Dynamics of holomorphic correspondences

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Dynamics of holomorphic correspondences

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(**In Memoriam**).

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Resumo

Generalizamos as noções de **estabilidade estrutural** e **hiperbolicidade** para a família de correspondências holomorfas

$$H_c(z) = z^r + c,$$

onde $r > 1$ é racional e $z^r = \exp r \log z$. Descobrimos que H_c é estruturalmente estável em todos os parâmetros hiperbólicos satisfazendo a **condição de fuga**. Tipicamente H_c possui **infinitos pontos periódicos atratores**, fato totalmente inesperado, uma vez que este número é sempre finito para aplicações racionais. O conjunto de tais pontos dá origem ao chamado **conjunto de Julia dual**, que é um conjunto de Cantor proveniente de um *Conformal Iterated Function System*.

Tanto o conjunto de Julia e quanto seu dual são **projeções de movimentos holomorfos** de sistemas definidos em subconjuntos compactos – denotados por X_c e W_c, respectivamente – de um **espaço de Banach.** Para todo c próximo de zero: (1) mostramos que J_c é reunião de **arcos quase-conformes** próximos do círculo unitário; (2) o conjunto X_c é um movimento holomorfo do **solenóide** X_0; (3) utilizando o formalismo dos **estados de Gibbs**, exibimos um limitante superior para a dimensão de Hausdorff de J_c. Consequentemente, J_c possui **medida de Lebesgue nula.**

Keywords: *1. Correspondências holomorfas. 2. Dinâmica complexa. 3. Conjunto de Julia. 4. Estabilidade estrutural. 5. Hiperbolicidade.*

RESUMO

Abstract

We generalize the notions of **structural stability** and **hyperbolicity** for the family of (multivalued) complex maps

$$H_c(z) = z^r + c,$$

where $r > 1$ is rational and $z^r = \exp r \log z$. We discovered that H_c is structurally stable at every hyperbolic parameter satisfying the **escaping condition**. Surprisingly, there may be **infinitely many attracting periodic points** for H_c. The set of such points gives rise to the **dual Julia set**, which is a Cantor set coming from a Conformal Iterated Funcion System.

Both the Julia set and its dual are **projections of holomorphic motions** of dynamical systems (single valued maps) defined on compact subsets of **Banach spaces,** denoted by X_c and W_c, respectively. For c close to zero: (1) we show that J_c is a union of **quasi-conformal arcs** around the unit circle; (2) the set X_c is an holomorphic motion of the **solenoid** X_0; (3) using the formalism of **Gibbs states** we exhibit an upper bound for the Hausdorff dimension of J_c, which implies that J_c has **zero Lebesgue measure.**

Keywords: *1. Holomorphic correspondences. 2. Complex Dynamics. 3. Julia set. 4. Structural stability. 5. Hyperbolicity.*

It is through science that we prove, but through intuition that we discover.

(Henri Poincaré)

Contents

CHAPTER 1

Introduction

In this thesis we present a detailed study of the dynamics of the holomorphic correspondence

$$H_c = \{(z, w) \in \mathbb{C}^2 : (w - c)^q = z^p\}.$$

Holomorphic correspondences have been studied since the middle 1980s, but as far as I know, this is the first work introducing the concept of structural stability to this subject. This notion has far reaching consequences in the dynamics of rational maps of the Riemann sphere. In spite of the fact that H_c is multi-valued, these far reaching results do also hold for H_c. Indeed, the main features of this thesis are:

- We define when H_c is hyperbolic and structural stable using the system

  $$\sigma : X_c \to X_c$$

  in the space of orbits (embedded in a infinity dinamensional Banach space). The projection of X_c is the Julia set of J_c.
- As usual, the limit set L_c of H_c is defined by taking accumulation points out of pre-orbits starting near $\infty \in \hat{\mathbb{C}}$. The Julia set J_c is the closure of repelling periodic orbits. If H_c is hyperbolic and satisfies the escaping condition, then L_c is written as a disjoint union

  $$L_c = J_c \cup E_c,$$

  where E_c is the dual Julia set of H_c. Typically E_c is a finite union of Cantor sets $\mathcal{K}_c^{(i)}$. The most surprising fact is that every point of E_c is a limit point of attracting periodic orbits! ($E_c = \phi$ for c close to 0).

- We also prove that each Cantor set $\mathcal{K}_c^{(i)}$ in E_c moves holomorphically with respect to c when H_c is hyperbolic and non-singular escaping.
- It is a remarkable result that when $c \sim 0$, the Julia set J_c of $z \mapsto z^2 + c$ is quasi-circle (image of $\mathbb{S}^1$ under a quasi-conformal map. In this thesis we prove that the Julia set J_c of H_c is an uncountable union of *quasi-conformal arcs* which are symmetrically placed around $\mathbb{S}^1$.
- Using the formalism of Gibbs states we give an upper bound to the Hausdorff dimension of $J_c = J(H_c)$ when $c \sim 0$. In particular, we obtain that J_c has zero area for $c \sim 0$, provided $q^2 < p$.

1.1. Motivation I: monotonicity of entropy conjecture

There are categories where the topological entropy map $f \mapsto h_{top}(f)$ is not even upper-semi continuous. However, in 1977 Milnor and Thurston [29] astonished the mathematical community proving that *the function $f \mapsto h_{top}(f)$ is continuous on the set $C^{2,b}$ of C^2 functions whose critical points are non-degenerate ($f''(c) \neq 0$).*

In this famous paper, it is proved that the topological entropy of the unimodal map

$$u_a(x) = ax(1 - x)$$

is monotonically increasing with $a \in \mathbb{R}$. This was just the starting point of a series of deep investigations which still occupy many present day eminent researchers. The monotonicity of $a \mapsto h_{top}(u_a)$ was proved in [29] using the Thurston rigidity theorem. Douady and Hubbard [15, 16] gave other proof using the univalent parametrization of a hyperbolic component. D. Sullivan gave a third proof using his *pullback argument*. M. Tsujii [41] gave an entirely real proof, but completely inspired in former results which were only discovered using complex methods.

It seems inevitable to deal with conformal extensions in this subject, although many struggle in a pure real approach.

The topological entropy of the family $f_c : \mathbb{R} \to \mathbb{R}$,

$$f_c(x) = |x|^r + c$$

with $r \in \mathbb{R}^+ \backslash \mathbb{Z}$, for example, has been investigated for the last thirty years. It is conjectured that is is monotone increasing, but no one knows how to prove it, mainly because there is no usual conformal extension of f_c as a map of $\mathbb{C}$.

1.1. **Remark**. Since the graph of $f_c(x) = |x|^{p/q} + c$ is contained in

$$H_c = \left\{(z, w) \in \mathbb{C}^2 : (w - c)^{2q} = z^{2p}\right\},$$

the correspondence is a conformal extension of f_c, but not in the usual sense. H_c is a *Riemann surface* with a single branch point at $(0, c)$.

1.2. Motivation II: Fatou conjecture

A rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is hyperbolic is the set of limit points $P(R)'$ of the post-critical set

$$P(R) = \overline{\bigcup_{n>0} \{R^n(c) : n > 0, R'(c) = 0\}}$$

is a finite union of attracting cycles. This is equivalent to say that R expands a conformal metric on its Julia set $J(R)$. In 1920 P. Fatou conjectured that hyperbolic maps are dense within the space of rational maps with fixed degree. This conjecture remains open, but in the 1980s R. Mañé, P. Sad and D. Sullivan gave one of the major contributions in understanding this problem **[30]**. They showed that if a rational map R is hyperbolic then it is structurally stable, and that structural stability holds in a open and dense set of parameters. In order to make more clear, we shall restrict to the quadratic family

$$q_c(z) = z^2 + c.$$

An holomorphic motion of $\Lambda \subset \mathbb{C}$ is family of injections $h_c : \Lambda \to \mathbb{C}$ parameterized in a neighborhood of 0 such that h_0 is the identity and $c \mapsto h_c(z)$ is holomorphic for every $z \in \Lambda$. If Λ is compact, then each h_c is a homeomorphism onto its image.

We say that q_a is structurally stable if every nearby map $q_c : J_c \to J_c$ is topologically conjugate to $q_a : J_a \to J_a$ by means of a conjugacy $h_c : J_a \to J_c$ which is a *holomorphic motion.*

Substantial part of this thesis is devoted to the generalization of this idea to H_c. It surprising that, in spite of the fact that H_c has uncountably many attracting periodic orbits, this notions still can be applied to H_c.

1.3. Motivation III: Quasi-Fuchsian groups and conformal repellers

The theory of Kleinian groups was founded by Felix Klein (1883) and Henri Poincaré (1883), who named them after Felix Klein.

Let M_b denote topological group of mobius transformations

$$\gamma(z) = \frac{az+b}{cz+c},$$

with determinant $ad - bc = 1$. A Kleinian group is discrete subgroup Γ of M_b which acts properly discontinuously. This means that any compact set K of $\mathbb{C}$ intersects only finitely many of its translates $\gamma(K)$ under the action of Γ. The set of accumulation points of an orbit $\Gamma.z$ is invariant under the action of Γ. It turns out that this set of accumulation points is independent of z. We shall denote it by $\Lambda(\Gamma)$, the *limit set* of Γ. Since Γ acts properly discontinuously on $\mathbb{C}$, the limit set is always a proper subset of the Riemann sphere.

A Kleinian group Γ is a Fuchsian group if there is an open disk U such that $\Gamma \subset \mathrm{Aut}(U)$, where $\mathrm{Aut}(U)$ is the set of conformal automorphisms of U. Most often one takes for U the upper half plane

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\},$$

or the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

In the first case the limit set $\Lambda(\Gamma)$ is the circle $\mathbb{R} \cup \{\infty\}$; in the second it is the unit circle $\mathbb{S}^1$.

We say that a subset A of the Riemann sphere invariant under a Kleinian group Γ if

$$\Gamma(A) = \{\gamma z = z \in A,\ \gamma \in \Gamma\} \subset A.$$

A quasi-Fuchsian group is a Kleinian group Γ which leaves invariant some Jordan curve ℓ in $\hat{\mathbb{C}}$. It follows that the limit set of Γ is contained in ℓ. The quasi-Fuchsian group Γ is of genus 1 if $\Lambda(\Gamma) \neq \ell$; otherwise we have $\Lambda(\Gamma) = \ell$ is the genus of Γ is 2.

Every finitely generated quasi-Fuchsian group is quasi-conformaly conjugate to Fuchsian group.

1.2. **Theorem**. *Let Γ be a finitely generated quasi-Fuchsian group (of genus 2). There is a Fuchsian group G and a quasiconformal homeomorphism $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that*

$$\Gamma = \varphi G \varphi^{-1}.$$

It follows from theorem 1.2 that the limit set of a quasi-Fuchsian group which is finitely generated of genus 2 is always a *quasicircle*.

The limit set of a quasifuchsian group is a *quasicircle* – the image the unit circle $\mathbb{S}^1$ under a quasiconformal map – and Bowen proved (see [11]) that *if the quasifuchsian group is not a fuchsian group, then its limit set must have Hausdorff dimension strictly greater* 1. This is a sort of geometric rigidity: either the limit set is a round circle or a fractal set. In order to prove this Bowen applied some concepts of Thermodynamic Formalism – such as Gibbs states –, certainly one the most successful ideas of the field.

Some years later D. Ruelle rediscovered the same property in the context of polynomial maps. The celebrated Ruelle's formula reads as follows (Ruelle, [34]): *If J_c denotes the Julia set of $z \mapsto z^p + c$ then its Hausdorff dimension is*

$$HD(J_c) = 1 + \frac{|c|^2}{4 \log p} + O(|c|^3),$$

for every c in a neighborhood V of the origin. We also have: $J_0 = \mathbb{S}^1$ and J_c is a quasicircle for $c \neq 0$ and $c \in V$.

1.3. **Remark**. There is a deep similarity of results concerning the apparently unrelated objects: (1) the limit set of a quasi-Fuchsian group; (2) the Julia set of the polynomial function $z \mapsto z^p + c$ and (3) (subject of this thesis) the Julia set of the holomorphic correspondence H_c. The first two are quasi-circles. The third is an uncountable union of quasi-conformal arcs obtained from 'motions' of the covering map $t \mapsto e^{it} \in \mathbb{S}^1$.

1.4. Motivation IV: Holomorphic correspondences

Holomorphic correspondences are interesting in themselves.

In 1988 S. Bullet investigated the dynamics of correspondences determined by implicit quadratic equations [6], which can be considered as a generalization of both quadratic maps and Kleinian groups with two generators.

Iterated holomorphic correspondences can be thought as a third field of complex dynamics, being the other two Kleinian groups and Rational maps. They are all interconnected and so there is no reason to threat them as separate subjects. As a matter of fact, holomorphic correspondences generalizes both Kleinian groups and Rational maps and serves to unify their dynamics in a single category (for more information about such relations, such as matings and the Sullivan dictionary, see [8] and [7]).

In 1994 S. Bullet and C. Penrose (Inventiones, [7]) showed that there is a non-empty set M of values of the parameter a for which the dynamics of the 2 : 2 correspondence

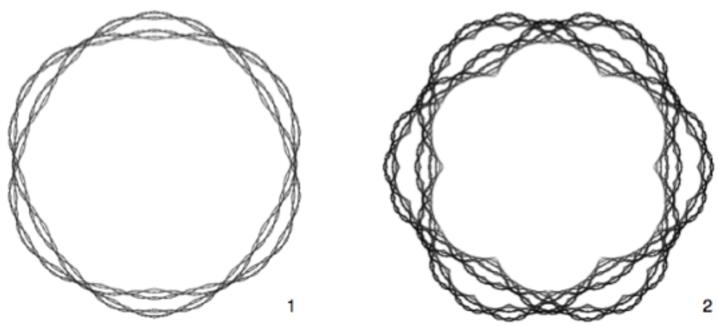
$$\left(\frac{az+1}{z+1}\right) + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3$$

is that of a mating of the modular group $PSL(2,\mathbb{Z})$ with the quadratic map $f_c(z) = z^2 + c$. This means that the Riemann sphere is partitioned into two subsets, each fully invariant under the correspondence: a regular domain Ω – a topological disk – on which the action of the correspondence resembles that of $PSL(2,\mathbb{Z})$ on the upper half plane $\mathbb{H}$; and a global attractor Λ, the point union of two subsets Λ_+, Λ_-, each resembling the filled-in Julia set K_c of f_c on each of which the actions of appropriate backward or forward branches of the correspondence resemble that of f_c on K_c. The set M is conjectured by the authors to be homeomorphic to the Mandelbrot set.

The results of this thesis are somehow independent; they are motivated by former results of the dynamics of rational maps.

1.4. **Remark**. The structure of this thesis is divided into part 1 and part 2. Except for the tools stated in the text, all the results that we present are new contributions and were developed from mid 2012 up to early 2015 by the author.

1.5. **Remark**. The following computer graphics illustrate the Julia set J_c of H_c. In (1) with $c = 0.2i$ and $(p, q) = (6, 2)$; for (2) the values of p and q are the same but $c = 0.35i$.



Part 1

The dynamics of H_c for c close to zero

CHAPTER 2

Structural Stability at the origin

We begin this chapter giving some basic concepts relating the dynamics of the correspondence

$$H_c = \{(z, w) \in \mathbb{C}^2 : (w - c)^q = z^p\}.$$

This correspondence may be treated as a multi-valued map of the Riemann sphere $\hat{\mathbb{C}}$: to every $z \in \hat{\mathbb{C}}$ we have q associated images w_i with $(z, w_i) \in H_c$. We define the Julia set J_c of H_c as the closure of repelling periodic points. This set is semi-invariant in the sense that every point of J_c has at least one image inside J_c; and every point of J_c has at least one pre-image inside J_c.

We consider the space of orbits O_c of the correspondence and define $X_c \subset O_c$ as the closure (in the product topology) of repelling periodic orbits $z = (z_i)_{i=0}^{\infty}$. The set X_c is invariant under the left shift $\sigma : X_c \to X_c$. We prove that the projection $\pi : X_c \to \mathbb{C}$ given by $\pi_i(z_n)_{n=0}^{\infty} = z_i$ is a semi-conjugacy from $\sigma : X_c \to X_c$ to $H_c : J_c \to J_c$. This means that $\pi_i(X_c) = J_c$ for every i and

$$(\pi_i(x), \pi_i\sigma(x)) \in H_c$$

for every $x \in X_c$.

The results of this chapter are proved for c close to the critical point 0. Some of them are extended to every parameter $c \in \mathbb{C}$ in part 2 of this thesis. So why do we not present them in their full generality since the beginning? For three reasons: (1) the technique for the general case is so much more sophisticated and uses the fact that post-critical set P_c has at least three points. (2) For $c = 0$ the set P_c has only one point. Therefore we need a separate proof for parameters close to the origin. (3) The language is simpler for $c \sim 0$ and we do not have to consider holomorphic motions of the *dual Julia set*. This set simply does not exist for rational maps. Furthermore, it is the structural stability at the origin which enables us describe J_c as an uncountable union of quasi-conformal arcs for $c \sim 0$. So we do not need full generality to obtain interesting results. This serves both as a motivation and also as a starting point to further generalizations.

In this chapter we develop a concept of structural stability for H_c and prove that H_c is structurally stable at origin. One of the consequences of this fact is that we can obtain J_c as holomorphic motions of $J_0 = \mathbb{S}^1$. Consequently, $c \mapsto J_c$ is continuous in the Hausdorff metric for compact sets.

2.1. **Remark**. The results of this chapter are proved under the condition $\frac{p}{q} > 1$ with $q \geq 2$. The case $q = 1$ is just $z \mapsto z^p + c$, which has been deeply studied for a long time.

2.1. The dynamics of H_c

We say that a sequence $z = (z_i)_{i=0}^{\infty}$ of complex numbers is an *orbit* of H_c if $(z_i, z_{i+1}) \in H_c$ for every i. As usual, the left shift map σ is defined on the set of orbits by $\sigma(z) = (z_i)_{i=1}^{\infty}$. We say that z is *periodic* with prime period $n > 0$ if $\sigma^n(z) = z$ and n is minimal for such a property. A periodic orbit is also referred as a *cycle.* If it happens that $z_i \in A$ for every i, then we say that *z is contained in A*.

If $\zeta = z_0$ is the first point of a periodic orbit $(z_i)_{i=0}^{\infty}$, then ζ is a *periodic point.* Since H_c is multivalued, it does not make sense to define the prime period of a periodic point as we do for orbits.

For any $A \subset \mathbb{C}$ we set

$$H_c(z) = \{w \in \mathbb{C} : (z, w) \in H_c\},$$

$$H_c^{-1}(w) = \{z \in \mathbb{C} : \ (z, w) \in H_c\},$$

$$H_c(A) = \bigcup_{z \in A} H_c(z),$$

$$H_c^{-1}(A) = \bigcup_{w \in A} H_c^{-1}(w).$$

2.2. **Definition**. We indicate $z \xrightarrow{H_c} w$ whenever $(z, w) \in H_c$. The notation $H_c(z)$ allows us to define the iterates H_c^n. By definition, $w \in H_c^n(z)$ if, and only if, there is a sequence

$$z = z_0 \xrightarrow{H_c} z_1 \xrightarrow{H_c} \cdots \xrightarrow{H_c} z_n = w.$$

Suppose $(z, w) \in H_c$, with $z \neq 0$. By the implicit function theorem, there is a unique bi-holomorphic map $\varphi : U \to V$ from a neighborhood U of z such that $(\zeta, \varphi(\zeta)) \in H_c$ for $\zeta \in U$, taking z into w. This map φ is the *univalent branch* of H_c *determined by* $z \xrightarrow{H_c} w$.

A cycle is a periodic orbit $\alpha : z_0 \to z_1 \cdots \to z_n = z_0$, where $(z_i, z_{i+1}) \in H_c$. Every cycle has a naturally associated complex number, called its *multiplier.* If the cycle contains no zero elements, then every point z_i determines an essentially unique branch φ_i of H_c (up to domain extensions) which takes z_i into z_{i+1}. The *multiplier* of this orbit (cycle) is

$$\lambda = \frac{d\varphi_{n-1} \circ \cdots \circ \varphi_0(z)}{dz}\bigg|_{z=z_0}.$$

If one of the elements of the cycle is 0, or ∞ (notice that ∞ is a fixed point) we set $\lambda = 0$, by convention.

A cycle is *attracting, repelling, neutral or super-attracting* according to whether $\lambda(\alpha)$ satisfies $|\lambda| < 1$, $|\lambda| > 1$, $|\lambda| = 1$ or $\lambda = 0$. Likewise, we can also speak of attracting periodic points and so on, using the obvious definitions.

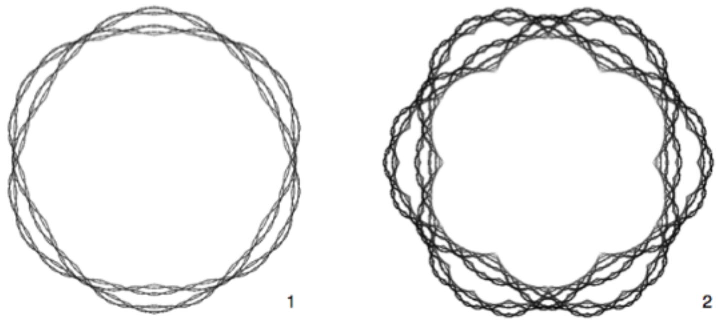
2.3. **Definition (Julia set)**. The Julia set J_c of H_c is the closure of the set of repelling periodic points of H_c.

2.4. **Theorem**. *We have $J_0 = \mathbb{S}^1$ and for every $\varepsilon > 0$ there is neighborhood V of $0 \in \mathbb{C}$ such that*

$$J_c \subset \{z \in \mathbb{C} : d(z, \mathbb{S}^1) < \varepsilon\},$$

for every $c \in V$.

The following computer graphics illustrate the possible motions of J_c. The union of all such motions gives J_c, which is shown in (1) with $c = 0.2i$ and $(p, q) = (6, 2)$; for (2) the values of p and q are the same but $c = 0.35i$.



PROOF. First we prove that $J_0 = \mathbb{S}^1$. It is clear that no periodic cycle may have a point outside $\mathbb{S}^1$. So $J_0 \subset \mathbb{S}^1$. Upon the other hand, every periodic orbit inside $\mathbb{S}^1$ is repelling. Indeed, if $(z, w) \in H_c$ and φ is the branch of H_c determined at (z, w), then

$$\varphi'(z) = \frac{p}{q}\left(\frac{\varphi(z) - c}{z}\right).$$

Hence the norm of the multiplier of a cycle of H_0 having period n inside $\mathbb{S}^1$ is always $(p/q)^n > 1$.

We only need to show that periodic cycles are dense in $\mathbb{S}^1$. But this is clear since H_0^n is given by $w^{q^n} = z^{p^n}$. Periodic cycles of H_0 correspond to roots of the equation $z^{p^n/q^n} = 1$, which are dense in $\mathbb{S}^1$.

Now we prove that J_c is contained in $\{z \in \mathbb{C} : d(z, \mathbb{S}^1) < \varepsilon\}$ for c sufficiently close to 0. The proof consists of a division of the plane into disjoint annuli in which the dynamics of the correspondence either increases or decreases the norm. In order to be more specific, let $\gamma = p/q - 1$. For $t \geq 0$, consider the function

$$f_t(x) = x^{p/q} - x + t,$$

which is defined on $[0, \infty)$. This function has a unique critical point at

$$\xi = \left(\frac{q}{p}\right)^{1/\gamma}.$$

Let

$$\delta = -f_0(\xi) = \left(\frac{q}{p}\right)^{1/\gamma} - \left(\frac{q}{p}\right)^{\frac{p}{\gamma q}}.$$

If $0 \leq t < \delta$, then f_t vanishes precisely at two points $a(t)$ and $b(t)$, with $0 < a(t) < \xi < b(t) < 1$. The complement of $\{z \in \mathbb{C} : |z| = t\}$ determines two simply connected sets $B_t(0)$ and $B_t(\infty)$, containing 0 and ∞ respectively. Let

$$A(s, t) = B_s(\infty) \cap B_t(0),$$

whenever $s < t$. Suppose $|c| \leq \delta$. If z belongs to the annulus $A(a(|c|), b(|c|))$ and w is an image of z, then

$$|w| \leq |w - c| + |c| = |z|^{p/q} + |c| < |z|.$$

Hence, H_c *decreases the norm on* $A(a(|c|), b(|c|))$, *if* $|c| < \delta$. We remark that every periodic orbit of H_c which is on $B_\xi(\infty)$ is necessarily repelling (this follows by direct computation of the derivatives of the branches at the points of the orbit). Moreover, H_c *expands the norm on* $\{|z| > (1 + |c|)^{1/\gamma}\}$, for then every image w of z satisfies

$$|w| \geq |w - c| - |c| = |z|^{p/q} - |c| > |z|.$$

Now we have a complete picture of the action of H_c when c is close to zero. Assume that $|c| < \delta/2$. We are going to prove that every repelling periodic orbit of H_c is contained in the set

$$\{z \in \mathbb{C} : \ b(|c|) \leq |z| \leq (1 + |c|)^{1/\gamma}\}.$$

Obviously, this will complete the proof. Let z_0 be some point of repelling periodic orbit of H_c. Since z_0 cannot be attracted to ∞, we have $|z_0| \leq (1 + |c|)^{1/\gamma}$. If $|z_0| \geq b(|c|)$, there is nothing to prove. Therefore we have two remaining possibilities: (*i*) $z_0 \in A(a(|c|), b(|c|)))$, and (*ii*) $z_0 \in B_{a(|c|)}(0)$. Let us suppose the period is N. There is $i < N$ such that

$$|z_0| > |z_1| > \cdots > |z_i|, \quad z_i \in \overline{B_{a(|c|)}(0)}.$$

The point z_i must comeback to $z_N = z_0$ under iteration. Hence, there is a $j > i$ such that z_j in in the set $\overline{B_{a(|c|)}(0)}$ while z_{j+1} is not. Since the distance between z_j and z_{j+1} is at most $|c|$, and since the assumption $|c| < \delta/2$ implies $|c| < \xi - a(|c|)$, it follows that $z_{j+1} \in A(a(|c|), \xi)$. The point z_{j+1} keeps being attracted to the center disk until is meets $\overline{B_{a(|c|)}(0)}$ again. The conclusion is that the whole orbit is contained in $B_\xi(0)$, which is a contradiction, for every cycle on this set is attracting (multiplier less than one in norm). The case (*ii*) is handled in a similar way. □

2.2. The relation between J_c and X_c

2.5. **Definition**. Consider the space of bounded orbits $\mathcal{O}_c$ of H_c. Each element of $\mathcal{O}_c$ is therefore a sequence $x = (x_i)$ for which $|x_i| \le M_x$ for some $M_x > 0$. The set $\mathcal{O}_c$ is equipped with the product topology and the left shift map σ. An element $x \in \mathcal{O}_c$ is a repelling periodic orbit if $\sigma^n(x) = x$ for some n and the multiplier $\lambda(x)$ of the orbit satisfies $|\lambda| > 1$.

2.6. **Theorem** (X_c). *Let X_c be the closure of the repelling periodic orbits in $\mathcal{O}_c$.*

(i) *For every c sufficiently close to* 0, *the set X_c is compact and for every projection $\pi_i : \mathcal{O}_c \to \mathbb{C}$ we have*

$$\pi_i(X_c) = J_c.$$

(ii) *If $\sigma : \mathcal{O}_c \to \mathcal{O}_c$ is the left shift, then $\sigma(X_c) = X_c$ and*

$$(\pi_i(x), \pi_i\sigma(x)) \in H_c, \tag{2.1}$$

for every $x \in X_c$.

The relation (2.1) reveals that H_c is semi-conjugate to $\sigma : X_c \to X_c$.

Proof. For c sufficiently close to 0, the Julia set J_c remains inside of an annulus $A = \{r \le |z| \le R\}$. This is proved in Theorem 2.4. The space of bounded complex sequences $A \times A \times \cdots$ is compact in the product topology. The closure of X_c in that space is X_c again. Hence X_c is compact.

Consequently, $\pi_i(X_c)$ is a closed set containing all repelling periodic points of H_c. Thus $J_c \subset \pi_i(X_c)$. On the other hand, it is clear that $\pi_i(X_c) \subset J_c$. Property (ii) is clear from the definitions. The proof is complete. □

We shall prove later that σ is expanding and topologically mixing on X_c. One of the consequences of such property is that J_c *has zero area if $q^2 < p$ and c is close to* 0. (A figure of J_c for $c \sim 0$ was given in the introduction). As we shall see, the expanding property of $\sigma : X_c \to X_c$ is a direct consequence of the hyperbolicity of H_c.

2.3. Hyperbolicity in the annulus

From Theorem 2.4 we know that J_c is contained in some annulus

$$A(\varepsilon) = \{z : \ d(z, \mathbb{S}^1) < \varepsilon\}$$

as $c \to 0$. The precise notion of hyperbolicity is defined later in this thesis, and we do not need to discuss it here in its full generality. One direct manifestation of this property for parameters close to zero is the following theorem. Recall that the univalent branch determined by $z \xrightarrow{H_c} w$ is the unique (up to domain extensions) univalent map $\varphi : U \to \mathbb{C}$ implicitly defined by $(\zeta, \varphi(\zeta)) \in H_c$, taking z into w.

2.7. **Theorem**. *Suppose $p/q > 1$. Then there are $\lambda > 1$, $\rho > 0$ and a neighborhood V of the origin for which $\varepsilon = \lambda\rho$ satisfies:*

(i) *If $c \in V$ and the entries z_i of an orbit $z = (z_i)_{i=0}^{\infty}$ of H_c are contained in $A(\varepsilon)$, then the domain of every branch φ_i determined by (z_i, z_{i+1}) contains the ball $B_\rho(z_i)$ of radius ρ, and the range of every composition*

$$g_N = \varphi_{i+N} \circ \cdots \circ \varphi_i$$

covers $B_\varepsilon(z_{i+N+1})$. Moreover,

$$\left|g_N'(z_i)\right| \geq \lambda^N.$$

(ii) *The branch φ_n determined by a pair of points (z_n, z_{n+1}) contained in $A(\varepsilon)$ satisfies*

$$|\varphi_n(x) - \varphi_n(y)| \geq \lambda|x - y|$$

whenever x, y belong to $B_\rho(z_i)$.

(iii) *If x_0 and x_1 are distinct preimages of a point $y \in A(\varepsilon)$, then $|x_0 - x_1| \geq \varepsilon$.*

(iv) *Assume that $z = (z_i)_{i=0}^{\infty}$ and $w = (w_i)_{i=0}^{\infty}$ are orbits of H_c whose elements are in $A(\varepsilon)$. If $|z_i - w_i| < \varepsilon$ for all i, then $z = w$.*

(v) *If $c \in V$, then every periodic orbit of H_c inside of $A(\varepsilon)$ is repelling.*

Proof. The local branches of H_c are given by the maps

$$\varphi_c(z) = \exp\left(\frac{1}{q}\log z^p\right) + c = \varphi(z) + c.$$

More specifically, if $(x_i, x_{i+1}) \in H_c$ and $x_i \neq 0$, then there is a branch of the logarithm defined in a region containing x_i^p such that $\varphi_c(x_i) = x_{i+1}$. The function φ_c is univalent on every sector $\theta < \arg(z) < \theta + \alpha$ with amplitude $\alpha < 2\pi/p$. Since

$$|\varphi_c'(z)| = \frac{p}{q}\frac{|\varphi(z)|}{|z|} = \frac{p}{q}|z|^{p/q-1},$$

there is $\varepsilon > 0$ such that

$$|\varphi_c'(z)| \geq \lambda > 1 \text{ on } A(\varepsilon). \tag{2.2}$$

It will be convenient to consider annuli of the form

$$A(r, s) = \{z : \ r < |z| < s\},$$

where $r < 1 < s$.

Suppose first that $c = 0$, so that $\varphi_c = \varphi$. After expressing φ in polar coordinates we conclude that φ maps $A(r, s)$ onto $A(r^{p/q}, s^{p/q})$, being injective (univalent) on every subset contained in a sector of amplitude $2\pi/p$. The main idea of the proof is to derive expansiveness from (2.2). First we choose $\delta > 0$ such that for any subset S of $A((1-\varepsilon/2)^{p/q},\ (1+\varepsilon/2)^{p/q})$ having diameter $|S| < \delta$, its convex hull is contained in $A((1-\varepsilon)^{p/q}, (1+\varepsilon)^{p/q})$. Then we choose a corresponding value of a for which ρ in the equation $\varepsilon/a = \lambda\rho$ satisfies

$$B_\rho(x) \subset A(\varepsilon/2) \text{ and } |\varphi(B_\rho(x))| < \delta$$

if $x \in A(\varepsilon/4)$. We also make the obvious assumption that $B_\rho(x)$ is contained in a sector of amplitude $2\pi/p$. Our first conclusion is that the local branches of H_c along the orbit $\alpha = (x_i)$ are always defined and univalent on $B_\rho(x_i)$. Now let x, y be two points of $B_\rho(x_i)$, where x_i is supposed to be in $A(\varepsilon/4)$. The line ζ joining $\varphi(x)$ and $\varphi(y)$ is still inside of

$A((1-\varepsilon)^{p/q},\ (1+\varepsilon)^{p/q})$. We pullback this arc and obtain the curve $\gamma = \varphi^{-1}(\zeta)$, which is contained in $A(\varepsilon)$. Denoting the length of an arc by ℓ, we have

$$
\begin{aligned}
|\varphi(x) - \varphi(y)| &= \ell(\zeta) = \ell(\varphi \circ \gamma) \\
&= \int_0^1 |(\varphi \circ \gamma)'(t)| dt \\
&= \int_0^1 |\varphi'(\gamma(t))| \cdot |\gamma'(t)| dt \\
&\geq \lambda \ell(\gamma) \geq \lambda |x - y|.
\end{aligned} \tag{2.3}
$$

It can be readily seen that c has no influence upon the preceding arguments (except for translations). So the conclusion is the same for φ_c and the first two items follow after the usual iteration arguments, with ε/a in place of ε. In order to obtain the third we make a second replacement of constants, with $\rho' = \rho^2/\varepsilon$ and $\varepsilon' = \rho$. Now ρ' and ε' works for the five conditions. □

2.8. **Definition (Expansive constant).** Any constant ε from Theorem 2.7 is, by definition, an *expansive constant* for the family H_c. We notice that if $\epsilon < \varepsilon$, then ϵ is also an expansive constant for H_c at $c = 0$.

