \tau)p_i(\tau) + \int_{\tau}^t L_i(t, s)P(s)F_i(s, p_i(s) + q_i(s)) ds \\ q_i(t) &= L_i(t, \tau)q_i(\tau) + \int_{\tau}^t L_i(t, s)Q(s)F_i(s, p_i(s) + q_i(s)) ds, \end{aligned} \quad (5.65)$$

where

$$F_i(t, x) = f_i(t, x + \xi_i(t)) - f_i(t, \xi_i(t)) - D_x f_i(t, \xi_i(t))x.$$

Now that we have set the necessary language, we can prove the continuity theorem.

Theorem 24. For $i = 1, 2$, suppose that

$$\|F_i(t, x)\|_X \leq \rho \quad \text{and} \quad \|F_i(t, x) - F_i(t, \tilde{x})\|_X \leq \rho \|x - \tilde{x}\|_X$$

for all $x, \tilde{x} \in X$, where ρ is such that (5.48) and (5.49) hold. Then there exist functions $\Sigma_i^u(t, \cdot) : P_i(t)X \rightarrow Q_i(t)X$ such that the unstable manifold $W_i^u(0)$ of the equilibrium solution 0 of (5.65) is given by

$$W_i^u(0)(t) = \{p + \Sigma_i^u(p) : p \in P(t)X\}.$$

If in addition

$$\frac{2\rho M}{2\omega - \rho M(1+L)} \leq \frac{1}{2},$$

then, for any $r > 0$,

$$\sup_{t \leq \tau} \sup_{\substack{x \in X \\ \|x\|_X \leq r}} \{\|P_2(t)x - P_1(t)x\|_X + \|\Sigma_2^u(t, P_2(t)x) - \Sigma_1^u(t, P_1(t)x)\|_X\} \rightarrow 0$$

as $\eta \rightarrow 0$.

Proof. We already know that the continuity of the projections follows from Theorem 21. It remains to prove that

$$\sup_{t \leq \tau} \sup_{x \in X: \|x\|_X \leq r} \|\Sigma_2^u(t, P_2(t)x) - \Sigma_1^u(t, P_1(t)x)\|_X \rightarrow 0 \quad \text{as} \quad \eta \rightarrow 0.$$

Under the conditions of the theorem,

$$\Sigma_i^u(t, p) = \int_{-\infty}^t L_i(t, s) Q_i(s) F_i(s, p_i(s; p) + \Sigma_i^u(p_i(s; p))) ds,$$

where we define, if $p \in R(P_i(t))$,

$$p_i(s; p) = L_i(s, t) p + \int_s^t L_i(t, r) P_i(r) F_i(r, p_i(r; p) + \Sigma_i^u(p_i(r; p))) ds.$$

For $p \in P_1(\tau)X$ with $\|p\|_X \leq r$ we have

$$\begin{aligned} & \Sigma_2^u(\tau, P_2(\tau)p) - \Sigma_1^u(\tau, p) \\ &= \int_{-\infty}^{\tau} [L_2(\tau, s) Q_2(s) - L_1(\tau, s) Q_1(s)] F_2(s, p_2(s) + \Sigma_2^u(s, p_2(s))) ds \\ & \quad + \int_{-\infty}^{\tau} L_1(\tau, s) Q_1(s) [F_2(s, p_2(s) + \Sigma_2^u(s, p_2(s))) - F_2(s, p_1(s) + \Sigma_1^u(s, p_1(s)))] ds \\ & \quad + \int_{-\infty}^{\tau} L_1(\tau, s) Q_1(s) [F_2(s, p_1(s) + \Sigma_1^u(s, p_1(s))) - F_1(p_1(s) + \Sigma_1^u(s, p_1(s)))] ds \\ &=: I_1(\eta) + I_2(\eta) + I_3(\eta). \end{aligned} \tag{5.66}$$

It follows from (5.40) that $I_3(\eta) \rightarrow 0$ as $\eta \rightarrow 0$. The term $I_1(\eta)$ is only moderately more difficult to deal with since it equals

$$\int_{-\infty}^{\tau} \{[L_2(\tau, s) - L_1(\tau, s)] Q_2(s) + L_1(\tau, s) [Q_2(s) - Q_1(s)]\} F_2(s, p_2(s) + \Sigma_2^u(s, p_2(s))) ds.$$

Once more, from Theorem 21, we know that

$$\sup_{s \in \mathbb{R}} \|Q_1(s) - Q_2(s)\|_{\mathcal{L}(X)} = \sup_{s \in \mathbb{R}} \|P_1(s) - P_2(s)\|_{\mathcal{L}(X)} \rightarrow 0 \quad \text{as} \quad \eta \rightarrow 0,$$

and since

$$L_2(t, \tau)Q_2(\tau) - L_1(t, \tau)Q_2(\tau) = \int_{\tau}^t L_1(t, s)B(s)L_2(s, \tau)Q_2(\tau) ds,$$

it follows that $I_1(\eta) \rightarrow 0$ as $\eta \rightarrow 0$.

