Exact multiplicity of solutions of differential equations via a computer-assisted method

## Mário Cesar Monteiro do Prado

Tese de Doutorado do Programa de Pós-Graduação em Matemática (PPG-Mat)

Data de Depósito:
Assinatura: $\qquad$

## Mário Cesar Monteiro do Prado

## Exact multiplicity of solutions of differential equations via a computer-assisted method

Thesis submitted to the Institute of Mathematics and Computer Sciences - ICMC-USP - in accordance with the requirements of the Mathematics Graduate Program, for the degree of Doctor in Science. FINAL VERSION<br>Concentration Area: Mathematics<br>Advisor: Prof. Dr. Marcio Fuzeto Gameiro

## USP - São Carlos

May 2019

Ficha catalográfica elaborada pela Biblioteca Prof. Achille Bassi e Seção Técnica de Informática, ICMC/USP, com os dados inseridos pelo(a) autor(a)

## M775e

Monteiro do Prado, Mário Cesar
Exact multiplicity of solutions of differential equations via a computer-assisted method / Mário Cesar Monteiro do Prado; orientador Marcio Fuzeto Gameiro. -- São Carlos, 2019.

82 p.

Tese (Doutorado - Programa de Pós-Graduação em Matemática) -- Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, 2019.

1. Differential equations. 2. Multiplicity of solutions. 3. Non-existence. 4. Computer-assisted methods. I. Gameiro, Marcio Fuzeto , orient. II. Título.

Bibliotecários responsáveis pela estrutura de catalogação da publicação de acordo com a AACR2:

## Mário Cesar Monteiro do Prado

# Multiplicidade exata de soluções de equações diferenciais via um método assistido por computador 

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação - ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências - Matemática. VERSÃO REVISADA<br>Área de Concentração: Matemática<br>Orientador: Prof. Dr. Marcio Fuzeto Gameiro

## ACKNOWLEDGEMENTS

I would like to thank Marcio for all his patience and support, to Paula for making my days better, to my mother for all her dedication to me and my sister. This work would never be accomplished without these people in my life.

I would like to thank to CAPES for the financial support without which this work would not even have started. ${ }^{1}$

Special thanks goes to my cat.

[^0]
## RESUMO

PRADO, M. C. Multiplicidade exata de soluções de equações diferenciais via um método assistido por computador. 2019. 82 p. Tese (Doutorado em Ciências - Matemática) - Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos - SP, 2019.

Neste trabalho, desenvolvemos um método assistido por computador para determinar todas as soluções de um determinado problema de valor de fronteira. Nossa abordagem combina estimativas de energia, validação rigorosa de soluções numéricas e um teste computacional rigoroso que verifica a inexistência de soluções nas regiões de interesse. O método obteve sucesso na obtenção da multiplicidade exata de soluções de equilíbrio das equações de SwiftHohenberg nas dimensões um e dois e na equação unidimensional de Cahn-Hilliard.

Palavras-chave: Equações diferenciais, Multiplicidade de soluções, Não existência, Métodos assistidos por computador.

## ABSTRACT

PRADO, M. C. Exact multiplicity of solutions of differential equations via a computerassisted method. 2019. 82 p. Tese (Doutorado em Ciências - Matemática) - Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos - SP, 2019.

In this work, we develope a computer-aided method to determine all solutions of a given boundary value problem. Our approach combines energy estimates, rigorous validation of numerical solutions and a rigorous computational test for verifying the non-existence of solutions in some regions of the solution space. The method was successful in obtaining the exact multiplicity of equilibria of a Swift-Hohenberg equation in dimensions one and two and of a one-dimensional Cahn-Hilliard equation, for some parameter values of these equations.

Keywords: Differential equations, Multiplicity of solutions, Non-existence, Computer-assisted methods.

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CHAPTER

## INTRODUCTION

In general, there are two usual (often mutually exclusive) points of view under which differential equations are investigated: numerically and analytically. On the one hand we have researchers who are dedicated to the elaboration, analysis and application of numerical methods to problems or classes of specific problems without confirming the results via formal proofs. Such an approach has in the computer one of the main tools of scientific production. On the other hand, we have those who work on a purely abstract level and whose work resources are often limited to pencil and paper.

However, this paradigm has become less prevalent in recent decades due, in part, to the increasing number of works produced from a third point of view. The latter proposes to use the computer in some step of a rigorous mathematical proof, thus producing the so-called textit computer-aided demonstrations, which, once made, can validate a theorem by simply running a computer program. Some of the results produced under this approach and which motivate and support the present research project can be found in (YAMAMOTO, 1998), (DAY et al., 2005), (DAY; LESSARD; MISCHAIKOW, 2007), (GAMEIRO; LESSARD; MISCHAIKOW, 2008) and (GAMEIRO; LESSARD, 2010) and in the papers to which they refer.

In differential equations, many problems can be reduced to the determination of zeros of functions or of fixed points of operators on a Banach space. However, numerical methods that deal with problems in infinite-dimensional spaces can provide spurious solutions, since the computational iterations inherent to these methods are executed over a finite-dimensional subspace and not on the entire space of possible solutions.

There may be, for example, the need to determine if a close to zero numerical solution corresponds to a non-trivial solution or if it is just a numerical approximation of the null solution. This raises the question of legitimacy of the obtained numerical solution. In this sense, a range of works (including those above) present methods for solving this problem, that is, with the objective of rigorously validating numerical solutions. We can find in (DAY; LESSARD;

MISCHAIKOW, 2007), for example, a method of validation of equilibrium solutions of PDE's that, besides guaranteeing with mathematical rigor the existence of an exact solution close to a given numerical solution, provides the error involved in this numerical approximation.

It is important to emphasize that the validation method proposed by (DAY; LESSARD; MISCHAIKOW, 2007) can be done with a feasible computational cost, less than twice that the one necessary to repeat the numerical algorithm in refined levels of parameters in order to verify the persistence of the obtained solution, what does not constitute a mathematical proof (despite providing strong evidences about behavior of the studied system).

The method used in (DAY; LESSARD; MISCHAIKOW, 2007) was obtained from small adaptations of the proposed ideas in (YAMAMOTO, 1998). In (GAMEIRO; LESSARD; MISCHAIKOW, 2008) the estimates for the truncation error are improved and it is included the FFT (Fast Fourier Transform) algorithm, decreasing the computational cost of the method. Finally, in (GAMEIRO; LESSARD, 2010) the generalization of these estimates occurs as well as the formulation of the method in dimension $d \geq 2$ in a Banach space different from that used in (DAY; LESSARD; MISCHAIKOW, 2007) and (GAMEIRO; LESSARD; MISCHAIKOW, 2008). In all these works the central idea is to obtain regions around numerical solutions (in a specific Banach space) in which a given fixed point operator, associated to the original problem, verifies the conditions of the Banach contraction mapping Theorem. More precisely, to each of these regions is associated a set of polynomials, whose coefficients explicitly depend on the considered numerical solution. The hypothesis of the Banach fixed point Theorem will be satisfied if all these polynomials assume negative values in some real number $r \geq 0$. This can be rigorously verified by using interval arithmetic in the calculation of the coefficients and in the evaluation of the polynomials at the number $r$. This ensures the existence of a unique exact solution close to the numerical solution.

The above works deal only with local uniqueness. It is presented by (DAY et al., 2005) a combination of a non-existence test based on a version of the mean value Theorem (Prop 4.3) and the method for local uniqueness presented by (DAY; LESSARD; MISCHAIKOW, 2007) in order to determine all the equilibria of a Swift-Hoenberg equation with unitary spatial dimension in a given region of interest.

In this work, we adopted a strategy inspired by (DAY et al., 2005) (but now with applications in dimensions 1 and 2 and with some reformulations) in order to determine all the solutions of a given boundary value problem. Not only in a predertermined region, but in all the solution space. For this, we start obtaining, from energy estimates, a bounded region in a given Banach space (the same as suggested in (GAMEIRO; LESSARD, 2010)) in which all solutions of the problem must be contained. Then we analyze this region, analytically obtained, by means of the following sequence of computer-aided steps:

1. We search, within an initial region (in the first iteration, this is the region obtained by energy
estimates), by numerical solutions via Galerkin's projection followed by the application of Newton's method.
2. In order to validate the results obtained in the previous step, we apply an adapted version of the method proposed in (GAMEIRO; LESSARD, 2013), obtaining regions around the numerical solutions in which exactly one true solution exists.
3. We eliminate from the initial region the union of the regions where we proved existence and uniqueness of solutions in the previous step. There may or may not be other solutions in the remaining region, since it is not expected that all solutions to the problem will be produced in a first application of the numerical method, despite the fact that this is the situation in all the results presented in this work.
4. In the remaining region of the elimination process previously described we apply a rigorous non-existence test using an algorithm of the type "Divide and conquer" in order to exclude, with mathematical rigor, as many areas as possible (according to some stopping criteria, which in practice is the diameter of the tested region) where no solutions exist.
5. If the non-existence test in the previous step is conclusive throughout the remaining region from the deletion process of step 3, the process ends and the only solutions are those validated in step 2 . Otherwise, we apply all the previous steps in the regions where the test was inconclusive until some stopping criteria is reached, in which case the method fails, or step 5 is conclusive, and in this case the method is successful.

## PRELIMINARIES

### 2.1 A few considerations about series

Throughout this section let $(E,|\cdot|)$ be a normed vector space.
Definition 2.1.1. Let $\left\{a_{j}\right\}_{j \in \mathbb{N}} \subset E$ be a sequence in $E$. We say that the series $\sum_{j=1}^{\infty} a_{j}$ converges to $s \in E$, or we just write $s=\sum_{j=1}^{\infty} a_{j}$, if the partial sums $\sum_{j=1}^{N} a_{j}$ converges to $s$ with respect to the norm of $E$ when $N \rightarrow \infty$.

Definition 2.1.2. Let $\Lambda$ be any enumerable set of indexes, $A=\left\{a_{n}, n \in \Lambda\right\} \subset E$ an enumerable subset of $E$, and $\Sigma(\Lambda)=\{\sigma: \mathbb{N} \longrightarrow \Lambda ; \sigma$ is bijective $\}$ the set of all the bijective maps from $\mathbb{N}$ to $\Lambda$. We call an element of $\Sigma(\Lambda)$ an ordering of $\Lambda$. When there exists $s \in E$ such that $s=\sum_{j=1}^{\infty} a_{\sigma(j)}$ independently of the choice of $\sigma \in \Sigma(\Lambda)$ we must write $s=\sum_{n \in \Lambda} a_{n}$ and we say that the series $\sum_{n \in \Lambda} a_{n}$ is absolutely convergent.

Proposition 2.1.1. Let $\Lambda$ be any enumerable set of indexes and $A=\left\{a_{n}, n \in \Lambda\right\} \subset E$ an enumerable subset of $E$. If $\sum_{j=1}^{\infty}\left|a_{\sigma(j)}\right|<\infty$ for some fixed ordering $\sigma \in \Sigma(\Lambda)$, then

$$
\sum_{j=1}^{\infty} a_{\sigma(j)}=\sum_{j=1}^{\infty} a_{\eta(j)}, \forall \eta \in \Sigma(\Lambda),
$$

that is, the series $\sum_{n \in \Lambda} a_{n}$ is absolutely convergent.
Proof. Suppose that $\sum_{j=1}^{\infty}\left|a_{\sigma(j)}\right|<\infty$ for some ordering $\sigma \in \Sigma(\Lambda)$ and let $\eta$ be any other ordering of $\Lambda$. Let us prove that

$$
\sum_{j=1}^{\infty} a_{\sigma(j)}=\sum_{j=1}^{\infty} a_{\eta(j)} .
$$

Given $\varepsilon>0$, choose $N_{0} \in \mathbb{N}$ such that

$$
\sum_{j=N_{0}}^{\infty}\left|a_{\sigma(j)}\right|<\varepsilon .
$$

Choose $M_{0} \in \mathbb{N}$ such that $\eta\left(\left\{1, \ldots, M_{0}\right\}\right) \supset \sigma\left(\left\{1, \ldots, N_{0}-1\right\}\right)$. For each $M \geq M_{0}$ define $S(M)=$ $\left\{j \in\{1, \ldots, M\} ; \eta(j) \notin \sigma\left(\left\{1, \ldots, N_{0}-1\right\}\right)\right\}$. Then, we can write

$$
\begin{align*}
\mid \sum_{j=1}^{\infty} a_{\sigma(j)}- & \sum_{j=1}^{M} a_{\eta(j)}\left|=\left|\sum_{j=N_{0}}^{\infty} a_{\sigma(j)}-\sum_{j \in S(M)} a_{\eta(j)}\right| \leq\right. \\
& \leq \sum_{j=N_{0}}^{\infty}\left|a_{\sigma(j)}\right|+\sum_{j \in S(M)}\left|a_{\eta(j)}\right| . \tag{2.1}
\end{align*}
$$

But observe that

$$
\left\{a_{\eta(j)}, j \in S(M)\right\} \subset\left\{a_{\sigma(j)}, j \geq N_{0}\right\}
$$

so that

$$
\sum_{j \in S(M)}\left|a_{\eta(j)}\right| \leq \sum_{j=N_{0}}^{\infty}\left|a_{\sigma(j)}\right| .
$$

Then we conclude that

$$
\left|\sum_{j=1}^{\infty} a_{\sigma(j)}-\sum_{j=1}^{M} a_{\eta(j)}\right| \leq 2 \sum_{j=N_{0}}^{\infty}\left|a_{\sigma(j)}\right|<2 \varepsilon, M \geq M_{0}
$$

that is,

$$
\sum_{j=1}^{\infty} a_{\sigma(j)}=\sum_{j=1}^{\infty} a_{\eta(j)}
$$

We finish this section with the following useful characterization of absolutely convergent series of real numbers.

Theorem 2.1.1. (TAO, 2006) Let $\Lambda$ be an at most countable set and let $f: \Lambda \mapsto \mathbb{R}$ be a function. Then the series $\sum_{n \in \Lambda} f(n)$ is absolutely convergent if and only if

$$
\sup \left\{\sum_{n \in A}|f(n)| ; A \subset \Lambda, A \text { finite }\right\}<\infty
$$

This Theorem will be useful in the proof of Corollary 2.3.2.

### 2.2 Coefficient space and discrete convolution operator

The class of differential equations addressed in this work admits a reformulation in a Banach space of sequences which is described in this section. We also introduce an operator that arises naturally from this reformulation.

We start with some definitions that must be used throughout this work.

Definition 2.2.1. Let $d \in \mathbb{N}$. For $M=\left(M_{1}, \ldots, M_{d}\right) \in \mathbb{N}^{d}$ define

$$
F_{M}=\left\{\left(k_{1}, \ldots, k_{d}\right) \in \mathbb{Z}^{d},\left|k_{i}\right|<M_{i}, i=1, \ldots, d\right\} .
$$

Definition 2.2.2. For $d \in \mathbb{N}, k=\left(k_{1}, \ldots, k_{d}\right) \in \mathbb{Z}^{d}$, and $j=\left(j_{1}, \ldots, j_{d}\right) \in \mathbb{Z}^{d}$ we define $k \pm j=$ $\left(k_{1} \pm j_{1}, \ldots, k_{d} \pm j_{d}\right)$. That is, the sum (subtraction) of $d$-dimensional indexes is defined by the term by term sum (subtraction) of the components of these indexes.

Definition 2.2.3. If $M=\left(M_{i, j}\right)$ is a real or complex matrix, we denote by $|M|$ the real matrix $\left(\left|M_{i, j}\right|\right)$.

Definition 2.2.4. For $q=\left(q_{1}, \ldots, q_{d}\right) \in \mathbb{R}^{d}$ and $s=\left(s_{1}, \ldots, s_{d}\right) \in \mathbb{R}^{d}$ we define $q^{s}:=q_{1}^{s_{1}} \cdots q_{d}^{s_{d}}$.
Definition 2.2.5. For $k \in \mathbb{Z}$ and $s \in \mathbb{R}$, define:

$$
\omega_{k}^{s}= \begin{cases}1 & \text { if } k=0 \\ |k|^{s} & \text { if } k \neq 0\end{cases}
$$

Definition 2.2.6. Let $d \in \mathbb{N}, k=\left(k_{1}, \ldots, k_{d}\right) \in \mathbb{Z}^{d}$ and $s=\left(s_{1}, \ldots, s_{d}\right) \in \mathbb{R}^{d}$, and define

$$
\omega_{k}^{s}=\omega_{k_{1}}^{s_{1}} \cdots \omega_{k_{d}}^{s_{d}}
$$

Definition 2.2.7. Denote by $\mathbb{R}^{\mathbb{Z}^{d}}$ the vector space of sequences of real numbers indexed by $\mathbb{Z}^{d}$ provided with the usual term-by-term summation and term-by-term scalar multiplication. That is, for $\left\{a_{k}\right\}_{k \in \mathbb{Z}^{d}} \in \mathbb{R}^{\mathbb{Z}^{d}},\left\{b_{k}\right\}_{k \in \mathbb{Z}^{d}} \in \mathbb{R}^{\mathbb{Z}^{d}}$ and $\lambda \in \mathbb{R}$ we have:

$$
\begin{gathered}
\left\{a_{k}\right\}_{k \in \mathbb{Z}^{d}}+\left\{b_{k}\right\}_{k \in \mathbb{Z}^{d}}=\left\{a_{k}+b_{k}\right\}_{k \in \mathbb{Z}^{d}} \\
\lambda\left\{a_{k}\right\}_{k \in \mathbb{Z}^{d}}=\left\{\lambda a_{k}\right\}_{k \in \mathbb{Z}^{d}} .
\end{gathered}
$$

Definition 2.2.8. Given $s=\left(s_{1}, \ldots, s_{d}\right)$, with $s_{i} \geq 2$ for $i \in\{1, \ldots, d\}$, define:

$$
X^{s}=\left\{a=\left\{a_{k}\right\}_{k \in \mathbb{Z}^{d}} \subset \mathbb{R}^{\mathbb{Z}^{d}} ; \sup _{k \in \mathbb{Z}^{d}}\left|a_{k} \omega_{k}^{s}\right|<\infty\right\}
$$

Endowing $X^{s}$ with the vector space structure given by $a+b=\left\{a_{k}+b_{k}\right\}_{k \in \mathbb{Z}^{d}}$ and $\lambda a=$ $\left\{\lambda a_{k}\right\}_{k \in \mathbb{Z}^{d}}, a, b \in X^{s}, \lambda \in \mathbb{R}$, we have the following.

Proposition 2.2.1. (GAMEIRO; LESSARD, 2010) The map

$$
X^{s} \ni a \mapsto\|a\|_{s}:=\sup _{k \in \mathbb{Z}^{d}}\left\{\left|a_{k}\right| \omega_{k}^{s}\right\} \in \mathbb{R}_{+}
$$

is a norm in the vector space $X^{s}$ and $\left(X^{s},\|\cdot\|_{s}\right)$ is a Banach space.
In what follows, unless any observation is made, let $d \in \mathbb{N}, s=\left(s_{1}, \ldots, s_{d}\right) \in \mathbb{R}^{d}$, with $s_{i} \geq 2, i=1, \ldots, d$.

Proposition 2.2.2. If $p \in \mathbb{N}, a^{(1)}, \ldots, a^{(p)} \in X^{s}$, and $k \in \mathbb{Z}^{d}$ then the series

$$
\sum_{\substack{k^{(1)} \\ k^{(1)}, \ldots, k^{(p)} \in \mathbb{Z}^{d}}} a_{k^{(1)}}^{(1)} \cdots a_{k^{(p)}}^{(p)}
$$

is absolutely convergent.

Proof. Set $\Lambda_{k}=\left\{\left(k^{(1)}, \ldots, k^{(p)}\right) ; k^{(1)}, \ldots, k^{(p)} \in \mathbb{Z}^{d}, k^{(1)}+\cdots+k^{(p)}=k\right\}$. Choose $\sigma=\left(\sigma^{(1)}, \ldots, \sigma^{(p)}\right) \in$ $\Sigma\left(\Lambda_{k}\right)$, with $\sigma^{(i)}=\left(\sigma_{1}^{(i)}, \ldots, \sigma_{d}^{(i)}\right), i=1, \ldots, p$. Then

$$
\begin{gathered}
\sum_{j=1}^{\infty}\left|a_{\sigma^{(1)}(j)}^{(1)} \cdots a_{\sigma^{(p)}(j)}^{(p)}\right| \leq\left\|a^{(1)}\right\|_{s} \cdots\left\|a^{(p)}\right\|_{s} \sum_{j=1}^{\infty} \frac{1}{\omega_{\sigma^{(1)}(j)}^{s}} \cdots \frac{1}{\omega_{\sigma^{(p)}(j)}^{s}} \leq \\
\leq\left\|a^{(1)}\right\|_{s} \cdots\left\|a^{(p)}\right\|_{s} \sum_{j=1}^{\infty} \frac{1}{\omega_{\sigma_{1}^{(1)}(j)}^{s_{1}}} \leq\left\|a^{(1)}\right\|_{s} \cdots\left\|a^{(p)}\right\|_{s} \sum_{j=1}^{\infty} \frac{1}{\omega_{\sigma_{1}^{(1)}(j)}^{2}}= \\
=\left\|a^{(1)}\right\|_{s} \cdots\left\|a^{(p)}\right\|_{s} \sum_{n=1}^{\infty} \frac{1}{n^{2}}=\frac{\pi^{2}}{6}\left\|a^{(1)}\right\|_{s} \cdots\left\|a^{(p)}\right\|_{s} .
\end{gathered}
$$

The result now follows from Theorem 2.1.1.

From proposition 2.2.2 we are allowed to make the following definition.
Definition 2.2.9. For $p \in \mathbb{N}$ and $a^{(1)}, \ldots, a^{(p)} \in X^{s}$ define

$$
a^{(1)} \cdots a^{(p)}=\left\{\left(a^{(1)} \cdots a^{(p)}\right)_{k}\right\}_{k \in \mathbb{Z}^{d}}
$$

with

$$
\left(a^{(1)} \cdots a^{(p)}\right)_{k}=\sum_{\substack{k^{(1)}+\cdots+k^{(p)}=k \\ k^{(1)}, \ldots, k^{(p)} \in \mathbb{Z}^{d}}} a_{k^{(1)}}^{(1)} \cdots a_{k^{(p)}}^{(p)} .
$$

If $a^{(1)}=\cdots=a^{(p)}=: a \in X^{s}$ we write $a^{p}=\left\{\left(a^{p}\right)_{k}\right\}_{k \in \mathbb{Z}^{d}}$ with $\left(a^{p}\right)_{k}=(a \cdots a)_{k} \in \mathbb{R}, k \in \mathbb{Z}^{d}$.
Theorem 2.2.1. (GAMEIRO; LESSARD, 2010) If $p \in \mathbb{N}$ and $a^{(1)}, \ldots, a^{(p)} \in X^{s}$ then $a^{(1)} \ldots a^{(p)} \in$ $X^{s}$.

Definition 2.2.10. The map

$$
X^{s} \times \cdots \times X^{s} \ni\left(a^{(1)}, \ldots, a^{(p)}\right) \mapsto a^{(1)} \cdots a^{(p)} \in X^{s}
$$

is called the discrete convolution operator of order $p$.

The discrete convolution will be necessary in the algebraic reformulation of the differential equations to be solved in this work.

### 2.3 Hilbert basis

Throughout this section let $H$ be a Hilbert space with inner product $\langle\cdot, \cdot\rangle$.
Definition 2.3.1. (BREZIS, 2010) Let $\Lambda$ be a contable set of indexes. We say that a contable set $\Phi=\left\{\phi_{n}, n \in \Lambda\right\} \subset H$ is an orthonormal Hilbert basis for $H$ when:

1. $\left\langle\phi_{n}, \phi_{m}\right\rangle=\delta_{m n}$, where $\delta_{m n}=\left\{\begin{array}{l}0 \text { if } m \neq n \\ 1 \text { if } m=n\end{array}\right.$, and
2. The linear space spanned by $\left\{\phi_{n}, n \in \Lambda\right\}$, that is, the subspace of $H$ formed by finite linear combinations of $\phi_{n}^{\prime} s$, is dense in $H$.

Theorem 2.3.1. (BREZIS, 2010) Let $\Phi=\left\{\phi_{n}, n \in \Lambda\right\}$ be an orthonormal Hilbert basis for H . Then for every $u \in H$, we have

$$
u=\sum_{n \in \Lambda}\left\langle u, \phi_{n}\right\rangle \phi_{n}
$$

and

$$
\|u\|_{H}^{2}=\sum_{n \in \Lambda}\left|\left\langle u, \phi_{n}\right\rangle\right|^{2} .
$$

Reciprocally, if $\sum_{n \in \Lambda}\left|a_{n}\right|^{2}<\infty$ then there exists $u \in H$ such that $\left\langle u, \phi_{n}\right\rangle=a_{n}, \forall n \in \Lambda$.
Definition 2.3.2. Let $\Lambda$ be a contable set and $\left\{\phi_{k}, k \in \Lambda\right\} \subset H$ an orthonormal Hilbert basis for $H$. For $u \in H$ define:

$$
\hat{u}(k)=\left\langle u, \phi_{k}\right\rangle, k \in \Lambda .
$$

The numbers $\hat{u}(k), k \in \Lambda$, are called the Fourier coefficients of $u$ with respect to the basis $\left\{\phi_{k}\right\}_{k \in \Lambda}$. When there is no ambiguity as to the basis that we are taking into account we just write $\hat{u}(k)$ to designate the Fourier coefficients of $u \in H$ with respect to this basis.

