A proof of Seidel's conjectures on the volume of ideal
tetrahedra in hyperbolic 3-space

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Omar Chavez Cussy

**Uma demonstração de conjecturas de Seidel sobre o
volume de tetraedros ideais no 3-espço hiperbólico**

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DE DEFESA*

Área de Concentração: Matemática

Orientador: Prof. Dr. Carlos Henrique Grossi Ferreira

USP – São Carlos
Maio de 2017

*Este trabalho é dedicado às crianças adultas que,
quando pequenas, sonharam em se tornar cientistas.
Em especial, ao pesquisadores do Instituto de Ciências Matemáticas e de Computação
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*“As invenções são, sobretudo,
o resultado de um trabalho de teimoso.”
(Santos Dumont)*

RESUMO

OMAR CHAVEZ C.. **A proof of Seidel's conjectures on the volume of ideal tetrahedra in hyperbolic 3-space**. 2017. 85 f. Master dissertation (Master student Program in Mathematics) – Instituto de Ciências Matemáticas e de Computação (ICMC/USP), São Carlos – SP.

Provamos duas conjecturas apresentadas por J. J. Seidel em “On the volume of a hyperbolic simplex”, Stud. Sci. Math. Hung. (21, 243–249, 1986). Estas conjecturas referem ao volume de tetraedros ideais no 3-espaço hiperbólico e estão relacionadas com o seguinte quadro geral. Como fórmulas explícitas para grandezas geométricas no espaço hiperbólico (distância, área, volume, etc.) tipicamente envolvem funções transcendentais sofisticadas, é desejável (e, na prática, bastante útil) expressar tais grandezas geométricas como aplicações monótonas de mapas algébricos. A Especulação 1 de Seidel diz que o volume de um tetraedro ideal no 3-espaço hiperbólico depende apenas do determinante e do permanente da matriz de Gram duplamente estocástica G de seus vértices; a Especulação 4 afirma que o referido volume é monótono tanto no determinante quanto no permanente de G . Damos respostas afirmativas às Especulações 1 e 4 ao parametrizar o espaço classificador de tetraedros ideais (marcados) de maneira adequada.

Palavras-chave: Espaço hiperbólico real, Volume de tetraedros ideais, Conjecturas de Seidel .

ABSTRACT

OMAR CHAVEZ C.. **A proof of Seidel's conjectures on the volume of ideal tetrahedra in hyperbolic 3-space.** 2017. 85 f. Master dissertation (Master student Program in Mathematics) – Instituto de Ciências Matemáticas e de Computação (ICMC/USP), São Carlos – SP.

We prove a couple of conjectures raised by J. J. Seidel in "On the volume of a hyperbolic simplex", Stud. Sci. Math. Hung. (21, 243–249, 1986). These conjectures concern the volume of ideal hyperbolic tetrahedra in hyperbolic 3-space and are related to the following general framework. Since explicit formulae for geometric quantities in hyperbolic space (distance, area, volume, etc.) typically involve sophisticated transcendental functions, it is desirable (and quite useful in practice) to express these geometric quantities as monotonic functions of algebraic maps. Seidel's Speculation 1 says that the volume of an ideal tetrahedron in hyperbolic 3-space depends only on the determinant and permanent of the doubly stochastic Gram matrix of its vertices; Speculation 4 claims that the mentioned volume is monotone in both the determinant and permanent. We are able to give affirmative answers to Speculations 1 and 4 by parameterizing the classifying space of (labelled) ideal tetrahedra in a suitable way.

Key-words: Real hyperbolic space, Volume of ideal tetrahedra, Seidel's conjectures.

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INTRODUCTION

It is well known that transcendental methods are typically involved in calculating even simple geometric invariants in hyperbolic geometry (say, distance, area, volume, etc.). This has been observed already by Gauss, who referred to volume-related problems in hyperbolic geometry as a ‘jungle’. One way to deal with this kind of difficulty is to express a given geometric invariant as a monotonic function of algebraic expressions.

As a toy example, let us consider the projective model of hyperbolic n -space. We take an $(n+1)$ -dimensional $\mathbb{R}$ -linear space V equipped with a bilinear symmetric form of signature $- \dots - +$; the hyperbolic n -space is nothing but the open n -ball of positive points

$$\mathbb{H}^n := \{p \in \mathbb{P}_{\mathbb{R}} V \mid \langle p, p \rangle > 0\}$$

inside the projective space. Hyperbolic n -space is endowed with the (squared) distance function

$$d^2(p, q) := \operatorname{arcCosh} \frac{\langle p, q \rangle^2}{\langle p, p \rangle \langle q, q \rangle}$$

for $p, q \in \mathbb{H}^n$. Clearly, distance is a monotonic function of the *tance*

$$\operatorname{ta}(p, q) := \frac{\langle p, q \rangle^2}{\langle p, p \rangle \langle q, q \rangle}.$$

In practical applications, it is usually much simpler to consider the tance instead of the distance.

J. J. Seidel’s conjectures concern applying an analogous idea to the case of the volume formula of non-degenerate ideal tetrahedra in $\mathbb{H}^n$. A non-degenerate ideal tetrahedron is nothing but an n -tuple of points $(v_1, \dots, v_n)$ in the ideal boundary (the *absolute*) of $\mathbb{H}^n$ such that the points $v_1, \dots, v_n$ do not belong to a same hyperplane. We will actually call $(v_1, \dots, v_n)$ a *labelled* non-degenerate ideal tetrahedron because the vertices are specified in a particular order. Choosing representatives $v_i \in V$, $i = 1 \dots, n$, we associate to a non-degenerate ideal tetrahedron a *Gram matrix* $G := (g_{ij})$, $g_{ij} := \langle v_i, v_j \rangle$. Clearly, a Gram matrix of an ideal tetrahedron depends on the choice of the representatives $v_i \in V$. Among all the Gram matrices of a given

non-degenerate ideal tetrahedron, there is a single one, DSG , that is *doubly stochastic* (a matrix is called doubly stochastic if all its entries are non-negative and the sum of entries in every row and every column equals 1). Seidel claims that:

Speculation 1. The determinant and the permanent¹ of DSG completely determine the volume of the associated ideal tetrahedron.

Speculation 4. The volume is a monotonic function of the determinant and of the permanent of DSG .

We prove that both conjectures are true in the case $n = 3$, i.e., in the case of 3-dimensional hyperbolic space.² In fact, we prove a stronger version of Speculation 1: the determinant and permanent of DSG determine not only the volume of the corresponding tetrahedron, they determine the tetrahedron itself (modulo isometries).

Roughly speaking, the proof goes as follows. We begin by describing a classifying (moduli) space of all labelled non-degenerate ideal tetrahedra in $\mathbb{H}^3$ (see Subsection 3.1.3). This is accomplished by invoking a simple linear algebra fact: Let $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ be n -tuples in a finite dimensional linear space equipped with a symmetric bilinear form. Assume that the kernel of the form restricted to the subspaces $\mathbb{R}v_1 + \dots + \mathbb{R}v_n$ and $\mathbb{R}w_1 + \dots + \mathbb{R}w_n$ is null. Then the n -tuples differ by an element I of the orthogonal group $O V := \{J \in GL V \mid \langle Jv, Jw \rangle = \langle v, w \rangle\}$, $Iv_i = w_i$, if and only if their Gram matrices are the same. This fact plus an appropriate choice of coordinates allow us to describe the space of non-degenerate labelled ideal tetrahedron as the interior of an equilateral triangle Δ in the Euclidean plane. The altitudes of Δ divide the triangle into six smaller triangles — each is a copy of the space of (non-labelled) non-degenerate ideal tetrahedra.

The fact that we obtain six copies of the space of non-degenerate ideal tetrahedra is simple to explain. There is a natural action of the 4-symmetric group S_4 on the space of labelled non-degenerate ideal tetrahedra (permutation of vertices). But this action has a kernel which is isomorphic to Klein's four group H and we arrive at an effective $S_3 = S_4/H$ action. This S_3 -action simply permutes the three dihedral angles of an ideal tetrahedron: an ideal tetrahedron in hyperbolic 3-space has only three possibly distinct dihedral angles because opposite angles in the tetrahedron are necessarily equal. At the level of Δ , the mentioned S_3 -action corresponds to reflections in the altitudes.

Once Δ is obtained, Speculation 1 can be proved as follows. It is easy to obtain a necessary and sufficient condition on a pair of real numbers α, β such that α and β are respectively the determinant and permanent of the doubly stochastic matrix of at least one non-degenerate ideal tetrahedron (this is an application of the so-called Sylvester's criterion in linear algebra).

¹ The permanent of a matrix $G = (g_{ij})$ is defined by the expression $\text{per } G := \sum_{\sigma \in S_n} g_{1\sigma(1)}g_{2\sigma(2)} \dots g_{n\sigma(n)}$, where S_n stands for the symmetric n -group.

² We also partially prove Speculation 3 which concerns the minimum possible value of the permanent of some doubly stochastic matrices.

Figure 1 – The open set Δ .

Source: Elaborated by the author.

Writing down the formulae stating that the determinant and permanent of the doubly stochastic matrix of an arbitrary non-degenerate ideal tetrahedron respectively equal α and β leads to solving (in the generic case) a polynomial of degree six. It turns out that, once a single solution of this polynomial is obtained, the other solutions are the obvious ones: they correspond to the Gram matrices of tetrahedra that differ only as *labelled* tetrahedra.³ Figuring out the region Δ was a crucial step in the solution of Speculation 1 since the equations relating the determinant and permanent of a doubly stochastic matrix to coordinates in other parameterizations of the classifying space of tetrahedra tend to be quite involved (say, it does not look feasible to solve this problem in the coordinates a, b of the region U , see Subsection 3.1.2).

Solving the above mentioned polynomial of degree six allows us to express the entries of the double stochastic matrix DSG of an ideal tetrahedron in terms of its determinant α and permanent β . In this way, one finds an explicit formula for the volume in terms of α and β . Indeed, one of the simplest ways to calculate the volume of an ideal hyperbolic tetrahedron was discovered by Milnor, who found an expression in terms of Lobachevsky's function and the dihedral angles of the tetrahedron:

$$\text{vol}(T) = \pi(\theta_1) + \pi(\theta_2) + \pi(\theta_3).$$

It is quite simple⁴ to express the dihedral angles in terms of the entries of DSG and there-

³ It could be that Seidel did not state Speculation 1 in its full generality because he did not make a clear distinction between labelled and non-labelled tetrahedra. His Speculation 1 was possibly made after an observation that there existed tetrahedra with different Gram matrices but with the same volume. However, these tetrahedra are geometrically the same as non-labelled tetrahedra.)

⁴ Curiously, the entries of DSG constitute lengths of the sides of an Euclidean triangle whose internal angles are

fore we arrive at a volume formula that depends only on α and β . Differentiation plus a (non-straightforward) manipulation of inequalities leads to a proof of Speculation 4.

It should be mentioned that Speculation 1 was given a ‘counter-example’ in [1]. Actually, the authors of [1] do not deal with the doubly stochastic Gram matrix of vertices of an ideal tetrahedron. Instead, they consider a normalized Gram matrix of the points which are *polar* to the faces of the tetrahedron.

Seidel’s speculations combine very well with some aspects of the elementary representation theory of the symmetric group which we recall below (our approach follows [4] almost literally).

Let $n \in \mathbb{N}$ be a natural number. A *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ of n is a choice of non-null natural numbers $\lambda_1, \dots, \lambda_k$ satisfying $n = \sum_{j=1}^k \lambda_j$ and $\lambda_i \geq \lambda_{i+1}$ for every i . The *Young diagram* of a partition λ is a union of squares organized as follows: the first horizontal line consists of λ_1 adjacent squares, the second horizontal line consists of λ_2 adjacent squares, etc.; the k -th horizontal line consists of λ_k adjacent squares. The squares in consecutive lines are drawn in such a way that the first square of the upper line is adjacent to the first square of the lower line. A *Young tableau* T_λ of a Young diagram is obtained by filling the squares of the diagram with the numbers $1, \dots, n$ (without repetition). The results we are interested in do not depend on a particular Young tableau associated to a Young diagram, so we assume that the diagram is always filled in some particular way (say, from 1 to n following lines). Below we illustrate the 5 Young tableau related to the partitions of the number 4.

Figure 2 – Young tableau

Source: Elaborated by the author.

Let T_λ be a Young tableau. We define two subgroups $L_\lambda, C_\lambda \leq S_n$ of the symmetric group as follows: the subgroup L_λ consists of all permutations that preserve the rows of the Young tableau and the subgroup C_λ consists of all permutations that preserve the columns of the Young tableau.

---

exactly the dihedral angles of the corresponding ideal tetrahedron, see Subsection 3.1.2.

In the group algebra $\mathbb{C}S_n$ we introduce the *Young projectors*

$$a_\lambda := \frac{1}{|L_\lambda|} \sum_{g \in L_\lambda} g, \quad b_\lambda := \frac{1}{|C_\lambda|} \sum_{g \in C_\lambda} (\text{sgn } g)g,$$

where $\text{sgn } g$ denotes the sign of the permutation $g \in S_n$. The *Schur symmetrizer* is defined to be the element $c_\lambda := a_\lambda b_\lambda \in \mathbb{C}S_n$. (Note that the Schur symmetrizer is non-null because $L_\lambda \cap C_\lambda = 1$.)

The subspace $V_\lambda := (\mathbb{C}S_n)c_\lambda \leq \mathbb{C}S_n$ is an irreducible representation of S_n (and every irreducible representation of S_n is isomorphic to some V_λ).

We are now ready to introduce the Schur functors.

Given a partition λ of n , we denote $\mathbb{S}^\lambda V := V^{\otimes n} c_\lambda$. In particular, when $\lambda = (n)$ we have $\mathbb{S}^\lambda = \mathbb{S}^n$, i.e., $\mathbb{S}^\lambda$ is the n -th symmetric power functor. When $\lambda = (1, 1, \dots, 1)$, we have $\mathbb{S}^\lambda = \bigwedge^n$, that is, $\mathbb{S}^\lambda$ is n -th exterior power functor. For other partitions, $\mathbb{S}^\lambda$ ‘interpolates’ between these extremal cases.

The space $\mathbb{S}^\lambda V$ can be visualized as follows: their elements are (finite) linear combinations of *decomposable* terms of the type

$$v_1 \dots v_n := (v_1 \otimes \dots \otimes v_n) c_\lambda.$$

For instance, in the exterior power case, $v_1 \dots v_n = \sum_{g \in S_n} (\text{sgn } g) v_{g1} \otimes \dots \otimes v_{gn}$. In the symmetric power case, $v_1 \dots v_n = \sum_{g \in S_n} v_{g1} \otimes \dots \otimes v_{gn}$.

One can readily see that $\mathbb{S}^\lambda$ is indeed a functor from the category of finite dimensional linear spaces to itself. Indeed, we already know how $\mathbb{S}^\lambda$ behaves at the level of objects. At the level of morphisms, let $f : V \rightarrow W$ be a linear map. It suffices to define $\mathbb{S}^\lambda f : \mathbb{S}^\lambda V \rightarrow \mathbb{S}^\lambda W$ for decomposable elements: $(\mathbb{S}^\lambda f)(v_1 \dots v_n) := f v_1 \dots f v_n$. We have just arrived at the *Schur functors*.

It should be noted that, if V is endowed with a symmetric bilinear form $\langle -, - \rangle$, then $\mathbb{S}^\lambda V$ has a naturally induced symmetric bilinear form. This is well known, for example, in the case of the exterior power functor: the induced symmetric bilinear form on $\bigwedge^n V$ is given, at the level of decomposable elements, by

$$\langle v_1 \wedge \dots \wedge v_n, w_1 \wedge \dots \wedge w_n \rangle := \det a_{ij},$$

where $a_{ij} := \langle v_i, w_j \rangle$. In general, we have:

4.1. Definition — Theorem [5] *Let λ be a partition of n , let V_λ be the associated representation of the symmetric group S_n , and let $\mathbb{S}^\lambda$ be the corresponding Schur functor. Then*

$$\langle v_1 \dots v_n, w_1 \dots w_n \rangle := \sum_{g \in S_n} \chi_g a_{1g(1)} \cdot a_{2g(2)} \cdot \dots \cdot a_{ng(n)}$$

gives a induced symmetric bilinear form on $\mathbb{S}^\lambda V$, where $a_{ij} := \langle v_i, w_j \rangle$ and χ is the character of the representation.

The above formula, in the cases of the exterior and symmetric powers, corresponds respectively the determinant and permanent of the matrix with entries a_{ij} . Littlewood-Richardson call

$$\sum_{g \in \mathcal{S}_n} \chi g \, a_{1g(1)} \cdot a_{2g(2)} \cdots a_{ng(n)}$$

the *immanant* of the matrix with entries a_{ij} (see [7]).

There are deep connections between Schur functors and the geometry of $\mathbb{H}^n$. For instance, let $p_1, p_2, p_3 \in \mathbb{H}^3$ be pairwise distinct points. Then $p_1 \wedge p_2 \wedge p_3 \in \wedge^3 V$ corresponds (under the Hodge star operator $\star : \wedge^3 V \rightarrow \wedge^1 V = V$) to the polar point of the plane generated by p_1, p_2, p_3 (see Subsection 2.2.4). In order to calculate the angle θ between the planes generated by, say, p_1, p_2, p_3 and q_1, q_2, q_3 , one essentially has to calculate the tance between $p := p_1 \wedge p_2 \wedge p_3$ and $q := q_1 \wedge q_2 \wedge q_3$ in $\wedge^3 V$; explicitly, the mentioned angle is given by

$$\cos^2 \theta = \frac{\langle p, q \rangle^2}{\langle p, p \rangle \langle q, q \rangle},$$

where the symmetric bilinear form in the previous formula is the one induced by the exterior power functor in $\wedge^3 V$.

Hence, is it no accident that the determinant and permanent (of a suitably normalized Gram matrix of ideal points) play an important role in volume problems. Since non-degenerate ideal tetrahedra in $\mathbb{H}^3$ form a 2-dimensional manifold, determinant and permanent suffice. In the general case of a tetrahedron whose vertices are all inside $\mathbb{H}^3$, the classifying space is 6-dimensional. There are, as we saw above, 5 Schur functors related to the representations of the symmetric group S_4 . Therefore, we make the following

Speculation. *Let T be a tetrahedron whose vertices (v_1, v_2, v_3, v_4) are all in $\mathbb{H}^3$. Then T is determined, up to isometry, by the trace and all immanants of the doubly stochastic Gram matrix associated to the vertices. The volume of T is a monotonic function of the trace and of all immanants of this matrix.*

PRELIMINARIES

2.1 The projective space

Let V be an $\mathbb{R}$ -linear space of dimension $n + 1$ endowed with the linear topology, i.e., the topology induced from any norm in V . We denote by $\mathbb{P}_{\mathbb{R}}V$ the *projective space*. We have the quotient map $\pi : V^* \rightarrow \mathbb{P}_{\mathbb{R}}V$, where $V^* := V \setminus \{0\}$ is the linear space punctured at the origin.

Let $b_0, b_1, \dots, b_n$ be an ordered basis in V . Let $x_0, x_1, \dots, x_n : V \rightarrow \mathbb{R}$ be linear coordinates in regard to this basis, that is, $x_i(v) = x_i$ where $v = \sum_{i=0}^n x_i b_i$. In this case, we also write $v = (x_0, x_1, \dots, x_n)$ and $\pi(v) = [x_0 : x_1 : \dots : x_n]$.

Finally, we introduce some notation. We frequently write $\mathbf{p}$ instead of $\pi(p)$, where $p \in V^*$. In such cases, by this notation we mean that our considerations do not change if we re-choose representatives in V of points in the projective space. One more convention. Given a subset $S \subset V$ we denote by $\mathbb{P}_{\mathbb{R}}S := \pi(S \setminus \{0\}) \subset \mathbb{P}_{\mathbb{R}}V$ the image of $S \subset V$ under the quotient map $\pi : V^* \rightarrow \mathbb{P}_{\mathbb{R}}V$. In this way, for every linear subspace $K \leq V$, we can consider the projective space $\mathbb{P}_{\mathbb{R}}K$ inside $\mathbb{P}_{\mathbb{R}}V$.

2.1.1 The real hyperbolic space $\mathbb{H}_{\mathbb{R}}^n$

From now on, we suppose that V is equipped with an hermitian form $\langle -, - \rangle : V \times V \rightarrow \mathbb{R}$ of signature $(n, 0, 1)$, see Definition A.1.23. As V is a real linear space we have that $\langle -, - \rangle$ is a bilinear symmetric form.

Definition 2.1.1. The *signature* of $\mathbf{p} \in \mathbb{P}_{\mathbb{R}}V$ is the sign of $\langle p, p \rangle$. It is well defined since, for another representative $rp \in V$, $r \in \mathbb{R}^*$, we have $\langle rp, rp \rangle = r^2 \langle p, p \rangle$. The projective space $\mathbb{P}_{\mathbb{R}}V$ is divided into three disjoint parts consisting of *positive*, *negative*, and *isotropic* points:

$$BV := \{\mathbf{p} \in \mathbb{P}_{\mathbb{R}}V \mid \langle p, p \rangle > 0\}, \quad EV := \{\mathbf{p} \in \mathbb{P}_{\mathbb{R}}V \mid \langle p, p \rangle < 0\}, \quad SV := \{\mathbf{p} \in \mathbb{P}_{\mathbb{R}}V \mid \langle p, p \rangle = 0\}.$$

The isotropic points constitute the *absolute* SV of $\mathbb{P}_{\mathbb{R}}V$.

Proposition 2.1.2. Let V be a space of signature $(n, 0, 1)$. The following are satisfied in $\mathbb{P}_{\mathbb{R}}V$:

- (i) The positive and negative points, BV and EV , are open subsets of $\mathbb{P}_{\mathbb{R}}V$. The absolute SV is a closed subset in $\mathbb{P}_{\mathbb{R}}V$.
- (ii) $\overline{BV} = BV \cup SV$ and $\overline{EV} = EV \cup SV$.
- (iii) The absolute SV is the boundary of BV and EV .

Proof. The function $V^* \rightarrow \mathbb{R}$, $p \mapsto \langle p, p \rangle$ is a composition of continuous functions. Then, the set $\{p \in V^* \mid \langle p, p \rangle > 0\}$ is open in V^* . As $\{p \in V^* \mid \langle p, p \rangle > 0\} = \pi^{-1}(BV)$ and π is a quotient map we have that BV is open in $\mathbb{P}_{\mathbb{R}}V$. In a same way, EV is an open set in $\mathbb{P}_{\mathbb{R}}V$. Then, $SV = \mathbb{P}_{\mathbb{R}}V \setminus (BV \cup EV)$ is a closed subset in $\mathbb{P}_{\mathbb{R}}V$.

It is clear that $\overline{BV} \subset BV \cup SV$ because $BV \cup SV = \mathbb{P}_{\mathbb{R}}V \setminus EV$ is a closed set that contains BV . Let $p \in BV \cup SV = \{p \in \mathbb{P}_{\mathbb{R}}V \mid \langle p, p \rangle \geq 0\}$. Let U be a neighborhood of p in $\mathbb{P}_{\mathbb{R}}V$. Then, $\pi^{-1}(U)$ is a neighborhood of p in V^* . Because of the signature of the space V , we can choose $b \in V$ such that $\langle b, b \rangle > 0$. Let $r \in \mathbb{R}^*$ such that $r\langle p, b \rangle \geq 0$ and $p + rb \in \pi^{-1}(U)$. Thus, $\langle p + rb, p + rb \rangle = \langle p, p \rangle + 2r\langle p, b \rangle + r^2\langle b, b \rangle > 0$, i.e., $\pi(p + rb) \in BV \cap U$. This shows that $p \in \overline{BV}$. We conclude that $\overline{BV} = BV \cup SV$. The proof of $\overline{EV} = EV \cup SV$ is analogous.

The boundary of BV is

$$\overline{BV} \cap \overline{\mathbb{P}_{\mathbb{R}}V \setminus BV} = \overline{BV} \cap (\overline{EV \cup SV}) = \overline{BV} \cap (\overline{EV} \cup \overline{SV}) = (BV \cup SV) \cap (EV \cup SV) = SV.$$

It is similar to show that the boundary of EV is SV . □

Proposition 2.1.3. $\overline{BV}$ is homeomorphic to the closed ball $\mathbb{B}^n$, BV is homeomorphic to the open ball $\mathbb{B}^n$ and SV is homeomorphic to the sphere $\mathbb{S}^{n-1}$.

