A post-Lie operad of rooted trees

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Uma operadad pós-Lie de árvores enraizadas

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*Às minhas avós
in memoriam*

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For non portuguese speakers, I would like to justify why I prefer to write my acknowledgments in portuguese. Indeed, the reasons are very simple: first, for me nothing better than my mother language to talk about feelings, second, the most of people cited here are portuguese speakers.

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ABSTRACT

SILVA, P. S. F. **A post-Lie operad of rooted trees**. 2018. 93 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

In this thesis we propose a description of the operad defining post-Lie algebras in terms of rooted trees and we discuss some applications of such a construction. In particular, we re-derive both the free post-Lie algebra defined in [22] and the main result of the paper [8]. Furthermore, a possible extension of the concept of symmetric brace algebra to the category of the post-Lie algebras is proposed.

Keywords: operads, post-Lie algebra, enveloping algebra, rooted trees.

RESUMO

SILVA, P. S. F. **Uma operad pós-Lie de árvores enraizadas**. 2018. 93 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

Nessa tese propomos a descrição da operad que define as álgebras pós-Lie em termos de árvores enraizadas e discutimos algumas aplicações dessa construção. Em particular, nós obtemos novamente a álgebra pós-Lie livre definida em [22] e o resultado principal do artigo [8]. Além disso, uma possível extensão do conceito de álgebra brace simétrica à categoria de álgebras pós-Lie é apresentada.

Palavras-chave: operads, álgebra pós-Lie, álgebra envolvente, árvores enraizadas.

LIST OF SYMBOLS

$x_1 x_2 \cdots x_n$ — The elements of $V^{\otimes n}$.

$V^{\otimes n}$ — Tensor product of n copies of V .

$\text{Hom}(V^{\otimes n}, V)$ — The vector space of the linear maps from $V^{\otimes n}$ to V .

$\langle X \rangle$ — Free vector space generated by X .

$\mathcal{R}\langle X \rangle$ — The free $\mathcal{R}$ -module generated by a set X .

$[n]$ — Set $\{1, \dots, n\}$.

$\mathbb{N}$ — Natural numbers.

$\mathbb{N}^*$ — $\mathbb{N} \setminus \{0\}$.

$\mathbb{S}_n$ — Permutation group.

$[\sigma(1)\sigma(2)\cdots\sigma(n)]$ — Permutation $\sigma \in \mathbb{S}_n$.

id_n — Identity permutation of $\mathbb{S}_n$.

x^σ — The image of the action of $\sigma \in G$ over $x \in V$.

$\tau_1 \oplus \cdots \oplus \tau_r$ — Direct sum of the permutations $\tau_i \in \mathbb{S}(k_i)$, $k_i \in \mathbb{N}$.

$\sigma \circ_i \tau$ — The permutation $\sigma \circ (id_1, \dots, id_1, \tau, id_1, \dots, id_1)$.

$\sigma \circ (\tau_1, \dots, \tau_r)$ — The inverse of the bloch permutation $\sigma(\tau_1^{-1} \oplus \cdots \oplus \tau_n^{-1})$.

$\mathcal{E}nd_V$ — Endoperads of V .

$a_{\triangleright}$ — The associator of the binary operation $\triangleright$.

$\mathcal{F}(\mathcal{S})$ — The free operad over the $\mathbb{S}$ -module $\mathcal{S}$.

$M(X)$ — The free magma constructed from a set X .

(R) — The operadic ideal generated by R .

$\mathbb{T}$ — Set of all rooted trees.

$\mathcal{N}(t)$ — The set of nodes of the rooted tree t or if t is a parenthesizing of $t_1, \dots, t_n$ is the set $\bigcup \mathcal{N}(t_i)$.

$\mathbb{T}(n)$ — The set of rooted trees with n nodes

$L(t, n)$ — The set of all bijections from $\mathcal{N}(t)$ to $[n]$.

$l_{t,n}$ — A bijection between $\mathbb{S}_n$ and $L(t,n)$.

$Sh(p,q)$ — The (p,q) -shuffles set.

$Sh(i_1 \dots i_k)$ — The $(i_1 \dots i_k)$ -shuffle set.

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INTRODUCTION

Informally speaking, an operad can be thought as an abstraction of a family, indexed by $n \in \mathbb{N}$, of composable functions of n -variables. In a more precise way, a (*linear*) *operad* is a collection of vector spaces $\{V_n\}_{n \geq 1}$ endowed with a collection of (linear) maps $\gamma_{n;m_1, \dots, m_n} : V_n \otimes V_{m_1} \otimes \dots \otimes V_{m_n} \rightarrow V_{m_1 + \dots + m_n}$ called *composition maps*, which satisfy suitable compatibility conditions (called associativity). An operad is called *symmetric* if, for each n , V_n is a module over the symmetric group (of a set with n -elements). Moreover, the operad is called *unital* if V_1 is endowed with a unit. A main example of a symmetric operad is the so called *Endomorphism Operad* ($\text{End}(V)$) defined by the vector space V . In this case $V_n = \text{Hom}(V^{\otimes n}, V)$, endowed with the action of the symmetric group given by changing the position of the variables, and the compositions map are the natural ones.

Even though the term *operad* was coined by J. Peter May at the beginning of the seventies of the last century, see [21], the first example of this mathematical object goes back at least to Lazard's work on formal groups, [19], where he introduced the structure of a *analyseur* to axiomatize the relevant composition laws. Among the many other examples in the pre-history of operads theory, it is worth to mention the one that can be found in the work of Gerstenhaber [11] on the deformation of associative algebras. In particular, in [15], the authors define a *brace structure* on a(ny) linear operad to prove a result analogous to the Deligne's conjecture on the Hochschild complex.¹

During the last two decades there have been a renewed interest in the operad theory, due both to the many applications of operads in the area of mathematical physics and to the growing importance of several classes of non-associative algebras both in pure and in applied mathematics. Just to mention a few examples, *pre-Lie algebras* play an important role in the approach of Connes and Kreimer to the theory of renormalization of qFT (see [12]) and in the Butcher theory of the geometric integration methods on $\mathbb{R}^n$ [13]. In the attempt to extend these methods to more general homogeneous spaces, Munthe-Kaas introduced the so called Lie-Butcher series, [14], whose composition is ruled by a new category of non-associative algebraic structures, which were called *post-Lie algebras*. While a model example of a pre-Lie algebra is provided by $(\mathfrak{X}(M), \nabla)$, where $\mathfrak{X}(M)$ is the vector space of vector fields defined on M and ∇ is a flat and torsion free connection on TM , the model example of a post-Lie algebra is the pair $(\mathfrak{X}(M), \nabla)$ where $\mathfrak{X}(M)$ is as before and ∇ is now a flat and constant torsion linear connection.

¹ Deligne conjecture: The Hochschild cochain complex has a natural structure of an algebra over a chain operad of the little squares operad.

The analysis of algebraic structures via operad theory still arouses the interest of many mathematicians see, for instance, [9]. The present work is meant to go along these lines of research and has as a main goal the analysis, via an operadic approach, of some of the existing relations among pre-Lie, post-Lie and brace algebras, the algebras associated to the brace structure of Gesternhaber and Voronov cited above, [15].

More in details, we will define the post-Lie operad ∇ as an extension of the pre-Lie operad $\mathcal{RT}$ of rooted trees defined by Chapoton and Livernet in [5], and we will present some of the properties of ∇ keeping in mind the analogue properties of the pre-Lie operad. The diagram below gives a pictorial summary of the main points addressed in the present work.

The post-Lie operad defined in this work is essential to verify the relations given by the diagram before. Since our main tool is an operad, we start our exposition presenting some background material on operad theory. We do this in Chapter 1. Chapter 2 is dedicated to the

construction of the operad ∇ . For this, we define a family of vector spaces with the bases given by a magma of labeled planar trees and prove that this family is an operad isomorphic to the post-Lie operad defined by Vallette in [24] (Proposition 5, Proposition 6, Theorem 1). In chapter 2, we also explain how post-Lie and pre-Lie operads are related and how we can use ∇ to describe the free post-Lie algebra obtained by Munthe-Kaas and Lundervold [22]. In Chapter 3, we use ∇ to prove some results related to the extension of the post-Lie product from $\mathfrak{g}$ to $\mathcal{U}(\mathfrak{g})$ (see [8]) and we present the problem that motivated us to develop this thesis.

PREREQUISITES

In this chapter, we present the basic tools used for the development of our work. In Section 1.1, some notations adopted along the text are given. In Section 1.2, besides defining algebras and operads, we discuss free and algebraic operads briefly. The reader familiar with algebraic operads, can skip to Chapter 2.

1.1 A word on notation

In this text, only vector spaces over a field $\mathbb{K}$ of characteristic zero are used. Let V be a vector space, the following notation is adopted:

- $V^{\otimes 0} := \mathbb{K}$ and $V^{\otimes n+1} := V \otimes V^{\otimes n}$. The elements of $V^{\otimes n}$ are denoted by $x_1 x_2 \cdots x_n$, with $x_i \in V$.
- $\text{Hom}(V^{\otimes n}, V)$, is the vector space of the linear maps from $V^{\otimes n}$ to V .
- $W \leq V$, for W a vector subspace of V .
- $\langle X \rangle$ is the free vector space generated by a set X .
- If $f: V^{\otimes n} \rightarrow V$ is a linear map, we use $f(x_1, \dots, x_n)$ to denote the range of $x_1 \cdots x_n \in V^{\otimes n}$.

If $\mathcal{R}$ is a ring with unit, the free $\mathcal{R}$ -module generated by a set X will be denoted by $\mathcal{R}\langle X \rangle$.

The set of the natural numbers is denoted by $\mathbb{N}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For each $n \in \mathbb{N}$, the permutation group $\mathbb{S}_n$ is the group of bijections from $[n] := \{1, \dots, n\}$ to $\{1, \dots, n\}$, the group operation is the composition and the elements are the permutations. A permutation $\sigma \in \mathbb{S}_n$ is denoted by:

$$\begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix} \text{ or } [\sigma(1)\sigma(2)\cdots\sigma(n)]$$

The identity permutation of $\mathbb{S}_n$ also will be denoted by id_n .

1.2 Operads and Algebras

Loday and Vallette in [20, p. vii] present the idea of what an operad is in the following way:

An operad is an algebraic device which encodes a type of algebras. Instead of studying the properties of a particular algebra, we focus on the universal operations that can be performed on the elements of any algebra of a given type. The information contained in an operad consists in these operations and all the ways of composing them.

In this thesis, we use the information contained in the operad ∇ to obtain some properties (Chapter 3) that arise from special compositions of the post-Lie product $\triangleright$ (Example 5 item 3). Due to this, in this section, we present the definition of operad (Subsection 1.2.1) and the operadic tools (Subsection 1.2.2) used in our work through a classical example of algebraic operad: the $\mathcal{A}ss$ operad encoding the associative algebras.

1.2.1 Definition of Operads

To introduce the notion of an operad properly, we need to talk a little about permutation groups.

Definition 1. If G is a group, a (right) G -**module** is a vector space V with a map $\cdot : V \times G \rightarrow V$, such that:

- $(kx + y) \cdot \sigma = k(x \cdot \sigma) + y \cdot \sigma$;
- $x \cdot 1 = x$;
- $y \cdot (\sigma\beta) = (y \cdot \sigma) \cdot \beta$,

for all $\sigma, \beta \in G$, $x, y \in V$ and $k \in \mathbb{K}$.

In other words, a G -module is a vector space V , with a right action of G that is compatible with the linear structure. To simplify the notation, we will write x^σ instead of $x \cdot \sigma$.

Remark 1. If we have $y \cdot (\sigma\beta) = (y \cdot \beta) \cdot \sigma$ instead of $y \cdot (\sigma\beta) = (y \cdot \sigma) \cdot \beta$, we say that V is a left G -module. In this case, the usual notation is $\sigma \cdot y$ with $\cdot : G \times V \rightarrow V$.

The only left G -modules that appear in this work are the left $\mathbb{S}_n$ -module $V^{\otimes n}$ for each $n \in \mathbb{N}$. If $x_1 \cdots x_n \in V^{\otimes n}$, then $\sigma \in \mathbb{S}_n$ acts by moving the factor of position i to position $\sigma(i)$, thus:

$$\sigma \cdot (x_1 \cdots x_n) = x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}. \quad (1.1)$$

However, it is important to be careful about equality (1.1). Indeed, if $\sigma, \beta \in \mathbb{S}_n$, then:

$$\sigma \cdot (\beta \cdot (x_1 \cdots x_n)) \neq (x_{\sigma^{-1}(\beta^{-1}(1))} \cdots x_{\sigma^{-1}(\beta^{-1}(n))}),$$

since, by definition, it is necessary a change of variables to compute $\sigma \cdot (\beta \cdot (x_1 \cdots x_n))$:

$$\begin{aligned} \sigma \cdot (\beta \cdot (x_1 \cdots x_n)) &= \sigma \cdot (x_{\beta^{-1}(1)} \cdots x_{\beta^{-1}(n)}) \\ &= y_{\sigma^{-1}(1)} \cdots y_{\sigma^{-1}(n)} \quad (\text{with } y_i = x_{\beta^{-1}(i)}) \\ &= x_{(\sigma \circ \beta)^{-1}(1)} \cdots x_{(\sigma \circ \beta)^{-1}(n)} = \sigma \circ \beta \cdot (x_1 \cdots x_n). \end{aligned}$$

This left action induces a right action on $\text{Hom}(V^{\otimes n}, V)$. In Example 3, the reader will see how this action helps us to understand some details of the operad definition.

Since most of the G -modules in this work are right G -modules, we will just write G -modules for right G -modules.

Definition 2. Let $(V, \cdot)$ be a G -module. We say that W is a G -submodule of V , if $W \leq V$, such that $\cdot|_W$ determines a G -module structure in W .

Definition 3 ($\mathbb{S}$ -modules). An $\mathbb{S}$ -module is a collection $\mathcal{S} = \{S(n)\}_{n \in \mathbb{N}}$, such that $S(i)$ is an $\mathbb{S}_i$ -module for all $i \in \mathbb{N}$. If $W = \{W(n)\}_{n \in \mathbb{N}}$ is a collection of vector spaces, such that $W(n)$ is an $\mathbb{S}_n$ -submodule of $S(n)$ for all n , then we say W is an $\mathbb{S}$ -submodule of $\mathcal{S}$.

