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Problem-solving techniques in infinite graphs

Lucas Silva Sinzato Real

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Lucas Silva Sinzato Real

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Técnicas de resoluções de problemas em grafos infinitos

Dissertação apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Mestre em Ciências – Matemática. *EXEMPLAR DE DEFESA*

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USP – São Carlos
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To everyone who already told me a story.

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"O acaso vai me proteger enquanto eu andar distraído."

(Titãs)

RESUMO

REAL, L. S. S. **Técnicas de resoluções de problemas em grafos infinitos**. 2024. 176 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

O estudo de grafos infinitos configura a uma área singular da teoria de grafos. Em geral, seus problemas não podem ser abordados por meio de princípios de contagem ou algoritmos otimizadores, ferramentas típicas da combinatória finita. De fato, uma gama de argumentos que sustentam demonstrações na teoria de grafos infinitos são provenientes de outros campos da matemática, principalmente daqueles em que a própria noção de infinito é um objeto de estudo. Nesta direção, este trabalho se insere na intersecção entre teoria dos grafos, teoria dos conjuntos e topologia, em que certos problemas da primeira área serão analisados sob uma ótica das duas últimas. Com especial profundidade, estudaremos a conjectura da partição não-amigável e seu estado da arte, bem como as noções de extremidades em grafos infinitos e suas aplicações. Inclusive, além de revisitar a literatura pertinente a estas discussões, esta dissertação contribui com resultados originais.

Palavras-chave: Grafos infinitos, Unfriendly partition, Árvores normais, Espaços de extremidades, Teorema de Menger.

ABSTRACT

REAL, L. S. S. **Problem-solving techniques in infinite graphs**. 2024. 176 p. Dissertação (Mestrado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

The study of infinite graphs consists in a singular area from graph theory. In general, its problems cannot be approached by counting principles or optimizing algorithms, typical tools from finite combinatorics. In fact, a sort of arguments that support proofs in infinite graph theory are inherited from other branches of mathematics, mainly those in which the notion of infinite itself is a matter of study. Regarding that, this work lies in the intersection between graph theory, set theory and topology, where some problems from first area will be analysed under a viewpoint of the others. With some special depth, we will study the unfriendly partition conjecture and its state of art, as well as the notion of ends in infinite graphs and their applications. Incidentally, besides revisiting the literature concerning these discussions, this dissertation contributes original results.

Keywords: Infinite graphs, Unfriendly partition, Normal trees, End spaces, Menger's Theorem.

LIST OF SYMBOLS

$V(G)$ — Vertex set of the graph G

$E(G)$ — Edge set of the graph G

$G[X]$ — Subgraph of G induced by the vertex set X

$N(v)$ — Neighborhood of the vertex v

$d(v)$ — Degree of the vertex v

$\lceil t \rceil$ — Set of nodes below (or equal to) t in some tree order

$\lfloor t \rfloor$ — Set of nodes above (or equal to) t in some tree order

$c * F$ — Coloring obtained by changing the values on F

$\overline{c}$ — Closure of the coloring c

$(2^\omega)^{+\omega}$ — Least limit cardinal greater than the continuum

$\varkappa$ — Minimum size of a graph whose vertices have infinite degree and that admits no unfriendly partitions

$\mathrm{dom}(c)$ — Domain of the function c

$\mathscr{R}(G)$ — Set of rays of G

$\Omega(G)$ — End space of the graph G

$[r]$ — End of the ray r

$|G|$ — Topological space of a graph G with its ends

$C(S,[r])$ — Connected component of $G \setminus S$ in which r has a tail

$\Omega(S,[r])$ — Set of ends that can not be separated from $[r]$ by S

$d(u,v)$ — Distance between vertices u and v

$h(t)$ — Height of the node t in an order tree

$\mathscr{L}_\alpha(T)$ — $\alpha-$level of an order tree T

$\sim_E$ — Edge-equivalence relation

$\Omega_E(G)$ — Edge-end space of the graph G

$[r]_E$ — Edge-end of a ray r

$\|G\|$ — Topological space of a graph G and its edge-ends

$C_E(F,[r]_E)$ — Connected component of $G \setminus F$ in which the ray r has a tail

$\Omega_E(F,[r]_E)$ — Set of edge-ends that cannot be separated from $[r]$ by F

CONTENTS

CHAPTER 1

INTRODUCTION

In 1736, Leonhard Euler announced its negative solution to the classical problem of the “Seven Bridges of Königsberg”, that asked whether one could find a walk visiting the seven bridges of the Königsberg city, in Prussia, precisely once. In the history of mathematics, this episode is now referred as the birth of Graph Theory, since Euler’s solution was based on a modelling for the problem in which the bridges were presented by edges. Almost three centuries later, Königsberg is now Kaliningrad, Russia, and only two of the bridges remain preserved. Euler’s ideas, on the other hand, spread over the mathematical community, so now Graph Theory is broadly recognized by its applications on computer sciences, complex networks and mathematical modelling.

Due to these impacts on solving “concrete problems”, however, the designation *Graph Theory* implicitly refers to the study of graphs on *finitely* many vertices, as one can deduce after analyzing the contents of most undergraduate courses on the subject. A remarkable exception of this standard, however, is found on Diestel’s *Graph Theory* book, whose eighth chapter has a concise introduction to the theory of Infinite Graphs. Indeed, that chapter provides a nice first contact for the objects more deeply investigated by this dissertation. Quoting Diestel, one of the purposes of the present work is to

> highlight the typical kinds of phenomena that will always appear when graphs are infinite, and to show how they can lead to deep and fascinating discussions (DIESTEL, 2018, p.209).

Nevertheless, these “phenomena” mentioned by Diestel may arise from distinct branches of mathematics. In particular, this dissertation lies on the intersection between graph theory, set theory and topology. Considering that, our aim is to exemplify how these two latter areas can provide tools for developing combinatorial arguments, while also contextualizing them within problems about infinite graphs. Structurally, this work is thus divided into two parts, briefly described as follows:

- Part I comprises a detailed study of the *unfriendly partition conjecture*, perhaps "one of the best-known open problems in infinite graph theory" (DIESTEL, 2018, p.275). Although not extensive, its literature provides a rich diversity of techniques inherited from set theory, as we discuss throughout Chapter 3. On the other hand, most results available in Chapter 4 are original, partially obtained by improving the tools just mentioned. Incidentally, Section 4.4 is extracted from the recently published paper (AURICHI; REAL, 2023);

- In its turn, Part II has a more topological flavour, since it formalizes the notion of "limit points" in infinite graphs. Following the approach given by the Hamburg group[1], Chapter 5 define the topological spaces $|G|$ and $\Omega(G)$ for a graph G, compiling their main properties and exemplifying their role in extending classical results from finite graph theory. On the other hand, in Chapter 6 we turn our attention to $\Omega_E(G)$, a third space introduced by Hahn, Laviolette and Širáň (1997). There, we present new applications regarding edge-connectivity problems, as well as we give original topological descriptions. In particular, Section 6.2 comprises the studies of our preprint (AURICHI; REAL, 2023), while Section 6.3 and Section 6.4, obtained in a joint work with Paulo Magalhães Júnior, follow our paper (AURICHI; REAL; JÚNIOR, 2023).

This dissertation was written in an attempt to be as self-contained as possible, so that Chapter 2 fix the basic results from graph theory that will be mentioned further on in the text. Looking its sections up is recommended to all the readers, since some concepts have no uniform notation or definition in the literature. Moreover, the (simple) proofs presented there already follow a style that is routine when dealing with infinite graphs.

Unfortunately, there is no similar preliminary chapter for a set-theoretic background. Concepts such as ordinal numbers or filters might be used without previous introduction, although we try to minimize these occurrences. In particular, specially in the first part, we almost always exhibit two version of a given result. The first one is a restriction to a countable setting, where the arguments involved can be understood by a wide range of mathematicians. This simplification is often enough to capture the main idea of a proof. Then, the second version states the given result in its general form, presenting their details via a more specific language from set theory, if needed.

Finally, some complementing exercises are proposed throughout the texts, but their solutions are not required to fulfill any other argument. They may be seen as remarks made after a discussion, whose aim is to bring some reflection or clarification. With a similar purpose, the more constructive definitions are frequently exemplified with figures, all them elaborated by the author. On the other hand, open problems are highlighted in certain sections, mainly in those that contains original contributions. In general, they correspond to questions that naturally arise when trying to improve some of the key results already obtained.

[1] See <https://www.math.uni-hamburg.de/spag/dm/projects/topgrth.html>.

1.1 A reading skeleton

We address this section to the readers that look forward overviewing the present work regardless its technicalities. After all, the discussions proposed by this dissertation are carried out with as many details as possible, so that some reasoning might seem longer than they could be. In fact, most main theorems of the next chapters could be explained even while suppressing other nearby statements, although we opted to draw a broad picture of each approached subject.

Therefore, we shall now point out which results from the following sections deserve a special attention, either due to its own importance or due to further applications. In particular, according to the notations in Table 1, some diagrams will illustrate the shortest trails that the reader can follow for an exposition of these main ideas.

Table 1 – Different types of prerequisites between results

When writing...	**we mean that...**
Result $X \rightarrow$ Result Y	...the proof of Result Y is *supported* by Result X. In this case, the reading of Result Y is compromised if the proof of Result X is skipped.
Result $X \dashrightarrow$ Result Y	...the proof of Result Y mentions the statement of Result X due to a *technical* purpose. In this case, this latter result can be assumed without proof.
Result $X \rightsquigarrow$ Result Y	...the proof of Result Y *resembles* the proof of Result X, but do not relies on it. Despite that, it might be clarifying to read the details of Result X.

Source: Elaborated by the author.

Once established this conventions, we recall that looking Chapter 2 up is suggested to everyone, as well as reading the introduction of each of the other chapters. Below, however, we briefly overview the core of the more specific sections, remarking which prerequisites are required:

- **Sections of Chapter 3:** These sections are extracted from the traditional literature regarding the unfriendly partition problem. First, the main motivation of Section 3.2 is to conclude Theorem 3.2.2, even though the restricted statement of Corollary 3.2.8 already illustrates the proof idea. The corresponding diagram for obtaining this latter result is drawn in Figure 1.

  In its turn, Section 3.3 is addressed to Theorem 3.3.2, which can be reached by the instructions in Figure 2. On the other hand, Section 3.4 is self-contained, despite written with a set-theoretic vocabulary. Although it concerns both Theorems 3.4.1 and 3.4.4, the details of the latter are closely inspired by the proof of the former. Similarly, Section 3.5 is motivated by Theorem 3.5.7, but the readers can sense its main arguments even when restricting themselves to Theorem 3.5.5. Finally, Section 3.6 is presented in order to prove Theorem 3.6.7, following the script outlined by Figure 3. However, at the begging of this section, Proposition 3.6.1 offers a simplified proof for Theorem 3.6.7 in its countable case.

- **Sections of Chapter 4:** all these sections bring new instances for the unfriendly partition problem. In particular, Theorem 4.2.1 and Proposition 4.2.5 are the core of Section 4.2. The diagram draw in Figure 4 outlines the prerequisites for this study. On the other hand, the aim of Section 4.3 is to prove Theorem 4.3.1, which is supported by the tools mentioned in Figure 5. In its turn, Section 4.4 follows our paper (AURICHI; REAL, 2023), from where Theorem 4.4.1 is a main result. The script of its proof is outlined by Figure 6.
- **Sections of Chapter 5:** These sections are extracted from diverse references regarding the end structure of infinite graphs. First, compactness properties of end spaces are the core of Section 5.2, in which Lemma 5.2.3 and Theorem 5.2.5 are main results. They can be understood with very few prerequisites, as suggested by Figure 7. On the other hand, the metric properties discussed in Section 5.3 are summarized by Theorems 5.3.1 and 5.3.4, which can be obtained after carrying out the instructions in Figure 8. In its turn, Section 5.4 is self-contained, but reading the proofs of Proposition 5.4.3 and Lemma 5.4.5 is advised. Finally, Section 5.5 motivates the main results of the following chapter, so that looking it up carefully might be helpful. There, the details of Lemmas 5.5.4 and 5.5.6 support further discussions.
- **Sections of Chapter 6:** these sections comprise our contributions to the study of edge-end spaces in infinite graphs. More precisely, Theorems 6.2.4 and 6.2.7 are the main results of Section 6.2, which were extracted from our preprint (AURICHI; REAL, 2023) and that can be approached by the instructions of Figure 9. In its turn, Section 6.3 is self-contained, but it develops the tools to be used in Section 6.3. There, in a joint work with Paulo Magalhães Júnior and following the diagram in Figure 10, we aim to prove Theorem 6.4.1 together with its Corollary 6.4.3.

Figure 1 – Diagram for concluding the main results of Section 3.2

$$\begin{tikzcd}
\text{Proposition 3.2.1} \arrow[d, rightsquigarrow] & \text{Rado's Lemma (3.2.5)} \arrow[dl, dashed] & \text{Lemma 3.2.3} \arrow[d, dashed] & \text{Lemma 3.2.4} \arrow[dl, dashed] \\
\text{Proposition 3.2.6} \arrow[r] & \text{Corollary 3.2.8} & \text{Lemma 3.2.7} \arrow[l, dashed] &
\end{tikzcd}$$

Source: Elaborated by the author.

Figure 2 – Diagram for concluding the main results of Section 3.3

$$\begin{tikzcd}
\text{Lemma 3.3.1} \arrow[d, squiggly] \arrow[dr, dashed] & \text{Theorem 3.2.2} \arrow[d, dashed] \\
\text{Theorem 3.3.3} \arrow[r] & \text{Theorem 3.3.2}
\end{tikzcd}$$

Source: Elaborated by the author.

Figure 3 – Diagram for concluding the main results of Section 3.6

$$\begin{tikzcd}
\text{Lemma 3.6.2} \arrow[d, dashed] & \text{Lemma 3.6.5} \arrow[d, dashed] & \text{Proposition 3.6.1} \arrow[d] \\
\text{Lemma 3.6.3} \arrow[r] \arrow[ur] & \text{Lemma 3.6.6} \arrow[r] & \text{Theorem 3.6.7}
\end{tikzcd}$$

Source: Elaborated by the author.

Figure 4 – Diagram for concluding the main results of Section 4.2

$$\begin{tikzcd}
\text{Corollary 3.2.8} \arrow[dr, dashed] & \text{Proposition 3.6.1} \arrow[d, squiggly] & \text{Lemma 4.2.3} \arrow[dl] \arrow[dr, squiggly] & \text{Lemma 3.3.1} \arrow[d, squiggly] \\
 & \text{Theorem 4.2.1} & & \text{Proposition 4.2.5}
\end{tikzcd}$$

Source: Elaborated by the author.

Figure 5 – Diagram for concluding the main result of Section 4.3

$$\begin{tikzcd}
\text{Proposition 2.2.2} \arrow[dr, squiggly] & \text{Proposition 2.3.1} \arrow[d, dashed] & \text{Proposition 3.2.6} \arrow[dl, dashed] \\
\text{Lemma 3.5.1} \arrow[r, squiggly] & \text{Theorem 4.3.1} & \text{Lemma 3.2.7} \arrow[l, dashed]
\end{tikzcd}$$

Source: Elaborated by the author.

Figure 6 – Diagram for concluding the main results of Section 4.4

$$\begin{tikzcd}
\text{Theorem 3.3.2} \arrow[r, dashed] & \text{Proposition 4.4.2} \arrow[r] & \text{Corollary 4.4.5} \arrow[r] & \text{Theorem 4.4.1} \\
\text{Proposition 3.3.3} \arrow[u] \arrow[r, dashed] & \text{Corollary 4.4.6} \arrow[urr] & \text{Proposition 4.4.3} \arrow[l, dashed] \arrow[u, dashed] &
\end{tikzcd}$$

Source: Elaborated by the author.

Figure 7 – Diagram for concluding the main results of Section 5.2

$$
\begin{tikzcd}
\text{Proposition 5.2.1} \arrow[d, dashed] \arrow[dr, dashed] & \text{Proposition 2.2.2} \arrow[d, squiggly] \\
\text{Lemma 5.2.3} \arrow[r, squiggly] & \text{Theorem 5.2.5}
\end{tikzcd}
$$

Source: Elaborated by the author.

Figure 8 – Diagram for concluding the main results of Section 5.3

$$
\begin{tikzcd}
 & \text{Proposition 5.3.10} \arrow[d, dashed] & \text{Theorem 5.2.5} \arrow[d, squiggly] & \text{Proposition 5.3.3} \arrow[dl, dashed] \\
\text{Proposition 5.3.11} \arrow[r] & \text{Theorem 5.3.1} & \text{Theorem 5.3.4} \arrow[l] & \text{Proposition 5.3.2} \arrow[l, dashed]
\end{tikzcd}
$$

Source: Elaborated by the author.

Figure 9 – Diagram for concluding the main results of Section 6.2

$$
\begin{tikzcd}
\text{Theorem 5.5.9} \arrow[dr, squiggly] & \text{Lemma 5.5.6} \arrow[l, dashed] \arrow[r, squiggly] & \text{Lemma 6.2.5} \arrow[dl] \arrow[d] & \text{Lemma 5.5.4} \arrow[d, dashed] \\
\text{Theorem 5.5.10} \arrow[u, dashed] \arrow[r, dashed] & \text{Theorem 6.2.7} & \text{Corollary 626} \arrow[r] & \text{Theorem 6.2.4}
\end{tikzcd}
$$

Source: Elaborated by the author.

Figure 10 – Diagram for concluding the main results of Section 6.4

$$
\begin{tikzcd}
\text{Proposition 2.2.2} \arrow[d, squiggly] & \text{Proposition 5.4.3} \arrow[r] \arrow[d, squiggly] & \text{Corollary 5.4.4} \arrow[d, squiggly] & \text{Theorem 6.3.13} \arrow[d, dashed] \\
\text{Proposition 5.3.6} \arrow[ur, dashed] \arrow[r, dashed] & \text{Proposition 6.4.4} \arrow[r] & \text{Corollary 6.4.5} \arrow[r] & \text{Theorem 6.4.1}
\end{tikzcd}
$$

Source: Elaborated by the author.

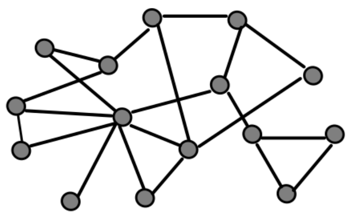
CHAPTER

2

GRAPH-THEORETIC PRELIMINARIES

Informally, a graph is understood as a main set of objects (called *vertices*) whose elements may be somehow related. If two vertices u and v are related, for example, we write uv in order to define one of its *edges*. As Figure 11 suggests, we often sketch a graph by representing its vertices with bullets and its edges with lines connecting some of them.

Figure 11 – Stardand example of a graph.



Source: Elaborated by the author.

As a general observation, the graphs in this text are undirected (i.e., there is no distinction between edges uv and vu) and simple (i.e., there is no edge connecting a vertex to itself, neither parallel edges), unless when the contrary is mentioned. In addition, if G is a graph, we denote by $V(G)$ and $E(G)$ its **vertex set** and **edge set** accordingly. When there is no possible misinterpretation, and the edge set is fixed by the context, we often identify G with its vertex set. For example, one might write "$v \in G$" to claim that v is a vertex of G, or write "$G \cap H = \emptyset$" to claim that the vertex sets of G and H are disjoint.

2.1 Canonical subgraphs

Within the above notation, a **subgraph** of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If the equality $V(H) = V(G)$ holds, we call H a **spanning** subgraph. On the other hand, if $E(H) = \{uv \in E(G) : u, v \in V(H)\}$, we call H an **induced** subgraph, often denoted via its vertex set $X \subseteq V(G)$ as $H = G[X]$. For a given subset (or even subgraph) $A \subset V(G) \cup E(G)$, we write by $G \setminus A$ the subgraph of G obtained by deleting the vertices of $V(G) \cap A$, the edges incident to them and the edges of $E(G) \cap A$. Below, we discuss some other canonical subgraphs that deserve special designations:

- **Path:** it is a subgraph P whose vertex set is a finite sequence $\{v_1, v_2, v_3, \dots, v_n\} \subseteq V(G)$ of distinct elements such that $v_i v_{i+1} \in E(G)$ for each $0 \leq i < n$. Then, $\{v_i v_{i+1} : 1 \leq i < n\}$ is defined as its edge set, while n is said to be its **length**. For short, P is often represented as $v_0 v_1 v_2 \dots v_n$. Then, we say that v_0 and v_n are the *endpoints* of P, or even that this path *connects* such vertices. If v_0 belongs to some set $A \subset V(G)$ and v_n belongs to some set $B \subset V(G)$, we might say that P is an $A - B$ path. In its turn, an $A - A$ path is written simply as an $A-$path if it intersects A precisely at its endpoints. When every pair of vertices of G can be connected by a path, we call G a **connected** graph. For many problems in graph theory, we can always assume that G is under this condition. Otherwise, we could restrict the study to its **connected components**, the maximal connected subgraphs of G. Finally, we also remark that the singleton $\{v\}$ fits in the definition of a path, for every $v \in V(G)$;

- **Cycle:** its vertex set is a finite sequence $\{v_0, v_1, v_2, \dots, v_n\} \subseteq V(G)$ of distinct elements such that $v_n v_1 \in E(G)$ and $v_i v_{i+1} \in E(G)$ for each $1 \leq i < n$. Then, $\{v_i v_{i+1} : 0 \leq i < n\} \cup \{v_n v_0\}$ is defined as its edge set. In other words, a cycle is obtained by a path after adding an edge that connects its endpoints;

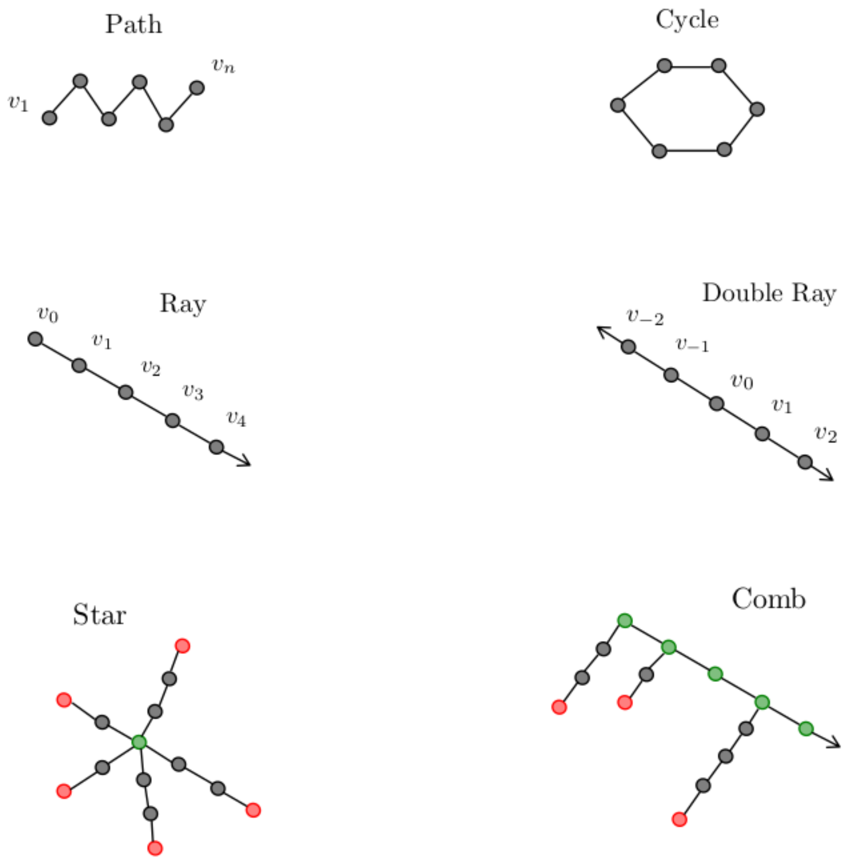
- **Ray:** it is an one-way infinite path r. More precisely, its vertex set is an infinite sequence $\{v_n\}_{n \in \mathbb{N}} \subseteq V(G)$ of distinct vertices such that $v_n v_{n+1} \in E(G)$ for every $n \in \mathbb{N}$. Then, its edge set is given by $\{v_n v_{n+1}\}_{n \in \mathbb{N}}$, so that $r = v_0 v_1 v_2 \dots$ is often a notation for presenting the ray. In this case, we often say that r *starts* at v_0. In addition, any infinite connected subgraph of a ray is called its **tail**. As we will broadly discuss in Part II of this dissertation, the rays are important objects when describing directions in infinite graphs. Even **rayless** graphs, i.e., those not containing rays as subgraphs, have their own applications, which we will approach in Section 3.5;

- **Double ray:** it is a two-way infinite path r. More precisely, its vertex set is an infinite sequence $\{v_n\}_{n \in \mathbb{Z}} \subseteq V(G)$ of distinct elements such that $v_n v_{n+1} \in E(G)$ for each $n \in \mathbb{Z}$. We often present the double ray by writing $r = \dots v_{-2} v_{-1} v_0 v_1 v_2 \dots$, considering also that $\{v_n v_{n+1}\}_{n \in \mathbb{Z}}$ is the edge set of r. In this case, for every $N \in \mathbb{Z}$, the rays described by the sequences $\{v_n\}_{n \geq N}$ and $\{v_n\}_{n \leq N}$ are called the **half-rays** of the double ray;

- **Star:** it is an arbitrary union of (finite) paths that pairwise intersects (precisely) at a distinguished vertex $v \in V(G)$. In this case, v is called the **center** of the star, while the other endpoints of each path define its **leaves**. For example, Figure 12 presents a star whose center is highlighted in green and whose leaves are drawn in red;

- **Comb:** it is a graph c obtained by adding, to a fixed ray, infinitely many disjoint finite paths. The given ray is called the **spine** of the comb, while the endpoints of the disjoint paths out of C are called its **teeth**. For example, Figure 12 presents a comb whose spine is highlighted in green and whose teeth are drawn in red.

Figure 12 – Examples of a path, a cycle, a ray, a double ray, a star and a comb.



Source: Elaborated by the author.

For a vertex $v \in V(G)$, the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ denotes its **neighborhood**, whose cardinal $d(v) = |N(v)|$ is called its **degree**. If $u \in N(v)$, we say that u is a **neighbor** of v or **adjacent** to v. Since we are interested in *infinite* graphs, namely, those of infinitely many vertices, $d(v)$ might be an infinite cardinal. Within this vocabulary, a classical compactness result from set theory can be restated in a graph-theoretic language:

Lemma 2.1.1 (König's Lemma)**.** *Let G be a connected infinite graph. Then, G has a vertex of infinite degree or contains a ray as a subgraph.*

Proof. Suppose that every vertex of G has finite degree and fix any $v_0 \in V(G)$. Since G is an infinite connected graph, there is an infinite connected component G_1 of $G \setminus \{v_0\}$, because $d(v_0)$ is finite. Then, fix $v_1 \in V(G_1)$ any neighbor of v_0. For some $n \geq 1$, suppose that we have defined $v_0, v_1, \ldots, v_n \in V(G)$ distinct vertices such that $v_i v_{i+1} \in E(G)$ for every $0 \leq i < n$. By induction, if $n \geq 1$, we assume that v_n is chosen within an infinite connected subgraph G_n of $G \setminus \{v_0, v_1, v_2, \dots, v_{n-1}\}$. Hence, $G_n \setminus \{v_n\}$ has finitely many connected components, once v_n has finite degree. Therefore, we can fix v_{n+1} a neighbor of v_n in an infinite connected component G_{n+1} of $G_n \setminus \{v_n\}$. At the end of this recursive process, $\{v_n\}_{n \in \mathbb{N}}$ defines the vertex set of a ray in G. □

In other words, König's Lemma claims that there are rays in infinite connected **locally finite** graphs, namely, graphs whose vertices have all finite degree. If we regard a vertex of infinite degree as a center of an infinite star, then the above proof of Lemma 2.1.1 can be slightly modified in order to present the following useful generalization:

Lemma 2.1.2 (Star-Comb Lemma)**.** *Let G be a connected graph and fix an infinite subset $U \subset V(G)$. Then, there is an infinite star whose leaves belong to U or there is a comb whose teeth belong to U.*

Proof. Fix P_0 any path in G connecting two distinct vertices of U, whose existence follows from the assumption that G is connected. If there are infinitely many connected components in $G \setminus P_0$ containing vertices of U, some $v \in P_0$ has neighbors in infinitely many of these components, because P_0 is finite. In this case, v is the center of an infinite star with leaves in U. Hence, since U is infinite, we can assume that some connected component G_1 of $G \setminus P_0$ has infinite intersection with U. Then, fix a path P_1 connecting one vertex $v_0 \in P_0$ to a vertex $u_1 \in U \cap V(G_1)$, assuming also that $V(P_1) \setminus \{v_0\} \subset V(G_1)$. By induction, for some $n \geq 1$, suppose that we have defined finitely many disjoint paths $P_0, P_1, \dots, P_n$ with the following properties:

1. For each $0 \leq i < n$, the intersection of P_i with $V(P_{i+1}) \cup V(P_{i+2}) \cup \dots \cup V(P_n)$ is precisely an endpoint v_i of P_{i+1}. The other endpoint of P_{i+1} is some vertex $u_{i+1} \in U$;
2. $V(P_{i+1}) \setminus \{v_i\}$ is contained in a connected component G_{i+1} of $G \setminus (V(P_0) \cup V(P_1) \cup \dots \cup V(P_i))$ that has infinite intersection with U, for each $0 \leq i < n$.

If there are infinitely many connected components of $G_n \setminus V(P_n)$ containing elements of U, then some vertex v of the finite set $V(P_n) \setminus \{v_{n-1}\}$ has neighbors in infinitely many of these components. As before, in this case, v is the center of an infinite star with leaves in U. Then, we can assume that there is a connected component G_{n+1} of $G_n \setminus V(P_n)$ whose intersection with U is infinite. This allows us to fix P_{n+1} a path connecting a vertex $v_n \in V(P_n) \setminus \{v_{n-1}\}$ to some vertex $u_{n+1} \in U$, also assuming that $P_{n+1} \setminus \{v_n\} \subset V(G_{n+1})$.

At the end of this recursive process, we have defined a family $\{P_n\}_{n \in \mathbb{N}}$ of finite paths satisfying items 1 and 2 above. In particular, the graph T given by $V(T) = \bigcup_{n \in \mathbb{N}} V(P_n)$ and $E(T) = \bigcup_{n \in \mathbb{N}} V(P_n)$ is connected. Moreover, by item 2, each vertex of T has degree at most 3.

Then, it follows from König's Lemma that there is a ray in T, being a spine of a comb with infinitely many teeth lying on $\{u_n\}_{n\in\mathbb{N}}$. $\square$

We finish this section with an estimate for the size of some connected graphs. In particular, the result bellow allows us to assume that every locally finite graph (connected) is countable:

Lemma 2.1.3. *Let G be a connected graph and κ be an infinite cardinal such that $d(v) \leq \kappa$ for every $v \in V(G)$. Then, $|V(G)| \leq \kappa$.*

Proof. Fix any vertex $v \in V(G)$. For each $i \in \mathbb{N}$, define the set

$$N^i(v) = \{u \in V(G) : \text{there is a path of length } i \text{ connecting } u \text{ and } v\},$$

so that $N^{i+1}(v) \subset \bigcup_{u\in N^i(v)} N(u)$. Then, since $|N^1(v)| = 1$ and $|N(u)| \leq \kappa$ for every $u \in V(G)$, it follows by induction on i that $|N^{i+1}(v)| \leq \kappa$. Therefore, $\left|\bigcup_{i=1}^{\infty} N^i(v)\right| \leq \kappa$. This finishes the proof, once $V(G) = \bigcup_{i=1}^{\infty} N^i(v)$ by the fact that G is connected. $\square$

2.2 Trees (for graph-theorists)

The structure of the graph T constructed in the proof of Lemma 2.1.2 is familiar for graph-theorists. Besides connected, we observe that T contains no cycles as subgraphs, or, equivalently, any two vertices of T are the endpoints of precisely one path. Due to this property, we say that T is a **tree**. Clearly, most graphs from Figure 12 are trees: paths, rays, stars and combs do not contains cycles as subgraphs.

In general, trees are important tools when describing algorithms to study graphs, since their vertices can be ordered in a rather natural way. More precisely, given a tree T, we fix any vertex $r \in V(T)$, said to be its **root**. Then, we write $u \leq_T v$ for vertices $u, v \in T$ if u belongs to the unique path in T connecting v and r. We easily check that $\leq_T$ is indeed an order relation, called the **tree-order** of T when the root r is fixed. The choice of r is often arbitrary, so that we write $\leq$ instead of $\leq_T$ when there is no doubt about the fixed tree T and its root. As useful notations, we write $\lceil v \rceil = \{u \in V(T) : u \leq v\}$ and $\lfloor v \rfloor = \{u \in V(T) : u \geq v\}$ for every vertex $v \in V(T)$. In particular, $\lceil v \rceil$ defines a path in T for every $v \in V(T)$, whose amount of edges is called the **height** of v.

Since cycles are finite graphs, it is also a tree the graph obtained by the union of a $\subseteq -$increasing family $\{T_n\}_{n\in\mathbb{N}}$ of trees contained in a fixed graph G. Then, Zorn's Lemma can be applied in order to conclude that G contains a $\subseteq -$maximal tree. Assuming that G is connected, it turns out that this is actually a spanning subgraph. In other words, every connected graph admits a spanning tree.

Exercise 2.2.1 (Usual proof of König's Lemma)**.** *Write a proof for Lemma 2.1.1 which relies on the fact that every graph has a spanning tree.*

On the other hand, it is a rather challenging task - sometimes impossible - to find a spanning tree whose tree order is compatible with the distribution of edges in the underlying graph. More precisely, we say that a tree T with a fixed tree order $\leq$ is **normal** in a graph G if, besides being a subgraph of G, every $T-$path has comparable endpoints regarding $\leq$. We illustrate this definition by saying that paths "run vertically" through normal trees, while they can "run horizontally" in arbitrary ones, as suggested by Figure 13. Similarly to the observation made in the above paragraph, $\subseteq-$maximal normal trees indeed exist by Zorn's Lemma, but we can not ensure that these are spanning subgraphs.

Figure 13 – Examples of arbitrary and normal trees, respectively.

Examples of two trees whose roots are denoted by r. Both are subgraphs of distinct underlying graphs, in which some paths are highlighted by dashed lines. At the left, the paths drawn in red certify that the corresponding tree is not normal.

Source: Elaborated by the author.

In fact, if K is an uncountable **complete graph** (or **clique**), i.e., a graph whose vertices are pairwise adjacent, then K does not admit a normal spanning tree. After all, in this case, the vertices of K should be pairwise comparable in some tree order, so that such tree would turn out to be a ray (and, therefore, countable). As we will discuss in Chapter 5, there are actually few characterizations of graphs that contain normal spanning trees. Most results in that direction are related to the distribution of the vertices of G along rays, a phenomena that has a topological interpretation. Under the graph-theoretic viewpoint, however, normal trees are often obtained via **depth-search** algorithms, a technique exemplified by the proof below:

Proposition 2.2.2. *Let G be a countable graph, i.e, such that $V(G)$ is countable. Then, G admits a normal spanning tree.*

Proof. Fix an enumeration $V(G) = \{v_n\}_{n\in\mathbb{N}}$. We will construct a $\subseteq-$increasing sequence $\{T_n\}_{n\in\mathbb{N}}$ of finite normal trees in G such that $v_n \in V(T_n)$ for each $n \in \mathbb{N}$. Naturally, we will ensure that the tree order of T_{n+1} extends the tree order of T_n, so that both orders will be denoted by $\leq$. Therefore, $T = \bigcup_{n\in\mathbb{N}} T_n$ will be the claimed normal tree.

We start this process by setting T_0 as the trivial tree containing only its root v_0. We now assume that the normal tree T_n is already defined, for some $n \in \mathbb{N}$. If $v_{n+1} \in V(T_n)$, we set $T_{n+1} = T_n$. Otherwise, v_{n+1} belongs to some connected component C of $G \setminus T_n$. Then, the set

$$N(C) = \{u \in V(T_n) : u \text{ has a neighbor in } C\}$$

is totally ordered by $\leq$, since T_n is a normal tree and C is connected. Moreover, $N(C)$ is finite, because so is T_n. We will now apply the *depth-search procedure* in order to define T_{n+1}. To this aim, we choose $v \in N(C)$ the $\leq -$maximal vertex of $N(C)$, also fixing $u \in C$ one of its neighbors. Let $uu_1u_2 \dots u_k v_{n+1}$ be a path in C connecting u and v_{n+1}. We define T_{n+1} by attaching this path to v.

Then, T_{n+1} is still a finite tree rooted at v_0. In order to show that this is a normal one, let P be any $T_{n+1}-$path. If its endpoints belong to T_n, then they are comparable by induction. If its endpoints belong to the path $uu_1u_2 \dots u_k v_{n+1}$, then they are comparable because $u \leq u_1 \leq u_2 \dots \leq u_k \leq v_{n+1}$. Finally, if one endpoint x of P belong to T_n and the other, say, y, belong to the path $uu_1u_2 \dots u_k v_{n+1}$, then $x \leq v < y$ due to the $\leq -$maximality of v. □

As observed in the above proof, if T is a normal tree in a graph G, then the neighbors of vertices in a fixed connected component of $G \setminus T$ define a totally ordered set. This property, convenient for approaching problems in algorithmic ways, is one of the main reasons for studying normal trees. Actually, in Chapter 5 we will better discuss how these trees might encode faithfully the connectivity of the underlying graphs. An example of this remark, for a while, is given by:

Proposition 2.2.3. *Let G be a graph that admits a normal spanning tree T, whose tree order is denoted by $\leq$. Then, the properties below are verified:*

i) If $u, v \in V(G)$ are incomparable with respect to $\leq$, then u and v belong to distinct connected components of $G \setminus \lceil x \rceil$, where $x = \max\{t \in T : t \leq u, v\}$;

ii) If r is a ray of G, there is r' a ray of T such that $r \cap r'$ is infinite.

Proof. If x is as in item i), consider $t_u = \min\{t \in T : x < t \leq u\}$ and $t_v = \min\{t \in T : x < t \leq v\}$. By the maximality of x, we must have t_v and t_u incomparable and, in particular, $t_u \neq t_v$. Even more, we observe that $\lfloor t_u \rfloor$ and $\lfloor t_v \rfloor$ are connected components of $G \setminus \lceil x \rceil$. In fact, being $w = u$ or $w = v$, neighbors of vertices of $\lfloor t_w \rfloor$ (in G) must belong to $\lfloor t_w \rfloor$ or to $\lceil x \rceil$, since T is normal. By the same reason, there is no edge connecting a vertex of $\lfloor t_u \rfloor$ to a vertex of $\lfloor t_v \rfloor$, so that $\lfloor t_u \rfloor$ and $\lfloor t_v \rfloor$ are indeed different connected components of $G \setminus \lceil x \rceil$. Observing that $u \in \lfloor t_u \rfloor$ and $v \in \lfloor t_v \rfloor$, item i) follows.

Now, suppose that $r = v_0v_1v_2 \dots$ denotes a ray of G. Since T is connected, we can apply the Star-Comb Lemma (2.1.2) with $U = V(r)$. For instance, suppose that there is a star in T, centered at some vertex $x \in T$, whose leaves define an infinite subset $\{v_{n_k}\}_{k \in \mathbb{N}} \subset V(r)$. In other words, x is connected to $\{v_{n_k}\}_{k\in\mathbb{N}}$ by paths that are disjoint unless by x. Hence, after possibly removing one element from $\{v_{n_k}\}_{k\in\mathbb{N}}$, this set is composed by pairwise incomparable vertices regarding $\leq$, since T is a tree. By the same argument, for $k \neq j$, we have $x = \max\{t \in T : t \leq v_{n_k}, v_{n_j}\}$. Therefore, it follows by item i) that each element from $\{v_{n_k}\}_{k\in\mathbb{N}}$ belongs to a different connected component of $G \setminus \lceil x \rceil$. This, however, contradicts the fact that $\lceil x \rceil$ is finite and r is a ray containing $\{v_{n_k}\}_{k\in\mathbb{N}}$.

Hence, there is C a comb in T whose teeth define some infinite subset $\{v_{n_k}\}_{k\in\mathbb{N}} \subset V(r)$. If an infinite subset of $\{v_{n_k}\}_{k\in\mathbb{N}}$ is contained in the spine r' of C, the result follows. If not, we can assume that $r' \cap \{v_{n_k}\}_{k\in\mathbb{N}} = \emptyset$. We also suppose that r' starts at the root z of T. Therefore, for every $k \in \mathbb{N}$, the vertex $x_k = \max\{t \in T : t \leq v_{n_k}, v_{n_{k+1}}\}$ must belong to r'. After all, since C is a comb and T is a tree containing C, the (unique) paths in T connecting v_{n_k} and $v_{n_{k+1}}$ to z must have intersection as a subset of r'. Then, it follows from item i) that the segment in r connecting v_{n_k} to $v_{n_{k+1}}$ intersects r'. In other words, there is $n_k < m_k < n_{k+1}$ such that $v_{m_k} \in r'$. Hence, $\{v_{m_k}\}_{k\in\mathbb{N}}$ is an infinite subset of both rays r and r'. $\square$

2.3 Separators

Most of the results presented so far in this chapter (such as Lemma 2.1.2 and Proposition 2.2.2) already suggest a routine procedure when dealing with infinite graphs. More precisely, we are often deleting some set of vertices and/or edges in order to analyze the remaining connected components. Due to the frequency of this heuristic, we call S a **separator** in G whether it is a subset $S \subset V(G) \cup E(G)$ to be further removed of the graph. This is not a formal definition, but rather it is an expression used to illustrate arguments regarding connectedness. The objects listed below, however, are indeed separators with some convenient properties, which justifies more careful definitions:

- **Cutvertex:** it is a vertex v in a connected graph G such that $G \setminus v$ is disconnected. A graph that is connected but admits no cutvertex is called **biconnected**;
- **Bridge:** it is an edge e in a connected graph G such that $G \setminus e$ is disconnected. In particular, a connected graph is a tree if, and only if, all its edges are bridges;
- **Cut:** in a graph G, it is an edge set of the form $\delta(X) = \{uv \in E(G) : u \in X, v \in V(G) \setminus X\}$, for some non-empty $X \subsetneq V(G)$. In general, finite minimal separators composed by edges are cuts, as we will properly approach in Chapter 6;
- *$A - B$* **separator:** for any pair of vertex sets $A, B \subset V(G)$, it is a separator $S \subset V(G) \cup E(G)$ such that there is no $A - B$ path in $G \setminus S$. Note that S might intersect $A \cup B$.

Since there are at least two connected components in any disconnected graph, it is easily seen that G is connected if, and only if, it admits no empty cut. On the other hand, we say that H

is a **block** in G if it is a maximal biconnected subgraph. In particular, any two distinct blocks H_1 and H_2 intersect in at most one vertex: otherwise, $H_1 \cup H_2$ would still define a biconnected graph, but containing both properly. Actually, this argument shows that, if $H_1 \cap H_2 \neq \emptyset$, then $H_1 \cap H_2$ is precisely one cutvertex of G.

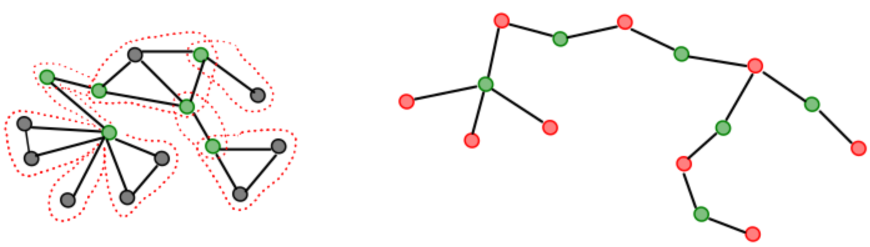
Therefore, if A denotes the set of cutvertices of G and $\mathscr{B}$ denotes the set of its blocks, a natural adjacency notion arises on $V(\check{G}) := A \cup \mathscr{B}$. More precisely, we declare $aB \in E(\check{G})$ if $a \in A$, $B \in \mathscr{B}$ and $a \in V(B)$. Therefore, it is defined the **block graph** $\check{G}$ of G, usually being a tree:

Proposition 2.3.1. *If G is a connected graph, its block graph $\check{G}$ is a tree.*

Proof. If G is biconnected, then $\check{G}$ has only one vertex, so that the result is immediate. Hence, we can assume that G has at least two blocks. Then, in order to show that $\check{G}$ is connected, it suffices to exhibit a path (in $\check{G}$) between given two blocks B and B' of G. In fact, since G connected, there is a path of the form $wu_1u_2\dots u_nv$ in G, for some fixed vertices $w \in V(B)$ and $v \in V(B')$. If B_i denotes the block in which u_i lie, the subgraph of $\check{G}$ induced by $\{B_i : 1 \leq i \leq n\} \cup \{V(B_i) \cap V(B_j) : 1 \leq i, j \leq n\}$ is connected, since $B_i = B_{i+1}$ or $V(B_i) \cap V(B_{i+1}) \in \{v_i, v_{i+1}\}$ for each $1 \leq i < n$. In particular, there is a path in $\check{G}$ connecting B and B'.

Finally, suppose that there is a cycle in $\check{G}$. Since there is no edges in $\check{G}$ between two distinct cutvertices or between two distinct blocks, this cycle has an even amount of vertices. In particular, the cycle contains two distinct blocks B and B'. Fixing two different vertices $u \in V(B)$ and $v \in V(B')$, we can now find a cycle C in G containing both, since blocks are connected. However, C is contained in some block of G, because it is a biconnected subgraph. This shows that $B = B'$, which is a contradiction. Hence, $\check{G}$ is a tree. □

Figure 14 – Example of a graph and its block graph



At the left, the blocks of a connected graph are involved by red balloons, while its cutvertices are drawn in green. At the right, we present the corresponding block graph, that is a tree.

Source: Elaborated by the author.

Alternatively, the above result claims that, if u and v are vertices that lie in different blocks of G, the paths connecting them must contain the same cutvertices. In particular, these paths are not disjoint. Hence, for finite graphs, we can see Proposition 2.3.1 as a restricted study of the following duality theorem:

Theorem 2.3.2 (Menger (1927), Menger's Theorem)**.** *Let G be a finite graph and fix $A, B \subset V(G)$ subsets. Consider $S \subset V(G)$ an $A-B$ separator of minimum size. Then, there exist $|S|$ disjoint $A-B$ paths.*

Although S is minimal, we remark that the family claimed by the above statement is maximal, since each element of the $A-B$ separator S lies on precisely one of the paths. Incidentally, we often illustrate Menger's Theorem with the metaphor that "by the most narrow neck it passes the maximum flow". Inspired by this observation, Erdős asked whether it could be generalized for infinite graphs, a problem that remained more than 30 years unsolved. Only in 2009, Aharoni and Berger in (AHARONI; BERGER, 2005) presented the answer below:

Theorem 2.3.3 (Aharoni and Berger (2005), Erdős-Menger Theorem)**.** *Let G be any graph and fix $A, B \subset V(G)$ subsets. Then, there exist a family $\mathscr{P}$ of disjoint $A-B$ paths and S an $A-B$ separator lying on it. In other words, S is obtained by the choice of precisely one vertex from each path of $\mathscr{P}$.*

The book (DIESTEL, 2018) contains five alternative proofs for Theorem 2.3.2, presented throughout its third and sixth chapters. At the eight chapter, Theorem 2.3.3 is proven for countable graphs, following the program carried out by Aharoni in (AHARONI, 1987). In general lines, Menger's Theorem has a wide range of applications, both theoretical (as we will exemplify further on in this dissertation) and in other sciences. Quoting Distel, this

> is probably the most-used classical result in graph theory (DIESTEL, 2018, p.86).

Although the separators of the two results above are vertex sets, in Section 6.2 we will be interested in edge-connectivity properties. To that aim, it is convenient to deduce an edge version of Theorem 2.3.3 as its corollary:

Corollary 2.3.4 (Erdős-Menger Theorem for edges)**.** *Let G be a graph and fix disjoint subsets $A, B \subset V(G)$. Then, there exist a family $\mathscr{P}$ of edge-disjoint $A-B$ paths and an $A-B$ separator F, which is a cut, lying on it. In other words, F is obtained by the choice of precisely one edge from each path of $\mathscr{P}$.*

Proof. We will define an auxiliary graph $\tilde{G}$. For every $v \in V(G)$, let K_v be a complete graph of $d(v)$ vertices. Then, the vertex set of $\tilde{G}$ will be the disjoint union $\bigcup_{v \in V(G)} V(K_v)$. For every edge $uv \in E(G)$, we define an edge $u'v'$ between the cliques K_u and K_v, referred as an *old edge*. Since $|K_v| = d(v)$, we can assume that every vertex of K_v is an endpoint of at most one old edge.

After applying Theorem 2.3.3 in order to separate the (disjoint) vertex sets $\tilde{A} = \bigcup_{v \in A} V(K_v)$ and $\tilde{B} = \bigcup_{u \in B} V(K_u)$, we fix $\tilde{\mathscr{P}}$ the family of disjoint $\tilde{A} - \tilde{B}$ paths and $\tilde{S}$ the separator lying on it. For every $\tilde{P} \in \tilde{\mathscr{P}}$ and every $v \in V(G)$, we can assume that $|\tilde{P} \cap V(K_v)| \leq 2$. In fact, if $\tilde{v}_1, \tilde{v}_2 \in V(K_v)$ are non-adjacent vertices in $\tilde{P}$, consider the path $\tilde{P}'$ obtained from $\tilde{P}$ after replacing the subpath connecting $\tilde{v}_1$ and $\tilde{v}_2$ by the edge $\tilde{v}_1\tilde{v}_2$. Being an $\tilde{A} - \tilde{B}$ path, $\tilde{P}'$ must meet $\tilde{S}$ in the unique vertex of $\tilde{P} \cap \tilde{S}$, since $\tilde{\mathscr{P}}$ is composed by disjoint paths. Therefore, $(\tilde{\mathscr{P}} \setminus \{\tilde{P}\}) \cup \{\tilde{P}'\}$ is also a family of disjoint $\tilde{A} - \tilde{B}$ paths in which $\tilde{S}$ lie.

By considering that $|\tilde{P} \cap V(K_v)| \leq 2$ for every $v \in V(G)$, a path P in G arises from $\tilde{P} \in \tilde{\mathscr{P}}$ after contracting the cliques $\{K_v : v \in V(G)\}$ to their original vertices. Then, $\mathscr{P} = \{P : \tilde{P} \in \tilde{\mathscr{P}}\}$ is a family of edge-disjoint $A - B$ paths. Moreover, each vertex $\tilde{v} \in \tilde{S}$ belongs to a clique of the form K_v, for some $v \in V(G)$. In addition, $\tilde{v}$ is the endpoint of an unique old edge $\theta(\tilde{v})$, originally incident in v. Note also that $\theta(\tilde{v})$ belongs to the path of $\tilde{\mathscr{P}}$ that contains $\tilde{v}$.

We observe that every $A - B$ path Q in G must passes through an edge from $\{\theta(\tilde{v}) : \tilde{v} \in \tilde{S}\}$. Otherwise, a minimal path in $\tilde{G}$ containing all the old edges of Q will not intersect $\tilde{G}$, contradicting the fact that $\tilde{S}$ separates $\tilde{A}$ and $\tilde{B}$. Therefore, $F = \{\theta(\tilde{v}) : \tilde{v} \in \tilde{S}\}$ is an edge set lying on $\mathscr{P}$ for which there is no $A - B$ path in $G \setminus F$. Although it is no difficult to see that $F = \delta(X)$ for some $X \subset V(G)$, this will later follow from Lemma 6.2.3. $\square$

Exercise 2.3.5. *Consider the following verbatim generalization of Menger's Theorem for infinite graphs: "in a graph G, for given subsets $A, B \subset V(G)$, there exist $\mathscr{P}$ a family of disjoint $A - B$ paths and a set of $|\mathscr{P}|-$many vertices that separate A and B". How does this statement differ from the Erdős-Menger Theorem?*

Part I

The Unfriendly Partition Conjecture

CHAPTER

3

PROBLEM BACKGROUND

As pointed out in the Introduction, the unfriendly partition conjecture is an open question whose literature, even though not extensive, is rich in different problem-solving techniques. The aim of this chapter is to bring the attention of the reader to such tools, while revisiting the main related results that are available so far. Although the proofs presented in the next sections are not original, some of them were slightly modified in order to be further mentioned with a more convenient statement.

3.1 Introduction

Before presenting the proper discussions of this chapter, we must fix the notation and the definitions that will be used from now on in this part of the dissertation. Given a graph G, a **partial coloring** of its vertices will almost always be a function of the form $c : D \to 2$, defined in a subset $D \subset V(G)$ and taking values on the set of two elements $2 = \{0,1\}$. The labels 0 and 1 are also referred as **colors**. Only in Section 4.1, however, we will study functions of the form $c : D \to 3$, taking values in a set of three elements. A **coloring**, in its turn, is a globally defined assignment $c : V(G) \to 2$.

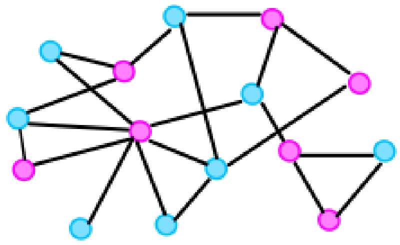
Regarding the natural bipartition arisen by a function of the form $c : D \to 2$, we say that two adjacent vertices $u, v \in D$ are **friends** if they belong to the same partition class, that is, if $c(u) = c(v)$. Otherwise, we call them **enemies** of each other. If $D = V(G)$, we ask whether a vertex $v \in V(G)$ has no less enemies than friends, or, in the current notation, whether

$$|\{u \in N(v) : c(u) \neq c(v)\}| \geq |\{u \in N(v) : c(u) = c(v)\}|.$$

If that inequality is verified, we say that c is **unfriendly in the vertex** v. Globally, we thus call c an **unfriendly partition** of G if it is unfriendly in each of its vertices.

In particular, verifying whether c is unfriendly in a vertex v of infinite degree is quite simple: it is necessary and sufficient that v has $d(v)$ enemies given by c. Inspired by that

Figure 15 – Example of an unfriendly partition



In this graph, we can check that every vertex has at least as many neighbors of opposite color as of its own. Hence, this is an example of an unfriendly partition.

Source: Elaborated by the author.

observation, we say that v has **almost all** its neighbors in a set $X \subset V(G)$ if $|N(v) \setminus X| < d(v)$. Intuitively, if we want to construct a coloring that is unfriendly in v, we might ignore its neighbors out of X.

Finally, supposing now that $c : D \to 2$ is a partial coloring defined in a proper subset $D \subsetneq V(G)$, we say that this coloring is **already unfriendly** in a vertex $v \in D$ if any extension of c to $V(G)$ is unfriendly in v as well. In particular, if v has infinite degree, this happens if, and only if, $|\{u \in N(v) \cap D : c(u) \neq c(v)\}| = d(v)$.

Now, we are ready to look forward answering a natural question: which graphs admit an unfriendly partition? Cowan and Emerson in their unpublished paper (COWAN; EMERSON, 1985) conjectured that every graph does, inspired by the simple fact that finite graphs can be colored this way:

Proposition 3.1.1. *Every finite graph admits an unfriendly partition.*

Proof. Let G be a finite graph. Consider c any **max-cut** of G, namely, any coloring that maximizes the amount of edges with different colors in its endpoints. In other words, choose c so that the cardinal $|\{uv \in E : c(u) \neq c(v)\}|$ is maximum. Observe that this is possible since there are only finitely many functions defined in the finite set $V(G)$ and taking values on 2.

We claim that c is an unfriendly partition. For instance, suppose that there is a vertex $v \in V(G)$ for which c is not unfriendly. Therefore, considering the sets $A = \{u \in N(v) : c(u) \neq c(v)\}$ and $B = \{u \in N(v) : c(u) = c(v)\}$, we have $|A| < |B|$. Now, define $c * v$ to be the coloring that agrees with c in $V \setminus \{v\}$ but such that $(c * v)(v) = 1 - c(v)$. In other words, $c * v$ differs from c only in the vertex v. Therefore,

$$\{ab \in E : (c*v)(a) \neq (c*v)(b)\} = \{ab \in E : c(a) \neq c(b)\} \cup B \setminus A,$$

so that

$$
\begin{aligned}
|\{ab \in E : (c*v)(a) \neq (c*v)(b)\}| &= |\{ab \in E : c(a) \neq c(b)\}| + |B| - |A| \\
&> |\{ab \in E : c(a) \neq c(b)\}|
\end{aligned}
$$

This, however, contradicts the choice of c. □

Exercise 3.1.2. *The above proof shows that every max-cut in a finite graph G is an unfriendly partition. Verifies that this is actually a stronger property. In other words, find an unfriendly partition over a finite graph that is not a max-cut.*

Despite this positive observation, Milner and Shelah in (SHELAH; MILNER, 1990) constructed a family of uncountable graphs that admit no unfriendly partitions. In fact, these are the only known graphs in the literature that cannot be colored in an unfriendly way. Hence, the original conjecture of Cowan and Emerson is now restricted to the countable case:

Unfriendly Partition Conjecture: *Every countable graph admits an unfriendly partition.*

Besides that, most of the affirmative results available in the literature do not use properly the countability hypothesis of the problem. Some of them, instead, describe unfriendly partitions by forbidding substructures of the underlying graphs, regardless their cardinality. Below, while explaining the organization of the next sections, we give a general overview of this literature:

- Section 3.2 studies graph with few vertices of infinite degree, closely supported by Proposition 3.1.1. In fact, unfriendly partitions for locally finite graphs are easily obtained by compactness principles, following routine proofs when dealing with this graph family. On the other hand, a more clever argument is needed to color graphs with finitely many vertices of infinite degree;
- Section 3.3 studies the other end of the above spectrum, i.e., graphs with few vertices of finite degree. In those cases, it is useful to describe unfriendly partitions via algorithms that, recursively, attribute enemies to vertices previously colored;
- Section 3.4 presents the only known graphs in the literature that admit no unfriendly partitions. These counterexamples rely on a set-theoretic obstruction, might being of particular interest for a reader that is familiar with fundamentals of mathematics. Curiously, the constructions presented in that section avoid vertices of finite degree, which justifies some hypothesis that will be assumed on Section 3.3;
- Section 3.5 deals with recursive characterizations of some graph families, specially in order to describe unfriendly partitions for rayless graphs. The techniques employed for this study are also mentioned by the literature of other problems in infinite graph theory;

- Section 3.6 shows that every graph admits an "unfriendly partition" when we are able to label vertices with an extra third color. The proof of this result is partially inspired by the techniques of Section 3.3.

Exercise 3.1.3. *Show that every tree admits an unfriendly partition.*

3.2 Finite-like scenarios

This section aims to describe unfriendly partitions for graphs with only finitely many vertices of infinite degree. Naturally, a first step in that direction is to well understand how these colorings are obtained for locally finite graphs. Already knowing that *finite* graphs can be colored in an unfriendly way, set theorists might claim the same conclusion for locally finite ones by applying compactness arguments. In fact, this is the usual treatment when dealing with locally finite graphs, which explains the frequent approach of this graph family by the literature regarding infinite graphs. According to Nash-Williams,

> The degree of additional difficulty involved when we try to extend work done for finite graphs to infinite graphs varies from one problem to another. Extension to enumerable graphs is usually easiest when it can be done by means of Kőnig's "Unendlichkeitslemma" (NASH-WILLIAMS, 1967).

Regarding mathematical logic, a Compactness Theorem informally claims that, if a statement is a consequence from a list of infinitely many axioms, then it is actually a consequence of some finitely many of them. Thus, when using the expression "compactness argument", we mean any proof technique which tries to verify a global property based on their finite approximations. Considering the unfriendly partition problem, the two proofs below exemplify this heuristic:

Proposition 3.2.1. *Every locally finite graph has an unfriendly partition.*

First proof: Fix G a locally finite graph. Without loss of generality, we can suppose that G is connected, so that $V(G)$ is a countable set by Lemma 2.1.3. Thus, fix an enumeration $V(G) = \{v_n\}_{n\in\mathbb{N}}$.

For each $n \in \mathbb{N}$, Proposition 3.1.1 allows us to fix an unfriendly partition $c_n : V(G_n) \to 2$ for the graph G_n induced from G by the finite set $V_n = \{v_0, v_1, \ldots, v_n\}$. We will now define a coloring $c : V(G) \to 2$ recursively as follows:

- By the pigeonhole principle, for some color $i \in \{0,1\}$, there is $S_0 \subset \mathbb{N}$ infinite so that $c_k(v_0) = i$ for every $k \in S_0$. Then, define $c(v_0) = i$.

- For some $n \in \mathbb{N}$, suppose that an infinite set $S_n \subset \mathbb{N}$ is already defined, as well as the color $c(v_n)$. Again, the pigeonhole principle guarantees that there is a color $i \in \{0,1\}$ and an infinite set $S_{n+1} \subset S_n$ such that $\min S_{n+1} \geq n+1$ and $c_k(v_{n+1}) = i$ for every $k \in S_{n+1}$. Then, define $c(v_{n+1}) = i$.