Proof. Let $b_0, b_1, \dots, b_n$ be an orthonormal basis in V such that b_0 is positive and the other vectors are negative. Let $p \in BV \cup SV$ and take any representative $p = x_0b_0 + x_1b_1 + \dots + x_nb_n$. It satisfies $\langle p, p \rangle = x_0^2 - x_1^2 - \dots - x_n^2 \geq 0$, thus $x_0 \neq 0$. We can therefore assume that $x_0 = 1$ and such a representative of p is unique. Then, the map $\overline{\mathbb{B}^n} \rightarrow BV \cup SV$, $(\alpha_1, \dots, \alpha_n) \mapsto [1 : \alpha_1 : \dots : \alpha_n]$ is a continuous bijection. As the closed ball $\overline{\mathbb{B}^n}$ is compact and $BV \cup SV$ is a Hausdorff space the map is a homeomorphism. By restricting the map to the open ball $\mathbb{B}^n$ and the sphere $\mathbb{S}^{n-1}$ we obtain the homeomorphism between BV and $\mathbb{B}^n$ and the homeomorphism between SV and $\mathbb{S}^{n-1}$ respectively. □

Let $p \in \mathbb{P}_{\mathbb{R}}V$ be nonisotropic. We introduce the following notation for the orthogonal decomposition:¹

$$V = \mathbb{R}p \oplus p^\perp, \quad v = \pi_1(p)v + \pi_2(p)v,$$

where

$$\pi_1(p)v := \frac{\langle v, p \rangle}{\langle p, p \rangle} p \in \mathbb{R}p, \quad \pi_2(p)v := v - \frac{\langle v, p \rangle}{\langle p, p \rangle} p \in p^\perp.$$

¹ We denote by $p^\perp$ the subspace $(\mathbb{R}p)^\perp = \{v \in V \mid \langle v, p \rangle = 0\}$.

The definitions of the functions $\pi_1(\mathbf{p})$ and $\pi_2(\mathbf{p})$ do not depend on the choice of a representative $p \in V$.

2.1.2 Tangent space

Let $\mathbf{p} \in \mathbb{P}_{\mathbb{R}}V$. Let $f \in C^\infty(U)$, where $U \subset \mathbb{P}_{\mathbb{R}}V$ is an open neighborhood of $\mathbf{p}$. Let $\tilde{f}$ denote the map $f \circ \pi|_{\pi^{-1}(U)} \in C^\infty(\pi^{-1}(U))$. The function $\tilde{f}$ satisfies $\tilde{f}(rp) = \tilde{f}(p)$, for all $r \in \mathbb{R}^*$. Since $\tilde{f}$ is smooth it defines the linear map $D_p \tilde{f} : V \rightarrow \mathbb{R}$ given by

$$D_p \tilde{f}(v) = \lim_{t \rightarrow 0} \frac{\tilde{f}(p + tv) - \tilde{f}(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} \tilde{f}(p + tv),$$

for all $v \in V$.

Proposition 2.1.4. The map $t : \text{Lin}(\mathbb{R}p, V) \rightarrow T_{\mathbf{p}}\mathbb{P}_{\mathbb{R}}V$, $\varphi \mapsto t_\varphi$, where $t_\varphi f := D_p \tilde{f}(\varphi p)$, for all $f \in C^\infty(U)$, is linear. Also, $t_\varphi = 0$ if and only if $\varphi p \in \mathbb{R}p$.

Proof. Note that $t_\varphi \in T_{\mathbf{p}}\mathbb{P}_{\mathbb{R}}V$ because $\widetilde{f+g} = \tilde{f} + \tilde{g}$, $\widetilde{rf} = r\tilde{f}$, and $\widetilde{fg} = \tilde{f}\tilde{g}$, for all $f, g \in C^\infty(\pi^{-1}(U))$ and $r \in \mathbb{R}$. The linearity of the map $D_p \tilde{f}$ guarantees the linearity of the map t .

Suppose that $t_\varphi = 0$. Let $b_0, b_1, \dots, b_n$ be any basis in V . Write $p = \alpha_0 b_0 + \alpha_1 b_1 + \dots + \alpha_n b_n$. For simplicity we assume that $\alpha_0 \neq 0$. So, $\mathbf{p} \in U_0 = \{[x_0 : x_1 : \dots : x_n] | x_0 \neq 0\}$. The function $f_i : U_0 \rightarrow \mathbb{R}$, $[x_0 : x_1 : \dots : x_n] \mapsto \frac{x_i}{x_0}$ is smooth for each $i = 1, \dots, n$, and $\tilde{f}_i : x_0 b_0 + x_1 b_1 + \dots + x_n b_n \mapsto \frac{x_i}{x_0}$. Let $\varphi p = \gamma_0 b_0 + \gamma_1 b_1 + \dots + \gamma_n b_n$. By hypothesis,

$$0 = t_\varphi f_i = \left. \frac{d}{dt} \right|_{t=0} \frac{\alpha_i + t\gamma_i}{\alpha_0 + t\gamma_0} = \frac{\gamma_i \alpha_0 - \alpha_i \gamma_0}{\alpha_0^2}.$$

Note that $\gamma_i = \frac{\gamma_0}{\alpha_0} \alpha_i$ for each $i = 0, 1, \dots, n$, then $\varphi p = \frac{\gamma_0}{\alpha_0} p \in \mathbb{R}p$. □

Corollary 2.1.5. The map $j : \text{Lin}(\mathbb{R}p, V/\mathbb{R}p) \rightarrow T_{\mathbf{p}}\mathbb{P}_{\mathbb{R}}V$, $\varphi \mapsto j(\varphi)$ such that $j(\varphi) : f \mapsto D_p \tilde{f}(\widehat{\varphi p})$, where $\widehat{\varphi p} \in V$ is any representative of the class $\varphi p \in V/\mathbb{R}p$, is a linear isomorphism.

Proof. Note that $j(\varphi)$ is well defined. The linearity of the map $D_p \tilde{f}$ guarantees the linearity of the map j . As the linear spaces $\text{Lin}(\mathbb{R}p, V/\mathbb{R}p)$ and $T_{\mathbf{p}}\mathbb{P}_{\mathbb{R}}V$ have the same dimension, the injectivity of j which comes from the last proposition which shows the isomorphism between them. □

Finally, for a nonisotropic $\mathbf{p} \in \mathbb{P}_{\mathbb{R}}V$, we have $V/\mathbb{R}p \cong p^\perp$ and conclude that

$$T_{\mathbf{p}}\mathbb{P}_{\mathbb{R}}V \cong \text{Lin}(\mathbb{R}p, p^\perp) \cong \langle -, p \rangle p^\perp,$$

where $\langle -, p \rangle p^\perp := \{ \langle -, p \rangle v : x \mapsto \langle x, p \rangle v \}$.

2.1.3 Metric

Let $\mathbf{p} \in \mathbb{P}_{\mathbb{R}}V$ be a nonisotropic point. Given $v \in p^{\perp}$, we define

$$t_{p,v} := \langle -, p \rangle v \in T_{\mathbf{p}}\mathbb{P}_{\mathbb{R}}V.$$

Note that $t_{p,v}$ does depend on the choice of a representative $p \in V$: if we pick a new representative $rp \in V$, $r \in \mathbb{R}^*$, then we must take $\frac{1}{r}v \in V$ in place of v in order to keep $t_{p,v}$ the same.

The tangent space $T_{\mathbf{p}}\mathbb{P}_{\mathbb{R}}V$ is equipped with the hermitian form

$$\langle t_{p,v_1}, t_{p,v_2} \rangle := -\langle p, p \rangle \langle v_1, v_2 \rangle. \quad (2.1)$$

This definition is correct as the formula is independent of the choice of representatives $p, v_1, v_2 \in V$ providing the same $\langle t_{p,v_1}, t_{p,v_2} \rangle$. One can readily see that this hermitian form, called a *hermitian metric*, depends smoothly on a nonisotropic $\mathbf{p}$.

We are going to work on the open set of positive points $BV \subset \mathbb{P}_{\mathbb{R}}V$. With the metric defined in equation 2.1, the space BV becomes a Riemannian manifold. We will denote this space by $\mathbb{H}_{\mathbb{R}}^n$ and call it *the hyperbolic space*.

2.2 Linear objects and duality

2.2.1 Geodesics

Definition 2.2.1. Let $W \leq V$ be a two-dimensional linear subspace such that the hermitian form, being restricted to W , is nonnull. We call $\mathbb{P}_{\mathbb{R}}W \subset \mathbb{P}_{\mathbb{R}}V$ a *geodesic*.

Let b_1, b_2 be an orthonormal basis in W . As the signature of V is $(n, 0, 1)$, by Claim A.1.27, the possible signatures of b_1, b_2 are $(2, 0, 0)$, $(1, 0, 1)$, and $(1, 1, 0)$. Also, we suppose that negative vectors, null vectors and positive vectors are listed in that order in the basis b_1, b_2 . Let $\mathbf{p} \in \mathbb{P}_{\mathbb{R}}W$ and write $p = rb_1 + sb_2$ for some $r, s \in \mathbb{R}$.

Let b_1, b_2 be of signature $(2, 0, 0)$. We have $\langle p, p \rangle = -r^2 - s^2 < 0$, then $\mathbf{p}$ is negative. Thus, the geodesic $\mathbb{P}_{\mathbb{R}}W$ is totally contained in EV .

Let b_1, b_2 be of signature $(1, 0, 1)$. The solutions of the equation $\langle p, p \rangle = -r^2 + s^2 = 0$ are $s = \pm r$. We can suppose that $r = 1$. Then, the isotropic points in the geodesic $\mathbf{p} \in \mathbb{P}_{\mathbb{R}}W$ are the classes of $b_1 + b_2$ and $b_1 - b_2$. These points are the *vertices* of the geodesic. Note that if we write an element $w \in W$ as $w = r(b_1 + b_2) + s(b_1 - b_2)$, where $r, s \in \mathbb{R}$, then the point $\mathbf{w} \in BV$ when r and s are of the same sign and $\mathbf{w} \in EV$ when r and s are of distinct sign.

Let b_1, b_2 be of signature $(1, 1, 0)$. We have $\langle p, p \rangle = -r^2$. The unique isotropic point in the geodesic $\mathbb{P}_{\mathbb{R}}W$ is $\mathbf{b}_2$. All the other points in $\mathbb{P}_{\mathbb{R}}W$ are in EV .

2.2.2 Totally geodesic planes

Definition 2.2.2. Let $W \leq V$ be a three-dimensional linear subspace. We call $\mathbb{P}_{\mathbb{R}}W \subset \mathbb{P}_{\mathbb{R}}V$ a *plane*.

Let b_1, b_2, b_3 be an orthonormal basis in W . As the signature of V is $(n, 0, 1)$, by Claim A.1.27, the possible signatures of b_1, b_2, b_3 are $(3, 0, 0)$, $(2, 0, 1)$, and $(2, 1, 0)$. Also, we suppose that the negative vectors, the null and the positive vectors are listed in that order in the basis b_1, b_2, b_3 . Let $\mathbf{p} \in \mathbb{P}_{\mathbb{R}}W$ and write $p = rb_1 + sb_2 + tb_3$ for some $r, s, t \in \mathbb{R}$.

Let b_1, b_2, b_3 be of signature $(3, 0, 0)$. We have $\langle p, p \rangle = -r^2 - s^2 - t^2 < 0$, then $\mathbf{p}$ is negative. Thus, the plane $\mathbb{P}_{\mathbb{R}}W$ is totally contained in EV .

Let b_1, b_2, b_3 be of signature $(2, 0, 1)$. The solutions for the equation $\langle p, p \rangle = -r^2 - s^2 + t^2 = 0$ are $t^2 = r^2 + s^2$. Thus, we can take the unique representative of $\mathbf{p}$ such that $t = 1$, i.e., $r^2 + s^2 = 1$. It follows that $\mathbb{P}_{\mathbb{R}}W \cap SV$ is homeomorphic to the sphere $\mathbb{S}^1$. In the same way we can show that $\mathbb{P}_{\mathbb{R}}W \cap BV$ is homeomorphic to the open ball $\mathbb{B}^2$.

Let b_1, b_2, b_3 be of signature $(2, 1, 0)$. We have $\langle p, p \rangle = -r^2 - s^2$. The unique isotropic point in the plane $\mathbb{P}_{\mathbb{R}}W$ is $\mathbf{b}_3$. All the other points in $\mathbb{P}_{\mathbb{R}}W$ are in EV .

2.2.3 Duality

Let $W \leq V$ be an n -dimensional linear subspace. As the signature of V is $(n, 0, 1)$, the possible signatures of W are $(n, 0, 0)$, $(n-1, 0, 1)$, and $(n-1, 1, 0)$. In the following we use frequently Claim A.1.6.

Let W be of signature $(n, 0, 0)$, then $W^\perp = \mathbb{R}p$, where $\mathbf{p} \in BV$. Also, if $\mathbf{p} \in BV$, we have $p^\perp = W$, where W is of signature $(n, 0, 0)$.

Let W be of signature $(n-1, 0, 1)$, then $W^\perp = \mathbb{R}p$, where $\mathbf{p} \in EV$. Also, if $\mathbf{p} \in BV$, we have $p^\perp = W$, where W is of signature $(n-1, 0, 1)$.

Let W be of signature $(n-1, 1, 0)$, by Claim A.1.5 we have $\dim W^\perp = 1$, then $W^\perp = \mathbb{R}p$. By Claim A.1.3, $(\mathbb{R}p)^\perp = (W^\perp)^\perp \supset W$. Assuming that p is positive or negative, we have that $\mathbb{R}p$ is nondegenerate and then $(\mathbb{R}p)^\perp$ has dimension n , thus $(\mathbb{R}p)^\perp = W$ which is a contradiction because W is degenerate. We conclude that $\mathbf{p} \in SV$. In the other sense, if $\mathbf{p} \in SV$. By Claim A.1.5 we know that $\dim p^\perp \geq 3$. As the form in V is nondegenerate we conclude that $\dim p^\perp = 3$. As $\mathbb{R}p \subset p^\perp$ and the possible signatures for $p^\perp$ are $(n, 0, 0)$, $(n-1, 0, 1)$, and $(n-1, 1, 0)$ we conclude that the signature of $p^\perp$ is $(n-1, 1, 0)$.

When $n = 2$, we have the duality between points and geodesics in $\mathbb{P}_{\mathbb{R}}V$.

When $n = 3$, we have the duality between points and planes in $\mathbb{P}_{\mathbb{R}}V$.

2.2.4 Dihedral angles

Let us apply the duality between points and planes in real hyperbolic space in order to find an expression for the dihedral angle between planes that we will apply in the proof of Seidel's Speculation 4. First, we need to discuss a few basic facts about the Hodge star operator. There is no need to specify the dimension of V neither the signature of the non-degenerate symmetric bilinear form until the very end of the subsection. The results in this subsection are related to Section 2.2 in [9].

Let V be an $\mathbb{R}$ -linear N -dimensional space equipped with a non-degenerate symmetric bilinear form $\langle -, - \rangle$ of signature $(n, 0, m)$, where $N = n + m$. Let $\sigma = 1$ if $n = 0 \pmod{2}$ and $\sigma = -1$ otherwise.

The k -th exterior power $\bigwedge^k V$, $1 \leq k \leq N$, is equipped with the symmetric bilinear form defined by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle := \det(g_{ij}),$$

where $g_{ij} := \langle v_i, w_j \rangle$. Note that this form on $\bigwedge^k V$ is non-degenerate. Indeed, let $(b_1, \dots, b_N)$ be an orthonormal basis for V , that is, $\langle b_i, b_i \rangle = \sigma_i$ and $\langle b_i, b_j \rangle = 0$ for $i \neq j$, where $\sigma_i = \pm 1$. Then $(b_{i_1} \wedge \cdots \wedge b_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq N)$, is an orthonormal basis for $\bigwedge^k V$ since $\langle b_{i_1} \wedge \cdots \wedge b_{i_k}, b_{i_1} \wedge \cdots \wedge b_{i_k} \rangle = \sigma_{i_1} \cdots \sigma_{i_k} = \pm 1$ and $\langle b_{i_1} \wedge \cdots \wedge b_{i_k}, b_{j_1} \wedge \cdots \wedge b_{j_k} \rangle$ vanishes for $i_1 < \cdots < i_k, j_1 < \cdots < j_k$ and $(i_1, \dots, i_k) \neq (j_1, \dots, j_k)$ because it is the determinant of a matrix that has at least one null row.

Fix $\omega \in \bigwedge^N V$ such that $\langle \omega, \omega \rangle = \sigma$. The Hodge star operator is the $\mathbb{R}$ -linear map $*$: $\bigwedge^k V \rightarrow \bigwedge^{N-k} V$, $b \mapsto *b$, defined by requiring that $a \wedge *b = \langle a, b \rangle \omega$ for every $a \in \bigwedge^k V$. Clearly, $*\omega = \sigma$ since $\omega \wedge *\omega = \langle \omega, \omega \rangle \omega = \sigma \omega$.

Lemma 2.2.3. The Hodge star operator is injective. It satisfies the following identities:

- $a \wedge *b = b \wedge *a$ for every $a, b \in \bigwedge^k V$
- $\langle a, b \rangle = \sigma \cdot *(a \wedge *b) = \sigma \cdot *(b \wedge *a)$ for every $a, b \in \bigwedge^k V$
- $\langle a, *b \rangle = (-1)^{k(N-k)} \langle *a, b \rangle$, for every $a \in \bigwedge^k V$ and $b \in \bigwedge^{N-k} V$
- $*^2 = (-1)^{k(N-k)} \sigma$

Proof. If $*b = 0$, then $a \wedge *b = \langle a, b \rangle \omega = 0$ for every $a \in \bigwedge^k V$ which implies $b = 0$ because the form on $\bigwedge^k V$ is nondegenerate.

The first identity is obvious. The second follows from $*(a \wedge *b) = \langle a, b \rangle \cdot *\omega = \sigma \langle a, b \rangle$. The third follows from the second:

$$\langle a, *b \rangle = \sigma \cdot *(*b \wedge *a) = (-1)^{k(N-k)} \sigma \cdot *(*a \wedge *b) = (-1)^{k(N-k)} \langle *a, b \rangle.$$

Finally, let us show the last equality. Take an orthonormal basis $b_1, \dots, b_N$ in V , that is, $\langle b_i, b_i \rangle = \sigma_i = \pm 1$ and $\langle b_i, b_j \rangle = 0$ for $i \neq j$. We can assume that $\omega = b_1 \wedge \dots \wedge b_N$. Fix $i_1 < i_2 < \dots < i_k$ and let $j_1 < j_2 < \dots < j_{N-k}$ be the indices complementary to $i_1 < i_2 < \dots < i_k$. Let us show that $*(b_{i_1} \wedge \dots \wedge b_{i_k}) = \text{sgn}(h) \cdot \sigma_{i_1} \dots \sigma_{i_k} \cdot b_{j_1} \wedge \dots \wedge b_{j_{N-k}}$, where h is the permutation $(i_1, i_2, \dots, i_k, j_1, j_2, \dots, j_{N-k}) \mapsto (1, 2, \dots, N)$. Indeed, for every $l_1 < \dots < l_k$,

$$\begin{aligned} (b_{l_1} \wedge \dots \wedge b_{l_k}) \wedge (\text{sgn}(h) \cdot \sigma_{i_1} \dots \sigma_{i_k} \cdot b_{j_1} \wedge \dots \wedge b_{j_{N-k}}) &= \begin{cases} 0, & \text{if } (l_1, \dots, l_k) \neq (i_1, \dots, i_k) \\ \sigma_{i_1} \dots \sigma_{i_k} \cdot \omega & \text{if } (l_1, \dots, l_k) = (i_1, \dots, i_k) \end{cases} \\ &= \langle b_{l_1} \wedge \dots \wedge b_{l_k}, b_{i_1} \wedge \dots \wedge b_{i_k} \rangle \omega. \end{aligned}$$

Finally, it remains to observe that

$$*(* (b_{i_1} \wedge \dots \wedge b_{i_k})) = \text{sgn}(h) \cdot \sigma_{i_1} \dots \sigma_{i_k} \cdot *(b_{j_1} \wedge \dots \wedge b_{j_{N-k}}) = (-1)^{k(N-k)} \sigma \cdot b_{i_1} \wedge \dots \wedge b_{i_k},$$

where $(-1)^{k(N-k)} = \text{sgn}(h) \cdot \text{sgn}(h')$ and h' is the permutation $(j_1, \dots, j_{N-k}, i_1, \dots, i_k) \mapsto (1, 2, \dots, N)$. $\square$

From now on, assume that V is of signature $(3, 0, 1)$. Let $v_1, v_2, v_3 \in SV$ be pairwise distinct isotropic points that generate the plane $P := \mathbb{P}_{\mathbb{R}} W_1$ for $W_1 = \mathbb{R}v_1 + \mathbb{R}v_2 + \mathbb{R}v_3$. We claim that $u := *(v_1 \wedge v_2 \wedge v_3)$ is the polar point to P . Indeed, by the previous lemma,

$$\langle v_i, *(v_1 \wedge v_2 \wedge v_3) \rangle \cdot \omega = v_i \wedge *(*(v_1 \wedge v_2 \wedge v_3)) = v_i \wedge v_1 \wedge v_2 \wedge v_3 = 0$$

for $i = 1, 2, 3$. We are now ready to calculate the expression for the angle between (half-)planes that we will need in Chapter 3.

Let $v_1, v_2, w_1, w_2 \in SV$ be pairwise distinct isotropic points. Let $u_1 := *(v_1 \wedge v_2 \wedge w_1)$ be the polar point to the plane P_1 generated by v_1, v_2, w_1 and let $u_2 := *(v_1 \wedge v_2 \wedge w_2)$ be the polar point to the plane P_2 generated by v_1, v_2, w_2 . The geodesic G joining v_1, v_2 is common to P_1 and P_2 ; let $p \in G$. We take a representative $p \in V$ such that $\langle p, p \rangle = 1$. Our intention is to measure the angle $\theta \in [0, \pi]$ between the half-plane in P_1 which contains w_1 and is determined by G and the half-plane in P_2 which contains w_2 and is determined by G .

By Lemma 4.2.2. in [3], $n_i := \langle -, p \rangle u_i$ is a normal vector to P_i at p , $i = 1, 2$. The angle $\theta \in [0, \pi]$ is exactly the angle between n_1 and n_2 , i.e.,

$$\cos \theta = \frac{\langle n_1, n_2 \rangle}{\sqrt{\langle n_1, n_1 \rangle} \sqrt{\langle n_2, n_2 \rangle}} = \frac{-\langle u_1, u_2 \rangle}{\sqrt{-\langle u_1, u_1 \rangle} \sqrt{-\langle u_2, u_2 \rangle}}$$

From Lemma 2.2.3 we obtain

$$-\langle u_1, u_2 \rangle = -\langle *(v_1, v_2, w_1), *(v_1, v_2, w_2) \rangle = \det \begin{pmatrix} 0 & \langle v_1, v_2 \rangle & \langle v_1, w_2 \rangle \\ \langle v_2, v_1 \rangle & 0 & \langle v_2, w_2 \rangle \\ \langle w_1, v_1 \rangle & \langle w_1, v_2 \rangle & \langle w_1, w_2 \rangle \end{pmatrix},$$

$$-\langle u_1, u_1 \rangle = -\langle * (v_1, v_2, w_1), * (v_1, v_2, w_1) \rangle = \det \begin{pmatrix} 0 & \langle v_1, v_2 \rangle & \langle v_1, w_1 \rangle \\ \langle v_2, v_1 \rangle & 0 & \langle v_2, w_1 \rangle \\ \langle w_1, v_1 \rangle & \langle w_1, v_2 \rangle & 0, \end{pmatrix},$$

and

$$-\langle u_2, u_2 \rangle = -\langle * (v_1, v_2, w_2), * (v_1, v_2, w_2) \rangle = \det \begin{pmatrix} 0 & \langle v_1, v_2 \rangle & \langle v_1, w_2 \rangle \\ \langle v_2, v_1 \rangle & 0 & \langle v_2, w_2 \rangle \\ \langle w_2, v_1 \rangle & \langle w_2, v_2 \rangle & 0 \end{pmatrix}.$$

since $*^2 = 1$.

In Chapter 3, we will calculate the above angle θ in terms of some specific representatives for $v_1, v_2, w_1, w_2 \in V$.

TETRAHEDRA

In this chapter, we introduce the space of *labelled ideal tetrahedra* as well as a couple of parameterizations of this space. The main objective is to arrive at the *doubly stochastic* parameterization which is used to formulate Seidel's conjecture.