Definition 4 (Direct sum of permutations). Let $\tau_i \in \mathbb{S}(k_i)$ be a permutation and $k_i \in \mathbb{N}$ be a natural number with $1 \leq i \leq r$, the **direct sum** $\tau_1 \oplus \cdots \oplus \tau_r$ is a permutation of $\mathbb{S}_{k_1 + \cdots + k_r}$, given by:

$$\begin{pmatrix} 1 & \cdots & k_1 & 1+k_1 & \cdots & k_2+k_1 & \cdots & 1+k_{r-1} & \cdots & k_r+k_{r-1} \\ \tau_1(1) & \cdots & \tau_1(k_1) & \tau_2(1)+k_1 & \cdots & \tau_2(k_2)+k_1 & \cdots & \tau_r(1)+k_{r-1} & \cdots & \tau_r(k_r)+k_{r-1} \end{pmatrix}.$$

Example 1. If $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, then

$$\sigma \oplus \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 5 & 4 & 2 & 6 & 8 & 7 \end{pmatrix}.$$

In particular, each direct sum of permutations can be divided into blocks

$$\tau_1 \oplus \cdots \oplus \tau_r = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdots & k_1 & 1+k_1 & \cdots & k_2+k_1 & \cdots & 1+k_{r-1} & \cdots & k_r+k_{r-1} \\ \tau_1(1) & \cdots & \tau_1(k_1) & \tau_2(1)+k_1 & \cdots & \tau_2(k_2)+k_1 & \cdots & \tau_r(1)+k_{r-1} & \cdots & \tau_r(k_r)+k_{r-1} \end{array} \right),$$

in which $\tau_1 \oplus \cdots \oplus \tau_r$ is denoted by

$$[B_{k_1} | B_{k_2} | \cdots | B_{k_r}],$$

where

- $B_{k_1} = \tau_1(1) \cdots \tau_1(k_1)$;
- $B_{k_2} = \tau_2(1) + k_1 \cdots \tau_2(k_2) + k_1$;

$$\bullet B_{k_r} = \tau_r(1) + k_{r-1} \cdots \tau_r(k_r) + k_{r-1}.$$

Definition 5 (Block permutations). Consider $\sigma, \tau_1, \dots, \tau_n$, with $\sigma \in \mathbb{S}_n$, $\tau_i \in \mathbb{S}(k_i)$, $k_i \in \mathbb{N}$, ($1 \leq i \leq n$). The **block permutation** of σ over $\tau_1 \oplus \cdots \oplus \tau_n$ is the permutation obtained by moving the block of position i to position $\sigma(i)$:

$$\sigma(\tau_1 \oplus \cdots \oplus \tau_n) := [B_{\sigma^{-1}(1)} | \cdots | B_{\sigma^{-1}(n)}] \in \mathbb{S}_{k_1 + \cdots + k_n}.$$

The analogous comments made in Note 1 apply to this definition.

Example 2. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $\tau_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $\tau_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ be permutations, so:

$$\begin{aligned} \tau_1 \oplus \tau_2 \oplus \tau_3 &= \left(\begin{array}{cc|cc|cc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 3 & 5 & 6 & 4 & 9 & 8 & 7 \end{array} \right) \Rightarrow \\ \sigma(\tau_1 \oplus \tau_2 \oplus \tau_3) &= [B_{k_2} | B_{k_3} | B_{k_1}] = \left(\begin{array}{cccc|ccc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 6 & 4 & 9 & 8 & 7 & 2 & 1 \end{array} \right). \end{aligned}$$

The inverse of the block permutation $\sigma(\tau_1^{-1} \oplus \cdots \oplus \tau_n^{-1})$ is denoted by $\sigma \circ (\tau_1, \dots, \tau_r)$. The special case $\sigma \circ (id_1, \dots, id_l, \underbrace{\tau}_{ith}, id_1, \dots, id_1) := \sigma \circ_i \tau$ (where id_1 is the unique element of $\mathbb{S}_1$).

Remark 2. In this work, we made some adjustments to the definitions of direct sums and block permutations that are in Fresse [10, p.13] in order to match them to the notations of our main reference Loday and Vallette [20].

Finally, operads can be defined!

Definition 6. An **operad** $\mathcal{P} = \{\mathcal{P}(n)\}_{n \in \mathbb{N}}$ is an $\mathbb{S}$ -module with a collection of linear maps

$$\begin{aligned} \circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) &\longrightarrow \mathcal{P}(n-1+m), \forall n, m \in \mathbb{N} - \{0\}, 1 \leq i \leq n \\ p \otimes r &\mapsto p \circ_i r \end{aligned}$$

that satisfy the following conditions:

$$(p \circ_i q) \circ_{j+i-1} r = p \circ_i (q \circ_j r) \quad 1 \leq i \leq l, 1 \leq j \leq m \quad (1.2)$$

$$(p \circ_i q) \circ_{k+m-1} r = (p \circ_k r) \circ_i q \quad 1 \leq i < k \leq l \quad (1.3)$$

$$(p^\sigma \circ_i q^\tau) = (p \circ_{\sigma(i)} q) \circ_{\sigma(i)}^\tau \quad (1.4)$$

where $p \in \mathcal{P}(l)$, $q \in \mathcal{P}(m)$, $r \in \mathcal{P}(n)$, $\sigma \in \mathbb{S}_l$, $\tau \in \mathbb{S}_m$.

Relation (1.2) is called **sequential composition axiom**, relation (1.3), **parallel composition axiom** and relation (1.4), **equivariance**. The maps $\circ_i$ are called **compositions** or **operadic compositions**. Moreover, there is $1 \in \mathcal{P}(1)$, the **unit**, such that:

$$p \circ_i 1 = 1 \circ_1 p = p, \forall n \in \mathbb{N}, p \in \mathcal{P}(n), 1 \leq i \leq n. \quad (1.5)$$

Historically, part of the development of the operads theory is given by the study of some topological function spaces. The main model for the theory of operads is given by the family $\{Hom(V^{\otimes n}, V)\}_{n \in \mathbb{N}}$, as we can see in the example below.

Example 3. The operad of endomorfisms or just **endoperads** is the family $\mathcal{E}nd_V = \{\mathcal{E}nd_V(n)\}_{n \in \mathbb{N}}$ with $\mathcal{E}nd_V(n) = Hom(V^{\otimes n}, V)$, in which $\mathcal{E}nd_V$ is an $\mathbb{S}$ -module with:

$$\begin{aligned} \cdot : \mathcal{E}nd_V(n) \times \mathbb{S}_n &\longrightarrow \mathcal{E}nd_V(n) \\ (f, \sigma) &\mapsto f^\sigma, \end{aligned} \tag{1.6}$$

where $f^\sigma(x_1, \dots, x_n) = f(\sigma \cdot (x_1 \cdots x_n)) = f(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$, $x_i \in V$ and $\sigma \cdot (x_1 \cdots x_n)$ is given by the left action of $\mathbb{S}_n$ on $V^{\otimes n}$ by changing the factor of position i to position $\sigma(i)$ (see Note 1).

The compositions in $\mathcal{E}nd_V$ are:

$$f \circ_i g := f \circ (id^{i-1} \otimes g \otimes id^{m-i}) \tag{1.7}$$

with $f \in \mathcal{E}nd_V(m)$, $1 \leq i \leq m$, $g \in \mathcal{E}nd_V(n)$ and id^j is the linear map $\underbrace{id \otimes \cdots \otimes id}_{j\text{-times}}$ ($id: V \rightarrow V$ the identity map). Note that id is the unit of the operad $\mathcal{E}nd_V$.

The $\mathcal{E}nd_V$ operad is a good example to understand the details of the operad definition. For instance, since some adjustments are made in the compositions (1.7), properties (1.2) and (1.3) tell us that it does not matter which composition occurs first:

$$\begin{aligned} (f \circ_2 g) \circ_4 h(x_1, x_2, x_3, x_4, x_5, x_6, x_7) &= (f \circ_2 g)(x_1, x_2, x_3, h(x_4, x_5, x_6), x_7) \\ &= f(x_1, g(x_2, x_3), h(x_4, x_5, x_6), x_7) \\ &= (f \circ_3 h) \circ_2 g(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \end{aligned}$$

with $x_i \in V$ ($1 \leq i \leq 7$), $f \in \mathcal{E}nd_V(4)$, $g \in \mathcal{E}nd_V(2)$, $h \in \mathcal{E}nd_V(4)$.

The equivariance property shows the operadic compositions are compatible with the $\mathbb{S}$ -module structure. In the following example, we enlighten how the action of $\mathbb{S}_n$ on $\mathcal{E}nd_V(n)$ works.

Example 4. Consider f and g as given before. If $\sigma = [1342] \in \mathbb{S}_4$ and $\tau = [21] \in \mathbb{S}_2$, then

$$\begin{aligned} f^\sigma \circ_2 g^\tau(x_1, x_2, x_3, x_4, x_5) &= f^\sigma(x_1, g^\tau(x_2, x_3), x_4, x_5) \\ &= f^\sigma(y_1, y_2, y_3, y_4) \quad (y_2 = g^\tau(x_2, x_3), y_1 = x_1, y_3 = x_4, y_4 = x_5) \\ &= f(y_{\sigma^{-1}(1)}, y_{\sigma^{-1}(2)}, y_{\sigma^{-1}(3)}, y_{\sigma^{-1}(4)}) \\ &= f(y_1, y_4, y_2, y_3) \\ &= f(x_1, x_5, g^\tau(z_1, z_2), x_4) \quad (z_1 = x_2, z_2 = x_3) \end{aligned}$$

$$\begin{aligned}
&= f(x_1, x_5, g(z_{\tau^{-1}(1)}, z_{\tau^{-1}(2)}), x_4) \\
&= f(x_1, x_5, g(z_2, z_1), x_4) \\
&= f(x_1, x_5, g(x_3, x_2), x_4).
\end{aligned}$$

Indeed, this action moves the factor in position i to position $\sigma(i)$. Thus, since $\mathbb{S}_n$ acts by changing the position of the variables, in case we want to compare $f^\sigma \circ_2 g^\tau$ with $(f \circ_k g)^\theta$, we need $k = \sigma(2)$. If the reader turns back to Definition 5, then he can see $[1324](id_1 \oplus [21] \oplus id_1 \oplus id_1) = [15324]$ describes the index of $f^\sigma \circ_2 g^\tau$ variables after the actions of σ and τ . If the aim is to obtain the permutation that describes these changes of position, we need the inverse of $[15324]$, i.e., $\theta = \sigma \circ_2 \tau = [15324]^{-1}$.

For other examples and equivalent definitions, see [20].

1.2.2 Free Operads and Algebraic Operads

Definition 7. An **algebra** is a pair $A = (V, \{\mu_i\}_{i \in I})$, where V is a vector space and $\{\mu_i\}_{i \in I}$ is a family of linear maps $\mu_i: V^{\otimes n} \rightarrow V$ (for some $n \in \mathbb{N}^*$) that satisfy some linear relations r_j , such that:

$$r_j := \sum_{\varphi} k_{\varphi} \varphi \equiv 0,$$

where $k_{\varphi} \in \mathbb{K}$, $\varphi = \mu_i^{\sigma}$ (for some $\sigma \in \mathbb{S}_n$) or $\varphi = (f_{j_1} \circ_{j_1} f_{j_2}) \cdots \circ_{j_{n-1}} f_{j_n})^{\theta}$, in which θ is a permutation, $f_{j_k} = \mu_{j_k}^{\beta}$ ($\mu_{j_k}: V^{\otimes m} \rightarrow V$ and some $\beta \in \mathbb{S}_m$). In this case, the action of the permutation group is given by (1.6) and $\circ_{j_k}$ are the operadic compositions (1.7) in $\mathcal{E}nd_V$. Each μ_i is called **operation** or **product**.

Example 5. 1. Associative algebra $(V, m: V^{\otimes 2} \rightarrow V)$:

$$r_1 := m \circ_1 m - m \circ_2 m \equiv 0, \text{ i. e.,} \quad (1.8)$$

$$m(m(x, y), z) - m(x, m(y, z)) = 0 \quad \forall x, y, z \in V.$$

2. Lie algebra $(V, [\cdot, \cdot]: V^{\otimes 2} \rightarrow V)$ ($[\cdot, \cdot]$ is called Lie bracket):

$$r_1 := [\cdot, \cdot] + [\cdot, \cdot]^{[21]} \equiv 0, \quad (1.9)$$

$$r_2 := [\cdot, \cdot] \circ_1 [\cdot, \cdot] - [\cdot, \cdot] \circ_2 [\cdot, \cdot] - ([\cdot, \cdot] \circ_1 [\cdot, \cdot])^{[132]}, \text{ i. e.,} \quad (1.10)$$

$$[x, y] + [y, x] = 0$$

$$[[x, y], z] - [x, [y, z]] - [[x, z], y] = 0 \quad \forall x, y, z \in V \text{ (Jacobi relation).}$$

3. Post-Lie algebra $(V, [\cdot, \cdot]: V^{\otimes 2} \rightarrow V, \triangleright: V^{\otimes 2} \rightarrow V)$:

$$r_1 := [\cdot, \cdot] + [\cdot, \cdot]^{[21]} \equiv 0, \quad (1.11)$$

$$r_2 := [\cdot, \cdot] \circ_1 [\cdot, \cdot] - [\cdot, \cdot] \circ_2 [\cdot, \cdot] - ([\cdot, \cdot] \circ_1 [\cdot, \cdot])^{[132]} \equiv 0, \quad (1.12)$$

$$r_3 := \triangleright \circ_2 [\cdot, \cdot] - [\cdot, \cdot] \circ_1 \triangleright - ([\cdot, \cdot] \circ_2 \triangleright)^{[213]} \equiv 0, \quad (1.13)$$

$$r_4 := \triangleright \circ_1 [\cdot, \cdot] - a_{\triangleright} + a_{\triangleright}^{[213]} \equiv 0, \quad (1.14)$$

in which $a_{\triangleright}$ denotes the associator of a binary operation, i. e.,

$$a_{\triangleright}(x, y, z) = x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z, \text{ with } x, y, z \in V.$$

Summing up, a post-Lie algebra is a Lie algebra, such that:

$$x \triangleright [y, z] - [x \triangleright y, z] - [y, x \triangleright z] = 0$$

$$[x, y] \triangleright z - a_{\triangleright}(x, y, z) + a_{\triangleright}(y, x, z) = 0, \quad \forall x, y, z \in V.$$

4. Pre-Lie algebra $(V, *: V^{\otimes 2} \rightarrow V)$:

$$r_1 := a_* - a_*^{[213]} \equiv 0, \text{ i.e.,} \quad (1.15)$$

$$a_*(x, y, z) - a_*(y, x, z) = 0, \quad \forall x, y, z \in V.$$

Remark 3. So far, we give examples of algebras in which the family of operations is finite. So, it is natural for the reader to enquire if there are algebras with an infinite family of operations. The answer is yes. However, these algebras normally have a more delicate description for their relations and, because of this, we prefer not to give these examples now. In Section 3.1, we define brace and symmetric brace algebras, whose family of operations is infinite.

Algebras and operads are closely related. Next, we give some examples that can help us understand this relation. The following definitions provide the necessary to present these examples.