2.4. Infinity dimensional holomorphic motion

The technique of holomorphic motions was originally introduced to study the structural stability of rational maps **[30]**. According to the standard definition, a subset Λ of the plane $\mathbb{C}$ *moves holomorphically* if there is a family of injections $h_c : \Lambda \to \mathbb{C}$ parameterized in a neighborhood of the origin such that h_0 is the identity and $c \mapsto h_c(z)$ is holomorphic. In this thesis the definition is the same except that Λ is allowed to be a subset of some Banach space.

Recall that a map $f : E \to F$ between Banach spaces E and F is holomorphic if it is Fréchet differentiable or, equivalently, if for every $x_0 \in E$ there is a power series which converges uniformly to f on a neighborhood of x_0.

2.9. **Definition**. Let Λ be a subset of a Banach space F, and U be a neighborhood of the origin in $\mathbb{C}$. Suppose $\Lambda \subset F$ is compact. We say that a one parameter family

$$h_c : \Lambda \to F$$

indexed in $c \in U$ is an holomorphic motion if

(i) h_0 is the identity;
(ii) h_c is a homeomorphism onto its image $h_c(\Lambda)$ for all $c \in U$;
(iii) $c \mapsto h_c(x)$ is holomorphic on U, for every $x \in \Lambda$ fixed.

There is only one difference from the classical definition: Λ is allowed to be any compact subset of a Banach space. According to the classical definition, $\Lambda \subset \mathbb{C}$ need not be compact and each h_c must be only injective. But it turns out that when $\Lambda \subset \mathbb{C}$ is compact, the mere fact that h_c is injective, together with (i) and (iii), implies that h_c is a homeomorphism onto its image. So our definition extends in a natural way the classical definition of Mañé, Sad and Sullivan [30].

2.10. **Remark**. Since the Julia set J_c is contained in an annulus $A = \{r \leq |z| \leq R\}$ for $c \sim 0$, the space $F = A^{\mathbb{N}_0}$ with the product topology and the compatible norm

$$||z|| = \sum_{i=0}^{\infty} 2^{-i}|z_i|$$

must contain X_c for $c \sim 0$.

2.11. **Theorem (Structural stability – recall 2.7, 2.6)**. *There is an holomorphic motion $h_c : X_0 \to A^{\mathbb{N}_0}$ parameterized in a neighborhood $U \subset V$ of the origin such that*

(i) *$h_c(X_0) = X_c$ and $h_c : X_0 \to X_c$ is a conjugacy between the shift spaces $\sigma : X_0 \to X_0$ and $\sigma : X_c \to X_c$;*
(ii) *If*

$$K = \sum_{i=0}^{\infty} \lambda^{-i} \tag{2.4}$$

then

$$\|h_c(x) - h_{c'}(x)\|_\infty \leq K|c - c'| \tag{2.5}$$

for every $x = (x_i)_{i=0}^\infty \in X_0$ *and* $c, c' \in U$. [1]

The proof is based on a shadowing argument.

2.12. **Definition** (**Shadowing – recall 2.5**). If $x = (x_i) \in \mathcal{O}_c$ and $y = (y_i) \in \mathcal{O}_{c'}$, we say that y is an η- shadowing of x if $|x_i - y_i| < \eta$, for every i.

2.13. **Lemma** (**Shadowing – recall (2.4), 2.7**). *Assume that* $c_0 \in V$ *and let*

$$\Omega(c_0, \varepsilon/n) = \left\{ c \in \mathbb{C} : \ |c - c_0| < \frac{\varepsilon}{nK}, \ c \in V \right\}. \tag{2.6}$$

Suppose the entries x_i *of* $x \in \mathcal{O}_{c_0}$ *are in* $A(\varepsilon/2) = \{z \in \mathbb{C} : d(z, \mathbb{S}^1) < \varepsilon/2\}$.

(i) *If* $n \geq 2$ *and* $c \in \Omega(c_0, \varepsilon/n)$, *then there is a* ε/n*-shadowing* $y \in \mathcal{O}_c$ *of* x. *The shadowing is unique in the following (stronger) sense: if* $c \in \Omega(c_0, \varepsilon/2)$ *and* $w, z \in \mathcal{O}_c$ *are* $\varepsilon/2$*-shadowings of* x, *then* $w = z$.

(ii) *Suppose* x *is a repelling periodic orbit of* H_{c_0}, $c \in \Omega(c_0, \varepsilon/n)$ *and* $y \in \mathcal{O}_c$ *is the* ε/n*-shadowing of* x. *Then* y *is also a repelling periodic orbit of* H_c.

(iii) *Assume that* $c \in \Omega(c_0, \varepsilon/2)$ *and let* $y(c) = (y_i(c)) \in \mathcal{O}_c$ *denote the* $\varepsilon/2$*-shadowing of* x. *Then the map* $c \mapsto y_i(c)$ *is holomorphic on* $\Omega(c_0, \varepsilon)$ *and satisfies*

$$|y_i(c) - x_i| < K|c - c_0|. \tag{2.7}$$

Proof. Let $x \in \mathcal{O}_{c_0}$, with $x_i \in A(\varepsilon/2)$ for all i. Suppose $c \in \Omega(c_0, \varepsilon/n)$, where $n \geq 2$. The range of the branch φ_i determined by (x_i, x_{i+1}) contains the ball $B_\varepsilon(x_{i+1})$ of radius ε and center x_{i+1}. Thus, for every $i \geq 0$, the sequence $y_i, y_{i-1}, \ldots, y_0$, inductively given by $y_i = x_i$ and $y_{j-1} = \varphi_{j-1}^{-1}(y_j + c - c_0)$ is well defined. Indeed, for $j \leq i - 1$,

$$\left|(y_j + c - c_0) - x_j\right| \leq |c - c_0| \left(1 + \lambda^{-1} + \cdots + \lambda^{-(i-j)}\right) < \frac{\varepsilon}{n}.$$

[1] We have $\|x\|_\infty = \sup_i |x_i|$ for every $x = (x_i)_{i=0}^\infty \in X_0$, which is not compatible with the product topology.

Since we are going to repeat this for every i, it is better to denote $a_{ij}(c) = y_j$. This is because y_j depends not only on j, but also on i and c. In this way, $a_{ij} : \Omega(c_0, \varepsilon/n) \to \mathbb{C}$ is a uniformly bounded sequence of analytic functions, for each j fixed. Therefore each j determines a sequence $\underline{i}_j = (i_1 < i_2 < \cdots)$ such that $a_{i_k j}$ converges locally uniformly to an analytic function g_j on $\Omega(c_0, \varepsilon/n)$. It is possible to take $\underline{i}_j$ such that $\underline{i}_{j-1}$ is a subsequece of $\underline{i}_j$. The standard diagonal method is applied to find a sequence $i_1 < i_2 < \cdots$ that works for all j :

$$\lim_{k\to\infty} a_{i_k j}(c) = g_j(c)$$

locally uniformly on $\Omega(c_0, \varepsilon/n)$. It is clear that

$$\left(g_{j+1}(c) - c\right)^q = \left(g_j(c)\right)^p ;$$

$$\left|g_j(c) - x_j\right| \leq K|c - c_0| < \varepsilon/n,$$

which proves simultaneously (iii) and the existence part of (i). Now we prove uniqueness. Assume that z and w in $\mathcal{O}_c$ are $\varepsilon/2$ shadowings of the point x. Then z_i and w_i are sequences in $A(\varepsilon)$ with $|z_i - w_i| < \varepsilon$ for every i. Theorem 2.7 – (iv) yields $z = w$.
If $y \in \mathcal{O}_c$ is an ε/n shadowing of a repelling periodic orbit x with prime period N, then $\sigma^N y$ is also a ε/n shadowing of x. Since the shadowing is unique, $\sigma^N y = y$. Theorem 2.7 shows that y is a repelling periodic orbit, for the sequence y_i is contained in $A(\varepsilon)$. The proof is complete. □

Proof of Theorem 2.11. Let h_c denote the map which assigns to every $x \in X_0$ its unique $\varepsilon/2$-shadowing $y \in \mathcal{O}_c$. Assume that $c \in \Omega(0, \varepsilon/2)$ and $|c| < \delta$ is such that $J_c \subset A(\varepsilon/2)$ whenever $|c| < \delta$. If we denote the set of repelling periodic orbits by $\mathcal{P}_c$, then $h_c(\mathcal{P}_0) \subset \mathcal{P}_c$. To prove the other inclusion we observe that if $y_0, \ldots, y_N = y_0$ is a repelling periodic orbit of H_c, then $y_i \in A(\varepsilon/2)$ for every i (because of the choice of δ). Lemma 2.13 is applied again to find a $\varepsilon/2$ shadowing $x \in \mathcal{O}_0$ of y, which is necessarily a repelling periodic orbit. Therefore $h_c(\mathcal{P}_0) = \mathcal{P}_c$, and the map h_c is a bijection between these sets. If h_c is continuous on $\mathcal{O}_0 \subset A(\varepsilon/2)^{\mathbb{N}_0}$, then obviously $h_c(X_0) = X_c$ (since X_0 is compact, h_c must be a homeomorphism). Suppose x and $\tilde{x}$ are in X_0, with $|x_i - \tilde{x}_i| < \eta$, for $i \leq N$. Let y

and $\tilde{y}$ denote their respective $\varepsilon/2$-shadowings in O_c. Theorem 2.7 gives an argument to prove continuity which reads as follows. Let φ_i and $\tilde{\varphi}_i$ denote the branches determined by (x_i, x_{i+1}) and $(\tilde{x}_i, \tilde{x}_{i+1})$, respectively. If η is small enough, then $\varphi_i^{-1} = \tilde{\varphi}_i^{-1}$ on the intersection D_i of their domains, for $i \le N-1$. If $\eta < \varepsilon/2 - |c|$, then the domain D_i contains both $y_i - c$ and $\tilde{y}_i - c$; and for $i \le N-1$,

$$y_{i-1} = \varphi_i^{-1}(y_i - c), \;\; \tilde{y}_{i-1} = \varphi_i^{-1}(\tilde{y}_i - c).$$

Theorem 2.7 – *(ii)* yields

$$|y_i - \tilde{y}_i| \le \lambda^{(i-N)}|y_N - \tilde{y}_N| \le \lambda^{(i-N)}(\varepsilon + \eta) \le \frac{3\lambda^{(i-N)}\varepsilon}{2}.$$

The continuity of h_c in the product topology follows from these observations letting $N \to \infty$. If $y = h_c(x)$, then $\sigma(y)$ is a $\varepsilon/2$-shadowing of $\sigma(x)$. Therefore $\sigma h_c = h_c\sigma$, and h_c is topological conjugacy. Lemma 2.13 – –*(iii)* finally shows that h_c is a holomorphic motion (in the product topology of $A(\varepsilon/2)^{\mathbb{N}_0}$, a function is Fréchet differentiable iff each coordinate is holomorphic). Now let $x \in X_0$ be fixed and consider $c, c_0 \in \Omega(0, \varepsilon/8)$. Then $c \in \Omega(c_0, \varepsilon/4)$ and the $\varepsilon/4$-shadowing $y \in O_c$ of $h_{c_0}(x)$ satisfy

$$\left|\pi_i h_{c_0}(x) - \pi_i(y)\right| \le K|c - c_0|. \tag{2.8}$$

Since $h_{c_0}(x)$ is the $\varepsilon/8$-shadowing of x, it follows that y is a $\varepsilon/2$-shadowing of x. Since the shadowing is unique, it follows that $y = h_c(x)$. The property (2.5) follows from (2.8). □

It is possible to analyze the continuity of J_c with respect to the parameter with the help of the Hausdorff distance d_H. If A and B are two compact subsets of the plane, let

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\},$$

where A_ε is the set of all points $z \in \mathbb{C}$ such that $d(z, A_\varepsilon) < \varepsilon$.

2.14. **Corollary (Continuity).** *The function $c \mapsto J_c$ is continuous on a neighborhood of the origin.*

PROOF. The continuity follows from the inequality

$$d_H(J_{c_0}, J_c) \le K|c - c_0|,$$

which we are going to prove using (2.5). If $z \in J_{c_0}$, then there is $x \in X_0$ such that $z = \pi_0 h_{c_0}(x)$, and therefore

$$d(z, J_c) \le d(z, \pi_0 h_c(x)) = \left|\pi_0 h_{c_0}(x) - \pi_0 h_c(x)\right| \le K|c - c_0|.$$

□

CHAPTER 3

Holomorphic motions

In the preceding chapter we described the sets X_c (cf. 2.6, 2.11) as holomorphic motions $h_c : X_0 \to X_c$ of X_0. As we are going to show next, the set X_0 is homemorphic to a solenoid contained in the solid torus $\mathbb{S}^1 \times \mathbb{D}$, where $\mathbb{S}^1 = \frac{[0,1]}{0\sim 1}$ and $\mathbb{D} = \{|z| < 1\}$. It should be noticed that for holomorphic motions $g_c : \Lambda \to \mathbb{C}$ of compact subsets of the plane, the function g_c is quasiconformal (cf. λ-Lemma in **[30]**). Hence the set X_c can be viewed as a quasiconformal image of X_0, i.e., a quasiconformal solenoid (but this is just an analogy).

There is another solenoid known as the Williams-Smale attractor. The one we present here is different in the sense that X_0 has infinitely many connected components. However, both are obtained from a similar construction which we briefly describe as follows. We consider the class $\mathcal{P} = \mathcal{P}(\mathbb{S}^1 \times \mathbb{D})$ of all subsets of the solid together with a transformation $\omega : \mathcal{P} \to \mathcal{P}$ which maps the solid torus onto $d = \gcd(p, q)$ homeomorphic copies of itself. As usual, we define the iterates ω^k of ω. The induced topology from $\mathbb{S}^1 \times \mathbb{D}$ makes

$$\bigcap_{k=1}^{\infty} \omega^k(\mathbb{S}^1 \times \mathbb{D})$$

homemorphic to X_0. As a consequence, X_0 is locally the product of a Cantor set with an interval, and X_0 is connected if, and only if, $d = 1$. It turns out that $J_c = \pi(X_c)$ is connected for $c \sim 0$ (cf. 2.6).

We can use the holomorphic motion of X_c to construct (plane) holomorphic motions of individual pieces of J_c. This is done in the second part of this chapter. Since $J_0 = \mathbb{S}^1$ (cf. 2.4), we consider an arbitrary interval $\Lambda \subset \mathbb{S}^1$. We can always "lift" Λ to a subset of X_0; in other words, there is an injective function $\psi : \Lambda \to X_0$ such that $\pi \circ \psi$ is the identity of Λ. Suppose for a moment that the projection π is injective on $h_c \circ \psi(\Lambda) \subset X_c$ for $c \sim 0$ (cf.

2.11). It turns out that $\pi \circ h_c \circ \psi : \Lambda \to \mathbb{C}$ is a holomorphic motion (of the plane). Using such fact we can describe J_c is an uncountable union of *quasiconformal arcs* (cf. 3.5, 3.6).

3.1. The solenoid homemorphic to X_0

A general element of X_0 is a sequence $z = (z_0, z_1, \ldots)$, where $z_i \xrightarrow{H_0} z_{i+1}$ (cf. 2.2). Alternatively, z may be viewed as pre-orbit of $G_0 = H_0^{-1}$, so that

$$z_0 \xleftarrow{G_0} z_1 \xleftarrow{G_0} z_2 \longleftarrow \cdots \tag{3.1}$$

Let $J_p = \{0, \ldots, p-1\}$. Since G_0 maps $\mathbb{S}^1$ into itself, for every $k \in J_p$ we consider the additive form of a branch of G_0 on $\mathbb{S}^1$, given as quotient of the maps $\theta_k : \mathbb{R} \to \mathbb{R}$,

$$\theta_k(t) = \frac{q}{p} t + \frac{k}{p}. \tag{3.2}$$

Until here we have worked under the assumption that $p > q$, but the reader will notice that the results of this section hold for arbitrary $p, q \geq 1$, with $d = \gcd(p, q)$ not necessarily equal to 1. Let $\mathbb{D} = \{|z| \leq 1\}$. The solid torus is $T = \mathbb{S}^1 \times \mathbb{D}$. Let $v_k : \mathbb{R} \times \mathbb{D} \to \mathbb{S}^1 \times \mathbb{D}$,

$$v_k(t, z) = \left([\theta_k(t)],\ \frac{1}{2}[t] + \lambda z \right) \tag{3.3}$$

where $[t] = \exp 2\pi i t$ and $\lambda \in (0, 1)$. If we choose λ small enough, then the function $u_k : T \to T$ given by $u_k([t], z) = v_k(t, z)$ is injective and the sets $u_k(T)$ are either disjoint or identical. While v_k is a homeomorphism from $[0, 1] \times D$ onto its image $v_k([0, 1] \times D)$, the same is not true for u_k, since v_k does not assign the point for $(0, z) \sim (1, z)$ in the solid torus. The geometry of the set

$$\omega(T) = \bigcup_{k=0}^{p-1} u_k(T) \tag{3.4}$$

is easily determined by observing that $u_k(T) = v_k([0, 1) \times D)$ is a cylinder C_k homeomorphic to $[0, 1) \times D$. By considering the map $\rho(i) = (i + q) \bmod p$ on J_p, it turns out that C_k pastes with $C_{\rho(k)}$, in the sense that $v_k(1, z) = v_{\rho(k)}(0, z)$ for every $z \in D$. We proceed iterating ρ until $C_{\rho^n(i)}$ pastes with C_i. It is then easy to conclude that the union of the cylinders

$C_i,\ C_{\rho(i)},\dots,C_{\rho^n(i)}$ is a homeomorphic copy of T inside of itself (winding around the origin in a certain number of times). As we see, the topology of $\omega(T)$ is intimately connected with the dynamics of ρ. The orbit set of 0 under ρ, for example, is $I = \{0, d, 2d, \dots, p-d\}$. From I we form the indexed partition $I_i = i + I$ of J_p, with $0 \le i < d$. The action of ρ on each I_i is a cyclic permutation, i.e., the orbit

$$i \mapsto \rho(i) \mapsto \rho^2(i) \mapsto \cdots \mapsto \rho^{p/d}(i) = i$$

has no repeated elements. The embedded Tori

$$T_i = \bigcup_{k\in I_i} C_k$$

are pairwise disjoints, thereby showing that $\omega(T)$ has precisely d connected components. The *solenoid* is defined by

$$\mathcal{S} = \bigcap_{n=1}^{\infty} \omega^n(T). \tag{3.5}$$

Since the intersection of a nested sequence of connected compact sets is again a nonempty connected and compact set, the Solenoid is nonempty and has uncountably many components A_τ ("infinite arcs") with index τ running in $J_d^{\mathbb{N}}$. (Recall that $\{0,1\}^{\mathbb{N}}$, for example, is uncountable). To be more specific, we first notice that each element of the sequence (3.1) is written $z_i = [t_i]$, with t_i in $[0, 1)$. Then each pair (t_i, t_{i+1}) determines a unique $k_i \in J_p$ for which $t_i = \theta_{k_i}(t_{i+1})$. Let us denote $\kappa(z) = (k_i)$ and consider

$$\varphi(z) = \bigcap_{n=1}^{\infty} f_{k_0} \circ f_{k_1} \circ \cdots \circ f_{k_n}(\{z_n\} \times \mathbb{D}), \tag{3.6}$$

defined for every $z = (z_i)$ in X_0, with $\kappa(z) = (k_i)$. The limit set $\varphi(z)$ consists of a single point in T, and it is not difficult to derive the analytic expression

$$\varphi(z) = \left(z_0, \sum_{i=0}^{\infty} \frac{\lambda^i}{2} z_{i+1}\right), \tag{3.7}$$

which allows us to conclude that φ is a homeomorphism from X_0 onto $\mathcal{S}$ (since φ is bijective and X_0 is compact). In proving (3.7) it becomes evident another important property of

$\mathcal{S}$: it is locally the product of a self-similar Cantor set $K \subset \mathbb{D}$ with an open interval. Now we use φ to determine an appropriate index $\tau = (t_i) \in J_d^{\mathbb{N}}$ for each connected component of $\mathcal{S}$. For every $x \in \mathcal{S}$ there corresponds the sequence $(k_i) = \kappa(z)$, where $\varphi(z) = x$. Considering the atoms I_j of the partition of J_p, we define A_τ as the set of all $x \in \mathcal{S}$ for which $k_i \in I_{t_i}$. Since every element of $\mathcal{S}$ is presented in the form (3.6), we conclude that A_τ is a connected component of $\mathcal{S}$ and that $\tau \mapsto A_\tau$ is bijective. The map $\varsigma = \varphi\sigma\varphi^{-1}$ on $\mathcal{S}$ makes $(\varsigma, \mathcal{S})$ conjugate to the shift space (σ, X_0). One may check that $y = \varsigma(x)$ is the unique point of $\mathcal{S}$ satisfying $x = u_k(y)$ for some $k \in J_p$. (In a rough sense, ς might be called the "unique pre-image" map). We cannot hope any component A_τ to be invariant under ς unless p and q are relatively prime, in which case there is only one connected component, the whole space $\mathcal{S}$. In any case, from (3.6) we have

$$\varsigma(A_\tau) = A_{\sigma(\tau)}.$$

We summarize as follows:

3.1. **Theorem** (Recall (3.5), 2.6). *Suppose $p, q \geq 1$ and let $d = \gcd(p, q)$,*

$$J_d = \{0, d, 2d, \ldots, p - d\}^{\mathbb{N}}.$$

(i) *For each $x \in \mathcal{S}$ there is a unique $y \in \mathcal{S}$ such that $x = u_k(y)$ for some $k \in J_p$. The function φ in (3.7) is a topological conjugacy between the shift (σ, X_0) and the the correspondence $\varsigma : x \mapsto y$ on $\mathcal{S}$.*

(ii) *The connected components A_τ of $\mathcal{S}$ may be indexed in $\tau \in J_d^{\mathbb{N}}$ in such a way that $\varsigma(A_\tau) = A_{\sigma\tau}$.*

(iii) *X_c is connected if $d = gcd(p, q) = 1$. Otherwise it has uncountably many components C_τ which may me indexed in $\tau \in J_d^{\mathbb{N}}$ so that $\sigma C_\tau = C_{\sigma\tau}$.*

(iv) *There is a self-similar Cantor set $K \subset \mathbb{D}$ with the property that to every $t \in \mathbb{S}^1$ there corresponds an open interval $t \in E \subset \mathbb{S}^1$ such that $(E \times \mathbb{D}) \cap \mathcal{S}$ is homeomorphic to $(0, 1) \times K$.*

3.2. **Remark**. Now the picture of J_c presented in Remark 1.5 is somehow predictable from the fact that X_0 is homeomorphic to $\mathcal{S}$: J_c is the projection of a Solenoid.

Once in the presence of such geometric result, we are ready to study the "motion" of the set J_c as we vary the parameter c near the origin.

3.2. Holomorphic motion of arcs in J_c

There is a strong evidence that J_c consists of uncountably many arcs: if we agree that the connected components of $\mathcal{S}$ are "arcs of infinite length", then we conclude the same for X_c from the homeomorphism $\varphi \circ h_c^{-1} : X_c \to \mathcal{S}$. In order to study the projection of these arcs we shall consider some specific continuous functions $\mathbb{R} \to X_c$. Obviously, their images are each subset of some connected component of X_c. The construction of these maps makes use of the auxiliary functions

$$\theta_k(t) = \frac{p}{q}t + \frac{k}{q},$$

where k is in $J_q = \{0, \ldots, q-1\}$. Given $\tau = (k_i)$ in $J_d^{\mathbb{N}}$, we define

$$\gamma^\tau(t) = (\exp 2\pi i t, \exp 2\pi\theta_1(t), \exp 2\pi i\theta_2 \circ \theta_1(t), \ldots)$$

and set $\gamma_c^\tau = h_c \circ \gamma_\tau$. If $0 < \alpha < 1$, then the projection under π_0 of each $\gamma^\tau([s, s+\alpha])$ is a sub-arc of $\mathbb{S}^1$ which is contained in

$$G(s,\alpha) = \{z \in \mathbb{C} : \ z \neq 0, \ 2\pi s \leq \arg(z) \leq 2\pi(s+\alpha)\}. \tag{3.8}$$

3.3. **Lemma**. *Consider a subset of A of the Riemann sphere whose complement has exactly two connected components C_0 and C_∞, containing 0 and ∞, respectively. Suppose Ω is a subset of $\widehat{\mathbb{C}}$ avoiding 0 and ∞. If the boundary of Ω is contained in A, then $\Omega \subset A$.*

Proof. The proof is simpler than the statement. If Ω is not contained in A, then one C_i must intersect both Ω and Ω^c; and therefore C_i meets $\partial\Omega$. But C_i is disjoint from A. □

3.4. **Theorem**. *Let γ be a continuous map from $\mathbb{R}$ into X_c. Suppose the projection $\pi_0(\gamma[a,b])$ is contained in some sector $G(s,\alpha)$, with $s \in \mathbb{R}$ and $\alpha \in (0,1)$. Then π_0 is an injective function from $\gamma[a,b]$ into $\mathbb{C}$.*

Proof. We denote the coordinate functions $\pi_i \circ \gamma$ by γ_i. In this way, all we need to prove is that, under the assumption $\gamma_0(t_0) = \gamma_0(t_1)$, we must have $\gamma_i(t_0) = \gamma_i(t_1)$ for all i. In order to do that, we first notice that γ_0 is a closed curve on the interval $[t_0, t_1]$, with trace

$$K := \gamma_0[t_0, t_1].$$

Step1 – Topological considerations. In analogy to the Jordan Curve Theorem, we would like to define an "interior" and "exterior" of K. Let V_i denote the connected components of the complement of K. They are connected open sets since K^c is open. We claim that each V_j is in fact simply connected. Indeed, the complement of the region V_j is a union of connected sets with a point in common,

$$V_j^c = \bigcup_{i \neq j} K \cup V_i,$$

and as such, it is connected. But an arbitrary region $U \subset \widehat{\mathbb{C}}$ is simply connected precisely when its complement is connected. Therefore V_j is simply connected. We define the *exterior* $E(K)$ of K to be the connected component V_i which contains ∞ (there is a V_i containing ∞ since K is a subset of $\mathbb{S}^1_\varepsilon$). The *interior* of K is the compact set

$$I(K) := E(K)^c.$$

We may exclude the case where $\{\gamma\}$ is a single point, for then the conclusion of the theorem holds trivially. The Riemann Mapping Theorem applies to $E(K)$. By considering a homeomorphism

$$\Phi : E(K) \to \{|z| < 1\}$$

we want to show that *for any $\beta > 0$ there is a simply connected set S with*

$$I(K) \subset S \subset I(K)_\beta. \tag{3.9}$$

We shall spend a little of time proving this regularity condition. The compact set $\Phi(I(K)_\beta^c)$ is contained in some disk $|z| < r < 1$. Denote by A the image of this disk through Φ^{-1}. It is a simply connected set containing $I(K)_\beta^c$ whose boundary is a Jordan curve ζ. By the Jordan Curve Theorem, the complement of $\{\zeta\}$ has precisely two connected components B_1 and B_2, which are necessarily simply connected and satisfy

$$\partial B_1 = \{\zeta\} = \partial B_2.$$

Since $I(K)$ is connected and does not intersect $\{\gamma\}$, it must be contained in some of the components B_i, say, $I(K) \subset B_1$. We claim that $A = B_2$. This will finish the proof of (3.9) since then

$$I(K) \subset B_1 \subset B_2^c \subset A^c \subset I(K)_\beta.$$

The set A is contained in some B_i; if the inclusion were proper, then B_i would contain a point of A^c, and therefore it would intersect ∂A, which is impossible. Hence $A = B_i$. Since A is disjoint from $I(K)$, we must have $i = 2$. This proves (3.9).

Step 2 – The main argument. We proceed inductively and construct a sequence of maps b_n with $b_n \circ \gamma_0 = \gamma_n$. Suppose $z \neq 0, \infty$. There are infinitely many values of $\log z^p$, and the expression

$$w = \exp\left(\frac{1}{q}\log z^p\right) + c \tag{3.10}$$

gives all the q values of w for which $(w - c)^q = z^p$. Chose $\beta > 0$ such that $I(K)_\beta \subset \mathbb{S}^1_\varepsilon$ and let $S^{(1)}$ be any simply connected set satisfying (3.9). Since $z^p \neq 0$ on $S^{(1)}$, for any value u_0 of $\log z_0^p$ there is an analytic function g defined on $S^{(1)}$ with

$$\exp g(z) = z^p, \quad g(z_0) = u_0.$$

For obvious reasons we shall refer to $g(z)$ as an analytic branch of $\log z^p$. By fixing an specific value of $\log \gamma_0(t_0)^p$ we conclude that there exists an analytic function

$$b_1 : S^{(1)} \to \mathbb{C}$$

such that

$$b_1(\gamma_0(t_0)) = \gamma_1(t_0)$$

and

$$(b_1(z) - c)^q = z^p, \quad z \in S^{(1)}.$$

For each z in $\mathbb{S}^1_\varepsilon$ let λ_z denote the set of all w with $(w - c)^q = z^p$. It has precisely q points which are at distance $> \rho_z$ from each other. The number ρ_z is independent of z provided z lies in a set bounded away from 0 and ∞. Using this property it is possible to show that

$$\Lambda = \{s \in [t_0, t_1] : \; b_1 \circ \gamma_0(s) = \gamma_1(s)\} \tag{3.11}$$

is open; furthermore, it is closed and contains t_0. Hence $b_1 \circ \gamma_0 = \gamma_1$ on $[t_0, t_1]$. We are ready to construct the sequence b_n with $b_n \circ \gamma_0 = \gamma_n$. Of course, the set X_c is defined for every c in a neighborhood U of the origin and, in view of Corollary 2.14, the Julia set J_c is contained in some annulus $\mathbb{S}^1_\varepsilon$ as $c \in U$. The first step is to show $b_1(I(K))$ is still contained in $\mathbb{S}^1_\varepsilon$, despite of the expanding behavior of b_1. In order Lemma 3.3 to b_1, we first observe that it is an open map (non-constant analytic function). Then

$$\begin{aligned} \partial b_1(I(K)) &\subset b_1(\partial I(K)) \\ &\subset b(\{\gamma_0\}) = \{\gamma_1\} \\ &\subset \pi_1(X_c) = J_c \subset \mathbb{S}^1_\varepsilon. \end{aligned} \tag{3.12}$$

There is $\beta > 0$ for which $I(K)_\beta$ in contained in $\mathbb{S}^1_\varepsilon$. One can also find a smaller β such that $f_1(I(K)_\beta)$ is contained in the same annulus. According to (2.14) it is therefore possible to choose a simply connected region $S^{(2)}$ between $I(K)$ and $I(K)_\beta$ with

$$S^{(2)} \subset S^{(1)}, \quad b_1(S^{(2)}) \subset \mathbb{S}^1_\varepsilon.$$

All these properties are used in the following induction process (although not explicitly exhibited in the proof). There is an analytic branch of $\log b_1(z)^p$ defined on $S^{(2)}$ for which

$$b_2(z) := \exp\left(\frac{1}{q} \log b_1(z)^p\right) + c \tag{3.13}$$

satisfies

$$b_2(\gamma_0(t_0)) = \gamma_2(t_0), \quad (b_2(z) - c)^q = z^p, \quad z \in S^{(2)}. \tag{3.14}$$

As before, $b_2 \circ \gamma_0 = \gamma_2$; and in fact, one can repeat the argument n times, obtaining simply connected regions

$$I(K) \subset S^{(n)} \subset S^{(n-1)} \subset \cdots \subset S^{(1)} \subset \mathbb{S}^1_\varepsilon \tag{3.15}$$

and analytic maps

$$b_n : S^{(2)} \to \mathbb{C}$$

with

$$(b_n(z) - c)^q = b_{n-1}(z)^p$$

on $S^{(n)}$ and $b_n \circ \gamma_0 = \gamma_n$ on $[t_0, t_1]$. This proves π_0 is injective on $\gamma[a, b]$. □

We are going to use this result to prove J_c consists of uncountably many quasi-arcs which move holomorphically with c. For the definition of holomorphic motion we consider a family of injections $i_\lambda : E \to \mathbb{C}$ of an arbitrary subset E of the plane. We assume that the parameter space is the open unit disk D and that i_0 is the identity. If i_λ depends analytically on λ, i.e., for each $z \in E$, the function $\lambda \mapsto i_\lambda(z)$ is holomorphic, then we say that (i_λ) is an *holomorphic motion* of E. According to the λ-Lemma in **[30]**, each i_λ has quasi-conformal extension from the closure $i_\lambda : \bar{E} \to \mathbb{C}$ which is a homeomorphism onto its image. The new injections do also depend analytically on $\lambda \in D$, and hence they constitute an holomorphic motion of $\bar{E}$. It can be shown that the correspondence $(\lambda, z) \mapsto i_\lambda(z)$ is continuous on $D \times \bar{E}$. In order to define an holomorphic motion of

$$\Lambda^\tau_s = \pi_0(\gamma^\tau[s, s + \alpha])$$

we fix an arbitrary $\alpha \in (0, 1)$ and consider the inverse map $\phi = \pi_0^{-1}$ defined on Λ^τ_s. This can be done because π_0 is injective on $\gamma^\tau([s, s + \alpha])$. Let U be the open set consisting of those c for which h_c is defined. Given $z \in \Lambda^\tau_s$ and c in U, we define

$$i_c(z) = \pi_0(h_c(\phi(z))).$$

The domain of i_0 is Λ_s^τ and i_0 is the identity. It follows that $c \mapsto i_c(z)$ is holomorphic for each fixed z. We may regard (i_c) as a "local holomorphic motion" of the sets Λ_s^τ. More specifically,

3.5. **Theorem**. *The family (i_c) is an holomorphic motion of Λ_s^τ, provided c lies in a corresponding neighborhood $V(s,\tau)$ of the origin. There is a uniform neighborhood V_0 (independent of s,τ and α) and $d > 0$ for which $(i_c)_{c\in V_0}$ is an holomorphic motion of each $S \subset \Lambda_s^\tau$ having diameter $< d$. The image of every i_c is contained in J_c.*

Proof. We claim that there is another neighborhood of $U_1 \subset U$ of the origin where each i_c is injective. If Λ is a sub-arc of Λ_s^τ of diameter $< d$ and C is the constant of then we conclude that

$$|i_c(\Lambda)| \leq d + 2C|U|,$$

where $|\cdot|$ denotes diameter. Hence $i_c(\Lambda)$ is contained in a sector $G(t,\beta)$, provided d and $|U|$ are small enough. (Notice that J_c is always contained in some annulus).