From now on, we deal only with an $\mathbb{R}$ -linear space V of signature $(3, 0, 1)$. As in the previous chapter, it gives rise to the extended real hyperbolic space divided into the proper hyperbolic space $\mathbb{H}^3$, its absolute SV and elsewhere EV .

3.1 Ideal and labelled ideal tetrahedra in $\mathbb{H}^3$

An *ideal tetrahedron* in $\mathbb{P}_{\mathbb{R}}V$ consists of four ideal points $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, $\mathbf{v}_i \in SV$. A *labelled ideal tetrahedron* is an ideal tetrahedron with ordered vertices, i.e., a 4-tuple $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ of ideal points. The set of ideal tetrahedra with ordered vertices is denoted by

$$\mathcal{T} := \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \mid \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \text{ are points in } SV\}.$$

3.1.1 A first classifying space

The main result in this subsection is Theorem 3.1.5, where a parameterization of the space of labelled non-degenerate ideal tetrahedra is given. The theorem is then applied in Proposition 3.1.8 to find, in an explicit form, the unique doubly stochastic matrix associated to a non-degenerate labelled ideal tetrahedron.

Definition 3.1.1. Given $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$, we denote the whole class of Gram matrices of representatives of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ by

$$\mathcal{M}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) := \{G(v_1, v_2, v_3, v_4) \mid v_i \text{ is a representative of } \mathbf{v}_i \text{ for } i = 1, 2, 3, 4\}.$$

Of course, rechoosing representatives, one can describe $\mathcal{M}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ using the Gram matrix $G(v_1, v_2, v_3, v_4)$ of any given representatives v_1, v_2, v_3, v_4 of the vertices. This simple fact is stated and proved below.

Proposition 3.1.2. *Let $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$. Fix representatives v_1, v_2, v_3, v_4 of the vertices of the labelled ideal tetrahedron. Then*

$$\mathcal{M}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \{DG(v_1, v_2, v_3, v_4)D \mid D = \text{diag}(x_1, x_2, x_3, x_4), x_i \in \mathbb{R}^* \text{ for } i = 1, 2, 3, 4\}.$$

Proof. Let u_1, u_2, u_3, u_4 be arbitrary representatives of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. This means that $u_i = x_i v_i$ where $x_i \neq 0, i = 1, 2, 3, 4$. The Gram matrix

$$\begin{aligned} G(u_1, u_2, u_3, u_4) &= G(x_1 v_1, x_2 v_2, x_3 v_3, x_4 v_4) \\ &= \begin{pmatrix} \langle x_1 v_1, x_1 v_1 \rangle & \langle x_1 v_1, x_2 v_2 \rangle & \langle x_1 v_1, x_3 v_3 \rangle & \langle x_1 v_1, x_4 v_4 \rangle \\ \langle x_2 v_2, x_1 v_1 \rangle & \langle x_2 v_2, x_2 v_2 \rangle & \langle x_2 v_2, x_3 v_3 \rangle & \langle x_2 v_2, x_4 v_4 \rangle \\ \langle x_3 v_3, x_1 v_1 \rangle & \langle x_3 v_3, x_2 v_2 \rangle & \langle x_3 v_3, x_3 v_3 \rangle & \langle x_3 v_3, x_4 v_4 \rangle \\ \langle x_4 v_4, x_1 v_1 \rangle & \langle x_4 v_4, x_2 v_2 \rangle & \langle x_4 v_4, x_3 v_3 \rangle & \langle x_4 v_4, x_4 v_4 \rangle \end{pmatrix} \\ &= DG(v_1, v_2, v_3, v_4)D, \end{aligned}$$

where D is the diagonal matrix $\text{diag}(x_1, x_2, x_3, x_4)$, $x_i \neq 0$. □

Observation 3.1.3. Suppose that v_i and u_i are representatives of the vertex $\mathbf{v}_i$ of the ideal tetrahedron $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$. Then (v_1, v_2, v_3, v_4) is linearly independent if and only if (u_1, u_2, u_3, u_4) is linearly independent.

A labelled tetrahedron $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$ is *degenerate* if its vertices lie in a common plane in $\mathbb{P}_{\mathbb{R}}^3 V$. In terms of representatives v_1, v_2, v_3, v_4 , this means exactly that (v_1, v_2, v_3, v_4) is linearly dependent.

Observation 3.1.4. Let $\mathbf{u}$ and $\mathbf{v}$ be different points in SV . Then $\langle u, v \rangle \neq 0$. Indeed, assuming $\langle u, v \rangle = 0$, the space spanned by u and v has induced null form and is two-dimensional; however, in the case of signature $(3, 0, 1)$, the highest possible dimension of a subspace with null form is 1 (see Claim A.1.26).

Let $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$ be a non-degenerate ideal tetrahedron. We are interested in doubly stochastic Gram matrices in $\mathcal{M}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$. (A matrix is called *doubly stochastic* if all its entries are non-negative and the sum of coefficients along any column or row equals 1. As we will see later, Seidel's conjecture is formulated in terms of doubly stochastic matrices.) As a preliminary step, we introduce the following parameterization of the space $\mathcal{T}$.

Theorem 3.1.5. *The classifying space (modulo isometries) of non-degenerate labelled ideal tetrahedra can be identified with the open region*

$$U := \{(a, b) \in \mathbb{R}^2 \mid (a-1)^2 < b < (a+1)^2\}$$

in $\mathbb{R}^2$.

Explicitly, this identification is as follows. Given a non-degenerate labelled ideal tetrahedron $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$, there exists a unique Gram matrix $G(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{M}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ of the form

$$G(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \begin{pmatrix} 0 & 1 & a & b \\ 1 & 0 & 1 & a \\ a & 1 & 0 & 1 \\ b & a & 1 & 0 \end{pmatrix}$$

with $(a, b) \in U$. Conversely, for $(a, b) \in U$, there exists a non-degenerate labelled ideal tetrahedron whose Gram matrix is of the above form. Tetrahedra with a same Gram matrix differ by an isometry.

Figure 3 – The open region U , domain of the parameters a and b .

Source: Elaborated by the author.

Proof. Let v_1, v_2, v_3, v_4 be representatives of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Since the ideal tetrahedron $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is non-degenerate its vertices are pairwise distinct points in SV , Observation 3.1.4 implies that $\langle v_i, v_j \rangle \neq 0$ when $i \neq j$. So we can take the new representatives

$$u_1 := \frac{\langle v_2, v_3 \rangle}{\langle v_1, v_2 \rangle \langle v_3, v_4 \rangle} v_1, \quad u_2 := \frac{\langle v_3, v_4 \rangle}{\langle v_2, v_3 \rangle} v_2, \quad u_3 := \frac{1}{\langle v_3, v_4 \rangle} v_3, \quad u_4 := v_4.$$

These satisfy $\langle u_i, u_{i+1} \rangle = 1$ for $i = 1, 2, 3$.

Now let $\alpha := \left(\left| \frac{\langle u_2, u_4 \rangle}{\langle u_1, u_3 \rangle} \right| \right)^{\frac{1}{4}} \neq 0$ and define

$$w_1 := \alpha u_1, \quad w_2 := \frac{1}{\alpha} u_2, \quad w_3 := \alpha u_3, \quad w_4 := \frac{1}{\alpha} u_4.$$

Clearly, $\langle w_i, w_{i+1} \rangle = 1$ for $i = 1, 2, 3$; note that

$$|\langle w_1, w_3 \rangle| = \alpha^2 |\langle u_1, u_3 \rangle| = \sqrt{|\langle u_1, u_3 \rangle \langle u_2, u_4 \rangle|}$$

and

$$|\langle w_2, w_4 \rangle| = \frac{1}{\alpha^2} |\langle u_2, u_4 \rangle| = \sqrt{|\langle u_1, u_3 \rangle \langle u_2, u_4 \rangle|}$$

imply $|\langle w_1, w_3 \rangle| = |\langle w_2, w_4 \rangle|$. We have just arrived at a Gram matrix

$$G(w_1, w_2, w_3, w_4) = \begin{pmatrix} 0 & 1 & a & b \\ 1 & 0 & 1 & c \\ a & 1 & 0 & 1 \\ b & c & 1 & 0 \end{pmatrix},$$

where $a, b, c \in \mathbb{R}^*$ and $|a| = |c|$.

In what follows we will repeatedly apply Sylvester's criterion (see Proposition A.1.24) to different ordered bases.

On one hand, the Gram matrix of the basis $\beta = (v_1 + v_2, v_2, v_3, v_4)$ is

$$G^{\beta\beta} = \begin{pmatrix} 2 & 1 & a+1 & b+c \\ 1 & 0 & 1 & c \\ a+1 & 1 & 0 & 1 \\ b+c & c & 1 & 0 \end{pmatrix}.$$

The determinants of the first three principal sub-matrices of $G^{\beta\beta}$ are

$$\det G_1^{\beta\beta} = 2, \det G_2^{\beta\beta} = -1, \det G_3^{\beta\beta} = 2a.$$

They are non-null. The determinant $\det G_4^{\beta\beta} = \det G^{\beta\beta}$ is also non-null because V is non-degenerate (see Lemma A.1.21). By Sylvester's criterion, three and one are respectively the amount of negative and positive numbers in the sequence

$$\det G_1^{\beta\beta} = 2, \frac{\det G_2^{\beta\beta}}{\det G_1^{\beta\beta}} = -\frac{1}{2}, \frac{\det G_3^{\beta\beta}}{\det G_2^{\beta\beta}} = -2a, \frac{\det G_4^{\beta\beta}}{\det G_3^{\beta\beta}}.$$

This implies that $a > 0$.

On the other hand, the Gram matrix of the basis $\gamma := (v_3 + v_4, v_4, v_2, v_1)$ is

$$G^{\gamma\gamma} = \begin{pmatrix} 2 & 1 & 1+c & a+b \\ 1 & 0 & c & b \\ 1+c & c & 0 & 1 \\ a+b & b & 1 & 0 \end{pmatrix}.$$

The determinants of the principal sub-matrices

$$\det G_1^{\gamma\gamma} = 2, \det G_2^{\gamma\gamma} = -1, \det G_3^{\gamma\gamma} = 2c, \det G_4^{\gamma\gamma} = \det G,$$

are non-null. Since three and one are respectively the amount of negative and positive numbers in the sequence

$$\det G_1^{\gamma\gamma} = 2, \quad \frac{\det G_2^{\gamma\gamma}}{\det G_1^{\gamma\gamma}} = -\frac{1}{2}, \quad \frac{\det G_3^{\gamma\gamma}}{\det G_2^{\gamma\gamma}} = -2c, \quad \frac{\det G_4^{\gamma\gamma}}{\det G_3^{\gamma\gamma}}$$

we obtain $c > 0$ and $\det G^{\gamma\gamma} = \det G_4^{\gamma\gamma} < 0$. We already knew that $|a| = |c|$ and $a > 0$, hence $a = c$. It remains to observe that

$$\begin{aligned} 0 > \det G^{\gamma\gamma} &= \begin{vmatrix} 2 & 1 & 1+a & a+b \\ 1 & 0 & a & b \\ 1+a & a & 0 & 1 \\ a+b & b & 1 & 0 \end{vmatrix} = a^4 + b^2 + 1 - 2a^2b - 2a^2 - 2b = \\ &= ((a+1)^2 - b)((a-1)^2 - b) \end{aligned}$$

implies the desired inequalities $(a-1)^2 < b < (a+1)^2$ because $(a-1)^2 - b < (a+1)^2 - b$ and the numbers $(a-1)^2 - b$, $(a+1)^2 - b$ have distinct signs.

In order to prove the uniqueness of the Gram matrix $G(v_1, v_2, v_3, v_4)$, suppose that $G(u_1, u_2, u_3, u_4) \in \mathcal{M}(v_1, v_2, v_3, v_4)$ has the form

$$G(u_1, u_2, u_3, u_4) = \begin{pmatrix} 0 & 1 & a' & b' \\ 1 & 0 & 1 & a' \\ a' & 1 & 0 & 1 \\ b' & a' & 1 & 0 \end{pmatrix}$$

with $(a', b') \in U$. By Proposition 3.1.2 we have $G(u_1, u_2, u_3, u_4) = DG(v_1, v_2, v_3, v_4)D$, where $D = \text{diag}(x_1, x_2, x_3, x_4)$, $x_i \neq 0$ for $i = 1, 2, 3, 4$. So

$$\begin{pmatrix} 0 & 1 & a' & b' \\ 1 & 0 & 1 & a' \\ a' & 1 & 0 & 1 \\ b' & a' & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_1x_2 & ax_1x_3 & bx_1x_4 \\ x_2x_1 & 0 & x_2x_3 & ax_2x_4 \\ ax_3x_1 & x_3x_2 & 0 & x_3x_4 \\ bx_4x_1 & ax_4x_2 & x_4x_3 & 0 \end{pmatrix},$$

i.e.,

$$\begin{cases} 1 = x_1x_2 = x_2x_3 = x_3x_4 \\ a' = ax_1x_3 = ax_2x_4 \\ b' = bx_1x_4 \end{cases}$$

It follows from the first line that $x_1 = x_3$ and $x_2 = x_4$ as well as that x_1, x_2 are of a same sign and x_2, x_3 are of a same sign. Since $a \neq 0$, the second line implies $x_1^2 = x_2^2$, that is, $x_1 = x_2$. We obtain $x_1 = x_2 = x_3 = x_4 = \pm 1$ and conclude, in both cases, that $a' = a$ and $b' = b$, i.e., $G(u_1, u_2, u_3, u_4) = G(v_1, v_2, v_3, v_4)$.

Given $(a, b) \in U$, let us show that there exists a non-degenerate labelled ideal tetrahedron $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$ such that

$$G(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \begin{pmatrix} 0 & 1 & a & b \\ 1 & 0 & 1 & a \\ a & 1 & 0 & 1 \\ b & a & 1 & 0 \end{pmatrix},$$

for some representatives v_1, v_2, v_3, v_4 . Let b_1, b_2, b_3, b_4 be an orthonormal basis in V with $\langle b_1, b_1 \rangle = 1$ and $\langle b_2, b_2 \rangle = \langle b_3, b_3 \rangle = \langle b_4, b_4 \rangle = -1$. We define

$$\begin{aligned} v_1 &:= b_1 + b_2, \\ v_2 &:= \frac{1}{2}b_1 - \frac{1}{2}b_2, \\ v_3 &:= \left(1 + \frac{a}{2}\right)b_1 + \left(1 - \frac{a}{2}\right)b_2 + \sqrt{2a}b_3, \\ v_4 &:= \left(a + \frac{b}{2}\right)b_1 + \left(a - \frac{b}{2}\right)b_2 + \frac{a^2 + b - 1}{\sqrt{2a}}b_3 + \\ &\quad \sqrt{\frac{(a + \sqrt{b} + 1)(a + \sqrt{b} - 1)(a - \sqrt{b} + 1)(-a + \sqrt{b} + 1)}{2a}}b_4. \end{aligned}$$

The inequalities $(a - 1)^2 < b < (a + 1)^2$ imply $a + \sqrt{b} - 1, a - \sqrt{b} + 1, -a + \sqrt{b} + 1 > 0$; in other words, v_4 is well defined. The Gram matrix of v_1, v_2, v_3, v_4 is the required one. Since its determinant is non-null (in fact, the inequalities $(a - 1)^2 < b < (a + 1)^2$ imply that it is negative), the corresponding labelled ideal tetrahedron $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$ is non-degenerate.

Finally, the fact that tetrahedra with a same Gram matrix differ by an isometry follows directly from [2], Lemma 4.8.1.

□

Observation 3.1.6. ?? The region $\{(a, b) \in \mathbb{R}^2 \mid a, b > 0 \text{ and } b = (a \pm 1)^2\}$ corresponds to all degenerate labelled ideal tetrahedra with pairwise distinct vertices. Indeed, note that if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are pairwise distinct points in SV , then $W := \mathbb{R}v_1 + \mathbb{R}v_2 + \mathbb{R}v_3$ is a space of signature $(2, 0, 1)$. The rest follows applying Sylvester's criterion as before. Therefore, the region

$$\overline{U} \setminus \{(0, 1), (1, 0)\} = \{(a, b) \in \mathbb{R}^2 \mid a, b > 0 \text{ and } (a - 1)^2 \leq b \leq (a + 1)^2\}$$

parameterizes all labelled ideal tetrahedra with pairwise distinct vertices. The remaining points $(0, 1)$ and $(1, 0)$ correspond respectively to the degenerate labelled tetrahedra $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_2)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1)$. In connection with this Observation, see also Subsection 3.1.3.

Definition 3.1.7. A doubly stochastic matrix is a square matrix $A = (a_{ij})$ of nonnegative real numbers such that $\sum_i a_{ij} = \sum_j a_{ij} = 1$.

Proposition 3.1.8. *Let $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$ be a labelled ideal tetrahedron with pairwise distinct vertices. There exists a unique doubly stochastic Gram matrix $M \in \mathcal{M}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$. Explicitly,*

$$M = \frac{1}{a + \sqrt{b} + 1} \begin{pmatrix} 0 & 1 & a & \sqrt{b} \\ 1 & 0 & \sqrt{b} & a \\ a & \sqrt{b} & 0 & 1 \\ \sqrt{b} & a & 1 & 0 \end{pmatrix},$$

where $(a, b) \in \overline{U} \setminus \{(0, 1), (1, 0)\}$. Conversely, given $(a, b) \in \overline{U} \setminus \{(0, 1), (1, 0)\}$, there exists a labelled ideal tetrahedron $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$ with pairwise distinct vertices whose doubly stochastic matrix is the above M .

Proof. By Observation ?? there exists a Gram matrix $G(v_1, v_2, v_3, v_4) \in \mathcal{M}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ of the form

$$G(v_1, v_2, v_3, v_4) = \begin{pmatrix} 0 & 1 & a & b \\ 1 & 0 & 1 & a \\ a & 1 & 0 & 1 \\ b & a & 1 & 0 \end{pmatrix}$$

with $(a, b) \in \overline{U} \setminus \{(0, 1), (1, 0)\}$. Using Proposition 3.1.2, we look for a diagonal matrix $D = \text{diag}(w, x, y, z)$ such that $w, x, y, z \in \mathbb{R}^*$ and

$$DG(v_1, v_2, v_3, v_4)D = \begin{pmatrix} 0 & wx & awy & bwz \\ wx & 0 & xy & axz \\ awy & xy & 0 & yz \\ bwz & axz & yz & 0 \end{pmatrix}$$

is doubly stochastic.

First, note that w, x, y, z are of a same sign because, otherwise, at least one entry in $DG(v_1, v_2, v_3, v_4)D$ would be negative. Also, $DG(v_1, v_2, v_3, v_4)D = (-D)G(v_1, v_2, v_3, v_4)(-D)$ and therefore we can take D with positive coefficients. Thus, we only have to solve

$$\begin{cases} wx + awy + bwz = 1 \\ wx + xy + axz = 1 \\ awy + xy + yz = 1 \\ bwz + axz + yz = 1 \end{cases}$$

for positive numbers w, x, y, z .

From the first equation, $w = \frac{1}{x + ay + bz}$. This implies

$$\begin{cases} \frac{1}{x + ay + bz}x + xy + axz = 1 \\ \frac{1}{x + ay + bz}ay + xy + yz = 1 \\ \frac{1}{x + ay + bz}bz + axz + yz = 1 \end{cases} \quad (3.1)$$

Summing these three equations we obtain $xy + yz + axz = 1$, that is, $y = \frac{1 - axz}{x + z}$. In particular, $1 - axz \neq 0$. Replacing y in the first and third equations of 3.1 leads to

$$\begin{cases} \frac{x+z}{(x+bz)(x+z)+a(1-axz)}x + \frac{1-axz}{x+z}x + axz = 1 \\ \frac{x+z}{(x+bz)(x+z)+a(1-axz)}bz + axz + \frac{1-axz}{x+z}z = 1 \end{cases}. \quad (3.2)$$

Multiplying the first equation by bz , the second equation by x , and subtracting the results, we obtain

$$\frac{1-axz}{x+z}bxz + abxz^2 - ax^2z - \frac{1-axz}{x+z}xz = bz - x$$

which implies

$$\frac{(1-axz)xz}{x+z}(b-1) + axz(bz-x) = bz-x$$

that is

$$\frac{(1-axz)xz}{x+z}(b-1) = (bz-x)(1-axz).$$

Since $1 - axz \neq 0$, we arrive at $xz(b-1) = (bz-x)(x+z)$, i.e., at $x^2 = bz^2$. In other words, $x = \sqrt{bz}$.

Replacing x in the first equation of 3.2 gives

$$\frac{\sqrt{bz}+z}{(\sqrt{bz}+bz)(\sqrt{bz}+z)+a(1-a\sqrt{bz}^2)}\sqrt{bz} + \frac{1-a\sqrt{bz}^2}{\sqrt{bz}+z}\sqrt{bz} + a\sqrt{bz}^2 = 1$$

which is nothing but

$$\left((b+\sqrt{b})z^2 + a\sqrt{bz}^2 - 1 \right) \left((b+\sqrt{b})z^2 - a\sqrt{bz}^2 + 1 \right) = 0. \quad (3.3)$$

We claim that $(b+\sqrt{b})z^2 - a\sqrt{bz}^2 + 1 \neq 0$. Indeed, if we assume otherwise, then

$$1 = \sqrt{b}(a-1-\sqrt{b})z^2;$$

this is a contradiction: the inequality $(a-1)^2 \leq b$ satisfied by $(a,b) \in \overline{U} \setminus \{(0,1), (1,0)\}$ implies $a-1-\sqrt{b} < 0$. By 3.3,

$$(b+\sqrt{b})z^2 + a\sqrt{bz}^2 - 1 = 0$$

and we arrive at

$$z = \frac{1}{\sqrt[4]{b}\sqrt{a+\sqrt{b}+1}}.$$

Therefore, the unique (positive) solution is

$$w = z = \frac{1}{\sqrt[4]{b}\sqrt{a+\sqrt{b}+1}}, \quad x = y = \frac{\sqrt[4]{b}}{\sqrt{a+\sqrt{b}+1}}$$

and the corresponding doubly stochastic matrix is

$$G(u_1, u_2, u_3, u_4) = DG(v_1, v_2, v_3, v_4)D = \frac{1}{a + \sqrt{b} + 1} \begin{pmatrix} 0 & 1 & a & \sqrt{b} \\ 1 & 0 & \sqrt{b} & a \\ a & \sqrt{b} & 0 & 1 \\ \sqrt{b} & a & 1 & 0 \end{pmatrix},$$

where $(a, b) \in \overline{U} \setminus \{(0, 1), (1, 0)\}$.

The converse is immediate. $\square$

Observation 3.1.9. Let $\mathbf{v}, \mathbf{u}$ be different points in SV . We can choose representatives such that $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}$, then the Gram matrix

$$G(\mathbf{v}, \mathbf{v}, \mathbf{u}, \mathbf{u}) = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

is doubly stochastic. It is not difficult to verify that it is the unique doubly stochastic Gram matrix in $\mathcal{M}(\mathbf{v}, \mathbf{v}, \mathbf{u}, \mathbf{u})$. Similarly, $G(\mathbf{v}, \mathbf{u}, \mathbf{v}, \mathbf{u})$ and $G(\mathbf{v}, \mathbf{u}, \mathbf{u}, \mathbf{v})$ are respectively unique doubly stochastic Gram matrices in $\mathcal{M}(\mathbf{v}, \mathbf{u}, \mathbf{v}, \mathbf{u})$ and $\mathcal{M}(\mathbf{v}, \mathbf{u}, \mathbf{u}, \mathbf{v})$. In Proposition 3.1.8, the points $(0, 1), (1, 0) \in \overline{U}$ can be associated respectively to the ideal tetrahedra $(\mathbf{v}, \mathbf{u}, \mathbf{v}, \mathbf{u})$ and $(\mathbf{v}, \mathbf{u}, \mathbf{u}, \mathbf{v})$, but there is no point in $\overline{U}$ which can be associated to the ideal tetrahedron $(\mathbf{v}, \mathbf{v}, \mathbf{u}, \mathbf{u})$.