Definition 8 ($\mathbb{S}$ -module morphism). Let $\mathcal{M} = \{M(n)\}_{n \in \mathbb{N}}$ and $\mathcal{C} = \{C(n)\}_{n \in \mathbb{N}}$ be $\mathbb{S}$ -modules. An **$\mathbb{S}$ -module morphism** $f: \mathcal{M} \rightarrow \mathcal{C}$ is a family $\{f_n: M(n) \rightarrow C(n)\}_{n \in \mathbb{N}}$ of linear maps, such that $\forall n \in \mathbb{N}, x \in M(n), \sigma \in \mathbb{S}_n$:

$$f_n(x^\sigma) = f_n(x)^\sigma.$$

Definition 9 (Operadic morphism). Let $\mathcal{P}$ and $\mathcal{Q}$ be two operads. An **operadic morphism** $f: \mathcal{P} \rightarrow \mathcal{Q}$ is an $\mathbb{S}$ -module morphism that satisfies the following conditions:

- $f(1_{\mathcal{P}}) = 1_{\mathcal{Q}}$, where $1_{\mathcal{P}}$ and $1_{\mathcal{Q}}$ are the units of $\mathcal{P}$ and $\mathcal{Q}$, respectively;
- $f_{n+m-1}(x \circ_i^{\mathcal{P}} y) = f_n(x) \circ_i^{\mathcal{Q}} f_m(y)$ with $x \in \mathcal{P}(n), y \in \mathcal{P}(m)$, $\circ_i^{\mathcal{P}}$ is the composition of $\mathcal{P}$ and $\circ_i^{\mathcal{Q}}$ is the composition of $\mathcal{Q}$.

Definition 10 (Operadic ideal). We say that $\mathcal{I} = \{\mathcal{I}(n)\}_{n \in \mathbb{N}}$ is an **operadic ideal** of the operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n \in \mathbb{N}}$ if $\mathcal{I}$ is an $\mathbb{S}$ -submodule of $\mathcal{P}$ and:

$$p \circ_i q, q \circ_j p \in \mathcal{I}(n+m-1)$$

$\forall n, m \in \mathbb{N}, p \in \mathcal{P}(n), q \in \mathcal{I}(m)$ with $1 \leq i \leq n, 1 \leq j \leq m$.

Definition 11 (Quotient operad). Let $\mathcal{P}$ be an operad and $\mathcal{I}$ be an operadic ideal of $\mathcal{P}$. The **quotient operad** is the $\mathbb{S}$ -module:

$$\mathcal{P}/\mathcal{I} = \{\mathcal{P}(n)/\mathcal{I}(n)\}_{n \in \mathbb{N}}.$$

As occurs in rings, the quotient of an operad by an operadic ideal is an operad with the induced compositions. Indeed, the operadic compositions $\circ_i: \mathcal{P}(n) \otimes \mathcal{P}(m) \rightarrow \mathcal{P}(n-1+m)$ induce the maps:

$$\begin{aligned} \overline{\circ}_i: \mathcal{P}(n)/\mathcal{I}(n) \otimes \mathcal{P}(m)/\mathcal{I}(m) &\longrightarrow \mathcal{P}(n-1+m)/\mathcal{I}(n+m-1) \\ (p + \mathcal{I}(n)) \otimes (r + \mathcal{I}(m)) &\mapsto p \circ_i r + \mathcal{I}(n+m-1) \end{aligned}$$

These maps are well-defined, since $p \in \mathcal{I}(n)$ and $r \in \mathcal{P}(m) \Rightarrow p \circ_i r \in \mathcal{I}(n+m-1)$, by definition of operadic ideal. We have a similar result for $p \circ_i r$ with $r \in \mathcal{I}(m)$. To simplify the notation, instead of $p + \mathcal{I}(n) \in \mathcal{P}(n)/\mathcal{I}(n)$ just p is used. We will also write $\circ_i$ instead of $\overline{\circ}_i$.

Definition 12 (Free operad). The **free operad** over the $\mathbb{S}$ -module $\mathcal{S}$ is an operad $\mathcal{F}(\mathcal{S})$ equipped with an $\mathbb{S}$ -module morphism $\eta(\mathcal{S}): \mathcal{S} \rightarrow \mathcal{F}(\mathcal{S})$, which satisfies the following universal condition: any $\mathbb{S}$ -module morphism $f: \mathcal{S} \rightarrow \mathcal{P}$, where $\mathcal{P}$ is an operad, extends uniquely into an operad morphism $\tilde{f}: \mathcal{F}(\mathcal{S}) \rightarrow \mathcal{P}$,

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{f} & \mathcal{P} \\ \eta(\mathcal{S}) \downarrow & \nearrow \tilde{f} & \\ \mathcal{F}(\mathcal{S}) & & \end{array}$$

Strictly speaking, it is necessary to prove the existence of a free operad. Building a free operad is something delicate and the reader can find this construction in [20]. However, we will soon give an idea of how to construct a special free operad that gives rise to some examples of **algebraic operads**.

Given an algebra, we can obtain an algebraic operad related to it. The construction of the algebraic operad is closely related to the description of its free algebra. Because of this, below we define a tool known for its use in the construction of free algebras that will be useful in our description of algebraic operads: magma.

Definition 13. A **magma** is a set M with a map $\diamond: M \times M \rightarrow M$ without any additional condition over $\diamond$. Let $(M_1, \diamond_1)$ and $(M_2, \diamond_2)$ be two magmas. A function $f: M_1 \rightarrow M_2$ is a magma morphism if $f(x_1 \diamond_1 x_2) = f(x_1) \diamond_2 f(x_2)$.

Proposition 1. Let X be a set. For every magma M and every function $f: X \rightarrow M$, there are a magma $M(X)$ (the free magma) and a magma morphism $\tilde{f}: M(X) \rightarrow M$ that extends the map f , i. e., that makes the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & M \\ \downarrow i & \nearrow \tilde{f} & \\ M(X) & & \end{array}$$

The reader can see in [2, Lemma 2.20] a proof of this result. Below, we give a brief idea of how to obtain a free magma on a set X .

Definition 14. To begin the construction, we inductively define:

$$\begin{aligned} M_0(X) &= \emptyset, \\ M_1(X) &:= X, \quad M_2(X) := X \times X, \\ M_3(X) &= M_2(X) \times M_1(X) \cup M_1(X) \times M_2(X), \\ &\vdots \\ M_n(X) &:= \bigcup_{p+q=n} M_p(X) \times M_q(X). \end{aligned}$$

An element of $M_n(X)$ with $n \geq 2$ will be called of **parenthesizing**.

Example 6. If x, y and $z \in X$, then:

$$x, y, (x, x), ((x, y), y), (x, (y, y)), ((x, y), (y, z)) \in M(X).$$

On $M(X)$ a natural map is defined:

$$\begin{aligned} \diamond: M(X) \times M(X) &\longrightarrow M(X) \\ (p, q) &\mapsto \diamond(p, q) = (p, q), \end{aligned}$$

such that $(M(X), \diamond)$ is a magma.

More details on magmas can be found in Bonfiglioli and Fulci [2].

Let V be a vector space. Now, consider the vector spaces $\langle M_n(V) \rangle$ and the subspaces $\langle R_n(V) \rangle \leq \langle M_n(V) \rangle$:

- $R_0(V) = R_1(V) = R_2(V) = \{0\}$;
- $R_3(V) = \{((x_1, x_2), x_3) - (x_1, (x_2, x_3)); x_1, x_2, x_3 \in V\}$;
- $R_4(V) = \{(((x_1, x_2), x_3), x_4) - ((x_1, x_2)(x_3, x_4)), (((x_1, x_2), x_3), x_4) - ((x_1, (x_2, x_3)), x_4), (x_1, (x_2, (x_3, x_4))) - (x_1, ((x_2, x_3), x_4)); x_1, x_2, x_3, x_4 \in V\}$;
- $R_n(V)$ ($n \geq 3$) is the set given by all the differences between two parenthesizings with n elements.

The free associative algebra over a vector space V is the vector space $F(V)$ that satisfies the following universal property. Given an associative algebra A and a linear map $f: V \rightarrow A$ there is a unique associative algebra morphism $\tilde{f}: F(V) \rightarrow A$ that extends the map f , i. e., that makes the following diagram commutative

$$\begin{array}{ccc} V & \xrightarrow{f} & A \\ \downarrow i & \nearrow \tilde{f} & \\ F(V) & & \end{array}$$

The associative algebra morphism is defined in a natural way, i. e., if (V_1, m_1) and (V_2, m_2) are associative algebras, then the linear map $f: V_1 \rightarrow V_2$ is an associative algebra morphism if

$$f(m_1(x_1, x_2)) = m_2(f(x_1), f(x_2)), \quad \forall x_1, x_2 \in V. \quad (1.16)$$

We leave as an exercise to the reader to prove that the free associative algebra on a vector space V is, up to isomorphism, the graded space:

$$F(V) = \bigoplus_{n \in \mathbb{N}} \frac{\langle M_n(V) \rangle}{\langle R_n(V) \rangle}, \quad (1.17)$$

with the graded linear map given by

$$\begin{aligned} m: F(V) \otimes F(V) &\longrightarrow F(V) \\ v \otimes w &\mapsto m(v, w) = (v, w), \text{ for all } v \text{ and } w \in M(V). \end{aligned}$$

In fact, in $F(V)$ every parenthesizing coincides with one another. So, for $x_1 \in M_{i_1}(V)$, $x_2 \in M_{i_2}(V)$ and $x_3 \in M_{i_3}(V)$, we have:

$$m(m(x_1, x_2), x_3) = ((x_1, x_2), x_3) = (x_1, (x_2, x_3)) = m(x_1, m(x_2, x_3)). \quad (1.18)$$

There are several ways to express the free associative algebra. However, we can view the operad related to the associative algebra, called operad $\mathcal{Ass}$, as a structure that describes how we obtain the free associative algebra, i.e., an structure encoding category of associative algebras.¹ Let's see how it works.

If we are interested in describing how we obtain the free associative algebra, then, at first, the vector space V does not matter. The stars of this process are the parenthesizings and the property that these parenthesizings need to verify (see (1.18)). So, we choose to work with a simpler way of describing the parenthesizings: $M(\bullet)$, the free magma over a unit set $\{\bullet\}$.

Next, we define the family $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$, with $\mathcal{P}_n = \langle M_n(\bullet) \rangle$ and the linear maps

$$\odot_i: \mathcal{P}_n \otimes \mathcal{P}_m \longrightarrow \mathcal{P}_{n-1+m}, \quad \forall n, m \in \mathbb{N} - \{0\}, \quad 1 \leq i \leq n$$

¹ In the category of associative algebras, the objects are the associative algebras and the arrows are the associative algebra morphisms (see (1.16)).

$$p \otimes r \mapsto p \odot_i r,$$

in which, for $p \in M_n(\bullet)$ and $r \in M_m(\bullet)$, the parenthesizing $p \odot_i r$ is given by replacing the i -th $\bullet$ of p (from left to right) by the parenthesizing r , as we can see in the next example.

Example 7.

$$\begin{aligned} (\bullet, \bullet) \odot_2 ((\bullet, \bullet), \bullet) &:= (\bullet, ((\bullet, \bullet), \bullet)) \\ (\bullet, (\bullet, \bullet)) \odot_3 ((\bullet, \bullet), \bullet) &:= (\bullet, (\bullet, ((\bullet, \bullet), \bullet))) \end{aligned}$$

These maps verify the sequential, parallel and unit axioms ($\bullet \in \mathcal{P}_1$ is the unit). A family of vector spaces $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ with a family of maps $\{\odot_i\}_{i \in I}$, in which the sequential, parallel and unit axioms hold, is called **non-symmetric operad**. Note that, an operad (also known as symmetric operad) is a non-symmetric operad. We can also obtain an operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n \in \mathbb{N}}$ from a non-symmetric operad $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$. Indeed, we define:

$$\mathcal{P}(n) = \mathcal{P}_n \times \langle \mathbb{S}_n \rangle.$$

The $\mathbb{S}$ -module structure in $\mathcal{P}$ is given by the composition between the elements of $\mathbb{S}_n$ at right, which are extended linearly. The operadic compositions in $\mathcal{P}$ are the linear maps given by:

$$\begin{aligned} \circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) &\longrightarrow \mathcal{P}(n-1+m), \quad \forall \sigma \in \mathbb{S}_n, \beta \in \mathbb{S}_m \\ (p, \sigma) \otimes (r, \beta) &\mapsto (p \odot_{\sigma(i)} r, \sigma \circ_i \beta), \end{aligned} \quad (1.19)$$

in which $\sigma \circ_i \beta$ is the inverse of the block permutation $\sigma(id_1 \oplus \cdots \oplus id_1 \oplus \underbrace{\beta^{-1}}_{ith} \oplus id_1 \oplus \cdots \oplus id_1)$ (see Definition 5). By definition, the equivariance axiom holds for these maps. The unit element is $(\bullet, id_1)$. The parallel and sequential axioms also hold. Indeed, we leave as exercise to prove that the maps $\circ_i : \langle \mathbb{S}_n \rangle \otimes \langle \mathbb{S}_m \rangle \rightarrow \langle \mathbb{S}_{n+m-1} \rangle$, given by $\sigma \otimes \beta \mapsto \sigma \circ_i \beta$, determine an operadic structure on $\{\langle \mathbb{S}_n \rangle\}_{n \in \mathbb{N}}$. By using this and the properties that the non-symmetric compositions $\odot_i$ satisfy, we have what it takes to ensure that $\circ_i$ are operadic compositions.

Example 8. We use the non-symmetric operad $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ with $\mathcal{P}_n = \langle M_n(\bullet) \rangle$ to give some examples that can enlighten the statements and definitions given before. To simplify the notation, we denote $(p, \sigma) := p^\sigma$ for $p \in M_n(\bullet)$.

Right action:

$$\begin{aligned} \left((\bullet, (\bullet, \bullet))^{[132]} + ((\bullet, \bullet), \bullet)^{id_3} \right)^{[321]} &= (\bullet, (\bullet, \bullet))^{[132] \circ [321]} + ((\bullet, \bullet), \bullet)^{id_3 \circ [321]} \\ &= (\bullet, (\bullet, \bullet))^{[231]} + ((\bullet, \bullet), \bullet)^{[321]}. \end{aligned}$$

Operadic composition:

$$(\bullet, (\bullet, \bullet))^{[231]} \circ_2 (\bullet, \bullet)^{[21]} = ((\bullet, (\bullet, \bullet)) \odot_3 (\bullet, \bullet))^{[231] \circ_2 [21]}$$

$$= (\bullet, (\bullet, (\bullet, \bullet)))^{[2431]},$$

in which,

$$[231](id_1 \oplus [21] \oplus id_1) = [4132] \Rightarrow [231] \circ_2 [21] = [4132]^{-1} = [2431].$$

Let R be a set and $\mathcal{P}$ be an operad, such that $\forall r \in R \Rightarrow r \in \mathcal{P}(n)$, for some $n \in \mathbb{N}$. Since $\mathcal{P}$ is an operadic ideal, the set $S = \{\mathcal{I} : \mathcal{I} \text{ is an operadic ideal and } \forall r \in R \Rightarrow r \in \mathcal{I}(n), \text{ for some } n \in \mathbb{N}\}$ is not empty. We leave as an exercise for the reader to show that

$$(R) := \bigcap_{\mathcal{I} \in S} \mathcal{I}, \quad (1.20)$$

therefore (R) is the least operadic ideal containing R .²

Definition 15. The ideal (R) is called the **operadic ideal generated by R** .