At the end of this inductive process, we claim that $c : V \to 2$ is an unfriendly partition. In fact, given $n \in \mathbb{N}$, there is $N \in \mathbb{N}$ large enough so that $N(v_n) \cup \{v_n\} \subset \{v_0, v_1, \dots, v_N\}$, since v_n has finite degree by hypothesis. Fix any $k \in S_N$, so that $k \geq N$. By the definition of $c(v_i)$ and the fact that $S_N \subset S_{N-1} \subset \cdots \subset S_0$, it holds that $c(v_i) = c_k(v_i)$ for each $0 \leq i \leq N$. In particular, as c_k is an unfriendly partition over G_k and $N(v_n) \cup \{v_n\} \subset \{v_0, v_1, \dots, v_N\} \subset \{v_0, v_1, \dots, v_k\} = V(G_k)$, the coloring c is also unfriendly in v_n. □

Second proof: Let G be a locally finite graph. As in the previous proofs, we can assume that G is connected and consider $V(G) = \{v_n\}_{n \in \mathbb{N}}$ an enumeration of its vertex set. For each $n \in \mathbb{N}$, denote by $G_n = G[v_0, v_1, \dots, v_{n-1}]$ the subgraph of G induced by the set $V_n = \{v_0, v_1, \dots, v_{n-1}\}$. We now say that a partial coloring $c : V_n \to 2$ is *extendable* if there are $m \geq n$ and $c' : V_m \to 2$ an unfriendly partition for G_m such that $c'|_{V_n} = c$. Denoting by T the set of extendable colorings, we define an order $\leq$ over T by declaring $c \leq c'$ if, and only if, c' extends c.

By Proposition 3.1.1, for each $n \in \mathbb{N}$ there is an extendable coloring whose domain is V_n. Moreover, by definition, $c|_{V_i}$ is an extendable coloring if so is $c : V_n \to 2$ and $i \leq n$. In this case, we also have $\emptyset \leq c|_{V_1} \leq c|_{V_2} \leq \cdots \leq c|_{V_{n-1}} \leq c|_{V_n}$. In other words, the pair $(T, \leq)$ describes an order of an infinite tree, rooted at the empty coloring $\emptyset$. More precisely, by setting an extendable function of the form $c : V_n \to 2$ as a neighbor of $c|_{V_{n-1}}$, T becomes a tree whose tree order, when fixing $\emptyset$ as a root, is $\leq$

Moreover, the colorings of height n in T are those defined on V_n. Since V_n is finite for every $n \in \mathbb{N}$, this means that every vertex of T has finite degree. By König's Lemma, then, there is $\{c_n\}_{n \in \mathbb{N}} \subset T$ an infinite branch. In this notation, c_n is defined on V_n for each n, so that the coloring $c : V(G) \to 2$ given by $c(v_n) = c_{n+1}(v_n)$ is globally defined.

Intuitively, c is a limit coloring approximated by the branch $\{c_n\}_{n \in \mathbb{N}}$, being a natural candidate for an unfriendly partition. In fact, given $v \in V(G)$, there is $n \in \mathbb{N}$ big enough such that $N(v) \cup \{v\} \subset V_n$, because v has finite degree. Hence, by definition of the order $\leq$, we have $c|_{N(v) \cup \{v\}} = c_n|_{N(v) \cup \{v\}}$. Since c_n is an extendable function, c_n is itself the restriction of an unfriendly partition $c : V_k \to 2$ to V_n, for some $k \geq n$. In particular, regarding c_k (and hence regarding both c_n and c), v has no more friends than enemies, so that c is unfriendly in v. □

We remark that the main hypothesis of Proposition 3.2.1 is applied at the very end of the above two proofs. In both cases, for a vertex $v \in V(G)$, the constructed coloring c agrees with some finite approximation c_k in the *finite* set $\{v\} \cup N(v)$, proving that c is indeed an unfriendly partition.

Moreover, both proofs presented are also supported by the fact that finite graphs admit an unfriendly partition, although it is possible to assume a slightly stronger property. We recall that, as Exercise 3.1.2 highlights, unfriendly partitions for finite graphs might be given by max-cuts. Therefore, one could use the compactness arguments to construct a coloring that is approximated by cuts of maximum size, instead of arbitrary unfriendly partitions. After exploring this suitable difference, in this section we may fix finitely many perturbations on colorings for locally finite graphs, following the program carried out by Aharoni, Milner and Prikry (1990). As a consequence, the result below will be verified:

Theorem 3.2.2 (Aharoni, Milner and Prikry (1990), Theorem 1)**.** *Let G be any graph and fix $c' : D \to 2$ a partial coloring of its vertices. If there are only finitely many vertices of infinite degree in $V(G) \setminus D$, then there is $c : V(G) \to 2$ an extension of c' that is unfriendly in every vertex of $V(G) \setminus D$.*

By considering $D = \emptyset$ in the above statement, it follows that every graph with finitely many vertices of infinite degree has an unfriendly partition. The above statement is not written this way due to its applicability as inductive hypothesis.

On the other hand, the proof of Theorem 3.2.2 requires some special notation, which will be recalled from its original proof in (AHARONI; MILNER; PRIKRY, 1990). First, we define the set

$$A_c(X,Y) = \{uv \in E : u \in X, \quad v \in Y \quad \text{and} \quad c(u) \neq c(v)\}$$

for every pair $X, Y \subset V(G)$. In other words, $A_c(X,Y)$ denotes the collection of edges with distinct colors in its endpoints and crossing the subsets X and Y. Analogously, we define

$$B_c(X,Y) = \{uv \in E : u \in X, \quad v \in Y \quad \text{and} \quad c(u) = c(v)\}$$

as the set of edges with endpoints of the same color that also cross the subsets X and Y. Similarly to the proof of Proposition 3.1.1, it is convenient to compare the sizes of $A_c(X,Y)$ and $B_c(X,Y)$, which will be denoted by $a_c(X,Y)$ and $b_c(X,Y)$ respectively. Simplifying this notation in some cases, we also highlight the following conventions:

- If $Y = V(G)$, the sets $A_c(X,Y)$ and $B_c(X,Y)$ are denoted by $A_c(X)$ and $B_c(X)$ accordingly. In this case, their cardinalities are given by $a_c(X) = a_c(X,Y)$ and $b_c(X) = b_c(X,Y)$.
- If $X = \{x\}$ is a singleton, we consider $A_c(X) = A_c(x)$, $B_c(X) = B_c(x)$, $a_c(X) = a_c(x)$ and $b_c(X) = b_c(x)$.

Now, if $c : D \to 2$ is a partial coloring and $F \subset V(G)$ is any subset, there is a natural coloring $c * F : D \to 2$ obtained when changing the labels of the vertices in $F \cap D$. Formally,

$$(c * F)(x) = \begin{cases} c(x), & \text{if } x \notin F, \\ 1 - c(x), & \text{if } x \in F. \end{cases}$$

for every $x \in D$. Again, if $F = \{v\}$ is a singleton, the coloring $c * F$ is also represented by $c * v$. As elementary combinatorial properties, the calculations below are verified:

Lemma 3.2.3 (Aharoni, Milner and Prikry (1990), Lemma 1)**.** *Fix G a graph and $c : V(G) \to 2$ any coloring. Then, given $K, F \subset V(G)$ and $x \in K$, the following statements hold:*

1. $a_{c*x}(K) + a_c(x) = a_c(K) + b_c(x)$.
2. *If F and K are disjoint, then $a_{c*F}(K) + a_c(F,K) = a_c(K) + b_c(F,K)$.*

Proof. In order to verify the first expression, we describe the set $A_{c*x}(K)$ in terms of the coloring c. By definition, this set comprises edges of G whose endpoints have distinct colors and such that one of them belongs to K. If none of them is x, these edges are also in $A_c(K)$, since c and $c * x$ differ only at x. Moreover, we must remove from $A_c(K)$ the edges of $A_c(x)$, whose endpoints have the same color regarding $c * x$. In its turn, also considering $c * x$, the edges from $B_c(x)$ have their endpoints as enemies of each other. Formally,

$$A_{c*x}(K) = A_c(K) \cup B_c(x) \setminus A_c(x),$$

from where we obtain item 1.

Aiming to conclude the second expression, we will also describe the set $A_{c*F}(K)$ in terms of the coloring c. However, we first observe that every edge of the sets $A_c(F,K)$ and $B_c(F,K)$ has exactly one end in F and the other in K, once $F \cap K = \emptyset$. As before, if no edge of $A_c(K)$ has an endpoint in F, then those edges are also elements of $A_{c*F}(K)$, since c and $c * F$ differ only in F. However, the edges of $B_c(F,K)$, that under c have their endpoints with the same color, have endpoints as enemies of each other when considering $c * F$. Finally, the edges of $A_c(F,K)$ are also elements of $A_c(K)$, but have their endpoints labeled the same regarding $c * F$. Therefore,

$$A_{c*F}(K) = A_c(K) \cup B_c(F,K) \setminus A_c(F,K),$$

from where item 2 is easily deduced. □

Finally, given any subset $F \subset V(G)$, we say that a coloring $c : V(G) \to 2$ is $F-$**good** if $a_c(F)$ is maximal among other colorings that are equal to c in $V(G) \setminus F$. In other words, c is $F-$good if $a_c(F) \geq a_{c'}(F)$ for every coloring $c' : V(G) \to 2$ that can differ from C only in F. In particular, if F is a finite set whose elements have finite degree, this implies that c is unfriendly in every vertex of F.

The next result points out that a $F-$good coloring is, in some sense, a coloring that is unfriendly in F as a set. After all, in this case, $a_c(F, V(G) \setminus F)$ cannot be increased by simply changing c to $c * F$. If this holds for every finite set $F \subset V(G)$ of vertices of finite degree, the converse, remarkably, is also true:

Lemma 3.2.4. *Fix G a graph, $D \subset V(G)$ a subset and $c : V(G) \to 2$ any coloring. Then, c is $F-$good for every finite set $F \subset V(G) \setminus D$ of vertices of finite degree if, and only if, $a_c(F, V(G) \setminus F) \geq b_c(F, V(G) \setminus F)$ for every such a F.*

Proof. If c is $F-$good for every finite set $F \subset V(G) \setminus D$ of vertices of finite degree, then, in particular,

$$a_c(F,F) + a_c(F, V(G) \setminus F) = a_c(F) \geq a_{c*F}(F) = a_{c*F}(F,F) + a_{c*F}(F, V(G) \setminus F).$$

Since $a_c(F,F) = a_{c*F}(F,F)$ and $a_{c*F}(F, V(G) \setminus F) = b_c(F, V(G) \setminus F)$, because c and $c*F$ differ precisely at F, it follows that $a_c(F, V(G) \setminus F) \geq b_c(F, V(G) \setminus F)$.

Conversely, suppose that $a_c(F, V(G) \setminus F) \geq b_c(F, V(G) \setminus F)$ holds for every finite set $F \subset V(G) \setminus D$ of vertices of finite degree. Then, fixed any such a F and $c' : V(G) \to 2$ a coloring satisfying $c|_{V(G)\setminus F} = c'|_{V(G)\setminus F}$, let $F' \subset F$ be the set of vertices in which c and c' differ. More precisely, $F' = \{v \in V(G) : c'(v) \neq c(v)\}$, so that $c' = c*F$. Being a subset of F, F' is itself finite and its members have finite degree. By definition of the sets $A_c(F)$, $A_{c'}(F)$, $B_c(F', V(G) \setminus F')$ and $A_c(F', V(G) \setminus F')$, we then have

$$a_{c'}(F) = a_c(F) + b_c(F', V(G) \setminus F') - a_c(F', V(G) \setminus F').$$

Since $a_c(F', V(G) \setminus F') \geq b_c(F', V(G) \setminus F')$ by hypothesis, the inequality $a_c(F) \geq a_{c'}(F)$ holds, verifying that c is $F-$good. □

Considering the statement of Theorem 3.2.2, our aim now is to extend some partial colorings to be $F-$good for every finite set F of uncolored vertices of finite degree. Inspired by the discussion done right before Proposition 3.2.1, this extension will be obtained via a compactness result called Rado's Selection Principle, that is presented below together with two proofs. The first one, relying on Zorn's Lemma, is somehow more constructive and direct, although the second, based on Tychonoff's Theorem, is shorter:

Lemma 3.2.5 (Rado's Selection Principle)**.** *Let $\{A_i\}_{i\in I}$ be any family of finite set. For each finite subset $F \subset I$, fix $\varphi_F : F \to \bigcup_{i\in F} A_i$ a choice function. In other words, $\varphi_F(i) \in A_i$ for every $i \in F$. Then, there exists a global choice function $\varphi : I \to \bigcup_{i\in I} A_i$ such that, for every finite $F \subset I$, there is another finite set $I \supset K \supset F$ such that $\varphi|_F = \varphi_K|_F$.*

First proof, due to tkf (2020). For any subset $X \subset I$ and any choice function $\varphi : X \to \bigcup_{i\in X} A_i$, consider the following approximation property:

($\star$) For every finite $F \subset I$, there is $K \supset F$ also finite such that $\varphi|_{F\cap X} = \varphi_K|_{F\cap X}$.

Let $\mathbb{P}$ be the set of choice functions that satisfies $(\star)$. If $\varphi, \varphi' \in \mathbb{P}$ are defined over $X, X' \subset I$ respectively, we write $\varphi \preceq \varphi'$ whenever $X \subseteq X'$ and φ' extends φ. Then, $\preceq$ is clearly an order relation for $\mathbb{P}$.

Now, consider $\{\varphi_\alpha\}_{\alpha\in\Lambda} \subset \mathbb{P}$ a totally ordered subset regarding $\preceq$. Writing $X_\alpha \subset I$ for the domain of φ_α, consider $X = \bigcup_{\alpha\in\Lambda} X_\alpha$ and $\varphi = \bigcup_{\alpha\in\Lambda} \varphi_\alpha$. More precisely, given $i \in X$, we set $\varphi(i) = \varphi_\alpha(i)$ for some (and, thus, every) $\alpha \in \Lambda$ such that $i \in X_\alpha$. Therefore, φ satisfies $(\star)$ as well: if $F \subset I$ is finite, then $F \cap X = F \cap X_\alpha$ for some $\alpha \in \Lambda$, so that

$$\varphi|_{F\cap X} = \varphi|_{F\cap X_\alpha} = \varphi_\alpha|_{F\cap X_\alpha} = \varphi_K|_{F\cap X_\alpha} = \varphi_K|_{F\cap X}$$

for some finite set $K \supset J$.

Hence, by Zorn's Lemma, there is $\varphi : X \to \bigcup_{i\in X} A_i$ a $\preceq -$maximal element from $\mathbb{P}$. The proof is finished if we conclude that $X = I$. For a contradiction, suppose that there is $i \in I \setminus X$. Then, for each $x \in A_i$, the choice function φ_x defined over $X \cup \{i\}$, extending φ by declaring $\varphi_x(i) = x$, does not satisfy $(\star)$, since φ is $\preceq -$maximal. This means that there is $F_x \subset I$ a finite set such that $\varphi_K|_{F_x\cap(X\cup\{i\})} \neq \varphi_x|_{F_x\cap(X\cup\{i\})}$ for every finite $K \supset F_x$. In particular, since φ satisfies $(\star)$, we must have $i \in F_x$.

However, the union $F = \bigcup_{x\in A_i} F_x$ is also finite and contains i. Hence, there indeed exists a finite set $K \supset F$ such that $\varphi_K|_{F\cap X} = \varphi|_{F\cap X}$. In particular, $\varphi_K|_{F_x\cap X} = \varphi|_{F_x\cap X} = \varphi_x|_{F_x\cap X}$ for every $x \in A_i$. We obtain a contradiction by choosing $x = \varphi_K(i) \in A_i$, so that $\varphi_K|_{F_x\cap(X\cup\{i\})} = \varphi_x|_{F_x\cap(X\cup\{i\})}$. $\square$

Second proof, due to Gottschalk (1951): Note that the set of globally defined choice functions can be identified with the cartesian product $\Phi = \prod_{i\in I} A_i$. If each A_i is endowed with the discrete topology, Φ becomes a compact space by Tychonoff's Theorem. In this case, for each finite $F \subset I$, the set

$$\Phi(F) = \{\varphi \in \Phi : \varphi|_F = \varphi_K|_F \text{ for some finite } K \supset F\}$$

is closed in Φ, by definition of the product topology. Moreover, given a finite collection $F_1, F_2, \ldots, F_n \subset I$ of finite subsets, we have $\varphi_F \in \Phi(F) \subset \bigcap_{i=1}^{n} \Phi(F_i)$, where $F = \bigcup_{i=1}^{n} F_i$. In other words, the family $\mathscr{F} = \{\Phi(F) : F \subset I \text{ finite}\}$ has the finite intersection property. Since Φ is compact, the intersection $\bigcap \mathscr{F}$ is non-empty. Then, any $\varphi \in \bigcap \mathscr{F}$ verifies the statement. $\square$

Back to our graph-theoretic discussion, any coloring over a graph G might be seen choice functions over finite sets. In fact, as suggested by the above second proof, we can identify a partially defined coloring $c : F \to 2$ with an element $c \in \prod_{v\in F} \{0,1\}$. Considering that, we are ready to prove the following:

Proposition 3.2.6 (Aharoni, Milner and Prikry (1990), Lemma 2)**.** *Let $c : D \to 2$ be a partial coloring of a graph G. If every vertex of $V(G) \setminus D$ has finite degree, there is an extension $c' : V(G) \to 2$ that is $F-$good for every finite set $F \subset V(G) \setminus D$.*

Proof. For every $F \subset V(G)$ finite, fix $c_F : F \to 2$ a partial coloring that agrees with c in $D \cap F$ and such that $a_{c_F}(F)$ maximal. Then, let $c' : V(G) \to 2$ be a coloring as in Rado's Selection Principle. In other words, for every finite set $F \subset V(G)$, there is $K \subset V(G)$ also finite such that $F \subset K$ and $c'(v) = c_K(v)$ for every $v \in F$.

We finish the proof by verifying that c' is the claimed coloring. For instance, suppose that c' is not $F-$good for some finite set $F \subset V(G) \setminus D$. Hence, there is $\tilde{c} : V(G) \to 2$ so that $a_{\tilde{c}}(F) > a_{c'}(F)$, even with $c'(v) = \tilde{c}(v)$ for every $v \in V(G) \setminus F$.

On the other hand, since every vertex of F has finite degree by hypothesis, the set $N(F) = \bigcup_{v \in F} N(v)$ is also finite. Therefore, there is a finite set $K \subset V$ containing $F \cup N(F)$ so that $c_K(v) = c'(v)$ for every $v \in F \cup N(F)$. By using the coloring $\tilde{c}$, however, we can increase $a_{c_K}(K)$. More precisely, consider the partial coloring $\tilde{c_K} : K \to 2$ given by

$$\tilde{c_K}(v) = \begin{cases} \tilde{c}(v), & \text{if } v \in F, \\ c_K(v), & \text{if } v \notin F. \end{cases}$$

Since $\tilde{c_K}$ and c_K differ only in $F \subset V(G) \setminus D$, these colorings are equal to c when restricted to $K \cap D$. We will now describe the set $A_{\tilde{c_K}}(K)$ in terms of $A_{c_K}(K)$ and $A_{c'}(F)$. First, we observe that the four colorings c', $\tilde{c}$, c_K and $\tilde{c_K}$ agree in $N(F) \setminus F$: by definition of $\tilde{c_k}$, by the fact that $N(F) \subset K$ and by the equality of $c|_{V(G) \setminus F}$ and $c'|_{V(G) \setminus F}$.

We now remark that, if an edge of $A_{\tilde{c_K}}(K)$ has none of its endpoints in F, then this is also an edge of $A_{c_K}(K)$ and conversely. Formally, $A_{\tilde{c_K}}(K) \setminus A_{\tilde{c_K}}(F) = A_{c_K}(K) \setminus A_{c_K}(F)$. The edges incident in F with different colors in its endpoints under c', however, are the same under c_K, by the choice of this latter coloring. Besides that, those are fewer than the edges with endpoints having distinct colors under $\tilde{c}$. In its turn, $\tilde{c}$ agrees with $\tilde{c_K}$ in $F \cup N(F)$. In other words, $a_{\tilde{c_K}}(F) = a_{\tilde{c}}(F) > a_{c'}(F) = a_{c_K}(F)$. Consequently,

$$a_{\tilde{c_K}}(K) = |A_{\tilde{c_K}}(K) \setminus A_{\tilde{c_K}}(F)| + a_{\tilde{c_K}}(F) > |A_{c_K}(K) \setminus A_{c_K}(F)| + a_{c_K}(F) = a_{c_K}(K),$$

contradicting the choice of c_K. □

Note that Proposition 3.2.6 encodes a third proof for Corollary 3.2.1: by taking $D = \emptyset$, this result provides a coloring $c : V(G) \to 2$ that is $F-$good for every finite set $F \subset V(G)$, when G is a locally finite graph. In particular, $a_c(x) \geq b_c(x)$, verifying that c is an unfriendly partition. However, the main motivation for the study of $F-$good colorings is given by the technical lemma below:

Lemma 3.2.7 (Fixing Lemma)**.** *Given G a countable graph, fix $D \subset V(G)$. Suppose that $c : V(G) \to 2$ is a coloring that is $F-$good for every finite set $F \subset V(G) \setminus D$ of vertices of finite degree. If c is not unfriendly in a vertex $v \in D$, one of the following statements holds:*

1. *If v has finite degree, then, for some finite set $H \subset (V(G) \setminus D) \cup \{v\}$ of vertices of finite degree, $c * H$ is $F-$good for every finite set $F \subset (V(G) \setminus D) \cup \{v\}$ of vertices of finite degree. In particular, $c * H$ is unfriendly in v.*
2. *If v has infinite degree, then, for some finite set $H \subset V(G) \setminus D$ of vertices of finite degree, $c * (\{v\} \cup H)$ is $F-$good for every finite set $F \subset V(G) \setminus D$ of vertices of finite degree. In particular, since c is not unfriendly in v and H is finite, $c * (\{v\} \cup H)$ is unfriendly in v.*

Revisited proof from Lemma 3 of Aharoni, Milner and Prikry (1990). Since G is countable and the coloring c is not unfriendly in v, this vertex has only finitely many enemies under c. Let k be this amount if v has infinite degree and $k = d(v)$ otherwise. First, set $c_1 = c * v$. If c_1 is $F-$good for every finite set $F \subset (V \setminus D) \cup \{v\}$ of vertices of finite degree, the proof is finished by setting $H = \emptyset$ if $d(v)$ is infinite and $H = \{v\}$ otherwise. If this is not the case, by Lemma 3.2.4 there is a finite set $F_1 \subset (V(G) \setminus D) \cup \{v\}$ of vertices of finite degree such that $a_{c_1}(F_1, V \setminus F_1) < b_{c_1}(F_1, V(G) \setminus F_1)$. If $c_2 = c_1 * F_1$ is $F-$good for every finite set $F \subset (V(G) \setminus D) \cup \{v\}$ of vertices of finite degree, we are done since c_2 and c differ only in finitely many vertices of finite degree. Otherwise, we can proceed with this algorithm and obtain, for each $n \in \mathbb{N}$, a coloring c_n and a finite set $F_n \subset (V(G) \setminus D) \cup \{v\}$ of vertices of finite degree such that:

- $c_{n+1} = c_n * F_n$.
- $a_{c_n}(F_n, V \setminus F_n) < b_{c_n}(F_n, V \setminus F_n)$.

For instance, suppose that c_{2k+2} and F_{2k+2} are defined. In this case, consider the finite set $F = \bigcup_{i=1}^{2k+1} F_i$. Since $F_i \subset (V(G) \setminus D) \cup \{v\}$ for every $1 \leq i \leq 2k+2$, we have $F \subset (V(G) \setminus D) \cup \{v\}$. Moreover, for each $1 \leq i \leq 2k+1$, by definition of the sets $A_{c_i}(F)$ and $A_{c_i}(F_i, V(G) \setminus F_i)$, the following inequality is also verified:

$$a_{c_{i+1}}(F) = a_{c_i}(F) + b_{c_i}(F_i, V(G) \setminus F_i) - a_{c_i}(F_i, V(G) \setminus F_i) > a_{c_i}(F) \geq a_{c_i}(F) + 1.$$

Therefore, $a_{c_{2k+2}}(F) \geq a_{c_1}(F) + 2k + 1$. Now, depending on whether v has finite degree or not, we consider the following cases:

1. First, suppose that v has infinite degree. In this case, $v \notin F_i$ for every $1 \leq i \leq 2k+2$, so that $c_i(v) = c_1(v) = 1 - c(v)$. Then, the coloring $c' = c_{2k+2} * v$ agrees with c in $V(G) \setminus F$. Applying Lemma 3.2.3 to the colorings c, c_1, c_{2k+2} and c', we conclude that

$$a_{c'}(F) = a_{c_{2k+2}}(F) + b_{c_{2k+2}}(\{v\}, F) - a_{c_{2k+2}}(\{v\}, F) \geq a_{c_{2k+2}}(F) - a_{c_{2k+2}}(\{v\}, F)$$

and $a_{c_1}(F)=a_c(F)+b_c(\{v\},F)-a_c(\{v\},F)$. But, $a_{c_{2k+2}}(\{v\},F)\leq a_c(\{v\},F)+b_c(\{v\},F)$, because all edges connecting a vertex of F to v are elements of $A_c(\{v\},F)\cup B_c(\{v\},F)$. Similarly, $A_c(\{v\},F)\subset A_c(v)$, so that $a_c(\{v\},F)\leq a_c(\{v\})=k$. Combining these inequalities, we contradict the fact that c is $F-$good:

$$\begin{aligned}
a_{c'}(F) &\geq a_{c_{2k+2}}(F)-a_{c_{2k+2}}(\{v\},F)\\
&\geq a_{c_1}(F)+2k+1-a_c(\{v\},F)-b_c(\{v\},F)\\
&\geq a_c(F)+b_c(\{v\},F)-a_c(\{v\},F)+2k+1-a_c(\{v\},F)-b_c(\{v\},F)\\
&\geq a_c(F)+1\\
&> a_c(F)
\end{aligned}$$

In other words, for some $1\leq i\leq 2k+2$, the coloring $c_{i+1}=c_i * F_i$ needs to be $F-$good for every finite set $F\subset (V(G)\setminus D)\cup\{v\}$ of vertices of finite degree. Since c_i differ from c_1 only in finitely many vertices of finite degree, the lemma follows.

2. If v has finite degree and $c_{2k+2}(v)=c_1(v)$, define $c'=c_{2k+2}*v$, so that $c'(v)=c(v)$. In this case, $c'|_{V(G)\setminus(F\setminus\{v\})}=c|_{V(G)\setminus(F\setminus\{v\})}$. However, by Lemma 3.2.3 again, $a_{c'}(F)=a_{c_{2k+2}}(F)+b_{c_{2k+2}}(\{v\},F)-a_{c_{2k+2}}(\{v\},F)\geq a_{c_{2k+2}}(F)-a_{c_{2k+2}}(\{v\},F)$ and $a_{c_1}(F)=a_c(F)+b_c(\{v\},F)-a_c(\{v\},F)$. On the other hand, $a_{c_{2k+2}}(\{v\},F)\leq a_c(\{v\},F)+b_c(\{v\},F)\leq k$, because k is chosen to be $d(v)$ if v has finite degree. Combining these inequalities, we conclude that

$$\begin{aligned}
a_{c'}(F) &\geq a_{c_{2k+2}}(F)-a_{c_{2k+2}}(\{v\},F)\\
&\geq a_{c_1}(F)+2k+1-k\\
&\geq a_{c_1}(F)+k+1\\
&= a_c(F)+b_c(\{v\},F)-a_c(\{v\},F)+k+1\\
&\geq a_c(F)+1\\
&> a_c(F)
\end{aligned}$$

Once $c'(v)=c(v)$, this contradicts the fact that c is $(F\setminus\{v\})-$good. As before, for some $1\leq i\leq 2k+2$, the coloring $c_{i+1}=c_i*F_i$ needs to be $F-$good for every finite set $F\subset (V(G)\setminus D)\cup\{v\}$ of vertices of finite degree. Since c_i differ from c_1 only in finitely many vertices of finite degree, the lemma follows.

3. If v has finite degree but $c_{2k+2}(v)\neq c_1(v)$, then $c_{2k+2}(v)=c(v)$. In particular, $c_{2k+2}|_{V(G)\setminus(F\setminus\{v\})}=c|_{V(G)\setminus(F\setminus\{v\})}$. Again, since $a_{c_1}(F)=a_c(F)+b_c(\{v\},F)-a_c(\{v\},F)$ by Lemma 3.2.3, we now conclude that

$$a_{c_{2k+2}}(F)\geq a_{c_1}(F)+2k+1=a_c(F)+b_c(\{v\},F)-a_c(\{v\},F)+2k+1\geq a_c(F)+k+1>a_c(F).$$

This also contradicts the fact that c is $(F\setminus\{v\})-$good. Therefore, as in the previous cases, for some $1\leq i\leq 2k+2$, the coloring $c_{i+1}=c_i*F_i$ needs to be $F-$good for every finite set $F \subset (V(G) \setminus D) \cup \{v\}$ of vertices of finite degree. The lemma follows because c_i differ from c_1 only in finitely many vertices of finite degree.

□

By taking $D = \emptyset$ in the corollary below, Proposition 3.2.6 and Lemma 3.2.7 verify that countable graphs with only finitely many vertices of infinite degree have unfriendly partitions:

Corollary 3.2.8 (Aharoni, Milner and Prikry (1990), Lemma 3)**.** *Let G be a countable graph and fix $c' : D \to 2$ a partial coloring, defined on a subset $D \subset V(G)$. If there are only finitely many vertices of infinite degree in $V(G) \setminus D$, then there is an extension $c : V(G) \to 2$ that is unfriendly in the vertices of $V(G) \setminus D$.*

Proof. Denote by I the (finite) set of vertices of $V(G) \setminus D$ that have infinite degree. Let $c'' : D \cup I \to 2$ be an arbitrary extension of c' over these vertices. Supported by Proposition 3.2.6, as $V \setminus (D \cup I)$ is composed only by vertices of finite degree, extend c'' to a coloring $c : V(G) \to 2$ that is $F-$good for every finite set $F \subset V(G) \setminus (D \cup I)$. Now, denote by $D_c \subset I$ the set of vertices of infinite degree that have only finitely many enemies regarding c. Since I is finite, we can choose c'' and c so that $|D_c|$ is minimum. Verifying that c is an unfriendly partition, we claim that $D_c = \emptyset$. For instance, suppose that there exists $v \in D_c$, namely, a vertex of infinite degree in which c is not unfriendly. Then, by Lemma 3.2.7, for some finite $H \subset V(G) \setminus (D \cup I)$, the coloring $\hat{c} = c * (\{v\} \cup H)$ is also $F-$good for every finite subset $F \subset V(G) \setminus (D \cup I)$. In particular, $\hat{c}$ is still unfriendly in the vertices of finite degree and, since $\hat{c}(v) = 1 - c(v)$, now unfriendly in v. Moreover, if c is unfriendly in a vertex $u \in D$, so is $\hat{c}$, since $c(u) = \hat{c}(u)$ and $\hat{c}$ is obtained from c after changing the colors of v and of only finitely many vertices of finite degree. Therefore, $D_{\hat{c}} = D_c \setminus \{v\}$, contradicting the minimality of $|D_c|$. □

As another consequence, Lemma 3.2.7 enables us to attach graphs by some finite subgraphs while preserving the existence of $F-$good colorings. More precisely, the following result will be helpful in future discussions:

Corollary 3.2.9 (Gluing Lemma for Colorings)**.** *Let G be a countable graph and $S \subset V(G)$ be a finite set of vertices of finite degree such that $G \setminus S$ has precisely two connected components, say, G_1 and G_2. For each $i = 1, 2$, suppose that there is $c_i : V(G_1) \cup S \to 2$ an unfriendly partition over $G[V(G_i) \cup S]$ that is $F-$good for every finite set $F \subset V(G_i) \cup S$ of vertices of finite degree. Then, there exists $c : V(G) \to 2$ an unfriendly partition over G that is $F-$good for every finite set $F \subset V(G)$ of vertices of finite degree.*

Proof. Let $N(S) = \{u \in V(G) : uv \in E(G) \text{ for some } v \in S\}$ be the neighborhood of the vertex set S. Since S is a finite set of vertices of finite degree, $N(S)$ is finite as well. Then, define $c : V(G) \to 2$ as $c(v) = c_1(v)$ if $v \in V(G_1) \cup S$ and $c(v) = c_2(v)$ if $v \in V(G) \setminus S$. By the hypothesis over c_1 and c_2, c is $F-$good for every finite set $F \subset V(G) \setminus N(S)$ of vertices of finite degree. Moreover, c is unfriendly in every vertex of infinite degree. After changing the colors of finitely many vertices, Lemma 3.2.7 guarantees that c can be chosen to be $F-$good for every finite set $F \subset V(G)$ of vertices of finite degree, since $N(S)$ is finite. □

Using Corollary 3.2.8 as a base case for an inductive argument, we can finally write down a proof for Theorem 3.2.2:

Proof of Theorem 3.2.2. We will show that the desired extension $c : V(G) \to 2$ exist by induction on $|V(G)|$, remarking that Corollary 3.2.8 provides a base case with $|V(G)| = \aleph_0$.

Then, suppose that $|V(G)| = \kappa > \aleph_0$ and denote by I the (possibly empty) set of vertices of $V(G) \setminus D$ that have degree precisely κ. If some $v \in I$ has κ neighbors in D, there is a color $i \in 2$ so that κ many of these neighbors have color $1-i$ under c'. By declaring $c'(v) = i$, we can assume that c' is defined on v and is unfriendly in this vertex.

So, without loss of generality, suppose that every element of I has almost all its neighbors in $V(G) \setminus D$ and denote $\mu = \max\{\sup\{d(v) : v \in V \setminus (D \cup I)\}, \aleph_0\}$. In other words, if there are vertices of infinite degree in $V(G) \setminus (D \cup I)$, then μ is the least upper bound of these degrees. If not, $G \setminus (D \cup I)$ is locally finite, and we consider $\mu = \aleph_0$. In both cases, $d(u) \leq \mu < \kappa$ for every $u \in V(G) \setminus (D \cup I)$.

Now, denote by $\mathscr{C}$ the collection of connected components of $G \setminus (D \cup I)$. Then, Lemma 2.1.3 shows that every $C \in \mathscr{C}$ has at most μ vertices. Since all them have degree at most μ, the set $N(C) = \bigcup_{u \in V(C)} N(u) \cap (D \cup I) = \{u \in D \cup I : u \text{ has a neighbor in } C\}$ has also at most μ vertices. Defining $\overline{C} = G[V(C) \cup N(C)]$ for every $C \in \mathscr{C}$, we conclude that the elements of $\overline{\mathscr{C}} = \{\overline{C} : C \in \mathscr{C}\}$ have size bounded by μ as well.

In particular, the inductive hypothesis guarantees that, for each $\overline{C} \in \overline{\mathscr{C}}$, the coloring $c'|_{N(C) \cap D}$ can be extended to a coloring $\pi_C : V(\overline{C}) \to 2$ which is unfriendly in every vertex of C. If $I = \emptyset$, then $N(C) = N(C) \cap D$ for every $C \in \mathscr{C}$. In other words, $c = \bigcup_{\overline{C} \in \overline{\mathscr{C}}} \pi_C$ is unfriendly in every $v \in V(G) \setminus D$, because $c|_{\overline{C}} = \pi_C$ and such a vertex lie in a connected component $C \in \mathscr{C}$.

Suppose now that $|I| > 0$ and that the Theorem follows for smaller values of $|I|$. Since an element $v \in I$ has κ neighbors in $V \setminus (D \cup I)$, we observe that $|\mathscr{C}| = \kappa$. After all, $\mu < \kappa$ and, by the Claim previously established, the vertex v has at most μ neighbors in a given component $C \in \mathscr{C}$. Furthermore, this proves that v has neighbors in κ different elements of $\mathscr{C}$. Then, fixing $v \in I$ and an arbitrary color $c'(v) \in 2$, by induction on $|I|$ we can extend c' to a coloring $c : V \to 2$ that is unfriendly in every vertex of $V \setminus (D \cup \{v\})$. If c is unfriendly in v, then c is the claimed coloring.

If not, regarding c, the vertex v has fewer than κ enemies. In particular, the set $\mathscr{C}' = \{C \in \overline{C} : c(v) \neq c(u) \text{ for some neighbor } u \in V(C) \text{ of } v\}$ has fewer than κ elements. This means that the coloring $c * v$ is unfriendly in v and possibly not unfriendly in vertices of components in $\mathscr{C}'$. For a component $C \in \mathscr{C}'$, consider $\pi_C : V(\overline{C}) \to 2$ any extension of $(c * v)|_{N(C)}$ that is unfriendly in all the vertices of C. We remark that this coloring exist by our first inductive hypothesis. Then, the desired coloring $\hat{c} : V(G) \to 2$ is given by

$$\hat{c}(u) = \begin{cases} c(u), & \text{if } u \in D \cup I \setminus \{v\}. \\ c(u), & \text{if } u \in V(C) \text{ for some } C \in \mathscr{C} \setminus \mathscr{C}' \\ 1 - c(u), & \text{if } u = v. \\ \pi_C(u), & \text{if } u \in V(C) \text{ for some } C \in \mathscr{C}'. \end{cases}$$

Since c and $\hat{c}$ differ only at v and at the connected components of $\mathscr{C}'$, it follows that $\hat{c}$ is also unfriendly in the vertices of $I \setminus \{v\}$. After all, $|\mathscr{C}'| < \kappa$ and these vertices have neighbors in κ distinct components of $\mathscr{C}$. Finally, $\hat{c}$ is unfriendly in the vertices of connected components of $\mathscr{C}$ by construction.

□

3.3 Greedy algorithms

While Chapter 3 studies graphs with few vertices of infinite degree, we will now approach the opposite direction: this section aims to find unfriendly partitions in graphs with restrictive conditions on vertices of finite degree. However, these vertices cannot be simply ignored without additional hypothesis, since Section 3.4 presents uncountable graphs not admitting unfriendly partitions neither vertices of finite degree.

Incidentally, the proofs in this section are rather different than those presented in the previous one. For finite-like graphs, the unfriendly partitions were determined by relying on the fact that max-cuts define suitable colorings in graphs on finitely many vertices. From now on, most results will be developed via algorithms that iteratively try to assigns enemies for already colored vertices. As a simple example, for a cardinal κ, we recall that a graph G is $\kappa-$**regular** if $d(v) = \kappa$ for every $v \in V(G)$. Then, the heuristic of the following proof will be adapted throughout the next paragraphs:

Lemma 3.3.1. *If G is a $\kappa-$regular graph, then it has an unfriendly partition.*

Proof. If κ is finite, then G is a locally finite graph, so that the result follows from Corollary 3.2.1. Then, suppose that κ is infinite. As in the Claim of Theorem 3.2.2, we can assume that $|V(G)| = \kappa$ by considering G connected. Therefore, fix an non-injective enumeration $\{v_i\}_{i<\kappa}$ for $V(G)$ with the following property: $|\{i < \kappa : v_i = v\}| = \kappa$ for every $v \in V(G)$. Then, for each $i < \kappa$, we define recursively a coloring $c : V(G) \to 2$ according to the cases below:

- If $v_i = v_j$ for some $j < i$, then $c(v_i) = c(v_j)$ is defined by induction. In this case, assuming that c is defined in $\{v_{i'} : i' < i\}$, there is $n < \kappa$ for which v_n is an uncolored neighbor of v_i regarding c. After all, $d(v_i) = \kappa$ by hypothesis. Then, we define $c(v_n) = 1 - c(v_i)$;
- If c is not defined for v_i, set $c(v_i) = 0$.

By the choice of the enumeration $\{v_i\}_{i<\kappa}$, the coloring c assigns $\kappa-$many neighbors to each vertex of G, being an unfriendly partition. □

In other words, Lemma 3.3.1 claims that, if we fix the degree of every vertex in a graph G, then G admits an unfriendly partition. Motivating the main discussions of this section, we will now allow that the vertices of G take degree within a finite list of regular cardinals. More precisely, approaching the techniques developed by Aharoni, Milner and Prikry (1990), the proof of the following result will be soon revisited:

Theorem 3.3.2 (Aharoni, Milner and Prikry (1990), Theorem 2)**.** *Let $\kappa_0 < \kappa_1 < \cdots < \kappa_n$ be a finite collection of infinite cardinals, with κ_i regular for each $1 \leq i \leq n$. Let G be a graph such that $|\{v \in V(G) : d(v) \text{ is finite}\}| < \kappa_0$ and $d(v) \in \{\kappa_0, \kappa_1, \ldots, \kappa_n\}$ for every vertex $v \in V(G)$ with infinite degree. Then, G admits an unfriendly partition.*

Originally, Theorem 3.3.2 was proven by induction over $n \in \mathbb{N}$, the size of the list of cardinals. Due to further applications in Chapter 4.4, we will rewrite this inductive argument, emphasizing that the regularity condition over κ_i (for $1 \leq i \leq n$) is the main hypothesis to be used in the proof. Moreover, we observe that the assumption $|\{v \in V(G) : d(v) \text{ is finite}\}| < \kappa_0$ claims that the amount of vertices of finite degree is despicable, since the vertices of infinite degree have almost all its neighbors attaining infinite degree as well.

Therefore, from now on in this section, we will assume that a fixed graph G has only vertices of infinite degree. Once proved Theorem 3.3.2 for this case, its general statement follows by extending a partial coloring to vertices of finite degree with the aid of Proposition 3.2.6, as we shall argue at the end of this section.

Before that, given a partial coloring $c : D \to 2$ of a graph G whose vertices have all infinite degree, it is convenient to globally extend that function in an unfriendly way, that is, to be unfriendly in the set $V(G) \setminus D$ of remaining vertices. To this aim, the **closure** of c is the function $\bar{c}$ defined by transfinite induction as follows:

1. Let $D_0 = \{v \in V(G) \setminus D : |N(v) \setminus D| < |N(v)|\}$ denote the vertices of $V(G) \setminus D$ with less uncolored neighbors than their degree. In particular, for each vertex $v \in D_0$ there is a color $c_v \in 2$ such that v has $d(v)$ neighbors colored $1 - c_v$ within D. Writing $c_0(v) = c(v)$ for all $v \in D$ and $c_0(v) = c_v$ for all $v \in D_0$, this extends c to a coloring $c_0 : W_0 \to 2$ unfriendly in the vertices of D_0, where $W_0 := D \cup D_0$.

2. For each ordinal $\alpha > 0$, suppose that $c_\beta : W_\beta \to 2$ is defined for each $\beta < \alpha$. Assume also that $W_\gamma \subset W_\beta$ and that c_β extends c_γ if $\gamma < \beta$. Therefore, a coloring $c'_\alpha = \bigcup_{\beta<\alpha} c_\beta$ is defined over $D'_\alpha = \bigcup_{\beta<\alpha} W_\beta$. As done with D_0, let
$$D_\alpha = \left\{v \in V(G) \setminus D'_\alpha : \left|N(v) \setminus D'_\alpha\right| < d(v)\right\}$$
be the set of uncolored vertices whose neighbors are almost all colored by c'_α. Then, for each $v \in D_\alpha$ there is a color $c_v \in 2$ such that $d(v)$ of its neighbors have color $1-c_v$. By setting $c_\alpha(v) = c'_\alpha(v)$ for every $v \in D'_\alpha$ and $c_\alpha(v) = c_v$ for every $v \in D_\alpha$, this extends c'_α to a partial coloring $c_\alpha : W_\alpha \to 2$ unfriendly in D_α, where $W_\alpha = D_\alpha \cup D'_\alpha$.
3. If Γ is the least ordinal such that $D_\Gamma = \emptyset$, this procedure defines a coloring $\overline{c} : \bigcup_{\alpha<\Gamma} W_\alpha \to 2$ that extends c and that is unfriendly in all the vertices of $W_\alpha \setminus D$, for every $\alpha < \Gamma$. Denoting its domain by $\overline{D}$, we observe that, by the choice of Γ, $|N(v) \setminus \overline{D}| = d(v)$ for all $v \in V(G) \setminus \overline{D}$. In particular, $\overline{\overline{c}} = \overline{c}$.

Inspired by the fact that $\overline{\overline{c}} = \overline{c}$, we say that a partial coloring $c : D \to 2$ is **closed** if $\overline{c} = c$. This definition plays a key role in the original proof of Theorem 3.3.2. Below, we revisit its details in order to provide a slightly more general statement for the result. These few modifications, however, will be useful for some independence discussions on Chapter 4.4:

Theorem 3.3.3. *Let $\mathscr{K}$ be a family of infinite cardinals such that the following property holds:*

Every graph whose vertices have degree as a cardinal of $\mathscr{K}$ has an unfriendly partition.

Then, if κ is a regular cardinal greater than every member of $\mathscr{K}$, every graph whose vertices have degree as a cardinal of $\mathscr{K} \cup \{\kappa\}$ admits an unfriendly partition as well.

Revisited proof from Theorem 2 of Aharoni, Milner and Prikry (1990). Let G be a graph with $d(v) \in \mathscr{K} \cup \{\kappa\}$ for every $v \in V(G)$. For each $X \subset V(G)$, we define $N(X) = \bigcup_{v \in X} N(v)$. To better apply our hypothesis, let $M \subset V(G)$ be the set of vertices of degree κ and $N = V(G) \setminus M$ be the set of vertices whose degree belongs to $\mathscr{K}$. Since κ is a regular cardinal, the proof of Lemma 2.1.3 can be adapted in order to show that $|V(C)| < \kappa$ and $|N(V(C))| < \kappa$ for each connected component C of $G[N]$.

To properly color some vertices of G, we say that two (disjoint) sets $F_0 \subset M$ and $F_1 \subset N$ define a *bipartite pair* (F_0, F_1) if, for every $v \in F_0$ and $u \in F_1$, we have $|N(v) \cap F_1| = d(v) = \kappa$ and $|N(u) \cap F_0| = d(u)$. In other words, for $i \in \{0, 1\}$, every member of F_i has its degree of neighbors in F_{1-i}. Note that this property is closed by unions, because, if (F'_0, F'_1) is another bipartite pair, every member of $F_i \cup F'_i$ has its degree as amount of neighbors in $F_{1-i} \cup F'_{1-i}$. Then, we denote by (F_0, F_1) the *maximal bipartite pair*, described by the unions $F_0 = \bigcup_{(F_0',F_1')\in\mathbb{BP}} F_0'$ and $F_1 = \bigcup_{(F_0',F_1')\in\mathbb{BP}} F_1'$, where $\mathbb{BP}$ is the set of all bipartite pairs of G.

That notation induces a natural unfriendly partial coloring $c' : F_0 \cup F_1 \to 2$, given by $c'(v) = 0$ and $c'(u) = 1$ for every $v \in F_0$ and $u \in F_1$. Denoting its closure by $\overline{c}' : \overline{D} \to 2$, it follows that $\overline{c}'$ is also an unfriendly partial coloring and that $|N(v)\setminus\overline{D}| = d(v)$ for every $v \in V(G)\setminus\overline{D}$. Therefore, it is enough to find a partial coloring $c : V(G)\setminus\overline{D} \to 2$ which is unfriendly in $G[V(G)\setminus\overline{D}]$, so $c\cup\overline{c}'$ will be the requested unfriendly partition. To this aim, the choice of the pair (F_0, F_1) guarantees the property below:

`Claim:` Let $S \subset M\setminus\overline{D}$ be any set with $|S| < \kappa$. Then, the set
$T = \{u \in N\setminus\overline{D} : |N(u)\cap S| = d(u)\}$ has fewer than κ vertices.

Proof of the claim. Define the sets $A = \{v \in S : |N(v)\cap T| < d(v) = \kappa\}$ and $B = \{u \in T : u \in N(x) \text{ for some } x \in A\} = \bigcup_{x\in A}(N(x)\cap T)$. Once $A \subset S$, we have that $|A| < \kappa$. Then, $|B| < \kappa$ by definition of A and the regularity of κ. Noticing that $(S\setminus A, T\setminus B)$ is a bipartite pair, we must have $T\setminus B = \emptyset$ by the fact that the maximal bipartite pair (F_0, F_1) is already colored. Therefore, $|T| = |B| < \kappa$. □

To construct the requested coloring of $G[V(G)\setminus\overline{D}]$, fix a non-injective enumeration $\{v_\alpha\}_{\alpha<\kappa}$ of $M\setminus\overline{D}$ such that every member is presented κ times, i.e., $|\{\alpha < \kappa : v_\alpha = v\}| = \kappa$ for every $v \in M\setminus\overline{D}$. Then, we will recursively define an unfriendly partition $c : V(G)\setminus\overline{D} \to 2$ according to the following algorithm:

1. We first define $c(v_0) = 0$. If v_0 has a neighbor $v \in M\setminus\overline{D}$, we define $c(v) = 1$. If not, once $|N(v_0)\setminus\overline{D}| = d(v_0) = \kappa$, there is a component C of $G[N\setminus\overline{D}]$ where v_0 has a neighbor v. For every $u \in N(C)\cap M\setminus(\overline{D}\cup\{v_0\})$, we set $c(u) = 0$. By sewing c as a coloring defined in $G[C\cup N(C)\setminus\overline{D}]$, we extend it to some vertices from $\overline{C} \subset C$ by taking its closure. Then, a vertex $u \in C\setminus\overline{C}$ satisfies $|N(u)\cap(C\setminus\overline{C})| = d(u) \in \mathscr{K}$. Hence, by hypothesis, we may extend c to the whole component C by adjoining an unfriendly partition of $G[C\setminus\overline{C}]$. Unless by changing the colors of all the vertices of $(C\cup N(C))\setminus(\overline{D}\cup\{v_0\})$, we can assume that $c(v) = 1$. This finishes the first iteration of the algorithm. Note that, besides those vertices of $\overline{D}$, we have colored a component of $G[N\setminus\overline{D}]$ and less than $\kappa-$many vertices from M, as the first claim guarantees.
2. For some ordinal $\alpha > 0$, denote by $S_\alpha \subset M\setminus\overline{D}$ the set of vertices of degree κ from $V(G)\setminus\overline{D}$ that we have colored so far by this algorithm. Since $\alpha < \kappa$ and κ is a regular cardinal, by transfinite induction we can suppose that $|S_\alpha| < \kappa$. First, consider the case in which $c(v_\alpha)$ is already defined. If v_α has a neighbor $v \in M\setminus(\overline{D}\cup S_\alpha)$, define $c(v) = 1 - c(v_\alpha)$. If not, then v_α has κ neighbors as elements of $N\setminus\overline{D}$. Moreover, once each connected

component of $G[N \setminus \overline{D}]$ has cardinality less than κ, v has neighbors in κ many of them. Regarding that less than κ of such components were colored so far, by the last claim we may find a connected component C of $G[N \setminus \overline{D}]$ such that $|N(u) \cap S_\alpha| < d(u)$ for every vertex u of C. In other words, every member of C has in $V(G) \setminus \overline{D}$ as many neighbors as in $D_\alpha := V(G) \setminus (\overline{D} \cup S_\alpha)$. Similarly to the procedure of the first iteration, define $c(u) = 0$ for each $u \in N(C) \cap M \cap D_\alpha$. Regarding c as a partial coloring of the graph $G[C \cup (N(C) \cap D_\alpha)]$, we define c on some vertex subset $\overline{C}$ of C by taking its closure. Hence, every remaining vertex $u \in C \setminus \overline{C}$ satisfies $|N(u) \cap (C \setminus \overline{C})| = d(u) \in \mathscr{K}$. Now, the hypothesis can be applied to extend c to $G[C \setminus \overline{C}]$ by adjoining an unfriendly partition of such subgraph. Again, up to changing the colors of all the vertices of $C \cup (N(C) \cap D_\alpha)$, we can assume that $c(v) = 1 - c(v_\alpha)$. Finally, if $c(v_\alpha)$ was not defined, we just set $c(v_\alpha) = 0$.

At the end of this transfinite process, c is defined for every vertex of M and for some connected components of $G[N]$. In such components, this is an unfriendly coloring: each vertex was colored by some closure throughout the procedure or by an unfriendly partition given by the hypothesis. A vertex of $M \setminus \overline{D}$, instead, received a neighbor of opposite color in κ many iterations after the moment his value by c was set, according to its indices of the enumeration $\{v_\alpha\}_{\alpha<\kappa}$. Therefore, every member of $M \setminus \overline{D}$ has κ neighbors of opposite color, verifying that c is unfriendly in those vertices.

It remains, however, to define colors for vertices of some connected components of $G[N \setminus \overline{D}]$ that were not analyzed by the steps above. If C is one of those components, as before we will see c as a partially defined coloring at $G[C \cup N(C)]$. In fact, it is only defined in $N(C) \cap M$. By taking its closure within this component, c is defined for some vertex set $\overline{C} \subset C$ and, therefore, it is unfriendly in such vertices. Again, every member $u \in C \setminus \overline{C}$ satisfies $|N(u) \cap (C \setminus \overline{C})| = d(u) \in \mathscr{K}$. Hence, the hypothesis can be applied to define c as an unfriendly partition for $G[C \setminus \overline{C}]$, coloring the entire component C.

□

Combining Lemma 3.3.1 with Theorem 3.3.3, Theorem 3.3.2 is recovered as an easy consequence:

Proof of Theorem 3.3.2. Let $F \subset V(G)$ be the set of vertices of finite degree. Due to the fact that $|F| < \kappa_0$ and $d(v) \in \{\kappa_0, \kappa_1, \dots, \kappa_n\}$, the graph $G \setminus F$ has only vertices of infinite degree. Actually, $|N(v) \setminus F| = d(v)$ for every $v \in V(G) \setminus F$. By Lemma 3.3.1, every κ_0-regular graph has an unfriendly partition. Then, since κ_i is regular for $1 \leq i \leq n$, we can apply Theorem 3.3.3 iteratively to conclude that every graph whose vertices have degree in $\{\kappa_i : 1 \leq i \leq n\}$ has an unfriendly partition. In particular, $G \setminus F$ has an unfriendly partition. By Proposition 3.2.6, there is an extension $c : V(G) \to 2$ of this coloring that is also unfriendly in F. □

3.4 The uncountable counterexamples

Written with a more specific vocabulary and notation from set theory, this section explains the constructions of the unique graphs known in the literature that do not admit unfriendly partition. They were obtained by Milner and Shelah in (SHELAH; MILNER, 1990) using combinatorial properties of ultrafilters. Curiously, the least of these graphs has $(2^{\omega})^{+\omega}$ vertices, none of them of finite degree. Here, $(2^{\omega})^{+\omega}$ means the first limit cardinal greater than the continuum. In addition, these graphs are **tripartite**, in the sense that their vertex sets can be partitioned into three subsets whose corresponding induced subgraphs have no edges.

Supposing an extra axiom of the set theory, Milner and Shelah initially describe in their paper a graph with $\aleph_{\omega}$ vertices, all them of infinite degree, that has no unfriendly partition. This is done in a very short construction, but motivates a more complex one under ZFC. As we will study in the Section 4.4, this consistent result also plays a theoretical role when discussing the least amount of vertices needed to provide a graph with no unfriendly partitions, none of them of finite degree. The mentioned independent assumption is the following:

($\dagger$) There exists a $p-$point $\mathscr{U}$ on ω of character ω_1. In particular, there is a family $\{A_{\xi} : \xi < \omega_1\} \subset \mathscr{U}$ so that, for every $A \in \mathscr{U}$, there exists a big enough $\eta < \omega_1$ for which $A \setminus A_{\xi}$ is finite whenever $\eta \leq \xi < \omega_1$.

In the above result, a $p-$point is an ultrafilter on $\mathbb{N}$ with some additional properties, closely related to the role it plays in the Stone-Čech compactification of the natural numbers. For the interested reader, the Master's dissertation of Zancul (2023) contains an introduction to the study of $p-$points and their applications. As we will recall in some next discussions, it is remarkable that $(\dagger)$ is consistent with ZFC $+ 2^{\omega} > \aleph_{\omega}$, as Theorem 8.0 (b) in (HART, 1989) guarantees. On the other hand, its negative can be established from Martin's Axiom (MA) and $\neg$CH, for example.

In order to construct the claimed graph G with no unfriendly partitions, let us fix $\mathscr{U}$ and $\{A_{\xi} : \xi < \omega_1\}$ as in the statement $(\dagger)$. The vertex set $V(G)$ will be taken as the union of the following three disjoint sets:

- $X = \{x_n : n < \omega\}$ is a copy of ω.
- $Y = \{y_{(\xi,\alpha)} : \xi < \omega_1, \alpha < \aleph_{\omega}\}$ is a copy of the cartesian product $\omega_1 \times \aleph_{\omega}$. Regarding an usual display of a grid, the set $\{y_{(\xi,\alpha)} : \alpha < \aleph_{\omega}\}$ will be referred as the $\xi-$**line** of Y for each $\xi < \omega_1$, while the set $\{y_{(\xi,\alpha)} : \xi < \omega_1\}$ is said to be its $\alpha-$**column** for every $\alpha < \aleph_{\omega}$.
- $Z = \{z_{\alpha} : \alpha < \aleph_{\omega}\}$ is a copy of $\aleph_{\omega}$.

Once $|X| = \aleph_0$, $|Y| = \max\{\aleph_1, \aleph_\omega\} = \aleph_\omega$ and $|Z| = \aleph_\omega$, it follows that G has $\aleph_\omega$ vertices (as promised). Now, the edges of G are defined according to also three rules:

- Set $x_n y_\alpha \in E$ for every $n < \omega$ and $\alpha < \aleph_\omega$. In other words, $G[X \cup Z]$ is a complete bipartite subgraph. Since $|Z| = |V| = \aleph_\omega$, it follows immediately that every $x_n \in X$ has degree $\aleph_\omega$.
- Set $z_\alpha y_{(\xi,\alpha)}$ for every $\alpha < \aleph_\omega$ and $\xi < \omega_1$. In other words, every element $z_\alpha \in Z$ is adjacent to the whole $\alpha-$column of Y. Since no more edges incident on vertices of Z will be defined, every element of Z has degree $\aleph_1$ and only countably many neighbors in X.
- For every $n < \omega$, $\xi < \omega_1$ and $\alpha \leq \aleph_n$, set $x_n y_{(\xi,\alpha)} \in E$ if, and only if, $n \in A_\xi$. Informally, x_n is a common neighbor of the ξ's-lines of Y such that $n \in A_\xi$, but until the elements of the $\aleph_n-$column. Conversely, every $y_{(\xi,\alpha)} \in Y$ is neighbor of (almost) all elements $x_n \in X$ with $n \in A_\xi$, except for finitely many of them whose indices are smaller than N, in which $N < \omega$ is the unique number such that $\aleph_N \leq \alpha < \aleph_{N+1}$. Since no more edges will be defined and $\mathscr{U}$ is a non-principal ultrafilter, every element of Y has degree $\aleph_0$: it is neighbor of infinitely many vertices of X and just one of Z.

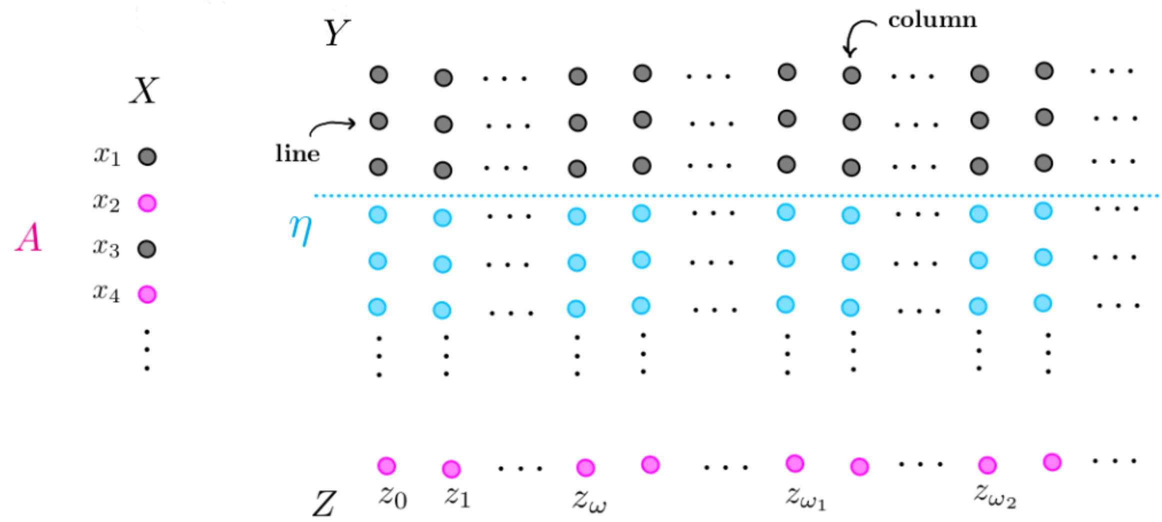
Therefore, G is a tripartite graph with parts X, Y and Z. Moreover, the countable cofinality of $\aleph_\omega$ is explored in order to guarantee that every vertex of X has almost all its neighbors in Z. On the other hand, every vertex of Z has almost all its neighbors in Y and every vertex of Y has almost all its neighbors in X. Although omitting the edges just described, Figure 16 shows a didactic way to present the vertex sets X, Y and Z, besides brief explaining why it is expected that G has no unfriendly partition.

In fact, suppose that there is an unfriendly partition $c : V(G) \to 2$. Then, ω is naturally partitioned into the sets $A = \{n < \omega : c(x_n) = 0\}$ and $B = \{n < \omega : c(x_n) = 1\}$. Without loss of generality, we can assume that $A \in \mathscr{U}$, because $\mathscr{U}$ is an ultrafilter on ω. By the choice of the sequence $\{A_\xi\}_{\xi<\omega_1}$, there is $\eta < \omega_1$ such that $A_\xi \setminus A$ is finite if $\eta \leq \xi < \omega_1$. Fixing $\eta \leq \xi < \omega_1$ and $\alpha < \aleph_\omega$, we remind that $y_{(\xi,\alpha)} \in Y$ is neighbor to all but finitely many elements $x_n \in X$ such that $n \in A_\xi$. Only finitely many of those vertices, then, have not their indices in A too. In other words, almost all the neighbors of $y_{(\xi,\alpha)}$ lie in $\{x_n : n \in A\}$. Since $c(\{x_n : n \in A\}) = \{0\}$ by definition of A, it follows that $c(y_{(\xi,\alpha)}) = 1$ because c is unfriendly in $y_{(\xi,\alpha)}$.