Corollary 2.3.1. Under the notations of definition 2.3.2, for $u, v \in H$, we have

$$
u=v \Leftrightarrow \hat{u}(k)=\hat{v}(k), \forall k \in \Lambda .
$$

Proof. For Theorem 2.3.1 we can write:

$$
u=v \Leftrightarrow\|u-v\|_{H}^{2}=0 \Leftrightarrow \sum_{k \in \Lambda}|\hat{u}(k)-\hat{v}(k)|^{2}=0 \Leftrightarrow \hat{u}(k)=\hat{v}(k) \forall k \in \Lambda .
$$

Theorem 2.3.2. Let $d \in \mathbb{N}$ and $H$ be a Hilbert space of functions with orthonormal Hilbert basis $\Psi=\left\{\psi_{k}\right\}_{k \in \mathbb{Z}^{d}}$. For $k \in \mathbb{Z}^{d}$ set

$$
\Lambda_{k}:=\left\{(m, n) \in \mathbb{Z}^{2 d}, m \in \mathbb{Z}^{d}, n \in \mathbb{Z}^{d}, m+n=k\right\} .
$$

Let $u, v \in H$ be such that $u v \in H$. If the basis functions satisfy the additive property

$$
\psi_{k} \psi_{j}=\psi_{k+j}, \forall k, j \in \mathbb{Z}^{d}
$$

then

$$
\begin{equation*}
\widehat{u v}(k)=\sum_{(m, n) \in \Lambda_{k}} \hat{u}(m) \hat{v}(n), k \in \mathbb{Z}^{d}, u, v \in D . \tag{2.2}
\end{equation*}
$$

Proof. Choose $k \in \mathbb{Z}^{d}$. Let us prove that the summation

$$
\sum_{(m, n) \in \Lambda_{k}} \hat{u}(m) \hat{v}(n)
$$

is well defined, that is, it does not depends on the ordering of $\Lambda_{k}$.
Indeed, given $\sigma \in \Sigma\left(\Lambda_{K}\right)$, with $\sigma(j)=\left(\sigma_{1}(j), \sigma_{2}(j)\right), j \in \mathbb{N}$, by Cauchy-Schwarz in $l^{2}$ and the Theorem 2.3.1 we can write:

$$
\begin{gather*}
\sum_{j=1}^{\infty}\left|\hat{u}\left[\sigma_{1}(j)\right] \hat{v}\left[\sigma_{2}(j)\right]\right| \leq\left(\sum_{j=1}^{\infty}\left|\hat{u}\left[\sigma_{1}(j)\right]\right|^{2}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty}\left|\hat{v}\left[\sigma_{2}(j)\right]\right|^{2}\right)^{\frac{1}{2}}= \\
=\|u\|_{H}\|v\|_{H} \tag{2.3}
\end{gather*}
$$

Then, the conclusion follows from (2.3) and Proposition 2.1.1.
Next we prove the identity (2.2). We divide the proof in four steps.
Step 1 For $u \in H$ define $u_{M}=\sum_{k \in F_{M}} \hat{u}(k) \psi_{k}, M \in \mathbb{N}$. Observe that $u_{M} \rightarrow u$ in $H$ when $M \rightarrow \infty$. Indeed, set an ordering $\sigma \in \Sigma\left(\mathbb{Z}^{d}\right)$, and let $\varepsilon>0$. For such $\varepsilon$ and $\sigma$, take $N_{0} \in \mathbb{N}$ such that $\sum_{j=N_{0}}^{\infty}|\hat{u}(\sigma(j))|^{2}<\varepsilon$. We can ensure the existence of such $N_{0}$ by Theorem 2.3.1. Now, take $M_{0} \in \mathbb{N}$ such that $F_{M_{0}} \supset \sigma\left(\left\{1, \ldots, N_{0}-1\right\}\right)$. So if $M \geq M_{0}$ we have:

$$
\begin{gathered}
\left|u-u_{M}\right|^{2}=\sum_{n \in \mathbb{Z}^{d} \backslash F_{M}}|\hat{u}(n)|^{2}=\sum_{j \in \mathbb{N} \backslash \sigma^{-1}\left(F_{M}\right)}|\hat{u}(\sigma(j))|^{2} \leq \\
\leq 2 \sum_{j=N_{0}}^{\infty}|\hat{u}(\sigma(j))|^{2} \leq 2 \varepsilon,
\end{gathered}
$$

that is, $\lim _{M \rightarrow \infty} u_{M}=u$ in $H$. Besides that, if $u, v \in H$, since multiplication $\Pi: D \times D \ni(u, v) \mapsto$ $u v \in H$ is a continuous bilinear form, we have that

$$
\begin{equation*}
u_{M} v_{M} \rightarrow u v \text { in } H \text { when } M \rightarrow \infty . \tag{2.4}
\end{equation*}
$$

Step 2 For a fixed $k \in \mathbb{Z}^{d}$, by Cauchy-Schwarz in $H$ we can write

$$
|\widehat{f g}(k)|=\left|\left\langle f g, \psi_{k}\right\rangle\right| \leq\|f g\|_{H}
$$

and since $H \times H \ni(f, g) \mapsto\|f g\|_{H}$ is continuous (because $\Pi$ and $\|\cdot\|_{H}$ are continuous), we conclude that the map

$$
\begin{equation*}
H \times H \ni(f, g) \mapsto \widehat{f g}(k) \tag{2.5}
\end{equation*}
$$

is continuous. From (2.5) and (2.4) we get

$$
\begin{equation*}
\widehat{u v}(k)=\lim _{M \rightarrow \infty} \widehat{u_{M} v_{M}}(k) . \tag{2.6}
\end{equation*}
$$

Step 3 By the additive property of the basis functions we can write

$$
\begin{equation*}
u_{M} v_{M}=\sum_{k \in F_{2 M}}\left(\sum_{\substack{m+n=k \\ m, n \in F_{M}}} \hat{u}(m) \hat{v}(n)\right) \Psi_{k}, M \in \mathbb{N} . \tag{2.7}
\end{equation*}
$$

For a fixed $k \in \mathbb{Z}^{d}$ choose $M_{0} \in \mathbb{N}$ such that $k \in F_{2 M_{0}}$. Then, for the uniqueness of the Fourier coefficients (see Corollary 2.3.1), we have

$$
\begin{equation*}
\widehat{u_{M} v_{M}}(k)=\sum_{\substack{m+n=k \\ m, n \in F_{M}}} \hat{u}(m) \hat{v}(n), M \geq M_{0} . \tag{2.8}
\end{equation*}
$$

From (2.6) and (2.8), we have:

$$
\begin{equation*}
\widehat{u v}(k)=\lim _{M \rightarrow \infty} \sum_{\substack{m+n=k \\ m, n \in F_{M}}} \hat{u}(m) \hat{v}(n) \tag{2.9}
\end{equation*}
$$

Step 4 For a fixed $k \in \mathbb{Z}$ observe that the inequality (2.3) implies that the map $\beta_{k}$ : $H \times H \rightarrow \mathbb{C}$ given by

$$
\beta_{k}(f, g)=\sum_{\substack{m+n=k \\ m, n \in \mathbb{Z}^{d}}} \hat{f}(m) \hat{g}(n)
$$

is continuous. Then, since $\widehat{u_{M}}(m)=\widehat{v_{M}}(n)=0$ if $m, n \notin F_{M}$, and $\widehat{u_{M}}(m)=\widehat{u}(m), \widehat{v_{M}}(n)=\widehat{v}(n)$ if $m, n \in F_{M}$, we can write:

$$
\begin{aligned}
\widehat{u v}(k) & =\lim _{M \rightarrow \infty} \sum_{\substack{m+n=k \\
m, n \in F_{M}}} \hat{u}(m) \hat{v}(n)=\lim _{M \rightarrow \infty} \sum_{\substack{m+n=k \\
m, n \in \mathbb{Z}^{d}}} \widehat{u_{M}}(m) \widehat{v_{M}}(n)= \\
& =\lim _{M \rightarrow \infty} \beta_{k}\left(u_{M}, v_{M}\right)=\beta_{k}(u, v)=\sum_{\substack{m+n=k \\
m, n \in \mathbb{Z}^{d}}} \hat{u}(m) \hat{v}(n),
\end{aligned}
$$

finishing the proof of (2.2).

By induction principle, Theorem 2.3.2 can be generalized as follows.

Corollary 2.3.2. Let $p, d \in \mathbb{N}$ and $H$ be a Hilbert space of functions with orthonormal Hilbert basis $\Phi=\left\{\psi_{k}\right\}_{k \in \mathbb{Z}^{d}}$. For $k \in \mathbb{Z}^{d}$ set

$$
\Lambda_{k}=\left\{\left(m_{1}, \ldots, m_{p}\right) \in \mathbb{Z}^{p d}, m_{1}+\cdots+m_{p}=k\right\}
$$

Let $u_{1}, \ldots, u_{p} \in H$ be such that $u_{1} \cdots u_{q} \in H, q=1, \ldots, p$. If the basis functions satisfies the additive property

$$
\psi_{k} \psi_{j}=\psi_{k+j}, \forall k, j \in \mathbb{Z}^{d},
$$

then

$$
\begin{equation*}
\widehat{u_{1} \cdots u_{p}}(k)=\sum_{\substack{m_{1}+\cdots+m_{p}=k \\ m_{1}, \ldots, m_{p} p \mathbb{Z}^{d}}} \widehat{u_{1}}\left(m_{1}\right) \cdots \widehat{u_{p}}\left(m_{p}\right), k \in \mathbb{Z}^{d} \tag{2.10}
\end{equation*}
$$

Proof. Theorem 2.3.2 provides the result for $p=2$. Let $q \in \mathbb{N}, q \geq 3$ and suppose the result is true for $p=q-1$. Let $v=u_{1} \cdots u_{q-1} \in H$. Define $\Lambda_{q}=\left\{\left(m_{1}, \ldots, m_{q}\right) \in \mathbb{Z}^{q d} ; m_{1}+\cdots+m_{q}=k\right\}$. Let us prove that the series

$$
\sum_{m \in \Lambda_{q}} \widehat{u_{1}}\left(m_{1}\right) \cdots \widehat{u_{q}}\left(m_{q}\right)
$$

is absolutely convergent using Theorem 2.1.1. Indeed, if $A \subset \Lambda_{q}$ is finite, take $M \in \mathbb{N}$ such that $A \subset\left(F_{M}\right)^{d} \cap \Lambda_{q}$. Then we can write:

$$
\begin{gathered}
\sum_{\left(m_{1}, \ldots, m_{q}\right) \in A}\left|\widehat{u}_{1}\left(m_{1}\right)\right| \cdots\left|\widehat{u_{q}}\left(m_{q}\right)\right| \leq \sum_{\left(m_{1}, \ldots, m_{q}\right) \in F_{M} \cap \Lambda_{q}}\left|\widehat{u_{1}}\left(m_{1}\right)\right| \cdots\left|\widehat{u_{q}}\left(m_{q}\right)\right|= \\
=\sum_{\substack{m_{1}+\cdots+m_{q}=k \\
m_{1}, \ldots, m_{q} \in F_{M}}}\left|\widehat{u_{1}}\left(m_{1}\right)\right| \cdots\left|\widehat{u_{q}}\left(m_{q}\right)\right|= \\
=\sum_{m_{q} \in F_{M}}\left[\sum_{\substack{m_{1}+\cdots+m_{q-1}=k-m_{q} \\
m_{1}, \ldots, m_{q-1} \in F_{M}}}\left|\widehat{u_{1}}\left(m_{1}\right)\right| \cdots\left|\widehat{u_{q-1}}\left(m_{q-1}\right)\right|\left|\widehat{u_{q}}\left(m_{q}\right)\right|=\right. \\
=\sum_{m_{q} \in F_{M}}\left|\hat{v}\left(k-m_{q}\right)\right|\left|\widehat{u_{q}}\left(m_{q}\right)\right| \leq\left(\|v\|_{H}\right)^{\frac{1}{2}}\left(\left\|u_{q}\right\|_{H}\right)^{\frac{1}{2}} .
\end{gathered}
$$

Remember that $v=u_{1} \cdots u_{p-1} \in H$, with

$$
\hat{v}\left(k_{1}\right)=\sum_{j_{1}+\cdots+j_{p-1}=k_{1}} \widehat{u_{1}}\left(j_{1}\right) \cdots \widehat{u_{p-1}}\left(j_{p-1}\right), k_{1} \in \mathbb{Z}^{d}
$$

so that we can write:

$$
\begin{gathered}
\widehat{u_{1} \cdots u_{p}}(k)=\widehat{v_{p}}(k)=\sum_{k_{1}+k_{2}=k} \widehat{v}\left(k_{1}\right) \widehat{u_{p}}\left(k_{2}\right)= \\
=\sum_{k_{1}+k_{2}=k}\left[\sum_{j_{1}+\cdots+j_{p-1}=k_{1}} \widehat{u_{1}}\left(j_{1}\right) \cdots \widehat{u_{p-1}}\left(j_{p-1}\right)\right] \widehat{u_{p}}\left(k_{2}\right)=
\end{gathered}
$$

$$
=\sum_{\left(j_{1}+\cdots+j_{p-1}\right)+k_{2}=k} \widehat{u_{1}}\left(j_{1}\right) \cdots \widehat{u_{p-1}}\left(j_{p-1}\right) \widehat{u_{p}}\left(k_{2}\right)
$$

where the last equality holds because of absolute convergence of the last series, as proved in the first part of this proof. For a proof of this fact we recommend, e.g., (TAO, 2006, Chap. 8).

### 2.3.1 Fourier Basis for $L^{2}$ spaces

In order to obtain the algebraic reformulation of the class of differential equations addressed in this work we will need some results about a specific Hilbert basis for $L^{2}$ spaces of functions defined on rectangular domains.

The next two Theorems are standard and can be found, e.g., in (BACHMANN; NARICI; BECKENSTEIN, 2012) and (SAXE, 2013).

Theorem 2.3.3. (SAXE, 2013) The set $\left\{\frac{e^{i k x}}{\sqrt{2 \pi}}\right\}_{k \in \mathbb{Z}}, x \in \mathbb{R}$, constitutes an orthonormal Hilbert basis for $L^{2}([0,2 \pi])$.

Changing variables we obtain the following.
Corollary 2.3.3. Let $l>0$. Then $\left\{\frac{e^{i k L x}}{\sqrt{2 l}}\right\}_{k \in \mathbb{Z}}$ and $\left\{\frac{e^{i k L x}}{\sqrt{l}}\right\}_{k \in \mathbb{Z}}$, where $x \in \mathbb{R}$, and $L=\frac{2 \pi}{l}$, constitute orthonormal Hilbert basis for $L^{2}([-l, l])$ and $L^{2}([0, l])$, respectively.

Theorem 2.3.4. (BACHMANN; NARICI; BECKENSTEIN, 2012) Let $a, b, c, d \in \mathbb{R}, a<b$ and $c<d$. If $\left\{f_{k}(x), k=0,1,2, \ldots\right\}$ and $\left\{g_{j}(y), j=0,1,2, \ldots\right\}$ are orthonormal Hilbert basis for $L^{2}([a, b])$ and $L^{2}([c, d])$, respectively, then

$$
\left\{\Psi_{(k, j)}(x, y)=f_{k}(x) g_{j}(y), k, j=0,1,2, \ldots\right\}
$$

is an orthonormal Hilbert basis for $L^{2}([a, b] \times[c, d])$.
Corollary 2.3.4. Let $d \in \mathbb{N}, l_{j}>0, j=1, \ldots, d$, and define $\Omega=\left[0, l_{1}\right] \times \cdots \times\left[0, l_{d}\right]$. Then an orthonormal Hilbert basis for $L^{2}(\Omega)$ is given by:

$$
\Psi=\left\{\frac{e^{i k_{1} L_{1} x_{1} \cdots e^{i k_{d} L_{d} x_{d}}}}{\sqrt{l_{1} \cdots l_{d}}}\right\}_{k=\left(k_{1}, \ldots k_{d}\right) \in \mathbb{Z}^{d}}
$$

where $x=\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{R}^{d}$, and $L_{j}=\frac{2 \pi}{l_{j}}, j=1, \ldots, d$.
Corollary 2.3.5. Consider $\Omega$ and $\Psi$ as in the previous Corollary. Let $u \in L^{\infty}(\Omega) \subset L^{2}(\Omega)$. Then

$$
\widehat{\left(u^{j}\right)}(k)=\frac{1}{\left(l_{1} \cdots l_{d}\right)^{\frac{j-1}{2}}} \sum_{\substack{m_{1}+\cdots+m_{j}=k \\ m_{1}, \ldots, m_{j} \in \mathbb{Z}^{d}}} \hat{u}\left(m_{1}\right) \cdots \hat{u}\left(m_{j}\right) .
$$

Proof. Apply Corollary 2.3.2 with $H=L^{2}(\Omega)$ and observe that $u \in L^{\infty}(\Omega) \Rightarrow u^{j} \in L^{2}(\Omega) \forall j \in$ $\mathbb{N}$.

Proposition 2.3.1. Let $d \in \mathbb{N}, l_{j}>0, j=1, \ldots, d$, and define $\Omega=\left[-l_{1}, l_{1}\right] \times \cdots \times\left[-l_{d}, l_{d}\right]$. If $f \in L^{2}(\Omega)$ is an even and real valued function, that is, $f(x)=f(|x|) \in \mathbb{R}$ for a.e. $x \in \Omega$, then $\hat{f}(k)=\hat{f}(|k|) \in \mathbb{R}, k \in \mathbb{Z}^{d}$.

Proof. Observe that for $k \in \mathbb{Z}^{d}$ there exists $\sigma_{i} \in\{-1,+1\}, i=1, \ldots, d$, such that $k=\left(\sigma_{1}\left|k_{1}\right|, \ldots, \sigma_{d}\left|k_{d}\right|\right)$. Consider the linear change of variables $\sigma: \Omega \longrightarrow \Omega$ given by $\sigma(x)=\left(\sigma_{1} x_{1}, \ldots, \sigma_{d} x_{d}\right)$. Then one can write

$$
\begin{gather*}
\hat{f}(k)=\int_{\Omega} f(x) e^{-i k_{1} L_{1} x_{1}} \cdots e^{-i k_{d} L_{d} x_{d}} d x=\int_{\Omega} f(x) e^{-i \sigma_{1}\left|k_{1}\right| L_{1} x_{1}} \cdots e^{-i \sigma_{d}\left|k_{d}\right| L_{d} x_{d}}= \\
=\int_{\Omega} f(\sigma(x)) e^{-i\left|k_{1}\right| L_{1} \sigma_{1} x_{1}} \cdots e^{-i\left|k_{d}\right| L_{d} \sigma_{1} x_{d}} d x \tag{2.11}
\end{gather*}
$$

where the last equality holds because $f(\sigma(x))=f(x)=f(|x|)$. Changing variables in (2.11), we obtain:

$$
\begin{align*}
& \int_{\Omega} f(\sigma(x)) e^{-i\left|k_{1}\right| L_{1} \sigma_{1} x_{1}} \cdots e^{-i\left|k_{d}\right| L_{d} \sigma_{1} x_{d}} d x= \\
& =\int_{\Omega} f(x) e^{-i\left|k_{1}\right| L_{1} x_{1}} \cdots e^{-i\left|k_{d}\right| L_{d} x_{d}} d x=\hat{f}(|k|) \tag{2.12}
\end{align*}
$$

since $\left|\operatorname{det}\left(D \sigma^{-1}\right)\right|=1$ and $\sigma^{-1}(\Omega)=\Omega$. From (2.11) and (2.12) we obtain

$$
\begin{equation*}
\hat{f}(k)=\hat{f}(|k|), \forall k \in \mathbb{Z}^{d} \tag{2.13}
\end{equation*}
$$

To see that $\hat{f}(k) \in \mathbb{R}$ one can write

$$
\begin{align*}
& \overline{\hat{f}(k)}=\overline{\int_{\Omega} f(x) e^{-i k_{1} L_{1} x_{1} \cdots e^{-i k_{d} L_{d} x_{d}} d x}}= \\
&=\int_{\Omega} f(x) e^{i k_{1} L_{1} x_{1}} \cdots e^{i k_{d} L_{d} x_{d}} d x=\hat{f}(-k)=\hat{f}(k) \tag{2.14}
\end{align*}
$$

where the second and the last equalities hold, respectively, because $f$ is real valued and by (2.13).
From (2.14) we conclude that $\hat{f}(k) \in \mathbb{R}, \forall k \in \mathbb{Z}^{d}$, and the proof is complete.

## FORMULATION OF THE PROBLEM

In this chapter we provide the formulation of the general problem addressed by the method proposed in this work and we discuss two concrete examples that fit in this abstract setting.

Let $\Omega$ be an open subset of $\mathbb{R}^{d}, d \geq 1, f(x, \lambda)=\sum_{j=2}^{p} q_{j}(\lambda) x^{j}$, where $x \in \mathbb{R}, \lambda \in I \subset \mathbb{R}$, with $I$ some open subset of $\mathbb{R}$ and $q_{j}(\lambda) \in \mathbb{C}, j=1, \ldots, p$.

Let $D$ be a subset of a given Hilbert space $H$ of functions defined in $\Omega$ and consider a one parameter family of linear differential operators $\{L(\cdot, \lambda): D \subset H \longrightarrow H, \lambda \in I\}$.

In this work, fixed a parameter value $\lambda$, we are concerned in developing rigorous numerical methods for finding all solutions of partial differential equations of the form:

$$
\begin{equation*}
L(u, \lambda)=f(u, \lambda) \text { in } \Omega, u \in D \tag{3.1}
\end{equation*}
$$

when the following hypothesis are satisfied.
Hypothesis 3.0.1. There exists an orthonormal Hilbert basis $\left\{\Psi_{k}\right\}_{k \in \mathbb{Z}^{d}}$ of $H$ such that, for each $\lambda \in I$ and $k \in \mathbb{Z}^{d}$ there exists $\mu_{k}(\lambda) \in \mathbb{C}$ satisfying $L\left(\Psi_{k}, \lambda\right)=\mu_{k}(\lambda) \Psi_{k}$ in the classical sense. Also, if $u=\sum_{k \in \mathbb{Z}^{d}} c_{k} \Psi_{k} \in D$ then:

1. $L(u, \lambda) \in H$ and $L(u, \lambda)=\sum_{k \in \mathbb{Z}^{d}} c_{k} \mu_{k}(\lambda) \Psi_{k}$, that is,

$$
\begin{equation*}
\widehat{L(u, \lambda)}(k)=\mu_{k}(\lambda) c_{k}, k \in \mathbb{Z}^{d}, u \in D \tag{3.2}
\end{equation*}
$$

2. $u^{j} \in H$ and $u^{j}=\sum_{k \in \mathbb{Z}^{d}}\left(c^{j}\right)_{k} \Psi_{k} j=2, \ldots, p$, that is

$$
\begin{equation*}
\widehat{u^{j}}(k)=\left(c^{j}\right)_{k}, j=2, \ldots, p, k \in \mathbb{Z}^{d}, u \in D . \tag{3.3}
\end{equation*}
$$

Hypothesis 3.0.2. There are constants $s \in \mathbb{R}_{+}^{d}$ and $C \geq 0$ such that, if $u=\sum_{k \in \mathbb{Z}^{d}} c_{k} \Psi_{k} \in H$ is a solution of (3.1) for a given $\lambda \in I$, then $c=\left\{c_{k}\right\}_{k \in \mathbb{Z}^{d}} \in X^{s}$ with $\left|c_{k}\right| \leq \frac{C}{\omega_{k}^{s}}, k \in \mathbb{Z}^{d}$.

It is a direct consequence of the Hypothesis 3.0.1 and 3.0.2 that if $u=\sum_{k \in \mathbb{Z}^{d}} c_{k} \Psi_{k}$ is a solution of (3.1) then the coefficients $c_{k} \in \mathbb{C}, k \in \mathbb{Z}^{d}$, satisfy the following infinit dimensional algebraic system of equations:

$$
\left\{\begin{array}{l}
\mu_{k}(\lambda) c_{k}-\sum_{j=2}^{p} q_{j}(\lambda)\left(c^{j}\right)_{k}=0, k \in \mathbb{Z}^{d}  \tag{3.4}\\
\left|c_{k}\right| \leq \frac{C}{\omega_{k}^{s}}, k \in \mathbb{Z}^{d}
\end{array}\right.
$$

On the other hand, observe that if $c \in X^{s}$ then

$$
\sum_{k \in \mathbb{Z}^{d}}\left|c_{k}\right|^{2} \leq\|c\|_{s}\left(\sum_{j \in \mathbb{Z}} \frac{1}{|j|^{2 s}}\right)^{d} \leq\|c\|_{s}\left(\frac{\pi^{4}}{45}\right)^{d}<\infty .
$$

Therefore, from Theorem 2.3.1 $u:=\sum_{k \in \Lambda} c_{k} \Psi_{k} \in H$. Furthermore, if $c \in X^{s}$ is a solution of (3.4) and we are able to prove that $u=\sum_{k \in \Lambda} c_{k} \Psi_{k} \in D$ then Hypothesis 3.0.1 implies that $u$ is a solution of (3.1). That is, under the above assumptions we have the equivalence of problems (3.1) and (3.4).

In this work we propose a computer assisted method for rigorously determining all the solutions of the problem (3.4).

For sake of clarity, in the next section we present some sample problems in the form (3.1) for which Hypotheses 3.0.1 and 3.0.2 are satisfied. Furthermore, we present their respective algebraic formulation in the form (3.4). Finally, the equivalence between the both algebraic and differential problems is established.

### 3.1 Sample problems

### 3.1.1 Equilibria of a Swift-Hohenberg-like equation

Throughout this section let $l_{i}>0, L_{i}=\frac{2 \pi}{l_{i}}, i \in\{1,2\}$, and $\Omega=\left[0, l_{1}\right] \times\left[0, l_{2}\right]$. We denote by $\Psi=\left\{\Psi_{k}\right\}_{k \in \mathbb{Z}^{2}}$ the orthonormal Hilbert basis of $L^{2}(\Omega)$ given by

$$
\Psi_{k}(x, y)=\frac{e^{i k_{1} L_{1}} e^{i k_{2} L_{2}}}{\sqrt{l_{1} l_{2}}}, k \in \mathbb{Z}^{2} .
$$

For a fixed parameter $\lambda \in \mathbb{R}_{+}$, consider the problem of finding all the solutions of

$$
\begin{equation*}
L(u, \lambda)=u^{3}, \text { in } \Omega, u \in D, \tag{3.5}
\end{equation*}
$$

where

$$
\begin{equation*}
L(u, \lambda):=\lambda u-(1+\Delta)^{2} u, \tag{3.6}
\end{equation*}
$$

and

$$
\begin{equation*}
D=\left\{u \in C ; u\left(x+l_{1}, y+l_{2}\right)=u(x, y)=u(|x|,|y|),(x, y) \in \mathbb{R}^{2}\right\} \tag{3.7}
\end{equation*}
$$

where $C$ is the set of all functions $u: \mathbb{R}^{2} \longmapsto \mathbb{R}$ such that the partial derivatives $u_{x x}, u_{y y},(\Delta u)_{x x}$ and $(\Delta u)_{y y}$ there exist in the classical sense over $\mathbb{R}^{2}$ and the restriction of these derivatives to $\Omega$ are square-integrable, that is,

$$
u_{x x}, u_{y y},(\Delta u)_{x x},(\Delta u)_{y y} \in L^{2}(\Omega)
$$

with all derivatives understood in the classical sense.
The solutions of the problem given by (3.5), (3.6) and (3.7) correspond to those classical equilibria of the Swift-Hohenberg-like equation

$$
\begin{equation*}
u_{t}=\lambda u-(1+\Delta)^{2} u-u^{3} \tag{3.8}
\end{equation*}
$$

that are even and periodic in each space variable.

Theorem 3.1.1. Let $D$ be as in (3.7). If $u \in D$ and $a=\left\{a_{k}\right\}_{k \in \mathbb{Z}^{2}}$ is the sequence of Fourier coefficients of the restriction of $u$ to $\Omega$ with respect to the orthonormal Hilbert basis $\Psi$ of $L^{2}(\Omega)$, that is, $u=\sum_{k \in \mathbb{Z}^{d}} a_{k} \Psi_{k}$, then

1. $L(u, \lambda) \in L^{2}(\Omega), \widehat{L(u, \lambda)}(k)=\mu_{k}(\lambda) a_{k}$, where

$$
\mu_{k}(\lambda)=L\left(\Psi_{k}, \lambda\right)=\lambda-\left(1-k_{1}^{2} L_{1}^{2}-k_{1}^{2} L_{2}^{2}\right)^{2}, k \in \mathbb{Z}^{2}
$$

2. $\widehat{u^{3}}(k)=\frac{1}{l_{1} l_{2}} \sum_{\substack{k_{1}+k_{2}+k_{3}=k \\ k_{1}, k_{2}, k_{3} \in \mathbb{Z}^{2}}} a_{k_{1}} a_{k_{2}} a_{k_{3}}, k \in \mathbb{Z}^{2}$;
3. $a_{k}=a_{|k|} \in \mathbb{R}, k \in \mathbb{Z}^{2}$.