If the reader needs an explicit description for (R) , we suggest to verify that

$$(R) = \mathcal{J} = \{\mathcal{J}(n)\}_{n \in \mathbb{N}},$$

$$\mathcal{J}(n) = \{0\} \text{ or } \mathcal{J}(n) = \langle X \rangle, \quad (1.21)$$

X is the set of elements $(p \circ_i q)^\sigma, (q \circ_j p)^\sigma, \forall \sigma \in \mathbb{S}_n, 1 \leq i \leq m, 1 \leq j \leq k$, such that:

- $p \in \mathcal{P}(m), q \in \mathcal{P}(k)$, with $k + m - 1 = n$;
- $q \in R$ or q is the range of compositions $((p_1 \circ_{n_1} q_1) \circ_{n_2} q_2) \cdots \circ_{n_k} q_k)^\theta$ (θ is a permutation) between elements of R and $\mathcal{P}$.

Example 9 (The operad $\mathcal{A}ss$). Now, we can construct the operad $\mathcal{A}ss$. For this, we use the operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n \in \mathbb{N}}$ with $\mathcal{P}(n) = \langle M_n(\bullet) \rangle \times \langle \mathbb{S}_n \rangle$. The operadic compositions are given by (1.19), as we can see in Example 8. We define the operadic ideal $\mathcal{J} = \{\mathcal{J}(n)\}_{n \in \mathbb{N}} = (R)$ with $R = \{((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3}\}$. Explicitly:

- $\mathcal{J}(0) = \mathcal{J}(1) = \mathcal{J}(2) = 0$;
- $\mathcal{J}(n) = \langle R_n(\bullet) \rangle$ with $R_n(\bullet) = \{p^\sigma - q^\sigma; p, q \in M_n(\bullet), \sigma \in \mathbb{S}_n\}$ and $n \geq 3$.

The operad $\mathcal{A}ss = \{\mathcal{A}ss(n)\}_{n \in \mathbb{N}} = \mathcal{P} / \mathcal{J}$.

² The intersection is defined in a natural way $\bigcap_{\mathcal{I} \in S} \mathcal{I} = \left\{ \bigcap_{\{I_n\}_{n \in \mathbb{N}} = \mathcal{I} \in S} I_n \right\}_{n \in \mathbb{N}}$, i. e., it is made respecting the grading.

The operadic ideal defined before gives us a good example of an operadic ideal generated by a set (see Definition 15, (1.20) and (1.21)).

Example 10. Some elements of $\mathcal{J}(3)$:

$$\begin{aligned} (((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3}) \circ_3 \bullet^{id_1} &= \bullet^{id_1} \circ_1 (((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3}) \\ &= ((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3} \in \mathcal{J}(3), \end{aligned}$$

$$\begin{aligned} (\bullet^{id_1} \circ_1 (((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3}))^{[312]} &= (((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3})^{[312]} \\ &= ((\bullet, \bullet), \bullet)^{[312]} - (\bullet, (\bullet, \bullet))^{[312]} \in \mathcal{J}(3). \end{aligned}$$

Example 11. Some elements of $\mathcal{J}(4)$:

$$(((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3}) \circ_2 (\bullet, \bullet)^{id_2} = ((\bullet, (\bullet, \bullet)), \bullet)^{id_4} - (\bullet, ((\bullet, \bullet), \bullet))^{id_4} \in \mathcal{J}(4),$$

$$\begin{aligned} (\bullet, \bullet)^{[21]} \circ_1 (((\bullet, \bullet), \bullet)^{[312]} - (\bullet, (\bullet, \bullet))^{[312]}) &= (\bullet, ((\bullet, \bullet), \bullet))^{[21] \circ_1 [312]} - (\bullet, (\bullet, (\bullet, \bullet)))^{[21] \circ_1 [312]} \\ &= (\bullet, ((\bullet, \bullet), \bullet))^{[3421]} - (\bullet, (\bullet, (\bullet, \bullet)))^{[3421]} \in \mathcal{J}(4). \end{aligned}$$

Having in mind the $\mathbb{S}$ -submodule structure and the need of $\mathcal{J}$ to be closed under the operadic compositions, the process to obtain $\mathcal{J} = (R)$ is inductive. Everything is made by respecting the grading.

One of the main objectives to be achieved in the construction of $(R) = \mathcal{J}$ is the range of compositions between elements of $\mathcal{P}$ and R in $\mathcal{J}$ (see Definition 10 and (1.20)). Thus,

$$\bullet^{id_1} \circ_1 (((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3}) \in \mathcal{J}(3),$$

$$(((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3}) \circ_2 (\bullet, \bullet)^{id_2} \in \mathcal{J}(4).$$

Note that, in Example 9, we have $\mathcal{J}(1) = \mathcal{J}(2) = 0$. It occurs because there is no composition $p \circ_i q$ with p or $q \in R$, such that $p \circ_i q \in \mathcal{P}(k)$ with $k \in \{1, 2\}$. Remember: (R) is the least operadic ideal containing R . Thus, in our description, we just considered the necessary to obtain this ideal. We also want $\mathcal{J}$ to be an $\mathbb{S}$ -submodule. So, we put

$$(\bullet^{id_1} \circ_1 (((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3}))^{[312]} \text{ in } \mathcal{J}(3).$$

It is important the compositions between elements of $\mathcal{P}$ and $\mathcal{J}$ are in $\mathcal{J}$. Thus, given

$$((\bullet, \bullet), \bullet)^{[312]} - (\bullet, (\bullet, \bullet))^{[312]} \in \mathcal{J}(3) \Rightarrow (\bullet, \bullet)^{[21]} \circ_1 (((\bullet, \bullet), \bullet)^{[312]} - (\bullet, (\bullet, \bullet))^{[312]}) \in \mathcal{J}(4).$$

Now, it is not so hard to see that

$$\mathcal{J}(n) = \langle R_n(\bullet) \rangle$$

with $R_n(\bullet) = \{p^\sigma - q^\sigma; p, q \in M_n(\bullet), \sigma \in \mathbb{S}_n\}$ and $n \geq 3$.

So far, the construction of the operad $\mathcal{A}ss$ is directly linked to the free associative algebra $F(V)$ for some vector space V . In fact, the idea was to describe the stages of the free associative algebra construction. However, considering the operadic composition (1.19), the elements of the operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n \in \mathbb{N}}$ with $\mathcal{P}(n) = \langle M_n(\bullet) \rangle \times \langle \mathbb{S}_n \rangle$ can be described in terms of compositions of $(\bullet, \bullet)^{[12]}$ or $(\bullet, \bullet)^{[21]}$.

Example 12.

$$\begin{aligned} (\bullet, (\bullet, \bullet))^{[312]} &= ((\bullet, \bullet) \odot_2 (\bullet, \bullet))^{[21] \circ_1 [12]} \\ &= (\bullet, \bullet)^{[21]} \circ_1 (\bullet, \bullet)^{[12]}. \end{aligned}$$

When we consider the vector space $\langle \mathbb{S}_n \rangle$ with an $\mathbb{S}_n$ -module structure given by the composition at right, $\langle \mathbb{S}_n \rangle$ is a non-commutative ring with unit. This ring is called **group algebra** and we will denote it by $\mathbb{K}[\mathbb{S}_n]$.³ In particular, we can also define an $\mathbb{S}$ -module as a collection of vector spaces $\{S(n)\}_{n \in \mathbb{N}}$, such that $S(i)$ is a right $\mathbb{K}[\mathbb{S}_i]$ -module.

Back to the operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n \in \mathbb{N}}$ with $\mathcal{P}(n) = \langle M_n(\bullet) \rangle \times \langle \mathbb{S}_n \rangle$, we have

$$\mathcal{P}(2) = \langle (\bullet, \bullet)^{[12]}, (\bullet, \bullet)^{[21]} \rangle$$

is the right $\mathbb{K}[\mathbb{S}_2]$ -module generated by $(\bullet, \bullet)^{[12]}$ and denoted by $\mathbb{K}[\mathbb{S}_2]\langle (\bullet, \bullet)^{[12]} \rangle$. Thus, $\mathcal{P} = \{\mathcal{P}(n)\}_{n \in \mathbb{N}}$ is the free operad $\mathcal{F}(\mathcal{S})$ with $\mathcal{S} = \{\mathcal{S}(n)\}_{n \in \mathbb{N}}$ given by $\mathcal{S}(2) = \mathbb{K}[\mathbb{S}_2]\langle (\bullet, \bullet)^{[12]} \rangle$ and by $\mathcal{S}(n) = 0$ for all $n \neq 2$.

It is necessary to prove that the free operad given before is isomorphic as an operad to the operad $\mathcal{P}$. However, considering what we presented so far, it is not so difficult to accept that it is enough to know the image of $(\bullet, \bullet)^{[12]}$ in an $\mathbb{S}$ -module morphism f to obtain an operadic morphism $\tilde{f}$ that extends f .

At end, with these new tools, the operad $\mathcal{A}ss$ can be described as:

$$\mathcal{A}ss = \mathcal{F}(\mathcal{S})/(R), \text{ with}$$

$\mathcal{S} = \{\mathcal{S}(n)\}_{n \in \mathbb{N}}$ given by $\mathcal{S}(2) = \mathbb{K}[\mathbb{S}_2]\langle (\bullet, \bullet)^{[12]} \rangle$ and by $\mathcal{S}(n) = 0$ for all $n \neq 2$ and $R = \{((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3}\}$.

The construction of the free associative algebra and the operad $\mathcal{P}$ was intended to provide an insight of how algebraic operads work. However, we can relate the algebraic operad to the corresponding algebra by using other resources.

Definition 16 ($\mathcal{P}$ -algebras). A vector space V is an **algebra over an operad** $\mathcal{P}$, or just a $\mathcal{P}$ -algebra, if there is an operadic morphism $f: \mathcal{P} \rightarrow \mathcal{E}nd_V$.

³ The group operation in $\mathbb{S}_n$ is associative with unit id_n . Since the group operation in $\langle \mathbb{S}_n \rangle$ extends linearly, it is distributive, which makes $\mathbb{K}[\mathbb{S}_n]$ a ring non-commutative with unit.

Example 13. An $\mathcal{A}ss$ -algebra is an associative algebra.

In fact, an $\mathcal{A}ss$ -algebra is a vector space V with an operadic morphism $f: \mathcal{A}ss \rightarrow \mathcal{E}nd_V$. Note that, in $\mathcal{A}ss$, $((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3} = 0$.⁴ Thus, if $f_2((\bullet, \bullet)^{[12]}) = m \in \mathcal{E}nd_V(2)$, then

$$\begin{aligned} 0 &= f_3(((\bullet, \bullet), \bullet)^{id_3} - (\bullet, (\bullet, \bullet))^{id_3}) \\ &= f_3((\bullet, \bullet)^{[12]} \circ_1 (\bullet, \bullet)^{[12]} - (\bullet, \bullet)^{[12]} \circ_2 (\bullet, \bullet)^{[12]}) \\ &= f_2((\bullet, \bullet)^{[12]}) \circ_1 f_2((\bullet, \bullet)^{[12]}) - f_2((\bullet, \bullet)^{[12]}) \circ_2 f_2((\bullet, \bullet)^{[12]}) \\ &= m \circ_1 m - m \circ_2 m. \end{aligned}$$

Therefore, (V, m) is an associative algebra.

Although, we use the free associative algebra to construct the operad $\mathcal{A}ss$ and, at some point, these two structures follow different paths. However, there is a tool that makes these paths meet again, i.e., from the operad $\mathcal{A}ss$, we get a description of the free associative algebra.

Definition 17. If G is a group and V is a G -module, we define the space of coinvariants of the action of G on V by :

$$V_G := V / \{x^\sigma - x \mid \sigma \in G, x \in V\}.$$

Definition 18. Given an operad $\mathcal{P}$ and a vector space V , we define the vector space

$$\mathcal{P}(V) = \bigoplus_{n \in \mathbb{N}} (\mathcal{P}(n) \otimes V^{\otimes n})_{\mathbb{S}_n},$$

in which the coinvariant construction $(-)_{\mathbb{S}_n}$ uses the diagonal right action given by:

$$(p \otimes x_1 \cdots x_n)^\sigma = p^\sigma \otimes x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

The right action on $V^{\otimes n}$ is the right action induced by the left action defined in Note 1, i. e., the permutation acts moving the factor of position i to position $\sigma^{-1}(i)$.

This construction allows us to identify the elements below:

$$p^\sigma \otimes x_1 \cdots x_n = (p \otimes x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)})^\sigma \equiv p \otimes x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}. \quad (1.22)$$

Example 14. Considering the operad $\mathcal{A}ss$, we define the map:

$$\begin{aligned} (\cdot, \cdot) : \mathcal{A}ss(V) \otimes \mathcal{A}ss(V) &\longrightarrow \mathcal{A}ss(V) \\ (p \otimes x_1 \cdots x_n) \otimes (q \otimes x_1 \cdots x_m) &\mapsto (((\bullet, \bullet)^{[12]} \circ_2 q) \circ_1 p) \otimes x_1 \cdots x_{n+m-1} \end{aligned} \quad (1.23)$$

with $p \in M_n(\bullet) \times \mathbb{S}_n$ and $q \in M_m(\bullet) \times \mathbb{S}_m$.

⁴ As we said before, in quotient operads, we will not change the notation to represent a class.

The map $(\cdot, \cdot)$ is associative. Indeed, given $v_1 = p \otimes x_1 \cdots x_n$, $v_2 = q \otimes x_1 \cdots x_m$ and $v_3 = s \otimes x_1 \cdots x_k$,

$$\begin{aligned}
 ((v_1, v_2), v_3) &= (((\bullet, \bullet)^{[12]} \circ_2 s) \circ_1 (((\bullet, \bullet)^{[12]} \circ_2 q) \circ_1 p)) \otimes x_1 \cdots x_{k+n+m-2} \\
 &= ((((((\bullet, \bullet)^{[12]} \circ_1 (\bullet, \bullet)^{[12]}) \circ_3 s) \circ_2 q) \circ_1 p) \otimes x_1 \cdots x_{k+n+m-2} \\
 &= ((((((\bullet, \bullet), \bullet)^{id_3} \circ_3 s) \circ_2 q) \circ_1 p) \otimes x_1 \cdots x_{k+n+m-2} \\
 &= (((((\bullet, (\bullet, \bullet))^{id_3} \circ_3 s) \circ_2 q) \circ_1 p) \otimes x_1 \cdots x_{k+n+m-2} \quad (\text{in } Ass, ((\bullet, \bullet), \bullet)^{id_3} = (\bullet, (\bullet, \bullet))^{id_3}) \\
 &= (v_1, (v_2, v_3)).
 \end{aligned}$$

The previous calculations are pretty annoying. We will not discuss their importance from a formal point of view, but as interesting as the calculations is the possibility of associating each element in $Ass(V)$ with a parenthesizing in $F(V)$ (due to (1.22)).