Therefore, we verified that the elements of any $\xi-$line of Y with $\xi \geq \eta$ receives color 1 by c. Consequently, each $z_\alpha \in Z$ has only countably many neighbors of color 0: some of then in the countable set X and some of then in countable set $\{y_{(\alpha,\xi)} : \xi < \eta\}$. Since z_α has degree $\aleph_1$, the fact that c is unfriendly in z_α guarantees that $c(z_\alpha) = 0$ for every $\alpha < \aleph_\omega$. If $n \in A$, however, $c(x_n) = 0$ and, as an element of X, x_n has almost all its neighbors in Z. Hence, c is not unfriendly in x_n, which is a contradiction.

In summary, the following consistence result is proven:

Figure 16 – Graph of size $\aleph_\omega$ without unfriendly partitions



The vertex set of G is the disjoint union of X, Y and Z. If there was an unfriendly partition $c : V(G) \to 2$, we could suppose that $A = \{n < \omega : c(x_n) = 0\} \in \mathscr{U}$. This would imply, by the statement (†), that $c(y_{(\alpha,\xi)}) = 1$ for every $\alpha < \omega_1$ and big enough ξ. Hence, we would have $c(z) = 0$ for every $z \in Z$, contradicting the fact that c is unfriendly in x_n for any $n \in A$.

Source: Elaborated by the author.

Theorem 3.4.1 (Shelah and Milner (1990), Theorem 3)**.** *Suppose that (†) holds. Therefore, there is a graph with $\aleph_\omega$ vertices, all them of infinite degree, that has no unfriendly partition. More specifically, every vertex of this graph has degree $\aleph_0$, $\aleph_1$ or $\aleph_\omega$.*

Now, our aim is to provide a similar construction under the usual axioms of set theory. As in Theorem 3.4.1, this will be a tripartite graph in which vertices of one part will have almost all its neighbors in another. With the aid of an ultrafilter, any coloring will fail to be unfriendly.

To this aim, for any ordinal α let us denote $\|\alpha\| = |\alpha|$ if α is infinite and $\|\alpha\| = 0$ if α is finite. We then consider the set $\mathscr{S} = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) : 2^\omega > \|\alpha_1\| > \|\alpha_2\| > \cdots > \|\alpha_n\|\}$ of finite sequences of ordinals that are decreasing in terms of $\|\cdot\|$. Under that definition, $\mathscr{S}$ has a natural tree structure: we denote $(\alpha_1, \alpha_2, \ldots, \alpha_i) \preceq (\alpha_1, \alpha_2, \ldots, \alpha_n)$ if for every $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathscr{S}$ and every $i \leq n$. For simplicity, we also consider that the empty sequence $\emptyset$ lies in $\mathscr{S}$, being its $\preceq$ −minimal element. Under a graph-theoretic viewpoint, we can see every $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathscr{S}$ as a neighbor of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ for $n \geq 1$, so that $\preceq$ is the corresponding tree order of $\mathscr{S}$ if the root $\emptyset$ is fixed.

Since there is no infinite decreasing sequence of cardinals, it is straightforward that $\mathscr{S}$ is a *rayless* tree, in the sense that it has no infinite chain. Moreover, the maximal elements of $(\mathscr{S}, \preceq)$ are precisely the sequences $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathscr{S}$ with α_n finite. For every $n < \omega$, we also denote by $\mathscr{S}_n$ the elements of $(\mathscr{S}, \preceq)$ with height n or, equivalently, the sequences of $\mathscr{S}$ with n elements.

Now, fix $\mathscr{U}$ any non-principal ultrafilter on ω. For each sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathscr{S}$, let us define an ordered subset $S(\alpha_1,\alpha_2,\dots,\alpha_n)\subset\mathscr{U}$ recursively by the following rules:

- $S(\emptyset)=\mathscr{U}$ is the entire ultrafilter $\mathscr{U}$ with an enumeration of its elements as $\{A_\xi\}_{\xi<2^\omega}$;
- In order to define $S(\alpha_1,\alpha_2,\dots,\alpha_n)$, we suppose that $S(\alpha_1,\alpha_2,\dots,\alpha_{n-1})$ is already defined and that its elements are well-ordered as $\{B_\xi\}_{\xi<|\alpha_{n-1}|}$. Since $\alpha_n<|\alpha_{n-1}|$ (because $\|\alpha_n\|<\|\alpha_{n-1}\|$ by definition of $\mathscr{S}$), it is possible to well-order $\{B_\xi\}_{\xi<\alpha_n}$ with order type $|\alpha_n|$. This defines $S(\alpha_1,\alpha_2,\dots,\alpha_n)$.

In particular, $S(\alpha_1,\alpha_2,\dots,\alpha_n)$ is a subset of $S(\alpha_1,\alpha_2,\dots,\alpha_{n-1})$, but its order is not necessarily inherited. That definition is useful, however, because every element of $\mathscr{U}$ becomes present in $S(v)$ for almost all leaves $v\in\mathscr{S}$. More precisely, the observation below holds:

Lemma 3.4.2. *Fix $(\alpha_1,\alpha_2,\dots,\alpha_n)\in\mathscr{S}$ with α_n infinite. Then, $|\{\alpha<|\alpha_n|:A\notin S(\alpha_1,\alpha_2,\dots,\alpha_n,\alpha)\}|<|\alpha_n|$ for every $A\in S(\alpha_1,\alpha_2,\dots,\alpha_n)$.*

Proof. Fix $\{B_\xi\}_{\xi<|\alpha_n|}$ as in the definition of $S(\alpha_1,\alpha_2,\dots,\alpha_n)$. If $A\in S(\alpha_1,\alpha_2,\dots,\alpha_n)$, then, $A=B_\eta$ for some $\eta<|\alpha_n|$. This means that $A\in\{B_\xi\}_{\xi<\alpha}$ if and only if $\eta<\alpha$. However, by construction, $\{B_\xi\}_{\xi<\alpha}$ and $S(\alpha_1,\alpha_2,\dots,\alpha_n,\alpha)$ are the same set, just possibly with different order types. Therefore

$$|\{\alpha<|\alpha_n|:A\notin S(\alpha_1,\alpha_2,\dots,\alpha_n,\alpha)\}|=|\{\alpha<|\alpha_n|:\alpha<\eta\}|<|\alpha_n|$$

$\square$

We are ready to explicit the vertices of the desired graph G. As previously mentioned, we will use $(2^\omega)^{+\omega}$ vertices. For simplicity, we will denote $\kappa=(2^\omega)^{+\omega}$. This is a singular cardinal with countable cofinality and a cofinal sequence in κ is given by $\{\kappa_n\}_{n<\omega}$, where $\kappa_n=(2^\omega)^{+n}$ is the $n-$th cardinal greater than 2^ω. More formally, if $\alpha>0$ is the ordinal such that $2^\omega=\aleph_\alpha$, then $\kappa=\aleph_{\alpha+\omega}$ and $\kappa_n=\aleph_{\alpha+n}$ for each $n<\omega$.

Similarly to the previous section, $V(G)$ is partitioned into three sets X, Y and Z. They are defined by the following rules:

- $X=\{x_n:n<\omega\}$ is (again) a copy of ω.
- Y is written as a disjoint union $Y=\bigcup\limits_{\substack{\alpha<\kappa\\ v\in\mathscr{S}}}Y_v^\alpha$. Each Y_v^α is, in its turn, a copy of the cartesian product
$$\omega_1^{n+1}:=\underbrace{\omega_1\times\omega_1\times\cdots\times\omega_1}_{n+1\text{ times}},$$
in which n is the length of v as a sequence. We highlight, therefore, that ω_1^{n+1} is not the usual exponentiation of ordinals in this context. Finally, for every $\alpha<\kappa$, we refer to the subset $Y^\alpha=\bigcup\limits_{v\in\mathscr{S}}Y_v^\alpha$ as the $\alpha-$**level** of Y.

With this definition, we observe that any two levels of Y are copies of the same set. This set, on the other hand, is obtained by the tree $\mathscr{S}$ when replacing every element of height n by a copy of ω_1^{n+1}.

- Z has a very similar definition to that of Y. First, it is written as a disjoint union $Z = \bigcup\limits_{\substack{\alpha<\kappa \\ v\in\mathscr{S}}} Z_v^\alpha$.

  Each Z_v^α is, in its turn, a copy of the cartesian product

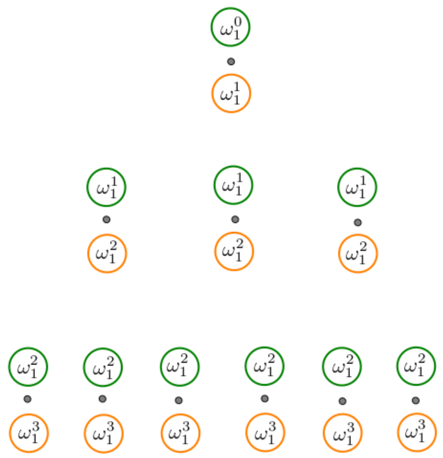
$$\omega_1^n := \underbrace{\omega_1 \times \omega_1 \times \cdots \times \omega_1}_{n \text{ times}},$$

  in which n is the length of v as a sequence. In particular, $Z_\emptyset^\alpha$ has exactly one element for every $\alpha < \kappa$, that we will define as the $\alpha-$**root**. As before, for every $\alpha < \kappa$, we refer to the subset $Z^\alpha = \bigcup\limits_{v\in\mathscr{S}} Z_v^\alpha$ as the $\alpha-$**level** of Z.

  In other words, any two levels of Z are copies of the same set, that is obtained by the tree $\mathscr{S}$ when replacing every element of height n by a copy of ω_1^n.

In Figure 17, we sketch a helpful way to visualize the vertices of any $\alpha-$level of Y and Z. The little black circles in the figure are not some of those vertices, but instead are vertices of the tree $\mathscr{S}$. If we fix v one of them, the green circle drawn above it illustrates the set Z_v^α, while the orange circle below it represents the set Y_v^α. Regarding this picture, it is rather natural to define the edge set E according to the following rules:

Figure 17 – $\alpha-$level of Y and Z



A display of the vertices of Y and Z in their $\alpha-$level, for some $\alpha < \kappa$. We stretch that the little black circles represent elements of $\mathscr{S}$ and do not belong to the graph in construction.

Source: Elaborated by the author.

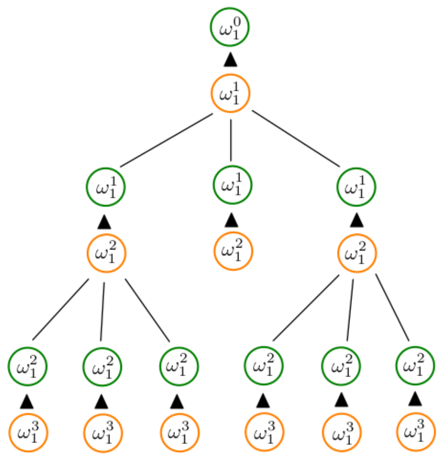
- For every $v \in \mathscr{S}$ and every $\alpha < \kappa$, a vertex of Z_v^α is an element $p \in \omega_1^n$, where n is height of v in the tree $\mathscr{S}$. We define p to be adjacent to every element of $\{(p,\delta) : \delta < \omega_1\} \subset Y_v^\alpha = \omega_1^{n+1}$. For further illustrations, we call the edges defined this way as **type A** edges.
- If $v \in \mathscr{S}$ is not a leaf of $\mathscr{S}$, then it is a sequence of the form $v = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, being α_n an infinite ordinal. Every successor of v in $\mathscr{S}$, therefore, has the form $v(\delta) = (\alpha_1, \alpha_2, \ldots, \alpha_n, \delta)$ for some $\delta < |\alpha_n|$. Then, we define a vertex $p \in Y_v^\alpha = \omega_1^{n+1}$ to be adjacent to its copy in $Z_{v(\delta)}^\alpha$, for every $\delta < |\alpha_n|$. The edges defined this way will be referred as **type B** edges.
- If $v \in \mathscr{S}$ is a leaf of $\mathscr{S}$, then $S(v) \subset \mathscr{U}$ is a finite subset of the ultrafilter $\mathscr{U}$. In particular, $\bigcap S(v) \in \mathscr{U}$ is an infinite subset of ω. Fixing $\alpha < \kappa$ and $N < \omega$ such that $\kappa_N \leq \alpha < \kappa_{N+1}$,

we define every vertex of Y_v^α to be adjacent of every element of the (infinite) set $\{x_n : n \geq N, n \in \bigcap S(v)\} \subset X$. The edges defined this way will be referred as **type C** edges.

- We set that every element of X is adjacent to every $\alpha-$root, that is, x_n and the vertex of Z^α are neighbors for every $n < \omega$ and every $\alpha < \kappa$.

Finished the definition of the edge set $E(G)$, it is not hard to verify that every vertex of G has infinite degree. First, we observe that every vertex of X has degree κ since it is adjacent to every $\alpha-$root with $\alpha < \kappa$. Moreover, $x_n \in X$ has fewer than κ_n neighbors that are not of this form, related to type C edges. In summary, every vertex of X has almost all[1] its neighbors as $\alpha-$roots, for $\alpha < \kappa$.

Figure 18 – Edges of type A and B in the $\alpha-$level of Y and Z



The symbol "▲" between green and orange circles represents a edge sets of type A. On the other hand, straight lines between green and orange circles illustrate edge sets of type B.

Source: Elaborated by the author.

If $v \in \mathscr{S}$ is not a leaf of $\mathscr{S}$, then it has the form $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ for some infinite ordinal α_n. In this case, for every $\alpha < \kappa$, every vertex of Y_v^α has $|\alpha_n|$ neighbors in Z given by edges of type B and one neighbor in Z given by an edge of type A. If $v \in \mathscr{S}$ is a leaf, then every vertex of Y_v^α has $\aleph_0$ neighbors in X, connected by edges of type C, and one neighbor in Z, connected by an edge of type A.

Finally, every vertex of Z has $\aleph_1$ neighbors in Y, given by edges of type A. The $\alpha-$roots have countably many neighbors in X, for each $\alpha < \kappa$, while the other elements of Z have one more neighbor in Y given by an edge of type B. In order to better visualize these adjacencies, Figure 18 is obtained by Figure 17 by including the edges of type A and B in a fixed $\alpha-$level of the sets Y and Z.

For instance, suppose that there exists an unfriendly partition $c : V(G) \to 2$. As in the proof of Theorem 3.4.1, ω is naturally partitioned into the sets $A = \{n < \omega : c(x_n) = 0\}$ and $B = \{n < \omega : c(x_n) = 1\}$. Since $\mathscr{U}$ is an ultrafilter, without loss of generality we can assume that $A \in \mathscr{U}$. By induction over the tree structure of $\mathscr{S}$, we will verify the following observation:

Proposition 3.4.3. *If $v \in \mathscr{S}$ is a sequence such that $A \in S(v)$, then $c(Y_v^\alpha) = \{1\}$ and $c(Z_v^\alpha) = \{0\}$ for every level $\alpha < \kappa$.*

[1] We observe that the countable cofinality of κ plays a relevant role in the construction precisely here, at the description of where do belong most of the neighbors of some $x \in X$. This is also an argument inspired by the consistent construction just studied.

Proof. Let $\alpha < \kappa$ be any level. If $v \in \mathscr{S}$ is a leaf in the tree structure of $\mathscr{S}$, then $S(v)$ is finite and $\bigcap S(v) \subset A$. Therefore, fixing $p \in Y_v^\alpha$, every neighbor of p given by an edge of type C received the color 0 by c. Since only one edge of type A is incident in p and c is unfriendly in this vertex, we must have $c(p) = 1$. This concludes that $c(Y_v^\alpha) = \{1\}$. Also, once every vertex of Z_v^α has almost all its neighbors in Y_v^α), we verify that $c(Z_v^\alpha) = \{0\}$ because c is an unfriendly partition.

Suppose now that v is not a leaf of $\mathscr{S}$. Therefore, it is of the form $(\alpha_1, \alpha_2, \dots, \alpha_n)$ with α_n infinite. In this case, Lemma 3.4.2 guarantees that $A \notin S(w)$ for fewer than $|\alpha_n|$ successors w of v in $\mathscr{S}$, because $A \in S(v)$. This means that every vertex of Y_v^α has, through edges of type B, almost all its neighbors among the elements of the set

$$\Omega = \{p \in Z_w^\alpha : w \text{ is a successor of } v \text{ in } \mathscr{S} \text{ and } A \in S(w)\}.$$

However, by induction on the tree structure of $\mathscr{S}$, we can assume that $c(\Omega) = \{0\}$. Hence, since c is unfriendly in every vertex of Y_v^α, we must have $c(Y_v^\alpha) = \{1\}$. As a consequence, $c(Z_v^\alpha) = \{0\}$, because c is an unfriendly partition and every vertex of Z_v^α has almost all its neighbors in Y_v^α. □

In particular, since $A \in \mathscr{U} = S(\emptyset)$, Proposition 3.4.3 verifies that $c(z) = 0$ for every $\alpha-$root $z \in Z$. However, any $x_n \in X$ with $n \in A$ has almost all its neighbors as elements of $\{z \in Z_\emptyset^\alpha : \alpha < \kappa\}$. Since $c(x_n) = 0$ by definition of A, the coloring c fails to be unfriendly in x. Facing this contradiction, the following statement is proved under ZFC:

Theorem 3.4.4 (Shelah and Milner (1990), Theorem 1)**.** *There exists a graph G with $|V(G)| = (2^\omega)^{+\omega}$ that has no unfriendly partition and such that all its vertices have infinite degree.*

With little changes in some cardinalities, it is not hard to generalize Theorem 3.4.4 and provide even bigger graphs that have no unfriendly partition. To that aim, fix λ any infinite cardinal. From now on to the end of this section, we will point out how the definitions previously made can be modified to construct a graph G with $\kappa = (2^\lambda)^{+\omega}$ vertices, all them of infinite degree, that has no unfriendly partition.

We start by remarking that κ is the first limit cardinal greater than 2^λ. Hence, it has also countable cofinality: the sequence $\{\kappa_n\}_{n<\omega}$ given by $\kappa_n = (2^\lambda)^{+n}$ for each $n < \omega$ is cofinal in κ. Again, if $\alpha > 0$ is the ordinal such that $\aleph_\alpha = 2^\lambda$, then $(2^\lambda)^{+\omega} = \aleph_{\alpha+\omega}$ and $(2^\lambda)^{+n} = \aleph_{\alpha+n}$.

The tree $(\mathscr{S}, \preceq)$ is now defined by

$$\mathscr{S} = \{(\alpha_1, \alpha_2, \dots, \alpha_n) : n < \omega, \quad 2^\lambda > \|\alpha_1\| > \|\alpha_2\| > \cdots > \|\alpha_n\|\}.$$

Meanwhile, the relation $\preceq$ is still given by the extension of sequences. Thus, $(\alpha_1, \alpha_2, \dots, \alpha_n) \preceq (\alpha_1, \alpha_2, \dots, \alpha_n, \delta)$ for any $\delta < \|\alpha_n\|$.

In its turn, the ultrafilter $\mathscr{U}$ is chosen on λ, being also non-principal in the sense that no subset of λ with fewer than λ elements belongs to $\mathscr{U}$. This is possible because the family $\mathscr{F}_1 = \{A \subset \lambda : |\lambda \setminus A| < \lambda\}$ has the finite intersection property, allowing us to choose $\mathscr{U}$

containing $\mathscr{F}_1$. Besides that, for each $n < \omega$, if $B_n = \{\omega\alpha + n : \alpha < \lambda\}$ denotes the set of ordinals with remainder n on division by ω, we also have that the family $\mathscr{F}_2 = \{\lambda \setminus B_n : n < \omega\}$ has the finite intersection property. Actually, the union $\mathscr{F}_1 \cup \mathscr{F}_2$ has the finite intersection property, so we will fix $\mathscr{U}$ containing $\mathscr{F}_1 \cup \mathscr{F}_2$.

For any sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathscr{S}$, we define $S(\alpha_1, \alpha_2, \ldots, \alpha_n) \subset \mathscr{U}$ exactly as before, so that Lemma 3.4.2 still holds. Now, in order to define the sets X, Y and Z that partition the vertex set $V(G)$, we present the following minor changes:

- $X = \{x_\xi : \xi < \lambda\}$ is a copy of λ.
- For a level $\alpha < \kappa$ and any $v \in \mathscr{S}$, we set Y_v^α to be a copy of $(\lambda^+)^{n+1}$, where n is the height of v in the tree structure of $\mathscr{S}$. Again, $(\lambda^+)^{n+1}$ denotes the cartesian product of λ^+ iterated $n+1$ times - and not the usual exponentiation of ordinals. We then define the disjoint union $Y = \bigcup\limits_{\substack{\alpha<\kappa \\ v\in\mathscr{S}}} Y_v^\alpha$ to be the set Y.
- For a level $\alpha < \kappa$ and any $v \in \mathscr{S}$, we set Z_v^α to be a copy of $(\lambda^+)^n$, where n is the height of v in the tree structure of $\mathscr{S}$. We then define the disjoint union $Z = \bigcup\limits_{\substack{\alpha<\kappa \\ v\in\mathscr{S}}} Z_v^\alpha$ to be the set Y. In particular, $Z_\emptyset^\alpha$ contains just one element, that we refer as the $\alpha-$root.

To describe the edge set $E(G)$, we first point out that the edges of type A and B can be defined as before whether we change ω_1 by λ^+ in the original definition. Also, we impose that x_ξ is adjacent to the $\alpha-$root, for every $\xi < \lambda$ and $\alpha < \kappa$. Finally, just one restriction is made when analogously defining the edges of type C. Now, if $v \in \mathscr{S}$ is a leaf in $\mathscr{S}$ and $p \in Y_v^\alpha$ for some $\alpha < \kappa$, we set $x_\xi p \in E$ if, and only if, $\xi \in \bigcap S(v)$ and there is $n < \omega$ such that $\alpha \leq \kappa_n$ and $\xi \in B_n$.

With this definition, any $x_\xi \in X$ is adjacent to at most κ_n vertices that do not belong to Z, where n is the remainder obtained when dividing ξ by ω. On the other hand, if v is a leaf of $\mathscr{S}$, we must have $\bigcap S(v) \in \mathscr{U}$, because $S(v) \subset \mathscr{U}$ is finite. Since $B_n \notin \mathscr{U}$ for any $n < \omega$ and every member of $\mathscr{U}$ has λ elements, every vertex of Y_v^α has degree λ. Under these conditions - and since Lemma 3.4.2 still holds -, an analogous statement of Proposition 3.4.3 can be proven in order to verify that G has no unfriendly partition.

3.5 Hierarchies

The results in this section are rather different from the previous ones, since they aim to describe unfriendly partitions by forbidding certain subgraphs. Now, we will revisit the paper of Bruhn *et al.* (2010) in order to verify the unfriendly partition conjecture for *rayless* graphs. Remarkably, the core of the proof is a characterization of this graph family via an hierarchy first formalized by Schmidt (1983).

Introducing this recursive structure, we set the finite graphs as graphs of lower complexity, calling them *rank* 0 graphs. If $\alpha > 0$ is an ordinal and the graphs of rank β are defined for every $\beta < \alpha$, we say that G is a graph of **rank** α if the following two properties are verified:

1. G has no rank $\beta < \alpha$;
2. There is a finite separator $S \subset V(G)$ such that each connected component of $V(G) \setminus S$ has a rank $\beta < \alpha$. Intuitively, the connected components of $G \setminus S$ have smaller complexity. In this case, we call S a **kernel** of G.

We first observe that not every graph has a rank: by deleting finitely many vertices of a ray, some remaining connected component has also a ray, so that an eventual rank would be decreased. In fact, the rays are the unique obstruction for the well definition of ranks, as one can verify by adapting the proof of König's Lemma (2.1.1):

Lemma 3.5.1. *A graph has a rank if, and only if, it is rayless.*

Proof. Clearly, finite graphs do not contain rays as subgraphs. Then, if G has rank $\alpha > 0$ and $S \subset V(G)$ is a corresponding kernel, we can assume by induction that the connected components of $G \setminus S$ are rayless as well. Hence, G also contains no ray, since S is finite.

Conversely, assume that G is a graph with no rank. Therefore, fixing $v_0 \in V(G)$, some connected component C_1 of $V(G) \setminus v_0$ also has no rank. Fix an arbitrary neighbor $v_1 \in V(C_1)$ of v_0. Suppose that it is already defined a $\subseteq -$decreasing family $C_1, C_2, \dots, C_n$ of connected subgraphs of G with no rank, as well as a path $v_0 v_1 \dots v_n$ with $v_i \in V(C_i)$ for each $1 \leq i \leq n$. Then, some connected component C_{n+1} of $C \setminus v_n$ has no rank too (and, in particular, it is an infinite graph). Considering $v_{n+1} \in V(C_{n+1})$ a neighbor of v_n, we finish a recursive process that defines a ray $v_0 v_1 v_2 \dots$ in G. □

Exercise 3.5.2. *Consider the tree* $\mathscr{S} = \{(\alpha_1, \alpha_2, \dots, \alpha_n) : 2^\omega > \|\alpha_1\| > \|\alpha_2\| > \cdots > \|\alpha_n\|\}$ *from the proof of Theorem 3.4.4. Recall that its tree order, when fixing* $\emptyset$ *as a root, is given by the extension of sequences. Moreover, we remarked on Section 3.4 that* $\mathscr{S}$ *is rayless. What is its rank?*

When Schmidt noticed the above description of the rayless graphs, his motivation was to give a partial positive result for the *reconstruction conjecture*, another well known open problem in (both finite and infinite) graph theory. Since them, other applications of his rank function were obtained. Besides the contributions for the unfriendly partition problem, we can also mention the existence of either none or infinitely many *twins* of rayless graphs, as approached by Bonato *et al.* (2011). In common, these discussions often relies on the useful properties below:

Lemma 3.5.3. *Consider* G *a rayless graph of rank* $\alpha > 0$*. Let* $S \subset V(G)$ *be a kernel of* G *and denote by* $\mathscr{C}$ *the set of connected components of* $G \setminus S$*. Hence,*

- *If $\mathscr{F} \subset \mathscr{C}$ is finite, then the subgraph induced by $S' \cup \bigcup \mathscr{F}$ has rank smaller than α, for any subset $S' \subseteq S$;*
- *If S is a $\subseteq -$minimal kernel, then its vertices have infinite degree. More precisely, they have neighbors in infinitely many connected components from $\mathscr{C}$.*

Proof. In order to prove the first item, write $\mathscr{F} = \{C_1, C_2, \ldots, C_n\}$ and let α_i denote the rank of C_i for each $1 \leq i \leq n$. Then, fix $S_i \subset V(C_i)$ a finite separator such that the connected components of $C_i \setminus S_i$ have rank strictly smaller than α_i. Therefore, $F = S' \cup \bigcup_{i=1}^{n} S_i$ is a kernel of the graph induced by $S' \cup \bigcup \mathscr{F}$. In fact, each connected component of $G[S' \cup \bigcup \mathscr{F}] \setminus F$ has rank smaller than $\max_{1 \leq i \leq n} \alpha_i < \alpha$.

For the second item, fix $v \in S$ be a vertex that has neighbors in only finitely many elements from $\mathscr{C}$. Let $\mathscr{F} \subset \mathscr{C}$ denote the set of these connected components. In particular, $\mathscr{C}' = \{G[\{v\} \cup \bigcup \mathscr{F}]\} \cup (\mathscr{C} \setminus \mathscr{F})$ is the set of connected components of $G \setminus (S \setminus \{v\})$. However, every subgraph in $\mathscr{C}'$ has rank smaller than α, by the previous item and the definition of $\mathscr{C}$. This shows that S is not a minimal kernel of G. $\square$

Exercise 3.5.4. *In addition, show that, if G is a rayless graph, then there is a unique $\subseteq -$minimal kernel.*

Lemma 3.5.3 is useful for combining inductive proofs over ranks with compactness arguments. Although in this section we aim to conclude that *every* rayless graph has an unfriendly partition, we shall first analyze the inspiring proof of the countable case:

Proposition 3.5.5. *Every countable rayless graph has an unfriendly partition.*

Proof, following Theorem 8.5.3 of Diestel (2018). Let G be a countable rayless graph and denote by α its rank. If $\alpha = 0$, then G is finite, admitting unfriendly partitions. Therefore, we can assume that $\alpha > 0$. Fix a minimal kernel $S \subset V(G)$ of G, so that, by Lemma 3.5.3, every vertex of S has *countable* infinite degree. Following the same lemma, denote by $\mathscr{C} = \{C_n\}_{n \in \mathbb{N}}$ the set of connected components of $G \setminus S$.

Let G_0 be the graph induced by $S \cup C_0$. Suppose that $G_k = G[S \cup C_0 \cup C_1 \cup \ldots C_{n_k}]$ is defined for some $k \in \mathbb{N}$. Choosing a large enough $n_{k+1} \in \mathbb{N}$, define $G_{k+1} = G[S \cup C_0 \cup C_1 \cup \ldots C_{n_k} \cup C_{n_k+1} \cup \cdots \cup C_{n_{k+1}}]$. The choice of n_{k+1} can be even done such that, for each $v \in S$, one of the following properties is verified:

- Either v has infinitely many neighbors in G_k;
- Or v has in $G_{k+1} \setminus G_k$ strictly more neighbors than in G_k.

By Lemma 3.5.3, G_k has rank strictly smaller than α, so that we can fix c_k an unfriendly partition for G_k. Passing $\{c_k\}_{k\in\mathbb{N}}$ to a subsequence if necessary, we can assume that $c_k|_S = c_i|_S$ for every $i,k \in \mathbb{N}$, since S is finite. Then, we define $c : V(G) \to 2$ as follows:

$$c(v) = \begin{cases} c_0(v), & \text{if } v \in S; \\ c_{k_v}(v), & \text{if } v \notin S, \text{ where } k_v = \min\{k \in \mathbb{N} : v \in V(G_k)\}. \end{cases} \tag{3.1}$$

In the above notation, we observe that $k_v = k_u$ if $u, v \in V(G) \setminus S$ belong to the same connected component of $G \setminus S$, by the definition of G_{k_v} and G_{k_u}. Hence, c is indeed unfriendly at these vertices, because so is c_{k_v} and we have $c_{k_v}|_S = c_0|_S = c|_S$. For a given $k \in \mathbb{N}$, if $v \in S$ has infinitely many neighbors in G_k (and thus in some connected component of $\{C_0, C_1, \ldots, C_{n_k}\}$), this same argument shows that c is unfriendly in v. On the other hand, if $v \in S$ has finite neighborhood in G_k, then v has strictly more neighbors in $G_{k+1} \setminus G_k$ than in G_k, by construction. In particular, since c_{k+1} is unfriendly at v, one of such neighbors is an enemy of v (considering both c and c_{k+1}). If this is the case for every $k \in \mathbb{N}$, it follows that v has infinitely many enemies regarding c, i.e., that c is unfriendly in v.

□

We observe that the above proof relies on the countability of G due to, at least, two purposes. First, in this case, to verify whether a coloring is unfriendly in a vertex of infinite degree turns out to check if it has infinitely many enemies. In general graphs, this is often not sufficient, since vertices might have uncountable degree. Second, after deleting a (minimal) kernel, an enumeration of the remaining connected components allows the definition of a global coloring via finite approximations.

However, with some clever adaptations, it is possible to replicate the proof of Proposition 3.5.5 for more general cases, as originally done in (BRUHN *et al.*, 2010). Following that work, let us fix G any rayless graph, denoting by α its rank. Since finite graphs have unfriendly partitions, we shall assume that $\alpha > 0$. Consider S a $\subsetneq -$minimal kernel of G. Writing $\mathscr{C}$ for the set of connected components of $G \setminus S$, we will prove that G has an unfriendly partition via two transfinite induction: first over α and, then, over κ. The base case $\kappa = \aleph_0$ is quite similar to Proposition 3.5.5:

Lemma 3.5.6. *If $\kappa = \aleph_0$, then G has an unfriendly partition.*

Proof. Write $\mathscr{C} = \{C_n\}_{n\in\mathbb{N}}$. Then, the neighborhood of a vertex $v \in S$ in $G \setminus S$ is given by $\bigcup_{n\in\mathbb{N}} N(v) \cap C_n$. Hence, precisely one of the following three items below is verified:

i) v has $d(v)$ neighbors in C_n for some $n \in \mathbb{N}$;

ii) v has countable degree, but finitely many neighbors in C_n for each $n \in \mathbb{N}$;

iii) $d(v)$ is a singular cardinal with countable cofinality. In this case, we fix $\{\kappa_v^n\}_{n\in\mathbb{N}}$ a cofinal sequence of cardinals for $d(v)$.

Now, let G_0 be the subgraph induced by $S\cup C_0$. Suppose that $G_k = G[S\cup C_0\cup C_1\cup\dots C_{n_k}]$ is defined for some $k\in\mathbb{N}$. Choosing a large enough $n_{k+1}\in\mathbb{N}$, define $G_{k+1} = G[S\cup C_0\cup C_1\cup \dots C_{n_k}\cup C_{n_k+1}\cup\dots\cup C_{n_{k+1}}]$. The choice of n_{k+1} is done such that, for each $v\in S$, one of the following properties is verified:

- v is as in the item i) above;
- If v is as in the item *ii*) above, then v has in $G_{k+1}\setminus G_k$ strictly more neighbors than in G_k;
- If v is as in the item *iii*) above, then v has in $G_{k+1}\setminus G_k$ at least κ_v^{k+1} neighbors. This is possible by considering n_{k+1} bigger than a given $n\in\mathbb{N}$ such that $|N(v)\cap C_n|\geq \kappa_v^j > \max\limits_{0\leq i\leq n_k}|N(v)\cap C_i|$ for some $j>k+1$.

By Lemma 3.5.3, we can fix c_k an unfriendly partition for G_k. Then, at the end of this recursive process, we defined an infinite sequence $\{c_k\}_{k\in\mathbb{N}}$ of partial colorings over G. Since S is finite, we can assume that $c_k|_S = c_i|_S$ for every $i,k\in\mathbb{N}$, unless by considering a subsequence of $\{c_k\}_{k\in\mathbb{N}}$. Now, let $c : V(G)\to 2$ be the coloring defined the same way as in the rule 3.1. In particular, c is unfriendly in $V(G)\setminus S$.

If $v\in S$ is as in *ii*), the second item above guarantees that c label an enemy for v in $G_{k+1}\setminus G_k$ for each $k\in\mathbb{N}$. Hence, c is unfriendly in v, since $d(v)$ is countable. If $v\in S$ follows item *iii*), then v has $\sup\limits_{k\in\mathbb{N}}|N(v)\cap V(G_k)| = \sup\limits_{k\in\mathbb{N}}\kappa_v^k = d(v)$ enemies regarding c, meaning that this coloring is unfriendly in v. Finally, if $v\in S$ is as in item i), there is some $n\in\mathbb{N}$ such that $d(v) = |N(v)\cap C_n|$. If n is minimal with that property and $k = \min\{i\in\mathbb{N} : C_n\subset V(G_k)\}$, then v has $d(v)$ enemies in G_k regarding c, because c_k is unfriendly in v as well.

□

The case in which κ is uncountable has some additional complexity. In order to approach it, we observe that, if G were a finite graph and $c_D : D\to 2$ were a partially defined coloring, then there would exist an extension $c : V(G)\to 2$ which is unfriendly in every vertex of $V(G)\setminus D$. In fact, we could apply Proposition 3.2.6 or choose c by maximizing the amount of edges with differently colored endpoints.

Then, being G a rayless graph with rank $\alpha > 0$, the above proof could be rewritten in order to mention this property. More precisely, Lemma 3.5.6 could be stated as "if $\kappa=\omega$ and $c_D : D\to 2$ is a partially defined coloring, there exists $c : V(G)\to 2$ an extension of c_D which is unfriendly in every vertex of $V(G)\setminus D$". To this aim, it would be enough to choose the sequence of colorings $\{c_k\}_{k\in\mathbb{N}}$ such that, by induction on α, c_k extends $c_D|_{U\cap V(G_k)}$. This observation was not pointed out previously since it will be employed only at the (already very technical) proof below. In other words, when claiming the existence of an unfriendly partition by transfinite induction, we might now assume that this coloring extends some partially defined map.

Theorem 3.5.7 (Bruhn *et al.* (2010), Theorem 1.1)**.** *If G is a rayless graph, then G has an unfriendly partition.*

Proof. Relying on Lemma 3.5.6, we can now assume that $\kappa > \omega$. Then, fix a well ordering $\{C_\beta\}_{\beta<\kappa}$ for $\mathscr{C}$. The neighborhood of a vertex $v \in S$ in $G \setminus S$ is thus given by $\bigcup_{\beta<\kappa} N(v) \cap C_\beta$. In particular, precisely one of the following cases is verified:

i) v has $d(v)$ neighbors in $\bigcup_{\beta<\kappa_v} C_\alpha$, for some $\kappa_v < \kappa$. In other words, v attains its degree in less than κ connected components of $G \setminus S$. Clearly, when $d(v) < \kappa$, we can even consider κ_v big enough so that $N(v) \bigcup_{\beta<\kappa_v} C_\alpha$;

ii) $d(v) = \kappa$, but v has fewer than κ neighbors in $\bigcup \mathscr{C}'$ for any subfamily $\mathscr{C}' \subset \mathscr{C}$ of size les than κ;

iii) $d(v) > \kappa$, but v has fewer than $d(v)$ neighbors in $\bigcup \mathscr{C}'$ for any subfamily $\mathscr{C}' \subset \mathscr{C}$ of size less than κ. In particular, $d(v)$ is a singular cardinal whose cofinality is κ. This allows us to fix $\{\kappa_v^i\}_{i<\kappa}$ a cofinal sequence of cardinals for $d(v)$;

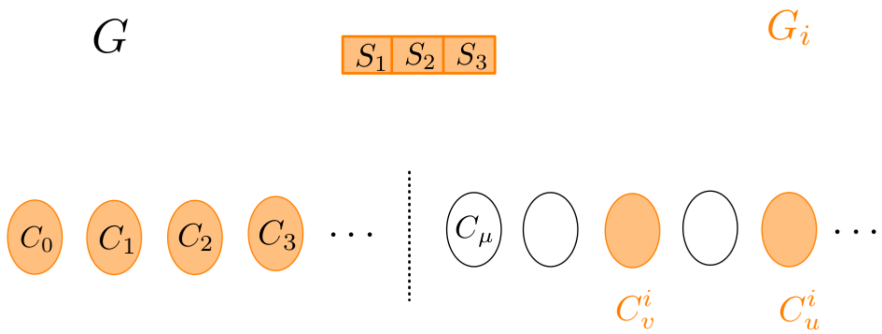
The subsets of S comprising those vertices that satisfy *i*), *ii*) and *iii*) will be denoted by S_1, S_2 and S_3 accordingly. Now, fix $\mu < \kappa$ an upper bound for $\{\kappa_v : v \in S_1\}$ and let G_0 denote the subgraph of G induced by $S \cup \bigcup_{\beta \leq \mu} C_\beta$. For each $v \in S_3$, let $\beta_v^1 > \mu$ be chosen such that $|N(v) \cap C_{\beta_v^1}| \geq \kappa_v^1$. By transfinite induction, suppose that, for some $i < \kappa$, an ordinal $\beta_v^j > \mu$ is defined for each $j < i$, chosen such that $|N(v) \cap C_{\beta_v^j}| \geq \kappa_v^j$. Hence, since $d(v) > \kappa$ has cofinality κ and v does not admit $d(v)$ neighbors within $\bigcup_{j<i} C_{\beta_v^j}$, we can find $\beta_v^i \in \kappa \setminus (\{\beta_v^j : j < i\} \cup \mu)$ such that $|N(v) \cap C_{\beta_v^i}| \geq \kappa_v^i$. At the end of this recursive process, for every $v \in S_3$, we defined a subset (but not necessarily a subsequence!) $\{\beta_v^i\}_{i<\kappa} \subset \kappa$ sstifying

$$|N(v) \cap C_{\beta_v^i}| \geq \kappa_v^i \text{ for every } i < \kappa.$$

Actually, since S is finite, we can refine the definition of these subsets in order to assume that $\{\beta_v^i\}_{i<\kappa} \cap \{\beta_u^i\}_{i<\kappa} \neq \emptyset$ whenever $u \neq v$.

For each $i < \kappa$, we then define G_i the subgraph induced by $V(G_0) \cup \{C_{\beta_v^i} : v \in S_3\}$, as illustrated by Figure 19. Next, the inductive hypothesis over κ guarantees that there is an unfriendly partition $c_i : V(G_i) \to 2$ for G_i. Unless by passing $\{c_i\}_{i<\kappa}$ to a subsequence, we assume (as in the two previous proofs) that $c_i|_S = c_j|_S =: c_S$ for every $i, j < \kappa$. Analogously, we

Figure 19 – Definition of the subgraph G_i



The separator $S \subset V(G)$ is represented by a rectangle, which is partitioned into the subsets S_1, S_2 and S_3. The connected components of $G \setminus S$, in its turn, are drawn as ellipses. Then, the subgraph G_i, sketched in orange, is induced by S and the connected components from $\{C_\alpha : \alpha < \mu \text{ or } \alpha = \alpha_v^i \text{ for some } v \in S_3\}$.

Source: Elaborated by the author.

can suppose that the definition of the set $\{v \in S_1 : v \text{ has } d(v) \text{ enemies regarding } c_i \text{ in } G_0\}$ does not depend on i. In other words, for $v \in S_1$,

$(\star)$ The vertex v has $d(v)$ enemies in G_0 regarding every c_i or v has $d(v)$ enemies in $G_i \setminus G_0$ regarding every c_i.

Given an index $\beta \in \kappa \setminus (\{\beta_v^j : v \in S_3 \text{ and } j < \kappa\} \cup \mu)$, if it exists, we fix an unfriendly partition c_β, which extends c_S, for the subgraph induced by $S \cup C_\beta$. Such a coloring is given by the inductive hypothesis over the rank of G. We now define a global coloring $c : V(G) \to 2$ by setting

$$c(v) = \begin{cases} c_S(v), & \text{if } v \in S; \\ c_1(v), & \text{if } v \in C_\beta \text{ for some } \beta < \mu; \\ c_i(v), & \text{if } v \in C_{\beta_u^i} \text{ for some } u \in S_3 \text{ and some } i < \kappa; \\ c_\beta(v), & \text{if } v \in C_\beta \text{ for some } \beta \in \kappa \setminus (\{\beta_u^j : u \in S_3 \text{ and } j < \kappa\} \cup \mu). \end{cases}$$

Hence, c is unfriendly in $G \setminus S$, since vertices in this set were labelled following colorings given by inductive hypothesis. If $v \in S_3$, then c is unfriendly in v as well, because $|N(v) \cap V(G_0)| < d(v)$, c_i is unfriendly in v and $|N(v) \cap C_{\beta_v^i}| \geq \kappa_v^i$ for every $i < \kappa$. Similarly, c is unfriendly in vertices of S_1, because so is c_1 and $|N(v) \cap G_1| = d(v)$ for every $i < \kappa$ and every $v \in S_1$.

Unfortunately, c might not be unfriendly for vertices in S_2. Then, we denote by $F \subset S_2$ the set of these vertices that have less than κ enemies regarding c. Thus, the coloring $c * F$, obtained by switching precisely the values of c in F, is unfriendly at F. However, $c * F$ is possibly not unfriendly in vertices of the connected components described by the following set of indices:

$$\Lambda_F = \{\beta < \kappa : \text{some } v \in F \text{ has an enemy in } C_\beta \text{ regarding } c\}$$

In particular, $|\Lambda_F| < \kappa$ by definition of F and the fact that F is finite. Moreover, for every index $\beta \in \Lambda_F$ with $\beta \geq \mu$, we can change the coloring $c * F$ on C_β by extending $c * F|_S$ to the remaining vertices of $G_\beta \setminus S$. This can be done so that the resulting (global) coloring, denoted by $\overline{c * F}$, is also unfriendly at vertices of $\{C_\beta : \beta \in \Lambda_F, \beta \geq \mu\}$. Now, still not being unfriendly at some vertices of $V(G_0) \setminus S = \bigcup\limits_{\beta < \mu} C_\beta$, the coloring $\overline{c * F}$ might not be unfriendly at some vertices of S_1 as well.

From now on to the end of this proof, we will work looking forward modifying $\overline{c * F}$ in these mentioned sets. To that aim, consider the cardinal

$$\gamma = \max\left\{\omega, \max_{s \in F} |\{u \in V(G_0) \setminus S : c(u) \neq c(v)\}|\right\} < \kappa,$$

where the above innequality follows from the fact that c is not unfriendly in vertices of F. The conclusion of the proof relies on the following claim:

`Claim:` Denote by $\hat{S} \subset S_1$ the set of vertices which have in $V(G_0)$, regarding c, less than their degree of enemies. Then, there is a coloring $\hat{c} : V(G_0) \to 2$ such that:

- $\hat{c}$ and $c * F$ agree on the vertices of $\hat{S} \cup S_2$;
- $\hat{c}$ is unfriendly in $V(G_0) \setminus (\hat{S} \cup S_2)$;
- For every $v \in V(G_0)$ with $|N(v) \cap V(G_0)| > \gamma$, we have $\hat{c}(v) = (c * F)(v)$.

Proof of the claim. The third condition states that $\hat{c}$ is obtained from $c * F$ by changes on vertices of degree at most γ in G_0. Then, let $\hat{\mathscr{C}}$ denote the set of connected components of $G_0[\{v \in V(G_0) : |N(v) \cap V(G_0)| \leq \gamma\}]$ that contains a vertex for which $\overline{c * F}$ is not unfriendly. These connected components are between the finitely many ones containing vertices of S_1 or the γ ones in which a vertex of F had an enemy regarding c. In other words, $|\hat{\mathscr{C}}| \leq \gamma$, so that the graph $\hat{G}_0$, comprising precisely the connected components of $\hat{\mathscr{C}}$, has at most γ vertices by Lemma 2.1.3. Therefore, by the inductive hypothesis over κ, there is a coloring $\hat{c} : V(G_0) \to 2$ which differs from $c * F$ possibly at $\hat{G}_0 \setminus (\hat{S} \cup S_2)$ and that is unfriendly in vertices of this subgraph.

Now, if v is a fixed vertex from $G_0 \setminus \hat{G}_0$ that also do not belong to $\hat{S} \cup S_2$, we analyze whether $d(v) \leq \gamma$ or $d(v) > \gamma$. In the first case, the neighbors of v also do not belong to $\hat{G}_0$, by definition of $\hat{\mathscr{C}}$. Hence, $\hat{c}$ agrees with $c * F$ in the neighborhood of v, proving that $\hat{c}$ is unfriendly in this vertex. Finally, if $d(v) > \gamma$, then $\hat{c}$ and $c * F$ disagrees in only fewer than $d(v)$ of its neighbors, since $|V(\hat{G}_0)| \leq \gamma$. However, $c * F$ is unfriendly at v even if $v \in S$, because $v \notin \hat{S}$ by assumption. Then, so is unfriendly at this vertex the coloring $\hat{c}$. $\square$

Finally, we will check that the coloring $h : V(G) \to 2$ defined below is an unfriendly partition for G.

$$h(v) = \begin{cases} c * F(v), & \text{if } v \in \hat{S} \cup S_2 \cup S_3; \\ \hat{c}(v), & \text{if } v \in S_1 \setminus \hat{S}; \\ \hat{c}(v), & \text{if } v \in V(G_0) \setminus S = \bigcup\limits_{\beta < \mu} C_\beta; \\ \overline{c * F}(v), & \text{if } v \in C_\beta \text{ for some } \beta \geq \mu. \end{cases}$$

In fact, the above claim guarantees that h is unfriendly in $S_1 \setminus \hat{S}$, besides also showing that h is unfriendly in vertices of $V(G_0) \setminus S$. Since h differs from $\overline{c * F}$ in fewer than κ connected components of $G \setminus S$, it follows that h is unfriendly in vertices of $S_2 \cup S_3$. Due to property $(\star)$,vertices of $\hat{S}$ have their degree as amount of enemies in $G_i \setminus G_0$ regarding c, for every $i < \kappa$. Hence, h is unfriendly in v, because $\overline{c * F}$ differs from c in F and less than κ connected components of $G \setminus S$.

It only remains to show that h is unfriendly in a vertex $v \in V(G) \setminus V(G_0)$. In this case, $v \in C_\beta$ for some $\beta \geq \mu$. For a contradiction, suppose that h is not unfriendly in v, meaning that $\overline{c * F}$ differs from $\hat{c}$ in some neighbor $u \in S_1 \setminus \hat{S}$. In particular, $d(u) = |N(u) \cap V(G_0)| \leq \gamma$, where the last inequality follows from the third property of $\hat{c}$ mentioned by the above claim. In this case, we have the inclusion $N(u) \subset \bigcup\limits_{\beta < \mu} C_\beta$, contradicting the fact that $v \notin V(G_0)$. $\square$

Once presented the whole proof that a rayless graph G has an unfriendly partition, we observe that the main hypothesis over G was only needed when well defining its rank. After that, the inductive steps of Theorem 3.5.7 were exclusively supported by the properties described on Lemma 3.5.3. This observation is mentioned in (BRUHN *et al.*, 2010), where the authors developed the above proof in order to obtain a more precise statement for Theorem 3.5.7.

Introducing their main result, we say that a class of graphs $\mathscr{U}$ is **finitely closed** if $\mathscr{U}$ is closed under finite unions and the addition of any finite set of vertices. In other words, for graphs $G_1, G_2, \ldots, G_n \in \mathscr{U}$, the graph whose set of connected components is precisely $\{G_i\}_{i=1}^n$ must belong to $\mathscr{U}$. Besides that, for every $G \in \mathscr{U}$ and every finite graph S, any definition of edges between S and G must construct another graph of U.

In particular, the finite graphs and the graphs with finitely many vertices of infinite degree describe two examples of finitely closed families. On the other hand, the class of locally finite graphs is not finitely closed: it is possible for an added new vertex to have infinitely many neighbors in the original graph.

If $\mathscr{U}$ is a finitely closed family, the previous discussions suggest a definition for an $\mathscr{U}-$**rank**. More precisely, we set the elements of $\mathscr{U}$ as graphs whose $\mathscr{U}-$rank is 0. If the graphs of $\mathscr{U}-$rank β are defined for every $\beta < \alpha$, we say that G has $\mathscr{U}-$rank $\alpha > 0$ if the following properties are verified:

- G is a graph that has no $\mathscr{U}-$rank β for any $\beta < \alpha$;

- There is a finite separator $S \subset V(G)$ such that the connected components of $G \setminus S$ have $\mathscr{U}-$rank smaller than α.

Then, we denote by $\overline{\mathscr{U}}$ the class of graphs whose $\mathscr{U}-$rank is well-defined. In particular, Lemma 3.5.1 claims that $\mathscr{U}$ is precisely the family of rayless graphs when considering $\mathscr{U} = \{\text{finite graphs}\}$. Adapting the proof of the Star-Comb Lemma (2.1.2), it is not hard to see that, if $\mathscr{U} = \{\text{graphs with finitely many vertices of infinite degree}\}$, then $\overline{\mathscr{U}}$ comprises the graphs that do not contain combs whose teeth are vertices of infinite degree.

Within the above language, Lemma 3.5.3 still holds for general $\mathscr{U}-$ranks, so that the main result of (BRUHN *et al.*, 2010) can be stated as:

Theorem 3.5.8 (Bruhn *et al.* (2010), Theorem 2.3)**.** *Let $\mathscr{U}$ be a finitely closed family of graphs. Suppose that, for every $G \in \mathscr{U}$, the property below is verified:*

Given a partial coloring $\tilde{c} : D \to 2$ in G, there is $c : V(G) \to 2$ an extension of $\tilde{c}$ that is unfriendly in $V(G) \setminus D$.

Then, this property also holds for graphs of $\overline{\mathscr{U}}$. In particular, by considering $D = \emptyset$, graphs of $\overline{\mathscr{U}}$ admit unfriendly partitions.

When $\mathscr{U} = \{\text{graphs with finitely many vertices of infinite degree}\}$, for example, the key property of the above statement is verified by Theorem 3.2.2. Hence, there exist unfriendly partitions for graphs that do not contain combs whose teeth are vertices of infinite degree. However, there is no mention in the literature for other characterizations of this resulting graph family. In fact, when relying on Theorem 3.5.8, one may be aware of facing the following two problems: (1) to verify its hypothesis and (2) to find a convenient description for the elements of $\overline{\mathscr{U}}$.

Exercise 3.5.9. *Consider the finitely closed family*

$$\mathscr{U} = \{\textit{graphs with finitely many vertices of finite degree}\}.$$

Which forbidden subgraphs describe $\overline{U}$?

3.6 An extra color

This section explore a singular result in the literature regarding unfriendly partitions, due to Shelah and Milner (1990). Now, we will discuss how coloring a graph with three colors, instead of only two, is considerably more comfortable. Formally, we call an **unfriendly** $3-$**partition** over a graph G any function $c : V(G) \to 3$ such that

$$|\{u \in N(v) : c(u) \neq c(v)\}| \geq |\{u \in N(v) : c(u) = c(v)\}|$$

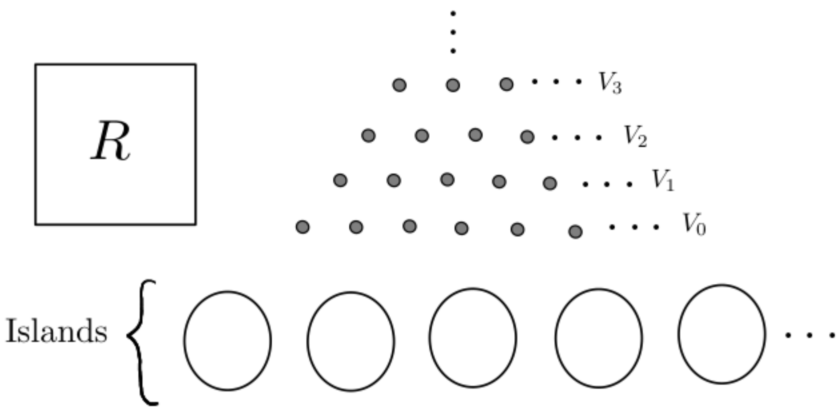
for every $v \in V(G)$. Here, $3 = \{0,1,2\}$ is a set of three elements. Naturally, we still say that c is *unfriendly in a vertex* v if the above inequality is verified. Motivating the next definitions, we point out that unfriendly $3-$partitions will be constructed following a recursive method similar to the *closure* of colorings presented in Section 3.3.

More precisely, for a fixed graph G, we denote by $\mathscr{I}$ the set of connected components of its subgraph induced by the vertices of finite degree. The elements of $\mathscr{I}$ are called the **islands** of G. Let $V_0 := \{v \in V(G) \setminus \bigcup \mathscr{I} : |N(v) \cap \bigcup \mathscr{I}| = d(v)\} \subset V(G)$ be the set of vertices of infinite degree that attain their degree within $\bigcup \mathscr{I}$. Now, suppose that, for some $\alpha > 0$, a set $V_\beta \subset V(G)$ of vertices of infinite degree is defined for every $\beta < \alpha$. We then denote by V_α the set of vertices not covered by $\{V_\beta : \beta < \alpha\} \cup \bigcup \mathscr{I}$ that attain their degree within $\bigcup_{\beta<\alpha} V_\beta$. In other words,

$$V_\alpha = \left\{ v \in V(G) \setminus \bigcup_{\beta<\alpha} V_\beta : \left| N(v) \cap \bigcup_{\beta<\alpha} V_\beta \right| = d(v) \right\}.$$

Finally, if Ω is the first ordinal such that $V_\Omega = \emptyset$, we define the **residual** graph (induced by) $R = V(G) \setminus \bigcup_{\alpha<\Omega} V_\alpha$. Then, we say that the triple $(\mathscr{I}, \{V_\alpha\}_{\alpha<\Omega}, R)$ is the **canonical decomposition** of G, whose construction is sketched by Figure 20.

Figure 20 – The canonical decomposition of a graph G



Source: Elaborated by the author.

If G is countable, it turns out that R is either empty or $\aleph_0-$regular: after all, any $v \in R$ must have infinitely many neighbors in R by the choice of Ω. In this latter case, there is $c_R : V(R) \to 2$ an unfriendly partition. Considering that, Milner and Shelah proved the following:

Proposition 3.6.1 (Shelah and Milner (1990), Theorem 4)**.** *Every countable graph admits an unfriendly* $3-$*partition.*

Proof. Let $(\mathscr{I}, \{V_\alpha\}_{\alpha<\Omega}, R)$ be the canonical decomposition of a countable graph G. Fix $c_R : V(R) \to \{0,1\}$ an unfriendly partition and consider $c_{V_0} : V_0 \to \{2\}$ the constant map, writing

$c_{V_0}(v) = c_v = 2$ for each $v \in V_0$. Given $\alpha > 0$, every vertex $v \in V_\alpha$ has infinitely many neighbors in $\bigcup_{\beta<\alpha} V_\beta$. Then, by induction on α, we can assume that there is a color $c_v \in \{0,1\}$ such that $\left|\left\{u \in N(v) \cap \bigcup_{\beta<\alpha} V_\beta : c_u \neq c_v\right\}\right| = \aleph_0$. In particular, the coloring $\hat{c}$ given by

$$\hat{c}(v) = \begin{cases} c_R(v), & \text{if } v \in V(R); \\ 2, & \text{if } v \in V_0; \\ c_v, & \text{if } v \in \bigcup_{\alpha \geq 1} V_\alpha. \end{cases}$$

for every $v \in V(R) \cup \bigcup_{\alpha<\Omega} V_\alpha$ is unfriendly in $V(R) \cup \bigcup_{\alpha \geq 1} V_\alpha$. Let $\hat{c} * V_0$ denote some coloring obtained from $\hat{c}$ by arbitrarily changing the colors of all the vertices in V_0. In particular, $\hat{c} * V_0 : V(R) \cup \bigcup_{\alpha<\Omega} V_\alpha \to \{0,1\}$ is a labeling with two colors. Hence, by Theorem 3.2.2, there exists $\tilde{c} : V(G) \to \{0,1\}$ an extension of $\hat{c}$ that is unfriendly in $\bigcup \mathscr{I}$. Therefore, after defining $c(v) = \tilde{c}(v)$ for every $v \in V(G) \setminus V_0$ and $c(v) = 2$ for every $v \in V_0$, the coloring $c : V(G) \to 3$ is an unfriendly $3-$partition. □

It is remarkable that, in the above proof, a third color was only necessary to label the vertices of V_0. However, obtaining an uncountable version of Proposition 3.6.1 requires additional efforts, as suggested by the experiences from the previous sections. In that direction, the following set-theoretic remark will be used for combinatorial purposes:

Lemma 3.6.2 (Shelah and Milner (1990), Lemma 1)**.** *Let $\{A_i\}_{I\in I}$ be an infinite family of sets such that $|A_i| \geq |I|$ for every $i \in I$. Then, there is a family $\{B_i\}_{i\in I}$ of pairwise disjoint sets such that $B_i \subset A_i$ and $|B_i| = |A_i|$ for every $i \in I$.*

Proof. First, denote $\mu = |I|$ and consider the set $I' = \{i \in I : |A_i| = \mu\}$. Then, we can fix a non-injective enumeration $\{i_\alpha\}_{\alpha<\mu}$ such that $|\{\alpha < \mu : i_\alpha = i\}| = \mu$ for every $i \in I'$. By induction, for any $\alpha < \mu$ it is possible to choose an element $a_\alpha \in A_{i_\alpha} \setminus \{a_\beta : \beta < \alpha\}$, since $|A_{i_\alpha}| = \mu$. Next, we define $B_i = \{a_\alpha : \alpha < \mu, i_\alpha = i\}$ for every $i \in I'$. By the choice of the enumeration $\{i_\alpha\}_{\alpha<\mu}$, the family $\{B_i\}_{i\in I'}$ is composed by pairwise disjoint sets such that $B_i \subset A_i$ and $|B_i| = \mu = |A_i|$ for every $i \in I'$. Finally, note that the union $B = \bigcup_{i\in I'} B_i$ has size $\mu = |I|$.

It remains to define B_i for each $i \in I \setminus I'$. To that aim, consider $\kappa = \sup_{i\in I\setminus I'} |A_i|$ and fix $\preceq$ a well ordering for $I \setminus I'$ of order type $|I \setminus I'|$. For some ordinal $\alpha \leq \kappa$, suppose that $a_i^\beta \in A_i \setminus B$ is defined for every $\beta < \alpha$ and every $i \in I \setminus I'$ such that $\beta < |A_i|$. By induction, assume that $a_i^\beta \neq a_j^\gamma$ if $i, j \in I \setminus I'$ are distinct or $\beta \neq \gamma$. For every $i \in I \setminus I'$ such that $\alpha = |A_i|$, we set $B_i = \{a_i^\beta : \beta < |A_i|\}$. Writing $J = \{i \in I \setminus I' : \alpha < |A_i|\}$ and supposing that $i \in J$ is a $\preceq -$minimal index for which a_i^α is not defined, we can choose

$$a_i^\alpha \in A_i \setminus (B \cup \{a_j^\beta : \beta = \alpha \text{ and } j \prec i \text{ or } \beta < \alpha \text{ and } \beta < |A_j|\}).$$

Such an element exists since $|A_i| > \alpha$ (because $i \in J$) and $|A_i| > |I|$ (because $i \notin I'$).

