Smooth perturbations of Lorenz-like flows

José Humberto Bravo Vidarte

SERVIÇO DE PÓS-GRADUAÇÃO DO ICMC-USP

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***Advisor:* Prof. Dr. Daniel Smania Bradão**

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Perturbações suaves de fluxos do tipo Lorenz

José Humberto Bravo Vidarte

***Orientador:* Prof. Dr. Daniel Smania Brandão**

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***This dissertation is dedicated for my
family, especially for my wonderful
Mother María.***

"A mathematical theory is not to be considered complete unless you made it so clear that you can explain it to the man in the street."

David Hilbert.

***Tradition becomes our security, and
when the mind is secure it is in decay.
Jiddu Krishnamurti.***

"Be -don't try to become"
Osho.

"Drop the idea of becoming someone, because you are already a masterpiece. You cannot be improved. You have only to come to it, to know it, to realize it."

Osho.

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Abstract

Given a Geometric Lorenz Flow X on $\mathbb{R}^{n+2}$ of class C^{k+1} , by definition there exists a Poincaré map P_X of class C^{k+1} ; often so-called Lorenz-type map [ABS83]. The main purpose in this dissertation is to show that under certain conditions the Lorenz-type map P_X can be associated to it a one-dimensional transformation f_X of class C^k (defined on an interval). This association is so-called the reduction transformation $\mathcal{R}$, so we have $\mathcal{R}P_X = f_X$. This association would allow us to study the dynamical properties for the original flow using techniques of one-dimensional dynamics of class C^k .

Resumo

Dado um Fluxo Geométrico de Lorenz X em $\mathbb{R}^{n+2}$ de classe C^{k+1} , por definição, existe uma aplicação de Poincaré P_X de classe C^{k+1} ; frequentemente chamado aplicação do tipo Lorenz [ABS83]. O objetivo principal desta tese é mostrar que, sob certas condições a aplicação do tipo Lorenz P_X pode ser associado a ele uma transformação unidimensional f_X de classe C^k (definida em um intervalo). Esta associação é chamada de transformação de redução $\mathcal{R}$, assim temos que $\mathcal{R}P_X = f_X$. Esta associação nos permitiria estudar as propriedades dinâmicas do fluxo original utilizando técnicas da dinâmica unidimensional de classe C^k .

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Introduction

In 1963, the meteorologist Edward Lorenz published a well-known paper called *Deterministic Nonperiodic Flow* [Lor63], attempted to set up a system of differential equations that would explain some the unpredictable behavior of the weather, after experimenting with several examples, derived of the equation from B. Saltzmann concerning thermal fluid convection, the following polynomial ordinary differential equations

$$(\dot{x}, \dot{y}, \dot{z}) = (\sigma(y - x), \rho x - y - xz, -\beta z + xy). \quad (1.1)$$

Setting the parameters (σ, ρ, β) to be at $(10, 20, \frac{8}{3})$. Lorenz has studied numerically this systems

Figure 1.1: Lorenz Attractor (1.1). Figure obtained by Oscar Lanford III.

and he found solutions which always remains bounded forever but behaves in a very compli-

cated manner.

This was one of the first examples of what is now known as "chaos". One of the first rigorous studies to understand the above "chaotic" dynamic was made independently by Guckenheimer [GH83], and, Afraimovich-Bykov-Shil'nikov [ABS77]. Inspired by the numerical simulation of the Eq. (1.1) (as described in Figure 1.1, due to Oscar Lanford III), they have constructed a Geometrical Model as an abstract three-dimensional vector field, and studied the dynamics of the "chaotic" attractor in the Geometrical model. Briefly they have proven the existence of a well-defined attractor in the Geometrical Model. Roughly speaking, a Geometric Lorenz Flow [ABS83] is a vector field X that admits a saddle equilibrium point O with a one-dimensional unstable manifold $W^u(O)$, codimension-one stable manifold $W^s(O)$, and a cross section D so that the Poincaré map $T : D \setminus D_0 \rightarrow D$ defined outside of the curve D_0 which is contained in the intersection of D with the stable manifold $W^s(O)$. The Figure 1.2 depicts the behavior of T . Since the domain of T is $(n+1)$ -dimensional, the map T is in some ways simpler than the flow φ_t ; however, because not all trajectories which begin in D necessarily return to D , the return map T is not defined on all of D . In particular, it may not be a continuous map on all of D ; this is the price we pay for the simplification.

Figure 1.2: Defining the Poincaré section on D .

1.1 Main Theorem

1.1.1 Statement of Result

To state our main result, we require some notation and definitions. Let n be a strictly positive integer and consider the Euclidean space $\mathbb{R}^{n+1} := \mathbb{R}^n \times \mathbb{R}$.

Define

$$\begin{aligned} D &:= \{(x, y) \in \mathbb{R}^{n+1} : \|x\| \leq 1, |y| \leq 1\}, \\ D_+ &:= \{(x, y) \in D : y > 0\}, \\ D_- &:= \{(x, y) \in D : -y < 0\}, \\ D_0 &:= \{(x, y) \in D : y = 0\}. \end{aligned} \tag{1.2}$$

Notice that the sets D_+ and D_- are separate by the hyperplane D_0 .

From now on, the symbol $\|\cdot\|$ denote a norm in $\mathbb{R}^n$, if applied to a vector or for the corresponding matrix norm if applied to a matrix.

We use the notation

$$\|\cdot\|_D := \sup_{(x,y) \in D_+ \cup D_-} \|\cdot\|$$

for norms of matrix and vector functions on $D_+ \cup D_-$.

We shall working with maps $T : D_+ \cup D_- \rightarrow D$ given by the equation

$$T(x, y) = (F(x, y), G(x, y)) = (\bar{x}, \bar{y}), \tag{1.3}$$

where the vector function F and the scalar function G are differentiables on $D_+ \cup D_-$.

Definition 1.1. Suppose that $\partial_y G(x, y)$ is non-vanishing on $D \setminus D_0$. Then, we define the following functions:

$$A(x, y) := \frac{\partial_x F(x, y)}{\partial_y G(x, y)}, \tag{1.4}$$

$$B(x, y) := \frac{\partial_y F(x, y)}{\partial_y G(x, y)}, \tag{1.5}$$

$$C(x, y) := \frac{\partial_x G(x, y)}{\partial_y G(x, y)}. \tag{1.6}$$

Here $A(x, y)$ is a $n \times n$ matrix, $B(x, y)$ is a n -column vector, and $C(x, y)$ is a n -row vector.

Assumption 1.2. We assume the following assumptions on T

(L₁) The function F and G have the form

$$F(x, y) = \begin{cases} x_+^* + |y|^\alpha [B_+^* + \varphi_+(x, y)] & y > 0, \\ x_-^* + |y|^\alpha [B_-^* + \varphi_-(x, y)] & y < 0 \end{cases}$$

$$G(x, y) = \begin{cases} y_+^* + |y|^\alpha [A_+^* + \psi_+(x, y)] & y > 0, \\ y_-^* + |y|^\alpha [A_-^* + \psi_-(x, y)] & y < 0 \end{cases}$$

in a neighborhood of D_0 , where, A_+^*, A_-^* are non zero and the functions $\varphi_\pm$ and $\psi_\pm$ are of class C^{k+1} . The derivatives of $\varphi_\pm$ and $\psi_\pm$ are uniformly bounded with respect to x and satisfy the following estimates:

$$\left\| \frac{\partial^{l+m} \varphi_\pm(x, y)}{\partial x^l \partial y^m} \right\| \leq K |y|^{\gamma-m}, \quad \left\| \frac{\partial^{l+m} \psi_\pm(x, y)}{\partial x^l \partial y^m} \right\| \leq K |y|^{\gamma-m}, \quad (1.7)$$

where $\gamma > k - 1$, K is a positive constant, $l = 0, 1, 2$, $m = 0, 1, 2$, and $l + m \leq k + 1$.

(L₂)

$$1 - \|A\|_D > 2\sqrt{\|B\|_D \|C\|_D}. \quad (1.8)$$

(L₃) The following relations hold:

(a)

$$\frac{(2)^2 (\|A\|_D + \|C\|_D \|B\|_D) \max_{m+n=1} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}}{(\|\partial_y G\|_D)^{-k} \left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D \|C\|_D}\right)^2} < 1. \quad (1.9)$$

(b) for $k \geq 2$

$$\frac{(2k!)^2 (\|A\|_D + \|C\|_D \|B\|_D) \max_{m+n=k} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}}{(\|\partial_y G\|_D)^{-k} \left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D \|C\|_D}\right)^2} < 1, \quad (1.10)$$

and

$$\|\partial_y G\|_D \geq \frac{1}{4} \quad \text{or} \quad \|\partial_x F\|_D \geq \frac{1}{4}. \quad (1.11)$$

1.1.1.1 Main Theorem: Existence of C^k -Invariant Foliation for Lorenz-Type Maps

In this dissertation we prove the following theorem:

Theorem 1.3. Suppose that the map T satisfies Assumption 1.2. Then, there is a C^k foliation $\mathcal{F}_D$, T -invariant, with C^{k+1} leaves, that is,

- (1) Each leaf $\mathcal{F}_{(x_0, y_0)} \in \mathcal{F}_D$ is the graph of a function $y = h(x)$; such that $y_0 = h(x_0)$ (see Figure 1.3).
- (2) The hyperplane Γ is a leaf of $\mathcal{F}_D$.
- (3) The foliation $\mathcal{F}_D$ is T -invariant, that is, for each leaf $\mathcal{F}_{(x_0, y_0)} \in \mathcal{F}_D$, $\mathcal{F}_{(x_0, y_0)} \neq D_0$, there is $\mathcal{F}_{T(x_0, y_0)} \in \mathcal{F}_D$ such that $T(\mathcal{F}_{(x_0, y_0)}) \subset \mathcal{F}_{T(x_0, y_0)}$ (see Figure 1.4).

Figure 1.3: Geometric interpretation of Foliation.

Figure 1.4: Geometric interpretation of the invariant foliation.

Remark 1.4. Our main influence for the Theorem 3.7 was the article of Shaskov and Shilnikov

[SS94], where was shown under three conditions the existence of a C^1 -stable invariant foliation for Lorenz-type map:

Condition 1: Assumption L_1 of Assumption 1.2 in the case $k = 1$.

Condition 2: Assumption L_2 of Assumption 1.2.

Condition 3: The following relation holds :

$$2(\|\partial_x f\|_D + \|\partial_x G\|_D \|B\|_D) < \frac{\left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D\|C\|_D}\right)^2}{\left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 + 4\|B\|_D\|C\|_D}\right)}. \quad (1.12)$$

A important consequence of the last Theorem 1.3 is the following corollary.

Corollary 1.5. Suppose that the map T satisfies Assumption 1.2. Then T can be associate with a one-dimensional map $\varphi : J \setminus \{0\} \rightarrow J$ of class C^k , such that, the following diagram is commutative

$$\begin{array}{ccc} D \setminus \Gamma & \xrightarrow{T} & D \\ \downarrow \pi & & \downarrow \pi \\ J \setminus \{0\} & \xrightarrow{\varphi} & J \end{array}$$

$\pi \circ T = \varphi \circ \pi$ on D .

The map $\varphi : J \setminus \{0\} \rightarrow J$ is called one-dimensional Lorenz-like transformation .

The Figures 1.5(a), 1.5(b), 1.5(c) show the possibles graphs of the one-dimensional Lorenz-like transformation $\varphi : J \setminus \{0\} \rightarrow J$.

Figure 1.5: Possibles graphs of the one-dimensional Lorenz-like transformation $\varphi : J \setminus \{0\} \rightarrow J$.

Geometric Lorenz Flow

The purpose of this chapter is give a brief description of the Lorenz equation , as well as define the geometric Lorenz flow. This material it follows of the books and articles , [ABS83], [ABS83], [ASB03] [J.G76], [GW79] [GH83] and [YC09].

2.1 A Little of History

In 1963, the meteorologist E.N. Lorenz published a well-known paper called *Deterministic Nonperiodic Flow* [Lor63], attempted to set up a system of differential equations that would explain some the unpredictable behaviour of the weather, after experimenting with several examples, derived of the equation from B. Saltzmann concerning thermal fluid convection, the following polynomial ordinary differential equations

$$\begin{aligned}\dot{x} &= \sigma(y - x), \sigma = 10, \\ \dot{y} &= \rho x - y - xz, \rho = 28, \\ \dot{z} &= -\beta z + xy, \beta = \frac{8}{3}.\end{aligned}\tag{2.1}$$

Lorenz numerically found solutions which remains bounded forever and the solutions started out very differently, but eventually they have more or less the same fate: The solutions seem to wind around a pair of points, alternating at times which point they encircle. This is the first important fact about the Lorenz systems: All non-equilibrium solutions tend eventually to same complicated set, the so-called *Lorenz Attractor*. Another fact that was noted by Lorenz was that the *Lorenz Attractor* have *sensitivity to initial conditions* or also know as *the butterfly effect*,

Figure 2.1: Attractor of the the Lorenz equation (2.1).

means, no matter how close two solution start, they will have different future behaviour, one of the main features of a “chaotic system”. This was one of the first examples of what is now know as “chaos”.

One of the first rigorous studies to understand the above “chaotic” dynamic in was made independently by [GH83], and, Afraimovich, Bykov and Shil’nikov [ABS77]. From the numerically simulation of (2.1), they have constructed a Geometrical Model as a abstract three-dimensional vector field, and studied the dynamics of the “chaotic” attractor in the Geometrical model. Briefly they have proven the existence of a well-defined attractor in the Geometrical Model.

2.2 Description of the Lorenz equation

In this section we will give a brief numerical description of Poincaré map of the Lorenz equation (2.1). For that we will follow closely the book and articles, [ABS77], [ABS77], [ABS83], [ASB03] [GH83], [YC09], [Via00] and [AP00].

2.2.1 Preliminares

2.2.2 Poincaré Map

The idea of reducing the study of continuous time systems(flow) to the study of an associated discrete time systems(map) is due to Henry Poincaré[1899], who was first to use it to study the problem of three body. Roughly speak, a first recurrence map or Poincaré map, is the

intersection of a periodic orbit in the state space of a continuous dynamical system with a certain lower dimensional subspace, called the Poincaré section, transversal to the flow of the system. More precisely, one considers a periodic orbit with initial conditions within a section of the space, which leaves that section afterwards, and observes the point at which this orbit first returns to the section. One then creates a map to send the first point to the second, hence the name first recurrence map. The transversality of the Poincaré section means that periodic orbits starting on the subspace flow through it and not parallel to it. Nowadays also there is a inverse process called suspension where every map is associated to a ordinary differential equation. The technique of Poincaré offers several advantages in the study of ordinary differential equations, here we mention two according S.Wiggins [Wig88, p. 77] of which are the following:

- 1) **Dimensional Reduction.** The Poicaré map allow us reduce at least one of the variables of the problem resulting in a lower dimensional problem to be studied, we can take as example the case the geometric Lorenz flow, that the Poincaré map allow us reduce the study of the flow in $\mathbb{R}^{n+2}$ for the Lorenz Poincaré in $\mathbb{R}^{n+1}$.
- 2) **Global and local Dynamics.** In some case the Poincaré Map allow us get a insightful and striking of the global dynamics of a system, by instance in the case of Geometric Lorenz Flow, all its properties are found starting from the Poincaré Lorenz Map. Also we can understand the behaviour local of flow, by instance problems of stability of flow can be studied in terms of Poincaré Map, this problem of the stability of a fixed point of the map which is simply characterized in terms of the eigenvalues of the map which is simply characterized in terms of the eigenvalues of the map linearized about the fixed point (see the book [Mei07, Ch. 4]).

We will now describe a useful relation between discrete-time (maps) and continuous-time (flows) dynamical systems.

Let $X : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field of class C^k , ($k \geq 1$), and let $\phi : \mathbb{R} \times \Omega \rightarrow \Omega$ be the flow generated by the differential equation

$$\dot{x} = X(x). \quad (2.2)$$

Definition 2.1. An $(n - 1)$ dimensional manifold $D \subset \mathbb{R}^n$ is transversal to the flow generated by (2.2) if for each $p \in D$, then $\langle X(p) \rangle \oplus T_p D = \mathbb{R}^n$.

Theorem 2.2 (Poincaré Map for flow of vector field). *Suppose that there exists a transversal section D to the flow (2.2), and let $\phi_t(x_0)$ a periodic solution of period T (that is, $\phi_{(t+T)}(x_0) = \phi_t(x_0)$), then there exists a subset $U \subset D$ containing x_0 such that for all $x \in U$ there exists*

a time of first return of point x to D $\tau(x)$ such that $\phi_{\tau(x)}(x) \in D$ and the Poincaré map $P : U \rightarrow D$ given by $P(x) = \phi_{\tau(x)}(x)$, is of class C^r .

Figure 2.2: Poincaré Map of the flow ϕ .

2.2.3 Stable Manifold Theorem

Let $X : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field of class C^k , ($k \geq 1$), and let $\varphi : \mathbb{R} \times \Omega \rightarrow \Omega$ be the flow generated by the differential equation

$$\dot{x} = X(x). \quad (2.3)$$

Suppose that 0 is a hyperbolic equilibrium point of (2.3), that is $X(0) = 0$ and the eigenvalues of $DX(0)$ all have part real different of zero.

We define the local stable and unstable manifold of 0, $W_{loc}^s(0)$, $W_{loc}^u(0)$ as follows

$$W_{loc}^s(0) = \{x \in U \mid \lim_{t \uparrow \infty} \varphi_t(x) = 0, \text{ and } \varphi_t(x) \in U \text{ for all } t \geq 0\}, \quad (2.4)$$

$$W_{loc}^u(0) = \{x \in U \mid \lim_{t \downarrow -\infty} \varphi_t(x) = 0, \text{ and } \varphi_t(x) \in U \text{ for all } t \leq 0\}, \quad (2.5)$$

where $U \subset \mathbb{R}^3$ is a neighborhood of the point 0.

Now, we state the stable manifold theorem for a hyperbolic fixed point

Theorem 2.3. Suppose that $X : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field of class C^k function such that 0 is a hyperbolic equilibrium point of the Eq. (2.3). Then there exist a local stable and unstable manifolds $W_{loc}^s(0)$, $W_{loc}^u(0)$, of the same dimension n_s, n_u as those the eigenspace E^s , E^u of

Figure 2.3: Sketch of Local Stable Manifold Theorem.

the linearized system, and tangent to E^s , E^u at 0. $W^s_{loc}(0)$, $W^u_{loc}(0)$ are as smooth as the vector field X .

Proof. The proof can be found in [HP69]. ■

We define the local stable and unstable manifold of 0, $W^s(0)$, $W^u(0)$ as follows

$$W^s(0) := \{x \in \Omega \mid \lim_{t \uparrow \infty} \varphi_t(x) = 0\}. \quad (2.6)$$

$$W^u(0) := \{x \in \Omega \mid \lim_{t \downarrow -\infty} \varphi_t(x) = 0\}. \quad (2.7)$$

Next, we state the global stable manifold theorem for a hyperbolic fixed point.

Theorem 2.4. *Suppose that $X : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^k function such that 0 is a hyperbolic equilibrium point of (2.3). Then the Global stable and unstable sets $W^s(0)$, $W^u(0)$ are C^r -immersed submanifold of $\mathbb{R}^n$ of the same dimension n_s, n_u as those the eigenspace $E^s(0)$, $E^u(0)$ of the linearized system, and tangent to $E^s(0)$ and, $E^u(0)$ at 0.*

Moreover,

$$W^s(0) = \bigcup_{t \leq 0} \varphi_t(W^s_{loc}(0)), \quad (2.8)$$

and

$$W^u(0) = \bigcup_{t \geq 0} \varphi_t(W^u_{loc}(0)). \quad (2.9)$$

2.2.4 Numerical Properties of the Poincaré Map of the Lorenz Equation

This section follows closely the books and articles, [ABS77], [ABS77], [ABS83], [ASB03] [GH83], [Via00] and [AP00].

Numerically, we can see that it is possible take a section D transversal to the lines of the flow of the Lorenz equation, and define Poincaré map on a subset of D (see Figure (2.4)). Since the domain of the Poincaré map T is two-dimensional, the map T is somewhat simpler than the flow φ_t ; however, because not all trajectories which begin in D necessarily return to D , the return map T may not be defined on all of D . In particular, it may not be a continuous map on all of D ; this is the price we pay for the simplification.

Figure 2.4: Defining the Poincaré section on D .

Indeed, the first difficulty arises when the stable manifold $W^s(0)$ crosses D . Thus, since D and $W^s(0)$ are transversal surfaces, and so $D_0 = D \cap W^s(0)$ is a curve. Therefore, to trajectories of points $x \in D_0$, we have $\lim_{t \rightarrow \infty} \varphi_t(x) = 0$, thus the trajectory $\varphi_t(x)$ approaches of 0. This means, every $x \in D_0$ never back to D , so $T(x)$ is undefined. Now, if we choose trajectories that started in points $x \in D \setminus D_0$, which lie just to one side of D_0 (see figure (2.4)), then the trajectories $\varphi_t(x)$ will follow the stable manifold $W^s(0)$ for time before diverging and following the unstable curve $W^u(0)$ out towards the edge of its range eventually passing outside the edge of D , and then intersecting D again in this time in other side of D_0 , as shown in Figure 2.4. When x which lies just to one side and comes close to D_0 , so the trajectory $\varphi_t(x)$ comes close the stable manifold $W^s(0)$, thus $\lim_{t \rightarrow \infty} \varphi_t \sim 0$. Since 0 is a homoclinic point, it follows that $T(x)$ approaches a_2 . Thus, any continuous extension of T to the line D_0 must have $T(x) = a_2$, for every $x \in D_0$. Nevertheless, we may make a identically argument for the other side of D_0 , in this case we require $T(x) = a_1$, for all $x \in D_0$. Therefore the Poincaré map T has no

continuous extension to the line D_0 , although such an extension exists if we approach the line from only one side or the other.

The Figure 2.1 takes of the book [GH83] shows numerically the geometric behaviour of the Attractor of the Lorenz attractor (??) with the classic parameters.

Now, we make a summary of the numerical properties of the Eq. (2.1), that served of source of inspiration for the definition of geometric Lorenz flow as we will see later on.

1. **[Equilibrium Point]** It easy to see that the Lorenz equation has an equilibrium point at 0 and

$$Spec(DX(0)) = \{\lambda_1 \approx 11.83, \lambda_2 \approx -22.83, \lambda_3 \approx -2.67\}.$$

Hence, one can see that

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1 < -\lambda_2, \quad (2.10)$$

where λ_3 is the eigenvalue of z -axis, which is invariant under the flow.

By using the Stable Manifold Theorem applied at point 0, we have that there are stable $W^s(0)$ and unstable $W^u(0)$ manifolds. Note that $W^u(0)$ has dimension one and so, it has two branches, $W^{u,+}(0)$ and $W^{u,-}(0)$ moving away from 0 in opposite directions as time increases.

Moreover, observe that $W^s(0)$ has dimension two, containing the equilibrium point and it is formed by solutions that converge to 0 as the time goes to $+\infty$.

2. **[Numerical existence of Poincaré map]** All the numerical description of the Poincaré map of the Lorenz equation (2.1) can be summarized analytically in following way:

There exists a transversal section to the lines of flow of the Lorenz Equation

$$D := \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\},$$

and sets

$$D_+ := \{(x, y) \in D : y > 0\}, D_- := \{(x, y) \in D : y < 0\},$$

$$D_0 := \{(x, y) \in D : y = 0\},$$

such that satisfy the following properties:

- (1) The stable manifold $W^s(0)$ crosses D on the hyperplane D_0 , that is, every trajectory beginning on D_0 never comes back to beat D for $t > 0$.

(2) The first return map $T : D_+ \cup D_- \rightarrow D$ is well-defined and can be expressed as

$$T_X(x, y) = (F_i(x, y), G_i(x, y)) = (\bar{x}, \bar{y}), i = \pm.$$

(3) F_i and G_i admit continuous extensions on D_0 , that is,

$$\lim_{y \rightarrow 0} F_i(x, y) = x_i^*, \lim_{y \rightarrow 0} G_i(x, y) = y_i^*, i = \pm.$$

Figure 2.5: Image of the Poincaré map.

3. **[Invariant contraction foliation]** For us, a foliation $\mathcal{F}_D$ of D , roughly speaking, is a decomposition of D into disjoint connected smooth curves $\mathcal{F}_{(x,y)}$ such that $D = \bigcup_{(x,y) \in D} \mathcal{F}_{(x,y)}$.

The simplest example of a foliation $\mathcal{F}_D$ of D , is given by the lines (see Figure 2.6).

$$\mathcal{F}_{(x,c)} = \{(\tilde{x}, c) : -1 \leq \tilde{x} \leq 1\}.$$

Later on we will give a formal definition of foliation.

Why is important the existence of the Invariant Contraction Foliation?

Suppose that there exists an Invariant Foliation under the first-return map T , that is, there is a family $\mathcal{F} = \{\mathcal{F}_{(x,y)}\}_{(x,y) \in D}$ of curves in D which contains D_0 such that, every leaf $\mathcal{F}_{(x,y)} \neq D_0$ is mapped by T completely inside of the leaf $\mathcal{F}_{T(x,y)}$, this means that $T(\mathcal{F}_{(x,y)}) \subset \mathcal{F}_{T(x,y)}$. Think the leaves as vertical lines in Figure 2.6. Moreover, the foliation should be a contraction, that is, give $x_1, x_2 \in \mathcal{F}_{(x,y)}$ the distance from $T^n(x_1)$ to $T^n(x_2)$ goes to zero exponentially fast as $n \rightarrow \infty$. The purpose of assuming the of invariant contraction foliation is reduce the dimension of problem to dimension 1. Roughly speaking, this goes as follows: points in the same leaf of the foliation have essentially the same behavior in the future, because their trajectories get closer and closer; so, for

understanding the dynamics of T it is enough to look at the trajectory of only one point on each leaf, for instance, the point where the leaf intersect a given horizontal segment as in the Figure 2.6.

Figure 2.6: Invariant contraction foliation.

2.3 The Geometric Lorenz Flow

The existence of the hyperbolic fixed point, and the numerical properties found in the Poincaré map of Equation of Lorenz (see Figure 2.1) inspired the rigorous mathematical development of a geometrically defined ordinary differential equation so-called geometric Lorenz flow, Guckenheimer [J.G76], Williams [Wil79], and Afraimovich-Bykov-Shilnikov [ABS77]) which seemed to have the same behaviour of the Lorenz equation .

Next, we give the analytical assumptions to define geometric Lorenz vector field according Afraimovich-Bykov-Shilnikov [ABS77].

Definition 2.5. A vector field X of class C^r , $r \geq 1$ on $\mathbb{R}^{n+2}$ is said to be a geometric Lorenz vector field according Afraimovich-Bykov-Shilnikov [ABS77] if it satisfies the following conditions L1)-L2):

L1) *Existence of a singularity.*

There is a singular point 0, that is, $X(0) = 0$ and

$$\text{Spec}(DX(0)) = \{\lambda_{n+2}, \lambda_{n+1}, \lambda_n, \dots, \lambda_1\}$$

satisfies the following relations

$$\operatorname{Re}\lambda_1 \leq \dots \leq \operatorname{Re}\lambda_{n-1} \leq \lambda_n < \lambda_{n+1} < 0 < -\lambda_{n+1} < \lambda_{n+2}. \quad (2.11)$$

Thus, by using Stable Manifold Theorem at singular point 0 we have that there are the stable manifold $W^s(0)$ and unstable manifold $W^u(0)$ of dimension $n + 1$ and dimension one respectively.

L2) Existence of a Poincaré map.

There exists a transversal section to the lines of flow of the Lorenz Equation

$$D := \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\},$$

and sets

$$D_+ := \{(x, y) \in D : y > 0\}, D_- := \{(x, y) \in D : y < 0\},$$

$$D_0 := \{(x, y) \in D : y = 0\},$$

such that satisfy the following properties:

(1) The stable manifold $W^s(0)$ crosses D on the hyperplane D_0 , that is, every trajectory beginning on D_0 never comes back to beat D for $t > 0$.

(2) The first return map $T : D_+ \cup D_- \rightarrow D$ is well-defined and can be expresses as

$$T_X(x, y) = (F_i(x, y), G_i(x, y)) = (\bar{x}, \bar{y}), i = \pm.$$

(3) F_i and G_i admit continuous extensions on D_0 , that is,

$$\lim_{y \rightarrow 0} F_i(x, y) = x_i^*, \lim_{y \rightarrow 0} G_i(x, y) = y_i^*, i = \pm.$$

(4) Hyperbolicity conditions

Let us impose the following restrictions on T called hyperbolicity conditions.

Denote $\|\cdot\| = \sup_{(x,y) \in D_+ \cup D_-} \|\cdot\|$.

$$(a) \quad \|(F_x)\| < 1, \|(G_y)^{-1}\| < 1,$$

$$(b) \quad 1 - \|(G_y)^{-1}\| \|F_x\| > 2\sqrt{\|(G_y)^{-1}\| \|G_x\| \|(G_y)^{-1} \cdot F_y\|},$$

$$(c) \quad \|(G_y)^{-1} \cdot F_y\| \|G_x\| < (1 - \|F_x\|)(1 - \|(G_y)^{-1}\|).$$

Remark 2.6. A map T with the property L2) of Definition 2.5 is called Lorenz-type map.

Now, we define the Geometric Lorenz flow according Afraimovich-Bykov-Shilnikov [ABS77].

Definition 2.7 ([ABS77]). An application $\varphi_X : \mathbb{R}^{n+2} \times \mathbb{R} \rightarrow \mathbb{R}^{n+2}$ is called *Geometric Lorenz flow* of class C^r , $r \geq 2$, if it is generated by a geometric Lorenz vector field X of class C^r , $r \geq 2$ (see Definition 2.5), that is,

$$\begin{aligned} \frac{\partial \varphi(x, t)}{\partial t} &= X(\varphi(x, t)), \\ \varphi(x, 0) &= x. \end{aligned}$$

Figure 2.7: Geometric interpretation of the Three dimensional Geometric Lorenz Flow.

Net, we give an example of a map T that satisfies the conditions L2) of Definition 2.5.

Example 2.8. [AP87, Theorem 11].

Suppose that the map

$$T(x, y) = (F_1(x, y), \dots, G_n(x, y), G(x, y)), \quad x = (x_1, \dots, x_n),$$

is given by the equalities

$$\begin{cases} F_j = (-B_j|y|^{\nu_0} + B_j x_j |y|^{\nu_j} + 1) \operatorname{sgn}(y), \\ G = ((1 + A)|y|^{\nu_0} + A) \operatorname{sgn}(y), \end{cases}$$

where $\operatorname{sgn}(y) = \frac{y}{|y|}$ denotes the sign of y , and the parameters lie in the following ranges:

$$0 < B_j < \frac{1}{2}, \nu_j > 1, j = 1, \dots, n, \frac{1}{2} < A \leq 1, (1 + A)^{-1} < \nu_0 < 1.$$

Then if we consider $\mathbb{R}^n$ with the sup-norm, that is, $\|x\| = \max\{|x_i| : 1 \leq i \leq n\}$ one can prove that T satisfies the properties L2) of Definition 2.5, in other words, T is a Lorenz-type Map.

Remark 2.9. The authors Guckenheimer [J.G76], [GH83] and Guckenheimer-Williams [GW79], who introduced the notion of the geometric Lorenz flow X in dimension three, that is, a flow on $\mathbb{R}^3$, similar to that defined by the authors Afraimovich-Bykov-Shilnikov [ABS77], but with the difference that they consider the function G independently of x , that is, the Poincaré Lorenz map of the geometric Lorenz flow X have the following form:

$$T(x, y) := (F(x, y), G(y)). \quad (2.12)$$

such that

$$G'(y) > \sqrt{2} \quad \text{and} \quad 0 < \partial_x F(x, y) < \delta < 1. \quad (2.13)$$

Thus, it easily seen that T is a Lorenz-type map. It was shown in [ABS83] and [ABS77] that all the main results related to the simplified situation still hold for more general case described in Definition 2.5.

2.3.1 Some Proprieties of Geometric Lorenz Flow

In this section, we state some important properties of geometric Lorenz flows.

Theorem 2.10. The conditions L1)-L2) of the Definition 2.5 hold for a small neighborhood U of the Geometric vector field X (see Definition 2.5).