Observation 3.1.10. Let $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$ be a labelled ideal tetrahedron. If the number of distinct vertices of $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is 1 or 3, there is no a doubly stochastic matrix in $\mathcal{M}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$. Indeed, if $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ has one distinct vertex, $\mathcal{M}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ consists only on the null matrix. If $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ has three distinct vertices (say $\mathbf{v}_1 = \mathbf{v}_2$ — the other cases are similar), assume that there exist representatives v_1, v'_1, v_3, v_4 of $\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4$ with doubly stochastic Gram matrix. Note that $v'_1 = rv_1$, where $r \in \mathbb{R}^*$. The doubly stochastic Gram matrix is

$$G(v_1, rv_1, v_3, v_4) = \begin{pmatrix} 0 & 0 & \langle v_1, v_3 \rangle & \langle v_1, v_4 \rangle \\ 0 & 0 & r\langle v_1, v_3 \rangle & r\langle v_1, v_4 \rangle \\ \langle v_1, v_3 \rangle & r\langle v_1, v_3 \rangle & 0 & \langle v_3, v_4 \rangle \\ \langle v_1, v_4 \rangle & r\langle v_1, v_4 \rangle & \langle v_3, v_4 \rangle & 0 \end{pmatrix}.$$

Then

$$\begin{cases} \langle v_1, v_3 \rangle + \langle v_1, v_4 \rangle = 1 \\ r\langle v_1, v_3 \rangle + r\langle v_1, v_4 \rangle = 1 \\ \langle v_1, v_3 \rangle + r\langle v_1, v_3 \rangle + \langle v_3, v_4 \rangle = 1 \\ \langle v_1, v_4 \rangle + r\langle v_1, v_4 \rangle + \langle v_3, v_4 \rangle = 1 \end{cases}.$$

Summing the third and forth equations, we have

$$\langle v_1, v_3 \rangle + \langle v_1, v_4 \rangle + r\langle v_1, v_3 \rangle + r\langle v_1, v_4 \rangle + 2\langle v_3, v_4 \rangle = 2.$$

Then from the first and the second equation we get $\langle v_3, v_4 \rangle = 0$, i.e., $\mathbf{v}_3 = \mathbf{v}_4$ which is a contradiction with the number of distinct vertices of the ideal tetrahedron.

Let $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$ be a labelled ideal tetrahedron with 2 or 4 distinct vertices. We denote by $DSG(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ the unique doubly stochastic Gram matrix in $\mathcal{M}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ and by

$$\mathcal{DS} := \{DSG(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \mid (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T} \text{ with an even number of distinct vertices}\}$$

the space of doubly stochastic Gram matrices of labelled ideal tetrahedra with 2 or 4 distinct vertices. Clearly, the space $\mathcal{DS}$ is simply another way of parameterizing labelled ideal tetrahedra with 2 or 4 distinct vertices via the injection (and surjective except for one point) $\varphi : \overline{U} \rightarrow \mathcal{DS}$ defined by

$$\varphi(a, b) = \frac{1}{a + \sqrt{b} + 1} \begin{pmatrix} 0 & 1 & a & \sqrt{b} \\ 1 & 0 & \sqrt{b} & a \\ a & \sqrt{b} & 0 & 1 \\ \sqrt{b} & a & 1 & 0 \end{pmatrix}.$$

The space of non-degenerate labelled ideal tetrahedra contains six components that are ‘copies’ of the space of (non-labelled) ideal tetrahedra; these copies correspond to those permutations of the vertices that cannot be accomplished by applying an isometry (see Section ???). They are not easy to describe as subsets of U .

In the next section, in order to solve Seidel’s first conjecture, we will essentially express a and b in terms of the permanent and the determinant of the doubly stochastic matrix associated to a non-degenerate labelled ideal tetrahedron. Curiously, this will lead to a suitable reparametrization of U where the mentioned six components will be quite easy to spot.

3.1.2 Determinant and permanent as coordinates

In the Introduction, we discussed that rational representations of the symmetric group give rise to *Schur functors*, which are functors from the category of finite-dimensional linear spaces to itself. Moreover, given a finite-dimensional linear space V equipped with a symmetric bilinear form and a Schur functor F , there is a natural way to endow FV with a symmetric bilinear form. This construction leads to the concept of *immanant* of a matrix. The particular cases corresponding to the trivial and to the alternating representations are respectively the *permanent* and the determinant. In Seidel’s conjecture for hyperbolic 3-space, only these two cases are needed and we remind the definition of the first (for the general definition of immanant, see the Introduction).

Definition 3.1.11. Let $A = (a_{ij})$ be an $n \times n$ matrix. The *permanent* of A is given by

$$\text{per} A = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where S_n is the symmetric group.

The first of Seidel's conjectures, Speculation 1 (see the Introduction), says that the determinant and the permanent of the doubly stochastic matrix associated to a labelled ideal tetrahedron uniquely determines its volume. This conjecture is solved by showing that there exists exactly six labelled non-degenerate ideal tetrahedra (hence, with different doubly stochastic matrices) with the same permanent and determinant; however, these six labelled tetrahedra differ only by the order of their vertices.

The fourth conjecture, Speculation 4 (see the Introduction), claims that the volume of an ideal tetrahedron is a monotonic function of the determinant (respectively, the permanent) of the doubly stochastic matrix associated to the tetrahedron. The volume of an ideal tetrahedron $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$ can be expressed in terms of the entries of any matrix in $\mathcal{M}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ corresponding to the tetrahedron. Indeed, it is easy to express the dihedral angles of the tetrahedron in terms of the coefficients of the Gram matrix and it is well-known that the volume of an ideal tetrahedron is a function of its dihedral angles — see Section ???.

Hence, our first step towards these Speculations will be trying to express a, b in terms of the determinant and the permanent of the corresponding doubly stochastic matrix (see Proposition 3.1.8).

Denote by $R \subset \mathbb{R}^2$ the set of ordered pairs of values of determinant and permanent of doubly stochastic Gram matrices of labelled ideal tetrahedra, i.e.,

$$R := \{(\det M, \text{per } M) \mid M \in \mathcal{DS}\}. \quad (3.4)$$

In terms of $(a, b) \in \overline{U}$ and the function φ introduced at the end of the previous subsection, R can be described as

$$R = \{(\det \varphi(a, b), \text{per } \varphi(a, b)) \mid (a, b) \in \overline{U}\} \cup \{(\det DSG(\mathbf{v}, \mathbf{v}, \mathbf{u}, \mathbf{u}), \text{per } DSG(\mathbf{v}, \mathbf{v}, \mathbf{u}, \mathbf{u}))\},$$

where $\mathbf{v}, \mathbf{u}$ are distinct points in SV (see Observation 3.1.9). We have

$$\begin{aligned} \det \varphi(a, b) &= \frac{1}{(a + \sqrt{b} + 1)^4} \cdot \det \begin{pmatrix} 0 & 1 & a & \sqrt{b} \\ 1 & 0 & \sqrt{b} & a \\ a & \sqrt{b} & 0 & 1 \\ \sqrt{b} & a & 1 & 0 \end{pmatrix} \\ &= \frac{a^4 + b^2 + 1 - 2a^2b - 2a^2 - 2b}{(a + \sqrt{b} + 1)^4} \\ &= -\frac{(a + \sqrt{b} + 1)(-a + \sqrt{b} + 1)(a - \sqrt{b} + 1)(a + \sqrt{b} - 1)}{(a + \sqrt{b} + 1)^4} \\ &= -\frac{(-a + \sqrt{b} + 1)(a - \sqrt{b} + 1)(a + \sqrt{b} - 1)}{(a + \sqrt{b} + 1)^3}. \end{aligned} \quad (3.5)$$

and

$$\begin{aligned}
 \text{per } \varphi(a, b) &= \frac{1}{(a + \sqrt{b} + 1)^4} \cdot \text{per} \begin{pmatrix} 0 & 1 & a & \sqrt{b} \\ 1 & 0 & \sqrt{b} & a \\ a & \sqrt{b} & 0 & 1 \\ \sqrt{b} & a & 1 & 0 \end{pmatrix} \\
 &= \frac{a^4 + b^2 + 1 + 2a^2b + 2a^2 + 2b}{(a + \sqrt{b} + 1)^4} \\
 &= \frac{(a^2 + b + 1)^2}{(a + \sqrt{b} + 1)^4}. \tag{3.6}
 \end{aligned}$$

Therefore, if we want to express a and b in terms of $\alpha := \det \varphi(a, b)$ and $\beta := \text{per } \varphi(a, b)$ for $(\alpha, \beta) \in R$, we need to solve

$$\begin{cases} -\frac{(-a + \sqrt{b} + 1)(a - \sqrt{b} + 1)(a + \sqrt{b} - 1)}{(a + \sqrt{b} + 1)^3} = \alpha \\ \frac{(a^2 + b + 1)^2}{(a + \sqrt{b} + 1)^4} = \beta \end{cases}. \tag{3.7}$$

It turns out that this is not a straightforward task. To accomplish this task, we will reparameterize the space $\mathcal{DS}$ in a more symmetric fashion.

Proposition 3.1.12. *Let $V := [-\frac{1}{2}, \frac{1}{2}] \times [1, \infty)$. The map $f = (f_1, f_2) : V \rightarrow \overline{U}$ defined by*

$$f(u, v) = \left(v - u - \frac{1}{2}, \left(v + u - \frac{1}{2} \right)^2 \right)$$

is bijective.

Proof. First, we show that f has the appropriate codomain, i.e., $f(u, v) \in \overline{U} = \{(a, b) \mid (a - 1)^2 \leq b^2 \leq (a + 1)^2\}$. We have

$$f_2(u, v)^2 - (f_1(u, v) - 1)^2 = \left(v + u - \frac{1}{2} \right)^2 - \left(v - u - \frac{3}{2} \right)^2 = 2(v - 1)(2u + 1) \geq 0$$

and

$$(f_1(u, v) + 1)^2 - f_2(u, v)^2 = \left(v - u + \frac{1}{2} \right)^2 - \left(v + u - \frac{1}{2} \right)^2 = 2v(-2u + 1) \geq 0.$$

Let $(a, b) \in \overline{U}$. The ordered pair $\left(\frac{\sqrt{b} - a}{2}, \frac{a + \sqrt{b} + 1}{2} \right) \in V$. Indeed, $a - 1 \leq \sqrt{b} \leq a + 1$

implies $-\frac{1}{2} \leq \frac{\sqrt{b} - a}{2} \leq \frac{1}{2}$ and $1 - a \leq \sqrt{b}$ implies $1 \leq \frac{a + \sqrt{b} + 1}{2}$. The inverse of f is given

$$\text{by } f^{-1}(a, b) = \left(\frac{\sqrt{b} - a}{2}, \frac{a + \sqrt{b} + 1}{2} \right).$$

□

Observation 3.1.13. In terms of the coordinates $(u, v) \in V$ of the above Proposition we have

$$\varphi \circ f(u, v) = \frac{1}{2v} \begin{pmatrix} 0 & 1 & v - u - \frac{1}{2} & v + u - \frac{1}{2} \\ 1 & 0 & v + u - \frac{1}{2} & v - u - \frac{1}{2} \\ v - u - \frac{1}{2} & v + u - \frac{1}{2} & 0 & 1 \\ v + u - \frac{1}{2} & v - u - \frac{1}{2} & 1 & 0 \end{pmatrix},$$

$$\det \circ \varphi \circ f(u, v) = \frac{(2u+1)(2u-1)(v-1)}{4v^3}, \quad \text{per} \circ \varphi \circ f(u, v) = \frac{(4u^2 + 4v^2 - 4v + 3)^2}{64v^4}. \square$$

Fix a point $(\alpha, \beta) \in R$. Now, our task became solving

$$\begin{cases} \frac{(2u+1)(2u-1)(v-1)}{4v^3} = \alpha \\ \frac{(4u^2 + 4v^2 - 4v + 3)^2}{64v^4} = \beta \end{cases}.$$

Define $F : V \rightarrow \mathbb{R}^2, (u, v) \mapsto \left(\frac{(2u+1)(2u-1)(v-1)}{4v^3}, \frac{(4u^2 + 4v^2 - 4v + 3)^2}{64v^4} \right)$. The Jacobian matrix of F at a point $(u, v) \in V$ is

$$J_F(u, v) = \begin{pmatrix} \frac{2u(v-1)}{v^3} & -\frac{(4u^2-1)(2v-3)}{4v^4} \\ \frac{u(4u^2+4v^2-4v+3)}{4v^4} & -\frac{(4u^2+4v^2-4v+3)(4u^2-2v+3)}{16v^5} \end{pmatrix}.$$

Its determinant equals

$$\det J_F(u, v) = \frac{u(2v-3+2u)(2v-3-2u)(4u^2+(2v-1)^2+2)}{16v^8}$$

and vanishes at the points $(u, v) \in V$ such that $u = 0$ or $v = \frac{3}{2} - u$ or $v = \frac{3}{2} + u$. So F has a local inverse at all points in V except at those lying on the three concurrent lines in Figure 4.

The central vertical line $u = 0$ is a visible line of symmetry in V ; this means that the points (u, v) and $(-u, v)$ correspond to different doubly stochastic matrices with the same determinant and permanent (in fact, they correspond to distinct ideal tetrahedra which are isometric as non-labelled ideal tetrahedron). We can already see in Figure 4 the above mentioned six ‘copies’ of the space of (non-labelled) nondegenerate ideal tetrahedra, but the corresponding symmetries around the lines $v = \frac{3}{2} - u$ and $v = \frac{3}{2} + u$ (this is line L in Figure 4) are not as simple to express as the one around $u = 0$. Let us show how to find a more ‘homogeneous’ classifying space.

Figure 4 – The open set V , domain of the parameters.

Source: Elaborated by the author.

Let $W \subset \mathbb{R}^2$ be the equilateral triangle (without the vertex on the top) in Figure 5 with basis the segment joining $\left(-\frac{1}{2}, 1\right)$ and $\left(\frac{1}{2}, 1\right)$. We want to find a bijective map from W to V . For this purpose, from some point on the vertical axis (in Figure 5, this is the point below the triangle basis) we will project segments joining $\left(0, 1 + \frac{\sqrt{3}}{2}\right)$ and $(s, 1)$ onto the vertical line $\{s\} \times [1, \infty)$, where $s \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. The blue dashed line in Figure 5 shows the idea of this projection. We would like the altitudes of the triangle to become the three concurrent lines in V ; in particular, the point $\left(\frac{1}{4}, 1 + \frac{\sqrt{3}}{4}\right)$ should map to $\left(\frac{1}{2}, 2\right)$. Thus, the point from which we must project is the intersection of the vertical axis with the straight line that joins $\left(\frac{1}{4}, 1 + \frac{\sqrt{3}}{4}\right)$ and $\left(\frac{1}{2}, 2\right)$, i.e., the point $\left(0, \frac{\sqrt{3}}{2}\right)$.

This construction is summarized in the following proposition.

Proposition 3.1.14. *Let W be the triangle (without the vertex on the top) in Figure 5 and let V be as in Proposition 3.1.12. The function $g : W \rightarrow V, (s, t) \mapsto \left(\frac{\sqrt{3}s}{2 + \sqrt{3} - 2t}, \frac{\sqrt{3}}{2 + \sqrt{3} - 2t}\right)$, is a bijection.*

Figure 5 – Bijection between W and V .

Source: Elaborated by the author.

Finally, we make the translation $W \rightarrow \Delta' := W - \left(0, 1 + \frac{\sqrt{3}}{6}\right)$ so that the orthocenter $\left(0, 1 + \frac{\sqrt{3}}{6}\right)$ of the triangle gets centred at the origin. The region Δ' is the classifying space of labelled ideal tetrahedra (except for one point ideal tetrahedron) that we are going to use in the proofs of Speculations 1 and 4.

Proposition 3.1.15. *Let*

$$\Delta := \left\{ (c, d) \in \mathbb{R}^2 \mid d \geq -\frac{\sqrt{3}}{6}, d \leq -\sqrt{3}c + \frac{\sqrt{3}}{3}, \text{ and } d \leq \sqrt{3}c + \frac{\sqrt{3}}{3} \right\}$$

stand for the triangle in Figure 6. The function $\phi : \Delta \rightarrow \mathcal{DS}$ defined by

$$\phi(c, d) = \frac{1}{6} \begin{pmatrix} 0 & 2 - 2\sqrt{3}d & 2 - 3c + \sqrt{3}d & 2 + 3c + \sqrt{3}d \\ 2 - 2\sqrt{3}d & 0 & 2 + 3c + \sqrt{3}d & 2 - 3c + \sqrt{3}d \\ 2 - 3c + \sqrt{3}d & 2 + 3c + \sqrt{3}d & 0 & 2 - 2\sqrt{3}d \\ 2 + 3c + \sqrt{3}d & 2 - 3c + \sqrt{3}d & 2 - 2\sqrt{3}d & 0 \end{pmatrix},$$

is a bijection. The determinant and permanent of the matrix $\phi(c, d)$ are

$$\det \phi(c, d) = \frac{(2\sqrt{3}d + 1)(3c + \sqrt{3}d - 1)(3c - \sqrt{3}d + 1)}{27}, \quad \text{per } \phi(c, d) = \left(\frac{3c^2 + 3d^2 + 2}{6} \right)^2.$$

Proof. First note that $\phi\left(0, \frac{\sqrt{3}}{3}\right) = \text{DSG}(\mathbf{v}, \mathbf{v}, \mathbf{u}, \mathbf{u})$. The function ϕ is a bijection by Proposition 3.1.14. The rest follows directly from Observation 3.1.13. $\square$

The non-null entries of the matrix $\phi(c, d)$ are expressions that will appear repeatedly in the future and we name them $r, s, t : \Delta \rightarrow \mathbb{R}$, where

$$r(c, d) := \frac{2 - 2\sqrt{3}d}{6}, \quad s(c, d) := \frac{2 - 3c + \sqrt{3}d}{6}, \quad t(c, d) := \frac{2 + 3c + \sqrt{3}d}{6}. \quad (3.8)$$

We have $\det \phi = -(r + s + t)(-r + s + t)(r - s + t)(r + s - t)$ and $\text{per } \phi = (r^2 + s^2 + t^2)^2$.

We found it a beautiful fact (see the next proposition) that the above three positive numbers constitute the sides of an Euclidean triangle. Another form of this fact can also be found in Seidel's work, at the end of Section 2 in [10]. It has the following interpretation. Opposite dihedral angles of a non-degenerate ideal tetrahedron are equal; hence, such a tetrahedron has at most three distinct dihedral angles, θ_1 , θ_2 , and θ_3 and these dihedral angles are exactly the internal angles of the Euclidean triangle with sides $r(c, d)$, $s(c, d)$, and $t(c, d)$. In particular, $\theta_1 + \theta_2 + \theta_3 = \pi$ (a fact discovered by Milnor, see [8], Lemma 2). All these facts will be proved in Section ???.

Proposition 3.1.16. *Let $(c, d) \in \Delta$. The positive numbers $r(c, d)$, $s(c, d)$, and $t(c, d)$ constitute the sides of a (non-degenerate) Euclidean triangle.*

Proof. We denote $r := r(c, d)$, $s := s(c, d)$, and $t := t(c, d)$. The matrix $\phi(c, d)$ is the Gram matrix of four linearly independent vectors (a basis for V). The signature of the hermitian form on V is $(3, 0, 1)$. By Proposition A.1.19, $\det(\phi(c, d))$ has the same sign as $\det(\text{diag}(-1, -1, -1, 1)) = -1$. Thus

$$\det(\phi(c, d)) < 0,$$

i.e.,

$$-(r + s + t)(-r + s + t)(r - s + t)(r + s - t) < 0,$$

that is,

$$(-r + s + t)(r - s + t)(r + s - t) > 0.$$

There is an even number of negative factors on the left side of the last inequation. If we assume that there are two negative factors (say, the first two ones — the other cases are similar) then $-r + s + t < 0$ and $r - s + t < 0$. But this implies $t < 0$ which is a contradiction. We conclude that there are no negative factors in the above expression. So, the triangle inequalities $r < s + t$, $s < r + t$, and $t < r + s$ are satisfied. $\square$

3.1.3 The classifying space Δ

The region Δ described in Proposition 3.1.15 is the classifying space of labelled non-degenerate ideal tetrahedra that we are going to use in order to prove Speculations 1 and 4. It is divided by its altitudes into six congruent triangles. Each such triangle is, as we show below, the space of non-labelled non-degenerate ideal tetrahedra. The reflections in the altitudes of Δ correspond to the relabellings that cannot be achieved by means of isometries. In this way,

a generic non-labelled tetrahedron (three distinct dihedral angles) has six copies in Δ while more symmetric tetrahedra (exactly two equal dihedral angles) have only three copies; finally, the regular tetrahedron has a single copy (since, in this case, there are enough symmetries to permute the vertices in every possible way). The generic case is illustrated in Figure 6.

Figure 6 – The open set Δ , domain of the parameters (c, d) .

Source: Elaborated by the author.

We call Δ_i , $i = 1, \dots, 6$, the six congruent triangles into which the equilateral triangle Δ is divided by its altitudes. We recall that Δ is an open region and does not contain its boundary (which correspond to some degenerate tetrahedra). Therefore, each Δ_i does not contain one of its legs (the longest). Explicitly,

$$\Delta_1 := \left\{ (c, d) \in \mathbb{R}^2 \mid c \geq 0, d \geq \frac{\sqrt{3}}{3}c, \text{ and } d < -\sqrt{3}c + \frac{\sqrt{3}}{3} \right\}. \quad (3.9)$$

For future reference, we display in the table below the reflections in altitudes of Δ of a point $(c, d) \in \Delta_1$:

Proposition 3.1.17. *Let $(c, d) \in \Delta_1$. Then*

- (i) $r(c, d) \leq s(c, d) \leq t(c, d)$
- (ii) $r(c, d) = s(c, d)$ if, and only if, (c, d) is on the smallest leg of Δ_1
- (iii) $s(c, d) = t(c, d)$ if, and only if, (c, d) is on the hypotenuse of Δ_1

Proof. We have

$$s(c, d) - r(c, d) = \frac{2 - 3c + \sqrt{3}d}{6} - \frac{2 - 2\sqrt{3}d}{6} = \frac{-c + \sqrt{3}d}{6} \geq 0$$

Triangle	Point
Δ_1	(c, d)
Δ_2	$\left(\frac{c+\sqrt{3}d}{2}, \frac{\sqrt{3}c-d}{2}\right)$
Δ_3	$\left(\frac{-c+\sqrt{3}d}{2}, \frac{-\sqrt{3}c-d}{2}\right)$
Δ_4	$\left(\frac{c-\sqrt{3}d}{2}, \frac{-\sqrt{3}c-d}{2}\right)$
Δ_5	$\left(\frac{-c-\sqrt{3}d}{2}, \frac{\sqrt{3}c-d}{2}\right)$
Δ_6	$(-c, d)$

Table 1 – Points obtained by reflecting $(c, d) \in \Delta_1$.

with the equality holding if, and only if, (c, d) is on the smallest leg of Δ_1 . Analogously,

$$t(c, d) - s(c, d) = \frac{2 + 3c + \sqrt{3}d}{6} - \frac{2 - 3c + \sqrt{3}d}{6} = c \geq 0,$$

with the equality holding if, and only if, (c, d) is on the hypotenuse of Δ_1 . $\square$

Let us see which permutations of vertices can always be performed through isometries.

Theorem 3.1.18. *Let $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$ be a non-degenerate labelled ideal tetrahedron. Let S_4 denote the 4-symmetric group and let $H := \{1, (12)(34), (13)(24), (14)(23)\} \triangleleft S_4$ be the Klein four group. The tetrahedra $(\mathbf{v}_{\sigma_1(1)}, \mathbf{v}_{\sigma_1(2)}, \mathbf{v}_{\sigma_1(3)}, \mathbf{v}_{\sigma_1(4)})$ and $(\mathbf{v}_{\sigma_2(1)}, \mathbf{v}_{\sigma_2(2)}, \mathbf{v}_{\sigma_2(3)}, \mathbf{v}_{\sigma_2(4)})$ are isometric if (and only if, in the generic case) $\sigma_1 \sigma_2^{-1} \in H$. Equivalently, at the level of Gram matrices, the quotient group $S_3 = S_4/H$ acts on the space $\mathcal{DS}$ by permuting the entries r , s , and t . At the level of the classifying space Δ , this action of S_3 is nothing but reflecting on the altitudes.*

Proof. Let $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{T}$ be a non-degenerate labelled ideal tetrahedron, let $\sigma \in S_4$ be an element of the 4-symmetric group, and let $G(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ be the doubly stochastic Gram matrix of representatives of the vertices of the tetrahedron, i.e., $DSG(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = G(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$. Since the doubly stochastic Gram matrix corresponding to $(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)})$ is unique, we have

$$DSG(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}) = G(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}).$$

By Proposition 3.1.15,

$$DSG(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \begin{pmatrix} 0 & r & s & t \\ r & 0 & t & s \\ s & t & 0 & r \\ t & s & r & 0 \end{pmatrix}$$

and

$$DSG(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}) = \begin{pmatrix} 0 & r' & s' & t' \\ r' & 0 & t' & s' \\ s' & t' & 0 & r' \\ t' & s' & r' & 0 \end{pmatrix},$$

for some $r, s, t, r', s', t' \in \mathbb{R}^{>0}$. We have $\{r, s, t\} = \{r', s', t'\}$, therefore σ simply permutes the entries r, s, t of $DSG(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

A direct calculation shows that $DSG(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}) = DSG(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ for $\sigma \in H = \{Id, (12)(34), (13)(24), (14)(23)\}$. In the generic case, r, s, t are pairwise distinct and, therefore, $DSG(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}) = DSG(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ only if $\sigma \in H$.