At the end of this recursive process, we have defined the set $B_i = \{a_i^\alpha : \alpha < |A_i|\} \subset A_i$ for every $i \in I \setminus I'$. Then, $\{B_i\}_{i \in I}$ is the claimed disjoint family. □

Carefully observing the above statement, we can sketch its applications for graph-theoretic problems: if I is an infinite set of vertices with at least $|I|$ neighbors each, we can assume that they attain their degree in pairwise disjoint subsets of their neighborhoods. As we will see further, this is useful for defining unfriendly colorings recursively. On the other hand, revisiting the original definition from Section 3.3, we note that there is more than one degree of freedom for calculating the closure of a $3-$partition $c : D \to 3$. After all, if v is an uncolored vertex of infinite degree and $d(v)$ of its neighbors are already colored by c, then there exists at least two choices of $c(v) \in 3$ which extends c to a coloring unfriendly in v. Then, which choice is more convenient?

In fact, the above choice to be done depends on the problem. Hence, instead of defining the closure of a coloring, we will now define the closure of its *domain*, in order to label the vertices *a posteriori*. Formally, given a subset $D \subset V(G)$, we call D **closed** if it contains any vertex $v \in V(G)$ such that $|N(v) \cap D| = d(v)$. If v has finite degree, this means that D contains its island as well. Then, the **closure** of a subset $D \subset V(G)$, denoted by $\overline{D}$, refers to the smallest closed set containing D. Recursively, $\overline{D}$ can be described as follows:

- We set $D = \overline{D}^0$;
- For an ordinal $\alpha > 0$, the set $\overline{D}^\alpha$ comprises the vertices v of $V(G)$ such that

$$\left| N(v) \cap \bigcup_{\beta < \alpha} \overline{D}^\beta \right| = d(v). \tag{3.2}$$

For some big enough ordinal Ω, we must have $\overline{D}^\Omega \setminus \bigcup_{\alpha < \Omega} \overline{D}^\alpha = \emptyset$, so that the closure $\overline{D} = \bigcup_{\alpha < \Omega} \overline{D}^\alpha$ is well defined. Considering that, the availability of a third color is useful for proving the technical lemma below:

Lemma 3.6.3 (Shelah and Milner (1990), Lemma 2)**.** *For a graph G, let $A, B \subset V(G)$ be disjoint sets with B infinite. Suppose that, for each $v \in B$, (precisely) one of the following properties holds:*

($\star$) *If $|N(v) \setminus A| \leq |B|$, then $N(v) \subset A \cup B$;*

($\star$) *If* $|N(v)\setminus A|>|B|$, *then* $|N(v)\cap B\setminus A|=|B|$.

Then, for any partial coloring $c: A\cup B\to 3$, *there is an extension* $\overline{c}:\overline{A\cup B}\to 3$ *which is, in the induced subgraph* $G[\overline{A\cup B}]$, *unfriendly in every vertex of*

$$(\overline{A\cup B}\setminus(A\cup B))\cup\{v\in B:|N(v)\cap\overline{A\cup B}\setminus(A\cup B)|>|N(v)\cap(A\cup B)|\}.$$

Proof. The definition of $\overline{A\cup B}$ suggests that we are able to describe an extension of c which is unfriendly in vertices of $\overline{A\cup B}\setminus(A\cup B)$. Then, we shall first analyze how an extension can be unfriendly in a vertex $v\in B$ for which $|N(v)\cap\overline{A\cup B}\setminus(A\cup B)|>|N(v)\cap(A\cup B)|$. In this case, we observe that $|N(v)\setminus A|>|B|$ (and, thus, $|N(v)\cap B\setminus A|=|B|$). Otherwise, we would have $N(v)\subset A\cup B$ by hypothesis, implying the contradiction that $|N(v)\cap\overline{A\cup B}\setminus(A\cup B)|=0$.

In other words, denoting $B'=\{v\in B:|N(v)\cap\overline{A\cup B}\setminus(A\cup B)|>|N(v)\cap(A\cup B)|\}$, we proved that every $v\in B'$ has infinite degree and $|N(v)\cap B\setminus A|=|B|$. In particular, $|N(v)\cap\overline{A\cup B}\setminus(A\cup B)|>|B|$. Therefore, applying Lemma 3.6.2, there is a disjoint family $\{N'(v)\}_{v\in B'}$ such that $N'(v)\subset N(v)\cap\overline{A\cup B}\setminus(A\cup B)$ and $|N'(v)|=|N(v)\cap\overline{A\cup B}\setminus(A\cup B)|$ for every $v\in B'$. Writing $\overline{A\cup B}=\bigcup_{\alpha<\Omega}\overline{A\cup B}^{\alpha}$ for some big enough ordinal Ω, the claimed coloring $\overline{c}:\overline{A\cup B}\to 3$ can be recursively described as follows:

- $\overline{c}(u)=c(u)$ for every $u\in A\cup B=\overline{A\cup B}^{0}$;
- Suppose that $\overline{c}|_{\overline{A\cup B}^{\beta}}$ is defined for every $\beta<\alpha$ (in a way that $\overline{c}|_{\overline{A\cup B}^{\beta}}$ extends $\overline{c}|_{\overline{A\cup B}^{\gamma}}$ if $\beta>\gamma$). Then, given $u\in\overline{A\cup B}^{\alpha}\setminus\bigcup_{\beta<\alpha}\overline{A\cup B}^{\beta}$, by equation (3.2) there is a color $j\in\{0,1,2\}$ such that
$$\left|N(u)\cap\bigcup_{\beta<\alpha}\overline{D}^{\beta}\cap\overline{c}^{-1}(j)\right|=d(u).$$
If $u\in N'(v)$ for some (unique) $v\in B'$, choose $\overline{c}(u)\in 3\setminus\{j,c(v)\}$. Otherwise, define $\overline{c}(u)\in 3\setminus\{i\}$ arbitrarily. In both cases, $\overline{c}$ is clearly unfriendly in v.

At the end of this recursive process, it is guaranteed that $\overline{c}:\overline{A\cup B}\to 3$ is unfriendly in vertices of $\overline{A\cup B}\setminus(A\cup B)$. Given $v\in B'$, we have $\overline{c}(u)\neq c(v)=\overline{c}(v)$ for every $u\in N'(v)$, by the second item above. Since $|N'(v)|=|N(v)\cap\overline{A\cup B}\setminus(A\cup B)|$, it follows that $\overline{c}$ is also unfriendly at v in the induced subgraph $G[\overline{A\cup B}]$.

□

On the next pages, we will follow the fifth section of (SHELAH; MILNER, 1990) in order to finally conclude that every uncountable graph G has an unfriendly $3-$partition. Before that, it is useful to sketch the proof and explain how the above definitions and results support the main idea:

1. First, we remark that the claimed partition c will be constructed recursively. Denoting by $A \subset V(G)$ its current *closed* domain, we might extend it for some vertex $x \in V(G) \setminus A$, hopefully being able to control its color. We will not extend c only to x, but also to an infinite set $B \subset V(G) \setminus A$ containing x. The size of B will be described based on how long we have been applying the algorithm that defines c. For example, consider the case in which B may be countable;

2. Then, we will close B under the operation of adding countably many uncolored neighbors. More precisely, if $v \in B$ has countably many neighbors outside A (finitely many or not), we add all them to B. If v has uncountably many neighbors outside A, we add only countably many of them for now. In both cases, this guarantees that A and B are under the hypothesis of Lemma 3.6.3. Therefore, we might extend c twice. First, by induction on $|B|$, we define c as an unfriendly $3-$partition for $G[A \cup B]$. After, Lemma 3.6.3 can extend it to $\overline{A \cup B}$, whose domain is closed in G once more;

3. Following this procedure, c is unfriendly in vertices of $\overline{A \cup B} \setminus (A \cup B)$, as well as in vertices of B having countable degree. However, c might not be unfriendly in a vertex $v \in B$ which have ω_1 neighbors in $V(G) \setminus (A \cup B)$, for example. Then, when further extending c, we must be aware of also labeling certain neighbors of v with colors from $3 \setminus \{c(v)\}$. Suspecting that $v \in B$ will not be unique for which this careful analysis will be needed, Lemma 3.6.2 is helpful for finding a suitable subset of $N(v)$ where we can define the required enemies.

The three items above draw a faithful overview for the first ω_1 steps of the algorithm that defines c. Formally, our aim now is to prove the inductive step below:

Theorem 3.6.4 (Shelah and Milner (1990), $\mathscr{P}_\mu$ statement)**.** *Fix $A, B \subset V(G)$ disjoint vertex subsets of a graph G such that A is closed and $A \cup B = V(G)$. Given $x \in B$, $i \in 3$ and $c : A \to 3$ a partial coloring, there is $\hat{c} : V(G) \to 3$ an extension which is unfriendly in every vertex of B and such that $\hat{c}(x) \neq i$.*

As suggested by the previous discussion, the proof of Theorem 3.6.4 is done by induction on $|B|$. Since A is closed, the base case $|B| = \omega$ is obtained by revisiting the proof of Proposition 3.6.1, considering that all the colorings defined there extends c to $G[B]$. In addition, the color for V_0 in the canonical decomposition of $G[B]$ is suitable chosen such that $\overline{c}(x) \neq i$. For example, $\overline{c}|_{V_0} \equiv i$ if $x \notin V_0$ and $\overline{c}|_{V_0} \equiv i \pm 1$ if $x \in V_0$.

Hence, we can assume that $|B|$ is uncountable, while another simplification is made by the remark below:

Lemma 3.6.5. *Let $A, B \subset V(G)$, $u \in B$, $i \in 3$ and $c : A \to 3$ be as in Theorem 3.6.4. If there is an infinite $B' \subset B$ such that $\overline{A \cup B'} = V(G)$ and $|B'| < |B|$, then there exists $\overline{c} : V(G) \to 3$ the claimed extension of c.*

Proof. Since $\overline{A \cup B'} = V(G)$, we have $\overline{A \cup D} = V(G)$ for any $D \subset B$ that also contains B'. Therefore, for each $v \in B'$ such that $|N(v) \setminus A| \leq |B'| < |B|$, we can assume that $N(v) \setminus A \subset B'$. Analogously, if $|N(v) \setminus A| > |B'|$, we can add $|B'|-$many elements of $N(v) \setminus A$ to B' in order to suppose that $|N(v) \cap B' \setminus A| = |B'|$. By induction, since $|B'| < |B|$, we observe that there is $c' : A \cup B' \to 3$ an extension of c which is unfriendly for every vertex of B' in $G[A \cup B']$. Moreover, assuming that $x \in B'$ without loss of generality, we have $c'(x) \neq i$.

On the other hand, by Lemma 3.6.3, there is $\bar{c} : V(G) \to 3$ an extension of c' that is unfriendly in every vertex of

$$(V(G) \setminus (A \cup B')) \cup \{v \in B' : |N(v) \setminus (A \cup B')| > |N(v) \cap (A \cup B')|\}.$$

Finally, if $v \in B'$ satisfies $|N(v) \setminus (A \cup B')| \leq |N(v) \cap (A \cup B')|$, we might consider two cases. If v has finite degree, then $N(v) \subset A \cup B'$ by the assumptions made in the first paragraph. Therefore, $\bar{c}$ is unfriendly in v because so is c'. By the same reason, $\bar{c}$ is unfriendly at v if it has infinite degree, since $d(v) = |N(v) \cap (A \cup B')|$ in this case. □

In other words, from now on we can assume the negative of the above result, i.e., that $\overline{A \cup B'} \subsetneq V(G)$ for every $B' \subset B$ with size strictly less than $|B|$. Denoting $\mu = |B|$, we fix $\preceq$ a well ordering for B of order type μ and consider the following recursive construction:

- Set $A_0 = A$. We choose $B_0 \subset B$ any infinite countable subset containing x_0;
- For $\alpha < \mu$, we suppose that $A_\beta \supset A$ and $B_\beta \subset B$ are already defined for every $\beta \leq \alpha$. We also assume that $|B_\beta| = \max\{|\beta|, \omega\}$ for every $\beta \leq \alpha$. Setting $A_{\alpha+1} = \overline{A_\alpha \cup B_\alpha}$, we suppose by induction that $A_{\alpha+1} = \overline{A \cup \bigcup_{\beta \leq \alpha} B_\beta}$. Hence, $|B \setminus A_{\alpha+1}| = |B|$: otherwise, $B' = (B \setminus A_{\alpha+1}) \cup \bigcup_{\beta \leq \alpha} B_\beta$ would have size less than B and would satisfy $\overline{A \cup B'} = V(G)$. Therefore, we can choose $B_{\alpha+1} \subset B \setminus A_{\alpha+1}$ any subset of size $|B_{\alpha+1}| = \max\{|\alpha+1|, \omega\}$ for which the conditions of Lemma 3.6.3 holds when applied to vertices of $\bigcup_{\beta \leq \alpha} B_\beta$. More precisely, for each $v \in \bigcup_{\beta \leq \alpha} B_\beta$, we can add enough vertices to $B_{\alpha+1}$ so that, for every $v \in \bigcup_{\beta \leq \alpha} B_\beta$, the following *construct conditions* holds:
  - ($\star$) If $|N(v) \setminus A_{\alpha+1}| \leq |B_{\alpha+1}|$, then $N(v) \subset A_{\alpha+1} \cup B_{\alpha+1}$. Moreover, we can assume that $B_{\alpha+1}$ contains a minimal element of $N(v) \setminus \overline{A_\alpha}$ (if there is some), regarding a previously fixed well ordering of $N(v)$ whose order type is $|N(v)|$;
  - ($\star$) If $|N(v) \setminus A_{\alpha+1}| > |B_{\alpha+1}|$, then $|N(v) \cap B_{\alpha+1} \setminus A_{\alpha+1}| = |B_{\alpha+1}|$.

  In addition, we suppose that $B_{\alpha+1}$ contains the $\preceq -$minimal element of $B \setminus A_{\alpha+1}$. Finally, we highlight that $A_{\alpha+1} \cap B_{\alpha+1} = \emptyset$;

- If α is a limit ordinal, we set $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and $B_\alpha = \emptyset$. Note that, if $v \in A_\alpha$ has finite degree, then $N(v) \subset A_\beta$ for any successor ordinal $\beta < \alpha$ such that $v \in A_\beta$. Finally, by the last paragraph of the above item, we have $A_\mu = V(G)$. Moreover, for $\gamma, \beta < \mu$, we have $B_\beta \cap B_\gamma = \emptyset$. After all, assuming that $\gamma < \beta$ for example, $B_\beta \subset V(G) \setminus A_{\gamma+1} = V(G) \setminus \overline{A_\gamma \cup B_\gamma}$.

For any cardinal $\kappa < \mu$, the set of vertices from $\bigcup_{\alpha<\kappa^+} B_\alpha$ that have (at least) κ^+ neighbors in $\bigcup_{\alpha<\kappa^+} B_\alpha$ will be denoted by Y_κ. Since $|B_\beta| = \beta$ for every infinite $\beta < \kappa^+$, it follows that $|Y_\kappa| \leq \kappa^+$. Moreover, for every vertex $v \in Y_\kappa$, the set $\{\kappa \leq \alpha < \kappa^+ : N(v) \cap B_\alpha \neq \emptyset\}$ contains κ^+ vertices, by the pigeonhole principle. Considering Y_κ as a set of indices, we will apply Lemma 3.6.2 to the family $\{\{\kappa \leq \alpha < \kappa^+ : N(v) \cap B_\alpha \neq \emptyset\}\}_{v \in Y_\kappa}$. Hence, there is $\{I_\kappa(v)\}_{v\in Y_\kappa}$ a disjoint family such that $I_\kappa(v) \subset \{\kappa \leq \alpha < \kappa^+ : N(v) \cap B_\alpha = \emptyset\}$ and $|I_\kappa(v)| = \kappa^+$ for every $v \in Y_\kappa$.

This definition suggests that we could, in a clever construction, label an enemy for $v \in Y_\kappa$ in every B_α with $\alpha \in I_\kappa(v)$. To this aim, for every $0 < \alpha < \mu$, fix $x_\alpha \in B_\alpha$ and, if $\alpha \in I_{|\alpha|}(v)$ for some (unique) $v \in Y_\kappa \cap \bigcup_{\beta<\alpha} B_\beta$, suppose even that x_α is a neighbor of v. For the case $\alpha = 0$, write $x_0 := x$. Then, the following result relies on Lemma 3.6.3 in order to construct a suitable increasing sequence of $3-$partitions:

Lemma 3.6.6. *There is $\{c_\alpha\}_{\alpha\leq\mu}$ a sequence of partially defined $3-$partitions such that:*

i) *$c_0 = c$ and $c_\alpha : A_\alpha \to 3$ for every $\alpha < \mu$. Moreover, $c_1(x) \neq i$ and $c_\alpha|_{A_\beta} = c_\beta$ if $\beta < \alpha$;*

ii) *$c_{\alpha+1}(x_{\alpha+1}) \neq c_{\alpha+1}(y)$ if α is not a limit ordinal and $\alpha + 1 \in I_{|\alpha+1|}(v)$ for some (unique) $v \in Y_{|\alpha+1|}$;*

iii) *For every $\alpha < \mu$, in the induced subgraph $G[A_{\alpha+1}]$, the coloring $c_{\alpha+1}$ is unfriendly in*

$$(A_{\alpha+1} \setminus (A_\alpha \cup B_\alpha)) \cup \{v \in B_\alpha : |N(v) \cap A_{\alpha+1} \setminus (A_\alpha \cup B_\alpha)| > |N(v) \cap (A_\alpha \cup B_\alpha)|\}.$$

Proof. The definition of c_0 is given by item *i*). If $\alpha > 0$ is a limit ordinal and we suppose that c_β satisfies the three conditions above for every $\beta < \alpha$, then so does the well-defined limit coloring $c_\alpha = \bigcup_{\beta<\alpha} c_\beta$. Now, suppose that $\alpha = \beta + 1$ for some ordinal β, so that the following cases arise:

- If β is 0 or also a successor ordinal, then A_β is closed by definition. Since $|A_\alpha| \leq |\alpha| < \mu$, by induction there is $c'_\alpha : A_\beta \cup B_\beta \to 3$ an extension of c_β which is unfriendly in every vertex of B_β within the subgraph $G[A_\beta \cup B_\beta]$. If $\beta = 0$, we can assume in addition that $c'_1(x) = c'_\alpha(x) \neq i$. Analogously, if $\beta > 0$, we suppose that $c'_\alpha(x) \neq c_\alpha(y)$ if $\alpha \in I_{|\alpha|}(v)$ for some (unique) $v \in Y_{|\alpha|}$. In both cases, by Lemma 3.6.3, there is $c_\alpha : A_\alpha \to 3$ an extension of c'_α for which item *iii*) is verified;

- If β is a limit ordinal, we observe that $A_\alpha = \overline{A_\beta}$, since $B_\beta = \emptyset$. In its turn, we can write A_β as the union $A_\beta = \left(A_\beta \setminus \bigcup_{\gamma<\beta} B_\gamma\right) \cup \left(\bigcup_{\gamma<\beta} B_\gamma\right)$, because $B_\gamma \subset A_{\gamma+1}$ for every $\gamma < \beta$. For a vertex $v \in \bigcup_{\gamma<\beta} B_\gamma$, if $\left|N(v) \setminus \left(A_\beta \setminus \bigcup_{\gamma<\beta} B_\gamma\right)\right| \le \left|\bigcup_{\gamma<\beta} B_\gamma\right| = \max\{|\beta|,\omega\}$, then $N(v) \setminus \left(A_\beta \setminus \bigcup_{\gamma<\beta} B_\gamma\right) \subset \bigcup_{\gamma<\beta} B_\gamma$. Otherwise, if $u \in N(v)$ but $u \notin \left(A_\beta \setminus \bigcup_{\gamma<\beta} B_\gamma\right)$ and $u \notin \bigcup_{\gamma<\beta} B_\gamma$, then $u \notin A_\beta$, which contradicts the construct conditions given by $(\star)$. Analogously, if $\left|N(v) \setminus \left(A_\beta \setminus \bigcup_{\gamma<\beta} B_\gamma\right)\right| \ge \left|\bigcup_{\gamma<\beta} B_\gamma\right| = \max\{|\beta|,\omega\}$, then
$$\left|\left(N(v) \cap \bigcup_{\gamma<\beta} B_\gamma\right) \setminus \left(A_\beta \setminus \bigcup_{\gamma<\beta} B_\gamma\right)\right| = \left|\bigcup_{\gamma<\beta} B_\gamma\right| = \max\{|\beta|,\omega\}.$$
  Therefore, by Lemma 3.6.3, there is $c_\alpha : \overline{A_\beta} \to 3$ an extension of c_β that verifies item *iii*).

$\square$

We will finish this section by showing that $\hat{c} = c_\mu$ satisfies the statement of Theorem 3.6.4. By item *i*) in the above lemma, we have $\hat{c}(x) \ne i$. Then, it only remains to prove that $\hat{c}$ is unfriendly in every vertex of B. If $v \in B \setminus \bigcup_{\beta<\mu} B_\beta$, then $v \in A_\alpha$ for some $\alpha > 0$. Choosing α minimum with that property, we must have $\alpha = \beta + 1$ for some ordinal β, as well as $v \in A_\alpha \setminus (A_\beta \cup B_\beta)$. Therefore, v lies on the closure of $A_\beta \cup B_\beta$, so that $|N(v)| = |N(v) \cap A_\alpha|$. It follows by item *iii*) of the above result that $\hat{c}$ is unfriendly in v.

Then, we will now suppose that $v \in \bigcup_{\alpha<\mu} B_\alpha$. In particular, $v \in B_\alpha$ for some (unique) successor ordinal α. Recalling that $A_\mu = V(G)$, fix β the first ordinal satisfying $|N(v) \cap A_\beta| = d(v)$. Note that $\alpha < \beta$, since $v \notin A_\alpha$ and A_α is closed. According to the construct conditions $(\star)$, v has infinite degree, because we would have $N(v) \subset A_\alpha \cup B_\alpha$ otherwise. The verification that $\hat{c}$ is unfriendly in v can be divided through the four cases below:

- Suppose that β is a successor ordinal, say $\beta = \gamma + 1$ for some $\alpha \le \gamma < \mu$, and that $d(v) = |N(v) \cap B_\gamma|$. We are done after proving that $\gamma = \alpha$, since $v \in B_\alpha$ and c'_α, as in the proof of Lemma 3.6.6, is unfriendly in v. But, in fact, $|N(v) \setminus A_\eta| > \max\{|\eta|,\omega\}$ if there were some $\alpha \le \eta < \gamma$: otherwise, $N(v) \subset A_\eta \cup B_\eta \subset A_\gamma$ by conditions $(\star)$, contradicting the choice of β. As a consequence,
$$|N(v) \cap A_\gamma| = \sup_{\alpha \le \eta < \gamma} |N(v) \cap A_\eta| \ge |\gamma| = |B_\gamma| \ge |N(v) \cap B_\gamma| = d(v),$$

which contradicts the minimality of β once more;

- Suppose that β is a successor ordinal, say $\beta = \gamma + 1$, but $d(v) > |N(v) \cap B_\gamma|$. By the minimality of β, we also have $d(v) > |N(v) \cap A_\gamma|$. Hence,

$$d(v) = |N(v) \cap A_\beta| = |N(v) \cap A_\beta \setminus (A_\gamma \cup B_\gamma)| > |N(v) \cap (A_\gamma \cup B_\gamma)|,$$

  proving that $\hat{c}$ is unfriendly in v because so is c_β, by property *iii*) of Lemma 3.6.6;

- Suppose that β is a limit ordinal and $d(v) \leq |\beta|$. Relying on the construct conditions $(\star)$ and on the minimality of β, we must have $|N(v) \setminus A_\gamma| > |B_\gamma| = |\gamma|$ for every successor $\gamma < \beta$. As a consequence, $|N(v) \cap B_\gamma \setminus A_\gamma| = |B_\gamma| = |\gamma|$. Hence, the minimality of β imposes now that $d(v) = |\beta| = \beta$. In particular, it follows that $v \in Y_{|\gamma|}$ for every $\alpha \leq \gamma < \beta$. Therefore,

$$|\{\alpha \leq \gamma < \beta : \gamma \in I_{|\gamma|}(v)\}| = \sup_{\alpha \leq \gamma < \beta} |I_{|\gamma|}(v)| = \sup_{\alpha \leq \gamma < \beta} |\gamma|^+ = \beta$$

  However, $\hat{c}(x_\gamma) = c_\gamma(x_\gamma) \neq \hat{c}(v)$ for every successor ordinal $\alpha \leq \gamma < \beta$ such that $\gamma \in I_{|\gamma|}(v)$, as guaranteed by item *ii*) Lemma 3.6.6. Hence, $\hat{c}$ is unfriendly in v;

- Finally, suppose that β is a limit ordinal such that $d(v) > |\beta|$. Since $d(v) = |N(v) \cap A_\beta| = \sup_{\gamma < \beta} |N(v) \cap A_\gamma|$, the degree of v is a singular cardinal in which $\{|N(v) \cap A_\gamma|\}_{\gamma < \beta}$ is a cofinal sequence. In particular, given a cardinal $\kappa < d(v)$, there is $\gamma < \beta$ such that $|N(v) \cap A_{\gamma+1}| > \kappa$. However, by the choice of β, we have $|N(v) \cap A_{\gamma+1} \setminus (A_\gamma \cup B_\gamma)| > |N(v) \cap (A_\gamma \cup B_\gamma)|$. Then, item *iii*) of Lemma 3.6.6 guarantees that

$$|\{u \in A_{\gamma+1} \cap N(v) : \hat{c}(u) = c_{\gamma+1}(u) \neq c_{\gamma+1}(v) = \hat{c}(v)\}| = |N(v) \cap A_{\gamma+1} \setminus (A_\gamma \cup B_\gamma)| > \kappa.$$

  Since $\kappa < d(v)$ was chosen arbitrarily, this proves that $|\{u \in N(v) \cap A_\beta : \hat{c}(u) \neq \hat{c}(v)\}| = d(v)$, concluding that $\hat{c}$ is unfriendly in v.

Hence, the details concerning Theorem 3.6.4 are now fulfilled. In particular, by considering $A = \emptyset$ in this result, we proved the main motivation for this section:

Theorem 3.6.7 (Shelah and Milner (1990), Theorem 4)**.** *Every graph has an unfriendly* $3-$*partition.*

CHAPTER 4

NEW INSTANCES

While Chapter 3 revisits the main literature concerning the unfriendly partition conjecture, in the next sections we will explore how some techniques already studied can be improved. In particular, most of the results to be presented in this chapter are original, which includes the recently published discussions in (AURICHI; REAL, 2023).

4.1 Introduction

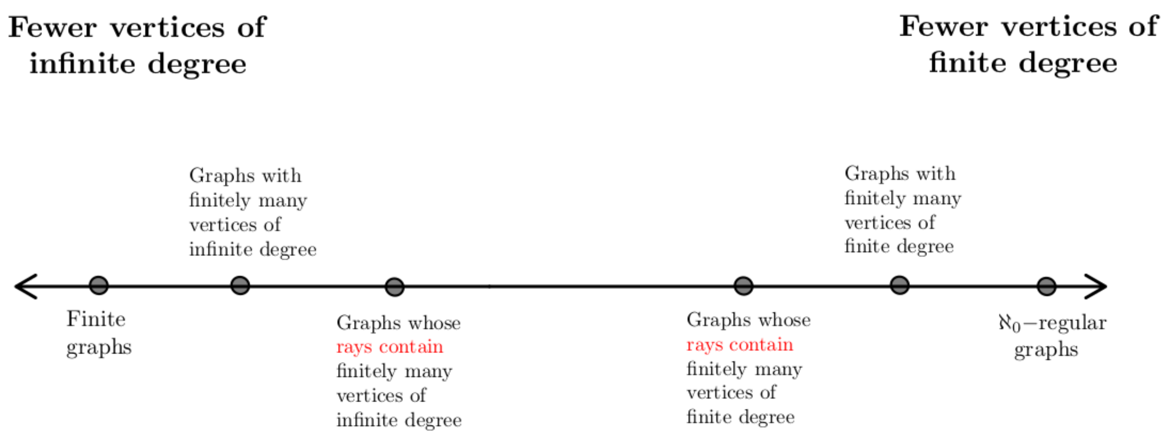
Although the unfriendly partition conjecture now states that every *countable* graph admits an unfriendly partitions, only simple observations in the literature make a strong use of that hypothesis. For example, as a particular case from Theorem 3.2.2, we easily check that every countable graph with finitely many vertices of finite degree has an unfriendly partition, relying on Lemma 3.3.1. Of course, the family of graphs studied in Section 3.4 shows that the countability condition of this remark cannot be simply dropped.

To summarize, compiling the results from the first three sections of Chapter 3, what is known about the unfriendly partition for countable graphs can be partitioned into two categories:

1. `Graphs with few vertices of finite degree`: countable graphs with finitely many vertices of finite degree have unfriendly partition. This can be proved by drawing a simple algorithm that attributes an enemy to every vertex of infinite degree;
2. `Graphs with few vertices of infinite degree`: graphs with finitely many vertices of infinite degree have unfriendly partitions. This is basically the statement of Theorem 3.2.2, that relies on the stronger notion of $F-$good colorings discussed by Section 3.2.

Clearly, the other positive results available in Section 3.5 restrict themselves to the countable case as well. In particular, as claimed by Theorem 3.5.7, we recall that every (not necessarily countable) rayless graph has an unfriendly partition. That work, actually, brings another possible meaning for the word "few" in the two observations above. In the next section, then, we will prove that every graph whose rays passes through only finitely many vertices of finite degree has an unfriendly partition. Dually, Section 4.3 verifies that every graph whose rays contains only finitely many vertices of infinite degree admits such a coloring. Therefore, the two categories above can be stretched as Figure 21 suggests.

Figure 21 – Criteria for the existence of unfriendly partitions in countable graphs



The known results related to the countable case of the unfriendly partition conjecture can be partitioned between graphs with few vertices of finite degree and graphs with few vertices of infinite degree.

Source: Elaborated by the author.

Finally, Section 4.4 is inspired by a natural question that arises after facing the uncountable counterexamples of Shelah and Milner (1990). As studied by Section 3.4, if we consider the constructions done only within ZFC, the more economical of these graphs has $(2^\omega)^{+\omega}$ vertices. When adding extra axioms for set theory, we can exhibit a graph with $\aleph_\omega$ vertices that has no unfriendly partition. In both cases, the cardinals $(2^\omega)^{+\omega}$ and $\aleph_\omega$ are far from being countable, although the graphs obtained have only vertices of infinite degree. Our paper (AURICHI; REAL, 2023), from where the last section of this chapter is extracted, verifies whether these amounts can be improved, even supposing that $2^\omega > \aleph_\omega$.

4.2 Few vertices of finite degree

In this dissertation, the main discussions regarding graphs with few vertices of finite degree were presented in Section 3.3. There, a remarkable statement is Theorem 3.2.2, obtained after adapting a greedy algorithm that assigns enemies to already colored vertices. This section works under a similar heuristic, but aiming to conclude the following criteria for finding unfriendly partitions in countable graphs:

Theorem 4.2.1. *Every countable graph whose rays pass through finitely many vertices of finite degree admits an unfriendly partition.*

The proof of the above result is supported by the canonical decomposition $(\mathscr{I}, \{V_\alpha\}_{\alpha<\Omega}, R)$ of a graph G, as introduced in Section 3.6. If G is countable, we recall that its residual graph $G[R]$ is either empty or $\aleph_0-$regular. In this latter case, by Lemma 3.3.1 for example, there is $c_R : R \to 2$ an unfriendly partition for $G[R]$. On the other hand, if G is under the main hypothesis of Theorem 4.2.1, then, by König's Lemma (2.1.1), its islands are finite graphs.

In an attempt to extend the coloring c_R to the remaining vertices, let us say that a subset $D \subset V(G)$ is **finitely covered** if, for every $I \in \mathscr{I}$ such that $I \cap D \neq \emptyset$, then $N(I) = \{u \in V(G) : uv \in E(G) \text{ for some } v \in I\} \subset D$.

Exercise 4.2.2. *Let G be a graph and suppose that $D \subset V(G)$ is closed, according to the definition presented right before Lemma 3.6.3. Show that D is finitely covered.*

We will thus define a partial ordering $\preceq$ over the set of pairs

$$\mathscr{D} = \{(D, c_D) | D \subset V(G) \text{ is finitely covered}, R \subset D, c_D : V(D) \to 2 \text{ is unfriendly in } D \text{ and extends } c_R\}$$

as follows: $(D, c_D) \preceq (D', c_{D'})$ if, and only if, $D \subset D'$ and $c_{D'}$ extends c_D. In particular, if $C \subset \mathscr{D}$ is a totally ordered subset, the pair $\left(\bigcup_{(D,c)\in C} D, \bigcup_{(D,c)\in C} c \right)$ is well-defined and belongs to $\mathscr{D}$. Therefore, by Zorn's Lemma, there is (D, c_D) a $\preceq -$maximal pair. Then, Theorem 4.2.1 follows if we argue that $D = V(G)$.

For a contradiction, suppose that $V(G) \setminus D \neq \emptyset$. In particular, we observe that there is an uncolored vertex of infinite degree. If not, by taking $I \in \mathscr{I}$ that has an uncolored vertex by c_D, any extension c_I of c_D to $D \cup I$, chosen so that $|\{uv \in E(G) : u \in I, c_I(u) \neq c_I(v)\}|$ attains maximum value, contradicts the choice of (D, c_D). Moreover, one can prove the following:

Lemma 4.2.3. *If $v \in V_\alpha \setminus D$ has infinite degree for some $\alpha > 0$, then there are infinitely many $\{v\} - V_0$ uncolored paths disjoint unless by v. More precisely, there is a collection $\{P_i\}_{i<\omega}$ of paths starting at v so that:*

- *$P_i \cap P_j = \{v\}$ if $i \neq j$;*
- *$P_i \subset V(G) \setminus D$ for every $i < \omega$;*
- *For every $i \in \mathbb{N}$, P_i contains precisely one vertex of V_0, that is its endpoint other than v.*

Proof. We will construct the family of paths claimed by the lemma by induction on α. To that aim, the maximality of (D, c_D) guarantees that v has infinitely many uncolored neighbors. Otherwise, we could extend c_D to v so that this vertex has infinitely many neighbors of color

$1 - c_{D\cup\{v\}}(v)$. In particular, if $\alpha = 1$, it is enough to consider $\{P_i\}_{i\in\mathbb{N}}$ as the paths of the form $\{vv_i\}_{i\in\mathbb{N}}$, where $\{v_i\}_{i\in\mathbb{N}}$ is an enumeration of the uncolored neighbors of v in V_0.

Suppose now that $\alpha > 1$ and let $v_0 \in V_{\beta_0} \setminus D$ be any neighbor of v with $0 < \beta_0 < \alpha$. By induction, fix P'_0 any uncolored $v_0 - \mathscr{I}$ path. Supposing that $P'_0, P'_1, \dots, P'_i$ is defined for some $i < \omega$, let $v_{i+1} \in V_{\beta_{i+1}} \setminus (D \cup P'_0 \cup P'_1 \cup \cdots \cup P'_i)$ be another neighbor of v, again chosen so that $\beta_{i+1} < \alpha$. By induction again, there is $P'_{i+1} \subset V_{\beta_{i+1}} \setminus (D \cup P'_0 \cup P'_1 \cup \cdots \cup P'_i)$ a $v_{i+1} - \mathscr{I}$ path. That procedure defines a family $\{P'_i\}_{i\in\mathbb{N}}$ of disjoint paths. By considering the concatenation $P_i = vP'_i$ for every $i < \omega$, the lemma follows. □

In particular, by Lemma 4.2.3, there is $v_0 \in V_0 \setminus D$ an uncolored vertex. Since D is finitely covered, no neighbor of finite degree of v_0 is colored. We claim that v_0 has a neighbor in an island $I_0 \in \mathscr{I}$ an island such that $N(I_0) \setminus (I_0 \cup D \cup \{v_0\}) \neq \emptyset$. For instance, suppose that this is not the case, i.e, $N(I) \setminus I \subset D \cup \{v_0\}$ for every island I in which v_0 has a neighbor. Denote by $\mathscr{I}_0$ the family of such islands. Then, by Corollary 3.2.8, there is c an extension of c_D to the graph $G[D \cup \{v\} \cup \bigcup \mathscr{I}_0]$ which is unfriendly in $\{v\} \cup \bigcup \mathscr{I}_0$. Noticing that $D \cup \{v\} \cup \bigcup \mathscr{I}_0$ is finitely covered, c contradicts the choice of (D, c_D).

Considering P_0 an edge between v_0 and some fixed neighbor in I_0, suppose that we have so far defined paths $P_0, P_1, \dots, P_n$ and islands $I_0, I_1, \dots, I_n$ such that:

- $P_i \cap P_j = \emptyset$ if $0 \leq i \leq n$ are distinct;
- For each $1 \leq i \leq n$, P_i is a path starting at u_i, an uncolored neighbor of some vertex in I_{i-1}, and ending in a vertex v_i of $N(I_i) \cap V_0$;
- $N(I_n) \setminus (D \cup P_0 \cup P_1 \cup \cdots \cup P_n) \neq \emptyset$.

By the third item above, there is $u_{n+1} \in N(I_n) \setminus (D \cup P_0 \cup P_1 \cup \cdots \cup P_n)$. Relying on Lemma 4.2.3, we fix P_{n+1} any $\{u_{n+1}\} - V_0$ uncolored path that is disjoint from $P_1, P_2, \dots, P_n$, denoting by $v_{n+1} \in V_0$ its endpoint other than u_{n+1}. Let $\mathscr{I}_{n+1} \subset \mathscr{I}$ be the collection of islands in which v_{n+1} has a neighbor. Since D is finitely covered, every member of $\mathscr{I}$ is uncolored by c_D.

Finishing a recursive construction, we claim that there is an island $I_{n+1} \in \mathscr{I}_{n+1}$ such that $N(I_{n+1}) \setminus (D \cup P_0 \cup P_1 \cup \cdots \cup P_{n+1}) \neq \emptyset$. For instance, suppose that this is not the case, i.e., $N(I) \setminus D \subset P_0 \cup P_1 \cup \cdots \cup P_{n+1}$. Therefore, by the pigeonhole principle, there is $A \subset P_0 \cup P_1 \cup \cdots \cup P_{n+1}$ satisfying $N(I) \setminus D = A$ for every island I in an infinite subfamily $\mathscr{I}' \subset \mathscr{I}_{n+1}$. In particular, $A \subset V_0$, or, in other words, every member of A has infinitely many neighbors of finite degree. Hence, by Corollary 3.2.8, c_D can be extended to an unfriendly coloring c over $G[D \cup A \cup \bigcup \mathscr{I}']$. By the choice of A, the set $D \cup A \cup \bigcup \mathscr{I}'$ is finitely covered, contradicting the maximality of (D, c_D).

Therefore, for each $n \in \mathbb{N}$, it is defined a path P_n between the vertices of infinite degree u_n and v_n. Moreover, u_n and v_{n-1} have neighbors in the island I_{n-1}. Let Q_{n-1} be a path between these two vertices of finite degree. Then, a concatenation $R = P_0Q_0P_1Q_1P_2Q_2\ldots$ defines a ray that passes through infinitely many vertices of finite degree, contradicting the main hypothesis over G. Therefore, we must have $D = V(G)$, proving Theorem 4.2.1.

Despite being mentioned at this very last stage of the proof, the assumption that the rays of G contains only finitely many vertices of finite degree is also used when, after applying König's Lemma, claiming that the islands of G are finite. By preserving only this latter property, we ask whether the following generalization of Theorem 4.2.1 holds:

Problem 4.2.4. *Let G be a countable graphs whose islands are finite. Does G admit an unfriendly partition?*

As a "toy result" for the above question, we finish this section by arguing that its answer is affirmative when we assume that all islands are singletons. This proof, closely similar to the one just presented for Theorem 4.2.1, also relies on a greedy algorithm:

Proposition 4.2.5. *Let G be a countable graph whose vertices of finite degree define an independent set, i.e., they are pairwise non-adjacent. Then, there is $c : V(G) \to 2$ an unfriendly partition.*

Proof. Consider the set

$$\mathscr{D} = \{(D,c) \mid c : D \to 2 \text{ is already unfriendly in } D\}$$

and endow it with the following partial order: $(D,c) \preceq (D',c')$ if, and only if, $D \subset D'$ and $c'|_D = c$. By Zorn's Lemma, there is $(D,c) \in \mathscr{D}$ a $\preceq$ −maximal element. As in the proof of Theorem 4.2.1, we will check that $D = V(G)$.

For instance, suppose that this is not the case. In particular, the arguments applied in the proof of Lemma 4.2.3 shows that there are vertices of finite degree in $V(G) \setminus D$. If $v \in V(G) \setminus D$ is a such a vertex, we denote $N_i(v) = N(v) \cap c^{-1}(i)$ for each color $i \in \{0,1\}$. By the maximality of (D,c), we must have $|N_i(v)| \leq \lfloor \frac{d(v)}{2} \rfloor$ for each i: otherwise, we could extend the pair (D,c) by setting $c(v) = 1-i$. Hence, for a set $F \subset V(G) \setminus D$ and a partial coloring $c_F : F \to 2$, we say that F (endowed with c_F) *fulfills* the vertex v if $|N_i(v)| + |N(v) \cap c_F^{-1}(i)| > \lfloor \frac{d(v)}{2} \rfloor$ for some $i \in \{0,1\}$. In this case, the extension of $c \cup c_F$ that colors v with $1-i$ is already unfriendly at this vertex. Thus, we even say that F fulfills v *with the fulfilling color* $1-i$. By the maximality of (D,c), the following claim holds:

`Claim`: Every finite set $F \subset V(G) \setminus D$ (with any coloring) of vertices of infinite degree fulfills only finitely many vertices of finite degree.

Proof of the claim: Suppose that a finite set $F \subset V(G) \setminus D$ of vertices of infinite degree fulfills an infinite set A of vertices of finite degree. Consider that F is minimal with that property. In this case, every $v \in F$ must have infinitely many neighbors in A: otherwise, $F \setminus \{v\}$ would also fulfills vertices in an infinite subset of A. Let $c_F : F \to 2$ denote the coloring for which F fulfills the vertices of A. Consider $\hat{c} : D \cup F \cup A \to 2$ the common extension of c and c_F given by $\hat{c}(v) = c_v$ for every $v \in A$, in which $c_v \in \{0,1\}$ is the color fulfilled in v by F. Then, $\hat{c}$ is already unfriendly in D because so is c. Moreover, $\hat{c}$ is already unfriendly in A by definition of fulfillment. Finally, if $\hat{c}$ is not unfriendly in a vertex $v \in F$, then $F \setminus \{v\}$ with the coloring $c_F|_{F\setminus\{v\}}$ fulfills the (infinitely many) vertices of $A \setminus \{u \in N(v) \cap A : \hat{c}(u) \neq \hat{c}(v)\}$, contradicting the minimality of F. Then, $\hat{c}$ is already unfriendly in the vertices of its domain, so that $(D \cup F \cup A, \hat{c})$ contradicts the maximality of (D,c) in $\mathscr{D}$. □

Relying on the above claim, we can adapt the greedy algorithm applied in Lemma 3.3.1. More precisely, we will construct a $\subseteq -$increasing sequence $\{D_n\}_{n\in\mathbb{N}}$ of finite subsets of $V(G) \setminus D$, together with a sequence of colorings $\{c_n : D \cup D_n \to 2\}_{n\in\mathbb{N}}$, as follows:

- We consider $D'_0 = \{v_0\} \subset V(G) \setminus D$ any singleton set and define $c'_0(v_0) = 0$. Let $c_0 : D \cup D_0 \to 2$ be the extension of c'_0 that labels the finitely many vertices of finite degree fulfilled by D'_0 with the fulfilling color. By the above claim, D_0 is finite;
- For each $n \in \mathbb{N}$, assume by induction that the vertices of finite degree fulfilled by D_n are contained in D_n as well. Enumerate the vertices of infinite degree in D_n as $\{v_0^n, v_1^n, \ldots, v_k^n\}$. Set $D_0^{n+1} = \emptyset$ and $c_0^{n+1} = c_n : D \cup D_0^{n+1} \to 2$. For some $0 \leq i \leq k$, suppose that it is defined a coloring $c_i^{n+1} : D \cup D_i^{n+1} \to 2$, where D_i^{n+1} does not fulfill any vertex of finite degree (regarding the coloring $c_i^{n+1}|_{D_i^{n+1}}$). Then, consider the following cases:
  - Suppose that v_i^n has a neighbor u of infinite degree in $V(G) \setminus (D \cup D_i^{n+1})$,. Write ${c_{i+1}^{n+1}}'$ for the assignment $u \mapsto 1 - c_n(v_i^n)$. Then, consider $c_{i+1}^{n+1} : D \cup D_{i+1}^{n+1} \to 2$ as the extension of c_i^{n+1} such that $c_{i+1}^{n+1}(u) = 1 - c_n(v_i^n)$ and that gives the fulfilling color to the finitely many vertices fulfilled by $D_i^{n+1} \cup \{u\}$ (regarding the coloring ${c_{i+1}^{n+1}}' \cup c_i^{n+1}$);
  - Suppose that v_i^n has a neighbor $u \in V(G) \setminus (D \cup D_i^{n+1})$ of finite degree. By induction on i, we can assume that D_i^{n+1} does not fulfill u, i.e., $|N_i(u)| + |N(u) \cap (c_i^{n+1})^{-1}(i)| \leq \lfloor \frac{d(v)}{2} \rfloor$ for each color $i \in \{0,1\}$. Then, consider ${c_{i+1}^{n+1}}' : \{u\} \cup (N(u) \setminus (D \cup D_i^{n+1})) \to 2$ the coloring given by:

$$
{c_{i+1}^{n+1}}'(x) = \begin{cases} 1 - c_n(v_i^n), & \text{if } x = u; \\ c_n(v_i^n), & \text{if } x \in N(u) \setminus (D \cup D_i^{n+1}) \end{cases}
$$

    Since D_i^{n+1} does not fulfill u, the coloring ${c_{i+1}^{n+1}}'$ is already unfriendly in this vertex, that has only neighbors of infinite degree. Let $c_{i+1}^{n+1} : D \cup D_{i+1}^{n+1} \to 2$ be the common extension of c_i^{n+1} and ${c_{i+1}^{n+1}}'$ that gives the fulfilling colors for the vertices fulfilled by $D_i^{n+1} \cup (N(u) \setminus (D \cup D_i^{n+1}))$. The above claim guarantees that the new set of colored vertices D_{i+1}^{n+1} is finite.

Then, consider $D_{n+1} = D^{n+1}_{k+1}$ and $c_{n+1} = c^{n+1}_{k+1}$.

At the end of this recursive process, it is defined a limit coloring $c' = \bigcup_{n\in\mathbb{N}} c_n$ over $D' = D \cup \bigcup_{n\in\mathbb{N}} D_n$. This coloring is already unfriendly at the vertices of finite degree, since $N(u) \subset D'$ for every $u \in \bigcup_{n\in\mathbb{N}} D_n$ of finite degree. Moreover, it is unfriendly in each vertex of infinite degree $v \in D' \setminus D$, since, for every big enough $n \in \mathbb{N}$, an enemy in $D_{n+1} \setminus D_n$ is assigned for v. However, it is now the pair (D', c') that contradicts the maximality of (D, c) in $\mathscr{D}$. Hence, we must have $D = V(G)$. $\square$

4.3 Few vertices of infinite degree

This section works in a dual direction of the previous one. While also studying countable graphs, we will suppose now that they have few vertices of infinite degree. Previously in this dissertation, this was also the setting for a sort of results presented by Section 3.2, some of them to be applied sooner. Considering that, following the diagram in Figure 21, the criteria below is a counterpart to Theorem 4.2.1. As we shall discuss at the end of this section, it could be directly obtained from the current literature concerning the unfriendly partition problem, but we take the opportunity to present an alternative proof:

Theorem 4.3.1. *Every countable graph whose rays pass through finitely many vertices of infinite degree admits an unfriendly partition.*

As a first simplification in order to conclude the above theorem, we will argue that it is enough to prove it for biconnected graphs. In fact, if G is a connected graph, its block graph $\check{G}$ as in Proposition 2.3.1 is a tree. We recall that the vertex set of $\check{G}$ is written as the disjoint union $A \cup \mathscr{B}$, where $A \subset V(G)$ is the set of cutvertices in G and $\mathscr{B}$ is the set of its blocks. Moreover, in $\check{G}$, there is no edge between two elements of A, neither edges between elements of B. Therefore, fixing a block $B_0 \in \mathscr{B}$ as a root for $\check{G}$, the family $\mathscr{B}$ is precisely the set of vertices of even height in the corresponding tree order $\leq$. Assuming that every graph in $\mathscr{B}$ admits an unfriendly partition, we define a coloring $c : V(G) \to 2$ recursively as follows:

- We set $c|_{B_0} : V(B_0) \to 2$ any unfriendly partition for $G[B_0]$;
- For some $n \in \mathbb{N}$, suppose that the coloring c is defined for every block of height $2n$ in $\check{G}$. Then, fix B an arbitrary block of height $2n+2$. Considering the tree structure of $\check{G}$, there is a unique block B' of height $2n$ which intersects B. Then, $B \cap B'$ consists in an unique cutvertex $a \in V(G)$. Hence, we consider $c|_B : B \to 2$ an unfriendly partition for $G[B]$ such that, after possibly switching all the colors, $c|_B(a) = c|_{B'}(a)$. This guarantees that c is still well defined after being extended to B.

The above construction of c certifies that this is indeed an unfriendly coloring in vertices of $V(G) \setminus A$. In fact, each $v \in V(G) \setminus A$ satisfies $\{v\} \cup N(v) \subset B$ for some block $B \in \mathscr{B}$, so that $c|_B$

is unfriendly in v. However, if $v \in A$ is a cutvertex, then its neighborhood is written as the disjoint union $\bigcup_{\substack{B\in\mathscr{B}\\ v\in B}} N(v)\cap B$. Denoting $c(v) = i$, we also have $|N(v)\cap B\cap c^{-1}(i)| \leq |N(v)\cap B\cap c^{-1}(1-i)|$ for every $B \in \mathscr{B}$ such that $v \in B$, since $c|_B$ is unfriendly at v in the induced subgraph $G[B]$. Therefore,

$$|N(v)\cap c^{-1}(i)| = \sum_{\substack{B\in\mathscr{B}\\ v\in B}} |N(v)\cap B\cap c^{-1}(i)| \leq \sum_{\substack{B\in\mathscr{B}\\ v\in B}} |N(v)\cap B\cap c^{-1}(1-i)| = |N(v)\cap c^{-1}(1-i)|,$$

where the above expression does not depend on the degree of v being finite. In other words, c is unfriendly in v.

If G is under the main hypothesis of Theorem 4.3.1, then its blocks also do not contain rays passing through infinitely many vertices of infinite degree. Hence, this previous discussion guarantees that Theorem 4.3.1 can be reduced to:

Lemma 4.3.2. *Let G be a countable biconnected graph. If the rays in G contains only finitely many vertices of infinite degree, then G admits an unfriendly partition.*

The proof of the above lemma is supported by the existence of a normal spanning tree T for G, as guaranteed by Theorem 2.2.2. If $\leq$ denotes its tree order, the main hypothesis of G certifies the well definition of the following hierarchy over its vertices of infinite degree:

- We say that a vertex $v \in V(G)$ of infinite degree has **rank** 1 if the subgraph induced by $\lfloor v\rfloor \setminus \{v\}$ is locally finite;
- For some $\alpha > 1$, and supposing that vertices of rank β are defined for every $\beta < \alpha$, we say that a vertex $v \in V(G)$ of infinite degree has **rank** α if $\lfloor v\rfloor \setminus \{v\}$ contains only vertices of finite degree or having rank strictly less than α.

We claim every vertex $v \in V(G)$ of infinite degree has a rank. If not, some $v_0 \in V(G)$ of infinite degree has no rank. Then, some $v_1 \in \lfloor v_0\rfloor \setminus \{v_0\}$ of infinite degree has no rank as well. In particular, $v_0 < v_1$. Inductively, therefore, we can construct a sequence $v_0 < v_1 < v_2 < \dots$ such that $v_i \in V(G)$ has infinite degree, but no rank, for every $i \in \mathbb{N}$. Being totally ordered by $\leq$, the set $\{v_n : n \in \mathbb{N}\}$ is then contained in some ray r of T. Since r is also a ray of G, this contradicts the main hypothesis over G in Lemma 4.3.2. Hence, a proof of this lemma can be done by induction on $\alpha = \sup\{\text{rank of } v : v \in V(G), d(v) = \omega\}$

$$\alpha = \begin{cases} 0, & \text{if } G \text{ is locally finite;} \\ \sup\{(\text{rank of } v) + 1 : v \in V(G), d(v) = \omega\}, & \text{otherwise.} \end{cases}$$

Proof of Lemma 4.3.2. If G is locally finite, then the existence of an unfriendly partition is due to Proposition 3.2.1. However, Proposition 3.2.6 can be applied to verify the following stronger property:

($\star$) Let K be any finite graph, disjoint of G, that is endowed with a coloring $c_K : V(K) \to 2$. Consider $G \oplus H$ some graph obtained by arbitrarily adding edges that connects vertices of K to vertices of G. Then, there is $c : V(G) \cup V(H) \to 2$ an extension of c_K which is unfriendly for vertices of $V(G)$ and $F-$good for every finite set $F \subset V(G)$ of vertices of finite degree.

Hence, we will prove the above statement, rather than the original thesis of Lemma 4.3.2, by induction on α. To that aim, if $\alpha > 0$, fix $A \subset V(G)$ the set comprising the $\leq -$minimal vertices of infinite degree in G. By definition, then, the elements of A are pairwise incomparable regarding $\leq$. Considering that, the core of the proof is given by the claim below:

`Claim:` A is finite.

Proof of the claim. For a contradiction, suppose that A is infinite. In particular, the root z of T must have finite degree: otherwise, $A = \{z\}$ by definition of A. Since T is normal and, for every $v \in A$, $\lceil v \rceil \setminus \{v\}$ is a finite set of vertices of finite degree, we also conclude that the subgraph of G induced by $\lceil A \rceil = \bigcup_{v \in A} \lceil v \rceil$ is locally finite. In particular, by the Star-Comb Lemma (2.1.2), there is a comb H in $T[\lceil A \rceil]$ whose teeth define an infinite subset $\{v_n\}_{n \in \mathbb{N}} \subset A$.

Denote by r the spine of H, which, without loss of generality, we suppose that starts at z. By definition of comb, for every $n \in \mathbb{N}$ there is a path $P_n \subset T$ connecting v_n to a vertex $u_n \in r$. Moreover, $P_n \cap P_m = \emptyset$ if $n \neq m$. We choose the ordering for $\{v_n\}_{n \in \mathbb{N}}$ so that $u_0 < u_1 < u_2 < \dots$. Unless by passing $\{v_n\}_{n \in \mathbb{N}}$ to a subsequence, we can also assume that, in G, no vertex of $\bigcup_{x \in P_{n+1} \setminus \{u_{n+1}\}} \lfloor x \rfloor$ has a neighbor in $\lceil u_n \rceil$. This is possible since $\lceil u_n \rceil$ is finite and every vertex of $\lceil u_n \rceil \subset \lceil v_n \rceil \setminus \{v_n\}$ has finite degree.

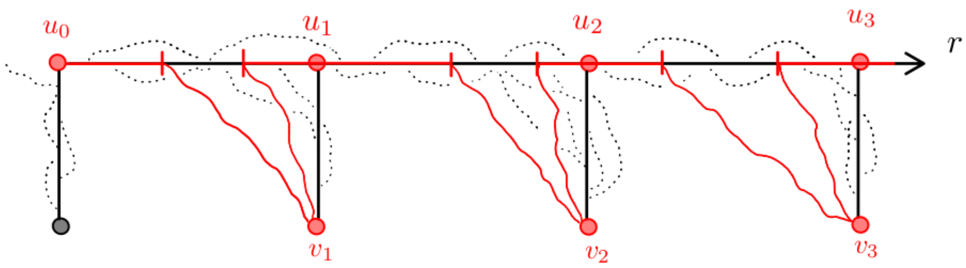
Once made the above simplifications, we are able to construct a ray in G containing the vertices from $\{v_n\}_{n \in \mathbb{N}}$, contradicting the main hypothesis of Lemma 4.3.2. To that aim, for each $n \geq 1$, let Q_n and Q_n' be two disjoint paths (in G) connecting v_n and z. These paths exist by the Erdős-Menger Theorem (2.3.3), since G is biconnected by hypothesis. Note that Q_n and Q_n' must intersect the segment in H connecting u_{n-1} and u_n, which we will denote by $[u_{n-1}, u_n]$. After all, T is a normal tree, but there is no edges between vertices of $\lceil u_{n-1} \rceil$ and $\bigcup_{x \in P_n \setminus \{u_n\}} \lfloor x \rfloor$. Informally, after starting at $v_n \in P_n$, both paths Q_n and Q_n' might contain some elements from $\bigcup_{x \in P_n \setminus \{u_n\}} \lfloor x \rfloor$, but need to intersect $[u_{n-1}, u_n]$ in order to reach z, that lies on $\lceil u_{n-1} \rceil$.

Therefore, presenting Q_n and Q_n' in terms of its vertices as $Q_n = v_n y_1 y_2 y_3 \dots y_k z$ and $Q_n' = v_n y_1' y_2' \dots y_l' z$, we can consider the instants of first intersection $i = \min\{1 \leq j \leq k : y_i \in [u_{n-1}, u_n]\}$ and $i' = \min\{1 \leq j \leq k : y_j' \in [u_{n-1}, u_n]\}$. Supposing that $y_i < y_{i'}'$ without loss of generality, we denote by $[u_{n-1}, y_i]$ and $[y_{i'}', u_n]$ the paths in H connecting u_{n-1} to y_i and $y_{i'}'$ to u_n, respectively. Then, the concatenation

$$R_n = [u_{n-1}, y_i] y_{i-1} y_{i-2} \dots y_1 v_n y_1' y_2' \dots y_{i'-1}' y_{i'}' [y_{i'}', u_n]$$

defines a $u_{n-1} - u_n$ path that contains a vertex of infinite degree. In its turn, proving the claim, the concatenation $R = R_1R_2R_3R_4\dots$ describes a ray that contains infinitely many vertices of infinite degree. $\square$

Figure 22 – A ray passing through infinitely many vertices of infinite degree



In red, we sketch the ray R constructed as in the claim within Lemma 4.3.2. The comb H is denoted by the straight lines. The dashed ones, on the other hand, suggest how edges of $G \setminus T$ are distributed while "respecting the tree order of" T, in an attempt to highlight that T is a normal tree.

Once established that A is finite, so is the set $\lceil A \rceil = \bigcup_{v\in A} \lceil v \rceil$. Then, consider the graph $G \oplus K$ as in statement $(\star)$, where K is any finite graph endowed with a coloring $c_K : V(K) \to 2$. Arbitrarily, we now extend c_K to a coloring $c_{K\cup\lceil A\rceil}$ over $K \cup \lceil A \rceil$.

Since A comprises all the $\leq -$minimal vertices of infinite degree in G, a connected component C of $G \setminus \lceil A \rceil$ is either a locally finite graph or has the form $G[\lfloor v \rfloor \setminus \{v\}]$ for some $v \in A$. In this latter case, the vertices of infinite degree in C have rank smaller than the rank of G. In both situations, we might see the subgraph of $G \oplus K$ induced by $C \cup K \cup \lceil A \rceil$ as a graph of the form $C \oplus (G[K \cup \lceil A \rceil])$. Then, there exists $\pi_C : V(C) \cup V(K) \cup \lceil A \rceil \to 2$ an extension of $c_{K\cup\lceil A\rceil}$ which is unfriendly in every vertex of $V(C)$ and $F-$good for every finite set $F \subset V(C)$ of vertices of finite degree. Considering that, define the coloring $c : V(G) \cup K \to 2$ as follows:

$$c(v) = \begin{cases} c_{K\cup\lceil A\rceil}(v), & \text{if } v \in V(K) \cup \lceil A \rceil; \\ \pi_C(v), & \text{if } v \in C \text{ for some connected component } C \text{ of } G \setminus \lceil A \rceil. \end{cases}$$

By construction, c is unfriendly in vertices lying on connected components of $G \setminus \lceil A \rceil$. Moreover, if $F \subset V(G) \setminus \lceil A \rceil$ is a finite set of vertices of finite degree, there are only finitely many such connected components $C_1, C_2, \dots, C_n$ which intersects F. Following the notation from Section 3.2, we have

$$a_{\pi_{C_i}}(F \cap C_i, (V(C_i) \cup V(K) \cup \lceil A \rceil) \setminus F \cap C_i) \geq b_{\pi_{C_i}}(F \cap C_i, (V(C_i) \cup V(K) \cup \lceil A \rceil) \setminus F \cap C_i),$$

because π_{C_i} is $F_i \cap C-$good for every $1 \leq i \leq n$. Since $F = \bigcup_{i=1}^{n} F \cap C_i$, we can sum the above inequalities over i and conclude that $a_c(F, V(G) \cup K \setminus F) \geq b_c(F, V(G) \cup K \setminus F)$. By Lemma 3.2.4, this proves that c is $F-$good for every finite set $F \subset V(G) \setminus \lceil A \rceil$ of vertices of finite degree. Since $\lceil A \rceil$ is finite, we can apply the Fixing Lemma (3.2.7) iteratively in order to obtain $H \subset V(G)$ a finite set such that $c * H$ is unfriendly in every vertex of $V(G)$, including those of $\lceil A \rceil$. In addition, H is chosen so that $c * H$ is $F-$good for every finite set $F \subset V(G)$ of vertices of finite degree, proving the statement $(\star)$. $\square$

In particular, the proof of Theorem 4.3.1 is concluded. However, for the sake of completeness, we remark that its statement could be obtained as a consequence of a deeper result in the literature. More precisely, in (BERGER, 2017) the following criteria is found:

Theorem 4.3.3 (Berger (2017))**.** *Every graph that does not contain a subdivision of an infinite clique as a subgraph admits an unfriendly partition.*

In the above theorem, we mean by **subdivision** of a graph G any graph that is constructed after subdividing edges from an arbitrary subset of $E(G)$. In its turn, *subdividing* an edge $e = uv$ of G consists in replacing e by a new path that connects u and v. In particular, any subdivision of an infinite complete graph K_ω contains a ray that passes through infinitely many vertices of infinite degree, from where Theorem 4.3.1 is deduced.