Proof. The proof can be found in [ABS83]. ■

Theorem 2.11. It follows from analysis of the behaviour of trajectories near $W^s(0)$ that in a small neighborhood of D_0 the following representation is valid :

$$F_i = x_i^* + \varphi_i(x, y)|y|^\alpha, i = \pm, \quad (2.14)$$

where the function φ_i is C^r -smooth in x ;

$$G_i = y_i^* + \psi_i(x, y)|y|^\alpha, i = \pm, \quad (2.15)$$

where the function ψ_i are C^r -smooth in x . Moreover,

$$|(\partial_y \varphi_i) y^{1-\beta_i}| < K_i, |(\partial_y \psi_i) y^{1-\alpha_i}| < K_i,$$

where K_i , are constants and β_i, α_i are positive numbers less than 1.

Proof. The proof can be found in [ABS83]. ■

Definition 2.12 (Separatrix values). *Let $\psi_i(X), i = \pm$ be functions as in Theorem 2.11. Let $A_i(X) = \lim_{y \rightarrow 0} \psi_i(X)(x, y), i = \pm$. The numbers A_i are called separatix values of X and we will assume that A_i does not vanish. Thus, it is natural to distinguish the following cases (see Definition [ABS83, p. 54]):*

- A. (Orientable): $A_1(X) > 0, A_2(X) > 0$;
- B. (Semiorientable): $A_1(X) > 0, A_2(X) < 0$;
- C. (Noorientable): $A_1(X) < 0, A_2(X) < 0$.

Theorem 2.13. *Suppose that the map T satisfies the condition L2) as in Definition 2.5. Then, there is a continuous stable foliation $\mathcal{F}_D$ which satisfies the following properties:*

- 1) *Each leaf $\mathcal{F}_{(x_0, y_0)} \in \mathcal{F}_D$ is the graph of Lypschtz fucntion h .*
- 2) *The hyperplane D_0 is a leaf of $\mathcal{F}_D$, and the preimages of the line of discontinuity are dense in $\mathcal{F}_D$.*
- 3) *The foliation $\mathcal{F}_D$ is T -invariant, that is, give a leaf $\mathcal{F}_{(x_0, y_0)} \in \mathcal{F}_D, \mathcal{F}_{(x_0, y_0)} \neq D_0$, there is $\mathcal{F}_{T(x_0, y_0)} \in \mathcal{F}$ such that $T(\mathcal{F}_{(x_0, y_0)}) \subset \mathcal{F}_{T(x_0, y_0)}$.*

Proof. The proof can be found in [ABS83]. ■

Remark 2.14. *From Theorem 2.13, we can deduce the following facts (see [ASB03] for more details):*

- (1) *Note that, one can identify a foliation $\mathcal{F}_D$ with a interval J . Indeed, every leaf $\mathcal{F}_{(x, y)} \in \mathcal{F}_D$ can be identified with the value*

$$\eta : (0, \eta) := \mathcal{F}_{(x, y)} \cap \{x = 0\}.$$

Considering $J : \{\eta : (0, \eta) := \mathcal{F}_{(x, y)} \cap \{x = 0\}\}$. Thus $\mathcal{F}_D$ can be identified with J .

- (2) *Consider a foliation $\mathcal{F}_D$ as in Theorem 2.13. Then, we have $T(\mathcal{F}_{(x, y)}) \subset \mathcal{F}_{T(x, y)}$. Set*

$$(0, \bar{\eta}) := T(x, y) \bigcap \{x = 0\},$$

Figure 2.8: Sketch of the C^0 -stable invariant foliation $\mathcal{F}_D$.

and

$$\varphi((0, \eta)) := (0, \tilde{\eta}).$$

Thus, one can write $y = h_\eta(x)$, $\eta := h_\eta(1)$. In other words, we may introduce new coordinate $\{(x, \eta)\}$ in D such that the map $\tilde{T}$ has the form

$$\bar{x} = \hat{F}(x, \eta), \bar{\eta} = \hat{G}(x, \eta) := \varphi(\eta),$$

where $\hat{F}(x, \eta) = \hat{F}(x, h_\eta(x))$.

From Theorem 2.13 we have $\hat{F}$ and $\hat{G}$ are continuous functions, unfortunately with this, we can not use the powerful tools of the one-dimensional smooth dynamics.

(3) The facts presented in the items (1), (2) can be interpreted graphically as follows.

Consider $\pi : D \setminus \Gamma \rightarrow \mathcal{F}_D$ given by $\pi(x, y) = \mathcal{F}_{(x,y)}$. By the T -invariance of $\mathcal{F}_D$ we can define the map

$$\varphi : \mathcal{F}_D \setminus \Gamma \rightarrow \mathcal{F}_D$$

uniquely defined so that

$$\begin{array}{ccc} D \setminus \Gamma & \xrightarrow{T} & D \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{F}_D \setminus \Gamma & \xrightarrow{\varphi} & \mathcal{F}_D \end{array}$$

commutes, that is, $\pi \circ T = \varphi \circ \pi$ on $\mathcal{F}_D \setminus \Gamma$.

By definition we have $\varphi(\mathcal{F}_{(x,y)}) = \mathcal{F}_{T(x,y)}$, from (1) we can identify the foliation $\mathcal{F}_D$ with a interval $J = [-1, 1]$, then we have

$$\varphi : J \setminus \{0\} \rightarrow J.$$

From the continuity of the foliation $\mathcal{F}_D$ we can see that g is continuous.

(4) The map $\varphi : J \setminus \{0\} \rightarrow J$ is called one-dimensional Lorenz-like transformation.

In [ABS83] was shown that the graphs of the unidimensional Lorenz-like Maps φ corresponding to the cases Orientable, Semiorientable and Noorientable are as in Figures 2.9(a), 2.9(b), 2.9(c) respectively.

(a) Graph of φ in the Orientable case.

(b) Graph of φ in the Semiorientable case.

(c) Graph of φ in the Noorientable case.

Figure 2.9: Possibles graphs of the one-dimensional Lorenz-like transformation $\varphi : J \setminus \{0\} \rightarrow J$.

Main Result

This chapter will be devoted to state and prove our main result, which provides sufficiency conditions for the existence of a C^k -invariant foliation for Lorenz type map. The main influence in this chapter was the article of Shaskov and Shilnikov [SS94], but the ideas in [Rob81], [Ryc90], [MPP00] and [AP00] were also quite useful.

3.1 Prelimares

This section is aimed to give some notations and definitions that are necessary to state our main result.

Let n be a strictly positive integer and consider the Euclidean space $\mathbb{R}^{n+1} := \mathbb{R}^n \times \mathbb{R}$.

Define

$$\begin{aligned} D &:= \{(x, y) \in \mathbb{R}^{n+1} : \|x\| \leq 1, |y| \leq 1\}, \\ D_+ &:= \{(x, y) \in D : y > 0\}, \\ D_- &:= \{(x, y) \in D : -y < 0\}, \\ D_0 &:= \{(x, y) \in D : y = 0\}. \end{aligned} \tag{3.1}$$

Notice that the sets D_+ and D_- are separate by the hyperplane D_0 .

From now on, the symbol $\|\cdot\|$ denote a norm in $\mathbb{R}^n$, if applied to a vector or for the corresponding matrix norm if applied to a matrix.

We will use the notation

$$\|\cdot\|_D = \sup_{(x,y) \in D \setminus D_0} \|\cdot\|$$

for norms of matrix and vector functions on $D \setminus D_0$.

The following set will be useful for defining the dominions of various maps:

$$D_x := \{x \in \mathbb{R}^n \text{ for which there exists a } y \in \mathbb{R} \text{ with } (x, y) \in D\}. \quad (3.2)$$

So D_x represent the projection of D onto $\mathbb{R}^n$.

Definition 3.1. A family of functions $\mathcal{F}_D = \{h(x)\}$ is called a foliation of D with C^m leaves ($m \geq 0$) given by the graphs of functions $y = h(x)$ if the following three conditions are satisfied:

- (1) The domain $Dom(h(x))$ of every function $h(x) \in \mathcal{F}_D$ is an open and connected set in D_x and its graph lies entirely in D ;
- (2) for every point $(x_0, y_0) \in D$ there is a unique function $h(x) \in \mathcal{F}_D$ such that $x_0 \in Dom(h(x))$ and $y_0 = h(x_0)$. [We denote this function by $h(x; x_0, y_0)$];
- (3) for every point $(x_0, y_0) \in D$ the function $x \rightarrow h(x; x_0, y_0)$ is of class C^m .

Figure 3.1: Geometric interpretation of Foliation.

The graphs of the functions $h(x)$ are called the leaves of $\mathcal{F}_D$ and the leaf that contain $(x_0, y_0) \in D$, will be denoted by $\mathcal{F}_{(x_0, y_0)}$.

Remark 3.2.

- (i) Note that the leaves are manifolds of class C^m , ($m \geq 0$). Indeed, take every point (x_0, y_0) , and the unique C^m function $h(\bullet; x_0, y_0) : Dom(h(x)) \rightarrow \mathbb{R}$ such that $(y_0 = h(x_0))$, then the graph of h is the subset the Cartesian product $\mathbb{R}^n \times \mathbb{R}$ given by

$$\mathcal{F}_{(x_0, y_0)} = \{(x, h(x; x_0, y_0)) : x \in Dom(h(x; x_0, y_0))\}.$$

The projection function $\Pi : \mathcal{F}_{(x_0, y_0)} \rightarrow \text{Dom}(h(x; x_0, y_0)) \subset \mathbb{R}^n$ is a C^m -diffeomorphism and provides a global chart on $\mathcal{F}_{(x_0, y_0)}$ making it a C^m -manifold.

(ii) Notice that, if we take $p = (a, h(a; x_0, y_0))$ then $T_p \mathcal{F}_{(x_0, y_0)}$ is the graph of the derivative $\partial_x h(x; x_0, y_0)_{x=a}$, in other words, $T_p \mathcal{F}_{(x_0, y_0)} = \{(v, \partial_x h(x; x_0, y_0)_{x=a} v) : v \in \mathbb{R}^n\}$.

(iii) Observe that By Definition 3.1(2) and (i), we have that the foliation $\mathcal{F}_D$ is a disjoint decomposition of D by C^m -manifolds $\mathcal{F}_{(x_0, y_0)}$.

Definition 3.3. A foliation $\mathcal{F}_D$ is called C^r -foliation ($r \geq 0$) if the function

$$(x; x_0, y_0) \rightarrow h(x; x_0, y_0)$$

is of class C^r .

Definition 3.4. A foliation $\mathcal{F}_D$ is called T -Invariant if

- 1) the hyperplane D_0 is a leaf of $\mathcal{F}$;
- 2) for each leaf $\mathcal{F}_{(x_0, y_0)} \in \mathcal{F}_D$, with $\mathcal{F}_{(x_0, y_0)} \neq D_0$, there is $\mathcal{F}_{T(x_0, y_0)} \in \mathcal{F}$ such that $T(\mathcal{F}_{(x_0, y_0)}) \subset \mathcal{F}_{T(x_0, y_0)}$.

Figure 3.2: Geometric interpretation of a foliation T -Invariant.

3.2 Statement of Main Result

Consider the map $T : D_+ \cup D_- \rightarrow D$ given by the equation

$$T(x, y) = (F(x, y), G(x, y)) = (\bar{x}, \bar{y}), \quad (3.3)$$

where the vector function F and the scalar function G are differentiables on $D_+ \cup D_-$.

Definition 3.5. Suppose that $\partial_y G(x, y)$ is non-vanishing on $D \setminus D_0$. Then, we define the following functions:

$$A(x, y) := \frac{\partial_x F(x, y)}{\partial_y G(x, y)}, \quad (3.4)$$

$$B(x, y) := \frac{\partial_y F(x, y)}{\partial_y G(x, y)}, \quad (3.5)$$

$$C(x, y) := \frac{\partial_x G(x, y)}{\partial_y G(x, y)}. \quad (3.6)$$

Here $A(x, y)$ is a $n \times n$ matrix, $B(x, y)$ is a n -column vector, and $C(x, y)$ is a n -row vector.

Assumption 3.6. We assume that the following assumptions on T :

(L_1) The function F and G have the form

$$F(x, y) = \begin{cases} x_+^* + |y|^\alpha [B_+^* + \varphi_+(x, y)], & y > 0, \\ x_-^* + |y|^\alpha [B_-^* + \varphi_-(x, y)], & y < 0 \end{cases}$$

$$G(x, y) = \begin{cases} y_+^* + |y|^\alpha [A_+^* + \psi_+(x, y)], & y > 0, \\ y_-^* + |y|^\alpha [A_-^* + \psi_-(x, y)], & y < 0 \end{cases}$$

in a neighborhood of D_0 , where, A_+^*, A_-^* are non zero and the functions, $\varphi_\pm$ and $\psi_\pm$ are the class C^{k+1} . The derivatives of $\varphi_\pm$ and $\psi_\pm$ are uniformly bounded with respect to x and satisfy the following estimates:

$$\left\| \frac{\partial^{l+m} \varphi_\pm(x, y)}{\partial x^l \partial y^m} \right\| \leq K |y|^{\gamma-m}, \quad \left\| \frac{\partial^{l+m} \psi_\pm(x, y)}{\partial x^l \partial y^m} \right\| \leq K |y|^{\gamma-m}, \quad (3.7)$$

where $\gamma > k - 1$, K is a positive constant, $l = 0, 1, 2$, $m = 0, 1, 2$, and $l + m \leq k + 1$.

(L_2)

$$1 - \|A\|_D > 2\sqrt{\|B\|_D \|C\|_D}. \quad (3.8)$$

(L_3) The following relations hold:

(a)

$$\frac{(2!)^2 (\|A\|_D + \|C\|_D \|B\|_D) \max_{m+n=1} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}}{(\|\partial_y G\|_D)^{-1} \left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D \|C\|_D}\right)^2} < 1. \quad (3.9)$$

(b) for $k \geq 2$

$$\frac{(2k!)^2 (\|A\|_D + \|C\|_D \|B\|_D) \max_{m+n=k} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}}{(\|\partial_y G\|_D)^{-k} \left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D \|C\|_D}\right)^2} < 1, \quad (3.10)$$

and

$$\|\partial_y G\|_D \geq \frac{1}{4} \quad \text{or} \quad \|\partial_x F\| \geq \frac{1}{4}. \quad (3.11)$$

We can now state our main result.

Theorem 3.7. *Suppose that the map T satisfies the assumptions L_1, L_2, L_3 of Assumption 3.6. Then, there is a C^k foliation $\mathcal{F}_D$, T -Invariant with C^{k+1} leaves.*

Remark 3.8. *Notice that if we assume that $\|C\|_D = 0$ then, in view of Definition 3.5, we have*

$$\partial_x G(x, y) = 0. \quad (3.12)$$

Whence, we obtain

$$G(x, y) = G(y). \quad (3.13)$$

Therefore, one can see that T takes the form $T(x, y) = (F(x, y), G(y))$. So, it is easily seen that the family of functions $h_c : U \rightarrow \mathbb{R}$ given by $h_c(x) = c$ define a foliation $\mathcal{F}_D := \{h_c\}_{c \in I}$ of D , which satisfies the statements of the Theorem 3.7. Thus, from now on we will assume that $\|C\|_D$ is non-zero.

The Theorem 3.7 will be proved in the Section 4.

Remark 3.9. *Our main influence for the Theorem 3.7 was the article of Shaskov and Shilnikov [SS94], where was shown under three conditions the existence of a C^1 -stable invariant foliation for Lorenz-type map:*

Condition 1: Assumption L_1 of Assumption 3.6 in the case $k = 1$.

Condition 2: Assumption L_2 of Assumption 3.6.

Condition 3: The following relation holds :

$$2(||\partial_x f||_D + ||\partial_x G||_D ||B||_D) < \frac{\left(1 + ||A||_D + \sqrt{(1 - ||A||_D)^2 - 4||B||_D ||C||_D}\right)^2}{\left(1 + ||A||_D + \sqrt{(1 - ||A||_D)^2 + 4||B||_D ||C||_D}\right)}. \quad (3.14)$$

A important consequence of the last Theorem 3.7 is the following Corollary.

Corollary 3.10. Suppose that the map T satisfies Assumption 3.6. Then T can be associate with a one-dimensional map $\varphi : J \setminus \{0\} \rightarrow J$ of class C^k , such that, the following diagram is commutative

$$\begin{array}{ccc} D \setminus \Gamma & \xrightarrow{T} & D \\ \downarrow \pi & & \downarrow \pi \\ J \setminus \{0\} & \xrightarrow{\varphi} & J \end{array}$$

$\pi \circ T = \varphi \circ \pi$ on D .

The map $\varphi : J \setminus \{0\} \rightarrow J$ is called one-dimensional Lorenz-like transformation.

Proof. For details see Remark 2.14. ■

3.2.1 Outlook and Discussion

In this work we give sufficient conditions on Lorenz-type map T of class C^k so that it can be associated with a skew product $\bar{T}(x, \eta) = (F(x, \eta), G(\eta))$ of class C^k .

There are currently two lines of future work planned:

1. Firstly, if we assumed that Lorenz-type map T_X is defined by a geometric Lorenz flow X of class C^{k+1} , we will want to prove that there exists a neighborhood U of X in the C^{k+1} topology so that for all $Y \in U$ the Lorenz-type map T_Y also can be associated with with a skew product $\bar{T}_Y(x, \eta) = (\bar{F}_Y(x, \eta), \bar{G}_Y(\eta))$ of class C^k . To do this, we intend to show that T_Y satisfy the assumptions L_1, L_2 and L_3 of the main theorem. Theorem 3.7. But, notice that since the application T_X as well derivatives are close of T_Y we conclude the map T_Y also satisfies the assumption L_2 and L_3 . Therefore, a future work will be to show that T_Y satisfies the assumption L_1 . So we would have a robust theorem. Furthermore, another future work will be seek the regularity of the function $Y \rightarrow \bar{T}_Y$.
2. Secondly, after carried out item 1, we will study the properties of the unidimensional Lorenz-like map $\bar{G}_Y$ and it is relation to geometric Lorenz flow.

3.3 Introducing the ideas

In this subsection following [Rob81] will give the ideas of how to build a foliation $\mathcal{F}_D$ with the statements of the main theorem. Theorem 3.7.

3.3.1 Preliminares

3.3.1.1 The Frobenius- Diudonné Theorem

Consider E, F two Banach spaces over K , U (resp. V) an open subset of E (resp. F).

Let $F : U \times V \rightarrow L(E; F)$ be a map. A differentiable map $\varphi : U \rightarrow V$ is a solution of the total differential equation

$$y' = F(x, y) \quad (3.15)$$

if, for every $x \in U$, we have

$$\varphi'(x) = F(x, \varphi(x)). \quad (3.16)$$

When $E = K$, $L(E; F)$ is identified to F and the total differential equation is thus an ordinary differential equation. When $E = K^n$ is finite dimensional, a linear mapping L of E into F is defined by its value at each of the n basis vectors of E , and by definition, (3.16) is thus equivalent to the system of n partial differential equations

$$D_i \varphi = F^i(x_1, \dots, x_n, y) \quad (1 \leq i \leq n). \quad (3.17)$$

In general, such a system will have no solution when $n > 1$, even if the right-hand sides F^i are continuously differentiable functions.

We say that an equation (3.15) is completely integrable in $U \times V$ if, for every point $(x_0, y_0) \in U \times V$, there is an open neighborhood W of x_0 in U such that there is a unique solution φ of (3.15), defined in W , with value in F , and such that $\varphi(x_0) = y_0$.

Theorem 3.11. *[Frobenius-Diudonné theorem for Banach spaces I] Let E and F be two Banach spaces and $F : U \times V \rightarrow L(E; F)$ a Frechet C^1 -map, with U open in E ; V open in F . Then the following are equivalent:*

- (a) *For each (x_0, y_0) in $U \times V$ the initial value problem $D\varphi(x) = F(x, \varphi(x))$ with $\varphi(x_0) = y_0$ has a unique local solution.*
- (b) *F satisfies the Frobenius condition, that is, for all (x, y) in $U \times V$ and all (h, k) in $E \times F$*

the following relation holds:

$$D_1F(x, y)(h, k) + D_2F(x, y).(F(x, y)h, k) = D_1F(x, y)(k, h) + D_2F(x, y).(F(x, y)k, h). \quad (3.18)$$

Proof. The proof this Corollary can be found in the books [Die69], [Mau76]. ■

Theorem 3.12. [Frobenious-Dioudoné Theorem for Banach spaces II] Let E and F be Banach spaces and $F : U \times E \rightarrow L(E; F)$ a Frechet C^1 -map, with U open in E ; V open in F . Suppose that F satisfies the Frobenious condition (3.18). Then, for every $(a, b) \in U \times V$ there is a open ball $B_{r_1}(a) \subset U$ of center a , and open ball $B_{r_2}(b) \subset V$ of center b , having the following properties:

(1) If $(x_0, y_0) \in B_{r_1}(a) \times B_{r_2}(b)$ the equation $y'(x) = F(x, y(x))$ has exactly one solution $x \rightarrow y(x, x_0, y_0)$, defined in $B_{r_1}(a)$ and take values in $B_{r_2}(b)$ such that $y(x_0) = y_0$.

(2) The map

$$\begin{aligned} y : B_{r_1}(a) \times B_{r_1}(a) \times B_{r_2}(b) &\rightarrow B_{r_2}(b) \\ (x, x_0, y_0) &\rightarrow y(x, x_0, y_0), \end{aligned}$$

is continuously differentiable on $B_{r_1}(a) \times B_{r_1}(a) \times B_{r_2}(b)$.

(3) If F is k -times continuously differentiable (respect.analytic, if E and F are finite dimensional)in $U \times V$, then the map

$$\begin{aligned} y : B_{r_1}(a) \times B_{r_1}(a) \times B_{r_2}(b) &\rightarrow B_{r_2}(b) \\ (x, x_0, y_0) &\rightarrow y(x, x_0, y_0), \end{aligned}$$

is k -times continuously differentiable (respect.analytic) in $B_{r_1}(a) \times B_{r_1}(a) \times B_{r_2}(b)$.

Proof. The proof this Theorem can be found in the books [Die69],[Mau76]. ■

3.3.1.2 Frobenius Condition in Euclidean Spaces

Consider $E = \mathbb{R}^m$, $F = \mathbb{R}^n$, is known that all linear transformation $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ can be identified with a matrix $T = [T_{i,j}]_{n \times m}$, where $1 \leq i \leq n$, $0 \leq j \leq m$, thus if consider a map $F : U \subset E \rightarrow L(E, F)$ we have $F(x) = [F_{i,j}]_{n \times m}$, $1 \leq i \leq n$, $0 \leq j \leq m$. Hence, the equation $y'(x) = F(x, y(x))$ reduce to a system of n differential equations with partial derivatives

$$\frac{\partial y_j}{\partial x_i} = F_{i,j}(x_1, \dots, x_n, y_1(x), \dots, y_m(x)),$$

$i = 1, 2, \dots, m, j = 1, 2, \dots, n.$

The integrability conditions

$$\frac{\partial^2 y_j}{\partial x_i \partial x_k} = \frac{\partial^2 y_j}{\partial x_k \partial x_i}, j = 1, \dots, n, i, k = 1, \dots, m,$$

reduce to a system of $\frac{1}{2}n(n-1)$ equations

$$\frac{\partial F_{i,j}(x, y)}{\partial x^k} + \sum_{s=1}^n \frac{\partial F_{i,j}(x, y)}{\partial x_s} F_{k,s}(x, y) = \frac{\partial F_{k,j}(x, y)}{\partial x^i} + \sum_{s=1}^n \frac{\partial F_{k,j}(x, y)}{\partial x_s} F_{i,s}(x, y),$$

which must hold for every $(x, y) \in U \times V$.

For more details see [Mau76, p. 361].

3.3.2 Outline of the Proof

In this section we will give a outline of the ideas to prove Theorem 3.7. To given a map $h : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$, we defined its graph to be

$$\text{graph}(h) := \{(x, h(x)) : x \in U\}. \quad (3.19)$$

Idea 1: Defining foliations through functions completely integrable.

Suppose that $\bar{\nu} : D \subset \mathbb{R}^{n+1} \rightarrow L(\mathbb{R}^n, \mathbb{R})$ is a C^k function completely integrable (C.I), that is, there exists solution for the initial value problem for the differential equation

$$\partial_x y(x) = \bar{\nu}(x, y(x)), y(x_0) = y_0, \quad (3.20)$$

for all $(x_0, y_0) \in D$, where $y : U(x_0) \subset D \rightarrow [-1, 1]$ and $U(x_0)$ is a neighborhood of x_0 . Then, by using Frobenius-Dieudonné Theorem we have that

$$\overline{\mathcal{F}}_D := \{\text{graph}(h) : h \text{ is solution of the Equation (3.63)}\}, \quad (3.21)$$

determines a foliation, that is, the leaves are the graphs of the solutions of the differential equation defined by the function $\nu : D \rightarrow \mathbb{R}^n$.

Remark 3.13. Notice that $L(\mathbb{R}^n, \mathbb{R})$ is naturally identified to $\mathbb{R}^n$ itself. Indeed, consider the function $\Xi : L(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R}^n$ defined by

$$A \rightarrow \Xi(A) := (A(e_1), \dots, A(e_n)),$$

Figure 3.3: Idea 1

where $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$. It is not difficult to see that Ξ is norm-preserving isomorphisms. Hence, one can see that every map $\bar{\nu} : D \rightarrow L(\mathbb{R}^n, \mathbb{R})$ can be identified to a map from D to $\mathbb{R}^n$. In particular, if $h : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a differentiable function, then, we identified the function $\partial_x h : U \rightarrow L(\mathbb{R}^n, \mathbb{R})$ to the gradient function $\nabla h : U \rightarrow \mathbb{R}^n$ that is, $\partial_x h \equiv \nabla h$.

Idea 2: How to build a T -invariant foliation?

Bearing in mind the Idea 1 and following the ideas of Robinson [Rob81]. The foliation $\mathcal{F}_D$ of the Theorem 3.7 will be obtained as the integral surfaces of a C^k completely integrable function $\nu : D \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$, which will be a fixed point of an appropriate graph transform Γ .

Next, we will give a brief outline of the idea behind the graph transform Γ , which is also illustrated in Figure 3.4. Our goal is to find a C^k integrable function $\nu^* : D \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$, so that every integral surface h its graphs is invariant under $T(x, y) = (F(x, y), G(x, y)) := (\bar{x}, \bar{y})$, which means that

$$\begin{aligned} F(x, h(x)) &= \bar{x}, \\ G(x, h(x)) &= \bar{h}(\bar{x}), \end{aligned} \tag{3.22}$$

where $\bar{h}$ is a integral surface of ν^* . To find ν^* we take any completely integrable function

Figure 3.4: Idea 2

$\bar{\nu} : D \rightarrow \mathbb{R}^n$ and seek a completely integrable function $\nu : D \rightarrow \mathbb{R}^n$ so that

$$\begin{aligned} F(x, h(x)) &= \bar{x}, \\ G(x, h(x)) &= \bar{h}(\bar{x}), \end{aligned} \quad (3.23)$$

where h is integral surface of ν and $\bar{h}$ is integral surface of $\bar{\nu}$.

If such a function exists, we define the graph transform of $\bar{\nu}$ via $\Gamma(\bar{\nu}) := \nu$ and note that the desired function ν^* is a fixed point of the graph transform so that $\Gamma(\nu^*) = \nu^*$. It is not difficult to see that

$$\Gamma(\bar{\nu})(x, y) = \begin{cases} \frac{\bar{\nu} \circ T(x, y) \partial_y G(x, y) - \partial_y F(x, y)}{\partial_x F(x, y) - \bar{\nu} \circ T(x, y) \partial_x G(x, y)}, & y \neq 0, \\ 0, & y = 0 \end{cases} \quad (3.24)$$

Indeed, choose any point $(x_0, y_0) \in D \setminus D_0$ and consider the following function

$$\mathcal{M}_{(x_0, y_0)} : B_\delta(x_0, y_0) \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R},$$

defined by

$$\mathcal{M}_{(x_0, y_0)}(x, y) = \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0) - G(x, y), \quad (3.25)$$

where $\bar{h}$ and G as in Definition (3.23). By Claim 3.22 (Eq. (3.62)) we have

$$\begin{aligned}\partial_x h(x; x_0, y_0) &= \frac{-\partial_x \mathcal{M}_{(x_0, y_0)}(x, y)}{\partial_y \mathcal{M}_{(x_0, y_0)}(x, y)} \\ &= -\frac{\partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0) \partial_x F(x, y) - \partial_x G(x, y)}{\partial_{\bar{x}} \bar{h}(\bar{y}; \bar{x}_0, \bar{y}_0) \partial_y F(x, y) - \partial_y G(x, y)},\end{aligned}\quad (3.26)$$

as consequence, one obtains

$$\partial_x h(x; x_0, y_0)|_{x=x_0} = -\frac{\partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0)|_{\bar{x}=\bar{x}_0} \partial_x F(x, y) - \partial_x G(x, y)}{\partial_{\bar{x}} \bar{h}(\bar{y}; \bar{x}_0, \bar{y}_0)|_{\bar{x}=\bar{x}_0} \partial_y F(x, y) - \partial_y G(x, y)}.\quad (3.27)$$

Since $\bar{h}$ is integral surface of $\bar{\nu}$ and h is integral surface of ν , then we have

$$\partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0)|_{\bar{x}=\bar{x}_0} = \bar{\nu}(\bar{x}_0, \bar{y}_0),\quad (3.28)$$

and

$$\partial_x h(x; x_0, y_0)|_{x=x_0} = \nu(x_0, y_0).\quad (3.29)$$

Therefore, by combining (3.29), (3.28) and (3.27) we obtain

$$\nu(x_0, y_0) = -\frac{\bar{\nu}(\bar{x}_0, \bar{y}_0) \partial_x F(x_0, y_0) - \partial_x G(x_0, y_0)}{\bar{\nu}(\bar{x}_0, \bar{y}_0) \partial_y F(x_0, y_0) - \partial_y G(x_0, y_0)}.\quad (3.30)$$

Thus, Eq.3.24 holds.

Notice that, in view of Definition 3.5, we can rewrite the operator Γ in the following way:

$$\Gamma(\bar{\nu})(x, y) = \begin{cases} \frac{(\bar{\nu} \circ TA - C)}{(1 - \bar{\nu} \circ TB)}(x, y), & y \neq 0, \\ 0, & y = 0, \end{cases}$$

Our goal is to show first that the graph transform Γ is well defined on a complete sub-space $\mathcal{A}_L$ of the continuous function from D to $\mathbb{R}^n$, and secondly that Γ has a C^k fixed point ν^* . For this, we endow $\mathcal{A}_L$ with the norm of supremum and prove that Γ is a contraction to this norm. Since $\mathcal{A}_L$ is complete space, Banach's fixed-point theorem would then imply that Γ has a unique fixed point ν^* . Afterward we prove that the point fixed of ν^* is a C^k function completely integrable. Then by the Idea 1 we have that the graphs of the integral surfaces given the desire foliation $\mathcal{F}_D$ of the main result. Theorem 3.7.

3.4 Rigorous Definition of the Operator Γ

Our goal in this section is to give a rigorous definition and to show some properties of the operator Γ described informally in the last section.

We begin by introducing the following definition and soon after will be defined the most important operator of our work.

Definition 3.14. *Let L be a real number such that $L \geq 0$. We define the space $\mathcal{A}_L$ as the set of all the functions $\nu : D \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{1 \times n}$ which satisfies the following three conditions:*

- 1) ν is continuous in D ;
- 2) $\|\nu\| \leq L$;
- 3) $\nu(x, 0) = 0$, for all $\|x\| \leq 1$.

Remark 3.15. *Since $\mathbb{R}^{n+1}$ is complete normed space, it is no difficult to show that $\mathcal{A}_L$ is a complete metric space with the norm of the supremum.*

Now we are ready to define the most important operator of our work. This operator is denoted by Γ , and is defined as in [SS94, Eq.(6)] by

Definition 3.16.

$$\begin{aligned} \Gamma : \mathcal{A}_L &\longrightarrow \Gamma(\mathcal{A}_L) \\ \bar{\nu} &\longmapsto \nu = \Gamma(\bar{\nu}), \end{aligned} \tag{3.31}$$

where the function $\Gamma(\bar{\nu}) : D \rightarrow \mathbb{R}^{1 \times n}$ is given by

$$\Gamma(\bar{\nu})(x, y) = \begin{cases} \frac{(\bar{\nu} \circ TA - C)}{(1 - \bar{\nu} \circ TB)}(x, y), & y \neq 0, \\ 0, & y = 0, \end{cases}$$

with the functions A, B, C as in Definition 3.5.

Next, we state and prove a important property of the operator Γ .