It remains to show that S_4/H acts on Δ by reflections on the altitudes. Let $(c, d) \in \Delta_1$. The matrices corresponding to the points in the table above are (the proof is a tedious calculation that can be seen below):

$$\begin{aligned} \phi(c, d) &= \begin{pmatrix} 0 & r & s & t \\ r & 0 & t & s \\ s & t & 0 & r \\ t & s & r & 0 \end{pmatrix} = DSG(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}), \sigma \in H \\ \phi\left(\frac{c + \sqrt{3}d}{2}, \frac{\sqrt{3}c - d}{2}\right) &= \begin{pmatrix} 0 & s & r & t \\ s & 0 & t & r \\ r & t & 0 & s \\ t & r & s & 0 \end{pmatrix} = DSG(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}), \sigma \in (23)H \\ \phi\left(\frac{-c + \sqrt{3}d}{2}, \frac{-\sqrt{3}c - d}{2}\right) &= \begin{pmatrix} 0 & t & r & s \\ t & 0 & s & r \\ r & s & 0 & t \\ s & r & t & 0 \end{pmatrix} = DSG(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}), \sigma \in (123)H \\ \phi\left(\frac{c - \sqrt{3}d}{2}, \frac{-\sqrt{3}c - d}{2}\right) &= \begin{pmatrix} 0 & t & s & r \\ t & 0 & r & s \\ s & r & 0 & t \\ r & s & t & 0 \end{pmatrix} = DSG(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}), \sigma \in (13)H \\ \phi\left(\frac{-c - \sqrt{3}d}{2}, \frac{\sqrt{3}c - d}{2}\right) &= \begin{pmatrix} 0 & s & t & r \\ s & 0 & r & t \\ t & r & 0 & s \\ r & t & s & 0 \end{pmatrix} = DSG(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}), \sigma \in (234)H, \\ \phi(-c, d) &= \begin{pmatrix} 0 & r & t & s \\ r & 0 & s & t \\ t & s & 0 & r \\ s & t & r & 0 \end{pmatrix} = DSG(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}, \mathbf{v}_{\sigma(4)}), \sigma \in (12)H. \end{aligned}$$

The first equality follows from 3.8. The entries (multiplied by 6) of the matrix corresponding to the point $\left(\frac{c+\sqrt{3}d}{2}, \frac{\sqrt{3}c-d}{2}\right) \in \Delta_2$ are the following:

$$6r\left(\frac{c+\sqrt{3}d}{2}, \frac{\sqrt{3}c-d}{2}\right) = 2 - 2\sqrt{3}\frac{\sqrt{3}c-d}{2} = 2 - 3c + \sqrt{3}d = 6s,$$

$$6s\left(\frac{c+\sqrt{3}d}{2}, \frac{\sqrt{3}c-d}{2}\right) = 2 - 3\frac{c+\sqrt{3}d}{2} + \sqrt{3}\frac{\sqrt{3}c-d}{2} = 2 - 2\sqrt{3}d = 6r,$$

$$6t\left(\frac{c+\sqrt{3}d}{2}, \frac{\sqrt{3}c-d}{2}\right) = 2 + 3\frac{c+\sqrt{3}d}{2} + \sqrt{3}\frac{\sqrt{3}c-d}{2} = 2 + 3c + \sqrt{3}d = 6t.$$

For the point $\left(\frac{-c+\sqrt{3}d}{2}, \frac{-\sqrt{3}c-d}{2}\right) \in \Delta_3$,

$$6r\left(\frac{-c+\sqrt{3}d}{2}, \frac{-\sqrt{3}c-d}{2}\right) = 2 - 2\sqrt{3}\frac{-\sqrt{3}c-d}{2} = 2 + 3c + \sqrt{3}d = 6t,$$

$$6s\left(\frac{-c+\sqrt{3}d}{2}, \frac{-\sqrt{3}c-d}{2}\right) = 2 - 3\frac{-c+\sqrt{3}d}{2} + \sqrt{3}\frac{-\sqrt{3}c-d}{2} = 2 - 2\sqrt{3}d = 6r,$$

$$6t\left(\frac{-c+\sqrt{3}d}{2}, \frac{-\sqrt{3}c-d}{2}\right) = 2 + 3\frac{-c+\sqrt{3}d}{2} + \sqrt{3}\frac{-\sqrt{3}c-d}{2} = 2 - 3c + \sqrt{3}d = 6s.$$

For the point $\left(\frac{c-\sqrt{3}d}{2}, \frac{-\sqrt{3}c-d}{2}\right) \in \Delta_4$,

$$6r\left(\frac{c-\sqrt{3}d}{2}, \frac{-\sqrt{3}c-d}{2}\right) = 2 - 2\sqrt{3}\frac{-\sqrt{3}c-d}{2} = 2 + 3c + \sqrt{3}d = 6t,$$

$$6s\left(\frac{c-\sqrt{3}d}{2}, \frac{-\sqrt{3}c-d}{2}\right) = 2 - 3\frac{c-\sqrt{3}d}{2} + \sqrt{3}\frac{-\sqrt{3}c-d}{2} = 2 - 3c + \sqrt{3}d = 6s,$$

$$6t\left(\frac{c-\sqrt{3}d}{2}, \frac{-\sqrt{3}c-d}{2}\right) = 2 + 3\frac{c-\sqrt{3}d}{2} + \sqrt{3}\frac{-\sqrt{3}c-d}{2} = 2 - 2\sqrt{3}d = 6r.$$

For the point $\left(\frac{-c-\sqrt{3}d}{2}, \frac{\sqrt{3}c-d}{2}\right) \in \Delta_5$,

$$6r\left(\frac{-c-\sqrt{3}d}{2}, \frac{\sqrt{3}c-d}{2}\right) = 2 - 2\sqrt{3}\frac{\sqrt{3}c-d}{2} = 2 - 3c + \sqrt{3}d = 6s,$$

$$6s\left(\frac{-c-\sqrt{3}d}{2}, \frac{\sqrt{3}c-d}{2}\right) = 2 - 3\frac{-c-\sqrt{3}d}{2} + \sqrt{3}\frac{\sqrt{3}c-d}{2} = 2 + 3c + \sqrt{3}d = 6t,$$

$$6t\left(\frac{-c-\sqrt{3}d}{2}, \frac{\sqrt{3}c-d}{2}\right) = 2 + 3\frac{-c-\sqrt{3}d}{2} + \sqrt{3}\frac{\sqrt{3}c-d}{2} = 2 - 2\sqrt{3}d = 6r.$$

Finally, for the point $(-c, d) \in \Delta_6$,

$$6r(-c, d) = 2 - 2\sqrt{3}d = 6r,$$

$$6s(-c, d) = 2 - 3(-c) + \sqrt{3}d = 6t,$$

$$6t(-c, d) = 2 + 3(-c) + \sqrt{3}d = 6s.$$

□

We arrived at the space of non-degenerate ideal tetrahedra modulo isometries:

Corollary 3.1.19. *Each Δ_i , $i = 1, \dots, 6$, is a copy of the space of nondegenerate non-labelled ideal tetrahedra.*

Corollary 3.1.20. *The doubly stochastic Gram matrices corresponding to the labelled ideal tetrahedra that differ by the action of S_3 in the previous theorem have the same determinant and permanent.*

Proof. The expressions for the determinant and permanent (see the formulae right after Proposition 3.1.15) are symmetric in r, s, t . □

We will prove in Corollary 3.2.3 that the function $G : \Delta \rightarrow R$ defined by

$$G(c, d) = (\det \phi(c, d), \text{per } \phi(c, d)) \quad (3.10)$$

is injective when restricted to Δ_1 . For now, let us show that G is locally invertible outside the altitudes. Indeed, the Jacobian matrix of G at a point (c, d) is

$$J_G(c, d) = \frac{1}{3} \begin{pmatrix} 2c(2\sqrt{3}d+1) & 2(\sqrt{3}c^2 - \sqrt{3}d^2 + d) \\ c(3c^2 + 3d^2 + 2) & d(3c^2 + 3d^2 + 2) \end{pmatrix}$$

and

$$\det J_G(c, d) = \frac{2\sqrt{3}c(\sqrt{3}d+c)(\sqrt{3}d-c)(3c^2+3d^2+2)}{9}.$$

Thus, $\det J_G \neq 0$ if, and only if, $(c, d) \in \Delta$ does not belong to an altitude.

3.2 Proof of Speculation 1

In Theorem 3.1.18 we saw that the points in Δ that differ by reflections on the altitudes of Δ (these points are listed in Table 1) correspond to doubly stochastic matrices in $\mathcal{DS}$ with the same determinant and permanent (see Corollary 3.1.20). This means that, for a given $(\alpha, \beta) \in R$ (see the definition of R in 3.4), the points in Table 1 are solutions of

$$\begin{cases} \frac{(2\sqrt{3}d+1)(3c+\sqrt{3}d-1)(3c-\sqrt{3}d+1)}{27} = \alpha \\ \left(\frac{3c^2+3d^2+2}{6} \right)^2 = \beta \end{cases}. \quad (3.11)$$

Speculation 1 claims that the volume of an ideal tetrahedron is determined by the determinant and permanent of its doubly stochastic Gram matrix. In fact, a stronger result holds: the determinant and permanent in question completely determine the non-degenerate ideal tetrahedron

up to isometry (see Theorem 3.2.5). This is a consequence of Corollary 3.1.19 and of the fact, proved below, that the points in Table 1 are *all* the solutions of 3.11.

We define

$$S := \left\{ (\alpha, \sqrt{\beta}) \mid (\alpha, \beta) \in R \right\}. \quad (3.12)$$

Lemma 3.2.1. *The solutions of 3.11 are the same as the solutions of*

$$\begin{cases} \frac{(2\sqrt{3}d+1)(3c+\sqrt{3}d-1)(3c-\sqrt{3}d+1)}{27} = \alpha \\ \frac{3c^2+3d^2+2}{6} = \omega \end{cases}, \quad (3.13)$$

where $\omega = \sqrt{\beta}$. There are at most six different solutions (c, d) of the above system of equations.

Proof. It is immediate that the system above is equivalent to 3.11.

The first equation implies

$$(2\sqrt{3}d+1) \left(9c^2 - (\sqrt{3}d-1)^2 \right) = 27\alpha.$$

Using the second equation, we may replace c^2 by $\frac{1}{3}(6\omega - 3d^2 - 2)$ in the previous expression. We obtain a cubic in d which has at most three different solutions. By the second equation, each such solution gives rise to at most two different solutions of the form (c, d) . $\square$

Let $\bar{\Delta}$ denote the closure of Δ in $\mathbb{R}^2$. We consider the function $H : \bar{\Delta} \rightarrow \mathbb{R}^2$ defined by

$$H(c, d) = \left(\frac{(2\sqrt{3}d+1)(3c+\sqrt{3}d-1)(3c-\sqrt{3}d+1)}{27}, \left(\frac{3c^2+3d^2+2}{6} \right)^2 \right).$$

Proposition 3.2.2. The function $H|_{\bar{\Delta}_i} : \bar{\Delta}_i \rightarrow H(\bar{\Delta}_i)$ is a homeomorphism.

Proof. Note that the formulas in Table 1 are also valid for the points in $\bar{\Delta}$. Let σ be an element of the reflection group S_3 , represented by one of the formulas in Table 1. We claim that $H(c, d) = H(\sigma(c, d))$ for all $(c, d) \in \bar{\Delta}_1$. Indeed, we can write $(c, d) = \lim_{n \rightarrow \infty} (c_n, d_n)$, where $(c_n, d_n) \in \Delta_1$ for each n . As σ is a continuous function $\sigma(c, d) = \lim_{n \rightarrow \infty} \sigma(c_n, d_n)$, where $\sigma(c, d) \in \bar{\Delta}_i$ and $(c_n, d_n) \in \bar{\Delta}_i$ for some $i = 1, \dots, 6$. Then

$$H(c, d) = \lim_{n \rightarrow \infty} H(c_n, d_n), \quad \text{and} \quad H(\sigma(c, d)) = \lim_{n \rightarrow \infty} H(\sigma(c_n, d_n)).$$

Since H coincides with G in Δ we have $H(c_n, d_n) = G(c_n, d_n)$ and $H(\sigma(c_n, d_n)) = G(\sigma(c_n, d_n))$. By Corollary 3.1.20 $G(c_n, d_n) = G(\sigma(c_n, d_n))$, so

$$H(c, d) = \lim_{n \rightarrow \infty} G(c_n, d_n) = \lim_{n \rightarrow \infty} G(\sigma(c_n, d_n)) = H(\sigma(c, d)).$$

Let $(c, d) \in \bar{\Delta}_1$, the points in Table 1 are solutions of the system 3.11 with $(\alpha, \beta) = H(c, d)$, and then they are solutions of the system 3.13 where $\omega = \sqrt{\beta}$

It suffices to show that $H|_{\bar{\Delta}_1} : \bar{\Delta}_1 \rightarrow H(\bar{\Delta}_1)$ is a homeomorphism. In order to show that $H|_{\bar{\Delta}_1}$ is injective, suppose that $H(c, d) = H(c', d')$ where $(c, d), (c', d') \in \bar{\Delta}_1$. In particular, (c, d) and (c', d') are solutions of 3.13. If some of the points, say (c, d) , is in the interior of the angle formed by the hypotenuse and the shortest leg, we have $(c, d) = (c', d')$ because otherwise the system of equations 3.13 would have more than six solutions, contradicting Lemma 3.2.1. We can therefore assume that (c, d) belongs to either the shortest leg of Δ_1 or to its hypotenuse.

Note that, according to the second equation of 3.13, the solutions of the system lie in a circumference centered at the origin. So, if $(c, d) = (0, 0)$, then $(c', d') = (0, 0)$. If $(c, d) \neq (0, 0)$ is, say, on the hypotenuse (we can arrive to this case if it lies on the shortest leg) of Δ_1 , then $c = 0$. Assume that $(c, d) \neq (c', d')$. Since (c', d') cannot be in the interior of Δ_1 , it has to be a point p on the intersection of the shortest leg of Δ_1 with the circumference of radius d . Reflecting p on the appropriate altitude of Δ we arrive at the point $(0, -d)$; therefore $G(0, d) = G(0, -d)$. In particular, the determinants of the doubly stochastic matrices in question are equal, that is,

$$(2\sqrt{3}d + 1)(\sqrt{3}d - 1)(-\sqrt{3}d + 1) = (2\sqrt{3}(-d) + 1)(\sqrt{3}(-d) - 1)(\sqrt{3}(-d) + 1),$$

i.e.,

$$-6\sqrt{3}d^3 + 9d^2 - 1 = 6\sqrt{3}d^3 + 9d^2 - 1.$$

leading to $d = 0$, which is a contradiction. $\square$

Corollary 3.2.3. *The restriction $G|_{\Delta_1} : \Delta_1 \rightarrow R$ is a bijection, where G and Δ_1 are respectively the function and the region defined in 3.10 and 3.9.*

Observation 3.2.4. Given $(\alpha, \beta) \in R$, there exists a unique $(c, d) \in \Delta_1$ such that $G(c, d) = (\alpha, \beta)$. The other solutions are obtained by reflecting (c, d) on the altitudes of Δ . Thus, if (c, d) is in the interior of Δ_1 , there are six different solutions in Δ ; if $(c, d) \neq (0, 0)$ is on the hypotenuse or on the shortest leg of Δ_1 , we have exactly three different solutions in Δ ; if $(c, d) = (0, 0)$, there is a unique solution.

We arrive at

Theorem 3.2.5 (Seidel, [10], Speculation 1). *For each $(\alpha, \beta) \in R$ there exists, up to isometry, a unique non-degenerate ideal tetrahedron $T \in \mathcal{T}$ such that $\alpha = \det DSG(T)$ and $\beta = \text{per} DSG(T)$. In particular, the volume of a non-degenerate ideal tetrahedron is completely determined by the determinant and permanent of its doubly stochastic matrix.*

Proof. By Observation 3.2.4, there exists a unique $(c, d) \in \Delta_1$ such that the determinant and permanent of the matrix $\phi(c, d)$ are respectively α and β . The conclusion follows from Corollary 3.1.19. $\square$

In Proposition 3.4.3 we will give an explicit expression for the volume of an ideal tetrahedron in terms of the determinant and permanent of the doubly stochastic Gram matrix of its ideal vertices.

3.3 Solving system 3.13. A few derivatives.

Let us find an explicit expression for $(c, d) \in \Delta$ in terms of given permanent and determinant of a doubly stochastic matrix of a labelled (non-degenerate) ideal tetrahedron. This is the expression that is going to be used in order to express the volume of the tetrahedron as a function of determinant and permanent.

We remind that the region S alluded to in this section is the one defined in 3.12 and described in Appendix B.

By Observation 3.2.4, for a given $(\alpha, \omega) \in S$, there exists a unique $(c, d) \in \Delta_1$ satisfying the system of equations 3.13:

$$\begin{cases} \frac{(2\sqrt{3}d+1)(3c+\sqrt{3}d-1)(3c-\sqrt{3}d+1)}{27} = \alpha \\ \frac{3c^2+3d^2+2}{6} = \omega \end{cases}$$

We will write c and d in terms of α and ω such that $(c, d) \in \Delta_1$. As we have seen, the other solutions can be obtained from this one.

Observation 3.3.1. Let $(\alpha, \omega) \in S$. Let (c, d) be the unique solution of 3.13 that belongs to Δ_1 . Then $(\alpha, \omega) = \left(-\frac{1}{27}, \frac{1}{3}\right)$ if, and only if, $(c, d) = (0, 0)$.

Lemma 3.3.2. Let $a, b \in \mathbb{R}$ be such that $a \neq 0$ and $\frac{b^2}{4} + \frac{a^3}{27} \leq 0$. The cubic equation $x^3 + ax + b = 0$ has the three real roots

$$x_k = 2\sqrt{\frac{-a}{3}} \cos\left(\frac{1}{3} \arccos \frac{-3\sqrt{3}b}{2\sqrt{-a^3}} + \frac{2k\pi}{3}\right), \quad k = 0, 1, 2.$$

Proposition 3.3.3. Let (α, ω) be a fixed point in the interior of $S \setminus \left\{\left(-\frac{1}{27}, \frac{1}{3}\right)\right\}$. The solution of 3.13 lying in Δ_1 is given by

$$c = \sqrt{\frac{2}{3}} \sqrt{3\omega - 1} \sin\left(\frac{1}{3} \arccos \frac{-27\alpha + 18\omega - 7}{4\sqrt{2}(3\omega - 1)^{\frac{3}{2}}}\right) \quad (3.14)$$

and

$$d = \sqrt{\frac{2}{3}} \sqrt{3\omega - 1} \cos\left(\frac{1}{3} \arccos \frac{-27\alpha + 18\omega - 7}{4\sqrt{2}(3\omega - 1)^{\frac{3}{2}}}\right). \quad (3.15)$$

Proof. From the second equation of the system we have

$$c^2 = 2\omega - d^2 - \frac{2}{3}.$$

As we are looking for $(c, d) \in \Delta_1$, we choose the positive root

$$c = \sqrt{2\omega - d^2 - \frac{2}{3}}. \quad (3.16)$$

From the first equation follows that

$$27\alpha = (2\sqrt{3}d + 1) \left(9c^2 - (\sqrt{3}d - 1)^2 \right).$$

Equation 3.16 implies that

$$27\alpha = (2\sqrt{3}d + 1) (18\omega - 9d^2 - 6 - 3d^2 + 2\sqrt{3}d - 1)$$

i.e.,

$$0 = 24\sqrt{3}d^3 + (1 - 3\omega) 12\sqrt{3}d + (27\alpha - 18\omega + 7).$$

We therefore get the cubic equation

$$0 = x^3 + ax + b$$

where $x := 2\sqrt{3}d$ and

$$a := 6(1 - 3\omega), \quad (3.17)$$

$$b := 27\alpha - 18\omega + 7. \quad (3.18)$$

In order to solve the cubic, we first have to determine the sign of $\frac{b^2}{4} + \frac{a^3}{27}$. Since

$$a = 6 \left(1 - 3 \frac{3c^2 + 3d^2 + 2}{6} \right) = -9(c^2 + d^2) < 0 \quad (3.19)$$

and

$$\begin{aligned} b &= 27 \frac{(2\sqrt{3}d + 1)(3c + \sqrt{3}d - 1)(3c - \sqrt{3}d + 1)}{27} - 18 \frac{3c^2 + 3d^2 + 2}{6} + 7 = \\ &= 6\sqrt{3}d(3c^2 - d^2), \end{aligned} \quad (3.20)$$

we get

$$\frac{b^2}{4} + \frac{a^3}{27} = -27c^2(3d^2 - c^2)^2 \leq 0. \quad (3.21)$$

Hence, by Lemma 3.3.2, the solutions of the cubic equation are

$$x_k = 2\sqrt{-\frac{a}{3}} \cos \left(\frac{1}{3} \arccos \frac{-3\sqrt{3}b}{2\sqrt{-a^3}} + \frac{2k\pi}{3} \right), \quad k = 0, 1, 2.$$

Defining $d_k := \frac{1}{2\sqrt{3}}x_k$, we have

$$d_k = \frac{\sqrt{-a}}{3} \cos \left(\frac{1}{3} \arccos \frac{-3\sqrt{3}b}{2\sqrt{-a^3}} + \frac{2k\pi}{3} \right), \quad k = 0, 1, 2.$$

From Equation 3.16, the c corresponding to each d_k is, for each $k = 0, 1, 2$,

$$\begin{aligned} c_k &= \sqrt{2\omega - d_k^2 - \frac{2}{3}} \\ c_k &= \sqrt{\frac{-a}{9} - d_k^2} \\ c_k &= \frac{\sqrt{-a}}{3} \sqrt{1 - \cos^2 \left(\frac{1}{3} \arccos \frac{-3\sqrt{3}b}{2\sqrt{-a^3}} + \frac{2k\pi}{3} \right)} \\ c_k &= \frac{\sqrt{-a}}{3} \left| \sin \left(\frac{1}{3} \arccos \frac{-3\sqrt{3}b}{2\sqrt{-a^3}} + \frac{2k\pi}{3} \right) \right|. \end{aligned}$$

Note that requiring (c, d) to be in the interior of Δ_1 implies that the point (d, c) , written in polar coordinates, has angular coordinate in the interval $(0, \pi/3)$. As the angular coordinates of the points (d_1, c_1) and (d_2, c_2) are not in $(0, \pi/3)$, the solution we are looking for is the one corresponding to $k = 0$. $\square$

From now on, we will see c and d as functions $c, d : S \rightarrow \mathbb{R}$ such that $(c(\alpha, \omega), d(\alpha, \omega))$ is the solution of 3.13 in Δ_1 , i.e.,

$$c(\alpha, \omega) = \begin{cases} 0 & , \text{ if } (\alpha, \omega) = \left(-\frac{1}{27}, \frac{1}{3}\right) \\ \sqrt{\frac{2}{3}}\sqrt{3\omega-1} \sin \left(\frac{1}{3} \arccos \frac{-27\alpha+18\omega-7}{4\sqrt{2}(3\omega-1)^{\frac{3}{2}}} \right) & , \text{ if } (\alpha, \omega) \neq \left(-\frac{1}{27}, \frac{1}{3}\right) \end{cases} \quad (3.22)$$

and

$$d(\alpha, \omega) = \begin{cases} 0 & , \text{ if } (\alpha, \omega) = \left(-\frac{1}{27}, \frac{1}{3}\right) \\ \sqrt{\frac{2}{3}}\sqrt{3\omega-1} \cos \left(\frac{1}{3} \arccos \frac{-27\alpha+18\omega-7}{4\sqrt{2}(3\omega-1)^{\frac{3}{2}}} \right) & , \text{ if } (\alpha, \omega) \neq \left(-\frac{1}{27}, \frac{1}{3}\right) \end{cases} \quad (3.23)$$

Note that both c and d are continuous in S and smooth in $\text{int } S$. We frequently denote $c := c(\alpha, \omega)$ and $d := d(\alpha, \omega)$.