Now we will estimate $I_2(\eta)$. Recall that

$$p_2(t) = L_2(t, \tau)p + \int_{\tau}^t L_2(t, s)P_2(s)F_2(s, p_2(s) + \Sigma_2^u(s, p_2(s))) ds,$$

from which similarly as in the case of (5.53) one can get

$$\|p_2(t)\|_X \leq Me^{(\omega - \rho M(1+L))(t-\tau)} \|p\|_X.$$

When $\eta = 0$, we may also assume, without loss of generality, that $M = 1$.

Now,

$$\begin{aligned} \|I_2\|_X &\leq \rho M \int_{-\infty}^{\tau} e^{-\omega(\tau-s)} [\|p_2(s) - p_1(s)\|_X + \|\Sigma_2^u(s, p_2(s)) - \Sigma_1^u(s, p_1(s))\|_X] ds \\ &\leq \rho M \int_{-\infty}^{\tau} e^{-\omega(\tau-s)} [(1+L)\|p_2(s) - p_1(s)\|_X \\ &\quad + \|\Sigma_2^u(s, P_2(s)p_1(s)) - \Sigma_1^u(s, p_1(s))\|_X] ds \\ &\leq \rho M(1+L) \int_{-\infty}^{\tau} e^{-\omega(\tau-s)} \|p_2(s) - p_1(s)\|_X ds + \frac{\rho M}{\omega} \|\Sigma_2^u - \Sigma_1^u\|_r, \end{aligned}$$

where

$$\|\Sigma_2^u - \Sigma_1^u\|_r = \sup_{s \leq \tau} \sup_{p \in P_1(s)X: \|p\| \leq r} \|\Sigma_2^u(s, P_2(s)p) - \Sigma_1^u(s, p)\|_X.$$

Placing this into (5.66) we have

$$\begin{aligned} \|\Sigma_2^u(\tau, P_2(\tau)p) - \Sigma_1^u(\tau, p)\|_X &\leq o(1) + \frac{\rho M}{\omega} \|\Sigma_2^u - \Sigma_1^u\|_r \\ &\quad + \rho M(1+L) \int_{-\infty}^{\tau} e^{-\omega(\tau-s)} \|p_1(s) - p_2(s)\|_X ds. \end{aligned} \quad (5.67)$$

We now estimate $\|p_2(t) - p_1(t)\|$ using $o(1)$ to denote any quantity that converges to zero as $\eta \rightarrow 0$:

$$\begin{aligned} \|p_2(t) - p_1(t)\|_X &\leq \|L_2(t, \tau)P_2(\tau)p - L_1(t, \tau)P_1(\tau)p\|_X \\ &\quad + \left\| \int_{\tau}^t [L_2(t, s)P_2(s)F_2(s, p_2(s) + \Sigma_2^u(s, p_2(s))) \right. \\ &\quad \left. - L_1(t, s)P_1(s)F_1(s, p_1(s) + \Sigma_1^u(s, p_1(s)))] ds \right\|_X \\ &\leq \|L_2(t, \tau)P_2(\tau)p - L_1(t, \tau)P_1(\tau)p\|_X \\ &\quad + \left\| \int_{\tau}^t L_2(t, s)P_2(s) [F_2(s, p_2(s) + \Sigma_2^u(s, p_2(s))) - F_2(s, p_1(s) + \Sigma_1^u(s, p_1(s)))] ds \right\|_X \\ &\quad + \left\| \int_{\tau}^t L_2(t, s)P_2(s) [F_2(s, p_1(s) + \Sigma_1^u(s, p_1(s))) - F_1(s, p_1(s) + \Sigma_1^u(s, p_1(s)))] ds \right\|_X \\ &\quad + \left\| \int_{\tau}^t [L_2(t, s)P_2(s) - L_1(t, s)P_1(s)] F_1(s, p_1(s) + \Sigma_1^u(s, p_1(s))] ds \right\|_X, \end{aligned}$$

and so

$$\begin{aligned}
& \|p_2(t) - p_1(t)\|_X \\
& \leq o(1) + \rho M \int_t^\tau e^{\omega(t-s)} [(1+L)\|p_2(s) - p_1(s)\|_X \\
& \quad + \|\Sigma_2^u(s, P_2(s)p_1(s)) - \Sigma_1^u(s, p_1(s))\|_X] ds \\
& \leq o(1) + \frac{\rho M}{\omega} \|\Sigma_2^u - \Sigma_1^u\|_r + \rho M(1+L) \int_t^\tau e^{\omega(t-s)} \|p_2(s) - p_1(s)\|_X ds.
\end{aligned}$$