Proposition 3.17. *There exists a real number $L \geq 0$ such that the operator $\Gamma : \mathcal{A}_L \rightarrow \mathcal{A}_L$ has a unique fixed point ν^* , and if you take every $\nu^1 \in \mathcal{A}_L$, then the sequence $\{\nu^n\}_{n \geq 1}$ given by*

$$\nu^{n+1}(x, y) = \begin{cases} \frac{(\nu^n \circ TA - C)}{(1 - \nu^n \circ TB)}(x, y), & y \neq 0, \\ 0, & y = 0 \end{cases} \tag{3.32}$$

converges to this field in $\mathcal{A}_L$.

The proof of the Proposition 3.17 will be given with the lemmas below.

The following simple lemma give us a explicit form for the functions $A(x, y)$, $B(x, y)$ and $C(x, y)$ of Definition 3.5 in a neighborhood of D_0 , this one will be very useful hereinafter.

Lemma 3.18. *Let $A(x, y)$, $B(x, y)$, and $C(x, y)$ be functions as in Definition 3.5. Then, the following relations hold:*

$$A(x, y) = \begin{cases} \frac{y\partial_x\varphi_+(x, y)}{\alpha(A_+^* + \partial_x\psi_+(x, y)) + y\partial_y\psi_+(x, y)}, & y > 0, \\ \frac{|y|\partial_x\varphi_+(x, y)}{-\alpha(A_-^* + \partial_x\psi_-(x, y)) + |y|\partial_y\psi_-(x, y)}, & y < 0. \end{cases} \quad (3.33)$$

$$B(x, y) = \begin{cases} \frac{\alpha(B_+^* + \partial_x\varphi_+(x, y)) + y\partial_y\varphi_+(x, y)}{\alpha(A_+^* + \partial_x\psi_+(x, y)) + y\partial_y\psi_+(x, y)}, & y > 0, \\ \frac{\alpha(B_-^* + \partial_x\varphi_-(x, y)) + y\partial_y\varphi_-(x, y)}{-\alpha(A_-^* + \partial_x\psi_-(x, y)) + |y|\partial_y\psi_-(x, y)}, & y < 0. \end{cases} \quad (3.34)$$

$$C(x, y) = \begin{cases} \frac{y\partial_x\psi_+(x, y)}{\alpha(A_+^* + \partial_x\psi_+(x, y)) + y\partial_y\psi_+(x, y)}, & y > 0, \\ \frac{|y|\partial_x\psi_-(x, y)}{-\alpha(A_-^* + \partial_x\psi_-(x, y)) + |y|\partial_y\psi_-(x, y)}, & y < 0, \end{cases} \quad (3.35)$$

in a neighborhood of D_0 . Moreover, for all $z^* \in D_0$ we have

$$\lim_{(x, y) \rightarrow z^*} A(x, y) = \lim_{(x, y) \rightarrow z^*} C(x, y) = 0, \quad (3.36)$$

and

$$\lim_{(x, y) \rightarrow z^*} B(x, y) = \begin{cases} \frac{B_+^*}{A_+^*}, & y > 0, \\ \frac{B_-^*}{A_-^*}, & y < 0. \end{cases} \quad (3.37)$$

Proof. We will prove the statement for $A(x, y)$. The others cases are analogous.

By Assumption 3.6, we have

$$F(x, y) = \begin{cases} x_+^* + |y|^\alpha[B_+^* + \varphi_+(x, y)], & y > 0, \\ x_-^* + |y|^\alpha[B_-^* + \varphi_-(x, y)], & y < 0 \end{cases}$$

$$G(x, y) = \begin{cases} y_+^* + |y|^\alpha[A_+^* + \psi_+(x, y)], & y > 0, \\ y_-^* + |y|^\alpha[A_-^* + \psi_-(x, y)], & y < 0 \end{cases}$$

Assume that $y > 0$, the case $y < 0$ is similar.

By calculating the partial derivative of $F(x, y)$ with respect x we get

$$\partial_x F(x, y) = y^\alpha \partial_x \varphi_+(x, y). \quad (3.38)$$

By calculating the partial derivative of $G(x, y)$ with respect x we get

$$\partial_y G(x, y) = \alpha y^{\alpha-1} (\alpha(B_+^* + \partial_x \psi_+(x, y)) + y(\partial_y \psi_+(x, y))). \quad (3.39)$$

By Definition 3.5 we have

$$A(x, y) := \frac{\partial_x F(x, y)}{\partial_y G(x, y)}. \quad (3.40)$$

Then, by replacing (3.39) and (3.38) into (3.40) we get

$$A(x, y) = \frac{y \partial_x \varphi_+(x, y)}{\alpha(A_+^* + \partial_x \psi_+(x, y)) + y \partial_y \psi_+(x, y)}. \quad (3.41)$$

To prove statement (3.36). By Assumption 3.6 we have

$$\left\| \frac{\partial^{l+m} \varphi_\pm(x, y)}{\partial x^l \partial y^m} \right\| \leq K|y|^{\gamma-m}, \quad \left\| \frac{\partial^{l+m} \varphi_\pm(x, y)}{\partial x^l \partial y^m} \right\| \leq K|y|^{\gamma-m}. \quad (3.42)$$

Hence, in view of Eq. (3.41) we get

$$\lim_{(x,y) \rightarrow z^*} A(x, y) = 0,$$

for every $z^* \in D_0$, and the proof is complete. ■

Lemma 3.19. *Let Γ be a operator as in Definition 3.16. Then, there exists a real number $L \geq 0$ satisfying the following properties:*

- (a) $\Gamma(\mathcal{A}_L) \subset \mathcal{A}_L$.
- (b) The operator $\Gamma : \mathcal{A}_L \rightarrow \mathcal{A}_L$ is a contraction.

Proof. The proof will be divided in two cases:

Case 1: Assume that $\|B\|_D \neq 0$.

Case 2: Assume that $\|B\|_D = 0$.

Let us start with the first case.

(a) We need to find $L \geq 0$ so that

$$\|\Gamma(\bar{\nu})\| \leq L, \quad \text{whenever} \quad \|\bar{\nu}\| \leq L. \quad (3.43)$$

By Definition 3.16 we have

$$\|\Gamma(\bar{\nu})\| \leq (1 - \|\bar{\nu}\| \|B\|_D)^{-1} (\|\bar{\nu}\| \|A\|_D + \|C\|_D). \quad (3.44)$$

Hence, we have Eq. (3.43) is equivalent to solve the following inequality:

$$(1 - L\|B\|_D)^{-1} L\|A\|_D + \|C\|_D \leq L, \quad (3.45)$$

or equivalent to solve the following system of inequalities :

$$\begin{aligned} 1 - L\|B\| &> 0, \\ L^2\|B\| + L(\|A\| - 1) + \|C\| &\leq 0. \end{aligned} \quad (3.46)$$

Since, by Assumption 3.6 we have $(\|A\| - 1)^2 - 4\|B\|\|C\| \geq 0$. Then, we can conclude that the system (3.46) is solvable; for instance one can take

$$L = \frac{-(\|A\| - 1) + \sqrt{(\|A\| - 1)^2 - 4\|B\|\|C\|}}{2\|B\|}. \quad (3.47)$$

Now, we have to prove that the function $\nu = \Gamma(\bar{\nu})$ is continuous on D . From Definition 3.16, we have ν is continuous on $(D \setminus D_0)$, so it remains to show that the function ν is continuous in the points $(x, 0) \in D_0$.

By Definition 3.16 we have

$$\nu(a, b) = \Gamma(\bar{\nu})(a, b) = \frac{(\bar{\nu} \circ TA - C)}{(1 - \bar{\nu}B)}(a, b), \quad \text{for } b \neq 0. \quad (3.48)$$

Hence, we get

$$\|\nu(a, b)\| \leq (1 - \|\bar{\nu}\|_D \|B\|)^{-1} (\|\bar{\nu}\| \|A(a, b)\| + \|C(a, b)\|). \quad (3.49)$$

Since $\bar{\nu} \in \mathcal{A}_L$ we have

$$\|\bar{\nu}\|_D \leq L. \quad (3.50)$$

Then, it follows from (3.50) and (3.49) that

$$\|\nu(a, b)\| \leq (1 - L\|B\|)^{-1}(L\|A(a, b)\| + \|C(a, b)\|). \quad (3.51)$$

From Lemma 3.18 we have

$$\lim_{(x,y) \rightarrow (a,0)} A(x, y) = \lim_{(x,y) \rightarrow (a,0)} C(x, y) = 0, \quad (3.52)$$

for every $(a, 0) \in D_0$.

Therefore, from (3.52) and (3.51) we get

$$\lim_{(a,b) \rightarrow (x,0)} \nu(x, y) = 0. \quad (3.53)$$

Thus, if we define

$$W(x, y) = \begin{cases} \nu(x, y), & y \neq 0, \\ 0, & y = 0. \end{cases}$$

Then, we get a continuous extension of ν on D , which completes the proof of (a).

(b) Consider $\bar{\nu}^1$ and $\bar{\nu}^2 \in \mathcal{A}_L$. By Definition 3.16 we have

$$\Gamma(\bar{\nu}^1) = \frac{(\bar{\nu}^1 \circ TA - C)}{(1 - \bar{\nu}^1 B)}, \quad (3.54)$$

$$\Gamma(\bar{\nu}^2) = \frac{(\bar{\nu}^2 \circ TA - C)}{(1 - \bar{\nu}^2 B)}. \quad (3.55)$$

Then, from (3.54) and (3.55), on account of add and subtract the term $\frac{(\bar{\nu}^2 \circ TA - C)}{(1 - \bar{\nu}^1 \circ TB)}$ we get

$$\begin{aligned} \Gamma(\bar{\nu}^1) - \Gamma(\bar{\nu}^2) &= \frac{(\bar{\nu}^1 \circ TA - C)}{(1 - \bar{\nu}^1 \circ B)} - \frac{(\bar{\nu}^2 \circ TA - C)}{(1 - \bar{\nu}^2 \circ TB)} \\ &= \frac{((\bar{\nu}^1 - \bar{\nu}^2) \circ T)A}{(1 - \bar{\nu}^1 \circ TB)} + \frac{(\bar{\nu}^2 \circ TA - C)((\bar{\nu}^1 - \bar{\nu}^2) \circ TB)}{(1 - \bar{\nu}^1 B)(1 - \bar{\nu}^2 \circ TB)}. \end{aligned} \quad (3.56)$$

Hence, we get

$$\|\Gamma(\bar{\nu}^1) - \Gamma(\bar{\nu}^2)\| \leq \left(\frac{\|A\|}{(1 - L\|B\|)} + \frac{(L\|A\| + \|C\|)\|B\|}{(1 - L\|B\|)^2} \right) \|\bar{\nu}^1 - \bar{\nu}^2\|. \quad (3.57)$$

Moreover, by using Eq. (3.46), it is easy to check that

$$\left(\frac{\|A\|}{(1 - L\|B\|)} + \frac{(L\|A\| + \|C\|)\|B\|}{(1 - L\|B\|)^2} \right) < 1. \quad (3.58)$$

Therefore, it follows from (3.58) and (3.57) that Γ is a contraction. Thus the lemma is proved in the case 1.

Case 2: Assume that $\|B\|_D = 0$. The same reasoning applies to the case 1 shows that

$$L = \frac{\|C\|}{1 - \|A\|_D}.$$

satisfies the conclusion of lemma. This finishes the proof of lemma. $\blacksquare$

Proof of the Proposition 3.17. By Remark 3.15, we have that $\mathcal{A}_L$ is complete, and from Lemma 3.19(b) we have that the operator Γ is a contraction. Then the operator Γ satisfies the assumptions of the Banach Contraction Principle. Therefore Γ has a unique attracting fixed point $\nu \in \mathcal{A}_L$, and satisfies Eq. (3.32), which concludes the proof of proposition. $\blacksquare$

3.5 Proof of Main Theorem 3.7

In this section we estate some importante proprieties about of the Operator Γ and after we will prove the main result.

Let us begin state our main proposition which will be proven in the next chapter. Chapter 4.

Proposition 3.20. *The attracting fixed point ν^* of the operator $\Gamma : \mathcal{A}_L \rightarrow \mathcal{A}_L$ is of class C^k .*

Proposition 3.21. *Let $\Gamma : \mathcal{A}_L \rightarrow \mathcal{A}_L$ be the operator as in Definition 3.16. Then Γ takes completely integrable function into completely integrable function, that is, if $\bar{\nu} \in \mathcal{A}_L$ is completely integrable then $\Gamma(\bar{\nu}) \in \mathcal{A}_L$ is also completely integrable.*

Moreover, if $\bar{\mathcal{F}}_D$ and $\mathcal{F}_D$ are foliations defined by the completely integrable functions $\bar{\nu}$ and $\Gamma(\bar{\nu})$ respectively. Then the following properties holds:

- (a) *T takes every leaf $\mathcal{F}_{(x_0, y_0)} \in \mathcal{F}_D$, $\mathcal{F}_{(x_0, y_0)} \neq D_0$, into a part of the leaf $\bar{\mathcal{F}}_{(T(x_0, y_0))} \in \bar{\mathcal{F}}_D$, that is, $T(\mathcal{F}_{(x_0, y_0)}) \subset \bar{\mathcal{F}}_{(T(x_0, y_0))}$.*

Proof. Fix $(x_0, y_0) \in D \setminus D_0$ and consider $(\bar{x}_0, \bar{y}_0) : T(x_0, y_0)$. Since $\bar{\nu}$ is a completely integrable function, then there exists a integral surface h of $\bar{\nu}$ so that $(\bar{x}_0, \bar{y}_0) \in \text{graph}(h)$. Now, we define the function $\mathcal{M}_{(x_0, y_0)} : B_\delta(x_0, y_0) \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ given by

$$\mathcal{M}_{(x_0, y_0)}(x, y) = \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0) - G(x, y), \quad (3.59)$$

The proof of the proposition will be given with the following claims:

Claim 3.22. *There exist numbers $\rho > 0$, $R > 0$ and a function*

$$h(\bullet, x_0, y_0) : B_\rho(\lambda_0) \rightarrow B_R(x_0) \quad (3.60)$$

of class C^k , such that, for all $x \in B_\rho(x_0)$ and for all $y \in B_R(y_0)$ we have

$$\mathcal{M}_{(x_0, y_0)}(x, y) = 0 \quad \text{if only if} \quad y = h(x, x_0, y_0). \quad (3.61)$$

In particular $\mathcal{M}_{(x_0, y_0)}(x, h(x, x_0, y_0)) = 0$, for all $x \in B_\rho(x_0)$, and

$$\partial_x h(x, x_0, y_0) = \frac{-\partial_x M(x, h(x, x_0, y_0))}{\partial_y \mathcal{M}_{(x_0, y_0)}(x, h(x, x_0, y_0))}, \quad (3.62)$$

for all $x \in B_\rho(x_0)$.

Figure 3.5: Geometric interpretation of Proposition 3.21.

Claim 3.23. The function $h(\bullet, x_0, y_0) : B_\rho(x_0) \rightarrow B_R(y_0)$ from Eq. 3.60 is a integral surface of $\Gamma(\bar{\nu})$, that is, h is a solution to the initial value problem

$$\partial_x y(x) = \Gamma(\bar{\nu})(x, y(x)), y(x_0) = y_0. \quad (3.63)$$

Recall that

$$D_x := \{x \in \mathbb{R}^n \quad \text{for which there exists a } y \in \mathbb{R} \quad \text{with} \quad (x, y) \in D\}. \quad (3.64)$$

Claim 3.24. *If $(x_0, y_0) \in D_0$. Then the function $h(\bullet, x_0, y_0) : D_x \rightarrow [-1, 1]$ defined by $h(x, x_0, y_0) = 0$ is a integral surface of $\Gamma(\bar{\nu})$.*

To prove that $\Gamma(\bar{\nu})$ is a completely integrable function it is a direct consequence from Claims 3.24 and 3.23.

To prove (a) it follows from Eqs. (3.61) and (3.59).

Now we will prove the Claims 3.22, 3.23 and 3.24.

To prove Claim 3.60. we will prove that the function $\mathcal{M}_{(x_0, y_0)}$ satisfies the following properties:

$$(1) \mathcal{M}_{(x_0, y_0)}(x_0, y_0) = 0,$$

$$(2) \partial_y \mathcal{M}_{(x_0, y_0)}(x_0, y_0) \neq 0.$$

Indeed, the condition (1) it follows from the definition of $\mathcal{M}_{(x_0, y_0)}$.

To prove (2). By Chain rule applied to $\mathcal{M}_{(x_0, y_0)}(x, y)$, we get

$$\begin{aligned} \partial_y \mathcal{M}_{(x_0, y_0)}(x, y) &= \partial_y \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0) - \partial_y G(x, y) \\ &= \partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0) \partial_y \bar{x}(x, y) - \partial_y G(x, y) \\ &= \partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0) \partial_y F(x, y) - \partial_y G(x, y). \end{aligned} \quad (3.65)$$

Hence, we obtain that $\partial_y \mathcal{M}_{(x_0, y_0)}(x_0, y_0) \neq 0$, otherwise, if $\partial_y \mathcal{M}_{(x_0, y_0)}(x_0, y_0) = 0$, then we have

$$\partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0) \partial_y F(x_0, y_0) = \partial_y G(x_0, y_0),$$

or equivalent

$$\partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0) (\partial_y G(x_0, y_0))^{-1} \partial_y F(x_0, y_0) = 1. \quad (3.66)$$

But, by Definition 3.5 we have

$$B(x_0, y_0) = \partial_y G(x_0, y_0)^{-1} \partial_y F(x_0, y_0). \quad (3.67)$$

Whence, in view of (3.66) we get

$$\partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0) B(x_0, y_0) = 1. \quad (3.68)$$

Since h is integral surface we have

$$\partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0) = \bar{\nu}(\bar{x}_0, \bar{y}_0). \quad (3.69)$$

Since $\bar{\nu} \in \mathcal{A}_L$. Then

$$\|\bar{\nu}\| \leq L. \quad (3.70)$$

From (3.70) and (3.68) we get

$$1 \leq L\|B\|. \quad (3.71)$$

But, by Lemma 3.19 Eq. (3.47). we have

$$1 - L\|B\| > 0, \quad (3.72)$$

Then, we get a contradiction between (3.72) and 3.71.

Therefore, from (1) and (2), on account of Implicit Function Theorem, Claim 3.60 holds.

To prove Claim 3.23. First, notice that, by Eq. (3.62) we have

$$\partial_x h(x; x_0, y_0) = \frac{-\partial_x \mathcal{M}(x, y)}{\partial_y \mathcal{M}(x, y)} = -\frac{\partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0) \partial_x F(x, y) - \partial_x G(x, y)}{\partial_{\bar{x}} \bar{h}(\bar{y}; \bar{x}_0, \bar{y}_0) \partial_y F(x, y) - \partial_y G(x, y)}. \quad (3.73)$$

Hence, it follows that

$$\partial_x h(x; x_0, y_0)|_{x=x_0} = -\frac{\partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0)|_{\bar{x}=\bar{x}_0} \partial_x F(x, y) - \partial_x G(x, y)}{\partial_{\bar{x}} \bar{h}(\bar{y}; \bar{x}_0, \bar{y}_0)|_{\bar{x}=\bar{x}_0} \partial_y F(x, y) - \partial_y G(x, y)}. \quad (3.74)$$

But, since $\bar{h}$ is a integral surface of $\bar{\nu}$ we have

$$\bar{\nu}(\bar{x}_0, \bar{y}_0) = \partial_{\bar{x}} \bar{h}(\bar{x}; \bar{x}_0, \bar{y}_0)|_{\bar{x}=\bar{x}_0}, \quad (3.75)$$

Thus, substituting (3.75) into (3.74), in view of Definitions 3.5 and 3.16 we get

$$\begin{aligned} \partial_x h(x; x_0, y_0)|_{x=x_0} &= -\frac{\bar{\nu}(\bar{x}_0, \bar{y}_0) \partial_x F(x_0, y_0) - \partial_x G(x_0, y_0)}{\bar{\nu}(\bar{x}_0, \bar{y}_0) \partial_y F(x_0, y_0) - \partial_y G(x_0, y_0)} \\ &:= (1 - \bar{\nu} \circ TB)^{-1} (\bar{\nu} \circ TA - C)(x_0, y_0) \\ &:= \Gamma(\bar{\nu})(x_0, y_0). \end{aligned} \quad (3.76)$$

which shows that $\Gamma(\bar{\nu})$ is completely integrable function.

To prove Claim 3.24 is just recall that if $(x_0, y_0) \in D_0$ then $\Gamma(\bar{\nu})(x_0, y_0) = 0$.

Thus, we conclude the proof of proposition. ■

In order to prove the next proposition, we need the following result

Proposition 3.25. *[Interchanging the order of differentiation and limit] Let E and F be normed spaces, F being complete, $U \subset E$ a non-empty connected open set, (f_n) a sequence of C^k maps*

from U to F . We suppose the following:

- (1) The sequence $(f_n(x_0))$ converges for at least one point $x_0 \in U$.
- (2) For every point $x \in U$ and for any j with $1 \leq j \leq k$, there are balls $B(x)$ contained in U and such that the sequence $D^j f_n$ converge uniformly to the map $g^j : B(x) \rightarrow L^s(E; F)$.

Then for each $x \in U$, the sequence (f_n) converges uniformly in $B(x)$. Letting

$$f(x) = \lim_{n \rightarrow \infty} f_n(x),$$

for all $x \in U$, we have that $f : U \rightarrow F$ is of class C^k and

$$D^j f(x) = \lim_{n \rightarrow \infty} D^j f_n(x),$$

for all $1 \leq j \leq k$, and $x \in U$.

Proof. The proof can found in [AMR88] and [Die69]. ■

Proposition 3.26. *The attracting fixed point ν^* of the operator $\Gamma : \mathcal{A}_L \rightarrow \mathcal{A}_L$ is completely integrable function.*

Proof. First, notice that the function $\bar{\nu} : D \rightarrow \mathbb{R}^{1 \times n}$ given by $\bar{\nu}(x, y) = 0$ is C^∞ completely integrable and their integral surfaces are the functions $h_c : U \rightarrow [-1, 1]$ given by $h_c(x) = c$, where $U = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Hence, we obtain that $\mathcal{F}_D := \{h_c\}_{c \in I}$ is a foliation of D .

Moreover, since the function $\bar{\nu}$ is C^∞ completely integrable then, by Proposition 3.21 it follows from that for $n \in \mathbb{N}$ there exists completely integrable function $\bar{\nu}^n := \Gamma^n(\bar{\nu})$. Thus, if we take a point $(x_0, y_0) \in D$, since the functions $\bar{\nu}^n$ are completely integrables for every $n \in \mathbb{N}$ then there exists solution for the initial value problem for the differential equation

$$\begin{aligned} \partial_x y^n(x) &= \bar{\nu}^n(x, y^n(x)), \\ y^n(x_0) &= y_0. \end{aligned} \tag{3.77}$$

But, by Lemma 3.19 we have $\|\bar{\nu}^n(x, y^n(x))\| \leq L$, for all $n \geq 1$ and on account of construction of the solutions $y^n(x)$ given in [Die69, 10.9.4] one can take a compact neighbourhood $B_r[x_0]$ of x_0 , so that it is contained in the domains of all the functions $y^n(x)$. Then, by (3.77) on account of Interchanging the order of differentiation and limit. Proposition 3.25. We have that

$$y(x) = \lim_{n \rightarrow \infty} y^n(x).$$

Figure 3.6: Geometric interpretation of the hyperplane field $\Gamma^n(0)$.

is a function of class C^k and

$$\partial_x y(x) = \lim_{n \rightarrow \infty} \partial_x y^n(x). \quad (3.78)$$

Therefore, since ν^* is a attracting fixed point of Γ , in view of (3.78) and (3.77) we get

$$\begin{aligned} \partial_x y(x) &= \nu^*(x, y(x)), \\ y(x_0) &= y_0, \end{aligned}$$

which proof that ν^* is a completely integrable function. The proposition is proved. $\blacksquare$

To close this section we present a proof of our main result. Theorem 3.7, which we recall here.

Theorem 3.27 (Existence of a C^k -Stable Invariant Foliation). *Suppose that the map T satisfies Assumption 3.6. Then, there is a C^k foliation $\mathcal{F}_D$ T -invariant, with C^{k+1} leaves.*

Proof of Theorem 3.7. By Proposition 3.26 we have that the the attracting fixed point ν^* of the operator $\Gamma : \mathcal{A}_L \rightarrow \mathcal{A}_L$ is integrable. Thus, the function ν^* defined a foliation $\mathcal{F}_D$, of class C^k and by Proposition 3.21(b) it follows that the foliation $\mathcal{F}_D$ is T -invariant, which finishes the proof the our main result. Theorem 3.7. $\blacksquare$

Proof of Proposition 3.20

We are going to proof the main proposition of this dissertation. Proposition 3.20 mentioned in the last chapter, which we recall here.

Proposition 4.1. *The attracting fixed point ν^* of the operator $\Gamma : \mathcal{A}_L \rightarrow \mathcal{A}_L$ is of class C^k .*

The proof of Proposition 3.20 will be divided into two steps:

Step 1: To show that the operator $\Gamma : \mathcal{A}_L \rightarrow \mathcal{A}_L$ takes functions of class C^k into functions of class C^k , that is, for all function $\mu \in \mathcal{A}_L$ of class C^k then the function $\Gamma(\mu)$ is of class C^k .

Step 2: To show that the limit

$$\lim_{n \rightarrow \infty} (\Gamma^n(\mu), D(\Gamma^n(\mu)), \dots, D^k(\Gamma^n(\mu)))$$

exists, for all $\mu \in \mathcal{A}_L$ of class C^k and $D^k\mu(x, 0) = 0$.

We begin with the Step 1.

4.1 Step 1.

The goal of this subsection is to show the following proposition:

Proposition 4.2. *Under the assumptions and Definitions of the Chapter 3. Let $\mu \in \mathcal{A}_L$ be a function of class C^k . Then the following statements hold:*

- (1) *The function $\Gamma(\mu) \in \mathcal{A}_L$ is of class C^k ,*
- (2) *$D^k\Gamma(\mu)(x, 0) = 0$, for all $(x, 0) \in D_0$.*

The proof is quite long and technical, so we divide it into three steps:

Step 1.1: To establish a formula for the k th order derivatives of the function $\Gamma(\nu)$ at (x, y) , for $y \neq 0$.

Step 1.2: To estimate the norms of the i th derivatives of the functions $A(x, y)$, $B(x, y)$, and $C(x, y)$, for the points (x, y) around of a neighborhood of D_0 .

Step 1.3: After carried out the above steps, in this step we will prove that $D^i(\Gamma(\mu))(a, b) \rightarrow 0$, when $b \rightarrow 0$, for $0 \leq i \leq k$.

4.1.1 Step 1.1

This section is devoted to establish a formula for the k th order derivatives of the function $\Gamma(\nu)$ at (x, y) , for $y \neq 0$.

4.1.1.1 Preliminares

In this subsection we state the Leinniz and Chain rule as well as will be give some definition pretty useful to establish the formula for k th order derivatives of $\Gamma(\nu)$.

4.1.1.1.1 The Leibniz and Chain Rules Here the explicit formulas are given for the k th order derivatives of products and compositions. This formulas will be useful in the next subsections.

From now on, S_k will denote the group of permutations on k elements.

Definition 4.3 (Symmetrizing operator). *The Symmetrizing operator Sym^k is defined by*

$$\begin{aligned} Sym^k : L^k(E; F) &\longrightarrow L^k(E; F) \\ A &\longmapsto Sym^k(A) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma A, \end{aligned}$$

where $(\sigma A)(e_1, \dots, e_k) = A(e_{\sigma(1)}, \dots, e_{\sigma(k)})$ and S_k is the group of permutations on k elements.

Remark 4.4. *The symmetrizing operator Sym^k satisfies the following properties:*

- (i) $Sym^k(L^k(E; F)) = L^k_s(E; F)$,
- (ii) $(Sym^k)^2 = Sym^k$,
- (iii) $\|Sym^k\| \leq 1$.

Definition 4.5. Let $\mathcal{B} : F_1 \times F_2 \rightarrow G$ be a bilinear map, we will denote by

$$\begin{aligned} \phi^{(i, (k-i))} : L^i(E; F_1) \times L^{(k-i)}(E; F_2) &\longrightarrow L^k(E; G) \\ (A_1, A_2) &\longmapsto [\phi^{(i, (k-i))}(A_1, A_2)] \end{aligned}$$

the bilinear map defined by

$$[\phi^{(i, (k-i))}(A_1, A_2)](e_1, \dots, e_k) = \mathcal{B}(A_1(e_1, \dots, e_i), A_2(e_{i+1}, \dots, e_k)). \quad (4.1)$$

Definition 4.6. Let $\bar{v}_i : U \rightarrow L^i(E; F_1)$, $\bar{v}_{(k-i)} : U \rightarrow L^{(k-i)}(E; F_2)$, we define the following map

$$\begin{aligned} \phi^{(i, (k-i))}(\bar{v}_i, \bar{v}_{(k-i)}) : U &\longrightarrow L^k(E; G) \\ p &\longrightarrow \phi^{(i, (k-i))}(\bar{v}_i(p), \bar{v}_{(k-i)}(p)). \end{aligned} \quad (4.2)$$

Definition 4.7. For every tuple $(q, r_1, r_2, \dots, r_q)$, where $q > 1$, and $r_1 + \dots + r_q = k$, we define the follows continuous multilinear map

$$\begin{aligned} \phi^{(q, r_1, \dots, r_q)} : L^q(F; G) \times L^{r_1}(E; F) \times \dots \times L^{r_q}(E; F) &\longrightarrow L^k(E; G) \\ (\bar{v}_q, \bar{v}_{r_1}, \dots, \bar{v}_{r_q}) &\longmapsto \phi^{(q, r_1, \dots, r_q)}(\bar{v}_q, \bar{v}_{r_1}, \dots, \bar{v}_{r_q}), \end{aligned} \quad (4.3)$$

where

$$\phi^{(q, r_1, \dots, r_q)}(\bar{v}_q, \bar{v}_{r_1}, \dots, \bar{v}_{r_q}) : \underbrace{E \times \dots \times E}_{k\text{-times}} \rightarrow G$$

is defined as

$$\begin{aligned} &\phi^{(q, r_1, \dots, r_q)}(\bar{v}_q, \bar{v}_{r_1}, \dots, \bar{v}_{r_q})(e_1, \dots, e_k) \\ &= \bar{v}_q(\bar{v}_{r_1}(e_1, \dots, e_{j_{r_1}}), \dots, \bar{v}_{r_q}(e_{(j_{r_1}+j_{r_2}+\dots+j_{r_{(q-1)}})+1}, \dots, e_{(j_{r_1}+j_{r_2}+\dots+j_{r_q})})). \end{aligned} \quad (4.4)$$

Definition 4.8. Let U be a subset of E . Let $\bar{v}_{r_i} : U \rightarrow L^{r_i}(E; F)$, $1 \leq i \leq q$ be maps, and suppose that $f : V \subset U \rightarrow U$ is a function. We define the function

$$\phi^{(q, r_1, \dots, r_q)} * ((\bar{v}_q \circ f) \times \bar{v}_{r_1} \times \dots \times \bar{v}_{r_q}) : U \rightarrow L^k(E; G) \quad (4.5)$$

given by

$$u \rightarrow \phi^{(q, r_1, \dots, r_q)}((\bar{v}_q \circ f(u)), \bar{v}_{r_1}(u), \dots, \bar{v}_{r_q}(u)),$$

with the function $\phi^{(q, r_1, \dots, r_q)}((\bar{v}_q \circ f(u)), \bar{v}_{r_1}(u), \dots, \bar{v}_{r_q}(u))$ as in Definition 4.7.

Theorem 4.9 (Leibnitz Rule for higher derivatives [AMR88]). Suppose that E, F_1, F_2 and

G are Banach Spaces, $U \subset E$ an open sets, $f : U \rightarrow F_1$ and $g : U \rightarrow F_2$ of class C^k . Let $\mathcal{B}$ be bilinear map from $F_1 \times F_2$ to G , that is, $\mathcal{B} \in L(F_1, F_2; G)$. Let $f \times g : U \rightarrow F_1 \times F_2$ denote the mapping $(f \times g)(x) = (f(x), g(x))$ and let $\mathcal{B}(f, g) = \mathcal{B} \circ (f \times g)$. Then $\mathcal{B}(f, g)$ is of class C^k , and we have following formula.

$$D^k \mathcal{B}(f, g) = \text{Sym}^k \circ \sum_{i=0}^k \binom{k}{i} \phi^{(i, k-i)}(D^i f, D^{k-i} g), \quad (4.6)$$

with the function $\phi^{(i, k-i)}(D^i f, D^{k-i} g)$ as in Definition 4.6 (Eq. (4.2)).