However, we were not able to explain Theorem 4.3.3 in details in this dissertation, justifying the study of the restricted case covered by Theorem 4.3.1. On the other hand, the construction of a ray R as in the proof of Lemma 4.3.2 suggests that some generalization is reachable:

Problem 4.3.4. *Let G be a countable graph that contains no alternating ray. Does G admit an unfriendly partition?*

In the above question, an **alternating ray** in a graph G is the designation we give for a ray that contains infinitely many vertices of finite degree and infinitely many vertices of infinite degree as well. The proof just presented for Theorem 4.3.1 does not fit for Problem 4.3.4 only because, under its weaker hypothesis, we cannot ensure that every vertex of infinite degree has a rank (as was argued right before the proof of Lemma 4.3.2).

However, the ray R sketched by Figure 22 is alternating, as well as the ray R constructed at the end of Section 4.2. There, neither the proof of Theorem 4.2.1 can be rewritten for solving the above problem. This because, when forbidding only alternating rays, König's Lemma guarantees no more that the islands of G are finite. Then, can an adaptation of both results, Theorems 4.2.1 and 4.3.1, answer positively Problem 4.3.4? In the affirmative case, this solution would complete the diagram of Figure 21 by drawing a new common generalization in its center.

4.4 A discussion about minimality

This section is extracted from the recently published paper (AURICHI; REAL, 2023), written with some set-theoretic vocabulary. However, if the results from Section 3.4 are assumed (even without their proofs), the text is quite accessible for all readers.

In that direction, we first recall that Theorem 3.4.1 constructs a graph of size $\aleph_\omega$ without unfriendly partitions, although relying on an extra axiom for set theory. Within ZFC, on the other hand, Theorem 3.4.4 is proven with similar ideas, but requiring $(2^\omega)^{+\omega}$ vertices. In any case, the literature concerning unfriendly partitions presents only the constructions of Shelah and Milner (1990) as examples of graphs that cannot be colored in an unfriendly way. Since both cardinals $\aleph_\omega$ and $(2^\omega)^{+\omega}$ are far from being countable, this section studies how economical are the counterexamples given by Theorems 3.4.1 and 3.4.4.

More precisely, from now on in this section, let us denote by $\varkappa$ the least cardinal for which there is a graph $G = (V,E)$ on $\varkappa$ vertices, all them of infinite degree, having no unfriendly partition. Then, the above discussions lead to the question of whether the equality $\varkappa = \aleph_\omega$ can be concluded within ZFC. More than that, we can analyze whether the statements $\varkappa = \aleph_\omega$ and $\varkappa = (2^\omega)^{+\omega}$ hold in models for ZFC $+\ \aleph_\omega < 2^\omega$, a theory in which the cardinals $\aleph_\omega$ and $(2^\omega)^{+\omega}$ are distinct. Rather surprisingly, we will prove the independence assertions below:

Theorem 4.4.1. *The following statements are independent from* ZFC $+\ \aleph_\omega < \mathfrak{c}$*:*

1. $\varkappa = \aleph_\omega$;
2. $\varkappa = (2^\omega)^{+\omega}$.

In particular, those statements are independent from the usual axioms of set theory.

In particular, by taking $\lambda = \omega$ in Theorem 3.4.4, it is straightforward that $\varkappa \leq (2^\omega)^{+\omega}$. In an attempt to discuss how sharp is this inequality, on the other hand, Theorem 3.3.2 easily provides a first lower bound for $\varkappa$:

Proposition 4.4.2. $\aleph_\omega \leq \varkappa$.

Proof. For instance, suppose that $\varkappa < \aleph_\omega$. Then, there is a graph $G = (V,E)$ with $|V| < \aleph_\omega$, whose vertices have infinite degree and that has no unfriendly partition. Fix $n \in \mathbb{N}$ so that $|V| = \aleph_n$. Since G has no vertices of finite degree, $|N(v)| \in \{\aleph_0, \aleph_1, \dots, \aleph_n\}$. Noticing that every successor cardinal is regular, G is under the hypothesis of Theorem 3.3.2. Therefore, contradicting its choice, G has an unfriendly partition. $\square$

Consistently, however, we have $\varkappa \leq \aleph_\omega$: this follows from Theorem 3.4.1 or from Theorem 3.4.4 when assuming the Continuum Hypothesis (CH), for example. On the other hand, as a less immediate conclusion, we will prove that the equality $\varkappa = \aleph_\omega$ is actually independent from ZFC $+\ \aleph_\omega < 2^\omega$, a theory in which the cardinals $\aleph_\omega$ and $(2^\omega)^{+\omega}$ are distinct. In particular, this statement is independent from the usual axioms of set theory.

First, we observe that Proposition 4.4.2 and Theorem 3.4.1 combined conclude that the statement $\varkappa = \aleph_\omega$ is consistent with ZFC $+ \aleph_\omega < 2^\omega$. In order to conclude that its negative is also consistent, we will introduce another extra axiom for set theory. Before that, fix $(\mathbb{P}, \leq)$ any partially ordered set. We call $D \subset \mathbb{P}$ a **dense** subset in $\mathbb{P}$ if, for every $p \in \mathbb{P}$, there is $d \in D$ such that $d \leq p$. Finally, we say that $\mathbb{P}$ satisfies the **countable chain condition** if every collection of pairwise incompatible elements of $\mathbb{P}$ is countable. In that terms, Martin's Axiom is the following statement:

Martin's Axiom (MA): Fix $(\mathbb{P}, \leq)$ a partially ordered set that satisfies the countable chain condition. If $\mathscr{D}$ is a family of dense subsets of $\mathbb{P}$ with $|\mathscr{D}| < 2^\omega$, then there exists a filter F of $\mathbb{P}$ such that $F \cap D \neq \emptyset$ for every $D \in \mathscr{D}$.

Convenient to our study, Martin's Axiom is independent from ZFC $+ \aleph_\omega < 2^\omega$. Moreover, in a theory where MA holds, 2^ω is a regular cardinal. Using some other properties that can be consulted in (JUST; WEESE, 1997), for example, we are ready to prove the following:

Proposition 4.4.3. *Suppose that Martin's Axiom holds. Then, every graph G having all its vertices of infinite degree and such that $|V(G)| < 2^\omega$ admits an unfriendly partition.*

Proof. We will apply Martin's Axiom to the Cohen forcing. More precisely, we will consider the partial order $\leq$ over $\mathbb{P} = \{c : D \to 2 \mid D \subset V(G) \text{ is finite}\}$ defined by: $c \leq c'$ if, and only if, c is an extension of c'. Let $\mathrm{dom}(c)$ be the domain of a partial coloring $c \in \mathbb{P}$. It is well know that $(\mathbb{P}, \leq)$ satisfies the countable chain condition. Moreover, by extending functions of finite domain accordingly, it is easily verified that the items below define dense sets in $\mathbb{P}$:

1. For each $v \in V(G)$, define $D_v = \{c \in \mathbb{P} : v \in \mathrm{dom}(c)\}$.
2. If $v \in V(G)$ is a vertex of regular degree κ_v, fix $\{v_\alpha\}_{\alpha < \kappa_v}$ an enumeration of its neighborhood. Now, for each $\alpha < \kappa_v$, define the set
$$R_v^\alpha = \{c \in \mathbb{P} : \text{ there are } \beta_0, \beta_1 > \alpha \text{ such that } c(v_{\beta_0}) = 0 \text{ and } c(v_{\beta_1}) = 1\}.$$
3. Similarly to the item above, if v is a vertex of singular degree κ_v, fix $\{v_\alpha\}_{\alpha < \kappa_v}$ an enumeration of its neighborhood and $\{\gamma_\xi\}_{\xi < cf(\kappa_v)}$ a cofinal sequence in κ_v of regular cardinals. Given $\alpha < \kappa_v$, let $\xi < cf(\kappa_v)$ be the index such that $\gamma_\xi \leq \alpha < \gamma_{\xi+1}$. Define then
$$S_v^\alpha = \{c \in \mathbb{P} : \text{ there are } \alpha < \beta_0, \beta_1 < \gamma_{\xi+1} \text{ such that } c(v_{\beta_0}) = 0 \text{ and } c(v_{\beta_1}) = 1\}.$$

Therefore, the sets $\mathscr{D} = \{D_v : v \in V(G)\}$, $\mathscr{R} = \{R_v^\alpha : v \in V(G) \text{ has regular degree } \kappa_v, \alpha < \kappa_v\}$ and $\mathscr{S} = \{S_v^\alpha : v \in V(G) \text{ has singular degree } \kappa_v, \alpha < \kappa_v\}$ are families of dense sets in $\mathbb{P}$. Since $|\mathscr{D}| = |V(G)| < 2^\omega$, $|\mathscr{R}| \leq |V(G)| \cdot |V(G)| = |V(G)| < 2^\omega$ and $|\mathscr{S}| \leq |V(G)| \cdot |V(G)| = |V(G)| < 2^\omega$, Martin's Axiom guarantees the existence of a filter $F \subset \mathbb{P}$ that intersects every dense set of $\mathscr{D} \cup \mathscr{R} \cup \mathscr{S}$.

We claim that the function $c = \bigcup_{f \in F} f$ is well-defined and that its domain is $V(G)$. In fact, for every $v \in V(G)$, once $F \cap D_v \neq \emptyset$, there is $f \in F$ with $v \in \mathrm{dom}(f)$. Moreover, if $g \in F$ is another coloring such that $v \in \mathrm{dom}(g)$, there is $h \in F$ satisfying $h \leq f, g$, because F is a filter. Hence, $f(v) = h(v) = g(v)$, concluding the well definition of c.

Now, we will verify that c is an unfriendly partition. For that, let $v \in V(G)$ be any vertex and denote by κ_v its degree. If κ_v is regular, let $\{v_\alpha\}_{\alpha<\kappa_v}$ be the enumeration of its neighborhood as fixed by the item 2 above. By the choice of F, for every $\alpha < \kappa_v$ there is $f \in F \cap R_v^\alpha$. Then, by definition of c, $c(v_{\beta_0}) = f(v_{\beta_0}) = 0$ and $c(v_{\beta_1}) = f(v_{\beta_1}) = 1$ for some ordinals $\beta_0, \beta_1 > \alpha$. This proves that $\sup\{\alpha < \kappa_v : c(v_\alpha) = 0\} = \sup\{\alpha < \kappa_v : c(v_\alpha) = 1\} = \kappa_v$. As κ_v is a regular cardinal, it follows that $|\{\alpha < \kappa_v : c(v_\alpha) = 0\}| = |\{\alpha < \kappa_v : c(v_\alpha) = 1\}| = \kappa_v$. In particular, c is unfriendly in v.

Finally, suppose that v is a singular cardinal. As done in the item 3 above, let $\{v_\alpha\}_{\alpha<\kappa_v}$ be an enumeration of its neighborhood and consider $\{\gamma_\xi\}_{\xi<cf(\kappa_v)}$ the cofinal sequence (of regular cardinals) fixed before. Hence, given $\xi < cf(\kappa_v)$ and $\gamma_\xi \leq \alpha < \gamma_{\xi+1}$, again the choice of F guarantees that there is $f \in F \cap S_v^\alpha$. This means that $c(v_{\beta_0}) = f(\beta_0) = 0$ and $c(v_{\beta_1}) = f(v_{\beta_1}) = 1$ for some $\alpha < \beta_0, \beta_1 < \gamma_{\xi+1}$. In other words, $\sup\{\gamma_\xi \leq \alpha < \gamma_{\xi+1} : c(v_\alpha) = 0\} = \sup\{\gamma_\xi \leq \alpha < \gamma_{\xi+1} : c(v_\alpha) = 1\} = \gamma_{\xi+1}$, implying that $|\{\gamma_\xi \leq \alpha < \gamma_{\xi+1} : c(v_\alpha) = 0\}| = |\{\gamma_\xi \leq \alpha < \gamma_{\xi+1} : c(v_\alpha) = 1\}| = \gamma_{\xi+1}$ by the fact that $\gamma_{\xi+1}$ is regular. Then, regarding the coloring c, we proved that v has at least $\gamma_{\xi+1}-$many neighbors of color 0 and at least $\gamma_{\xi+1}$ neighbors of color 1, for every $\xi < cf(\kappa_v)$. Therefore, it has $\sup\{\gamma_{\xi+1} : \xi < cf(\kappa_v)\} = \kappa_v$ neighbors of each color. In particular, c is unfriendly in v.

□

Exercise 4.4.4. *In the above proof, verify that the sets of the form D_v, R_v^α and S_v^α are, in fact, dense.*

Corollary 4.4.5. *The statement $\varkappa = \aleph_\omega$ is independent from* ZFC $+ \aleph_\omega < 2^\omega$.

Proof. We already argued that $\varkappa = \aleph_\omega$ is consistent with ZFC $+ \aleph_\omega < 2^\omega$. Verifying the consistency of its negative, we observe that $\varkappa \neq \aleph_\omega$ under ZFC $+$ MA $+ \aleph_\omega < 2^\omega$. In fact, if G is a graph without vertices of finite degree and $|V(G)| = \aleph_\omega < 2^\omega$, then, by Proposition 4.4.3, G admits an unfriendly partition. Therefore, $\varkappa \neq \aleph_\omega$ by the definition of $\varkappa$. □

On the other hand, Proposition 4.4.3 combined with Theorem 3.3.3 can prove that $(2^\omega)^{+\omega}$ vertices are needed to describe a graph with no unfriendly partitions whose vertices have infinite degree. Corollary 4.4.5 and the next result, then, prove Theorem 4.4.1:

Corollary 4.4.6. *$\varkappa$ is a singular cardinal. In particular, the statement $\varkappa = (2^\omega)^{+\omega}$ is independent from* ZFC $+ \aleph_\omega < \mathfrak{c}$.

Proof. For instance, suppose that $\varkappa$ is a regular cardinal. Then, if G is a connected graph whose vertices have degree less than $\varkappa$, all them of infinite degree, the first claim of the above proof shows that $|V(G)| < \varkappa$. By the minimality of $\varkappa$, there is an unfriendly partition for G. Hence, according to Theorem 3.3.3, there is also an unfriendly partition for every graph G such that $\aleph_0 \leq |N(v)| \leq \varkappa$ for each $v \in V(G)$. In particular, contradicting the definition of $\varkappa$, there is no graph with $\varkappa$ vertices, all them of infinite degree, that has no unfriendly partitions. Therefore, $\varkappa$ must be a singular cardinal.

In addition, the inequality $2^\omega \leq \varkappa$ holds in $\mathrm{ZFC} + \mathrm{MA} + \aleph_\omega < 2^\omega$, as guaranteed by Proposition 4.4.3. Since successor cardinals are regular and 2^ω is regular under MA, the above paragraph actually proves that $(2^\omega)^{+\omega} \leq \varkappa$. Note that this lower bound is sharp, because the construction of Milner and Shelah given by Theorem 3.4.4 has precisely $(2^\omega)^{+\omega}$ vertices. Then, the statement $\varkappa = (2^\omega)^{+\omega}$ is consistent with $\mathrm{ZFC} + \aleph_\omega < 2^\omega$.

On the other hand, the axioms used by Milner and Shelah to prove Theorem 3.4.1 are also consistent with the statement $\aleph_\omega < 2^\omega$. Hence, it is consistent with $\mathrm{ZFC} + \aleph_\omega < 2^\omega$ that there is a graph with less than $(2^\omega)^{+\omega}$ vertices, all of them of infinite degree, that does not admit an unfriendly partition. Thus, the statement $\varkappa \neq (2^\omega)^{+\omega}$ is also consistent with $\mathrm{ZFC} + \aleph_\omega < 2^\omega$.

Therefore, the equality $\varkappa = (2^\omega)^{+\omega}$ is independent from $\mathrm{ZFC} + \aleph_\omega < 2^\omega$. □

The fact that $\varkappa$ is a singular cardinal was already suggested by Theorems 3.4.1 and 3.4.4, because both uncountable cardinals $\aleph_\omega$ and $(2^\omega)^{+\omega}$ have cofinality ω. Moreover, this countable cofinality, in fact, played an important role for the constructions given by Milner and Shelah. Then, we finish this section by asking if this property is unavoidable:

Problem 4.4.7. *Can we prove, within* ZFC*, that $\varkappa$ has countable cofinality?*

Part II

Topological approach for infinite graphs

CHAPTER

5

THE END SPACE

Inspired by the contributions of the Hamburg group [1] this chapter discuss their topological approach for infinite graph theory. The key object to be studied, the *ends* of a graph, was actually introduced by Halin (1964/65), but most improvements in the area were obtained in the past twenty years. The literature on this subject is quite extensive, compiling papers from several authors, but many first statements can be actually redrawn by more modern tools.

In Sections 5.2 and 5.3, for example, we point out the main properties of the spaces $|G|$ and $\Omega(G)$, to be sooner defined. The results presented there were obtained still in the 90s, but the recent work of Kurkofka, Melcher and Pitz (2021), as we shall see, unified most proofs. Nevertheless, not every claim in this chapter will be completely detailed: sounding like Section 1.1 suggests, we tend to prove only the most important statements or those that motivate the studies in Chapter 6, where new instances are introduced. Finally, despite the topological flavour of the next discussions, we remark that Section 5.5 outlines some combinatorial applications of an end structure.

5.1 Introduction

The topological approach for the study of infinite graphs arise when trying to formalize the idea of directions on them. Intuitively, for a fixed graph G, this notion is brought by its rays. However, regarding their connectedness, some distinct rays might have a similar behaviour. This inspires the following equivalence relation over the set $\mathscr{R}(G)$ of rays in G: for every pair $r,s \in \mathscr{R}(G)$, we write $r \sim s$ whenever r and s are *infinitely connected*. In its turn, the expression "infinitely connected" means any of the following equivalent conditions:

i) There exists an infinite family of disjoint $r-s$ paths;

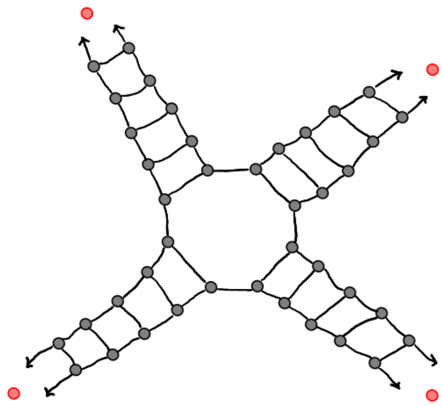
ii) There exists a ray $t \in \mathscr{R}(G)$ such that $r \cap t$ and $s \cap t$ are infinite;

[1] See <https://www.math.uni-hamburg.de/spag/dm/projects/topgrth.html>.

iii) No finite set $S \subset V(G)$ **separates** r and s, i.e., r and s have its tails in the same connected component of $G \setminus S$.

In particular, it is easily verified that $\sim$ is indeed an equivalence relation over $\mathscr{R}(G)$. The corresponding equivalence classes are called the **ends** of G, of which consists its **end space** $\Omega(G) = \mathscr{R}(G)/\sim$. Given a ray $r \in \mathscr{R}(G)$, we write $[r]$ when denoting its end. This definition is back to Halin (1964/65), but similar notions were developed by Hopf (1943) and Freudenthal (1942) in order to approach representation problems in group theory.

Figure 23 – A locally finite graph and its four ends



The ends of this locally finite graph are highlighted by the red circles.

Source: Elaborated by the author.

As Figure 23 suggests, we may understand $[r]$ as a limit point of the graph at which r aims. This convergence notion, however, requires a topology to be formalized. Then, we will now describe a topological space $|G|$ (together with its point set) as follows:

- An edge uv is identified with the unit segment $[0,1]$ and its usual topology. The identification may maps, say, u to 0 and v to 1, so we write $uv = [u,v]$. For any $0 < \varepsilon < 1$, then, we denote by $[u,\varepsilon)$ the open segment in uv starting at u whose length is ε;
- A vertex $v \in V(G)$ belongs to $|G|$. A basic open set around it has the form

$$V_\varepsilon(v) := \bigcup_{u \in N(v)} [v,\varepsilon)\,. \tag{5.1}$$

  for some $\varepsilon > 0$, as sketched by Figure 24. In particular, this union is disjoint unless by v;
- An end $[r] \in \Omega(G)$ also belongs to $|G|$. A basic open set around it has a finite set $S \subset V(G)$ and some $\varepsilon > 0$ as parameters. The connected component of $G \setminus S$ that contains a tail of r is denoted by $C(S,[r])$, while the set of edges with an endpoint in S and the other in $C(S,[r])$ is written as $E(S,[r])$. Finally, we define

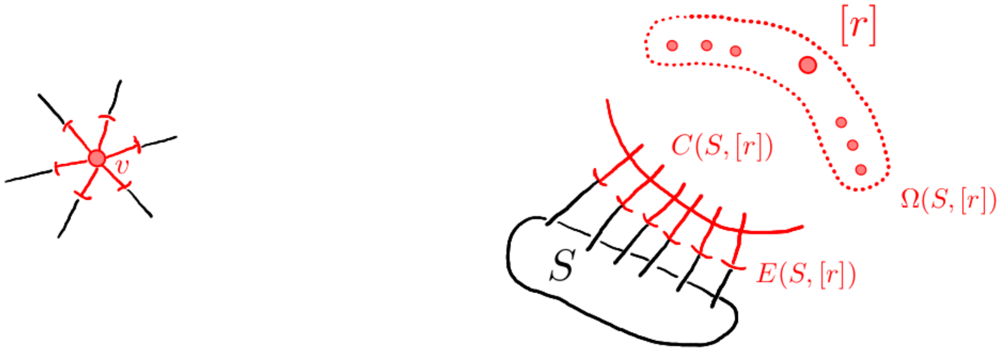
$$\Omega(S,[r]) = \{[s] \in \Omega(G) : S \text{ does not separate } s \text{ and } r\}. \tag{5.2}$$

  Hence, the claimed open neighborhood around $[r]$ is described by

$$\hat{C}(S,[r],\varepsilon) := C(S,[r]) \cup \Omega(S,[r]) \cup \bigcup_{\substack{u \in C(S,[r]) \\ uv \in E(S,[r])}} [u,\varepsilon) \tag{5.3}$$

In this description, $C(S,[r])$ contains the vertices of the corresponding connected component of $G\setminus S$ as well as its edges, which are under the identification made in the first item.

Figure 24 – Basic open sets of $|G|$



At the left, in red, we sketch a basic open neighborhood for a vertex v in $|G|$. At the right, in its turn, we present a basic open neighborhood for an end $[r]$.

Source: Elaborated by the author.

Exercise 5.1.1. *Consider G the graph from Figure 23. Suppose that it is drawn in an open bounded subset of $\mathbb{R}^2$ precisely as in the figure (where vertices are points and edges are segments). Let $X \subset \mathbb{R}^2$ be the set given by this representation of G. Show that $|G|$ turns out to be $\overline{X}$ with its subspace topology.*

This definition for $|G|$ is motivated by combinatorial issues, being broadly used when extending classical theorems from finite graph theory to locally finite graphs. The interested reader can find examples of these applications in the survey of Diestel (2010). Despite that, Section 5.5 also discuss the role that ends play in some generalizations of Menger's Theorem and other near results regarding connectivity properties.

However, topologists might be more interested in the structure of the *end space* $\Omega(G) \subset |G|$ with its inherited topology. In fact, when G is a tree, it well known that $\Omega(G)$ (called its *branch space* in set-theoretic contexts) is an ultrametric space and conversely: every complete ultrametric space is the end space of some tree. This is the core of Propositions 5.3.2 and 5.4.7, to be presented throughout the next sections, but categorical discussions regarding this equivalence can be consulted in (HUGHES, 2004) or in the master's dissertation of Boska (2021). On the other hand, a similar characterization of arbitrary end spaces concerns Problem 5.1 of Diestel (1992):

Characterization of end spaces: *Which topological spaces can be represented as $\Omega(G)$ for some graph G?*

Although Diestel's question was formally stated in 1992, other older conjectures, specially from Halin (1964/65), could provide its answer. Hence, the end structure of graphs has been studied for more than fifty years. Only recently, Pitz (2023) compiled a sort of topological conditions that precisely describe the family of end spaces of graphs, solving the above problem. An overview of his answer, focusing on the main tools employed, is given by Section 5.4.

Before that, the next sections discuss the main properties obtained by the early literature that approached Diestel's question. For instance, Section 5.2 deals with covering hypothesis in end spaces, while Section 5.3 gives a broad description for the metric case. In particular, we will revisit the main result of (KURKOFKA; MELCHER; PITZ, 2021), which, in both studies, also obtain conclusions for the global space $|G|$. Throughout the proofs to be presented, we will often rely on the observations below, possibly without previous mention. Intuitively, they explain how the definition of $|G|$ formalizes the idea that a ray converges to its end:

Lemma 5.1.2. *Let G be a connected graph. Then, the statements below hold:*

- *Given $A \subset V(G)$ and $[r] \in \Omega(G)$, we have $[r] \in \overline{A}$ if, and only if, there is a comb in G whose teeth belong to A and whose spine is (a ray equivalent to) r;*
- *For a ray r, we have $\overline{V(r)} \cap \Omega(G) = \{[r]\}$;*
- *If $S \subset V(G)$ is a finite set and r is a ray, we have $\overline{C(S,[r])} = C(S,[r]) \cup \Omega(S,[r])$.*

Proof. In order to prove the first item, suppose first that there is a comb C whose teeth belong to A. Let r denote its spine. Observe that every finite set $S \subset V(G)$ intersects only finitely many of the infinite disjoint paths that connects r to its teeth. Then, in the connected component $C(S,[r])$ there must lie infinitely many vertices from A as well, proving that $[r] \in \overline{A}$.

Conversely, assume that $[r] \in \overline{A}$. Let P_0 be a path connecting the ray r to some vertex $v_0 \in A$. Suppose that finitely many disjoint paths $P_0, P_1, P_2, \dots, P_n$ are defined, assuming that every P_i connects r to some vertex $v_i \in A$. The set $S = V(P_0) \cup V(P_1) \cup V(P_2) \cup \dots \cup V(P_n)$ is also finite. Hence, since $[r] \in \overline{A}$, we fix $v_{n+1} \in C(S,[r])$. Let P_{n+1} be a path connecting a tail of r to v_{n+1} in $C(S,[r])$. By the choice of S, we clearly have $P_{n+1} \cap P_i = \emptyset$ for every $0 \leq i \leq n$. At the end of this recursive definition, the comb defined by the vertex set $r \cup \bigcup_{n=0}^{\infty} P_n$ has r as spine and $\{v_n\}_{n \in \mathbb{N}} \subset A$ as teeth.

In order to prove the second item, fix $[s] \in \Omega(G) \cap \overline{V(r)}$ for some ray r. According to the item just proven, s is the spine of a comb whose teeth belong to $V(r)$. This means that there is an infinite family of disjoint paths connecting r and s, so that $[s] = [r]$.

Aiming to prove the third item, fix $S \subset V(G)$ a finite set and $r \in \mathscr{R}(G)$ a ray. We clearly have $C(S,[r]) \subset \overline{C(S,[r])}$, while $\Omega(S,[r]) \subset \overline{C(S,[r])}$ follows from the item just verified. Conversely, if $x \in \overline{C(S,[r])}$ is a vertex of G or an inner point of one of its edges, we must have $x \in C(S,[r])$ by definition of its systems of open neighborhoods. Finally, if $x = [s]$ for some ray

s, the first item proven shows that s is the spine of a comb whose teeth belong to $C(S,[r])$. In particular, in $G \setminus S$, a tail of s is contained in $C(S,[r])$ as well. This means that $[s] \in \Omega(S,[r])$ by (5.2). □

Exercise 5.1.3. *Show that, for every graph G, the spaces $|G|$ and $\Omega(G)$ are Hausdorff and Fréchet-Urysohn.*

Finally, we observe that many results to be presented in the next sections could be simplified for locally finite (connected) graphs. For example, Proposition 8.6.1 in the book (DIESTEL, 2018) contains a short proof for the fact that $|G|$ is compact in this case. Indeed, $|G|$ and $\Omega(G)$ are rather familiar spaces when G is locally finite: the former is the Freudenthal compactification of G when it is seen as an unidimensional complex, while the latter is a compact and complete ultrametric space. However,

> for arbitrary G, the spaces $|G|$ and $\Omega(G)$ are usually non-compact and far from being completely understood (KURKOFKA; MELCHER; PITZ, 2021, p.174).

5.2 Covering properties

We first remark that the notion of directions brought by the end structure can be understood in an alternative (but equivalent) way. For example, in their study of a pursuit-evasion game in an infinite graph G, Robertson, Seymour and Thomas (1991) relied on the following definition[2]: we say that a **direction** in G is a function f of the form

$$f : \{\text{finite subsets of } V(G)\} \to \{\text{connected subgraphs of } G\}$$

such that $f(S)$ is a (non-empty) connected component of $G \setminus S$ for every finite $S \subset V(G)$ and, if $S' \supseteq S$ is also finite, then $f(S') \subseteq f(S)$. In particular, $f(S)$ is always infinite: after all, for any other finite set $S' \subset V(G)$, we must have $\emptyset \neq f(S' \cup S) \subset f(S)$.

For example, when an end $[r] \in \Omega(G)$ is fixed, the map $C(\cdot,[r])$ defines a direction. In fact, for every finite $S \subset V(G)$, the subgraph $C(S,[r])$ is a connected component of $G \setminus S$ by definition, being the one which contains a tail of r. Hence, given $S' \supseteq S$ also finite, the tail of r in $C(S',[r])$ is contained in $C(S,[r])$, so that $C(S',[r]) \subset C(S,[r])$. Curiously, every direction of G can be described this way:

Proposition 5.2.1 (Robertson, Seymour and Thomas (1991), 2.5)**.** *Let f be a direction in an infinite graph G. Then, there is a ray $r \in \mathscr{R}(G)$ such that $f = C(\cdot,[r])$.*

[2] Originally, in (ROBERTSON; SEYMOUR; THOMAS, 1991), this notion of direction is called an "$\aleph_0$–haven".

Proof, following Theorem 2.2 of Diestel and Kühn (2003). Consider the set

$$S^* = \{v \in V(G) : v \text{ has a neighbor in } f(S) \text{ for every finite } S \subset V(G)\}.$$

Suppose first that S^* is infinite. Then, declaring P_0 as a path containing only a single vertex $v_0 \in V_0$, consider the following recursive definition:

- If P_n is a defined path, which connects v_0 to some vertex $v_n \in S^*$, let $u_n \in f(P_n)$ be one of its neighbors. Its existence follows from the definition of S^*, as well as from the fact that $S^* \setminus P_n \subset f(P_n)$. Then, consider P_{n+1} as a path in $f(P_n)$ which connects u_n to some $v_{n+1} \in S^*$.

At the end of this inductive proccess, it follows that the concatenation $r = P_0P_1P_2\dots$ defines a ray in G containing $\{v_n\}_{n\in\mathbb{N}}$. In order to show that $f = C(\cdot, [r])$, fix any finite set $S \subset V(G)$. By definition of S^*, the vertex v_n has a neighbor in $f(S)$ for every $n \in \mathbb{N}$, meaning that $v_n \in f(S)$ if $n > \max\{i \in \mathbb{N} : v_i \in S\}$. Therefore, $f(S) = C(S, [r])$, since $f(S)$ contains a tail of r in $G \setminus S$.

On the other hand, assume now that S^* is empty. Then, define $S_0 = \{v\}$ for an arbitrary $v \notin V(G)$. For some $n \in \mathbb{N}$, suppose that $S_n \subset V(G)$ is a finite non-empty set already defined. For every $s \in S_n$, consider a finite set $S_s \subset V(G)$ such that s has no neighbor in $f(S_s)$. Note that S_s exists since $S^* = \emptyset$. Writing $U_n = \bigcup_{s\in S_n} S_s$, define

$$S_{n+1} = \{s \in U_n : s \text{ has a neighbor in } f(U_n)\},$$

which has the properties below:

- $f(S_{n+1}) = f(U_n)$. In fact, $f(U_n) \subseteq f(S_{n+1})$ because $S_{n+1} \subset U_n$ and f is a direction. By its definition, however, S_{n+1} comprises all the vertices of U_n that have neighbors in $f(U_n)$. Hence, since $f(S_{n+1})$ is the connected component of $G \setminus S_{n+1}$ containing $f(U_n)$, we must have the equality $f(S_{n+1}) = f(U_n)$;
- $S_n \cap (S_{n+1} \cup f(S_{n+1})) = \emptyset$. This because, given $s \in S_n$, this vertex has no neighbor in $f(S_s)$. In particular, s has no neighbor in $f(U_n) = f(S_{n+1})$, since $S_s \subset U_n$ and f is a direction (so that $f(U_n) \subset f(S_s)$). In particular, $s \notin S_{n+1}$ and $s \notin f(S_{n+1})$;
- $f(S_n)$ contains both S_{n+1} and $f(S_{n+1})$. In fact, vertices of S_{n+1} are neighbors of some vertices of $f(S_{n+1}) = f(U_n)$, meaning that $G_{n+1} := G[S_{n+1} \cup f(S_{n+1})]$ is a connected subgraph of G. Since G_n does not intersect S_n by the previous item, it follows that G_{n+1} is contained in some connected component of $G \setminus S_n$. As a property of directions, it is already known that $f(S_n) \cap f(S_{n+1}) \neq \emptyset$. Therefore, we must have $V(G_n) \subset f(S_n \setminus S^*)$.

At the end of this recursive definition, we claim that $\bigcap_{n\geq 1} V(G_n) = \emptyset$. For instance, suppose that there is a vertex u in this intersection. Once $V(G_{n+1}) \subset f(S_n)$, we actually have $u \in \bigcap_{n\in\mathbb{N}} f(S_n)$.

Since G is connected, there is a path P connecting $v \in S_0$ to u. Fixing $i = \max\{n \in \mathbb{N} : P \cap S_n \neq \emptyset\}$, a subpath of P connects a vertex of S_i to u, that also belongs to $f(S_{i+1})$. Then, $P \cap S_{i+1} \neq \emptyset$, because S_i and $f(S_{i+1})$ are contained in distinct connected components of $G \setminus S_{i+1}$ by the above second item. This contradicts the choice of i, concluding that $\bigcap_{n \geq 1} V(G_n) = \emptyset$.

Now, fix $u_n \in S_n$ for every $n \in \mathbb{N}$. Suppose that there is a star centered at some vertex $z \in V(G)$ whose leaves belong to $\{u_n\}_{n \in \mathbb{N}}$. Since $S^* = \emptyset$, there must exist a finite set $S \subset V(G)$ such that z has no neighbor in $f(S)$. Let n be big enough so that $S_m \cap S = \emptyset$ if $m \geq n$. Observing that $f(S_m) \cap f(S) \neq \emptyset$, because f is a direction, we have $f(S_m) \subset f(S) \cup S$. Hence, $\{u_m\}_{m \geq n} \subset f(S)$. Then, the infinitely many paths connecting z to $\{u_m\}_{m \geq n}$ must intersect the finite set S, contradicting the fact that they are disjoint unless by z.

Therefore, by the Star-Comb Lemma (2.1.2), there is a comb C in G whose teeth belong to $\{u_n\}_{n \in \mathbb{N}}$. Let r denote its spine. If $S \subset V(G)$ is any finite set, we have $\{u_m\}_{m \geq n} \subset f(S)$ for some big enough $n \in \mathbb{N}$, as in the above paragraph. Hence, $f(S)$ contains a tail of r as well: otherwise, the infinitely many disjoint paths connecting r to $\{u_m\}_{m \geq n}$ would intersect the finite set S. Hence, $f(S) = C(S, [r])$, proving that $f = C(\cdot, [r])$.

Finally, consider the case in which S^* is finite. Then, in the infinite connected graph $G_0 = f(S^*)$ we define the direction f' given by $f'(S) = f(S \cup S^*)$ for every finite $S \subset f(S^*)$. In this case, the set

$$S_0^* = \{v \in f(S^*) : v \text{ has a neighbor in } f'(S) \text{ for every finite } S \subset V(G)\}$$

is empty. This because, if $v \in S_0^*$ and $S \subset V(G)$ is finite, then v has a neighbor in $f(S \cup S^*) \subseteq f(S)$, proving that $v \in S^*$. By the case just analyzed, therefore, there is r a ray in G_0 such that, for every finite $S \subset V(G)$, the subgraph $f'(S)$ is the connected component of $G_0 \setminus S$ containing a tail of r. Hence, $f(S) \supseteq f(S \cup S^*) = f'(S)$ contains also a tail of r, concluding that $f(S) = C(S, [r])$. In other words, $f = C(\cdot, [r])$. □

Exercise 5.2.2. *Find a graph G and a direction f on it for which S^*, as defined in the above proof, is not empty.*

Although the above definition of directions in infinite graphs is a more abstract notion than the one given by the ends, it is useful for concluding covering properties. Below, for example, we give a criteria for verifying whether $\Omega(G)$ is a compact topological space. As a consequence, we can also extend this result for spaces of the form $|G|$:

Lemma 5.2.3 (Diestel (2006), Theorem 4.1)**.** *Let G be any graph. Then, its end space $\Omega(G)$ is compact if, and only if, for every finite set $S \subset V(G)$, there are only finitely many connected components in $G \setminus S$ which contain rays.*

Proof. First, suppose that $\Omega(G)$ is compact and fix $S \subset V(G)$ a finite set. Note that $\mathscr{C} = \{\Omega(S, [r]) : r \in \mathscr{R}(G)\}$ defines an open covering for $\Omega(G)$ whose elements are equal or disjoint.

Since $\mathscr{C}$ admits a finite subcover, we conclude that only finitely many connected components of $G \setminus S$ contain rays.

Now, for every finite set $S \subset V(G)$, suppose that $G \setminus S$ contains finitely many connected components with rays. In addition, let $\mathscr{C}$ be any open cover for $\Omega(G)$. For a finite set $S \subset V(G)$, we call a connected component C of $G \setminus S$ *bounded* if there is $U_C \in \mathscr{C}$ such that $\overline{C} \cap \Omega(G) \subset U_C$, where the closure of C is taken in $|G|$. Otherwise, we call C an unbounded component, which contains a ray since $\overline{C} \cap \Omega(G) \neq \emptyset$. If every C is bounded, then

$$\{U_C : C \text{ is a connected component of } G \setminus S \text{ containing a ray}\}$$

is a finite subcover of $\mathscr{C}$. Hence, in order to conclude that $\Omega(G)$ is compact, we can suppose for a contradiction that, for every finite set $S \subset V(G)$, there is at least one *unbounded* connected component in $G \setminus S$. Denote by D_S the union of such components.

In particular, $D_S \supseteq D_{S'}$ if $S \subseteq S'$, because every unbounded connected component of $G \setminus S'$ is contained in some (unbounded, thus) connected component of $G \setminus S$. Then, the map $S \mapsto D_S$ only fails to be a direction because D_S is not a single connected component of $G \setminus S$. However, it sill holds that $D_{S \cup S'} \subset D_S \cap D_{S'}$ for each pair of finite sets $S, S' \subset V(G)$. In particular, this shows that the family $\{A \subset V(G) : A \supset D_S \text{ for some finite } S \subset V(G)\}$ is a filter on the power set $\wp(V(G))$ considering the inclusion. Thus, we can fix $\mathscr{U}$ an ultrafilter containing it. If $S \subset V(G)$ is finite, then

$$\bigcup \{C : C \text{ is a component of } G \setminus S \text{ containing a ray}\}$$

clearly belongs to $\mathscr{U}$, by containing D_S. Hence, since this union is finite by hypothesis and $\mathscr{U}$ is an ultrafilter, there is a unique connected component $f(S)$ of $G \setminus S$ that contains a ray and belongs to $\mathscr{U}$. By containing an unbounded connected component from the complement of some (possible another) finite set, $f(S)$ itself is unbounded.

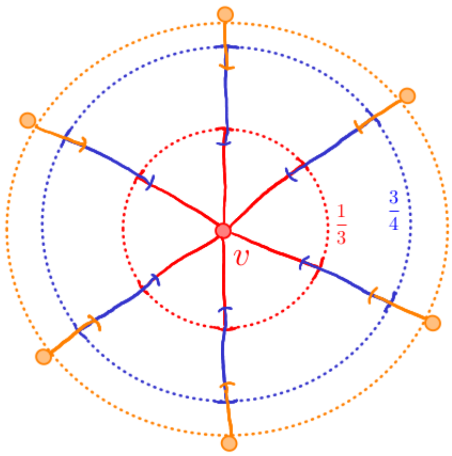
Moreover, for finite sets $S, S' \subset V(G)$, since $f(S') \cap f(S) \neq \emptyset$ (because both components are elements of $\mathscr{U}$), we have $f(S') \subset f(S)$ if $S' \supseteq S$. Therefore, the map f describes a direction over G. It follows from Proposition 5.2.1 that $f = C(\cdot, [r])$ for some ray $r \in \mathscr{R}(G)$. However, since $\mathscr{C}$ is an open cover for $\Omega(G)$, there is $U \in \mathscr{C}$ an open set containing the end $[r]$. In particular, for some finite set $S \subset V(G)$, we must have $\Omega(S, [r]) \subset U$. Since $\Omega(S, [r]) = \overline{C(S, [r])} \cap \Omega(G)$, we conclude that $f(S) = C(S, [r])$ is a bounded connected component of $G \setminus S$, which is a contradiction. □

Corollary 5.2.4. *A graph G is locally finite if, and only if, $|G|$ is a compact topological space.*

Proof. If $|G|$ is compact, then G must be locally finite due to an obstruction when defining the open neighborhoods of its vertices. More precisely, suppose that there is $v \in V(G)$ a vertex of infinite degree κ. Let $\{v_\alpha\}_{\alpha < \kappa}$ be an enumeration of its neighborhood. Consider the open cover $\mathscr{C}$ of $|G|$ whose elements are described by the items below:

- The open set containing v has the form $\bigcup_{\alpha<\kappa}\left[v,\frac{1}{3}\right)$, as sketched in red in Figure 25;
- In each edge $vv_\alpha \simeq [0,1]$, we fix the open interval $\left(\frac{1}{4},\frac{3}{4}\right)$, as drawn in blue in Figure 25;
- Finally, as presented in orange in Figure 25, the open set $|G| \setminus \bigcup_{\alpha<\kappa}\left[v,\frac{2}{3}\right]$ is an element of $\mathscr{C}$.

Figure 25 – A clever open cover for $|G|$



Source: Elaborated by the author.

Described this way, for each $\alpha < \kappa$, the inner point $\frac{1}{2} \in [v, v_\alpha]$ is contained in precisely one covering set: the interval contained in $[v, v_\alpha]$ that has the form $\left(\frac{1}{4},\frac{3}{4}\right)$. Therefore, $\mathscr{C}$ cannot have a finite subcover, since the degree of v is infinite.

Conversely, suppose that G is locally finite and let $\mathscr{C}$ be any open cover for $|G|$. Note that $G \setminus S$ has finitely many connected components for every finite set $S \subset V(G)$, because every vertex of S has finite degree. Hence, we are under the conditions of Lemma 5.2.3. In particular, there are finitely many finite sets $S_1, S_2, \ldots, S_n \subset V(G)$ and finitely many ends $[r_1], [r_2], \ldots, [r_n] \in \Omega(G)$ such that:

- For each $1 \leq i \leq n$, there is $U_i \in \mathscr{C}$ an open set containing $\overline{C(S_i,[r_i])} = C(S_i,[r_i]) \cup \Omega(S_i,[r_i])$;
- $\Omega(G) = \bigcup_{i=1}^{n} \Omega(S_i,[r_i])$.

In particular, by König's Lemma (2.1.1) and the above second item, the vertices that do not belong to $\bigcup_{i=1}^{n} C(S_i,[r_i])$ define a graph whose connected components are finite. Moreover, these connected components are those (finitely many) that contains vertices of $S_1 \cup S_2 \cup \cdots \cup S_n$ or those (finitely many, since G is locally finite) in which vertices from $S_1 \cup S_2 \cup \cdots \cup S_n$ have neighbors.

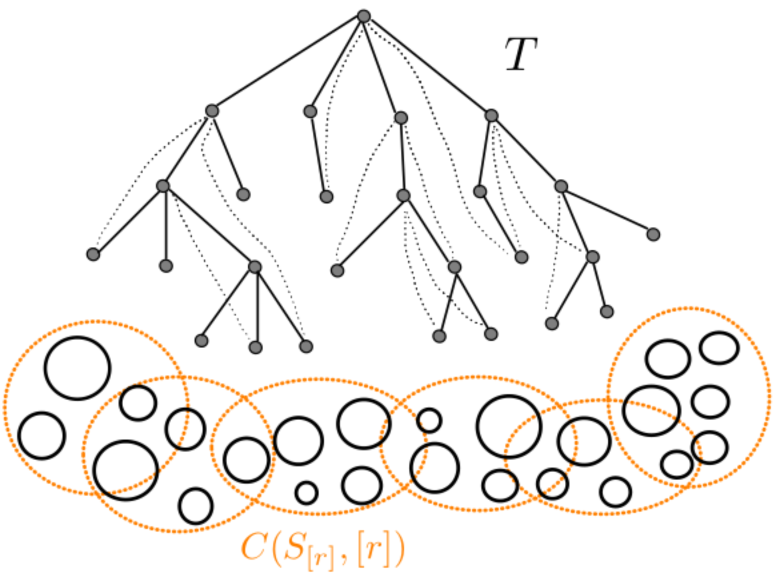
To summarize, the vertices that do not belong to $\bigcup_{i=1}^{n} C(S_i,[r_i])$ define a finite set S. Hence, it is also finite the set $F \subset E(G)$ comprising the edges incident to vertices in S. For each $v \in S$, let $U_v \in \mathscr{C}$ be any open set containing it. For each edge $e \in F$, once the unit segment $[0,1]$ is compact, let $\mathscr{C}_e \subset \mathscr{C}$ be a finite subcover for e. Then, $\{U_i : 1 \leq i \leq n\} \cup \{U_v : v \in S\} \cup \bigcup \{\mathscr{C}_e : e \in F\}$ is a finite subcover of $\mathscr{C}$ for $|G|$, proving that this is a compact space.

□

Corollary 5.2.4 addresses a message that was already pointed out in the introduction: the topological approach for the study of infinite graphs is suitable for locally finite ones. In fact, Proposition 8.6.1 from (DIESTEL, 2018) presents a simpler proof for the above result, without relying on the more abstract definition of directions. However, this alternative interpretation for the end structure still supports the following approximation theorem:

Theorem 5.2.5 (Kurkofka, Melcher and Pitz (2021), Theorem 1)**.** *Fix any collection of connected components $\mathscr{C} = \{C(S_{[r]},[r]) : [r] \in \Omega(G)\}$. Then, there is a normal tree T in G such that the connected components of $G \setminus T$ refines $\mathscr{C}$. In other words, for every connected component C of $G \setminus T$, there is an end $[r] \in \Omega(G)$ such that $C \subseteq C(S_{[r]},[r])$.*

Figure 26 – A normal tree T as in Theorem 5.2.5



The paths presented by dashed black lines suggests that T is a normal tree in an underlying graph G. Drawn as black circles, the connected components of $G \setminus T$ refine the family $\mathscr{C} = \{C(S_{[r]},[r]) : [r] \in \Omega(G)\}$, whose elements are sketched by the orange ellipsis.

Source: Elaborated by the author.

Proof. As in Lemma 5.2.3, we say that a connected subgraph H of G is *bounded* if there is $[r] \in \Omega(G)$ such that H is contained in $C(S_{[r]},[r])$. Otherwise, we say that H is *unbounded*. Via depth-search algorithms, we will recursively construct a sequence of normal trees $T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$ with the following properties:

- The tree order of T_{i+1} extends the tree order of T_i for every $i < n$;
- T_i is a rayless tree for every $i < n$;

As usual, we consider T_0 any tree that contains only its root z. Supposing that T_n is defined for some $n \in \mathbb{N}$, let C be any connected component from $G \setminus T_n$.

For a contradiction, suppose that, for every finite set $S \subset C$, there is exactly one unbounded connected component $f(S)$ in $G \setminus S$. Hence, if $S' \supseteq S$ is another finite subset of C, the unbounded component $f(S')$ must be contained in the unique unbounded component $f(S)$ of $G \setminus S$. In other words, f is a direction in C. Then, by Proposition 5.2.1, there is $r \in \mathscr{R}(G)$ a ray such that, for every finite set $S \subset C$, $f(S)$ is the connected component of $C \setminus S$ containing a tail of r. On the other hand, $f((S \cup S_{[r]}) \cap C)$ is a connected subgraph of $G \setminus (S \cup S_{[r]})$ which contains a tail of r, so that $f(S \cup S_r) \subset C(S_r, [r])$. This, however, contradicts the fact that $f(S \cup S_r)$ is chosen as an unbounded subgraph.

In other words, we proved that, for every connected component C of $G \setminus T_n$, there is $S_C \subset C$ a finite set for which $C \setminus S$ has none or at least two unbounded connected components. Then, by applying the depth-search procedure[3] finitely many times in each *unbounded* connected component C, we can define T_{n+1} as a normal extension of T_n that contains $\bigcup\{S_C : C$ is an unbounded connected component of $G \setminus T_n\}$. Since S_C is finite for each C, so is finite the intersection $T_{n+1} \cap C$, besides being connected by the application of the depth-search procedure. In particular, T_{n+1} is also a rayless tree. Note that $T_{n+1} \cap C = \emptyset$ if C is a *bounded* connected component of $G \setminus T_n$.

At the end of this recursive construction, $T = \bigcup_{n \in \mathbb{N}} T_n$ is a normal tree. For instance, suppose that there is r a ray in T, which we can assume that starts at the root z. Since T_n is rayless for every $n \in \mathbb{N}$, we have $r \cap (T_{n+1} \setminus T_n) \neq \emptyset$. In particular, r meets an unbounded connected component C_n of $G \setminus T_n$. Moreover, S_{C_n} was chosen so that $C_n \setminus S_{C_n}$ has at least two unbounded connected components, because, once $r \cap (T_{n+2} \setminus T_{n+1}) \neq \emptyset$, the ray r meets some of them. Hence, we can fix $P_n \subset T_{n+1} \setminus T_n$ a path connecting r to some vertex $u_n \in S_{C_n}$ that has a neighbor in D_{n+1}, a fixed connected component of $C_n \setminus S_{C_n}$ other than the one that contains a tail of r.

For distinct $n, m \in \mathbb{N}$, note that $P_n \cap P_m = \emptyset$, because $P_n \subset T_{n+1} \setminus T_n$ and $P_m \subset T_{m+1} \setminus T_m$. In other words, the subgraph of T induced by $r \cup \bigcup_{n \in \mathbb{N}} P_n$ is a comb whose teeth belong to $\{u_n\}_{n \in \mathbb{N}}$. Then, since $S_{[r]}$ is finite, choose $N \in \mathbb{N}$ big enough so that $S_{[r]} \cap T \subset T_N$ and $S_{[r]} \cap D_n = \emptyset$ for every $n \geq N$. Therefore, for $n \geq N$, there is a path in $G \setminus S_{[r]}$ connecting a tail of r to u_n, which,

[3] See the proof of Proposition 2.2.2.

in its turn, has a neighbor in D_n. In other words, $D_n \subset C(S_{[r]}, [r])$, contradicting the fact that D_n is unbounded. To summarize, T is rayless.

Finally, it remains to show that T refines the family $\mathscr{C}$. In fact, if C is a connected component of $G \setminus T$, then $N(C) = \{v \in T : v \text{ has a neighbor in } C\}$ is a chain in T, since T is normal. However, T has no rays, so that $N(C)$ is finite. If $n \in \mathbb{N}$ is large enough for which $N(C) \subset T_n$, then C is also a connected component of $G \setminus T_n$. Therefore, C is bounded, because $T_{n+1} \cap C = \emptyset$ (otherwise, some vertex of C would have a neighbor in $T_{n+1} \setminus T_n$). In other words, $C \subset C(S_{[r]}, [r])$ for some end $[r] \in \Omega(G)$, finishing the proof. □

Exercise 5.2.6. *Why did we need to define the family of connected components $\{D_n\}_{n\in\mathbb{N}}$? More precisely, let $N \in \mathbb{N}$ and r be as in this penultimate paragraph of the above proof. Can we conclude that $C_n \subset C(S_{[r]}, [r])$ for any $n \geq N$?*

Theorem 5.2.5 suggests a close correlation between normal trees and end spaces. In fact, in Section 5.4 we will discuss the role that normality plays in the recent characterization of spaces of the form $\Omega(G)$, due to Pitz (2023). In its turn, we will finish this section by compiling other covering properties that follows from the theorem just proven.

Actually, some of the next statements, as well as most key results from Section 5.3, were first obtained in the literature before (KURKOFKA; MELCHER; PITZ, 2021). However, that recent paper unified several proofs:

Corollary 5.2.7 (Kurkofka, Melcher and Pitz (2021), Corollary 3.1)**.** *Every end space is **ultra-paracompact**. In other words, if G is a graph and $\mathscr{C}$ is an open cover for $\Omega(G)$, then there is $\mathscr{C}'$ a disjoint refinement for $\mathscr{C}$. Namely, $\mathscr{C}'$ is another open cover for $\Omega(G)$, with pairwise disjoint covering sets, such that every $U' \in \mathscr{C}'$ is contained in some $U \in \mathscr{C}$.*

Proof. Given a graph G and $\mathscr{C}$ an open cover for $\Omega(G)$, we can assume that elements of $\mathscr{C}$ are basic open sets as in (5.2). Hence, for some index set I, we write $\mathscr{C} = \{\Omega(S_i, [r_i])\}_{i\in I}$. For each end $[r] \in \Omega(G)$, there is $i \in I$ such that $[r] \in \Omega(S_i, [r_i])$, because $\mathscr{C}$ covers $\Omega(G)$. In this case, $C(S_i, [r]) = C(S_i, [r_i])$ by definition of $\Omega(S_i, [r])$. Choosing $S_{[r]} = S_i$, we can apply Theorem 5.2.5 in order to find a rayless normal tree T such that, for every connected component C of $G \setminus T$, there is $i \in I$ for which $C \subseteq C(S_i, [r_i])$.

Since T is normal and rayless, the set $N(C) = \{v \in T : v \text{ has a neighbor in } C\}$ is a finite chain in T. This means that $\overline{C} \cap \Omega(G)$ is either empty or a basic open set of the form (5.2). If D is another connected component of $G \setminus T$, then $D \cap C = \emptyset$, so that $\overline{D} \cap \overline{C} = \emptyset$. In addition, every ray $r \in \mathscr{R}(G)$ has a tail in some connected component of $G \setminus T$, since, once more, T is normal and rayless. Hence,

$$\mathscr{C}' = \{\overline{C} \cap \Omega(G) : C \text{ is a connected component of } G \setminus T\}$$

verifies the statement. □

Corollary 5.2.8 (Kurkofka, Melcher and Pitz (2021), Corollary 3.2)**.** *For every graph G, the space $|G|$ is* ***paracompact****, i.e., every open cover $\mathscr{C}$ admits a locally finite open refinement $\mathscr{C}'$. In other words, every $U' \in \mathscr{C}'$ is contained in some $U \in \mathscr{C}$ and $\{U' \in \mathscr{C}' : x \in U'\}$ is finite for every $x \in |G|$.*

Proof. Given a graph G and $\mathscr{C}$ an open cover for $|G|$, we assume that the elements of $\mathscr{C}$ are basic open sets. In particular, $\mathscr{C}$ contains an open cover for $\Omega(G)$, which, following the notation from (5.3), we denote by $\{\hat{C}(S_i,[r_i],\varepsilon_i)\}_{i\in I}$ for some index set I. Hence, every ray r of G has a tail in $C(S_i,[r_i])$ for some $i \in I$. Writing $S_{[r]} = S_i$, we now apply Theorem 5.2.5 in order to find T a rayless normal tree in G such that every connected component of $G \setminus T$ is contained in some $C(S_i,[r_i])$.

As in the previous proof, since T is normal and rayless, every connected component C of $G \setminus T$ that contains a ray (say, r) can be written as $C = C(S,[r])$ for some finite set $S \subset V(G)$ (actually, $S = N(C)$). Then, write $\{C(S_j,[r_j])\}_{j\in J}$ for the set of connected components of $G \setminus T$. For each $j \in J$, set $\varepsilon_j = \varepsilon_i$, in which $i \in I$ is chosen such that $C(S_j,[r_j]) \subset C(S_i,[r_i])$. Considering that, $\{\hat{C}(S_j,[r_j],\varepsilon_j)\}_{j\in J}$ is a disjoint refinement for $\{\hat{C}(S_i,[r_i],\varepsilon_i)\}_{i\in I}$.

Now, let H denote the quotient space from $|G|$ obtained by contracting, for each $j \in J$, the set $\overline{C(S_j,[r_j])} = C(S_j,[r_j]) \cup \Omega(S_j,[r_j])$ to an artificial point x_j. Regarding H as a "graph", there might be parallel edges connecting x_j to vertices of T. Due to this, we call H a **multigraph**. Even though, the quotient topology on H is closely similar to the one presented in the Introduction.

For vertices of $H \cap T$ and inner points of edges of H, the basic open neighborhoods in H are precisely the ones of $|G|$. In its turn, for given $j \in J$ and $\varepsilon > 0$, an open basic neighborhood around x_j has the form $\displaystyle\bigcup_{e\in E(x_j)} [x_j,\varepsilon)$, where $E(x_j)$ denotes the set of all edges incident to x_j in H as a multigraph. In fact, this open basic neighborhood is obtained by passing the open basic set $\hat{C}(S_j,[r_j],\varepsilon)$ to the quotient that defines H. Roughly speaking, the quotient topology in H would be $|H|$ if H were a graph rather than a multigraph (or if we have defined the structure $|\cdot|$ also for multigraphs).

In any case, since H is a rayless, the described quotient topology is metrizable [4]. In particular, H is paracompact. Then, we are able to find $\mathscr{C}_H$ an open cover for H such that:

- $\{U \in \mathscr{C}_H : x \in U\}$ is finite for every $x \in H$. Here, x might be a vertex (even of the form x_j for some $j \in J$) or the inner point of an edge;
- If $U \in \mathscr{C}_H$ does not intersect $\{x_j\}_{j\in J}$, then U is also an open set in $|G|$ and there is $V \in \mathscr{C}$ such that $U \subseteq V$;
- If $U \in \mathscr{C}_H$ contains x_j for some $j \in J$, then U is contained in the open basic neighborhood of x_j obtained after passing $\hat{C}(S_j,[r_j],\varepsilon_j)$ to the quotient that defines H.

[4] See the introduction of Section 5.3 for more details

Hence, the claimed refinement of $\mathscr{C}$ in $|G|$ can be taken as

$$\mathscr{C}' = \{\hat{C}(S_j,[r_j],\varepsilon_j) : j \in J\} \cup \{U \in \mathscr{C}_H : U \cap \{x_j\}_{j\in J} \neq \emptyset\}.$$

$\square$

Corollary 5.2.9 (Polat (1996a), Lemma 4.14)**.** *Every end space is collectionwise normal.*

Proof. This a topological consequence from the fact that $\Omega(G)$ is ultra-paracompact (and, in particular, paracompact) for every graph G. $\square$

Corollary 5.2.10 (Sprüssel (2008), Theorem 4.1)**.** *$|G|$ is a normal topological space for every graph G.*

Proof. Follows from the fact that every paracompact topological space is normal. $\square$

Exercise 5.2.11 (Kurkofka, Melcher and Pitz (2021), Lemma 5.1)**.** *This is a guide for showing that end spaces are actually **hereditarily ultra-paracompact**, in the sense that all its subspaces also ultra-paracompact. Actually, it is sufficient to show that open subspaces have this latter property. Then, fix G a graph and $V \subset \Omega(G)$ an open set:*

- *Consider $\mathscr{M}$ a $\subseteq -$maximal family of disjoint rays whose ends belong to $\Omega(G)\setminus V$. Define $M = \bigcup \mathscr{M}$ and note that $\overline{M} \cap \Omega(G) \subset \Omega(G)\setminus V$;*
- *Show that, for every ray r in $G\setminus M$, we have $[r] \in V$. Conversely, show that every end in V has a ray in $G\setminus M$ as a representative;*
- *Finally, prove that the map*

$$\begin{array}{rccc} f: & \Omega(G\setminus M) & \to & V \\ & [r]_{G\setminus M} & \mapsto & [r] \end{array}$$

is a well-defined homeomorphism, where $[r]_{G\setminus M}$ denotes the equivalence class of the ray r in $\Omega(G\setminus M)$. Conclude from Corollary 5.2.8 that V is ultra-paracompact.

5.3 Metric properties

The reader with some background in algebraic topology might strange the given definition for $|G|$. After all, vertices are $0-$dimensional complexes, so that, when identifying edges with the unit interval, a graph G has the structure of a $1-$complex. The topology of G as an unidimensional CW$-$complex, however, is often not first countable. In fact, if there is in G a vertex v of infinite degree, a routine diagonalization argument shows that v has no countable system of open neighborhoods in this topology.

Figure 27 – Different systems of open neighborhoods for a vertex

In red, we sketch two open basic neighborhoods for a vertex v, but regarding two different topologies. At the left, we consider the topology of the underlying graph G as an unidimensional complex. In this case, v has no countable system of open neighborhoods if $d(v)$ is infinite. At the right, we represent a neighborhood for v considering the space $|G|$.

Source: Elaborated by the author.