It now follows from Gronwall's inequality that

$$\|p_2(t) - p_1(t)\|_X \leq (o(1) + \frac{\rho M}{\omega} \|\Sigma_2^u - \Sigma_1^u\|_r) e^{(\omega - \rho M(1+L))(t-\tau)}. \quad (5.68)$$

Substituting (5.68) into (5.67) we obtain

$$\begin{aligned}
& \|\Sigma_2^u(\tau, P_2(\tau)p) - \Sigma_1^u(\tau, p)\|_X \leq o(1) + \frac{\rho M}{\omega} \|\Sigma_2^u - \Sigma_1^u\|_r \\
& \quad + \rho M(1+L) \int_{-\infty}^\tau e^{-(2\omega - \rho M(1+L))(\tau-s)} \left[o(1) + \frac{\rho M}{\omega} \|\Sigma_2^u - \Sigma_1^u\|_r \right] ds \\
& \leq o(1) + \left[\frac{\rho M}{\omega} + \frac{\rho^2 M^2(1+L)}{\omega(2\omega - \rho M(1+L))} \right] \|\Sigma_2^u - \Sigma_1^u\|_r \\
& =: o(1) + \tilde{\theta} \|\Sigma_2^u - \Sigma_1^u\|_r.
\end{aligned}$$

Using (5.49) we have

$$\|\Sigma_2^u - \Sigma_1^u\|_r \leq o(1) + \tilde{\theta} \|\Sigma_2^u - \Sigma_1^u\|_r,$$

where $\tilde{\theta} \in (0, 1)$, and we conclude the proof. $\square$

Remark 17. The proof for the stable manifold follows similarly and we conclude that there is a ball around zero such that the local unstable and stable manifolds are given as a graph in this ball and behave continuously within it.

THE CASCADE SYSTEM

In this chapter we will apply the theory studied in the previous chapters to understand the behavior of the cascade system. In order to solve this, we need one more definition.

Definition 43. Let $f : \mathbb{R}^+ \times X \rightarrow X$ be such that the nonlinear problem

$$\begin{cases} \dot{y} = f(t, x), & t > \tau \\ x(\tau) = x_0 \in X \end{cases} \quad (6.1)$$

generates a dynamical system $\mathcal{S}$ with a pullback attractor $\hat{A} = \{A(t)\}_{t \in \mathbb{R}}$. Given $v > 0$, define the maps f_v and f^v as

$$f_v(t, x) = \begin{cases} f(t, x), & \text{if } t \leq -v, \\ f(-v, x), & \text{if } t > -v \end{cases} \quad (6.2)$$

and

$$f^v(t, x) = \begin{cases} f(v, x), & \text{if } t < v, \\ f(t, x), & \text{if } t \geq v. \end{cases} \quad (6.3)$$

We say that the problem (6.1) is compatible from the left if exists $v_0 > 0$ such that, for all $v \geq v_0$, the problem

$$\begin{cases} \dot{y} = f_v(t, x), & t > \tau \\ x(\tau) = x_0 \in X \end{cases} \quad (6.4)$$

generates a dynamical system $\mathcal{S}_v$ with a pullback attractor $\hat{A}_v = \{A_v(t)\}_{t \in \mathbb{R}}$ such that $\bigcup_{v \geq v_0} \bigcup_{t \in \mathbb{R}} A_v(t)$ is relatively compact.

Similarly, (6.1) is compatible from the right if exists $v_0 > 0$ such that, for all $v \geq v_0$, the problem

$$\begin{cases} \dot{y} = f^v(t, x), & t > \tau \\ x(\tau) = x_0 \in X \end{cases} \quad (6.5)$$

generates a dynamical system $\mathcal{S}^v$ with a pullback attractor $\hat{A}^v = \{A^v(t)\}_{t \in \mathbb{R}}$ such that $\bigcup_{v \geq v_0} \bigcup_{t \in \mathbb{R}} A^v(t)$ is relatively compact.

The following theorem is based on (ARAGÃO-COSTA *et al.*, 2013).