Explicitly, taking into account the symmetry of higher order derivatives, this formula is

$$\begin{aligned} D^k \mathcal{B}(f, g)(p) \cdot (e_1, \dots, e_k) \\ = \sum_{\sigma} \sum_{i=0}^k \binom{k}{i} B(D^i f(p)(e_{\sigma(1)}, \dots, e_{\sigma(i)}), D^{k-i} g(p)(e_{\sigma(i+1)}, \dots, e_{\sigma(k)})), \end{aligned} \quad (4.7)$$

where the outer sum is over all permutations $\sigma \in S_k$ such that

$$\sigma(1) < \dots < \sigma(i) \quad \text{and} \quad \sigma(i+1) < \dots < \sigma(k).$$

Theorem 4.10 (The Chain Rule for Higher Derivatives [AMR88]). Suppose that E, F , and G are Banach spaces and $U \subset F, V \subset F$ open sets, and $f : U \rightarrow V, g : V \rightarrow G$ function of class C^k . Then the function $g \circ f : U \rightarrow G$ is of class C^k and we have following formula.

$$D^k(g \circ f) = \text{Sym}^k \circ \sum_{i=1}^k \sum_{j_1 + \dots + j_i = k} \frac{k!}{j_1! \dots j_i!} \phi^{(i, j_1, \dots, j_i)} * (D^i g \circ f \times D^{j_1} f \times \dots \times D^{j_i} f), \quad (4.8)$$

with the function $\phi^{(i, j_1, \dots, j_i)} * (D^i g \circ f \times D^{j_1} f \times \dots \times D^{j_i} f)$ as in Definition 4.8.

Taking into account the symmetry of higher order derivatives, the explicit formula for $p \in U$ and $e_1, \dots, e_k \in E$, is

$$\begin{aligned} D^k(g \circ f) \cdot (e_1, \dots, e_k) = \\ \sum_{i=1}^k \sum_{j_1 + \dots + j_i = k} \sum D^i g(f(p))(D^{j_1} f(p) \cdot (e_{l_1})), \dots, D^{j_i} f(p) \cdot (e_{(l_{j_1} + \dots + j_{i-1} + 1), \dots, e_{l_k}}). \end{aligned} \quad (4.9)$$

where the third sum is taken over indices satisfying $l_1 < \dots < j_1 \dots l_{j_1} + \dots + j_{i-1} + 1 < \dots < l_k$.

4.1.1.1.2 Useful Definitions Here we give definitions that will be generalizations for the k th order derivatives of products and compositions of function.

Next, we define generalizations of the k th derivative of the composition of two functions (See Theorem 4.10).

Definition 4.11. Let k_1, k_2, k_3, q be integers such that $k_1 \geq k_3 \geq k_2 \geq 1$. Assume the functions $\bar{v}_q : V \subset F \rightarrow L^q(F, G)$, for $k_2 \leq q \leq k_3$ and $\bar{v}_i : U \subset E \rightarrow L^i(E; F)$ for $1 \leq i \leq k_1 - k_2 + 1$ and suppose that $f : U \subset E \rightarrow V \subset F$ is a continuous function. Then, we define the function

$$\mathcal{DC}^{(k_1, k_2, k_3)}((\bar{v}_{k_2}, \dots, \bar{v}_{k_3}), f, \bar{v}_1, \dots, \bar{v}_{(k_3 - k_2 + 1)}) : U \rightarrow L^{k_1}(E; F)$$

given by

$$\begin{aligned} & \mathcal{DC}^{(k_1, k_2, k_3)}((\bar{v}_{k_2}, \dots, \bar{v}_{k_3}), f, \bar{v}_1, \dots, \bar{v}_{(k_3 - k_2 + 1)})(p) \\ &:= \text{Sym}^{k_1} \left(\sum_{n=k_2}^{k_3} \sum_{r_1 + \dots + r_n = k_1} \frac{k_1!}{r_1! \dots r_n!} (\phi^{(n, r_1, \dots, r_n)} * ((\bar{v}_n \circ f) \times \bar{v}_{r_1} \times \dots \times \bar{v}_{r_n})) \right) (p), \end{aligned} \quad (4.10)$$

with the function $(\phi^{(n, r_1, \dots, r_n)} * ((\bar{v}_n \circ f) \times \bar{v}_{r_1} \times \dots \times \bar{v}_{r_n}))$ as in Definition 4.8.

And when $k_1 = k_3 = 0$ and $k_2 = 1$, we define the function

$$\mathcal{DC}^{(0, 1, 0)}(\bar{v}_0, f) : U \rightarrow F$$

given by

$$\mathcal{DC}^{(0, 1, 0)}(\bar{v}_0, f)(p) := (\bar{v}_0 \circ f)(p). \quad (4.11)$$

Definition 4.12. Let k_1, k_2, k_3, q be integers such that $k_1 \geq k_3 \geq k_2 \geq 1$. Assume the functions $\bar{v}_q : V \subset F \rightarrow L^q(F, G)$, for $k_2 \leq q \leq k_3$. Suppose that $f : U \rightarrow V$ is a function of class $C^{k_1 - k_2 + 1}$ and that $D^i f : U \rightarrow L^i(E, F)$ are the derivatives of f for $0 \leq i \leq k_1 - k_2 + 1$. Then, we define the function

$$\mathcal{DC}^{(k_1, k_2, k_3)}((\bar{v}_{k_2}, \dots, \bar{v}_{k_3}), f) : U \rightarrow L^{k_1}(E, F)$$

given by

$$\mathcal{DC}^{(k_1, k_2, k_3)}((\bar{v}_{k_2}, \dots, \bar{v}_{k_3}), f) := \mathcal{DC}^{(k_1, k_2, k_3)}((\bar{v}_{k_2}, \dots, \bar{v}_{k_3}), f, Df, \dots, D^{k_1 - k_2 + 1} f), \quad (4.12)$$

with the function $\mathcal{DC}^{(k_1, k_2, k_3)}((\bar{v}_{k_2}, \dots, \bar{v}_{k_3}), f, Df, \dots, D^{k_1 - k_2 + 1} f)$ as in Definition 4.11.

Definition 4.13. Suppose that $\bar{v} : V \subset F \rightarrow G$ and $f : U \subset E \rightarrow V \subset F$ are functions of class C^{k_3} and $C^{k_1-k_2+1}$ respectively. Then, we define the function

$$\mathcal{DC}^{(k_1, k_2, k_3)}(\bar{v}, f) : U \rightarrow L^{k_1}(E, F)$$

given by

$$\mathcal{DC}^{(k_1, k_2, k_3)}(\bar{v}, f) := \mathcal{DC}^{(k_1, k_2, k_3)}(D^{k_2}(\bar{v}), \dots, D^{k_3}(\bar{v}), f), \quad (4.13)$$

with the function $\mathcal{DC}^{(k_1, k_2, k_3)}(D^{k_2}(\bar{v}), \dots, D^{k_3}(\bar{v}), f)$ as in Definition 4.12.

Remark 4.14. Consider $\bar{v} : D \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{1 \times n}$ and $T : D^* \subset \mathbb{R}^{n+1} \rightarrow D \subset \mathbb{R}^{n+1}$ functions of class C^k . Then, on account of Chain rule applied to the function $\bar{v} \circ T$, we obtain

$$D^k(\bar{v} \circ T) = \text{Sym}^k \left(\sum_{i=1}^k \sum_{j_1 + \dots + j_i = k} \frac{k!}{j_1! \dots j_i!} \phi^{(i, j_1, \dots, j_i)} * ((D^i \bar{v} \circ T \times D^{j_1} T \times \dots \times D^{j_i} T)) \right). \quad (4.14)$$

Hence, in view of Definition 4.13 it follows that

$$D^k(\bar{v} \circ T) := \mathcal{DC}^{(k, 1, k)}(\bar{v}, T). \quad (4.15)$$

Therefore, we conclude that Definitions 4.11, 4.11, 4.12 and 4.13 are generalizations of the k th derivative of the composite of two functions.

Remark 4.15. Notice that, by Definition 4.13, we have

$$\mathcal{DC}^{(k, k, k)}(\bar{v}, T) := \text{Sym}^k \left(k!(D^k \bar{v}) \circ T \underbrace{DT \dots DT}_{k\text{-times}} \right). \quad (4.16)$$

Therefore, since $D^k \bar{v}$ is symmetric one can conclude that

$$\text{Sym}^k \left(k!(D^k \bar{v}) \circ T \underbrace{DT \dots DT}_{k\text{-times}} \right) = k!(D^k \bar{v}) \circ T \underbrace{DT \dots DT}_{k\text{-times}}. \quad (4.17)$$

Then, combining (4.17) and (4.16) and (4.17) we get

$$\mathcal{DC}^{(k, k, k)}(\bar{v}, T) := k!(D^k \bar{v}) \circ T \underbrace{DT \dots DT}_{k\text{-times}}. \quad (4.18)$$

Next, we define generalizations of the k th derivative of product of the map $(f \circ g)$ with h .

(See Theorem 4.9).

Definition 4.16. Let $\mathcal{B} : F_1 \times F_2 \rightarrow G$ be a bilinear map. Let $k_1, k_2, \geq k_3$ be integers such that $k_1 \geq k_3 \geq 1, k_2 \geq 0$. Let $\bar{v}_i : U \rightarrow L^i(E; F), \bar{v}_i : V \subset F \rightarrow L^i(F, F_1), 0 \leq i \leq k_3$ be maps. Let $f : U \subset E \rightarrow V \subset F$, be a function and Let $\bar{u}_i : U \rightarrow L^i(E, F_2), k_1 - k_3 \leq i \leq k_1 - k_2$. be a maps. We define the map.

$$\mathcal{DCP}^{(k_1, k_2, k_3)}(\bar{v}, T, B) : V \rightarrow L^{k_1}(E; G) \quad (4.19)$$

given by

$$\begin{aligned} \mathcal{DCP}^{(k_1, k_2, k_3)}((\bar{v}_0, \dots, \bar{v}_{k_3}), f, (\bar{v}_1, \dots, \bar{v}_{k_3}), (\bar{u}_{k_1-k_3}, \dots, \bar{u}_{k_1-k_2}))(p) := \\ \text{Sym}^{k_1} \left(\sum_{n=k_2}^{k_3} \binom{k_1}{n} \phi^{(n, k_1-n)} \left(\mathcal{DC}^{(n, 1, n)}((\bar{v}_1, \dots, \bar{v}_n), f, (\bar{v}_1, \dots, \bar{v}_n)), u_{k_1-n} \right) \right) (p), \end{aligned} \quad (4.20)$$

with the function $\phi^{(n, k_1-n)}$ as in Definition 4.6.

Definition 4.17. Let $\mathcal{B} : F_1 \times F_2 \rightarrow G$ be a bilinear map. Let k_1, k_2, k_3 be integers such that $k_1 \geq k_3 \geq 1, k_2 \geq 0$. Suppose that $f : U \subset E \rightarrow V \subset F$ is a function of class C^{k_3} , and that $D^i f : U \rightarrow L^i(E, F)$ are the derivatives of f , for $0 \leq i \leq k_3$. Assume every functions $\bar{v}_i : V \subset F \rightarrow L^i(F, F_1)$, for $0 \leq i \leq k_3$ and $\bar{u}_i : U \rightarrow L^i(E, F_2)$, for $k_1 - k_3 \leq i \leq k_1 - k_2$. Then, we define the function

$$\mathcal{DCP}^{(k_1, k_2, k_3)}(f, (\bar{v}_1, \dots, \bar{v}_{k_3}), (\bar{u}_{k_1-k_3}, \dots, \bar{u}_{k_1-k_2})) : V \rightarrow L^{k_1}(E; G)$$

$$\begin{aligned} \mathcal{DCP}^{(k_1, k_2, k_3)}(f, (\bar{v}_1, \dots, \bar{v}_{k_3}), (\bar{u}_{k_1-k_3}, \dots, \bar{u}_{k_1-k_2})) \\ := \mathcal{DCP}^{(k_1, k_2, k_3)}((Df, \dots, D^{k_3} f), f, (\bar{v}_1, \dots, \bar{v}_{k_3}), (\bar{u}_{k_1-k_3}, \dots, \bar{u}_{k_1-k_2})), \end{aligned} \quad (4.21)$$

with the function $\mathcal{DCP}^{(k_1, k_2, k_3)}((Df, \dots, D^{k_3} f), f, (\bar{v}_1, \dots, \bar{v}_{k_3}), (\bar{u}_{k_1-k_3}, \dots, \bar{u}_{k_1-k_2}))$ as in Definition 4.16.

Definition 4.18. Let $\mathcal{B} : F_1 \times F_2 \rightarrow G$ be a bilinear map. Let k_1, k_2, k_3 be integers such that $k_1 \geq k_3 \geq 1, k_2 \geq 0$. Let $f : U \subset E \rightarrow V \subset F$ be a function of class C^{k_3} . Let $B : U \rightarrow F_2$ be a function of class $C^{k_1-k_2}$, their derivative $D^i B : U \rightarrow L^i(E, F_2)$, for $0 \leq i \leq k_1 - k_2$. Let $\bar{v}_i : V \subset F \rightarrow L^i(F, F_1)$ be functions, where $0 \leq i \leq k_3$. Then, we define the map

$$\mathcal{DCP}^{(k_1, k_2, k_3)}(f, (\bar{v}_1, \dots, \bar{v}_{k_3}), B) : V \rightarrow L^{k_1}(E; G),$$

given by

$$\begin{aligned} \mathcal{DCP}^{(k_1, k_2, k_3)}(f, (\bar{\nu}_1, \dots, \bar{\nu}_{k_3}), (B^{k_1-k_3}, \dots, B^{k_1-k_2})) \\ := \mathcal{DCP}^{(k_1, k_2, k_3)}(f, (\bar{\nu}_1, \dots, \bar{\nu}_{k_3}), (\bar{u}_{k_1-k_3}, \dots, \bar{u}_{k_1-k_2})), \end{aligned} \quad (4.22)$$

with the function $\mathcal{DCP}^{(k_1, k_2, k_3)}(f, (\bar{\nu}_1, \dots, \bar{\nu}_{k_3}), (\bar{u}_{k_1-k_3}, \dots, \bar{u}_{k_1-k_2}))$, as in Definition 4.17.

Definition 4.19. Let $\mathcal{B} : F_1 \times F_2 \rightarrow G$ be a bilinear map. Let k_1, k_2, k_3 be integers such that $k_1 \geq k_3 \geq 1, k_2 \geq 0$. Let $\bar{\nu} : U \subset E \rightarrow F$ be a function of class C^{k_3} , their derivatives $D^i(\bar{\nu}) : U \subset E \rightarrow L^i(E, F), 0 \leq i \leq k_3$. Let $f : U \subset E \rightarrow V \subset F$ be a function of class C^{k_3} . Let $B : U \rightarrow F_2$ be a function of class $C^{k_1-k_2}$. Then, we define the function

$$\mathcal{DCP}^{(k_1, k_2, k_3)}(\bar{\nu}, f, B) : V \rightarrow L^{k_1}(E; G)$$

give by

$$\mathcal{DCP}^{(k_1, k_2, k_3)}(\bar{\nu}, f, B) := \mathcal{DCP}^{(k_1, k_2, k_3)}(f, (\bar{\nu}, D(\bar{\nu}), \dots, D^{k_3}(\bar{\nu})), B), \quad (4.23)$$

with the function $\mathcal{DCP}^{(k_1, k_2, k_3)}(f, (\bar{\nu}, D(\bar{\nu}), \dots, D^{k_3}(\bar{\nu})), B)$ as in Definition 4.18.

Remark 4.20. Let F_1 and F_2 be the space of the n -columns and n -rows respectively. Then, we define the multilinear map $\mathcal{B} : F_1 \times F_2 \rightarrow \mathbb{R}$ given by

$$\mathcal{B}(A, B) = A \times B,$$

where $A \times B$ is the usual product of matrices.

Assume that $\bar{\nu} : D \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, T : D^* \subset \mathbb{R}^{n+1} \rightarrow D$ and $B : D^* \rightarrow F_1$ are functions of class C^k . Then, by Leibniz and Chain rule applied to the functions $((\bar{\nu} \circ T) \cdot B)$ and $(\bar{\nu} \circ T)$, respectively, and in view of Eq.(4.15) and Definition 4.19 we get the following chain of equalities.

$$D^k((\bar{\nu} \circ T)B) = \text{Sym}^k \left(\sum_{n=0}^k \binom{k}{n} \phi^{(n, k-n)} (D^n(\bar{\nu} \circ T), D^{(k-n)}(B)) \right) \quad (4.24)$$

$$= \text{Sym}^k \left(\sum_{n=0}^k \binom{k}{n} \phi^{(n, k-n)} \left(\mathcal{DC}^{(n, 1, n)}(\bar{\nu}, T), D^{(k-n)}(B) \right) \right) \quad (4.25)$$

$$:= \mathcal{DCP}^{(k, 0, k)}(\bar{\nu}, T, B). \quad (4.26)$$

Thus, one can see that the Definitions 4.16, 4.17, 4.18 and 4.19 are generalizations of the k th derivative of product of the map $(\nu \circ T)$ with B . This remark will be very useful later on.

Next, we define generalizations of the k th derivative of the map $(1 - \nu \circ B)^{-1}$, where B is a function as Definition 3.5, and $\nu \in \mathcal{A}_L$ is a function of class C^k .

Definition 4.21. Let $(q, r_1, \dots, r_q)$ be a tuple such that $q \geq 1$ and $r_1 + \dots + r_q = k$. Assume the functions $f : U \subset E \rightarrow V \subset F$, $\bar{v}_q : V \subset F \rightarrow L^q(F, G)$ and $\bar{v}_s : V \subset F \rightarrow L^s(F, F_1)$, for $0 \leq s \leq r_q$. Suppose that $T : U \subset E \rightarrow V \subset F$, and $B : U \rightarrow F_2$ are functions of class C^{r_q} . Then, we define the map

$$\prod_{(r_1 \leq \dots \leq r_q)} (\bar{v}_q, f, (T, \bar{v}_0, \dots, \bar{v}_{r_q}), B) : U \rightarrow L^k(E, G),$$

given by

$$\begin{aligned} & \prod_{(r_1 \leq \dots \leq r_q)} (\bar{v}_q, f, T, (\bar{v}_0, \dots, \bar{v}_{r_q}), B) := \\ & \phi^{(q, r_1, \dots, r_q)} * ((\bar{v}_q \circ f) \times \mathcal{DCP}^{(r_1, 0, r_1)}(T, (\bar{v}_0, \dots, \bar{v}_{r_1}), B) \times \dots \times \mathcal{DCP}^{(r_q, 0, r_q)}(T, (\bar{v}_0, \dots, \bar{v}_{r_q}), B)), \end{aligned} \quad (4.27)$$

with the function $\mathcal{DCP}^{(r_i, 0, r_i)}(T, (\bar{v}_0, \dots, \bar{v}_{r_i}), B)$, $1 \leq i \leq q$ as in Definition 4.18.

Definition 4.22. Under the notations of Definition 4.21. Suppose that k_1, k_2, k_3 are integers such that $k_1 \geq k_3 \geq k_2 \geq 1$. Then, we define the map

$$\mathcal{DICP}^{(k_1, k_2, k_3)}((\bar{v}_{k_2}, \dots, \bar{v}_{k_3}), f, T, (\bar{v}_0, \dots, \bar{v}_{(k_1 - k_2) + 1}), B) : U \rightarrow L^{k_1}(E, G)$$

given by

$$\begin{aligned} & \mathcal{DICP}^{(k_1, k_2, k_3)}((\bar{v}_{k_2}, \dots, \bar{v}_{k_3}), f, (\bar{v}_0, \dots, \bar{v}_{(k_1 - k_2) + 1}), T, B) \\ & := \text{Sym}^{k_1} \left(\sum_{q=k_2}^{k_3} \sum_{r_1 + \dots + r_q = k_1} \frac{k!(-1)^q q!}{r_1! \dots r_q!} \prod_{(r_1 \leq \dots \leq r_q)} ((\bar{v}_q \circ f), (\bar{v}_0, \dots, \bar{v}_{r_q}), T, B) \right), \end{aligned} \quad (4.28)$$

with the function $\prod_{(r_1 \leq \dots \leq r_q)} ((\bar{v}_q \circ f), (\bar{v}_0, \dots, \bar{v}_{r_q}), T, B)$ as in Definition 4.21.

Definition 4.23. Under the notations of Definition 4.22. Let $\bar{v} : V \subset F \rightarrow G$ be a map of class C^{k_3} . Let $D^i(\bar{v}) : V \rightarrow L^i(F, G)$ be the derivatives of $\bar{v}$, where $0 \leq i \leq k_3$. Then, we define the map

$$\mathcal{DICP}^{(k_1, k_2, k_3)}((\bar{v}, f, T, (\bar{v}_0, \dots, \bar{v}_{(k_1 - k_2) + 1}), B) : U \rightarrow L^{k_1}(E, G)$$

given by

$$\begin{aligned} & \mathcal{DICP}^{(k_1, k_2, k_3)}((\bar{v}, f, T, (\bar{v}_0, \dots, \bar{v}_{(k_1-k_2)+1}), B) \\ & := \mathcal{DICP}^{(k_1, k_2, k_3)}((D^{k_2}(\bar{v}), \dots, D^{k_3}(\bar{v})), f, (\bar{v}_0, \dots, \bar{v}_{(k_1-k_2)+1}), T, B), \end{aligned} \quad (4.29)$$

with the function $\mathcal{DICP}^{(k_1, k_2, k_3)}((D^{k_2}(\bar{v}), \dots, D^{k_3}(\bar{v})), f, (\bar{v}_0, \dots, \bar{v}_{(k_1-k_2)+1}), T, B)$ as in Definition 4.22.

Definition 4.24. Under the notations of Definition 4.23. Let $\bar{v} : V \subset E \rightarrow F_1$ be a map of class $C^{k_1-k_2+1}$. Let $D^i(\bar{v}) : V \rightarrow L^i(E, F_1)$ be the derivatives of $\bar{v}$, where $0 \leq i \leq k_1 - k_2 + 1$, and assume that $\bar{v} : V \subset F \rightarrow G$ is a function of class C^{k_3} . Let $D^i(\bar{v}) : V \rightarrow L^i(F, G)$, $0 \leq i \leq k_3$ be the derivatives of $\bar{v}$. Then, we define the map

$$\mathcal{DICP}^{(k_1, k_2, k_3)}(\bar{v}, f, \bar{v}, T, B) : U \rightarrow L^{k_1}(E, G)$$

given by

$$\begin{aligned} & \mathcal{DICP}^{(k_1, k_2, k_3)}(\bar{v}, f, \bar{v}, T, B) \\ & := \mathcal{DICP}^{(k_1, k_2, k_3)}((\bar{v}, f, (\bar{v}, D(\bar{v}), \dots, D^{k_1-k_2+1}(\bar{v})), T, B), \end{aligned} \quad (4.30)$$

with the function $\mathcal{DICP}^{(k_1, k_2, k_3)}((\bar{v}, f, (\bar{v}, D(\bar{v}), \dots, D^{k_1-k_2+1}(\bar{v})), T, B)$ as in Definition 4.23.

Remark 4.25. Let $\mathcal{A}_L$ be a set as in Definition 3.14, and suppose that $\nu \in \mathcal{A}_L$ is a function of class C^k . Assume that B is a function of class C^k as in Definition 3.14. Let T be a function as in Definition 3.3. Consider the function $\mathcal{I} : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$ given by

$$\mathcal{I}(x) = \frac{1}{x}.$$

It is not difficult to show that

$$D^q \mathcal{I} : \mathbb{R} - \{0\} \rightarrow L^q(\mathbb{R}, \mathbb{R}),$$

is given by

$$D^q \mathcal{I}(a)(x_1, \dots, x_q) = \frac{(-1)^q q! x_1 \times \dots \times x_q}{a^{q+1}}, \quad (4.31)$$

for every $a \in \mathbb{R} - \{0\}$.

Let E be a metric space. Assume that the function $f : E \rightarrow \mathbb{R}$ is of class C^k and does not

vanish on E . Then, by Chain rule applied to $\mathcal{I} \circ f$ and (4.31) we get

$$\begin{aligned} D^k(f^{-1}) = D^k(\mathcal{I} \circ f) &= \text{Sym}^k \circ \left(\sum_{q=1}^k \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} D^q(\mathcal{I}) \circ a D^{r_1}(f) \dots D^{r_q}(f) \right) \\ &= \text{Sym}^k \left(\sum_{q=1}^k \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} \frac{(-1)^q q!}{a^{(q+1)}} D^{r_1}(a) \dots D^{r_q}(a) \right). \end{aligned} \quad (4.32)$$

We set $f := (1 - \bar{\nu} \circ TB)$. Then, in view of (4.32) and Definition 4.24 one obtains

$$\begin{aligned} D^k(1 - \bar{\nu} \circ TB)^{-1} &= \text{Sym}^k \left(\sum_{q=1}^k \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} \prod_{(r_1 \leq \dots \leq r_q)} (D^q \mathcal{I}, f, \bar{\nu}, T, B) \right) \\ &:= \mathcal{DICP}^{(k,1,k)}((\mathcal{I}, f, \bar{\nu}, T, B)). \end{aligned} \quad (4.33)$$

Thus, one can see that Definitions 4.22, 4.23 and 4.24 are generalizations of the function k th derivative of the map $(1 - \nu \circ B)^{-1}$. This remark will be quite useful later on.

Remark 4.26. Under the assumption of Remark 4.25. Then, we define the map

$$\mathcal{DICP}^{(k_1, k_2, k_3)}(\nu, T, B) : U \rightarrow L^k(\mathbb{R}^{n+1}, \mathbb{R}),$$

given by

$$\mathcal{DICP}^{(k_1, k_2, k_3)}(\nu, T, B) := \mathcal{DICP}^{(k_1, k_2, k_3)}(\mathcal{I}, (1 - \bar{\nu} \circ TB)^{-1}, \nu, T, B), \quad (4.34)$$

with the function $\mathcal{DICP}^{(k_1, k_2, k_3)}(\mathcal{I}, (1 - \bar{\nu} \circ TB)^{-1}, \nu, T, B)$ as in Definition 4.24.

4.1.2 Formula for Derivatives

The purpose of this sub-section is give some lemmas, which will give us the formula for the k th derivative of the function $\Gamma(\nu)$ at point (x, y) , for $y \neq 0$.

We start by noticing the following simple but very useful lemma.

Lemma 4.27. Under Definitions 3.5, 3.14 and 3.16. Let us assume that $\bar{\nu} \in \mathcal{A}_L$ is a function of

class C^k . Then, for $y \neq 0$ the following formulas hold:

$$\begin{aligned} D(\Gamma(\bar{\nu}))(x, y) &= (\bar{\nu} \circ TA - C)D(1 - \bar{\nu} \circ TB)^{-1}(x, y) \\ &\quad + D(\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-1}(x, y) \\ &:= (U_1^1(\bar{\nu}, T, A, B, C) + U_2^1(\bar{\nu}, T, A, B, C))(x, y). \end{aligned} \quad (4.35)$$

For $k \geq 2$

$$\begin{aligned} D^k(\Gamma(\bar{\nu}))(x, y) &= Sym^k \circ \bar{\nu} \circ TA - CD^k(1 - \bar{\nu} \circ TB)^{-1}(x, y) \\ &\quad + Sym^k \circ D^k(\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-1}(x, y) \\ &\quad + Sym^k \circ \sum_{q=1}^{k-1} \binom{k}{q} D^q(\bar{\nu} \circ TA - C)D^{k-q}(1 - \bar{\nu} \circ TB)^{-1}(x, y). \\ &:= Sym^k \circ (U_1^k(\bar{\nu}, T, A, B, C) + U_2^k(\bar{\nu}, T, A, B, C))(x, y) \\ &\quad + Sym^k \circ (U_3^k(\bar{\nu}, T, A, B, C))(x, y). \end{aligned} \quad (4.36)$$

Proof. By Definition 3.16, we have

$$\Gamma(\bar{\nu})(x, y) = ((\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-1})(x, y), \quad \text{for } y \neq 0.$$

By using Leibnitz rule inequality, it follows that

$$D^k(\Gamma(\bar{\nu}))(x, y) = Sym^k \circ \left(\sum_{q=0}^k \binom{k}{q} D^q(\bar{\nu} \circ TA - C)D^{k-q}(1 - \bar{\nu} \circ TB)^{-1} \right)(x, y). \quad (4.37)$$

Hence, if $k = 1$ then, the sum in Eq. (4.37) can be decomposed in two parts as follows:

$$D(\Gamma(\bar{\nu}))(x, y) = ((\bar{\nu} \circ TA - C)D(1 - \bar{\nu} \circ TB)^{-1} + D(\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-1})(x, y). \quad (4.38)$$

which concludes the formula (4.35), and

If $k \geq 2$ then, the sum in Eq. (4.37) can be decomposed in three parts as follows:

$$\begin{aligned} D^k(\Gamma(\bar{\nu}))(x, y) &= Sym^k \circ (\bar{\nu} \circ TA - C)D^k(1 - \bar{\nu} \circ TB)^{-1}(x, y) \\ &\quad + Sym^k \circ D^k(\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-1}(x, y) \\ &\quad + Sym^k \circ \sum_{q=1}^{k-1} \binom{k}{q} D^q(\bar{\nu} \circ TA - C)D^{k-q}(1 - \bar{\nu} \circ TB)^{-1}(x, y), \end{aligned}$$

which concludes the formula (4.36). ■

Lemma 4.28. *Under Definitions 3.5 and 3.14. Let us assume that $\bar{\nu} \in \mathcal{A}_L$ is a function of class C^k and the function*

$$U_1^k(\bar{\nu}, T, A, B, C) : D \setminus D_0 \rightarrow L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$$

as in Lemma 4.27, that is,

$$U_1^k(\bar{\nu}, T, A, B, C)(x, y) := (\bar{\nu} \circ TA - C)D^k(1 - \bar{\nu} \circ TB)^{-1}(x, y). \quad (4.39)$$

Then the following formulas hold:

$$U_1^1(\bar{\nu}, T, A, B, C) = (\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-2}(\bar{\nu} \circ TDB + D\bar{\nu} \circ T.DT.B), \quad (4.40)$$

for $k \geq 2$

$$\begin{aligned} U_1^k(\bar{\nu}, T, A, B, C) &= (\bar{\nu} \circ TA - C)k!(1 - \bar{\nu} \circ TB)^{-2}Sym^k \circ \left(\mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)B \right) \\ &+ (\bar{\nu} \circ TA - C)k!(1 - \bar{\nu} \circ TB)^{-2}Sym^k \circ \left(\mathcal{DC}^{(k,1,(k-1))}(\bar{\nu}, T) \right) \\ &+ (\bar{\nu} \circ TA - C)k!(1 - \bar{\nu} \circ TB)^{-2}Sym^k \circ \left(\mathcal{DCP}^{(k,0,(k-1))}(\bar{\nu}, T, B) \right) \\ &+ (\bar{\nu} \circ TA - C)\mathcal{DICP}^{(k,2,k)}(\bar{\nu}, T, B), \end{aligned} \quad (4.41)$$

with the functions $\mathcal{DC}^{(k_1,k_2,k_3)}(\bar{\nu}, T)$, $\mathcal{DCP}^{(k_1,k_2,k_3)}(\bar{\nu}, T, B)$, $\mathcal{DICP}^{(k_1,k_2,k_3)}(\bar{\nu}, T, B)$, as in Definitions 4.13, 4.19 and 4.25 respectively.