Since this restriction on $|U|$ is independent of τ and s, we may assume that, for every $c \in U_1$, the map i_c is injective on each subset of Λ_s^τ of diameter less than d. Hence $(i_c)_{c\in U_1}$ is an holomorphic motion of such sets. If i_c fails to be injective on the whole Λ_s^τ, then $i_c(z_1) = i_c(z_2)$ for two points with $|z_1 - z_2| \geq d$. The fact is that there is $U_1 = V(s,\tau)$ so that this cannot happen, mainly because of the uniform continuity of the holomorphic motion. Hence $(i_c)_{c\in V}$ is an holomorphic motion of Λ_s^τ. □

Using this result we can describe J_c as union of quasi-arcs. Here, a quasi-arc is any curve of the form $f \circ \gamma$, where f is quasi-conformal and γ is piecewise C^1. As we vary s, the union of the sets $i_c(\Lambda_s^\tau)$ gives the projection of $\gamma_c^\tau(\mathbb{R})$. Since this can be done for every index τ, the whole Julia set J_c can be obtained as holomorphic motions of the arc e^{it}, as explained in the following Characterization

3.6. **Corollary** (**Quasi-conformal arcs in J_c**). *Let V_0 be as in Theorem 3.5. Given $\tau = (k_i)$ in $J_d^{\mathbb{N}}$, we define*

$$\gamma^\tau(t) = (\exp 2\pi i t, \exp 2\pi\theta_1(t), \exp 2\pi i\theta_2 \circ \theta_1(t), \dots)$$

and set $\gamma_c^\tau = h_c \circ \gamma_\tau$. Consider the function $\zeta_c^\tau(t) = \pi_0(\gamma_c^\tau(t))$, $t \in \mathbb{R}$.

(i) *For τ and $c \in V_0$, the curve ζ_c^τ is a quasi-arc.*

(ii) *The winding number $n(\zeta_c^\tau, 0) \to \infty$ as $c \to 0$.*

(iii) *The union of the curves $\{\zeta_c^\tau\}$ is J_c when $c \in V_0$.*

(iv) *For each t and τ fixed, $c \mapsto \zeta_c^\tau(t)$ is holomorphic on V_0.*

CHAPTER 4

Hausdorff dimension

In this chapter we give an upper bound for the Hausdorff dimension of J_c, for $c \sim 0$, using the formalism of Gibbs states.

4.1. Expanding maps

Let (X, d) be a compact metric space. A continuous map T of X into itself is *expanding* if there is a constant $\eta > 1$ with the following property: every $x \in X$ has a neighborhood U such that $T^{-1}(U)$ can be written as finite union of open sets

$$U_1, \ldots, U_n, \tag{4.1}$$

each of which is mapped homeomorphically onto U, with

$$d(Tx, Ty) \geq \eta d(x, y)$$

for every $x, y \in U_i$.
Although the number n depends of x, using compactness we can choose U to be a ball of constant radius $B_\rho(x)$ (independent of x). However, the U_i need not to be the connected components of $T^{-1}U$. Hence such sets are not uniquely determined unless ρ is sufficiently small. In fact, there is ρ such that T restricted to every ball o radius ρ is a homeomorphism onto its image; and with some extra effort it can be shown that this ρ can be made even smaller so that:

4.1. **Proposition**. *The open sets U_i in (4.1) are uniquely[1] determined by the conditions*

$$T^{-1}B_\rho(x) = U_1 \sqcup \cdots \sqcup U_n;$$

[1] We use $\sqcup$ for disjoint unions.

$$x_i \in U_i \subset B_\rho(x_i).$$

PROOF. Follows from the definition. □

4.2. **DEFINITION** (**Injective constant**). If ρ satisfy the properties of Proposition 4.1, we shall refer to ρ as an *injective constant* of T. It can be shown that for every expanding system (T, X) there is another constant ε, now called an *expansive constant* of T, such that "$d(T^n x, T^n y) < \varepsilon$ for all n" implies $x = y$.

4.3. **DEFINITION** (**Topologically mixing**). The system (T, X, d) is topologically mixing if for every pair of open sets $U, V \subset X$ there is $n_0 \geq 1$ such that $T(U) \cap V$ is nonempty for every $n \geq n_0$. This definition makes sense for every topological dynamical system. In our case, it is equivalent to a much stronger condition, sometimes referred as *eventually onto maps:* the expanding map of T is topologically mixing if, and only if, every open set $U \subset X$ is eventually mapped onto the whole space ($T^n U = X$, for some n).

4.2. Mixing properties

In this section we are going to prove that the shift map σ is expanding and topologically mixing. Perhaps the easiest way of doing it is by considering the auxiliary sets Y_c. If K is a compact set of the plane, let

$$Y_c(K) := \{x = (x_0, x_1, \ldots) : \ (x_i, x_{i+1}) \in H_c, x_i \in K\}.$$

4.4. **PROPOSITION**. *The set $Y_0(\mathbb{S}^1)$ is invariant under the shift σ and there is $\mu > 1$ such that the system (σ, Y_0) is expanding for the metric*

$$d_\mu(x, y) = \sum_{i=0}^{\infty} \mu^{-i} |x_i - y_i|. \tag{4.2}$$

PROOF. This is an easy consequence of Theorem 2.7. So let λ and ε denote the constants of that theorem and take $\mu > 1$ such that $\mu^{-1} + \lambda^{-1} < 1$. Let $\eta = 1/(\mu^{-1} + \lambda^{-1})$. A sufficiently small neighborhood U of a point $x = (x_i)$ in Y_0 satisfy $\pi_0(U) \subset B_\varepsilon(x_0)$. The point x_0 has precisely p pre images $z_1, \dots, z_p$ in $\mathbb{S}^1$. Let φ_i denote the branch determined by (z_i, x_0). Theorem 2.7 says that $\sigma^{-1}(U)$ is the union of the sets

$$U_i = \{(\varphi_i^{-1}(x_0), x_0, x_1, \dots) : \; x \in U\},$$

with

$$d_\mu(\sigma x, \sigma y) \geq \eta d_\mu(x, y)$$

for every $x, y \in U_i$. □

4.5. **Proposition**. *The system $\sigma : X_0 \to X_0$ is topologically mixing, $X_0 = Y_0(\mathbb{S}^1)$, and $J_0 = \mathbb{S}^1$.*

Proof. It will be convenient to consider iterates of the correspondence. By definition, (z, w) belongs to H_c^n iff there is a finite orbit $x_0 = z, x_1, \dots, x_n = w$ of H_c connecting z to w. If A is a subset of the plane, let $H_c^n(A)$ denote the set of all w for which there is $z \in A$ with $(z, w) \in H_c^n$. The mixing property on $\mathbb{S}^1$ means that for every open subset U of $\mathbb{S}^1$ there exists n with $H_0^n(U) = \mathbb{S}^1$. (Notice that $\mathbb{S}^1$ is invariant under H_c). This property holds for H_0 since high iterates $H_0^n(z)$ of any point become dense in $\mathbb{S}^1$, and since the image of every branch covers a ball of constant radius ε. (Take ε as the expansive constant of Theorem 2.7). The mixing property is therefore proved on $\mathbb{S}^1$. Now if U is an open subset of $Y_0(\mathbb{S}^1)$ then there is n_1 and some interval $I \subset \mathbb{S}^1$ such that

$$\sigma^{n_1}(U) \supset I_\infty := \{(x_0, x_1, \dots) : \; x_0 \in I\}.$$

Since the correspondence is mixing on $\mathbb{S}^1$, there is n_2 such that $\sigma^{n_2}(I_\infty) = Y_0(\mathbb{S}^1)$. This proves that σ is topologically mixing on $Y_0(\mathbb{S}^1)$. Every expanding and topologically mixing dynamical system is the closure of its periodic points (in fact, transitivity is enough, as a consequence of the shadowing property). Since every periodic orbit in $\mathbb{S}^1$ is repelling, from the definition of X_0 it follows that $Y_0(\mathbb{S}^1) = X_0$. The proof is complete. □

The figure of J_c in the introduction suggested some symmetry of J_c, which can be described as the invariance $\omega(J_c) = J_c$ under the maps $z \mapsto \omega z$, where is any pth root of unity. If $(z, w) \in H_c$, we say that z is a pre-image of w or, equivalently, that w is an image of z.

4.6. **Corollary (Symmetry – recall 2.11)**. *Suppose $\omega^p = 1$. Let U denote the parametrization domain of the holomorphic motion h_c. If $c \in U$, then the Julia set J_c of the correspondence $H_c : (w - c)^q = z^p$ satisfies*

$$\omega(J_c) = J_c.$$

Moreover, J_c is backward invariant: if $w \in J_c$ and z is pre-image of w under H_c, then $z \in J_c$. Every $z \in J_c$ has at least one image w which is in J_c.

Proof. Let $c \in U$. A point of X_c is an orbit of H_c, and since $\pi_i(X_c) = J_c$ for every i, it follows that every point of J_c has at least one image in J_c. The backward invariance follows from the fact that (σ, X_c) is conjugate to the action of σ on $X_0 = Y_0(\mathbb{S}^1)$. A point of X_c must have exactly p pre-images under the shift, but this can happen only if J_c is backward invariant. Notice that if z is a pre image of w, then the same is true for ωz. Since J_c is backward invariant, it follows that $\omega(J_c) \subset J_c$. The other inclusion is trivial: for every z in J_c there is $\zeta = \omega^{p-1} z \in J_c$ such that $\omega\zeta = z$. □

4.7. **Theorem (X_c is expanding and mixing – recall 2.11)**. *Let U denote the parametrization domain of the holomorphic motion h_c, and let d_μ be as in (4.2). If $c \in U$, then (σ, X_c, d_μ) is topologically mixing, expanding, and $Y_c(J_c) = X_c$.*

Proof. Suppose $c \in U$. The map h_c is a topological conjugacy between the systems X_0 and X_c, and since the first is mixing, so must be the second. The mixing property has immediate counterpart with respect to the dynamics of the correspondence: if an open set V of the plane intersects J_c, then there is n such that $H_c^n(V \cap J_c) \supset J_c$. This is applied to show that $(\sigma, Y_c(J_c))$ is topologically mixing. Indeed, let U be an open subset of $Y_c(J_c)$ (in the product topology). It can be shown that some iterate $\sigma^n(U)$ contains a set of the form

$$V_\infty = \{(x_0, x_1, \ldots) \in Y_c(J_c) : \ x_0 \in V\}$$

where V is an open subset of the plane which intersects J_c. The mixing property on J_c yields $\sigma^N(V_\infty) = Y_c(J_c)$, for some N. Therefore σ is topologically mixing on $Y_c(J_c)$. The arguments used to the case of X_0 can be extended to a general parameter $c \in U$ to show that the function d_μ is still an expanding metric on X_c and $Y_c(J_c)$. Now the system $Y_c(J_c)$ is expanding and topologically mixing and, as such, it is the closure of the periodic points contained in $Y_c(J_c)$. Since all periodic orbits contained in J_c must be repelling, it follows that $X_c = Y_c(J_c)$. □

4.8. **Corollary (Topologically mixing on J_c).** *Suppose c belongs to the parametrization domain of h_c. If V is an open set of the plane which intersects J_c, then there is n such that $H_c^n(V \cap J_c) \supset J_c$.*

4.3. Gibbs state

Let (T, X, d) be an expanding system. Every point $x \in X$ gives rise to an orbit $x = x_0, x_1, x_2 \ldots$, and a sequence of locally defined inverse branches g_i of T taking x_i into x_{i-1}. If ρ is an injective constant of σ, we may assume that g_i is uniquely determined as a homeomorphism from $B_\rho(x_i)$ onto a neighborhood $V(x_{i-1})$, which is contained in $B_\rho(x_{i-1})$. We call

$$B_n(x, \rho) = g_1 \circ g_2 \circ \cdots \circ g_n(B_\rho(x_n)) \tag{4.3}$$

of a *dynamic ball* of T. The point x is the center, n is the length and ρ is the radius of the ball. Any continuous function ϕ from X into $\mathbb{R}$ is a *potential* of (T, X). The Birkhoff sums

$$\phi(x) + \phi(Tx) + \phi(T^2x) + \cdots + \phi(T^{n-1}x)$$

are denoted by $S_n\phi(x)$. Let ε be an expansive constant of T. We say that an invariant probability measure μ on X is a *Gibbs state* for ϕ (with respect to T) if for every $\rho \in (0, \varepsilon)$ there is a constant $C_\rho > 0$ such that

$$C_\rho^{-1} \leq \frac{\mu(B_n(x, \rho))}{\exp(S_n\phi(x) - nP(\phi))} \leq C_\rho$$

for every $n \geq 1$ and $x \in X$. It is a remarkable result that: *If the function ϕ is Hölder continuous on X and T is topologically mixing, then there is a unique Gibbs state for ϕ.* For a proof we refer to [42].

4.4. The Ruelle operator

We denote the topological pressure of potential ϕ with respect to the system T by $P(\phi, T)$, or simply $P(\phi)$. (For the original definition using coverings, see [10]). By a potential we mean any continuous, real valued function on X. The pressure can be defined either by functional analytic methods, or by direct topological computations (see [10]). For the analytic one, we consider the bounded linear operator $\mathcal{L}_\phi$ from the space $C(X, \mathbb{C})$ of continuous and complex valued functions into itself. The explicit formula for $\mathcal{L}_\phi$ is

$$(\mathcal{L}_\phi g)(x) = \sum_{T(y)=x} e^{\phi(y)} g(y),$$

where $g \in C(X, \mathbb{C})$. If $\psi = S_n\phi$, then $\mathcal{L}^n_\phi$ equals to the Ruelle operator $\mathcal{L}_\psi$ with respect to the system σ^n. This is particularly useful when studying the convergence properties of the iterates $\mathcal{L}^n_\phi g$. The dual of $C(X, \mathbb{C})$ is the space of complex measures on X. Hence any eigenvector of the dual operator $\mathcal{L}^*_\phi$ must be a complex measure. If $c = \exp P(\phi, T)$, then according to the the following result [2]

4.9. **Theorem** (**Ruelle-Perron-Frobenius**). *There is a probability measure ν on X and a continuous function h from X into $(0, \infty)$ such that*

(i) $\mathcal{L}^*_\phi \nu = c\nu$;
(ii) $\mathcal{L}_\phi h = ch$;
(iii) $\int h d\nu = 1$;
(iv) $\|\lambda^{-n} \mathcal{L}^n_\phi g - h \int g d\nu\|_\infty \to 0$ *as* $n \to \infty$.

The $\|\cdot\|_\infty$ indicates the supremum norm. A proof can be found in [42] or in any standard reference of the subject.

[2] This is an incomplete version of the so called Ruelle-Perron-Frobenius' Theorem.

4.5. Hausdorff dimension

Let A be a Borel subset of $\mathbb{C}$. If $C = \{U_i\}$ is a countable cover of A consisting of arbitrary subsets U_i of the plane, let $m_t(C) = \Sigma_i |U_i|^t$. The diameter $|C|$ of the cover C is the supremum of all $|U_i|$. For $t \geq 0$ fixed, the quantity

$$\mu^t_\delta(A) = \inf\{m_t(C) : \ C \text{ is a countable cover of } A \text{ with } |C| < \delta\}$$

is monotone increasing with δ; and hence converges to the limit $\mu^t(A)$ as $\delta \to 0$. The set function μ^t is a measure on the class of Borel subsets of the plane. It is not difficult to prove that $t \mapsto \mu^t(A)$ has a unique singularity $d \in [0, \infty)$ characterized by the fact that $\mu^t(A) = \infty$ for $0 \leq t < d$ while $\mu^t(A) = 0$ if $t > d$. The number $d = HD(A)$ is the *Hausdorff dimension* of the Borel set A.

4.6. An upper bound for $HD(J_c)$.

Let $x = (x_0, x_1, \ldots)$ be an element of X_c. The first two points of x determines the univalent branch $f_c(x_0, x_1)$, whose derivative at x_0 we denote by $f_c(x_0, x_1)'$. We also denote

$$f_c(x, m) = f_c(x_0, x_1) \circ f_c(x_1, x_2) \circ \cdots \circ f_c(x_{m-1}, x_m). \tag{4.4}$$

The expression

$$\phi_c(\alpha) = -\log |f_c(x_0, x_1)'|$$

defines a continuous map on X_c. Hence this is a potential for (σ, X_c). We shall prove the parameter t_c in the following result is an upper bound for $HD(J_c)$.

4.10. **Theorem** (**Recall** (4.2)). *The function ϕ_c defined above is Hölder continuous with respect to the metric d_μ. Moreover, for each c in a neighborhood of the origin, the equation*

$$P(t\phi_c, \sigma) = 0 \tag{4.5}$$

has a unique solution in the interval $[0, \infty)$. *This parameter, denoted by t_c, is never zero.*

Proof. The Hölder continuity can be verified using the standard metric estimates (some tricky estimates, with no advanced tools). Another important property about $P(t\phi_c)$ concerns monotonicity. The proof is based on a explicit topological computation of $P(t\phi_c)$ using the notation of [10]. In our context (cf. (4.4)), the Birkhoff sum assume the following form for $x = (x_0, x_1, \ldots)$,

$$S_m t\phi_c(x) = \log |f_c(x,m)'|^{-t}.$$

Consider a finite open cover $\mathcal{U}$ of X_c. (At this stage we invite the reader to check the notation used of pressure calculations in Chapter 2.B of [10]). Letting

$$Z_m(t\phi_c, \mathcal{U}) = \inf \sum_{\underline{U}\in\Gamma} \exp S_m t\phi_c(\underline{U}) \tag{4.6}$$
$$= \inf \sum_{\underline{U}\in\Gamma} |f_c(x,m)'|^{-t},$$

and taking into account that $|f_c(x,m)'| \geq \lambda^n$ (cf. 2.7) it is readily seen that

$$Z_m((t+s)\phi_c, \mathcal{U}) \leq \lambda^{-sm} Z_m(t\phi_c, \mathcal{U});$$

and hence

$$P((t+s)\phi_c, \mathcal{U}) = \lim_{m\to\infty} \frac{\log Z_m((t+s)\phi_c, \mathcal{U})}{m} \tag{4.7}$$
$$\leq -s\log\lambda + P(t\phi_c, \mathcal{U}).$$

We conclude that

$$P((t+s)\phi_c, \mathcal{U}) \leq -s\log\lambda + P(t\phi_c).$$

Hence $P(t\phi_c)$ is strictly decreasing with t and $P(t\phi_c) \to -\infty$ as $t \to \infty$.
The topological entropy of an expanding and topologically mixing system which is d to 1 is always $\log d$. From this fact we conclude that there is a unique root $t \geq 0$ of the equation $P(t\phi_c) = 0$, and that this root is > 0. This completes the proof. □

The estimate of $HD(J_c)$ is related to the dymics of σ on X_c, mainly because of the existence of a Gibbs state μ_c for the potential $t_c\phi_c$, whose pressure is zero. Following the general rule to determine whether $HD(J_c) \leq t_c$, we exhibit a sequence of coverings C_n of J_c with diameter $|C_n| \to 0$ for which

$$m_{t_c}(C_n) \leq B < \infty \tag{4.8}$$

for all n. We shall choose C_n as a projection of dynamic balls in X_c. The motivating idea is that if ρ is an expansive constant for (σ, X_c), then for all $x = (x_i) \in X_c$ and $n \geq 0$ we have (cf. (4.4), (4.3)):

$$C_\rho^{-1} \leq \frac{\mu_c(B_n(x,\rho))}{|f_c(\alpha,n)'|^{-t_c}} \leq C_\rho.$$

There is a second link between t_c and diameter of the sets $\pi_0 B_n(\alpha,\rho)$, which cover J_c. This is given by Koebe's Theorem and the observation that $\pi_0 B_n(\alpha,\rho)$ is the image of the disk $|z - x_n| < \rho$ under $B_c(\alpha,n)^{-1}$. We replace ρ by $\rho/4$ in order to apply this result and obtain, for some universal constant L,

$$|\pi_0 B_n(\alpha,\rho)| \leq \rho L |f_c(\alpha,n)'|^{-1} \leq \frac{\rho L}{\lambda^n}. \tag{4.9}$$

Some important assumptions must be made on ρ. It is required that ρ is an injective constant of (σ, X_c), so that (4.3) maybe used to determine $B_n(x,n)$ as (cf. (4.1)):

$$B_n(x,\rho) = \{y \in X_c :\ d_\mu(\sigma^i x, \sigma^i y) < \rho \text{ for } i \leq n\}. \tag{4.10}$$

In the second inequality of (4.9) it is implicitly assumed that $J_c \subset A(\varepsilon)$ as we vary c in a neighborhood U of 0 (ε is an expansive constant of H_c. Great advantage is attained if we choose the centers of the dynamic balls to lie in a (n,ρ)-separated set. By definition, two points $x = (x_i)$ and $y = (y_i)$ are said to be (n,ρ)-separated if there is $i \leq n$ with $d_\mu(\sigma^i x, \sigma^i y) \geq \rho$ (cf. (4.2)). A subset E of X_c is (n,ρ)-separated if every two points of E has the same property. Considering all (n,ρ)-separated subsets of X_c we choose one which is maximal for the inclusion. Denote it by E_n. From (4.10) we conclude that

$$C_n^* = \{B_n(x,\rho) : x \in E_n\}$$

is a cover of X_c with the property that $B_n(x, \rho/2)$ is disjoint from $B_n(y, \rho/2)$ whenever $x \neq y$ are in E_n. We shall prove the projected cover $C_n = \pi_0 C_n^*$ satisfies (4.8) thereby showing that

4.11. **Theorem.** *For every c in a neighborhood of the origin,* $1 \leq HD(J_c) \leq t_c$.

Proof. Property (4.9) implies $|C_n| \to 0$ as $n \to \infty$. Furthermore,

$$\begin{aligned} m_{t_c}(C_n) &\leq \rho L \sum_{x \in E_n} |f_c(x,n)'|^{-t_c} \\ &\leq \rho L \sum_{x \in E_n} C_{\rho/2} \mu_c(B_n(x, \rho/2)) \qquad (4.11) \\ &= \rho L C_{\rho/2} < \infty. \end{aligned}$$

This completes the proof. □

4.7. How good is the estimate

Let us test the preceding estimate. The simplest case is when $c = 0$, for then J_c is the unit circle $\mathbb{S}^1$. The value of t_0 can be computed directly using the Ruelle operator and Theorem 4.9. Let $\mathcal{L} = \mathcal{L}_{t_0 \phi_0}$. After evaluating $\mathcal{L}^n$ at the constant function 1, we find that

$$\begin{aligned} \mathcal{L}^n(1)(x) &= \sum_{\sigma^n(y)=x} |f_0(\beta, n)'|^{-t_0} \\ &= \sum_{\sigma^n(y)=x} \left(\frac{p}{q}\right)^{-nt_0} \qquad (4.12) \\ &= p^n \left(\frac{p}{q}\right)^{-nt_0}. \end{aligned}$$

In particular, there is a real constant ω such that $\mathcal{L}(1) = \omega \cdot 1$. From Theorem 4.9, we have $\omega^n \cdot 1 \to h$. Obviously, this implies $h = 1$. The explicit form of the equation $\mathcal{L}(1) = 1$ is $p\,(p/q)^{-t_0} = 1$, and since $p > q$, it follows that $t_0 > 1$. Therefore t_0 has no relevance as a good approximation of $HD(J_0) = 1$. The situation is even worse for higher values of p and q, when p/q is very close to 1, in which case $t_0 \to \infty$. The main cause of this discrepancy is due to the fact that t_c is not directly connected with the geometry of the Julia set. Recall that t_c was obtained from (σ, X_c), and the relevance of t_c as good approximation of $HD(J_c)$ depends also on π_0. Fortunately, nothing worse than the case $c = 0$ may happen: the entire solenoid structure of X_0 projects onto a single closed curve of dimension 1. For other values of c we have proved J_c consists of uncountably many quasi-arcs obtained as holomorphic motions of small pieces of $\mathbb{S}^1$. Hence the value of $HD(J_c)$ can be significantly bigger than 1, and it makes sense to ask whether there are values of c near the origin where $HD(J_c) = 2$. The answer to this question is no if assume that $p > q^2$. A simple application of the estimate by t_c yields

4.12. **Theorem**. *If $q^2 < p$, then $t_c < 2$ for all $c \sim 0$. Consequently, J_c has zero area.*

Proof. Under the assumption $q^2 < p$ we have $\lambda^2 > p$, if λ is sufficiently close to p/q. The number λ from Theorem 2.7 satisfies this condition provided we choose $c \sim 0$. For such values of c we shall prove $t_c < 2$. If $\mathcal{L} = \mathcal{L}_{t_c\phi_c}$, then

$$\begin{aligned} \mathcal{L}^n(1) &= \sum_{\sigma^n(\beta)=\alpha} |f_c(\beta, n)'|^{-t_c} \\ &\leq (p\lambda^{-t_c})^n; \end{aligned} \tag{4.13}$$

and since $\mathcal{L}^n(1)$ converges to the positive function h of Theorem 4.9, we must have $\lambda^{t_c} \leq p$. Hence $t_c < 2$. □

Part 2

General structural stability

CHAPTER 5

Iterated branch systems around Cantor sets

5.1. Conformal metrics

By a *Riemann surface* we mean a connected complex analytic manifold of complex dimension 1. A Riemannian metric on an open subset of $\mathbb{C}$ can be described as an expression of the form (using classical notation)

$$ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2,$$

where (g_{ik}) is a positive definite matrix which depends smoothly on the point $z = x + iy$ (by smooth we mean C^∞). Such a metric is said to be conformal if $g_{11} = g_{22}$ and $g_{12} = 0$. In other words, a conformal metric is one which can be written as

$$ds^2 = \gamma(x + iy)^2(dx^2 + dy^2),$$

or briefly as $d\gamma = \gamma(z)|dz|$, where the function $\gamma(z)$ is smooth and strictly positive. By definition, such a metric is invariant under a conformal automorphism $w = f(z)$ if, and only if, it satisfies the identity

$$\gamma(w)|dw| = \frac{\gamma(z)}{|f'(z)|}.$$

Every function f satisfying this condition is called an *isometry* (with respect to the metric). It is possible to define these notions on every Riemann surface using local coordinate charts.

5.2. Geodesically complete surfaces

Let $d\gamma$ be a conformal metric on a Riemann surface $\mathcal{R}$. The length of a vector $v \in T_z\mathcal{R}$ is

$$||v||_{z,\gamma} = \langle v, v\rangle_{z,\gamma}.$$

The length of any piecewise smooth curve $c : [a, b] \to \mathcal{R}$ is defined by

$$L(c) = \int_0^1 \left\| \frac{dc}{dt}(t) \right\|_{c(t),\gamma} dt.$$

The associated Riemannian distance on $\mathcal{R}$ is the metric

$$d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$$

defined by

$$d(z, w) = \inf\{L(c);\ c : [0, 1] \to \mathcal{R} \text{ is a piecewise smooth curve beween } z \text{ and } w\}.$$

The Riemannian distance defines a metric whose topology agrees with the topology of the surface.

5.1. **Definition**. Let $\mathcal{R}$ be a Riemann surface with a conformal metric $d\gamma$. We say that $\mathcal{R}$ is (geodesically) complete if the exponential map exp_z at an arbitrary point $z \in \mathcal{R}$ is defined in the whole tangent space $T_z\mathcal{R}$.

Intuitively, geodesics in a complete Riemann surface go on indefinitely, i.e., each geodesic is isometric to the real line. For example, the plane $\mathbb{C}$ with the euclidean metric is complete, but the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} :\ |z| < 1\}$$

with the euclidean metric is not complete.

5.2. **Theorem** (**Hopf-Rinow**). *A Riemann surface with a conformal metric $d\gamma$ is geodesically complete if, and only if, it is complete with respect to the Riemannian distance.*

5.3. Hyperbolic Riemann surfaces

The *Gaussian curvature* of a conformal metric $d\gamma = \gamma(z)|dz|$ is given by

$$K(z) = \frac{\gamma_x^2 + \gamma_y^2 - \gamma(\gamma_{xx} + \gamma_{yy})}{\gamma^4},$$

where $z = x + iy$ and the subscripts stand for partial derivatives.

5.3. **Theorem (Surfaces with constant curvature).** *Every Riemann surface admits a complete conformal metric with constant curvature which is either positive, negative, or zero according to whether the surface is* **spherical, hyperbolic**, *or* **Euclidean**.

Proof. See [43]. □

5.4. **Theorem.** *Let $\mathcal{R}$ be a Riemann surface. Suppose there is an analytic function*

$$f : \mathcal{R} \to \hat{\mathbb{C}}$$

omitting three points. Then $\mathcal{R}$ is hyperbolic.

Proof. See [43]. □

5.5. **Definition (Poincaré metric).** If $\mathcal{R}$ is a hyperbolic Riemann surface, then there is a unique complete conformal metric of constant curvature $K = -1$ on $\mathcal{R}$. This is the Poincaré metric of the surface. We denote the corresponding Riemannian distance by $\mathrm{dist}_{\mathcal{R}}$.

We say that a map $f : \mathcal{S} \to \mathcal{R}$ between Riemann surfaces is a *conformal isomorphism* if f is a homeomorphism and both f and its inverse are holomorphic. The word *isometry* is used for maps which preserve distance. When we have a linear map

$$A : T_z\mathcal{S} \to T_w\mathcal{R}$$

between tangent spaces of Riemann surfaces $\mathcal{S}$ and $\mathcal{R}$ we define its *norm* with respect to a pair of conformal metrics $d\rho$ on $\mathcal{R}$ and $d\mu$ on $\mathcal{S}$ to be

$$||A||_{\mu,\rho} = \sup_{v \in T_z\mathcal{R}\setminus\{0\}} \frac{|A(v)|_{\rho,w}}{|v|_{\mu,z}},$$

where $|\cdot|_{\rho,z}$ denotes the norm at the tangent space $T_z\mathcal{R}$ with respect to the metric $d\rho$. The differential $Df(z)$ of a holomorphic map f at a point z of a Riemann surface is an example of linear map between tangent spaces.

The Poincaré metric is of fundamental importance because of its marvelous property of never increasing under holomorphic maps.

5.6. **Theorem** (**Schwarz-Pick**). *If $f : \mathcal{S} \to \mathcal{R}$ is a holomorphic map between hyperbolic Riemann surfaces, then exactly one of the following statements is valid:*

(i) *f is a conformal isomorphism from $\mathcal{S}$ onto $\mathcal{R}$, and it maps $\mathcal{S}$ with its Poincaré metric isometrically onto $\mathcal{R}$ with its Poincaré metric.*

(ii) *f is a covering map but is not one-to-one. In this case, it is locally but not globally a Poincaré isometry. Every smooth path $P : [0,1] \to \mathcal{S}$ of arc-length ℓ in $\mathcal{S}$ maps to a smooth path $f \circ P$ of the same length ℓ in $\mathcal{R}$, and it follows that*

$$dist_{\mathcal{R}}(f(z), f(w)) \leq dist_{\mathcal{S}}(z, w)$$

for every $z, w \in \mathcal{S}$. Here equality holds whenever z is sufficiently close to w, but no strict inequality will hold, for example, if $f(z) = f(w)$ with $z \neq w$.

(iii) *In all other cases, f strictly decreases all nonzero distances. In fact, for any compact set $K \subset \mathcal{S}$ there is a constant $c_K < 1$ so that*

$$dist_{\mathcal{R}}(f(z), f(w)) \leq c_K dist_{\mathcal{S}}(z, w)$$

for every $z, w \in K$ and so that every smooth path in K with arc length ℓ (using the Poincaré metric for $\mathcal{S}$) maps to a path of Poincaré arc length $\leq c_K \ell$ in $\mathcal{R}$.