Example 15.

$$\begin{aligned}
 (\bullet, \bullet)^\beta \otimes x_1 x_2 &\mapsto (x_{\beta^{-1}(1)}, x_{\beta^{-1}(2)}) = (x_2, x_1) \quad (\beta = [21]), \\
 (\bullet, (\bullet, (\bullet, \bullet)))^\sigma \otimes x_1 x_2 x_3 x_4 &\mapsto (x_{\sigma^{-1}(1)}, (x_{\sigma^{-1}(2)}, (x_{\sigma^{-1}(3)}, x_{\sigma^{-1}(4)}))) = (x_4, (x_1, (x_3, x_2))) \quad (\sigma = [2431]).
 \end{aligned}$$

If we had not worked with the spaces of coinvariants, we would have had problems like this:

$$(\bullet, (\bullet, \bullet))^{id_3} \otimes cba \mapsto (c, (b, a)),$$

$$(\bullet, (\bullet, \bullet))^{[321]} \otimes abc \mapsto (c, (b, a))$$

and $(\bullet, (\bullet, \bullet))^{id_3} \otimes cba \neq (\bullet, (\bullet, \bullet))^{[321]} \otimes abc$ with $a, b, c \in V$.

The reader already has enough elements to verify that $(Ass(V), (\cdot, \cdot))$ and $(F(V), m)$ are isomorphic, i. e., $(Ass(V), (\cdot, \cdot))$ is the free associative algebra. However, Loday and Vallette [20] show that all the process used for associative algebras till now can be applied to any algebra (Definition 7). We will briefly summarize how Loday and Vallette prove this.

1. An algebra of type P determines a category and they called it P-*alg*. For instance, the associative-*alg* category is the category of associative algebras.
2. Given an algebra of type P, the free P-algebra $\mathcal{P}(V)$ over a vector space V determines a functor $\mathcal{P}$ from the category of vector spaces Vec to itself. They prove the following lemma:

Lemma 1 ([20, Lemma 5.7.1]). The functor $\mathcal{P}$ is a Schur functor whose arity n component is the multilinear part of the free P-algebra $\mathcal{P}[\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n]$. Moreover, $\mathcal{P}$ is an algebraic operad.

For the reader who is not familiar with the operadic theory, it is important to say something about the monoidal definition of operads. In this definition, an operad can be described in

terms of a special functor, the Schur functor (see [20]). We do not recommend spending much energy on this lemma at this moment. In fact, for us, the most important point is that this lemma allows to use a free algebra to obtain an operad.

3. As we made for the associative algebra, we can define the operad generated by the product(s) of the algebra of type P.

Consider the naturals n , such that P has a product $\mu : V^{\otimes n} \rightarrow V$. Let $\mathcal{S} = \{S(n)\}_{n \in \mathbb{N}}$ be the $\mathbb{S}$ -module given by: $S(n)$ is the $K[\mathbb{S}_n]$ -module generated by the symbols that represent all the operations from $V^{\otimes n}$ to V of P, which are called generated operations. If P does not have an operation from $V^{\otimes m}$ to V (for some $m \in \mathbb{N}$), then $S(m) = 0$. We define the set of relators R given by the relations the algebra holds. Thus, we get the operad $\mathcal{F}(\mathcal{S})/(R)$, in which $\mathcal{F}(\mathcal{S})$ is the free operad generated by $\mathcal{S}$.

Example 16 (The operad $\mathcal{PL}$). To give another example of algebraic operad, consider the pre-Lie algebras (Example 5 item (4)). The operad encoding the category of pre-Lie algebras is:

$$\mathcal{PL} = \mathcal{F}(\mathcal{S})/(R),$$

with the $\mathbb{S}$ -module

$$\mathcal{S} = \{S(n)\}_{n \in \mathbb{N}} \text{ given by } S(2) = \mathbb{K}[\mathbb{S}_2] \langle (\bullet * \bullet) \rangle \text{ and } S(m) = 0 \text{ for all } m \neq 2,$$

$$R = \{a_* - a_*^{[213]}\} \text{ and } a_* = (\bullet * \bullet) \circ_2 (\bullet * \bullet) - (\bullet * \bullet) \circ_1 (\bullet * \bullet) = (\bullet * (\bullet * \bullet)) - ((\bullet * \bullet) * \bullet).$$

The operadic product and the $\mathbb{S}$ -module structure are analogous to the ones defined for $\mathcal{Ass}$.

Remark 4. From now on, if p is a parenthesizing, then $p^{id_n} := p$.

4. The following result compares the operads $\mathcal{P}$ and $\mathcal{F}(\mathcal{S})/(R)$.

Lemma 2 ([20, Lemma 5.7.2]). Let P be a type of algebras defined by the $\mathbb{S}$ -module of generating operations $\mathcal{S}$ and the $\mathbb{S}$ -module of relators $R \subset \mathcal{F}(\mathcal{S})$. Then, the operad $\mathcal{P}$ constructed above coincides with the quotient operad $\mathcal{F}(\mathcal{S})/(R)$.

With this, we have:

Proposition 2 ([20, Proposition 5.7.3]). In characteristic zero, a type of algebras whose relations are multilinear determines an operad. The category of algebras over this operad is equivalent to the category of algebras of the given type.

In the proof of Lemma 1, Loday and Vallette show:

$$\mathcal{P}(V) = \bigoplus_{n \in \mathbb{N}} \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n} \simeq \bigoplus_{n \in \mathbb{N}} (\mathcal{P}(n) \otimes V^{\otimes n})_{\mathbb{S}_n}. \quad (1.24)$$

The symbol $\otimes_{\mathbb{S}_n}$ denotes the tensor product on $\mathbb{K}[\mathbb{S}_n]$ and the left action that turns $V^{\otimes n}$ into a left $\mathbb{K}[\mathbb{S}_n]$ -module is the action defined in Note 1. Since $\mathcal{P}$ and $\mathcal{F}(\mathcal{S})/(R)$ coincide, this guarantees what we did for $\mathcal{Ass}(V)$ and $F(V)$ is true and it can be done for any algebra.

Roughly, operads can also be used to describe some properties contained in all the ways of composing the algebra operations. For instance, given a pre-Lie algebra $(V, *)$, we obtain the following linear maps by recurrence:

$$T_{1,1}(x_1; x_2) = x_1 * x_2,$$

$$T_{2,1}(x_1, x_2; x_3) := T_{1,1}(x_1; T_{1,1}(x_2; x_3)) - T_{1,1}(T_{1,1}(x_1; x_2); x_3) = a_*(x_1, x_2, x_3),$$

$$T_{n,1}(x_1, \dots, x_n; x_{n+1}) := T_{1,1}(x_1; T_{n-1,1}(x_2, \dots, x_{n+1})) - \sum_{2 \leq i \leq n} T_{n-1,1}(x_2, \dots, T_{1,1}(x_1; x_i), \dots, x_{n+1})$$

with $x_i \in V$.

Due to the pre-Lie relation (see (1.15)), these maps verify some properties. For example, $T_{2,1}(x_1, x_2; x_3) = a_*(x_1, x_2, x_3) = a_*(x_2, x_1, x_3) = T_{2,1}(x_1, x_2; x_3)$. We present and prove these properties in Section 3.2 by using an operadic approach. However, to do this, we do not use the operad $\mathcal{PL}$. We use an operad isomorphic to the operad $\mathcal{PostLie}$ encoding the category of post-Lie algebras.

Example 17 (The operad $\mathcal{PostLie}$).

$$\mathcal{PostLie} = \mathcal{F}(\mathcal{S})/(R),$$

with the $\mathbb{S}$ -module $S = \{S(n)\}_{n \in \mathbb{N}}$ given by

$$S(2) = \mathbb{K}[\mathbb{S}_2] \langle (\bullet \triangleright \bullet), [\bullet, \bullet] \rangle \text{ and } S(m) = 0 \text{ for all } m \neq 2,$$

$R = \{r_1, r_2, r_3, r_4\}$, in which

$$r_1 = [\bullet, \bullet] + [\bullet, \bullet]^{[21]},$$

$$\begin{aligned} r_2 &= [\bullet, \bullet] \circ_1 [\bullet, \bullet] - [\bullet, \bullet] \circ_2 [\bullet, \bullet] - ([\bullet, \bullet] \circ_1 [\bullet, \bullet])^{[132]} \\ &= [[\bullet, \bullet], \bullet] - [\bullet, [\bullet, \bullet]] - [[\bullet, \bullet], \bullet]^{[132]}, \end{aligned}$$

$$\begin{aligned} r_3 &= (\bullet \triangleright \bullet) \circ_2 [\bullet, \bullet] - [\bullet, \bullet] \circ_1 (\bullet \triangleright \bullet) - ([\bullet, \bullet] \circ_2 (\bullet \triangleright \bullet))^{[213]} \\ &= (\bullet \triangleright [\bullet, \bullet]) - [(\bullet \triangleright \bullet), \bullet] - [\bullet, (\bullet \triangleright \bullet)]^{[213]}, \end{aligned}$$

$$\begin{aligned} r_4 &= (\bullet \triangleright \bullet) \circ_1 [\bullet, \bullet] - a_{\triangleright} + a_{\triangleright}^{[213]} \\ &= ([\bullet, \bullet] \triangleright \bullet) - a_{\triangleright} + a_{\triangleright}^{[213]} \end{aligned}$$

and $a_{\triangleright} = (\bullet \triangleright \bullet) \circ_2 (\bullet \triangleright \bullet) - (\bullet \triangleright \bullet) \circ_1 (\bullet \triangleright \bullet) = (\bullet \triangleright (\bullet \triangleright \bullet)) - ((\bullet \triangleright \bullet) \triangleright \bullet)$. The operadic product and the $\mathbb{S}$ -module structure are analogous to the ones defined for $\mathcal{Ass}$.

In fact, a post-Lie algebra $(V, [\cdot, \cdot], \triangleright)$ with $[\cdot, \cdot] \equiv 0$ is a pre-Lie algebra $(V, \triangleright)$, if $[\cdot, \cdot] \equiv 0$, then relation (1.14)

$$\triangleright \circ_1 [\cdot, \cdot] - a_{\triangleright} + a_{\triangleright}^{[213]} \equiv 0 \Rightarrow a_{\triangleright} - a_{\triangleright}^{[213]} \equiv 0. \quad (1.25)$$

So, it is natural to investigate if we can extend the results obtained for pre-Lie algebras to post-Lie algebras. In this work, we do this for the operad $\mathcal{RT}$, an operad of rooted trees defined by Chapoton and Livernet that is isomorphic to the operad $\mathcal{PL}$. We construct an operad ∇ that extends this operad (Chapter 2). The ∇ -algebras are an extension of the algebras $T_{n,1}$ and we use their properties to prove the particular results given by the special compositions of the post-Lie product (Chapter 3).

THE OPERAD ∇

Post-Lie algebras first occur in an operadic context from the operad $\mathcal{PostLie}$ (Example 17) defined by Vallette in [24].¹ In this chapter, we construct the operad $\nabla \simeq \mathcal{PostLie}$ that, in some sense, is an extension of the pre-Lie operad $\mathcal{RT}$ defined by Chapoton and Livernet in [5] (Proposition 7).² In Section 2.1, we present our alphabet: the magma $M(\mathbb{T})$ built from the set of rooted trees $\mathbb{T}$. The operad ∇ is the quotient of the operad $\mathcal{FT} = \{\mathcal{FT}(n)\}_{n \in \mathbb{N}^*}$ by an operadic ideal $\mathcal{I}$, as we will see in Section 2.4. In Section 2.2, we define the $\mathbb{S}$ -module $\{\mathcal{FT}(n)\}_{n \in \mathbb{N}^*}$. The operadic products of $\mathcal{FT}$ are defined in Section 2.3. In Section 2.5, we prove the isomorphism $\nabla \simeq \mathcal{PostLie}$ and give some corollaries of this isomorphism.

2.1 Magmas and Rooted Trees

As we said before, our alphabet is the magma of rooted trees. So, it is necessary to define what a rooted tree is (Subsection 2.1.1). However, even if the definition is important, we do not use it along the work. In fact, we use a representation that depends if the set that defines the rooted tree is ordered or not, as we will see in Definition 22 (Subsection 2.1.2).

2.1.1 Rooted Trees

To define the operad ∇ , we need sets with properties that give us a “branching” relationship among their elements, much like the one found in natural trees.

Definition 19 ([18, p. 308]). A **rooted tree** t is a finite set of one or more nodes such that:

- ¹ Strictly speaking, the $\mathcal{PostLie}$ operad defined in Example 17 is different from the original defined by Vallette. We adopted an equivalent definition more appropriate for our context.
- ² The symbol ∇ , used to name the operad described below, was chosen because on a manifold M endowed with a flat and constant torsion linear connection ∇ , the vector space $\mathfrak{X}(M)$ of the vector fields on M becomes a post-Lie algebra if one defines the product $\circ$ by $X \circ Y = \nabla_X Y$, for all $X, Y \in \mathfrak{X}(M)$. (see [22]).

1. There is one specially designated node called the **root** of the rooted tree and
2. the remaining nodes (excluding the root) are partitioned into $n \geq 0$ disjoint sets $t_1, \dots, t_n$, and each of these sets, in turn, is a rooted tree. The rooted trees $t_1, \dots, t_n$ are called the subtrees of the root or just subtrees when there is no risk of confusion.

Definition 20. The set of rooted trees will be denoted by $\mathbb{T}$. If $t \in \mathbb{T}$, the set of nodes of t is called $\mathcal{N}(t)$. The set of rooted trees with n nodes will be denoted by $\mathbb{T}(n)$.

Note that,

$$\mathbb{T} = \bigcup_{n \in \mathbb{N}^*} \mathbb{T}(n).$$

Example 18. If we consider the sets $\{H\}$, $\{\mathbb{A}, \mathbb{B}, \mathbb{C}\}$ and $\{\mathbb{A}, \mathbb{B}, C, D, \mathbb{E}, F, G\}$, we have:

- The rooted tree $t = \{H\} \in \mathbb{T}(1)$, since $\{H\}$ is a unit set;
- The rooted tree $q = \{\mathbb{A}, \mathbb{B}, \mathbb{C}\} \in \mathbb{T}(3)$, in which $\mathbb{A}$ is the root of q and the subsets $\{\mathbb{B}\}$ and $\{\mathbb{C}\}$ are the subtrees of the root;
- The rooted tree $s = \{\mathbb{A}, \mathbb{B}, C, D, \mathbb{E}, F, G\} \in \mathbb{T}(7)$, in which $\mathbb{A}$ is the root of s , the subsets $s_1 = \{\mathbb{B}, C\}$ and $s_2 = \{D, \mathbb{E}, F, G\}$ are the root subtrees of s , with $\mathbb{B}$ the root of s_1 and $\mathbb{E}$ the root of s_2 .