Proof. Recall that, from Remark 4.25, if $f : E \rightarrow \mathbb{R}$ is a function of class C^k and does not vanish on E , we have

$$D^k(f^{-1}) = Sym^k \left(\sum_{q=1}^k \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} (-1)^q q! a^{-(q+1)} D^{r_1}(a) \dots D^{r_q}(a) \right). \quad (4.42)$$

By assumption we have

$$\begin{aligned} U_1^1(\bar{\nu}, T, A, B, C) &= (\bar{\nu} \circ TA - C)D(1 - \bar{\nu} \circ TB)^{-1}. \\ &:= (\bar{\nu} \circ TA - C)\Delta_1 \end{aligned} \quad (4.43)$$

Since the function $\Delta_1 := (1 - \bar{\nu} \circ TB)^{-1}$ is of class C^k and does not vanish on $D \setminus D_0$, from

(4.42) it follows that

$$\begin{aligned}\Delta_1 &= (1 - \bar{\nu} \circ TB)^{-2} D(\bar{\nu} \circ TB). \\ &:= (1 - \bar{\nu} \circ TB)^{-2} \Delta_2\end{aligned}\tag{4.44}$$

By Leibniz rule applied to the function $(\bar{\nu} \circ T)B$, after by applying the Chain rule to the function $\bar{\nu} \circ T$, it follows that

$$\Delta_2 = \bar{\nu} \circ TDB + D\bar{\nu} \circ T.DT.B.\tag{4.45}$$

Moreover, going back to (4.44), we get

$$\Delta_1 = (1 - \bar{\nu} \circ TB)^{-2} (\bar{\nu} \circ TDB + D\bar{\nu} \circ T.DT.B),\tag{4.46}$$

Therefore, substituting (4.46) into (4.43), we get

$$U_1^1(\bar{\nu}, T, A, B, C) = (\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-2} ((\bar{\nu} \circ T)D(B) + D(\bar{\nu}) \circ T.DT.B),\tag{4.47}$$

which concludes the formula (4.40).

Now we shall prove the formula (4.41). By assumption we have

$$U_1^k(\bar{\nu}, T, A, B, C) = (\bar{\nu} \circ TA - C)D^k(1 - \bar{\nu} \circ TB)^{-1}.\tag{4.48}$$

$$:= (\bar{\nu} \circ TA - C)I_1^k.\tag{4.49}$$

Again, by (4.42) it follows that

$$I_1^k = Sym^k \left(\sum_{q=1}^k \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} (1 - \bar{\nu} \circ TB)^{-(q+1)} D^{r_1}(\bar{\nu} \circ TB) \dots D^{r_q}(\bar{\nu} \circ TB) \right).\tag{4.50}$$

Now our goal is to find one formula for I_1^k . Notice that, by splitting the last sum in two parts, it is easy to see that

$$\begin{aligned}I_1^k &= Sym^k \circ (k!(1 - \bar{\nu} \circ TB)^{-2} D^k(\bar{\nu} \circ TB)) \\ &+ Sym^k \circ \left(\sum_{q=2}^k \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} (1 - \bar{\nu} \circ TB)^{-(q+1)} D^{r_1}(\bar{\nu} \circ TB) \dots D^{r_q}(\bar{\nu} \circ TB) \right) \\ &:= Sym^k \circ (k!(1 - \bar{\nu} \circ TB)^{-2} I_2^k) \\ &+ Sym^k \circ \left(\sum_{q=2}^k \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} (1 - \bar{\nu} \circ TB)^{-(q+1)} I_3^k \right).\end{aligned}\tag{4.51}$$

We will find the formula of I_2^k . By using the Leibniz rule to the function $(\bar{\nu} \circ T)B$, we get

$$I_2^k := D^k((\bar{\nu} \circ T)B) = \text{Sym}^k \circ \left(\sum_{q=0}^k \binom{k}{q} D^q(\bar{\nu} \circ T) \cdot D^{k-q}(B) \right). \quad (4.52)$$

Whence, it follows that

$$I_2^k = \text{Sym}^k \circ (D^k(\bar{\nu} \circ T)B) + \text{Sym}^k \circ \left(\sum_{q=0}^{k-1} \binom{k}{q} D^q(\bar{\nu} \circ T) \cdot D^{k-q}(B) \right). \quad (4.53)$$

$$:= \text{Sym}^k \circ (I_{2,1}^k B) + \text{Sym}^k \circ \left(\sum_{q=0}^{k-1} \binom{k}{q} I_{2,2}^k \right). \quad (4.54)$$

By using the Chain rule to the function $I_{2,1}^k := D^k(\bar{\nu} \circ T)$, we get

$$I_{2,1}^k = \text{Sym}^k \circ \left(\sum_{q=1}^k \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} (D^q \bar{\nu}) \circ T \cdot D^{r_1} T \dots D^{r_q} T \right). \quad (4.55)$$

whence, and by decomposing the last sum in two parts, it follows that

$$\begin{aligned} I_{2,1}^k &= \text{Sym}^k \circ \left(k! (D^k \bar{\nu}) \circ T \underbrace{DT \dots DT}_{k\text{-times}} \right) \\ &+ \text{Sym}^k \circ \left(\sum_{q=1}^{k-1} \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} (D^q \bar{\nu}) \circ T \cdot D^{r_1} T \dots D^{r_q} T \right). \end{aligned} \quad (4.56)$$

Next, we find the formula for $I_{2,2}^k$. From (4.53) we have

$$I_{2,2}^k = D^q(\bar{\nu} \circ T) \cdot D^{k-q}(B). \quad (4.57)$$

By Applying Chain rule, we get

$$D^q(\bar{\nu} \circ T) = \text{Sym}^q \circ \left(\sum_{n=1}^q \sum_{r_1+\dots+r_n=q} \frac{q!}{r_1! \dots r_n!} (D^n \bar{\nu}) \circ T \cdot D^{r_1} T \dots D^{r_n} T \right). \quad (4.58)$$

Thus, by replacing (4.58) into (4.57), we obtain

$$I_{2,2}^k = \text{Sym}^q \circ \left(\sum_{n=1}^q \sum_{r_1+\dots+r_n=q} \frac{q!}{r_1! \dots r_n!} (D^n \nu) \circ T \cdot D^{r_1} T \dots D^{r_n} T \right) \cdot D^{k-q}(B). \quad (4.59)$$

Therefore, by Replacing (4.59) and (4.56) into (4.53), and in view of $Sym^k \circ Sym^k = Sym^k$, we get

$$\begin{aligned}
I_2^k &= Sym^k \left(k!(D^k \bar{\nu}) \circ T \underbrace{DT \dots DT}_{k\text{-times}} B \right) \\
&+ Sym^k \circ \left(\sum_{q=1}^{k-1} \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} (D^q \bar{\nu}) \circ F.D^{r_1}T \dots D^{r_q}TB \right) \\
&+ Sym^k \circ \sum_{q=0}^{k-1} \binom{k}{q} Sym^q \left(\sum_{n=1}^q \sum_{r_1+\dots+r_n=q} \frac{q!(D^n \nu) \circ T.D^{r_1}T \dots D^{r_n}T}{r_1! \dots r_n!} . D^{k-q}(B) \right). \tag{4.60}
\end{aligned}$$

Thus, on account of Definitions 4.13 and 4.19 we have

$$I_2^k = Sym^k(\mathcal{DC}^{(k,1,k)}(\bar{\nu}, T)B) + \mathcal{DC}^{(k,1,(k-1))}(\bar{\nu}, T, B) + \mathcal{DCP}^{(k,0,(k-1))}(\bar{\nu}, T, B). \tag{4.61}$$

By similar computation as above, in view of Definition 4.13 and 4.21 we reach that

$$I_3^k = (1 - \bar{\nu} \circ TB)^{-(q+1)} \mathcal{DCP}^{(r_1,0,r_1)}(\bar{\nu}, T, B) \dots \mathcal{DCP}^{(r_q,0,r_q)}(\bar{\nu}, T, B). \tag{4.62}$$

Hence, by using Definition 4.21, Eq. (4.62) becomes

$$I_3^k = \prod_{(r_1 \leq \dots \leq r_q)} (D^q \mathcal{I}, (1 - \bar{\nu} \circ TB), \bar{\nu}, T, B). \tag{4.63}$$

Whence, on account of Definition 4.25, we get

$$Sym^k \circ \left(\sum_{q=2}^k \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} (1 - \bar{\nu} \circ TB)^{-(q+1)} I_3^k \right) := \mathcal{DICP}^{(k,2,k)}(\bar{\nu}, T, B). \tag{4.64}$$

Then, by replacing (4.64) and (4.61) into (4.51), we get

$$\begin{aligned}
I_1^k &= Sym^k \circ \left(k!(1 - \bar{\nu} \circ TB)^{-2} (\mathcal{DC}^{(k,1,k)}((\bar{\nu}, T)B)) \right) \\
&+ Sym^k \circ \left(k!(1 - \bar{\nu} \circ TB)^{-2} (\mathcal{DC}^{(k,1,(k-1))}(\bar{\nu}, T, B)) \right) \\
&+ Sym^k \circ \left(k!(1 - \bar{\nu} \circ TB)^{-2} (\mathcal{DCP}^{(k,0,(k-1))}(\bar{\nu}, T, B)) \right) \\
&+ \mathcal{DICP}^{(k,2,k)}(\bar{\nu}, T, B). \tag{4.65}
\end{aligned}$$

Therefore, by replacing (4.65) into (4.48), and on account of $Sym^k \circ Sym^k = Sym^k$, we get

$$\begin{aligned} U_1^k(\bar{\nu}, T, A, B, C) &= (\bar{\nu} \circ TA - C)k!(1 - \bar{\nu} \circ TB)^{-2} Sym^k \circ \left(\mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)B \right) \\ &+ (\bar{\nu} \circ TA - C)k!(1 - \bar{\nu} \circ TB)^{-2} Sym^k \circ \left(\mathcal{DC}^{(k,1,(k-1))}(\bar{\nu}, T) \right) \\ &+ (\bar{\nu} \circ TA - C)k!(1 - \bar{\nu} \circ TB)^{-2} Sym^k \circ \left(\mathcal{DCP}^{(k,0,(k-1))}(\bar{\nu}, T, B) \right) \\ &+ (\bar{\nu} \circ TA - C)\mathcal{DICP}^{(k,2,k)}(\bar{\nu}, T, B), \end{aligned}$$

which establishes the formula. ■

Lemma 4.29. *Under Definitions 3.5 and 3.14. Let us assume that $\bar{\nu} \in \mathcal{A}_L$ is a function of class C^k and the function*

$$U_2^k(\bar{\nu}, T, A, B, C) : D \setminus D_0 \rightarrow L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$$

as in Lemma 4.27, that is,

$$U_2^k(\bar{\nu}, T, A, B, C)(x, y) = D^k(\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ B)^{-1}. \quad (4.66)$$

Then, the following formulas hold:

$$U_2^1(\bar{\nu}, T, A, B, C) = ((D\bar{\nu}) \circ TDTA + \bar{\nu} \circ TDA - D(C))(1 - \bar{\nu} \circ B)^{-1}, \quad (4.67)$$

for $k \geq 2$

$$\begin{aligned} U_2^k(\bar{\nu}, T, A, B, C) &= Sym^k \left(\mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)A \right) (1 - \bar{\nu} \circ B)^{-1} \\ &+ Sym^k \left(\mathcal{DC}^{(k,1,k-1)}(\bar{\nu}, T)A \right) (1 - \bar{\nu} \circ B)^{-1} \\ &+ Sym^k \left(\mathcal{DCP}^{(k,0,k-1)}(\bar{\nu}, T, A) \right) (1 - \bar{\nu} \circ B)^{-1} \\ &- Sym^k (D^k(C)) \cdot (1 - \bar{\nu} \circ B)^{-1}, \end{aligned} \quad (4.68)$$

with the functions $\mathcal{DC}^{(k_1,k_2,k_3)}(\bar{\nu}, T)$, $\mathcal{DCP}^{(k_1,k_2,k_3)}(\bar{\nu}, T, B)$, $\mathcal{DICP}^{(k_1,k_2,k_3)}(\bar{\nu}, T, B)$ as in Definitions 4.13, 4.19 and 4.25 respectively.

Proof. From Eq. (4.66), by using Leibniz rule and Chain rule, we have the following chain of

equalities

$$\begin{aligned}
U_1^1(\bar{\nu}, T, A, B, C) &= (1 - \nu \circ B)^{-1} D(\bar{\nu} \circ TA - C). \\
&= (1 - \nu \circ B)^{-1} (D(\bar{\nu} \circ TA) - D(C)) \\
&= (1 - \nu \circ B)^{-1} (D(\bar{\nu} \circ T)A + \bar{\nu} \circ TDA - DC). \\
&= (1 - \nu \circ B)^{-1} ((D\bar{\nu}) \circ TDTA + \bar{\nu} \circ TDA - D(C)) \quad (4.69)
\end{aligned}$$

which establishes the formula (4.67). To establish the formula (4.68). By assumption we have

$$\begin{aligned}
U_2^k(\bar{\nu}, T, A, B, C) &= (1 - \nu \circ B)^{-1} (D^k(\bar{\nu} \circ TA) - D^k(C)) \\
&:= (1 - \nu \circ B)^{-1} (II_1^k - D^k(C)). \quad (4.70)
\end{aligned}$$

Now our goal is to find one formula for II_1^k . By using Leibniz rule to the function $((\nu \circ T)A)$, we have the following chain of equalities.

$$\begin{aligned}
II_1^k &= Sym^k \circ \left(\sum_{q=0}^k \binom{k}{q} D^q(\bar{\nu} \circ T) \cdot D^{k-q}(A) \right). \\
&= Sym^k \circ (D^k(\bar{\nu} \circ T)A) + \sum_{q=0}^{k-1} \binom{k}{q} D^q(\bar{\nu} \circ T) \cdot D^{k-q}(A). \quad (4.71)
\end{aligned}$$

$$:= Sym^k \circ (II_{1,1}^k A) + II_{1,2}^k. \quad (4.72)$$

Now, we will find the formula for $II_{1,1}^k$. By using Chain rule to the function $\bar{\nu} \circ T$, we get

$$\begin{aligned}
II_{1,1}^k := D^k(\bar{\nu} \circ T) &= Sym^k \circ \left(\sum_{q=1}^k \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} (D^q \nu) \circ T \cdot D^{r_1} T \dots D^{r_q} T \right) \\
&= Sym^k \circ \left(k! (D^k \bar{\nu}) \circ DT \underbrace{T \dots DT}_{k\text{-times}} \right) \\
&+ Sym^k \circ \left(\sum_{q=1}^{k-1} \sum_{r_1+\dots+r_q=k} \frac{k!}{r_1! \dots r_q!} (D^q \nu) \circ T \cdot D^{r_1} T \dots D^{r_q} T \right). \quad (4.73)
\end{aligned}$$

Then, by using Definition 4.13, Eq. (4.73) becomes

$$II_{1,1}^k = \mathcal{DC}^{(k,1,k)}(\bar{\nu}, T) + \mathcal{DC}^{(k,1,k-1)}(\bar{\nu}, T). \quad (4.74)$$

By similar computation as above, in view of Definition 4.13 we reach that

$$II_{1,2}^k = \sum_{q=0}^{k-1} \binom{k}{q} \left(\mathcal{DC}^{(q,1,q)}(\bar{\nu}, T) \right) . D^{k-q}(A), \quad (4.75)$$

Therefore, by substituting (4.75) and (4.74) into (4.71) it follows that

$$II_1^k = \mathcal{DC}^{(k,1,k)}(\bar{\nu}, T)A + \mathcal{DC}^{(k,1,k-1)}(\bar{\nu}, T)A + \sum_{q=0}^{k-1} \binom{k}{q} \left(\mathcal{DC}^{(q,1,q)}(\bar{\nu}, T) \right) . D^{k-q}(A). \quad (4.76)$$

which establishes the formula for II_2^k .

Consequently, by replacing (4.76) into (4.70) we obtain

$$\begin{aligned} U_2^k(\bar{\nu}, T, A, B, C) &= (1 - \bar{\nu} \circ B)^{-1} Sym^k \circ \left((\mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)A) \right) \\ &+ (1 - \bar{\nu} \circ B)^{-1} Sym^k \circ \left(\mathcal{DC}^{(k,1,k-1)}(\bar{\nu}, T)A \right) \\ &+ (1 - \bar{\nu} \circ B)^{-1} Sym^k \circ \left(\mathcal{DCP}^{(k,0,k-1)}(\bar{\nu}, T, A) \right) \\ &- (1 - \bar{\nu} \circ B)^{-1} Sym^k \circ (D^k(C)), \end{aligned}$$

which establishes the formula for U_2^k , $k \geq 2$. The proof of Lemma 4.29 is thereby complete. ■

Lemma 4.30. *Under Definitions 3.5 and 3.14. Let us assume that $\bar{\nu} \in \mathcal{A}_L$ is a function of class C^k and the function*

$$U_3^k(\bar{\nu}, T, A, B, C) : D \setminus D_0 \rightarrow L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$$

as in Lemma 4.27, that is,

$$U_3^k(\bar{\nu}, T, A, B, C)(x, y) = \sum_{q=1}^{k-1} \binom{k}{q} D^q(\bar{\nu} \circ TA - C) D^{k-q}(1 - \bar{\nu} \circ TB)^{-1}(x, y). \quad (4.77)$$

Then the following equality holds:

$$\begin{aligned} U_3^k(\bar{\nu}, T, A, B, C) &= \\ &Sym^k \left(\sum_{q=1}^{k-1} \binom{k}{q} \phi^{(q,k-q)}((\mathcal{DCP}^{(q,0,q)}(\bar{\nu}, T, A) - D^q(C)), \mathcal{DICP}^{(k-q,1,k-q)}(\bar{\nu}, T, B)) \right), \end{aligned} \quad (4.78)$$

with the functions $\mathcal{DCP}^{(k_1, k_2, k_3)}(\bar{\nu}, T, B)$, $\phi^{(q, k-q)}$, $\mathcal{DICP}^{(k_1, k_2, k_3)}(\bar{\nu}, T, B)$ as in Definitions 4.19, 4.8 and 4.25 respectively.

Proof. The proof is similar to that of the Lemmas 4.28 and 4.29. For more details, see all the developments of the formulas I_1^k (Eq. (4.50)) and I_2^k (Eq. (4.73)). ■

As a direct consequence of the Lemmas 4.27, 4.28, 4.29 and 4.30 we obtain a formula for a k th derivative of the function $\Gamma(\bar{\nu})$ at the point (x, y) , for $y \neq 0$.

Lemma 4.31. *Under Definitions 3.5, 3.14 and 3.16. Let us assume that $\bar{\nu} \in \mathcal{A}_L$ is a function of class C^k . Then, for $y \neq 0$ the following formulas hold:*

$$\begin{aligned} D(\Gamma(\bar{\nu}))(x, y) &= ((\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-2} (\bar{\nu} \circ TDB + D\bar{\nu} \circ T.DT.B) \\ &\quad + (\bar{\nu} \circ TDTA + \bar{\nu} \circ TDA - DC)(1 - \bar{\nu} \circ TB)^{-1})(x, y). \end{aligned} \quad (4.79)$$

for $k \geq 2$

$$\begin{aligned} D^k(\Gamma(\bar{\nu}))(x, y) &= \left((\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \circ \left(\mathcal{DC}^{(k, k, k)}(\bar{\nu}, T)B \right) \right. \\ &\quad + (\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \circ \left(\mathcal{DC}^{(k, 1, (k-1))}(\bar{\nu}, T) \right) \\ &\quad + (\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \circ \left(\mathcal{DCP}^{(k, 0, (k-1))}(\bar{\nu}, T, B) \right) \\ &\quad + (\bar{\nu} \circ TA - C) \mathcal{DICP}^{(k, 2, k)}(\bar{\nu}, T, B) \\ &\quad + (1 - \bar{\nu} \circ TB)^{-1} \text{Sym}^k \circ \left(\mathcal{DC}^{(k, k, k)}(\bar{\nu}, T)A \right) \\ &\quad + (1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \circ \left(\mathcal{DC}^{(k, 1, k-1)}(\bar{\nu}, T)A \right) \\ &\quad + (1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \circ \left(\mathcal{DCP}^{(k, 0, k-1)}(\bar{\nu}, T, A) \right) \\ &\quad - (1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \circ (D^k(C)) \\ &\quad \left. + U_3^k(\bar{\nu}, T, A, B, C) \right). \end{aligned} \quad (4.80)$$

with the functions $\mathcal{DC}^{(k_1, k_2, k_3)}(\bar{\nu}, T)$, $\mathcal{DCP}^{(k_1, k_2, k_3)}(\bar{\nu}, T, B)$, $\mathcal{DICP}^{(k_1, k_2, k_3)}(\bar{\nu}, T, B)$, $\phi^{(q, k-q)}$ as in Definitions 4.13, 4.19, 4.25, 4.8 respectively and the function $U_3^k(\bar{\nu}, T, A, B, C)$ as in Lemma 4.30.

4.1.3 Step 1.2

The goal of this sub-section is to estimate the norms of i th derivatives of the function $A(x, y)$, $B(x, y)$ and $C(x, y)$ around of a neighborhood of D_0 .

We start by noticing the following simple but useful lemma.

Lemma 4.32. *Assume that the functions*

$$d(x, y) = \begin{cases} \alpha(A_+^* + \partial_x \psi_+(x, y)) + |y| \partial_y \psi_+(x, y), & y > 0, \\ \alpha(A_-^* + \partial_x \psi_-(x, y)) + |y| \partial_y \psi_-(x, y), & y < 0, \end{cases} \quad (4.81)$$

and

$$\rho(x, y) = \begin{cases} \frac{1}{\min_{0 \leq j \leq i} |(\alpha A_+^* + \psi_+(x, y) + y \partial_y \psi_+(x, y))|^{j+1}}, & y > 0, \\ \frac{1}{\min_{0 \leq j \leq i} |(\alpha A_-^* + \psi_-(x, y) + |y| \partial_y \psi_-(x, y))|^{j+1}}, & y < 0, \end{cases} \quad (4.82)$$

are defined in a neighborhood of D_0 .

Then, there exist a constant $C \geq 0$ such that the following estimative holds:

$$\|D^i(d(x, y)^{-1})\| \leq C \rho(x, y) |y|^{\gamma-i}, \quad (4.83)$$

in a neighborhood of D_0 .

Moreover, the limits

$$\lim_{(x, y) \rightarrow (a, 0^\pm)} \rho(x, y) \quad (4.84)$$

exist, for all $(a, 0) \in D_0$.

Proof. Assume that $y > 0$, the case $y < 0$ is similar. By Eq.(3.7) we have

$$\lim_{(x, y) \rightarrow (a, 0^\pm)} d(x, y) = A_\pm^*. \quad (4.85)$$

for all $(a, 0) \in D_0$.

Since $A_\pm^* \neq 0$ then there exists a neighborhood Ω of D_0 such that $d : \Omega \rightarrow \mathbb{R}$ does not vanish. Then, on account of Example 4.25, we have

$$D^i(d^{-1}(x, y)) = Sym^i \left(\sum_{q, \beta} c_{i, \beta} \frac{D^{r_1}(d)(x, y) \dots D^{r_q}(d)(x, y)}{(\alpha(A_+^* + \partial_x \psi_+(x, y)) + y \partial_y \psi_+(x, y))^{q+1}} \right), \quad (4.86)$$

where the summation extends on $1 \leq q \leq i$ and all $\beta = (\beta_1, \dots, \beta_q \in \mathbb{Z}_+^q)$, such that $\sum_{j=1}^q \beta_j = i$.

By Assumption 3.6 (Eq.(3.7)), we have

$$\left\| \frac{\partial^{l+m} \psi_{\pm}(x, y)}{\partial x^l \partial y^m} \right\| \leq K|y|^{\gamma-m}, \quad l+m \leq k+1 \quad (4.87)$$

Hence, by deriving i -times the function $d(x, y)$, we get

$$\begin{aligned} D^i(d(x, y)) &= D^i(\alpha A_+^* + \psi_+(x, y) + y \partial_y \psi_+(x, y)) = D^i(\psi_+(x, y)) + D^i(y \partial_y \psi_+(x, y)) \\ &\leq K\|y\|^{\gamma-i} + K\|y\|^{\gamma-i} \\ &\leq 2K\|y\|^{\gamma-i}. \end{aligned} \quad (4.88)$$

Thus, by replacing (4.88) into (4.86), we obtain

$$\|D^i(d^{-1}(x, y))\| \leq Sym^i \left(\sum_{q,\beta} \frac{c_{i,\beta} 2K \|y\|^{\gamma-i}}{(\alpha[A_+^* + \partial_x \psi_+(x, y)] + y \partial_y \psi_+(x, y))^{q+1}} \right). \quad (4.89)$$

Hence, by using Eq. (4.82) and considering $C := \max\{c_{i,\beta}\}2K$, we reach that From (4.89), we have

$$\|D^i(d^{-1}(x, y))\| \leq C \cdot \rho(x, y) |y|^{\gamma-i}. \quad (4.90)$$

Then, estimative (4.83) holds.

Now we prove the statement (4.84)

From Assumption 3.6(L1), it follows that

$$\lim_{(x,y) \rightarrow (a,0)} (\partial_x \psi_{\pm}(x, y) + y \partial_y \psi_{\pm}(x, y)) = 0.$$

Hence, we get

$$\lim_{(x,y) \rightarrow (a,0^+)} \rho(x, y) = \min_{0 \leq j \leq i} |(\alpha A_+^*)^{-(j+1)}| \quad (4.91)$$

and

$$\lim_{(x,y) \rightarrow (a,0^-)} \rho(x, y) = \min_{0 \leq j \leq i} |(\alpha A_-^*)^{-(j+1)}|. \quad (4.92)$$

But $A_{\pm}^* \neq 0$. Then, the limits $\lim_{(x,y) \rightarrow (a,0^{\pm})} \rho(x, y)$ exist. This finishes the proof of Lemma. $\blacksquare$

Corollary 4.33. *Let i be a integer such that $i \geq 0$. Let A, C be functions as in Definition 3.5, and assume the function ρ as in Lemma 4.32. Then there is a constant $C \geq 0$ such that the following inequalities hold:*

$$\|D^i A(x, y)\| \leq C \rho(x, y) |y|^{\gamma-i+1}, \quad (4.93)$$

$$\| D^i C(x, y) \| \leq C \rho(x, y) |y|^{\gamma-i+1}, \quad (4.94)$$

in a neighborhood of D_0 .

Proof. Through of the proof we assume we assume that $y > 0$, the case $y < 0$ is similar.

We distinguish two cases:

Case $i = 0$. According to Lemma 3.18 and Assumption 3.6(L_1) we have

$$A(x, y) = \frac{y \partial_x \varphi_+(x, y)}{\alpha(A_+^* + \partial_x \psi_+(x, y)) + y \partial_y \psi_+(x, y)} \quad (4.95)$$

and

$$\|\partial_x \varphi_\pm(x, y)\| \leq K |y|^\gamma. \quad (4.96)$$

Whence, we get the following estimative

$$\|A(x, y)\| \leq \frac{K |y|^\gamma}{|\alpha(A_+^* + \partial_x \psi_+(x, y)) + y \partial_y \psi_+(x, y)|}. \quad (4.97)$$

But, by (4.82) we have

$$\rho(x, y) \leq \frac{1}{|\alpha(A_+^* + \partial_x \psi_+(x, y)) + y \partial_y \psi_+(x, y)|}. \quad (4.98)$$

Therefore, it follows from (4.98) and (4.97) that

$$\|A(x, y)\| \leq \rho(x, y) K |y|^{\gamma+1}. \quad (4.99)$$

Case $i \geq 1$. By Lemma 3.18 we have

$$A(x, y) = \frac{y \partial_x \varphi_+(x, y)}{\alpha(A_+^* + \partial_x \psi_+(x, y)) + y \partial_y \psi_+(x, y)}, \quad (4.100)$$

in a neighborhood of D_0 .

Letting

$$a(x, y) := y \partial_x \varphi_+(x, y). \quad (4.101)$$

and

$$d(x, y) := \alpha(A_+^* + \partial_x \psi_+(x, y)) + y \partial_y \varphi_+(x, y) \quad (4.102)$$

By Chain rule applied to function $a(x, y)$, we get

$$\begin{aligned} \|D^i a(x, y)\| &= \left\| \text{Sym}^i \sum_{j=0}^i \binom{j}{i} D^j(y) D^{i-j}(\partial_x \varphi_+(x, y)) \right\| \\ &\leq \left\| \text{Sym}^i (y D^i(\partial_x \varphi_+(x, y))) \right\| + \left\| \text{Sym}^i (D(y) D^{i-1}(\partial_x \varphi_+(x, y))) \right\|. \end{aligned} \quad (4.103)$$

But, by Assumption 3.6 (Eq. (3.7)) we have

$$\left\| \frac{\partial^{l+m} \varphi_{\pm}(x, y)}{\partial x^l \partial y^m} \right\| \leq K |y|^{\gamma-m}. \quad l + m \leq k + 1 \quad (4.104)$$

Then, in view of (4.104) and (4.103) we get

$$\begin{aligned} \|D^i a(x, y)\| &\leq \left\| \text{Sym}^i (y D^i(\partial_x \varphi_+(x, y))) + D(y) D^{i-1}(\partial_x \varphi_+(x, y)) \right\| \\ &\leq \|y\| K \|y\|^{\gamma-i} + K \|y\|^{\gamma-i+1} \\ &\leq 2K \|\gamma\|^{\gamma-i+1}. \end{aligned} \quad (4.105)$$

Moreover, by Leibniz rule applied to function $A(x, y)$, we get

$$D^i A(x, y) = D^i \left(\frac{a(x, y)}{d(x, y)} \right) = \sum_{q=0}^i \binom{i}{q} D^q(a)(x, y) D^{i-q}(d^{-1})(x, y). \quad (4.106)$$

But, by Lemma 4.32 we have

$$\|D^i(d^{-1}(x, y))\| \leq |y|^{\gamma-i}, \quad \text{for } 0 \leq i \leq k. \quad (4.107)$$

Hence, on account of (4.105) and (4.106) we obtain

$$\|D^i A(x, y)\| \leq C \rho(x, y) |y|^{\gamma-q+1} |y|^{\gamma-(i-q)} \quad (4.108)$$

$$\leq C \rho(x, y) |y|^{\gamma-i+1}. \quad (4.109)$$

By similar arguments one can show the estimate (4.94). This proves the corollary. ■

Corollary 4.34. *Let i be a integer such that $1 \leq i \leq k$. Let B be a function as in Definition 3.5, and assume the function ρ as in Lemma 4.32. Then there exists a $\tilde{C}$ such that the following inequality holds:*

$$\|D^i B(x, y)\| \leq \tilde{C} \rho(x, y) |y|^{\gamma-i}, \quad (4.110)$$

in a neighborhood of D_0 .

Proof. The proof is identical to that last corollary. ■

Corollary 4.35. *Let $T : D_+ \cup D_- \rightarrow D$ be a map given by the equation*

$$T(x, y) = (F(x, y), G(x, y)) = (\bar{x}, \bar{y}). \quad (4.111)$$

Assume that T satisfies Assumption 3.6(L_1). Then the following relation holds:

$$\|D^k T(a, b)\| \leq \text{const} \|b\|^{\alpha-k}, \quad (4.112)$$

in a neighborhood of D_0 , where const denotes a constant positive.

Proof. Recall that, from Assumption 3.6(L_1) we have F and G have the form

$$F(x, y) = \begin{cases} x_+^* + |y|^\alpha [B_+^* + \varphi_+(x, y)], & y > 0, \\ x_-^* + |y|^\alpha [B_-^* + \varphi_-(x, y)], & y < 0, \end{cases} \quad (4.113)$$

$$G(x, y) = \begin{cases} y_+^* + |y|^\alpha [A_+^* + \psi_+(x, y)], & y > 0, \\ y_-^* + |y|^\alpha [A_-^* + \psi_-(x, y)], & y < 0, \end{cases} \quad (4.114)$$

in a neighborhood of D_0 .

Claim 4.36. *The following relations hold:*

$$\|D^k F(x, y)\| \leq \text{const} |y|^{\alpha-k}. \quad (4.115)$$

$$\|D^k G(x, y)\| \leq \text{const} |y|^{\alpha-k}. \quad (4.116)$$

To prove the claim. Assume that $y > 0$, the case $y < 0$ is similar.

By Leibniz rule applied to the function $F(x, y)$ we get

$$D^k F(x, y) = \sum_{m=0}^k \binom{k}{m} D^m(y^\alpha) \cdot D^{k-m}((B_+^* + \varphi_+(x, y))). \quad (4.117)$$

Moreover, by Assumption 3.6(L_1) we have

$$\left\| \frac{\partial^{l+m} \varphi_\pm(x, y)}{\partial x^l \partial y^m} \right\| \leq K |y|^{\gamma-m}, \quad \text{for } l + m \leq k + 1, \quad (4.118)$$

and it is easily seen that

$$\|D^j(y^\alpha)\| \leq |y|^{\alpha-j}, \quad \text{for } j \geq 0. \quad (4.119)$$

Thus, it follows from (4.119) and (4.118) that

$$\|D^k F(x, y)\| \leq \text{const} |y|^{\alpha-k}. \quad (4.120)$$

We now apply this argument again, with G replaced by F , to obtain

$$\|D^k G(x, y)\| \leq \text{const} |y|^{\alpha-k}. \quad (4.121)$$

This concludes the proof of the claim. Hence, we get

$$\|D^k T(a, b)\| \leq \text{const} \|b\|^{\alpha-k}, \quad (4.122)$$

which finishes the proof of lemma. ■

4.1.4 Step 1.3

The aim of this section is to prove the following corollary:

Corollary 4.37. *Under the assumptions and with the definitions used on Chapters 3 and 4. Assume that $\mu \in \mathcal{A}_L$ is a function of class C^k . Then*

$$\lim_{(a,b) \rightarrow (x,0)} D^i(\Gamma(\mu))(a, b) = 0, \quad (4.123)$$

for each $0 \leq i \leq k$ and for every $(x, 0) \in D_0$.