If f is a covering map (this includes isomorphisms), then

$$\|Df(z)\|_{\mu,\rho} = 1 \quad (z \in \mathcal{S}).$$

In all other cases we have

$$\|Df(z)\|_{\mu,\rho} < 1 \quad (z \in \mathcal{S}),$$

where $d\mu$ and $d\rho$ denote the Poincaré metrics of $\mathcal{S}$ and $\mathcal{R}$, respectively.

Proof. See [5] □

5.4. Iterated branch systems of first type

We shall consider three different types of iterated branch systems (a concept to be defined later). For single valued maps the natural IBS (iterated branch system) is determined by attracting periodic points. In the present case, H_c is not a multivalued function and other types of attracting regions may appear. We shall deal first with IBS which correspond to periodic cycles, or *iterated branch systems of first type.* The precise definition is as follows.

Suppose $\varphi : U \to V$ is a homeomorphism between two regions U and V of the plane (region means open and connected). If

$$\overline{\mathbb{D}} = \{z \in \mathbb{C} : \ |z| \leq 1\}$$

is contained in U, then $\overline{D} = \varphi(\overline{\mathbb{D}})$ is by definition a *topological disk.* It is a convention that the interior of $\overline{D}$ should be denoted by D. This notation is coherent and similar to the case of the unit disk $\mathbb{D}$ (the interior of $\overline{\mathbb{D}}$), for then

$$D = \varphi(\mathbb{D}),$$

$$\partial D = \partial \overline{D} = \varphi(\mathbb{S}^1)$$

and

$$\mathrm{int}_{\mathbb{C}}(D) = \mathrm{int}_{\mathbb{C}}(\overline{D}) = \varphi(\mathbb{D}),$$

and ∂ denotes the boundary of the set and $\mathrm{int}_{\mathbb{C}}$ indicates the interior with respect to $\mathbb{C}$.

5.7. **Remark**. IBS stands for iterated branch system.

5.8. **Definition** (**IBS of first type**). A IBS $\mathcal{A}$ (of first type) for H_c is determined by a sequence of biholomorphic maps $F_i : U_i \to U_{i+1}$ between regions U_i of the plane,

$$U_0 \xrightarrow{F_0} U_1 \xrightarrow{F_1} U_2 \xrightarrow{\cdots} U_{N-1} \xrightarrow{F_{N-1}} U_N,$$

such that:

(i) Each F_i is a branch of H_c, i.e.,

$$(x, F_i(x)) \in H_c,$$

for every $x \in U_i$;

(ii) There are topological disks $\overline{D}_i \subset U_i$ such that F_i maps $\overline{D}_i$ onto $\overline{D}_{i+1}$ and $\overline{D}_N$ is contained in D_0.

The attraction is determined by property (*ii*). Indeed, the function

$$F = F_{N-1} \circ F_{N-2} \circ \cdots \circ F_0$$

maps D_0 onto a pre compact region contained in D_0 and therefore it must strictly contract the hyperbolic distance $dist_{D_0}$ on D_N. As we shall see, under iteration every point in D_N is asymptotic to an attracting periodic orbit. Of course, since H_c is not single valued, iteration must be restricted to the holomorphic branches F_i determined by the IBS. By definition,

$$z \in \mathcal{A} \leftrightarrow z \in \bigcup_{i=0}^{N-1} D_i.$$

If $z \in \mathcal{A}$, then $\beta(z)$ is the sequence of iterates of P with respect to the maps F_i; more explicitly, we have

$$\beta(z) = (z_i)_{i=0}^{\infty}$$

where $z_0 = z$ and

$$F_{(i \bmod N)}(z_i) = z_{i+1} \quad (i \geq 0).$$

5.9. **Proposition** (**Hyperbolic attraction**). *Let $\mathcal{A}$ be a IBS of first type, determined by biholomorphic maps*

$$F_i : U_i \to U_{i+1} \quad (0 \leq i \leq N-1),$$

with topological disks $\overline{D}_i \subset U_i$.

(i) *The map*

$$F = F_{N-1} \circ \cdots F_1 \circ F_0 : D_0 \to D_0$$

has a unique fixed point $z_0 \in D_0$ which is necessarily attracting, i.e., $|F'(z_0)| < 1$.

(ii) *There is a constant $a < 1$ such that for any $y \in \mathcal{A}$,*

$$|y_{kN+i} - z_i| \leq a^i \qquad (k \geq 0,\ 0 \leq i \leq N-1),$$

where $(y_i) = \beta(y)$ and $(z_i) = \beta(z_0)$ is the unique periodic orbit in the region.

Proof. Let

$$d\rho = \rho(z)|dz|$$

be the Poincaré metric of D_0. Since F is a holomorphic map which maps D_0 into a compact subset of D_0 – in fact, the closure of $F(D_0)$ is

$$F(\overline{D}_0) = \overline{D}_N \subset D_0,$$

we conclude that there is a constant $a < 1$ such that

$$\|F'(z)\|_\rho \leq a < 1,$$

for every $z \in F(D_0)$. Hence

$$\operatorname{diam}_\rho\left(F^n\left(\overline{D}_0\right)\right) \leq \lambda^{n-1}\operatorname{diam}_\rho\left(F\left(\overline{D}_0\right)\right) \to 0,$$

as $n \to \infty$, where diam_ρ denotes diameter with respect to $d\rho$. The intersection of the nested sequence of compact sets

$$F^{n+1}\left(\overline{D}_0\right) \subset F^n\left(\overline{D}_0\right)$$

consists of a single point $z_0 \in D_0$, which is a fixed point of F and satisfies $\|F'(z_0)\|_\rho < 1$. Since any two conformal metrics are equivalent on compact sets, this proves $|F'(z_0)| < 1$. More explicitly, for any holomorphic map $f : D_0 \to D_0$ and any conformal metric $d\gamma$ on D_0, if $K \subset D_0$ is compact subset with $f(K) \subset K$, then there is a constant c_K such that

$$\frac{1}{c_K} \leq \frac{\|f'(z)\|_\gamma}{|f'(z)|} \leq c_K \quad (z \in K). \tag{5.1}$$

The value of the constant c_K is

$$c_K = \frac{\sup_K \gamma}{\inf_K \gamma}. \tag{5.2}$$

In our case, the compact invariant set is $\overline{D}_0$ and

$$\frac{1}{c_{\overline{D}_0}} \le \frac{\|(F^n)'(z_0)\|_\rho}{|(F^n)'(z_0)|} \le c_{\overline{D}_0}. \tag{5.3}$$

Since $\|(F^n)'(z_0)\|_\rho \to 0$, it follows that $|(F^n)'(P)| < 1$. The proof is complete. □

The inequality (5.1) can be proved in the following way. Let $z \in K$, the invariant compact set. Then

$$\begin{aligned} \frac{\|f'(z)\|_\gamma}{|f'(z)|} &= \sup_{v\neq 0} \frac{\|f'(z)\cdot v\|_\gamma}{|v|_\gamma} \cdot \frac{1}{|f'(z)|} \\ &= \sup_{v\neq 0} \frac{|f'(z)\cdot v| \cdot \gamma(f(z))}{|v|\cdot\gamma(z)} \cdot \frac{1}{|f'(z)|} \\ &= \frac{\gamma(f(z))}{\gamma(z)}. \end{aligned} \tag{5.4}$$

With c_K as indicated in (5.2) it follows at once the estimate given in (5.1).

The general principle that was used in the preceding result (and will be used in different formulations in the sequel) reads as follows:

A. **General Principle**. *If Ω is any connected open subset of $\mathbb{C}$ whose complement has at least three points and $f : \Omega \to \Omega$ is a holomorphic map with*

$$\overline{f(\Omega)} \subset \Omega,$$

then f has a unique fixed point $z_0 \in \Omega$ which is necessarily attracting, in the sense that $|f'(z_0)| < 1$. *Moreover, there are* $0 < a < 1$ *and* $C > 0$ *such that*

$$|f^n(z) - z_0| \le C\lambda^n \to 0$$

as $n \to \infty$, for every $z \in \Omega$.

5.5. Iterated branch systems of second type

If for IBS of first type every orbit $\beta(y)$ is asymptotic to an attracting cycle (see Proposition 5.9), for IBS of second type these orbits are asymptotic to a cycle of Cantor sets. This is the main difference, and as we shall see, it is due to the fact that the critical point 0 belongs to such IBS.

We say that an open simply connect subset D of the plane is a *univalent (open) disk* if there are q univalent branches φ_k of the correspondence H_c such that the images $\varphi_k(D)$ are pairwise disjoint and

$$H_c(D) = \bigcup_{i=1}^{q} \varphi_i(D).$$

Recall that for any set S, $H_c(S)$ consists of all w for which there is some $z \in S$ with $(z, w) \in H_c$.

The main ingredients for an IBS of second type are topological disks

$$\overline{D}_0 \xrightarrow{F_0} \overline{D}_1 \xrightarrow{F_1} \overline{D}_2 \xrightarrow{\cdots} \overline{D}_{N-1} \xrightarrow{F_{N-1}} \overline{D}_N \subset D_0$$

such that $0 \in D_0$ and

$$0 \notin \bigcup_{i=1}^{N} \overline{D}_i,$$

where F_i maps a neighborhood U_i of $\overline{D}_i$ biholomorphically onto a neighborhood U_{i+1} of $\overline{D}_{i+1}$, for $1 \leq i < N$. The first map in the above sequence, F_0, is multivalued. In fact, F_0 is the restriction of H_c to any neighborhood U_0 of $\overline{D}_0$. Hence, for every $Z \in U_0$ there corresponds q complex numbers under F_0,

$$W_0, W_1, \ldots, W_{q-1},$$

which satisfy

$$(W_i - c)^q = Z^p.$$

These points are symmetric with respect to c, in the sense that

$$(W_i - c) = \omega^i (W_0 - c),$$

where ω is the primitive q-th root of unit. Recall that for any correspondence G of the plane and any set $S \subset \mathbb{C}$,

$$G(S) = \{y \in \mathbb{C} : (x, y) \in G, \text{ for some } x \in S\}.$$

Now we give the precise definition of IBS of second type. The conditions are very natural and in no sense restrictive (a rather technical to describe, but yet very simple in its essence: it is just the branch point 0 which, under iteration, gives rise to q^n conformal disks whose diameter decrease exponentially fast on each step n. The corresponding limit set is a Cantor set).

5.10. **Definition** (**IBS of second type**). A IBS of second type $\mathcal{A}$ for H_c consists of $(N+1)$ topological disks $\overline{D}_0, \dots, \overline{D}_N$; $(N-1)$ biholomorphic maps

$$F_i : U_i \to U_{i+1} \ (0 < i < N);$$

and a multivalued, surjective map $F_0 : U_0 \to U_1$ such that:

(i) The disks $\overline{D}_0, \dots, \overline{D}_{N-1}$ may overlap, but

$$0 \notin \bigcup_{i=1}^{N} \overline{D}_i;$$

(ii) U_i is a region containing $\overline{D}_i$, and $\overline{D}_N \subset D_0$;

(iii) The critical point 0 belongs to the first disk D_0;

(iv) $F_i(\overline{D}_i) = \overline{D}_{i+1}$ for $0 \leq i < N$;

(v) $\overline{D}_N$ is contained in a univalent open disk. In other words, $F_0(\overline{D}_N)$ consists of q disjoint topological (closed) disks inside of D_1. (This always happens if the diameter of D_N is sufficiently small).

(vi) $F_0 = H_c$ on D_0, and F_0 maps $\overline{D}_0$ into D_1.

Let $\mathcal{A}$ be a IBS of second type, determined by maps (F_0 multivalued)

$$F_i : U_i \to U_{i+1} \ \ (0 \leq i < N).$$

By condition (ii), there are q univalent branches$\varphi_0, \varphi_1, \dots, \varphi_{q-1}$ of H_c defined on a certain region V containing $\overline{D}_N$ such that the open sets $\varphi_k(V)$ are pairwise disjoint and

$$F_0(V) = \bigcup_{i=0}^{N-1} \varphi_i(V). \tag{5.5}$$

We let

$$F = F_{N-1} \circ F_{N-2} \circ \cdots \circ F_1 \circ F_0.$$

Notice that F is a multivalued map (strictly speaking, a correspondence) which maps $\overline{D}_0$ onto $\overline{D}_N$. The second iterate $F^2(\overline{D}_0)$ consists of q disjoint topological disks inside of D_N and so on. It is expected that that this procedure should yield an invariant Cantor set in D_0. We shall prove it using the branches

$$T_i = F_{N-1} \circ \cdots \circ F_1 \circ \varphi_i : V \to D_N$$

of the correspondence F_0. In fact, it easy to see that

$$F(V) = \bigcup_{i=0}^{N-1} T_i(V). \tag{5.6}$$

Each map T_i is well defined. Indeed, $T_i(\overline{D}_N)$ is a compact subset of D_N; hence for a small neighborhood V of $\overline{D}_N$ we have $T_i(V) \subset D_N$, for every i. Therefore, $F(\overline{D}_N)$ consists of q (closed) topological disks $T_i(\overline{D}_N)$ inside of D_N. Taking into account the general principle A, we consider the limit-set map

$$\psi(k) = \bigcap_{n=1}^{\infty} T_{k_0} \circ T_{k_1} \circ \cdots \circ T_{k_n}(D_N), \tag{5.7}$$

where

$$k = (k_i) \in \Sigma_q = \{(k_0, k_1, \dots) : \quad k_i = 0, 1, \dots, (q-1)\}.$$

5.11. **Theorem.** *Let ψ be as in (5.7). For every $k \in \Sigma_q$, $\psi(k)$ is single point in in D_N. We denote $\mathcal{K} = \psi(\Sigma_q)$. The function*

$$\psi : \Sigma_q \to \mathcal{K}$$

is a homeomorphism.[1] *Hence $\mathcal{K}$ is a Cantor set contained in D_N, and $F(\mathcal{K}) = \mathcal{K}$.*

[1]We consider the product topology on Σ_q.

PROOF. Let $d\rho = \rho(z)|dz|$ be the Poincaré metric of V. Since $\overline{D}_N \subset V$ is compact set which is forward invariant under T_i, from the Schwarz-Pick lemma it follows that there are $\mu_i \in (0,1)$ such that

$$\mathrm{dist}_V\,(T_i(P), T_i(Q)) \leq \mu_i\ \mathrm{dist}_V(P, Q)$$

for every $P, Q \in \overline{D}_N$ and $0 \leq i < N$. For

$$\mu = \max\,\{\mu_0, \mu_1, \ldots, \mu_{(q-1)}\},$$

we have

$$\mathrm{diam}_\rho\,(T_{k_0} \circ T_{k_1} \circ \cdots \circ T_{k_n}(D_N)) \leq \mu^{(n+1)}\mathrm{diam}_\rho\,(D_N)\,.$$

Since

$$T_{k_0} \circ T_{k_1} \circ \cdots \circ T_{k_n} \circ T_{k_{n+1}}(D_N) \subset T_{k_0} \circ T_{k_1} \circ \cdots \circ T_{k_n}(D_N),$$

the intersection of the nested sequence of pre-compact sets, $\psi(k)$, consists of a single point $\{W\}$. We also write $\psi(k) = W$. As obvious, ψ is a surjective function onto $\mathcal{K}$. It remains to show that ψ is continuous and injective. In order to prove that ψ is injective, let

$$m = (m_0, m_1, m_2, \ldots) \neq n = (n_0, n_1, n_2, \ldots)$$

be two different sequences in Σ_q. Assume

$$m_0 = n_0,$$

$$m_1 = n_1,$$

$$\vdots$$

$$m_k = n_k,$$

$$m_{k+1} \neq n_{k+1}.$$

Then

$$T_{n_0} \circ \cdots \circ T_{n_k} \circ T_{n_{k+1}}(D_N)$$

and

$$T_{m_0} \circ \cdots \circ T_{m_k} \circ T_{m_{k+1}}(D_N)$$

are two disjoint conformal (open) disks contained in

$$T_{n_0} \circ \cdots \circ T_{n_k}(D_N) = T_{m_0} \circ \cdots \circ T_{m_k}(D_N).$$

Since

$$\psi(n) \in T_{n_0} \circ \cdots \circ T_{n_k} \circ T_{n_{k+1}}(D_N)$$

and

$$\psi(m) \in T_{m_0} \circ \cdots \circ T_{m_k} \circ T_{m_{k+1}}(D_N),$$

it follows that $\psi(n) \neq \psi(m)$. This proves that ψ is injective. The set Σ_q is compact with respect to the product topology. The basic sets are given by *cylinders*

$$C(m_0, m_1, \ldots, m_k) = \left\{n \in \Sigma_q :\ n_0 = m_0, n_1 = m_1, \ldots, n_k = m_k\right\}.$$

In order to prove continuity, let $\varepsilon > 0$. We are going to prove that for any $m \in \Sigma_q$ there is a cylinder V containing m for which the diameter of $\psi(V)$ is less than ε. Let $L \in \mathbb{N}$ be such that

$$\operatorname{diam}_\rho \left(T_{m_0} \circ T_{m_1} \circ \cdots \circ T_{m_L}(D_N)\right) < \varepsilon.$$

Consider the open set

$$V = C(m_0, m_1, \ldots, m_L).$$

Notice that $\psi(m) \in \psi(V)$. We are going to show that

$$\operatorname{diam}_\rho \psi(V) < \varepsilon. \tag{5.8}$$

This proves continuity with respect to dist_U, but any two conformal metrics are equivalent on invariant compact sets (see (5.4)). Hence, ψ will be continuous with respect to the standard euclidean metric provided we show (5.8). So let $n \in V$. Then n is presented in the form

$$n = (m_0, m_1, \ldots, m_L, n_{L+1}, n_{L+2}, \ldots);$$

and therefore

$$\begin{aligned}\psi(n) &= \bigcap_{Q=0}^{\infty} T_{m_0} \circ \cdots \circ T_{m_L} \circ T_{n_{L+1}} \circ \ldots \circ T_{n_{L+Q}}(D_N) \\ &\subset T_{m_0} \circ \cdots \circ T_{m_L}(D_N),\end{aligned} \tag{5.9}$$

whose diameter diam_ρ is less than ε. This proves (5.8). Now since Σ_q is compact and ψ is continuous, it must be an open map; hence ψ is a homeomorphism from Σ_q onto $\mathcal{K}$. Since

$$\mathcal{K} = \bigcap_{n=1}^{\infty} F^n(D_N),$$

$$F(\mathcal{K}) = \bigcap_{n=2}^{\infty} F^n(D_N) = \mathcal{K}.$$

The proof is complete. □

If $\mathcal{A}$ is an IBS of second type determined by topological disks $\overline{D}_0, \ldots, \overline{D}_N$ and maps $F_0, \ldots, F_{N-1}$, we shall indicate it briefly as

$$\mathcal{A} = (D_0, D_1, \ldots, D_N, F_0, F_1, \ldots, F_{N-1}).$$

By definition,

$$P \in \mathcal{A} \leftrightarrow P \in \bigcup_{i=0}^{N-1} D_i.$$

The key fact about IBS of second type is that they generate an invariant Cantor set $\mathcal{K} \subset D_N$, as described in Theorem 5.11. By invariant we mean that $F(\mathcal{K}) = \mathcal{K}$, where F is the composition of all maps of $\mathcal{A}$. Indeed, we have a *cycle of Cantor sets*

$$\mathcal{K}_0 \xrightarrow{F_0} \mathcal{K}_1 \xrightarrow{F_1} \mathcal{K}_2 \xrightarrow{\cdots} \mathcal{K}_{N-1} \xrightarrow{F_{N-1}} \mathcal{K}_N = \mathcal{K}_0, \tag{5.10}$$

where

$$\mathcal{K}_i = F_{i-1} \circ F_{i-2} \circ \cdots \circ F_1 \circ F_0(\mathcal{K}).$$

Let $P \in \mathcal{A}$. Without loss of generality, we may suppose that $P \in D_0$. The point P can be iterated inside of $\mathcal{A}$ using the maps F_i. According to (5.5), F_0 have precisely q univalent branches $\varphi_0, \dots, \varphi_{N-1}$ defined on a neighborhood of $\overline{D}_N$. Each sequence $n = (n_i)$ in Σ_q determines a sequence of maps φ_{n_i} and a sequence of iterates

$$\eta(P, n) = (Z_i)_{i=0}^{\infty}$$

where the points $Z_i \in \mathbb{C}$ are given by

$$\begin{aligned} Z_0 &= P, \\ Z_1 &= \varphi_{n_0}(P), \\ Z_2 &= F_1 \circ \varphi_{n_0}(P), \\ Z_3 &= F_2(Z_2), \\ &\vdots \\ Z_N &= F_{N-1}(Z_{N-1}), \\ Z_{N+1} &= \varphi_{n_1}(Z_N), \\ Z_{N+2} &= F_1(Z_{N+1}), \\ &\vdots \\ Z_{2N} &= F_{N-1}(Z_{2N-1}), \\ Z_{2N+1} &= \varphi_{n_2}(Z_{2N}), \\ &\vdots \end{aligned}$$

5.12. **Theorem** (**Hyperbolic attraction**). *Let $\mathcal{A}$ be an IBS of second type, with the associated cycle of Cantor sets*

$$\mathcal{K}_0 \xrightarrow{F_0} \mathcal{K}_1 \xrightarrow{F_1} \mathcal{K}_2 \xrightarrow{\cdots} \mathcal{K}_{N-1} \xrightarrow{F_{N-1}} \mathcal{K}_N = \mathcal{K}_0$$

and topological disks $\overline{D}_0, \dots, \overline{D}_{N-1}$.

(i) *Every orbit of a point in $\mathcal{A}$ is still contained in $\mathcal{A}$. In symbols, if $P \in \mathcal{A}$, $n \in \Sigma_q$ and $(Z_i) = \eta(P, n)$, then $Z_i \in \mathcal{A}$ for every $i \geq 0$.*

(ii) *Every orbit of a point in the cycle of Cantor sets is still contained in this cycle. In symbols, if* $P \in \mathcal{K}_j$, $n \in \Sigma_q$ *and* $(Z_i) = \eta(P, n)$, *then*

$$Z_i \in \mathcal{K}_{(i+j) \ mod \ N} \quad (i \geq 0).$$

(iii) *There are constants* $C > 0$ *and* $\lambda \in (0, 1)$ *such that the following holds for every* $P \in \mathcal{A}$. *If* $P \in D_j$, *then for every* $Q \in \mathcal{K}_j$ *and every* $n \in \Sigma_q$, *the sequences*

$$(Z_i)_{i=0}^{\infty} = \eta(P, n)$$

and

$$(W_i)_{i=0}^{\infty} = \eta(Q, n)$$

satisfy

$$|W_{kN+i} - Z_{kN+i}| \leq C\lambda^k \quad (0 \leq i < N, \ k \geq 0).$$

Proof. Suppose

$$\mathcal{A} = (D_0, D_1, \ldots, D_N, F_0, F_1, \ldots, F_{N-1}).$$

Statement (*i*) follows directly from the definition of $\eta(P, n)$. The same is true for the second, for $F_i(\mathcal{K}_i) = \mathcal{K}_{i+1}$ for every $0 \leq i < N$. According to the third assertion, suppose $P \in D_j$, $Q \in \mathcal{K}_j$, and consider the sequences

$$Z = (Z_i)_{i=0}^{\infty} = \eta(P, n),$$

$$W = (W_i)_{i=0}^{\infty} = \eta(Q, n),$$

where $n \in \Sigma_q$. Without loss of generality, we shall assume that $j = 0$. The sequence $n = (n_i)$ determines a sequence of branches φ_{n_i} of F_0 defined on a neighborhood U of $\overline{D}_N$, as described in (5.5). It turns out that

$$Z_1 = \varphi_{n_0}(Z_0),$$

$$Z_{N+1} = \varphi_{n_1}(Z_N),$$

$$Z_{2N+1} = \varphi_{n_2}(Z_{2N}),$$

$$\vdots$$

Since[2]

$$T_{n_i} = F_{N-1} \circ F_{N_2} \circ \cdots \circ F_1 \varphi_{n_i} : U \to D_N,$$

T_{n_i} maps the compact set $\overline{D}_N$ into its interior D_N. We also notice that from the definition of $\eta(P, n)$,

$$Z_{(k+1)N} = T_{n_k}(Z_{kN}) \quad (k \geq 0).$$

Therefore, there are constants $\mu_i \in (0, 1)$ such that

$$\mathrm{dist}_U(T_{n_i}(x), T_{n_i}(y)) \leq \mu_i \ \mathrm{dist}_U(x, y),$$

for every $x, y \in \overline{D}_N$. The same sequence of maps T_{n_k} which determine Z_{kN} does also determine W_{kN} from the initial point $W_0 = Q$. In other words, $W_{(k+1)N} = T_{n_k}(W_{kN}),\ \ k \geq 0$. Since both sequences $(W_{kN})_k$ and (Z_{kN}) are contained in D_N, it follows that

$$\mathrm{dist}_U(W_{kN}, Z_{kN}) \leq \mu^{(k-1)} \mathrm{dist}_U(W_N, Z_N) \quad (k \geq 0). \tag{5.11}$$

Any two conformal metrics are equivalent on compact sets. Since the two sequences involved are contained in $\overline{D}_N \subset U$, there is $C > 0$ (which only on dist_U) such that

$$\frac{1}{C} |z_1 - z_2| \leq \mathrm{dist}_U(z_1, z_2) \leq C\, |z_1 - z_2| \qquad (z_1, z_2 \in \overline{D}_N).$$

Combining this with (5.11) we get

$$\begin{aligned} |W_{kN} - Z_{kN}| &\leq C\, \mathrm{dist}_U(W_{kN}, Z_{kN}) \\ &\leq C\, \mu^{(k-1)}\, \mathrm{dist}_U(W_N, Z_N) \\ &\leq C^2\, \mu^{(k-1)}\, |W_N - Z_N| \\ &\leq \left(\frac{C^2}{\mu}\ |W_N - Z_N| \right) \mu^k \\ &\leq C_0\, \mu^k, \end{aligned} \tag{5.12}$$

where

$$C_0 = \frac{C^2 \cdot \mathrm{diam}\ D_N}{\mu}.$$

[2]Compare (5.6)

We now notice that $\varphi_{n_i} : U \to D_1$ is Lipschitz on the compact set $\overline{D}_N \subset U$, i.e., there is a constant B such that

$$\left|\varphi_{n_i}(x) - \varphi_{n_i}(y)\right| \leq B|x - y|, \quad (x, y \in \overline{D}_N).$$

(Since there are only a finite number of φ_i, we may take $B = B_i$ independent of i).
From

$$W_{kN+1} = \varphi_{n_k}(W_N) \quad (k \geq 0),$$

we have

$$\begin{aligned} |W_{kN+1} - Z_{kN+1}| &= \left|\varphi_{n_k}(W_{kN}) - \varphi_{n_k}(Z_{kN})\right| \\ &\leq B\,|W_{kN} - Z_{kN}| \\ &\leq B \cdot C_0\,\mu^k. \end{aligned} \tag{5.13}$$

Similarly, F_1 has a Lipschitz constant on the compact set $F(\overline{D}_N)$, and the same argument carries out for

$$W_{kN+2} = F_1(W_{kN+1}) \quad (k \geq 0).$$

Indeed, if L_1 is the Lipschitz constant of F_1, then

$$\begin{aligned} |W_{kN+2} - Z_{kN+2}| &= |F_1(W_{kN+1}) - F_1(Z_{kN+1})| \\ &\leq L_1\ |W_{kN+1} - Z_{kN+1}| \\ &\leq (L_1\,B\,C_0)\ \mu^k. \end{aligned} \tag{5.14}$$

Inductively,

$$|W_{kN+i+1} - Z_{kN+i+1}| \leq (L_1 \cdot L_2 \cdots L_i)\ B\,C_0\,\mu^k.$$

We may assume all constants L_i, B and C_0 are greater than 1, so that for

$$C = (L_1 \cdots L_{N-2})\ B\,C_0,$$

we have

$$|Z_{kN+i} - W_{kN+i}| \leq C\,\mu^k.$$

The conclusion of (*iii*) follows with $\lambda = \mu$. □

5.6. Attracting region of infinity

We say that a subset Ω of $\mathbb{C}$ is invariant under the dynamics of H_c if

$$H_c(\Omega) = \{w \in \mathbb{C} : \ (z,w) \in H_c \ \text{ for some } z \in \Omega\} \subset \Omega.$$

Given a parameter $c \in \mathbb{C}$ and $\lambda > 1$ there is $R > 0$ such that

$$R^{\frac{p}{q}} - |c| > \lambda R.$$

The region

$$B_\infty(R) = \{z \in \mathbb{C} : \ |z| > R\}$$

is invariant under H_c, for if $z \in B_\infty(R)$ and $(z,w) \in H_c$, then $|w| > \lambda |z|$.
It follows that

$$H_c(B_\infty(R)) = B_\infty(\lambda R).$$

Under iteration of H_c, the diameter of the sets $H_c^n(B_\infty(R))$ in the spherical metric tends to zero as $n \to \infty$. For obvious reasons, we shall refer to $B_\infty(R)$ as an *attracting region of* ∞. Let $\Omega(R)$ be an attracting region of ∞. The dynamics of H_c on $B_\infty(R)$ can be replaced by that of the shift σ on the space (with the product topology)

$$X_c(R) = \{(x_0, x_1, \ldots) : \ x_i \in B_\infty(R), \ \ (x_i, x_{i+1}) \in H_c\},$$

where

$$\sigma(x_0, x_1, \ldots) = (x_1, x_2, \ldots).$$

It is clear that

$$\sigma(X_c(R)) \subset X_c(R).$$

The projection onto first coordinate $\pi(x_0, x_1, \ldots) = x_0$ can be treated as a semiconjugacy since

$$\pi : X_c(R) \to B_\infty(R)$$

is surjective and the diagram

$$\begin{array}{ccc} X_c(R) & \xrightarrow{\ \sigma\ } & X_c(R) \\ \pi \downarrow & \searrow & \downarrow \pi \\ B_\infty(R) & \xrightarrow[\ H_c\ ]{} & B_\infty(R) \end{array}$$

is commutative in the sense that

$$(\pi(x),\ \pi\sigma(x)) \in H_c,\ \text{for } x \in X_c(R).$$

Before we state our following result, it will be necessary to define two terms which specify speed divergence (though they can be used for convergence as well). Let A_n be a sequence of positive real numbers. We say that A_n diverges *exponentially fast* if there are constants $a > 1$ and $C > 0$ such that

$$A_n \geq Ca^n \quad (n \geq 0).$$

We say that A_n diverges *double-exponentially fast* if there are $a, b > 1$ and $C > 0$ such that

$$A_n \geq Ca^{b^n} \quad (n \geq 0).$$

The next result reveals that the dynamics of H_c near infinity is always the same, no matter what parameter we choose.

5.13. **Theorem** (**Dynamics near infinity**). *Given two parameters $a, b \in \mathbb{C}$, there are $R > 1$, two sequences of positive real numbers $(T_n)_0^\infty$ and $(S_n)_0^\infty$, and a homeomorphism*

$$h : X_a(R) \to Y_b \subset X_b(T_0)$$

such that

(i) *Y_b is invariant under the (left) shift: $\sigma(Y_b) \subset Y_b$.*
(ii) *The map h is a topological conjugacy from $\sigma : X_a(R) \to X_a(R)$ to $\sigma : Y_b \to Y_b$.*

(iii) *The sequence T_n diverges exponentially fast and S_n diverges double-exponentially fast. Moreover,*

$$X_b(S_n) \subset \sigma^n(Y_b) \subset X_b(T_n) \quad (n \geq 0).$$

With some imagination, we may think of Y_b, $X_a(R)$ and $X_b(R_n)$ as neighborhoods of the point at infinity in the Riemann sphere, with the property that $X_b(R_n)$ "shrinks" to infinity as $n \to \infty$. The theorem says that T is a homeomorphism between the sets $X_a(R)$ and Y_b, and that $\sigma^n(Y_b)$ does also shrink to infinity as $n \to \infty$. With this analogy, H_a and H_b are topologically conjugate when restricted to $X_a(R)$ and Y_b, respectively.

Proof. Let $c \in \mathbb{C}$. If φ is branch of H_c defined in some open set Ω, then for every $z \in \Omega$ we have

$$|\varphi'(z)| = \frac{p}{q}|z|^{\frac{p}{q}-1}.$$

Recall that since φ is branch of H_c, by definition it satisfies $(z, \varphi(z)) \in H_c$ for every z in its domain.

We may therefore choose $R^* > 1$ such that

$$|\varphi'(z)| \geq \lambda > 1 \quad (|z| > R^*).$$

Then take

$$\varepsilon = |a - b| \sum_{i=0}^{\infty} \lambda^{-i}.$$

Step 1 We state here a couple of preliminary properties which are needed for the the proof. Let $\mu > 1$ be given. There is $R_1^* > R^*$ such that $\mu R_1^* - R_1^* > 2\varepsilon$ and $|w| > \mu|z|$, whenever $(z, w) \in H_a$ or $(z, w) \in H_b$ with $|z| > R_1^*$. We shall also assume that $R_1^* > |a|, |b|$.

For any point d of the plane we choose the symbol $S_d(\theta)$ to denote any open sector of amplitude θ centered at d. Therefore, whenever we specify the initial angle α, $S_d(\theta)$ is determined as a set of the form

$$S_d(\theta) = \{z \in \mathbb{C} : z \neq d,\ \alpha < \arg(z - d) < \alpha + \theta\}.$$

In most cases, it will not be necessary to specify α.