Proof. It follows from the definition of the set $C$ that $L(u, \lambda) \in L^{2}(\Omega)$. Integrating by parts the integrals that provide the Fourier coefficients of $u_{x x}$ related to the basis $\Psi$ and taking into account that $u(x, y)=u\left(x+l_{1}, y+l_{2}\right),(x, y) \in \mathbb{R}^{2}$, we obtain:

$$
\begin{gathered}
\widehat{u_{x x}}(k)=\int_{\Omega} u_{x x} \bar{\Psi}_{k}= \\
=\frac{1}{\sqrt{l_{1} l_{2}}} \int_{\left[0, l_{2}\right]} e^{-i k_{2} L_{2} y}\left(\int_{\left[0, l_{1}\right]} u_{x x} e^{-i k_{1} L_{1} x} d x\right) d y= \\
=\frac{1}{\sqrt{l_{1} l_{2}}} \int_{\left[0, l_{2}\right]} e^{-i k_{2} L_{2} y}\left[-\int_{\left[0, l_{1}\right]}-i k_{1} L_{1} u_{x} e^{-i k_{1} L_{1} x} d x+u_{x}\left(l_{1}, y\right)-u_{x}(0, y)\right] d y= \\
=\frac{i k_{1} L_{1}}{\sqrt{l_{1} l_{2}}} \int_{\left[0, l_{2}\right]} e^{-i k_{2} L_{2} y}\left(\int_{\left[0, l_{1}\right]} u_{x} e^{-i k_{1} L_{1} x} d x\right) d y=
\end{gathered}
$$

$$
\begin{gathered}
=\frac{i k_{1} L_{1}}{\sqrt{l_{1} l_{2}}} \int_{\left[0, l_{2}\right]} e^{-i k_{2} L_{2} y}\left[-\int_{\left[0, l_{1}\right]}-i k_{1} L_{1} u e^{-i k_{1} L_{1} x} d x+u\left(l_{1}, y\right)-u(0, y)\right] d y= \\
=\frac{\left(i k_{1} L_{1}\right)^{2}}{\sqrt{l_{1} l_{2}}} \int_{\left[0, l_{2}\right]} \int_{\left[0, l_{1}\right]} u e^{-i k_{1} L_{1} x} e^{-i k_{2} L_{2} y} d x d y= \\
=\left(i k_{1} L_{1}\right)^{2} a_{k}, k \in \mathbb{Z}^{2} .
\end{gathered}
$$

Similarly, we have:

$$
\widehat{u_{y y}}(k)=\left(i k_{2} L_{2}\right)^{2} a_{k}, k \in \mathbb{Z}^{2}
$$

Therefore,

$$
\begin{equation*}
\widehat{\Delta u}(k)=\left[\left(i k_{1} L_{1}\right)^{2}+\left(i k_{2} L_{2}\right)^{2}\right] a_{k}, k \in \mathbb{Z}^{2} . \tag{3.9}
\end{equation*}
$$

On the other hand, since $\Delta u(x, y)=\Delta u\left(x+l_{1}, y+l_{2}\right),(x, y) \in \mathbb{R}^{2}$, we can calculate as follows:

$$
\begin{gathered}
\widehat{(\Delta u)_{x x}}(k)=\int_{\Omega}(\Delta u)_{x x} \bar{\Psi}_{k}= \\
=\frac{1}{\sqrt{l_{1} l_{2}}} \int_{\left[0, l_{2}\right]} e^{-i k_{2} L_{2} y}\left(\int_{\left[0, l_{1}\right]}(\Delta u)_{x x} e^{-i k_{1} L_{1} x} d x\right) d y= \\
=\frac{1}{\sqrt{l_{1} l_{2}}} \int_{\left[0, l_{2}\right]} e^{-i k_{2} L_{2} y}\left[-\int_{\left[0, l_{1}\right]}-i k_{1} L_{1}(\Delta u)_{x} e^{-i k_{1} L_{1} x} d x+(\Delta u)_{x}\left(l_{1}, y\right)-(\Delta u)_{x}(0, y)\right] d y= \\
=\frac{i k_{1} L_{1}}{\sqrt{l_{1} l_{2}}} \int_{\left[0, l_{2}\right]} e^{-i k_{2} L_{2} y}\left(\int_{\left[0, l_{1}\right]}(\Delta u)_{x} e^{-i k_{1} L_{1} x} d x\right) d y= \\
=\frac{i k_{1} L_{1}}{\sqrt{l_{1} l_{2}}} \int_{\left[0, l_{2}\right]} e^{-i k_{2} L_{2} y}\left[-\int_{\left[0, l_{1}\right]}-i k_{1} L_{1}(\Delta u) e^{-i k_{1} L_{1} x} d x+(\Delta u)\left(l_{1}, y\right)-(\Delta u)(0, y)\right] d y= \\
=\frac{\left(i k_{1} L_{1}\right)^{2}}{\sqrt{l_{1} l_{2}}} \int_{\left[0, l_{2}\right]} \int_{\left[0, l_{1}\right]}(\Delta u) e^{-i k_{1} L_{1} x} e^{-i k_{2} L_{2} y} d x d y= \\
=\left(i k_{1} L_{1}\right)^{2} \widehat{\Delta u}(k)=\left(i k_{1} L_{1}\right)^{2}\left[\left(i k_{2} L_{2}\right)^{2}+\left(i k_{2} L_{2}\right)^{2}\right] a_{k}, k \in \mathbb{Z}^{2} .
\end{gathered}
$$

Similarly,

$$
\widehat{(\Delta u)_{y y}}(k)=\left(i k_{2} L_{2}\right)^{2}\left[\left(i k_{2} L_{2}\right)^{2}+\left(i k_{2} L_{2}\right)^{2}\right] a_{k}, k \in \mathbb{Z}^{2} .
$$

Therefore

$$
\begin{equation*}
\widehat{\Delta(\Delta u)}(k)=\left[\left(i k_{2} L_{2}\right)^{2}+\left(i k_{2} L_{2}\right)^{2}\right]^{2} a_{k}, k \in \mathbb{Z}^{2} \tag{3.10}
\end{equation*}
$$

Then,

$$
\begin{align*}
& \widehat{L(\lambda, u)}(k)=(\lambda-1) a_{k}-2 \widehat{\Delta u}(k)-\widehat{\Delta(\Delta u)}(k)= \\
& =\left[\lambda-\left(1-k_{1}^{2} L_{1}^{2}-k_{1}^{2} L_{2}^{2}\right)^{2}\right] a_{k}=\mu_{k} a_{k}, k \in \mathbb{Z}^{2} \tag{3.11}
\end{align*}
$$

what conclude the proof of item 1 .
Item 2 follows immediately from Corollary 2.3.5.
Now let us prove the last item. Since $u\left(x+l_{1}, y+l_{2}\right)=u(x, y)=u(|x|,|y|),(x, y) \in \mathbb{R}^{2}$, we can write:

$$
\hat{u}(k)=\int_{\left[0, l_{2}\right]} \int_{\left[0, l_{1}\right]} u(x, y) e^{-i k_{1} L_{1} x} e^{-i k_{2} L_{2} y} d x d y=
$$

$$
\begin{aligned}
& =\int_{\left[0, l_{2}\right]} \int_{\left[0, \frac{1}{2}\right]} u(x, y) e^{-i k_{1} L_{1} x} e^{-i k_{2} L_{2} y} d x d y+\int_{\left[0, l_{2}\right]} \int_{\left[\frac{l_{1}}{2}, l_{1}\right]} u(x, y) e^{-i k_{1} L_{1} x} e^{-i k_{2} L_{2} y} d x d y= \\
& =\int_{\left[0, l_{2}\right]} \int_{\left[0, \frac{\left.l_{1}\right]}{}\right]} u\left(l_{1}-x, y\right) e^{-i k_{1} L_{1} x} e^{-i k_{2} L_{2} y} d x d y+\int_{\left[0, l_{2}\right]} \int_{\left[\frac{l_{1}}{2}, l_{1}\right]} u(x, y) e^{-i k_{1} L_{1} x} e^{-i k_{2} L_{2} y} d x d y= \\
& =\int_{\left[0, l_{2}\right]} \int_{\left[\frac{l_{1}}{2}, l_{1}\right]} u(x, y) e^{-i k_{1} L_{1}\left(l_{1}-x\right)} e^{-i k_{2} L_{2} y} d x d y+\int_{\left[0, l_{2}\right]} \int_{\left[\frac{l_{1}}{2}, l_{1}\right]} u(x, y) e^{-i k_{1} L_{1} x} e^{-i k_{2} L_{2} y} d x d y= \\
& =\int_{\left[0, l_{2}\right]} \int_{\left[\frac{l_{1}}{2}, l_{1}\right]} u(x, y) e^{i k_{1} L_{1} x} e^{-i k_{2} L_{2} y} d x d y+\int_{\left[0, l_{2}\right]} \int_{\left[\frac{l_{1}}{2}, l_{1}\right]} u(x, y) e^{-i k_{1} L_{1} x} e^{-i k_{2} L_{2} y} d x d y= \\
& =\int_{\left[0, l_{2}\right]} \int_{\left[\frac{l_{1}}{2}, l_{1}\right]} u(x, y) 2 \cos \left(k_{1} L_{1} x\right) e^{-i k_{2} L_{2} y} d x d y .
\end{aligned}
$$

Applying the same routine to the integration variable $y$ we obtain

$$
\begin{aligned}
& \hat{u}(k)=\int_{\left[0, l_{2}\right]} \int_{\left[\frac{l_{1}}{2}, l_{1}\right]} u(x, y) 2 \cos \left(k_{1} L_{1} x\right) e^{-i k_{2} L_{2} y} d x d y= \\
& =\int_{\left[\frac{l_{1}}{2}, l_{1}\right]} 2 \cos \left(k_{1} L_{1} x\right) \int_{\left[0, l_{2}\right]} u(x, y) e^{-i k_{2} L_{2} y} d y d x= \\
& =\int_{\left[\frac{l_{1}}{2}, l_{1}\right]} 2 \cos \left(k_{1} L_{1} x\right) \int_{\left[\frac{[2}{2}, l_{2}\right]} u(x, y) 2 \cos \left(k_{2} L_{2} y\right) d y d x= \\
& =4 \int_{\left[\frac{l_{1}}{2}, l_{1}\right]} \int_{\left[\frac{l_{2}}{2}, l_{2}\right]} u(x, y) \cos \left(k_{1} L_{1} x\right) \cos \left(k_{2} L_{2} y\right) d y d x .
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
\hat{u}(k)=\hat{u}(|k|) \in \mathbb{R}, k \in \mathbb{Z}^{2} \tag{3.12}
\end{equation*}
$$

The following theorems aims to produce global bounds for the Fourier coefficients of solutions of problem 3.5. ${ }^{1}$

Proposition 3.1.1. (Energy estimates for two-dimensional Swift-Hohenberg equation.) If $u \in D$ is solution of (3.5) then

$$
\|u\|_{L^{2}(\Omega)} \leq \sqrt{v l_{1} l_{2}} \text { and }\|\Delta u\|_{L^{2}(\Omega)} \leq \sqrt{2 v(v+1) l_{1} l_{2}}
$$

Proof. From the fact that $u$ is a solution of (3.5) we have:

$$
\begin{equation*}
v u-[u+2 \Delta u+\Delta(\Delta u)]-u^{3}=0 \tag{3.13}
\end{equation*}
$$

Multiplying the right hand side of (3.13) by $u$ and integrating over $\Omega$ taking into account that $u \in D$, we obtain:

[^1]\[

$$
\begin{align*}
& 0=\int_{\Omega}(v-1) u^{2}-u^{4} d x+\int_{\Omega}-2 u \Delta u d x-\int_{\Omega}(\Delta u)^{2} d x \leq \\
& \leq \int_{\Omega}(v-1) u^{2}-u^{4} d x+\int_{\Omega} \frac{u^{2}}{\varepsilon}+\varepsilon(\Delta u)^{2} d x-\int_{\Omega}(\Delta u)^{2} d x \tag{3.14}
\end{align*}
$$
\]

Setting $\varepsilon=1$, we get:

$$
\begin{equation*}
0 \leq \int_{\Omega} v u^{2}-u^{4} d x \tag{3.15}
\end{equation*}
$$

On the other hand,

$$
\begin{equation*}
\int_{\Omega} u^{2} \leq\left(\int_{\Omega} u^{4} d x\right)^{\frac{1}{2}} \cdot\left(\int_{\Omega} 1 d x\right)^{\frac{1}{2}}=\left(\int_{\Omega} u^{4} d x\right)^{\frac{1}{2}} \cdot \sqrt{l_{1} l_{2}} \tag{3.16}
\end{equation*}
$$

From (3.15) and (3.16), we can write:

$$
\begin{gather*}
0 \leq \int_{\Omega} v u^{2}-u^{4} d x \leq \int_{\Omega} v u^{2} d x-\frac{1}{l_{1} l_{2}} \cdot\left(\int_{\Omega} u^{2} d x\right)^{2}= \\
=v\|u\|_{L^{2}(\Omega)}^{2}-\frac{1}{l_{1} l_{2}} \cdot\|u\|_{L^{2}(\Omega)}^{4} \Rightarrow \\
\Rightarrow v-\frac{1}{l_{1} l_{2}} \cdot\|u\|_{L^{2}(\Omega)}^{2} \geq 0 \Rightarrow \\
\Rightarrow\|u\|_{L^{2}(\Omega)} \leq \sqrt{v l_{1} l_{2}} \tag{3.17}
\end{gather*}
$$

Furthermore, setting $\varepsilon=\frac{1}{2}$ in (3.14), we have

$$
\begin{gather*}
\frac{1}{2} \int_{\Omega}(\Delta u)^{2} d x \leq \int_{\Omega}(v+1) u^{2}-u^{4} d x \leq \\
\leq(v+1) \int_{\Omega} u^{2} \Rightarrow \\
\Rightarrow\|\Delta u\|_{L^{2}(\Omega)} \leq \sqrt{2 v(v+1) l_{1} l_{2}} \tag{3.18}
\end{gather*}
$$

Corollary 3.1.1. If $u$ is a solution of (3.5), then

$$
\int_{\Omega}|\Delta(\Delta u)| d x \leq(|v-1|+2 \sqrt{2(v+1)}+v) \sqrt{v} l_{1} l_{2}
$$

Proof. Simply integrate (3.13), use the estimates $\|u\|_{L^{2}(\Omega)}$ and $\|\Delta u\|_{L^{2}(\Omega)}$, and observe that

$$
\int_{\Omega}\left|u^{3}\right| d x \leq\left(\int_{\Omega}\left|u^{4}\right| d x\right)^{\frac{1}{2}}\left(\int_{\Omega} u^{2} d x\right)^{\frac{1}{2}} \leq \sqrt{v} \int_{\Omega} u^{2} d x
$$

where the last inequality is justified by (3.15).
Theorem 3.1.2 (Decay of Fourier Coefficients). Let $L=\left(L_{1}, L_{2}\right) \in \mathbb{R}^{2}$, with $L_{1}>0, L_{2}>0$, and $N=\left(N_{1}, N_{2}\right) \in \mathbb{Z}^{2}, N_{1} \geq \frac{2 L_{2}}{L 1}, N_{2} \geq \frac{2 L_{1}}{L 2}$. If $u$ is a solution of (3.5) and $a=\left\{a_{k}\right\}_{k \in \mathbb{Z}^{2}}$ is the sequence of Fourier coefficients of the restriction of $u$ to $\Omega$ with respect to the basis $\Psi$ then $a \in X^{(2,2)}$ with:

$$
\begin{equation*}
\left|a_{k}\right| \leq c_{k}(v, L), k \in \mathbb{Z}^{2}, \tag{3.19}
\end{equation*}
$$

where

$$
c_{k}(v, L)= \begin{cases}\sqrt{v l_{1} l_{2}} & \text { if } k=(0,0)  \tag{3.20}\\ \frac{(|v-1|+2 \sqrt{2(v+1)}+v) \sqrt{v l_{1} l_{2}}}{\left(k_{1}^{2} L_{1}^{2}+k_{2}^{2} L_{2}^{2}\right)^{2}} & \text { if } k \in F_{N} \mathrm{e} k \neq(0,0) \\ \frac{(|v-1|+2 \sqrt{2(v+1)}+v) \sqrt{v l_{1} l_{2}}}{4 L_{1}^{2} L_{2}^{2} \omega_{k}^{5}} & \text { if } k \notin F_{N} .\end{cases}
$$

Proof. If $k=(0,0)$, we have:

$$
a_{(0,0)}=\frac{1}{\sqrt{l_{1} l_{2}}} \int_{\Omega} u d x \leq\|u\|_{L^{2}(\Omega)} \leq \sqrt{v l_{1} l_{2}}
$$

On the other hand, observe that:

1. $\Delta(\Delta u)=\sum_{k \in \mathbb{Z}^{2}} b_{k} \Psi_{k}$, where $b_{k}=a_{k}\left(k_{1}^{2} L_{1}^{2}+k_{2}^{2} L_{2}^{2}\right)^{2}$,
2. $b_{k}=\int_{\Omega} \Delta(\Delta u) \bar{\Psi}_{k}$, with $\bar{\Psi}_{k}(x, y) \leq \frac{1}{\sqrt{l_{1} l_{2}}}, \forall k \in \mathbb{Z}^{2}$,
3. $\int_{\Omega}|\Delta(\Delta u)| d x \leq(|v-1|+2 \sqrt{2(v+1)}+v) \sqrt{v} l_{1} l_{2}$,
4. $k_{1}^{2} L_{1}^{2}+k_{2}^{2} L_{2}^{2} \geq 2 k_{1} L_{1} k_{2} L_{2}$
5. $k_{i}^{2} L_{i}^{4} \geq 4 L_{1}^{2} L_{2}^{2}$ if $k_{i} \geq N, i=1,2$.

Define $b\left(v, l_{1}, l_{2}\right)=(|v-1|+2 \sqrt{2(v+1)}+v) \sqrt{v l_{1} l_{2}}$. For itens 1,2 and 3 we conclude that:

$$
a_{k} \leq \frac{b\left(v, l_{1}, l_{2}\right)}{k_{1}^{4} L_{1}^{4}+k_{2}^{4} L_{2}^{4}+2 k_{1}^{2} L_{1}^{2} k_{2}^{2} L_{2}^{2}}, k \neq(0,0) .
$$

In particular, if $k_{1} \neq 0$ and $k_{2} \neq 0$ then:

- for item 4:

$$
a_{\left(k_{1}, k_{2}\right)} \leq \frac{b\left(v, l_{1}, l_{2}\right)}{k_{1}^{4} L_{1}^{4}+k_{2}^{4} L_{2}^{4}+2 k_{1}^{2} L_{1}^{2} k_{2}^{2} L_{2}^{2}} \leq \frac{b\left(v, l_{1}, l_{2}\right)}{4 L_{1}^{2} L_{2}^{2} \omega_{k}^{s}}
$$

- for item 5 :

$$
\begin{aligned}
& a_{\left(k_{1}, 0\right)} \leq \frac{b\left(v, l_{1}, l_{2}\right)}{k_{1}^{4} L_{1}^{4}}=\frac{b\left(v, l_{1}, l_{2}\right)}{k_{1}^{2} L_{1}^{4} \omega_{k}^{s}} \leq \frac{b\left(v, l_{1}, l_{2}\right)}{4 L_{1}^{2} L_{2}^{2} \omega_{k}^{s}}, \text { if } k_{1} \geq N, \\
& a_{\left(0, k_{2}\right)} \leq \frac{b\left(v, l_{1}, l_{2}\right)}{k_{2}^{4} L_{2}^{4}}=\frac{b\left(v, l_{1}, l_{2}\right)}{k_{2}^{2} L_{2}^{4} \omega_{k}^{s}} \leq \frac{b\left(v, l_{1}, l_{2}\right)}{4 L_{1}^{2} L_{2}^{2} \omega_{k}^{s}}, \text { if } k_{2} \geq N .
\end{aligned}
$$

Therefore,

$$
a_{k} \leq \frac{b\left(v, l_{1}, l_{2}\right)}{4 L_{1}^{2} L_{2}^{2} \omega_{k}^{s}}, k \notin F_{N},
$$

which concludes the proof.

We summarize Theorems (3.1.1) and (3.1.2) in the following.
Theorem 3.1.3. Let $L$ and $N$ as in Theorem (3.1.2). If $u$ is a solution of (3.5) and $a=\left\{a_{k}\right\}_{k \in \mathbb{Z}^{2}}$ is the sequence of Fourier coefficients of the restriction of $u$ to $\Omega$ with respect to the basis $\Psi$ then

$$
\left\{\begin{array}{l}
l_{1} l_{2} \mu_{k}(\lambda) a_{k}-\left(a^{3}\right)_{k}=0, k \in \mathbb{Z}^{2}  \tag{3.21}\\
a_{k}=a_{|k|},\left|a_{k}\right| \leq c_{k}(v, L), k \in \mathbb{Z}^{2}
\end{array}\right.
$$

where

$$
\begin{equation*}
\mu_{k}(\lambda)=\lambda-\left(1-k_{1}^{2} L_{1}^{2}-k_{1}^{2} L_{2}^{2}\right)^{2}, k \in \mathbb{Z}^{2} \tag{3.22}
\end{equation*}
$$

and

$$
c_{k}(v, L)= \begin{cases}\sqrt{v l_{1} l_{2}} & \text { if } k=(0,0)  \tag{3.23}\\ \frac{(|v-1|+2 \sqrt{2(v+1)}+v) \sqrt{v l_{1} l_{2}}}{\left(k_{1}^{2} L_{1}^{2}+k_{2}^{2} L_{2}^{2}\right)^{2}} & \text { if } k \in F_{N} \mathrm{e} k \neq(0,0) \\ \frac{(|v-1|+2 \sqrt{2(v+1)}+v) \sqrt{v l_{1} l_{2}}}{4 L_{1}^{2} L_{2}^{2} \omega_{k}^{5}} & \text { if } k \notin F_{N} .\end{cases}
$$

In particular, $a \in X^{(2,2)}$.

Reciprocally, we have the following Theorem.
Theorem 3.1.4. Let $a=\left\{a_{k}\right\}_{k \in \mathbb{Z}^{d}}$ be a solution of the following problem:

$$
\begin{equation*}
l_{1} l_{2} \mu_{k}(\lambda) a_{k}-\left(a^{3}\right)_{k}=0, k \in \mathbb{Z}^{2}, a \in X^{(2,2)} \tag{3.24}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu_{k}(\lambda)=\lambda-\left(1-k_{1}^{2} L_{1}^{2}-k_{1}^{2} L_{2}^{2}\right)^{2}, k \in \mathbb{Z}^{2} . \tag{3.25}
\end{equation*}
$$

Then, we have

$$
u(x, y)=\sum_{k \in \mathbb{Z}^{d}} a_{k} \Psi_{k}(x, y) \in C^{\infty}\left(\mathbb{R}^{2}\right)
$$

and we can obtain their derivatives by term-by-term differentiation. That is:

$$
\begin{equation*}
\frac{\partial^{\alpha} u}{\partial x^{\alpha}}(x, y)=\sum_{k \in \mathbb{Z}^{d}} i^{|\alpha|} k^{\alpha} L^{\alpha} a_{k} \Psi_{k}(x, y), \alpha \in \mathbb{Z}_{+}^{2} \tag{3.26}
\end{equation*}
$$

with absolute and uniform convergence on $\mathbb{R}^{2}$, and

$$
\begin{equation*}
u(x, y)=u(|x|,|y|)=u\left(x+l_{1}, y+l_{2}\right),(x, y) \in \mathbb{R}^{2} . \tag{3.27}
\end{equation*}
$$

In particular, (3.26) and (3.27) imply that $u$ is a solution of (3.5).
Proof. Observe that $\delta:=\left\{\frac{1}{\mu_{k}(\lambda)}\right\}_{k \in \mathbb{Z}^{d}} \in X^{(2,2)}$. From Theorem (2.2.1) we know that $a^{3} \in X^{(2,2)}$. With this we get

$$
\mu_{k}(\lambda) a_{k}-\left(a^{3}\right)_{k}=0 \Rightarrow\left|a_{k}\right| \leq \frac{\left\|a^{3}\right\|_{s}\|\boldsymbol{\delta}\|_{s}}{\omega_{k}^{(4,4)}} .
$$

Therefore $a \in X^{(4,4)}$. Repeating this procedure we obtain $a \in X^{(2 n, 2 n)} \forall n \in \mathbb{N}$. Then, for $\alpha \in \mathbb{Z}_{+}^{d}$ we can take $n \in \mathbb{N}$ such that $\frac{|k|^{\alpha}}{\omega_{k}^{2(2,2 n)}} \leq \frac{1}{\omega_{k}^{2,2}}$. This implies the absolute convergence of the series (3.26).

On the other hand, given $(x, y) \in \mathbb{R}^{2}$ and constants $\sigma_{1}, \sigma_{2} \in\{-1,1\}$ such that $(x, y)=$ $\left(\sigma_{1}|x|, \sigma_{2}|y|\right)$, since the series of $u$ converges uniformly on $\mathbb{R}^{2}$ and

$$
a_{k}=a_{\left(\sigma_{1} k_{1}, \sigma_{2} k_{2}\right)}=a_{|k|} \forall k=\left(k_{1}, k_{2}\right) \in \mathbb{Z}^{d}
$$

we can calculate as follows:

$$
\begin{gathered}
u(x, y)=\frac{1}{\sqrt{l_{1} l_{2}}} \sum_{k \in \mathbb{Z}^{d}} a_{k} e^{k_{1} L_{1} x} e^{k_{2} L_{2} y}= \\
=\frac{1}{\sqrt{l_{1} l_{2}}} \sum_{k \in \mathbb{Z}^{d}} a_{k} e^{\left(\sigma_{1} k_{1}\right) L_{1}|x|} e^{\left(\sigma_{2} k_{2}\right) L_{2}|y|}=\frac{1}{\sqrt{l_{1} l_{2}}} \sum_{k \in \mathbb{Z}^{d}} a_{\left(\sigma_{1} k_{1}, \sigma_{2} k_{2}\right)} e^{\left(\sigma_{1} k_{1}\right) L_{1}|x|} e^{\left(\sigma_{2} k_{2}\right) L_{2}|y|}= \\
=\frac{1}{\sqrt{l_{1} l_{2}}} \sum_{k \in \mathbb{Z}^{d}} a_{k} e^{k_{1} L_{1}|x|} e^{k_{2} L_{2}|y|}=u(|x|,|y|) .
\end{gathered}
$$

Furthermore, we have:

$$
\begin{gathered}
u\left(x+l_{1}, y+l_{2}\right)=\frac{1}{\sqrt{l_{1} l_{2}}} \sum_{k \in \mathbb{Z}^{d}} a_{k} e^{k_{1} L_{1}\left(x+l_{1}\right)} e^{k_{2} L_{2}\left(y+l_{2}\right)}= \\
=\frac{1}{\sqrt{l_{1} l_{2}}} \sum_{k \in \mathbb{Z}^{d}} a_{k} e^{k_{1} L_{1} x+2 \pi k_{1}} e^{k_{2} L_{2} y+2 \pi k_{2}}=\frac{1}{\sqrt{l_{1} l_{2}}} \sum_{k \in \mathbb{Z}^{d}} a_{k} e^{k_{1} L_{1} x} e^{k_{2} L_{2} y}=u(x, y),
\end{gathered}
$$

which concludes the proof of (3.27).
To see that $u$ solves (3.5) observe that (3.26) implies that

$$
v u-[u+2 \Delta u+\Delta(\Delta u)]=\sum_{k \in \mathbb{Z}^{2}} \mu_{k}(\lambda) a_{k} \Psi_{k},
$$

and by (3.24) we conclude that

$$
\sum_{k \in \mathbb{Z}^{2}} \mu_{k}(\lambda) a_{k} \Psi_{k}=\frac{1}{l_{1} l_{2}} \sum_{k \in \mathbb{Z}^{2}}\left(a^{3}\right)_{k} \Psi_{k}=u^{3}
$$

that is,

$$
v u-[u+2 \Delta u+\Delta(\Delta u)]=u^{3},
$$

which concludes the proof that $u$ is a solution of (3.5).
We now state the main result of this section, which summarizes the last two Theorems.
Corollary 3.1.2. If $u$ is a solution of (3.5) then $u \in C^{\infty}\left(\mathbb{R}^{2}\right)$. Furthermore, $u=\sum_{k \in \mathbb{Z}^{d}} a_{k} \Psi_{k}$ is a solution of (3.5) if and only if $a$ is a solution of the problem given by (3.21), (3.22) and (3.23).

This work is devoted to present a method to find all solutions of the algebraic problem like is the one given by (3.21), (3.22) and (3.23). As a consequence of the correspondence given by Corollary (3.1.2) we will be able to find all solutions of some differential equations that assume that kind of algebraic reformulation.

### 3.1.2 Cahn-Hilliard

Throughout this section let $l>0, L=\frac{2 \pi}{l}, \Omega=(-l, l)$ and $\Omega_{+}=(0, l)$. We denote by $\Gamma=\left\{\Gamma_{k}\right\}_{k \in \mathbb{Z}}$ the orthonormal Hilbert basis of $L^{2}(\Omega)$ given by $\Gamma_{k}(x)=\frac{e^{i k L x}}{\sqrt{2 l}}, x \in \Omega, k \in \mathbb{Z}$. If $A$ is any subset of $\mathbb{R}^{n}$, denote by $\operatorname{int}(A)$ the interior of $A$ and by $\bar{A}$ the closure of $A$ in $\mathbb{R}^{n}$. For a fixed parameter $\varepsilon \in \mathbb{R}_{+}$, consider the problem of finding all the solutions of

$$
\begin{equation*}
L(u, \varepsilon)=u^{3}, \text { in } \Omega_{+}, u \in D, \tag{3.28}
\end{equation*}
$$

where

$$
\begin{equation*}
L(u, \varepsilon)=\varepsilon^{2} \Delta u+u, \tag{3.29}
\end{equation*}
$$

and

$$
\begin{equation*}
D=\left\{u \in C ; u_{x}(0)=u_{x}(l)=0\right\}, \tag{3.30}
\end{equation*}
$$

where $C$ is the set of all functions $u: \overline{\Omega_{+}} \longrightarrow \mathbb{R}$ such that the classical derivative $u_{x x}$ exists in $\Omega_{+}$, and $u_{x x} \in L^{2}\left(\Omega_{+}\right)$. The first order derivative in (3.30) is built by taking lateral limits as follows:

$$
\begin{equation*}
u_{x}(0)=\lim _{t \rightarrow 0^{+}} \frac{u(t)-u(0)}{t}, u_{x}(l)=\lim _{t \rightarrow 0^{-}} \frac{u(l+t)-u(l)}{t}, \tag{3.31}
\end{equation*}
$$

that is, if $u \in D$ then $u$ has null normal derivatives over the boundary of $\Omega_{+}$. Next we obtain a new setting of the previous problem more suitable for the energy estimates.

Proposition 3.1.2. Let $u$ be a solution of (3.28). Define $\tilde{\Omega}=\{x \in \Omega ; x \neq 0\}$, and define $v$ : $\bar{\Omega} \longrightarrow \mathbb{R}$ by $v(x)=u(|x|)$. Then $v$ is differentiable and satisfies:

$$
\begin{equation*}
\varepsilon^{2} \Delta v(x)=v^{3}(x)-v(x), x \in \tilde{\Omega} \tag{3.32}
\end{equation*}
$$

and

$$
\begin{equation*}
v_{x}(-l)=v_{x}(l)=0 \tag{3.33}
\end{equation*}
$$

Reciprocally, if $v: \bar{\Omega} \rightarrow \mathbb{R}$ is differentiable, $v(x)=v(|x|), x \in \bar{\Omega}$, and if $v$ is a solution of (3.32) and (3.33) with $v_{x x} \in L^{2}(\Omega)$ then $u:=\left.v\right|_{\overline{\Omega_{+}}}$is a solution of (3.28).

Proof. Let $u$ be a solution of (3.28)and define $v(x)=u(|x|), x \in \bar{\Omega}$. It is obvious that $v$ is differentiable at $x \neq 0$. Also, by the hypothesis we can calculate as follows:

- $\lim _{t \rightarrow 0^{+}} \frac{v(t)-v(0)}{t}=\lim _{t \rightarrow 0^{+}} \frac{u(t)-u(0)}{t}=0 ;$
- $\lim _{t \rightarrow 0^{-}} \frac{v(t)-v(0)}{t}=\lim _{t \rightarrow 0^{+}} \frac{v(-t)-v(0)}{-t}=-\lim _{t \rightarrow 0^{+}} \frac{v(t)-v(0)}{t}=0$.