We will now find the partial derivatives of c and d with respect to α and ω . The derived expressions will be used in the proof of Speculation 4.

Proposition 3.3.4. Let $(\alpha, \omega) \in \text{int } S$. Then

$$\frac{\partial c}{\partial \alpha}(\alpha, \omega) = \frac{d}{6c(3d^2 - c^2)}, \quad (3.24)$$

$$\frac{\partial d}{\partial \alpha}(\alpha, \omega) = -\frac{1}{6(3d^2 - c^2)}. \quad (3.25)$$

Proof. The partial derivative of c with respect to α is

$$\frac{\partial c}{\partial \alpha}(\alpha, \omega) = \frac{\sqrt{2}\sqrt{3\omega-1}}{k} \cos\left(\frac{1}{3} \arccos \frac{-27\alpha+18\omega-7}{4\sqrt{2}(3\omega-1)^{\frac{3}{2}}}\right),$$

where

$$k := \sqrt{\left(4\sqrt{2}(3\omega-1)^{\frac{3}{2}}\right)^2 - (-27\alpha+18\omega-7)^2}.$$

From Equation 3.15, $\sqrt{2}\sqrt{3\omega-1} \cos\left(\frac{1}{3} \arccos \frac{-27\alpha+18\omega-7}{4\sqrt{2}(3\omega-1)^{\frac{3}{2}}}\right) = \sqrt{3}d(\alpha, \omega)$. So,

$$\frac{\partial c}{\partial \alpha}(\alpha, \omega) = \frac{\sqrt{3}}{k}d(\alpha, \omega). \quad (3.26)$$

Similarly, the partial derivative of d with respect to α is

$$\frac{\partial d}{\partial \alpha}(\alpha, \omega) = -\frac{\sqrt{2}\sqrt{3\omega-1}}{k} \sin\left(\frac{1}{3} \arccos \frac{-27\alpha+18\omega-7}{4\sqrt{2}(3\omega-1)^{\frac{3}{2}}}\right),$$

and, from Equation 3.14, we obtain

$$\frac{\partial d}{\partial \alpha}(\alpha, \omega) = -\frac{\sqrt{3}}{k}c(\alpha, \omega). \quad (3.27)$$

We can write k in terms of $c(\alpha, \omega)$ and $d(\alpha, \omega)$ as follows. First, it is possible to write k in terms of a and b because equations 3.54 and 3.18 imply that $3\omega - 1 = -\frac{a}{6}$ and $-27\alpha + 18\omega - 7 = -b$. Therefore,

$$\begin{aligned} k &= \sqrt{32(3\omega-1)^3 - (-27\alpha+18\omega-7)^2}, \\ k &= \sqrt{32\left(-\frac{a}{6}\right)^3 - (-b)^2} \\ k &= 2\sqrt{-\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}. \end{aligned}$$

From Equation 3.21,

$$k = 2\sqrt{27c^2(3d^2 - c^2)^2}.$$

Since $(c, d) \in \Delta_1$, we have $3d^2 \geq c^2$. So

$$k = 6\sqrt{3}c(3d^2 - c^2). \quad (3.28)$$

Finally, replacing k in equations 3.26 and 3.27, we arrive at the desired formulae. $\square$

Proposition 3.3.5. Let $(\alpha, \omega) \in \text{int } S$. Then

$$\frac{\partial c}{\partial \omega}(\alpha, \omega) = \frac{3d^2 - 3c^2 - \sqrt{3}d}{3c(3d^2 - c^2)} \quad (3.29)$$

$$\frac{\partial d}{\partial \omega}(\alpha, \omega) = \frac{6d + \sqrt{3}}{3(3d^2 - c^2)}. \quad (3.30)$$

Proof. The partial derivative of c respect to ω is

$$\begin{aligned} \frac{\partial c}{\partial \omega}(\alpha, \omega) &= \frac{\sqrt{3}}{\sqrt{2}\sqrt{3\omega-1}} \sin \left(\frac{1}{3} \arccos \frac{-27\alpha + 18\omega - 7}{4\sqrt{2}(3\omega-1)^{\frac{3}{2}}} \right) - \\ &\quad - \frac{3\sqrt{3}(9\alpha - 2\omega + 1)}{k\sqrt{2}\sqrt{3\omega-1}} \cos \left(\frac{1}{3} \arccos \frac{-27\alpha + 18\omega - 7}{4\sqrt{2}(3\omega-1)^{\frac{3}{2}}} \right). \end{aligned}$$

As in the proof of the previous proposition, we write

$$\begin{aligned} \frac{\partial c}{\partial \omega}(\alpha, \omega) &= \frac{3}{2(3\omega-1)} c(\alpha, \omega) - \frac{9(9\alpha - 2\omega + 1)}{2(3\omega-1)k} d(\alpha, \omega) = \\ &= \frac{3}{2(3\omega-1)k} (kc(\alpha, \omega) - 3(9\alpha - 2\omega + 1)d(\alpha, \omega)). \end{aligned} \quad (3.31)$$

Since $(c(\alpha, \omega), d(\alpha, \omega))$ is a solution of 3.13, we obtain

$$\begin{aligned} 9\alpha - 2\omega + 1 &= 9 \frac{(2\sqrt{3}d + 1)(3c + \sqrt{3}d - 1)(3c - \sqrt{3}d + 1)}{27} - 2 \frac{3c^2 + 3d^2 + 2}{6} + 1 \\ &= 2(3\sqrt{3}c^2d + c^2 - \sqrt{3}d^3 + d^2). \end{aligned} \quad (3.32)$$

From 3.32 and 3.28, we get

$$\begin{aligned} \frac{\partial c}{\partial \omega}(\alpha, \omega) &= \frac{3}{2(3\omega-1)6\sqrt{3}c(3d^2 - c^2)} \left(6\sqrt{3}c^2(3d^2 - c^2) - 6d(3\sqrt{3}c^2d + c^2 - \sqrt{3}d^3 + d^2) \right) \\ &= \frac{\sqrt{3}}{2(3\omega-1)c(3d^2 - c^2)} (c^2 + d^2) (\sqrt{3}d^2 - \sqrt{3}c^2 - d). \end{aligned}$$

We arrive at

$$\frac{\partial c}{\partial \omega}(\alpha, \omega) = \frac{3d^2 - 3c^2 - \sqrt{3}d}{3c(3d^2 - c^2)}$$

because $2(3\omega - 1) = 3(c^2 + d^2)$.

The partial derivative of d with respect to ω is

$$\begin{aligned} \frac{\partial d}{\partial \omega}(\alpha, \omega) &= \frac{\sqrt{3}}{\sqrt{2}\sqrt{3\omega-1}} \cos \left(\frac{1}{3} \arccos \frac{-27\alpha + 18\omega - 7}{4\sqrt{2}(3\omega-1)^{\frac{3}{2}}} \right) \\ &\quad + \frac{3\sqrt{3}(9\alpha - 2\omega + 1)}{k\sqrt{2}\sqrt{3\omega-1}} \sin \left(\frac{1}{3} \arccos \frac{-27\alpha + 18\omega - 7}{4\sqrt{2}(3\omega-1)^{\frac{3}{2}}} \right). \end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial d}{\partial \omega}(\alpha, \omega) &= \frac{3}{2(3\omega - 1)}d(\alpha, \omega) + \frac{9(9\alpha - 2\omega + 1)}{2(3\omega - 1)k}c(\alpha, \omega) \\ &= \frac{3}{2(3\omega - 1)k}(kd(\alpha, \omega) + 3(9\alpha - 2\omega + 1)c(\alpha, \omega)).\end{aligned}$$

From 3.32 and 3.28, we obtain

$$\begin{aligned}\frac{\partial d}{\partial \omega}(\alpha, \omega) &= \frac{3}{2(3\omega - 1)6\sqrt{3}c(3d^2 - c^2)}\left(6\sqrt{3}cd(3d^2 - c^2) + 6c\left(3\sqrt{3}c^2d + c^2 - \sqrt{3}d^3 + d^2\right)\right) \\ &= \frac{3}{2(3\omega - 1)\sqrt{3}c(3d^2 - c^2)}c(c^2 + d^2)(2\sqrt{3}d + 1) = \frac{\partial d}{\partial \omega}(\alpha, \omega) = \frac{6d + \sqrt{3}}{3(3d^2 - c^2)}.\end{aligned}$$

□

3.4 Volume formulae

There are several known formulae for the volume of an ideal tetrahedron in hyperbolic 3-space. Perhaps the simplest one was obtained by Milnor. It expresses the volume as a function of the dihedral angles of the tetrahedron. In order to present Milnor's form, we introduce the Lobachevsky function:

Definition 3.4.1. The Lobachevsky function $\pi : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\pi(x) := -\int_0^x \log |2 \sin t| dt.$$

Clearly, π is differentiable at each $x \neq k\pi$, $k \in \mathbb{Z}$, and

$$\frac{d\pi}{dx}(x) = -\log |2 \sin x|.$$

If $0 < x < \pi$, then $\frac{d\pi}{dx}(x) = -\log(2 \sin x)$.

Theorem 3.4.2 (Milnor, [8] Lemma 4). *The volume of an ideal non-degenerate tetrahedron T with dihedral angles $\theta_1, \theta_2, \theta_3$ is given by*

$$\text{vol}(T) = \pi(\theta_1) + \pi(\theta_2) + \pi(\theta_3),$$

where π is the Lobachevsky function.

It follows from section ??? that the dihedral angles of the labelled tetrahedron (v_1, v_2, v_3, v_4) in terms of the entries r, s, t of its doubly stochastic matrix are given by

$$a_1 = \arccos \frac{-r^2 + s^2 + t^2}{2st}, \quad a_2 = \arccos \frac{r^2 - s^2 + t^2}{2rt}, \quad a_3 = \arccos \frac{r^2 + s^2 - t^2}{2rs}.$$

Indeed, let us calculate the dihedral angle between the faces generated by v_1, v_2, v_3 and v_1, v_2, v_4 .

Note that the dihedral angles of the polyhedron are, by the law of cosines, exactly the internal angles of the Euclidean triangle with sides of lengths r, s, t .

Summarizing, we have the following

Proposition 3.4.3. The volume function $\text{vol} : S \rightarrow \mathbb{R}$ is given by

$$\text{vol}(\alpha, \omega) = \pi(a_1) + \pi(a_2) + \pi(a_3), \quad (3.33)$$

where

$$\begin{aligned} a_1 &= \arccos \frac{-r^2 + s^2 + t^2}{2st}, \quad a_2 = \arccos \frac{r^2 - s^2 + t^2}{2rt}, \quad a_3 = \arccos \frac{r^2 + s^2 - t^2}{2rs}, \\ r &= \frac{2 - 2\sqrt{3}d}{6}, \quad s = \frac{2 - 3c + \sqrt{3}d}{6}, \quad t = \frac{2 + 3c + \sqrt{3}d}{6}, \\ c &= \begin{cases} 0 & , \text{ if } (\alpha, \omega) = (-\frac{1}{27}, \frac{1}{3}) \\ \sqrt{\frac{2}{3}}\sqrt{3\omega - 1} \sin\left(\frac{1}{3}\arccos \frac{-27\alpha + 18\omega - 7}{4\sqrt{2}(3\omega - 1)^{\frac{3}{2}}}\right) & , \text{ if } (\alpha, \omega) \neq (-\frac{1}{27}, \frac{1}{3}) \end{cases}, \\ d &= \begin{cases} 0 & , \text{ if } (\alpha, \omega) = (-\frac{1}{27}, \frac{1}{3}) \\ \sqrt{\frac{2}{3}}\sqrt{3\omega - 1} \cos\left(\frac{1}{3}\arccos \frac{-27\alpha + 18\omega - 7}{4\sqrt{2}(3\omega - 1)^{\frac{3}{2}}}\right) & , \text{ if } (\alpha, \omega) \neq (-\frac{1}{27}, \frac{1}{3}) \end{cases}. \end{aligned}$$

3.5 Proof of a particular case of Speculation 3

Recall the definition of R from 3.4. Given $(\alpha, \beta) \in R$, we have

$$\alpha = -(r+s+t)(-r+s+t)(r-s+t)(r+s-t) \text{ and } \beta = (r^2 + s^2 + t^2)^2,$$

for some $r, s, t \geq 0$ such that $r+s+t = 1$.

Note that β is the size (to the forth grade) of the vector $(r, s, t) \in \mathbb{R}^3$. Then the permanent has a unique minimum when $r = s = t = \frac{1}{3}$.

From Proposition 3.1.16 we also have that r, s, t are the sides of an euclidean triangle. Let A denote the area of this triangle. By the Heron's formula $\alpha = -16A^2$. Thus, the determinant function has a unique minimum when $r = s = t = \frac{1}{3}$.

3.6 Proof of Speculation 4

Let O be the open set $O := \{(r, s, t) \in \mathbb{R}^3 \mid r < s+t, s < r+t, t < r+s\}$. Consider the functions $(c, d) : S \rightarrow \Delta_1 \subset \Delta$ and $(r, s, t) : \Delta \rightarrow O$ as they are in Definition 3.4.3, i.e., as in equations 3.14, 3.15. Consider the function $f : O \rightarrow \mathbb{R}$ defined by

$$f(r, s, t) = \pi(a_1) + \pi(a_2) + \pi(a_3), \quad (3.34)$$

where

$$a_1 = \arccos \frac{-r^2 + s^2 + t^2}{2st}, \quad a_2 = \arccos \frac{r^2 - s^2 + t^2}{2rt}, \quad a_3 = \arccos \frac{r^2 + s^2 - t^2}{2rs}.$$

The function equality $\text{vol} = f \circ (r, s, t) \circ (c, d)$ is satisfied.

We can calculate the partial derivatives of the volume functions at all the points of $\text{int}S$. The Chain rule ensures that

$$\begin{aligned} \frac{\partial \text{vol}}{\partial \alpha} &= \frac{\partial (f \circ (r, s, t))}{\partial c} \frac{\partial c}{\partial \alpha} + \frac{\partial (f \circ (r, s, t))}{\partial d} \frac{\partial d}{\partial \alpha} \\ \frac{\partial \text{vol}}{\partial \alpha} &= \left(\frac{\partial f}{\partial r} \frac{\partial r}{\partial c} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial c} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial c} \right) \frac{\partial c}{\partial \alpha} + \left(\frac{\partial f}{\partial r} \frac{\partial r}{\partial d} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial d} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial d} \right) \frac{\partial d}{\partial \alpha}. \end{aligned} \quad (3.35)$$

In the same way

$$\frac{\partial \text{vol}}{\partial \omega} = \left(\frac{\partial f}{\partial r} \frac{\partial r}{\partial c} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial c} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial c} \right) \frac{\partial c}{\partial \omega} + \left(\frac{\partial f}{\partial r} \frac{\partial r}{\partial d} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial d} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial d} \right) \frac{\partial d}{\partial \omega}. \quad (3.36)$$

The partial derivatives of r, s, t respect to c are

$$\frac{\partial r}{\partial c}(c, d) = 0, \quad \frac{\partial s}{\partial c}(c, d) = -\frac{1}{2}, \quad \frac{\partial t}{\partial c}(c, d) = \frac{1}{2}.$$

And the partial derivatives of r, s, t respect to d are

$$\frac{\partial r}{\partial d}(c, d) = -\frac{\sqrt{3}}{3}, \quad \frac{\partial s}{\partial d}(c, d) = \frac{\sqrt{3}}{6}, \quad \frac{\partial t}{\partial d}(c, d) = \frac{\sqrt{3}}{6}.$$

Then, replacing these values in equations 3.35 and 3.36, we obtain

$$\frac{\partial \text{vol}}{\partial \alpha} = \frac{1}{2} \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial c}{\partial \alpha} + \frac{\sqrt{3}}{6} \left(-2\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial d}{\partial \alpha}, \quad (3.37)$$

and

$$\frac{\partial \text{vol}}{\partial \omega} = \frac{1}{2} \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial c}{\partial \omega} + \frac{\sqrt{3}}{6} \left(-2\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial d}{\partial \omega}. \quad (3.38)$$

Proposition 3.6.1. The partial derivatives of f respect to the variables r, s, t are

$$\frac{\partial f}{\partial r}(r, s, t) = 2 \frac{-r \log r + \cos a_3 s \log s + \cos a_2 t \log t}{\sqrt{(r+s+t)(-r+s+t)(r-s+t)(r+s-t)}}, \quad (3.39)$$

$$\frac{\partial f}{\partial s}(r, s, t) = 2 \frac{\cos a_3 r \log r - s \log s + \cos a_1 t \log t}{\sqrt{(r+s+t)(-r+s+t)(r-s+t)(r+s-t)}}, \quad (3.40)$$

$$\frac{\partial f}{\partial t}(r, s, t) = 2 \frac{\cos a_2 r \log r + \cos a_1 s \log s - t \log t}{\sqrt{(r+s+t)(-r+s+t)(r-s+t)(r+s-t)}}. \quad (3.41)$$

Proof. From Observation ?? and the Chain rule, we have that

$$\frac{d}{dx} (\pi(\arccos x)) = \frac{1}{\sqrt{1-x^2}} \log \left(2\sqrt{1-x^2} \right).$$

Denote by $h := \sqrt{(r+s+t)(-r+s+t)(r-s+t)(r+s-t)}$. The following identities will be used.

$$\begin{aligned}\sqrt{1 - \left(\frac{-r^2 + s^2 + t^2}{2st}\right)^2} &= \frac{\sqrt{(r+s+t)(-r+s+t)(r-s+t)(r+s-t)}}{2st} = \frac{h}{2st}, \\ \sqrt{1 - \left(\frac{r^2 - s^2 + t^2}{2rt}\right)^2} &= \frac{\sqrt{(r+s+t)(-r+s+t)(r-s+t)(r+s-t)}}{2rt} = \frac{h}{2rt}, \\ \sqrt{1 - \left(\frac{r^2 + s^2 - t^2}{2rs}\right)^2} &= \frac{\sqrt{(r+s+t)(-r+s+t)(r-s+t)(r+s-t)}}{2rs} = \frac{h}{2rs}.\end{aligned}$$

Let us find the partial derivative of the f respect to r in Equation 3.34. We have that

$$\begin{aligned}\frac{\partial}{\partial r}(\pi(a_1)) &= \frac{\partial}{\partial r} \left(\pi \left(\arccos \frac{-r^2 + s^2 + t^2}{2st} \right) \right) = -\frac{2r}{h} \log \left(\frac{h}{st} \right), \\ \frac{\partial}{\partial r}(\pi(a_2)) &= \frac{\partial}{\partial r} \left(\pi \left(\arccos \frac{r^2 - s^2 + t^2}{2rt} \right) \right) = \frac{r^2 + s^2 - t^2}{rh} \log \left(\frac{h}{rt} \right), \\ \frac{\partial}{\partial r}(\pi(a_3)) &= \frac{\partial}{\partial r} \left(\pi \left(\arccos \frac{r^2 + s^2 - t^2}{2rs} \right) \right) = \frac{r^2 - s^2 + t^2}{rh} \log \left(\frac{h}{rs} \right).\end{aligned}$$

Adding these equations we get

$$\begin{aligned}\frac{\partial f}{\partial r}(r, s, t) &= \frac{1}{h} \left(-2r \log r + \frac{r^2 + s^2 - t^2}{r} \log s + \frac{r^2 - s^2 + t^2}{r} \log t \right) \\ \frac{\partial f}{\partial r}(r, s, t) &= 2 \frac{-r \log r + \cos a_3 s \log s + \cos a_2 t \log t}{\sqrt{(r+s+t)(-r+s+t)(r-s+t)(r+s-t)}}.\end{aligned}$$

Because of the symmetry of f respect to the variables we get the other two identities. $\square$

Observation 3.6.2. We need the values of $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial t}$ only at the points that are image by the function (r, s, t) , i.e., $(r, s, t) = (r, s, t)(c(\alpha, \omega), d(\alpha, \omega))$, for some $(\alpha, \omega) \in \text{int} S$. In this case, the expression

$$(r+s+t)(-r+s+t)(r-s+t)(r+s-t) = -\det \begin{pmatrix} 0 & r & s & t \\ r & 0 & t & s \\ s & t & 0 & r \\ t & s & r & 0 \end{pmatrix} = -\alpha. \quad (3.42)$$

Also, for these kind of points (r, s, t) , we have that $r+s+t = 1$. Thus,

$$\cos a_1 = \frac{-r^2 + s^2 + t^2}{2st} = \frac{-r^2 + (s+t)^2 - 2st}{2st} = \frac{-r^2 + (1-r)^2}{2st} - 1 = \frac{1-2r}{2st} - 1.$$

Similarly,

$$\begin{aligned}\cos a_2 &= \frac{1-2s}{2rt} - 1, \\ \cos a_3 &= \frac{1-2t}{2rs} - 1.\end{aligned}$$

Then, equations 3.39, 3.40, 3.41 become

$$\frac{\partial f}{\partial r}(r, s, t) = \frac{2}{\sqrt{-\alpha}} \left(-r \log r + \left(\frac{1-2t}{2r} - s \right) \log s + \left(\frac{1-2s}{2r} - t \right) \log t \right), \quad (3.43)$$

$$\frac{\partial f}{\partial s}(r, s, t) = \frac{2}{\sqrt{-\alpha}} \left(\left(\frac{1-2t}{2s} - r \right) \log r - s \log s + \left(\frac{1-2r}{2s} - t \right) \log t \right), \quad (3.44)$$

$$\frac{\partial f}{\partial t}(r, s, t) = \frac{2}{\sqrt{-\alpha}} \left(\left(\frac{1-2s}{2t} - r \right) \log r + \left(\frac{1-2r}{2t} - s \right) \log s - t \log t \right). \quad (3.45)$$

Proposition 3.6.3. Let $(\alpha, \omega) \in \text{int } S$. Then,

$$\frac{\partial \text{vol}}{\partial \alpha}(\alpha, \omega) = \frac{1}{1944c(3d^2 - c^2)rst\sqrt{-\alpha}} \left(M \log \frac{st}{r^2} + N \log \frac{t}{s} \right),$$

where c, d, r, s, t are as in Definition 3.4.3 and

$$M := 27c^3d - 9\sqrt{3}c^3 + 63cd^3 - 9\sqrt{3}cd^2 - 18cd + 2\sqrt{3}c,$$

$$N := -27\sqrt{3}c^4 + 27\sqrt{3}c^2d^2 + 27c^2d + 9\sqrt{3}c^2 + 18\sqrt{3}d^4 + 27d^3 - 9\sqrt{3}d^2 - 6d.$$

Proof. By equations 3.44 and 3.45, the factor of $\frac{\partial c}{\partial \alpha}$ in Equation 3.37 is

$$\begin{aligned} \frac{1}{2} \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) &= \frac{1}{\sqrt{-\alpha}} \left(\left(-\frac{1-2t}{2s} + \frac{1-2s}{2t} \right) \log r + \frac{1-2r}{2t} \log s - \frac{1-2r}{2s} \log t \right) \\ &= \frac{r(s(1-2s) - t(1-2t)) \log r + rs(1-2r) \log s - rt(1-2r) \log t}{2rst\sqrt{-\alpha}}. \end{aligned}$$

Then,

$$\frac{1}{2} \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) = \frac{A \log r + B \log s + C \log t}{2rst\sqrt{-\alpha}}, \quad (3.46)$$

where $A := r(s(1-2s) - t(1-2t))$, $B := rs(1-2r)$, and $C := -rt(1-2r)$.

We can write A, B, C in terms of c and d replacing $r = \frac{2-2\sqrt{3}d}{6}$, $s = \frac{2-3c+\sqrt{3}d}{6}$, and $t = \frac{2+3c+\sqrt{3}d}{6}$.