Theorem 25. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a map such that the autonomous problem

$$\begin{cases} \dot{x} = g(x), & t > \tau, \\ x(\tau) = x_0 \in \mathbb{R}^n \end{cases} \quad (6.6)$$

generates a gradient semigroup $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ relative to $\mathcal{E} = \{x_1^*, \dots, x_l^*\}$ (see Examples 1 and 2). Consider the cascade system:

$$\begin{cases} \dot{y} = f(x, y), & t > \tau \\ \dot{x} = g(x), & t > \tau, \\ x(\tau) = x_0 \in \mathbb{R}^n, \\ y(\tau) = y_0 \in \mathbb{R}^m, \end{cases} \quad (6.7)$$

where $f \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m)$ is a function such that, for each $j = 1, \dots, l$, the problem

$$\begin{cases} \dot{y} = f(x_j^*, y), & t > \tau, \\ y(\tau) = y_0 \in \mathbb{R}^m \end{cases} \quad (6.8)$$

generates a dynamically gradient semigroup $\mathcal{T}_{x_j^*}$ relative to $\mathcal{E}_{x_j^*} = \{y_{j,1}^*, \dots, y_{j,n_j}^*\}$, where $y_{j,k}^*$ is a hyperbolic equilibrium, for all $k = 1, \dots, n_j$.

Defining the map $\mathbb{R}^{n+m} \ni z = (x, y) \mapsto h(z) := (g(x), f(x, y)) \in \mathbb{R}^{n+m}$ and $z_0 = (x_0, y_0)$, the problem (6.7) can be rewritten as

$$\begin{cases} \dot{z} = h(z), & t > \tau, \\ z(\tau) = z_0 \in \mathbb{R}^{n+m} \end{cases} \quad (6.9)$$

Suppose that, for each global solution ϕ of $\mathcal{T}$, the nonautonomous problem

$$\begin{cases} \dot{y} = \hat{f}(t, y), & t > \tau \\ y(\tau) = y_0 \in \mathbb{R}^m \end{cases} \quad (6.10)$$

generates a dynamical system $\mathcal{S}$ with pullback attractor $\hat{A} = \{A(t)\}_{t \in \mathbb{R}}$ and is compatible from the left and from the right, where $\hat{f}(t, y)$ denotes $f(\phi(t), y)$.

For each solution $\zeta = (\phi, \psi)$ of (6.9), we know that exists x_i^* and x_j^* in $\mathcal{E}$ such that $\lim_{t \rightarrow -\infty} \phi(t) = x_j^*$ and $\lim_{t \rightarrow \infty} \phi(t) = x_i^*$. Assume that for each $r > 0$

$$\lim_{t \rightarrow -\infty} \sup_{\|y\| \leq r} \{\|f(t, y) - f(x_j^*, y)\| + \|D_y f(t, y) - D_y f(x_j^*, y)\|\} = 0 \quad (6.11)$$

and

$$\lim_{t \rightarrow +\infty} \sup_{\|y\| \leq r} \{\|f(t, y) - f(x_i^*, y)\| + \|D_y f(t, y) - D_y f(x_i^*, y)\|\} = 0. \quad (6.12)$$

Then, the semigroup related to (6.9) is gradient.

Proof. We know from Theorem 6 that is sufficient to show that the semigroup generated by (6.9) is dynamically gradient. First we will show (G1). Let $\zeta = (\phi, \psi)$ be a global bounded solution of (6.9). Then ϕ is a solution of (6.6) and there exist $x_j^*, x_i^* \in \mathcal{E}$ such that $\lim_{t \rightarrow -\infty} \phi(t) = x_j^*$ and $\lim_{t \rightarrow \infty} \phi(t) = x_i^*$. Since (6.10) is compatible from the left, there exists v_0 such that, for all $v \geq v_0$ the system

$$\begin{cases} \dot{y} = f_v(t, y), & t > \tau \\ y(\tau) = y_0 \in \mathbb{R}^m \end{cases} \quad (6.13)$$

generates a dynamical system $\mathcal{S}_v$ with pullback attractor $\hat{A}_v = \{A_v(t)\}_{t \in \mathbb{R}}$ and

$$\overline{\bigcup_{v \geq v_0} \bigcup_{t \in \mathbb{R}} A_v(t)} \text{ is compact.} \quad (6.14)$$

Note that from (6.11), given $\varepsilon > 0$ and $r > 0$ there exists $t_0 \in \mathbb{R}$ that satisfies

$$t \leq t_0 \implies \{\|f(t, y) - f(x_j^*, y)\| + \|D_y f(t, y) - D_y f(x_j^*, y)\|\} < \varepsilon. \quad (6.15)$$

Taking $v_0 > -t_0$ it follows that

$$v \geq v_0 \implies \sup_{t \in \mathbb{R}} \{\|f_v(t, y) - f(x_j^*, y)\| + \|D_y f_v(t, y) - D_y f(x_j^*, y)\|\} < \varepsilon \quad (6.16)$$

This shows that

$$\lim_{v \rightarrow \infty} \sup_{t \in \mathbb{R}} \sup_{\|y\| \leq r} \{\|f_v(t, y) - f(x_j^*, y)\| + \|D_y f_v(t, y) - D_y f(x_j^*, y)\|\} = 0. \quad (6.17)$$