Therefore, $v$ is differentiable at $x=0$ and $v_{x}(0)=0$.
Now, let us prove (3.32) and (3.33). Since $0 \notin \tilde{\Omega}$, if $B$ is an open ball contained in $\tilde{\Omega}$ then there exists a constant $\sigma \in\{-1,1\}$ such that

$$
|x|=\sigma x, \forall x \in B
$$

Then, we can write:

$$
\begin{gathered}
v(x)=u(|x|)=u(\sigma x), x \in B \Longrightarrow \\
\Longrightarrow \Delta v(x)=\sigma^{2} u_{x x}(\sigma x)=\Delta u(|x|)=\frac{1}{\varepsilon^{2}}\left[u^{3}(|x|)-u(|x|)\right], x \in B \Longrightarrow \\
\Longrightarrow \varepsilon^{2} \Delta v(x)=u^{3}(|x|)-u(|x|)=v^{3}(x)-v(x), x \in B .
\end{gathered}
$$

Therefore,

$$
\begin{equation*}
\varepsilon^{2} \Delta v(x)=v^{3}(x)-v(x), x \in \tilde{\Omega} \tag{3.34}
\end{equation*}
$$

what proves (3.32). Also, since $v(-x)=v(x), \forall x \in \bar{\Omega}$, we can write:

$$
\begin{aligned}
& v_{x}(-l):=\lim _{t \rightarrow 0^{+}} \frac{v(-l+t)-v(-l)}{t}=\lim _{t \rightarrow 0^{+}} \frac{v(l-t)-v(l)}{t}= \\
= & \lim _{t \rightarrow 0^{-}} \frac{v(l+t)-v(l)}{-t}=-\lim _{t \rightarrow 0^{-}} \frac{v(l+t)-v(l)}{t}=-u_{x}(l)=0 .
\end{aligned}
$$

The boundary condition $v_{x}(l)=0$ follows directly from the fact that $v_{x}(l)=u_{x}(l)=0$, what concludes the proof of (3.33).

Reciprocally, let $v: \bar{\Omega} \rightarrow \mathbb{R}$ satisfying (3.32) and (3.33) with $v_{x x} \in L^{2}(\Omega)$ and $v(x)=$ $v(|x|), x \in \bar{\Omega}$. All the necessary conditions for $u:=\left.v\right|_{\overline{\Omega_{+}}}$to be a solution of (3.28), except that $u_{x}(0)=0$, follows immediately from the properties of $v$. To see that $u_{x}(0)=0$, since $v$ is differentiable we can calculate as follows:

$$
v(x)=v(-x) \Rightarrow v_{x}(0)=-v_{x}(0) \Rightarrow v_{x}(0)=0 \Rightarrow u_{x}(0)=0
$$

Remark 3.1.1. Proposition (3.1.2) says that the solutions of problem (3.28) are restrictions to $\Omega_{+}$of solutions of the problem given by:

$$
\begin{equation*}
L(v, \boldsymbol{\varepsilon})=v^{3}, \text { in } \tilde{\Omega}, v \in \tilde{D} \tag{3.35}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{D}=\left\{v \in \tilde{C} ; v_{x}(-l)=v_{x}(l)=0\right\} \tag{3.36}
\end{equation*}
$$

where $\tilde{C}$ is the set of all differentiable functions $v: \bar{\Omega} \longrightarrow \mathbb{R}$ such that the classical partial derivative $v_{x x}$ exists in $\tilde{\Omega}$, and $v_{x x} \in L^{2}(\tilde{\Omega})$.

In what follows we provide the algebraic formulation (3.4) for the reformulated differential problem (3.35).

Theorem 3.1.5 (Energy estimates for two-dimensional Cahn-Hilliard equation). If $v$ is a solution of (3.35) then:

$$
\|v\|_{L^{2}(\Omega)} \leq \sqrt{l_{1} l_{2}},\|\Delta v\|_{L^{2}(\Omega)} \leq \frac{\sqrt{l_{1} l_{2}}}{\varepsilon^{2}}
$$

Proof. From (3.32) and (3.33) we can calculate as follows:

$$
\begin{align*}
& \varepsilon^{2} \Delta v=v^{3}-v \text { in } \tilde{\Omega} \Rightarrow \varepsilon^{2} \int_{\Omega} v \Delta v=\int_{\Omega} v^{4}-\int_{\Omega} v^{2} \Rightarrow \\
& -\varepsilon^{2} \int_{\Omega}|\nabla v|^{2}=\int_{\Omega} v^{4}-\int_{\Omega} v^{2} \Rightarrow \int_{\Omega} v^{4}-\int_{\Omega} v^{2} \leq 0 \tag{3.37}
\end{align*}
$$

On the other hand,

$$
\begin{equation*}
\int_{\Omega} v^{2} \leq\left(\int_{\Omega} v^{4}\right)^{\frac{1}{2}}\left(\int_{\Omega} 1\right)^{\frac{1}{2}} \Rightarrow \int_{\Omega} v^{4} \geq \frac{1}{2 l}\left(\int_{\Omega} v^{2}\right)^{2} \tag{3.38}
\end{equation*}
$$

From (3.37) and (3.38) we obtain

$$
\begin{gather*}
\frac{1}{2 l}\left(\int_{\Omega} v^{2}\right)^{2}-\int_{\Omega} v^{2} \leq \int_{\Omega} v^{4}-\int_{\Omega} v^{2} \leq 0 \Rightarrow \\
\Rightarrow \frac{1}{2 l}\|v\|_{L^{2}(\Omega)}^{4}-\|v\|_{L^{2}(\Omega)}^{2} \leq 0 \Rightarrow \\
\Rightarrow\|v\|_{L^{2}(\Omega)} \leq \sqrt{2 l} \tag{3.39}
\end{gather*}
$$

From (3.32) we can write:

$$
\varepsilon^{2} \Delta v=v^{3}-v \Rightarrow \varepsilon^{4}(\Delta v)^{2}=v^{6}-2 v^{4}+v^{2} \Rightarrow
$$

$$
\begin{equation*}
\varepsilon^{4}\|\Delta v\|_{L^{2}(\Omega)}^{2}=\int_{\Omega} v^{6}-2 \int_{\Omega} v^{4}+\int_{\Omega} v^{2} \tag{3.40}
\end{equation*}
$$

From (3.32) and (3.33) we can write:

$$
\begin{align*}
\varepsilon^{2} v^{3} \Delta v=v^{6}-v^{4} \Rightarrow & -\varepsilon^{2} \int_{\Omega} 3 v^{2} \nabla v \cdot \nabla v=\int_{\Omega} v^{6}-\int_{\Omega} v^{4} \Rightarrow \\
& \Rightarrow \int_{\Omega} v^{6} \leq \int_{\Omega} v^{4} \tag{3.41}
\end{align*}
$$

From (3.40) and (3.41) we obtain

$$
\begin{align*}
\varepsilon^{4}\|\Delta v\|_{L^{2}(\Omega)}^{2} & \leq \int_{\Omega} v^{2}-\int_{\Omega} v^{4} \leq\|v\|_{L^{2}(\Omega)}^{2} \leq 2 l \Rightarrow \\
& \Rightarrow\|\Delta v\|_{L^{2}(\Omega)} \leq \frac{\sqrt{2 l}}{\varepsilon^{2}} \tag{3.42}
\end{align*}
$$

Corollary 3.1.3. If $v$ is a solution of (3.35) and $|v(x, y)| \leq M \forall(x, y) \in \Omega$, for some $M \in \mathbb{R}$, then:

$$
\int_{\Omega}|\Delta(\Delta v)| \leq \frac{6(2 M+1) l}{\varepsilon^{2}}
$$

Proof. First, observe that the statement $|v(x, y)| \leq M \forall(x, y) \in \Omega$, for some $M \in \mathbb{R}$, makes sense because $v \in C^{1}(\bar{\Omega})$. From (3.32), we can write:

$$
\begin{equation*}
\varepsilon^{2} \Delta v=v^{3}-v \text { in } \tilde{\Omega} \Rightarrow \varepsilon^{2} \Delta(\Delta v)=\Delta\left(v^{3}-v\right)=6 v|\nabla v|^{2}+3 v^{2} \Delta v \text { in } \tilde{\Omega} \tag{3.43}
\end{equation*}
$$

But, for (3.37) we have $\int_{\Omega}|\nabla v|^{2} \leq \frac{\|v\|_{L^{2}(\Omega)}^{2}}{\varepsilon^{2}}$. Therefore,

$$
\begin{equation*}
\int_{\Omega}|\Delta(\Delta v)| \leq 6 M \frac{\|v\|_{L^{2}(\Omega)}^{2}}{\varepsilon^{2}}+3\left(\int_{\Omega} v^{4}\right)^{\frac{1}{2}}\|\Delta v\|_{L^{2}(\Omega)} \tag{3.44}
\end{equation*}
$$

From (3.37) we know that $\int_{\Omega} v^{4} \leq \int_{\Omega} v^{2}$. Plugging this estimate in (3.44) and using the estimates for $\|v\|_{L^{2}(\Omega)}$ and $\|\Delta v\|_{L^{2}(\Omega)}$, we obtain:

$$
\begin{equation*}
\int_{\Omega}|\Delta(\Delta v)| \leq 6 M \frac{2 l}{\varepsilon^{2}}+3\|v\|_{L^{2}(\Omega)}\|\Delta v\|_{L^{2}(\Omega)} \leq \frac{6(2 M+1) l}{\varepsilon^{2}} \tag{3.45}
\end{equation*}
$$

Theorem 3.1.6. If $v \in \tilde{D}$ and $\{\hat{v}(k)\}_{k \in \mathbb{Z}^{2}}$ is the sequence of Fourier coefficients of $v$ with respect to the orthonormal Hilbert basis $\Gamma$ of $L^{2}(\Omega)$, then:

1. $\widehat{L(v, \varepsilon)}(k)=\mu_{k} \hat{v}(k), k \in \mathbb{Z}^{2}$, where

$$
\mu_{k}(\varepsilon)=L\left(\Gamma_{k}, \varepsilon\right)=1-\varepsilon^{2}\left(k^{2} L^{2}\right), k \in \mathbb{Z}
$$

2. $\widehat{v^{3}}(k)=\frac{1}{2 l} \sum_{\substack{k_{1}+k_{2}+k_{3}=k \\ k_{1}, k_{2}, k_{3} \in \mathbb{Z}}} \hat{v}\left(k_{1}\right) \hat{v}\left(k_{2}\right) \hat{v}\left(k_{3}\right)=0, k \in \mathbb{Z}$;
3. $\hat{v}(k)=\hat{v}(|k|) \in \mathbb{R}, k \in \mathbb{Z}$.

Proof. Let $v$ be a solution of (3.35), with

$$
\begin{equation*}
v=\sum_{k \in \mathbb{Z}} \hat{v}(k) \Gamma_{k}, \text { in } L^{2}(\Omega) . \tag{3.46}
\end{equation*}
$$

Also, from (3.33) and since $v$ is an even function, we can integrate by parts as follows:

$$
\begin{align*}
& \widehat{\Delta u}(k)=\int_{\Omega} \Delta v \overline{\Gamma_{k}}=\frac{1}{\sqrt{2 l}} \int_{\Omega} v_{x x} e^{-i k L x}=\frac{-i k L}{\sqrt{2 l}} \int_{\Omega} v_{x} e^{-i k L x}= \\
& =\frac{(-i k L)^{2}}{\sqrt{2 l}} \int_{\Omega} v e^{-i k L x}=-k^{2} L^{2} \int_{\Omega} v \overline{\Gamma_{k}}=-k^{2} L^{2} \hat{v}(k), k \in \mathbb{Z} \tag{3.47}
\end{align*}
$$

what proves the first item.
The second item follows from Corollary 2.3.5. Finally, for Proposition 2.3.1 we know that $\hat{v}(k)=\hat{v}(|k|) \in \mathbb{R}$, what concludes the proof.

Theorem 3.1.7 (Decay of Fourier Coefficients). Let $s=2, L>0$, and $N \in \mathbb{N}$. If $v$ is a solution of (3.35) with $|v(x)| \leq M, x \in \Omega$, and $a=\left\{a_{k}\right\}_{k \in \mathbb{Z}}$ is the sequence of Fourier coefficients of $v$ with respect to the basis $\Gamma$, then $a \in X^{s}$ and:

$$
\begin{equation*}
\left|a_{k}\right| \leq c_{k}(v, L), k \in \mathbb{Z} \tag{3.48}
\end{equation*}
$$

where

$$
c_{k}(\varepsilon, L)= \begin{cases}\sqrt{2 l} & \text { if } k=0  \tag{3.49}\\ \frac{6(2 M+1) \sqrt{l}}{\sqrt{2}(k L)^{4} \varepsilon^{2}} & \text { if } 0<k<N \\ \frac{6(2 M+1) \sqrt{l}}{\sqrt{2} N^{2} L^{4} \varepsilon^{2} \omega_{k}^{s}} & \text { if } k \geq N\end{cases}
$$

Proof. From (3.32) we have $\Delta u(-l)=\Delta u(l)$. Deriving (3.32) we conclude that $(\Delta u)_{x}(-l)=$ $(\Delta u)_{x}(l)=0$, since $v$ satisfies the conditions given by (3.33). These information plus equation (3.47) allow us calculate as follows:

$$
\begin{gather*}
\int_{\Omega}(\Delta v)_{x x} \overline{\Gamma_{k}}=\frac{1}{\sqrt{2 l}} \int_{\Omega}(\Delta v)_{x x} e^{-i k l L x}=\frac{i k L}{\sqrt{2 l}} \int_{\Omega}(\Delta v)_{x} e^{-i k L x}= \\
=\frac{(i k L)^{2}}{\sqrt{2 l}} \int_{\Omega} \Delta v e^{-i k L x}=\frac{(i k L)^{4}}{\sqrt{2 l}} \int_{\Omega} v e^{-i k L x}=(k L)^{4} \int_{\Omega} v \overline{\Gamma_{k}}= \\
=(k L)^{4} \hat{v}(k), k \in \mathbb{Z} . \tag{3.50}
\end{gather*}
$$

From (3.50) and Corollary (3.1.3) we obtain

$$
\begin{equation*}
|\hat{v}(k)| \leq \frac{6(2 M+1) \sqrt{l}}{\sqrt{2}(k L)^{4} \varepsilon^{2}}, k \in \mathbb{Z}, k \neq 0 . \tag{3.51}
\end{equation*}
$$

We summarize Theorems 3.1.6 and 3.1.7 in the following.
Theorem 3.1.8. If $v$ is a solution of (3.35) and $a=\left\{a_{k}\right\}_{k \in \mathbb{Z}}$ is the sequence of Fourier coefficients of $v$ with respect to the basis $\Gamma$ then

$$
\left\{\begin{array}{l}
2 l \mu_{k}(\varepsilon) a_{k}-\left(a^{3}\right)_{k}=0, k \in \mathbb{Z}  \tag{3.52}\\
a \in X^{2}, a_{k}=a_{|k|},\left|a_{k}\right| \leq \frac{C_{k}}{\omega_{k}^{2}}, k \in \mathbb{Z}
\end{array}\right.
$$

where

$$
\begin{equation*}
\mu_{k}(\varepsilon)=1-\varepsilon^{2} k^{2} L^{2}, k \in \mathbb{Z} \tag{3.53}
\end{equation*}
$$

and

$$
C_{k}= \begin{cases}\sqrt{2 l} & \text { if } k=0  \tag{3.54}\\ \frac{6(2 M+1) \sqrt{l}}{\sqrt{2}(k L)^{4} 4^{2}} & \text { if } 0<k<N \\ \frac{6(2 M+1) \sqrt{l}}{\sqrt{2} N^{2} L^{4} \varepsilon^{2}} & \text { if } k \geq N .\end{cases}
$$

Analogously to the previous section we can prove the following reciprocal result.
Corollary 3.1.4. If $v$ is a solution of (3.35) then $v \in C^{\infty}\left(\mathbb{R}^{2}\right)$. Furthermore, $v=\sum_{k \in \mathbb{Z}^{d}} a_{k} \Gamma_{k}$ is a solution of (3.35) if and only if $a$ is a solution of the problem given by (3.52), (3.53) and (3.54).

## DEVELOPMENT OF THE METHOD

In the previous chapter we saw that solving some differential equations in some subsets of $\mathbb{R}^{d}$ is equivalent to solve a system of algebraic equations of the form:

$$
\left\{\begin{array}{l}
\mathscr{F}_{k}(a, \lambda)=0, k \in \mathbb{Z}^{d}  \tag{4.1}\\
\left|a_{k}\right| \leq \frac{C_{k}}{\omega_{k}^{*}} \text { if } k \in F_{m} \\
\left|a_{k}\right| \leq \frac{C}{\omega_{k}^{*}} \text { if } k \notin F_{m}
\end{array}\right.
$$

for some constants $s \in \mathbb{R}^{d}, m \in \mathbb{N}^{d}$, and $C_{k}>0, k \in F_{m}$, and $C>0$, with

$$
\begin{gather*}
\mathscr{F}_{k}(\cdot, \lambda): X_{s} \longrightarrow \mathbb{R}, \lambda \in \mathbb{R} \\
\mathscr{F}_{k}(a, \lambda)=\mu_{k}(\lambda) a_{k}-\sum_{j=2}^{p} q_{j}(\lambda)\left(a^{j}\right)_{k}, k \in \mathbb{Z}^{d} . \tag{4.2}
\end{gather*}
$$

Definition 4.0.1. Define one parameter family of maps $\mathscr{F}(\cdot, \lambda): X^{s} \longrightarrow \mathbb{R}^{Z^{d}}$ by

$$
\begin{equation*}
\mathscr{F}(a, \lambda)=\left\{\mathscr{F}_{k}(a, \lambda)\right\}_{k \in \mathbb{Z}^{d}}, \tag{4.3}
\end{equation*}
$$

where $\mathscr{F}_{k}(a, \lambda)$ is given by (4.2).

Then, the problem given by (4.1) can be written in the form:

$$
\left\{\begin{array}{l}
\mathscr{F}(a, \lambda)=0, a \in X^{s},  \tag{4.4}\\
\left|a_{k}\right| \leq \frac{C_{k}}{\omega_{k}^{s}} \text { if } k \in F_{m}, \\
\left|a_{k}\right| \leq \frac{C}{\omega_{k}^{s}} \text { if } k \notin F_{m} .
\end{array}\right.
$$

In this chapter we present our approach for determining all solutions of the problem (4.4) for suitable values of $\lambda$ and under the assumption that

$$
\begin{equation*}
\inf _{k \notin F_{m}}\left|\mu_{k}(\lambda)\right|>0 . \tag{4.5}
\end{equation*}
$$

Remark 4.0.1. In the applications presented in this work it happens that $\left|\mu_{k}\right| \rightarrow \infty$ when $k \rightarrow \infty$, so that hypothesis (4.5) is fulfilled for some $m \in \mathbb{N}^{d}$ big enough.

We start with the following important definition.
Definition 4.0.2. Let $m \in \mathbb{N}^{d}$. Define the finite dimensional subspace $X_{(m)}^{s}$ of $X^{s}$ by

$$
X_{(m)}^{s}=\left\{x \in X^{s} ; x_{k}=0 \text { if } k \notin F_{m}\right\} .
$$

Denote by $\Pi_{(m)}: X^{s} \longrightarrow X_{(m)}^{s}$ the projection over $X_{(m)}^{s}$, that is:

$$
\Pi_{(m)} u=\left\{\begin{array}{ll}
u_{k} & \text { if } k \in F_{m} \\
0 & \text { if } k \notin F_{m}
\end{array}, u \in X^{s}\right.
$$

We will write $u_{(m)}$ to refer to $\Pi^{(m)} u$ and $u^{(m)}$ to refer to $u-u_{(m)}$, that is, $u=u_{(m)}+u^{(m)}, u \in X^{s}$. If $M \in \mathbb{Z}^{d}$ is such that $F_{m} \subset F_{M}$ then we define the sequence $u_{m}^{M}$ by

$$
\left[u_{m}^{M}\right]_{k}= \begin{cases}u_{k} & \text { if } k \in F_{M} \backslash F_{m} \\ 0 & \text { if } k \notin F_{M} \backslash F_{m}\end{cases}
$$

Finally, given $M \in \mathbb{N}^{d}$ such that $F_{m} \subset F_{M}$, real numbers $r>0$ and $c>0$ and a set $C=\left\{c_{k}, k \in F_{M} \backslash F_{m}\right\}$, of positive real numbers, we define the following neighborhood of $0 \in X^{s}$ :

$$
\begin{equation*}
B(r, c, C)=\left\{x \in X^{s} ;\left|\left[x_{(m)}\right]_{k}\right| \leq \frac{r}{\omega_{k}^{s}},\left|\left[x_{m}^{M}\right]_{k}\right| \leq \frac{c_{k}}{\omega_{k}^{s}},\left|\left[x^{(M)}\right]_{k}\right| \leq \frac{c}{\omega_{k}^{s}}\right\} . \tag{4.6}
\end{equation*}
$$

Consider $\lambda$ fixed and let $\bar{x}_{1}, \ldots, \bar{x}_{q} \in X_{(m)}^{s}$ be some numerical solution obtained by applying Newton's method to the truncated problem:

$$
\begin{equation*}
\mathscr{F}^{(m)}(x, \lambda)=0, \text { with } \mathscr{F}^{(m)}(x, \lambda)=\left\{\mathscr{F}_{k}(x, \lambda)\right\}_{k \in F_{m}} \text { and } x \in X_{(m)}^{s} . \tag{4.7}
\end{equation*}
$$

Looking for such numerical solutions is the first step of the method. The second step of the method consists in proving the existence of a unique exact solution of problem (4.4) in a neighborhood of each approximated solution $\bar{x}_{i}, i=1, \ldots, q$.

In order to prove the existence of a unique zero of $\mathscr{F}(\cdot, \lambda)$ around a numerical solution $\bar{x} \in\left\{\bar{x}_{1}, \ldots, \bar{x}_{q}\right\}$ we introduce a Newton-like operator $T$ (depending on $\bar{x}$ ) whose fixed points correspond to zeros of $\mathscr{F}(\cdot, \lambda)$. Then we provide conditions that can be rigorously checked by a computer for the mentioned fixed point operator be a contraction in a small neighborhood of $\bar{x}$ of the form $\bar{x}+B(r, c, C)$, for some $B(r, c, C)$ of the form (4.6). Once these conditions are checked we have, by the Banach's fixed point Theorem, the existence of a unique fixed point of $T$, i.e. a zero of $\mathscr{F}(\cdot, \lambda)$, inside the neighborhood $\bar{x}+B(r, c, C)$. These ideas are developed in details in the next section.

### 4.1 Local uniqueness

### 4.1.1 Fixed point formulation

In what follows consider $\lambda$ fixed and $\bar{x} \in\left\{\bar{x}_{1}, \ldots, \bar{x}_{q}\right\}$ a numerical solution of the truncated problem (4.7).

Let $A=A(\bar{x}, \lambda): \mathbb{R}^{\mathbb{Z}^{d}} \longrightarrow \mathbb{R}^{\mathbb{Z}^{d}}$ be a linear map, whose detailed construction is made next, such that $A v \in X^{s}$ if $v \in \mathscr{F}\left(X^{s}, \lambda\right)$. Then we can associate to $\mathscr{F}$ the Newton-like operator

$$
\begin{equation*}
T_{\mathscr{F}}=T_{\mathscr{F}}(\bar{x}, \lambda):=I-A \mathscr{F}: X^{s} \longrightarrow X^{s}, \tag{4.8}
\end{equation*}
$$

where $I$ is the identity of $X^{s}$.
Observe that if $A$ is injective then the fixed points of $T$ correspond to zeros of $\mathscr{F}(\cdot, \lambda)$.

### 4.1.1.1 Construction of A

For $\bar{x} \in X_{(m)}^{s}$ define the linear operator $A^{\dagger}=A^{\dagger}(\bar{x}): X^{s} \longrightarrow \mathbb{R}^{\mathbb{Z}^{d}}$ by

$$
\left(A^{\dagger} v\right)_{k}=\left\{\begin{array}{cl}
\mathscr{F}_{k}^{\prime}(\bar{x}) v_{(m)} & \text { if } k \in F_{m}  \tag{4.9}\\
\mu_{k}(\lambda) v_{k} & \text { if } k \notin F_{m}
\end{array} .\right.
$$

In the attempting of to emulate the acting of the abstract finite dimensional operator $\left.A^{\dagger}(\bar{x})\right|_{X_{(m)}^{s}}$ in terms of concrete operations to be performed in the computer, we start by choosing a bijection

$$
\mathscr{J}: F_{m} \longrightarrow\left\{1, \ldots, \# F_{m}\right\}
$$

and to it we associate the isomorfism

$$
J: X_{(m)}^{s} \longrightarrow \mathbb{R}^{\# F_{m}}
$$

given by

$$
(J w)_{i}=w_{\mathcal{J}^{-1}(i)}, w \in X_{(m)}^{s}, i \in\left\{1, \ldots, \# F_{m}\right\} .
$$

Observe that

$$
\left(J^{-1} w\right)_{k}=w_{\mathscr{J}(k)}, w \in \mathbb{R}^{\# F_{m}}, k \in F_{m}
$$

Both $\mathscr{J}$ and $J$ and their respective inverses can be easily implemented in a computer. Also, we can write the finite dimensional linear operator

$$
\left.A^{\dagger}(\bar{x})\right|_{X_{(m)}^{s}}: X_{(m)}^{s} \ni w \mapsto\left\{\mathscr{F}_{k}^{\prime}(\bar{x}) w\right\}_{k \in F_{m}} \in X_{(m)}^{s}
$$

as the following composition:

$$
\left.A^{\dagger}(\bar{x})\right|_{X_{(m)}^{s}}=J^{-1} \circ\left[\left.J \circ A^{\dagger}(\bar{x})\right|_{X_{(m)}^{s}} \circ J^{-1}\right] \circ J=J^{-1} \circ B \circ J,
$$

where we are denoting by $B: \mathbb{R}^{\# F F_{m}} \longrightarrow \mathbb{R}^{\# F_{m}}$ the map defined by

$$
B=\left.J \circ A^{\dagger}(\bar{x})\right|_{X_{(m)}^{s}} \circ J^{-1}
$$

Then, $\left.A^{\dagger}(\bar{x})\right|_{X_{(m)}^{s}}$ and $B$ are conjugated operators, as the following diagram illustrates.

$$
\begin{array}{ccc}
X_{(m)}^{s} & \xrightarrow{A^{\dagger}} & X_{(m)}^{s}  \tag{4.10}\\
J^{-1} \uparrow & & \downarrow J \\
\mathbb{R}^{\# F_{m}} & \xrightarrow{B} & \mathbb{R}^{\# F_{m}}
\end{array}
$$

The operator $B$ can be concretely performed as a matrix multiplication, as we show next for the case of cubic non-linearity, that is when

$$
\mathscr{F}_{k}(x, \lambda)=\mu_{k}(\lambda) x_{x}-\left(x^{3}\right)_{k}, x \in X^{s}, k \in \mathbb{Z}^{d} .
$$

In this case, for $n \in\left\{1, \ldots, \# F_{m}\right\}$ and $v \in \mathbb{R}^{\# F_{m}}$ we can calculate as follows:

$$
\begin{gather*}
{[B(v)]_{n}=\left[J\left(A^{\dagger} J^{-1} v\right)\right]_{n}=\left[J\left(\left\{\mathscr{F}_{k}^{\prime}(\bar{x}) J^{-1} v\right\}_{k \in F_{m}}\right)\right]_{n}=} \\
=\left[J\left(\left\{\mu_{k}(\lambda)\left(J^{-1} v\right)_{k}-3\left(\bar{x}^{2} J^{-1} v\right)_{k}\right\}_{k \in F_{m}}\right)\right]_{n}= \\
=\mu_{\mathscr{J}^{-1}(n)}(\lambda)\left(J^{-1} v\right)_{\mathscr{J}^{-1}(n)}-3\left(\bar{x}^{2} J^{-1} v\right)_{\mathscr{J}^{-1}(n)}= \\
=\mu_{\mathscr{J}^{-1}(n)}(\lambda) v_{n}-3 \sum_{j \in F_{m}}\left(\bar{x}^{2}\right)_{\mathscr{J}^{-1}(n)-j}\left(J^{-1} v\right)_{j}= \\
=\mu_{\mathscr{J}^{-1}(n)}(\lambda) v_{n}-3 \sum_{i=1}^{\# F_{m}}\left(\bar{x}^{2}\right)_{\mathscr{J}^{-1}(n)-\mathscr{J}^{-1}(i)}\left(J^{-1} v\right)_{\mathscr{J}^{-1}(i)}= \\
=\mu_{\mathscr{J}^{-1}(n)}(\lambda) v_{n}-3 \sum_{i=1}^{\# F_{m}}\left(\bar{x}^{2}\right)_{\mathscr{J}^{-1}(n)-\mathscr{J}^{-1}(i) v_{i}=}^{v_{i}} \\
=\sum_{i=1}^{\# F_{m}}\left[\delta_{i, n} \mu_{\mathscr{J}^{-1}(n)}(\lambda)-3\left(\bar{x}^{2}\right)_{\left.\mathscr{J}^{-1}(n)-\mathscr{J}^{-1}(i)\right]} v_{i}=\sum_{i=1}^{\# F_{m}} b_{n, i} v_{i},\right. \tag{4.11}
\end{gather*}
$$

where

$$
\begin{equation*}
b_{n, i}=b_{n, i}(\lambda, \bar{x}):=\delta_{i, n} \mu_{\mathscr{J}^{-1}(n)}(\lambda)-3\left(\bar{x}^{2}\right)_{\mathscr{J}^{-1}(n)-\mathscr{J}^{-1}(i)}, \quad i, n \in\left\{1, \ldots, \# F_{m}\right\} . \tag{4.12}
\end{equation*}
$$

Then, denoting by $\mathbf{B}$ the square matrix with entries $b_{n, i}, n, i \in\left\{1, \ldots, \# F_{m}\right\}$, we can write

$$
B(v)=\mathbf{B} v, v \in \mathbb{R}^{\# F_{m}}
$$

and

$$
A^{\dagger} w=J^{-1} \mathbf{B} J w, w \in X_{(m)}^{s} .
$$

Observe that if $\mu_{k}(\lambda) \neq 0 \forall k \notin F_{m}$ then $A^{\dagger}$ is an invertible map if and only if $\mathbf{B}$ is an non-singular matrix. In this case the inverse of $A^{\dagger}$ is given by

$$
\left[\left(A^{\dagger}\right)^{-1} v\right]_{k}=\left\{\begin{array}{cl}
{\left[J^{-1} \mathbf{B}^{-1} J v_{(m)}\right]_{k}} & \text { if } k \in F_{m} \\
\frac{1}{\mu_{k}(\lambda)} v_{k} & \text { if } k \notin F_{m}
\end{array}\right.
$$

Let $\mathscr{B}$ be a numerically obtained inverse of the matrix $\mathbf{B}$. Then, we define the linear operator $A: \mathbb{R}^{\mathbb{Z}^{d}} \longrightarrow \mathbb{R}^{\mathbb{Z}^{d}}$ as follows:

$$
(A v)_{k}=[A(\bar{x}, \lambda) v]_{k}=\left\{\begin{array}{cl}
{\left[J^{-1} \mathscr{B} J v_{(m)}\right]_{k}} & \text { if } k \in F_{m}  \tag{4.13}\\
\frac{1}{\mu_{k}(\lambda)} v_{k} & \text { if } k \notin F_{m},
\end{array}\right.
$$

which is well defined because of hypothesis (4.5).
Observe that Theorem (2.2.1) and hypothesis (4.5) give that $A \mathscr{F}(v) \in X^{s}$ if $v \in X^{s}$, that is:

$$
A \mathscr{F}: X^{s} \longrightarrow X^{s} .
$$

It is a straightforward task to show that $A \mathscr{F}: X^{s} \longrightarrow X^{s}$ is a non-linear Frechet differentiable map with:

$$
(A \mathscr{F})^{\prime}(x) v=A G_{\mathscr{F}}(x) v, x, v \in X^{s},
$$

where $G_{\mathscr{F}}(x): X^{s} \longrightarrow \mathbb{R}^{\mathbb{Z}^{d}}$ is the linear map defined by:

$$
\left[G_{\mathscr{F}}(x) v\right]_{k}=\mathscr{F}_{k}^{\prime}(x) v=\mu_{k}(\lambda) v_{k}-\sum_{j=2}^{p} j q_{j}(\lambda)\left(x^{j-1} v\right)_{k}, k \in \mathbb{Z}^{d} .
$$

Observe that

$$
A G_{\mathscr{F}}(x) v \in X^{s}, \forall x, v \in X^{s} .
$$

Once the fixed-point operator $T_{\mathscr{F}}$ associated to the zero finding problem $\mathscr{F}(\cdot, \lambda)=0$ in a neighborhood of a point $\bar{x} \in X_{(m)}^{s}$ is built, the next step is to provide conditions that can be rigorously checked by a computer for $T_{\mathscr{F}}$ to be a contraction in a small neighborhood of $\bar{x}$.