After operating, we get

$$\begin{aligned} A &= \frac{1}{9} \left(-6cd^2 + \sqrt{3}cd + c \right), \\ B &= \frac{1}{54} \left(18cd^2 - 3\sqrt{3}cd - 3c - 6\sqrt{3}d^3 - 9d^2 + 3\sqrt{3}d + 2 \right), \\ C &= \frac{1}{54} \left(18cd^2 - 3\sqrt{3}cd - 3c + 6\sqrt{3}d^3 + 9d^2 - 3\sqrt{3}d - 2 \right). \end{aligned}$$

From Equation 3.24, the first term of the sum in Equation 3.37 is

$$\begin{aligned} \frac{1}{2} \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial c}{\partial \alpha} &= \frac{A \log r + B \log s + C \log t}{2rst\sqrt{-\alpha}} \frac{d}{6c(3d^2 - c^2)} \\ \frac{1}{2} \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial c}{\partial \alpha} &= \frac{dA \log r + dB \log s + dC \log t}{12c(3d^2 - c^2)rst\sqrt{-\alpha}}. \end{aligned} \quad (3.47)$$

On the other hand, by equations 3.43, 3.44, and 3.45, the factor of $\frac{\partial d}{\partial \alpha}$ in Equation 3.37 is

$$\begin{aligned} \frac{\sqrt{3}}{6} \left(-2 \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) &= \frac{\sqrt{3}}{3\sqrt{-\alpha}} \left(\left(\frac{1-2t}{2s} + \frac{1-2s}{2t} \right) \log r + \left(-\frac{1-2t}{r} + \frac{1-2r}{2t} \right) \log s \right. \\ &\quad \left. + \left(-\frac{1-2s}{r} + \frac{1-2r}{2s} \right) \log t \right) \\ &= \frac{\sqrt{3}}{6rst\sqrt{-\alpha}} (r(t(1-2t) + s(1-2s)) \log r + \\ &\quad s(r(1-2r) - 2t(1-2t)) \log s + t(r(1-2r) - 2s(1-2s)) \log t). \end{aligned}$$

Then,

$$\frac{\sqrt{3}}{6} \left(-2 \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) = \frac{D \log r + E \log s + F \log t}{6rst\sqrt{-\alpha}}, \quad (3.48)$$

where we call $D := \sqrt{3}r(t(1-2t) + s(1-2s))$, $E := \sqrt{3}s(r(1-2r) - 2t(1-2t))$, and $F := \sqrt{3}t(r(1-2r) - 2s(1-2s))$.

As before, we can write D, E, F in terms of c and d . After operating, we obtain

$$\begin{aligned} D &= \frac{1}{27} (27c^2d - 9\sqrt{3}c^2 + 9d^3 - 9d + 2\sqrt{3}), \\ E &= \frac{1}{54} (-27\sqrt{3}c^3 - 27c^2d + 9\sqrt{3}c^2 + 27\sqrt{3}cd^2 + 27cd + 9\sqrt{3}c - 9d^3 + 9d - 2\sqrt{3}), \\ F &= \frac{1}{54} (27\sqrt{3}c^3 - 27c^2d + 9\sqrt{3}c^2 - 27\sqrt{3}cd^2 - 27cd - 9\sqrt{3}c - 9d^3 + 9d - 2\sqrt{3}). \end{aligned}$$

From Equation 3.25, the second term of the sum in Equation 3.37 is

$$\begin{aligned} \frac{\sqrt{3}}{6} \left(-2 \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial d}{\partial \alpha} &= \frac{D \log r + E \log s + F \log t}{6rst\sqrt{-\alpha}} \left(-\frac{1}{6(3d^2 - c^2)} \right) \\ \frac{\sqrt{3}}{6} \left(-2 \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial d}{\partial \alpha} &= \frac{-cD \log r - cE \log s - cF \log t}{36c(3d^2 - c^2)rst\sqrt{-\alpha}} \end{aligned} \quad (3.49)$$

Summing equations 3.47 and 3.49, we obtain

$$\frac{\partial \text{vol}}{\partial \alpha} = \frac{(3dA - cD) \log r + (3dB - cE) \log s + (3dC - cF) \log t}{36c(3d^2 - c^2)rst\sqrt{-\alpha}},$$

where

$$\begin{aligned} 3dA - cD &= \frac{1}{27} (-27c^3d + 9\sqrt{3}c^3 - 63cd^3 + 9\sqrt{3}cd^2 + 18cd - 2\sqrt{3}c), \\ 3dB - cE &= \frac{1}{54} (27\sqrt{3}c^4 + 27c^3d - 9\sqrt{3}c^3 - 27\sqrt{3}c^2d^2 - 27c^2d - 9\sqrt{3}c^2 + 63cd^3 - 9\sqrt{3}cd^2 \\ &\quad - 18cd + 2\sqrt{3}c - 18\sqrt{3}d^4 - 27d^3 + 9\sqrt{3}d^2 + 6d), \\ 3dC - cF &= \frac{1}{54} (-27\sqrt{3}c^4 + 27c^3d - 9\sqrt{3}c^3 + 27\sqrt{3}c^2d^2 + 27c^2d + 9\sqrt{3}c^2 + 63cd^3 - 9\sqrt{3}cd^2 \\ &\quad - 18cd + 2\sqrt{3}c + 18\sqrt{3}d^4 + 27d^3 - 9\sqrt{3}d^2 - 6d). \end{aligned}$$

At this point, note that

$$3dA - cD = -\frac{M}{27}, \quad 3dB - cE = \frac{M-N}{54}, \quad 3dC - cF = \frac{M+N}{54}.$$

Then,

$$\begin{aligned}\frac{\partial \text{vol}}{\partial \alpha}(\alpha, \omega) &= \frac{1}{36c(3d^2 - c^2)rst\sqrt{-\alpha}} \left(-\frac{M}{27} \log r + \frac{M-N}{54} \log s + \frac{M+N}{54} \log t \right) \\ \frac{\partial \text{vol}}{\partial \alpha}(\alpha, \omega) &= \frac{1}{1944c(3d^2 - c^2)rst\sqrt{-\alpha}} \left(M \log \frac{st}{r^2} + N \log \frac{t}{s} \right).\end{aligned}$$

□

Proposition 3.6.4. Let $(\alpha, \omega) \in \text{int } S$. Then,

$$\frac{\partial \text{vol}}{\partial \omega}(\alpha, \omega) = \frac{2\sqrt{3}d + 1}{324c(3d^2 - c^2)rst\sqrt{-\alpha}} \left(P \log \frac{st}{r^2} + Q \log \frac{t}{s} \right),$$

where c, d, r, s, t are as in Definition 3.4.3 and

$$\begin{aligned}P &:= -18\sqrt{3}c^3d + 18c^3 + 6\sqrt{3}cd^3 - 18cd^2 + 6\sqrt{3}cd - 2c, \\ Q &:= 27c^4 - 36c^2d^2 - 12\sqrt{3}c^2d - 3c^2 + 9d^4 - 9d^2 + 2\sqrt{3}d.\end{aligned}$$

Proof. The first term in the sum of Equation 3.38 is obtained replacing equations 3.29 and 3.46 as follows:

$$\begin{aligned}\frac{1}{2} \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial c}{\partial \omega} &= \frac{A \log r + B \log s + C \log T}{2rst\sqrt{-\alpha}} \frac{3d^2 - 3c^2 - \sqrt{3}d}{3c(3d^2 - c^2)} \\ \frac{1}{2} \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial c}{\partial \omega} &= \frac{3(3d^2 - 3c^2 - \sqrt{3}d)(A \log r + B \log s + C \log T)}{18c(3d^2 - c^2)rst\sqrt{-\alpha}}.\end{aligned}\tag{3.50}$$

The second term in the sum of Equation 3.38 is obtained equations 3.30 and 3.48 as follows:

$$\begin{aligned}\frac{\sqrt{3}}{6} \left(-2\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial d}{\partial \omega} &= \frac{D \log r + E \log s + F \log t}{6rst\sqrt{-\alpha}} \frac{6d + \sqrt{3}}{3(3d^2 - c^2)} \\ \frac{\sqrt{3}}{6} \left(-2\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial d}{\partial \omega} &= \frac{c(6d + \sqrt{3})(D \log r + E \log s + F \log t)}{18c(3d^2 - c^2)rst\sqrt{-\alpha}}.\end{aligned}\tag{3.51}$$

Summing equations 3.50 and 3.51, we obtain

$$\frac{\partial \text{vol}}{\partial \omega} = \frac{J \log r + K \log s + L \log t}{18c(3d^2 - c^2)rst\sqrt{-\alpha}},$$

where

$$\begin{aligned}J &:= 3(3d^2 - 3c^2 - \sqrt{3}d)A + c(6d + \sqrt{3})D, \\ K &:= 3(3d^2 - 3c^2 - \sqrt{3}d)B + c(6d + \sqrt{3})E, \\ L &:= 3(3d^2 - 3c^2 - \sqrt{3}d)C + c(6d + \sqrt{3})F.\end{aligned}$$

We replace the values of A, B, C, D, E, F in terms of c and d obtained in the proof of Proposition 3.6.3. After operating, we obtain

$$\begin{aligned} J &= \frac{2\sqrt{3}d+1}{9} \left(18\sqrt{3}c^3d - 18c^3 - 6\sqrt{3}cd^3 + 18cd^2 - 6\sqrt{3}cd + 2c \right), \\ K &= \frac{2\sqrt{3}d+1}{18} \left(-27c^4 - 18\sqrt{3}c^3d + 18c^3 + 36c^2d^2 + 12\sqrt{3}c^2d + 3c^2 + 6\sqrt{3}cd^3 - 18cd^2 \right. \\ &\quad \left. + 6\sqrt{3}cd - 2c - 9d^4 + 9d^2 - 2\sqrt{3}d \right), \\ L &= \frac{2\sqrt{3}d+1}{18} \left(27c^4 - 18\sqrt{3}c^3d + 18c^3 - 36c^2d^2 - 12\sqrt{3}c^2d - 3c^2 + 6\sqrt{3}cd^3 - 18cd^2 \right. \\ &\quad \left. + 6\sqrt{3}cd - 2c + 9d^4 - 9d^2 + 2\sqrt{3}d \right). \end{aligned}$$

And note that

$$J = \frac{2\sqrt{3}d+1}{9} (-P), \quad K = \frac{2\sqrt{3}d+1}{18} (P-Q), \quad L = \frac{2\sqrt{3}d+1}{18} (P+Q).$$

Finally, we have that

$$\begin{aligned} \frac{\partial \text{vol}}{\partial \omega} &= \frac{2\sqrt{3}d+1}{324c(3d^2-c^2)rst\sqrt{-\alpha}} (-2P \log r + (P-Q) \log s + (P+Q) \log t) \\ \frac{\partial \text{vol}}{\partial \omega} &= \frac{2\sqrt{3}d+1}{324c(3d^2-c^2)rst\sqrt{-\alpha}} \left(P \log \frac{st}{r^2} + Q \log \frac{t}{s} \right). \end{aligned}$$

□

Observation 3.6.5. We claim that for a given $(\alpha, \omega) \in \text{int} S$, we have that $(c, d) \in \text{int} \Delta_1$. Indeed, if we suppose that $(c, d) \Delta_1 \setminus \text{int} \Delta_1$, Proposition B.1.3 implies that $(\alpha, \omega) = I(c, d) \notin \text{int} S$. This fact explains that the factor of $M \log \frac{st}{r^2} + N \log \frac{t}{s}$ in Proposition 3.6.3 and the factor of $P \log \frac{st}{r^2} + Q \log \frac{t}{s}$ in Proposition 3.6.4 are both positive. With this, the signs of $\frac{\partial \text{vol}}{\partial \alpha}$ and $\frac{\partial \text{vol}}{\partial \omega}$ are determined by the signs of $M \log \frac{st}{r^2} + N \log \frac{t}{s}$ and $P \log \frac{st}{r^2} + Q \log \frac{t}{s}$ respectively.

Proposition 3.6.6. Let $(\alpha, \omega) \in \text{int} S$. Then,

$$M \log \frac{st}{r^2} + N \log \frac{t}{s} < 0.$$

Proof. We will write c and d in terms of r, s, t . Since $r = \frac{1-\sqrt{3}d}{3}$, $s = \frac{2-3c+\sqrt{3}d}{6}$, $t = \frac{2+3c+\sqrt{3}d}{6}$, we have that $c = t - s$ and $d = \frac{1-3r}{\sqrt{3}}$. Note that there are other formulas to express c and d in terms of r, s, t because $r + s + t = 1$, but we will use the mentioned formulas.

Now,

$$\begin{aligned} M &= 27c^3d - 9\sqrt{3}c^3 + 63cd^3 - 9\sqrt{3}cd^2 - 18cd + 2\sqrt{3}c \\ M &= c \left(3d - \sqrt{3} \right) \left(9c^2 + 21d^2 + 4\sqrt{3}d - 2 \right) \\ M &= (t-s) \left(3\frac{1-3r}{\sqrt{3}} - \sqrt{3} \right) \left(9(t-s)^2 + 21 \left(\frac{1-3r}{\sqrt{3}} \right)^2 + 4\sqrt{3}\frac{1-3r}{\sqrt{3}} - 2 \right) \\ M &= -27\sqrt{3}r(t-s) (7r^2 + s^2 + t^2 - 2st - 6r + 1). \end{aligned} \tag{3.52}$$

The same for

$$\begin{aligned}
N &= -27\sqrt{3}c^4 + 27\sqrt{3}c^2d^2 + 27c^2d + 9\sqrt{3}c^2 + 18\sqrt{3}d^4 + 27d^3 - 9\sqrt{3}d^2 - 6d \\
N &= 3 \left(-9\sqrt{3}c^4 + 3c^2 \left(3\sqrt{3}d^2 + 3d + \sqrt{3} \right) + d \left(2\sqrt{3}d + 1 \right) \left(\sqrt{3}d + 2 \right) \left(\sqrt{3}d - 1 \right) \right) \\
N &= 3 \left(-9\sqrt{3}(t-s)^4 + 3(t-s)^2 \left(3\sqrt{3} \left(\frac{1-3r}{\sqrt{3}} \right)^2 + 3\frac{1-3r}{\sqrt{3}} + \sqrt{3} \right) \right. \\
&\quad \left. + \frac{1-3r}{\sqrt{3}} \left(2\sqrt{3}\frac{1-3r}{\sqrt{3}} + 1 \right) \left(\sqrt{3}\frac{1-3r}{\sqrt{3}} + 2 \right) \left(\sqrt{3}\frac{1-3r}{\sqrt{3}} - 1 \right) \right) \\
N &= -27\sqrt{3} \left((t-s)^4 - (t-s)^2 (3r^2 - 3r + 1) + r(1-3r)(1-r)(1-2r) \right). \tag{3.53}
\end{aligned}$$

From Proposition 3.1.16, the positive numbers r, s, t are the sides of an Euclidean triangle. Also, $(\alpha, \omega) \in \text{int } S$ implies that $(c, d) \in \text{int } \Delta_1$, then by Proposition 3.1.17, we have that $r < s < t$.

We define $a, \varepsilon \in \mathbb{R}$ as

$$a := \frac{s}{r}, \tag{3.54}$$

$$\varepsilon := \frac{t-s}{r}. \tag{3.55}$$

Then, $a > 1$ and $0 < \varepsilon < 1$. Also, we have that

$$\frac{t}{r} = a + \varepsilon. \tag{3.56}$$

The equation $1 = r + s + t$ implies that

$$\frac{1}{r} = 1 + 2a + \varepsilon. \tag{3.57}$$

Now, we write Equation 3.52 as

$$\begin{aligned}
M &= -27\sqrt{3}r^4 \frac{t-s}{r} \left(7 + \frac{s^2}{r^2} + \frac{t^2}{r^2} - \frac{2st}{r^2} - \frac{6}{r} + \frac{1}{r^2} \right) \\
M &= -27\sqrt{3}r^4 \varepsilon \left(7 + a^2 + (a + \varepsilon)^2 - 2a(a + \varepsilon) - 6(1 + 2a + \varepsilon) + (1 + 2a + \varepsilon)^2 \right) \\
M &= -54\sqrt{3}r^4 (2a^2\varepsilon + 2a\varepsilon^2 - 4a\varepsilon + \varepsilon^3 - 2\varepsilon^2 + \varepsilon).
\end{aligned}$$

Then

$$M \log \frac{st}{r^2} = -54\sqrt{3}r^4 (2a^2\varepsilon + 2a\varepsilon^2 - 4a\varepsilon + \varepsilon^3 - 2\varepsilon^2 + \varepsilon) \log a(a + \varepsilon). \tag{3.58}$$

In the same way for Equation 3.53.

$$\begin{aligned}
N &= -27\sqrt{3}r^4 \left(\left(\frac{t-s}{r} \right)^4 - \left(\frac{t-s}{r} \right)^2 \left(3 - \frac{3}{r} + \frac{1}{r^2} \right) + \left(\frac{1}{r} - 1 \right) \left(\frac{1}{r} - 2 \right) \left(\frac{1}{r} - 3 \right) \right) \\
N &= -27\sqrt{3}r^4 \left(\varepsilon^4 + \varepsilon^2 (3 - 3(1 + 2a + \varepsilon) + (1 + 2a + \varepsilon)^2) \right. \\
&\quad \left. + (1 + 2a + \varepsilon - 1)(1 + 2a + \varepsilon - 2)(1 + 2a + \varepsilon - 3) \right) \\
N &= -54\sqrt{3}r^4 (4a^3 - 2a^2\varepsilon^2 + 6a^2\varepsilon - 6a^2 - 2a\varepsilon^3 + 4a\varepsilon^2 - 6a\varepsilon + 2a + \varepsilon^3 - 2\varepsilon^2 + \varepsilon).
\end{aligned}$$

Then

$$N \log \frac{t}{s} = -54\sqrt{3}r^4 (4a^3 - 2a^2\varepsilon^2 + 6a^2\varepsilon - 6a^2 - 2a\varepsilon^3 + 4a\varepsilon^2 - 6a\varepsilon + 2a + \varepsilon^3 - 2\varepsilon^2 + \varepsilon) \log \frac{a+\varepsilon}{a}. \quad (3.59)$$

Summing equations 3.58 and 3.59, we have that

$$M \log \frac{st}{r^2} + N \log \frac{t}{s} = -54\sqrt{3}r^4 X,$$

where

$$\begin{aligned} X &:= (2a^2\varepsilon + 2a\varepsilon^2 - 4a\varepsilon + \varepsilon^3 - 2\varepsilon^2 + \varepsilon) \log a(a+\varepsilon) + \\ &\quad (4a^3 - 2a^2\varepsilon^2 + 6a^2\varepsilon - 6a^2 - 2a\varepsilon^3 + 4a\varepsilon^2 - 6a\varepsilon + 2a + \varepsilon^3 - 2\varepsilon^2 + \varepsilon) \log \frac{a+\varepsilon}{a} \\ X &= (4a^3 - 2a^2\varepsilon^2 + 8a^2\varepsilon - 6a^2 - 2a\varepsilon^3 + 6a\varepsilon^2 - 10a\varepsilon + 2a + 2\varepsilon^3 - 4\varepsilon^2 + 2\varepsilon) \log(a+\varepsilon) + \\ &\quad (-4a^3 + 2a^2\varepsilon^2 - 4a^2\varepsilon + 6a^2 + 2a\varepsilon^3 - 2a\varepsilon^2 + 2a\varepsilon - 2a) \log a \\ X &= 2((a+\varepsilon)(a-1)(2a-\varepsilon^2+\varepsilon-1) \log(a+\varepsilon) - a(a+\varepsilon-1)(2a-\varepsilon^2-1) \log a). \end{aligned} \quad (3.60)$$

We claim that $X > 0$. Indeed, we will show in a first step that

$$X' := (a+\varepsilon)(a-1) \log(a+\varepsilon) - a(a+\varepsilon-1) \log a \quad (3.61)$$

is a positive number. For this, we use the following series for the natural logarithm. For each $x \geq 1$,

$$\log x = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x-1}{x} \right)^n.$$

Then,

$$\begin{aligned} X' &= (a+\varepsilon)(a-1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{(a+\varepsilon-1)^n}{(a+\varepsilon)^n} - a(a+\varepsilon-1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{(a-1)^n}{a^n} \\ X' &= (a-1)(a+\varepsilon-1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{(a+\varepsilon-1)^{n-1}}{(a+\varepsilon)^{n-1}} - (a-1)(a+\varepsilon-1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{(a-1)^{n-1}}{a^{n-1}} \\ X' &= (a-1)(a+\varepsilon-1) \sum_{n=1}^{\infty} \frac{1}{n} \left(\left(\frac{a+\varepsilon-1}{a+\varepsilon} \right)^{n-1} - \left(\frac{a-1}{a} \right)^{n-1} \right), \end{aligned}$$

is a positive number because $\frac{a+\varepsilon-1}{a+\varepsilon} - \frac{a-1}{a} = \frac{\varepsilon}{a(a+\varepsilon)} > 0$.

For a second step, $X' > 0$ implies that

$$(a+\varepsilon)(a-1) \log(a+\varepsilon) > a(a+\varepsilon-1) \log a.$$

As $2a - \varepsilon^2 + \varepsilon - 1 > 2a - \varepsilon^2 - 1 > 0$, we have that

$$(a+\varepsilon)(a-1)(2a-\varepsilon^2+\varepsilon-1) \log(a+\varepsilon) > a(a+\varepsilon-1)(2a-\varepsilon^2-1) \log a.$$

Seeing Equation 3.60, we have that X is a positive number.

Finally,

$$M \log \frac{st}{r^2} + N \log \frac{t}{s} = -54\sqrt{3}r^4X < 0.$$

□

The last Proposition shows that the volume function is decreasing in the determinant.

Proposition 3.6.7. Let $(\alpha, \omega) \in \text{int } S$. Then,

$$P \log \frac{st}{r^2} + Q \log \frac{t}{s} > 0.$$

Proof. We implement the same idea used to prove Proposition 3.6.6. So, we consider a and ε as they were defined in equations 3.54 and 3.55 respectively and their consequences.

Remember that $c = t - s$ and $d = \frac{1-3r}{\sqrt{3}}$. So,

$$\begin{aligned} P &= -18\sqrt{3}c^3d + 18c^3 + 6\sqrt{3}cd^3 - 18cd^2 + 6\sqrt{3}cd - 2c \\ P &= -2c \left(9c^2 \left(\sqrt{3}d - 1 \right) - 3\sqrt{3}d^3 + 9d^2 - 3\sqrt{3}d + 1 \right) \\ P &= -2(t-s) \left(9(t-s)^2 \left(\sqrt{3} \frac{1-3r}{\sqrt{3}} - 1 \right) - 3\sqrt{3} \left(\frac{1-3r}{\sqrt{3}} \right)^3 + 9 \left(\frac{1-3r}{\sqrt{3}} \right)^2 \right. \\ &\quad \left. - 3\sqrt{3} \frac{1-3r}{\sqrt{3}} + 1 \right) \\ P &= -54r(t-s)(r-s+t)(r+s-t) \\ P &= -54(r-s+t)(r+s-t)r^2 \left(\frac{t-s}{r} \right) \\ P &= -54(r-s+t)(r+s-t)r^2\varepsilon. \end{aligned}$$

Then,

$$P \log \frac{st}{r^2} = -54(r-s+t)(r+s-t)r^2\varepsilon \log a(a+\varepsilon). \quad (3.62)$$

The same for

$$\begin{aligned} Q &= 27c^4 - 36c^2d^2 - 12\sqrt{3}c^2d - 3c^2 + 9d^4 - 9d^2 + 2\sqrt{3}d \\ Q &= 27c^4 - 3c^2 \left(12d^2 + 4\sqrt{3}d + 1 \right) + 9d^4 - 9d^2 + 2\sqrt{3}d \\ Q &= 27(t-s)^4 - 3(t-s)^2 \left(12 \left(\frac{1-3r}{\sqrt{3}} \right)^2 + 4\sqrt{3} \frac{1-3r}{\sqrt{3}} + 1 \right) + 9 \left(\frac{1-3r}{\sqrt{3}} \right)^4 \\ &\quad - 9 \left(\frac{1-3r}{\sqrt{3}} \right)^2 + 2\sqrt{3} \frac{1-3r}{\sqrt{3}} \\ Q &= 54(r-s+t)(r+s-t)(2st - rs - rt) \\ Q &= 54(r-s+t)(r+s-t)r^2 \left(\frac{2st}{r^2} - \frac{s}{r} - \frac{t}{r} \right) \\ Q &= 54(r-s+t)(r+s-t)r^2 (2a(a+\varepsilon) - a - (a+\varepsilon)) \\ Q &= 54(r-s+t)(r+s-t)r^2 (2a^2 + 2a\varepsilon - 2a - \varepsilon). \end{aligned}$$

Then,

$$Q \log \frac{t}{s} = 54(r-s+t)(r+s-t)r^2(2a^2+2a\varepsilon-2a-\varepsilon) \log \frac{a+\varepsilon}{a}. \quad (3.63)$$

Summing equations 3.62 and 3.63, we have that

$$P \log \frac{st}{r^2} + Q \log \frac{t}{s} = 54(r-s+t)(r+s-t)r^2Y,$$

where

$$\begin{aligned} Y &:= -\varepsilon \log a(a+\varepsilon) + (2a^2+2a\varepsilon-2a-\varepsilon) \log \frac{a+\varepsilon}{a} \\ Y &= (2a^2+2a\varepsilon-2a-2\varepsilon) \log(a+\varepsilon) + (-2a^2-2a\varepsilon+2a) \log a \\ Y &= 2((a+\varepsilon)(a-1) \log(a+\varepsilon) - a(a+\varepsilon-1) \log a). \end{aligned}$$

From equation 3.61, we note that $Y = 2X'$. Since $X' > 0$, we conclude that

$$P \log \frac{st}{r^2} + Q \log \frac{t}{s} = 54(r-s+t)(r+s-t)r^2Y > 0.$$

□

The last Proposition shows that the volume function is increasing in the permanent of the matrix.