From the variation of constants formula we know that

$$\mathcal{S}_v(t, \tau)x_0 = e^{A(t-\tau)}x_0 + \int_{\tau}^t e^{A(t-s)}f_v(s, S_v(s, \tau)x_0)ds \quad (6.18)$$

is solution of

$$\begin{cases} \dot{x} = Ax + f(t, y), & t > \tau \\ x(\tau) = x_0 \in \mathbb{R}^m, \end{cases}$$

where $A \in \mathcal{L}(\mathbb{R}^m)$ and $f : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is $\mathcal{C}^1$. Hence, we have

$$\lim_{v \rightarrow \infty} \sup_{t \in [0, T]} \sup_{\|y\| \leq r} \{\|\mathcal{S}_v(t, t-\tau)y - \mathcal{S}_{x_j^*}(t - (t-\tau))y\|\} = 0. \quad (6.19)$$

We will use (6.17) to satisfy conditions 3 and 4 from Theorem 14.

Define $B = \max_{k=1, \dots, n_j} \|y_{j,k}^*\|$. From the continuity of f it follows that

$$\sup_{\|y\| \leq B} \{\|f(x_j^*, y)\| + \|D_y f(x_j^*, y)\|\} < \infty. \quad (6.20)$$

Moreover, if $M > 1$ and $0 < \omega < 1$ are the constants related to the exponential dichotomy of the process generated by (6.8), it follows from the fact that f is differentiable that there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$

$$\sup_{\|x\| \leq \varepsilon} \|f(x_j^*, y_{j,k}^* + x) - f(x_j^*, y_{j,k}^*) - D_y f(x_j^*, y_{j,k}^*)x\| < \frac{\varepsilon}{4M\omega^{-1}}, \forall k = 1, \dots, n_j. \quad (6.21)$$

Also, from (6.17) there exists $v_1 > 0$ such that

$$v \geq v_1 \implies \sup_{t \in \mathbb{R}} \sup_{\|y\| \leq r} \{ \|f_v(t, y) - f(x_j^*, y)\| + \|D_y f_v(t, y) - D_y f(x_j^*, y)\| \} < \frac{\varepsilon}{4M\omega^{-1}}. \quad (6.22)$$

Conditions (6.20), (6.21) and (6.22) guarantee, together with Lemma 27, that for each $v \geq v_1$ and $k = 1, \dots, n_j$ there exists a unique ξ_v^k global solution of $\mathcal{S}_v$ such that

$$\sup_{t \in \mathbb{R}} \|\xi_v^k(t) - y_k^*\| < \varepsilon. \quad (6.23)$$

Moreover, ξ_v^k is a global hyperbolic solution. Since we can make $\varepsilon \rightarrow 0$, we have

$$\lim_{v \rightarrow \infty} \sup_{t \in \mathbb{R}} \|\xi_v^k(t) - y_k^*\| = 0, \forall k = 1, \dots, n_j. \quad (6.24)$$

Note that, for each $v \geq v_1$, the set $\xi_v^k(\mathbb{R})$ is a maximal invariant set for $\mathcal{S}_v$. Indeed, if we take $\varepsilon > 0$ as (6.23), we will show that $\xi_v^k(\mathbb{R})$ is the maximal invariant in $\mathcal{O}_{\frac{\varepsilon}{2}}(\xi_v^k(\mathbb{R}))$. Suppose that Ξ is an invariant set in $\mathcal{O}_{\frac{\varepsilon}{2}}(\xi_v^k(\mathbb{R}))$. Since Ξ is a union of global orbits ξ (Remark 4) with $\sup_{t \in \mathbb{R}} \|\xi(t) - y_k^*\| < \delta$, Lemma 27 implies that $\xi = \xi_v^k$ and consequently $\Xi \subset \xi_v^k(\mathbb{R})$. This fact and (6.24) will be used to satisfy condition 3 from Theorem 14.

It remains to show condition 4 from Theorem 14. Define $\delta_0 = \min_{1 \leq i < j \leq n_j} d(y_i^*, y_j^*)$. From Remark 17, there exists $\delta' \in (0, \frac{\delta_0}{2})$ satisfying the condition mentioned on Remark 16 uniformly for $k = 1, \dots, n$ and $v \geq v_2$, for some $v_2 \geq v_1$. For this δ' , condition 4 is clearly satisfied to $\mathcal{T}_{x_j^*}$, since it is dynamically gradient. To show that is also satisfied by the family $\{\mathcal{S}_v\}_{v \geq v_2}$, suppose that ξ_v is a solution of $\mathcal{S}_v$, $v \geq v_2$, such that