### 4.1.1.2 Theorem for local existence and uniqueness of fixed point

The next Theorem gives sufficient conditions for local existence and uniqueness of a fixed point of Frechet differentiable operators in $X^{s}$ in a neighborhood of an approximated solution $\bar{x}$ of problem (4.4).

Theorem 4.1.1 (Local existence and uniqueness). Let $T: X^{s} \longrightarrow X^{s}$ be a Frechet differentiable map, and let be fixed $m, M \in \mathbb{N}^{d}$, with $F_{m} \subset F_{M}$, real numbers $c>0$ and $\delta \in(0,1)$, a set of positive real numbers $C=\left\{c_{k}>0, k \in F_{M} \backslash F_{m}\right\}$, and $\bar{x} \in X^{s}$. Define the boxes $B(r, c, C), r>0$, as in (4.6). Suppose there exist constants

$$
Y_{k} \geq 0, k \in \mathbb{Z}^{d}
$$

and real functions

$$
Z_{k}^{(1)}(r), r>0, k \in \mathbb{Z}^{d}
$$

and

$$
Z_{k}^{(2)}(r), r>0, k \in \mathbb{Z}^{d}
$$

satisfying the following:

$$
\begin{gather*}
\left|T_{k}(\bar{x})-\bar{x}_{k}\right| \leq Y_{k}, k \in \mathbb{Z}^{d},  \tag{4.14}\\
\sup _{v, w \in B(r, c, C)}\left|T_{k}^{\prime}(\bar{x}+v) w\right| \leq Z_{k}^{(1)}(r), r>0, \forall k \in \mathbb{Z}^{d},  \tag{4.15}\\
\sup _{\substack{v \in B(r, c, C) \\
\|w\|_{s \leq 1} \leq}}\left|T_{k}^{\prime}(\bar{x}+v) w\right| \leq Z_{k}^{(2)}(r), r>0, \forall k \in \mathbb{Z}^{d} \tag{4.16}
\end{gather*}
$$

In this case, if there exists $\bar{r}>0$ such that :

$$
Y_{k}+Z_{k}^{(1)}(\bar{r}) \leq\left\{\begin{array}{l}
\frac{\bar{r}}{\omega_{k}^{5}}
\end{array} \quad \forall k \in F_{m},\left\{\begin{array}{c}
\frac{c_{k}}{\omega_{k}^{5}}
\end{array} \forall k \in F_{M} \backslash F_{m}, \begin{array}{c}
\frac{c}{\omega_{k}^{5}} \tag{4.17}
\end{array} \forall k \notin F_{M}\right.\right.
$$

and

$$
\begin{equation*}
Z_{k}^{(2)}(\bar{r}) \leq \frac{\delta}{\omega_{k}^{s}} \forall k \in \mathbb{Z}^{d} \tag{4.18}
\end{equation*}
$$

then there exists a unique $x \in \bar{x}+B(r, c, C)$ such that $T(x)=x$.

Proof. First, let us prove that

$$
T[\bar{x}+B(\bar{r}, c, C)] \subset \bar{x}+B(\bar{r}, c, C)
$$

Taking $x=\bar{x}+w, w \in B(\bar{r}, c, C)$, we have:

$$
\begin{gathered}
\left|T_{k}(\bar{x}+w)-\bar{x}_{k}\right| \leq \\
\leq\left|T_{k}(\bar{x})-\bar{x}_{k}\right|+\left|T_{k}(\bar{x}+w)-T_{k}(\bar{x})\right|= \\
=\left|T_{k}(\bar{x})-\bar{x}_{k}\right|+\left|T_{k}^{\prime}(\bar{x}+v) w\right|, \text { for some } v \in B(\bar{r}, c, C) .
\end{gathered}
$$

So,

$$
\begin{aligned}
& \left|T_{k}(x)-\bar{x}_{k}\right| \leq Y_{k}+\sup _{v, w \in B(\bar{r}, c, C)}\left|T_{k}^{\prime}(\bar{x}+v) w\right| \leq \\
& \leq Y_{k}+Z_{k}^{(1)}(\bar{r}) \leq \begin{cases}\frac{\bar{r}}{\omega_{k}^{s}} & \forall k \in F_{m} \\
\frac{c_{k}}{\omega_{k}^{s}} & \forall k \in F_{M} \backslash F_{m}, \forall x \in \bar{x}+B(\bar{r}, c, C), \\
\frac{c}{\omega_{k}^{s}} & \forall k \notin F_{M}\end{cases}
\end{aligned}
$$

that is:

$$
\begin{equation*}
T(x) \in \bar{x}+B(\bar{r}, c, C), \forall x \in \bar{x}+B(\bar{r}, c, C) . \tag{4.19}
\end{equation*}
$$

Next, let us prove that

$$
\|T(x)-T(y)\|_{X^{s}} \leq \delta\|x-y\|_{X^{s}}, \quad \forall x, y \in \bar{x}+B(\bar{r}, c, C)
$$

Indeed,

$$
T_{k}(x)-T_{k}(y)=T_{k}^{\prime}(\bar{x}+v) \cdot(x-y), \text { for some } v \in B(\bar{r}, c, C)
$$

that is,

$$
\begin{equation*}
\left|T_{k}(x)-T_{k}(y)\right| \leq\left(\sup _{\substack{v \in B(\bar{r}, c, C) \\\|w\| \leq 1}}\left|T_{k}^{\prime}(\bar{x}+v) w\right|\right)\|x-y\|_{X^{s}} \tag{4.20}
\end{equation*}
$$

However by hypotheses, we have:

$$
\begin{equation*}
\sup _{\substack{v \in B(\bar{r}, C, C) \\ \text { } w \mid s \leq 1}}\left|T_{k}^{\prime}(\bar{x}+v) w\right| \leq Z_{k}^{(2)}(\bar{r}) \leq \frac{\delta}{\omega_{k}^{s}}, \forall k \in \mathbb{Z}^{d} \tag{4.21}
\end{equation*}
$$

From (4.20) and (4.21) we conclude that

$$
\begin{align*}
\omega_{k}^{s}\left|T_{k}(x)-T_{k}(y)\right| & \leq \omega_{k}^{s}\left(\sup _{\substack{v \in B(\bar{r},, C) \\
\|w\| s \leq 1}}\left|T_{k}^{\prime}(\bar{x}+v) w\right|\right)\|x-y\|_{X^{s}} \leq \\
& \leq \delta\|x-y\|_{X^{s}}, \forall k \in \mathbb{Z}^{d} . \tag{4.22}
\end{align*}
$$

Therefore,

$$
\begin{equation*}
\|T(x)-T(y)\|_{X^{s}} \leq \delta\|x-y\|_{X^{s}}, \forall x, y \in \bar{x}+B(\bar{r}, c, C) \tag{4.23}
\end{equation*}
$$

From (4.19) and (4.23), we can apply the Banach Fixed Point Theorem to conclude that there exists unique $x \in \bar{x}+B(\bar{r}, c, C)$ such that $T(x)=x$, as we wanted.

In what follows we consider $\mu_{k}(\lambda) \in \mathbb{C}, k \in \mathbb{Z}^{d}, \lambda \in \mathbb{R}$. Fix $\lambda$ and take $m \in \mathbb{N}^{d}$ such that $\mu_{k}(\lambda) \neq 0$ if $k \notin F_{m}$. Also, let $\mathscr{F}=\mathscr{F}(\cdot, \lambda)$ as in (4.2) and $\bar{x} \in X_{(m)}^{s}$ be a numerical solution of the truncated problem (4.7). Use these values of $\lambda, m$ and $\bar{x}$ to build the linear operator $A$ like in (4.13).

Next we give explicit formulas for the bounds $Y_{k}, Z_{k}^{(1)}(r)$ and $Z_{k}^{(2)}(r)$ in Theorem 4.1.1 when $T$ is given by

$$
T=T_{\mathscr{F}}=I-A \mathscr{F}=I-A(\bar{x}, \lambda) \mathscr{F}(\cdot, \lambda),
$$

where $I$ is the identity of $X^{s}$ and $\mathscr{F}_{k}(x, \lambda)$ is of the form:

$$
\begin{equation*}
\mathscr{F}_{k}(x, \lambda)=\mu_{k}(\lambda) x_{x}-\left(x^{3}\right)_{k}, x \in X^{s}, k \in \mathbb{Z}^{d} . \tag{4.24}
\end{equation*}
$$

### 4.1.2 Estimate of $Y_{k}$

The definition of $A$ gives

$$
[\mathscr{F}(\bar{x})-\bar{x}]_{k}=[A \mathscr{F}(\bar{x})]_{k}= \begin{cases}\left\{J^{-1} \mathscr{B} J[\mathscr{F}(\bar{x})]_{(m)}\right\}_{k}, & k \in F_{m} \\ \frac{\mathscr{\mathscr { F }}_{k}(\bar{x})}{\mu_{k}(\lambda)}, & k \notin F_{m}\end{cases}
$$

Observe that

$$
\mathscr{F}_{k}(\bar{x})=0, \text { if } k \notin F_{\left(d\left(m_{1}-1\right)+1, \ldots, d\left(m_{d}-1\right)+1\right)} .
$$

Therefore we have the following explicit formulas for the bound $Y_{k}$ :

$$
Y_{k}= \begin{cases}\left\{\left|J^{-1} \mathscr{B} J[\mathscr{F}(\bar{x})]_{(m)}\right|\right\}_{k}, & k \in F_{m},  \tag{4.25}\\ \left|\frac{\mathscr{F}_{k}(\bar{x})}{\mu_{k}(\lambda)}\right|, & k \in F_{\left(d\left(m_{1}-1\right)+1, \ldots, d\left(m_{d}-1\right)+1\right)} \backslash F_{m}, \\ 0, & k \notin F_{\left(d\left(m_{1}-1\right)+1, \ldots, d\left(m_{d}-1\right)+1\right)} .\end{cases}
$$

Observe that all the expressions in (4.25) can be rigorously computed with interval arithmetic.

### 4.1.3 Derivative estimates

In this section we get explicit formulas for the bounds $Z_{k}^{(1)}(r)$ and $Z_{k}^{(2)}(r)$ for cubic non-linearity, that is when $\mathscr{F}_{k}(x, \lambda)$ is of the form (4.24).

We start obtaining the general formula for the derivatives of the operator $T$ in a neighborhood of $\bar{x} \in X_{(m)}^{s}$ as follows:

$$
\begin{align*}
T^{\prime}(\bar{x}+v) w & =I w-(A \mathscr{F})^{\prime}(\bar{x}+v) w=I w-A G_{\mathscr{F}}(\bar{x}+v) w= \\
& =\left(I-A A^{\dagger}\right) w-A\left[G_{\mathscr{F}}(\bar{x}+v)-A^{\dagger}\right] w . \tag{4.26}
\end{align*}
$$

By construction of $A$ and $A^{\dagger}$, we have:

$$
\left[\left(I-A A^{\dagger}\right) w\right]_{k}=\left\{\begin{array}{cl}
{\left[J^{-1}(I-\mathscr{B} \mathbf{B}) J w_{(m)}\right]_{k}} & \text { if } k \in F_{m}  \tag{4.27}\\
0 & \text { if } k \notin F_{m}
\end{array}\right.
$$

If $k \in F_{m}$, we can write:

$$
\begin{align*}
& {\left[\left(G_{\mathscr{F}}(\bar{x}+v)-A^{\dagger}\right) w\right]_{k}=\mathscr{F}_{k}^{\prime}(\bar{x}+v) w-\mathscr{F}_{k}^{\prime}(\bar{x}) w_{(m)}=} \\
& \quad=-3\left(\bar{x}^{2} w^{(m)}\right)_{k}-6(\bar{x} v w)_{k}-3\left(v^{2} w\right)_{k}, k \in F_{m}, \tag{4.28}
\end{align*}
$$

If $k \notin F_{m}$, we can write:

$$
\begin{align*}
& {\left[\left(G_{\mathscr{F}}(\bar{x}+v)-A^{\dagger}\right) w\right]_{k}=\mathscr{F}_{k}^{\prime}(\bar{x}+v) w-\mu_{k}(\lambda) w_{k}=} \\
& \quad=-3\left(\bar{x}^{2} w\right)_{k}-6(\bar{x} v w)_{k}-3\left(v^{2} w\right)_{k}, k \notin F_{m} . \tag{4.29}
\end{align*}
$$

Therefore, defining

$$
\begin{equation*}
\xi(\bar{x}, v, w):=\left[G_{\mathscr{F}}(\bar{x}+v)-A^{\dagger}\right] w \in X^{s}, \tag{4.30}
\end{equation*}
$$

formulas (4.28) and (4.29) give

$$
\begin{equation*}
[\xi(\bar{x}, v, w)]_{k}=-3\left[\bar{x}^{2}\left(\theta_{k} w_{(m)}+w^{(m)}\right)\right]_{k}-6(\bar{x} v w)_{k}-3\left(v^{2} w\right)_{k}, k \in \mathbb{Z}^{2} \tag{4.31}
\end{equation*}
$$

where

$$
\theta_{k}= \begin{cases}0 & \text { if } k \in F_{m} \\ 1 & \text { if } k \notin F_{m}\end{cases}
$$

Putting together the above identities we get:

$$
T_{k}^{\prime}(\bar{x}+v) w= \begin{cases}{\left[J^{-1}(I-\mathscr{B} \mathbf{B}) J w_{(m)}\right]_{k}-\left\{J^{-1} \mathscr{B} J[\xi(\bar{x}, v, w)]_{(m)}\right\}_{k},} & k \in F_{m} \forall v v w \in X^{s} .  \tag{4.32}\\ \frac{1}{\mu_{k}(\lambda)}[\xi(\bar{x}, v, w)]_{k}, & k \notin F_{m}\end{cases}
$$

Before proceeding to the estimates of the bounds $Z_{k}^{(i)}(r), i=1,2$, we go through the following intermediary estimates.

### 4.1.3.1 $\xi$-Estimates

Fix $m, M \in \mathbb{N}^{d}$ such that $F_{m} \subset F_{M}, \bar{x} \in X_{(m)}^{s}, c>0$ and let $C=\left\{c_{k}>0, k \in F_{M} \backslash F_{m}\right\}$ be a set of positive real numbers. For each $r>0$ let $B(r, c, C)$ be as in (4.6). Also, denote by $\omega$ the sequence given by $\left\{\frac{1}{\omega_{k}^{s}}\right\}_{k \in \mathbb{Z}^{d}} \in X^{s}$. Finally, define the sequence $W=\left\{W_{k}\right\}_{k \in \mathbb{Z}^{d}}$, with $W_{k}=0$ if $k \notin F_{M} \backslash F_{m}$ and $W_{k}=c_{k} \omega_{k}$ if $k \in F_{M} \backslash F_{m}$. In this case, if $v, w \in B(r, c, C)$, then we can write:

$$
\begin{equation*}
v=r v_{1}+v_{2}+c v_{3}, w=r w_{1}+w_{2}+c w_{3} \tag{4.33}
\end{equation*}
$$

with

$$
\begin{align*}
& \left|\left(v_{1}\right)_{k}\right|,\left|\left(w_{1}\right)_{k}\right| \leq\left[\omega_{(m)}\right]_{k} \\
& \left|\left(v_{2}\right)_{k}\right|,\left|\left(w_{2}\right)_{k}\right| \leq W_{k} \quad k \in \mathbb{Z}^{d}  \tag{4.34}\\
& \left|\left(v_{3}\right)_{k}\right|,\left|\left(w_{3}\right)_{k}\right| \leq\left[\omega^{(M)}\right]_{k}
\end{align*}
$$

Substituting (4.33) into (4.31) and using the bounds (4.34) we get:

$$
\begin{gather*}
\left|[\xi(\bar{x}, v, w)]_{k}\right| \leq 3\left[\omega_{(m)}^{3}\right]_{k} r^{3}+3\left[2\left(|\bar{x}| \omega_{(m)}^{2}\right)_{k}+3 c\left(\omega_{(m)}^{2} \omega^{(M)}\right)_{k}+3\left(\omega_{(m)}^{2} W\right)_{k}\right]^{2}+ \\
+3\left[\theta_{k}\left(\left|\bar{x}^{2}\right| \omega_{(m)}\right)_{k}+4 c\left(|\bar{x}| \omega_{(m)} \omega^{(M)}\right)_{k}+4\left(|\bar{x}| \omega_{(m)} W\right)_{k}+3 c^{2}\left(\omega_{(m)} \omega^{(M)} \omega^{(M)}\right)_{k}+\right. \\
\left.+6 c\left(\omega_{(m)} \omega^{(M)} W\right)_{k}+3\left(\omega_{(m)} W^{2}\right)_{k}\right] r+ \\
+3\left(W^{3}\right)_{k}+9 c\left(W^{2} \omega^{(M)}\right)_{k}+9 c^{2}\left(W \omega^{(M)} \omega^{(M)}\right)_{k}+3 c^{3}\left(\omega^{(M)} \omega^{(M)} \omega^{(M)}\right)_{k}+ \\
+6\left(|\bar{x}| W^{2}\right)_{k}+12 c\left(|\bar{x}| \omega^{(M)} W\right)_{k}+6 c^{2}\left(|\bar{x}| \omega^{(M)} \omega^{(M)}\right)_{k}+3\left(\left|\bar{x}^{2}\right| W\right)_{k}+ \\
+3 c\left(\left|\bar{x}^{2}\right| \omega^{(M)}\right)_{k}, k \in \mathbb{Z}^{d} . \tag{4.35}
\end{gather*}
$$

Therefore, for $d=1,2$ we can use the bounds $\alpha_{i}^{(d)}, i=0,1,2$, and $\beta_{i}^{(d)}, i=0,1$, constructed in the Appendix of this work, to get:

$$
\begin{equation*}
\left|[\xi(\bar{x}, v, w)]_{k}\right| \leq \frac{\left[\xi_{1}^{(d)}(\bar{x}, r)\right]_{k}}{\omega_{k}^{s}}, k \in \mathbb{Z}^{d} \tag{4.36}
\end{equation*}
$$

where:

$$
\begin{gather*}
{\left[\xi_{1}^{(d)}(\bar{x}, r)\right]_{k}=3 \omega_{k}^{s}\left[\omega_{(m)}^{3}\right]_{k} r^{3}+} \\
+3\left[2 \omega_{k}^{s}\left(|\bar{x}| \omega_{(m)}^{2}\right)_{k}+3 c\left[\alpha_{2}^{(d)}\left(\omega_{(m)}, \omega_{(m)}, M\right)\right]_{k}+3 \omega_{k}^{s}\left(\omega_{(m)}^{2} W\right)_{k}\right] r^{2}+ \\
+3\left[\theta_{k} \omega_{k}^{s}\left(\left|\bar{x}^{2}\right| \omega_{(m)}\right)_{k}+4 c\left[\alpha_{2}^{(d)}\left(|\bar{x}|, \omega_{(m)}, M\right)\right]_{k}+4 \omega_{k}^{s}\left(|\bar{x}| \omega_{(m)} W\right)_{k}+\right. \\
\left.+3 c^{2}\left[\alpha_{1}^{(d)}\left(\omega_{(m)}, M\right)\right]_{k}+6 c\left[\alpha_{2}^{(d)}\left(W, \omega_{(m)}, M\right)\right]_{k}+3 \omega_{k}^{s}\left(\omega_{(m)} W^{2}\right)_{k}\right] r+  \tag{4.37}\\
+3 \omega_{k}^{s}\left(W^{3}\right)_{k}+9 c\left[\alpha_{2}^{(d)}(W, W, M)\right]_{k}+9 c^{2}\left[\alpha_{1}^{(d)}(W, M)\right]_{k}+3 c^{3}\left[\alpha_{0}^{(d)}(M)\right]_{k}+ \\
+6 \omega_{k}^{s}\left(|\bar{x}| W^{2}\right)_{k}+12 c\left[\alpha_{2}^{(d)}(|\bar{x}|, W, M)\right]_{k}+6 c^{2}\left[\alpha_{1}^{(d)}(|\bar{x}|, M)\right]_{k}+3 \omega_{k}^{s}\left(\left|\bar{x}^{2}\right| W\right)_{k}+ \\
+3 c\left[\beta_{1}^{(d)}\left(\left|\bar{x}^{2}\right|, M\right)\right]_{k}, k \in \mathbb{Z}^{d} .
\end{gather*}
$$

Now consider $v \in B(r, c, C)$ and $\|w\|_{s} \leq 1$, that is $\left|w_{k}\right| \leq \frac{1}{\omega_{k}^{s}}, k \in \mathbb{Z}^{d}$. In this case, we can write

$$
\begin{equation*}
v=r v_{1}+v_{2}+c v_{3}, w=w_{(M)}+w^{(M)} \tag{4.38}
\end{equation*}
$$

with

$$
\begin{align*}
& \left|\left(v_{1}\right)_{k}\right| \leq\left[\omega_{(m)}\right]_{k}, k \in \mathbb{Z}^{d}, \\
& \left|\left(v_{2}\right)_{k}\right| \leq W_{k}, k \in \mathbb{Z}^{d}, \\
& \left|\left(v_{3}\right)_{k}\right| \leq\left[\omega^{(M)}\right]_{k}, k \in \mathbb{Z}^{d},  \tag{4.39}\\
& w_{(M)} \leq\left[\omega_{(M)}\right]_{k}, k \in \mathbb{Z}^{d}, \\
& w^{(M)} \leq\left[\omega^{(M)}\right]_{k}, k \in \mathbb{Z}^{d} .
\end{align*}
$$

Substituting (4.38) into (4.31) and using the estimates (4.39) we get:

$$
\begin{gather*}
\left|[\xi(\bar{x}, v, w)]_{k}\right| \leq 3\left[\left(\omega_{(m)} \omega_{(m)} \omega_{(M)}\right)_{k}+\left(\omega_{(m)} \omega_{(m)} \omega^{(M)}\right)_{k}\right] \cdot r^{2}+ \\
+6\left[\left(|\bar{x}| \omega_{(m)} \omega_{(M)}\right)_{k}+\left(|\bar{x}| \omega_{(m)} \omega^{(M)}\right)_{k}+3\left(\omega_{(m)} W \omega_{(M)}\right)_{k}+\right. \\
\left.+3\left(\omega_{(m)} W \omega^{(M)}\right)_{k}+3\left(\omega_{(m)} \omega^{(M)} \omega_{(M)}\right)_{k} \cdot c+3\left(\omega_{(m)} \omega^{(M)} \omega^{(M)}\right)_{k} \cdot c\right] \cdot r+ \\
+3 \theta_{k}\left(\left|\bar{x}^{2}\right| \omega_{(m)}\right)_{k}+3\left(\left|\bar{x}^{2}\right| \omega_{(m)}^{(M)}\right)_{k}+3\left(\left|\bar{x}^{2}\right| \omega^{(M)}\right)_{k}+6\left(|\bar{x}| W \omega_{(M)}\right)_{k}+  \tag{4.40}\\
+6\left(|\bar{x}| W \omega^{(M)}\right)_{k}+6\left(|\bar{x}| \omega^{(M)} \omega_{(M)}\right)_{k} \cdot c+6\left(|\bar{x}| \omega^{(M)} \omega^{(M)}\right)_{k} \cdot c+ \\
+3\left(W^{2} \omega_{(M)}\right)_{k}+3\left(W^{2} \omega^{(M)}\right)_{k}+3\left(\omega^{(M)} \omega^{(M)} \omega_{(M)}\right)_{k} \cdot c^{2}+ \\
+3\left(\omega^{(M)} \omega^{(M)} \omega^{(M)}\right)_{k} \cdot c^{2}+6\left(W \omega^{(M)} \omega_{(M)}\right)_{k} \cdot c+6\left(W \omega^{(M)} \omega^{(M)}\right)_{k} \cdot c
\end{gather*}
$$

Again, for $d=1,2$ we can use the $\alpha^{(d)}$-bounds from the Appendix to get:

$$
\begin{equation*}
\left|[\xi(\bar{x}, v, w)]_{k}\right| \leq \frac{\left[\xi_{2}^{(d)}(\bar{x}, r)\right]_{k}}{\omega_{k}^{s}}, k \in \mathbb{Z}^{d} \tag{4.41}
\end{equation*}
$$

where:

$$
\begin{gather*}
{\left[\xi_{2}^{(d)}(\bar{x}, r)\right]_{k}=3\left[\omega_{k}^{s}\left(\omega_{(m)} \omega_{(m)} \omega_{(M)}\right)_{k}+\left[\alpha_{2}^{(d)}\left(\omega_{(m)}, \omega_{(m)}, M\right)\right]_{k}\right] \cdot r^{2}+} \\
+6\left[\omega_{k}^{s}\left(|\bar{x}| \omega_{(m)} \omega_{(M)}\right)_{k}+\left[\alpha_{2}^{(d)}\left(|\bar{x}|, \omega_{(m)}, M\right)\right]_{k}+3 \omega_{k}^{s}\left(\omega_{(m)} W \omega_{(M)}\right)_{k}+\right. \\
\left.+3\left[\alpha_{2}^{(d)}\left(\omega_{(m)}, W, M\right)\right]_{k}+3\left[\alpha_{2}^{(d)}\left(\omega_{(m)}, \omega_{(M)}, M\right)\right]_{k} \cdot c+3\left[\alpha_{1}^{(d)}\left(\omega_{(m)}, M\right)\right]_{k} \cdot c\right] \cdot r+ \\
+3 \theta_{k} \omega_{k}^{s}\left(\left|\bar{x}^{2}\right| \omega_{(m)}\right)_{k}+3 \omega_{k}^{s}\left(\left|\bar{x}^{2}\right| \omega_{(m)}^{(M)}\right)_{k}+3\left[\beta_{1}^{(d)}\left(\left|\bar{x}^{2}\right|, M\right)\right]_{k}+6 \omega_{k}^{s}\left(|\bar{x}| W \omega_{(M)}\right)_{k}+  \tag{4.42}\\
+6\left[\alpha_{2}^{(d)}(|\bar{x}|, W, M)\right]_{k}+6\left[\alpha_{2}^{(d)}\left(|\bar{x}|, \omega_{(M)}, M\right)\right]_{k} \cdot c+6\left[\alpha_{1}^{(d)}(| | \bar{x} \mid, M)\right]_{k} \cdot c+ \\
\quad+3 \omega_{k}^{s}\left(W^{2} \omega_{(M)}\right)_{k}+3\left[\alpha_{2}^{(d)}(W, W, M)\right]_{k}+3\left[\alpha_{1}^{(d)}\left(\omega_{(M)}, M\right)\right]_{k} \cdot c^{2}+ \\
\quad+3\left[\alpha_{0}^{(d)}(M)\right]_{k} \cdot c^{2}+6\left[\alpha_{2}^{(d)}\left(W, \omega_{(M)}, M\right)\right]_{k} \cdot c+6\left[\alpha_{1}^{(d)}(W, M)\right]_{k} \cdot c
\end{gather*}
$$

Observe that the uniform estimates of the $\alpha^{(d)}$-bounds, $d=1,2$, in the Appendix imply the following uniform estimates for the $\xi$-bounds:

$$
\begin{equation*}
\left[\xi_{i}^{(1)}(\bar{x}, r)\right]_{k}=\left[\xi_{i}^{(1)}(\bar{x}, r)\right]_{3 M} \forall k \geq 3 M, i=1,2 \tag{4.43}
\end{equation*}
$$

$$
\left[\xi_{i}^{(2)}(\bar{x}, r)\right]_{\left(k_{1}, k_{2}\right)}= \begin{cases}{\left[\xi_{i}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, k_{2}\right)}} & \text { if }\left|k_{1}\right| \geq 3 M_{1}, 0 \leq\left|k_{2}\right| \leq 3 M_{2}-1  \tag{4.44}\\ {\left[\xi_{i}^{(2)}(\bar{x}, r)\right]_{\left(k_{1}, 3 M_{2}\right)}} & \text { if }\left|k_{2}\right| \geq 3 M_{2}, 0 \leq\left|k_{1}\right| \leq 3 M_{1}-1, i=1,2 \\ {\left[\xi_{i}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, 3 M_{2}\right)}} & \text { if }\left|k_{1}\right| \geq 3 M_{1},\left|k_{2}\right| \geq 3 M_{2}\end{cases}
$$

Now we are finally ready to write explicit formulas for the bounds $Z_{k}^{(i)}(r), i=1,2$.