The proof of this Corollary will be given after the following lemmas.

Lemma 4.38. *Let A, B, C be functions as in Definition 3.5, and assume that $\bar{\nu} \in \mathcal{A}_L$ (see Definition 3.14) is a functions of class C^k . Let $U_1^k(\bar{\nu}, T, A, B, C) : D^* \rightarrow L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ be a function as in Lemma 4.27, that is,*

$$U_1^k(\bar{\nu}, T, A, B, C)(x, y) = (\bar{\nu} \circ TA - C)D^k(1 - \bar{\nu} \circ TB)^{-1}(x, y). \quad (4.124)$$

Then

$$\lim_{(a,b) \rightarrow (x,0)} U_1^k(\bar{\nu}, T, A, B, C)(a, b) = 0, \quad (4.125)$$

for every $(x, 0) \in D_0$.

Proof. We will give the proof for the case $k \geq 2$, the case $k = 1$ is similar.

Recall that by Lemma 4.28($k \geq 2$) we have

$$\begin{aligned}
U_1^k(\bar{\nu}, T, A, B, C)(a, b) &= \frac{(\bar{\nu} \circ TA - C)}{(1 - \bar{\nu} \circ TB)^2} k! \cdot \text{Sym}^k \left(\mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)B \right) (a, b) \\
&+ \frac{(\bar{\nu} \circ TA - C)}{(1 - \bar{\nu} \circ TB)^2} \text{Sym}^k \left(k! \cdot \mathcal{DC}^{(k,1,(k-1))}(\bar{\nu}, T) \right) (a, b) \\
&+ \frac{(\bar{\nu} \circ TA - C)}{(1 - \bar{\nu} \circ TB)^2} \text{Sym}^k \left(k! \left(\mathcal{DCP}^{(k,0,(k-1))}(\bar{\nu}, T, B) \right) \right) (a, b) \\
&+ (\bar{\nu} \circ TA - C) \mathcal{DICP}^{(k,2,k)}(\bar{\nu}, T, B)(a, b). \tag{4.126}
\end{aligned}$$

According Definition 3.14 and Remark 4.4(iii) we have

$$\|\nu\| \leq L \tag{4.127}$$

and

$$\|\text{Sym}^k\| \leq 1. \tag{4.128}$$

Thus, on account of (4.128)-(4.126), we get

$$\begin{aligned}
\|U_1^k(\bar{\nu}, T, A, B, C)\| &\leq \frac{(L\|A(a, b)\| + \|C(a, b)\|)}{(1 - L\|B\|)^2} \|k! \mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)(a, b)\| \|B\| \\
&+ \frac{(L\|A(a, b)\| + \|C(a, b)\|)}{(1 - L\|B\|)^2} \|k! \mathcal{DC}^{(k,1,(k-1))}(\bar{\nu}, T)(a, b)\| \\
&+ \frac{(L\|A(a, b)\| + \|C(a, b)\|)}{(1 - L\|B\|)^2} \|\mathcal{DCP}^{(k,0,(k-1))}(\bar{\nu}, T, B)(a, b)\| \\
&+ (L\|A(a, b)\| + \|C(a, b)\|) \|\mathcal{DICP}^{(k,2,k)}(\bar{\nu}, T, B)(a, b)\|. \tag{4.129}
\end{aligned}$$

Next, we will estimate the first expression of (4.129). Recall that, by Example 4.14 (Eq. (4.18)) we have

$$\mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)(a, b) = k!(D^k \bar{\nu}) \circ T \underbrace{DT(a, b) \dots DT(a, b)}_{k\text{-times}}. \tag{4.130}$$

This implies that

$$\|\mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)(a, b)\| \leq \|k!(D^k \bar{\nu}) \circ T\| \|DT(a, b)\|^k. \tag{4.131}$$

Since ν is of class C^k , then the function $k!(D^k \bar{\nu}) \circ T$ is bounded, that is,

$$\|k!(D^k \bar{\nu}) \circ T\| \leq \text{const}, \tag{4.132}$$

and by Corollary 4.35 we have

$$\|D^j T(a, b)\| \leq \text{const} |b|^{\alpha-j}, \quad \text{for } 0 \leq j \leq k \quad (4.133)$$

Therefore, it follows from (4.133) to (4.131) that

$$\|\mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)(a, b)\| \leq \text{const} |b|^{\gamma-k}. \quad (4.134)$$

Whence, in view of Corollary 4.33 we get

$$\frac{(L\|A(a, b)\| + \|C(a, b)\|)}{(1 - L\|B\|)^2} \|k! \mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)(a, b)\| \leq \text{const} \frac{|b|^{\alpha-k+\gamma+1}}{(1 - L\|B\|)^2}. \quad (4.135)$$

Now, we will estimate the second expression of (4.129). By Definition 4.13 and in view of (4.128) it follows that

$$\begin{aligned} \|\mathcal{DC}^{(k_1, k_2, (k_3))}(\bar{\nu}, T)(a, b)\| &:= \|Sym^{k_1} \circ \left(\sum_{q=k_2}^{k_3} c_{k, \beta} D^q v \circ T \cdot D^{\beta_1} T \dots D^{\beta_q} T \right) (a, b)\| \\ &\leq \sum_{q=1}^{k-1} c_{k, \beta} \|D^q v \circ T\| \cdot \|D^{\beta_1} T(a, b)\| \dots \|D^{\beta_q} T(a, b)\|, \end{aligned} \quad (4.136)$$

where $\beta = (\beta_1, \dots, \beta_q) \in \mathbb{Z}_+^q$ such that $\sum_{j=1}^q \beta_j = k_1$.

Hence, on account of (4.133) and (4.132), Eq. (4.136) becomes

$$\|\mathcal{DC}^{(k_1, k_2, k_3)}(\bar{\nu}, T)(a, b)\| \leq \text{const} |b|^{\alpha-k_1}. \quad (4.137)$$

Moreover, recall that by Corollary 4.33 we have

$$\|DA(a, b)\| \leq C |\rho(a, b)| |b|^{\gamma+1}, \quad (4.138)$$

and

$$\|DC(a, b)\| \leq C |\rho(a, b)| |b|^{\gamma+1}. \quad (4.139)$$

Thus, it follows from (4.138), (4.139) and (4.137) that

$$\frac{(L\|A(a, b)\| + \|C(a, b)\|)}{(1 - L\|B\|)^2} \|k! \mathcal{DC}^{(k, 1, (k-1))}(\bar{\nu}, T)(a, b)\| \leq |b|^{\alpha-k+\gamma+1} (1 - L\|B\|)^2 \text{const}. \quad (4.140)$$

Now, we will estimate the third expression of (4.129). By Remark 4.20

$$\begin{aligned} \mathcal{DCP}^{(k_1, k_2, k_3)}(\bar{\nu}, T, B)(a, b) &:= \\ Sym^{k_1} \left(\sum_{n=0}^k \binom{k}{n} \phi^{(n, k_1-n)} \left(\mathcal{DC}^{(n, 1, n)}(\bar{\nu}, T)(a, b), D^{(k_1-n)} B(a, b) \right) \right). \end{aligned} \quad (4.141)$$

Hence, since $\phi^{(n, k_1-n)}$ is multilinear map, in view of Eq. (4.128) we obtain

$$\|\mathcal{DCP}^{(k_1, k_2, k_3)}(\bar{\nu}, T, B)(a, b)\| \leq \sum_{n=0}^k \binom{k}{n} \|\mathcal{DC}^{(n, 1, n)}(\bar{\nu}, T)(a, b)\| \|D^{(k_1-n)} B(a, b)\|. \quad (4.142)$$

Furthermore, by Corollary 4.34 we have

$$\|D^j B(a, b)\| \leq const |b|^{\gamma-j}. \quad (4.143)$$

Therefore, it follows from (4.143) and (4.142) that

$$\|\mathcal{DCP}^{(k_1, k_2, k_3)}(\bar{\nu}, T, B)(a, b)\| \leq const |b|^{\gamma-k_1}. \quad (4.144)$$

Thus, combining (4.144), (4.138) and (4.13) we obtain

$$\frac{(L\|A(a, b)\| + \|C(a, b)\|)}{(1 - L\|B\|_D)^2} \|\mathcal{DCP}^{(k, 0, (k-1))}(\bar{\nu}, T, B)(a, b)\| \leq \frac{|b|^{\gamma-k+1}}{(1 - L\|B\|_D)^2}. \quad (4.145)$$

Next, we will estimate the fourth expression of (4.129). By Definition 4.25, in view of (4.128) and Triangle Inequality we obtain

$$\|\mathcal{DICP}^{(k, 2, k)}(\bar{\nu}, T)(a, b)\| \leq \sum_{q=2}^k c_{k, \beta} \frac{\|\mathcal{DICP}^{(\beta_1, 0, \beta_1)}(\bar{\nu}, T, B)\| \dots \|\mathcal{DCP}^{(\beta_q, 0, \beta_q)}(\bar{\nu}, T, B)\|}{(1 - L\|B\|_D)^{-(q+1)}}, \quad (4.146)$$

where $\beta = (\beta_1, \dots, \beta_q) \in \mathbb{Z}_+^q$ such that $\sum_{j=1}^q \beta_j = k$.

Moreover, since $(1 - L\|B\|_D) \leq 1$ we have

$$(1 - L\|B\|_D)^{q+1} \leq (1 - L\|B\|_D), \quad q \geq 0. \quad (4.147)$$

Thus, combining (4.147) and (4.146) we obtain

$$\|\mathcal{DICP}^{(K, 2, k)}(\bar{\nu}, T)(a, b)\| \leq const |b|^{\alpha-k}. \quad (4.148)$$

Hence, in view of (4.138) and (4.13) we get

$$(L\|A(a, b)\| + \|C(a, b)\|)\|\mathcal{DICP}^{(k,2,k)}(\bar{\nu}, T, B)(a, b)\| \leq \text{const}|b|^{\alpha-k+\gamma+1}. \quad (4.149)$$

which finishes the estimative fourth expression of (4.129).

Therefore, combining the four estimatives (4.149), (4.145), (4.140) and (4.135) with (4.129) we obtain

$$\|U_1^k(\bar{\nu}, T, A, B, C)\| \leq \text{const}|b|^{\alpha-k+\gamma+1}. \quad (4.150)$$

Then, by letting $b \rightarrow 0$ on both sides of the Eq (4.150), and on account of $\gamma > k - 1$ we reach that

$$\lim_{(a,b) \rightarrow (x,0)} \|U_1^k(\bar{\nu}, T, A, B, C)(a, b)\| = 0. \quad (4.151)$$

This concludes the corollary. ■

Lemma 4.39. *Let A, B, C be functions as in Definition 3.5, and assume that $\bar{\nu} \in \mathcal{A}_L$ (see Definition 3.14) is a function of class C^k . Let $U_2^k(\bar{\nu}, T, A, B, C) : D \rightarrow L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ be a function as Lemma 4.27, that is,*

$$U_2^k(\bar{\nu}, T, A, B, C)(x, y) = D^k(\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ B)^{-1}(x, y). \quad (4.152)$$

Then

$$\lim_{(a,b) \rightarrow (x,0)} U_2^k(\bar{\nu}, T, A, B, C)(a, b) = 0. \quad (4.153)$$

Proof. The proof is quite similar to that of Lemma 4.38. ■

Lemma 4.40. *Let A, B, C be functions as in Definition 3.5, and assume that $\bar{\nu} \in \mathcal{A}_L$ (see Definition 3.14) is a function of class C^k . Let $U_3^k(\bar{\nu}, T, A, B, C) : D^* \rightarrow L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ be a function as Lemma 4.27, that is,*

$$U_3^k(\bar{\nu}, T, A, B, C)(x, y) = \sum_{q=1}^{k-1} \binom{k}{q} D^q(\bar{\nu} \circ TA - C) D^{k-q}(1 - \bar{\nu} \circ TB)^{-1}(x, y). \quad (4.154)$$

Then

$$\lim_{(a,b) \rightarrow (x,0)} U_3^k(\bar{\nu}, T, A, B, C)(a, b) = 0. \quad (4.155)$$

Proof. The proof is quite similar to that of Lemma 4.38. ■

Proof of Corollary 4.37. This is a direct consequence from Lemmas 4.37, 4.38, 4.39 and 4.40. ■

4.1.5 Proof of Proposition 4.2

We are going to proof the Proposition 4.2 mentioned in the Section 4.1, which we recall here.

Proposition 4.41. *Assume that $\mu \in \mathcal{A}_L$ is a function of class C^k . Then the following statements hold:*

- (1) *The function $\Gamma(\mu) \in \mathcal{A}_L$ is of class C^k ,*
- (2) *$D^k\Gamma(\mu)(x, 0) = 0$, for all $(x, 0) \in D_0$.*

Prove of Proposition 4.41. We proceed by induction on k . Assume the assertion is proved for $0 \leq j \leq k-1$, we wish to show that statement is true for $j = k$. By definition of operator Γ (see Definition 3.16), we have that $D^k\Gamma(\mu)$ is differentiable in $D \setminus D_0$, only remains to show the differentiability at point $(x, 0) \in D_0$. To do this, by definition to prove that the function $D^k\Gamma(\mu)$ is differentiable in $(x, 0) \in D_0$ we have to show that the function

$$\psi : D \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$$

given by

$$\psi(v) = \begin{cases} \frac{D^{k-1}\Gamma(u)((x, 0) + v) - D^{k-1}\Gamma(u)(x, 0)}{\|v\|}, & \text{if } v \in D \setminus D_0, \\ 0, & \text{if } v \in D_0. \end{cases} \quad (4.156)$$

is continuous at zero.

First, note that ψ is well defined by inductive hypothesis. Moreover, from Corollary 4.37, we have

$$\lim_{(a,b) \rightarrow (x,0)} D^k(\Gamma(\mu))(a, b) = 0. \quad (4.157)$$

Therefore, given $\epsilon > 0$, there exists $\delta > 0$, such that for every $v \in D \setminus D_0, 0 < \|v\| < \delta$, we have

$$\|D(D^{k-1}(\Gamma)(\mu))(x, 0) + v\| < \epsilon. \quad (4.158)$$

Whence, on account of Mean Value Theorem applied to the function

$$D^{k-1}\Gamma(\mu) : D \rightarrow L^{k-1}(\mathbb{R}^{n+1}, \mathbb{R}^n),$$

it follows that

$$\frac{D^{k-1}\Gamma(u)((x, 0) + v) - D^{k-1}\Gamma(u)(x, 0)}{\|v\|} < \epsilon, \quad (4.159)$$

whenever $D \setminus D_0$, $0 < \|v\| < \delta$.

Thus, it follows from (4.159) and (4.156) that

$$\psi(v) < \epsilon, \quad (4.160)$$

whenever $v \in D_+ \cup D_-$, $0 < \|v\| < \delta$.

Hence, since ϵ was arbitrary, we obtain

$$\lim_{v \rightarrow 0} \psi(v) = 0. \quad (4.161)$$

Whence, in view of (4.156), we obtain that ψ is continuous at zero, which proves that $D^k \Gamma(\mu)$ is differentiable at the point $(x, 0)$, and $D^k \Gamma(\mu)(x, 0) = 0$. Therefore, $D^k \Gamma(\mu)$ is differentiable on D . Thus, the statement of Theorem holds for $j = k$. That completes the inductive argument, which completes the proof. ■

4.2 Step 2.

The aim of this step is to prove the following Proposition:

Proposition 4.42. *Let $\mathcal{A}_L$ be a set as in Definition 3.14. Let $\Gamma : \mathcal{A}_L \rightarrow \mathcal{A}_L$ be a Operator as in Definition 3.16. Then limit*

$$\lim_{n \rightarrow \infty} (\Gamma^n(\mu), D(\Gamma^n(\mu)), \dots, D^k(\Gamma^n(\mu))) \quad (4.162)$$

exists, for all $\mu \in \mathcal{A}_L$ of class C^k and $D^k \mu(x, 0) = 0$.

The proof of Proposition 4.42 was influenced by the ideas contained in the articles [Rob81, p. 313] and [SS94, Eq. (3)]. The proof is quite long and technical, so we divide it into two steps. Before that, we may remark that by Proposition 4.2 we have if $\mu \in \mathcal{A}_L$ is of class C^k then the function

$$D^i(\Gamma^m(\mu)) : D \rightarrow L^i(\mathbb{R}^{n+1}, \mathbb{R}^n) \quad (4.163)$$

is continuous and $D^i(\Gamma^m(\mu))(x, 0) = 0$, for all $m \in \mathbb{N}$ and $0 \leq i \leq k$. This remark leads us to the following definition.

Definition 4.43. *We define the set $\mathcal{D}_i$ of all the continuous function $\nu_i : D \rightarrow L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ such that $\nu_i(x, 0) = 0$, that is,*

$$\mathcal{D}_i := \{\nu_i : D \rightarrow L_s^i(\mathbb{R}^{n+1}, \mathbb{R}^n)^1 : \nu_i(x, 0) = 0; \nu_i \text{ is continuous}\}, \quad (4.164)$$

¹We use the notation $L_s^i(E, F)$ for the space of symmetric bounded i -linear maps from E to F

for every $1 \leq i \leq k$, and ,

$$\mathcal{D}_0 := \mathcal{A}_L. \quad (4.165)$$

The steps to prove Proposition 4.42 are the following:

Step 2.1: To find functions

$$\Psi^i : \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i \rightarrow D_i,$$

for all $0 \leq i \leq k$, so that

$$D^i(\Gamma(\bar{\nu}_0)) = \Psi^i(\bar{\nu}_0, D(\bar{\nu}_0), \dots, D^i(\bar{\nu}_0)), \quad (4.166)$$

for all $0 \leq i \leq k$.

Step 2.2: To show that the function

$$\tilde{N}_i : \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i \rightarrow \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i \quad (4.167)$$

given by

$$\tilde{N}_i((\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i) = (\Gamma(\bar{\nu}_0), \Psi^1((\bar{\nu}_0, \bar{\nu}_1), \dots, \Psi^i((\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i)). \quad (4.168)$$

have a global attracting fixed point $(A_0, A_1, \dots, A_i)$, for all $0 \leq i \leq k$.

We start by Step 2.1.

4.2.1 Step 2.1

The purpose of this subsection is to define the functions

$$\Psi^i : \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i \rightarrow D_i$$

that satisfy the Eq. (4.166). For that, first, we will define functions that generalize U_1^k, U_2^k, U_3^k , (see Corollaries 4.28, 4.29 and 4.30).

We start by define the generalization of the function U_1^k .

Definition 4.44. Let i be a integer such that $1 \leq i \leq k$. Let $D_0, \mathcal{D}_i$ be sets as in Definition 4.43. We define the function

$$U_1^i : \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i \rightarrow D_i \quad (4.169)$$

given by

$$U_1^1(\bar{\nu}_0, \bar{\nu}_1) = (\bar{\nu}_0 \circ TA - C)(1 - \bar{\nu}_0 \circ TB)^{-2}(\bar{\nu}_0 \circ TDB + \bar{\nu}_1 \circ T.DT.B), \quad (4.170)$$

for $i \geq 2$

$$\begin{aligned} U_1^i(\bar{\nu}_0, \dots, \bar{\nu}_i) &= \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} \text{Sym}^i \circ \mathcal{DC}^{(i,i,i)}(\bar{\nu}_i, T)B \\ &+ \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} \text{Sym}^i \circ \mathcal{DC}^{(i,1,(i-1))}(\bar{\nu}_1, \dots, \bar{\nu}_{(i-1)}, T) \\ &+ \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} \text{Sym}^i \circ \mathcal{DCP}^{(i,0,(i-1))}(\bar{\nu}_0, \dots, \bar{\nu}_{(i-1)}, T, B) \\ &+ (\bar{\nu}_0 \circ TA - C)\mathcal{DICP}^{(i,2,i)}(\bar{\nu}_0, \dots, \bar{\nu}_{(i-1)}, T, B), \end{aligned} \quad (4.171)$$

with the functions $\mathcal{DC}^{(k_1,k_2,k_3)}(\bar{\nu}, T)$, $\mathcal{DCP}^{(k_1,k_2,k_3)}(\bar{\nu}, T, B)$, $\mathcal{DICP}^{(k_1,k_2,k_3)}(\bar{\nu}, T, B)$, as in Definitions 4.13, 4.19 and 4.25 respectively.

Next, we define the generalization of the function U_2^k , (see Corollary 4.29).

Definition 4.45. Let i be a integer such that $1 \leq i \leq k$. Let $D_0, \mathcal{D}_i$ be sets as in Definition 4.43. We define the function

$$U_2^i : \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i \rightarrow D_i \quad (4.172)$$

given by

$$U_2^1(\nu_0, \nu_1, T) = \nu_1 \circ TDTA + \nu_0 \circ TDA, \quad (4.173)$$

for $i \geq 2$

$$\begin{aligned} (U_2^i)(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i) &= (1 - \bar{\nu}_0 \circ B)^{-1} \text{Sym}^i \circ \left(\mathcal{DC}^{(i,i,i)}(\bar{\nu}_i, T)A \right) \\ &+ (1 - \bar{\nu}_0 \circ B)^{-1} \text{Sym}^i \circ \left(\mathcal{DC}^{(i,1,i-1)}(\bar{\nu}_1, \dots, \bar{\nu}_{i-1}, T)A \right) \\ &+ (1 - \bar{\nu}_0 \circ B)^{-1} \text{Sym}^i \left(\mathcal{DCP}^{(i,0,i-1)}(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_{(i-1)}, T, A) \right) \\ &- (1 - \bar{\nu}_0 \circ B)^{-1} \text{Sym}^i \circ (D^i(C)), \end{aligned} \quad (4.174)$$

with the functions $\mathcal{DC}^{(k_1,k_2,k_3)}(\bar{\nu}, T)$, $\mathcal{DCP}^{(k_1,k_2,k_3)}(\bar{\nu}, T, B)$, as in Definitions 4.13 and 4.19 respectively.

Next, we define the generalization of the function U_3^k (see Corollary 4.30).

Definition 4.46. Let i be a integer such that $2 \leq i \leq k$. Let $D_0, \mathcal{D}_i$ be sets as in Definition 4.43. We define the function

$$U_3^i : \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i \rightarrow D_i \quad (4.175)$$

given by

$$(U_3^i)(\bar{\nu}_0, \dots, \bar{\nu}_i) = \text{Sym}^i \circ \sum_{q=1}^{i-1} \binom{i}{q} \phi^{(q, i-q)}(\mathcal{DCP}^{(q, 0, q)}(\bar{\nu}_0, \dots, \bar{\nu}_q, T, A) - D^i C, \mathcal{DICP}^{(i-q, 1, i-q)}(\bar{\nu}_0, \dots, \bar{\nu}_{i-q-1}, T, B)), \quad (4.176)$$

with the functions $\mathcal{DCP}^{(k_1, k_2, k_3)}(\bar{\nu}, T, B)$, $\mathcal{DICP}^{(k_1, k_2, k_3)}(\bar{\nu}, T, B)$, $\phi^{(q, i-q)}$ as in Definitions 4.19, 4.25 and 4.8 respectively.

Next, we define the generalization of the function $D^k(\Gamma(\nu))$ (see Lemma 4.31).

Definition 4.47. Let i be a integer such that $1 \leq i \leq k$. Let $D_0, \mathcal{D}_i$ be sets as in Definition 4.43, and assume the functions U_1^i, U_2^i, U_3^i as in Definitions 4.44, 4.45 and 4.46 respectively. We define the function

$$\Psi^i : \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i \rightarrow D_i \quad (4.177)$$

given by

$$\Psi^1(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i) = (U_1^1 + U_2^1)(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i), \quad (4.178)$$

and for $i \geq 2$

$$\Psi^i(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i) = (U_1^i + U_2^i + U_3^i)(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i). \quad (4.179)$$

Proposition 4.48. The function Ψ^i given in the last definition is well-defined.

Moreover, if $\bar{\nu}_0 \in \mathcal{A}_L$ is of class C^i , then

$$\Psi^i(\bar{\nu}_0, D\bar{\nu}_0, \dots, D^i\bar{\nu}_0) = D^i\Gamma(\bar{\nu}_0). \quad (4.180)$$

Proof. To prove that the function Ψ^i is well-defined we have to show that

$$\Psi^i(\bar{\nu}_0, \dots, \bar{\nu}_i) \in \mathcal{D}_i, \quad (4.181)$$

whenever $\bar{\nu}_j \in \mathcal{D}_j$, for each $0 \leq j \leq i$.

To prove statement (4.181), by Definition 4.43 we have to show that

- 1) $\Psi^i(\bar{\nu}_0, \dots, \bar{\nu}_i)$ is continuous on D , and

2) $\Psi^i(\bar{\nu}_0, \dots, \bar{\nu}_i)(x, 0) = 0$, for every $x \in \mathbb{R}^n$, $\|x\| \leq 1$,

whenever $\bar{\nu}_j \in \mathcal{D}_j$, for each $0 \leq j \leq i$.

By Definition 4.47 we have that $\Psi^i(\bar{\nu}_0, \dots, \bar{\nu}_i)$ is continuous on $D \setminus D_0$, so it remains to show the continuity of $\Psi^i(\bar{\nu}_0, \dots, \bar{\nu}_i)$ at points $(x, 0) \in D_0$.

Analysis similar to that in the proof of Corollary 4.38 shows that

$$\lim_{(a,b) \rightarrow (x,0)} \psi^i(\bar{\nu}_0, \dots, \bar{\nu}_i)(a, b) = 0, \quad (4.182)$$

for all $(x, 0) \in D_0$.

Therefore, if we define

$$W^i(x, y) = \begin{cases} \Psi^i(\bar{\nu}_0, \dots, \bar{\nu}_i)(x, y), & y \neq 0, \\ 0, & y = 0. \end{cases}$$

Then, we get a continuous extension of $\Psi^i(\bar{\nu}_0, \dots, \bar{\nu}_i)$ on D , which completes the proof of (4.2.1), so $\Psi^i(\bar{\nu}_0, \dots, \bar{\nu}_i) \in \mathcal{D}_i$. Therefore Ψ^i is well-defined.

Now we will prove the equality

$$\Psi^i(\bar{\nu}_0, D(\bar{\nu}_0), \dots, D^i(\bar{\nu}_0)) = D^i(\Gamma(\bar{\nu}_0)).$$

The proof of this equality will be given for the case $i \geq 2$, the case $i \geq 2$ is similar.

By Definition 4.47($i \geq 2$) we have

$$\Psi^i(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i) = (U_1^i + U_2^i + U_3^i)(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i), \quad (4.183)$$

where

$$\begin{aligned} U_1^i(\bar{\nu}_0, \dots, \bar{\nu}_i) &= \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} \text{Sym}^i \circ \mathcal{DC}^{(i,i,i)}(\bar{\nu}_i, T)B \\ &+ \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} \text{Sym}^i \circ \mathcal{DC}^{(i,1,(i-1))}(\bar{\nu}_1, \dots, \bar{\nu}_{(i-1)}, T) \\ &+ \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} \text{Sym}^i \circ \mathcal{DCP}^{(i,0,(i-1))}(\bar{\nu}_0, \dots, \bar{\nu}_{(i-1)}, T, B) \\ &+ (\bar{\nu}_0 \circ TA - C)\mathcal{DICP}^{(i,2,i)}(\bar{\nu}_0, \dots, \bar{\nu}_{(i-1)}, T, B), \end{aligned}$$

$$\begin{aligned}
(U_2^i)(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i) &= (1 - \bar{\nu}_0 \circ B)^{-1} Sym^i \circ \left(\mathcal{DC}^{(i,i,i)}(\bar{\nu}_i, T)A \right) \\
&+ (1 - \bar{\nu}_0 \circ B)^{-1} Sym^i \circ \left(\mathcal{DC}^{(i,1,i-1)}(\bar{\nu}_1, \dots, \bar{\nu}_{i-1}, T)A \right) \\
&+ (1 - \bar{\nu}_0 \circ B)^{-1} Sym^i \left(\mathcal{DCP}^{(i,0,i-1)}(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_{i-1}, T, A) \right) \\
&- (1 - \bar{\nu}_0 \circ B)^{-1} Sym^i \circ (D^i(C)),
\end{aligned}$$

and

$$\begin{aligned}
(U_3^i)(\bar{\nu}_0, \dots, \bar{\nu}_i) &= \\
Sym^i \circ \sum_{q=1}^{i-1} \binom{i}{q} \phi^{(q,i-q)} &(\mathcal{DCP}^{(q,0,q)}(\bar{\nu}_0, \dots, \bar{\nu}_q, T, A) - D^i C, \mathcal{DICP}^{(i-q,1,i-q)}(\bar{\nu}_0, \dots, \bar{\nu}_{i-q-1}, T, B)).
\end{aligned}$$

Therefore, letting $\nu_0 := \bar{\nu}$ and $\nu_j := D^j \bar{\nu}$, for $1 \leq j \leq i$ it follows from (4.183) that.

$$\begin{aligned}
\Psi^i(\bar{\nu}, D\bar{\nu}, \dots, D^i \bar{\nu}) &= \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} Sym^i \circ \mathcal{DC}^{(i,i,i)}(\bar{\nu}, T)B \\
&+ \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} Sym^i \circ \mathcal{DC}^{(i,1,i-1)}(\bar{\nu}, T) \\
&+ \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} Sym^i \circ \mathcal{DCP}^{(i,0,i-1)}(\bar{\nu}, T, B) \\
&+ (\bar{\nu}_0 \circ TA - C) \mathcal{DICP}^{(i,2,i)}(\bar{\nu}, T, B) \\
&+ (1 - \bar{\nu}_0 \circ B)^{-1} Sym^i \circ \left(\mathcal{DC}^{(i,i,i)}(\bar{\nu}, T)A \right) \\
&+ (1 - \bar{\nu}_0 \circ B)^{-1} Sym^i \circ \left(\mathcal{DC}^{(i,1,i-1)}(\bar{\nu}, T)A \right) \\
&+ (1 - \bar{\nu}_0 \circ B)^{-1} Sym^i \left(\mathcal{DCP}^{(i,0,i-1)}(\bar{\nu}_0, \bar{\nu}, T, A) \right) \\
&- (1 - \bar{\nu}_0 \circ B)^{-1} Sym^i \circ (D^i(C)) \\
&+ (U_3^i)(\bar{\nu}, D\bar{\nu}, \dots, D^i \bar{\nu}).
\end{aligned} \tag{4.184}$$

From Lemma 4.31 we have

$$\begin{aligned}
D^k(\Gamma(\bar{\nu}))(x, y) &= \left((\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \circ \left(\mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)B \right) \right. \\
&+ (\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \circ \left(\mathcal{DC}^{(k,1,(k-1))}(\bar{\nu}, T) \right) \\
&+ (\bar{\nu} \circ TA - C)(1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \circ \left(\mathcal{DCP}^{(k,0,(k-1))}(\bar{\nu}, T, B) \right) \\
&+ (\bar{\nu} \circ TA - C) \mathcal{DICP}^{(k,2,k)}(\bar{\nu}, T, B) \\
&+ (1 - \bar{\nu} \circ TB)^{-1} \text{Sym}^k \circ \left(\mathcal{DC}^{(k,k,k)}(\bar{\nu}, T)A \right) \\
&+ (1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \left(\mathcal{DC}^{(k,1,k-1)}(\bar{\nu}, T)A \right) \\
&+ (1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \circ \left(\mathcal{DCP}^{(k,0,k-1)}(\bar{\nu}, T, A) \right) \\
&- (1 - \bar{\nu} \circ TB)^{-2} \text{Sym}^k \circ (D^k(C)) \\
&+ U_3^k(\bar{\nu}, T, A, B, C).
\end{aligned} \tag{4.185}$$

with the functions $\mathcal{DC}^{(k_1,k_2,k_3)}(\bar{\nu}, T)$, $\mathcal{DCP}^{(k_1,k_2,k_3)}(\bar{\nu}, T, B)$, $\mathcal{DICP}^{(k_1,k_2,k_3)}(\bar{\nu}, T, B)$, $\phi^{(q,k-q)}$ as in Definitions 4.13, 4.19, 4.25, 4.8 respectively and the function $U_3^k(\bar{\nu}, T, A, B, C)$ as in Lemma 4.30.