As usual, we dente by $B(x,r)$ the open ball of radius r and center $x \in \mathbb{C}$. If E is a bounded open subset of $\mathbb{C}$ and $x \in E$, then we let $r = \sup|z - x|$ as z varies in E. By $\mathrm{cov}_x(E)$ we mean the open ball $B(x,r)$, which necessarily contains E.

Let

$$B_\infty(R) = \{z \in \mathbb{C} : |z| > R\}.$$

CLAIM A. *We may choose the previous constant R_1^* in such a way that whenever a ball $B(x,2\varepsilon)$* is contained in $B_\infty(R_1^*)$, *there is a sector $S_0(\pi/p)$ containing $B(x,2\varepsilon)$.*

In order to prove this we let θ be minimal for the property $B(x,2\varepsilon) \subset S_0(\theta)$. We notice in such case there is only one such sector which is minimal for this property. The size of θ may computed either by elementary properties of the argument function, or by trigonometry. We have

$$\sin(\theta/2) = \frac{2\varepsilon}{|x|}.$$

The claim follows easily from this.

Let $c = a$ or $c = b$. As the set $B(x,2\varepsilon)$ is contained in a sector $S_0(\pi/p)$, there are q univalent branches of H_c defined on $B(x,2\varepsilon)$. Let φ be any of them.

CLAIM B. *We may choose R_1^* above in such a way that*

$$\mathrm{cov}_{\varphi(x)}(\varphi(B(x,2\varepsilon)) \subset S_c(\pi/q),$$

for every univalent branch φ and every ball $B(x,2\varepsilon)$ that is contained in the region $B_\infty(R_1^*)$.

Using the mean value inequality we see that

$$|\varphi(z) - \varphi(w)| \leq \frac{p}{q}\,(|x| + 2\varepsilon)^{\frac{p}{q}-1}\,|z - w|, \tag{5.15}$$

for every z and w in $B(x,2\varepsilon)$. From this we conclude that $\varphi(B(x,2\varepsilon))$ is contained in the ball $B(\varphi(x),r)$ of radius

$$r = 2\frac{p}{q}\,(|x| + \varepsilon)^{\frac{p}{q}-1}\,\varepsilon.$$

In particular, $\mathrm{cov}_{\varphi(x)}\varphi B(x,2\varepsilon)$ is contained in $B(\varphi(x),r)$.

Now let θ be minimal for the property $B(\varphi(x),r) \subset S_c(\theta)$.

It is clear that in this case there is a unique sector $S_c(\theta)$ satisfying this inclusion. The value of θ can be computed as in claim A. We have

$$\sin(\theta/2) = 2\frac{\frac{p}{q}\,(|x|+\varepsilon)^{\frac{p}{q}-1}\,\varepsilon}{|\varphi(x)-c|} = 2\frac{\varepsilon\frac{p}{q}\,(|x|+\varepsilon)^{\frac{p}{q}-1}}{|x|^{\frac{p}{q}}} \to 0, \tag{5.16}$$

as $|x| \to \infty$. This proves the claim.

We say that a finite collection of sets $A_1, A_2, \ldots, A_n$ is ϵ-sparse if

$$\inf_{i\neq j} d(A_i, A_j) > \epsilon,$$

where

$$d(A,B) = \inf\{d(x,y) :\; x \in A,\; y \in B\}.$$

Let $c = a$ or $c = b$. Whenever a ball $B(x, 2\varepsilon)$ of radius ε is contained in $B_\infty(R_1^*)$, we already know that there are q univalent branches φ_i of H_c defined on this ball; the union of the images $\varphi_i(B(x, 2\varepsilon)$ is $H_c(B(x, 2\varepsilon))$. We may assume that R_1^* of claim A satisfies the following additional property.

Claim C. *By taking R_1^* larger, if necessary, we may assume that for every ball $B(x, 2\varepsilon)$ contained in $B_\infty(R_1^*)$, the image sets $\varphi_i(B(x, 2\varepsilon))$ are 2ε-sparse; and also that every point y* in $\mathrm{cov}_{\varphi_i(x)}(\varphi_i(B(x, 2\varepsilon)))$ *satisfies*

$$|y| \geq |x| + 2\varepsilon. \tag{5.17}$$

Since $|x|+2\varepsilon$ is an upper bound for the norm $|z|$ of every $z \in B(x, 2\varepsilon)$, the last inequfality says that, under iteration, the images of balls of radius less than 2ε are disjoint and *move* to infinity by passing through disjoint annuli which partition the region $B_\infty(R_1^*)$.

In order to prove claim C, let $B(x, 2\varepsilon)$ be a ball which is contained in $B_\infty(R_1^*)$. Let φ be a univalent branch of H_c defined on $B(x, 2\varepsilon)$. The image point $y = \varphi(x)$ satisfies

$$|y| \geq |y-c| - |c| = |x|^{\frac{p}{q}} - |c|.$$

By (5.15) the ball of center y and radius

$$r = 2\varepsilon \frac{p}{q} (|x| + \varepsilon)^{\frac{p}{q}-1}$$

contains $\mathrm{cov}_y \varphi B(x, 2\varepsilon)$. If z is any point of $B(y, r)$, then

$$\begin{aligned} |z| &\geq |y| - r \\ &\geq |x|^{\frac{p}{q}} - |c| - 2\varepsilon \frac{p}{q} (|x| + \varepsilon)^{\frac{p}{q}-1} \\ &\geq |x| + 2\varepsilon, \end{aligned} \tag{5.18}$$

for large enough R_1^* (which is supposed to work for both $c = a$ and $c = b$), since $|x| > R_1^*$. This proves (5.17).

If ω is a primitive qth root of unity and ζ is any complex number such that $\zeta^q = x^p$, it follows that all the images of x under the correspondence are determined by the equation

$$y_k = \zeta \omega^k + c,$$

as we vary k from 0 to $(q-1)$. It follows that the distance between any two different images of z is bounded bellow by $|\zeta|\delta$, where δ is the infimum of $|\omega^i - \omega^j|$ for $i \neq j$. In other words, for every $i \neq j$ we have

$$\left| y_i - y_j \right| \geq \delta |x|^{\frac{p}{q}}.$$

Suppose R_1^* is sufficiently large so that

$$(R_1^*)^{\frac{p}{q}} \delta - 4\varepsilon \frac{p}{q} (R_1^*)^{\frac{p}{q}-1} - 2\varepsilon > 0.$$

Then

$$\begin{aligned} d(\varphi_i B(x, \varepsilon), \varphi_j B(x, \varepsilon)) &\geq d(B(y_i, r), B(y_j, r)) \\ &\geq \delta |x|^{\frac{p}{q}} - 4\varepsilon \frac{p}{q} (|x| + \varepsilon)^{\frac{p}{q}-1} \\ &\geq (R_1^*)^{\frac{p}{q}} \delta - 4\varepsilon \, (R_1^* + \varepsilon)^{\frac{p}{q}-1} \\ &> 2\varepsilon \end{aligned} \tag{5.19}$$

and we conclude from it that the image sets are 2ε-sparse, as desired. Claim C is proved.

The notation $B_\infty(M) \stackrel{f}{\leftarrow} B_\infty(N)$ used for two constants $M < N$ and $f = H_c$ for some parameter c indicates that the following property holds: if w is in $B_\infty(N)$ and z is a pre-image of w through H_c – this means $(z, w) \in H_c$ – then necessarily we have $z \in B_\infty(M)$.

Claim D. *Given $a, b \in \mathbb{C}$, there are R_1^* sufficiently large and constants R_1, R_2, R_3 and R which satisfy*

$$B_\infty(R_3) \stackrel{f}{\leftarrow} B_\infty(R_2) \stackrel{f}{\leftarrow} B_\infty(R_1) \stackrel{f}{\leftarrow} B_\infty(R),$$

$$2\varepsilon < \min\left(|R - R_1|,\ |R_1 - R_2|,\ |R_2 - R_3|\right),$$

$$R_1^* < R_3 < R_2 < R_1 < R$$

for $f = H_a, H_b$.

We only need to take a certain $\rho \in (0, 1)$ and R_1^* so that

$$R_1^* - (\rho R_1^*)^{\frac{q}{p}} > \varepsilon$$

and

$$R_1^* - d \geq \rho R_1^*,$$

where d is the greatest between the norms of the two given parameters $|a|$ and $|b|$. Let $R > R_1^*$ and denote by c either a or b. If $w \in B_\infty(R)$ and z is any pre-image of w under H_c we have

$$|z| = |w - c|^{\frac{q}{p}} \geq (|w| - |c|)^{\frac{q}{p}} \geq (\rho|w|)^{\frac{q}{p}} \geq (\rho R)^{\frac{q}{p}} =: R_1.$$

Notice that $R_1 < R$. This argument may be repeated inductively. For example, if w is any point of $B_\infty(R_1)$ and R_1 is still greater than R_1^*, then we conclude that any pre-image of w must be in $B_\infty(R_2)$, where $R_2 = (\rho R_1)^{\frac{q}{p}}$. A similar assertion is true for $R_3 = (\rho R_2)^{\frac{q}{p}}$. So in order to complete the argument we only need to take R large enough so that after three steps we have $R_3 > R_1^*$. It is easy to see that the difference $R_{i+1} - R_i$ become very large when $R \to \infty$; thus they become greater than 2ε, and the second set of inequalities is immediately fulfilled. Claim D is proved.

Step 2. We complete the proof of the theorem using shadowing properties on $B_\infty(R_1^*)$. We first notice that for any $x \in B_\infty(R_2)$ the set $B(x, 2\varepsilon)$ is contained in $B_\infty(R_3)$; and that for any branch φ of H_c [3] $\mathrm{cov}_{\varphi(x)}\varphi(B(x,2\varepsilon))$ is contained in $B_\infty(R_2)$, by claims C and D. In fact, by claim B the set $\mathrm{cov}_{\varphi(x)}\varphi(B(x,2\varepsilon))$ is contained in a sector $S_c(\pi/q)$. There is a branch of inverse of H_c defined on every such sector. The unique inverse branch which coincides with φ^{-1} when restricted to $\mathrm{cov}_{\varphi(x)}\varphi(B(x,2\varepsilon))$ is again denoted by φ^{-1}. By claim D,

$$D = \varphi^{-1}\left(\mathrm{cov}_{\varphi(x)}\varphi B(x,2\varepsilon)\right) \subset B_\infty(R_3).$$

The region D is chosen as the domain of φ. As the image of this function is convex, for every two points $\varphi(z)$ and $\varphi(w)$ in it there is a straight line ζ inside of $\varphi(D)$ which connects these two points. Let $\gamma = \varphi^{-1}(\zeta)$ be the pre-image curve, contained in D. Since the norm of φ' is bounded below by λ,

$$\begin{aligned} |\varphi(z)-\varphi(w)| &= \int_0^1 |\zeta'(t)|dt \\ &= \int_0^1 |\varphi'(\gamma(t))|\cdot|\gamma'(t)|dt \\ &\geq \int_0^1 \lambda|\gamma'(t)|dt \\ &= \lambda\ell(\gamma) \\ &\geq \lambda|z-w|. \end{aligned} \tag{5.20}$$

This says that for every $x \in B_\infty(R_2)$ and every image y of x through H_c (by the choice made on R_2, the point y must be also in $B_\infty(R_2)$) there is branch φ of H_c, with domain D and image B such that (i) D contains the open ball $B(x,2\varepsilon)$; (ii) B is itself an open ball which contains a smaller ball $B(y,2\lambda\varepsilon)$, and (iii) D is contained in $B_\infty(R_3)$, $B \subset B_\infty(R_2)$ and $\varphi : D \to B$ is biholomorphic. The radius $2\lambda\varepsilon$ of the image-ball has obviously a larger radius than that of $B(x,2\varepsilon)$; and in fact, the branch φ expands distances by the same factor λ on D. This property plays a central role in the following *shadowing* argument.

[3]Although the arguments apparently treat a single c, the conclusions do not depend on the choice of c in a set of two *fixed* parameters (in this case a and b).

Claim E (**Shadowing**). *For every orbit* $x = (x_i)_0^\infty$ *in* $X_a(R)$ *there is a unique orbit* $y = (y_i)_0^\infty$ *of* H_b *such that* $|x_i - y_i| < \varepsilon \ (i \geq 0)$. *The function* $h(x) = y$ *so defined satisfies*

$$h(X_a(R)) = Y_b \subset X_b(R_1).$$

Let $\varphi_i : D_i \to B_i$ be a univalent branch of H_a which takes x_i into x_{i+1}. As we have seen, the domain D_i contains $B(x_i, 2\varepsilon)$ and B_i contains $B(x_{i+1}, 2\lambda\varepsilon)$. Whenever $(w - b + a)$ belongs to B_i the inverse image $\varphi_i^{-1}(w - b + a) = z$ is such that $(z, w) \in H_b$. So we can construct a finite orbit of H_b using the maps φ_i (which are determined from the orbit (x_i) of H_a). Given $n \geq 0$, define $y_n(k)$ for $0 \leq k \leq n$ as follows: $y_n(n) = x_n$ and

$$y_n(k-1) = \varphi_{k-1}^{-1}(y_n(k) - b + a).$$

The sequence $(y_n(k))_{k=0}^n$ is a finite orbit of H_b. For $n < k$ we let $y_n(k)$ be any fixed constant, say, 0. The point $y_n(k)$ is always within the ε-neighborhood of x_k. In fact, the argument of successively applying φ_i^{-1} is possible only because

$$|y_n(k) - x_k| \leq |a - b| \left(\lambda^{-1} + \lambda^{-2} + \ldots + \lambda^{-(n-k)}\right) < \varepsilon,$$

for every $n \geq k$.

For a fixed k, the sequence $(y_n(k))_n$ is bounded, and as such, it has a convergent subsequence $y_{n(k,i)}(k)$, where $n(k,i)$ is a sequence indexed in i, for each fixed k. Now what we have is a sequence of sequences which may be chosen so that $(n(k+1,i))_i$ is a subsequence of $n(k,i)$, with $n(k+1,i) > n(k,k)$ for every i, k. Let y_k denote the limit of $y_{n(k,i)}(k)$ as $i \to \infty$. The diagonal sequence $n_i = n(i,i)$ is a subsequence of every sequence $n(k,0), n(k,1), \ldots, n(k,i), \ldots$ Hence $y_{n_i}(k)$ converges to y_k as $i \to \infty$ for every k. Since $(y_{n_i}(k), y_{n_i}(k+1)) \in H_b$ for all i, it follows by continuity that $(y_k, y_{k+1}) \in H_b$. We conclude that (y_k) is an orbit of H_b which satisfies $|y_k - x_k| < \varepsilon$ for every k. It remains to prove that an orbit of H_b with this property is unique.

Suppose there is another orbit (z_i) of H_b with $|z_i - x_i| < \varepsilon$ for every i. The terms of these two orbits are contained in $B_\infty(R_1)$, since $\varepsilon < (R - R_1)$. Let φ_i denote the univalent branch of H_b which takes y_i to y_{i+1}. According to claim C, the image of the ball $B(y_i, 2\varepsilon)$

under H_b is a collection of q sets which are 2ε sparse. These sets are the images of the q branches determined by the correspondence at y_i; by claim A, these branches are defined on a domain which includes the ball $B(y_i, 2\varepsilon)$. Since $|y_i - z_i| < 2\varepsilon$, the same branch φ_i which takes y_i to y_{i+1} must also take z_i to z_{i+1}. Since φ_i expands distances by the factor λ, it follows that

$$|z_i - y_i| \leq \lambda^{-k} |z_{i+k} - y_{i+k}| \leq \lambda^{-k} 2\varepsilon \to 0,$$

which implies $z_i = y_i$, for every i. This proves claim E.

Claim F. *The function $h : X_a(R) \to Y_b$ of claim E is a homeomorphism.*

The function $h : X_a(R) \to Y_b$ has a natural inverse, given by the shadowing. In fact, let $h(x) = y \in Y_b$. The sequence y is contained in $X_b(R_1)$. The same argument of claim E applies: there is a unique orbit $z = (z_i)$ of H_a such that $|z_i - y_i| < \varepsilon$ for every i. The unique difference is that now this orbit belongs to $X_a(R_2)$, and *a priori* we cannot say that it is in $X_a(R)$. But since $|z_i - x_i| < 2\varepsilon$, we have $x = z$, and the conclusion is that, indeed, $x \in X_a(R)$.

In this way we have constructed a map $g : Y_b \to X_a(R)$ which satisfies $g \circ h(x) = x$. Hence h is injective and its inverse on Y_b is g. Since both h and g are given by the shadowing of a sequence, in order to prove that h is a homeomorphism it is sufficient to show that the shadowing of a sequence $x = (x_i)$ depends continuously upon x in the product topology (whether $x \in X_a(R)$ or $x \in X_b(R_1)$; we shall deal only with the former case). If $c \in \mathbb{C}$, $\delta > 0$, $n \geq 0$ and $x = (x_i)_0^\infty$ is an orbit of H_c, then we define

$$C_c^n(x, \delta) = \left\{z = (z_i)_0^\infty : \ z \text{ is an orbit of } H_c \text{ and } |z_i - x_i| < \delta \text{ for } 0 \leq i \leq n\right\}.$$

Notice the collection of neighborhoods $C_a^n(x, \delta) \cap X_a(R)$ is a local base at $x \in X_a(R)$, if we consider all $n \geq 0$ and $\delta > 0$. To prove continuity at an arbitrary point $x^{(1)} \in X_a(R)$, let $y^{(2)} = h(x^{(1)})$. We are going to prove that for any given $\epsilon > 0$ and $n \geq 0$, there are $N \geq 0$ and $\delta > 0$ such that whenever $x^{(2)}$ is in $C_a^N(x^{(1)}, \delta) \cap X_a(R)$, the corresponding $y^{(2)} = h(x^{(1)})$ must be in $C_b^n(y^{(1)}, \epsilon)$.

We first take $k \geq 1$ such that $\lambda^{-k}\varepsilon < \epsilon$. Then let $N = n + k$ and $\delta = \varepsilon$. Supoose $x^{(2)}$ is in $C_a^N(x^{(1)}, \delta) \cap X_a(R)$. Since $|x_i^{(1)} - x_i^{(2)}| < \varepsilon$ for $0 \leq i \leq N$, the same univalent branch φ_i

which takes $x_i^{(1)}$ to $x_{i+1}^{(1)}$ must also take $x_i^{(2)}$ to $x_{i+1}^{(2)}$. Let $y^{(2)} = h(x^{(2)})$. From the definition of the map h, we have

$$\varphi_i^{-1}(y_{i+1}^{(1)} - b + a) = y_i^{(1)},$$

$$\varphi_i^{-1}(y_i^{(2)} - b + a) = y_i^{(2)}.$$

And since φ_i expands distances by the factor λ, we have

$$\left|y_i^{(1)} - y_i^{(2)}\right| \leq \lambda^{-k}\left|y_{i+k}^{(1)} - y_{i+k}^{(2)}\right| \leq \lambda^{-k}\varepsilon < \epsilon,$$

for $0 \leq i \leq n$. In other words, $y^{(2)}$ is in $C_b^n(y^{(1)}, \epsilon)$, as desired. Claim F is proved.

Claim G. *The space Y_b is invariant under the unilateral shift σ (to the left) and*

$$X_b(S_n) \subset \sigma^n(Y_b) \subset X_b(T_n),$$

as in the statement of the theorem. The function h of claim F is a topological conjugacy from $(\sigma, X_a(R))$ to (σ, Y_b).

Since the shadowing is unique, we have $h\sigma(x) = \sigma h(x)$ for every $x \in X_a(R)$. This proves that $\sigma(Y_b) \subset Y_b$ and also that the homeomorphism h is a topological conjugacy between the systems $X_a(R)$ and Y_b.

Now let $T_n = \mu^n R - \varepsilon$, where $\mu > 1$ is such that $|w| > \mu|z|$, whenever $z \in B_\infty(R_1^*)$ and $(z, w) \in H_a$. We are going to prove that

$$\sigma^n(Y_b) \subset X_b(T_n),$$

for every $n \geq 0$. In fact, every $y \in Y_b$ is written $h(x) = y$ for some $x = (x_i)_0^\infty$ in $X_a(R)$. The points of this last sequence satisfy $|x_n| \geq \mu^n R$; and since $|y_i - x_i| < \varepsilon$, we conclude that $\sigma^n(y) \in X_b(T_n)$, which proves the assertion.

Recall that whenever w is $B_\infty(R_1^*)$ and z is a pre-image of w under H_a we have

$$|z| \geq (\rho|w|)^{q/p},$$

since from the definition of ρ it satisfies $|w| - |a| \geq \rho|w|$ and $0 < \rho < 1$, for every $w \in B_\infty(R_1^*)$.

Let

$$K = \sum_{n=1}^{\infty} \left(\frac{q}{p}\right)^n, \quad S_n = \left(\frac{R}{\rho^K}\right)^{\frac{p^n}{q^n}} + \varepsilon.$$

We are going to prove that

$$\sigma^n(Y_b) \supset X_b(S_n) \qquad (n \geq 0).$$

In this way, the iterate $\sigma^n(Y_b)$ is always an open set between $X_b(S_n)$ and $X_b(T_n)$.

Every sequence $\overline{y}$ of $X_b(S_n)$ can be written in the form $\overline{y} = (y_n, y_{n+1}, \ldots)$. The inverse shadowing (cf. claim F) is a well defined map $g : Y_b \to X_a(R)$ which is nothing but the inverse of h. Let $(x_n, x_{n+1}, \ldots) = g(\overline{y})$. Since the terms of this sequences are within distance ε from the corresponding terms of the sequence $\overline{y}$, we conclude that $|x_n| \geq S_n - \varepsilon$. We aim at completing the sequence so as to form $x = (x_0, x_1, \ldots, x_n, \ldots) \in X_a(R)$. Indeed, the inverse correspondence H_a^{-1} maps each $B_\infty(S)$ into $B_\infty((\rho S)^{q/p})$. Starting with the radius $S_n - \varepsilon$, the first backward iterate is in $B_\infty(L_1)$, where $L_1 = (\rho(S_n - \varepsilon))^{q/p}$. By induction, after n backward iterates we reach $L_n = (\rho L_{n-1})^{q/p}$. Hence

$$L_n = \rho^{q/p+(q/p)^2+\cdots+(q/p)^n} S_n^{\frac{q^n}{p^n}} > \rho^K S_n^{\frac{q^n}{p^n}} = R,$$

so that the inverse of H_a^n maps $B_\infty(S_n - \varepsilon)$ into $B_\infty(R)$. By taking successive pre-images of x_n we form a sequence $x \in X_a(R)$ as indicated above. Let $y = h(x) \in Y_b$. Since h is a topological conjugacy and $g = h^{-1}$ we have

$$\sigma^n(y) = \sigma^n(h(x)) = h(\sigma^n(x)) = \overline{y}.$$

In other words, $X_b(S_n) \subset \sigma^n(Y_b)$. This completes the proof of the theorem. □

5.7. The Limit set

A *backward orbit* of H_c starting at y_0 is a sequence $(y_i)_{i=0}^{\infty}$ such that $(y_{i+1}, y_i) \in H_c$.

It is natural to define the Limit set L_c of the correspondence H_c as the closure the accumulations points of backward orbits of any point in an attracting region of infinity $B_\infty(R)$. The definition makes sense since any backward orbit is actually bounded: if we start with a point y_0 which is in $B_\infty(R)$, where $B_\infty(R)$ is an attracting region of infinity, then

it is clear that any backward orbit starting at y_0 is contained in the ball $\{|z| < |y_0|\}$.
Another definition is given by the closure of repelling periodic orbits. Recall that a periodic orbit of H_c is a finite sequence $z_0, \dots z_n = z_0$ together with branches φ_i of H_c taking z_i into z_{i+1}. The *multiplier* of the orbit is the derivative of the composition $\varphi_{n-1} \circ \cdots \circ \varphi_0$ at z_0. When the multiplier λ of the cycle satisfies $|\lambda| > 1$, the orbit is said to be *repelling*.

These two definitions yields the same set in the case of rational maps. But here the first one tends to be more general, and in some cases the set closure of repelling periodic orbits is strictly contained in the set that is obtained by taking pre-images out of an attracting region of ∞. This will be clear when we discuss questions related to hyperbolicity.

Moreover, taking accumulation points of pre-orbits has the advantage that $L_c \neq \phi$ is immediately fulfilled.

5.14. **Definition** (**Limit set**). We write $z \in G_c(y)$ if z is a sub sequential limit of a backward orbit starting at y. The Limit set L_c is the closure of the union of all $G_c(y)$, with y belonging to an attracting region of infinity.

This definition allows us to draw L_c using computer algorithms.

Recall that any point of $H_c(z)$ is called an *image* of z. Similarly, every point z with $(z, w) \in H_c$ is a *pre-image* of w. We say that A is *forward semi-invariant* under H_c if every point in A has at least one image which is still in A. If every point in A has at least one pre-image which is still in A we say that A is *backward semi-invariant.* The term *semi-invariant* alone indicates that A is both forward and backward semi-invariant.

5.15. **Remark**. Notice that the Limit set is always a compact set contained in $\mathbb{C}$. Although we have not yet defined the concept of hyperbolicity, we anticipate that when H_c is hyperbolic and satisfies the escaping condition (to be defined later), the Limit set consist of two disjoint semi-invariant compact sets: one is the closure of repelling periodic orbits, the *Julia set* J_c; the other is a cycle of Cantor sets obtained from a IBS of second type, the *dual Julia set* E_c

Hence, hyperbolicity still implies expanding behavior on the Julia set (as for rational maps), but also the coexistence of both attracting and expanding properties which partition L_c (as in the case of stable and unstable manifolds for diffeomorphisms). This is one of the most surprising facts about the dynamics of the correspondence H_c.

5.16. **Theorem (Invariance).** *The Limit set L_c of H_c is semi-invariant, in the sense that every $z \in L_c$ has at least one image $w \in L_c$, and that for every $w \in L_c$ there is at least one pre-image $z \in L_c$.*

Proof. In fact, we are going to prove that the set of subsequential limits $G_c(y)$ (defined together with L_c) is semi-invariant. Let y be a point of an attracting region of ∞.
Let $z \in G_c(y)$, i.e., there is a pre-orbit $y_0 = y, \ldots, y_n, \ldots$ staring at y and a subsequence $y_{n(k)}$ which satisfy

$$\left|z - y_{n(k)}\right| < 1/k.$$

In the first case we assume that $z \neq 0, c$, so that the images and pre-images of points near z are determined by q forward branches $\varphi_i : D_z \to \mathbb{C}$ and p backward branches $\psi_j : D_z \to \mathbb{C}$. The sequence y_n leaves and enters the domain D_z infinitely often. Hence, there is a subsequence, which we again denote by $y_{n(k)}$, so that

$$\varphi\left(y_{n(k)}\right) = y_{n(k)-1},$$

for every $k \geq k_0$, for the same forward branch φ. Similarly, there is a branch ψ of H_c^{-1} such that $\psi(y_{n(k)}) = y_{n(k)+1}$. Now the subsequence $y_{n(k)-1}$ converges to $\varphi(z) \in G_c(y)$, while $y_{n(k)+1}$ converges to $\psi(z) \in B_c(z)$.

The case $z = 0$ is even simpler. Although there is no single valued branch at $z = 0$, the correspondence maps points near to $z = 0$ to points which are near to $w = c$. Hence, whenever 0 belongs to $G_c(y)$, so does c. The same argument applies to pre-images of points near c, and we conclude that the two assertions $0 \in G_c(y)$ and $c \in B_c(y)$ occur simultaneously. This proves that $G_c(y)$ is semi-invariant.

Since L_c consists of the closure the union of all such $G_c(y)$, it follows that L_c is also semi-invariant. Indeed, let z be a point of L_c, with a sequence z_n of points $z_n \in B_n(y_n)$

converging to z. Suppose z is neither zero, nor c. The correspondence H_c at z is determined by finitely many forward branches; likewise, H_c^{-1} is also determined by finitely many backward branches. For each z_n there is one forward branch φ_n which takes z_n to a point inside of $G_c(y_n)$. As there are only finitely many possible choices among these maps, by taking subsequences we may suppose (without loss of generality) φ_n is always the same branch φ and the conclusion is that

$$\varphi(z) = \lim_{n\to\infty} \varphi(z_n) \in \bigcup_n \overline{{}_c(y_n)} \subset L_c.$$

This proves that L_c forward semi-invariant. The same reasoning shows that L_c is also backward semi-invariant (the cases 0 and c are handled in the same manner). $\square$

CHAPTER 6

Structural stability at hyperbolic parameters

A cycle is a periodic orbit $z_0 \to z_1 \cdots \to z_n = z_0$, where $(z_i, z_{i+1}) \in H_c$. Every cycle has a naturally associated complex number, called its *multiplier.* If the cycle contains no zero elements, then every point z_n determines an essentially unique branch φ_n of H_c (up to domain extensions) which takes z_n into z_{n+1}. The *multiplier* of this orbit is

$$\lambda = \frac{d\varphi_{n-1} \circ \cdots \circ \varphi_0(z)}{dz}\bigg|_{z=z_0}.$$

If one of the elements of the cycle is 0, or ∞ (notice that ∞ is a fixed point) we set $\lambda = 0$, by convention. This convention, however, has a meaningful dynamic justification. If the first point $z_0 = 0$ is zero, for example, then the composition of branches (instead of branch, at the nonzero element we consider $\varphi_0 = H_c$) yields a multivalued map $f : D \to D$ from a neighborhood D of zero. There is a constant C such that

$$|f(z) - 0| < C|z|^{p/q} \text{ on } D.$$

Since $p/q > 1$, this shows that f becomes more contractive the closer the point z is from zero. The same effect happens at ∞ if we consider the coordinate change $\zeta = 1/z$ for z near zero. The cycle is *attracting* if $|\lambda| < 1$. We call a cycle *super-attractive* whenever its multiplier λ is zero.

6.1. **Definition** (**Hyperbolic** H_c). If H_c has an attracting cycle, then we say that H_c is hyperbolic.

6.2. **Remark**. In particular, if H_c has a IBS of first type, then H_c is hyperbolic. On the other hand, there are cases where H_c is hyperbolic in the absence of IBS of first type: $c = 0$ is one example.

We are going to see in the next theorem that every IBS of second type contain a Conformal Iterated Function System (CIFS) in a natural way. As a consequence, we have

6.3. **Theorem (Infinitely many attracting cycles)**. *Every IBS of second type contains infinitely many attracting cycles.*

Proof. Let

$$\mathcal{A}: \ \overline{D}_0 \xrightarrow{F_0} \overline{D}_1 \xrightarrow{F_1} \overline{D}_2 \xrightarrow{F_2} \cdots \xrightarrow{F_{N-1}} \overline{D}_N \subset D_0$$

be a IBS of second type, where F_0 is the restriction of the correspondence H_c to $\overline{D}_0$. Since $\overline{D}_N$ is a univalent disk, there is a simply connected open set V containing $\overline{D}_N$ such that

$$F_0(V) = \bigcup_{k=0}^{q-1} \varphi_k(V)$$

is a disjoint union, being φ_k the q univalent branches of H_c determined on V. Each composition

$$T_k = F_{N-1} \circ F_{N-2} \cdots \circ F_1 \circ \varphi_k$$

maps $\overline{D}_N$ into its interior D_N. In fact, $\{T_k(\overline{D}_N)\}_{k=0}^{q-1}$ is a disjoint collection of closed topological disks inside of D_N. This constitutes a conformal iterated function system on D_N, since each map T_k uniformly contracts the Poincaré metric on D_N (from the second iterate on).

Let $\Sigma_q = \{k = (k_0, k_1, \ldots, k_n, \ldots) : \ k_i = 0, \ldots, (q-1)\}$. Consider the map

$$\psi(k) = \bigcap_{n=0}^{\infty} T_{k_0} \circ T_{k_1} \circ \cdots \circ T_{k_n}(\overline{D}_N),$$

from Σ_q into the the first Cantor set $\mathcal{K}_0$ contained in the IBS $\mathcal{A}$. For a periodic sequence $k \in \Sigma_q$ with period n, notice that

$$f = T_{k_0} \circ T_{k_1} \circ \cdots \circ T_{k_{n-1}}$$

satisfies

$$f\psi(k) = \psi(k).$$

Hence, $\psi(k)$ is a fixed point of the correspondence H_c^{nN}; in other words, a periodic point. As there are infinitely many periodic points in Σ_q (under the shift map), the same must be true on $\mathcal{K}_0$, since ψ is injective (recall theorem 5.11). Since f is defined on V and f maps the compact set $\overline{D}_N$ into the interior D_N, this function uniformly contracts the hyperbolic metric of V on $\overline{D}_N$. Hence, the multiplier λ of the periodic point $\psi(k)$ has norm less than 1. In this way, we obtain infinitely many attracting periodic orbits inside any IBS of second type. □

6.1. Normal families

It is now time state one of the most important tools in the study of iteration of holomorphic maps: Montel's Theorem.

Consider the Riemann sphere $\hat{\mathbb{C}}$ with its spherical metric. Let U be a connected open subset of $\hat{\mathbb{C}}$. We say that a sequence of functions $f_n : U \to \hat{\mathbb{C}}$ converges *locally uniformly* to some $f : U \to \hat{\mathbb{C}}$ if every point of U has a neighborhood V on which $f_n|_V$ converges uniformly to $f|_V$. Equivalently, f_n converges locally uniformly to f if, and only if, $f_n|_K$ restricted to any compact set $K \subset U$ converges uniformly to $f|_K$.