### 4.1.3.2 Estimate of $Z_{k}^{(1)}(r)$

From (4.27), (4.33) and (4.34) we obtain:

$$
\left|\left[\left(I-A A^{\dagger}\right) w\right]_{k}\right|=\left\{\begin{array}{cl}
r \mid J^{-1}\left[(I-\mathscr{B} B) J w_{1} \mid \leq r\left[J^{-1}|I-\mathscr{B} B| J \omega_{(m)}\right]_{k}\right. & \text { if } k \in F_{m}  \tag{4.45}\\
0 & \text { if } k \notin F_{m}
\end{array}\right.
$$

Therefore, from (4.32), (4.36), (4.43), (4.44) and (4.45) we obtain the following:
In the one-dimensional case:

$$
Z_{k}^{(1)}(r)=\left\{\begin{array}{cl}
r\left[J^{-1}|I-\mathscr{B} B| J \omega_{(m)}\right]_{k}+\left\{J^{-1}|\mathscr{B}| J\left[\xi_{1}^{(1)}(\bar{x}, r)\right]_{(m)}\right\}_{k} & \text { if } k \in F_{m}  \tag{4.46}\\
\frac{1}{\mu_{k}(\lambda)} \frac{\left[\xi_{1}^{(1)}(\bar{x}, r)\right]_{k}}{\omega_{k}^{s}} & \text { if } k \in F_{3 M} \backslash F_{m} \\
\frac{1}{\mu_{k}(\lambda)} \frac{\left[\xi_{1}^{(1)}(\bar{x}, r)_{3 M}\right.}{\omega_{k}^{5}} & \text { if }|k| \geq 3 M .
\end{array}\right.
$$

In the two-dimensional case:

$$
Z_{k}^{(1)}(r)=\left\{\begin{array}{cl}
r\left[J^{-1}|I-\mathscr{B} B| J \omega_{(m)}\right]_{k}+\left\{J^{-1}|\mathscr{B}| J\left[\xi_{1}^{(2)}(\bar{x}, r)\right]_{(m)}\right\}_{k} & \text { if } k \in F_{m}  \tag{4.47}\\
\frac{1}{\mu_{k}(\lambda)}\left[\xi_{1}^{(2)}(\bar{x}, r)\right]_{k} & \omega_{k}^{s} \\
\frac{1}{\mu_{k}(\lambda)} \frac{\left[\xi_{1}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, k_{2}\right)}}{\omega_{k}} & \text { if }\left|k_{1}\right| \geq 3 M_{M}, 0 \leq\left|k_{2}\right|<3 M_{2} \\
\frac{1}{\mu_{k}(\lambda)} \frac{\left[\xi_{1}^{(2)}(\bar{x}, r)\right]_{\left(k_{1}, 3 M_{2}\right)}}{\omega_{k}^{s}} & \text { if }\left|k_{2}\right| \geq 3 M_{2}, 0 \leq\left|k_{1}\right|<3 M_{1} \\
\frac{1}{\mu_{k}(\lambda)} \frac{\left[\xi_{1}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, 3 M_{2}\right)}}{\omega_{k}^{s}} & \text { if }\left|k_{1}\right| \geq 3 M_{1},\left|k_{2}\right| \geq 3 M_{2} .
\end{array}\right.
$$

4.1.3.3 Estimate of $Z_{k}^{(2)}(r)$

From (4.27), (4.38) and (4.39) we obtain:

$$
\left|\left[\left(I-A A^{\dagger}\right) w\right]_{k}\right|=\left\{\begin{array}{cl}
\mid J^{-1}\left[(I-\mathscr{B} B) J w_{1} \mid \leq\left[J^{-1}|I-\mathscr{B} B| J \omega_{(m)}\right]_{k}\right. & \text { if } k \in F_{m}  \tag{4.48}\\
0 & \text { if } k \notin F_{m}
\end{array}\right.
$$

Therefore, from (4.32), (4.41), (4.43), (4.44) and (4.48) we obtain the following:
In the one-dimensional case:

$$
Z_{k}^{(2)}(r)=\left\{\begin{array}{cl}
{\left[J^{-1}|I-\mathscr{B} B| J \omega_{(m)}\right]_{k}+\left\{J^{-1}|\mathscr{B}| J\left[\xi_{2}^{(1)}(\bar{x}, r)\right]_{(m)}\right\}_{k}} & \text { if } k \in F_{m}  \tag{4.49}\\
\frac{1}{\mu_{k}(\lambda) \frac{\left[\xi_{2}^{(1)}(\bar{x}, r)\right]_{k}}{\omega_{k}^{k}}} & \text { if } k \in F_{3 M} \backslash F_{m} \\
\frac{1}{\mu_{k}(\lambda)} \frac{\left[\xi_{2}^{(1)}(\bar{x}, r)\right]_{3 M}}{\omega_{k}^{s}} & \text { if }|k| \geq 3 M
\end{array}\right.
$$

In the two-dimensional case:

$$
Z_{k}^{(2)}(r)=\left\{\begin{array}{cl}
{\left[J^{-1}|I-\mathscr{B} B| J \omega_{(m)}\right]_{k}+\left\{J^{-1}|\mathscr{B}| J\left[\xi_{2}^{(2)}(\bar{x}, r)\right]_{(m)}\right\}_{k}} & \text { if } k \in F_{m}  \tag{4.50}\\
\frac{1}{\mu_{k}(\lambda)} & \text { if } k \in F_{M} \backslash F_{m} \\
\left.\frac{1}{\mu_{k}(\lambda)}(\bar{x} r)\right]_{k} & \frac{\left[\xi_{2}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, k_{2}\right)}^{s}}{\omega_{k}^{s}} \\
\frac{1}{\mu_{k}(\lambda)} \frac{\left[\xi_{2}^{(2)}(\bar{x}, r)\right]_{\left(k_{1}, 3 M_{2}\right)}^{\left(\omega_{k}\right.}}{\omega_{k}^{s}} & \text { if }\left|k_{1}\right| \geq 3 M_{1}, 0 \leq\left|k_{2}\right|<3 M_{2} \\
\frac{1}{\mu_{k}(\lambda)} \frac{\left[\xi_{2}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, 3 M_{2}\right)}^{\omega_{k}^{s}}}{\omega_{k}^{s}} & \text { if }\left|k_{2}\right| \geq 3 M_{2}, 0 \leq\left|k_{1}\right|<3 M_{1} \\
& \text { if }\left|k_{1}\right| \geq 3 M_{1},\left|k_{2}\right| \geq 3 M_{2} .
\end{array}\right.
$$

### 4.1.4 Local uniqueness theorem for cubic non-linearity

Now we apply Theorem 4.1.1 for cubic non-linearity in dimension $d=1$ and $d=2$. To be precise let $\mathscr{F}_{k}(a, \lambda)=\mu_{k}(\lambda) a_{k}-\left(a^{3}\right)_{k} \in \mathbb{C}$ with, $\lambda \in \mathbb{R}, k \in \mathbb{Z}^{d}, a \in X^{s}, \mu_{k}(\lambda) \in \mathbb{C}$. Suppose that there exist $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}^{d}$ such that $\mu_{k}(\lambda) \neq 0$ if $k \notin F_{m}$. Given $\bar{x} \in X_{(m)}^{s}$, define
the Newton-like operator $T=T_{\mathscr{F}}=I-A(\bar{x}, \lambda) \mathscr{F}(\cdot, \lambda)$, where $\mathscr{F}(\cdot, \lambda)=\left\{\mathscr{F}_{k}(\cdot, \lambda)\right\}_{k \in \mathbb{Z}^{d}}$ and $A(\bar{x}, \lambda)$ is as given by (4.13). Let be given $M \in \mathbb{N}^{d}$ such that $F_{m} \subset F_{M}$, real numbers $\delta>0, c>0$ and a set of positive real numbers $C=\left\{c_{k}>0, k \in F_{M} \backslash F_{m}\right\}$. For each $r>0$ construct the box $B(r, c, C)$ as in (4.6) and the bounds $Y_{k}$, given by the formula in (4.25), $Z_{k}^{(i)}(r), i=1,2, k \in \mathbb{Z}^{2}$, given by the formulas (4.46) and (4.49), respectively, if $d=1$, and by the formulas (4.47) and (4.50), if $d=2$.

Lastly, if $d=2$ make the following definitions:

$$
\begin{aligned}
& \zeta_{M_{2}}\left(k_{1}, \lambda\right)=\min _{\left|k_{2}\right| \geq 3 M_{2}}\left|\mu_{\left(k_{1}, k_{2}\right)}(\lambda)\right|,\left|k_{1}\right|<M_{1} \\
& \zeta_{M_{1}}\left(k_{2}, \lambda\right)=\min _{\left|k_{1}\right| \geq 3 M_{1}}\left|\mu_{\left(k_{1}, k_{2}\right)}(\lambda)\right|,\left|k_{2}\right|<M_{2} \\
& \zeta_{M}(\lambda)=\min _{\substack{\left|k_{1} \geq 3 M_{1}\\
\right| k_{2} \mid \geq 3 M_{2}}}\left|\mu_{\left(k_{1}, k_{2}\right)}(\lambda)\right| .
\end{aligned}
$$

If $d=1$ we just define:

$$
\zeta_{M}(\lambda)=\min _{|k| \geq 3 M}\left|\mu_{k}(\lambda)\right|
$$

In this case, if $d=2$ we have the following.
Corollary 4.1.1 (Of Theorem 4.1.1). Suppose that there exists $0<r<1$ satisfying the following finite set of inequalities:

$$
\begin{align*}
& Y_{k}+Z_{k}^{(1)}(r) \leq \begin{cases}\frac{r}{\omega_{k}^{s}} & \forall k \in F_{m} \\
\frac{c k}{\omega_{k}^{s}} & \forall k \in F_{M} \backslash F_{m} \\
\frac{c}{\omega_{k}^{s}} & \forall k \in F_{3 M} \backslash F_{M}\end{cases}  \tag{4.51}\\
& Z_{k}^{(2)}(r) \leq \frac{\delta}{\omega_{k}^{s}} \forall k \in F_{3 M},  \tag{4.52}\\
& \frac{\left[\xi_{1}^{(2)}(\bar{x}, r)\right]_{\left(k_{1}, 3 M_{2}\right)}}{\zeta_{M_{2}}\left(k_{1}, \lambda\right)} \leq c,\left|k_{1}\right| \leq 3 M_{1}-1 \\
& \frac{\left[\xi_{1}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, k_{2}\right)}}{\zeta_{M_{1}}\left(k_{2}, \lambda\right)} \leq c,\left|k_{2}\right| \leq 3 M_{2}-1  \tag{4.53}\\
& \frac{\left[\xi_{1}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, 3 M_{2}\right)}}{\zeta_{M}(\lambda)} \leq c \\
& \frac{{ }_{\left[\xi_{2}^{(2)}(\bar{x}, r)\right]_{\left(k_{1}, 3 M_{2}\right)}}^{\zeta_{M_{2}( }\left(M_{2}, \lambda\right)} \leq \delta,\left|k_{1}\right| \leq 3 M_{1}-1}{} \\
& \frac{\left[\xi_{2}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, k_{2}\right)}}{\zeta_{M_{1}}\left(k_{2}, \lambda\right)} \leq \delta,\left|k_{2}\right| \leq 3 M_{2}-1  \tag{4.54}\\
& \frac{\left[\xi_{2}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, 3 M_{2}\right)}}{\zeta_{M}(\lambda)} \leq \delta
\end{align*}
$$

Then there exists a unique $x \in \bar{x}+B(r, c, C)$ such that $\mathscr{F}(x, \lambda)=0$.

Proof. Let $k=\left(k_{1}, k_{2}\right) \notin F_{3 M}$. Suppose that $k_{1} \geq 3 M_{1}$. In this case, since $k_{1} \geq 3 M_{1}>3 m_{1}-2$ we have $Y_{k}=0$. This and (4.61) give:

$$
\begin{align*}
& Y_{k}+Z_{k}^{(1)}(r)=Z_{k}^{(1)}(r)=\frac{1}{\mu_{k}(\lambda)} \frac{\left[\xi_{1}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, k_{2}\right)}}{\omega_{k}^{s}} \leq \\
&  \tag{4.55}\\
& \leq \frac{1}{\zeta_{M_{1}}\left(k_{2}, \lambda\right)} \frac{\left[\xi_{1}^{(2)}(\bar{x}, r)\right]_{\left(3 M_{1}, k_{2}\right)}}{\omega_{k}^{s}} \leq \frac{c}{\omega_{k}^{s}} \forall k \notin F_{3 M} .
\end{align*}
$$

Similarly we get

$$
\begin{equation*}
Z_{k}^{(2)}(r) \leq \frac{\delta}{\omega_{k}^{s}} \forall k \notin F_{3 M} . \tag{4.56}
\end{equation*}
$$

Now observe that, (4.58), (4.59), (4.55), (4.56) fulfill the hypothesis of Theorem 4.1.1, what proves the existence of a unique fixed point of $T$ in $\bar{x}+B(r, c, C)$.

To see that fixed points of $T$ correspond to zeros of $\mathscr{F}$ we calculate as follows:

$$
\begin{gather*}
Y_{k}+Z_{k}^{(1)}(r) \leq \frac{r}{\omega_{k}^{s}} \forall k \in F_{m} \Rightarrow\left|\left[\left(I-A A^{\dagger}\right) w\right]_{k}\right| \leq \frac{r}{\omega_{k}^{s}} \forall\|w\|_{s} \leq 1 \forall k \in \mathbb{Z}^{2} \Rightarrow \\
\Rightarrow\left\|I-A A^{\dagger}\right\|_{\mathscr{L}\left(X^{s}, X^{s}\right)} \leq r<1 \Rightarrow A A^{\dagger} \text { is invertible } \Rightarrow A \text { is injective. } \tag{4.57}
\end{gather*}
$$

Therefore,

$$
T(x)=x \Rightarrow x-A \mathscr{F}(x)=x \Rightarrow A \mathscr{F}(x)=0 \Rightarrow \mathscr{F}(x)=0(A \text { is injective }) .
$$

On the other hand, it is easy to see that if $\mathscr{F}(x)=0$ then $T(x)=x$.

If $d=1$ we have the following.
Corollary 4.1.2. Suppose that there exists $0<r<1$ satisfying the following finite set of inequalities:

$$
\begin{gather*}
Y_{k}+Z_{k}^{(1)}(r) \leq\left\{\begin{array}{cl}
\frac{r}{\omega_{k}^{5}} & \forall k \in F_{m} \\
\frac{c_{k}^{s}}{\omega_{k}^{s}} & \forall k \in F_{M} \backslash F_{m} \\
\frac{c}{\omega_{k}^{5}} & \forall k \in F_{3 M} \backslash F_{M}
\end{array}\right.  \tag{4.58}\\
Z_{k}^{(2)}(r) \leq \frac{\delta}{\omega_{k}^{s}} \forall k \in F_{3 M},  \tag{4.59}\\
\frac{\left[\xi_{1}^{(1)}(\bar{x}, r)\right]_{3 M}}{\zeta_{M}(\lambda)} \leq c  \tag{4.60}\\
\frac{\left[\xi_{2}^{(1)}(\bar{x}, r)\right]_{3 M}}{\zeta_{M}(\lambda)} \leq \delta \tag{4.61}
\end{gather*}
$$

then there exists a unique $x \in \bar{x}+B(r, c, C)$ such that $\mathscr{F}(x, \lambda)=0$.

### 4.2 Non-existence

Let $m \in \mathbb{N}^{2}$ and remember that $\omega=\left\{\frac{1}{\omega_{k}^{s}}\right\}_{k \in \mathbb{Z}^{2}} \in X^{s}$ and $\omega=\omega_{(m)}+\omega^{(m)}$, where $\omega^{(m)}$ is given component wise by $\left[\omega_{(m)}\right]_{k}=\left\{\begin{array}{ll}\omega_{k}, & \text { if } k \in F_{m} \\ 0, & \text { if } k \notin F_{m}\end{array}\right.$.

Given real numbers $b_{k}^{1}<b_{k}^{2}, k \in F_{m}, c_{k}>0, k \in F_{M} \backslash F_{m}$ and $c>0$, define the following subset of $X^{s}$ :

$$
\begin{equation*}
B=\prod_{k \in F_{m}}\left[b_{k}^{1}, b_{k}^{2}\right] \times \prod_{k \in F_{M} \backslash F_{m}}\left[-c_{k}, c_{k}\right] \times \prod_{k \notin F_{m}}\left[-\frac{c}{\omega_{k}^{s}}, \frac{c}{\omega_{k}^{s}}\right] \subset X^{s} . \tag{4.62}
\end{equation*}
$$

A subset of this form is called a box in $X^{s}$. The first factor is called the main part of the box, the second one is called the middle part and the last one is called the tail of the box.

If $a \in B$, one can write

$$
\begin{equation*}
a=a_{(m)}+a^{(m)}=a_{(m)}+a_{m}^{M}+a^{(M)} \tag{4.63}
\end{equation*}
$$

where

$$
\begin{gather*}
\left|\left(a^{(M)}\right)_{k}\right| \leq\left(\omega^{(M)}\right)_{k}, k \in \mathbb{Z}^{d} \\
\left(a_{m}^{M}\right)_{k} \leq W_{k}, k \in \mathbb{Z}^{d} \tag{4.64}
\end{gather*}
$$

where $W=\left\{W_{k}\right\}_{k \in \mathbb{Z}^{d}}$ with

$$
W_{k}=\left\{\begin{array}{cl}
\frac{c_{k}}{\omega_{k}^{s}} & \text { if } k \in F_{M} \backslash F_{m}  \tag{4.65}\\
0 & \text { if } k \notin F_{M} \backslash F_{m}
\end{array}\right.
$$

For $k \in F_{m}$, the identity (4.63) gives

$$
\begin{gather*}
\mathscr{F}_{k}(a, \lambda)=\mu_{k}(\lambda) a_{k}-\left(a^{3}\right)_{k}=\mu_{k}(\lambda) a_{k}-\left[\left(a_{(m)}+a_{m}^{M}+a^{(M)}\right)^{3}\right]_{k}= \\
=\mu_{k}(\lambda) a_{k}-\left[a_{(m)} a_{(m)} a_{(m)}\right]_{k}-\left[a_{m}^{M} a_{m}^{M} a_{m}^{M}\right]_{k}-\left[a^{(M)} a^{(M)} a^{(M)}\right]_{k}- \\
-3\left[a_{(m)} a_{(m m} a_{m}^{M}\right]_{k}-3\left[a_{(m)} a_{m}^{M} a_{m}^{M}\right]_{k}-3\left[a_{(m)} a_{(m)} a^{(M)}\right]_{k}-  \tag{4.66}\\
-3\left[a_{m}^{M} a_{m}^{M} a^{(M)}\right]_{k}-6\left[a_{(m)} a_{m}^{M} a^{(M)}\right]_{k}-3\left[a_{(m)} a^{(M)} a^{(M)}\right]_{k}-3\left[a_{m}^{M} a^{(M)} a^{(M)}\right]_{k}=: \\
=: \mathscr{F}_{k}\left(a_{(m)}, \lambda\right)+R(a, k),
\end{gather*}
$$

Define $b=\left\{b_{k}\right\}_{k \in \mathbb{Z}^{d}}$, with

$$
b_{k}=\left\{\begin{array}{cc}
\max \left\{\left|b_{k}^{1}\right|,\left|b_{k}^{2}\right|\right\} & \text { if } k \in F_{m}  \tag{4.67}\\
0 & \text { if } k \notin F_{m}
\end{array}\right.
$$

Applying the bounds given by (4.64) and (4.67) in the formula (4.66) gives the following estimate

$$
\begin{gather*}
|R(a, k)| \leq\left[\omega^{(M)} \omega^{(M)} \omega^{(M)}\right]_{k} c^{3}+3\left[b \omega^{(M)} \omega^{(M)}\right]_{k} c^{2}+3\left[W \omega^{(M)} \omega^{(M)}\right]_{k} c^{2}+ \\
+3\left[b^{2} \omega^{(M)}\right]_{k} c+3\left[W^{2} \omega^{(M)}\right]_{k} c+6\left[b W \omega^{(M)}\right]_{k} c+  \tag{4.68}\\
+\left(W^{3}\right)_{k}+3\left[b^{2} W\right]_{k}+3\left[b W^{2}\right]_{k}
\end{gather*}
$$

From the estimates in the Appendix we can write:

$$
\begin{align*}
& |R(a, k)| \leq \frac{\left[\alpha_{0}^{(d)}(M)\right]_{k}}{\omega_{k}^{s}} c^{3}+3 \frac{\left[\alpha_{1}^{(d)}(b, M)\right]_{k}}{\omega_{k}^{s}} c^{2}+3 \frac{\left[\alpha_{1}^{(d)}(W, M)\right]_{k}}{\omega_{k}^{s}} c^{2}+ \\
& \quad+3 \frac{\left[\alpha_{2}^{(d)}(b, b, M)\right]_{k}}{\omega_{k}^{s}} c+3 \frac{\left[\alpha_{2}^{(d)}(W, W, M)\right]_{k}}{\omega_{k}^{s}} c+3 \frac{\left[\alpha_{2}^{(d)}(b, W, M)\right]_{k}}{\omega_{k}^{s}} c+  \tag{4.69}\\
& \quad+\left(W^{3}\right)_{k}+3\left[b^{2} W\right]_{k}+3\left[b W^{2}\right]_{k}=: \mathscr{E}_{k}(B, c)
\end{align*}
$$

The above computations can be summarized in the following.
Theorem 4.2.1. Let $\mathscr{F}(a, \lambda)=\left\{\mathscr{F}_{k}(a, \lambda)\right\}_{k \in \mathbb{Z}^{2}}=\left\{\mu_{k}(\lambda) a_{k}-\left(a^{3}\right)_{k}\right\}_{k \in \mathbb{Z}^{2}} \in \mathbb{R}^{\mathbb{Z}^{2}}, a \in X^{s}$, and let $B$ and $\mathscr{E}_{k}(B, c)$ be as defined above. Define the finite sequence of intervals $\tilde{b}=\left\{\left[b_{k}^{1}, b_{k}^{2}\right]\right\}_{k \in F_{m}}$. Then

$$
\mathscr{F}_{k}(B, \lambda) \subset\left[-\mathscr{E}_{k}(B, c), \mathscr{E}_{k}(B, c)\right]+\mathscr{F}_{k}(\tilde{b}, \lambda)=: I(B, \lambda) .
$$

In particular, if

$$
\inf \left\{\mathscr{F}_{k}(\tilde{b}, \lambda)\right\}-\mathscr{E}_{k}(B, c)>0 \text { for some } k \in F_{m}
$$

then $0 \notin \mathscr{F}_{k}(B, \lambda)$ and the problem $\mathscr{F}(\cdot, \lambda)=0$ has no solutions in the box $B$.
Remark 4.2.1. Theorem 4.2 .1 provides a test to verify the non existence of solutions in a box of the form (4.62). It says that the image of the box $B$ by the real valued function $F_{k}(\cdot, \lambda)$ is contained in the real interval $I(B, \lambda)$. It is important to notice that by using interval arithmetic the interval $I(B, \lambda)$ can be rigorously calculated. In practice, if the test given by Theorem 4.2.1 is not successful in the box $B$, we split it into two smaller boxes, and we check each of the new boxes. We discard the sub-boxes where the test is successful and subdivide the others. We repeat this process until the box $B$ is fully exhausted. Observe that the smaller the box $B$, the closer to the actual image $\mathscr{F}_{k}(B, \lambda)$ is the interval $\mathscr{F}_{k}(\tilde{b}, \lambda)$ calculated with interval arithmetcs. Also, the bigger the projection $m$, the smaller the truncation error $\mathscr{E}_{k}(B, c)$. This idea, already used in (DAY et al., 2005), provides an algorithm for proving the non-existence of solutions inside a box in the form of $B$. This algorithm is encoded in the MATLAB function ExhaustingBox.m.

Next we explain how to use Theorems 4.1.1 and 4.2.1 to obtain the exact multiplicity of solutions of problem (4.4).

Once again, consider problem (4.4) with a fixed parameter value $\lambda$. Let be given $M \in \mathbb{N}^{d}$, positive real numbers $c>0, c_{k}>0, k \in F_{M}$, and define

$$
B=\prod_{k \in F_{M}}\left[-\frac{c_{k}}{\omega_{k}^{s}}, \frac{c_{k}}{\omega_{k}^{s}}\right] \times \prod_{k \notin F_{M}}\left[-\frac{c}{\omega_{k}^{s}}, \frac{c}{\omega_{k}^{s}}\right] .
$$

Our proofs in the next section consist in applying the following steps:
Step 1 Find numerical solutions $x_{1}, \ldots, x_{q} \in B, q \in \mathbb{N}$, of problem (4.4) using Newton method.

Step 2 Find positive real numbers $r_{1}, \ldots, r_{q}$ such that the equation in (4.4) possesses exactly one solution in each of the boxes:

$$
B_{i}=x_{i}+\prod_{k \in F_{m}}\left[-\frac{r_{i}}{\omega_{k}^{s}}, \frac{r_{i}}{\omega_{k}^{s}}\right] \times \prod_{k \in F_{M} \backslash F_{m}}\left[-\frac{c_{k}}{\omega_{k}^{s}}, \frac{c_{k}}{\omega_{k}^{s}}\right] \times \prod_{k \notin F_{M}}\left[-\frac{c}{\omega_{k}^{s}}, \frac{c}{\omega_{k}^{s}}\right] \subset B, i=1, \ldots, q .
$$

Step 3 Check that the remaining region $R:=B-\cup_{i=1}^{q} B_{i}$ has no solutions.

Step 3.1 We start decomposing this remaining region $R$ into boxes; more precisely, we need to construct boxes $R_{i}=\prod_{k \in F_{m}}\left[\frac{b_{i, k}^{1}}{\omega_{k}^{5}}, \frac{b_{i, k}^{2}}{\omega_{k}^{5}}\right] \times \prod_{k \in F_{M} \backslash F_{m}}\left[-\frac{c_{k}}{\omega_{k}^{5}}, \frac{c_{k}}{\omega_{k}^{5}}\right] \times \prod_{k \notin F_{m}}\left[-\frac{c}{\omega_{k}^{5}}, \frac{c}{\omega_{k}^{5}}\right], i=1, \ldots, n$, for some $n \in \mathbb{N}$, such that $R=B-\cup_{i=1}^{q} B_{i}=\cup_{i=1}^{n} R_{i}$. This decomposition is made by the algorithm encoded in the MATLAB function PROCESSING.m presented in the second section of the Appendix. The next figure shows an example of this decomposition in the case of a twodimensional box containing three sub-boxes.