Thus, it follows from (4.185) and (4.184) that

$$\Psi^i(\bar{\nu}_0, D\bar{\nu}_0, \dots, D^i\bar{\nu}_0) = D^i\Gamma(\bar{\nu}_0). \tag{4.186}$$

This concludes the proof of Proposition. ■

4.2.2 Step 2.2

The goal of this section is to show that the following Proposition:

Proposition 4.49. *Let i be a integer such that $1 \leq i \leq k$. Let $\Psi^j, 1 \leq j \leq i$ be functions as in Definition 4.47. Then the function*

$$\tilde{N}_i : \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i \rightarrow \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i \tag{4.187}$$

given by

$$\tilde{N}_i((\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i) = (\Gamma(\bar{\nu}_0), \Psi^1((\bar{\nu}_0, \bar{\nu}_1), \dots, \Psi^i((\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i))), \tag{4.188}$$

have a global attracting fixed point $(A_0, A_1, \dots, A_i)$.

4.2.2.1 Preliminaries

4.2.2.1.1 Fiber Contraction Theorem In this paragraph we state and prove a important theorem called Fiber Contraction Theorem [HP69] often used to prove that functions obtained as fixed points of contractions are smooth. We will use this technique as one method to prove Proposition 4.63.

Let $(X, d_X), (Y, d_Y)$ be metric spaces. A map $\Upsilon : X \times Y \rightarrow X \times Y$ of the form

$$\Upsilon(x, y) = (\Gamma(x), \Psi(x, y)),$$

where $\Gamma : X \rightarrow X$ and $\psi : X \times Y \rightarrow Y$, called a bundle map over the base Γ with part principal ψ . Here, the triple $(X \times Y, X, \pi)$, where $\pi : X \times Y \rightarrow X$ is the projection $\pi(x, y) = x$, is called the trivial bundle over X with fiber Y .

For notational convenience, let us denote the map $y \rightarrow \Psi(x, y)$ by Ψ_x , and by $d := d_X + d_Y$ the metric the space of $X \times Y$ where with it is space is metric.

Definition 4.50 (Lipschitz constant). *Let $f : X \rightarrow Y$ be a Lipschitz map, then we defined the Lipschitz constant $L(f) = \inf\{k : d_Y(f(x_1), f(x_2)) \leq kd_X(x_1, x_2) \text{ for all } x_1, x_2 \in X\}$. If $X = Y$ and $L(f) < 1$ then f is called contractive with constant of contraction $L(f)$. If f is no Lipschitz then $L(f) = \infty$.*

Theorem 4.51 (Fiber Contraction Theorem I [HP69], [Chi06]). *Let $(X, d_X), (Y, d_Y)$ be metric spaces, and Υ a bundle map over the base $\Gamma : X \rightarrow X$ with principal part $\psi : X \times Y \rightarrow Y$, with the following properties:*

(a) Γ has a globally attractor point fixed point x_∞ , that is,

$$\lim_{n \rightarrow \infty} \Gamma^n(x) = x_\infty, \quad \text{for all } x \in X; \quad (4.189)$$

(b) $\Psi_{x_\infty} = \Psi(x_\infty, \cdot) : Y \rightarrow Y$ has a point fixed y_∞ ;

(c) the function $\Psi(\cdot, y_\infty) : X \rightarrow Y$ is continuous in the point x_∞ ;

(d) $\lim_{n \rightarrow \infty} \sup L(\Psi_{\Gamma^n(x)}) < 1$, for all $x \in X$.

Then (x_∞, y_∞) is a global attracting fixed point of Υ , that is,

$$\lim_{n \rightarrow \infty} \Upsilon^n(x, y) = (x_\infty, y_\infty).$$

Proof. We must prove that for each $(x, y) \in X \times Y$ we have

$$\lim_{n \rightarrow \infty} \Upsilon^n(x, y) = (x_\infty, y_\infty). \quad (4.190)$$

By using the definition of $\limsup$ and hypotheses (d) one can conclude that there exists k_0 so that $\sup_{k \geq k_0} \{L(\Psi_{\Gamma^n(x)})\} < 1$. Hence, we obtain $L(\Psi_{\Gamma^n(x)}) \leq \lambda < 1$, for all $k \geq k_0$. If we set

$$\lambda_1 = \sup_{k \geq k_0} \{L(\Psi_{\Gamma^n(x)})\}, \quad \lambda = \max\{\lambda_1, L(\Psi_{x_\infty})\}$$

and

$$\tilde{X} = \{x_\infty\} \cup \{\Gamma^k(x) : k \geq k_0\}.$$

Then

$$L(\Psi_{\tilde{x}}) \leq \lambda < 1, \text{ for all } \tilde{x} \in \tilde{X}. \quad (4.191)$$

Defining the sequence $(x_n, y_n) = \Upsilon^n(x, y)$, as we used the metric $d := d_X + d_Y$, then, the theorem is proved if we show that $\lim_{n \rightarrow \infty} y_n = y_\infty$.

By induction we have

$$\Upsilon^n(x, y) = (\Gamma^n(x), \Psi_{\Gamma^n(x)} \circ \Psi_{\Gamma^{n-1}(x)} \dots \circ \Psi_x(y)).$$

Hence,

$$\pi_2 \Upsilon^{n+1}(x, y) = \Psi_{\Gamma^n(x)} \circ \Psi_{\Gamma^{n-1}(x)} \dots \circ \Psi_x(y). \quad (4.192)$$

Using Triangle inequality we get

$$d_Y(\pi_2 \Upsilon^{n+1}(x, y_\infty), y_\infty) \leq d_Y(\pi_2 \Upsilon^{n+1}(x, y), \Psi_{\Gamma^n(x)}(y_\infty)) + d_Y(\Psi_{\Gamma^n(x)}(y_\infty), y_\infty). \quad (4.193)$$

Notice that, on account of (4.191) we get

$$d_Y(\pi_2 \Upsilon^{n+1}(x, y), \Psi_{\Gamma^n(x)}(y_\infty)) \leq \lambda d_Y(\pi_2 \Upsilon^n(x, y_\infty), y_\infty). \quad (4.194)$$

Thus, combining (4.193) and (4.194) we get

$$d_Y(\pi_2 \Upsilon^{n+1}(x, y_\infty), y_\infty) \leq \lambda d_Y(\pi_2 \Upsilon^n(x, y_\infty), y_\infty) + d_Y(\Psi_{\Gamma^n(x)}(y_\infty), y_\infty). \quad (4.195)$$

And by using induction one can prove that

$$d_Y(\pi_2 \Upsilon^{n+1}(x, y_\infty), y_\infty) \leq \sum_{j=0}^n \lambda^{n-j} d_Y(\Psi_{\Gamma^j(x)}(y_\infty), y_\infty).$$

Setting $\delta_n := d_Y(\Psi_{\Gamma^n(x)}(y_\infty), y_\infty)$, then

$$\sum_{j=0}^n \lambda^{n-j} d_Y(\Psi_{\Gamma^j(x)}(y_\infty), y_\infty) = \sum_{j=0}^n \lambda^{n-j} \delta_j.$$

We claim that $\lim_{n \rightarrow \infty} \sum_{j=0}^n \lambda^{n-j} \delta_j = 0$, to show this claim, we have the following considerations:

- First show that $\lim_{n \rightarrow \infty} \delta_n = 0$. Indeed, since $\lim_{n \rightarrow \infty} \Gamma^n(x) = x_\infty$ and $\Psi(\cdot, y_\infty)$ is continuous at x_∞ , by the characterization of the continuity by sequences we have

$$\lim_{n \rightarrow \infty} d_Y(\Psi_{\Gamma^n(x)}(y_\infty), \Psi(x_\infty, y_\infty)) = 0.$$

Since y_∞ is a fixed point of Ψ_{x_∞} , then $\Psi_{x_\infty}(y_\infty) = y_\infty$, and so

$$\lim_{n \rightarrow \infty} \delta_n = 0. \quad (4.196)$$

- Since the sequence $\{\delta_n\}_{n=0}^\infty$ converge to zero and is therefore bounded. If A is an upper bound for the element of this sequence, then for any $\delta_n, n \geq 0$, we have $0 \leq \delta_n < A$. Furthermore for any $\epsilon > 0$ there exists k_0 such that $\delta_n < \frac{1-\lambda}{2}\epsilon$, as $n \geq k_0$.
- Furthermore, as $\lambda < 1$, then $\lim_{n \rightarrow \infty} \lambda^n = 0$ and $\lim_{n \rightarrow \infty} \sum_{i=0}^n \lambda^i = \frac{1}{1-\lambda}$. Then for any $\epsilon > 0$ there is k_0 such that $\lambda^n < \frac{1-\lambda}{2A}\epsilon$, as $n \geq k_0$.
- With the consideration above we have that for $n \geq 2k_0$

$$\begin{aligned} \sum_{j=0}^n \lambda^{n-j} \delta_j &= \sum_{j=0}^{k_0-1} \lambda^{n-j} \delta_j + \sum_{j=k_0}^n \lambda^{n-j} \delta_j \\ &= (\lambda^n + \dots + \lambda^{n-k_0+1})A + (1 + \dots + \lambda^{n-k_0}) \sup_{n \geq k_0} \delta_n \\ &= \lambda^{n-k_0} \frac{A}{1-\lambda} + \frac{\epsilon(1-\lambda)}{2} \\ &= \epsilon. \end{aligned} \quad (4.197)$$

Hence, we obtain

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n \lambda^{n-j} \delta_j = 0. \quad (4.198)$$

Therefore, by letting $n \rightarrow \infty$, in view of Eqs. (4.198) and (4.2.2.1.1) we get

$$\lim_{n \rightarrow \infty} d_Y(\pi_2 \Upsilon^{n+1}(x, y_\infty), y_\infty) = 0. \quad (4.199)$$

By applying Triangle inequality we get

$$d_Y(\pi_2 \Upsilon^n(x, y), y_\infty) \leq d_Y(\pi_2 \Psi^n(x, y), \pi_2 \Psi^n(x, y_\infty)) + d_Y(\pi_2 \Upsilon^n(x, y_\infty, y_\infty)). \quad (4.200)$$

But, From Eqs. (4.192) and (4.191) we reach

$$d_Y(\pi_2 \Psi^n(x, y), \pi_2 \Psi^n(x, y_\infty)) \leq \lambda^n d(y, y_\infty). \quad (4.201)$$

Replacing (4.201) into (4.200) we obtain

$$d_Y(\pi_2 \Upsilon^n(x, y), y_\infty) \leq \lambda^n d(y, y_\infty) + d_Y(\pi_2 \Upsilon^n(x, y_\infty, y_\infty)). \quad (4.202)$$

Thus, by letting $n \rightarrow \infty$ on account of Eq. (4.199) we get

$$\lim_{n \rightarrow \infty} d_Y(\pi_2 \Upsilon^n(x, y), y_\infty) = 0. \quad (4.203)$$

The theorem is proved. ■

Theorem 4.52 (Fiber Contraction Theorem II). *Let $(X, d_X), (Y, d_Y)$ be two complete metric spaces, and let $\Upsilon : X \times Y \rightarrow X \times Y$ be a map of the form*

$$\Upsilon(x, y) = (\Gamma(x), \Psi(x, y)).$$

Assume that

(a) Γ has an attracting fixed point x_∞ , that is,

$$\Gamma(x_\infty) = x_\infty, \quad \lim_{n \rightarrow \infty} \Gamma^n(x) = x_\infty, \quad \text{for all } x \in X;$$

(b) the function $\Psi(\cdot, y) : X \rightarrow Y$ is continuous in X for every $y \in Y$;

(c) for every $x \in X$ the map $\Psi_x := \Psi(x, \cdot) : Y \rightarrow Y$ defined by $\Psi_x(y) := \Psi(x, y)$ is a λ -contraction, with $\lambda < 1$. This mean that

$$d_Y(\Psi_x(y_1), \Psi_x(y_2)) \leq \lambda d_Y(y_1, y_2),$$

for all $x \in X$ and $y_1, y_2 \in Y$.

Then if y_∞ denotes the unique fixed point of Ψ_{x_∞} , the point $(x_\infty, y_\infty) \in X \times Y$ is an attracting fixed point of Υ , that is,

$$\lim_{n \rightarrow \infty} \Upsilon^n(x, y) = (x_\infty, y_\infty).$$

Proof. The proof of this Theorem is an immediate consequence of Theorem 4.51. ■

4.2.2.1.2 Perron-Frobenius Theorem for positive matrices Now we state the Perron-Frobenius Theorem for positive matrices.

Theorem 4.53. *Let $A = [a_{i,j}]_{n \times n}$ be a real $n \times n$ positive matrix: $a_{i,j} > 0$, for $1 \leq i, j \leq n$. Then the following statements hold:*

- *A has a positive eigenvalue r which is equal to the spectral radius of A , (r is called the **Perron root**).*
- *r is a simple.*
- *There exists an eigenvector $x > 0$ such that $Ax = rx$ (x is called the **Perron vector**).*
- *The **Perron vector** is the unique vector define by*

$$Ap = rp, p > 0, \text{ and } \|p\|_1 = 1, \text{ where } \|p\|_1 = \sum_{i=1}^n |p_i|,$$

and, except for positive multiples of p , there are no other nonnegative eigenvector for A , regardless of the eigenvalue.

- *r is the only eigenvalue on the spectral circle of A ,*
- *$r = \max_{x \in \mathcal{N}} f(x)$ (**the Collatz-Wielandt formula**),*

$$\text{where } f(x) = \min_{\substack{1 \leq i \leq n \\ x_i \neq 0}} \frac{[Ax]_i}{x_i} \text{ and } \mathcal{N} = \{x : x \geq 0, x \neq 0\}.$$

- *an estimate of r is given by inequalities:*

$$\min_i \sum_j a_{ij} \leq r \leq \max_i \sum_j a_{ij}.$$

Proof. The proof can be found in [Mey01]. ■

4.2.2.1.3 Some Properties of Multilinear Maps Here will be given some elementary properties of Multilinear Maps. Before that, let us fix some notations.

From now on, the set $\{1, \dots, n\}$ will be denoted by $[n]$, $E := \mathbb{R}^n$. If $F := \mathbb{R}$, the set of all the functions $f : [k] \rightarrow \{E, F\}$ will be denoted $\mathcal{F}([k], \{E, F\})$. Notice that the cardinality of $\mathcal{F}([k], \{E, F\})$ is 2^k .

Definition 4.54. Let $f \in \mathcal{F}([k], \{E, F\})$. We define the function $g_{f,E} : [n+1] \rightarrow \{E, F\}$ given by

$$g_{f,E}(i) = \begin{cases} f(i), & \text{if } i \in [n], \\ E, & \text{if } i = n+1, \end{cases}$$

and the set of all this functions will be denoted by $\Omega([n+1], \{E\})$.

Definition 4.55. Let $f \in \mathcal{F}([k], \{E, F\})$. We define the function $g_{f,F} : [n+1] \rightarrow \{E, F\}$ given by

$$g_{f,F}(i) = \begin{cases} f(i), & \text{if } i \in [n], \\ F, & \text{if } i = n+1. \end{cases}$$

and the set of all this functions will be denoted by $\Omega([n+1], \{F\})$.

We will denote by $A \uplus B$ the usual disjoint intersection between sets.

Lemma 4.56. Under the notations above. The following statement holds:

$$(a) \quad \Omega([n+1], \{E\}) \uplus \Omega([n+1], \{F\}) = \mathcal{F}([n+1], \{E, F\}).$$

Proof. The proof it follows immediately from the Definitions (4.55) and (4.54). ■

We have that $\mathbb{R}^{n+1} = E \oplus F$, where $E := \mathbb{R}^n$ and $F := \mathbb{R}$. We define $\pi_E : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ as follows, for $x \in \mathbb{R}^{n+1}$ write $x = e + f$, where $e \in E$ and $f \in F$, then define $\pi_E(x) := e$. By definition one can see that π_E satisfies the following properties:

1. π_E is well defined,
2. π_E is linear,
3. $\text{Im}\pi_E = E$, $\text{Ker}\pi_E = F$
4. $\pi_E \circ \pi_E = \pi_E$.

π_E is called the projection of $\mathbb{R}^{n+1}$ on E along F . In the same manner we can define the function $\pi_F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$.

Let $L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ be the space of all the k -linear map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$.

Let $f \in \mathcal{F}([k], \{E, F\})$. We will denote by

$$L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n) \quad (4.204)$$

the set of all k -linear map b such that

1. $b(\pi_{g(1)}(x_1), \pi_{g(2)}(x_2), \dots, \pi_{g(k)}(x_k)) = 0$, for every $g \in \mathcal{F}([k], \{E, F\})$, $g \neq f$ and for each k -tuple $(x_1, \dots, x_k) \in \underbrace{\mathbb{R}^{n+1} \times \dots \times \mathbb{R}^{n+1}}_{k\text{-times}}$.

Lemma 4.57. *Let $L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ be the space of all the k -linear map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. Then the followings properties hold:*

(i) *For all $b \in L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$. We have*

$$b(x_1, x_2, \dots, x_k) = \sum_{f \in \mathcal{F}([k], \{E, F\})} b(\pi_{f(1)}(x_1), \pi_{f(2)}(x_2), \dots, \pi_{f(k)}(x_k)), \quad (4.205)$$

for every $(x_1, \dots, x_k) \in \underbrace{\mathbb{R}^{n+1} \times \dots \times \mathbb{R}^{n+1}}_{k\text{-times}}$.

The function $(x_1, x_2, \dots, x_k) \rightarrow b(\pi_{f(1)}(x_1), \pi_{f(2)}(x_2), \dots, \pi_{f(k)}(x_k))$ will be denoted by b_f .

(ii) *If $f \in \mathcal{F}([k], \{E, F\})$ and $b \in L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$. Then*

$$b(x_1, x_2, \dots, x_k) = b(\pi_{f(1)}(x_1), \pi_{f(2)}(x_2), \dots, \pi_{f(k)}(x_k)), \quad (4.206)$$

for every $(x_1, \dots, x_k) \in \underbrace{\mathbb{R}^{n+1} \times \dots \times \mathbb{R}^{n+1}}_{k\text{-times}}$.

(iii) *$L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ can be decomposed into a direct sum of 2^k k -linear map, that is,*

$$L^k(\mathbb{R}^{n+1}, \mathbb{R}^n) = \bigoplus_{f \in \mathcal{F}([k], \{E, F\})} L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n), \quad (4.207)$$

where $L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ as in 4.204.

Proof. To prove (i) we will use induction on k . Suppose that the statement 4.205 holds for k , we will prove it for $k+1$. Take $b \in L^{k+1}(\mathbb{R}^{n+1}, \mathbb{R}^n)$. Since $\mathbb{R}^{n+1} := E \oplus F$ and b is $(k+1)$ -linear map. Then, we have

$$b(x_1, x_2, \dots, x_k, x_{k+1}) = b(x_1, x_2, \dots, x_n, e_{k+1}) + b(x_1, x_2, \dots, x_n, f_{k+1}), \quad (4.208)$$

where $x_{k+1} := e_{k+1} + f_{k+1}$, $e_{k+1} \in E$ and $f_{k+1} \in F$. Letting

$$\tilde{b}_1(x_1, x_2, \dots, x_k) := b(x_1, x_2, \dots, x_n, e_{k+1}). \quad (4.209)$$

and

$$\tilde{b}_2(x_1, x_2, \dots, x_k) := b(x_1, x_2, \dots, x_n, f_{k+1}). \quad (4.210)$$

Thus, Eq. (4.208) becomes

$$b(x_1, x_2, \dots, x_k, x_{k+1}) := \tilde{b}_1(x_1, x_2, \dots, x_k) + \tilde{b}_2(x_1, x_2, \dots, x_k). \quad (4.211)$$

Since $\tilde{b}_1$ is k -linear map, by inductive hypothesis we have

$$\tilde{b}_1(x_1, x_2, \dots, x_k) = \sum_{f \in \mathcal{F}([k], \{E, F\})} \tilde{b}_1(\pi_{f(1)}(x_1), \pi_{f(2)}(x_2), \dots, \pi_{f(k)}(x_k)). \quad (4.212)$$

Hence, on account of (4.209) we obtain

$$\tilde{b}_1(x_1, x_2, \dots, x_k) = \sum_{f \in \mathcal{F}([k], \{E, F\})} b(\pi_{f(1)}(x_1), \pi_{f(2)}(x_2), \dots, \pi_{f(k)}(x_k), e_{k+1}). \quad (4.213)$$

But, by using Definition 4.54, E.q 4.213 becomes

$$\tilde{b}_1(x_1, x_2, \dots, x_k) = \sum_{g \in \Omega([k+1], \{E\})} b(\pi_{g(1)}(x_1), \pi_{g(2)}(x_2), \dots, \pi_{g(k)}(x_k), \pi_{g(k+1)}(x_{k+1})). \quad (4.214)$$

We now apply this argument again, with e_{k+1} replaced with f_{k+1} , to obtain

$$\tilde{b}_2(x_1, x_2, \dots, x_k) = \sum_{g \in \Omega([k+1], \{F\})} b(\pi_{g(1)}(x_1), \pi_{g(2)}(x_2), \dots, \pi_{g(k)}(x_k), \pi_{g(k+1)}(x_{k+1})). \quad (4.215)$$

Therefore, by replacing (4.215) and (4.214) into (4.211) we get

$$\begin{aligned} b(x_1, x_2, \dots, x_k, x_{k+1}) &= \sum_{g \in \Omega([k+1], \{E\})} b(\pi_{g(1)}(x_1), \pi_{g(2)}(x_2), \dots, \pi_{g(k)}(x_k), \pi_{g(k+1)}(x_{k+1})) \\ &+ \sum_{g \in \Omega([k+1], \{F\})} b(\pi_{g(1)}(x_1), \pi_{g(2)}(x_2), \dots, \pi_{g(k)}(x_k), \pi_{g(k+1)}(x_{k+1})). \end{aligned} \quad (4.216)$$

Recall that by Lemma 4.56 we have

$$\mathcal{F}([k+1], \{E, F\}) = \Omega([k+1], \{E\}) \uplus \Omega([k+1], \{F\}).$$

Thus, Eq. (4.216) becomes

$$b(x_1, x_2, \dots, x_k, x_{k+1}) = \sum_{g \in \mathcal{F}([k+1], \{E, F\})} b(\pi_{g(1)}(x_1), \pi_{g(2)}(x_2), \dots, \pi_{g(k)}(x_k), \pi_{g(k+1)}(x_{k+1})). \quad (4.217)$$

Hence, we conclude that Eq. (4.205) holds for $k+1$. This finishes the proof of (i)

To prove (ii). Take $b \in L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$. Since b is k -linear, by (i) we have

$$b(x_1, x_2, \dots, x_k, x_{k+1}) = \sum_{g \in \mathcal{F}([k+1], \{E, F\})} b(\pi_{g(1)}(x_1), \pi_{g(2)}(x_2), \dots, \pi_{g(k)}(x_k), \pi_{g(k+1)}(x_{k+1})). \quad (4.218)$$

But, by Definition 4.204 we have

$$b(\pi_{g(1)}(x_1), \pi_{g(2)}(x_2), \dots, \pi_{g(k)}(x_k)) = 0, \quad \text{for every } g \in \mathcal{F}([k], \{E, F\}), g \neq f. \quad (4.219)$$

Therefore, combining (4.219) and (4.218) we get

$$b(x_1, x_2, \dots, x_k, x_{k+1}) = b(\pi_{g(1)}(x_1), \pi_{g(2)}(x_2), \dots, \pi_{g(k)}(x_k), \pi_{g(k+1)}(x_{k+1})). \quad (4.220)$$

The proof of (iii) it follows immediately from (i) and (ii). This concludes the proof of lemma. ■

4.2.2.2 Proof of Proposition 4.63

In order to prove the Proposition 4.63 we begin state and prove the following proposition.

Proposition 4.58. *Under the notation of Definitions 4.43 and 4.47. Let i be a integer such that $1 \leq i \leq k$, fix a point $(\bar{\nu}_0, \dots, \bar{\nu}_{i-1}) \in \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i$. Then, the space $\mathcal{D}_i$ can be endowed with a norm $|\cdot|_{i,D}$ equivalent to the original norm $\|\cdot\|_D$ so that the function*

$$\Psi^i(\bar{\nu}_0, \dots, \bar{\nu}_{i-1}, \bullet) : \mathcal{D}_i \rightarrow \mathcal{D}_i$$

it is a contraction with constant of contraction independent of the point $(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_{i-1})$.

The proof of the Proposition 4.58 will be given with some lemmas. From now on, we denote

$$\widehat{DT}(x, y) := \begin{bmatrix} A(x, y) & B(x, y) \\ C(x, y) & 1 \end{bmatrix}_{(n+1) \times (n+1)}, \quad (4.221)$$

where the functions $A(x, y)$, $B(x, y)$ and $C(x, y)$ are as in Definition 3.5.

Lemma 4.59. *Let $L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ be the space of all the i -linear map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. Then $L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ can be endowed with a norm $|\cdot|_i$ equivalent to $\|\cdot\|$ such that the map*

$$M^i : L^i(\mathbb{R}^{n+1}, \mathbb{R}^n) \rightarrow L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$$

defined by

$$M^i(b)(x_1, \dots, x_i) = b(\widehat{DT}x_1, \dots, \widehat{DT}x_i) \quad (4.222)$$

satisfies the following relation:

$$\frac{|M^i(b)|_i}{|b|_i} \leq \max_{\substack{m, n \in \mathbb{N} \\ m+n=i}} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}. \quad (4.223)$$

Proof of Lemma 4.59. Through of the proof, we deal with the case that $\|B\|_D$ is different from zero, the other the case is similar.

We will endow $L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ a new norm $|\cdot|_i$ in the following way:

We set

$$c_{g,f} := \|\pi_{g(1)} \widehat{DT} \pi_{f(1)}\| \dots \|\pi_{g(i)} \widehat{DT} \pi_{f(i)}\|, \quad (4.224)$$

where g and $f \in \mathcal{F}([i], \{E, F\})$ while

$$\pi_{g(j)} \widehat{DT} \pi_{f(j)} := \begin{cases} A, & \text{if } g(j) = E \text{ and } f(j) = E, \\ B, & \text{if } g(j) = E \text{ and } f(j) = F, \\ C, & \text{if } g(j) = F \text{ and } f(j) = E, \\ 1, & \text{if } g(j) = F \text{ and } f(j) = F. \end{cases} \quad (4.225)$$

Next, consider the matrix

$$\Delta := [c_{g,f}]_{2^i \times 2^i}. \quad (4.226)$$

Notice that since, by assumption $\|A\|_D$, $\|B\|_D$ and $\|C\|_D$ are different of zero, then, in view of (4.224) and (4.225) it follows that

$$c_{g,f} > 0, \quad \text{for } g, f \in \mathcal{F}([i], \{E, F\}). \quad (4.227)$$

Thus, the matrix Δ is positive.

In addition, by Perron- Frobenius theorem. Theorem 4.53 applied to matrix Δ , we get the following properties:

- (a) The matrix Δ has a positive eigenvalue λ .
- (b) The matrix Δ has an eigenvector V with entries k_f such that

$$\sum_{f \in \mathcal{F}([i], \{E, F\})} k_f = 1. \quad (4.228)$$

- (c) An estimate of λ is given by inequalities

$$\min_g \sum_f c_{g,f} \leq \lambda \leq \max_g \sum_f c_{g,f}. \quad (4.229)$$

Let $b \in L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ be a map different from zero, then in view of Lemma 4.205(i) we can write

$$b = \sum_{f \in \mathcal{F}([i], \{E, F\})} b_f, \quad (4.230)$$

where b_f as in Definition 4.204. Thus, we define the norm $|\cdot|_i$ on $L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ by

$$|b|_i := \sum_{f \in \mathcal{F}([i], \{E, F\})} k_f \|b_f\|. \quad (4.231)$$

It is easily to seen that $|\cdot|_i$ is a norm on $L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ equivalent to the norm $\|\cdot\|$.

Next, we will prove that

$$\frac{|M^i(b)|_i}{|b|_i} \leq \max_{\substack{m, n \in \mathbb{N} \\ m+n=i}} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}. \quad (4.232)$$

Indeed, by definition one has $M^i(b)$ is i -linear map, then on account of Lemma 4.205(iii) we can write

$$M^i(b) = \sum_{f \in \mathcal{F}([i], \{E, F\})} M^i(b)_f. \quad (4.233)$$

where $M^i(b)_f$ as in Definition 4.204.

Hence, in view of (4.231) we have

$$|M^i(b)|_i := \sum_{f \in \mathcal{F}([i], \{E, F\})} k_f \|M^i(b)_f\|. \quad (4.234)$$

But, by using Lemma 4.205(ii) we have

$$M^i(b)_f(x_1, \dots, x_i) = M^i(b)(\pi_{f(1)}(x_1), \dots, \pi_{f(i)}(x_i)). \quad (4.235)$$

and by assumption (4.222) we have

$$M^i(b)(x_1, \dots, x_i) = b(\widehat{DT}x_1, \dots, \widehat{DT}x_i). \quad (4.236)$$

Thus, combining (4.236) and (4.235) we get

$$M^i(b)_f(x_1, \dots, x_i) = b(\widehat{DT}\pi_{f(1)}x_1, \dots, \widehat{DT}\pi_{f(i)}x_i). \quad (4.237)$$

Furthermore, by using Lemma 4.205(i) we can write

$$b(\widehat{DT}\pi_{f(1)}, \dots, \widehat{DT}\pi_{f(i)}) = \sum_{g \in \mathcal{F}([i], \{E, F\})} b_g(\pi_{g(1)}\widehat{DT}\pi_{f(1)}, \dots, \pi_{g(i)}\widehat{DT}\pi_{f(i)}). \quad (4.238)$$

Therefore, it follows from (4.238) and (4.237), that

$$M^i(b)_f(x_1, \dots, x_i) = \sum_{g \in \mathcal{F}([i], \{E, F\})} b_g(\pi_{g(1)}\widehat{DT}\pi_{f(1)}(x_1), \dots, \pi_{g(i)}\widehat{DT}\pi_{f(i)}(x_i)). \quad (4.239)$$

Hence, on account of (4.234) we get

$$\begin{aligned} |M^i(b)|_i &= \sum_{f \in \mathcal{F}([i], \{E, F\})} k_f \left\| \sum_{g \in \mathcal{F}([i], \{E, F\})} b_g(\pi_{g(1)}\widehat{DT}\pi_{f(1)}, \dots, \pi_{g(i)}\widehat{DT}\pi_{f(i)}) \right\| \\ &\leq \sum_{f \in \mathcal{F}([i], \{E, F\})} k_f \sum_{g \in \mathcal{F}([i], \{E, F\})} \|b_g(\pi_{g(1)}\widehat{DT}\pi_{f(1)}, \dots, \pi_{g(i)}\widehat{DT}\pi_{f(i)})\|. \end{aligned} \quad (4.240)$$

But, since b_g is i -linear map, then we have

$$\|b_g(\pi_{g(1)}\widehat{DT}\pi_{f(1)}, \dots, \pi_{g(i)}\widehat{DT}\pi_{f(i)})\| \leq \underbrace{\|\pi_{g(1)}\widehat{DT}\pi_{f(1)}\| \dots \|\pi_{g(i)}\widehat{DT}\pi_{f(i)}\|}_{:=c_{g,f}} \cdot \|b_g\|. \quad (4.241)$$

Therefore, Eq. (4.240) becomes

$$\begin{aligned} |M^i(b)|_i &\leq \sum_{f \in \mathcal{F}([i], \{E, F\})} k_f \sum_{g \in \mathcal{F}([i], \{E, F\})} c_{g,f} \|b_g\| \\ &\leq \sum_{g \in \mathcal{F}([i], \{E, F\})} \|b_g\| \sum_{f \in \mathcal{F}([i], \{E, F\})} c_{g,f} k_f. \end{aligned} \quad (4.242)$$

Moreover, since $V = [k_f]_{f \in \mathcal{F}([i], \{E, F\})}$ is a eigenvector of matrix Δ then we have

$$\Delta[k_f] = \lambda[k_g], \quad (4.243)$$

where $\Delta = [c_{g,f}]$ while g and $f \in \mathcal{F}([i], \{E, F\})$. Hence, it is easily seen that

$$\sum_{g \in \mathcal{F}([i], \{E, F\})} c_{g,f} k_f = \lambda k_g. \quad (4.244)$$

Then, by replacing (4.244) into (4.242) we get

$$\begin{aligned} |M^i(b)|_i &\leq \sum_{g \in \mathcal{F}([i], \{E, F\})} \|b_g\| \lambda k_g \\ &\leq \lambda \sum_{g \in \mathcal{F}([i], \{E, F\})} \|b_g\| k_g. \end{aligned} \quad (4.245)$$

Furthermore, by definition we can write

$$|b|_i = \sum_{g \in \mathcal{F}([i], \{E, F\})} \|b_g\| k_g. \quad (4.246)$$

Thus, from (4.246) and (4.245) one obtains

$$|M^i(b)|_i \leq \lambda |b|_i. \quad (4.247)$$

Through the remainder of the proof, we denote by $\#(S)$ the cardinality of the set S .