As usual, we denote the higher order derivatives of a complex function f inductively by

$$f^{(n)} = \left(f^{(n-1)}\right)'.$$

6.4. **Theorem (Weierstrass)**. *If a sequence of analytic functions $f_k : U \to \hat{\mathbb{C}}$ from a connected open set U converges locally uniformly to $f : U \to \hat{\mathbb{C}}$, then f is also analytic. The sequence of derivatives $f_k^{(n)}$ of fixed order n converges locally uniformly to $f^{(n)}$ on U for every n.*

6.5. **Definition (Normality)**. Let U be a connected open subset of $\hat{\mathbb{C}}$. A sequence of holomorphic functions $f_n : U \to \hat{\mathbb{C}}$ is said to be normal if every subsequence of f_n has another subsequence which converges locally uniformly to some function $U \to \hat{\mathbb{C}}$.

Although the case of normal families $f_n : U \to \mathbb{C}$ is included in the case of maps onto $\hat{\mathbb{C}}$, in some cases we need an alternative definition which does not involve the spherical metric. In fact, it may happen that a sequence of a sequence of maps $U \to \mathbb{C}$ which converge locally uniformly to some function $U \to \hat{\mathbb{C}}$ does not converge locally uniformly to any map $U \to \mathbb{C}$.

We say that a sequence $f_n : U \to \mathbb{C}$ *escape to infinity* if for every compact set $K \subset U$ and every compact set $K' \subset \mathbb{C}$ we have $f_n(K) \cap K' = \phi$ for n sufficiently large.

6.6. **Proposition** (**Normality for maps onto** $\mathbb{C}$). *Let U be a connected open subset of the Riemann sphere $\hat{\mathbb{C}}$. A sequence of holomorphic maps $f_n : U \to \mathbb{C}$ is normal if, and only if, every subsequence of f_n contains either a subsequence which converges locally uniformly to some function $U \to \mathbb{C}$, or a subsequence which escape to infinity.*

A sequence of maps $f_n : U \to \hat{\mathbb{C}}$ *omits three points* if there is a set $Q \subset \hat{\mathbb{C}}$ containing three points such that $f_n(U) \subset \hat{\mathbb{C}} \setminus Q$ for every n.

6.7. **Theorem** (**Montel**). *Let U be a connected open subset of $\hat{\mathbb{C}}$. Every sequence of holomorphic maps $f_n : U \to \hat{\mathbb{C}}$ omitting three points is normal.*

6.8. **Remark**. This Theorem has one immediate surprising consequence: if $f_n : U \to \hat{\mathbb{C}}$ is not normal in a small neighborhood U of a point, then there are at least two points a and b in $\hat{\mathbb{C}}$ such that

$$\bigcup_{n=1}^{\infty} f_n(U) \supset \hat{\mathbb{C}} \setminus \{a, b\}.$$

6.2. Critical IBS

Recall that if

$$\mathcal{A} = (D_0, D_1, \ldots, D_N, F_0, \ldots, F_{N-1})$$

is a IBS of first type, then none of the topological disks D_i contains the critical point 0. As a consequence, none of $D_1, \ldots, D_N$ contain the critical value c. However, nothing prevents that $c \in D_0$. In fact, such IBS play a very important role. Unless there is no attracting cycle for the correspondence, they always exist. Furthermore, they are responsible for the existence of invariant Cantor sets whenever H_c is hyperbolic.

6.9. **Definition (Critical IBS).** Let $\mathcal{A} = (D_0, \dots D_N, F_0, \dots, F_{N-1})$ be a IBS of first type. We say that $\mathcal{A}$ is a critical if $c \in D_0$.

It should be noticed that every IBS of second type contains a critical IBS (just disregard the first disk D_0; we invite the reader to check from the definition). The converse is also true: if $\mathcal{A} = (D_0, \dots, D_N, F_0, \dots, F_{N-1})$ is a critical IBS, then by introducing the new topological disk $D_{-1} = H_c^{-1}(D_0)$ we get a IBS of second type, namely,

$$D_{-1} \xrightarrow{H_0} D_0 \xrightarrow{F_0} D_1 \cdots \xrightarrow{F_{N-2}} D_{N-1} \subset D_{-1}.$$

Hence,

Every critical IBS may be identified with a IBS of second type.

Another way of expressing an IBS of first type $\mathcal{A} = (\overline{D}_0, \dots, \overline{D}_N, F_0, \dots, F_{N-1})$ takes into account the following sequence of maps

$$\mathcal{A} : D_0 \xrightarrow{\varphi_0} D_1 \xrightarrow{\varphi_1} D_2 \cdots \xrightarrow{\varphi_n} D_{n+1} \xrightarrow{\varphi_{n+1}} \cdots$$

where the regions are defined inductively by $D_{n+1} = \varphi_n(D_n)$. Therefore $D_{k+N} \subset D_k$ and, by definition,

$$\varphi_0 = F_0,$$
$$\vdots$$
$$\varphi_{N-1} = F_N.$$

For all the other maps, the restriction of φ_k to D_{k+N} is the bi-holomorphic map

$$\varphi_{k+N} : D_{k+N} \to D_{k+N+1}.$$

There is an advantage in doing so since it offers a better language for dealing with extensions. The number N is the *period* of the IBS.
The formal shift map σ is defined as $\sigma(\mathcal{A}) = D_1 \xrightarrow{\varphi_1} D_2 \xrightarrow{\varphi_2} D_3 \cdots$

6.10. **Definition (Extension).** For any two IBS of first type $\mathcal{B}$ and C, we say that C is an extension of $\mathcal{B}$ (and write $C \succ \mathcal{B}$) if $\sigma^k(C) = \mathcal{B}$ for some integer $k \geq 0$.

6.11. **Lemma**. *Suppose $\mathcal{A}, \mathcal{B}$ and C are IBS of first type.*

(i) *If $\mathcal{A} \succ \mathcal{B}$ and $C \succ \mathcal{B}$, then either $\mathcal{A} \succ C$ or $C \succ \mathcal{A}$.*

(ii) *If $\mathcal{B}$ and C are critical IBS and $\mathcal{B} \succ C$, then $\mathcal{B} = C$.*

Proof. Let us denote the IBS $\mathcal{B}$ by $D_0 \xrightarrow{\varphi_0} D_1 \cdots$, with period N.

Suppose there are two extensions

$$D_{-1} \xrightarrow{\alpha_1} D_0 \xrightarrow{\varphi_0} D_1 \xrightarrow{\varphi_1} D_2 \cdots$$

and

$$\check{D}_{-1} \xrightarrow{\beta_1} D_0 \xrightarrow{\varphi_0} D_1 \xrightarrow{\varphi_1} D_2 \cdots$$

(Recall that every map in the extension must be bi-holomorphic, by definition). It follows that both $D_{-1}, \check{D}_{-1}$ contain D_{N-1} and $D_0 \supset D_N$. The restriction of α_1^{-1} to D_N equals φ_{N-1}^{-1}, as well as the restriction of β_1^{-1} to D_N equals φ_{N-1}^{-1}. Since the maps involved are holomorphic, we conclude that $\alpha_1^{-1} = \beta_1^{-1}$. Therefore $D_{-1} = \check{D}_{-1}$ and $\alpha_1 = \beta_1$. This argument may be carried out for any two finite extensions

$$D_{-n} \xrightarrow{\alpha_n} \cdots D_{-2} \xrightarrow{\alpha_2} D_{-1} \xrightarrow{\alpha_1} D_0 \xrightarrow{\varphi_0} D_1 \xrightarrow{\varphi_1} D_2 \cdots$$

and

$$\check{D}_{-n} \xrightarrow{\beta_n} \cdots \check{D}_{-2} \xrightarrow{\beta_2} \check{D}_{-1} \xrightarrow{\beta_1} D_0 \xrightarrow{\varphi_0} D_1 \xrightarrow{\varphi_1} D_2 \cdots$$

As soon as the length in both extensions is the same n, the conclusion is that $\alpha_i = \beta_i$ and $D_{-i} = \check{D}_{-i}$ for any $0 < i \leq n$. The item (i) follows easily from this.

The second assertion follows from that fact that no critical IBS can be further extended without including the critical point 0, which is a contradiction, since no IBS of first type is allowed to include 0 in any of its disks. This completes the proof. □

6.12. **Theorem**. *Let $\mathcal{B}$ be a IBS of first type of H_c. There is a unique critical IBS C with $C \succ \mathcal{B}$.*

Notice that in assuming that H_c has a IBS of first type it is implicit that $c \neq 0$, because H_0 has no such IBS. The proof of this theorem requires the following:

6.13. **Lemma**. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain which does not contain the critical value c. For every $a \in \Omega$ and every $b \in H_c^{-1}(a)$ there is a unique branch φ of H_c^{-1} defined on Ω with $\varphi(a) = b$. This branch is necessarily injective (univalent).*

Proof. Let us consider the Riemann surface

$$W = \{(z, w) : \ (w - c)^q = z^p, \ z \neq 0\}.$$

The function $\sigma(z, w) = w$ defines a covering map $W \to \mathbb{C} \setminus \{c\}$. Let $g : \Omega \to W$ be the unique lift of the identity $I : \Omega \to \Omega$ with $g(a) = (b, a)$. It is clear that $g(\Omega)$ is an open set in W and that $g : \Omega \to g(\Omega)$ is bi-holomorphic. If we consider the projection $\tau(z, w) = z$ defined on W, then

$$\varphi = \tau \circ g : \Omega \to \mathbb{C}$$

is a branch of H_c^{-1} with $\varphi(a) = b$.

Uniqueness. Any branch ψ of H_c^{-1} which is defined on Ω and takes a into b is equal to φ. In fact, the map $f(w) = (\psi(w), w)$ – defined on Ω – is a lift of the identity to the covering space W which takes a into (b, a). Since the lift is unique (once fixed the base-points), it follows that $f = g$ and, consequently, $\varphi = \psi$.

It remains to show that φ is injective. Of course, this is the same thing as showing that φ is a bi-holomorphic map onto its image (which is necessarily an open set). There is a branch $\theta(w)$ of the multi-valued function $\arg(w - c)$ of the complex variable w which is defined on Ω. The range of the function $\theta(w)$ is some open interval (s, t) of length $t - s \leq 2\pi$. All these choices are possible due to the fact that Ω is simply connected and does not contain c.

Let $w_0 \neq w_1$ in Ω. We are going to show that $z_0 = \varphi(w_0)$ is different from $z_1 = \varphi(w_1)$. Join the points w_0 and w_1 by a smooth arc $\gamma : [0, 1] \to \Omega$. Since

$$\varphi(\gamma(t))^p = (\gamma(t) - c)^q = |\gamma(t) - c|^q \cdot e^{i\theta(\gamma(t))},$$

it follows that

$$\delta(t) = \arg \varphi(\gamma(t)) - \frac{q}{p}\theta(\gamma(t)) = \frac{2k_t\pi}{p} + 2\pi\mathbb{Z} \subset \mathbb{R},$$

for some[1] integer k_t. The sets $\delta(t)$ vary continuously with respect to t. This implies $k_0 = k_t$ for every t. Therefore,

$$\arg\varphi(w_i) = \frac{q}{p}\theta(\gamma(i)) + \frac{2k_0\pi}{p} + 2\pi\mathbb{Z},$$

and

$$\arg\varphi(w_1) - \arg\varphi(w_0) = \frac{q}{p}(\theta_0 - \theta_1) + 2\pi\mathbb{Z},$$

where $\theta_i = \theta(\gamma(i))$. Since the quotient $q(\theta_0 - \theta_1)/p$ is nonzero and strictly less than 2π, it follows that z_0 and z_1 have different arguments modulo 2π. Hence $z_0 \neq z_1$. □

Proof of Theorem 6.12. The proof is based on successive applications of Lemma 6.13. Suppose we have a IBS of first type $\mathcal{B}$, with period N, given by

$$D_0 \xrightarrow{\varphi_0} D_1 \xrightarrow{\varphi_1} D_2 \xrightarrow{\varphi_2} \cdots$$

Existence. Consider the function $D_{N-1} \xrightarrow{\varphi_{N-1}} D_N$. Choose an arbitrary $a \in D_N$ and let $b = \varphi_{N-1}^{-1}(a)$. By Lemma 6.13 there is branch $g_1 : D_0 \to \mathbb{C}$ of H_c^{-1} which is defined on D_0 and satisfies $g_1(a) = b$. This is possible since D_0 is simply connected. Let $\alpha_1 = g_1^{-1}$ and set $D_{-1} = g_1(D_0)$. It should be noticed that the restriction of g_1 to $D_N \subset D_0$ is the original map φ_{N-1}^{-1}, since the branch is uniquely determined by the property $a \mapsto b$. Because of this fact, the following sequence of maps

$$D_{-1} \xrightarrow{\alpha_1} D_0 \xrightarrow{\varphi_0} D_1 \xrightarrow{\varphi_1} D_2 \xrightarrow{\varphi_2} \cdots$$

is now another IBS of first type (which extends $\mathcal{B}$). The procedure may continue indefinitely unless we reach a sequence

$$D_{-n} \xrightarrow{\alpha_n} D_{1-n} \xrightarrow{\alpha_{n-1}} \cdots D_{-2} \xrightarrow{\alpha_2} D_{-1} \xrightarrow{\alpha_1} D_0 \xrightarrow{\varphi_0} D_1 \xrightarrow{\varphi_1} D_2 \xrightarrow{\varphi_2} \cdots \tag{6.1}$$

which cannot be further extended because $c \in D_{-n}$. All disks up to D_{-n} are simply connected, and there is always a branch of H_c^{-1} defined on these sets provided none includes c. The topological disk D_{-n} is the unique disk which includes c, and the IBS(6.1) is critical.

[1]The argument function is multi-valued. So for each $z \neq 0$ in the plane, we may consider $\arg(z)$ as a set of the form $\alpha + 2\pi\mathbb{Z}$. This terminology works better here than the notation $\bmod 2\pi$.

But how to guarantee that there is always some D_{-n} containing c? The proof is by reduction to absurd. If D_{-n} does not contain c, then we can construct the next simply connected region $D_{-(n+1)}$ and the corresponding bi-holomorphic map $D_{-(n+1)} \xrightarrow{\alpha_{n+1}} D_{-n}$ as before. Since these regions give rise to a IBS of first type, we have

$$D_0 \subset D_{-N} \subset D_{-2N} \subset D_{-3N} \cdots$$

and each of these sets contains neither c, nor 0. (Notice that $c \neq 0$ because H_0 has no IBS of first type). Let

$$f_k = \alpha_{-kN} \circ \cdots \circ \alpha_{(1-k)N} : D_{-kN} \to D_{(1-k)N}.$$

There is a common fixed point $z_0 \in D_0$ of all maps f_k. This follows from proposition 5.9. In fact, each map f_k is an extension of the original map

$$\phi = \varphi_{N-1} \circ \cdots \circ \varphi_0 : D_0 \to D_N,$$

and the latter has a unique fixed point due to proposition 5.9. The multiplier $\lambda = \phi'(z_0)$ satisfies $|\lambda| < 1$. The sequence

$$h_k : (f_1 \circ \cdots \circ f_k)^{-1} : D_0 \to D_{-kN}$$

is a normal family because $\{0, c, \infty\}$ is outside its range (Montel's theorem). Hence, either h_k scape to infinity or h_k converges locally uniformly to some holomorphic function $h : D_0 \to \mathbb{C}$. The latter turns out to be the case since the sequence fixes z_0. By Weierstrass theorem, h_k' converges locally uniformly h' on D_0. On the other hand, $h_k'(z_0) = \lambda^{-k} \to \infty$, and because of this fact the family is not normal. This is a contradiction. Hence some disk D_{-n} must contain c.

Uniqueness. Suppose $\mathcal{C}$ and $\mathcal{A}$ are two IBS which extend $\mathcal{B}$. Then one of them must extend the other, say, $\mathcal{C} \succ \mathcal{A}$. From Lemma 6.11 we have $\mathcal{C} = \mathcal{A}$. □

That IBS of second type contain IBS of first type is obvious. In fact, any IBS of second type gives rise to a critical IBS. What the above Theorem reveals is that any IBS of first type also gives rise to a IBS of second type.

6.14. **Definition (Critical cycles).** We say that a cycle

$$z_0 \mapsto z_1 \mapsto \cdots \mapsto z_n = z_0$$

is *critical* if one of its members z_i is a critical point (either 0 or ∞).

6.15. **Definition** ($\mathcal{P}^{\odot}$, $\mathcal{P}^*$). Let $\mathcal{P}^{\odot}$ denote the set of super-attracting cycles of H_c (those which have multiplier $\lambda = 0$). Let $\mathcal{P}^*$ denote the set of attracting cycles which are not super-attracting (multiplier satisfy $0 < |\lambda| < 1$).

There is only one critical cycle containing ∞. A *non-critical cycle* is by definition a cycle which does not contain any critical point.

6.16. **Corollary.** *IBS of first and second type occur simultaneously, i.e., each one implies the existence of the other. Moreover,*

(i) *$\mathcal{P}^{\odot}$ is the set of critical cycles.*
(ii) *Every member of $\mathcal{P}^*$ is a non-critical cycle and the cardinality of $\mathcal{P}^*$ is either* 0 *or* ∞.

Proof. A cycle which does not contain any critical point has nonzero multiplier. On the other hand, every cycle which contains 0 or ∞ has zero multiplier. Therefore, if A has one attracting cycle, then this cycle is associated with a IBS of first type. By Theorem 6.12 this IBS is extend to a IBS of second type. From Theorem 6.3 H_c has infinitely many attracting periodic orbits inside the IBS. The multiplier of each of these orbits is never zero because they are given by the derivatives of the univalent branches which determine the IBS. Hence the cardinality of A is ∞ in this case. □

6.17. **Remark.** This shows how rich can be the periodic orbit structure of H_c. The number of attracting periodic orbits of a rational function is always finite.

6.2.1. The post-critical set. The post-critical set P_c is defined as the closure of the positive forward orbits of the critical point 0. Put in different terms, let S^+ denote the set of all $y \in \mathbb{C}$ for which there are $N > 0$ and $y_0, \ldots, y_N$ such that $(y_i, y_{i+1}) \in H_c$, with $y_0 = 0$ and $y = y_N$.

6.18. **Definition (Post-critical set).** The *post-critical set* P_c is the closure of S^+.

It should be noticed that, unless 0 is periodic, S^+ does not contain the critical point 0. Since H_c is multi-valued, it turns out that the structure of P_c may be very complicated, even when H_c is hyperbolic. We shall examine this set under a natural condition on the branches of the correspondence: the escaping condition. Under this condition, P_c is a Cantor set and $\hat{\mathbb{C}} \setminus P_c$ is a hyperbolic Riemann surface.

This condition may be introduced in two different levels: for critical cycles and attracting non-critical cycles. In either case, if one assumes that a critical cycle is escaping, then there is only one such cycle and no attracting non-critical cycle exists.

Similarly, if there is one attracting non-critical cycle α which is escaping, then all attracting but not super-attracting cycle comes from the same IBS of second type $\mathcal{A}$ which determines α. Hence every attracting but not super-attracting cycle will be escaping. In this case, there is no critical cycle except $\infty \mapsto \infty$. We are going to define this condition precisely in the following section.

6.3. Escaping condition

Suppose $\alpha : z_0 \mapsto z_1 \mapsto \cdots \mapsto z_n = z_0$ is a critical cycle of H_c. We may assume that $z_0 = 0$. For each point $z_i \neq 0$ there is a unique bi-holmorphic branch of the correspondence $\varphi_i : D_i \mapsto D_{i+1}$ which takes $z_i \mapsto z_{i+1}$. The critical point $z_0 = 0$ is the exception; in this case, the map φ_0 is the correspondence H_c which maps 0 onto c and every nearby point of 0 onto q different images near to c.

6.19. **Definition (Escaping cycles in $\mathcal{P}^{\odot}$).** Let α be a critical cycle determined by bi-holomorphic maps $\varphi_i : D_i \mapsto D_{i+1}$, for $0 < i < n$. Suppose the first point of α is the critical point 0 and $\varphi_0 = H_0$. We say that α is *escaping* if any other univalent branch

$$\psi_i : \check{D}_i \mapsto \check{D}_{i+1}$$

at z_i, with $\check{D}_i \subset D_i$ and $\psi_i(z_i) \neq z_{i+1}$ for $i > 0$ has the property that $\psi_i(\check{D}_i) \subset B_\infty(R)$, for some attracting region of infinity $B_\infty(R)$.

Notice that in this definition no restriction is made on the first map $\varphi_0 = H_c$. The case $c = 0$ is included as escaping although the maps φ_i for $i > 0$ do not exist in this case.

6.20. **Proposition**. *Suppose there is a critical cycle $\alpha \in \mathcal{P}^{\odot}$ which is escaping. Then there is no finite critical cycle $\beta \neq \alpha$ and no IBS of second type. Consequently,*

$$\mathcal{P}^{\odot}(H_c) = \{\alpha,\ \infty \mapsto \infty\} \text{ and } \mathcal{P}^*(H_c) = \phi.$$

Proof. Let us denote the critical cycle α by $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_n = z_0$, with $z_0 = 0$ and univalent branches $\varphi_i : D_i \to D_{i+1}$ taking z_i onto z_{i+1}. Suppose there is another critical cycle β determined by $w_0 \mapsto \cdots \mapsto w_k = w_0 = 0$, with $k \geq n$ and $\psi_i : \check{D}_i \to \check{D}_{i+1}$ taking w_i onto w_{i+1}. Of course, no point of β can be mapped to an attracting region of infinity. Since $w_1 = z_1 = c$, it follows that $\psi_1 = \varphi_1$ in a common neighborhood of z_1. In particular, $z_2 = w_2$. We use this argument repeatedly until $z_{n-1} = w_{n-1}$. The conclusion is that $\psi_{n-1} = \varphi_{n-1}$ in a common neighborhood of z_{n-1}, since α is escaping. Hence $\alpha = \beta$.

Recall that every attracting cycle which is not super-attracting gives rise to a IBS of first type which can be extended to a critical IBS. A critical IBS, on its turn, is identified with an attracting IBS of second type. Once in the presence of a IBS of second type, there is an orbit $0 \mapsto \zeta_1 \mapsto \zeta_2 \mapsto \cdots$ with $\zeta_i \neq 0$ for every $i > 0$. Hence, for every $i > 0$ there is a unique univalent branch ψ_i of H_c which takes ζ_i into ζ_{i+1}. Since α is escaping and $\zeta_1 = c = z_1$, we conclude that $\varphi_1 = \psi_1$ in a common neighborhood of z_1. The repetition of this argument yields a contradiction: that $\zeta_n = 0$. Therefore, there is no IBS of second type. □

In view of this proposition, the structure of P_c is the simplest when there is a finite super-attracting cycle which is escaping. Unless $c = 0$ and P_c consists of a single point, in all other cases where there is a escaping critical cycle α, we have

$$P_c \cap \{z \in \mathbb{C} : |z| < R\} = \alpha,$$

where $B_\infty(R)$ is an attracting region of infinity.

6.3.0.1. *Essentially unique IBS.* Let

$$\alpha : z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots \mapsto z_n = z_0$$

be an attracting cycle which is not critical. There is an *essentially unique IBS of first type*

$$\mathcal{A} : \overline{D}_0 \xrightarrow{\varphi_0} \overline{D}_1 \xrightarrow{\varphi_1} \overline{D}_2 \cdots \xrightarrow{\varphi_{n-1}} \overline{D}_n \subset D_0$$

associated with α. By essentially unique we mean that any other IBS of first type $\mathcal{B}$ given by

$$\mathcal{B} : \overline{E}_0 \xrightarrow{\phi_0} \overline{E}_1 \xrightarrow{\phi_1} \overline{E}_2 \cdots \xrightarrow{\phi_{n-1}} \overline{E}_n \subset E_0$$

with $z_i \in E_i$ and $\phi(z_i) = z_{i+1}$ must satisfy the property that $\varphi_i = \phi_i$ on neighborhood of z_i contained in the intersection $D_i \cap E_i$. Given a connected open set U and a holomorphic map $\rho : U_0 \to \mathbb{C}$ from a smaller open set $U_0 \subset U$, there is unique extension of ρ to a holomorphic map $U \to \mathbb{C}$. Hence, the fact that ϕ_i and φ_i coincide on their common subdomain implies they must considered the same function up to domain extension. This justifies the name *essentially unique*. For any two IBS of first type $\mathcal{A}$ and $\mathcal{B}$ which are related in this way, we write

$$\mathcal{A} \simeq_\alpha \mathcal{B}.$$

We say that a IBS of first type $\mathcal{A} = (\overline{D}_0, \ldots, \overline{D}_N, F_0, \ldots, F_{N-1})$ contains a cycle $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_n = z_0$ if $z_i \in D_i$ and $F_i(z_i) = z_{i+1}$ for all i.

6.21. **Remark.** In the notation $\mathcal{A} \simeq_\alpha \mathcal{B}$ it is implicit that $\mathcal{A}$ and $\mathcal{B}$ are IBS of first type containing the cycle α.

The same concept applies to IBS of second type C, since the critical IBS associated $\sigma(C)$ is a IBS of first type.

For an attracting cycle $\alpha \in \mathcal{P}^\star(H_c)$ there is an essentially unique critical IBS $\mathcal{A}$ containing α.

6.22. **Definition** ($\mathscr{C}_\alpha, \mathcal{A}^\bullet$). Let $\mathscr{C}_\alpha$ denote the class of all critical IBS $\mathcal{B}$ containing α such that $\mathcal{A} \simeq_\alpha \mathcal{B}$.

(i) The set $\mathscr{C}_\alpha$ does not depend on the initial choice of $\mathcal{A}$.

(ii) If $\mathcal{A} \in \mathscr{C}_\alpha$, then $\mathcal{A}^\bullet$ denotes the critical IBS associated.

The definition (*ii*) before is explained as follows: let

$$\alpha : z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots \mapsto z_n = z_0$$

be an attracting cycle which is not critical. There is an essentially unique IBS of first type

$$\mathcal{A} : \overline{D}_0 \xrightarrow{\varphi_0} \overline{D}_1 \xrightarrow{\varphi_1} \overline{D}_2 \cdots \xrightarrow{\varphi_{n-1}} \overline{D}_n \subset D_0.$$

The IBS $\mathcal{A}$ has a unique critical extension (Theorem 6.12). This extension has the same period of $\mathcal{A}$ and for this reason we denote it by the same letter $\mathcal{A}$. Now let $D_{-1} = H_c^{-1}(D_0)$ and $\varphi_{-1} = H_c$. The correspondence φ_{-1} maps $\overline{D}_{-1}$ onto $\overline{D}_0$. If we set inductively $D_{i+1} = \varphi_i(D_i)$, where φ_{i+n} is the restriction of φ_i to $D_{i+n} \subset D_i$, then the following sequence of maps

$$\mathcal{A}_{-1} : D_{-1} \xrightarrow{\varphi_{-1}} D_0 \xrightarrow{\varphi_0} D_1 \xrightarrow{\varphi_1} D_2 \cdots \xrightarrow{\varphi_{n-1}} D_n \xrightarrow{\varphi_n} D_{n+1} \xrightarrow{\varphi_{n+1}} \cdots D_{2n-1} \xrightarrow{\varphi_{2n-1}} D_{2n} \xrightarrow{\varphi_{2n}} \tag{6.2}$$

becomes a IBS of second type with some period ℓn. The value of ℓ is that necessary to make $D_{\ell n-1}$ into a univalent disk. There is no a priori reasoning which implies that $\ell = 1$. In fact, in order to obtain a IBS of second type we have to iterate the disks until reach a small disk $D_{\ell n-1} \subset D_{-1}$ which is univalent.

6.23. **Definition** (**Escaping critical IBS**). Let $\mathcal{B}$ be a critical IBS of first type defined by

$$\overline{D}_0 \xrightarrow{\varphi_0} \overline{D}_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} \overline{D}_n \subset D_0.$$

We say that $\mathcal{B}$ is escaping if there is an attracting region of infinity $B_\infty(R)$ such that

$$\bigcup_{i=0}^{n} \overline{D}_i \subset D_R := \{z \in \mathbb{C} : |z| < R\}$$

and

$$H_c(\overline{D}_i) \cap D_R = \varphi_i(\overline{D}_i),$$

for $i = 0, \ldots, n-2$.

We say that cycle $\alpha \in \mathcal{P}^*(H_c)$ is *escaping* if there is $\alpha \in \mathscr{C}_\alpha$ such that α is escaping.

6.24. **Definition** (**Escaping condition for H_c**). Suppose H_c is hyperbolic. We say that H_c satisfies the escaping condition if there is $\alpha \in \mathcal{P}^{\odot}(H_c) \cap \mathcal{P}^*(H_c)$ which is escaping.

The escaping condition may happen in two different situations. In the first, there is a critical cycle $\alpha \in \mathcal{P}^{\odot}(H_c)$. In the second there is a cycle in $\mathcal{P}^*(H_c)$ which is escaping. The two cases do not happen simultaneously. In the presence of escaping critical cycles, we say that H_c is *singular escaping*. In the presence of escaping cycle $\alpha \in \mathcal{P}^*(H_c)$ we say that H_c is *non-singular escaping*.

If α is given by $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_n = z_0$, then with a certain abuse of notation we denote

$$\alpha = \{z_0, \ldots, z_{n-1}\}.$$

6.25. **Theorem**. *Let H_c be hyperbolic, satisfying the escaping condition.*

(i) *If H_c is singular escaping, then there is an attracting region of infinity $B_\infty(R)$ such that*

$$P_c \cap \{z \in \mathbb{C} : \ |z| < R\} = \alpha.$$

(ii) *Suppose H_c is non-singular escaping. Let $\alpha \in \mathcal{P}^*(H_c)$ be escaping. For any $\mathcal{A} \in \mathscr{C}_\alpha$, with associated IBS type given by*

$$\mathcal{A}^\bullet : \overline{D}_0 \xrightarrow{H_0=\varphi_0} \overline{D}_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} \overline{D}_n \subset D_0,$$

there is an attracting region of infinity $B_\infty(R)$ such that

$$P_c \cap \{z \in \mathbb{C} : \ |z| < R\} = D_1 \cup \cdots \cup D_n.$$

Proof. Follows directly form the definition of IBS of second type and the escaping condition. □

6.26. **Corollary**. *Suppose H_c is hyperbolic. If H_c is non-singular escaping, then*

$$\mathcal{P}^{\odot}(H_c) = \{\infty \mapsto \infty\}.$$

If H_c is singular escaping, then

$$\mathcal{P}^{\odot}(H_c) = \{\alpha, \infty \mapsto \infty\} \ \textit{and} \ \mathcal{P}^*(H_c) = \phi.$$

Proof. If H_c is non-singular escaping, then by the item (ii) of the previous result, we have (using the same notation of this item)

$$0 \notin \bigcup_{i=1}^{n} D_i,$$

$$P_c \subset \bigcup_{i=1}^{n} D_i,$$

and so it it is impossible to have any periodic orbit starting at the critical point 0. Therefore, if H_c is non-singular escaping, we must have $\mathcal{P}^{\odot}(H_c) = \{\infty \mapsto \infty\}$.

Part of the second assertion was already proved in Propostion 6.20. It remains to show that if H_c is singular escaping, then $\mathcal{P}^*(H_c) = \phi$. In order to do that we suppose the opposite, that $P^*(H_c)$ is non-empty and let $\alpha \in \mathcal{P}^*(H_c)$. Let $\mathcal{A} \in \mathscr{C}_\alpha$ and let the associated IBS of second type be denote by

$$\mathcal{A}^\bullet : \overline{D}_0 \xrightarrow{\varphi_0 = H_c} \overline{D}_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} \overline{D}_n \subset D_0.$$

The obvious conclusion is that there are infinitely many points of P_c inside $D_n \subset \{|z| < R\}$. This is a contradiction since whenever H_c is escaping singular, $P_c \cap \{|z| < R\} = \alpha$. □

When H_c is hyperbolic and non-singular escaping, there a naturally associated IBS of second type which determines the shape of the post-critical set. In more specific terms, assume there is $\alpha \in \mathcal{P}^*(H_c)$, let $\mathcal{A} \in \mathscr{C}_\alpha$ and consider the IBS of second type $\mathcal{A}^\bullet$.

Theorem 5.12 implies that the every point inside of $\mathcal{A}^\bullet$ has infinitely many orbits which are asymptotic to a cycle of Cantor sets

$$\mathcal{K}_0 \xrightarrow{\varphi_0} \mathcal{K}_1 \xrightarrow{\varphi_1} \mathcal{K}_2 \xrightarrow{\cdots} \mathcal{K}_{n-1} \xrightarrow{\varphi_{n-1}} \mathcal{K}_n = \mathcal{K}_0.$$

Each Cantor set $\mathcal{K}_i$ is contained the corresponding topological disk D_i.

Since this cycle is associated with the parameter c of H_c, it will be convenient to change the notation a little bit and denote the cycle by

$$\mathcal{K}_c^{(0)} \xrightarrow{\varphi_0} \mathcal{K}_c^{(1)} \xrightarrow{\varphi_1} \mathcal{K}_c^{(2)} \xrightarrow{\cdots} \mathcal{K}_c^{(n-1)} \xrightarrow{\varphi_{n-1}} \mathcal{K}_c^{(n)} = \mathcal{K}_c^{(0)}.$$

Nothing prevents the overlapping of these cycles.

We shall denote

$$\mathcal{K}_c = \bigcup_{i=1}^{n} \mathcal{K}_c^{(i)}.$$

6.27. **Theorem**. *Let H_c be hyperbolic, satisfying the escaping condition.*

(i) *If $c = 0$, then $P_c = \{0\}$.*

(ii) *If H_c is singular escaping, then there is $\alpha \in \mathcal{P}^{\odot}(H_c)$ and an attracting region of infinity $B_\infty(R)$ such that*

$$P_c \cap \{z \in \mathbb{C} : \ |z| < R\} = \alpha.$$

Consequently, the points of P_c are isolated, the unique limit point of P_c is ∞, and every bounded intersection $\{|z| < r\} \cap P_c$ is a finite set.