For graph-theorists, however, it is interesting to preserve the notion of distance that paths suggest, which justifies the given definition for $|G|$. More precisely, if G is connected, between two of its vertices u and v we set

$$d(u,v) = \min\{n \in \mathbb{N} : \text{there is a path of length } n \text{ connecting } u \text{ and } v\}.$$

Note that d is indeed a metric over $V(G)$, that we can naturally extend to $|G| \setminus \Omega(G)$ if we consider that edges have diameter 1. More precisely, if x and y are points from edges $u_1u_2 = [u_1,u_2]$ and $v_1v_2 = [v_1,v_2]$ respectively, we set

$$d(x,y) = \begin{cases} |x-y|, & \text{if } u_1v_1 = u_2v_2 \\ \min\limits_{i,j\in\{1,2\}} [|x-u_i| + d(u_i,v_j) + |y-v_j|], & \text{otherwise.} \end{cases}$$

Considering that, the open basic neighborhoods for a vertex v in $|G|$ are precisely the open balls around it considering the above metric d. Therefore, the subspace topology of $|G| \setminus \Omega(G)$ is induced by d. In particular, $|G|$ is a metric space if $\Omega(G) = \emptyset$, i.e., if G is a rayless graph.

However, if $\Omega(G) \neq \emptyset$ (even if $\Omega(G)$ is singleton), $|G|$ might not be metrizable. For instance, every two rays in an infinite clique K are equivalent, so that $\Omega(K) = \{*\}$ is the one point space. By definition of $|K|$, the family $\{C(S,*) : S \subset V(K)\}$ is a system of open neighborhoods for $*$, but it has no countable subsystem if K is uncountable. Hence, $|K|$ is not first-countable and, then, neither a metric space. On the other hand, this counterexample is already familiar: in Section 2.2, we observed that uncountable complete graphs also do not admit normal spanning trees. In fact, as one of the goals of this section, we will prove the equivalence below:

Theorem 5.3.1 (Diestel (2006), Theorem 3.1 (i))**.** *Fix G an infinite graph. Then, G has a normal spanning tree if, and only if, $|G|$ is a metric space.*

Following Kurkofka, Melcher and Pitz (2021), we will present their proof for the above result by relying on the approximation theorem (5.2.5) and some other intermediate technical arguments in the literature. In particular, we will first establish a characterization for metrizability in end spaces. Historically, this was first approached by Polat (1996a), who described when end spaces are ultrametric[5]. The following observation motivates his study:

Proposition 5.3.2. *Let T be a tree. Then, $\Omega(T)$ is a complete ultrametric space.*

Proof. Suppose that T is rooted at some vertex z and denote by $\leq$ the corresponding tree order. Throughout this proof, we will always assume that the rays start at z, which fix a representative r for every end $[r] \in \Omega(T)$. In particular, if r and s are non-equivalent rays in T, then it is well defined the vertex $v_{s,r} = \max r \cap s = \max\{t \in T : t \in r \cap s\}$. In fact, $s \cap r$ is a totally ordered set regarding T because it is a subset of a ray, being finite since $[r] \neq [s]$. Then, we consider the map

$$d([r],[s]) = \begin{cases} 0, & \text{if } r = s; \\ \frac{1}{n+1}, & \text{if } r \neq s, \text{ where } n \text{ denotes the height of } v_{s,r} \text{ in } T. \end{cases}$$

Hence, clearly $d([r],[s]) = 0$ if, and only if, $[s] = [r]$. In addition, $d([r],[s]) = d([s],[r])$ for every pair of ends $[r],[s] \in \Omega(T)$. In order to show that d is an ultrametric in $\Omega(T)$, let $[w]$ be a third (distinct) end in T. If the height of $v_{w,s}$ is strictly smaller than the height of $v_{r,s}$, then $v_{w,s} < v_{r,s}$, since $v_{w,s} \in s$ and, thus, is comparable to $v_{r,s}$. Hence, $w \cap s \subseteq r \cap s$, so that $v_{w,s} \in r \cap w$. In particular, we have $v_{w,r} \geq v_{w,s}$. Note that this must be an equality: otherwise, since $v_{w,r} \in s$, we would contradict the definition of $v_{w,s}$. To summarize, if the height of $v_{w,s}$ is strictly smaller than the height of $v_{r,s}$, then $v_{w,r} = v_{w,s}$, from where we verify that $d([r],[s]) < \max\{d([r],[w]), d([w],[s])\}$. Clearly, by symmetry, this conclusion also holds if the height of $v_{w,r}$ is strictly smaller than the height of $v_{r,s}$.

Showing that d is a complete metric is quite similar to the proof of König's Lemma (2.1.1). In fact, consider $\{[r_n]\}_{n\in\mathbb{N}}$ a Cauchy sequence of ends in $\Omega(T)$. In $T \setminus \{z\}$, all but finitely many elements of this sequence belong to a same connected component C_0. In fact, there is $n_0 \in \mathbb{N}$ such that $d([r_n],[r_m]) \leq \frac{1}{2}$ for every $n,m \geq n_0$, or, equivalently, $v_{r_n,r_m} > z$. Let v_1 be a neighbor of z in C_0. Now, suppose that we have defined a path $P_k = zv_1v_2v_3\dots v_k$ in T for some $k \in \mathbb{N}$. In particular, the height of v_k is k. Moreover, since $\{[r_n]\}_{n\in\mathbb{N}}$ is a Cauchy sequence, there is $n_k \in \mathbb{N}$ such that $d([r_n],[r_m]) \leq \frac{1}{k+2}$ for every $n,m \geq n_k$. By induction, we suppose that n_k is big enough so that r_n contains the path $zv_1v_2v_3\dots v_k$ whenever $n \geq n_k$. Hence, for $n,m \geq n_k$, we must have $v_{r_m,r_n} > v_k$, proving that the rays r_n and r_m have their tails in a common connected component C_k of $T \setminus P_k$. Thus, we choose $v_{k+1} \in C_k$ as a neighbor of v_k in C_k. Due to this recursive process, it is defined a ray $r = zv_1v_2v_3\dots$ whose end is the limit of the sequence $\{[r_n]\}_{n\in\mathbb{N}}$ by construction.

[5] We recall that a metric space X is **ultrametric** if the triangle inequality is strengthened as follows: $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ for every $x,y,z \in X$

Suppose now that the heights of $v_{w,r}$ and $v_{w,s}$ are greater than or equal to the height of $v_{r,s}$, besides both vertices being comparable with $v_{r,s}$. If these three heights are the same, then $d([r],[s]) = d([r],[w]) = d([w],[s])$, because $v_{r,s} = v_{r,w} = v_{w,s}$. If not, without loss of generality, assume that $v_{w,r} > v_{r,s}$. If we had $v_{w,s} > v_{r,s}$, then $v_{w,s}$ and $v_{w,r}$ would be incomparable, because $v_{w,s}$ would not belong to r by the definition of $v_{s,r}$. However, this contradicts the fact that $v_{w,r}$ and $v_{w,s}$ belong to w, which, being a ray, is totally ordered by $\leq$. In other words, we proved the equality $v_{w,s} = v_{r,s}$ while supposing that $v_{w,r} > v_{r,s}$. In this case, $d([r],[s]) = d([w],[s])$ and $d([r],[s]) \geq d([w],[r])$, which also proves that $d([r],[s]) = \max\{d([r],[w]), d([w],[s])\}$. This finishes the verification that d is an ultrametric.

Finally, we will prove that d induces the topology of $\Omega(T)$ as the end space of T. First, let $\Omega(S,[r])$ be any basic open set in $\Omega(T)$, parameterized by a finite subset $S \subset V(T)$ and an end $[r] \in \Omega(T)$. Since T is a tree, we observe that there is precisely one edge between the infinite connected component $C(S,[r])$ and a vertex $v \in S$. Indeed, we can also write $\Omega(S,[r]) = \Omega(\{v\},[r])$. Note that $v \in r$ once we assumed that r starts at the root z. If n denotes the height of v, we claim that $\left\{[s] \in \Omega(T) : d([r],[s]) < \frac{1}{n+1}\right\} \subset \Omega(\{v\},[r])$. In fact, if $d([s],[r]) < \frac{1}{n+1}$ for an end $[s] \in \Omega(T)$, then the height of $v_{s,r}$ is greater than or equal to n. Therefore, since $v_{s,r}, v \in r$, we must have $v_{s,r} > v$. Hence, the tails of r and s belong to the same connected component of $G \setminus \{v\}$, namely, the one that contains $v_{s,r}$. This proves that $[s] \in \Omega(\{v\},[r])$.

Conversely, fix some $n \in \mathbb{N}$ and an end $[r] \in \Omega(T)$. Let $v \in r$ denotes the vertex of height n in r and consider an end $[s] \in \Omega(\{v\},[r])$. Since some tail of s intersects some tail of r in $G \setminus \{v\}$, because T is a tree, we must have $v_{s,r} > v$ by definition of $v_{s,r}$. Then, $d([r],[s]) < \frac{1}{n+1}$, since the height of $v_{s,r}$ is greater than the height of v. In other words, we proved that $\Omega(\{v\},[r]) \subset \left\{[s] \in \Omega(T) : d([r],[s]) < \frac{1}{n+1}\right\}$. This concludes the verification that the topology of $\Omega(T)$ is induced by d. □

The description of Polat for ultrametric end spaces is done via graphs that admit suitable trees which encode the global end structure. Formally, we say that a subgraph H of a graph G is **end-faithful** if the inclusion map $\iota : \Omega(H) \to \Omega(G)$ is bijective. More precisely, ι sends the equivalence class of a ray r in H to the equivalence class of r in G. Note that ι is indeed well-defined: if r' is connected to r via an infinite family of disjoint paths in H, this same family of paths shows that $[r] = [r']$ in $\Omega(G)$.

Roughly speaking, an end-faithful subgraph contains a representative of every $[r] \in \Omega(G)$, but also do not separate rays that belong to the same equivalence class in G. Normality condition over trees implies that this identification can be also topological:

Proposition 5.3.3 (Diestel (1992), Proposition 5.5)**.** *Let G be a graph which admits an end-faithful normal tree T. Then, the inclusion map $\iota : \Omega(T) \to \Omega(G)$ is an homeomorphism.*

Proof. Let $\leq$ denotes the tree order of T. Aiming to show that ι is continuous, fix a finite set

$S \subset V(G)$ and an end $[r] \in \Omega(G)$. Note that the connected component C of $T \setminus S$ containing a tail of r is also a connected subgraph of $G \setminus S$, since T is a subgraph of G. Hence, C is contained in the connected component of $G \setminus S$ in which r has a tail, proving that ι is continuous.

In order to prove that ι is an open map, fix a finite set $S \subset V(T)$ and an end $[r] \in \Omega(G)$. Note that $\lceil S \rceil = \bigcup_{s \in S} \lceil s \rceil$ is also finite, since $\lceil s \rceil$ is finite for every $s \in S$. Let $[r']$ be an end in $\Omega(G) \setminus \{[r]\}$ such that r and r' have their tails in a common connected component of $G \setminus \lceil S \rceil$. Since ι is surjective, we assume that both rays are contained in T and start at its root. By contradiction, however, suppose that r and r' do not have their tails in a same connected component of $T \setminus S$. Since T is a tree, this means that there is $s \in S$ which belongs to one of the rays and do not belong to the other. In any case, we have $s > x = \max r \cap r' = \max\{t \in T : t \in r \cap r'\}$. On the other hand, let P be a path in $G \setminus \lceil S \rceil$ connecting vertices from the tails of r and r' in $T \setminus S$. By Proposition 2.2.3, however, the path P must intersect $\lceil x \rceil \subset \lceil s \rceil \subset \lceil S \rceil$, which is a contradiction. Therefore, r and r' have their tails in the same connected component of $T \setminus S$, proving that ι is an open map. □

Therefore, Propositions 5.3.3 and 5.3.2 combined prove that every graph containing an end-faithful normal tree has a complete ultrametric end space. The converse statement is rather more involving and difficult to prove, as first obtained by Polat (1996a). In addition, he asked whether the metrizability of end spaces would be an enough condition to find end-faithful normal trees. Theorem 5.2.5 addresses this question positively, also presenting Polat's result in a simpler way:

Theorem 5.3.4 (Kurkofka, Melcher and Pitz (2021), Theorem 4.1)**.** *Let G be a graph. Then, the following conditions are equivalent:*

i) G contains an end-faithful normal tree;

ii) The end space $\Omega(G)$ is completely ultrametrizable;

iii) The end space $\Omega(G)$ is metrizable.

Proof. The implication $i) \Rightarrow ii)$ was already discussed, while $ii) \Rightarrow i)$ is trivial. Then, supposing that the topology of $\Omega(G)$ is induced by some metric d, it remains to construct an end-faithful normal tree for G. Similarly to previous proofs relying on the depth-search procedure, this will be done after constructing a sequence of normal trees $\{T_n\}_{n\in\mathbb{N}}$ such that T_{n+1} extends T_n and its tree order $\leq$. First, let T_0 denote the tree that contains only its root z. Supposing that T_n is a rayless normal tree defined for some $n \in \mathbb{N}$, fix C a connected component of $G \setminus T_n$. As usual, we observe that $N(C) = \{v \in T_n : v \text{ has a neighbor in } C\}$ is finite, by being a totally ordered subset of T_n. Then, let $u_C \in C$ be some neighbor of the $\leq -$maximal element $v_C \in N(C)$.

For every $[r] \in \overline{C} \cap \Omega(G)$ (if it exists), we fix $B_{\frac{1}{n+1}}[r]$ the open ball around $[r]$ of radius $\frac{1}{n+1}$. By applying Theorem 5.2.5, we can define in C a rayless normal tree rooted at u_C, abusively denoted by $T_{n+1} \cap C$, with the property below:

($\star$) Every connected component D of $C \setminus T_{n+1}$ satisfies $\overline{D} \cap \Omega(G) \subset B_{\frac{1}{n+1}}[r]$ for some end $[r] \in \overline{C} \cap \Omega(G)$.

After this being done for every connected component C of $G \setminus T_n$, the definition of T_{n+1} is finished. Since the choice of the vertices u_C and v_C follows the depth-search procedure [6], T_{n+1} is indeed a normal tree.

At the end of this recursive definition, let $T' = \bigcup_{n \in \mathbb{N}} T_n$ be the limit normal tree. Although we can not conclude that T' is end-faithful, the following observation holds:

`Claim:` If a ray r in G is non-equivalent to any ray of T' (i.e., r attests that T' is not end-faithful), then it has a tail in a connected component C of $G \setminus T'$ whose rays are all equivalent to r. Moreover, the set $N(C) = \{v \in T' : v \text{ has a neighbor in } C\}$ is finite.

Proof of the claim. Since T' is normal and r is non-equivalent to any ray of T', the intersection $r \cap T'$ is finite. Hence, r indeed has a tail in some connected component C of $G \setminus T'$. Let s be any other ray in C. Observing that, for every $n \geq 1$, C is contained in some connected component of $G \setminus T_n$, it follows from ($\star$) that $[r], [s] \in B_{\frac{1}{n}}[r']$ for some end $[r'] \in \Omega(G)$. Therefore, $d([r],[s]) < \frac{2}{n}$ by the triangle inequality, proving that $[r] = [s]$ when considering $n \to \infty$.

Finally, suppose that $N(C)$ is infinite. Hence, it is contained in some ray s of T'. By ($\star$), for every $n \geq 1$, note that C and a tail of s are contained in the same connected component of $G \setminus T_n$. Thus, as in the previous paragraph, $d([r],[s]) < \frac{2}{n}$. By considering $n \to \infty$, we conclude that $[r] = [s]$. This, however, contradicts the fact that r and s are not equivalent rays, since $s \subset T'$. □

Finally, for every ray r as in the above claim, denote by C_r the connected component of $G \setminus T'$ containing its tail. Consider u_r a neighbor of the $\leq -$maximal element $v_r \in N(C_r)$, whose existence follows from the fact that $N(C_r)$ is finite. In C_r, let P_r be a ray equivalent to r which starts at u_r. Finishing the proof, an end-faithful tree of G can be given by

$$T = T' \cup \bigcup \{P_r : C_r \text{ is a connected component of } G \setminus T' \text{ which contains a ray } r \text{ as in the claim}\}.$$

Note that T is indeed normal by the choice of u_r and v_r, as well as by the fact that $C_r \neq C_s$ whenever s and r are non-equivalent rays ruled by the claim. □

Exercise 5.3.5. *Find a graph whose end space is not metrizable. Can it be countable? Can it have a normal spanning tree?*

[6] Compare with the proof of Proposition 2.2.2.

However, in face of Theorem 5.3.4, a natural question also arises: which graphs do admit an end-faithful normal tree? A stronger problem in that direction was proposed by Halin (2000), who asked if graphs with normal spanning trees can be characterized via forbidden minors[7]. In fact, a normal spanning tree T in a graph G is also end-faithful: if $\iota : \Omega(T) \to \Omega(G)$ denotes the inclusion map, the injection and surjection of ι follow from items $i)$ and $ii)$ of Proposition 2.2.3, respectively.

In that direction, Diestel and Leader (2001) claimed that the graphs without normal spanning trees could be described via two suitable classes of forbidden minors, which was later shown incorrect by Pitz (2021). Moreover, in this latter paper, Pitz concluded that any list of forbidden minors whose aim is to characterize graphs without normal spanning trees must contain graphs of arbitrarily large size. In other words, to separate the graphs which do admit normal spanning trees from the graphs which do not is far from being a completely understood problem. As a consequence of the constructive result below, Corollary 5.3.7 is the more general criteria in the literature that approach this goal:

Proposition 5.3.6 (Pitz (2020), Theorem 3)**.** *Let G be any graph and fix $K \subset V(G)$ finite. Denote $G' = G \setminus K$. Then, G' has a maximal normal tree T such that, for every connected component C of $G' \setminus T$:*

- *The neighborhood $N(C) = \{v \in G \setminus C : v \text{ has a neighbor in } C\}$ is infinite. In particular, $N(C) \cap T$ is contained in a infinite ray r_C of T;*
- *Every $v \in N(C)$ **dominates** r_C, i.e., v is the center of a star whose leaves belong to r_C.*

Proof. As usual, we will construct an increasing sequence $T_0 \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq \dots$ of rayless normal trees in G', by first fixing $T_0 = \{r\}$ an arbitrary root. Suppose that T_n is defined for some $n \in \mathbb{N}$. Let $\mathscr{C}_n$ denote the collection of the connected components of $G' \setminus T_n$. Since T_n is rayless and normal, $N(C) = \{v \in T_n \cup K : v \text{ has a neighbor } x \in C\}$ is finite for every $C \in \mathscr{C}_n$, because K is finite and $N(C)$ is a chain in the tree order $\leq$ of T_n.

For every vertex $v \in N(C)$, fix $x_v^n \in C$ a neighbor of it. Then, by the previous observation, $F_C = \{x_v^n : v \in N(C)\}$ is finite. Applying the depth-search procedure finitely many times [8], we can extend T_n within C via a normal tree that contains F_C. With an abusive notation, this extension is denoted by $T_{n+1} \cap C$, meaning that T_{n+1} is indeed defined after we search for F_C in every connected component C of $G \setminus T_n$.

Once finished this inductive process, we claim that $T = {}_{n<\omega}T_n$ satisfies the statement. Since T_n is a normal tree for each $n \in \mathbb{N}$, so is T. For instance, suppose that $N(C)$ is finite for some $C \in \mathscr{C}$, in which $\mathscr{C}$ denote the family of connected components of $G' \setminus T$. Hence, for some

[7] A minor of a graph G is a graph obtained from it by removing vertices, edges and contracting connected vertex sets. For formal definitions and basic properties, see (DIESTEL, 2018, p.19).

[8] See the proof of Proposition 2.2.2

$n \in \mathbb{N}$ we must have $N(C) \cap T \subset T_n$. In this case, C is also a connected component of $G' \setminus T_n$, contradicting the fact that $F_C \subset T_{n+1} \setminus T_n$. Therefore, the first item of the proposition holds.

Now, consider $T' \supset T$ a tree in G' extending the tree order of T. Let $v \in T' \setminus T$ be minimal, i.e., such that $\lceil v \rceil \setminus \{v\} \subset T$. Then, there is a path in $G' \setminus \lceil \mathring{v} \rceil$ connecting v to a vertex of T, because the connected component $C \in \mathscr{C}$ containing v has infinitely many neighbors in T. Since T' extends the tree order of T, this verifies that T' is not normal. In other words, T is a maximal normal tree.

Finally, fixed $C \in \mathscr{C}$ and $v \in N(C)$, let r_C denote the ray of T containing $N(C)$. For a finite set $S \subset V(G) \setminus \{v\}$, let $n \in \mathbb{N}$ be big enough so that $v \in T_n$ and $S \cap T = S \cap T_n$. Consider $C_n \in \mathscr{C}_n$ the connected component of $G' \setminus T_n$ for which $C \subset C_n$ and, hence, $v \in N(C_n)$. In T_{n+1}, then, there is a path connecting x_v^n (which is a neighbor of v in C_n) to a vertex v' of r_C, because $T_{n+1} \cap C_n \subset G' \setminus S$ is connected. Therefore, v dominates the ray r_C. □

Corollary 5.3.7 (Halin (1978))**.** *Every graph not containing a subdivision of an infinite clique has a normal spanning tree.*

Proof. Let G be a graph not containing a subdivision of an infinite clique. Considering $K = \emptyset$, let T be a maximal normal tree as in the above result. For a contradiction, suppose that T is not spanning and fix $v \in V(G) \setminus V(T)$. Denote also by C the connected component of $G \setminus T$ containing this vertex. Then, there is r a ray in T containing the infinite set $N(C) = \{v \in T : v \text{ has a neighbor in } C\}$.

Consider any $v_1 \in N(C)$. By induction on n, suppose that there are defined vertices $v_1, v_2, v_3, \ldots, v_n \in N(C)$ and, for each distinct pair $1 \leq i < j \leq n$, a path $P_{i,j}$ such that:

- $P_{i,j}$ connects v_i to v_j;
- $P_{i',j'} \cap P_{i,j} = \emptyset$ if $i' \neq i$ and $j' \neq j$;
- $P_{i,j'} \cap P_{i,j} = \{v_i\}$ for distinct $j, j' \in \{1, 2, \ldots, n\} \setminus \{i\}$.

In other words, the graph $G_n = \bigcup_{1 \leq i < j \leq n} P_{i,j}$ is a subdivision of a clique on n vertices. In particular, G_n is finite, so that there is $v_{n+1} \in r \setminus V(G_n)$. Moreover, both v_1 and v_{n+1} dominate the ray r. Hence, there is in $r \setminus (V(G_n) \cup \{v_{n+1}\})$ a path $P_{1,(n+1)}$ connecting a neighbor of v_1 and a neighbor of v_{n+1}. Similarly, if $P_{1,n+1)}, P_{2,(n+1)}, \ldots, P_{i,(n+1)}$ are defined for some $i < n$, consider $P_{(i+1),(n+1)}$ as a path in $r \setminus \left(V(G_n) \cup \bigcup_{j=1}^{i+1} P_{j,(n+1)} \right)$ connecting a neighbor of v_{i+1} to a neighbor of v_{n+1}. Again, this path indeed exists because $v_{i+1}, v_{n+1} \in N(C)$ dominate r.

At the end of this recursive process, the union $\bigcup_{n \in \mathbb{N}} G_n$ is an infinite clique contained in G, contradicting the main hypothesis over it. Therefore, T is a normal spanning tree. □

Exercise 5.3.8. *Compare the proofs of Propositions 5.2.1 and 5.3.6.*

Besides the above proof for Corollary 5.3.7, this criteria was also obtained via other works in infinite graph theory, mainly those due to Polat (1996b) and Robertson, Seymour and Thomas (1991). However, all these approaches are somehow equivalent, once they rely on the characterization below for graphs that contain subdivisions of infinite cliques. Theorem 12.6.9 of the book (DIESTEL, 2018) discusses these similarities.

Exercise 5.3.9. *Prove that a graph G contains a subdivision of an infinite clique if, and only if, there is a ray r in G which is dominated by infinitely many vertices. Note that the proof of Corollary 5.3.7 provides the less trivial implication.*

We will now turn our attention back to the spaces of the form $|G|$, aiming to finally prove Theorem 5.3.1. The existence of a normal spanning tree in $|G|$ will follow from a clever application of Proposition 5.3.4, which is inspired by the identification below:

Lemma 5.3.10 (Polat (1996a), 4.16)**.** *Let G be a graph. For every $v \in V(G)$, fix a "new" artificial ray r_v that starts at v and such that $r_v \cap V(G) = \{v\}$. Then, $\overline{V(G)} = V(G) \cup \Omega(G) \subset |G|$ endowed with its subspace topology is homeomorphic to $\Omega(G^+)$, where G^+ is the graph given by*

$$G^+ = G \cup \bigcup_{v \in V(G)} r_v$$

Proof. Since G is a subgraph of G^+, the rays which are equivalent in G are also equivalent in G^+. Now, if $S \subset V(G)$ is any finite set, suppose that two rays r and r' have their tails in distinct connected components C and C' of $G \setminus S$. Then, in $G^+ \setminus S$, the tails of r and r' lie on the disjoint connected components given by $C \cup \bigcup_{v \in C} r_v$ and $C' \cup \bigcup_{v \in C'} r_v$. This proves that the inclusion map $\iota : \Omega(G) \to \Omega(G^+)$ is indeed well defined and injective.

For a vertex $v \in V(G)$, in its turn, the connected component of $G^+ \setminus \{v\}$ containing a tail of r_v is precisely that tail. As a consequence, r_v is equivalent to no other ray of G neither of the form r_u with $u \neq v$. This means that $\Omega(G^+) = \Omega(G) \cup \{[r_v] : v \in V(G)\}$. Moreover, the function $\varphi : V(G) \cup \Omega(G) \to \Omega(G^+)$ defined by

$$\varphi(x) = \begin{cases} \iota(x), & \text{if } x \in \Omega(G); \\ [r_x], & \text{if } x \in V(G). \end{cases}$$

is thus a bijection. We observe that φ is continuous in vertices of G, since the inherited topology on $V(G) \subset |G|$ is discrete. Now, given an end $[r] \in \Omega(G)$, fix a representative r and a finite set $S \subset V(G^+)$. Assume even that $r \cap (V(G^+) \setminus V(G)) = \emptyset$. Clearly, it is also finite the set $S' = (S \cap V(G)) \cup \{v \in V(G) : S \cap r_v \neq \emptyset\}$. Denote by C the connected component of $G \setminus S'$ containing a tail of r. If $v \in C$, the ray r_v is contained in the connected component of $G^+ \setminus S$ in which r has its tail, since $S \cap r_v = \emptyset$. Analogously, if $[s] \in \overline{C}$ is an end, fix a representative s that is contained in G. Then, in $G \setminus S'$, the tails of r and s also belong to the same connected component. This analysis proves that φ is continuous in the end $[r]$.

Conversely, for every $v \in V(G)$, since $r_v \setminus \{v\}$ is precisely the connected component of $G^+ \setminus \{v\}$ that contains a tail of r_v, the singleton $\{[r_v]\}$ is an open set in $\Omega(G^+)$. In particular, φ^{-1} is continuous in $[r_v]$. Now, consider an end $[r] \in \Omega(G^+) \cap \Omega(G)$ and a finite set $S \subset V(G)$. Again, we fix a representative r which is contained in G. Let $[s] \in \Omega(G^+)$ be an end that has also a representative in the connected component of $G^+ \setminus S$ where r has its tail.

If $[s] = [r_v]$ for some $v \in V(G)$, this means that there is a path P in $G^+ \setminus S$ connecting v to a tail of r. Since $P \cap r_u \subseteq \{u\}$ for every $u \in V(G)$, we observe that $P \subset V(G)$. Therefore, v and a tail of r belong to the same connected component of $G \setminus S$. Now, suppose that $[s] \in \Omega(G^+) \cap \Omega(G)$. As before, we can choose a representative of $[s]$ which is contained in G. Fix P a path in $G^+ \setminus S$ connecting the tails of r and s. Again, $P \cap (r_u \setminus \{u\}) = \emptyset$ for every $u \in V(G)$, because both r and s are contained in G. Then, P itself is contained in G, proving that r and s have their tails in the same connected component of $G \setminus S$. This verifies that φ^{-1} is continuous in $[r]$.

□

Note that Theorem 5.3.4 and Lemma 5.3.10 combined proves a first implication which Theorem 5.3.1 claims. In fact, if $|G|$ is a metric space, then so is the subspace $\overline{V(G)}$. But, if G^+ is constructed as in the above result, this means that $\Omega(G^+)$ is a metrizable topological space as well. By Theorem 5.3.4, then, there is an end-faithful normal tree T^+ in G^+. In particular, for every $v \in V(G)$, the tree T contains a ray r which is equivalent to r_v. Once T is connected, r must contain the vertex v. In other words, $T \cap V(G) = V(G)$, so that $T[V(G)]$ is a normal spanning tree for G. Hence, aiming to conclude Theorem 5.3.1, it only remains to verify the observation below:

Proposition 5.3.11. *If G admits a normal spanning tree, then $|G|$ is metrizable.*

Proof, following Theorem 3.1 (i) of Diestel (2006). For every $\varepsilon > 0$, consider the metric d_ε over $[0,1]$ given by $d_\varepsilon(x,y) = \varepsilon|x-y|$ for every pair $x,y \in [0,1]$. Note that d_ε induces the usual Euclidean topology on $[0,1]$. Now, fix a normal spanning tree T for G, whose tree order will be denoted by $\leq$. Note that T is end-faithful, as discussed after the proof of Theorem 5.3.4. We will first define a metric d over the vertices of T and its edges: given $uv \in E(T)$ with $u < v$, we set $d(u,v) = \frac{1}{2^{n+1}}$, where n denotes the height of u in T. For inner points $x,y \in [u,v]$, we consider $d(x,y) = d_{\frac{1}{2^{n+1}}}(x,y)$, observing that $d(u,v) = d_{\frac{1}{2^{n+1}}}(u,v)$ when taking $x = u$ and $u = v$. Then, for arbitrary vertices $u,v \in V(T) = V(G)$, we set

$$d(u,v) = \sum_{i=1}^{k} d(v_{i-1},v_i),$$

in which $v_0v_1v_2\ldots v_k$ is a presentation of the unique path in P connecting $v_0 = u$ and $v_k = v$. In its turn, for an end $[r] \in \Omega(G) \simeq \Omega(T)$, fix a representative $r = v_0v_1v_2\ldots$ which starts at the root

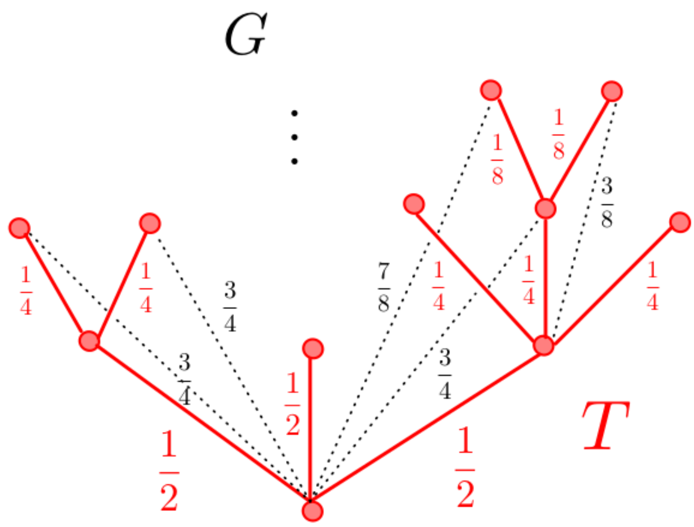
$v_0 = z$ of T. If $v \in V(T) = V(G)$ is any vertex, we define $k = \max\{i \in \mathbb{N} : v_i \leq v\}$ and set

$$d(v,[r]) = d(v,v_k) + \lim_{i\to\infty} d(v_k,v_i) = d(v,v_k) + \sum_{i=k}^{\infty} d(v_i,v_{i+1}).$$

Since r is a ray in T, the series $\sum_{i=k}^{\infty} d(v_i,v_{i+1})$ converges, because it is geometric of ratio $\frac{1}{2}$. Thus, $d(v,[r])$ is indeed well-defined.

Now, if $[s] \in \Omega(G)$ is chosen to be distinct of r, we also consider the representative s that starts at z and define $x = \max r \cap s = \max\{v \in V(T) : v \in r \cap s\}$. Then, we set $d([r],[s]) = d([r],x) + d(x,[s])$.

Figure 28 – A metric for $|G|$



If T is a normal spanning tree in a graph G, we first define a metric over its vertices. This is done so that edges connecting vertices of height n to vertices of height $n+1$ have length $\frac{1}{2^{n+1}}$. Next, while respecting the triangle inequality, this metric is extended to the remaining edges of G. Finally, the distances to ends of G are calculated via geometric series.

Source: Elaborated by the author.

Finally, we will extend d to the edges of $E(G) \setminus E(T)$ as Figure 28 suggests. Given $uv \in E(G) \setminus E(T)$, we can assume that $u < v$, since T is a normal spanning tree in G. Once $d(u,v)$ is already defined, we set $d|_{[u,v]} = d_{d(u,v)}$, i.e., $d(x,y) = d(u,v)|x-y|$ for every $x,y \in [u,v] \simeq [0,1]$. More generally, if x and y are points from edges $u_1u_2 = [u_1,u_2]$ and $v_1v_2 = [v_1,v_2]$, we consider $d(x,y) = \min_{i,j\in\{1,2\}} [|x-u_i| + d(u_i,v_j) + |y-v_j|]$. If $[r] \in \Omega(G)$, in its turn, we set $d(x,[r]) = \min_{i\in\{1,2\}} d(u_i,[r])$. This finishes the definition of d on $|G|$, which turns out to be a metric by construction. Below, we briefly argue how d induces the topology of $|G|$:

- Under both $|G|$ (with its usual topology) and $(|G|,d)$, the open neighborhoods around inner points of edges are open Euclidean intervals;

- If v is a vertex, let n denote its height in T. Following the notation of (5.1), if $V_\varepsilon(v)$ is a basic open neighborhood of v in $|G|$, then $\left\{x \in |G| : d(x,v) < \frac{1}{m}\right\} \subset V_\varepsilon(v)$ whether m is chosen to be bigger than $\frac{n}{\varepsilon}$. This means that $V_\varepsilon(v)$ is also a basic open neighborhood for v considering the metric d. On the other hand, clearly, every open ball around v of radius less than $\frac{1}{n}$ is open in $|G|$;
- Let $[r] \in \Omega(G)$ be an end of G. As usual, fix the representative $r = v_0v_1v_2\ldots$ of $[r]$ which is a ray of T that starts in its root. Then, v_n has height n for each $n \in \mathbb{N}$, so that, for every $0 < \varepsilon < 1$ and following the notation of (5.3), $\hat{C}(\lceil v_{n+1}\rceil, [r], \varepsilon)$ is contained in the open ball of radius $\frac{1}{2^{n+1}}$ around $[r]$. In other words, the open balls around $[r]$ are indeed open neighborhoods for this end in $|G|$. Conversely, if $S \subset V(G)$ is finite, then so is $\lceil S\rceil = \bigcup_{s\in S} \lceil s\rceil$. Fixing $n = \min\{i \in \mathbb{N} : v_i \notin \lceil S\rceil\}$, we observe that the open ball of radius $\frac{1}{2^{n+1}}$ around $[r]$ is contained in $\hat{C}(\lceil S\rceil, [r], \varepsilon)$ for every $0 < \varepsilon < 1$. After all, if uv is an edge that has an endpoint in $\lceil S\rceil$ and the other in $C(\lceil S\rceil, [r])$, then $d(u,v) \geq \frac{1}{2^n}$ by construction. Therefore, every basic open neighborhood around $[r]$ as in (5.3) is also an open basic neighborhood for this end regarding the metric d.

□

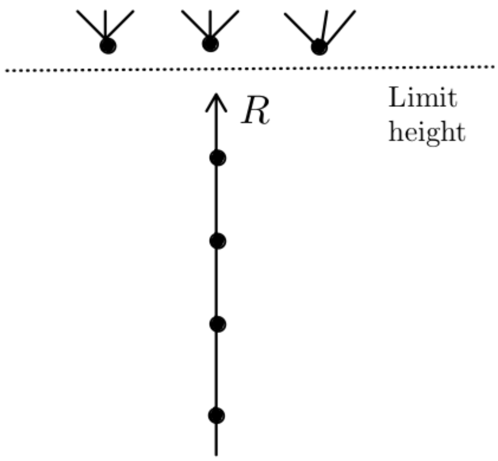
5.4 A closer look to $\Omega(G)$

As pointed out in the Introduction, Diestel's question regarding the characterization of end spaces was solved only recently by Pitz (2023). This section brings an overview of his answer, but omitting some technical details that can be found in the original paper. In particular, we also discuss the representation result available in (KURKOFKA; PITZ, 2023), which strongly supports the mentioned solution. This latter work shows that every end space arises from a generalized notion of tree, extending the observations made by Theorem 5.3.4 and Proposition 5.3.3 about the description of metric spaces of the form $\Omega(G)$.

Hence, introducing the main notation of this section, we recall that, in a set-theoretic context, an **(order) tree** is a partially ordered set $\langle T, \leq\rangle$ where $\lceil \mathring{t}\rceil = \{s \in T : s < t\}$ is well-ordered by $\leq$ for every **node** $t \in T$. The order type $\mathrm{h}(t)$ of $\lceil \mathring{t}\rceil$ is called the **height** of the node $t \in T$, so that, for a given ordinal α, the set $\mathscr{L}_\alpha(T) = \{t \in T : t \text{ has height } \alpha\}$ defines the $\alpha-$**level** of T. If there is a **predecessor** $s = \max\lceil \mathring{t}\rceil$, we say that $t \in T$ is a **successor** point. Otherwise, the cofinality of $\lceil \mathring{t}\rceil$ is infinite and we say that t is a **limit** point of T. For the least ordinal α such that $\mathscr{L}_\alpha(T) = \emptyset$, we say that α is the **height of** T. Finally, we also denote $\lceil t\rceil = \{s \in T : s \leq t\}$ and $\lfloor t\rfloor = \{s \in T : s \geq t\}$.

Throughout this section, trees will always be **rooted**, i.e., will always satisfy $|\mathscr{L}_0(T)| = 1$. Moreover, a set $R \subset T$ in which the order $\leq$ is total is called a **chain** of T, while a maximal one is called a **branch**. On the other hand, an **antichain** in T is a set of pairwise incomparable elements regarding $\leq$. If T has no infinite branches, we say that T is **rayless**. Now, following the notation of (KURKOFKA; PITZ, 2023), we say that $R \subset T$ is a **high-ray** if it is a down-closed chain of cofinality ω.

Figure 29 – Example of an infinite chain and three of its tops.



Source: Elaborated by the author.

In particular, when endowed with its usual order after fixing a root, a *graph-theoretic* tree T as defined in Section 2.2 is also a tree in this set-theoretic notion. In this case, since every vertex has a finite distance to the root, the tree T has height bounded by ω, so that every infinite chain is a high-ray and a branch. However, in arbitrary tree orders, we might find ramifications at limit levels, meaning that some infinite chains are not maximal. More precisely, if R is an infinite chain of T, we say that $t \in T$ is a **top** of R if it is a $\leq -$minimal upper bound of this set. In other words, $t > s$ for every $s \in R$ and, if $t' < t$, there is $s \in R$ such that $t' < s < t$. In particular, every top has limit height, as well as every node t of limit height is the top of the infinite chain $\lceil \mathring{t} \rceil$.

Although order trees are defined as abstract posets, Figure 29 suggests that they can be seen as graphs somehow. Within a notation first set by Diestel and Leader (2001), we say that a $T-$**graph** is any graph obtained from an order tree T after the following definition of edges: we declare every successor node as neighbor of its predecessor and, for a limit $t \in T$, we fix an edge set connecting it to a cofinal sequence in $\lceil \mathring{t} \rceil$. In addition, if G is a $T-$graph, we call it **uniform** if, for every limit node $t \in T$, there is a finite set $S_t \subset \lceil \mathring{t} \rceil$ such that $N(t') \cap \lceil \mathring{t} \rceil \subseteq S_t$ for every $t > t'$.

In particular, the above paragraph shows how to construct $T-$graphs for every order tree T. Describing uniform $T-$graphs, on the other hand, is not a trivial task. Actually, some set-theoretic obstructions do not allow the existence of uniform $T-$graphs. For example, let G be any ω_1-graph. After applying the Pressing-Down Lemma iteratively[9], we can find an infinite subset $\{\alpha_n\}_{n\in\mathbb{N}}$ and a limit ordinal $\xi > \sup\limits_{n\in\mathbb{N}}$ such that: for every $n \in \mathbb{N}$, there is an edge of the form $\alpha_n\eta$ for some $\eta > \xi$. In other words, G is not uniform. In fact, a criteria for the existence of uniform graphs is given by:

Lemma 5.4.1 (Kurkofka and Pitz (2023), Theorem 4.5(i) and Proposition 5.4)**.** *Fix T an order*

[9] See Lemma III.6.14 in (KUNEN, 2011).

tree. Then, there exists an uniform $T-$graph if, and only if, T is a special tree. In this case, any two uniform $T-$graphs have homeomorphic end spaces.

In the above statement, we recall that an order tree T is **special** if we can write $T = \bigcup_{n\in\mathbb{N}} A_n$ for some countable family of antichains $\{A_n : n \in \mathbb{N}\}$ in T. Then, as the main result of the paper Kurkofka and Pitz (2023), the special order trees codify the family of all end spaces:

Theorem 5.4.2 (Kurkofka and Pitz (2023), Theorem 1)**.** *For every graph G, there exists a special order tree T such that any uniform $T-$graph has its end space homeomorphic to $\Omega(G)$.*

If $\Omega(G)$ is a metric space, then T as in the above result might be chosen to be any end-faithful normal tree of G, as previously observed by Theorem 5.3.1 and Proposition 5.3.3. If this is not the case, however, we must search for a suitable (special order) tree whose height is even greater than ω. Following the original construction of Kurkofka and Pitz (2023), this is done when approximating the claimed tree T by iterative applications of depth-search algorithms. More precisely, for an (order) tree T, a pair $(T,\mathscr{V})$ is called a **partition tree** of G if $\mathscr{V} = \{V_t : t \in T\}$ is a partition of $V(G)$ into connected subsets satisfying the properties below:

- $|V_t| = 1$ if $t \in T$ is not a limit point;
- The graph $\dot{G} := \frac{G}{\mathscr{V}}$ obtaining by contracting each part of $\mathscr{V}$ to a single vertex is a $T-$graph;
- For each successor $t \in T$, the neighborhood

$$N(V_{\lfloor t\rfloor}) = \left\{u \in V(G)\setminus V_{\lfloor t\rfloor} : u \text{ has a neighbor in } V_{\lfloor t\rfloor}\right\}$$

  is finite, where $V_{\lfloor t\rfloor} = \bigcup_{s\geq t} V_s$. In this case, we say that $(T,\mathscr{V})$ has **finite adhesion**.

Hence, by tracking some rays of G, we are able to describe high-rays of T. In other words, given an end $[r] \in \Omega(G)$, it is well-defined the set

$$\Theta([r]) = \{t \in T : r \text{ has a tail in } V_{\lfloor t\rfloor}\}. \tag{5.4}$$

This is clearly a down-closed chain of T and, by Lemma 6.2 of (KURKOFKA; PITZ, 2023), it has countable cofinality. When this cofinality is indeed infinite, we say that $\Theta([r])$ **corresponds** to the end $[r]$, because $\Theta([r])$ is thus an element of $\mathscr{R}(T) = \{\text{high-rays of } T\}$. If every $[r] \in \Omega(G)$ corresponds to precisely one high-ray of T, in the sense that $\Theta : \Omega(G) \to \mathscr{R}(T)$ is a bijection, we even say that the partition tree $(T,\mathscr{V})$ **displays all the ends** of G. In this case, comparing to the definition of end-faithful subgraph presented in Section 5.3, T may be understood as an "end-faithful normal (order) tree" of G. A first step when proving Theorem 5.4.2 is to ensure the existence of such decomposition:

Proposition 5.4.3 (Kurkofka and Pitz (2023), Theorem 7.3)**.** *Every graph admits a partition tree that display all its ends.*

Sketch of the proof. We will only describe how the claimed partition tree $(T,\mathscr{V})$ can be constructed. The proof that it displays all the ends of G can be consulted in the original paper of Kurkofka and Pitz (2023). For some cardinal κ, we will recursively define a sequence of partition trees $\{(T_\alpha,\mathscr{V}_\alpha)\}_{\alpha<\kappa}$ for G.

Let T_0' be a maximal normal tree for this graph, whose tree-order is denoted by $\leq$. Consider F_0 the set of connected components of $G\setminus T_0'$, so that each $C\in F_0$ has its infinite neighborhood $N(C)$ contained in a ray r_C of T_0'. Then, the order $\leq$ can be extended to $T_0 := T_0'\cup F_0$, by declaring $C>t$ for every $C\in F_0$ and every $t\in r_C$. If we set $V_t^0=\{t\}$ for each $t\in T_0'$ and $V_C^0=V(C)$ for each $C\in F_0$, we partition G with the family $\mathscr{V}_0=\{V_t^0 : t\in T_0'\}\cup\{V_C^0 : C\in F_0\}$. It is easily verified that $(T_0,\mathscr{V}_0)$ is indeed a partition tree for G.

Now, for some $\alpha<\kappa$, suppose that we have defined a partition tree $(T_\beta,\mathscr{V}_\beta)$ for every $\beta<\alpha$. In addition, we denote by κ_β the height of T_β. If $\alpha=\beta+1$ for some $\beta<\alpha$, we assume by induction that the final level $F_\beta := \mathscr{L}_{\kappa_\beta}(T_\beta)$ has the following form:

$$C\in F_\beta \iff C \text{ is a connected component of } G\setminus \bigcup_{t\in T_\beta\setminus F_\beta} V_t. \tag{5.5}$$

Moreover, for each $C\in F_\beta$, we suppose that its neighborhood $N(C)=\{v\in V(G)\setminus C : v \text{ has a neighbor in } C\}$ is contained in a ray r_C such that, following the notation of (5.4), $\Theta([r_C])$ is a high-ray of T_β. Then, for every $C\in F_\beta$ we can apply Lemma 7.2 from (KURKOFKA; PITZ, 2023) to obtain $U_C\subset V(C)$ a connected vertex set that encodes suitable topological properties, to be further detailed in this section. In particular, we mention:

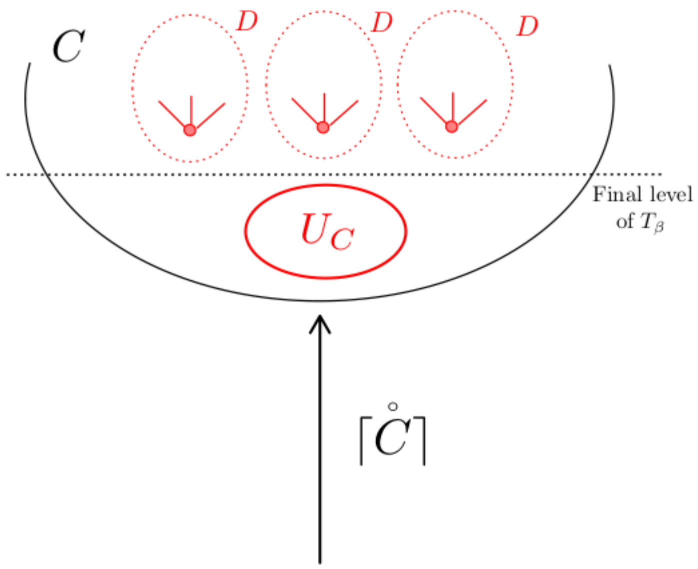
`Fact:` Any connected component D of $C\setminus U_C$ has finite neighborhood in G. In other words, the set $N(D)=\{v\in G\setminus D : v \text{ has a neighbor in } D\}$ is finite.

Then, rooted at a vertex that has some neighbor in U_C, we can fix T_D a maximal normal tree[10] for D. Hence, every connected component D' of $D\setminus T_D$ has infinitely many neighbors within a branch $r_{D'}$ of T_D. Then, writing $\mathscr{V}_\beta=\{V_t^\beta\}_{t\in T_\beta}$ and $\mathscr{V}_\alpha=\{V_t^\alpha\}_{t\in T_\alpha}$, the partition tree $(T_\alpha,\mathscr{V}_\alpha)$ can be described as follows:

- The tree T_α extends the order tree T_β, obtained after adding, for each $C\in F_\beta$, the nodes of T_D for every connected component D of $C\setminus U_C$. In this case, we set $t>C$ for every $t\in T_D$, as Figure 30 suggests. Moreover, for each connected component D' of $D\setminus T_D$, we also see D' as a node of T_α, defining $D'>t$ for every $t\in r_{D'}$;
- We set $V_t^\alpha=V_t^\beta$ for every $t\in T_\beta\setminus F_\beta$. Given $C\in F_\beta$ and a connected component D of $C\setminus U_C$, however, we define $V_C^\alpha=U_C$ and $V_t^\alpha=\{t\}$ for every $t\in T_D$. Finally, we set $V_{D'}^\alpha=V(D')$ for every connected component D' of $D\setminus T_D$.

[10] Note that T_D could be obtained, for example, by applying Proposition 5.3.6 to $G[D\cup N(D)]$ with $K=N(D)$. This observation will further support the main discussion in Section 6.4.

Figure 30 – Construction of T_α from T_β



In black, we draw a node C in the final level of a partition tree $(T_\beta, \mathscr{V}_\beta)$. In red, we sketch how $T_{\beta+1}$ is obtained from T_β.

Source: Elaborated by the author.

Then, $\frac{G}{\mathscr{V}_\alpha}$ is indeed a $T_\alpha-$graph and $|V_t^\alpha| = 1$ for every successor $t \in T_\alpha$. Relying on the above `Fact`, the choice of V_t^α for a limit node $t \in F_\beta$ guarantees that $(T_\alpha, \mathscr{V}_\alpha)$ has finite adhesion.

Now, suppose that α is a limit ordinal. Assuming that T_γ extends the tree order of T_β if $\beta < \gamma < \alpha$, it is well defined the limit tree $T'_\alpha = \bigcup\limits_{\beta<\alpha} T_\beta$. Then, we can also define $\mathscr{V}'_\alpha = \{V_t^\alpha\}_{t\in T'_\alpha}$ as follows: given $t \in T'_\alpha$, there is $\beta < \alpha$ a successor ordinal such that $t \in T_\beta \setminus F_\beta$, by the construction on the previous paragraphs. Thus, we set $V_t^\alpha = V_t^\beta$, observing that $V_t^\alpha = V_t^\gamma$ for every $\beta \leq \gamma < \alpha$.

Finally, if C is a connected component of $G \setminus \bigcup\limits_{t\in T'_\alpha} V_t^\alpha$, we observe by (5.5) that, for every $\beta < \alpha$, there is $C_\beta \in F_\beta$ which contains C. Also by the construction on the previous paragraphs, $\{C_\beta\}_{\beta<\alpha}$ (as a set o nodes) is a chain in T'_α, meaning that it is contained in some infinite branch R_C. Hence, we extend T'_α to an order tree T_α by declaring C as a top of R_C for every such C. In its turn, considering $V_C^\alpha = C$, this extends the family $\mathscr{V}'_\alpha$ to the partition of $V(G)$ given by $\mathscr{V}_\alpha = \{V_t^\alpha\}_{t\in T_\alpha}$.

In order to conclude that $(T_\alpha, \mathscr{V}_\alpha)$ is indeed a partition tree for G, we must verify whether $\dot{G} = \frac{G}{\mathscr{V}_\alpha}$ is a $T_\alpha-$graph. By induction on α, it suffices to show that, for each connected component C of $G \setminus \bigcup\limits_{t\in T'_\alpha} V_t^\alpha$, the set $T_C := \{t \in R_C : V_t$ contains a vertex that has a neighbor in $C\}$ is cofinal in R_C. In fact, given $v \in T_C$, we have $v \in T_\beta$ for some $\beta < \alpha$. In addition, v has a neighbor in C_β, which proves that $v \in R_C$ since $\frac{G}{\mathscr{V}_\beta}$ is a $T_\beta-$graph. In other words, we indeed have $T_C \subset R_C$. For instance, suppose that T_C is not cofinal in R_C. Hence, there is $t \in R_C$ an upper bound for T_C.

Choosing t as a successor node in $T_\beta \setminus F_\beta$ for some $\beta < \alpha$, we conclude that C is a connected component of $G \setminus \bigcup_{t \in T_\beta \setminus F_\beta} V_t^\beta$. This implies that $C \in F_\beta$, which is a contradiction. Hence, T_C is in fact cofinal in R_C, finishing the verification that $(T_\alpha, \mathscr{V}_\alpha)$ is a partition tree for G.

Finally, the claimed end-faithful partition tree $(T, \mathscr{V})$ arises when considering $T = T_\alpha$ and $\mathscr{V} = \mathscr{V}_\alpha$ for the first ordinal α such that $T_\alpha = T_{\alpha+1}$.

□

Corollary 5.4.4. *Let G be a graph and fix $(T', \mathscr{V}')$ a partition tree which displays all its ends. Then, there is T a special order tree such that the end space of any uniform T−graph is homeomorphic to $\Omega(G)$. Moreover, T can be chosen to have the same height as T'.*

Sketch of the proof. Let T be obtained from T' according to the procedure below:

1. For every limit node $t \in T'$, denote by $S(t)$ the set of its successors, if there are some. Then, since $(T', \mathscr{V}')$ has finite adhesion, the set $N_s := N(V_{\lfloor s \rfloor})$ is finite for each $s \in S(t)$;
2. Now, for every limit node $t \in T'$ that has a successor and every finite $X \subset \lceil t \rceil$, we declare a new node $v(t, X)$ to be a successor of t and a predecessor of each $s \in S(t)$ with $N_s = X$. We then remove t.

Since only non-empty levels of T' were modified to construct T, the heights of these two trees are the same. The verification that the end space of every uniform T−graph is homeomorphic to $\Omega(G)$ can be consulted in the original proof of Theorem 1 in (KURKOFKA; PITZ, 2023). □

Corollary 5.4.4 is a detailed rephrase of Theorem 5.4.2, which is, unless by the omitted steps, now concluded. However, from now on in this section, we will mention other results that are either used to fulfill the details of the above arguments or obtained from them with some additional efforts.

First, we shall discuss which techniques are employed when choosing U_C as in the `Fact` within the proof of Theorem 5.4.3. In the verification that the partition tree just constructed indeed display all the ends of G, one of the roles played by U_C is to isolate the corresponding end $[r_C]$ from the others. Inspired by this idea, for a given set of vertices $U \subset V(G)$ of a graph G, we say that a superset $U^* \supseteq U$ is an **envelope** for U if the following two properties are verified:

- U^* has finite adhesion, in the sense that $N(C) = \{v \in U^* : v \text{ has a neighbor in } C\}$ is finite for every connected component C of $G \setminus U^*$;
- $\overline{U^*} \setminus U^* = \overline{U} \setminus U$, i.e., the boundaries of U and U^* in the topological space $|G|$ define the same set of ends.

Roughly speaking, U^* may be obtained from U after adding vertices and rays that are infinitely connected to its elements, which is formalized by the below result:

Lemma 5.4.5 (Kurkofka and Pitz (2023), Theorem 3.2)**.** *Every infinite subset $U \subset V(G)$ admits an envelope.*

Proof. Let $\mathscr{W}$ be a $\subseteq -$maximal family of disjoint spines of combs whose teeth belong to U. Note that $\mathscr{W}$ exists by Zorn's Lemma. In its turn, denote by S be the set of all vertices which are infinitely connected to U. More precisely, S comprises the centers of infinite stars whose leaves belong to U. We will prove that $U^* = U \cup S \cup \bigcup \mathscr{W}$ holds as an envelope for U.

For a while, suppose that U^* does not have finite adhesion. This means that $N(C)$ is infinite for some connected component C of $G \setminus U^*$. Then, consider the connected graph $\hat{C}$ obtained from C by adding, for every $v \in N(C)$, only one edge connecting v to some of its neighbors in C. Hence, by the Star-Comb Lemma (2.1.2), one of the following items must hold, but both lead to contradictions:

- There is a vertex $v \in C$ which is a center of an infinite star in $\hat{C}$ whose set of leaves L belong to $N(C)$. In particular, there is a path P_0 in G, containing some $v_0 \in L$, that connects v to a vertex $u_0 \in U$. For some $n \in \mathbb{N}$, suppose that we have defined a sequence of paths $P_0, P_1, \ldots, P_n$ which are disjoint unless by v. Observe that the set $S = \bigcup_{i=0}^{n} V(P_i) \setminus \{v\}$ is finite. Since the elements of $\mathscr{W}$ are disjoint, we suppose that $r \setminus S$ is a tail of r for every $r \in \mathscr{W}$, unless by adding to S initial segments of finitely many rays from $\mathscr{W}$. In $\hat{C} \setminus S$, then, there is a path P'_{n+1} connecting v to some vertex $v_{n+1} \in L \setminus S$. If $v_{n+1} \in S$, we can extend P'_{n+1} to a path P_{n+1} in $G \setminus S$ whose endpoint other than v is some $u_{n+1} \in U$. Analogously, if $v_{n+1} \in r$ for some $r \in \mathscr{W}$, we can extend P'_{n+1} to a path P_{n+1} in $G \setminus S$ whose endpoint other than v is some $u_{n+1} \in U$. In this case, P'_{n+1} contains a finite subpath of $r \setminus S$ which starts at v_{n+1}, while u_{n+1} is some teeth of the comb in which r is a spine. In other words, at the end of this inductive construction, the graph $\bigcup_{n \in \mathbb{N}} P_n$ describes a star centered at v whose leaves are $\{u_n\}_{n \in \mathbb{N}} \subset U$, contradicting the fact that $v \notin S$;
- There is a ray s which is the spine of a comb in $\hat{C}$ whose teeth belong to U. In particular, there is a path P_0 in G, containing some $v_0 \in L$, that connects s to a vertex $u_0 \in U$. For some $n \in \mathbb{N}$, suppose that we have defined a finite sequence of disjoint paths $P_0, P_1, \ldots, P_n$. Observe that the set $S = \bigcup_{i=0}^{n} V(P_i)$ is also finite. Since the elements of $\mathscr{W}$ are disjoint, we suppose that $r \setminus S$ is a tail of r for every $r \in \mathscr{W}$, unless by adding to S initial segments of finitely many rays from $\mathscr{W}$. In $\hat{C} \setminus S$, then, there is a path P'_{n+1} connecting a tail of s to some vertex $v_{n+1} \in L \setminus S$. If $v_{n+1} \in S$, we can extend P'_{n+1} to a path P_{n+1} in $G \setminus S$ whose endpoint other than the one in s is some $u_{n+1} \in U$. Analogously, if $v_{n+1} \in r$ for some $r \in \mathscr{W}$, we can extend P'_{n+1} to a path P_{n+1} in $G \setminus S$ whose endpoint other than the one in s

is some $u_{n+1} \in U$. In this latter case, P'_{n+1} contains a finite subpath of $r \setminus S$ which starts at v_{n+1}, while u_{n+1} is some teeth of the comb for which r is a spine. In other words, at the end of this inductive construction, the graph $s \cup \bigcup_{n\in\mathbb{N}} P_n$ describes a comb whose spine is s and whose teeth are $\{u_n\}_{n\in\mathbb{N}} \subset U$. However, since $s \subset C$ by the definition of $\hat{C}$, the family $\mathscr{W} \cup \{s\}$ contradicts the $\subseteq -$maximality of $\mathscr{W}$.

Therefore, U^* has finite adhesion. It remains to show that $\overline{U^*} \setminus U^* = \overline{U} \setminus U$ in the topological space $|G|$. In fact, if $[s] \in \overline{U}$ is an end, then the representative s is the spine of a comb whose teeth belong to U. In particular, these teeth belong to U^*, so that $[s] \in \overline{U^*}$. This proves that $\overline{U^*} \setminus U^* \supseteq \overline{U} \setminus U$. Conversely, suppose for a contradiction that $[s] \notin \overline{U}$ for some $[s] \in \overline{U^*}$. Then, there is a finite set $F \subset V(G)$ such that $U \cap C(F,[s]) = \emptyset$. Since the elements of $\mathscr{W}$ are disjoint, we suppose that $r \setminus F$ is a tail of r for every $r \in \mathscr{W}$, unless by adding to F initial segments of finitely many rays from $\mathscr{W}$. However, in $G \setminus F$ there is a path P' connecting a tail of s to some vertex $v \in U^*$, since $[s] \in \overline{U^*}$. If $v \in S$, then P' can be extended to a path P in $G \setminus F$ which connects a tail of s to a vertex $u \in U$, by definition of S. Analogously, if $v \in r \setminus F$ for some $r \in \mathscr{W}$, then P' can also be extended to a path P in $G \setminus F$ that connects v to a vertex $u \in U$. In this case, P' contains a finite subpath of r, while u is some teeth of the comb for which r is a spine. In both scenarios, however, we find $u \in U \cap C(F,[r])$, which is a contradiction. Therefore, the inclusion $\overline{U^*} \setminus U^* \subseteq \overline{U} \setminus U$ is verified. □

Fixed a special tree T, we observe that Lemma 5.4.1 suggests the definition of a topology over $\mathscr{R}(T) = \{\text{high-rays of } T\}$: the **ray space** of T now consists in the end space of some (and, thus, any) uniform $T-$graph G. Indeed, considering the partition tree of G given by $(T, \{t\}_{t\in T})$, the map $\Theta : \Omega(G) \to \mathscr{R}(T)$ as defined in (5.4) is a bijection, so that the point set of $\mathscr{R}(T)$ can be identified with $\Omega(G)$. Within this notation, Theorem 5.4.2 claims that the topological spaces which arise as end spaces of graphs are precisely the ones which arise as ray spaces of special trees.