$$d(\xi_v(t), \xi_v^k(\mathbb{R})) < \delta', \forall t \leq t_0 \quad (6.25)$$

for some $t_0 \in \mathbb{R}$ and $k \in \{1, \dots, n_j\}$. It follows directly from Remark 16 that

$$\lim_{t \rightarrow -\infty} d(\xi_v(t), \xi_v^k(t)) = 0. \quad (6.26)$$

Finally, from (6.19), (6.14), (6.24) and (6.26), the family $\{\mathcal{G}_{\frac{v_2}{v}}\}_{v \geq v_2}$ with $\mathcal{G}_0 = \mathcal{T}_{x_j^*}$ and $\mathcal{G}_{\frac{v_2}{v}} = \mathcal{S}_v$, satisfies the conditions from Theorem 14. Therefore there exists $v_3 \geq v_2$ such that $\mathcal{S}_v$ is dynamically gradient relative to $\Xi_v = \{\{\xi_v^1(t)\}_{t \in \mathbb{R}}, \dots, \{\xi_v^{n_j}(t)\}_{t \in \mathbb{R}}\}$, for all $v \geq v_3$.

Note that ψ is a bounded solution of (6.10) and

$$\psi_v(t) = \begin{cases} \psi(t), & t \leq -v \\ \mathcal{S}_v(t, -v)\psi(-v), & t > -v \end{cases} \quad (6.27)$$

is a bounded solution of $\mathcal{S}_v$. Hence, for each $v \geq v_3$, there exists $\xi_v^k(\mathbb{R}) \in \Xi_v$ such that

$$\lim_{t \rightarrow -\infty} d(\psi_v(t), \xi_v^k(\mathbb{R})) = 0. \quad (6.28)$$

From (6.27) and (6.28) we have

$$\lim_{t \rightarrow -\infty} d(\psi(t), \xi_v^k(\mathbb{R})) = 0, \forall v \geq v_2. \quad (6.29)$$

It follows from (6.24) and (6.29) that

$$\lim_{t \rightarrow -\infty} d(\psi(t), y_k^*) = 0. \quad (6.30)$$

This proves that $\zeta = (\phi, \psi)$ converges to an equilibrium when $t \rightarrow -\infty$. The proof of the convergence when $t \rightarrow +\infty$ is analogous. Therefore, the semigroup related to (6.9) satisfies (G1).

We will prove (G2) by contradiction. Suppose that the semigroup $\mathcal{L} = \{L(t) : t \in \mathbb{R}^+\}$ generated by (6.9) has a homoclinic structure, given by the global solutions $\{\zeta_i = (\phi_i, \psi_i) : i = 1, \dots, k\}$ and equilibrium points $\{z_i^* = (x_i^*, y_i^*) : i = 1, \dots, k\}$. Remember that each ϕ_i is a global solution of (6.6) and each ψ_i is a global solution of (6.10). Therefore, each ϕ_i must be constant with value $x_i^* = x^*$, for all $i = 1, \dots, n$, otherwise the semigroup $\mathcal{T}$ related to (6.6) has a homoclinical structure. Therefore, the maps ψ_i solves the same equation

$$\begin{cases} \dot{y} = f(x^*, y), & t > \tau \\ y(\tau) = y_0 \in \mathbb{R}^m \end{cases} \quad (6.31)$$

and jointly with the equilibrium points are part of a homoclinial structure in $\mathcal{T}_{x^*}$, contradicting that $\mathcal{T}_{x^*}$ is dynamically gradient. Therefore, (6.9) generates a semigroup $\mathcal{L} = \{L(t) : t \in \mathbb{R}^+\}$ which is dynamically gradient. $\square$

Remark 18. Note that the semigroup related to (6.9) is gradient but $J(h)$ (Jacobian of h) is given by the $n \times m$ matrix

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \dots & \frac{\partial g_n}{\partial x_n} & 0 & \dots & 0 \\ \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial x_{n+1}} & \dots & \frac{\partial f_1}{\partial x_{n+m}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} & \frac{\partial f_m}{\partial x_{n+1}} & \dots & \frac{\partial f_m}{\partial x_{n+m}} \end{pmatrix}$$

which is, in general, not symmetrical when $m = n$. This shows that gradient semigroups are a bigger class than the semigroups obtained in Remark 7.

Now we will use the skew-product approach, seen in Section 4.1, to prove a similar result about the asymptotically behavior of the cascade system.