Definition 21. A **corolla** is a rooted tree t , such that either t or each root subtree of t is a unit set. A **forest** is a set of zero or more disjoint trees.

Example 19. The rooted trees t and q of Example 18 are corollas.

From the definition of rooted trees, we have the following result.

Proposition 3. Any rooted tree that is not a corolla can be built from corollas.

2.1.2 Representing Rooted Trees

As important as the definition of rooted trees, is the way we describe a rooted tree. The rooted trees given by a unit set are represented by $\bullet$. The description for the other cases is made by recurrence and in some steps.

First, if t is a corolla and $t_1, t_2, \dots, t_n$ are its subtrees, then t is represented by drawing an edge from the root of t to the root of each subtree. In all cases, we draw the rooted trees with the root at the bottom.

Example 20. The rooted trees of Example 18 can be represented in the following way:

The attentive reader probably notice that there is more than one representation for some rooted trees t .

Example 21. Considering s in Example 20,

are both possible.

To solve this problem, consider the following definition.

Definition 22. If, in Definition 19 item 2, the relative order of the subtrees $t_1, \dots, t_n$ is important, then we say the rooted tree is an ordered tree or **planar** tree. If the respective ordering of subtrees does not matter, the rooted tree is said to be a **non-planar** tree.

In a planar tree, it makes sense to call t_2 the “second subtree” of the root, etc. The name planar tree comes from the relevance of the way of embedding the rooted tree on a plane.

We can finish with the description of the representation of rooted trees adopted in this thesis. This representation considers if the rooted trees are planar or non-planar. If the rooted tree is planar, we draw its subtrees from left to right by using its order. If the rooted tree is non-planar, all possible representations coincide.

Example 22. Back to Example 20, if s is the planar tree in which s_1 is the first subtree and s_2 is the second subtree, then the representation of s is

In this case, the order adopted in s for the subtree consider $\{D\}$ the first subtree, $\{F\}$ the second subtree and $\{G\}$ the third one.

If s is a non-planar tree, then

coincide.

Remark 5. From now on we will not distinguish the rooted tree of its representation. We will make it clear in the context whether we are working with planar or non-planar trees.

The definitions bellow fit into the both cases of rooted trees (planar or non-planar).

Definition 23. Consider Definition 19, if r is the root of t and r_i is the root of t_i ($1 \leq i \leq n$), then $r_1, \dots, r_n$ are **children** of r and r is a **parent** of $r_1, \dots, r_n$. The edge that attaches r_i to r is called **outgoing edge** of r_i and the **incoming edge** of r .

It follows from the definition that every node of a rooted tree is the root of a subtree of the whole rooted tree.

Definition 24. The number of subtrees of a node is called the **degree** of that node. A node of degree zero is called an **external node** or a **leaf**. A non-external node is called an **internal node** or a **branch node**.

Example 23. For instance,

More informations about rooted trees can be found in Knuth [18] .

2.1.3 The Magma $M(\mathbb{T})$

The operad that is constructed in this work is also an operad of a Lie algebra. Thus, it is necessary to define symbols that plays the role of the Lie bracket. These symbols are given by the magma $M(\mathbb{T})$ (Definitions 13 and 14). As it occurred before, the definitions bellow fit into the both cases of rooted trees (planar or non-planar).

Back to the construction in Definition 14, we have a grading given by $M(\mathbb{T}) = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{T})$.

However, we also need another grading for our work. We define a new grading based on the total number of nodes that each parenthesizing and each rooted tree contain.

Consider $p \in M(\mathbb{T})$ a parenthesizing of $t_1, \dots, t_k \in \mathbb{T}$ and $\mathcal{N}(t_i)$ the set of nodes of t_i (Definition 20). The set $\bigcup \mathcal{N}(t_i)$ is also denoted by $\mathcal{N}(p)$.

Remark 6. We denote by $\mathbb{M}(n)$ the set of the elements $p \in M(\mathbb{T})$, such that $p \in \mathbb{T}(n)$ or p is the parenthesizing of $t_1, \dots, t_k \in \mathbb{T}$ with $|\mathcal{N}(p)| = n$.

Note that,

$$M(\mathbb{T}) = \bigcup_{n \in \mathbb{N}^*} \mathbb{M}(n).$$

Example 25. Back to Example 24, with the new grading, we have:

$$\left[\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] \in \mathbb{M}(7), \begin{array}{c} \bullet \\ | \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \in \mathbb{M}(5) \text{ and } \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \in \mathbb{M}(3).$$

$$\text{Example 26. } \mathbb{M}(2) = \left\{ \begin{array}{c} \bullet \\ | \\ \bullet \end{array}, [\bullet, \bullet] \right\}, \mathbb{M}(3) = \left\{ \begin{array}{c} \bullet \\ | \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet \end{array}, [[\bullet, \bullet], \bullet], [\bullet, [\bullet, \bullet]], \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array}, \bullet \right], \left[\bullet, \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] \right\}.$$

Definition 25. A **label** of $p \in \mathbb{M}(n)$ is a function from the set of nodes $\mathcal{N}(p)$ to a set Y . If t is a rooted tree with a label, we called it a **labeled rooted tree**.

We also use the representations of rooted trees to represent an element p of $M(\mathbb{T})$ with a label. It is represented by the element p with the image next to the corresponding node. This representation also consider if the rooted trees are planar or non-planar trees as we will see in the next example.

Example 27. Considering the Example 18, given the rooted tree q and the bijection

$$\begin{aligned} f: \{\mathbf{A}, \mathbf{B}, \mathbf{C}\} &\longleftrightarrow \{1, 2, 3\} \\ \mathbf{A} &\mapsto 3 \\ \mathbf{B} &\mapsto 1 \\ \mathbf{C} &\mapsto 2, \end{aligned}$$

we have (q, f) is a labeled rooted tree.

If q is a planar tree in which $\{\mathbf{B}\}$ is the first subtree and $\{\mathbf{C}\}$ is the second one, we represent the labeled planar tree by:

$$\begin{array}{c} \bullet \quad 1 \quad \bullet \quad 2 \\ \diagdown \quad \diagup \\ \bullet \quad 3 \end{array} . \quad (2.1)$$

If q is a non-planar tree we have:

$$\begin{array}{c} \bullet \quad 1 \quad \bullet \quad 2 \\ \diagdown \quad \diagup \\ \bullet \quad 3 \end{array} = \begin{array}{c} \bullet \quad 2 \quad \bullet \quad 1 \\ \diagdown \quad \diagup \\ \bullet \quad 3 \end{array} .$$

For $p = \left[\begin{array}{c} \bullet \\ \bullet \end{array}, \bullet \right]$ with $\bullet$ representing the rooted tree t (Example 18) and representing the planar tree s (Example 22), we have

$$\left[\begin{array}{c} 2 \quad 4 \quad 5 \quad 6 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 3 \quad 7 \\ \bullet \quad \bullet \\ 1 \quad \bullet \end{array}, \bullet \right] \quad (2.2)$$

corresponds to parenthesizing $[t, s]$ with the label

$$\begin{aligned} \{H, \mathbb{A}, \mathbf{B}, C, D, \mathbf{E}, F, G\} &\longleftrightarrow \{1, 2, 3\} \\ H &\mapsto 1 \\ \mathbb{A} &\mapsto 8 \\ \mathbf{B} &\mapsto 3 \\ C &\mapsto 2 \\ D &\mapsto 4 \\ \mathbf{E} &\mapsto 7 \\ F &\mapsto 5 \\ G &\mapsto 6. \end{aligned}$$

The labels are used to define the $\mathbb{S}$ -module structure in $\mathcal{FT}$. So, the most important labels are when $Y = [n]$. Although in this thesis we also work with non-planar trees, our main tools are planar trees. Thus we define a special notation for labels from a planar tree or a parenthesizing of planar trees to $[n]$.

Definition 26. Consider $p \in \mathbb{M}(n)$ and $Q = \{f : f \text{ is a label from } p \text{ to } [n]\}$. We denote by $L(p, [n])$ the set in which its elements are the representations of (p, f) with $f \in Q$ and p a planar tree or a parenthesizing of planar trees.

Example 28. $L([\bullet, \bullet], [2]) = \{[\bullet 1, \bullet 2], [\bullet 2, \bullet 1]\}$;

$$L\left(\begin{array}{c} \bullet \quad \bullet \\ \bullet \end{array}, [3]\right) = \left\{ \begin{array}{c} \bullet \quad 3 \quad 2 \\ \bullet \quad \bullet \quad \bullet \\ 1 \end{array}, \begin{array}{c} \bullet \quad 2 \quad 3 \\ \bullet \quad \bullet \quad \bullet \\ 1 \end{array}, \begin{array}{c} \bullet \quad 3 \quad 1 \\ \bullet \quad \bullet \quad \bullet \\ 2 \end{array}, \begin{array}{c} \bullet \quad 1 \quad 3 \\ \bullet \quad \bullet \quad \bullet \\ 2 \end{array}, \begin{array}{c} \bullet \quad 1 \quad 2 \\ \bullet \quad \bullet \quad \bullet \\ 3 \end{array}, \begin{array}{c} \bullet \quad 2 \quad 1 \\ \bullet \quad \bullet \quad \bullet \\ 3 \end{array} \right\};$$

$$\begin{aligned} L\left(\left[\begin{array}{c} \bullet \\ \bullet \end{array}, \bullet\right], [3]\right) &= \left\{ \left[\begin{array}{c} \bullet \quad 2 \\ \bullet \quad \bullet \\ 1 \end{array}, \bullet \quad 3 \right], \left[\begin{array}{c} \bullet \quad 1 \\ \bullet \quad \bullet \\ 2 \end{array}, \bullet \quad 3 \right], \left[\begin{array}{c} \bullet \quad 2 \\ \bullet \quad \bullet \\ 3 \end{array}, \bullet \quad 1 \right], \left[\begin{array}{c} \bullet \quad 3 \\ \bullet \quad \bullet \\ 2 \end{array}, \bullet \quad 1 \right], \left[\begin{array}{c} \bullet \quad 3 \\ \bullet \quad \bullet \\ 1 \end{array}, \bullet \quad 2 \right], \right. \\ &\quad \left. \left[\begin{array}{c} \bullet \quad 1 \\ \bullet \quad \bullet \\ 3 \end{array}, \bullet \quad 2 \right] \right\}. \end{aligned}$$

2.2 The $\mathbb{S}$ -module $\mathcal{FT}$

Let $\mathcal{L}(n)$ be the set defined by

$$\mathcal{L}(n) = \bigcup_{p \in \mathbb{M}(n)} L(p, [n]), \quad n \in \mathbb{N}^*. \quad (2.3)$$

Example 29. $\mathcal{L}(1) = \{ \bullet_1 \},$

$$\mathcal{L}(2) = \left\{ \begin{array}{c} 1 \\ \vdots \\ 2 \end{array}, \begin{array}{c} 2 \\ \vdots \\ 1 \end{array}, [\bullet_1, \bullet_2], [\bullet_2, \bullet_1] \right\},$$

$$\mathcal{L}(3) = \left\{ \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \bullet_3 \end{array}, \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ \bullet_2 \end{array}, \dots, \begin{array}{c} 1 \quad 3 \\ \vdots \quad \vdots \\ 2 \quad 1 \\ \vdots \quad \vdots \\ 3 \quad 2 \end{array}, \dots, \left[\begin{array}{c} 1 \\ \vdots \\ 2 \end{array}, \bullet_3 \right], \dots, \left[\left[\bullet_2, \bullet_1 \right], \bullet_3 \right] \right\}.$$

We denote by $\mathcal{FT} = \{\mathcal{FT}(n)\}_{n \in \mathbb{N}^*}$, the family of vector spaces $\mathcal{FT}(n) = \langle \mathcal{L}(n) \rangle$. Roughly, $\mathcal{FT}(n)$ is the vector space generated by planar trees and parenthesizings of planar trees.

Since we have a family of vector spaces, it is necessary to define an $\mathbb{S}$ -module structure on $\mathcal{FT}$. The idea is $\mathbb{S}_n$ acts by changing the position of the labels. However, this approach brings us two problems:

- First, formally speaking, what do we mean by “changing the position of the labels”?
- Second, is this change of position compatible with the operadic compositions that we define next?

To help us with this, given $y \in L(p, [n])$, we correspond y to an unique permutation $\sigma = [\sigma(1) \cdots \sigma(n)] \in \mathbb{S}_n$ by using a special label.

For each planar tree $t \in \mathbb{T}(n)$ (Definition 22), we define a label $\mathfrak{l}_t(x_1, \dots, x_n)$ between a sequence $(x_1, \dots, x_n)$ and t by recurrence. If t is a root or a corolla then:

$$\mathfrak{l}_t(x_1) := \bullet_{x_1}, \quad \mathfrak{l}_t(x_1, \dots, x_n) := \begin{array}{c} x_1 \quad x_2 \quad x_3 \quad \dots \quad x_{n-1} \\ \diagdown \quad \diagup \quad \diagup \quad \dots \quad \diagup \\ \bullet_{x_n} \end{array} \quad (2.4)$$

In other cases, the definition is by recurrence. Consider $t_1, \dots, t_m$ the subtrees of the root $r \in t$ in which t_i is the i -th subtree given by the order defined in t . We denote the root of t_j by r_j and $k_j = |\mathcal{N}(t_j)|$ ($1 \leq j \leq m$). The following partition of the set $\{x_1, \dots, x_n\}$ is done:

$$\left\{ x_1 \quad \dots \quad x_{k_1} \mid x_{k_1+1} \quad \dots \quad x_{k_1+k_2} \mid \dots \mid x_{k_{m-1}+1} \quad \dots \quad x_{k_{m-1}+k_m} \mid x_n \right\}. \quad (2.5)$$

Example 30. Consider the planar tree $t =$ and the sequence $(x_1, x_2, x_3, x_4, x_5, x_6)$. We have two root subtrees of t : $t_1 =$, $t_2 =$. So, we partition the set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ in the following way: $\{x_1, x_2 | x_3, x_4, x_5 | x_6\}$.

Suppose all the t_i 's are corollas. The labeled rooted tree $\mathfrak{l}_t(x_1, \dots, x_n)$ is given by the following rule:

1. We associate x_n to r ;
2. The subtree t_j is labeled by $\mathfrak{l}_{t_j}(x_{k_{j-1}+1}, \dots, x_{k_{j-1}+k_j})$ as in 2.4.