Although this is not a purely topological description of the class of all end spaces, such characterization allows alternative approaches for studying that family. For example, the topology of $\mathscr{R}(T)$ can also be declared intrinsically in terms of the tree T. More precisely, an open basic neighborhood around a high-ray $R \in \mathscr{R}(T)$ can be chosen as a set of the form

$$[t,F] = \{s \in \mathscr{R}(T) : t \in s \text{ and } t' \notin s \text{ for every } t' \in F\}, \tag{5.6}$$

where $t \in R$ and $F \subset T$ is a finite collection of tops of R. In subsection 2.4 of (PITZ, 2023), Pitz formalizes the equivalence between these two ways of defining a topology on $\mathscr{R}(T)$.

In addition, the above system of open basic neighborhoods defines a basis $\mathscr{B}$ for $\mathscr{R}(T)$ which is generated by a natural clopen subbase for this space. In fact, $\mathscr{B}$ is obtained when closing the family $\mathscr{C} = \{[t] : t \in T\}$ under complements and finite intersections, where $[t] :=$

$\{R \in \mathscr{R}(T) : t \in R\}$ is a clopen set in $\mathscr{R}(T)$ for every $t \in T$. Since T is a special tree, it is rather easy to verify the axioms below for this family $\mathscr{C}$:

- $\mathscr{C}$ is **nested**, i.e., for every pair $U, V \in \mathscr{C}$ such that $U \cap V \neq \emptyset$, we have $U \subseteq V$ or $V \subseteq U$;
- $\mathscr{C}$ is **noetherian**, i.e., if $U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots$ is an $\subseteq -$increasing sequence of elements from $\mathscr{C}$, then there is $n \in \mathbb{N}$ such that $U_n = U_{n+k}$ for every $k \in \mathbb{N}$;
- $\mathscr{C}$ is **hereditarily complete**, i.e., the family $\mathscr{C}_Y = \{U \cap Y : U \in \mathscr{C}\}$ is complete for every closed subspace $Y \subset \mathscr{R}(T)$. More precisely, every subfamily of $\mathscr{C}_Y$ which satisfies the finite intersection property has non-empty intersection itself;
- $\mathscr{C}$ is $\sigma-$**disjoint**, i.e., we can write $\mathscr{C} = \bigcup_{n \in \mathbb{N}} \mathscr{A}_n$ for some countable family $\{\mathscr{A}_n\}_{n \in \mathbb{N}}$ of antichains[11] in $\mathscr{C}$.

On the other hand, Pitz (2023) noticed that the above properties are precisely the ones required to characterize all the end spaces of graphs, finally answering Diestel's question presented in the Introduction. Among other descriptions of similar spaces (such as path and branch spaces of trees), a main result in his paper (PITZ, 2023) is recalled below:

Theorem 5.4.6 (Pitz (2023), Theorems 1.1 and 1.2)**.** *For a Hausdorff topological space X, the following properties are equivalent:*

i) $X \simeq \Omega(G)$ for some graph G;

ii) $X \simeq \mathscr{R}(T)$ for some special tree T;

iii) X admits a clopen subbase which is nested, noetherian, hereditarily complete and $\sigma-$disjoint.

In our preprint (AURICHI; REAL; JÚNIOR, 2023), written in a joint work with Paulo Júnior, we give an alternative interpretation for the above result in terms of a topological game. This is a clever approach, since games are often a simplified way to encode a tree. Intuitively, under suitable game, some winning strategy when playing over a topological space X as in item *iii*) can be used to construct a tree T as in item *ii*). As a poset, the elements of T are open sets of X, which are ordered via reverse inclusion. The restriction of Theorem 5.4.6 to the metric case illustrates this kind of construction:

Proposition 5.4.7. *X is a complete ultrametric space if, and only if, it is the end space of some (graph-theoretic) tree.*

Proof. Proposition 5.3.2 precisely claims that end spaces of (graph-theoretic) trees are complete ultrametric spaces. Conversely, fix (X, d) a complete ultrametric space. For every $x \in X$ and $n \geq 1$, denote by $B_{\frac{1}{n}}(x)$ the open ball of radius $\frac{1}{n}$ around x. For distinct points $x, y \in X$, we observe

[11] In this topological context, we recall that an *antichain* is a collection of pairwise disjoint open sets.

that, if there is $z \in B_{\frac{1}{n}}(x) \cap B_{\frac{1}{n}}(y)$, then $d(x,y) \leq \frac{1}{n}$ because d is an ultrametric. Hence, we must have $B_{\frac{1}{n}}(x) = B_{\frac{1}{n}}(y)$. In other words, we proved that the family $\mathscr{B}_n = \{B_{\frac{1}{n}}(x) : x \in X\}$ is an open cover for X whose elements are equal or disjoint.

Then, consider the tree T whose vertex set is given by $\{X\} \cup \bigcup_{n \in \mathbb{N}} \mathscr{B}_n$. We assume that T is rooted at X and we define an edge $XU \in E(T)$ for every $U \in \mathscr{B}_1$. Given $n \geq 1$ and $x \in X$, we also declare an edge in T connecting $B_{\frac{1}{n+1}}(x) \in \mathscr{B}_{n+1}$ to $B_{\frac{1}{n}}(x) \in \mathscr{B}_n$. This finishes the description of T, remaining to show that $X \simeq \Omega(T)$.

In fact, for every $x \in X$, denote by ray r_x in T starting at X and containing the vertices from $\{B_{\frac{1}{n}}(x)\}_{n \geq 1}$. Since the metric d is complete, it follows that, if r is a ray of T starting at X, then $\bigcap r = \bigcap_{U \in r} U \neq \emptyset$. Therefore, $r = r_x$ for some $x \in \bigcap r$. This proves that the map $\varphi : X \to \Omega(T)$ given by $x \mapsto [r_x]$ is surjective. It is injective as well: if $x \neq y$, there is $n \geq 1$ big enough so that $B_{\frac{1}{n}}(x) \cap B_{\frac{1}{n}}(y) = \emptyset$. In this case, r_x and r_y belong to distinct connected components of $T \setminus \{U \in T : U \supseteq B_{\frac{1}{n}}(x), B_{\frac{1}{n}}(y)\}$.

In order to show that φ is continuous, fix $\{x_k\}_{k \in \mathbb{N}} \subset X$ a sequence which converges to a given $x \in X$. Let $S \subset T$ be any finite set, so that, for some $n_0 \in \mathbb{N}$, we have $B_{\frac{1}{k_0}}(y) \notin S$ for all $y \in X$. In particular, for every $k \in \mathbb{N}$ such that $x_k \in B_{\frac{1}{n_0}}(x)$, the tail of r_{x_k} starting at $B_{\frac{1}{n_0}}(x)$ does not intersect S. Since $x_k \to x$ as $k \to \infty$, this proves that $\{[r_{x_k}]\}_{k \in \mathbb{N}}$ converges to $[r_x]$, concluding the continuity of φ in x.

Conversely, suppose that $\{[r_{x_k}]\}_{k \in \mathbb{N}}$ is a sequence in $\Omega(T)$ which converges to an end $[r_x] \in \Omega(T)$. In particular, given $n \geq 1$, all but finitely many rays from $\{r_{x_k}\}_{k \in \mathbb{N}}$ must contain the open ball $B_{\frac{1}{n}}(x)$. In other words, there is $k_0 \in \mathbb{N}$ such that $x_k \in B_{\frac{1}{n}}(x)$ if $k \geq k_0$, proving that $\{x_k\}_{k \in \mathbb{N}}$ converges to x. Hence, φ^{-1} is continuous in the end $[r_x]$.

Therefore, φ is an homeomorphism. □

5.5 Applications and remarks

As pointed out in the Introduction, the end structure of infinite graphs is useful for extending classical theorems from finite graph theory. With special attention to connectivity results, in this section we will discuss some of these applications, although most details will be omitted. Nevertheless, it is remarkable that a given statement regarding finite graphs might admit more than one infinite version. When setting an infinite Menger-type theorem, for example, different references in the literature present distinct definitions for connecting paths and separators, although all the notions are coincident for locally finite graphs.

Before comparing these concepts, it is useful to distinguish the role played by some vertices of infinite degree. More precisely, fixed a graph G, we recall from Theorem 5.3.6 that $v \in V(G)$ **dominates** a ray $r \in \mathscr{R}(G)$ if it is infinitely connected to r, in the following sense:

$v \in C(S,[r])$ for every finite set $S \subset V(G) \setminus \{v\}$, i.e., the vertex v and a tail of r belong to the same connected component of $G \setminus S$. Equivalently, v is the center of a star whose leaves belong to r. In this case, v dominates any other ray equivalent to r, allowing us to say that v dominates the *end* $[r]$. Then, combining notations from (POLAT, 1991) and (JACOBS *et al.*, 2023), we set the following definitions:

Definition 5.5.1 (Connecting paths)**.** *Let G be a graph. Depending on whether we are considering connectivities of the form "vertex-vertex", "vertex-end" and "end-end", we define the* ***paths*** *and the* ***graphic paths*** *as follows:*

- `Connectivity between vertices:` *for fixed vertex sets $A,B \subset V(G)$, an $A-B$* ***path*** *or* ***graphic path*** *is a finite path P in which one endpoint belongs to A, the other belongs to B and no inner vertex of the path belongs to $A \cup B$, as defined in Section 2.1. If these endpoints are $a \in A$ and $b \in B$ respectively, we say that P is an $a-b$* ***path****;*
- `Connectivity between vertices and ends:` *Fix $v \in V(G)$ and $[r] \in \Omega(G)$. A* ***graphic*** *$v-[r]$* ***path*** *is a ray equivalent to r starting at v. A* ***$v-[r]$ path*** *is either a graphic $v-[r]$ path or a $v-u$ path, in which $u \in V(G)$ is a vertex that dominates $[r]$;*
- `Connectivity between ends:` *Fix $[r_1],[r_2] \in \Omega(G)$. A* ***graphic*** *$[r_1]-[r_2]$* ***path*** *is a double ray in which one half-ray is equivalent to r_1 and the other is equivalent to r_2. In its turn, a $[r_1]-[r_2]$* ***path*** *is either a graphic one or a $v-[r_i]$ path for some $i \in \{1,2\}$ and some vertex $v \in V(G)$ that dominates r_{3-i}.*

More generally, for sets $A,B \subset V(G) \cup \Omega(G)$, a ***graphic*** *$A-B$* ***path*** *is a graphic $a-b$ path P for some $a \in A$ and some $b \in B$ such that $\{a,b\} = \overline{P} \cap (A \cup B)$, where this latter closure is taken in $|G|$. In addition, we say that two graphic paths P and Q are* ***strongly disjoint*** *if $\overline{P} \cap \overline{Q} = \emptyset$. In its turn, if $A,B \subset \Omega(G)$ or $A \subset V(G)$ and $B \subset \Omega(G)$, an $A-B$* ***path*** *is simply an $a-b$ path for some $a \in A$ and some $b \in B$.*

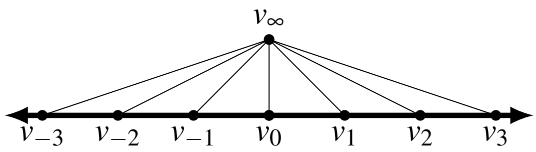
Roughly speaking, the paths in Definition 5.5.1 differ from the graphic ones by allowing dominating vertices to represent some reachable end. If we consider only graphic paths, Bruhn, Diestel and Stein (2005) generalized Theorem 2.3.3 somehow verbatim, under the following condition of topological separation:

Theorem 5.5.2 (Bruhn, Diestel and Stein (2005), Theorem 1.1)**.** *In a given connected graph G, fix $A,B \subset V(G) \cup \Omega(G)$ two sets that are separated topologically, i.e., which satisfy $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. Then, there exist $S \subset V(G) \cup \Omega(G)$ and a family $\mathscr{P}$ of strongly disjoint graphic $A-B$ paths such that:*

- *$S = \bigcup_{P \in \mathscr{P}} S \cap \overline{P}$ and $|\overline{P} \cap S| = 1$ for every $P \in \mathscr{P}$;*
- *If Q is any graphic $A-B$ path, then $\overline{Q} \cap S \neq \emptyset$.*

Comparing Theorem 5.5.2 with the original statement of the Erdős-Menger Theorem, the subset $S \subset V(G) \cup \Omega(G)$ of the above thesis plays the role of an "$A - B$ separator obtained by the choice of precisely one element from each $P \in \mathscr{P}$". Nevertheless, when applied to the graph G of Figure 31, there are no two disjoint graphic paths connecting the only two ends $[r_1], [r_2] \in \Omega(G)$. But, since v_∞ is adjacent to v_n for each $n \in \mathbb{Z}$, at least two *vertices* are needed to separate $[r_1]$ and $[r_2]$. The "$\{[r_1]\} - \{[r_2]\}$ separator" claimed by Theorem 5.5.2, thus, is given by either $\{[r_1]\}$ or $\{[r_2]\}$ in this case.

Figure 31 – Double rays in which both half-rays are dominated by a same vertex



Source: Elaborated by the author.

However, if we aim to forbid ends in separating sets, the broader notion of connecting paths in Definition 5.5.1, rather than the graphic ones, is useful. Also considering the role played by dominating vertices, this definition was first introduced by Polat (1991). In his work, the separators allowed are composed only by vertices, although with reasonable topological restrictions:

Definition 5.5.3 (Separators)**.** *Fix G a graph and $X \subset \Omega(G)$ a set of ends. We say that a vertex set $S \subset V(G)$ is* ***dispersed*** *regarding X if, for every $[r] \in X$, there is a finite set $F \subset V(G)$ such that $C(F, [r]) \cap S = \emptyset$. In this case, there are representatives of the end $[r]$ in $G \setminus S$, as well as every vertex that dominates $[r]$ belongs to $C(F, [r]) \cup F$.*

For subsets $A, B \subset \Omega(G)$, we say that S is an $A - B$ ***separator*** *if S is dispersed regarding $A \cup B$ and there is no graphic $A - B$ path in $G \setminus S$. In particular, there is no $A - B$ path in $G \setminus S$.*

The existence of separators as in the above definition is, similar to Theorem 5.5.2, closely related to the topological separation of the sets A and B. This is the core of the result below, that was originally proved using the notion of *multiendings* developed by Polat (1996b). However, we will concluded it based on the more recent technique of enveloping a set o vertices, as formalized by Lemma 5.4.5.

Lemma 5.5.4 (Polat (1991), Theorem 1.2)**.** *Let $A, B \subset \Omega(G)$ be sets of ends in an infinite graph G such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Then, there exists an $A - B$ separator $S \subset V(G)$ with the following property:*

($\star$) *S has finite adhesion, namely, for every end $[r] \in A \cup B$, the connected component C of $G \setminus S$ in which r has a tail has finite neighborhood in S. In other words,*

$$N(C) = \{v \in S : v \text{ has a neighbor in } C\} \text{ is finite.}$$

Proof. Let $\mathscr{W}$ be a maximal set of pairwise disjoint rays in G whose ends belong to $\overline{A}$. Considering $U = \bigcup_{W \in \mathscr{W}} V(W)$, we claim that $\overline{U} \setminus U = \overline{A}$. In fact, given an end $[r] \in \overline{A}$, the ray r must intersect infinitely many vertices of U: otherwise, a tail of r would contradict the maximality of $\mathscr{W}$ when added to this family. In particular, $C(F,[r]) \cap U \neq \emptyset$ for every finite subset $F \subset V(G)$, so that $[r] \in \overline{U} \setminus U$. Conversely, let $[r] \in \overline{U}$ be any end. Then, for every finite set $F \subset V(G)$, the connected component of $G \setminus F$ containing a tail of r also contains a tail of a ray $w \in \mathscr{W}$. Since $[w] \in \overline{A}$ and $\overline{A}$ is closed in $\Omega(G)$, it follows that $[r] \in \overline{A}$.

Relying on Lemma 5.4.5, fix U^* an envelope for U. In particular, $\overline{U^*} \setminus U^* = \overline{A}$. For an end $[r] \in B$, we observe that its representative r has a tail in a connected component $C_{[r]}$ of $G \setminus U^*$. If not, then r intersects U^* in infinitely many vertices, so that $[r] \in \overline{U^*} \setminus U^* = \overline{A}$. However, this contradicts the hypothesis that $\overline{A} \cap B = \emptyset$.

For every $[r] \in B$, then, denotes by $S_{[r]}$ the neighborhood of $C_{[r]}$ in U^*, which is finite since U^* is an envelope. Recalling that $U \subset U^*$, by the maximality of $\mathscr{W}$ there is no ray in $C_{[r]}$ whose end belongs to A. Therefore, setting $S' = \bigcup_{[r] \in B} S_{[r]}$, there is no graphic $A - B$ path in $G \setminus S'$. We will now show that every end of $A \cup B$ can be separated from S' by finitely many vertices.

Indeed, $C(S_{[r]},[r]) \cap S' = \emptyset$ for every end $[r] \in B$, since $S' \subset U^*$. For a contradiction, fixed $[r] \in A$, suppose that $C(F,[r]) \cap S' \neq \emptyset$ for every finite $F \subset V(G)$. In particular, S' is infinite, so that there is an infinite family $\{C_{[r_i]}\}_{i \in \mathbb{N}} \subset \{C_{[s]} : [s] \in B\}$ of connected components of $G \setminus U^*$ that have a neighbor in $C(F,[r])$. Hence, by choosing n big enough such that $C_{[r_n]} \cap F = \emptyset$, we must have $C(F,[r_n]) = C(F,[r])$. Since $\{[r_i]\}_{i \in \mathbb{N}} \subset B$, this proves that $[r] \in A \cap \overline{B}$, which contradicts the disjunction $A \cap \overline{B} = \emptyset$. Hence, $(A \cup B) \cap \overline{S'} = \emptyset$.

Finally, let S be an envelope for S'. We will verify that S is an $A - B$ separator as claimed. Indeed, the connected components of $G \setminus S$ have finite neighborhood in S. Moreover, $\overline{S} \setminus S = \overline{S'} \setminus S' \subset \Omega(G) \setminus (A \cup B)$, so that every end $[r] \in A \cup B$ has a representative in some connected component of $G \setminus S$. In particular, S is dispersed regarding $A \cup B$. Finally, since $S' \subset S$, there is no $A - B$ graphic path in $G \setminus S$.

□

Relying on Lemma 5.5.4, Polat (1991) proves the next Menger-type theorem. Its statement differs from Theorem 5.5.2 in the sense that the paths and the separators are now those considered by Definitions 5.5.1 and 5.5.3, respectively. Moreover, the claimed separator S is not required to be obtained by the choice of precisely one vertex from each path of the family $\mathscr{P}$:

Theorem 5.5.5 (Polat (1991), Theorem 3.2). *Let $A, B \subset \Omega(G)$ be two sets of ends such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Consider a maximum-sized family $\mathscr{P}$ of disjoint $A - B$ paths and fix an $A - B$ separator S of minimum size. Then, $|\mathscr{P}| = |S|$.*

We will not present a proof for Theorem 5.5.5 in the next paragraphs since it will be revisited by Theorem 6.2.4 in Section 6.2. Instead, we will finish this section by discussing other minor connectivity results regarding ends. Although some of them are consequences from the above generalizations of Menger's Theorem, we might fulfill its details for further didactic references. Considering that, we start with the following example:

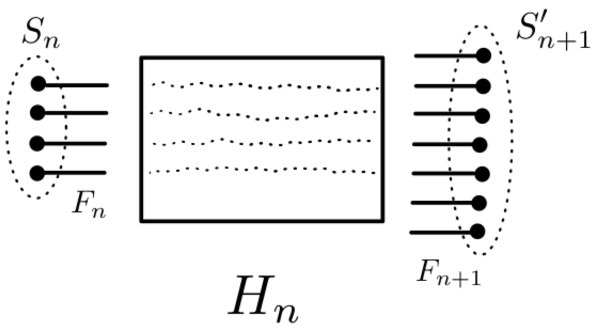
Lemma 5.5.6 (Bruhn and Stein (2007), Lemma 10)**.** *Let G be a locally finite connected graph. Fix an end $[r] \in \Omega(G)$ and a finite set $S \subset V(G)$. Then, the maximum number of rays starting at S which are equivalent to r is equal to the minimum size of a cut separating S and r.*

Proof. Consider $F \subset E(G)$ a minimum cut separating r and the vertices of S, i.e., such that no vertex of S belongs to the connected component of $G \setminus F$ which contains a tail of r. Once G is locally finite, the cut $\delta(S)$ satisfies this property, so that a minimum such F indeed exists and it is finite. By this minimality assumption, every edge of F has an endpoint in the connected component C_0 of $G \setminus F$ in which r has its tail. Consider the subgraph G_0 obtained from C_0 after adding to it precisely the edges of F and its endpoints. Incidentally, let $S_0 = V(G_0) \setminus V(C_0)$ denote the set of endpoints of the edges from $F_0 := F$ other the the ones that belong to C_0.

By induction, suppose that a finite cut F_n in a connected subgraph G_n of G is defined for some $n \in \mathbb{N}$. We write as C_n for the connected component of $G \setminus F_n$ in which r has a tail. In addition, we assume that $S_n = V(G_n) \setminus V(C_n)$ is a finite set which is separated from r by F_n, that also has minimum size with such property. We suppose even that F_n is precisely the edge set connecting S_n to C_n in G_n.

Under these assumptions, denote by S'_n the set of endpoints of edges from F_n that lie in C_n. Since C_n is connected and locally finite, there is F_{n+1} a cut on this graph separating S'_n and r. We assume that F_{n+1} has minimum size with this property, so that all edges from F_{n+1} have an endpoint in the connected component C_{n+1} of $G_n \setminus F_{n+1}$ which contains a tail of r. In addition, the cut F_{n+1} also separates S_n from r in G_n, so that $|F_n| \leq |F_{n+1}|$ by the minimality of F_n when defined in G_n. Finally, we construct the connected subgraph G_{n+1} by adding to C_{n+1} the edges of F_{n+1} and its endpoints. Then, set $S_{n+1} := V(G_{n+1}) \setminus V(C_{n+1})$.

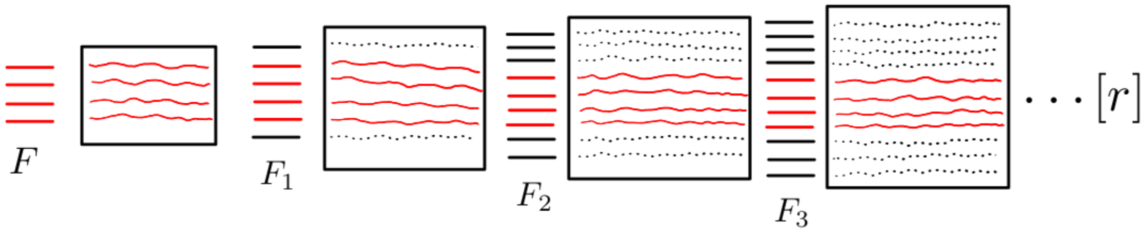
Figure 32 – Definition of H_n



The paths of the family $\mathscr{P}_n$ are presented by dashed lines.

Now, consider the graph H_n obtained from $G_n \setminus C_{n+1}$ by adding the edges of F_{n+1} and its endpoints, as Figure 32 suggests. Then, let $\mathscr{P}_n$ be a maximum family of edge-disjoint paths in H_n connecting S_n to S'_{n+1}, as given by the Erdős-Menger Theorem for edges (see Corollary 2.3.4). Observing that F_n is a cut of minimum size in H_n separating S_n from S'_{n+1}, because it has minimum size while separating S_n from r, we must have $|\mathscr{P}_n| = |F_n|$. At the end of this recursive process, the union $\bigcup_{n\in\mathbb{N}} \bigcup \mathscr{P}_n$ contains $|F_0| = |F|$ many edge-disjoint rays starting at S_0, since H_n and H_{n+1} intersects precisely at the edges of F_n. These rays define a family $\mathscr{P}$ as in Figure 33, being equivalent to r due to the fact that r has a tail in C_n for every $n \in \mathbb{N}$.

Figure 33 – Construction of a family of $|F|$ many edge-disjoint rays



In red, we present the family $\mathscr{P}$ of $|F|$ many edge-disjoint rays starting at vertices of S_0. They are obtained by concatenating suitable paths from the family $\bigcup_{n\in\mathbb{N}} \mathscr{P}_n$.

Source: Elaborated by the author.

Finally, consider G_- the graph obtained from $G \setminus C_0$ after adding the edges of F and its endpoints. Note that F separates the vertices of S and S'_0 in G_-, besides having minimum size with that property. Then, by relying again on the Erdős-Menger Theorem for edges, we can also extend the paths from $\mathscr{P}$ within G_- so that they start at S. $\square$

Exercise 5.5.7. *Consider the family of subgraphs $\{C_n : n \in \mathbb{N}\}$ as defined in the above proof. Show that $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$. If needed, search for a hint in the proof of Proposition 5.2.1.*

Following the work of Jacobs *et al.* (2023), we will now apply Lemma 5.5.6 in order to obtain a version of the Lovász-Cherkassky Theorem for locally finite graphs and their ends. Before introducing it, we recall that a $T-$**path** in a graph G is a $T-T$ path as in Definition 5.5.1, namely, a path that meets the set $T \subset V(G)$ in precisely its endpoints. On the other hand, we say that G is **inner-Eulerian** for T if every vertex of $V(G) \setminus T$ has even degree. Considering that, the classical Lovász-Cherkassky Theorem is stated below:

Theorem 5.5.8 (Lovász-Cherkassky)**.** *In a given finite graph G, fix $T \subset V(G)$. If G is inner-Eulerian for T, the maximum number o pairwise disjoint edge-disjoint $T-$paths in G is equal to*

$$\frac{1}{2}\sum_{t\in T} \lambda(t, T \setminus \{t\}),$$

where $\lambda(t, T \setminus \{t\})$ denotes the minimum size of a cut separating t from $T \setminus \{t\}$.

Roughly speaking, Theorem 5.5.8 claims that there is a family of edge-disjoint $T-$paths $\mathscr{P}$ attaining an optimal size. This because, in a family of at least $1+\frac{1}{2}\sum_{t\in T}\lambda(t,T\setminus\{t\})$ paths, two of them must contain a common edge of a minimum cut separating some vertex t from $T\setminus\{t\}$. In particular, for every $t\in T$, the maximum family $\mathscr{P}$ contains precisely $\lambda(t,T\setminus\{t\})$ paths that have t as an endpoint. Considering this property, Jacobs *et al.* (2023) proposed the following generalization of the Lovász-Cherkassky Theorem for locally finite graphs and their ends:

Theorem 5.5.9 (Jacobs *et al.* (2023), Theorem 1)**.** *Let G be a locally finite graph and fix $T\subset V(G)\cup\Omega(G)$ a discrete subset in the topological space $|G|$. Suppose that $|\delta(X)|$ is even or infinite for every $X\subset V(G)$ in which $T\subset\overline{X}$. Then, there is a family $\mathscr{P}$ of edge-disjoint graphic $T-$paths such that, for every $t\in T$, the number of $\{t\}-(T\setminus\{t\})$ paths is equal to $\lambda(t,T\setminus\{t\})$.*

In the above statement, a **graphic** $T-$**path** is a graphic $T-T$ path, following the notation from Definition 5.5.1. Analogously, $\lambda(t,T\setminus\{t\})$ denotes the minimum size of an edge set $F\subset E(G)$ that separates t from $T\setminus\{t\}$, in the sense that the connected component of $G\setminus F$ which contains t does not contain a vertex from $T\setminus\{t\}$ neither the tail of a ray whose end belongs to $T\setminus\{t\}$.

Comparing the Theorems 5.5.8 and 5.5.9, the "inner-Eulerian" hypothesis over G is, in the locally finite generalization, replaced by a parity condition over some finite cuts. Despite that, the new assumption restricts to the original hypothesis for finite graphs. More precisely, if G is finite and every vertex of $G\setminus T$ has even degree, let $\tilde{G}$ be the multigraph obtained by contracting a set $X\supset T$ to a new vertex v. Since $\tilde{G}$ has an even number of vertices of odd degree, the degree of v must be even by the well-known handshaking lemma. Noticing that $|\delta(X)|$ is the degree of v in $\tilde{G}$, the above result is, in fact, a generalization of the classical Lovász-Cherkassky Theorem for finite graphs. On the other hand, its proof relies also on a countable version of Theorem 5.5.8 previously obtained by Joó (2023):

Theorem 5.5.10 (Joó (2023), Theorem 1.3)**.** *Let G be a (multi)graph and fix $T\subset V(G)$ a countable subset. Suppose that $|\delta(X)|$ is even or infinite for every $X\subset V(G)$ which contains T. Then, there is a family $\mathscr{P}$ of edge-disjoint $T-$paths such that, for every $t\in T$, there is a cut separating t from $T\setminus\{t\}$ obtained by the choice of precisely one edge from each path of $\mathscr{P}_t=\{P\in\mathscr{P}: t \text{ is an endpoint of } P\}$.*

In fact, by applying Lemma 5.5.6, Theorem 5.5.9 is reduced to the hypothesis of Theorem 5.5.10:

Proof of Theorem 5.5.9. Since G is a connected locally finite graph, we argued in the previous sections that $|G|$ is a compact metric space and, in particular, second countable. Therefore, the subset $T\subset V(G)\cup\Omega(G)$ is countable, because it is discrete. Hence, fix an enumeration $\{[r_i]\}_{i<\kappa}$ for $T\cap\Omega(G)$, where κ is a countable (possibly finite) cardinal.

Once G is locally finite and T is discrete, there is a finite set $F_0 \subset V(G)$ such that, in the connected component C_0 of $G \setminus F_0$ containing a tail of r_0, there is no vertex of T neither a tail of r_i for some $i > 0$. We choose F_0 with minimum size satisfying that property.

Analogously, suppose that disjoint finite sets $F_0, F_1, \ldots, F_i \subset E(G)$ are defined for some $i < \kappa$. For each $j \leq i$, we assume that the connected component C_j of $G \setminus F_j$ in which r_j has a tail does not contain a vertex from T neither the tail of r_k whether $k \in \kappa \setminus \{j\}$. If $\kappa = i+1$, we finish this recursive process. Otherwise, we denote by $\dot{G}$ the (multi)graph that arises from G after contracting the connected component C_j to an artificial point x_j for every $j \leq i$. Endowing $\dot{G}$ with the corresponding quotient topology from $|G|$, the set $T_{i+1} = (T \setminus \{[r_j] : j \leq i\}) \cup \{x_j : j \leq i\}$ is also discrete. Then, there is a finite set $F_{i+1} \subset E(\dot{G})$ separating r_{i+1} and T_j, in the following sense: the connected component C_{i+1} of $\dot{G} \setminus F_{i+1}$ in which r_{i+1} has its tail does not contain a vertex from T_j neither a tail of r_k if $k > i+1$. Again, we choose F_{i+1} of minimum size with that property. Hence, as a subgraph of G, the connected component C_{i+1} does not contain a vertex of T neither a tail of r_k if $k \in \kappa \setminus \{i+1\}$. Moreover, C_{i+1} is indeed the connected component of $G \setminus F_{i+1}$ in which r_{i+1} has its tail.

Hence, at the end of this recursive process, C_i is defined for every $i < \kappa$ and satisfies $\overline{C_i} \cap T = \{[r_i]\}$. Denote by $\hat{C}_i$ the subgraph obtained from C_i after adding to it the edges of F_i and its endpoints. By the choice of F_i, this edge set has minimum size while separating the vertices of $S_i := V(\hat{C}_i) \setminus V(C_i)$ from a tail of r_i. Hence, by Lemma 5.5.6, there is a family $\mathscr{P}_i$ comprising $|F_i|$ many edge-disjoint rays starting at vertices of S_i and which are equivalent to r_i.

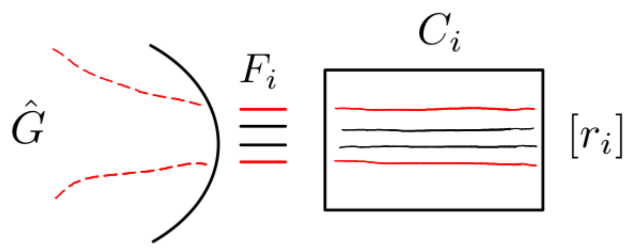
On the other hand, after contracting each connected subgraph C_i to an artificial vertex v_i, we define a multigraph $\hat{G}$. Now, we can apply Theorem 5.5.10 to the vertex set $\hat{T} = (T \setminus \Omega(G)) \cup \{v_i : i < \kappa\}$ in $\hat{G}$, since, by the main hypothesis over G, the cardinal $|\delta(X)|$ is even or infinite for every $X \subset V(\hat{G})$ that contains $\hat{T}$. Hence, there is a family $\hat{\mathscr{P}}$ of edge-disjoint $\hat{T}-$paths in $\hat{G}$ with the following property:

$(*)$ For every $t \in \hat{T}$, there is a cut separating t from $\hat{T} \setminus \{t\}$ obtained by the choice of precisely one edge from each path of $\hat{\mathscr{P}}_t = \{P \in \hat{\mathscr{P}} : t \text{ is an endpoint of } P\}$.

Therefore, the claimed family $\mathscr{P}$ of graphic $T-$paths for G is obtained from $\hat{P}$ after concatenating some of its elements with suitable paths in $\bigcup_{n \in \mathbb{N}} \mathscr{P}_n$. Explicitly, if $P \in \hat{\mathscr{P}}$ is a path such that none of its endpoints in $\hat{G}$ belong to $\{v_i : i < \kappa\}$, we set P as an element of $\mathscr{P}$ as well. If not, P has an endpoint of the form v_i for some $i < \kappa$. In this case, the edge e in P which is incident at v_i belongs to F_i. Hence, e is the first edge of an unique ray $R' \in \mathscr{P}_i$. Therefore, as suggested by Figure 34 we set the concatenation PR' as an element of $\mathscr{P}$ if the other endpoint of P does not have the form v_j for some $j < \kappa$. Otherwise, we proceed similarly in order to extend PR' to a double ray $R''PR'$, where R'' is the unique ray of $\mathscr{P}_j$ that contains the single edge from $F_j \cap P$. In this case, we set $R''PR'$ as, in fact, an element of $\mathscr{P}$.

Finally, since $\hat{\mathscr{P}}$ satisfies property $(*)$ and $|\mathscr{P}_i| = |F_i|$ for every $i < \kappa$, we note that $\mathscr{P}$ defined this way indeed verifies the statement of Theorem 5.5.9. □

Figure 34 – Construction of an element of $\mathscr{P}$



In dashed red lines, we sketch some paths of $\hat{\mathscr{P}}$ which have a vertex v_i as one of their endpoints. These paths are extended after being concatenated with suitable paths from $\mathscr{P}_i$.

Source: Elaborated by the author.

CHAPTER

6

THE EDGE-END SPACE

The last section of the previous chapter presented two extensions of the Lovász-Cherkassky Theorem for infinite graphs. Due to Joó (2023), the first one states it for *countable* graphs, as a result regarding the connectivity between its vertices. The second one, obtained by Jacobs *et al.* (2023), studies the connectivity between *ends* as well, but it is restricted to the locally finite case. In fact, this latter generalization is obtained via a reduction to the hypothesis of the former, so that a natural question arises: can we extend the Lovász-Cherkassky Theorem for *countable graphs and their ends*?

As we shall discuss in Section 6.2, the answer is "somehow". After all, Lemma 5.5.6 (that supports the proof of Theorem 5.5.9) is not applicable for arbitrary graphs. For example, as is the case in Figure 31, there might be an end which cannot be separated from a given (finite) set of vertices by finitely many edges. However, we can get around this obstruction by considering a slightly different notion of end, despite the same when restricted to locally finite graphs. Hence, following the work of Hahn, Laviolette and Širáň (1997), this chapter brings combinatorial and topological properties of *edge-ends* in infinite graphs, more suitable objects for the improvement of edge-connectivity results. In particular, the next sections were extracted from our preprints (AURICHI; REAL, 2023) and (AURICHI; REAL; JÚNIOR, 2023), which are on final stage of writing.

6.1 Introduction

While ends are equivalence classes of rays that are infinitely (vertex-)connected, the **edge-ends** are obtained after identifying rays which are infinitely edge-connected. More precisely, if G is a graph and we fix $r, s \in \mathscr{R}(G)$, we say that r and s are **edge-equivalent**, writing $r \sim_E r$, whenever the tails of r and s belong to the same connected component of $G \setminus F$ for every finite set $F \subset E(G)$. If this is not the case, we say that some such $F \subset E(G)$ **separates** r and s. It is easily verified that $\sim_E$ is an equivalence relation over $\mathscr{R}(G)$, which is also highlighted by the following observation:

Lemma 6.1.1 (Hahn, Laviolette and Širáň (1997), Lemma 2)**.** *For two rays r and s of a graph G, we have $r \sim_E s$ if, and only if, there is an infinite family of pairwise edge-disjoint $r-s$ paths whose endpoints in r and s are distinct.*

Proof. Suppose first that there is a family $\mathscr{P}$ of $r-s$ paths as in the statement. If $F \subset E(G)$ is a finite edge set, then some path $P \in \mathscr{P}$ does not intersect F. Moreover, we can choose P such that its endpoints $u \in s$ and $v \in r$ are far from F in the following sense: the tails of r and s starting at v and u respectively do not intersect F as well. Hence, these tails belong to the same connected component of $G \setminus F$. Since F is arbitrary, we proved that $r \sim_E s$.

Conversely, suppose that r and s are edge-equivalent. In particular, there is a path P_0 connecting r and s. Suppose that we have defined edge-disjoint $r-s$ paths $P_0, P_1, \ldots, P_n$ whose endpoints in r and s, denoted by $v_0, v_1, \ldots, v_n$ and $u_0, u_1, \ldots, u_n$ respectively, are distinct. Note that the set $F = \bigcup_{i=0}^{n} E(P_i)$ is finite. Hence, since $r \sim_E s$, the tails of r and s in $G \setminus F$ belong to the same connected component. In these tails, we fix vertices $v_{n+1} \in r \setminus \{v_0, v_1, \ldots, v_n\}$ and $u_{n+1} \in s \setminus \{u_0, u_1, \ldots, u_n\}$. Then, let P_{n+1} be any path connecting v_{n+1} to u_{n+1} in $G \setminus F$. At the end of this recursive process, the family $\mathscr{P} = \{P_n\}_{n \in \mathbb{N}}$ verifies the statement. □

Now, the quotient $\mathscr{R}(G)/\sim_E$, denoted by $\Omega_E(G)$, is said to be the **edge-end space** of the graph G. Then, the edge-end of a ray r refers to its equivalence class in $\Omega_E(G)$, represented by $[r]_E$. Comparing this notion with the definition of end presented in the previous chapter, we observe that, if r and s are equivalent under the usual relation $\sim$, then they are also edge-equivalent. If G is locally finite, the converse also holds. In this case, if no finite edge set separates two rays r and s in G, then neither a finite set of vertices does, because only finitely many edges are incident to them. However, when there are vertices of infinite degree, $\sim_E$ might identify more rays: although the graph in Figure 31 has two ends, for example, it has only one edge-end.

Despite that, the definition of the topological space $|G|$ suggests a similar topology on G and its edge-ends. Indeed, consider the space $\|G\|$ whose point set comprises the vertices of G, its edges (identified with the unit interval as in Section 5.1) and $\Omega_E(G)$. In $\|G\|$, the open basic neighborhoods around vertices and inner points of edges are precisely the same as in $|G|$. However, by fixing $\varepsilon > 0$ and a finite set $F \subset E(G)$, an open basic neighborhood around an edge-end $[r]_E \in \Omega_E(G)$ assumes now the following form:

$$\hat{C}_E(F,[r]_E,\varepsilon) := C_E(F,[r]_E) \cup \Omega_E(F,[r]_E) \cup \bigcup_{\substack{u \in C_E(F,[r]_E) \\ uv \in E(F,[r]_E)}} [u,\varepsilon) \tag{6.1}$$

In the above expression, $C_E(F,[r]_E)$ denotes the connected component of $G\setminus F$ in which r has a tail, while $E(F,[r]_E)$ refers to the cut $\delta(C_E(F,[r]_E))$. Finally, $\Omega_E(F,[r]_E)$ comprises the edge-ends of G that have a representative in $C_E(F,[r]_E)$.

Exercise 6.1.2. *Is $\|G\|$ a Hausdorff space for every graph G?*

In particular, $\{\Omega_E(F,[r]_E) : F \subset E(G) \text{ finite}\}$ is a local basis at $[r]_E$ in $\Omega_E(G)$, seen as a topological subspace of $\|G\|$. With this inherited substructure, the topology of $\Omega_E(G)$ is a matter of interest by itself. Following the spirit of Diestel's question as stated in Introduction, Section 6.3 and Section 6.4 investigate the topological spaces that can be written in the form $\Omega_E(G)$ for some graph G. More precisely, the former section brings a graph-theoretic characterization of this family via the topology of (usual) end spaces, while the latter one sets $\{\Omega_E(G) : G \text{ graph}\}$ as a proper subclass of $\{\Omega(G) : G \text{ graph}\}$. Extracted from our preprint (AURICHI; REAL; JÚNIOR, 2023), part of this discussion deeply relies in the recent paper of Kurkofka and Pitz (2023), whose main results were overviewed by Section 5.4.

Actually, the contributions in (AURICHI; REAL; JÚNIOR, 2023) were obtained as an intersection of two studies. On one hand, Paulo Júnior was looking forward a *topological description* of end spaces, which he obtained via a topological game. On the other, the author of the dissertation applied the *edge-end structure* in order to generalize the Lóvasz-Cherkassky Theorem, as we shall detail among other connectivity results in Section 6.2. In a joint work, a search for *topological properties of edge-end spaces* naturally arisen.

Exercise 6.1.3. *For a given graph G, show that its edge-end space $\Omega_E(G)$ is Hausdorff and Fréchet-Urysohn.*

In general lines, this chapter will often compare edge-end spaces of given graphs with end spaces of possible other ones. Thus, we finish this section by presenting a natural definition in that direction: we consider the **line graph** G' of a given graph G as the one obtained by setting $V(G') = E(G)$ and

$$ef \in E(G') \text{ if, and only if, } e \text{ and } f \text{ are adjacent edges.}$$

Here, we mean by *adjacent edges* those that share a common endpoint. Then, the following observation easily translates paths in G to paths in G' and conversely:

Lemma 6.1.4. *Fix e and f two non-adjacent edges in a connected graph G. Therefore, the two statements below holds:*

i) If $v_0v_1v_2\dots v_n$ is a path in G such that $e = v_0v_1$ and $f = v_{n-1}v_n$, then $(v_iv_{i+1})_{i<n}$ defines a path in G' connecting e and f;

ii) Conversely, if $e_0e_1e_2\dots e_n$ is a $\subsetneq$-minimal path in G' connecting the edges $e = e_0$ and $f = f_n$ of G, then there is a path $v_0v_1v_2\dots v_{n+1}$ in G such that $e_i = v_iv_{i+1}$ for each $0 \le i \le n$.

Proof. The first item is trivial. In fact, if $v_0v_1v_2\dots v_n$ is a path in G, then $v_{i_1}v_i$ and v_iv_{i+1} are distinct and adjacent edges for every $1 < i < n$, defining a path in G'.

Now, suppose that $e_0e_1e_2\dots e_n$ is a $\subseteq-$minimal path in G' connecting the edges $e = e_0$ and $f = f_n$. Hence, for each $i < n$, e_i and e_{i+1} share an endpoint v_{i+1} in G, since those are adjacent edges. Moreover, $v_{i+1} \neq v_{j+1}$ for every $j < n$ distinct from i. Otherwise, if $v_{i+1} = v_{j+1}$ and $i < j$ for instance, the edges e_i and e_{j+1} would be adjacent, so that the path $e_0e_1\dots e_ie_{j+1}\dots e_n$ would contradict the minimality of $e_0e_1e_2\dots e_n$. □

Exercise 6.1.5. *Does it hold $\Omega_E(G) \simeq \Omega(G')$ for every graph G?*

6.2 Edge-connectivity results

As previously pointed out, this section aims to extend the Lovász-Cherkassky Theorem for countable graphs and its edge-ends. Since ends and edge-ends are the same object in locally finite graphs, the claimed extension generalizes both Theorems 5.5.10 and 5.5.9. In fact, we will proceed similarly to the proof of this latter result, that consists in a reduction to the hypothesis of the former. However, we shall replace Lemma 5.5.6 by another Menger-type result, which must now be applicable to non-locally finite graphs.

This new tool will be obtained by restating Theorem 5.5.5 to edge-ends. In this edge-related setting, however, we need to also adapt the definitions of paths and separators from 5.5.1 and 5.5.3, respectively. Then, we first say that a vertex $v \in V(G)$ of a graph G **edge-dominates** an edge-end $[r]_E \in \Omega_E(G)$ if v is infinitely edge-connected to r (and, thus, to any other representative of $[r]_E$). More precisely, $v \in C_E(F, [r]_E)$ for every finite set $F \subset E(G)$. In this case, following the proof of Lemma 6.1.1, there is an infinite family of edge-disjoint paths connecting v to infinitely many vertices of r. Considering that, we state the definitions below:

Definition 6.2.1 (Connecting paths - edge version)**.** *Let G be a graph. For vertex sets $A, B \subset V(G)$, the definition $A - B$ paths given by 5.5.1 is preserved. For connectivities between edge-ends and vertices or edge-ends and edge-ends, we consider the following criteria:*

- `Connectivity between vertices and edge-ends:` *Fix $v \in V(G)$ and $[r]_E \in \Omega_E(G)$. A **graphic** $v - \omega$ **path** is a ray which is edge-equivalent to r and starts at v. Then, a $v - [r]_E$ **path** is either a graphic one or a $\{v\} - \{u\}$ path for some vertex $u \in V(G)$ that edge-dominates $[r]_E$;*
- `Connectivity between edge-ends:` *For edge-ends $[r_1]_E, [r_2]_E \in \Omega_E(G)$, an $[r_1]_E - [r_2]_E$ **path** is one of objects below:*
  - *i) A $v - [r_i]_E$ path, for some $i \in \{1, 2\}$ and some vertex $v \in V(G)$ that edge-dominates $[r_{3-i}]_E$;*

*ii) A double ray in which $[r_1]_E$ and $[r_2]_E$ are the edge-ends of its half-rays. This case defines a **graphic** $[r_1]_E - [r_2]_E$ **path**.*

*Finally, given $A, B \subset \Omega_E(G)$ or $A \subset V(G)$ and $B \subset \Omega_E(G)$, an $A - B$ **path** is an $a - b$ path for some $a \in A$ and some $b \in B$.*

Definition 6.2.2 (Separators - edge version)**.** *Fix a graph G and vertex sets $A, B \subset V(G)$. Then, we say that an edge set $S \subset E(G)$ is an $A - B$ **separator** if there is no $A - B$ path in $G \setminus S$. Now, let $X \subset \Omega_E(G)$ be a set of edge-ends. We say that $S \subset E(G)$ is **dispersed** regarding X if, for every $[r]_E \in X$, there is a finite set $F \subset E(G)$ such that no edge of S belongs to the subgraph induced by $C_E(F, [r]_E)$. In this case, we write $S \cap C_E(F, [r]_E) = \emptyset$.*

*For subsets $A, B \subset \Omega_E(G)$, we say that S is an $A - B$ **separator** if it is dispersed regarding $A \cup B$ and, for every $[r_1]_E \in A$ and every $[r_2]_E \in B$, there is no graphic $[r_1]_E - [r_2]_E$ path in $G \setminus S$. In particular, there is no $A - B$ path in $G \setminus F$.*

When discussing Menger-type results, we are interested in small separators, often minimal or with minimum size. Then, in future proofs, we shall rely on the following remark without explicit mention:

Lemma 6.2.3. *Consider sets A and B such that either $A \subset V(G)$ or $A \subset \Omega_E(G)$, as well as either $B \subset V(G)$ or $B \subset \Omega_E(G)$. Suppose that there is a finite $A - B$ separator $F \subset E(G)$, which we assume to be $\subsetneq -$minimal. Then, there is $X \subset V(G)$ such that $F = \delta(X)$. In other words, F is a cut.*

Proof. Considering the topology of $\|G\|$, define the family

$$\mathscr{C} = \{C : C \text{ is a connected component of } G \setminus F \text{ such that } A \cap \overline{C} \neq \emptyset\}.$$

We will show that the claimed subset of $V(G)$ can be chosen as $X = \bigcup_{C \in \mathscr{C}} V(C)$. In fact, for every edge $e \in F$, there is an $A - B$ path in $G \setminus (F \setminus \{e\})$ by the minimality of F. Since F is an $A - B$ separator, this path contains e, although it passes through no other edge from F. Therefore, e has an endpoint in X and the other in $V(G) \setminus X$, so that $e \in \delta(X)$.

Conversely, by definition of $\mathscr{C}$, an edge $e \in \delta(X)$ has endpoints in two distinct connected components of $G \setminus F$. Hence, we must have $e \in F$. $\square$

Now, fixing G' the line graph of an infinite graph G, we observe that item $i)$ in Lemma 6.1.4 provides a natural identification between edge-ends of G and some elements from $\Omega(G')$. More precisely, given a ray $r = v_0 v_1 v_2 \ldots$, we denote by $\theta(r)$ the ray in G' whose vertex set is $\theta(r) = \{v_i v_{i+1}\}_{i \in \mathbb{N}}$. We thus consider the induced map

$$\begin{array}{rccc} \Theta : & \Omega_E(G) & \to & \Omega(G') \\ & [r]_E & \mapsto & [\phi(r)] \end{array} \tag{6.2}$$

We claim that Θ is well defined. In fact, if r and s are edge-equivalent rays in G, then there are infinitely many edge-disjoint paths connecting (infinitely many) vertices of r to (infinitely many) vertices of s. By Lemma 6.1.4, those paths correspond in G' to infinitely many disjoint paths connecting $\theta(r)$ to $\theta(s)$, so that $[\theta(r)] = [\theta(s)]$. Similarly, we argue that Θ is injective. Indeed, suppose that r and s are rays in G satisfying $[\theta(r)] = [\theta(s)]$. In other words, $\theta(r)$ and $\theta(s)$ are connected by infinitely many vertex-disjoint paths. Unless by passing to proper connected subgraphs, we can assume that these paths are $\subseteq -$minimal while connecting their endpoints. Thus, by Lemma 6.1.4, they correspond in G to infinitely many edge-disjoint paths connecting (infinitely many vertices from) r to (infinitely many vertices from) s, so that $[r]_E = [s]_E$.

Now, fix sets $A, B \subset \Omega_E(G)$ such that $\overline{A} \cap B = \overline{B} \cap A = \emptyset$. Considering $\Omega(G')$ with its end space topology, we claim that $\overline{\Theta(A)} \cap \Theta(B) = \emptyset$ and, by symmetry, that $\Theta(A) \cap \overline{\Theta(B)} = \emptyset$. In order to prove this, fix r a ray in G such that $[r]_E \in B$. Since $\overline{A} \cap B = \emptyset$, there is a finite edge set $F \subset E(G)$ which separates r from A, i.e., $A \cap \Omega_E(F, [r]_E) = \emptyset$. For a while, suppose that $\Omega(F, [\theta(r)]) \cap \Theta(A) \neq \emptyset$ in G'. In other words, a tail of $\theta(s)$ is contained in $C(F, [\theta(r)])$ for some ray s in G with $[s] \in A$. Then, there is a (minimal) path in G' connecting the tails of $\theta(r)$ and $\theta(s)$. Again, by Lemma 6.1.4, this defines a path in G connecting r to s, but avoiding the edges from F. However, this contradicts the fact that F separates r from A. Therefore, $\Omega(F, [\theta(r)]) \cap \Theta(A) = \emptyset$ in G', proving that $\overline{\Theta(A)} \cap \Theta(B) = \emptyset$. Supported by this remark, Theorem 5.5.5 can be written in its edge analogous:

Theorem 6.2.4 (Erdős-Menger Theorem for edge-ends)**.** *Let $A, B \subset \Omega_E(G)$ be two sets of ends such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Hence,*

i) There is F an $A - B$ separator, which we can choose to have minimum cardinality;

ii) If $\mathscr{P}$ is a family of edge-disjoint $A - B$ paths with maximum size, then $|\mathscr{P}| = |F|$.

Proof. Let $A, B \subset \Omega_E(G)$ be the two sets of edge-ends such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Then, consider the cardinal

$$\kappa = \sup\{|\mathscr{P}| : \mathscr{P} \text{ is a family of edge-disjoint } A - B \text{ paths}\}.$$

If κ is finite, the above supremum is clearly attained, even by a family $\mathscr{P}$ that we can suppose to be $\subseteq -$maximal. If κ is infinite, in its turn, we fix a $\subseteq -$maximal family $\mathscr{P}$ of edge-disjoint $X - Y$ paths and consider the following two cases:

- Suppose that κ is countable but that $\mathscr{P}$ is finite. Given $n \in \mathbb{N}$, by definition of κ there is a family $\mathscr{P}_n$ of edge-disjoint $A - B$ paths satisfying $|\mathscr{P}_n| \geq 2n \cdot |\mathscr{P}|$. However, since $\mathscr{P}$ is $\subseteq -$maximal, $Q \cap \bigcup_{P \in \mathscr{P}} E(P) \neq \emptyset$. In particular, there is $r_n \in \mathscr{P}$ an $A - B$ path such that $|\{Q \in \mathscr{P}_n : Q \cap E(r_n) \neq \emptyset\}| \geq 2n$. Considering a subsequence of $\{r_n\}_{n \in \mathbb{N}}$ if necessary, we can even assume that $r_n = r_m =: r$ for every $n, m \in \mathbb{N}$. Thus, by being infinite, r is either a ray or a double ray. In this latter case, unless by passing $\{\mathscr{P}_n\}_{n\in\mathbb{N}}$ to a subsequence, r contains a ray r' such that $|\{Q \in \mathscr{P}_n : Q \cap E(r') \neq \emptyset\}| \geq n$. Therefore, we can assume that r is a ray such that $|\{Q \in \mathscr{P}_n : Q \cap E(r) \neq \emptyset\}| \geq n$. Once $r \in \mathscr{P}$ is a $A-B$ path, we have $[r]_E \in A \cup B$. On the other hand, for every finite set $F \subset E(G)$, a tail of r intersects an $A-B$ path Q from $\mathscr{P}_{|F|+1}$. Hence, in $C_E(F,[r]_E)$ there are representatives of ends in A and B. This proves that $[r]_E \in \overline{A} \cap \overline{B}$, contradicting the topological separation hypothesis. Therefore, $|\mathscr{P}| = \aleph_0$ if $\kappa = \aleph_0$;

- Suppose that κ is uncountable but $|\mathscr{P}| < \kappa$. In particular, there is $\mathscr{Q}$ a family of edge-disjoint $A-B$ paths such that $|\mathscr{P}| < |\mathscr{Q}|$. However, $|\mathscr{P}| = \left|\bigcup_{P\in\mathscr{P}} E(P)\right|$, because $A-B$ paths are countable. Hence, there is $Q \in \mathscr{Q}$ such that $Q \cap \bigcup_{P\in\mathscr{P}} E(P) = \emptyset$. In this case, $\mathscr{P} \cup \{Q\}$ contradicts the maximality of $\mathscr{P}$. Thus, we must have $|\mathscr{P}| = \kappa$ if κ is uncountable.

In any case, $\mathscr{P}$, being $\subseteq$ −maximal, has maximum size. However, as discussed previously, $\overline{\Theta(A)} \cap \Theta(B) = \Theta(A) \cap \overline{\Theta(B)} = \emptyset$ in the line graph G'. From now on, then, let $S \subset V(G')$ be a $\Theta(A)-\Theta(B)$ separator satisfying $(\star)$ as in Lemma 5.5.4.

Regarding S as an edge set in G, we observe that there is no graphic $A-B$ path in $G \setminus S$: otherwise, the edge set of such a double ray would be the vertex set of a graphic $\Theta(A)-\Theta(B)$ path in $G' \setminus S$, contradicting the fact that S is a $\Theta(A)-\Theta(B)$ separator. Moreover, S is a dispersed set regarding $A \cup B$ in G. To see this, fix $[r]_E \in A \cup B$ any edge-end. Since S is a dispersed set in G' regarding $\Theta(A) \cup \Theta(B)$, there is a finite set of vertices $F \subset V(G')$ such that $C(F,[\theta(r)]) \cap F = \emptyset$. Hence, by seeing F as a set of edges in G, Lemma 6.1.4 guarantees that there is no path connecting an edge of $S \subset E(G)$ to a tail of r in $G \setminus F$. Therefore, S is an $A-B$ separator.

Finally, for each $P \in \mathscr{P}$, we define a finite set of edges T_P as follows:

- If P is a finite path, we set $T_P = E(P)$;
- If P is a ray whose edge-end belongs to $A \cup B$, consider $C'_{\Theta([P]_E)}$ the connected component of $G' \setminus S$ in which $\theta(P)$ has a tail. According to Lemma 6.1.4, $C'_{\Theta([P]_E)}$ is indeed the line graph of the connected component $C_{[P]_E}$ of $G \setminus S$ in which P has a tail. Moreover, by property $(\star)$, the neighborhood of $C'_{\Theta([P]_E)}$ in S is as a finite set $S_{[P]_E}$. Considering $S_{[P]_E}$ as an edge subset of G, note that $S_{[P]_E}$ is the cut $\delta(C_{[P]_E})$. In this case, we define T_P to be $S_{[P]_E} \cup (E(P) \cap S)$;
- Now, suppose that P is a double ray such that one half-ray P_1 has its edge-end in A and the other, named P_2, has its edge-end as an element of B. Denote by $C_{[P_1]_E}$ and $C_{[P_2]_E}$ the connected components of $G \setminus S$ in which P_1 and P_2 have their tails, respectively. As in the previous case, the cuts $S_{[P_1]_E} = \delta(C_{[P_1]_E})$ and $S_{[P_2]_E} = \delta(C_{[P_2]_E})$ are finite subsets of S. Then, we set $T_P = S_{[P_1]_E} \cup S_{[P_2]_E} \cup (E(P) \cap S)$.

By the maximality of $\mathscr{P}$, any graphic $A-B$ path R must intersect $F = \bigcup_{P\in\mathscr{P}} T_P$: in $S\cap E(P)$ for some $P\in\mathscr{P}$ or in $S_{[r]}$ for some edge-end $[r]\in\overline{V(R)}$. In other words, F is an $A-B$ separator. Moreover, F has the size of the family $\mathscr{P}$ if $\kappa = |\mathscr{P}|$ is infinite, since T_P is finite for every $P\in\mathscr{P}$. By the same reason, F is finite if so is κ. In this latter case, Theorem 6.2.4 is reduced to Corollary 6.2.6, that will be sooner stated. $\square$

Then, the above proof of Theorem 6.2.4 is complete unless by studying the case in which the sets A and B can be separated by finitely many edges. This discussion is isolated from the whole proof because, besides being rather wide when written in details, it will be further mentioned when generalizing the Lovász-Cherkassky Theorem for countable graphs and its edge-ends. In that study, the following intermediate result, whose proof is an adaptation of the one drawn for Lemma 5.5.6, will be useful:

Lemma 6.2.5. *Let G be a connected graph and fix $A\subset\Omega_E(G)$. Let $F\subset E(G)$ be a finite edge set that separates a given vertex set $S\subset V(G)$ from A. Suppose that F is minimal with that property. Then, there is a family $\mathscr{P}$ of $|F|$ many edge-disjoint $S-A$ paths.*

Proof. We first consider the case in which A is closed in $\Omega_E(G)$. Setting $k=|F|$, we shall construct the required family $\mathscr{P}$ recursively, as the limit object from a sequence $\{\mathscr{P}_n\}_{n\in\mathbb{N}} = \{P_1^n, P_2^n, \ldots, P_k^n\}_{n\in\mathbb{N}}$ of families of edge-disjoint finite paths. By its minimality, we can write $F=\delta(V_0)$ for some $V_0\subset V(G)$, denoting $V_0' = V(G)\setminus V_0$. Without loss of generality, we assume that $S\subset V_0$ and $A\subset\overline{V_0'}$.

Let S' be the endpoints of the edges in F that belong to V_0'. Consider $\hat{G}$ the subgraph of G obtained by adjoining to $G[V_0]$ precisely S' and the edges from F. Hence, by the minimality of F, the Erdős-Menger Theorem for edges guarantees the existence in $\hat{G}$ of a family $\mathscr{P}$ of k edge-disjoint $S'-S$ paths. In particular, each edge from F belongs to precisely one path of $\mathscr{P}$.

Setting $G_0 := G$, $\mathscr{P}_0 := P$ and $F_0 := F$, suppose by induction that we have defined the following objects and its properties:

- The family of paths $\mathscr{P}_n = \{P_i^n : 1\leq i\leq k\}$. For each $1\leq i\leq k$, denote by v_i^n the endpoint of P_i^n other than the one in S;
- G_n is a subgraph of G in which every end of A has a representative;
- A cut F_n in G_n of minimum size that separates a vertex set $S_n\subset V(G_n)$ from A. We assume also that the last edge from each path of $\mathscr{P}_n$ is an element of F_n.

First, since F_n is a cut of G_n, let us write $F_n=\delta(V_n)$ and $V_n' := V(G_n)\setminus V_n$ for some $V_n\subset V(G_n)$. We assume that $G_{n+1} := G_n[V_n]$ is the subgraph induced by the part of the bipartition $\{V_n, V_n'\}$ in which every edge-end of A has a representative. Then, define

$$S_{n+1} = \{v_j^n : \text{ finitely many edges separate } v_j^n \text{ from } A\}.$$

For every $1 \leq i \leq k$ such that $v_i^n \notin S_{n+1}$, we set $P_i^{n+1} = P_i^n$. Since S_{n+1} is finite, its definition allows us to find a finite set $F_{n+1} \subset E(G_{n+1})$ that separates S_{n+1} from A. By choosing F_{n+1} of minimum size with that property, F_{n+1} is a cut in G_{n+1}. Hence, we write $F_{n+1} = \delta(V_{n+1})$ for some $V_{n+1} \subset V(G_{n+1})$, considering $V'_{n+1} := V(G_{n+1}) \setminus V_{n+1}$ the part of the bipartition $\{V_{n+1}, V'_{n+1}\}$ containing S_{n+1}.