Theorem 26. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a map such that the autonomous problem

$$\begin{cases} \dot{x} = g(x), & t > 0, \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \quad (6.32)$$

generates a gradient semigroup $\theta = \{\theta(t) : t \in \mathbb{R}^+\}$ relative to $\mathcal{E} = \{x_1^*, \dots, x_k^*\}$. Consider the cascade system

$$\begin{cases} \dot{y} = f(x, y), & t > 0 \\ \dot{x} = g(x), & t > 0, \\ x(0) = x_0 \in \mathbb{R}^n, \\ y(0) = y_0 \in \mathbb{R}^m \end{cases} \quad (6.33)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$. Rewriting (6.33) as

$$\begin{cases} \dot{y}(t) = f(\theta(t)x_0, y(t)), & t > 0, \\ y(0) = y_0 \in \mathbb{R}^m \end{cases} \quad (6.34)$$

and assuming that (6.34) has a unique solution $\mathcal{K}(\cdot, x_0)y_0$ in $[0, \infty)$, we obtain that $\mathcal{K} = \{\mathcal{K}(t, x) : t \geq 0, x \in \mathbb{R}^n\}$ is a cocycle relative to θ . If the skew-product Π related to θ and $\mathcal{K}$ has a global attractor $\mathbb{A}$, then the semigroup $\mathcal{T}_{x_j^*} = \{T_j(t) : t \geq 0\}$ related to

$$\begin{cases} \dot{y} = f(x_j^*, y), & t > 0, \\ y(0) = y_0 \in \mathbb{R}^m \end{cases} \quad (6.35)$$

has a global attractor $\mathcal{A}_j$, for all $j = 1, \dots, k$. Moreover, if $\mathcal{T}_{x_j^*}$ is a gradient semigroup, for each $j = 1, \dots, k$, then Π is a gradient semigroup relative to $\{\mathcal{A}_1 \times x_1^*, \dots, \mathcal{A}_k \times x_k^*\}$.

Proof. From Theorem 12, we know that

$$\mathbb{A} = \bigcup_{\eta} \bigcup_{t \in \mathbb{R}} \{A_{\eta}(t)\} \times \{\eta(t)\},$$

where $\{A_{\eta}\}_{t \in \mathbb{R}}$ is the pullback attractor of the dynamical system S_{η} , given by

$$S_{\eta}(t, s) = \mathcal{K}(t - s, \eta(s)).$$

Note that, for each $j = 1, \dots, k$ we have $S_{x_j^*}(t, s) = \mathcal{K}(t - s, x_j^*) = T_j(t - s)$. Hence $S_{x_j^*} = \mathcal{T}_{x_j^*}$ and from Theorem 6 the pullback attractor $\{A_{x_j^*}(t)\}_{t \in \mathbb{R}}$ satisfies $\mathcal{A}_j := A_{x_j^*}(t)$, for all $t \in \mathbb{R}$. Thus, given $(y(\cdot), x(\cdot))$ a bounded global solution of Π , there exists $1 \leq i \leq j \leq k$ such that

$$\lim_{t \rightarrow -\infty} d((y(t), x(t)), \mathcal{A}_j \times x_j^*) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} d((y(t), x(t)), \mathcal{A}_i \times x_i^*) = 0.$$

It is easy to see that $\{\mathcal{A}_1 \times x_1^*, \dots, \mathcal{A}_k \times x_k^*\}$ is a disjoint collection of isolated Π -invariant sets. Proceeding as Theorem 25 we conclude (G2) and that Π is a gradient semigroup. $\square$

Example 4. Consider the autonomous system

$$\begin{cases} \dot{x} = x - x^3, & t > 0, \\ x(0) = x_0 \in \mathbb{R}. \end{cases} \quad (6.36)$$

First we will show that (6.36) generates a semigroup θ which is gradient. Note that there exists $M > 0$ such that

$$\langle x, x - x^3 \rangle = x^2 - x^4 < -1, \forall |x| \geq M. \quad (6.37)$$

It follows from Example 2 that (6.36) generates a semigroup θ , with global attractor $[-1, 1]$, which is gradient (as Example 1) relative to the set of equilibriums $\{-1, 1, 0\}$ (already ordered to be a Morse decomposition).

Figure 1 – Local Attractors of (6.36)

Consider the cascade system

$$\begin{cases} (y_1, y_2) = (xy_2 - y_1^3, xy_1 - y_2^3), t > 0 \\ \dot{x} = x - x^3, t > 0, \\ x(0) = x_0 \in \mathbb{R}, \\ y(0) = y_0 \in \mathbb{R}^2, \end{cases} \quad (6.38)$$

that can be rewritten as

$$\begin{cases} (y_1(t), y_2(t)) = (\theta(t)x_0 \cdot y_2(t) - y_1(t)^3, \theta(t)x_0 \cdot y_1(t) - y_2(t)^3), t > 0 \\ y(0) = y_0 \in \mathbb{R}^2. \end{cases} \quad (6.39)$$