Claim 4.60. *Let f and $g \in \mathcal{F}([i], \{E, F\})$ such that $\#(g^{-1}(E)) = m$ and $\#(g^{-1}(F)) = n$. Then the following equality holds:*

$$\sum_{f \in \mathcal{F}([i], \{E, F\})} c_{g,f} = (\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n, \quad (4.248)$$

where $c_{g,f}$ as in Eq. (4.224).

To prove Claim . Since $\#(g^{-1}(E)) = m$ and $\#(g^{-1}(F)) = n$, then one can consider

$$g^{-1}(E) := \{a_1, a_2, \dots, a_m\}, \quad (4.249)$$

$$g^{-1}(F) := \{b_1, b_2, \dots, b_n\}. \quad (4.250)$$

Thus, by definition we have

$$\pi_{g(a_i)} \widehat{DT} \pi_{f(a_i)} := \begin{cases} A, & \text{if } f(a_i) = E, \\ B, & \text{if } f(a_i) = F, \end{cases} \quad (4.251)$$

and

$$\pi_{g(b_i)} \widehat{DT} \pi_{f(b_i)} := \begin{cases} C, & \text{if } f(b_i) = E, \\ 1, & \text{if } f(b_i) = F. \end{cases} \quad (4.252)$$

Now, consider integers s, t such that $0 \leq s \leq m$, $0 \leq t \leq n$ and fix $f \in \mathcal{F}([i], \{E, F\})$ so that

$$\#(g^-(E) \cap f^{-1}(E)) = s \quad \text{and} \quad \#(g^-(F) \cap f^{-1}(E)) = t.$$

Then, since

$$c_{g,f} := \|\pi_{g(1)} \widehat{DT} \pi_{f(1)}\| \dots \|\pi_{g(i)} \widehat{DT} \pi_{f(i)}\|, \quad (4.253)$$

it follows from (4.252) and (4.251) that

$$c_{g,f} = \|A\|_D^s \|B\|_D^{m-s} \|C\|_D^t 1^{n-t}. \quad (4.254)$$

In addition, since $\#g^-(E) = m$ and $\#g^-(F) = n$, it is not difficult to see that the cardinality of the sets

$$\mathcal{F}_{g,s}([i], \{E, F\}) := \{f \in \mathcal{F}([i], \{E, F\}) : \text{card}(g^-(E) \cap f^{-1}(E)) = s\} \quad (4.255)$$

and

$$\mathcal{F}_{g,t}([i], \{E, F\}) := \{f \in \mathcal{F}([i], \{E, F\}) : \text{card}(g^-(F) \cap f^{-1}(E)) = t\} \quad (4.256)$$

are $\binom{n}{t}$ and $\binom{m}{s}$ respectively

Thus, from (4.256) and (4.255), on account of Rule Product we deduce that the cardinality of

$$\mathcal{F}_{g,st}([i], \{E, F\}) := \mathcal{F}_{g,s}([i], \{E, F\}) \cap \mathcal{F}_{g,t}([i], \{E, F\}) \quad (4.257)$$

is $\binom{m}{s} \cdot \binom{n}{t}$.

Finally, notice that

$$\mathcal{F}([i], \{E, F\}) = \biguplus_{\substack{0 \leq s \leq m \\ 0 \leq t \leq n}} \mathcal{F}_{g,st}([i], \{E, F\}). \quad (4.258)$$

Whence, on account of (4.254) and Binomial Theorem we get the following chain of equalities

$$\begin{aligned}
 \sum_{f \in \mathcal{F}([i], \{E, F\})} c_{g,f} &= \sum_{s=0}^m \sum_{t=0}^n \left(\sum_{f \in \mathcal{F}_{g,st}([i], \{E, F\})} c_{g,f} \right) \\
 &= \sum_{s=0}^m \sum_{t=0}^n \binom{m}{s} \cdot \binom{n}{t} \|A\|_D^s \|B\|_D^{m-s} \|C\|_D^t 1^{n-t}. \\
 &= (\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n.
 \end{aligned} \tag{4.259}$$

Thus the Claim 4.60 is proved.

To finish the proof of lemma. Recall that, from (4.229) we have

$$\min_g \sum_f c_{g,f} \leq \lambda \leq \max_g \sum_f c_{g,f}. \tag{4.260}$$

Hence, on account from (4.259) and (4.247) we have

$$\frac{|M^i(b)|_i}{|b|_i} \leq \max_{\substack{m, n \in \mathbb{N} \\ m+n=i}} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}, \tag{4.261}$$

for all $b \in L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$.

This finishes the proof of lemma. ■

Now, we are going to proof the Proposition 4.58 mentioned in the beginning of the section, which we recall here.

Theorem 4.61. *Under the notation of Definitions 4.43 and 4.47. Let i be a integer such that $1 \leq i \leq k$, fix a point $(\bar{\nu}_0, \dots, \bar{\nu}_{i-1}) \in \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i$. Then, the space $\mathcal{D}_i$ can be endowed with a norm $|\cdot|_{i,D}$ equivalent to the original norm $\|\cdot\|_D$ so that the function*

$$\Psi^i(\bar{\nu}_0, \dots, \bar{\nu}_{i-1}, \bullet) : \mathcal{D}_i \rightarrow \mathcal{D}_i$$

it is a contraction with constant of contraction independent of the point $(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_{i-1})$.

Proof of Theorem 4.58. Consider $\nu_i \in \mathcal{D}_i$. We define the norm $|\cdot|_{i,D}$ of the map

$$\nu_i : D \rightarrow L_s^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$$

by

$$|\nu_i|_{i,D} := \sup\{|\nu_i(x, y)|_i : (x, y) \in D\}, \tag{4.262}$$

where $|\cdot|_i$ is the norm $|\cdot|_i$ on $L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ as in Lemma 4.59. It is easy to check that $|\cdot|_{i,D}$ is a norm on $\mathcal{D}_i$ equivalent to $\|\cdot\|_D$.

Take $\nu_i^1 = \Psi^i(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_{i-1}, \mu^1)$ and $\nu_i^2 = \Psi^i(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_{i-1}, \mu^2)$, where $\mu^1, \mu^2 \in \mathcal{D}_i$. Recall that, by Definition 4.47 we have

$$\begin{aligned}
\Psi^i(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_{i-1}, \mu) &= \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} \text{Sym}^i \circ \mathcal{DC}^{(i,i,i)}(\mu, T)B \\
&+ \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} \text{Sym}^i \circ \mathcal{DC}^{(i,1,(i-1))}(\bar{\nu}_1, \dots, \bar{\nu}_{i-1}, T) \\
&+ \frac{(\bar{\nu}_0 \circ TA - C)i!}{(1 - \bar{\nu}_0 \circ TB)} \text{Sym}^i \circ \mathcal{DCP}^{(i,0,(i-1))}(\bar{\nu}_0, \dots, \bar{\nu}_{i-1}, T, B) \\
&+ (\bar{\nu}_0 \circ TA - C) \mathcal{DICP}^{(i,2,i)}(\bar{\nu}_0, \dots, \bar{\nu}_{i-1}, T, B) \\
&+ (1 - \bar{\nu}_0 \circ B)^{-1} \text{Sym}^i \circ \left(\mathcal{DC}^{(i,i,i)}(\mu, T)A \right) \\
&+ (1 - \bar{\nu}_0 \circ B)^{-1} \text{Sym}^i \circ \left(\mathcal{DC}^{(i,1,i-1)}(\bar{\nu}_1, \dots, \bar{\nu}_{i-1}, T)A \right) \\
&+ (1 - \bar{\nu}_0 \circ B)^{-1} \text{Sym}^i \left(\mathcal{DCP}^{(i,0,i-1)}(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_{i-1}, T, A) \right) \\
&- (1 - \bar{\nu}_0 \circ B)^{-1} \text{Sym}^i \circ (D^i(C)) \\
&+ (U_3^i)(\bar{\nu}_0, \dots, \bar{\nu}_{i-1}, \mu). \tag{4.263}
\end{aligned}$$

Hence, since the point $(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_{i-1})$ is fixed and just the first and fifth term depend of μ , then one can deduce that

$$\begin{aligned}
\nu_i^1 - \nu_i^2 &= (\bar{\nu}_0 \circ TA - C)i!(1 - \bar{\nu}_0 \circ TB)^{-2} \cdot \mathcal{DC}^{(i,i,i)}((\bar{\nu}_i^1 - \bar{\nu}_i^2), T)B \\
&+ (1 - \bar{\nu}_0 \circ B)^{-1} \mathcal{DC}^{(i,i,i)}((\bar{\nu}_i^1 - \bar{\nu}_i^2), T)A. \tag{4.264}
\end{aligned}$$

Recall that, by Eq. 4.18 we have

$$\mathcal{DC}^{(i,i,i)}((\bar{\nu}_i^1 - \bar{\nu}_i^2), T)(x, y) := i! \partial_y G(x, y)(\bar{\nu}_i^1 - \bar{\nu}_i^2) \circ T(x, y) \underbrace{\widehat{DT}(x, y) \dots \widehat{DT}(x, y)}_{k\text{-times}}, \tag{4.265}$$

where $\widehat{DT}(x, y)$ as in Eq. (4.221). Hence, in view of (4.264), (4.262) and Lemma 4.59 we get

$$\begin{aligned}
|\nu_i^1 - \nu_i^2|_{i,D} &\leq |\nu_i^1 - \nu_i^2|_{i,D} \frac{(L\|A\|_D + \|C\|_D)\|B\|_D}{\|\partial_y G(x, y)\|^{-i}(1 - L\|B\|_D)^2} (i!)^2 \Lambda(i) \\
&+ |(\bar{\nu}_i^1 - \bar{\nu}_i^2)|_{i,D} \frac{(\|A\|_D)(1 - L\|B\|_D)}{\|\partial_y G(x, y)\|^{-i}(1 - L\|B\|_D)^2} (i!)^2 \Lambda(i) \\
&= (i!)^2 |(\bar{\nu}_i^1 - \bar{\nu}_i^2)|_D \frac{\|A\|_D + \|C\|_D\|B\|_D}{\|\partial_y G(x, y)\|^{-i}(1 - L\|B\|_D)^2} \Lambda(i), \tag{4.266}
\end{aligned}$$

where

$$\Lambda(i) := \max_{\substack{m, n \in \mathbb{N} \\ m+n=i}} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}. \quad (4.267)$$

But, since $2\|B\|_D L := 1 - \|A\|_D - \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D\|C\|_D}$, then Eq. (4.266) becomes

$$|\nu_i^1 - \nu_i^2|_{i,D} \leq |(\bar{\nu}_i^1 - \bar{\nu}_i^2)|_{i,D} \Theta(i), \quad (4.268)$$

where

$$\Theta(i) := \frac{(\|A\|_D + \|C\|_D\|B\|_D) \max_{m+n=i} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}}{(2i!)^{-2} \|\partial_y G\|^{-i} \left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D\|C\|_D}\right)^2}. \quad (4.269)$$

Claim 4.62. *The following inequality holds:*

$$\Theta(i) < 1, \quad 1 \leq i \leq k. \quad (4.270)$$

Indeed, if $i = 1$ then it is a direct consequence of Assumption 3.6(L_3).

If $i \geq 2$, then $k \geq i \geq 2$, thus by Assumption 3.6(L_3) we have that

$$\Theta(k) < 1, \quad (4.271)$$

and

$$\|\partial_y G\|_D \geq \frac{1}{4} \quad \text{or} \quad \|\partial_x F(x, y)\| \geq \frac{1}{4}. \quad (4.272)$$

Hence, we obtain

$$(2j!)^2 (\|\partial_y G\|_D)^j \leq (2(j+1)!)^2 (\|\partial_y G\|_D)^{j+1}, \quad 2 \leq j < k. \quad (4.273)$$

Notice that, since $(\|C\|_D + 1) \geq 1$, then, we have

$$\max_{m+n=j} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\} \leq \max_{m+n=j+1} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}. \quad (4.274)$$

Therefore, it follows from (4.274) and (4.273) that

$$\Theta(j) \leq \Theta(j+1), \quad 2 \leq j < k. \quad (4.275)$$

This finishes the Claim 4.62. Then, on account of Claim 4.62 and Eq. (4.268) one obtains that the function

$$\Psi^i(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_{i-1}, \bullet) : \mathcal{D}_i \rightarrow \mathcal{D}_i$$

is a contraction independent of the point $(\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_{i-1})$, which finishes the proof. $\blacksquare$

Before proceeding to state and prove the following proposition, it is convenient to introduce some useful notation. Consider the following norm-spaces $X_1, \dots, X_n$ with norm $\|\cdot\|_i$, for $0 \leq i \leq k$ respectively and let $X := X_1 \times \dots \times X_n$. Then the norm of the space X will be denoted $\|\cdot\|_X$ and defined by $\|\cdot\|_X := \max\{\|\cdot\|_i : 1 \leq i \leq n\}$.

Proposition 4.63. *Under Definition 4.47. Let i be a integer such that $0 \leq i \leq k$. Suppose that the sets $\mathcal{D}_j$, for $1 \leq j \leq i$ are endowed with the norm $|\cdot|_{j,D}$ of Proposition 4.58 and the set $X_i := \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i$ is endowed with the norm $|\cdot|_{X_i}$. Then, the function*

$$\tilde{N}_i : X_i \rightarrow X_i$$

defined by

$$\tilde{N}_i((\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i) = (\Gamma(\bar{\nu}_0), \Psi^1((\bar{\nu}_0, \bar{\nu}_1), \dots, \Psi^i((\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_i))) \quad (4.276)$$

have a global attracting fixed point $(A_0, A_1, \dots, A_i)$.

Proof. We proceed by induction on i . Suppose that statement holds for j with $0 \leq j < i$. We wish to show the statement holds for i . To do this we will use the Fiber contraction theorem. By definition of $\tilde{N}_i$, it follows that $\tilde{N}_i = (\tilde{N}_{i-1}, \Psi^i)$. We must now verify that the function $\tilde{N}_i : X_i \times Y \rightarrow X \times Y$, where $X_{i-1} = \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_{i-1}$ and $Y = \mathcal{D}_i$, satisfies the three conditions of Fiber contraction theorem. Theorem 4.52.

- (a) By inductive hypothesis $\tilde{N}_{i-1} : \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_{i-1} \rightarrow \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_i$ have a global attracting fixed point $(A_0, \dots, A_{i-1}) \in \mathcal{D}_0 \times \mathcal{D}_1 \times \dots \times \mathcal{D}_{i-1}$.
- (b) By using Theorem 4.58 applied to $(A_0, \dots, A_{i-1})$, we have

$$\Psi^i(A_0, \dots, A_{i-1}, \bullet) : \mathcal{D}_i \rightarrow \mathcal{D}_i$$

is a contraction. Then by the Banach fixed-point theorem $\Psi^i(A_0, \dots, A_{i-1}, \bullet)$ have a attracting fixed point A_i .

- (c) It is easily seen that $\Psi^i(\cdot, A_i) : X \rightarrow Y$ is continuous (for more details see Eq. (4.183)).

Therefore, from (a), (b), (c), we deduce that $\tilde{N}_i$ satisfies the three conditions of Fiber contraction theorem. Then we conclude that there exists a global attracting fixed point $(A_0, A_1, \dots, A_i)$ to the function $\tilde{N}_i$, which completes the proof. $\blacksquare$

Lemma 4.64. *Under the assumptions of Proposition 4.2. Given integers $n \in \mathbb{N}$, and i with $0 \leq i \leq k$, and a function $\bar{\nu} \in \mathcal{A}_L$ of class C^k . Then the following relation holds:*

$$\tilde{N}_i^n(\bar{\nu}, D\bar{\nu}, \dots, D^i\bar{\nu}) = (\Gamma^n(\bar{\nu}), D(\Gamma^n(\bar{\nu})), \dots, D^i(\Gamma^n(\bar{\nu}))). \quad (4.277)$$

Proof. Fix i . Then, we will proceed by induction on n .

For $n = 1$. Since the function $\bar{\nu} \in \mathcal{A}_L$ is of class C^i , then by Proposition 4.2 we have $D^j\bar{\nu} \in \mathcal{D}_j$, for very $0 \leq j \leq i$. Hence, on account of Eq.(4.276) one can deduce

$$\tilde{N}_i((\bar{\nu}, D(\bar{\nu}), \dots, D^i\bar{\nu})) = (\Gamma(\bar{\nu}), \Psi^1(\bar{\nu}, D(\bar{\nu})), \bar{\nu}, \dots, \Psi^i((\bar{\nu}, D^1(\bar{\nu}), \dots, D^i\bar{\nu})). \quad (4.278)$$

and recall that by using Proposition 4.47 we have

$$\Psi^i(\bar{\nu}_0, D(\bar{\nu}_0), \dots, D^i(\bar{\nu}_0)) = D^i(\Gamma(\bar{\nu}_0)). \quad (4.279)$$

Thus, combining (4.278) and (4.279) we get

$$\tilde{N}_i(\bar{\nu}, D\bar{\nu}, \dots, D^i\bar{\nu}) = (\Gamma(\bar{\nu}), D(\Gamma^n(\bar{\nu})), \dots, D^i(\Gamma^n(\bar{\nu}))). \quad (4.280)$$

For $n + 1$. For that, it is just using inductive hypothesis and Eq. (4.280) to get

$$\tilde{N}_i^{n+1} = (\Gamma^{n+1}(\bar{\nu}), D(\Gamma^{n+1}(\bar{\nu})), \dots, D^i(\Gamma^{n+1}(\bar{\nu}))),$$

which completes the proof by induction, this concludes the proof of the lemma. ■

We are going to proof our main proposition. Proposition 3.20, which we recall here.

Proposition 4.65. *The attracting fixed point ν^* of the operator $\Gamma : \mathcal{A}_L \rightarrow \mathcal{A}_L$ is of class C^k .*

Proof the Main Proposition 3.20. Choose a function $\bar{\nu} \in \mathcal{A}_L$ of class C^k . Then, since ν^* is attracting fixed point of Γ , we have

$$\lim_{n \rightarrow \infty} \Gamma^n(\bar{\nu}) = \nu^*. \quad (4.281)$$

Hence, setting $f_n := \Gamma^n(\bar{\nu})$ we get

$$\lim_{n \rightarrow \infty} f_n(a) = \nu^*(a), \quad a \in D. \quad (4.282)$$

Moreover, on account of Lemma 4.64 and Proposition 4.63 one obtains

$$\lim_{n \rightarrow \infty} (\Gamma^n(\bar{\nu}), D(\Gamma^n(\bar{\nu})), \dots, D^k(\Gamma^n(\bar{\nu}))) = (\nu, A_1, A_2, \dots, A_k), \quad (4.283)$$

where $A_j \in \mathcal{D}_j$, for all $1 \leq j \leq k$.

Therefore, in view of (4.283) we obtain

$$\lim_{n \rightarrow \infty} D^j \Gamma^n(\bar{\nu}) = A_j, \quad (4.284)$$

for $1 \leq j \leq k$.

Thus, from Eqs. (4.282) and (4.284) one can deduce that the sequence $f_n = \Gamma^n(\bar{\nu})$ satisfies the assumptions of Interchanging the order of differentiation and limit. Proposition 3.25. Then, we have

$$D^j(\lim_{n \rightarrow \infty} \Gamma^n(\bar{\nu})) = A_j, \quad (4.285)$$

for $0 \leq j \leq k$.

Therefore, it follows from (4.285) and (4.281) that

$$D^j(\nu) = A_j, \quad (4.286)$$

for $0 \leq j \leq k$.

Hence, since A_j is a continuous function, it follows that the function ν^* is of class C^k , which concludes the proof of our main proposition. ■

Illustrative Examples

To illustrate our main result, in this chapter we present two examples of Lorenz-type maps that satisfy the assumptions of Main Theorem 3.7. These ones were motivated by [AP87, Theorem 11].

From now on we consider $\mathbb{R}^n$ with the sup-norm, that is,

$$\|x\| := \|x\|_\infty = \max\{|x_i|, 1 \leq i \leq n\}.$$

Lemma 5.1. *Consider $A = [a_{ij}] \in K^{m \times n}$ a matrix $m \times n$. Then*

$$\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|. \quad (5.1)$$

Proof. The proof can be found in the book [Mey01]. ■

Example 5.2. *Under the notation used in the Chapter 3. Let $\tilde{T} : D \setminus D_0 \rightarrow D$ be a map defined by $\tilde{T}(x, y) = (\tilde{F}(x, y), \tilde{G}(x, y))$, such that the functions $\tilde{F} = (\tilde{F}_1, \dots, \tilde{F}_n)$, $\tilde{G}$ are given by the equalities*

$$\tilde{F}_j(x, y) = ((-\beta_j|y|^{\omega_0} + \beta_j x_j |y|^{\omega_j} + 1)) \operatorname{sgn} y, \quad (5.2)$$

$$\tilde{G}(x, y) = (|y|^{\omega_0}(1 + \alpha) - \alpha) \operatorname{sgn} y, \quad (5.3)$$

where $\operatorname{sgn}(y) = \frac{y}{|y|}$ denotes the sign of y .

Suppose that the numbers $\alpha, \beta_j, \omega_0, \omega_j$, satisfy the following relations:

$$\beta_j < \min \left\{ \frac{1}{(k!)^2 2^{k+1}}, \frac{1}{2(\omega_0 + \omega_j)} \right\}, \quad (5.4)$$

$$\omega_j - \omega_0 > (k-1), \quad j = 1, \dots, n, \quad (1 + \alpha)^{-1} < \omega_0 < 1, \quad 0 < \alpha < 1. \quad (5.5)$$

Then the following statements hold:

$$(a) \quad 1 - \|A\|_D > 2\sqrt{\|B\|_D \|C\|_D}.$$

$$(b) \quad 1 \leq \|\partial_y \tilde{G}\|_D \leq 2,$$

$$(c) \quad \frac{4(k!)^2 (\|A\|_D + \|C\|_D \|B\|_D) \|\partial_y \max_{m+n=k} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}\|}{\|\tilde{G}(x, y)\|^{-k} \left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D \|C\|_D}\right)^2} < 1.$$

Proof. (a) By Eq. (5.4) we have

$$\tilde{G}(x, y) = ((1 + \alpha)|y|^{\omega_0} - \alpha) \operatorname{sgn}(y), \quad (5.6)$$

and hence by calculating the partial derivative of $\tilde{G}(x, y)$ with respect to y we get

$$\partial_y \tilde{G}(x, y) = \operatorname{sgn}(y)(1 + \alpha)\omega_0 |y|^{\omega_0-1}. \quad (5.7)$$

Then, it follows from Eq (5.7), assumptions (5.5) and (5.4) that

$$1 \leq \|\partial_y \tilde{G}\|_D \leq 2. \quad (5.8)$$

This completes the proof of part (a).

(b) By Eq. (5.5) we have

$$\tilde{F}_j(x, y) = ((-\beta_j |y|^{\omega_0} + \beta_j x_j |y|^{\omega_j} + 1)) \operatorname{sgn}(y). \quad (5.9)$$

Here, calculating the partial derivative of $\tilde{F}_j(x, y)$ with respect to x_j we get

$$\partial_{x_j} \tilde{F}_j(x, y) = \beta_j |y|^{\omega_j} \operatorname{sgn}(y), \quad \text{for } 1 \leq j \leq n. \quad (5.10)$$

By Definition 3.5 (Eq. (3.4)) we have

$$A(x, y) := \frac{\partial_x \tilde{F}_x(x, y)}{\partial_y \tilde{G}(x, y)}. \quad (5.11)$$

Then, by replacing (5.10) and (5.7) into (5.11) we get

$$A(x, y) = \begin{bmatrix} \frac{\beta_1 |y|^{\omega_1 - \omega_0 + 1}}{(1+\alpha)\omega_0} & 0 & \cdots & 0 \\ 0 & \frac{\beta_2 |y|^{\omega_2 - \omega_0 + 1}}{(1+\alpha)\omega_0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\beta_n |y|^{\omega_n - \omega_0 + 1}}{(1+\alpha)\omega_0} \end{bmatrix}_{n \times n}. \quad (5.12)$$

Hence, since $\|\cdot\|$ is the sup-norm, it follows from Lemma 5.1 that

$$\|A(x, y)\| = \max_{1 \leq j \leq n} \left\{ \frac{\beta_j |y|^{\omega_j - \omega_0 + 1}}{(1+\alpha)\omega_0} \right\}. \quad (5.13)$$

Then, from Eqs. (5.4) and (5.5) we obtain that

$$\|A(x, y)\| < \frac{1}{(k!)^2 2^k} < \frac{1}{2}. \quad (5.14)$$

Since $\|A\|_D = \sup_{(x,y) \in D_0} \|A(x, y)\|$, it follows from (5.14) that

$$\|A\|_D < \frac{1}{(k!)^2 2^k} < \frac{1}{2}. \quad (5.15)$$

By similar computation as above, we reach that

$$\|B\|_D < \frac{1}{2}, \quad (5.16)$$

and

$$\|C\|_D = 0. \quad (5.17)$$

Therefore, from (5.17), (5.16) and (5.15) we get

$$1 - \|A\|_D > 2\sqrt{\|B\|_D \|C\|_D}.$$

This proves part (b).

(c) From (5.17), (5.16) and (5.15) it follows that

$$\max_{m+n=k} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\} = 1. \quad (5.18)$$

Here, in view of (5.17), (5.16) and (5.8) we get

$$\frac{(\|A\|_D + \|C\|_D\|B\|_D) \max_{m+n=k} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}}{(2k!)^{-2} \|\partial_y \tilde{G}(x, y)\|^{-k} \left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D\|C\|_D}\right)^2} < 1.$$

■

Next we present a example of Lorenz-type map with C^k foliation invariant to it. Our strategy is to guarantee that this one satisfies the assumptions of our main theorem. Theorem 3.7.

Example 5.3. *Under the notation used in the Chapter 3. Let $T : D \setminus D_0 \rightarrow D$ be a map defined by $T(x, y) = (F(x, y), G(x, y))$, where the functions $F = (F_1, \dots, F_n)$, G are given by the equalities*

$$F_j(x, y) = ((-\beta_j |y|^{\omega_0} + \beta_j x_j |y|^{\omega_j} + 1)) \operatorname{sgn} y, \quad (5.19)$$

$$G(x, y) = |y|_0^\omega (1 + \alpha + \psi(x, y)) - \alpha \operatorname{sgn} y, \quad (5.20)$$

where $\operatorname{sgn} = \frac{y}{|y|}$ denotes the sign of y .

Assume that the following conditions are satisfied

(1) The function ψ is of class C^{k+1} in $D \setminus D_0$ and satisfies the following estimate:

$$\left\| \frac{\partial^{l+m} \psi(x, y)}{\partial x^l \partial y^m} \right\| \leq K |y|^{\gamma-m}. \quad (5.21)$$

(2) $\|G - \tilde{G}\|_{C^1} < \epsilon$, where $\tilde{G}(x, y) = (|y|^{\omega_0} (1 + \alpha) - \alpha)$ and ϵ small enough.

(3) The numbers $\alpha, \beta_j, \omega_0, \omega_j$, satisfy the following estimates:

$$\beta_j < \min \left\{ \frac{1}{(k!)^2 2^k}, \frac{1}{2(\omega_0 + \omega_j)} \right\}, \quad (5.22)$$

$$\omega_j - \omega_0 > (k-1), \quad j = 1, \dots, n, \quad (1 + \alpha)^{-1} < \omega_0 < 1, \quad 0 < \alpha < 1. \quad (5.23)$$

Then, there is a C^k foliation T -invariant with C^{k+1} leaves.

Proof. By Eq. (5.19) it follows that

$$F(x, y) = (1, \dots, 1) + |y|^{\omega_0} (\beta^*(x, y) + \varphi(x, y)), \quad (5.24)$$

where

$$\varphi(x, y) := (\beta_1 x_1 |y|^{\omega_1 - \omega_0}, \dots, \beta_n x_n |y|^{\omega_n - \omega_0}) \operatorname{sgn}(y), \quad (5.25)$$

and

$$\beta^*(x, y) := (-\beta_1, \dots, -\beta_n)(\operatorname{sgn}(y)). \quad (5.26)$$

Claim 5.4. *The function φ is of class C^{k+1} on $D \setminus D_0$ and satisfies the following estimate:*

$$\left\| \frac{\partial^{l+m} \varphi(x, y)}{\partial x^l \partial y^m} \right\| \leq K |y|^{\gamma-m}, \quad (5.27)$$

where $\gamma = \min\{\gamma_j : 1 \leq j \leq n\} > (k-1)$.

To prove the claim, consider

$$\varphi(x, y) = (\varphi_1(x, y), \dots, \varphi_n(x, y)) \operatorname{sgn}(y), \quad (5.28)$$

where

$$\varphi_j(x, y) = \beta_j x_j |y|^{\omega_j - \omega_0}, \quad \text{for } 1 \leq j \leq n. \quad (5.29)$$

Calculating partial derivatives of the functions $\varphi_j(x, y)$, for $1 \leq j \leq n$, we obtain that

$$\frac{\partial^m \varphi_j}{\partial y^m}(x, y) = \operatorname{sgn}(y)^{m+1} x_j |y|^{\gamma_j - m} \gamma_j (\gamma_j - 1) \dots (\gamma_j - m + 1), \quad (5.30)$$

$$\begin{aligned} \frac{\partial^m \varphi_j}{\partial y^m \partial x_j}(x, y) &= \operatorname{sgn}(y)^{m+1} |y|^{\gamma_j - m} \gamma_j (\gamma_j - 1) \dots (\gamma_j - m + 1), \\ \vdots &\quad \quad \quad \vdots \\ \frac{\partial^m \varphi_j}{\partial y^m \partial x_j^2}(x, y) &= \dots = \frac{\partial^m \varphi_j}{\partial y^m \partial x_j^l} = 0, \end{aligned}$$

where $\gamma_j = \omega_j - \omega_0$.

Hence, it follows that

$$\left\| \frac{\partial^{l+m} \varphi(x, y)}{\partial x^l \partial y^m} \right\| \leq K |y|^{\gamma-m}, \quad (5.31)$$

where $\gamma = \min\{\gamma_j : 1 \leq j \leq n\} > (k-1)$, $l, m \geq 0$, and $l + m \leq k + 1$. This proves the claim.

Recall that, by Definition 3.5 we have

$$A(x, y) := \frac{\partial_x F(x, y)}{\partial_y G(x, y)}, \quad (5.32)$$

$$B(x, y) := \frac{\partial_y F(x, y)}{\partial_y G(x, y)}, \quad (5.33)$$

$$C(x, y) := \frac{\partial_x G(x, y)}{\partial_y G(x, y)}. \quad (5.34)$$

Claim 5.5. *The following estimates hold:*

$$(a) \quad 1 - \|A\|_D > 2\sqrt{\|B\|_D\|C\|_D}.$$

$$(b) \quad 1 \leq \|\partial_y G\|_D \leq 2$$

$$(c) \quad \frac{(\|A\|_D + \|C\|_D\|B\|_D) \max_{m+n=k} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n\}}{(2k!)^{-2} \|\partial_y G(x, y)\|^{-k} \left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D\|C\|_D}\right)^2} < 1.$$

Indeed, it follows from condition (2) that

$$\|\tilde{T} - T\|_{C^1} < \epsilon, \tag{5.35}$$

where the application $\tilde{T}$ is as Theorem 5.2 and ϵ small enough.

Moreover, by Lemma 5.2 we have that the map $\tilde{T}$ satisfies the estimates (a), (b) and (c). Thus, since the application T as well derivatives are close of $\tilde{T}$ as describe the Eq. (5.35) one can conclude that the relations (a), (b) and (c) hold for the map T , which proves the claim .

Therefore, it follows from Claims 5.5, 5.4 and Eq. (5.31) that the application T satisfies the assumptions of main theorem. Theorem 3.7. Then, there is a C^k -invariant foliation $\mathcal{F}_{D,T}$, with C^{k+1} leaves, which proves the theorem. ■

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