(iii) *If H_c is non-singular escaping, let $\alpha \in \mathcal{P}^*(H_c)$ and $\mathcal{A} \in \mathcal{C}_\alpha$. Let*

$$\mathcal{K}_c^{(0)} \xrightarrow{\varphi_0} \mathcal{K}_c^{(1)} \xrightarrow{\varphi_1} \mathcal{K}_c^{(2)} \xrightarrow{\cdots} \mathcal{K}_c^{(n-1)} \xrightarrow{\varphi_{n-1}} \mathcal{K}_c^{(n)} = \mathcal{K}_c^{(0)}$$

denote the cycle of Cantor sets associated with $\mathcal{A}^\bullet$. There is an attracting region of infinity $B_\infty(R)$ such that for every $\varepsilon > 0$ given, the set $P_c \cap \{|z| < R\}$ is contained in

$$(\mathcal{K}_c)_\varepsilon = \{z \in \mathbb{C} : \ d_e(z, \mathcal{K}_c) < \varepsilon\},$$

except for finitely many points of $P_c \cap \{|z| < R\}$ which are in $\{|z| < R\} - (\mathcal{K}_c)_\varepsilon$. [2]

(iv *Suppose H_c is non-singular escaping. If the associated IBS of second type (which ultimately determine the cycle of Cantor sets) has only one map $\varphi_0 = H_c$, then the entire post-critical set P_c is contained in $\{|z| < R\}$. Otherwise, if there is a second map φ_1 – a single-valued one – then $B_\infty(R)$ contains uncountably many points of P_c. Therefore, $\infty \in P_c$ in the latter case.*

Proof. Compare Theorems 5.12, 6.25 and Corollary 6.26. □

[2]The symbol d_e denotes the Euclidean distance.

Some important remarks are in order with respect to the item (iv) of the preceding result. The case of non-singular escaping H_c may be divided into to disjoint classes, one in which $\infty \notin P_c$, the other when $\infty \in P_c$. The latter class is responsible for the existence of the **dual Julia set**, one of the most striking features of the dynamics of H_c (if compared to rational maps).

6.28. **Theorem.** *Suppose H_c is hyperbolic and satisfies the escaping condition. Provided P_c contains at least three points, $\hat{\mathbb{C}} - P_c$ is a hyperbolic Riemann surface.*

The cases where P_c has only one or two points are exceptional. The post-critical set P_c has at least three points in the following situations:

- When $H_c : (w - c)^q = z^p$, $q \geq 3$ and $c \neq 0$.
- When H_c is hyperbolic, nonsingular escaping, and $\infty \in P_c$.
- When $c \neq 0$ is sufficiently close to the critical point 0, for then H_c is hyperbolic and non-singular escaping. In this case $\infty \notin P_c$, but P_c is a Cantor set close to 0.
- When $q \geq 2$ and H_c is hyperbolic escaping we have $\#(P_c) \geq 3$.

The set P_c has at most two points only in a few exception cases, of which we list a two:

- $c = 0$;
- $q = 2$, $0 \mapsto c \mapsto \{0, c\}$.

Proof of Theorem 6.28. Recall that any Riemann surface $\mathcal{R}$ for which there is an analytic map

$$\rho : \mathcal{R} \to \hat{\mathbb{C}}$$

omitting three points is necessarily hyperbolic (admits a complete conformal metric of constant curvature -1, compare Theorem 5.4). In this case it is the identity map which omits three points since P_c contains at least three points. In this way we only need to show that $\hat{\mathbb{C}} - P_c$ is connected.

We may suppose that P_c is non-singular escaping. Therefore, the part of P_c contained in the complement of an attracting region of infinity $B_\infty(R)$ is asymptotic to a cycle of

Cantor sets

$$\mathcal{K}_c^{(0)} \xrightarrow{\varphi_0} \mathcal{K}_c^{(1)} \xrightarrow{\varphi_1} \mathcal{K}_c^{(2)} \xrightarrow{\cdots} \mathcal{K}_c^{(n-1)} \xrightarrow{\varphi_{n-1}} \mathcal{K}_c^{(n)} = \mathcal{K}_c^{(0)}.$$

If $\infty \notin P_c$, then the length n of this cycle is $n = 0$ and $P_c \subset \{|z| < R\}$. For any curve $\gamma : [0,1] \to \mathbb{C}$, let

$$\|\gamma\| = \sup_{t\in[0,1]} |\gamma(t)|.$$

If $\gamma(t) \in A$ for every $t \in [0,1]$, then we denote $\gamma \subset A$. Given $\varepsilon > 0$, the Cantor set $K_c^{(i)}$ is covered by disjoint conformal disks (image of a the close unit disk $\{|z| \leq 1\}$ under a bi-holomorphi map) $\check{D}_1, \ldots, \check{D}_{n_\varepsilon}$, each one having diameter less than ε. Using this fact it can be shown that

($\star$) *For any curve $\gamma \subset \mathbb{C}$ and every $\varepsilon > 0$ there is another curve $\zeta \subset (\hat{\mathbb{C}} - P_c)$ with*

$$\|\zeta - \gamma\| < \varepsilon.$$

Once there is a curve $\zeta \subset (\hat{\mathbb{C}} - P_c)$, every small perturbation of ζ is still contained in this set (using the fact that the image of ζ is compact). Hence, successive applications of ($\star$) shows that $\hat{\mathbb{C}} - P_c$ is path connected. □

6.29. Corollary (**Branches expand the hyperbolic metric**). *Suppose H_c is hyperbolic and satisfies the escaping condition, with P_c having at least three points. Let d_c denote the Riemannian distance from the hyperbolic metric of $\hat{\mathbb{C}} - P_c$.*

(i) *If $\varphi : U \to V$ is a univalent branch of H_c with $V \subset (\hat{\mathbb{C}} - P_c)$, then*

$$d_c(\varphi(z), \varphi(w)) > d_c(z, w), \quad z, w \in U.$$

(ii) *For any compact set $K \subset (\hat{\mathbb{C}} - P_c)$, there is a constant $\lambda < 1$ such that whenever the range $V = \varphi(U) \subset K$, we have*

$$d_c(\varphi(z), \varphi(w)) \geq \lambda d_c(z, w).$$

Proof. Consider the Riemann surface

$$\mathcal{R}_c = \{(z, w) \in \mathbb{C}^2 : \ (w - c)^q = z^p, \ w \notin P_c\}.$$

We know that $\mathcal{R}_c$ is a Riemann surface because $\hat{\mathbb{C}} - P_c$ is a Riemann surface. In fact, it easy to see that $\mathcal{R}_c$ is a hyperbolic Riemann surface using the same criterion of the preceding theorem.

Now consider the $\sigma : \mathcal{R}_c \to (\hat{\mathbb{C}} - P_c)$ given by $\sigma(z, w) = w$ and $\tau : \mathcal{R}_c \to (\hat{\mathbb{C}} - P_c)$ given by $\tau(z, w) = z$.

The point is that σ is a covering map and $\tau(\mathcal{R}_c)$ is strictly contained in $\hat{\mathbb{C}} - P_c$. Hence τ is an isometry and τ is a contraction with respect to the hyperbolic metrics of $\mathcal{R}_c$ and $\hat{\mathbb{C}} - P_c$. In the next paragraph we are going to show why $\tau(\mathcal{R}_c)$ is strictly contained in $\hat{\mathbb{C}} - P_c$. Let us finish the argument first. If $\varphi : U \to V$ is any univalent branch of H_c with $V \subset (\hat{\mathbb{C}} - P_c)$, then

$$\varphi = \sigma \circ \tau^{-1}|_U.$$

Since τ is a contraction, it follows that

$$d_c(\varphi(z), \varphi(w)) > d_c(z, w),$$

for $z, w \in U$, with the existence of a $\lambda < 1$ on compact sets, as described in the statement of the corollary.

To prove that $\tau(\mathcal{R}_c)$ is strictly contained in $\hat{\mathbb{C}} - P_c$ is equivalent to prove that $H_c^{-1} = Q_c$ strictly contains P_c. If $0 \notin P_c$ then there is nothing to prove. So we may assume that $0 \in P_c$.

Since $c \neq 0$ and H_c satisfies the escaping condition, $H_c^{-1}(0)$ consists of p points inside of $D_R = \{|z| < R\}$, where $B_\infty(R)$ is an attracting region of ∞. Suppose first that H_c is singular escaping. Then

$$P_c \cap D_R = \alpha,$$

for some critical cycle $\alpha \in \mathcal{P}^{\odot}(H_c)$. If α is given by

$$a_0 \mapsto a_1 \mapsto a_1 \mapsto \cdots \mapsto a_{n-1} \mapsto a_n = a_0 = 0,$$

then since $H_c(a_i) \cap D_R = \{a_{i+1}\}$, we conclude that $H_c^{-1}(0)$ consists of a_{n-1} plus $(p-1)$ points in $D_R \setminus \alpha$. Of course, since $p > 1$, this implies that $H_c^{-1}(0)$ is not contained in P_c.

In the second case H_c is non-singular escaping and we want to prove the $H_c^{-1}(0)$ is not contained in P_c.

Let $\alpha \in \mathcal{P}^*(H_c)$ with associated IBS of second type

$$\overline{D}_0 \xrightarrow{\varphi_0 = H_c} \overline{D}_1 \xrightarrow{\varphi_1} \overline{D}_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} \overline{D}_{n-1} \subset D_0.$$

This IBS is escaping. Since

$$0 \notin \bigcup_{i=1}^{n} D_i$$

this translates easily into

$$H_c^{-1}(0) \subset D_R - \bigcup_{i=0}^{n} D_i.$$

But

$$P_c \cap D_R \subset \bigcup_{i=0}^{n} D_i.$$

We have shown in either case that P_c is strictly contained in Q_c. □

Suppose H_c is hyperbolic and satisfies the escaping condition. If H_c is singular escaping, then there is unique finite critical cycle $\alpha \in \mathcal{P}^{\odot}(H_c)$, with an associated sequence of maps

$$D_0 \xrightarrow{\varphi_0 = H_c} D_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} D_n \subset D_0$$

where $\varphi_i : D_i \to D_{i+1}$ is bi-holomorphic for $i > 0$, with $0 \in D_n$ and

$$0 \notin \bigcup_{i=1}^{n-1} D_i.$$

We denote

$$\mathcal{N}(\alpha) = \bigcup_{i=0}^{n} D_i.$$

Notice, however, that the set $\mathcal{N}(\alpha)$ is not uniquely determined. If H_c is non-singular escaping then there is $\beta \in \mathcal{P}^*(H_c)$, whose corresponding critical IBS $\mathcal{A}$ is escaping. We are allowed to construct sets $\mathcal{N}(\beta)$ in the same way using the IBS of second type $\mathcal{A}^{\bullet}$. What is essential about the sets $\mathcal{N}$ is that they contain P_c.

6.30. **Theorem**. *Let H_c be hyperbolic, satisfying the escaping condition. Let $B_\infty(R)$ be an attracting region of ∞ and set D_R to be its complement. For every $y \in (B_\infty(R) - P_c)$, the set $G_c(y)$ is compact and contained in $D_R - \mathcal{N}$, where $\mathcal{N}$ is any $\mathcal{N}(\alpha)$ obtained from a escaping $\alpha \in \mathcal{P}^*(H_c) \cap \mathcal{P}^{\odot}(H_c)$. Since $P_c \subset \mathcal{N}$, in particular we have*

$$G_c(y) \cap P_c = \phi.$$

Proof. The case $c = 0$ is handled separately and is shown that $G_c(y) = \mathbb{S}^1$ which $P_c = \{0\}$ for $c = 0$.

Assume $c \neq 0$. Then either H_c is singular or non-singular escaping. Suppose first that H_c is non-singular escaping and let $\beta \in \mathcal{P}^*(H_c)$. There is a naturally associated IBS of second type

$$\overline{D}_0 \xrightarrow{\varphi_0 = H_c} \overline{D}_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} \overline{D}_n \subset D_0.$$

Since D_n is a univalent disk, there is a connected neighborhood $V \supset \overline{D}_n$ such that $H_c(V)$ can be written as a disjoint union

$$H_c(V) = \bigcup_{j=0}^{q-1} \psi_j(V),$$

where ψ_j are univalent branches of H_c. So we have a system of maps

$$\mathcal{S}_\beta = \{\varphi_i : D_i \to \mathbb{C},\ \ \psi_j : V \to \mathbb{C};\ 0 < i < n,\ 0 \leq j < q\},$$

and

$$\mathcal{N}(\beta) = \bigcup_{i=0}^{n} D_i.$$

In order to get a contradiction, suppose that there is $z \in G_c(y) \cap \mathcal{N}(\beta)$. There is a pre-orbit

$$y = y(0) \xleftarrow{H_c} y(1) \xleftarrow{H_c} y(2) \xleftarrow{H_c} \cdots$$

with $y(n_k) \to z$ for some subsequence (n_k). Since $H_c(P_c) \subset P_c$ and $y \notin P_c$, none of the points $y(i)$ of the pre-orbit belongs to P_c. We conclude that $y(i)$ visit $\mathcal{N}(\beta)$ infinitely often; and in fact, there is k_0 such that $y(n_k) \in \mathcal{N}(\beta)$ for $k \geq k_0$. A simple argument involving the escaping property shows that for $i \geq n_{k_0}$ the point $y(i)$ is always inside of $\mathcal{N}(\beta)$, otherwise no further iterate would visit $\mathcal{N}(\beta)$ again. As a conclusion we have that for each piece $y(n_k) \xleftarrow{H_c} y(n_{k+1})$ of the sequence, with $k \geq k_0$, there is a unique $\eta_k \in \mathcal{S}_\beta$ such that

$$\eta_k(y(n_{k+1})) = y(n_k).$$

From the definition of

$$\mathcal{K}_c = \bigcup_{i=0}^{n-1} \mathcal{K}_c^{(i)},$$

the point $y(n_{k_0})$ must be in $\mathcal{K}_c \subset P_c$, which is a contradiction. We conclude that

$$G_c(y) \cap \mathcal{N}(\beta) = \phi.$$

A similar argument is applied to the case where H_c is singular escaping. □

6.4. The Julia set

We say that a periodic orbit α is *repelling* if its multiplier λ satisfies $|\lambda| > 1$.

6.31. **Definition (Julia set)**. The Julia set J_c of H_c is defined as the closure of the repelling periodic orbits of H_c.

6.32. **Proposition (Julia set is non-empty)**. *Suppose H_c is hyperbolic and escaping, with $c = 0$ or $\#(P_c) \geq 3$. Then for every $y \in B_\infty(R) - P_c$, we have $J_c \supset G_c(y)$. In particular it follows that $J_c \neq \phi$.*

Proof. The case $c = 0$ is handled separately, using slightly different methods (independent from the results developed so far) in another section of this thesis. So let us concentrate on the case $\#(P_c)$. We know that $\hat{\mathbb{C}} - P_c$ is a hyperbolic Riemann surface, and that the corresponding Riemannian distance d_c from the Poincaré metric is expanded by univalent branches of H_c on the outside of P_c.

Let $y \in B_\infty - P_c$. There is $\alpha \in \mathcal{P}^*(H_c) \cap \mathcal{P}^{\odot}(H_c)$ and an associated $\mathcal{N} = \mathcal{N}(\alpha)$ such that $P_c \cap D_R \subset \mathcal{N}$, where D_R is the complement of $B_\infty(R)$. From the previous results, we get $G_c(y) \subset D_R - \mathcal{N}$. Let $z \in G_c(y)$. We are going to prove that $z \in J_c$, thus completing the proof of the theorem.

There is no loss of generality in treating only the case where H_c is non-singular escaping, since the singular is case is handled in a similar way, with easier arguments. In this case, the set $\mathcal{N}$ comes from a IBS of second type

$$\mathcal{A}^\bullet : \overline{D}_0 \xrightarrow{\varphi_0 = H_c} \overline{D}_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} \overline{D}_n \subset D_0.$$

which is escaping. In fact, $\mathcal{A} \in \mathcal{C}_\alpha$ for some $\alpha \in \mathcal{P}^*(H_c)$. As usual, the IBS $\mathcal{A}^\bullet$ determines a system of maps $\mathcal{S}_\alpha$ as in the proof of Theorem 6.30. Since there is an open set V such that

$$D_R \cap P_c \subset \overline{V} \subset \mathcal{N},$$

there is a constant $\delta > 0$ such that whenever $U \subset (\hat{\mathbb{C}} - P_c)$ and $\mathrm{diam}_c(U) < \delta$, [3] with $U \cap \overline{V} \neq \phi$, we have $U \subset \mathcal{N}$.

There is a pre-orbit

$$y = y(0) \xleftarrow{H_c} y(1) \xleftarrow{H_c} \cdots$$

with $y(n_k) \to z$ for some subsequence n_k. Choose a simply connected set $U_0 \subset D_R - P_c$ containing the point z, with $\mathrm{diam}_c(U_0) < \delta$, and choose k_0 so that

$$d_c(z, y(n_{k_0})) < \frac{1}{9} d_c(z, \partial U_0).$$

It follows that $y(n_{k_0})$ is contained in U_0. Since the critical value c is not in U_0, and since U_0 is simply connected, there is a unique univalent branch $\eta_0 : U_0 \to \mathbb{C}$ of H_c^{-1} such that $\eta_0(y(n_{k_0})) = y(n_{k_0} + 1)$. The image $\eta_0(U_0) = U_1$ is a simply connected set inside $\hat{\mathbb{C}} - P_c$ with diameter

$$\mathrm{diam}_c(U_1) \leq \mathrm{diam}_c(U_0) < \delta.$$

The procedure may continue determining simply connected sets $U_j \subset \hat{\mathbb{C}} - P_c$ with diameter $\mathrm{diam}_c(U_j) < \delta$ and bi-holomorphic maps

$$\eta_j : U_j \to U_{j+1}$$

such that $y(n_{k_0} + j) \in U_j$ and $\eta_j(y(n_{k_0} + j)) = y(n_{k_0} + j + 1)$.

[3] diam_c indicates diameter with respect to d_c, where d_c is the hyperbolic metric of $\hat{\mathbb{C}} - P_c$.

The closure K of the union of all U_j for $j \geq 0$ is a compact set. We claim that K and P_c are disjoint. If K meets P_c at some point, then this point must be an accumulation point of the union of U_j, and hence the sets U_j would visit P_c infinitely often. Suppose it is the case (to get a contradiction). Since the $\mathrm{diam}_c(U_j) < \delta$, whenever U_j intersects P_c, it must be contained in $\mathcal{N}$. Using the same reasoning of the proof of Theorem 6.30, the conclusion is that from the first time $U_{j_0} \subset \mathcal{N}$, we have $U_j \subset \mathcal{N}$ for every $j \geq j_0$. Furthermore, $\eta_j \in \mathcal{S}_\alpha$ for every $j \geq j_0$. The Euclidean diameter of U_{j_0} must be 0, and the unique point contained in U_{j_0} is actually a point of the cycle $\mathcal{K}_c$ of Cantor sets. So the conclusion is that $U_{j_0} \subset P_c$, which is clearly a contradiction aroused from the assumption $K \cap P_c \neq \phi$. Therefore K is disjoint from P_c. The branches η_j uniformly contracts the hyperbolic metric d_c by a factor $\lambda < 1$. Therefore,

$$\mathrm{diam}_c(U_j) \leq \lambda^j \mathrm{diam}_c(U_0) \leq \lambda^j \delta.$$

Hence some U_s is compactly contained in U_0, and we conclude from the General Principal A that there is a repelling periodic orbit of H_c inside U_s. Since U_0 is an arbitrary neighborhood of $z \in G_c(y)$, it follows that

$$G_c(y) \subset J_c.$$

Notice that the assumption $\#(P_c) \geq 3$ was essential to obtain d_c on $\hat{\mathbb{C}} - P_c$. □

6.33. **Theorem**. *Suppose H_c is hyperbolic and satisfies the escaping condition, with $c = 0$ or $\#(P_c) \geq 3$. Then for some $\mathcal{N} = \mathcal{N}(\alpha) \supset P_c$, with $\alpha \in \mathcal{P}^{\odot}(H_c) \cap \mathcal{P}^*(H_c)$, we have*

$$J_c \subset D_R - \mathcal{N},$$

where D_R is the complement of an attracting region of infinity. In particular, $J_c \cap P_c = \phi$.

Proof. The case $c = 0$ will deserved a special attention in the preceding chapters; we have $J_0 = \mathbb{S}^1$ and $P_0 = \{0\}$. In this case we have $L_c = P_c$ for every c near to the critical point 0. (But we are going to prove it later using independent techniques).

Assume $c \neq 0$. Suppose H_c is hyperbolic and satisfy the escaping condition. By the same reasoning of the proof of Theorem 6.30, we have that $J_c \subset D_R - \mathcal{N}$, for if some element of a cycle

$$\beta : z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots \mapsto z_n = z_0$$

enters $\mathcal{N}(\alpha)$, with $\alpha \in \mathcal{P}^*(H_c) \cap \mathcal{P}^{\odot}(H_c)$ being escaping, then the whole cycle must be given by the system of maps naturally associated with $\mathcal{N}$. This implies that either $\beta = \alpha$ is a critical cycle (in the case where H_c is singular escaping) or that β is contained in the cycle of Cantor sets associated with $\mathcal{N}(\alpha)$. In both situations the obvious conclusion is that $\beta \subset P_c$ and the multiplier $\lambda(\beta)$ satisfies $|\lambda| < 1$, since it is given by the derivative of a composition of maps from an IBS of second type. Hence, there is no doubt that $J_c \subset D_R \subset \mathcal{N}$ in either case. □

6.34. **Theorem**. *Suppose H_c is hyperbolic and satisfies the escaping condition, with $\#(P_c) \neq 2$. Then for every $y \in B_\infty(R) - P_c$ we have*

$$J_c = G_c(y).$$

Proof. The condition $\#(P_c) \neq 2$ is equivalent to say that either $c = 0$ or $\#(P_c) \geq 3$. The first case was handled before. We have $G_0(y) = J_0 = \mathbb{S}^1$. For c close to zero the set P_c is uncountable and is included in the following arguments.

One side of the inclusion was already proved: $G_c(y) \subset J_c$. Now let

$$\alpha : z_0 \mapsto z_1 \mapsto \cdots \mapsto z_n = z_0$$

be a repelling periodic orbit. We know that since the points of this orbit are in J_c, they do not belong to P_c. We are going to prove that $z_i \in G_c(y)$ for every i.

For every pair of points $\{z, w\} \subset \hat{\mathbb{C}} - P_c$ there is a simply connected set $D \supset \{z, w\}$ such that $D \subset \hat{\mathbb{C}} - P_c$.

In our case, we consider a simply connected set $D \subset \hat{\mathbb{C}} - P_c$ containing both z_0 and y. The critical value c does not belong to D; there is a unique univalent branch $f_1 : D \to \mathbb{C}$ of H_c^{-1} which takes z_0 into z_{n-1}. The set $D_1 = f_1(D)$ is simply connected and we obtain a second bi-holomorphic map $f_2 : D_1 \to D_2$ taking z_{n-1} into z_{n-2}. This procedure may be repeated indefinitely, producing simply connected sets and maps $f_j : D_{j-1} \to D_j$. The sequence $y, f_1(y), f_2 \circ f_1(y), \ldots$ together with its sub-sequential limits are contained in a compact set disjoint from P_c. From this fact we conclude that there exists $\lambda < 1$ such that

$$d_c(f_n \cdots f_2 f_1(y),\ f_n \cdots f_2 f_1(z_0)) \le \lambda^n d_c(z_0, y) \to 0,$$

as $n \to \infty$. The obvious conclusion is that $\alpha \subset G_c(y)$. □

6.5. The dual Julia set

Now we introduce the dual Julia set. This is a subset of the limit set L_c which concentrates the stable part of the dynamics of H_c on L_c. It may sound strange at a first moment, but the fact is that for every hyperbolic H_c which is escaping and $\infty \in P_c$, the limit set L_c contains infinitely many attracting periodic orbits!

It does not happen for rational maps. Indeed, a rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has only finitely many attracting cycles. These cycles are contained in the Fatou set $F(R)$.

The limit set $L(f_c)$ of the quadratic map $f_c(z) = z^2 + c$, for example, is equally defined; it turns out that the limit set equals the Julia set $L(f_c) = J(f_c)$.
It is impossible for the Limit set of a quadratic map f_c to contain attracting periodic orbits because they are in the Fatou set.

What the reader should keep in mind in order to avoid any confusion is that for H_c, in general:

- $L_c \neq J_c$.
- $\infty \in P_c$.

So there is no contradiction in having attracting cycles in L_c because L_c is not supposed to be J_c. This happens for c close to the origin, but in general we have $L_c \neq J_c$.
In the case of H_c the set of attracting periodic orbits deserves a special attention because it may contain invariant Cantor sets when the map is hyperbolic and escaping. What is surprising about the dynamics of H_c on L_c is that it is still well understood despite of this huge generality.

We shall prove in this section that L_c is splitted into a 'stable' and 'unstable' set for hyperbolic parameters where H_c is escaping and $\infty \in P_c$. The stable set is the dual Julia set, to be defined in the sequel.
Let $y \in P_c \cap B_\infty(R)$. We write $z \in \check{G}_c(y)$ if there is a pre-orbit

$$y_0 = y \xleftarrow{H_c} y_1 \xleftarrow{H_c} y_2 \xleftarrow{H_c} \cdots$$

with $y_i \in P_c$ for every i, such that $y_{n_k} \to z$ as $k \to \infty$, for some subsequence n_k.

6.35. **Definition** (**dual Julia set**). Suppose $B_\infty(R) \cap P_c \neq \phi$. The dual Julia set of H_c, denoted by E_c, is the closure of the union of all $\check{G}_c(y)$ with $y \in B_\infty(R) \cap P_c$.

If P_c does not intersect $B_\infty(R)$, then we set $E_c = \phi$ by convention.

Recall that $\alpha \in \mathcal{P}^*(H_c)$ is escaping, then there is an essentially unique critical IBS $\mathcal{A}$ containing α which is escaping. Let us denote the associated IBS of second type by

$$\mathcal{A}^\bullet : \overline{D}_0 \xrightarrow{\varphi_0} \overline{D}_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} \overline{D}_n \subset D_0.$$

We have already defined

$$\mathcal{N}(\alpha) = \bigcup_{i=0}^{n} D_i$$

as well as the system of maps $\mathcal{S}_\alpha$, which contains all φ_i and all univalent branches of H_c determined at the univalent disk $\overline{D}_n$. The set $\mathcal{N}(\alpha)$ is invariant under the action of $\mathcal{S}_\alpha$. By

$$\mathcal{S}_\alpha : \mathcal{N}(\alpha) \to \mathcal{N}(\alpha)$$

we mean the correspondence naturally associated with the action of $\mathcal{S}_\alpha$.

6.36. **Theorem**. *Suppose $P_c \cap B_\infty(R) \neq \phi$. If H_c is hyperbolic and satisfies the escaping condition, then*

(i) *If H_c is singular escaping, then there is $\alpha \in \mathcal{P}^{\odot}(H_c)$ such that $E_c = \alpha$.*

(ii) *If H_c is non-singular escaping, then there is a escaping $\alpha \in \mathcal{P}^*(H_c)$ such that E_c is the closure of attracting periodic orbits of $\mathcal{S}_\alpha : \mathcal{N}(\alpha) \to \mathcal{N}(\alpha)$. In this case,*

$$E_c = \mathcal{K}_c = \bigcup_{i=1}^{n} \mathcal{K}_c^{(i)}$$

is the cycle of Cantor sets associated with α.

In both cases (i) and (*ii*) above $E_c \subset P_c$. Recall that the set $\mathcal{K}_c$ has the unique pre-image property: for every $w \in \mathcal{K}_c$ there is a unique $z \in \mathcal{K}_c$ such that $(z, w) \in H_c$.

Proof. Suppose H_c is singular escaping. Let α be the unique finite critical cycle in $\mathcal{P}^{\odot}(H_c)$. If y_n is a pre-orbit of $y \in P_c \cap B_\infty(R)$ then some backward iterate y_k must be in a point of $D_R = \{|z| < R\}$. This point still belongs to P_c; and since $P_c \cap D_R = \alpha$, we have $y_k \in \alpha$. It is clear from the definition of E_c that $E_c = \alpha$ in this case.

Let us prove (ii). Suppose H_c is non-singular escaping and let $\alpha \in \mathcal{P}^*(H_c)$. Take a point $y \in P_c \cap B_\infty(R)$ and let (y_n) be a pre-orbit of y in P_c. There is k_0 such that $y_n \in P_c \cap D_R$ for every $n \geq k_0$. From the definition of $\mathcal{K}_c$ we conclude that $y_{k_0} \in \mathcal{K}_c$, as well as all the other backward iterates $y_n \in \mathcal{K}_c$ for $n \geq k_0$. Since the set $\mathcal{K}_c$ is closed, it follows that $E_c \subset \mathcal{K}_c$.

When we proved that there is a homeomorphism $\psi : \Sigma_q \to \mathcal{K}_c^{(i)}$ it was implicit that $\mathcal{K}_c^{(i)}$ is the closure of periodic orbits contained in $\mathcal{K}_c^{(i)}$. In fact, the shift map σ on Σ_q has this property, and since ψ is a topological conjugacy with the unique pre-image map on $\mathcal{K}_c^{(i)}$ – something that was implicit in the construction of ψ –, we conclude that $\mathcal{K}_c^{(i)}$ is indeed the closure of periodic points inside of $\mathcal{K}_c^{(i)}$. Every such periodic point is attracting, since the multiplier is given by the composition of maps in $\mathcal{S}_\alpha$. We collect all these information to conclude that $\mathcal{K}_c$ is the closure of repelling periodic orbits of $\mathcal{S}_\alpha : \mathcal{N}(\alpha) \to \mathcal{N}(\alpha)$.

We have shown that $E_c \subset \mathcal{K}_c$. Every attracting cycle β of $\mathcal{S}_\alpha : \mathcal{N} \to \mathcal{N}$ is contained in E_c, from the simple fact that β can be sent to $B_\infty(R)$ by some composition of branches of H_c. Since E_c is closed, it follows that $\mathcal{K}_c$, the closure of attracting cycles of $\mathcal{S}_\alpha$, is contained in E_c. Thus $E_c = \mathcal{K}_c$. □

6.37. **Theorem** (**Hyberbolic Limit set**). *Suppose $H_c : (w-c)^q = z^p$ is hyperbolic and satisfies the escaping condition, with $q \geq 2$. Then L_c can be written as a disjoint union of compact sets*

$$L_c = J_c \cup E_c.$$

Furthermore, the sets J_c and E_c satisfy the following invariance properties:

(i) *For every $z \in J_c$ there is $w \in J_c$ such that $(z, w) \in H_c$. For every $w \in J_c$ there is $z \in J_c$ such that $(z, w) \in J_c$.*

(ii) *For every $w \in E_c$ (if $E_c \neq \phi$) there is a unique $z \in E_c$ such that $(z, w) \in H_c$. For every $z \in E_c$ there is $w \in E_c$ such that $(z, w) \in H_c$.*

If $P_c \cap B_\infty(R) \neq \phi$, the dual Julia set E_c is non-empty. If $P_c \cap B_\infty(R) = \phi$, then $E_c = \phi$.

Proof. The proof is entirely based on the previous results. Let us summarize the ideas. The invariance properties (i) and (ii) were already proved. For E_c, for example, provided $E_c \neq \phi$, it must be either be a critical cycle or the union of a cycle of Cantor sets associated with an escaping critical IBS. In both cases we have shown that these sets satisfy the invariance properties stated. So let us concentrate on the equation $L_c = J_c \cup E_c$.
If $\#(P_c) = 1$, then $c = 0$ and $E_c = \phi$. Since $J_0 = L_0 = \mathbb{S}^1$, the equation $L_c = J_c \cup E_c$ holds trivially. The case $\#(P_c) = 2$ is inconsistent with the hypothesis that H_c is escaping. Indeed, if P_c contains only two points, then it must be contained in D_R (the complement of an attracting region of infinity). The correspondence H_c cannot by non-singular escaping, for in this case P_c would contain infinitely many points. So P_c is a escaping critical cycle which cannot be mapped to $B_\infty(R)$, otherwise P_c would contain infinitely many points. We conclude that P_c is a escaping cycle with only one element, $P_c = \{0\}$. This proves that P_c can never have only two elements when H_c is hyperbolic escaping.

The case $\#(P_c) \geq 3$ is splitted into other two cases:

(1) $P_c \cap B_\infty(R) = \phi$, and the case

(2) $P_c \cap B_\infty(R) \neq \phi$.