Example 31. Back to Example 30, the label $\mathfrak{l}_t(x_1, x_2, x_3, x_4, x_5, x_6)$ is obtained by corresponding

x_6 to the root of t , $\mathfrak{l}_{t_1}(x_1, x_2) =$ and $\mathfrak{l}_{t_2}(x_3, x_4, x_5) =$. Finally,

$$\mathfrak{l}_t(x_1, x_2, x_3, x_4, x_5, x_6) =$$

If t is a planar tree and $t_1, \dots, t_n$ are its root subtrees in which, for some $j \in [n]$, t_j is not a corolla, again we partitioned the set $\{x_1, \dots, x_n\}$ as we made in (2.5) and label the root with x_n . For the subtrees that are corollas we label by usign (2.4) as we did before. For the subtrees t_j we partition the set $\{x_{k_{j-1}+1}, \dots, x_{k_{j-1}+k_j}\}$, considering the number of nodes of each root subtree in t_j , and repeat the stages (1) and (2).

Example 32. If $s =$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$, we have two root subtrees: $s_1 =$, $s_2 =$. Then, we partition: $\{x_1, x_2 | x_3, x_4, x_5, x_6 | x_7\}$

The first tree is a corolla: $l_{s_1}(x_1, x_2) =$. For the second rooted tree, we do: $\{x_3 | x_4, x_5 | x_6\}$.

So, $l_{s_2}(x_3, x_4, x_5, x_6) =$. Finally, $l_s(x_1, x_2, x_3, x_4, x_5, x_6, x_7) =$.

Note that this label is built based on the fact that every rooted tree is a corolla or is built from corollas.

If $p \in \mathbb{M}(n) \setminus \mathbb{T}(n)$, a label between $(x_1, \dots, x_n)$ and p is similarly defined. Suppose that p is a parenthesizing of $t_1 \in \mathbb{T}(k_1), \dots, t_m \in \mathbb{T}(k_m)$, whose trees are ordered from left to right. The set $\{x_1, \dots, x_n\}$ is partitioned:

$$\left\{ x_1 \cdots x_{k_1} \mid x_{k_1+1} \cdots x_{k_1+k_2} \mid \cdots \mid x_{k_{m-1}+1} \cdots x_{k_{m-1}+k_m} \right\},$$

with $k_j = |\mathcal{N}(t_j)|$, $1 \leq j \leq m$. Thus, the label is represented by the parenthesizing of the planar trees t_j , in which each rooted tree is labeled by $l_{t_j}(x_{k_{j-1}+1}, \dots, x_{k_{j-1}+k_j})$. To simplify the notation we also denoted the label between p and $(x_1, \dots, x_n)$ by $l_p(x_1, \dots, x_n)$.

Example 33. Considering $p =$ and the sequence $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$, we have two planar trees $t_1 =$, $t_2 =$ $\Rightarrow$ the partition $\{x_1, x_2, x_3 | x_4, x_5, x_6, x_7\}$

$$\Rightarrow l_{t_1}(x_1, x_2, x_3) =$$
 , $l_{t_2}(x_4, x_5, x_6, x_7) =$.

Thus, $l_p(x_1, x_2, x_3, x_4, x_5, x_6, x_7) =$.

Definition 27. Since each permutation determines a unique sequence, for each $t \in \mathbb{M}(n)$ (Definition 6), we define the bijections:

$$\begin{aligned} l_{t,n} : \mathbb{S}_n &\longrightarrow L(t, [n]) \\ \sigma &\mapsto l_t(\sigma(1), \dots, \sigma(n)). \end{aligned}$$

Example 34. If $t =$ (see Example 32) and $\sigma = [1346527]$, then $l_{t,7}(\sigma) =$.

Given $p =$, (see Example 33) and the same σ :

$$l_{p,7}(\sigma) = \left[\begin{array}{c} 1 \\ \bullet \\ 3 \\ \bullet \\ 4 \end{array}, \begin{array}{c} 6 \\ \bullet \\ 7 \end{array}, \begin{array}{c} 5 \\ \bullet \\ 2 \end{array} \right].$$

The $\mathbb{S}$ -module structure of $\mathcal{FT}$ is given by the following right action:

$$\begin{aligned} \cdot : \mathbb{S}_n \times \mathcal{FT}(n) &\longrightarrow \mathcal{FT}(n) \\ (\sigma, s) &\mapsto s^\sigma = l_{t,n}(\sigma^{-1} \circ \beta) \end{aligned} \quad (2.6)$$

where $l_{t,n}(\beta) = s$, $t \in \mathbb{M}(n)$.

Of course, it is sufficient to set in $\mathcal{L}(n)$ and extend to $\mathcal{FT}$ by linearity.

Example 35. For $l_{t,3}([132]) =$, $l_{p,3}([321]) = \left[\begin{array}{c} 3 \\ \bullet \\ 2 \end{array}, \bullet^1 \right] \in \mathcal{L}(3)$. Given $\sigma = [321]$:

$$\begin{aligned} \begin{array}{c} 1 \\ \bullet \\ 2 \end{array} \begin{array}{c} 3 \\ \bullet \end{array} &\stackrel{\sigma}{=} l_{t,3}(\sigma^{-1} \circ [132]) = l_{t,3}([312]) = \begin{array}{c} 3 \\ \bullet \\ 2 \end{array} \begin{array}{c} 1 \\ \bullet \end{array}, \\ \left[\begin{array}{c} 3 \\ \bullet \\ 2 \end{array}, \bullet^1 \right] &\stackrel{\sigma}{=} l_{c,3}(id_3) = \left[\begin{array}{c} 1 \\ \bullet \\ 2 \end{array}, \bullet^3 \right]. \end{aligned}$$

Proposition 4. For any $s \in L(t, [m])$ there is $\beta \in \mathbb{S}_m$, such that $s = (l_{t,m}(id_m))^\beta$.

Proof. In fact, by definition of $l_{t,m}$ there is $\sigma \in \mathbb{S}_m$ such that: $l_{t,m}([\sigma(1)\sigma(2)\cdots\sigma(m)]) = s$. So,

$$(l_{t,m}(id_m))^{\sigma^{-1}} = l_{t,m}(\sigma \circ id_m) = l_{t,n}(\sigma) = s.$$

□

Remark 7. With this proposition we conclude that (2.6) is a transitive group action. It allows us to prove some properties just for $l_{t,m}(id_m)$ and this argument is used along the text without further mentions.

2.3 The operad $\mathcal{FT}$

The operadic product defined in $\mathcal{FT}$ is based on the products defined by Chapoton and Livernet in [5] over a family of non-planar trees $\mathcal{RT} = \{\mathcal{RT}(n)\}$ (Definition 22).

Definition 28. The operad $\mathcal{RT} = \{\mathcal{RT}(n)\}$ is given by the free $\mathbb{Z}$ -modules $\mathcal{RT}(n)$ generated by labeled non-planar trees t of n nodes with labels from $\mathcal{N}(t)$ (Definition 20) to $[n]$. $\mathbb{S}_n$ acts on $\mathcal{RT}(n)$ by changing the position of the labels.

Let $s \in \mathcal{RT}(n)$ and $u \in \mathcal{RT}(m)$ be non-planar trees, i a node of s . Consider $C_{i,s}$ the set of all subtrees of i in s , if ϕ is a function from $C_{i,s}$ to the set $\mathcal{N}(u)$, we denote by

$$s \circ_{\phi}^i u \in \mathcal{RT}(n+m-1)$$

the non-planar tree obtained by replacing the node i of s by the rooted tree u . In case the outgoing edge of i exists, it is attached to the root of u . In case the incoming edges of i exists (Definition 23), they are connected to the nodes of u following the map ϕ . The root of $s \circ_{\phi}^i u$ is the root of u if i is a root of s and it is the root of s in other cases.

The composition $s \circ_i u$ is given by:

$$s \circ_i u = \sum_{\phi: C_{i,s} \rightarrow \mathcal{N}(u)} s \circ_{\phi}^i u.$$

Example 36. Consider $\begin{array}{c} 3 \quad 1 \\ \diagdown \quad \diagup \\ 2 \end{array} \in \mathcal{RT}(3)$ and $\begin{array}{c} 1 \\ | \\ 2 \end{array} \in \mathcal{RT}(2)$. The steps below describes how to obtain the operadic composition $\begin{array}{c} 3 \quad 1 \\ \diagdown \quad \diagup \\ 2 \end{array} \circ_2 \begin{array}{c} 1 \\ | \\ 2 \end{array}$:

1. The summands of $\begin{array}{c} 3 \quad 1 \\ \diagdown \quad \diagup \\ 2 \end{array} \circ_2 \begin{array}{c} 1 \\ | \\ 2 \end{array}$ are in $\mathcal{RT}(4)$. So, it is necessary to relabel $\begin{array}{c} 3 \quad 1 \\ \diagdown \quad \diagup \\ 2 \end{array}$ and $\begin{array}{c} 1 \\ | \\ 2 \end{array}$. Roughly, this is done considering that the node labeled by 2 “will be changed by the

rooted tree $\begin{array}{c} 1 \\ | \\ 2 \end{array}$ ”. The two rooted trees relabeled are: $\begin{array}{c} 4 \quad 1 \\ \diagdown \quad \diagup \\ 2 \end{array}$ and $\begin{array}{c} 2 \\ | \\ 3 \end{array}$.

2. We consider the forest obtained by removing the node labeled by 2 in $\begin{array}{c} 4 \quad 1 \\ \diagdown \quad \diagup \\ 2 \end{array}$ and all the

functions ϕ from the subtrees of $\bullet 2$ to the nodes of $\begin{array}{c} 2 \\ | \\ 3 \end{array}$. This functions determine all the summands of the composition, since they indicate where the incoming edges of 2 are connected:

$$\begin{array}{c} 3 \quad 1 \\ \diagdown \quad \diagup \\ 2 \end{array} \circ_2 \begin{array}{c} 1 \\ | \\ 2 \end{array} = \begin{array}{c} 4 \quad 1 \quad 2 \\ \diagdown \quad \diagup \quad | \\ 3 \end{array} + \begin{array}{c} 1 \\ | \\ 4 \quad 2 \\ \diagdown \quad \diagup \\ 3 \end{array} + \begin{array}{c} 4 \\ | \\ 1 \quad 2 \\ \diagdown \quad \diagup \\ 3 \end{array} + \begin{array}{c} 4 \quad 1 \\ \diagdown \quad \diagup \\ 2 \\ | \\ 3 \end{array}.$$

$\phi(1)=3 \quad \phi(1)=2 \quad \phi(1)=3 \quad \phi(1)=2$
 $\phi(4)=3 \quad \phi(4)=3 \quad \phi(4)=2 \quad \phi(4)=2$

3. As we can see, the root of each summand is the root of $\begin{array}{c} 2 \\ \bullet \\ 3 \end{array}$.

In Example 36, each summand of $\begin{array}{c} 3 & 1 \\ \bullet & \bullet \\ 2 \end{array} \circ_2 \begin{array}{c} 1 \\ \bullet \\ 2 \end{array}$ is a label rooted tree of $\mathcal{RT}(4)$. Due to the relabel given by (1), except for $\bullet_2 \in \begin{array}{c} 3 & 1 \\ \bullet & \bullet \\ 2 \end{array}$, there is a bijection between the nodes of $\begin{array}{c} 3 & 1 \\ \bullet & \bullet \\ 2 \end{array}$ and $\begin{array}{c} 1 \\ \bullet \\ 2 \end{array}$ and the nodes of each summand of $\begin{array}{c} 3 & 1 \\ \bullet & \bullet \\ 2 \end{array} \circ_2 \begin{array}{c} 1 \\ \bullet \\ 2 \end{array}$. This relabel is not only intuitive but also very important to guarantee that the operadic compositions in $\mathcal{RT}$ are compatible with the $\mathbb{S}$ -module sutructure. For the operad $\mathcal{FT}$, we follow a similar idea.

If the aim is to define an operadic product $s \circ_i v$, with $s \in \mathcal{L}(n)$, $v \in \mathcal{L}(m)$ and $i \in [n]$, we change the labels of v by $i, i+1, \dots, i+m-1$ and the labels of s that are greater than or equal to $i+1$ by $i+m, i+m+1, \dots, n+m-1$. In other words, this is done by the bijections defined below.

Definition 29. Consider an ordered pair $(s, v) \in \mathcal{L}(n) \times \mathcal{L}(m)$ and $i \in [n]$. The **relabel** of (s, v) , denoted by (s', v'') , is a change of label given by the following bijections. Let $f: [n] \rightarrow [m+n-1] \setminus \{i+1, i+2, \dots, i+m-1\}$, $g: [m] \rightarrow \{i, i+1, \dots, i+m-1\}$ defined by:

$$f(x) = \begin{cases} x, & 1 \leq x \leq i \\ x+m-1, & i+1 \leq x \leq n \end{cases} \quad \text{and } g(x) = x+i-1, \forall n, m \in \mathbb{N}^*, i \in [n].$$

By using the maps $l_{t,n}$ (Definition 27), if $s \in L(t_1, [n])$, $v \in L(t_2, [m])$, then there are $\sigma \in \mathbb{S}_n$ and $\beta \in \mathbb{S}_m$ such that, $l_{t_1,n}(\sigma) = s$, $l_{t_2,m}(\beta) = v$. Thus, s' is given by $l_{t_1}(f(\sigma(1)), \dots, f(\sigma(n)))$ and v'' by $l_{t_2}(g(\beta(1)), \dots, g(\beta(m)))$.

Indeed, they are the relabels needed to define $s \circ_i v$.

Example 37. If $s = l_{t_1,5}([52341]) = \begin{array}{c} 5 \\ \bullet \\ 2 \end{array} \begin{array}{c} 3 & 4 \\ \bullet & \bullet \\ 1 \end{array}$, $[u_1, u_2] = l_{t_2,3}([312]) = \begin{array}{c} \bullet \\ \bullet \\ 1 \end{array} \begin{array}{c} 3 \\ \bullet \\ 2 \end{array}$, $i = 2$ and (s, u) is our pair, then

$$s' = l_{t_1}(f(5), f(2), f(3), f(4), f(1)) = \begin{array}{c} 7 \\ \bullet \\ 2 \end{array} \begin{array}{c} 5 & 6 \\ \bullet & \bullet \\ 1 \end{array} \quad \text{and } [u_1'', u_2''] = l_{t_2}(g(3), g(1), g(2)) = \begin{array}{c} \bullet \\ \bullet \\ 2 \end{array} \begin{array}{c} 4 \\ \bullet \\ 3 \end{array}.$$

We have tools enough to define the operadic products in $\mathcal{FT}$. First, we consider the $\mathbb{S}_n$ submodule $\langle B(n) \rangle$ of $\mathcal{FT}(n)$ (Definition 2) with

$$B(n) = \{s \in \mathcal{L}(n); s \in L(t, [n]), \text{ with } t \in \mathbb{T}(n) \text{ or } t \in M_2(\mathbb{T})\},$$

i.e., the subset of labeled planar trees and brackets of two labeled planar trees.³

³ From this moment we use both ways of graduating the magma $M(\mathbb{T})$, see Definitions 14 and 6.