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HERMITIAN FORMS

A.1 Hermitian forms

We deal with finite-dimensional linear spaces over $\mathbb{R}$ or $\mathbb{C}$. To cover both cases, denote the scalars by $\mathbb{K}$. The symbol $\bar{k}$ stands for the *conjugate* to the (complex) number $k \in \mathbb{K}$.

Definition A.1.1. Let V be a $\mathbb{K}$ -linear space. A *hermitian form* is a map $\langle -, - \rangle : V \times V \rightarrow \mathbb{K}$, $(x, y) \mapsto \langle x, y \rangle$ linear in x and such that $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$. In particular, $\langle x, y + y' \rangle = \langle x, y \rangle + \langle x, y' \rangle$ and $\langle x, ky \rangle = \bar{k} \langle x, y \rangle$.

If $W \leq V$ is a subspace, we can restrict the form $\langle -, - \rangle$ to $W \times W$, getting a linear space W equipped with the *induced* hermitian form.

Definition A.1.2. Let V be a linear space equipped with a hermitian form and let $W \leq V$ be a subspace. We define¹ $W^\perp := \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}$, the *orthogonal* to W . We call $V^\perp$ the *kernel of the form* on V . If the kernel vanishes, we say that the form is *nondegenerate*. If the induced form on a subspace $W \leq V$ is nondegenerate, W is said to be *nondegenerate*. For $U, W \leq V$ the *orthogonal of W relatively to U* is given by $W^{\perp_U} := W^\perp \cap U$.

Claim A.1.3. If $W \leq V$, then $W^\perp \leq V$ and $W \subset W^{\perp\perp}$. If $W_1, W_2 \leq V$, then $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ and $W_1^\perp + W_2^\perp \subset (W_1 \cap W_2)^\perp$.

Claim A.1.4. It is possible to define an induced form on $V/V^\perp$ by $\langle [x], [y] \rangle := \langle x, y \rangle$ for $[x], [y] \in V/V^\perp$. The linear space $V/V^\perp$ equipped with the induced hermitian form is nondegenerate. Decomposing $V = V^\perp \oplus W$, the linear function $W \rightarrow V/V^\perp$, $w \mapsto [w]$ is a natural isomorphism that preserves the forms.

Claim A.1.5. If $W \leq V$, then $\dim W + \dim W^\perp \geq \dim V$.

¹ We also write $\langle v, W \rangle = 0$ to denote $v \in W^\perp$.

Indeed, using induction on $\dim W$, decompose $W = W' \oplus \mathbb{K}w$ for some $w \in W^*$. Being $W'^\perp \cap (\mathbb{K}w)^\perp$ the kernel of the functional $W'^\perp \rightarrow \mathbb{K}$ given by the rule $x \mapsto \langle x, w \rangle$. By the Claim A.1.3 we have $\dim W^\perp = \dim(W'^\perp \cap (\mathbb{K}w)^\perp)$ and the Rank-nullity theorem implies that $\dim(W'^\perp \cap (\mathbb{K}w)^\perp) \geq \dim W'^\perp - 1$, then $\dim W^\perp + 1 \geq \dim W'^\perp$. The rest follows from $\dim W - 1 = \dim W'$ and by induction.

Claim A.1.6. If $W \geq V$ is a nondegenerate subspace, then $V = W \oplus W^\perp$.

Claim A.1.7. Suppose that both W and V are nondegenerate, where $W \geq V$. Then $W^{\perp\perp} = W$.

Claim A.1.8. Suppose that both W and V are nondegenerate, where $W \geq V$. Then $W^\perp$ is nondegenerate.

Claim A.1.9. If the form on V is not identically null, then there exists a *nonisotropic*² $v \in V$, i.e., $\langle v, v \rangle \neq 0$.

Indeed, assuming that $\langle v, v \rangle = 0$ for all $v \in V$, we obtain $\langle v_1 + v_2, v_1 + v_2 \rangle = 0$ and, hence, $\operatorname{Re} \langle v_1, v_2 \rangle = 0$ for all $v_1, v_2 \in V$. If $\mathbb{K} = \mathbb{C}$, it remains to apply the last identity to iv_1, v_2 in order to get $\operatorname{Im} \langle v_1, v_2 \rangle = 0$.

Claim A.1.10. Suppose that both W and V are nondegenerate, where $W \not\leq V$. Then, there exists a nondegenerate subspace $W' \leq V$ such that $W \leq W'$ and $\dim W' = \dim W + 1$.

Definition A.1.11. A *flag* of subspaces is a chain of subspaces $V_0 \leq V_1 \leq V_2 \leq \dots \leq V_n$ such that $V_n = V$ and $\dim V_i = i$ for all i . If V is equipped with a hermitian form, a flag is *nondegenerate* when all V_i 's are nondegenerate.

Claim A.1.12. Every nondegenerate linear space admits a nondegenerate flag of subspaces.

Definition A.1.13. A linear basis $\beta : b_1, b_2, \dots, b_n$ is *orthonormal* if $\langle b_i, b_i \rangle \in \{-1, 0, 1\}$ and $\langle b_i, b_j \rangle = 0$ for all i and j such that $i \neq j$. Denote by $\beta_-, \beta_0, \beta_+$ the amount of elements in the basis β such that $\langle b_i, b_i \rangle = -1, \langle b_i, b_i \rangle = 0, \langle b_i, b_i \rangle = 1$, respectively. The triple $(\beta_-, \beta_0, \beta_+)$ is the *signature* of the basis.

Claim A.1.14. Let $\beta : b_1, b_2, \dots, b_n$ be an orthonormal basis in V . Then the set of isotropic vectors in β is a basis for $V^\perp$. In particular, the dimension of the kernel of the form in V is β_0 , $\dim V^\perp = \beta_0$.

Proposition A.1.15 (Gram-Schmidt orthogonalization). Let $V_0 \leq V_1 \leq V_2 \leq \dots \leq V_n$ be a nondegenerate flag of subspaces in V . Then there exists an orthonormal basis $b_1, b_2, \dots, b_n$ in V such that $b_1, b_2, \dots, b_k$ is a basis in V_k for all k .

Corollary A.1.16. Every linear space with a hermitian form admits an orthonormal basis.

² If $\langle v, v \rangle = 0$, v is called *isotropic*.

Definition A.1.17. Let $v_1, v_2, \dots, v_k \in V$. The matrix $G := G(v_1, v_2, \dots, v_k) := [g_{ij}]$, where $g_{ij} := \langle v_i, v_j \rangle$, is called the *Gram matrix* of $v_1, v_2, \dots, v_k$.

Obviously, $\overline{G}^t = G$, where M^t denotes the transpose matrix of M and $\overline{M}$ denotes the matrix M with conjugate entries. In other words, G is *hermitian* (symmetric when $\mathbb{K} = \mathbb{R}$). Thus $\det G$ is always a real number.

The following Proposition shows that the Gram matrix of some basis in V determines the hermitian form on V .

Proposition A.1.18. Let $G^{\beta\beta} := G(b_1, b_2, \dots, b_n)$ be the Gram matrix of some basis $\beta : v_1, v_2, \dots, v_n$ in V . Then

$$\langle v, v' \rangle = [v]_{\beta}^t G^{\beta\beta} \overline{[v']}_{\beta}$$

for all $v, v' \in V$, where $[v]_{\beta}$ denotes the column matrix whose entries are the coefficients c_i appearing in the linear combination $v = \sum_{i=1}^n c_i b_i$.

A basis is orthonormal if and only if its Gram matrix is diagonal with diagonal entries $-1, 0, 1$.

Proposition A.1.19. Let $\alpha : a_1, a_2, \dots, a_n$ and $\beta : b_1, b_2, \dots, b_n$ be bases in V and let $M_{\alpha}^{\beta} = [m_{ij}]$ be the matrix representing a change of basis from α to β , that is, $b_j = \sum_{i=1}^n m_{ij} a_i$ for all j . Then

$$G^{\beta\beta} = (M_{\alpha}^{\beta})^t G^{\alpha\alpha} \overline{M_{\alpha}^{\beta}}.$$

In particular, it follows that the sign of $\det G^{\beta\beta}$ does not depend on the choice of the basis because $\det G^{\beta\beta} = |\det M_{\alpha}^{\beta}|^2 \det G^{\alpha\alpha}$.

Corollary A.1.20. Let $\beta : b_1, b_2, \dots, b_n$ be any basis in V . Then $\dim V^{\perp} = n - \text{rk } G^{\beta\beta}$, where $\text{rk } G^{\beta\beta}$ is the rank of the matrix $G^{\beta\beta}$.

Lemma A.1.21. Let $G^{\beta\beta}$ be the Gram matrix of a basis in a linear space V . Then V is degenerate if, and only, if $\det G^{\beta\beta} = 0$.

Proposition A.1.22 (Sylvester's law of inertia). The signature does not depend on the choice of an orthonormal basis.

Definition A.1.23. The *signature* of a space V is the signature of any orthonormal basis in V .

How do we measure it? By Claim A.1.4 we begin by finding the signature of the non-degenerate space $W \simeq V/V^{\perp}$ where $V = V^{\perp} \oplus W$. By Claim A.1.14 this signature will be of the type $(n_-, 0, n_+)$ and the signature of $V^{\perp}$ is $(0, \dim V^{\perp}, 0)$, where $\dim V^{\perp}$ can be found using Corollary A.1.20. Noting that the signature of V is the sum of the signatures of W and $V^{\perp}$, the signature of V is $(n_-, \dim V^{\perp}, n_+)$. So the problem can be reduced to the case of a nondegenerate V .

Proposition A.1.24 (Sylvester criterion). Let $\gamma : c_1, c_2, \dots, c_n$ be a basis in V with a known matrix $G^{\gamma\gamma}$. Let $G_k^{\gamma\gamma}$ be the matrix of $c_1, c_2, \dots, c_k$ for every k such that $1 \leq k \leq n$. If $\det G_k^{\gamma\gamma} \neq 0$ for every k , then the signature of the space equals $(n_-, 0, n_+)$, where n_- is the amount of negative numbers in the sequence

$$\det G_1^{\gamma\gamma}, \frac{\det G_2^{\gamma\gamma}}{\det G_1^{\gamma\gamma}}, \frac{\det G_3^{\gamma\gamma}}{\det G_2^{\gamma\gamma}}, \dots, \frac{\det G_n^{\gamma\gamma}}{\det G_{n-1}^{\gamma\gamma}}$$

and n_+ is the amount of positive numbers in the same sequence.

The following results concern the study of the possible signatures of a subspace when the signature of the space is given.

Claim A.1.25. Let V be a space of signature (n_-, n_0, n_+) . If V contains a subspace W of signature (m_-, m_0, m_+) , then the space $V/V^\perp$ (of signature $(n_-, 0, n_+)$) possesses a subspace of signature $(m_-, m_0 - m, m_+)$, where $m = \dim W \cap V^\perp$. If the space $V/V^\perp$ contains a subspace U of signature $(m_-, m_0 - m, m_+)$, where $m = \dim \pi^{-1}(U) \cap V^\perp$ and the map $\pi : V \rightarrow V/V^\perp$ takes v to its class, then V contains a subspace of signature (m_-, m_0, m_+) .

Claim A.1.26. Let V be a space of signature $(n_-, 0, n_+)$. The highest possible dimension of a subspace W with the null induced form is $\min(n_-, n_+)$.

Claim A.1.27. Let V be a space of signature $(n_-, 0, n_+)$. Then V contains a subspace W of signature (m_-, m_0, m_+) if and only if

$$m_- \leq n_-, \quad m_+ \leq n_+, \quad m_0 \leq n_- - m_-, \quad m_0 \leq n_+ - m_+.$$

Claim A.1.28. Let V be a space of signature (n_-, n_0, n_+) . Then V contains a subspace of signature (m_-, m_0, m_+) if and only if

$$m_- \leq n_-, \quad m_+ \leq n_+, \quad m_- + m_0 \leq n_- + n_0, \quad m_0 + m_+ \leq n_0 + n_+.$$

AN EXPLICIT DESCRIPTION OF R AND S

B.1 An explicit description of R and S

In what follows we use the notation from Section 3.1.3; in particular, we need definitions 3.9, 3.10, and 3.4.

We will give explicit descriptions of the sets R and S introduced in 3.4 and 3.12.

The following proposition describes the sets R and $\text{int} R$.

Proposition B.1.1. Let $\bar{\Delta}_1$ be the closure of Δ_1 in $\mathbb{R}^2$ and let $H : \bar{\Delta}_1 \rightarrow \mathbb{R}^2$ be defined by

$$H(c, d) = \left(\frac{(2\sqrt{3}d + 1)(3c + \sqrt{3}d - 1)(3c - \sqrt{3}d + 1)}{27}, \left(\frac{3c^2 + 3d^2 + 2}{6} \right)^2 \right).$$

Then

- (i) H is a homeomorphism onto its image.
- (ii) $C_1 \cup C_2 \cup C_3$ is a simple closed curve in $\mathbb{R}^2$, where

$$C_1 := H \left[(0, 0), \left(0, \frac{\sqrt{3}}{3} \right) \right), \quad C_2 := H \left[(0, 0), \left(\frac{1}{4}, \frac{\sqrt{3}}{12} \right) \right),$$

$$C_3 := H \left[\left(\frac{1}{4}, \frac{\sqrt{3}}{12} \right), \left(0, \frac{\sqrt{3}}{3} \right) \right].$$

- (iii) The bounded component of $\mathbb{R}^2 \setminus (C_1 \cup C_2 \cup C_3)$ is

$$A := \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{1}{9} < y < \frac{1}{4}, x > -\frac{1}{27} \left(2\sqrt{6\sqrt{y}-2} + 1 \right) \left(1 - \sqrt{6\sqrt{y}-2} \right)^2 \right\} \cap$$

$$\left(\left\{ (x, y) \in \mathbb{R}^2 \mid \frac{1}{9} < y \leq \frac{9}{64}, x < -\frac{1}{27} \left(1 - 2\sqrt{6\sqrt{y}-2} \right) \left(\sqrt{6\sqrt{y}-2} + 1 \right)^2 \right\} \right.$$

$$\left. \cup \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{9}{64} \leq y < \frac{1}{4}, x < 0 \right\} \right).$$

(iv) $R = A \cup C_1 \cup C_2$.

(v) $\text{int} R = A$.

Proof. (i) The injectivity of H can be shown as in the proof of Proposition 3.2.3. Since $H : \bar{\Delta}_1 \rightarrow H(\bar{\Delta}_1)$ is a continuous bijective function and $\bar{\Delta}_1$ is compact, we have that H is a homeomorphism.

(ii) The boundary $\text{bd} \bar{\Delta}_1 = \left[(0,0), \left(0, \frac{\sqrt{3}}{3}\right) \right] \cup \left[(0,0), \left(\frac{1}{4}, \frac{\sqrt{3}}{12}\right) \right] \cup \left[\left(\frac{1}{4}, \frac{\sqrt{3}}{12}\right), \left(0, \frac{\sqrt{3}}{3}\right) \right]$, so $H(\text{bd} \bar{\Delta}_1) = C_1 \cup C_2 \cup C_3$. Parameterizing the hypotenuse $h := \left[(0,0), \left(0, \frac{\sqrt{3}}{3}\right) \right]$ of $\bar{\Delta}_1$ by

$$\left\{ t \left(0, \frac{\sqrt{3}}{3} \right) \mid 0 \leq t < 1 \right\},$$

one obtains

$$\begin{aligned} H(h) = C_1 &= \left\{ H \left(0, \frac{\sqrt{3}}{3} t \right) \mid 0 \leq t < 1 \right\} = \\ &= \left\{ \left(-\frac{1}{27} (2t+1)(1-t)^2, \frac{1}{36} (t^2+2)^2 \right) \mid 0 \leq t < 1 \right\}. \end{aligned}$$

Reparameterizing, we get

$$C_1 = \left\{ \left(-\frac{1}{27} \left(2\sqrt{6\sqrt{y}-2} + 1 \right) \left(1 - \sqrt{6\sqrt{y}-2} \right)^2, y \right) \mid \frac{1}{9} \leq y < \frac{1}{4} \right\}. \quad (\text{B.1})$$

Writing the smallest leg $l_1 := \left[(0,0), \left(\frac{1}{4}, \frac{\sqrt{3}}{12}\right) \right]$ of $\bar{\Delta}_1$ as $\left\{ t \left(\frac{1}{4}, \frac{\sqrt{3}}{12} \right) \mid 0 \leq t < 1 \right\}$, we have

$$\begin{aligned} H(l_1) = C_2 &= \left\{ H \left(\frac{t}{4}, \frac{\sqrt{3}}{12} t \right) \mid 0 \leq t < 1 \right\} = \\ &= \left\{ \left(-\frac{1}{27} (1-t) \left(\frac{t}{2} + 1 \right)^2, \frac{1}{36} \left(\frac{t^2}{4} + 2 \right)^2 \right) \mid 0 \leq t < 1 \right\} = \\ &= \left\{ \left(-\frac{1}{27} \left(1 - 2\sqrt{6\sqrt{y}-2} \right) \left(\sqrt{6\sqrt{y}-2} + 1 \right)^2, y \right) \mid \frac{1}{9} \leq y < \frac{9}{64} \right\}. \end{aligned}$$

Parameterizing the largest leg $l_2 := \left[\left(\frac{1}{4}, \frac{\sqrt{3}}{12}\right), \left(0, \frac{\sqrt{3}}{3}\right) \right]$ of $\bar{\Delta}_1$ by

$$\left\{ (1-t) \left(\frac{1}{4}, \frac{\sqrt{3}}{12} \right) + t \left(0, \frac{\sqrt{3}}{3} \right) \mid 0 \leq t \leq 1 \right\},$$

we get

$$\begin{aligned} C_3 &= \left\{ H \left(\frac{1-t}{4}, \frac{\sqrt{3}}{12} (3t+1) \right) \mid 0 \leq t \leq 1 \right\} = \left\{ \left(0, \frac{(t^2+3)^2}{64} \right) \mid 0 \leq t \leq 1 \right\} = \\ &= \left\{ (0, y) \mid \frac{9}{64} \leq y \leq \frac{1}{4} \right\}. \end{aligned}$$

Figure 7 – The set R .

Source: Elaborated by the author.

The boundary of $\bar{\Delta}_1$ is a simple closed curve, so $H(\text{bd}\bar{\Delta}_1) = C_1 \cup C_2 \cup C_3$ is a simple closed curve in $\mathbb{R}^2$ that joins the points $H(0,0) = (-\frac{1}{27}, \frac{1}{9})$, $H(\frac{1}{4}, \frac{\sqrt{3}}{12}) = (0, \frac{9}{64})$, $H(0, \frac{\sqrt{3}}{3}) = (0, \frac{1}{4})$.

(iii) By Jordan's theorem, $\mathbb{R}^2 \setminus (C_1 \cup C_2 \cup C_3)$ has exactly two connected components B_1, B_2 with B_1 bounded and B_2 unbounded; the curve $C_1 \cup C_2 \cup C_3$ is the boundary of each component (see Figure 7). In this way, B_1 is the open region A delimited by the curves C_1 , C_2 , and C_3 .

(iv) We claim that $A = H(\text{int}\bar{\Delta}_1)$. First, note that $A \subset H(\bar{\Delta}_1)$. Indeed, if

$$A \cap (\mathbb{R}^2 \setminus H(\bar{\Delta}_1)) \neq \emptyset,$$

then the simple closed curve $C_1 \cup C_2 \cup C_3 = H(\text{bd}\bar{\Delta}_1) \subset H(\bar{\Delta}_1)$ is not null homotopic in $H(\bar{\Delta}_1)$. This is a contradiction because, by (i), $H(\bar{\Delta}_1)$ is simply connected.

Now, we have $A \subset H(\bar{\Delta}_1) \setminus H(\text{bd}\bar{\Delta}_1) = H(\bar{\Delta}_1 \setminus \text{bd}\bar{\Delta}_1) = H(\text{int}\bar{\Delta}_1)$. Conversely, $H(\text{int}\bar{\Delta}_1) = H(\bar{\Delta}_1) \setminus H(\text{bd}\bar{\Delta}_1) \subset \mathbb{R}^2 \setminus (C_1 \cup C_2 \cup C_3) = A \cup B$ and $H(\text{int}\bar{\Delta}_1)$ is a connected space that intersects A . It follows that $H(\text{int}\bar{\Delta}_1) \subset A$ and, therefore, $H(\text{int}\bar{\Delta}_1) = A$.

Finally, Proposition 3.2.3 implies $R = G(\Delta_1)$. Since H coincides with G in Δ_1 , we have

$$R = H(\Delta_1) = H\left(\text{int}\bar{\Delta}_1 \cup \left[(0,0), \left(0, \frac{\sqrt{3}}{3}\right)\right] \cup \left[(0,0), \left(\frac{1}{4}, \frac{\sqrt{3}}{12}\right)\right]\right) = A \cup C_1 \cup C_2.$$

(v) $C_1 \cup C_2 \subset \text{bd}A$ implies $\text{int}R = \text{int}(A \cup C_1 \cup C_2) = A$. $\square$

Observation B.1.2. The bijective function $G|_{\Delta_1} : \Delta_1 \rightarrow R$ of Proposition 3.2.3 is a homeomorphism because $G|_{\Delta_1} = H|_{\Delta_1}$.

The following result describes the sets S and $\text{int}S$. The proof is very similar to that of Proposition B.1.1 and will be omitted.

Proposition B.1.3. Let $I : \bar{\Delta}_1 \rightarrow \mathbb{R}^2$ be defined by

$$I(c, d) = \left(\frac{(2\sqrt{3}d + 1)(3c + \sqrt{3}d - 1)(3c - \sqrt{3}d + 1)}{27}, \frac{3c^2 + 3d^2 + 2}{6} \right).$$

Then

- (i) I is a homeomorphism onto its image.
- (ii) $D_1 \cup D_2 \cup D_3$ is a simple closed curve in $\mathbb{R}^2$, where

$$D_1 = I\left[\left(0,0\right), \left(0, \frac{\sqrt{3}}{3}\right)\right], \quad D_2 = I\left[\left(0,0\right), \left(\frac{1}{4}, \frac{\sqrt{3}}{12}\right)\right],$$

$$D_3 = I\left[\left(\frac{1}{4}, \frac{\sqrt{3}}{12}\right), \left(0, \frac{\sqrt{3}}{3}\right)\right].$$

- (iii) The bounded component of $\mathbb{R}^2 \setminus (D_1 \cup D_2 \cup D_3)$ is

$$B = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{1}{3} < y < \frac{1}{2}, x > -\frac{1}{27} \left(2\sqrt{6y-2} + 1 \right) \left(1 - \sqrt{6y-2} \right)^2 \right\} \cap$$

$$\left(\left\{ (x, y) \in \mathbb{R}^2 \mid \frac{1}{3} < y \leq \frac{3}{8}, x < -\frac{1}{27} \left(1 - 2\sqrt{6y-2} \right) \left(\sqrt{6y-2} + 1 \right)^2 \right\} \right.$$

$$\left. \cup \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{3}{8} \leq y < \frac{1}{2}, x < 0 \right\} \right).$$

- (iv) $S = B \cup D_1 \cup D_2$.
- (v) $\text{int}S = B$.

Figure 8 displays the set S .

Figure 8 – The set S .

Source: Elaborated by the author.