Figure 1 - Decomposition of the region $[0,4] \times[0,4]-\{[0.5,3] \times[0.5,1] \cup[1.25,3] \times[2,3.5] \cup[1,1.5] \times[0.75,3]\}$ given by the function PROCESSING.m

Step 3.2 Check the non-existence of solutions in each box $R_{i}$ using the test given by Theorem 4.2.1 and the splitting strategy of Remark 4.2.1. The main algorithm to perform this step is encoded in the MATLAB function ExhaustingBox.m presented in the second section of the Appendix.

## RESULTS AND FINAL CONSIDERATIONS

### 5.1 Results

Next we present some results obtained by the application of the method described in the last chapter to the sample problems presented in chapter 3. In what follows, two functions $u$ and $v$ are called symmetric to each other if $u=-v$.

Theorem 5.1.1 (Two-dimensional Swift-Hoenberg equation). For $\lambda=0.001, L_{1}=1$, and $L_{2}=4$, problem (3.5) has exactly three solutions: the null solution and a pair of symmetric (non-constant) solutions whose graphics of its numerical approximations are presented in Figure 3.

The proof was performed using $m=(8,2), M=(8,8)$. In Figure 2 the blue boxes are two-dimensional projections of the "uniqueness boxes" obtained by the Radii Polynomial approach. Excluding these boxes from the existence box and splitting this remaining region into sub-boxes we get 247 16-dimensional boxes whose two-dimensional projections are plotted in red in the Figure 2. These 247 boxes were partitioned into 2416 other sub-boxes in which the non existence test was successful.

Theorem 5.1.2 (One-dimensional Swift-Hoenberg equation). For $\lambda \in \mathbb{R}_{+}$, consider the problem

$$
\begin{equation*}
L(u, \lambda)=u^{3}, \text { in } \Omega, u \in D \tag{5.1}
\end{equation*}
$$

where

$$
\begin{equation*}
L(u, \lambda)=\lambda u-(1+\Delta)^{2} u \tag{5.2}
\end{equation*}
$$

and

$$
\begin{equation*}
D=\{u \in C ; u(x+l)=u(x)=u(|x|), x \in \mathbb{R}\}, \tag{5.3}
\end{equation*}
$$

where $C$ is the set of all functions $u: \mathbb{R} \longmapsto \mathbb{R}$ such that the partial derivative $(\Delta u)_{x x}=u_{x x x x}$ exists in the classical sense over $\mathbb{R}$ and its restriction to $\Omega$ is square-integrable, that is,

$$
u_{x x x x}, \in L^{2}(\Omega) .
$$



Figure 2 - Projection into the first two dimensions of the existence boxes, in blue, and the partitioned remaining region, in red, for $\lambda=0.001, L_{1}=1$ and $L_{2}=4$.


Figure 3 - The pair of symmetric solutions of problem (3.5) for $\lambda=0.001, L_{1}=1$ and $L_{2}=4$.

1. If $l=1.1$ and $\lambda=1.4$ then the problem given by (5.1), (5.2) and (5.3) has exactly five solutions: three constant solutions and one pair of symmetric non-constant solutions, whose graphics are plotted in Figure 5.


Figure 4 - Projection into the first two dimensions of the existence boxes, in blue, and the partitioned remaining region, in red, for $\lambda=1.4, l=2 \pi / 1.1$
2. If $l=2 \pi / 1.1$ and $\lambda=1.6$ then the problem given by (5.1), (5.2) and (5.3) has exactly nine


Figure 5 - The five solutions of problem (5.1) for $\lambda=1.4$ and $l=2 \pi / 1.1$
solutions. The same multiplicity holds for $\lambda=2$. In both cases we have three constant solutions and three pairs of symmetric (non-constant) solutions whose graphics of its numerical approximations are plotted in Figure 7.


Figure 6 - Projection into the first two dimensions of the existence boxes, in blue, and the partitioned remaining region, in red, for $\lambda=2, l=2 \pi / 1.1$


Figure 7 - The pairs of symmetric solutions of problem (5.1) for $\lambda=2$ and $l=2 \pi / 1.1$

The proof the first item was performed using $m=6, M=40$. In Figure 4 the blue boxes are two-dimensional projections of the "uniqueness boxes" obtained by the Radii Polynomial approach. Excluding these boxes from the "existence box" and splitting this remaining region into sub-boxes we get 836 -dimensional boxes whose two-dimensional projections are plotted in red in the Figure 4 . These 83 boxes were partitioned into 12323 other sub-boxes in which the non existence test was successful after about 59 minutes.

The proof the second item was performed using $m=5, M=40$ in both parameters cases. For $\lambda=2$, in Figure 6 the blue boxes are two-dimensional projections of the "uniqueness boxes" obtained by the Radii Polynomial approach. Excluding these boxes from the "existence box" and splitting this remaining region into sub-boxes we get 1345 -dimensional boxes whose two-dimensional projections are plotted in red in the Figure 6. These 134 boxes were partitioned into 12038 other sub-boxes in which the non existence test was successful after about 51 minutes.

Theorem 5.1.3 (One-dimensional Cahn-Hilliard equation). Problem (3.28) with $\varepsilon=0.4$ and $l=2$ has exactly five solutions, three constant solutions and one pair of symmetric solutions whose graphics of its numerical approximations are presented in Figure 9. If $\varepsilon=0.6$ and $l=2$ problem (3.28) has no non-constant solutions and it has exactly three constant solutions, $u=0$, $u=1$ and $u=-1$.

For $\varepsilon=0.4$ the proof were performed using $m=6, M=60$. In Figure 8 the blue boxes are two-dimensional projections of the "uniqueness boxes" obtained by the Radii Polynomial approach. Excluding these boxes from the "existence box" and splitting this remaining region into sub-boxes we get 816 -dimensional boxes whose two-dimensional projections are plotted in red in the Figure 8. These 81 boxes was partitioned into 8550 other sub-boxes in which the non existence test was successful after about 35 minutes.


Figure 8 - Projection into the first two dimensions of the existence boxes, in blue, and the partitioned remaining region, in red, for $\varepsilon=0.4, l=\pi$

### 5.2 Final considerations

This work provides a method formulated from the combination of elementary mathematical tools and the careful use of the computer to produce rigorous results on existence, non-existence and exact multiplicity of solutions for some differential equations. More precisely, those that have an algebraic reformulation of type (4.4).

Briefly, the method proposed here start by bounding the Fourier coefficients of a solution of the differential equation via energy estimates in the equation itself. This produces


Figure 9 - The five solutions of problem (3.28) for $l=p i$ and $\varepsilon=0.4$
the $C_{k}$ and $C$ constants that define the algebraic problem (4.4) to be solved. Next, we obtain as many numerical solutions as possible to the (4.4) problem and we ensure the existence and uniqueness of an exact solution around each of the numerical solutions. For this, we apply the method of radial polynomials, already described in many works, such as (LESSARD; JAMES; REINHARDT, 2014), (HUNGRIA; LESSARD; JAMES, 2016), (LESSARD; REINHARDT, 2014), (GAMEIRO; LESSARD, 2011), (GAMEIRO; LESSARD, 2013), (LESSARD; JAMES; RANSFORD, 2016) among others, which basically consists of associating a Newton-like operator whose construction is made in the subsection 4.1.1, to the problem given by (4.4) and of obtaining computationally verifiable conditions given by Theorem 4.1.1, so that this operator is a contraction in the neighborhood of a numerical solution. This stage triggers a series of convolution estimates, presented in the appendix of this paper, in order to obtain, among others, estimates for the derivatives of the fixed-point operator in the regions where it is desired to prove the occurrence of the contraction.

The method of the radial polynomials provides regions of existence and uniqueness of solutions of the problem (4.4). We exclude these regions from the region of existence of the solutions obtained in the first step of the method. We split the remaining region of this process into boxes in which we apply a non-existence test presented in Theorem 4.2.1, which basically consists of estimating an interval (via interval arithmetic) containing the range of these boxes by one of the component functions that define the operator of the problem (4.4), then verifying that these intervals are strictly to the right of zero.

The larger the diameter of the box tested, the coarser the estimation of its image, what makes necessary the successive splitting in smaller sub-boxes in which the test is reapplied. In addition to the box diameter, the accuracy of its range estimation depends explicitly on the constant $C$, determined by the parameters and the domain of the differential equation, which limits the application of the method to certain parameter values and domains. Furthermore, some domains produce equations whose solutions are close to each other, so that the regions of uniqueness and existence obtained by the method of the radial polynomials, which isolate the solutions from each other, are also small. This makes the remaining region referred to in the previous paragraph contain points close to the solutions of the problem, which makes difficult
verifying the non existence of solutions in the boxes that contains such points.
The method was successful in obtaining the exact multiplicity of equilibria of the one and two-dimensional Swift-Hohenberg equation and of one-dimensional Chan-Hilliard equation. Since the determination of the equilibria is of fundamental importance for the study of the dynamics of the system, we believe that this work contributes to the validation of a method for relevant studies in differential equations.

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## APPENDIX

A

## CONVOLUTION ESTIMATES

## A. 1 One-dimensional estimates

Throughout this work, for $d \in \mathbb{N}, m \in \mathbb{N}^{d}$, and $s \in \mathbb{R}^{d}$ we have adopted the notation

$$
\omega=\left\{\frac{1}{\omega_{k}^{s}}\right\}_{k \in \mathbb{Z}^{d}}, \omega=\omega_{(m)}+\omega^{(m)}, m \in \mathbb{N}^{d}
$$

where

$$
\left[\omega_{(m)}\right]_{k}=\left\{\begin{array}{cc}
\omega_{k} & \text { if } k \in F_{m} \\
0 & \text { if } k \notin F_{m}
\end{array} \text { and }\left[\omega^{(m)}\right]_{k}=\left\{\begin{array}{cl}
\omega_{k} & \text { if } k \notin F_{m} \\
0 & \text { if } k \in F_{m}
\end{array}\right.\right.
$$

In this section we consider the one-dimensional setting, that is when $d=1$.
Let $m_{i} \in \mathbb{N}, M \in \mathbb{N}, M>m_{i}, i=1,2, a \in X_{\left(m_{1}\right)}^{s}, b \in X_{\left(m_{2}\right)}^{s}$ with $a_{k}=a_{|k|}, k \in \mathbb{Z}$, and $b_{k}=b_{|k|}, k \in \mathbb{Z}$. In what follows we obtain estimates of the forms:

$$
\begin{gather*}
\left|\left[a \omega^{(M)}\right]_{k}\right| \leq \frac{\left[\beta_{1}^{(1)}(a, M)\right]_{k}}{\omega_{k}^{s}}, k \in \mathbb{Z}  \tag{A.1}\\
\left|\left[\omega^{(M)} \omega^{(M)}\right]_{k}\right| \leq \frac{\left[\beta_{0}^{(1)}(M)\right]_{k}}{\omega_{k}^{s}}, k \in \mathbb{Z}  \tag{A.2}\\
\left|\left[a b \omega^{(M)}\right]_{k}\right| \leq \frac{\left[\alpha_{2}^{(1)}(a, b, M)\right]_{k}}{\omega_{k}^{s}}, k \in \mathbb{Z}  \tag{A.3}\\
\left|\left[a \omega^{(M)} \omega^{(M)}\right]_{k}\right| \leq \frac{\left[\alpha_{1}^{(1)}(a, M)\right]_{k}}{\omega_{k}^{s}}, k \in \mathbb{Z} \tag{A.4}
\end{gather*}
$$

$$
\begin{equation*}
\left|\left[\omega^{(M)} \omega^{(M)} \omega^{(M)} \omega^{(M)}\right]_{k}\right| \leq \frac{\left[\alpha_{0}^{(1)}(M)\right]_{k}}{\omega_{k}^{s}}, k \in \mathbb{Z} \tag{A.5}
\end{equation*}
$$

Since it is easy to verify the symmetries

$$
\begin{gathered}
{\left[a \omega^{(M)}\right]_{k}=\left[a \omega^{(M)}\right]_{|k|},\left[\omega^{(M)} \omega^{(M)}\right]_{k}=\left[\omega^{(M)} \omega^{(M)}\right]_{|k|}, k \in \mathbb{Z}} \\
{\left[a b \omega^{(M)}\right]_{k}=\left[a b \omega^{(M)}\right]_{|k|},\left[a \omega^{(M)} \omega^{(M)}\right]_{k}=\left[a \omega^{(M)} \omega^{(M)}\right]_{|k|}, k \in \mathbb{Z},} \\
{\left[\omega^{(M)} \omega^{(M)} \omega^{(M)}\right]_{k}=\left[\omega^{(M)} \omega^{(M)} \omega^{(M)}\right]_{|k|}, k \in \mathbb{Z},}
\end{gathered}
$$

we limit ourselves to bound the above convolutions with non-negative indexes.

## A.1.1 Estimates for $\left[a \omega^{(M)}\right]_{k}, k \in \mathbb{Z}_{+}$

Let $\tilde{M} \geq M+m$. For $k \in\{0,1, \ldots, \tilde{M}-1\}$ we compute explicitly:

$$
\begin{equation*}
\left[a \omega^{(M)}\right]_{k}=\sum_{\substack{|j|<m \\|k-j| \geq M}}\left|a_{j}\right| \frac{1}{|k-j|^{s}}=\frac{1}{\omega_{k}^{s}} \sum_{\substack{|j|<m \\|k-j| \geq M}} \frac{\left|a_{j}\right| \omega_{k}^{s}}{|k-j|^{s}}=: \frac{\beta_{1}^{(1)}(a, M)}{\omega_{k}^{s}} \tag{A.6}
\end{equation*}
$$

For $k \geq \tilde{M}$ we can write

$$
\begin{aligned}
{\left[a \omega^{(M)}\right]_{k} } & =\sum_{\substack{|j|<m \\
|k-j| \geq M}}\left|a_{j}\right| \frac{1}{|k-j|^{s}}=\frac{1}{k^{s}} \sum_{|j|<m}\left|a_{j}\right|\left|\frac{k}{k-j}\right|^{s} \leq \\
& \leq \frac{1}{k^{s}} \sum_{|j|<m}\left|a_{j}\right| \gamma(j, \tilde{M})^{s}=: \frac{\tilde{\beta}_{1}^{(1)}(a, \tilde{M})}{k^{s}},
\end{aligned}
$$

where, from now on, we define:

$$
\gamma(j, n)=\left\{\begin{array}{cc}
1 & \text { if } j \leq 0  \tag{A.7}\\
\frac{n}{n-j} & \text { if } j>0
\end{array}, j \in \mathbb{Z}, n \in \mathbb{N}, n>j\right.
$$

So, we can take the uniform bound:

$$
\left[\beta_{1}^{(1)}(a, M)\right]_{k}=\tilde{\beta}_{1}^{(1)}(a, \tilde{M}), \forall k \geq \tilde{M}
$$

## A.1.2 Estimates for $\left[\omega^{(M)} \omega^{(M)}\right]_{k}, k \in \mathbb{Z}_{+}$

By definition:

$$
\left[\omega^{(M)} \omega^{(M)}\right]_{k}=\sum_{\substack{|j| \geq m \\|k-j| \geq m}} \frac{1}{|j|^{s}} \frac{1}{|k-j|^{\mid}} .
$$

So, if $s \geq 2$ we get:

$$
\begin{equation*}
\left[\omega^{(M)} \omega^{(M)}\right]_{0}=\sum_{|j| \geq m} \frac{1}{|j|^{2 s}}=2 \sum_{j \geq m} \frac{1}{j^{2 s}} \leq \frac{2}{(2 s-1)(m-1)^{2 s-1}}=:\left[\beta_{0}^{(1)}(M)\right]_{k} \tag{A.8}
\end{equation*}
$$

For $k \geq 1$ we have:

$$
\begin{gathered}
{\left[\omega^{(M)} \omega^{(M)}\right]_{k}=\sum_{\substack{j \geq M \\
|k-j| \geq M}} \frac{1}{j^{s}} \frac{1}{|k-j|^{s}}+\sum_{\substack{j \geq M \\
|k+j| \geq M}} \frac{1}{j^{s}} \frac{1}{|k+j|^{s}}=} \\
=\sum_{\substack{j \geq M \\
k-j \geq M}} \frac{1}{j^{s}} \frac{1}{(k-j)^{s}}+\sum_{\substack{j \geq M \\
j-k \geq M}} \frac{1}{j^{s}} \frac{1}{(j-k)^{s}}+\sum_{j \geq M} \frac{1}{j^{s}} \frac{1}{(k+j)^{s}}= \\
=\sum_{j=M}^{k-M} \frac{1}{j^{s}} \frac{1}{(k-j)^{s}}+2 \sum_{j=M}^{\infty} \frac{1}{j^{s}} \frac{1}{(k+j)^{s}}= \\
=\frac{1}{k^{s}} \sum_{j=M}^{k-M}\left(\frac{1}{j}+\frac{1}{k-j}\right)^{s}+\frac{2}{k^{s}} \sum_{j=M}^{\infty}\left(\frac{1}{j}-\frac{1}{k+j}\right)^{s} .
\end{gathered}
$$

Therefore, given any $\tilde{M} \in \mathbb{N}, \tilde{M} \geq 8$, for $k \in\{1, \ldots, \tilde{M}-1\}$ we have:

$$
\begin{gather*}
{\left[\omega^{(M)} \omega^{(M)}\right]_{k} \leq \frac{1}{k^{s}} \sum_{j=M}^{k-M}\left(\frac{1}{j}+\frac{1}{k-j}\right)^{s}+} \\
+\frac{2}{k^{s}}\left[\sum_{j=M}^{N}\left(\frac{1}{j}-\frac{1}{k+j}\right)^{s}+\frac{1}{(s-1) N^{s-1}}\right]=: \frac{\left[\beta_{0}^{(1)}(M)\right]_{k}}{k^{s}} \tag{A.9}
\end{gather*}
$$

For $k \geq \tilde{M}$, if $s \geq 2$ we have:

$$
\begin{array}{r}
{\left[\omega^{(M)} \omega^{(M)}\right]_{k} \leq \frac{1}{k^{s}}\left(\frac{2}{M}\right)^{s-2} \sum_{j=M}^{k-M}\left(\frac{1}{j}+\frac{1}{k-j}\right)^{2}+\frac{2}{k^{s}} \frac{1}{(s-1)(M-1)^{s-1}} \leq} \\
\leq \frac{2}{k^{s}}\left(\frac{2}{M}\right)^{s-2} \sum_{j=M}^{k-M}\left(\frac{1}{j^{2}}+\frac{2}{k j}\right)+\frac{2}{k^{s}} \frac{1}{(s-1)(M-1)^{s-1}} \leq \\
\leq \frac{2}{k^{s}}\left(\frac{2}{M}\right)^{s-2}\left[\frac{\pi^{2}}{6}-\sum_{j=1}^{M-1} \frac{1}{j^{2}}+\frac{2 \ln (k-M)}{k}\right]+\frac{2}{k^{s}} \frac{1}{(s-1)(M-1)^{s-1}} \leq \\
\leq \frac{2}{k^{s}}\left\{\left(\frac{2}{M}\right)^{s-2}\left[\frac{\pi^{2}}{6}-\sum_{j=1}^{M-1} \frac{1}{j^{2}}+\frac{2 \ln (\tilde{M})}{\tilde{M}}\right]+\frac{1}{(s-1)(M-1)^{s-1}}\right\}=: \frac{\tilde{\beta}_{0}^{(1)}(M, \tilde{M})}{\omega_{k}^{s}} . \tag{A.10}
\end{array}
$$

The last inequality holds because the function $\frac{\ln (x)}{x}$ is decreasing for $x \geq \tilde{M}$ if $\tilde{M} \geq 8$. So, we can take the uniform bound

$$
\left[\beta_{0}^{(1)}(M)\right]_{k}=\tilde{\beta}_{0}^{(1)}(M, \tilde{M}), k \geq \tilde{M}
$$

## A.1.3 Estimates for $\left[a b \omega^{(M)}\right]_{k}, k \in \mathbb{Z}_{+}$

For $0 \leq k \leq 3 M-1$, we have:

$$
\begin{equation*}
\left[a b \omega^{(M)}\right]_{k}=\frac{1}{\omega_{k}^{s}} \sum_{\substack{|j| \leq 2 m-2 \\|k-j| \geq m}}(a b)_{j} \frac{\omega_{k}^{s}}{\omega_{k-j}^{s}}=: \frac{\alpha_{2}^{(1)}(a, b, M)}{\omega_{k}^{s}} \tag{A.11}
\end{equation*}
$$

For $k \geq 3 M$, we have:

$$
\begin{align*}
& {\left[a b \omega^{(M)}\right]_{k}=\frac{1}{\omega_{k}^{s}} \sum_{|j| \leq 2 m-2}(a b)_{j}\left|\frac{k}{k-j}\right|^{s} \leq } \\
\leq & \frac{1}{\omega_{k}^{s}} \sum_{|j| \leq 2 m-2}(a b)_{j} \gamma(j, \tilde{M})^{s}=: \frac{\tilde{\alpha}_{2}^{(1)}(a, b, \tilde{M})}{\omega_{k}^{s}} . \tag{A.12}
\end{align*}
$$

So, we can take the uniform bound

$$
\alpha_{2}^{(1)}(a, b, M)=\tilde{\alpha}_{2}^{(1)}(a, b, \tilde{M}), k \geq \tilde{M}
$$

## A.1.4 Estimates for $\left[a \omega^{(M)} \omega^{(M)}\right]_{k}, k \in \mathbb{Z}_{+}$

Let $\tilde{M} \in \mathbb{N}$ with $\tilde{M} \geq 2 M+m$. For $k \in \mathbb{Z}_{+}$, we can write:

$$
\left[a \omega^{(M)} \omega^{(M)}\right]_{k}=\sum_{|j|<m}\left|a_{j}\right|\left[\omega^{(M)} \omega^{(M)}\right]_{k-j} \leq \sum_{|j|<m}\left|a_{j}\right| \frac{\left[\beta_{0}^{(1)}(M)\right]_{k}}{\omega_{k-j}^{s}}, k \in \mathbb{Z}_{+}
$$

Therefore, for $k \in\{0, \ldots, \tilde{M}-1\}$ we have

$$
\left[a \omega^{(M)} \omega^{(M)}\right]_{k} \leq \frac{1}{\omega_{k}^{s}} \sum_{|j|<m}\left|a_{j}\right| \frac{\left[\omega_{k}^{s} \beta_{0}^{(1)}(M)\right]_{k-j}}{\omega_{k-j}^{S}}=: \frac{\left[\alpha_{1}^{(1)}(a, M)\right]_{k}}{\omega_{k}^{s}} .
$$

For $k \geq \tilde{M}$, if $|j|<m$ we have $|k-j| \geq \tilde{M}-m \geq 2 M$, so that

$$
\left[\beta_{0}^{(1)}(M)\right]_{k-j} \leq \tilde{\beta}_{0}^{(1)}(2 M), k \geq \tilde{M},|j|<m
$$

and we can write

$$
\begin{aligned}
& {\left[a \omega^{(M)} \omega^{(M)}\right]_{k} \leq \frac{\tilde{\beta}_{0}^{(1)}(2 M)}{\omega_{k}^{s}} \sum_{|j|<m}\left|a_{j}\right| \frac{\omega_{k}^{s}}{\omega_{k-j}^{s}} \leq } \\
\leq & \frac{\tilde{\beta}_{0}^{(1)}(2 M)}{\omega_{k}^{s}} \sum_{|j|<m}\left|a_{j}\right| \gamma(j, \tilde{M})^{s}=: \frac{\left[\alpha_{1}^{(1)}(a, M)\right]_{k}}{\omega_{k}^{s}} .
\end{aligned}
$$

## A.1.5 Estimates for $\left[\omega^{(M)} \omega^{(M)} \omega^{(M)}\right]_{k}, k \in \mathbb{Z}_{+}$

$$
\begin{gathered}
{\left[\omega^{(M)} \omega^{(M)} \omega^{(M)}\right]_{k}=\sum_{\substack{|j| \in \mathbb{Z}^{2} \\
|k-j| \geq M}}\left[\omega^{(M)} \omega^{(M)}\right]_{j} \frac{1}{|k-j|^{s}}=} \\
=\sum_{\substack{|j|<M \\
|k-j| \geq M}}\left[\omega^{(M)} \omega^{(M)}\right]_{j} \frac{1}{|k-j|^{s}}+\sum_{\substack{|j| \geq M \\
|k-j| \geq M}}\left[\omega^{(M)} \omega^{(M)}\right]_{j} \frac{1}{|k-j|^{s}} \leq \\
\leq \sum_{\substack{|j|<M \\
|k-j| \geq M}}\left[\omega^{(M)} \omega^{(M)}\right]_{j} \frac{1}{|k-j|^{s}}+\tilde{\beta}_{0}^{(1)}(M, M) \sum_{\substack{|j| \geq M \\
|k-j| \geq M}} \frac{1}{\omega_{j}^{s}} \frac{1}{|k-j|^{s}} \leq \\
\leq \sum_{\substack{|j|<M \\
|k-j| \geq M}} \frac{\left[\beta_{0}^{(1)}(M)\right]_{j}}{\omega_{j}^{S}} \frac{1}{|k-j|^{s}}+\tilde{\beta}_{0}^{(1)}(M, M) \frac{\left[\beta_{0}^{(1)}(M)\right]_{k}}{\omega_{k}^{S}} .
\end{gathered}
$$

Therefore, given $\tilde{M} \geq 8$, for $k \in\{0, \ldots \tilde{M}-1\}$ we can define:

$$
\left[\alpha_{0}^{(1)}(M)\right]_{k}=\sum_{\substack{|j|<M \\|k-j| \geq M}} \frac{\left[\beta_{0}^{(1)}(M)\right]_{j}}{\omega_{j}^{s}} \frac{\omega_{k}^{s}}{|k-j|^{s}}+\tilde{\beta}_{0}^{(1)}(M, M)\left[\beta_{0}^{(1)}(M)\right]_{k},
$$

and, for $k \geq \tilde{M}$ we can take:

$$
\left[\omega^{(M)} \omega^{(M)} \omega^{(M)}\right]_{k} \leq \sum_{\substack{|j|<M \\|k-j| \geq M}} \frac{\left[\beta_{0}^{(1)}(M)\right]_{j}}{\omega_{j}^{S}} \gamma(j, \tilde{M})^{s}+\tilde{\beta}_{0}^{(1)}(M, M) \tilde{\beta}_{0}^{(1)}(M, \tilde{M})
$$

## A. 2 Two-dimensional estimates

Now let us consider estimates in the two-dimensional setting, that is when $d=2$. In this section, we set $m=\left(m_{1}, m_{2}\right) \in \mathbb{N}^{2}, M=\left(M_{1}, M_{2}\right) \in \mathbb{N}^{2}$ with $M_{i}>m_{i}, i=1,2, s=\left(s_{1}, s_{2}\right) \in$ $\mathbb{R}_{+}^{2}, s_{i}>=2, i=1,2$. We will use the same symbol $\omega$ to denote the sequence $\left\{1 / \omega_{k}^{s}\right\}_{k \in \mathbb{Z}^{d}}$ independently of the dimension $d$. This will cause no confusion because the dimension of the index of the convolution corresponds to the dimension of the sequences involved in the convolution.

For example, we know that $\left[\omega^{\left(M_{1}\right)} \omega_{\left(m_{2}\right)}\right]_{m_{1}}$ is a convolution of one-dimensional sequences, while $\left[\omega^{(M)} \omega_{(m)}\right]_{(2,5)}$ is a convolution of two-dimensional sequences.