For (1) we have $L_c = G_c(y) = J_c$ for every $y \in B_\infty$ and $E_c = \phi$. Hence $L_c = J_c \cup E_c$.
For (2) we consider a pre-orbit y_n of some $y \in B_\infty(R)$. If $y \notin P_c$, then since H_c is hyperbolic

escaping with $\#(P_c) \geq 3$, we have $G_c(y) = J_c$. If $y \in P_c$ but some element of the pre-orbit leave P_c, say, $y_k \notin P_c$, then it never enters P_c again and every sub-sequential limit of y_n must be a point of J_c, which is disjoint from P_c. Finally, if y_n remain inside of P_c for every n, then every sub-sequential limit of y_n is in P_c. Of course, these considerations lead to the conclusion $L_c = J_c \cup E_c$, with $J_c \cap E_c = \phi$. □

6.6. Holomorphic motions of the dual Julia set

Suppose H_c is hyperbolic and non-singular escaping. Then there is an IBS of second type

$$\mathcal{A}_c : \overline{D}_0 \xrightarrow{\varphi_0^{(c)}} \overline{D}_1(c) \xrightarrow{\varphi_1^{(c)}} \cdots \xrightarrow{\varphi_{n_c-1}^{(c)}} \overline{D}_{n_c}(c) \subset D_0(c) \tag{6.3}$$

satisfying the escaping property, with

$$0 \notin \bigcup_{j=1}^{n_c} D_j(c)$$

and

$$H_c\left(V^{(c)}\right) = \bigcup_{i=0}^{q-1} \psi_i\left(V^{(c)}\right),$$

where $V^{(c)} \subset \overline{D}_{n_c}(c)$ is a univalent disk and ψ_i are the branches of H_c determined at $\overline{D}_{n_c}(c)$. Therefore the sets $\psi_i^{(c)}(V^{(c)})$ are pairwise disjoint. Given a parameter $c = c_0$ for which H_{c_0} is hyperbolic and non-singular escaping, we may choose the IBS $\mathcal{A}_c$ so as to vary continuously, in the sense that $n_c = n_{c_0}$ is constant and[4]

$$c \mapsto d_H\left(\overline{D}_i(c), \overline{D}_i(c_0)\right) \in \mathbb{R}$$

is continuous for every $c \in$ in a neighborhood of U of c_0. The maps $\varphi_i^{(c)}$ associated with $\mathcal{A}_c$ vary holomorphically with c in the sense that $(c, z) \mapsto \varphi_i^{(c)}(z)$ is holomorphic on $U \times D_i$, for any open set D_i contained the the intersection of all $D_i(c),\ c \in U$.

[4] d_H denotes the Hausdorff distance between compact sets.

It should be noticed that the cycle of Cantor sets

$$\mathcal{K}_c = \bigcup_{i=0}^{n_c} \mathcal{K}_c^{(i)}$$

associated with $\mathcal{A}_c$ satisfy $\mathcal{K}_c^{(i)} \subset D_i(c)$.

Recall that a function $h : U \times \Lambda \to \mathbb{C}$ defined on the product of a connected open set U with an arbitrary $\Lambda \subset \mathbb{C}$ is an holomorphic motion with *base point* $c_0 \in U$ if

- $h(c_0, \cdot) : \Lambda \to \mathbb{C}$ is the identity;
- Each $h(c, \cdot) : \Lambda \to \mathbb{C}$ is an injection; and
- $h(\cdot, z) : U \to \mathbb{C}$ is holomorphic for every $z \in \Lambda$.

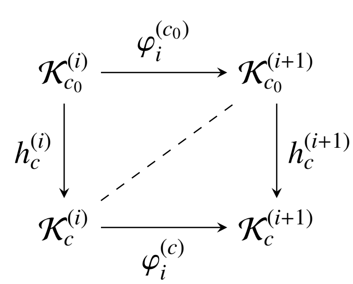
6.38. **Theorem** (**Holomorphic motion of** E_c). *Suppose H_{c_0} is hyperbolic and non-singular escaping. Then there is a connected neighborhood U of c_0 such that H_c is hyperbolic and non-singular escaping for every $c \in U$. Let $\mathcal{K}_c$ be the cycle of Cantor sets associated with a IBS of second type $\mathcal{A}_c$ satisfying the escaping condition, as in (6.3).*

The set U may be chosen so that for each $0 < i \leq n$ there is an holomorphic motion

$$h^{(i)} : U \times \mathcal{K}_{c_0}^{(i)} \to \mathbb{C}$$

for which $h_c^{(i)} = h^{(i)}(c, \cdot)$ satisfy the follows conjugacy equations:

(i) $h_c^{(i)}(\mathcal{K}_{c_0}^{(i)}) = \mathcal{K}_c^{(i)}$;
(ii) $\psi_j^{(c)} \circ h_c^{(n)} = h_c^{(1)} \circ \psi_j^{(c_0)}$ *on* $\mathcal{K}_{c_0}^{(0)}$*; and*
(iii) $\varphi_i^{(c)} \circ h_c^{(i)} = h_c^{(i+1)} \circ \varphi_i^{(c_0)}$ *on* $\mathcal{K}_c^{(i)}$ *for* $0 < i < n$.



Notice that the number n is independent from c and that the same U works for all $h^{(i)}$.

The maps $\psi_j^{(c)}$ and $\varphi_i^{(c)}$ come from $\mathcal{A}_c$, and by definition we have

$$\bigcup_{j=0}^{q-1} \psi_j^{(c)}\left(\mathcal{K}_c^{(n)}\right) = \mathcal{K}_c^{(1)};$$

$$\varphi_i^{(c)}\left(\mathcal{K}_c^{(i)}\right) = \mathcal{K}_c^{(i+1)} \text{ for } 0 < i < n.$$

The first equation is a disjoint union.

Proof. We must choose U so that

$$\bigcup_{c \in U} \mathcal{K}_c^{(i)} \subset \bigcap_{c \in U} D_i(c) =: D_i, \ \ 0 < i \leq n$$

for then the functions $\varphi_i^{(c)}(z) : U \times D_i \to \mathbb{C}$, $\psi_j^{(c)}(z) : U \times D_n \to \mathbb{C}$ are holomorphic. The open set U may be chosen so that $D_n(c) \subset K(n) \subset \mathbb{C}$ remain in a compact set K independent from $c \in U$.

Let ξ be any point in D_n. For an specific sequence

$$\tau = (k_0, k_1, \ldots) \in \{0, \ldots, (q-1)\}^{\mathbb{N}_0} = \Sigma_q,$$

let τ_j denote the first $j+1$ elements $(k_0, \ldots, k_j)$. Accordingly, we have the associated maps

$$T_j^{(c)} = \varphi_{n-1}^{(c)} \circ \cdots \circ \varphi_1^{(c)} \circ \psi_j^{(c)} : D_n(c) \to D_n(c),$$

$$f_{\tau_j}(c) = T_{k_0}^{(c)} \circ \cdots \circ T_{k_j}^{(c)}(\xi).$$

For a given $c \in U$, the sequence $f_{\tau_j}(c)$ converges to a point of $\mathcal{K}_c^{(n)}$ as $j \to \infty$. In fact, as we vary $\tau \in \Sigma_q$ we obtain the entire Cantor set $\mathcal{K}_c^{(n)}$ in this way. As family of functions $f_{\tau_n} : U \to K$ is uniformly bounded; hence they constitute a normal family. We have a convergent subsequence (which we keep denoting by f_{τ_n} to avoid over indexation), such that f_{τ_n} converges locally uniformly to some holomorphic function $f_\tau(c)$ on U. But since the former sequence is point-wise convergent, what we have obtained is that the limit function $\lim f_{\tau_n}(c) = f_\tau(c)$ is holomorphic (without taking subsequences).

For each $c \in U$ there is a homeomorphism between Σ_q (product topology) and $\mathcal{K}_c^{(n)}$. So every point $z \in \mathcal{K}_{c_0}^{(n)}$ has a unique corresponding $\tau \in \Sigma_q$, and the association

$$h_c^{(n)} : z \mapsto \tau(z) \mapsto f_{\tau(z)}(c)$$

defines an holomorphic motion $h^{(n)}(c, z) = f_c^{(n)}(z)$ from $U \times \mathcal{K}_{c_0}^{(n)}$ into $\mathcal{K}_c^{(n)}$.
The other holomorphic motions $h_c^{(i)}$ are constructed so as to fulfill equations (i)-(iii). □

6.39. **Corollary**. *Suppose H_{c_0} is hyperbolic and satisfies the escaping condition. Then there is a neighborhood V of c_0 in the space of parameters such that $c \mapsto P_c$ is continuous on V.*

Proof. If $c_0 = 0$ or H_c is singular escaping with $c_0 \neq 0$ – in which case P_c has infinitely many points ($q \geq 2$) and intersects an attracting region of infinity $B_\infty(R)$ –, then every perturbation of c_0 produces a Cantor set P_c very close to P_{c_0} (with respect to the Hausdorff distance of compact sets, using the spherical metric of $\hat{\mathbb{C}}$). For example, if $c_0 = 0$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that $P_c \subset \{|z| < \varepsilon\}$ for $|c| < \delta$.

Therefore it suffices to deal with the non-singular case. Let D_R denote the complement of $B_\infty(R)$. If $\mathcal{K}_c$ denotes the union of the cycle of Cantor sets of H_c, then for every $\varepsilon > 0$, the bounded part of the post-critical set $D_R \cap P_c$ is contained in $(\mathcal{K}_c)_\varepsilon$, except for finitely many points. The function $c \mapsto \mathcal{K}_c$ is obviously continuous (since its individual pieces $\mathcal{K}_c^{(i)}$ move holomorphically). Hence $c \mapsto P_c \cap D_R$ is continuous. Since ∞ is a super-attracting fixed point of H_c, we also have that $c \mapsto P_c \cap B_\infty(R)$ is continuous. (Notice that the points of $P_c \cap B_\infty(R)$ are obtained from copies of $P_c \cap D_R$ inside the attracting region of infinity). □

6.7. The attractor $W(P_c)$

Suppose H_c is hyperbolic and non-singular escaping. There is a escaping cycle $\alpha_c \in \mathcal{P}^*(H_c)$ and a corresponding critical IBS (of first type) $\mathcal{A}_c$. As usual, let $\mathcal{A}_c^\bullet$ denote the IBS of second type associated to $\mathcal{A}_c$. It is presented in the form (6.3), with maps

$$\varphi_i^{(c)} : D_i(c) \to D_{i+1}(c),$$

$$\psi_j^{(c)} : V^{(c)} \to D_1(c).$$

The system of maps $\mathcal{S}_{\alpha_c} = \left\{\psi_j^{(c)}, \varphi_i^{(c)}\right\}$ leaves the set

$$\mathcal{N}(\alpha_c) = \bigcup_{i=0}^{n} D_i(c)$$

invariant, in the sense that if $z \in \mathcal{N}$ and $\eta \in \mathcal{S}_{\alpha_c}$ then $\eta(z) \in \mathcal{N}$. A closer analysis shows that $\mathcal{N}(\alpha_c)$ and $\mathcal{S}_{\alpha_c}$ depends upon P_c, and that the choice of α_c is irrelevant. It is for this reason that we shall write $\mathcal{S}(P_c)$ and $\mathcal{N}(P_c)$ instead.

6.40. **Definition** (**The attractor** WP_c). Suppose H_c is hyperbolic and non-singular escaping, with $\mathcal{A}_c^\bullet$ escaping as in (6.3). Then

$$W(P_c) = \left\{(z_i)_{i=0}^{\infty} : \ z_i \in \mathcal{K}_c \ \text{ and } \ z_{i+1} = f_i(z_i) \ \text{ for some } \ f_i \in \mathcal{S}(P_c)\right\}.$$

6.8. E-Stability.

If H_{c_0} is hyperbolic and non-singular escaping, then

$$E_{c_0} = \bigcup_{i=0}^{n} \mathcal{K}_{c_0}^{(i)}$$

is the cycle of Cantor sets associated to P_{c_0}. Since there is an holomorphic motion of $\mathcal{K}_{c_0}^{(i)}$, the set E_c moves continuously at $c = c_0$. We would like to give a dynamic meaning to these motions. We cannot develop, however, any concept of structural stability using conjugacy classes of functions on E_c. In fact, if $z \in E_{c_0}$ then there may be more then one motion $h_c^{(i)}(z) \in E_c$.

This ambiguity with the choice of the motion is overcome with introduction of a new dynamical system in the space of orbits

$$\sigma : W_c \to W_c.$$

In fact, if $W_c = W(P_c)$, then it is clear that W_c is invariant under the left shift σ and that $\pi_i(W_c) = E_c$, where π_i is the projection $(z_0, z_1 \ldots) \mapsto z_i$ onto the i-th coordinate. In a certain sense, the function π_i is a semi-conjugacy from (σ, W_c) to (H_c, E_c).

6.41. **Theorem** (**E-stability**). *Suppose H_{c_0} is hyperbolic and non-singular escaping. Then there is a connected neighborhood U of c_0 in the space of parameters such that H_c is hyperbolic and non-singular escaping for every $c \in U$. The set U can be chosen so that there is a function*

$$h : U \times W(P_{c_0}) \to W(P_c)$$

with the following properties:

(i) *Each $h_c = h(c, \cdot) : W_{c_0} \to W_c$ is a homeomorphism;*
(ii) *h_{c_0} is the identity;*
(iii) *For each $z \in W_{c_0}$ and each projection π_i the composition*

$$c \mapsto \pi_i(h(c, z))$$

is holomorphic.
(iv) *h_c is a topological conjugacy from (σ, W_{c_0}) to (σ, W_c).*

Strictly speaking, holomorphic motions are defined only for subsets of $\mathbb{C}$, but the function h above should be treated as holomorphic motion of the set $W(P_{c_0})$ because of the properties just mentioned.

Proof. Let $h^{(i)} : U \times \mathcal{K}^{(i)}_{c_0} \to \mathcal{K}^{(i)}_c$ denote the holomorphic motion of $\mathcal{K}_{c_0}(i)$. Suppose $z = (z_i)_{i=0}^{\infty}$ is in W_{c_0}. Without loss of generality we may assume that $z_0 \in \mathcal{K}^n_{c_0}$ (by checking the next argument). Each piece $z_i \mapsto z_{i+1}$ of the sequence determines a unique $f_i^{(c_0} \in \mathcal{S}(P_c)$ such that $f_i(z_i) = z_{i+1}$, with $z_i \in D_{k_i}(c)$. We then define

$$h_c(z) = \left(h_c^{(k_0)}(z_0), h_c^{(k_1)}(z_1), h_c^{(k_2)}(z_2), \ldots\right).$$

Notice that $f_i^{(c)}$ takes $h_c^{(k_i)}(z_i)$ into $h_c^{(k_{i+1})}(z_{i+1})$, and therefore $h_c(z)$ is an element of W_c. This function from W_{c_0} to W_c is injective because each component is.

The sequence $(\eta_z)_i = f_i^{(c_0)} \in \mathcal{S}(P_c)$ obtained from $z \in W_{c_0}$ varies continuously with respect to the product topology. Said differently, we have $(\eta_z)_i = (\eta_w)_i$ for all $i \geq 0$ up to a certain order $i \leq N$ provided $z \in W_{c_0}$ is sufficiently close to $w \in W_{c_0}$ in the product topology. The ultimate consequence of this fact is that $h_c : W_{c_0} \to W_c$ is continuous. The same argument applies to h_c^{-1}, which shows that h_c is a homeomorphism.

Properties (ii) and (iii) follows from the definition of $h_c^{(i)}$.

By definition we have

$$\sigma h_c(z) = \left(h_c^{k_1}(z), h_c^{(k_2)}(z), \ldots\right) = h_c(z_1, z_2, \ldots) = h_c\sigma(z),$$

which proves (iv). □

6.9. J-Stability

J- Stability means stability on the Julia set. We are going to define it precisely later in this section. First we prove that the Julia set J_c varies continuously at every hyperbolic parameter c for which H_c is escaping (no matter singular or non-singular). Notice that the case $c = 0$ is not included here, but we have already proved this fact using different arguments in another chapter. The reason $c = 0$ is not included is because $\hat{\mathbb{C}} - P_0$ is no longer a hyperbolic Riemann surface.

6.42. **Theorem.** *Let Ω denote the set of parameters $c \in \mathbb{C} - \{0\}$ for which the correspondence H_c is hyperbolic and satisfies the escaping condition. This set is open and the function $c \mapsto J_c$ is continuous on it. Moreover, for every $c \in \Omega$*

$$J_c \cap P_c = \phi.$$

Proof. Notice that $\#(P_c) \geq 3$ for every $c \in \Omega$. Let $c_0 \in \Omega$. Let $B_\infty(R)$ be an attracting region of infinity and consider the set $\mathcal{G}_c$ of all pre-orbits $y = (y_i)$ of H_c starting at a point $y_0 \in B_\infty(R)$. Every pre-orbit $y = (y_i)_{i=0}^\infty$ in $\mathcal{G}_c$ intersects $\mathcal{N}(P_c)$ only at finitely many points $y_{i_1}, \ldots, y_{i_n}$; otherwise there would be a sub-sequential limit of (y_n) in $\mathcal{N}(P_c)$. This sub-sequential limit is a point of $G_c(y_0) = J_c$. But the fact is that J_c does not intersect $\mathcal{N}(P_c)$. We denote

$$\eta(y) = \{y_0, y_1, y_2, \ldots\} - \{y_{i_1},\ y_{i_2}, \ldots, y_{i_n}\}$$

and let $\eta(\mathcal{G}_c)$ be the union of all $\eta(y)$ with $y \in \mathcal{G}_c$. This set never intersects $\mathcal{N}(P_c)$ when H_c is hyperbolic and satisfies the escaping condition.

As the set $\eta(\mathcal{G}_c)$ never intersects $\mathcal{N}(P_c)$ and $\mathcal{N}(P_c)$ varies continuously with c, there is $\varepsilon_1 > 0$ and a neighborhood V_1 of c_0 such that $\eta(\mathcal{G}_c) \cap (P_{c_0})_{\varepsilon_1} = \phi$ for every $c \in V_1$.

Since $c \mapsto P_c$ is continuous at $c = c_0$, there is another neighborhood of c_0, $V_2 \subset V_1$, such that $P_c \subset (P_{c_0})_{\varepsilon_1}$ for every $c \in V_2$. Hence there are disjoint compact subsets K_1 and K_2 of $\hat{\mathbb{C}}$ such that $\eta(\mathcal{G}_c) \subset K_1$ and $P_c \subset K_2$ for every $c \in V_2$.

Cover K_1 by simply connected sets $U_1, \ldots, U_k$ in a such a way that the closure $\check{K}_1$ of the union of U_i does not intersect K_2. Let 2δ be the Lebesgue number of this cover, so that if $x \in K$, then $B(x, \delta) \subset U_i$ for some U_i.

In general, if $\varphi : U \to \mathbb{C}$ is a branch of H_c^{-1}, then $\zeta \mapsto \varphi(\zeta - c_0 + c)$ is a branch of $H_{c_0}^{-1}$, provided $(\zeta - c_0 + c) \in U$. Let $d_c = \mathrm{dist}_{(\hat{\mathbb{C}} - P_c)}$ denote the hyperbolic metric of $\hat{\mathbb{C}} - P_c$. This metric is defined on $\check{K}_1$. There is a constant $C > 0$ such that for every $c \in V_2$ and every $\zeta \in \check{K}_1$ we have

$$d_c(\zeta - c_0 + c, \zeta) \leq C|c - c_0|.$$

In view of Corollary 6.29, there is also $\lambda \in (0, 1)$ such that for every branch $\varphi : U_i \to \mathbb{C}$ of H_c^{-1}, with $c \in V_2$,

$$d_c(\varphi(x), \varphi(y)) \leq \lambda d_c(x, y), \quad x, y \in U_i. \tag{6.4}$$

Let $\varepsilon < \delta$ and pick any point

$$y_0 \in \bigcap_{c \in V_2} (B_\infty(R) - P_c).$$

We know that $J_c = G_c(y_0)$ for every $c \in V_2$. Now let

$$V_3 = \left\{ c \in V_2 : \ |c - c_0| < \frac{\varepsilon}{2C \sum_{i=0}^{\infty} \lambda^{-i}} \right\}.$$

Let $z \in J_c$, with $c \in V_3$. We are going to show that there is $w \in J_{c_0}$ such that $d_c(z, w) < \varepsilon$. Since both ε and $z \in J_c$ are arbitrary and d_c is equivalent to the spherical metric on compact sets disjoint from P_c, it follows that $c \mapsto J_c$ is continuous at $c = c_0$.

There is a pre-orbit $y = (y_i) \in \mathcal{G}_c$ of y_0 such that $y_{i_k} \to z$, for some subsequence (i_k). The set $\eta(y)$ is contained in K_1. If $y_i \in \eta(y)$, then $B(y_i, \delta)$ is contained in some U_i; and so there is a univalent branch $\varphi_i : U_i \to \mathbb{C}$ of H_c^{-1} taking y_i into y_{i+1}, with

$$d_c(\varphi_i(x), \varphi_i(\zeta)) \le \lambda d_c(x, \zeta),$$

for any $x, \zeta \in B(y_i, \delta)$. We may assume, without loss of generality, than no element of y enters $\mathcal{N}(P_c)$, so that $\eta(y)$ is just the set of terms of y. We are going to construct a sequence (w_i) as follows. First set $w_0 = y_0$ and $w_1 = \varphi_0(w_0 - c_0 + c)$. Since

$$d_c(w_1, y_1) \le \lambda d_c(w_0 - c_0 + c, w_0) \le \lambda C|c - c_0| < \frac{\varepsilon}{2} < \delta,$$

we have $w_1 \in B(y_1, \delta)$. Hence the procedure may be applied again to w_1, yielding

$$w_2 = \varphi_1(w_1 - c_0 + c)$$

$$w_3 = \varphi_2(w_2 - c_0 + c)$$

$$\vdots$$

and so on. The conclusion is that $w = (w_i)$ is a pre-orbit of H_c

$$d_c(w_k, y_k) \le (\lambda^k + \cdots + \lambda^2 + \lambda)\, C|c - c_0| < \frac{\varepsilon}{2} < \delta$$

which justifies the induction process. For every i we have $d_c(w_i, -y_i) < \frac{\varepsilon}{2}$. Therefore the points w_i visit the ball $B_c(z, \varepsilon) = \{x \in \mathbb{C} : d_c(x, z) < \varepsilon\}$ infinitely often. Hence there is an accumulation point w_* of the sequence w with $d_c(z, w_*) < \varepsilon$. This accumulation point belongs to $G_c(y_0) = J_c$. The proof is complete. □

6.43. **Corollary**. *Suppose H_c is hyperbolic and satisfies the escaping condition. Then the limit L_c can be written as disjoint union*

$$L_c = J_c \cup E_c$$

with $a \mapsto J_a$ and $a \mapsto E_a$ continuous at $c = a$.

Proof. We have already proved that $a \mapsto E_a$ is continuous at $c = a$. If the post-critical set P_c does not intersect the attracting region of infinity, however, then $E_c = \phi$. This is no big deal, for then $E_a = \phi$ for every a close to c and we have the continuity of $a \mapsto E_a$ anyway. □

6.10. X-Stability: holomorphic motions in Banach spaces

Suppose H_a is hyperbolic and satisfies the escaping condition. Since the Julia set J_c varies continuously at $c = a$, we would like to describe this stability property in terms of the dynamics of H_c. As in the case of the dual Julia set, we cannot expect to find an holomorphic motion of the entire Julia set. Furthermore, the correspondence H_c is multi-valued on J_c and the usual notion of structural stability does not apply in this case. The point is that every $x_a \in J_a$ may have two images in J_a. Depending on this choice, we may consider different motions $x_c \in J_c$ of the initial point x_a.

This idea becomes so much clear with the introduction of a new dynamical system whose projection is $H_c : J_c \to J_c$. To be more specific, consider the space of bounded orbits O_c of H_c. Each element of O_c is therefore a sequence $x = (x_i)$ for which $|x_i| \leq M_x$ for some $M_x > 0$. The set O_c is equipped with the product topology and the left shift map σ. An element $x \in O_c$ is a repelling periodic orbit if $\sigma^n(x) = x$ for some n and the multiplier $\lambda(x)$ of the orbit satisfies $|\lambda| > 1$.

6.44. **Theorem** (**The repeller** X_c). *Suppose H_c is hyperbolic and satisfies the escaping condition. Let X_c be the closure of the repelling periodic orbits in O_c. The set X_c is compact and for every projection $\pi_i : O_c \to \mathbb{C}$ we have*

$$\pi_i(X_c) = J_c.$$

The map π_i can be thought as a semi-conjugacy, for

$$(\pi_i(x), \pi_i(\sigma(x)) \in H_c$$

for every $x \in X_c$.

Proof. Recall that π_i is the map $(x_0, x_1, \ldots) \mapsto x_i$. When H_c is hyperbolic and satisfies the escaping condition the set of repelling periodic orbits remains inside of an annulus $A = \{r \leq |z| \leq R\}$. (In fact, J_c is contained in the outside of attracting region of infinity and is disjoint form P_c. Sine J_c is compact and $0 \notin J_c$, then we have $J_c \subset A$ for some annulus

A). The space of bounded complex sequences $A \times A \times \cdots$ is compact in the product topology. The closure of X_c in that space is X_c again. Hence X_c is compact.

Consequently, $\pi_i(X_c)$ is a closed set containing all repelling periodic points of H_c. Thus $J_c \subset \pi_i(X_c)$. On the other hand, it is clear that $\pi_i(X_c) \subset J_c$. The proof is complete. □

6.10.1. Holomorphic motions in Banach spaces. Suppose U is a connected open subset of $\mathbb{C}$ and Z is a complex Banach space. We say that a function $h : U \times \Lambda \to Z$ is an holomorphic motion of a compact $\Lambda \subset Z$ if

(i) For every $c \in U$, the map $h_c = h(c, \cdot) : \Lambda \to Z$ is a homeomorphism onto its image $h_c(\Lambda)$.
(ii) There is $c_0 \in U$ such that $h(c_0, \cdot)$ is the identity on Λ.
(iii) For every $z \in \Lambda$, the function $h(\cdot, z) : U \to Z$ is holomorphic.

Recall that a function $f : U \to Z$ (U a region of $\mathbb{C}$) is holomorphic if it is Fréchet differentiable at every point $z_0 \in U$. This means that there is $a \in Z$ for which

$$\left\| \frac{f(z_0 + h) - f(z_0)}{h} - ah \right\|_Z \to 0$$

as $h \to 0$.

6.10.2. The Banach space Z_A. If H_c is hyperbolic and satisfies the escaping condition, then the Julia set J_c is bounded and avoid the critical point 0; hence it is contained in some annulus

$$A = \{z \in \mathbb{C} : r \le |z| \le R\}.$$

Since $c \mapsto J_c$ is continuous at such parameters, the annulus is locally constant, i.e., independent of c.

Consider the set

$$Z(A) = \left\{(z_i)_{i=0}^{\infty} : z_i \in A,\ i \ge 0\right\}.$$

This set is turned into a complex Banach space with the norm

$$||z||_A = \sum_{i=0}^{\infty} 2^{-i} |z_i|.$$

Notice that a function $f : U \to Z_A$ is holomorphic if, and only if, every projection

$$\pi_i \circ f : U \to \mathbb{C}$$

is holomorphic.

6.45. **Theorem** (**X-Stability**). *Suppose H_{c_0} is hyperbolic and satisfies the escaping condition. Then there is an open connected $U \subset \mathbb{C}$ neighborhood of c_0 such that H_c is hyperbolic and satisfies the escaping condition for every $c \in U$. The set U may be chosen so that:*

(i) *The Julia set J_c is contained in some annulus A as $c \in U$.*
(ii) *There is an holomorphic motion*

$$h_c(z) : U \times X_{c_0} \to Z_A$$

such that $h_c(X_{c_0}) = X_c$ and h_c is a topological conjugacy from (σ, X_{c_0}) to (σ, X_c).

Proof. The case $c_0 = 0$ was already proved and involve slightly different techniques (mainly because there is no hyperbolic metric on the outside of P_c).

The case $c_0 \neq 0$ is proved using the fact that $\#(P_{c_0}) \geq 3$. (Recall that $q \geq 2$, since the beginning). Since both $c \mapsto P_c$ and $c \mapsto J_c$ are continuous at $c = c_0$, there is a neighborhood V_1 of c_0 and two disjoint compact sets K_1 and K_2 such that $J_c \subset K_1$ and K_2 contains P_c for every $c \in V_1$. We may in fact assume that there is $\varepsilon > 0$ such that

$$\{z \in \mathbb{C} : d_{c_0}(z, J_{c_0}) < \varepsilon\} \subset K_1.$$

Let U_i be a finite open cover of K_1 such that (i) each U_i is simply connected and (ii) the closure of the union of U_i, denoted by $\check{K}_1$, is disjoint from K_2.

Let 2δ be the Lebesgue number of this cover with respect to the metric d_{c_0}. There is $\lambda \in (0, 1)$ such that for every $c \in V_1$ and every branch $\varphi : U_i \to \mathbb{C}$ of H_c^{-1}, we have

$$d_c(\varphi(x), \varphi(y)) \leq \lambda d_c(x, y),$$

for $x, y \in U_i$, where d_c is the Poincaré metric of $\hat{\mathbb{C}} - P_c$. There is a constant $\varepsilon_0 < \delta, \varepsilon$ with the following property: for every $c \in V_1$, if $x = (x_i) \in O_c$ and $y = (y_i) \in O_c$ are two sequences in $\check{K}_1$ with

$$d_c(x_i, y_i) < 2\varepsilon_0$$

for every i, then necessarily $x = y$. There is also a constant $C > 0$ such that

$$d_c(\zeta - c + c_0, \zeta) \leq C|z - z_0|,$$

for $c \in V_1$ and $\zeta \in \check{K}_1$.

Now let

$$V_2 = \left\{ c \in V_1 : |c - c_0| < \frac{\varepsilon_0}{C \sum_{i=0}^{\infty} \lambda^{-i}} \right\}.$$

For each sequence $z = (z_i)_{i=0}^{\infty}$ in X_{c_0} we are going to construct another sequence $w \in X_c$. Next we show that this association is uniquely determined and defines a map $X_{c_0} \to X_c$.

Notice that every term z_i of z is contained in K_1 and therefore $B_{c_o}(z_i, \delta)$ is contained in some U_i. As a consequence there is a unique branch $\varphi_i : B(z_i, \delta) \to \mathbb{C}$ of $H_{c_0}^{-1}$ which takes z_i into z_{i-1}. This branch satisfies

$$d_{c_0}(\varphi_i(x), \varphi_i(y)) \leq \lambda d_{c_0}(x, y),$$

for every $x, y \in B_{c_0}(z_i, \delta)$.
We are going to construct a double sequence $w_{kn}(c)$, with $k \leq n$ and $c \in V_1$, as follows. Given $k \geq 0$, let $w_{kk}(c) = z_k$. Then let

$$w_{(k-1)k}(c) = \varphi_k(w_{kk}(c) - c + c_0).$$

Notice that

$$d_{c_0}(w_{(k-1)k}(c), z_{k-1}) \leq \lambda C|c - c_0| < \varepsilon_0 < \delta.$$

Therefore we are allowed to repeat the argument:

$$w_{(k-2)k}(c) = \varphi_{k-1}(w_{(k-1)k}(c) - c + c_0);$$

$$d_{c_0}(w_{(k-2)k}(c), z_{k-2}) \leq \lambda^2 C|c - c_0| + \lambda C|c - c_0| < \varepsilon_0$$

$$\vdots$$

As a result we obtain a finite orbit ε_0-close to z :

$$w_{0k}(c) \xrightarrow{H_c} w_{1k}(c) \xrightarrow{H_c} w_{2k}(c) \xrightarrow{H_c} \cdots$$

with $d_{c_0}(w_{jk}(c), z_j) < \varepsilon_0$. For a fixed j, the sequence of holomorphic functions $g_k : V_1 \to \mathbb{C}$ given by $g_k(c) = w_{jk}(c)$ maps V_1 onto some set contained in $B_{c_0}(z_j, \varepsilon_0)$, and as such, it constitutes a normal family $\{g_k\}_{k=j}^{\infty}$. For each j there is a sequence $(k_{js})_{s=0}^{\infty}$ such that $w_{jk_{js}}(c)$ converges locally uniformly to some holomorphic function $f_j : V_1 \to \mathbb{C}$ on V_1, as $s \to \infty$. We may take these sequences in such a way that $(k_{js})_s$ is a subsequence of $(k_{(j+1)s})_s$. The diagonal sequence $\Delta_s = k_{ss}$ is a subsequence of every $(k_{js})_s$. Consequently,

$$w_{j\Delta_s}(c) \to f_j(c)$$

locally uniformly on V_1 as $s \to \infty$. From this fact it follows that

$$h_c(z) = (f_j(c))_{j=0}^{\infty} \in X_c.$$

Notice that $h_c(z)$ is characterized as the unique orbit $(w_0, w_1, \ldots) \in O_c$ with $d_{c_0}(z_i, w_i) < \varepsilon_0$, for every $i \geq 0$. (ε_0 was chosen so as to satisfy this property). It is this same property that is used to show that h_c maps periodic orbits of X_{c_0} into periodic orbits of X_c. The periodic orbits obtained in this way are repelling since they are contained in the compact set K_1 which does not intersect P_c. Hence we may say that h_c maps repelling periodic orbits of X_{c_0} into repelling periodic orbits of X_c. It is clear that h_c is continuous. Since the set of repelling periodic orbits of X_{c_0} are dense in X_{c_0}, it follows that $h_c(X_{c_0}) \subset X_c$.

The fact is that under these conditions we are allowed to construct a continuous map $g_{c_0} : X_c \to X_{c_0}$ using the same technique, so that

$$g_{c_0} \circ h_c = Id_{X_{c_0}}; \;\; h_c \circ g_{c_0} = Id_{X_c}.$$

Hence $h_c : X_{c_0} \to X_c$ is a homeomorphism. By the way it was construct, we have

$$h_c(\sigma x) = \sigma h_c(x)$$

for every $x \in X_{c_0}$ and $\pi_i h_c(x) = f_i(c)$ is an holomorphic function of $c \in V_1$. The theorem is proved. □

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