First, we will show that (6.39) generates a semigroup, that we will denote by $\mathcal{H} = \{\mathcal{H}(t, x_0)y_0 : t \geq 0\}$. Note that $\{x(t) : t \geq 0\}$ is bounded because θ has a global attractor and dimension of $\mathbb{R}$ is finite. Thus, there exists $M_1 > 0$ such that

$$\begin{aligned} \langle (y_1, y_2), (\theta(t)x_0 \cdot y_2(t) - y_1(t)^3, \theta(t)x_0 \cdot y_1(t) - y_2(t)^3) \rangle &= 2xy_1y_2 - y_1^4 - y_2^4 \\ &< -1, \forall t \in \mathbb{R}, \forall \|(y_1, y_2)\| > M_1. \end{aligned}$$

Again, from Example 2, we guarantee that (6.36) generates a semigroup.

Now we will show that the skew product Π in $\mathbb{R}^2 \times \mathbb{R}$ has a global attractor $\mathbb{A}$. Note that $\Pi(t)(y_0, x_0) = (\mathcal{H}(t, x_0)y_0, \theta(t)x_0)$ is the solution of the autonomous equation

$$\begin{cases} \dot{z} = h(z), t > 0, \\ z(0) = z_0 \in \mathbb{R}^2 \times \mathbb{R} \end{cases} \quad (6.40)$$

where $z_0 = (y_0, x_0)$ and $h((y_1, y_2, x)) = (xy_2 - y_1^3, xy_1 - y_2^3, x - x^3)$ for each $z = (y_1, y_2, x)$. Again, there is $M_2 > 0$ such that

$$\begin{aligned} \langle (x, y_1, y_2), (xy_2 - y_1^3, xy_1 - y_2^3, x - x^3) \rangle \\ = x^2 + 2y_1y_2 - x^4 - y_1^4 - y_2^4 \\ < -1, \forall \|(x, y_1, y_2)\| > M_2 \end{aligned}$$

implying that Π has a global attractor $\mathbb{A}$.

Finally, we will analyze if the semigroups generated by $\dot{y} = f(x^*, y)$, where $x^* \in \{-1, 1, 0\}$, are gradient. First, we will consider $x^* = -1$, which bring us to the system

$$\begin{cases} \dot{y} = (-y_2 - y_1^3, -y_1 - y_2^3), t > 0, \\ y(0) = y_0 \in \mathbb{R}^2. \end{cases} \quad (6.41)$$

We already know from Theorem 26 that (6.41) generates a semigroup $\mathcal{T}_{-1}$ with a global attractor $\mathcal{A}_{-1}$. Since

$$\frac{\partial(-y_2 - y_1^3)}{\partial y_2} = -1 = \frac{\partial(-y_1 - y_2^3)}{\partial y_1},$$

it follows from Remark 7 that $\mathcal{T}_{-1}$ is a gradient semigroup. It is easy to see that the set of equilibria is $\mathcal{E}_{-1} = \{(-1, 1), (1, -1), (0, 0)\}$ (already ordered to be a Morse decomposition), implying, from Theorem 3, that

$$\mathcal{A}_{-1} = W^u((-1, 1)) \cup W^u((1, -1)) \cup W^u((0, 0)).$$

The same happens to the system

$$\begin{cases} \dot{y} = (-y_1^3, -y_2^3), t > 0, \\ y(0) = y_0 \in \mathbb{R}^2 \end{cases} \quad (6.42)$$

that generates a gradient semigroup $\mathcal{T}_0$ relative to $\mathcal{E}_0 = \{(0, 0)\}$ and with global attractor $\mathcal{A}_0 = (0, 0)$.

Lastly, for $x^* = 1$, the system

$$\begin{cases} \dot{y} = (y_2 - y_1^3, y_1 - y_2^3), t > 0, \\ y(0) = y_0 \in \mathbb{R}^2 \end{cases} \quad (6.43)$$

generates a gradient semigroup $\mathcal{T}_1$, relative to $\mathcal{E}_1 = \{(-1, -1), (1, 1), (0, 0)\}$ (already ordered to be a Morse decomposition), which global attractor is

$$\mathcal{A}_1 = W^u((-1, -1)) \cup W^u((1, 1)) \cup W^u((0, 0)).$$

Figure 2 – Global Attractor of (6.43)

Therefore we have fulfilled the conditions from Theorem 26 and we concluded that Π is a gradient semigroup relative to the set $\{\mathcal{A}_{-1} \times \{-1\}, (0, 0, 0), \mathcal{A}_1 \times \{1\}\}$, which is already ordered to be a Morse decomposition. Moreover, from Theorem 3 we have

$$\mathbb{A} = W^u(\mathcal{A}_{-1} \times \{-1\}) \cup W^u((0, 0, 0)) \cup W^u(\mathcal{A}_1 \times \{1\}).$$

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