Denote by $K_n \subset F_n$ the set of edges of F_n whose endpoints belong to S_{n+1}. Let A' and B' be the set of endpoints of edges from F_n and F_{n+1}, respectively, that do not belong to $G_{n+1}[V'_{n+1}]$. Define $\hat{G}_n$ as the graph obtained after adding to $G_{n+1}[V'_{n+1}]$ the vertices of $A' \cup B'$ and the edges of $K_n \cup F_{n+1}$. By the edge version of the Erdős-Menger Theorem (2.3.4), there is a family $\mathscr{P}'$ of edge-disjoint $A' - B'$ paths in $\hat{G}_n$ with a cut C' obtained by the choice of precisely one edge from each path of $\mathscr{P}'$. We observe that $|C'| \geq |K_n|$. Otherwise, $F'_n = (F_n \setminus K_n) \cup C'$ has strictly fewer edges than F_n and separates A from S_n, contradicting the definition of F_n. Therefore, each edge from K_n lies in precisely one path of $\mathscr{P}'$. By concatenating these paths with the previous paths of $\mathscr{P}_n$ that end in $K_n \subset F_n$, we finish the definition of $\mathscr{P}_{n+1}$.

After finishing this recursive process, consider $P'_i = \bigcup_{n \in \mathbb{N}} P_i^n$ for each $1 \leq i \leq k$. We observe that, if P'_i is a ray, its edge-end belongs to A, so that P'_i is a $S - A$ path. Otherwise, since A is closed, there would be a finite set $F' \subset E(G)$ such that $A \cap C_E(F', [P'_i]_E) = \emptyset$. However, F' is not contained in G_n for some $n \in \mathbb{N}$ big enough, contradicting the construction of P'_i. Similarly, if P'_i is a finite path for some $1 \leq i \leq k$, then $P'_i = P_i^n$ for all but finitely many indices $n \in \mathbb{N}$. If its endpoint v_i edge-dominates an edge-end from A, then P'_i is also a $S - A$ path by definition.

Then, consider the set of indices $I = \{1 \leq i \leq k : P'_i \text{ is not a } S - A \text{ path}\}$. Let $J \subset I$ be a subset of maximum size for which there is a family $\mathscr{P}_J = \{P_j : j \in J\}$ of edge-disjoint $S - A$ paths satisfying the following properties:

- $\{P'_i : i \notin J\} \cup \{P_j : j \in J\}$ is a set of edge-disjoint rays or finite paths;
- P'_j is an initial subpath of P_j for every $j \in J$.

If we prove that $J = I$, then $\mathscr{P} = \{P'_i : i \notin I\} \cup \{P_i : i \in I\}$ is the claimed family of edge-disjoint $S - A$ paths. Indeed, suppose that there exists $l \in I \setminus J$. Then, P'_l is a finite path whose endpoint v_l (other than the one in S) does not edge-dominate an end from A. In particular, v_l does not edge-dominate an edge-end from $\{[P'_i]_E : P'_i \text{ is a ray}, i \notin I\} \cup \{[P_j]_E : P_j \text{ is a ray}, j \in J\}$. Hence, there is $L \subset E(G)$ a finite set of edges such that, in $G \setminus L$, the connected component C containing v_l does not intersect $\bigcup_{i \notin J} V(P'_i) \cup \bigcup_{j \in J} V(P_j)$. However, by construction, v_l cannot be separated from A by finitely many edges, since P'_l is a finite path. Hence, in C there is a ray starting at v'_l whose edge-end belongs to A. Concatenating this ray with the finite path P'_l, we contradict the maximality of J.

Now, suppose that $A \subset \Omega_E(G)$ is any subset that can be separated from S by the finite set $F \subset E(G)$. If F has minimum size with that property, we observe that F also separates $\overline{A}$ from S. In fact, suppose that r is a graphic $S - \overline{A}$ path that misses F. Since $[r]_E \in \overline{A}$, fix $[r']_E \in A \cap \Omega_E(F,[r])$. Then, the tail of r' in $G \setminus F$ is contained in $C_E(F,[r]')$. Assuming that r' starts at S, we contradict the main hypothesis over A.

Hence, by the case just analyzed, there is a family $\mathscr{P}$ of $k = |F|$ edge-disjoint $S - \overline{A}$ paths. Then, for every $P \in \mathscr{P}$ there is an edge-end $[r_P]_E \in \overline{A}$ such that P is either a ray which is edge-equivalent to r_P or a finite path such that one of its endpoints edge-dominates $[r_P]_E$. In this latter case, by shortening the path P if necessary, we assume that its endpoint is the unique vertex of P that edge-dominates the edge-end $[r_P]_E$.

Consider then the set $X = \{P \in \mathscr{P} : [r_P]_E \in \overline{A} \setminus A\}$. Suppose that $\mathscr{P}$ is chosen so that X has minimum size. Finishing the proof, we claim that $X = \emptyset$. Otherwise, fix $P \in \mathscr{P}$. Since $\Omega_E(G)$ is a Hausdorff space, there is $L \subset E(G)$ a finite set for which $[r_Q]_E \notin \Omega_E(L,[r_P]_E)$ for every $Q \in \mathscr{P}$ such that $[r_Q]_E \neq [r_P]_E$. Moreover, by possibly adding (finitely many) edges to L, we can assume that no edge from $\bigcup_{\substack{Q\in\mathscr{P}\\ [r_Q]_E\neq[r_P]_E}} E(Q)$ lies in $C_E(L,[r_P]_E)$. However, there is a ray r' in $C_E(L,[r_P]_E)$ whose edge-end $[r']_E$ belongs to A, since $[r_P]_E \in \overline{A} \setminus A$. Unless by changing the choice of P within $\{Q \in \mathscr{P} : [r_Q]_E = [r_P]_E\}$, we can assume that r' starts in a vertex of P and does not intersect $\bigcup_{\substack{Q\in\mathscr{P}\\ [r_Q]_E=[r_P]_E}} V(Q)$ in any other point, because $[r']_E \neq [r_P]_E$. This defines a $S - A$ path P' by concatenating r' with an initial segment of P, so that $(\mathscr{P} \setminus \{P\}) \cup \{P'\}$ contradicts the minimality of X.

$\square$

After applying the above result for both sides of a bipartition of $V(G)$ given by a minimum cut, the following Corollary finishes the proof of Theorem 6.2.4:

Corollary 6.2.6. *Let $A, B \subset \Omega_E(G)$ be two sets of edge-ends of a graph G. Suppose that there is a finite $A - B$ separator $F \subset E(G)$. If F has minimum size with that property, there is a family $\mathscr{P}$ of $|F|$−many edge-disjoint $A - B$ paths.*

Proof. Since F has minimum size, we can write $F = \delta(V_1)$ for some $V_1 \subset V(G)$. Consider $V_2 = V(G) \setminus V_1$ and let S_i be the set of endpoints of edges of F in V_i, for $i = 1,2$. Assume that the representatives of edge-ends in A have its tails in $G[V_1]$, while the representatives of edge-ends in B have its tails in $G[V_2]$. Hence, F is an edge set of minimal size separating A from S_2 in $G[V_1 \cup S_2]$. By Lemma 6.2.5, there is a family $\mathscr{P}_1 = \{P_1^1, P_2^1, \ldots, P_{|F|}^1\}$ of edge-disjoint $S_2 - A$ paths in this graph. Analogously, there is $\mathscr{P}_2 = \{P_1^2, P_2^2, \ldots, P_{|F|}^2\}$ a set of edge-disjoint $S_1 - B$ paths in $G[V_2 \cup S_1]$. After changing the enumeration of the elements in $\mathscr{P}_2$ if necessary, we observe that P_i^1 and P_i^2 intersects in a common edge from F for every $1 \leq i \leq |F|$. Therefore, the concatenation $P_i^1P_i^2$ is a well defined $A-B$ path, so that $\mathscr{P} = \{P_i^1P_i^2 : 1 \leq i \leq |F|\}$ is the claimed family. $\square$

We are now ready to state a generalization of the Lovász-Cherkassky Theorem for countable graphs and their edge-ends. This is done by combining Lemma 6.2.5 to the main idea for proving Theorem 5.5.9, which reduces the problem to hypothesis of Theorem 5.5.10. To that aim, in the below result, a $T-$**path** means a $t_1 - t_2$ **path** P as in Definition 6.2.1 satisfying $\overline{P} \cap T = \{t_1, t_2\}$:

Theorem 6.2.7 (Lovász-Cherkassky for countable graphs and their edge-ends)**.** *Let G be a countable graph and $T \subset V(G) \cup \Omega_E(G)$ be a discrete subspace of $\|G\|$. Suppose that $|\delta(X)|$ is even or infinite for every $X \subset V(G)$ in which $T \subset \overline{X}$. Then, there exists a collection $\mathscr{P}$ of edge-disjoint $T-$paths such that, for every $t \in T$, there is a cut separating t from $T \setminus \{t\}$ obtained by the choice of precisely one edge from each path of $\mathscr{P}_t = \{P \in \mathscr{P} : t$ is an endpoint of $P\}$.*

Revisited proof of Theorem 5.5.9. Without loss of generality, we assume that G is connected. Since G is countable and the graph $G \setminus F$ has finitely many connected components for every finite set of edges $F \subset E(G)$, the topological space $V(G) \cup \Omega_E(G)$ has a countable basis. Therefore, T is also countable, by being a discrete subspace. Thus, for some countable (possibly finite) cardinal κ, fix an enumeration $\{t_i\}_{i<\kappa}$ for $T \cap \Omega_E(G)$. Since T is discrete, we can find a finite edge set $F_0 \subset E(G)$ so that $C_E(F_0, t_0) \cap T = \{t_0\}$. By taking F_0 of minimum size with that property and denoting $C_0 = C(F_0, t_0)$, the separator F_0 is actually the cut $\delta(C_0)$.

By induction, suppose that we have defined finitely many cuts $F_0, F_1, \ldots, F_i$ and disjoint connected subgraphs $C_0, C_1, C_2, \ldots, C_i$ of G with the following properties:

- For every $j < i$, the cut $F_j \subset E(G)$ is finite. Moreover, $C_j = C_E(F_j, t_j)$;
- For every $j < i$, $\overline{C_j} \cap T = \{t_j\}$.

In order to define F_{i+1} and C_{i+1}, consider G_i' the (multi)graph obtained from G after contracting the connected subgraph C_j to a vertex v_j for every $j \leq i$. Endowing G_i' with the quotient topology that arises from $\|G\|$, the set $T_i' = \{v_0, v_1, \ldots, v_i\} \cup (T \setminus \{t_1, t_2, \ldots, t_i\})$ is also discrete. Therefore, we can find a finite set $F_{i+1} \subset E(G_i')$ so that F_{i+1} separates t_{i+1} from the other elements of T_i'. More precisely, by choosing F_{i+1} with minimum size under these conditions, F_{i+1} is a cut in G with $\overline{C_E(F_{i+1}, t_{i+1})} \cap T = \{t_{i+1}\}$. We then denote $C_{i+1} = C_E(F_{i+1}, t_{i+1})$.

Once C_i is defined for every $i < \kappa$, let $\tilde{G}$ be the (multi)graph obtained by contracting each C_i to a vertex v_i. Setting $T' = \{v_i : i < \kappa\} \cup (T \cap V(G))$, by hypothesis $|\delta(X)|$ is even or infinite for every $X \subset V(\tilde{G})$ containing T'. Hence, Theorem 5.5.10 guarantees the existence of a collection $\mathscr{P}'$ of edge-disjoint $T'-$paths such that, for every $t \in T'$, there is a cut C_t separating t from $T' \setminus \{t\}$ which is obtained by the choice of precisely one edge from each path of $\mathscr{P}_t' = \{P \in \mathscr{P}' : t$ is an endpoint of $P\}$.

On the other hand, for each $i < \kappa$, fix $S_i = \{a \in V(G) \setminus C_i : ab \in F_i \text{ for some } b \in V(G)\}$ the set of endpoints of edges in the cut F_i that lie on $V(G) \setminus C_i$. By applying Lemma 6.2.5 in the connected (induced) subgraph $H_i = C_i \cup S_i$, we obtain $\mathscr{P}_i$ a collection of $|F_i|$ edge-disjoint $S_i - \{t_i\}$ paths. In particular, every edge of F_i belongs to precisely one path of $\mathscr{P}_i$.

Therefore, we describe the claimed collection $\mathscr{P}$ of edge-disjoint $T-$paths as follows: for every $i < \kappa$, we concatenate different paths from $\mathscr{P}'_{v_i}$ with different paths from the family $\mathscr{P}_i$. Note that $|\mathscr{P}'_{v_i}| = |C_{v_i}| \leq |F_i| = |\mathscr{P}_i|$, because F_i separates v_i from $T' \setminus \{v_i\}$ in $\tilde{G}$. Then, this construction of $\mathscr{P}$ is well defined, although some paths of $\mathscr{P}_i$ are not extended to paths of $\mathscr{P}$ if $|C_{v_i}| < |F_i|$. □

The extensions presented in this section for Menger's and Lovász-Cherkassky Theorems suggest that other results from finite graph theory might be generalized in terms of edge-ends. Despite that, the literature concerning edge-end spaces is still not broad, and, in fact, the original paper of Hahn, Laviolette and Širáň (1997) is practically the unique reference on the subject. Hence, we address the following problem for future investigations:

Problem 6.2.8. *Which other connectivity results from finite graph theory can be extended to infinite graphs and their edge-ends? In particular, which similar results concerning locally finite graphs and their ends, such as those overviewed by Diestel (2010), can be stated also for countable graphs when considering its edge-ends?*

6.3 Edge-end spaces via end spaces

Since the definition of edge-ends is somehow inspired by the definition of ends, it is natural to compare the classes of topological spaces $\{\Omega(G) : G \text{ graph}\}$ and $\{\Omega_E(G) : G \text{ graph}\}$. Indeed, as pointed out in the Introduction, equivalent rays in a graph are also edge-equivalent, attesting the weakness of this latter equivalence relation when compared with the former. This suggests that the edge-end space of a graph G can be seen as the end space of possibly another graph H, which might be obtained from G by operations that improve its connectivity. The program that we shall first carry out in this section formalizes this idea, aiming to conclude the observation below:

Theorem 6.3.1. *Let G be a graph. Then, there is a graph H whose each vertex edge-dominates at most one end and such that $\Omega(H) \simeq \Omega_E(G)$.*

In the above result, we say that a vertex $v \in V(H)$ *edge*-dominates an *end* $[r] \in \Omega(H)$ if it edge-dominates the corresponding edge-end $[r]_E$. Hence, the graph H claimed by Theorem 6.3.1 has the following property: if $[r], [s] \in \Omega(H)$ are distinct, then there is no vertex $v \in V(H)$ which edge-dominates both $[r]_E$ and $[s]_E$.

In order to construct such H, fix, for each vertex $v \in V(G)$ that edge-dominates some end, a complete graph K_v on $d(v)$ vertices. Then, fix also a bijection $\rho_v : N(v) \to V(K_v)$ between $N(v) := \{u \in V(G) : uv \in E(G)\}$ and the vertices of K_v. Denoting by D the set of vertices of G that edge-dominate some ray, the vertex set of H is written as $V(H) := (V(G) \setminus D) \cup \bigcup_{v \in D} V(K_v)$, so that the elements of the (disjoint) union $\bigcup_{v \in D} V(K_v)$ are called *expanded vertices*.

The edge set of H, in its turn, is given by the (disjoint) union $\bigcup_{v \in D} E(K_v)$ and the range of the function $\rho : E(G) \hookrightarrow E(H)$ defined as follows:

- $\rho(uv) = uv$ if $u, v \in V(G) \setminus D$;
- $\rho(uv) = u\rho_v(u)$ if $u \in V(G) \setminus D$ but $v \in D$;
- $\rho(uv) = \rho_u(v)\rho_v(u)$ if $u, v \in D$.

Roughly speaking, ρ is a map that attaches different edges incident in a vertex $v \in D$ to different vertices of K_v. Since adjacencies between vertices of $V(G) \setminus D$ are preserved, we regard ρ as an *inclusion map* from $E(G)$ to $E(H)$. For a vertex $x \in K_v$, hence, we define its *canonical edge* to be the unique edge of $\rho(E(G))$ that has x as an endpoint.

This inclusion map also translate rays of G into rays of H in a natural sense. In fact, if $r = v_0v_1v_2v_3\ldots$ is a ray in G, we consider the ray $\rho(r)$ in H whose presentation by its edges is $\rho(v_0v_1)s_1\rho(v_1v_2)s_2\rho(v_2v_3)\ldots$, where, for every $i \geq 1$,

- $s_i = \emptyset$ is the empty edge if $v_i \notin D$;
- $s_i = \rho_{v_i}(v_{i-1})\rho_{v_i}(v_{i+1})$ is the edge in K_{v_i} which is adjacent to the canonical edges $\rho(v_{i-1}v_i)$ and $\rho(v_iv_{i+1})$.

Clearly, finite paths of G can be recovered in H by the same construction. In other words, if $P = v_0v_1\ldots v_n$ is a path in G, we denote by $\rho(P)$ the path in H whose edges are $\rho(v_0v_1)s_1\rho(v_1v_2)s_2\rho(v_2v_3)\ldots s_{n-1}\rho(v_{n-1}v_n)$. Conversely, a path or a ray r in H has the form $\rho(s)$ for some path or some ray s of G, accordingly, whenever $|E(r) \cap E(K_v)| \leq 1$ for every $v \in D$. In this sense, via ρ, the next technical result allows us to map edge-disjoint families of paths in G to disjoint families of paths in H:

Lemma 6.3.2. *Fix r a ray in G. Let R be a vertex or a ray that cannot be separated from r by finitely many edges. Then, there is an infinite family $\{P_n\}_{n\in\mathbb{N}}$ of edge-disjoint paths such that:*

i) For every $n \in \mathbb{N}$, P_n connects r and R, i.e., it has one endpoint in r and the other in R;

ii) If $n \neq m$, then $V(P_n) \cap V(P_m) \subset D$.

Proof. We first remark that, if R is a vertex, then the main hypothesis above actually says that R dominates r. In this case, if another vertex $v \in V(G)$ can not be separated from R by finitely many edges, then v belongs to D as well. This because v, R and a tail of r belong to the same connected component of $G \setminus F$ for every finite set $F \subset E(G)$.

Now, let P_0 be a path connecting r and s. For some $n \geq 1$, suppose that we have defined finitely many edge-disjoint paths $P_0, P_1, \ldots, P_{n-1}$ that connect r and R. Moreover, we assume by induction that $V(P_i) \cap V(P_j) \subset D$ whenever $i \neq j$. Then, by the observation made in the previous paragraph, any vertex from $V(G) \setminus D$ can be separated from r and R by a finite set of edges. In particular, there is a finite set $F \subset E(G)$ that separates every vertex of $\bigcup_{i=0}^{n-1} V(P_i) \setminus D$ from r and R. By hypothesis, in $G \setminus F$ there is a path P_n that connects the tail of r to R (if it is a vertex) or to its tail (if it is a ray). Moreover, $V(P_n) \cap V(P_i) \subset D$ for every $i < n$ by the choice of F.

At the end of this recursive process, $\{P_n\}_{n \in \mathbb{N}}$ is the claimed family of paths. □

In particular, if $R = s$ is a ray that is edge-equivalent to r, then the above family $\{P_n\}_{n \in \mathbb{N}}$ comprises edge-disjoint paths connecting r and s whose elements intersect only (possibly) at dominating vertices. Hence, in H, the family $\{\rho(P_n)\}_{n \in \mathbb{N}}$ turns out to be a family of vertex-disjoint paths connecting $\rho(r)$ to $\rho(s)$, since every expanded vertex is an endpoint of precisely one canonical edge. This means that the map

$$\begin{aligned} \Phi : \quad \Omega_E(G) &\rightarrow \Omega(H) \\ [r]_E &\mapsto [\rho(r)] \end{aligned} \tag{6.3}$$

is well-defined. Proving Theorem 6.3.1, our aim now is to verify that Φ is an homeomorphism. First, we remark that its surjection also follows from Lemma 6.3.2:

Proposition 6.3.3. *The map Φ is surjective.*

Proof. Fix $r = x_0 x_1 x_2 \ldots$ a ray in H. First, consider the case in which $V(r) \cap V(K_v)$ is finite for every $v \in D$. We will recursively define a ray $r' = x'_0 x'_1 x'_2 \ldots$ of H as follows:

- We declare $x'_0 = x_0$. If $x_0 \in V(K_v)$ for some $v \in D$, we also define $x'_1 = x_{i_1}$, where $i_1 = \max\{j \in \mathbb{N} : x_j \in V(K_v)\}$. Since K_v is a complete graph, x'_0 and x'_1 are indeed neighbors;
- We suppose that $x'_0, x'_1, x'_2, \ldots, x'_n$ are already defined for some $n \in \mathbb{N}$. By induction, we can write $x'_k = x_{i_k}$ for every $1 \leq k \leq n$ and certain indices $i_1 < i_2 < i_3 < \cdots < i_n$. Moreover, we assume that, if $x'_n \in V(K_v)$ for some $v \in D$, then $i_n = \max\{j \in \mathbb{N} : x_j \in V(K_v)\}$. We thus set $x'_{n+1} = x_{i_n+1}$. In addition, if $x_{i_n+1} \in V(K_v)$ for some $v \in D$, we also set $x'_{n+2} = x_{i_{n+2}}$, in which $i_{n+2} = \max\{j \in \mathbb{N} : x_j \in V(K_v)\}$.

Defined this way, the ray r' is actually a subgraph of r, so that $[r'] = [r]$. Note that $r' = \rho(s)$ for some ray s in H, because $|E(r) \cap E(K_v)| \leq 1$ for every $v \in D$ by construction. Therefore, $\Phi([s]_E) = [r'] = [r]$.

Now, suppose that $V(r) \cap V(K_v)$ is infinite for some $v \in D$. By definition of D, there is a ray s in G that v edge-dominates. Then, there is $\{P_n\}_{n \in \mathbb{N}}$ an infinite family of edge-disjoint paths connecting v and s. By Lemma 6.3.2, we can assume that $P_n \cap P_m \subset D$ if $n \neq m$. Since expanded vertices are endpoints of precisely one canonical edge, $\{\rho(P_n)\}_{n \in \mathbb{N}}$ is a family of vertex-disjoint paths that verifies the equivalence between $\rho(s)$ and a ray r_v composed by expanded vertices of K_v. On the other hand, since $V(r) \cap V(K_v)$ is infinite and K_v is a complete graph, we have $r \sim r_v$. Hence, $\Phi([s]_E) = [\rho(s)] = [r_v] = [r]$.

□

Rather strongly than showing that Φ is injective, we can conclude that distinct edge-ends of G are mapped to ends of H which are not infinitely edge-connected:

Proposition 6.3.4. *If r and s are rays of G such that $[r]_E \neq [s]_E$, then $\rho(r)$ and $\rho(s)$ are not edge-equivalent in H. In particular, $[\rho(r)] \neq [\rho(s)]$.*

Proof. Fix r and s two rays that can be separated by a finite set $F \subset E(G)$. In particular, $\rho(F)$ is also finite. For a while, suppose that, in $H \setminus \rho(F)$, there is a path $P = x_0x_1 \ldots x_n$ connecting the tails of $\rho(s)$ and $\rho(r)$. Suppose that $|V(P)|$ is minimum with that property. Then, given distinct $x_i, x_j \in K_v$ for some $v \in D$, we must have $j = i+1$ if $i < j$, because the subpath $P' = x_0x_1 \ldots x_ix_j \ldots x_n$ is well defined and also connects $\rho(s)$ to $\rho(r)$. Hence, if $x_i \in D$, the edges $x_{i-1}x_i$ and $x_{i+1}x_{i+2}$, if exist, are canonical. In other words, $P = \rho(Q)$ for some path Q that connects r and s in G. However, this contradicts the fact that F separates the rays r and s, because $E(P) \subset E(H) \setminus \rho(F)$ and, thus, $E(Q) \subset E(G) \setminus F$. Therefore, $\rho(F)$ separates $\rho(r)$ and $\rho(s)$. □

Corollary 6.3.5. *Every vertex of H edge-dominates at most one end of $\Omega(H)$.*

Proof. Fix $v \in V(H)$ a vertex that edge-dominates two distinct ends of H. Fix r and s representatives for those ends. In particular, if $F \subset E(H)$ is finite, there is a path P in $H \setminus F$ connecting v to a tail of r. Then, $F' = F \cup E(P)$ is also a finite set of edges, so that there is a path P' in $H \setminus F'$ connecting v to a tail of s. Therefore, by concatenating P and P', we obtain a path in $H \setminus F$ connecting the tails of r and s. This proves that r and s are infinitely edge-connected, although they belong to different ends of H. Since Φ is surjective, this contradicts Proposition 6.3.4. □

Proving Theorem 6.3.1, thus, we will now show that Φ is open and continuous:

Proposition 6.3.6. Φ *is an homeomorphism.*

Proof. We will first verify that Φ is continuous. To this aim, for some $[r] \in \Omega(H)$ and some finite $S \subset V(H)$, let $\Omega(S,[r])$ be a basic open set in the end space of H. Since Φ is a bijection, we can write the representative r as $r = \rho(s)$ for some ray s in G. If a fixed vertex $u \in S$ belongs to D, let $F_u = \{f_u\}$ denote the singleton set containing its canonical edge f_u. If not, u does not edge-dominate the ray s, so that there is a finite set $F_u \subset E(G)$ that separates s and u. Hence, $F = \bigcup_{u\in S} F_u$ is a finite set of edges. We then claim that the basic open set $\Omega_E(F,[s]_E)$ for the edge-end space of G is contained in $\Phi^{-1}(A)$. In fact, if F does not separate s and a ray s', there is P a path connecting s and s' in $G \setminus F$. By the choice of F, therefore, $\rho(P)$ is a path connecting $\rho(s)$ and $\rho(s')$ in $G \setminus S$. In other words, S does not separate $\rho(s)$ and $\rho(s')$, proving that $\Phi(\Omega_E(F,[s]_E)) \subset \Omega(S,[r])$.

Conversely, in order to show that Φ is an open map, let $\Omega_E(F,[s]_E)$ be a basic open set containing an edge-end $[s]_E \in \Omega_E(G)$ for some finite $F \subset E(G)$. Hence, it is also finite the set $S = \{x \in V(H) : x \text{ is endpoint of } \rho(e) \text{ for some } e \in F\}$. Then, it is enough to verify that $\Omega(S,[\rho(s)]) \subset \Phi(\Omega(F,[s]_E))$. To this aim, again by the fact that Φ is surjective, an element $[r] \in \Omega(S,[\rho(s)])$ has a representative of the form $r = \rho(s')$ for some ray s' of G. Since S does not separate $\rho(s')$ and $\rho(s)$, there is P a path connecting the tails of these two rays in $H \setminus S$. As in Proposition 6.3.4, if we consider P to have as few vertices as possible, we can write $P = \rho(Q)$ for some path Q in $G \setminus F$ connecting s and s'. Hence, F does not separate s and s', so that $[s']_E \in \Omega_E(F,[s]_E)$ and, therefore, $[r] = [\rho(s')] \in \Phi(\Omega_E(F,[s]_E))$. □

Once established the proof of Theorem 6.3.1, we observe that, in particular, it fits the edge-end spaces as subclass of $\{\Omega(H) : H \text{ graph}\}$. On the other hand, Section 6.4 shall verify that the reverse inclusion does not hold: there exists a graph H for which $\Omega(H)$ is not the edge-end space of any other graph. Before that, we will finish this section by discussing how Theorem 6.3.1 is actually a characterization of edge-end spaces, since we can conclude its converse:

Theorem 6.3.7. *Let G be a graph in which every vertex edge-dominates at most one end of $\Omega(G)$. Then, there is a graph H such that $\Omega_E(H) \simeq \Omega(G)$.*

Following the opposite direction of the construction that supports Theorem 6.3.1, H as in the above result will be defined from G by operations that weaken its connectivity. In order to start this construction, we first extract from the proof of Lemma 5.4.5 the following observation:

Lemma 6.3.8. *Let $[s] \in \Omega(G)$ be an end of G and fix $\mathscr{R}_{[s]} \subset [s]$ a maximal family of pairwise (vertex-)disjoint rays. Denote by $D_{[s]}$ the set of vertices that dominate $[s]$. Then,*

$$E_{[s]} = D_{[s]} \cup \bigcup_{r\in\mathscr{R}_{[s]}} V(r) \tag{6.4}$$

has finite adesion, i.e., for each connected component C of $G \setminus E_{[s]}$, the set $N(C) = \{v \in E_{[s]} : v \text{ has a neighbor in } C\}$ is finite. Moreover, if r is a ray such that $V(r) \cap E_{[s]}$ is infinite, then r and s are equivalent.

Then, for each end $[s] \in \Omega(G)$, we will fix $E_{[s]}$ as defined by (6.4) and, following the notation from Section 5.4, we will call it an *envelope* for $[s]$. By the main hypothesis of Theorem 6.3.7, each vertex $v \in V(G)$ edge-dominates at most one such end $[s] \in \Omega(G)$. If this is the case, we define a new vertex v' and a bipartition $\tau_v : N(v) \to \{v, v'\}$ according to the following rules:

$$\tau_v(u) = \begin{cases} v, & \text{if } u \in E_{[s]}; \\ v', & \text{if } u \in V(G) \setminus E_{[s]}. \end{cases}$$

From now on in this section, $D \subset V(G)$ will denote the set of vertices that edge-dominates an end of G. Then, we consider $V(G) \cup \{v' : v \in D\}$ as the vertex set of the claimed graph H. Its edge set is $\{vv' : v \in D\} \cup \tau(E(H))$, in which $\tau : E(G) \hookrightarrow E(H)$ is the injective map given by:

$$\tau(uv) = \begin{cases} uv, & \text{if } u, v \in V(G) \setminus D; \\ u\tau_v(u), & \text{if } u \in V(G) \setminus D \text{ and } v \in D; \\ \tau_u(v)\tau_v(u), & \text{if } u, v \in D. \end{cases}$$

In other words, H is constructed from G after duplicating vertices of D, rearranging the edges of G according to τ and defining an edge between a vertex and its copy. In particular, by tracking the edges of G, the map τ also allows us to include paths and rays of G into H. For example, given a ray $r = v_0v_1v_2\dots$ in G, we denote by $\tau(r)$ the ray in H whose presentation by its edges is $\tau(v_0v_1)s_1\tau(v_1v_2)s_2\tau(v_2v_3)s_3\dots$, in which, for $i \geq 1$:

- $s_i = \emptyset$ is the empty edge if $v_i \notin D$ or $v_i \in D$ and $\tau_{v_i}(v_{i+1}) = \tau_{v_i}(v_{i-1})$;
- $s_i = v_iv_i'$ if $v_i \in D$ and $\tau_{v_i}(v_{i+1}) \neq \tau_{v_i}(v_{i-1})$.

Regarding the above notation, a path $P = v_0v_1v_2\dots v_n$ in G can also be seen within H: we naturally define the path $\tau(P)$ whose edges are given by $\tau(v_0v_1)s_1\tau(v_1v_2)s_2\tau(v_2v_3)\dots s_{n-1}\tau(v_{n-1}v_n)$. In particular, if $\{P_n\}_{n\in\mathbb{N}}$ is an infinite family of (vertex-)disjoint paths connecting the rays r and s in G, then $\{\tau(P_n)\}_{n\in\mathbb{N}}$ are (vertex-)disjoint paths connecting $\rho(r)$ and $\rho(s)$. Therefore, the map below is well-defined and it is a natural candidate to be an homeomorphism between $\Omega(G)$ and $\Omega_E(H)$:

$$\begin{array}{rccc} \Psi : & \Omega(G) & \to & \Omega_E(H) \\ & [r] & \mapsto & [\tau(r)]_E \end{array} \tag{6.5}$$

In order to verify that Ψ is indeed a bijection, the following technical observation is helpful:

Lemma 6.3.9. *Fix K' a connected subgraph of H and let K be the subgraph of G whose edges are given by $\tau^{-1}(E(K'))$ (and whose vertices are its endpoints). Then, K is connected.*

Proof. We will prove that there is no empty cut in K. To that aim, let $\{A_1, A_2\}$ be a bipartition of $V(K)$ into non-empty parts. For a contradiction, assume that there is no edge of K with an endpoint in A_1 and the other in A_2. However, there are edges $e = u_1v_1$ and $f = u_2v_2$, for some $u_1, v_1 \in A_1$ and $u_2, v_2 \in A_2$. Since K' is connected, there is a path in this graph whose presentation via its edges is $e_0e_1e_2 \ldots e_n$, where $e_0 = \tau(e)$ and $e_n = \tau(f)$. Consider the index

$$i = \max\{0 \leq j \leq n : e_j \in \tau(E(G)) \text{ and } \tau^{-1}(e_j) \text{ is an edge of } K[A_1]\}.$$

This index is well-defined by the choice of $e_0 = e$, but $i < n$ by the choice of $f = e_n$. Then, e_{i+1} has one of the following forms, both contradicting the fact that $A_1 \cap A_2 = \emptyset$:

- If $e_{i+1} = vv'$ for some $v \in D$, then, by construction, $e_{i+2} \in \tau(E(G))$. Moreover, both $\tau^{-1}(e_i)$ and $\tau^{-1}(e_{i+2})$ have v as an endpoint. However, $\tau^{-1}(e_{i+2})$ is an edge of $K[A_2]$ by the maximality of i. Therefore, we should have $v \in A_1 \cap A_2$;
- If $e_{i+1} \in \tau(E(G))$, then $\tau^{-1}(e_{i+1})$ is an edge of $K[A_2]$ by the maximality of i. On the other hand, since $e_i \in \tau(E(G))$ is adjacent to e_{i+1} in H, the edges $\tau^{-1}(e_i)$ and $\tau^{-1}(e_{i+1})$ have a common endpoint $v \in V(G)$. Then, we should have again $v \in A_1 \cap A_2$.

□

The fact that Ψ is surjective can be seen as an application of König's Lemma, while Ψ is injective due to the main hypothesis over G:

Proposition 6.3.10. *Ψ is bijective.*

Proof. Let r' be a ray in H, whose presentation via edges might be written as $f_0f_1f_2\ldots$. Since the edges from $\{vv' : v \in D\} \subset E(H)$ are pairwise non-adjacent, $\tau^{-1}(E(r'))$ is an infinite set of edges in G. Consider K the subgraph of G that contains precisely these edges and its endpoints, being connected by the above lemma. We observe that every vertex $v \in V(K)$ has degree at most 4. In fact, if $v \notin D$, then $v \in V(H)$ has at most two neighbors in r'. Similarly, v and v' have at most two neighbors each in r' if $v \in D$, so that v is the endpoint of at most four edges in K in this case. By König's Lemma (2.1.1), then, there is r a ray in K. Since $E(K) = \tau^{-1}(E(r'))$, the ray $\tau(r)$ meets r' in infinitely many edges, so that $[\tau(r)]_E = [r']_E$. This verifies the surjection of Ψ.

In order to conclude that Ψ is injective, let s and r be non-equivalent rays in G. In other words, there is a finite set $S \subset V(G)$ that separates r and s. Since $[s] \neq [r]$, by our main hypothesis over G no vertex of S edge-dominates both r and s. Hence, for each $v \in S$ there is a finite set $F_v \subset E(G)$ that separates v from r or s. We thus claim that the finite set $\bigcup_{v \in S} \tau(F_v)$ separates $\tau(r)$ and $\tau(s)$ in H. For instance, suppose that the tails of $\tau(s)$ and $\tau(r)$ belong to the same connected component K' of $H \setminus \bigcup_{v \in S} \tau(F_v)$. Then, by Lemma 6.3.9, there is a path P in G connecting r and s

such that $\tau(E(P)) \subset H \setminus \bigcup_{v \in S} \tau(F_v)$. However, there is $v \in V(P) \cap S$, because S separates r and s. Since $E(P) \cap F_v = \emptyset$, this contradicts the fact that F_v separates the vertex v from r or s. □

The continuity of Ψ follows easily from the fact that any edge-separator has a natural vertex-separator associated, an argument also employed in Proposition 6.3.6 when we verified that Φ is an open map:

Proposition 6.3.11. Ψ *is continuous.*

Proof. Let $F \subset E(H)$ be a finite set of edges and fix $[r']_E \in \Omega_E(H)$ an edge-end of H. Since Ψ is surjective, there is a ray r in G such that $[\tau(r)]_E = [r']_E$. For each $e \in F \cap \tau(E(G))$, consider S_e the set of endpoints of $\tau^{-1}(e)$. If $e \in F$ has the form $e = vv'$ for some $v \in D$, define $S_e = \{v\}$. Then, $S = \bigcup_{e \in F} S_e$ is finite.

Let C be the connected component of $G \setminus S$ in which there is a tail of r. If s is any other ray in C and P is a path connecting r and s, then $\tau(P)$ connects $\tau(r)$ to $\tau(s)$ in $H \setminus F$ by construction. This argument verifies the inclusion $\Psi(\Omega(S,[r])) \subset \Omega_E(F,[r']_E)$, proving that Ψ is continuous. □

Whether $v \in D$, we observe that the criteria for defining the neighbors of v and v' in H was not mentioned in the proofs of the previous propositions. In fact, finally concluding Theorem 6.3.7, it is employed only to show that Ψ is an open map:

Proposition 6.3.12. Ψ *is an open map.*

Proof. Let $S \subset V(G)$ be finite and fix C the connected component of $G \setminus S$ in which there is a ray s. Recall that we fixed a set $E_{[s]} \subset V(G)$ as in (5.4.5) in order to define τ. Choosing the representative s so that $V(s) \subset E_{[s]}$, we will now show that $\Psi(\Omega(S,[s]))$ is open in $\Omega_E(H)$.

First, for each $v \in S \setminus E_{[s]}$, we observe that there is C_v a connected component in $G \setminus E_{[s]}$ containing v. By Lemma 6.3.8, the set $N(C_v) = \{u \in E_{[s]} : u \text{ has a neighbor in } C_v\}$ is finite. Moreover, for each $u \in N(C_v)$, one of the options below is verified:

- If $u \in D$, define the singleton set $F_u = \{uu'\} \subset E(H)$;
- If $u \notin D$, there is a finite set $F'_u \subset E(G)$ that separates u from r. Hence, we define $F_u = \tau(F'_u)$.

In any case, $F_1 = \bigcup_{v \in S \setminus E_{[s]}} \bigcup_{u \in N(C_v)} F_u$ is a finite set of edges of H. Relying on the following claim, a similar set can be defined:

`Claim`: It is finite the set

$$S' = \{v \in E_{[s]} : \text{there is no path connecting } v \text{ and a tail of } s \text{ in } G \setminus S\}.$$

Proof of the Claim. By definition, we first observe that S separates vertices of $S' \setminus S$ from s. Therefore, $D_\varepsilon \cap S' \subset S$, where $D_{[s]}$, as in (5.4.5), is the set of vertices of G that dominate $[s]$. If we assume for a contradiction that S' is infinite, then $S' \cap \bigcup_{r \in \mathscr{R}_{[s]}} V(r)$ is infinite as well, where $\mathscr{R}_{[s]}$ is also defined by Lemma 6.3.8. If $S' \cap V(r)$ is infinite for some $r \in \mathscr{R}_{[s]}$, then there is a path connecting s to a vertex of $S' \cap V(r)$ in $G \setminus S$, since $[r] = [s]$. This contradicts the definition of S', so that $S' \cap V(r)$ must be finite for every $r \in \mathscr{R}_{[s]}$. In particular, $\mathscr{R}'_{[s]} = \{r \in \mathscr{R}_{[s]} : S' \cap V(r) \neq \emptyset\}$ is infinite and composed by pairwise disjoint rays. Hence, there is also some ray $r \in \mathscr{R}'_{[s]}$ such that $S \cap V(r) = \emptyset$, because S is finite. Therefore, due to the equivalence between the rays r and s, there is in $G \setminus S$ a path connecting a tail of s to a vertex of $S' \cap V(r)$. This contradicts the definition of S' once more, so that S' must be infinite. $\square$

If $v \in S'$ edge-dominates s, we define the set $J_v = \{vv'\} \subset E(H)$. Otherwise, there is a finite set $J'_v \subset E(G)$ that separates s and v. In this case, we denote $J_v = \tau(J'_v)$. Then, $F_2 = \bigcup_{v \in S'} J_v$ is a finite set of edges of H. Thus, it is finite the set $F = F_1 \cup F_2$.

Let C' be the connected component of $H \setminus F$ in which $\tau(s)$ has a tail. If another ray in that component has the form $\tau(s')$ for some ray s' in G, then, by Lemma 6.3.9, there is a path P in G connecting s and s' such that $\tau(E(P)) \subset E(H) \setminus F$.

On the other hand, if $[s] \neq [s']$, by Lemma 6.3.8 there is a connected component $C_{s'}$ in $G \setminus E_{[s]}$ in which s' has a tail. Hence, writing P in terms of its vertices as $v_0 v_1 v_2 \dots v_n$, also assuming that $v_0 \in E_{[s]}$ and $v_n \in C_{s'}$, fix $i = \min\{0 \leq j \leq n : v_j \in E_{[s]} \text{ and } v_{j+1} \in C_{s'}\}$. Since $v_i \in E_{[s]}$, one of the following cases must hold:

- If v_i edge-dominates s, the edge $v_i v'_i$ is defined in H. Moreover, by definition of τ_{v_i}, the path P must contain this edge, because $v_{i+1} \in C_{s'}$. Hence, $v_i v'_i$ does not belong to F. By definition of F_1, this means that $S \cap C_{s'} = \emptyset$, while, by definition of F_2, $v_i \notin S'$;
- Supposing now that v_i can be separated from S by finitely many edges, we have $S \cap C_{s'} = \emptyset$. Otherwise, $E(P) \cap F'_{v_i} \neq \emptyset$, because P contains v_i and connects the tails of s and s', contradicting the fact that $\tau(E(P)) \subset H \setminus F_1$. Analogously, if $v \in S'$ for instance, then we have $E(P) \cap J_{v'_i} \neq \emptyset$ by the same reason, contradicting the fact that $\tau(E(P)) \subset H \setminus F_2$.

In both cases, we conclude that $v \notin S'$ and $S \cap C_{s'} = \emptyset$. Then, $C_{s'}$ is a connected subgraph of $G \setminus S$ containing v_{i+1} and a tail of s', while there is also a path in $G \setminus S$ connecting v_i and a tail of s. Therefore, the tails of s and s' belong to the same connected component of $G \setminus S$, proving that $\Omega_E(F, [s]_E) \subset \Psi(\Omega(S, [s]))$. $\square$

The reader might have noticed some similarities between the proofs that Φ (as in 6.3) and Ψ (as in 6.5) are homeomorphisms. Indeed, Table 2 summarizes the main arguments which support the details of Propositions 6.3.3 - 6.3.12. Moreover, it addresses the idea that Ψ is, in fact, obtained via a converse strategy than the one employed to define Φ. After all, checking whether Φ is open and whether Ψ is continuous are two instances of a same heuristic, for example.

Table 2 – Comparison between the proofs that Φ and Ψ are homeomorphisms

	Φ	Ψ
Well-definition	By the choice of vertices to blow up.	Since equivalent rays are also edge-equivalent.
Injection	Choose the endpoints of a separating edge set to define a vertex-separator.	By the main hypothesis over G.
Surjection	By the choice of vertices to blow up.	By contracting the edges of $E(H)\setminus\psi(E(G))$.
Continuity	By construction, vertex-separators of H are associated to edge-separators in G.	Choose the endpoints of a separating edge set to define a vertex-separator.
Open mapping	Choose the endpoints of a separating edge set to define a vertex-separator.	By properties of envelopes, vertex-separators of G are associated to edge-separators in H.

Therefore, as the main conclusion of this section, Theorems 6.3.1 and 6.3.7 combined state the following representation result for edge-end spaces:

Theorem 6.3.13. *Let X be a topological space. Then, $X \simeq \Omega_E(G)$ for some graph G if, and only if, $X \simeq \Omega(H)$ for some graph H whose each vertex edge-dominates at most one end.*

6.4 Topological consequences

Even though Theorem 6.3.13 characterizes the edge-end spaces, this result does not highlight any topological behaviour that distinguishes this family from the usual end spaces. On the contrary, interesting properties of edge-end spaces, such as ultraparacompactness, now follows from Theorem 6.3.13 when combined to the studies of Chapter 5. However, this section aims to better explore the end spaces of graphs whose vertices edge-dominates at most one end, especially by revisiting the discussions from Section 5.4. In particular, we shall conclude the proposition below, which is a key result of our paper (AURICHI; REAL; JÚNIOR, 2023):

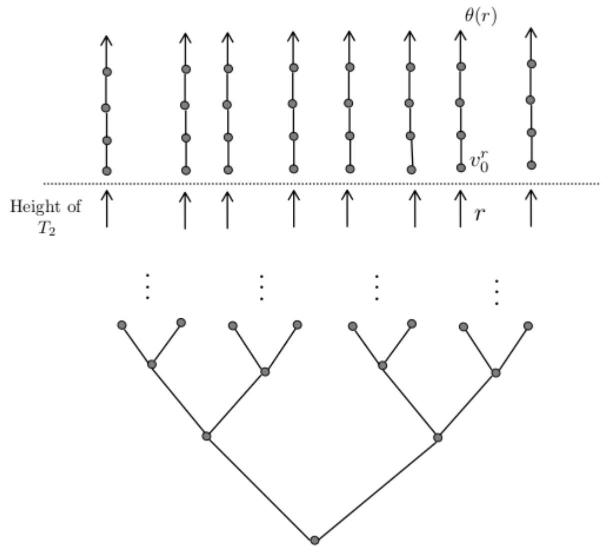
Theorem 6.4.1. *Let X be a first-countable and Lindelöf topological space. If X is the edge-end space of some graph, then X is metrizable.*

Before carrying out its proof, we will argue how Theorem 6.4.1 can detect graphs whose end spaces are not edge-end spaces of other graphs. In fact, we recall that the **binary tree** is the tree T_2 in which the root has degree 2 and any other vertex has degree 3. For a given ray r of T_2, fix an "artificial ray" $\theta(r) = v_0^r v_1^r v_2^r \dots$, so that $\theta(r) \cap T_2 = \emptyset$. As sketched by Figure 35, consider the order tree T obtained from T_2 as follows: for every ray r of T_2, we set v_0^r as a top of r and $v_i^r < v_{i+1}^r$ for every $i \in \mathbb{N}$.

Then, let G be the uniform $T-$graph in which, for every ray $r \in \mathscr{R}(T)$, the vertex v_0^r is adjacent to every vertex of r. In addition, every successor node of T is, in G, a neighbor of its predecessor. The end space of G defined this way is, thus, given by $\Omega(G) = \Omega(T_2) \cup \{[\theta(r)] : r \in \mathscr{R}(T_2)\}$. After all, the high-rays of T are those from T_2 or those that contain $\theta(r)$ for some $r \in \mathscr{R}(T_2)$.

In particular, considering (5.6) as a system of open basic neighborhoods for the ray space of T, we observe that $\Omega(G) \simeq \mathscr{R}(T)$ is a first-countable space. Moreover, according to Lemma 5.2.3, $\Omega(G)$ is also a compact space, since $G \setminus S$ has only finitely many connected components for every finite set $S \subset V(G)$. Observing that compact metric spaces are separable, $\Omega(G)$ must not be metrizable, once $\{\{\theta(r)\} : r \in \mathscr{R}(T)\}$ is an antichain of size continuum. For topologists, as Pitz (2023) remarked through its Example 2.6, $\Omega(G)$ is often known as the *Alexandroff duplicate* of the Cantor space $2^\omega \simeq \Omega(T_2)$.

Figure 35 – The Alexandroff duplicate of a Cantor space



In the order tree T sketched above, any uniform $T-$graph G has a compact and first-countable end space. However, due to the uncountable antichain $\{\{\theta(r)\} : r \in \mathscr{R}(T_2)\}$, the space $\Omega(G)$ is not metrizable.

Source: Elaborated by the author.

Exercise 6.4.2. *Let T be a special Aronszajn tree (see (KUNEN, 2011, p.204)) and consider G any uniform $T-$graph. Show that $\Omega(G)$ is also first-countable but not metrizable.*

From this example, Theorem 6.4.1 concludes the following:

Corollary 6.4.3. *There are graphs whose end spaces cannot be written as edge-end spaces of possibly other graphs. In other words, $\{\Omega_E(G) : G \text{ graph}\}$ is a proper subfamily of $\{\Omega(G) : G \text{ graph}\}$.*

The proof of Theorem 6.4.1, in its turn, is done by revisiting the construction of partition trees that display all the ends of a given graph G, as detailed in Proposition 5.4.3. Due to the characterization for edge-end spaces obtained by the previous section, we are interested in the case where every vertex of G edge-dominates at most one end. Under the hypothesis of Theorem 6.4.1, we then have:

Proposition 6.4.4. *Let G be a connected graph whose each vertex dominates at most one end. Suppose that $\Omega(G)$ is a first-countable Lindelöf topological space. Then, G has a partition tree $(T,\mathscr{V})$ that display all its ends and such that:*

i) T has countable height, bounded by $\omega\cdot\omega$;

ii) The subtree $\hat{T}=\{t\in T : t$ belongs to a high-ray of $T\}$ is countable.

Proof. Consider the sequence of partition trees $\{(T_\alpha,\mathscr{V}_\alpha)\}_{\alpha<\kappa}$ as constructed in Proposition 5.4.3. Recall that, for an ordinal $\alpha>0$, the vertices on the final level of T_α defined a set of nodes F_α with the following property: $C\in F_\alpha$ if, and only if, C is a connected component of $G\setminus \bigcup_{t\in T_\alpha\setminus F_\alpha} V_t$. In each such component C, we fixed a suitable vertex set U_C such that the connected components of $C\setminus U_C$ had finite neighborhood in G. In the partition tree $(T_{\alpha+1},\mathscr{V}_{\alpha+1})$, then, U_C became the top of the high-ray of T_α given by $\lceil \mathring{C}\rceil$, as suggested by Figure 30. In its turn, for each connected component D of $C\setminus U_C$, we fixed a maximal normal (graph-theoretic) tree T_D. Now, we observe that T_D could be chosen as in Proposition 5.3.6 when applied to the subgraph $G[D\cup N(D)]$ with $K=N(D)=\{v\in V(G)\setminus D : v$ has a neighbor in $D\}$. Therefore, if D' is a connected component of $D\setminus T_D$, the following property now holds:

($\star$) The neighborhood $N(D')=\{v\in V(G)\setminus D' : v$ has a neighbor in $D'\}$ is infinite. Moreover, all but finitely many of its elements lie in a ray $r_{D'}$ of T_D. In this case, every $v\in N(D')$ dominates (and, in particular, edge-dominates) the end $[r_{D'}]$.

After pointed out this improvement, the definition of $(T_{\alpha+1},\mathscr{V}_{\alpha+1})$ is done as in the proof of Proposition 5.4.3. In particular, the height of $T_{\alpha+1}$ is at most $h_\alpha+\omega$, where h_α is the height of T_α.

Relying on the fact that vertices of G edge-dominates at most one end, we now claim that the final level F_ω of T_ω is empty. For instance, suppose that there is a vertex $v\in V(G)\setminus \bigcup_{t\in T_\omega\setminus F_\omega} V_t$. Once $(T_n,\mathscr{V}_n)$ is a partition tree for each $n<\omega$, we must have $v\in C_n$ for some $C_n\in F_n$. By considering a big enough $n_0<\omega$, let $u\in \bigcup_{t\in T_{n_0}} V_t^{n_0}$ be a neighbor of v. By the above property ($\star$), the vertex u dominates the ray r_{C_n} for every $n\geq n_0$. However, if $n>m\geq n_0$, the choice of U_{C_m} guarantees that r_{C_n} and r_{C_m} can be separated by finitely many vertices. After all, r_{C_n} is contained in a connected component of $C_m\setminus U_{C_m}$. In other words, u dominates infinitely many non-equivalent rays, which is a contradiction.

Hence, we can finish the definition of the sequence $\{(T_\alpha, \mathscr{V}_\alpha)\}_{\alpha<\kappa}$ at the ordinal $\kappa = \omega$, so that $T := T_\omega$ and $\mathscr{V} := \mathscr{V}_\omega$ describes a partition tree $(T, \mathscr{V})$ which displays all the ends of G. Moreover, by induction, T_n has height bounded by $\omega \cdot n$, meaning that $T = \bigcup_{n<\omega} T_n$ has height bounded by $\omega \cdot \omega$. Thus, it remains to show that the subtree $\hat{T} = \{t \in T : t \text{ belongs to a high-ray of } T\}$ is countable.

For instance, suppose that $\hat{T}$ is uncountable and fix

$$\eta = \min\{\xi < \omega \cdot \omega : \text{the } \alpha \text{ level of } \hat{T} \text{ is uncountable}\}.$$

This ordinal exists because $\hat{T}$ has countable height, while we have $\eta > 0$ since $\hat{T}$ is rooted. If η is a successor ordinal, written as $\eta = \xi + 1$, let $t \in \mathscr{L}_\xi(\hat{T})$ be a predecessor of an uncountable family of nodes $\{t_i\}_{i<\omega_1} \subset \mathscr{L}_\eta(\hat{T})$. By passing $\{t_i\}_{i<\omega}$ to another uncountable subsequence if necessary, we can assume that $N(V_{\lfloor t_i \rfloor}) = N(V_{\lfloor t_j \rfloor}) =: S$ for every $i, j < \omega_1$, since $h(t)$ is countable and $(T, \mathscr{V})$ has finite adhesion. Then, $\{\Omega(S, [r]) : [r] \in \Omega(G)\}$ is an open cover for $\Omega(G)$ whose distinct elements are disjoint. However, by definition of $\hat{T}$, there is a ray r_i in G that has a tail in $V_{\lfloor t_i \rfloor}$, for each $i < \omega_1$. Hence, S separates r_i and r_j if $i \neq j$. This means that $\{\Omega(S, [r_i]) : i < \omega_1\} \subset \{\Omega(S, [r]) : [r] \in \Omega(G)\}$ is an uncountable subfamily whose elements are pairwise disjoint, contradicting the assumption that $\Omega(G)$ is a Lindelöf topological space.

Therefore, η must be a limit ordinal, so that $\mathscr{L}_\eta(T) = F_n$ for some $n < \omega$. We argue that $n \neq 0$. Otherwise, fix an uncountable subset $\{C_i\}_{i<\omega_1} \subset F_0 \cap \hat{T}$. Recall that these are connected components of $G \setminus T'_0$, and, thus, are pairwise disjoint. Then, one of the following cases is verified, but both lead to contradictions:

- If there is an uncountable subset $I \subset \omega_1$ such that $r_{C_i} = r_{C_j} =: r$ for every $i, j \in I$, then each C_i has infinitely many neighbors in the branch r of T'_0. As before, let r_i be a ray in $V_{\lfloor C_i \rfloor}$ for each $i \in I$, whose existence is guaranteed by the definition of $\hat{T}$ and by the fact that $(T, \mathscr{V})$ displays the ends of G. Then, given a finite subset $S \subset V(G)$, we have $[r_i] \in \Omega(S, [r])$ for all but finitely many indices $i \in I$. Once I is uncountable, this contradicts the fact that $\Omega(G)$ is a first-countable topological space;
- Then, there is an uncountable subset $I \subset \omega_1$ such that $r_{C_i} \neq r_{C_j}$ for every $i, j \in I$. In this case, $[r_{C_i}] \neq [r_{C_j}]$, because $(T, \mathscr{V})$ displays all the ends of G. Since r_{C_i} is a branch of T'_0, we have $r_{C_i} \subset \hat{T}$ for each $i \in I$. Fix $v_i \in r_{C_i}$ a neighbor of the connected component C_i. Being $\hat{T} \cap T'_0$ countable by the minimality of η, there must be $v \in T'_0$ such that $v = v_i$ for uncountably many indices $i \in I$. According to Proposition 5.3.6, this means that v dominates uncountably many non-equivalent rays, contradicting the main hypothesis over G.

Then, we must have $n > 0$. Moreover, $F_{n-1} \cap \hat{T}$ is countable, since this is a smaller (limit) level of $\hat{T}$. Hence, for some $C \in F_{n-1} \cap \hat{T}$, the set $\{t \in F_n \cap \hat{T} : t > C\}$ is uncountable. As an element of $\hat{T}$, the node C has only countably many successors, because η is a limit ordinal. Therefore, we can fix $v_0 \in \hat{T}$ a successor of C such that $\{t \in F_n \cap \hat{T} : t > v_0\}$ is uncountable. By construction, we recall that v_0 is the root of a normal tree T_D for a connected component D of $C \setminus V_C^t$. Fixing an uncountable family $\{D_i'\}_{i<\omega_1} \subset \{t \in F_n \cap \hat{T} : t > v_0\}$, then, each D_i' is a connected component of $D \setminus T_D$. Analogously to the above discussion, one of the following cases is verified, but both also lead to contradictions:

- Suppose that there is an uncountable set $I \subset \omega_1$ such that $r_{D_i'} = r_{D_j'} =: r$ for every $i, j \in I$. Since $\{D_i'\}_{i<\omega_1} \in F_n \cap \hat{T}$, the branch r of T_D is contained in $\hat{T}$. For each $i \in I$, let r_i be a ray in $V_{\lfloor C_i \rfloor}$, whose existence is guaranteed by the definition of $\hat{T}$ and by the fact that $(T, \mathscr{V})$ displays the ends of G. Then, given a finite subset $S \subset V(G)$, we have $[r_i] \in \Omega(S, [r])$ for all but finitely many indices $i \in I$. This, however, contradicts the fact that $\Omega(G)$ is a first-countable topological space;
- Then, there is an uncountable set $I \subset \omega_1$ such that $r_{D_i'} \neq r_{D_j'}$ for every $i, j \in I$. In this case, $[r_{D_i'}] \neq [r_{D_j'}]$, because $(T, \mathscr{V})$ displays the ends of G. Since $r_{D_i'}$ is a branch of T_D for every $i \in I$, we have $r_{D_i'} \subset \hat{T}$, allowing us to choose $v_i \in \hat{T} \cap T_D$ a neighbor of D_i'. Observing that $\hat{T} \cap T_D$ is countable by the minimality of η, there is $v \in \hat{T} \cap T_D$ a vertex such that $v = v_i$ for every i within some uncountable subset of I. By Proposition 5.3.6, this means that v dominates uncountably many distinct ends of G, contradicting a main hypothesis over this graph.

Therefore, $\hat{T}$ is countable.

□

Corollary 6.4.5. *Let G be a graph as in Proposition 6.4.4. Then, there is an order tree T such that $\Omega(G)$ is the end space of any uniform $T-$graph. Moreover, T can be chosen so that $\hat{T} = \{t \in T : t$ belongs to a high-ray of $T\}$ is countable.*

Proof. Let $(T', \mathscr{V}')$ denote the partition tree for G claimed by the previous result. Consider the order tree T obtained from T' precisely as instructed in the proof of Corollary 5.4.4. In particular, the height of T is also countable.

Now, let $t \in T$ be a node that belongs to a high-ray of T. If t is a successor node in T, then it is also a successor node in T' by construction of T. In particular, t also belongs to a high-ray of T', so that $t \in \hat{T}$. If t is a limit point, however, then $t = v(t', X)$ for some $t' \in T'$ that has at least one successor and some finite subset $X \subset \lceil t' \rceil$. Actually, the node t' must lie on a high-ray of T' (i.e., $t' \in \hat{T}'$), because t itself belongs to a high-ray of T. Hence, since $\hat{T}'$ is countable and there are countably many finite subsets of $\lceil t' \rceil$ for every $t' \in \hat{T}'$, it follows that $\hat{T}$ is also countable. □

Relying on the above corollary, we now have the complete machinery to write a proof for Theorem 6.4.1:

Proof of Theorem 6.4.1. Let G be a graph whose end space is first-countable and Lindelöf. According to Corollary 6.4.5, there is T an order tree such that $\Omega(G)$ is the end space of any uniform $T-$graph. Moreover, T can be chosen so that $\hat{T} = \{t \in T : t$ belongs to a high-ray of $T\}$ is countable. However, $\mathscr{R}(T)$ and $\mathscr{R}(\hat{T})$ describe the same topological space, since $\mathscr{R}(T) = \mathscr{R}(\hat{T})$ (as point sets) and the basic open neighborhoods given by (5.6) are coincident for these order trees. In particular, if G' is a uniform $\hat{T}-$graph, its end space is homeomorphic to $\Omega(G) \simeq \mathscr{R}(T)$. However, G' is countable, because so is $\hat{T}$. Therefore, $\Omega(G')$ is metrizable by Theorem 5.3.1, since, as in Proposition 2.2.2, G' has a normal spanning tree. $\square$

Finally, the conclusion established by Corollary 6.4.3 also brings another problem to our attention: once the edge-end spaces do not comprise all the end spaces, which additional topological properties describes this former family? In other words, can the answer of Pitz (2023) to Diestel's question stated in Chapter 5 be extended to also characterize edge-end spaces? The representation criteria in Theorem 6.3.13 might light this investigation, since it was useful for detecting a topological behaviour of edge-end spaces in this section. To summarize, we address the following problem for future works:

Problem 6.4.6. *Which topological properties precisely describe the family $\{\Omega_E(G) : G$ graph$\}$? In particular, inspired by Theorem 6.4.1, how the topology of $\Omega(G)$ is affected when the vertices of an arbitrary graph G edge-dominates at most one end?*

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