## A.2.1 Estimates for $\left[a b \omega^{(M)}\right]_{k}, k \in \mathbb{Z}_{+}^{2}$

Let be given $a \in X_{m}^{s}$ and $b \in X_{n}^{s}$ with $m=\left(m_{1}, m_{2}\right) \in \mathbb{N}^{2}$ and $n=\left(n_{1}, n_{2}\right) \in \mathbb{N}^{2}$. For $\tilde{M}=\left(\tilde{M}_{1}, \tilde{M}_{2}\right) \mathbb{N}^{2}$ with $\tilde{M}_{i}>m_{i}+n_{i}-2, i=1,2$, we have:

$$
\begin{gathered}
\left|\left[a b \omega^{(M)}\right]_{k}\right|=\sum_{\substack{j \in F_{m+n-(1,1)} \\
k-j \notin F_{M}}}\left|(a b)_{j}\right| \frac{1}{\omega_{k-j}^{s}} \leq \\
\leq \frac{1}{\omega_{k}^{s}} \sum_{\substack{j \in F_{m+n-(1,1)} \\
k-i \neq F_{1}}}\left|(a b)_{j}\right| \frac{\omega_{k}^{s}}{\omega_{k-j}^{s}}=: \frac{\left[\alpha_{2}^{(2)}(a, b, M)\right]_{k}}{\omega_{k}^{s}}, k \in F_{\tilde{M}} .
\end{gathered}
$$

For $k \notin F_{\tilde{M}}$ we have the following uniform estimates.
If $k_{1} \geq \tilde{M}_{1}$ and $k_{2} \in\left\{0, \ldots, \tilde{M}_{2}-1\right\}$ then:

$$
\left|\left[a b \omega^{(M)}\right]_{k}\right| \leq \frac{1}{\omega_{k}^{s}} \sum_{\substack{j \in F_{m+n-(1,1)} \\ k-j \notin F_{M}}}\left|(a b)_{j}\right| \gamma\left(j_{1}, \tilde{M}_{1}\right)^{s_{1}} \frac{\omega_{k_{2}}^{s_{2}}}{\omega_{k_{2}-j_{2}}^{s_{2}}}=: \frac{\left[\alpha_{2}^{(2)}(a, b, M)\right]_{k}}{\omega_{k}^{s}} .
$$

If $k_{2} \geq \tilde{M}_{2}$ and $k_{1} \in\left\{0, \ldots, \tilde{M}_{1}-1\right\}$ then:

$$
\left|\left[a b \omega^{(M)}\right]_{k}\right| \leq \frac{1}{\omega_{k}^{s}} \sum_{\substack{j \in F_{m+n-(1,1)} \\ k-j \notin F_{M}}}\left|(a b)_{j}\right| \gamma\left(j_{2}, \tilde{M}_{2}\right)^{s_{2}} \frac{\omega_{k_{1}}^{s_{1}}}{\omega_{k_{1}-j_{1}}^{s_{2}}}=: \frac{\left[\alpha_{2}^{(2)}(a, b, M)\right]_{k}}{\omega_{k}^{s}}
$$

Finally, if $k_{1} \geq \tilde{M}_{1}$ and $k_{2} \geq \tilde{M}_{2}$ then:

$$
\left|\left[a b \omega^{(M)}\right]_{k}\right| \leq \frac{1}{\omega_{k}^{s}} \sum_{\substack{j \in F_{m+n-(1,1)} \\ k-j \notin F_{M}}}\left|(a b)_{j}\right| \gamma\left(j_{2}, \tilde{M}_{2}\right)^{s_{2}} \gamma\left(j_{1}, \tilde{M}_{1}\right)^{s_{1}}=: \frac{\left[\alpha_{2}^{(2)}(a, b, M)\right]_{k}}{\omega_{k}^{s}} .
$$

To get bounds for convolutions of the form $\left[a \omega^{(M)} \omega^{(M)}\right]_{k}, k \in \mathbb{Z}^{2}$, we need to bound, via reduction to one-dimensional estimates, the convolutions of the form $\left[\omega^{(M)} \omega^{(M)}\right]_{k}, k \in \mathbb{Z}^{2}$, as we see in the next section.

## A.2.2 Reduction to one-dimensional estimates

From the definition of two-dimensional convolution, for a given $k=\left(k_{1}, k_{2}\right) \in \mathbb{Z}_{+}^{2}$, we can write:

$$
\begin{gathered}
{\left[\omega^{(M)} \omega^{(M)}\right]_{k}=\sum_{\substack{j \notin F_{M} \\
k-j \notin F_{M}}} \frac{1}{\omega_{j}^{S} \omega_{k-j}^{S}}=} \\
=\sum_{\substack{\left|j_{1}\right|<M_{1},\left|j_{2}\right| \geq M_{2} \\
\left|k_{1}-j_{1}\right|<M_{1},\left|k_{2}-j_{2}\right| \geq M_{2}}} \frac{1}{\omega_{k_{1}}^{S_{1}} \omega_{k_{1}-j_{1}}^{S_{1}}} \frac{1}{\omega_{k_{2}}^{S_{2}} \omega_{k_{2}-j_{2}}^{S_{2}}}+\sum_{\substack{\left|j_{1}\right|<M_{1},\left|j_{2}\right| \geq M_{2} \\
\left|k_{1}-j_{1}\right| \geq M_{1},\left|k_{2}-j_{2}\right|<M_{2}}} \frac{1}{\omega_{k_{1}}^{S_{1}} \omega_{k_{1}-j_{1}}^{S_{1}}} \frac{1}{\omega_{k_{2}}^{S_{2}} \omega_{k_{2}-j_{2}}^{S_{2}}}+
\end{gathered}
$$

$$
\begin{aligned}
& +\sum_{\substack{\left|j_{1}\right|<M_{1},\left|j_{2}\right| \geq M_{2} \\
\left|k_{1}-j_{1}\right| \geq M_{1},\left|k_{2}-j_{2}\right| \geq M_{2}}} \frac{1}{\omega_{k_{1}}^{s_{1}} \omega_{k_{1}-j_{1}}^{s_{1}}} \frac{1}{\omega_{k_{2}}^{s_{2}} \omega_{k_{2}-j_{2}}^{s_{2}}}+\sum_{\substack{\left|j_{1}\right| \geq M_{1},\left|j_{2}\right|<M_{2} \\
\left|k_{1}-j_{1}\right|<M_{1},\left|k_{2}-j_{2}\right| \geq M_{2}}} \frac{1}{\omega_{k_{1}}^{s_{1}} \omega_{k_{1}-j_{1}}^{s_{1}}} \frac{1}{\omega_{k_{2}}^{S_{2}} \omega_{k_{2}-j_{2}}^{s_{2}}}+ \\
& +\sum_{\substack{\left|j_{1}\right| \geq M_{1},\left|j_{2}\right|<M_{2} \\
\left|k_{1}-j_{1}\right| \geq M_{1},\left|k_{2}-j_{2}\right|<M_{2}}} \frac{1}{\omega_{k_{1}}^{s_{1}} \omega_{k_{1}-j_{1}}^{s_{1}}} \frac{1}{\omega_{k_{2}}^{s_{2}} \omega_{k_{2}-j_{2}}^{s_{2}}}+\sum_{\substack{\left|j_{1}\right| \geq M_{1},\left|j_{2}\right|<M_{2} \\
\left|k_{1}-j_{1}\right| \geq M_{1},\left|k_{2}-j_{2}\right| \geq M_{2}}} \frac{1}{\omega_{k_{1}}^{s_{1}} \omega_{k_{1}-j_{1}}^{s_{1}}} \frac{1}{\omega_{k_{2}}^{S_{2}} \omega_{k_{2}-j_{2}}^{s_{2}}}+ \\
& +\sum_{\substack{\left|j_{1}\right| \geq M_{1},\left|j_{2}\right| \geq M_{2} \\
\left|k_{1}-j_{1}\right|<M_{1},\left|k_{2}-j_{2}\right| \geq M_{2}}} \frac{1}{\omega_{k_{1}}^{s_{1}} \omega_{k_{1}-j_{1}}^{s_{1}}} \frac{1}{\omega_{k_{2}}^{s_{2}} \omega_{k_{2}-j_{2}}^{s_{2}}}+\sum_{\substack{\left|j_{1}\right| \geq M_{1},\left|j_{2}\right| \geq M_{2} \\
\left|k_{1}-j_{1}\right| \geq M_{1},\left|k_{2}-j_{2}\right|<M_{2}}} \frac{1}{\omega_{k_{1}}^{s_{1}} \omega_{k_{1}-j_{1}}^{s_{1}}} \frac{1}{\omega_{k_{2}}^{s_{2}} \omega_{k_{2}-j_{2}}^{s_{2}}}+ \\
& +\sum_{\substack{\left|j_{1}\right| \geq M_{1},\left|j_{2}\right| \geq M_{2} \\
\left|k_{1}-j_{1}\right| \geq M_{1},\left|k_{2}-j_{2}\right| \geq M_{2}}} \frac{1}{\omega_{k_{1}}^{s_{1}} \omega_{k_{1}-j_{1}}^{s_{1}}} \frac{1}{\omega_{k_{2}}^{s_{2}} \omega_{k_{2}-j_{2}}^{s_{2}}}= \\
& =\left[\omega_{\left(M_{1}\right)} \omega_{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+\left[\omega_{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+ \\
& +\left[\omega_{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+\left[\omega_{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+ \\
& +\left[\omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega_{\left(M_{2}\right)}\right]_{k_{2}}+\left[\omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+ \\
& +\left[\omega_{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+\left[\omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+ \\
& +\left[\omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}= \\
& =\left[\omega_{\left(M_{1}\right)} \omega_{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+\left[\omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega_{\left(M_{2}\right)}\right]_{k_{2}}+ \\
& +2\left[\omega_{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+2\left[\omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+ \\
& +2\left[\omega_{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+\left[\omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}} \leq
\end{aligned}
$$

$$
\begin{align*}
& \leq \frac{1}{\omega_{k}^{s}}\left\{\omega_{k_{1}}^{s_{1}}\left[\omega_{\left(M_{1}\right)} \omega_{\left(M_{1}\right)}\right]_{k_{1}}\left[\beta_{0}^{(1)}\left(M_{2}\right)\right]_{k_{2}}+\omega_{k_{2}}^{s_{2}}\left[\omega_{\left(M_{2}\right)} \omega_{\left(M_{2}\right)}\right]_{k_{2}}\left[\beta_{0}^{(1)}\left(M_{1}\right)\right]_{k_{1}}+\right. \\
& \quad+2\left[\beta_{1}^{(1)}\left(\omega_{\left(M_{1}\right)}, M_{1}\right)\right]_{k_{1}}\left[\beta_{0}^{(1)}\left(M_{2}\right)\right]_{k_{2}}+2\left[\beta_{0}^{(1)}\left(M_{1}\right)\right]_{k_{1}}\left[\beta_{1}^{(1)}\left(\omega_{\left(M_{2}\right)}, M_{2}\right)\right]_{k_{2}}+  \tag{A.13}\\
& \left.+2\left[\beta_{1}^{(1)}\left(\omega_{\left(M_{1}\right)}, M_{1}\right)\right]_{k_{1}}\left[\beta_{1}^{(1)}\left(\omega_{\left(M_{2}\right)}, M_{2}\right)\right]_{k_{2}}+\left[\beta_{0}^{(1)}\left(M_{1}\right)\right]_{k_{1}}\left[\beta_{0}^{(1)}\left(M_{2}\right)\right]_{k_{2}}\right\}=: \frac{\beta_{k}(M)}{\omega_{k}^{s}}
\end{align*}
$$

Observe that, if $\tilde{M}=\left(\tilde{M}_{1}, \tilde{M}_{2}\right) \in \mathbb{N}^{2}$ with $\tilde{M}_{i}>2 M_{i}-2, i=1,2$, then we have the following uniform bounds.

If $k_{1} \geq \tilde{M}_{1}$ and $k_{2} \in\left\{0, \ldots, \tilde{M}_{2}-1\right\}$ then:

$$
\beta_{k}(M) \leq \tilde{\beta}_{1}\left(M, \tilde{M}_{1}, k_{2}\right)
$$

where

$$
\begin{gathered}
\tilde{\beta}_{1}\left(M, \tilde{M}_{1}, k_{2}\right)=\omega_{k_{2}}^{s_{2}}\left[\omega_{\left(M_{2}\right)} \omega_{\left(M_{2}\right)}\right]_{k_{2}} \tilde{\beta}_{0}^{(1)}\left(M_{1}, \tilde{M}_{1}\right)+ \\
+2 \tilde{\beta}_{1}^{(1)}\left(\omega_{\left(M_{1}\right)}, \tilde{M}_{1}\right)\left[\beta_{0}^{(1)}\left(M_{2}\right)\right]_{k_{2}}+2 \tilde{\beta}_{0}^{(1)}\left(M_{1}, \tilde{M}_{1}\right)\left[\beta_{1}^{(1)}\left(\omega_{\left(M_{2}\right)}, M_{2}\right)\right]_{k_{2}}+ \\
+2 \tilde{\beta}_{1}^{(1)}\left(\omega_{\left(M_{1}\right)}, \tilde{M}_{1}\right)\left[\beta_{1}^{(1)}\left(\omega_{\left(M_{2}\right)}, M_{2}\right)\right]_{k_{2}}+\tilde{\beta}_{0}^{(1)}\left(M_{1}, \tilde{M}_{1}\right)\left[\beta_{0}^{(1)}\left(M_{2}\right)\right]_{k_{2}} .
\end{gathered}
$$

If $k_{2} \geq \tilde{M}_{2}$ and $k_{1} \in\left\{0, \ldots, \tilde{M}_{1}-1\right\}$ then:

$$
\beta_{k}(M) \leq \tilde{\beta}_{2}\left(M, \tilde{M}_{2}, k_{1}\right)
$$

where

$$
\begin{gathered}
\tilde{\beta}_{2}\left(M, \tilde{M}_{2}, k_{1}\right)=\omega_{k_{1}}^{s_{1}}\left[\omega_{\left(M_{1}\right)} \omega_{\left(M_{1}\right)}\right) k_{1} \tilde{\beta}_{0}^{(1)}\left(M_{2}, \tilde{M}_{2}\right)+ \\
+2 \tilde{\beta}_{1}^{(1)}\left(\omega_{\left(M_{2}\right)}, \tilde{M}_{2}\right)\left[\beta_{0}^{(1)}\left(M_{1}\right)\right]_{k_{1}}+2 \tilde{\beta}_{0}^{(1)}\left(M_{2}, \tilde{M}_{2}\right)\left[\beta_{1}^{(1)}\left(\omega_{\left(M_{1}\right)}, M_{1}\right)\right]_{k_{1}}+ \\
+2 \tilde{\beta}_{1}^{(1)}\left(\omega_{\left(M_{2}\right)}, \tilde{M}_{2}\right)\left[\beta_{1}^{(1)}\left(\omega_{\left(M_{1}\right)}, M_{1}\right)\right]_{k_{1}}+\tilde{\beta}_{0}^{(1)}\left(M_{2}, \tilde{M}_{2}\right)\left[\beta_{0}^{(1)}\left(M_{1}\right)\right]_{k_{1}} .
\end{gathered}
$$

Finally, if $k_{1} \geq \tilde{M}_{1}$ and $k_{2} \geq \tilde{M}_{2}$ then

$$
\beta_{k}(M) \leq \tilde{\beta}_{3}(M, \tilde{M})
$$

where

$$
\begin{gathered}
\tilde{\beta}_{3}(M, \tilde{M})=+2 \tilde{\beta}_{1}^{(1)}\left(\omega_{\left(M_{2}\right)}, \tilde{M}_{2}\right) \tilde{\beta}_{0}^{(1)}\left(M_{1}, \tilde{M}_{1}\right)+2 \tilde{\beta}_{0}^{(1)}\left(M_{2}, \tilde{M}_{2}\right) \tilde{\beta}_{1}^{(1)}\left(\omega_{\left(M_{1}\right)}, \tilde{M}_{1}\right)+ \\
\quad+2 \tilde{\beta}_{1}^{(1)}\left(\omega_{\left(M_{2}\right)}, \tilde{M}_{2}\right) \tilde{\beta}_{1}^{(1)}\left(\omega_{\left(M_{1}\right)}, \tilde{M}_{1}\right)+\tilde{\beta}_{0}^{(1)}\left(M_{2}, \tilde{M}_{2}\right) \tilde{\beta}_{0}^{(1)}\left(M_{1}, \tilde{M}_{1}\right) .
\end{gathered}
$$

Analogously, we can calculate as follows

$$
\left[\omega^{(M)} \omega^{(M)} \omega^{(M)}\right]_{k}=\left[\omega_{\left(M_{1}\right)} \omega_{\left(M_{1}\right)} \omega_{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+
$$

$$
\begin{align*}
& +3\left[\omega_{\left(M_{1}\right)} \omega_{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+3\left[\omega_{\left(M_{1}\right)} \omega_{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+ \\
& +3\left[\omega_{\left(M_{1}\right)} \omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega_{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+3\left[\omega_{\left(M_{1}\right)} \omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+ \\
& +\left[\omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+6\left[\omega_{\left(M_{1}\right)} \omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+ \\
& +\left[\omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega_{\left(M_{2}\right)} \omega_{\left(M_{2}\right)}\right]_{k_{2}}+3\left[\omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega_{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}}+ \\
& +3\left[\omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)} \omega^{\left(M_{1}\right)}\right]_{k_{1}}\left[\omega_{\left(M_{2}\right)} \omega^{\left(M_{2}\right)} \omega^{\left(M_{2}\right)}\right]_{k_{2}} \leq \\
& \leq \frac{1}{\omega_{k}^{s}}\left\{\omega_{k_{1}}^{s_{1}}\left[\omega_{\left(M_{1}\right)} \omega_{\left(M_{1}\right)} \omega_{\left(M_{1}\right)}\right]_{k_{1}}\left[\alpha_{0}^{(1)}\left(M_{2}\right)\right]_{k_{2}}+3\left[\alpha_{2}^{(1)}\left(\omega_{\left(M_{1}\right)}, \omega_{\left(M_{1}\right)}, M_{1}\right)\right]_{k_{1}}\left[\alpha_{1}^{(1)}\left(\omega_{\left(M_{2}\right)}, M_{2}\right)\right]_{k_{2}}+\right. \\
& +3\left[\alpha_{2}^{(1)}\left(\omega_{\left(M_{1}\right)}, \omega_{\left(M_{1}\right)}, M_{1}\right)\right]_{k_{1}}\left[\alpha_{0}^{(1)}\left(M_{2}\right)\right]_{k_{2}}+3\left[\alpha_{1}^{(1)}\left(\omega_{\left(M_{1}\right)}, M_{1}\right)\right]_{k_{1}}\left[\alpha_{2}^{(1)}\left(\omega_{\left(M_{2}\right)}, \omega_{\left(M_{2}\right)}, M_{2}\right)\right]_{k_{2}}+ \\
& +3\left[\alpha_{1}^{(1)}\left(\omega_{\left(M_{1}\right)}, M_{1}\right)\right]_{k_{1}}\left[\alpha_{0}^{(1)}\left(M_{2}\right)\right]_{k_{2}}+\left[\alpha_{0}^{(1)}\left(M_{1}\right)\right]_{k_{1}}\left[\alpha_{0}^{(1)}\left(M_{2}\right)\right]_{k_{2}}+ \\
& +6\left[\alpha_{1}^{(1)}\left(\omega_{\left(M_{1}\right)}, M_{1}\right)\right]_{k_{1}}\left[\alpha_{1}^{(1)}\left(\omega_{\left(M_{2}\right)}, M_{2}\right)\right]_{k_{2}}+3\left[\alpha_{0}^{(1)}\left(M_{1}\right)\right]_{k_{1}}\left[\alpha_{2}^{(1)}\left(\omega_{\left(M_{2}\right)}, \omega_{\left(M_{2}\right)}, M_{2}\right)\right]_{k_{2}}+ \\
& \left.+3\left[\alpha_{0}^{(1)}\left(M_{1}\right)\right]_{k_{1}}\left[\alpha_{1}^{(1)}\left(\omega_{\left(M_{2}\right)}, M_{2}\right)\right]_{k_{2}}+\omega_{k^{2}}^{s_{2}}\left[\omega_{\left(M_{2}\right)} \omega_{\left(M_{2}\right)} \omega_{\left(M_{2}\right)}\right]_{k_{2}}\left[\alpha_{0}^{(1)}\left(M_{1}\right)\right]_{k_{1}}\right\}=: \\
& =: \frac{\alpha_{0}^{(2)}(M, k)}{\omega_{k}^{s}} . \tag{A.14}
\end{align*}
$$

## A.2.3 Estimates for $\left[a \omega^{(M)} \omega^{(M)}\right]_{k}, k \in \mathbb{Z}_{+}^{2}$

Let $k_{1} \geq \tilde{M}_{1}$ and $k_{2} \in\left\{0, \ldots, \tilde{M}_{2}-1\right\}$. If $j \in F_{m}$ then $k_{1}-j_{1} \geq \tilde{M}_{1}-m_{1}+1$, what implies (if $\tilde{M}_{1}-m_{1}+1>2 M_{1}-2$, i.e. $\tilde{M}_{1}>2 M_{1}+m_{1}-3$ ) that $\left[\omega^{(M)} \omega^{(M)}\right]_{k-j} \leq \frac{\tilde{\beta}_{1}\left(\tilde{M}_{1}-m_{1}+1, k 2-j 2\right)}{\omega_{k-j}^{s}}$ Therefore, we can write:

$$
\begin{gathered}
\left|\left[a \omega^{(M)} \omega^{(M)}\right]_{k}\right|=\sum_{j \in F_{m}}\left|a_{j}\right|\left[\omega^{(M)} \omega^{(M)}\right]_{k-j} \leq \\
\leq \frac{1}{\omega_{k}^{s}} \sum_{j \in F_{m}}\left|a_{j}\right| \frac{\tilde{\beta}_{1}\left(\tilde{M}_{1}-m_{1}-1, k_{2}-j_{2}\right) \omega_{k}^{s}}{\omega_{k-j}^{s}} \leq \\
\leq \frac{1}{\omega_{k}^{s}} \sum_{j \in F_{m}}\left|a_{j}\right| \frac{\tilde{\beta}_{1}\left(\tilde{M}_{1}-m_{1}-1, k_{2}-j_{2}\right) \gamma\left(j_{1}, \tilde{M}_{1}\right)^{s_{1}} \omega_{k_{2}}^{s_{2}}}{\omega_{k_{2}-j_{2}}^{s_{2}}} .
\end{gathered}
$$

## A. 3 Algorithms

In the algorithms of this section, a box $\prod_{j=1}^{n}\left[a_{j}, b_{j}\right]$ is represented by the $n$-by- 2 matrix $B=\left(B_{j, k}\right)_{n \times 2}, B_{j, 1}=a_{j}, B_{j, 2}=b_{j}, j=1, \ldots, n$.

The next function exclude from the box "Theoretical_Box" the boxes "Boxes_of_Roots" and splits the remaining region into sub-boxes. It makes use of the functions MULTIPLE_DECOMPOSITION.m, INTERSECT.m, EXCLUDE.m and Split.m.

```
function [ Boxes_Without_Roots ] = PROCESSING(Boxes_of_Roots, Theoretical_Box )
```

```
n_roots = length(Boxes_of_Roots(1,1,:));
```

TESTING_SET = Theoretical_Box;
for $k=1$ : n_roots
[ Boxes ] = MULTIPLE_DECOMPOSITION( TESTING_SET,Boxes_of_Roots (:, : ,k) );
TESTING_SET = Boxes;
end
Boxes_Without_Roots = Boxes;
end

If the box " $B_{i} i n$ " is strictly contained in the box " $B$ " the function Split.m returns "Splittable" $=1$ and the boxes " $C$ " and " $D$ " where only one of them contains the sub-box " $B_{-} i n$ ". If " $B \_i n$ " is the same as " $B$ " then the function returns "Splittable" $=0$ and no splitting is made on the box " $B$ ".

```
function [ Splittable, C, D ] = Split( B_in,B )
```

$\mathrm{n}=\operatorname{length}(\mathrm{B}(:, 1))$;
Splittable = 1;
C = B;
D = B;
$d=[0,0]$;
for $\mathrm{j}=1$ : n
if $B_{-}$in $(j, 1)>B(j, 1)$
$d=[j, 1] ;$
break
end
if $B_{-}$in $(j, 2)<B(j, 2)$
$d=[j, 2] ;$
break
end
end
if $d(2)==0$
Splittable $=0$;
return
end
if $d(2)==1$
Splittable = 1; $\quad \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \%$
$C(j, 2)=B \_i n(j, 1) ; \%$ The sub-box Bin stays within the box D. $\%$
D (j, 1) = B_in(j, 1) ; $\% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \%$
return
end
if $d(2)==2$
Splittable = 1; $\% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \%$
C(j,1) = B_in $(j, 2) ; \%$ The sub-box Bin stays within the box C. $\%$
D (j,2)= B_in(j,2); $\% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \%$
return
end
end

Given box " $B$ " and a sub-box " $B \_i n$ " of " $B$ ", the next function excludes the box " $B \_i n$ " from the box " $B$ " and splits the remaining region into "Sub_Boxes". It just iterates the function Split.m until there is no box strictly containing the box " $B \_i n$ ".

```
function [ Sub_Boxes,number_subboxes ] = EXCLUDE( Bin,B)
n =length(B(:,1));
aux = zeros(n,2,2*n);
number_subboxes = 0;
BO = B;
Splittable = 1;
while Splittable == 1
    [ Splittable, C, D ] = Split( Bin, BO );
    if Splittable == 1
        number_subboxes = number_subboxes + 1;
        aux(:,:,number_subboxes) = C;
        BO = D;
    end
end
Sub_Boxes = aux(:,:,1:number_subboxes);
```

end

The next function returns the intersection " $C$ " of the boxes " $A$ " and " $B$ " if the intersection is not empty. In this case it returns "INTERSECTION" $=1$. If the intersection is empty, it returns a degenerated box and "INTERSECTION" $=0$.

```
function [C, INTERSECTION] = INTERSECT( A, B )
n = length(A(:, 1));
C = zeros(n,2);
for j=1:n
    if (B(j,2) <= A(j,1) || B (j,1) >= A(j, 2))
        INTERSECTION = 0;
        return
    end
end
INTERSECTION = 1;
for j=1:n
    C(j,1) = max([A(j,1) B (j,1)]);
    C(j,2) = min([A(j,2) B(j,2)]);
end
end
```

The function "MULTIPLEDECOMPOSITION.m" excludes the box "Box" from each box of the set of boxes "TESTINGBOXES", and returns the resulting set of boxes named "Boxes".

```
function [ Boxes ] = MULTIPLEDECOMPOSITION(TESTINGBOXES, Box )
N = length(TESTINGBOXES(1,1,:));
n = length(Box(:, 1));
Boxes = zeros(n,2,N*2*n); % When one box is excluded from another one,
N_Boxes = 0; % it is necessary at most 2*n sub-boxes
for j = 1:N % to cover the remaining region.
    [C, INTERSECTION] = INTERSECT( TESTINGBOXES(:,:,j), Box );
    if INTERSECTION == 1
            [ Sub_Boxes, number_subboxes ] = EXCLUDE( C, TESTINGBOXES(:,:,j));
            Boxes(:,:,N_Boxes + 1:N_Boxes + number_subboxes) = Sub_Boxes;
            N_Boxes = N_Boxes + number_subboxes;
    end
    if INTERSECTION == 0
```

```
    Boxes(:,:,N_Boxes + 1) = TESTINGBOXES(:,:,j);
    N_Boxes = N_Boxes + 1;
    end
end
Boxes = Boxes(:,:,1: N_Boxes);
end
```

In the next function, the procedure "CheckBox.m" is an implementation of the test given by Theorem 4.2.1. That is, it uses interval arithmetic to rigorously check the non existence of solutions in a box. The function "ExhaustingBox.m" starts checking a single box. If the checking fails, the algorithm splits the initial box into two smaller boxes to be tested again. The process is stopped successfully if the check succeeds in all the tested boxes. The process is stopped unsuccessfully if the check does not succeed in very small boxes. The input "Bounds" provides the neccessary convolution estimates to apply the test given by Theorem 4.2.1.

```
function [r_boxes, proved, n_tested] = ExhaustingBox( box, Bounds )
m = length(box(:,1));
t_boxes = zeros(m,2,100000); %it keeps the boxes to be tested
n_t_boxes = 1; %it starts with only one box
t_boxes(:,:,1) = box; %
r_boxes = zeros(m,2,100000); %it keeps the very small boxes
n_r_boxes=0; %where the test do not work
proved = 0;
verified_boxes = 0;
n_tested=0;
while ( proved == 0 && n_t_boxes > 0 )
    if n_t_boxes + 1 > length(t_boxes(1,1,:))
        temp=zeros(m,2,2*length(t_boxes (1, 1,:)));
        temp(:,:,1:length(t_boxes(1,1,:)))=t_boxes;
        t_boxes=temp;
    end
    if n_r_boxes + 1 > length(r_boxes(1,1,:))
        temp=zeros(m,2,2*length(r_boxes (1, 1,:)));
        temp(:,:,1:length(r_boxes(1,1,:)))=r_boxes;
        r_boxes=temp;
    end
    B=t_boxes(:,:,1); % the test is applied always to
    check = CheckBox(B,Bounds); % the first box of the list
```

```
    if ( check == 1 ) % test succeeds
    n_tested=n_tested+1;%count the number of checked boxes
    if ( n_t_boxes == 1 )%no more boxes to be tested
        if n_r_boxes==0 %test did not failed in very
            proved = 1; %small boxes
            disp('box verified') %the algorithm
            return %is stopped successfully
        else
            r_boxes=r_boxes(:,:,1:n_r_boxes); %returns
            disp('Check failed in very small boxes')%a set
            return %of very small boxes where the test
        end %did not work
    else
        t_boxes(:,:,1:n_t_boxes-1) = t_boxes(:,:,2:n_t_boxes);
        n_t_boxes = n_t_boxes - 1; %exclude the tested box
        verified_boxes = verified_boxes+1;%from the list of
        continue %boxes to be tested
    end %and go to the next box of the list
end
    if ( check == 0 ) %
    [maximo,direction] = max(t_boxes(:,2,1)-t_boxes(:,1,1));
    if maximo > 1.e-8 %if the box is not to much small, split it
        [C,D] = partition(t_boxes(:,:,1),direction); %along its
        t_boxes(:,:,1) = C; %bigger direction
        t_boxes(:,:,n_t_boxes+1) = D;%this gives rise to two
        n_t_boxes = n_t_boxes + 1; %sub-boxes to be tested
    else
        n_r_boxes=n_r_boxes+1; %it rises a very small very box
        r_boxes(:,:,n_r_boxes)=B;%where the test did not work
        t_boxes(:,:,1:n_t_boxes-1) = t_boxes(:,:,2:n_t_boxes);
        n_t_boxes = n_t_boxes - 1;
    end
    end
end
```

end


[^0]:    1 This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001"

[^1]:    1 Here we would like to thank Sarah ((DAY et al., 2005)) for contributing with the integration strategy used to get the global bounds for the Fourier coefficients.