Example 38. Considering the Example 29, we have $B(1) = \mathcal{L}(1)$, $B(2) = \mathcal{L}(2)$ and

$$B(3) = \left\{ \begin{array}{c} \bullet 1 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 3 \end{array}, \begin{array}{c} \bullet 1 \quad \bullet 3 \\ \diagdown \quad \diagup \\ \bullet 2 \end{array}, \dots, \begin{array}{c} \bullet 1 \quad \bullet 3 \\ | \quad | \\ \bullet 2 \quad \bullet 1 \\ | \quad | \\ \bullet 3 \quad \bullet 2 \end{array}, \dots, \left[\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array}, \bullet 3 \right] \right\} \subsetneq \mathcal{L}(3),$$

since, for instance, $[[\bullet 1, \bullet 2], \bullet 3] \notin B(3)$.

Next, we define the bilinear maps $\circ_i: \langle B(n) \rangle \times \langle B(m) \rangle \rightarrow \mathcal{FT}(n+m-1)$. After, in these maps, we prove that equivariance, unit, parallel and sequential axioms hold. This enables to extend the maps $\circ_i$ to $\mathcal{FT}$, i. e., we use $\{B(n)\}_{n \in \mathbb{N}^*}$ to generates the operad $\mathcal{FT}$.

As in the operad $\mathcal{RT}$ (Definition 28) we define the operadic compositions in $\mathcal{FT}$ as sums of the elements in $\mathcal{L}(n)$. Next, we define each summand for the main case: when the elements are in $B(n)$.

Definition 30. For $s \in L(t_1, [n])$, $v \in L(t_2, [m])$ ($t_1 \in \mathbb{T}(n)$, $t_2 \in \mathbb{T}(m)$) labeled planar trees and $u = [u_1, u_2] \in B(k)$ a bracket of two labeled planar trees, let (s, v) and (s, u) be ordered pairs and $i \in [n]$ fixed. Consider $C_{i,s'}$ the set of all subtrees of i in s' , if ϕ is a function from $C_{i,s'}$ to $\mathcal{N}(v'')$, we denote by

$$s' \circ_{\phi}^i v'' \quad (2.7)$$

the tree obtained by replacing the node i of s' by the tree v'' . Defined by the following rules.

In case the outgoing edge of i exists, it is attached to the root of v'' .

$$\begin{array}{c} \bullet 2 \\ | \\ \bullet 2 \quad \bullet 3 \\ | \quad | \\ \bullet 1 \quad \bullet 2 \end{array} \circ_{\phi}^2 \begin{array}{c} \bullet 2 \\ | \\ \bullet 3 \end{array} = \begin{array}{c} \bullet 2 \\ | \\ \bullet 3 \\ | \\ \bullet 1 \end{array}$$

For instance, $s = \begin{array}{c} \bullet 2 \\ | \\ \bullet 1 \end{array}$, $v = \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array}$, ϕ is the empty function.

In case the incoming edges of i exists, they are connected to the nodes of v'' following the map ϕ . For each node of v'' , all the connections are at left of the first child of $\phi(j)$ ($j \in [n]$) and preserves the order, from left to right, which the subtrees of i appear in s' .

$$\begin{array}{c} \bullet 4 \quad \bullet 3 \\ \diagdown \quad \diagup \\ \bullet 1 \end{array} \circ_{\phi}^1 \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} = \begin{array}{c} \bullet 4 \quad \bullet 3 \quad \bullet 1 \\ \diagdown \quad | \quad \diagup \\ \bullet 2 \end{array}$$

For instance, $s = \begin{array}{c} \bullet 3 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 1 \end{array}$, $v = \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array}$, $\phi: C_1 \rightarrow \mathcal{N}(v'')$ is given by $\phi(3) = \phi(4) = 2$.

The root of $s' \circ_{\phi}^i v''$ is the root of v'' if i is a root of s' and it is the root of s' in other cases.

Let θ be a function from $C_{i,s'}$ to $\mathcal{N}(u'')$, consider the forest obtained by removing the root i . If i is a root of s' , then

$$s' \circ_{\theta}^i [u_1'', u_2''] \quad (2.8)$$

is a bracket of two rooted trees obtained by connecting the incoming edges of i to the nodes of u_1'' and u_2'' by following the map θ . The attachings involving the function θ are done as in 2.7.

For example, $s = \begin{array}{c} 4 \\ | \\ 2 \\ | \backslash \\ 3 \quad 1 \end{array}$, $[u_1, u_2] = \left[\begin{array}{c} 2 \\ | \backslash \\ 4 \quad 3 \end{array}, \begin{array}{c} 1 \end{array} \right]$ and $\theta: C_3 \rightarrow \mathcal{N}(u'')$ given by $\theta(x) = \begin{cases} 6, & x=2 \\ 3, & x=1 \end{cases}$.

If i is not a root of s' , then

$$s' \circ_{\theta}^i (u_1'', u_2'') \quad (2.9)$$

is the tree obtained by replacing the node i and its outgoing edge by the trees u_1'' and u_2'' , both with two new edges connected to the parent of i (Definition 23), in the same order they appear in the ordered pair (left $\rightarrow$ right). The incoming edges of i are connected to the nodes of u_1'' and u_2'' by following the map θ . The root of $s' \circ_{\theta}^i (u_1'', u_2'')$ is the root of s' .

For instance, $s = \begin{array}{c} 5 \\ | \backslash \\ 2 \quad 3 \\ | \backslash \\ 1 \quad 4 \end{array}$, $[u_1, u_2] = \left[\begin{array}{c} 2 \\ | \backslash \\ 4 \quad 3 \end{array}, \begin{array}{c} 1 \end{array} \right]$ $\theta: C_2 \rightarrow \mathcal{N}(u_1'') \cup \mathcal{N}(u_2'')$ given by $\theta(8) = 5$.

The attachings involving the function θ are done as in 2.7.

Definition 31. Let $s \in L(t_1, [n])$, $v \in L(t_2, [m])$ ($t_1 \in \mathbb{T}(n)$, $t_2 \in \mathbb{T}(m)$), $u = [u_1, u_2] \in B(k)$ and $i \in [n]$ fixed, we define:

$$s \circ_i v = \sum_{\phi} s' \circ_{\phi}^i v'', \quad (2.10)$$

$$s \circ_i [u_1, u_2] = \begin{cases} \sum_{\theta} s' \circ_{\theta}^i [u_1'', u_2''], & \text{if } i \text{ is the root of } s' \\ \sum_{\theta} (s' \circ_{\theta}^i (u_1'', u_2'') - s' \circ_{\theta}^i (u_2'', u_1'')), & \text{if } i \text{ is not the root of } s' \end{cases} \quad (2.11)$$

in which the sums are over all the functions $\phi: C_{i,s'} \rightarrow \mathcal{N}(v'')$ and $\theta: C_{i,s'} \rightarrow \mathcal{N}(u'')$.

If $s = [w_1, w_2]$ is a bracket of two labeled rooted trees, the compositions are similar to 2.10 and 2.11. In case the node belongs to w_2 , each summand is a bracket of w_1' with the

composition of $w'_2 \circ_\phi^i v''$ ($\circ_\phi^i$ given by 2.10 or 2.11). In case the node belongs to w_1 , it is analogous:

$$[w_1, w_2] \circ_i [u_1, u_2] = \begin{cases} \sum_{\theta} [w'_1 \circ_{\theta} [u''_1, u''_2], w'_2], & i \text{ is the root} \\ \sum_{\theta} ([w'_1 \circ_{\theta} (u''_1, u''_2), w'_2] - [w'_1 \circ_{\theta} (u''_2, u''_1), w'_2]), & i \text{ is not the root} \end{cases} \quad (2.12)$$

Example 39.

$$\begin{aligned} & \left[\begin{array}{c} 6 \quad 7 \\ \diagdown \quad \diagup \\ 2 \quad 3 \\ \diagup \quad \diagdown \\ 4 \end{array}, \begin{array}{c} 5 \\ | \\ 1 \end{array} \right] \circ_2 \begin{array}{c} 2 \\ | \\ 1 \end{array} = \sum_{\phi} \left[\begin{array}{c} 7 \quad 8 \\ \diagdown \quad \diagup \\ 2 \quad 4 \\ \diagup \quad \diagdown \\ 5 \end{array} \circ_{\phi}^2 \begin{array}{c} 3 \quad 6 \\ | \quad | \\ 2 \quad 1 \end{array} \right] \\ &= \left[\begin{array}{c} 7 \quad 8 \quad 3 \\ \diagdown \quad \diagup \quad | \\ 2 \quad 4 \\ \diagup \quad \diagdown \\ 5 \end{array}, \begin{array}{c} 6 \\ | \\ 1 \end{array} \right]_{\substack{\phi(7)=2 \\ \phi(8)=2}} + \left[\begin{array}{c} 7 \\ | \\ 8 \quad 3 \\ \diagdown \quad \diagup \\ 2 \quad 4 \\ \diagup \quad \diagdown \\ 5 \end{array}, \begin{array}{c} 6 \\ | \\ 1 \end{array} \right]_{\substack{\phi(7)=3 \\ \phi(8)=2}} \\ &+ \left[\begin{array}{c} 8 \\ | \\ 7 \quad 3 \\ \diagdown \quad \diagup \\ 2 \quad 4 \\ \diagup \quad \diagdown \\ 5 \end{array}, \begin{array}{c} 6 \\ | \\ 1 \end{array} \right]_{\substack{\phi(7)=2 \\ \phi(8)=3}} + \left[\begin{array}{c} 7 \quad 8 \\ \diagdown \quad \diagup \\ 3 \\ | \\ 2 \quad 4 \\ \diagup \quad \diagdown \\ 5 \end{array}, \begin{array}{c} 6 \\ | \\ 1 \end{array} \right]_{\substack{\phi(7)=3 \\ \phi(8)=3}} \\ & \text{For } s = \begin{array}{c} 6 \quad 5 \quad 3 \\ \diagdown \quad \diagup \quad | \\ 2 \quad 4 \\ \diagup \quad \diagdown \\ 1 \end{array} \text{ and } u = [\bullet 2, \bullet 1], i = 1, \text{ we have } s' = \begin{array}{c} 7 \quad 6 \quad 4 \\ \diagdown \quad \diagup \quad | \\ 3 \quad 5 \\ \diagup \quad \diagdown \\ 1 \end{array}. \text{ The forest obtained} \\ & \text{by removing 1 is : } \left\{ \begin{array}{c} 4 \\ | \\ 5 \end{array}, \begin{array}{c} 7 \quad 6 \\ \diagdown \quad \diagup \\ 3 \end{array} \right\}. \text{ Thus:} \end{aligned}$$

$$\begin{aligned} & \begin{array}{c} 6 \quad 5 \quad 3 \\ \diagdown \quad \diagup \quad | \\ 2 \quad 4 \\ \diagup \quad \diagdown \\ 1 \end{array} \circ_1 [\bullet 2, \bullet 1] = \sum_{\theta} \begin{array}{c} 7 \quad 6 \quad 4 \\ \diagdown \quad \diagup \quad | \\ 3 \quad 5 \\ \diagup \quad \diagdown \\ 1 \end{array} \circ_{\theta}^1 [\bullet 2, \bullet 1] \\ &= \left[\begin{array}{c} 4 \\ | \\ 5 \\ | \\ 2 \end{array}, \begin{array}{c} 7 \quad 6 \\ \diagdown \quad \diagup \\ 3 \\ | \\ 1 \end{array} \right]_{\substack{\theta(5)=2 \\ \theta(3)=1}} + \left[\begin{array}{c} 7 \quad 6 \\ \diagdown \quad \diagup \\ 3 \\ | \\ 2 \end{array}, \begin{array}{c} 4 \\ | \\ 5 \\ | \\ 1 \end{array} \right]_{\substack{\theta(5)=1 \\ \theta(3)=2}} \end{aligned}$$

$$\begin{aligned}
& \left[\begin{array}{c} \bullet 6 \quad \bullet 7 \\ \diagdown \quad \diagup \\ \bullet 3 \\ | \\ \bullet 2 \\ | \\ \bullet 8 \quad \bullet 4 \\ \diagup \quad \diagdown \\ \bullet 5 \end{array} \right] \circ_2 \left[\begin{array}{c} \bullet 3 \\ | \\ \bullet 2 \end{array} \right] = \sum_{\theta} \left(\left[\begin{array}{c} \bullet 8 \quad \bullet 9 \\ \diagdown \quad \diagup \\ \bullet 5 \\ | \\ \bullet 2 \\ | \\ \bullet 10 \quad \bullet 6 \\ \diagup \quad \diagdown \\ \bullet 7 \end{array} \right] \circ_{\theta}^2 \left(\begin{array}{c} \bullet 4 \\ | \\ \bullet 3 \end{array} \right) \right) \\
& - \left[\begin{array}{c} \bullet 8 \quad \bullet 9 \\ \diagdown \quad \diagup \\ \bullet 5 \\ | \\ \bullet 2 \\ | \\ \bullet 10 \quad \bullet 6 \\ \diagup \quad \diagdown \\ \bullet 7 \end{array} \right] \circ_{\theta}^2 \left(\begin{array}{c} \bullet 4 \\ | \\ \bullet 3 \end{array} \right) \right) \\
& = \left(\left[\begin{array}{c} \bullet 8 \quad \bullet 9 \\ \diagdown \quad \diagup \\ \bullet 5 \quad \bullet 4 \\ | \quad | \\ \bullet 2 \quad \bullet 3 \\ \diagup \quad \diagdown \\ \bullet 10 \quad \bullet 6 \\ \diagup \quad \diagdown \\ \bullet 7 \end{array} \right] - \left[\begin{array}{c} \bullet 8 \quad \bullet 9 \\ \diagdown \quad \diagup \\ \bullet 4 \quad \bullet 5 \\ | \quad | \\ \bullet 3 \quad \bullet 2 \\ \diagup \quad \diagdown \\ \bullet 10 \quad \bullet 6 \\ \diagup \quad \diagdown \\ \bullet 7 \end{array} \right] \right) \\
& \quad \theta(5)=2 \\
& + \left(\left[\begin{array}{c} \bullet 8 \quad \bullet 9 \\ \diagdown \quad \diagup \\ \bullet 5 \quad \bullet 4 \\ \diagdown \quad \diagup \\ \bullet 2 \quad \bullet 3 \\ \diagup \quad \diagdown \\ \bullet 10 \quad \bullet 6 \\ \diagup \quad \diagdown \\ \bullet 7 \end{array} \right] - \left[\begin{array}{c} \bullet 8 \quad \bullet 9 \\ \diagdown \quad \diagup \\ \bullet 5 \quad \bullet 4 \\ \diagdown \quad \diagup \\ \bullet 3 \quad \bullet 2 \\ \diagup \quad \diagdown \\ \bullet 10 \quad \bullet 6 \\ \diagup \quad \diagdown \\ \bullet 7 \end{array} \right] \right) \\
& \quad \theta(5)=3 \\
& + \left(\left[\begin{array}{c} \bullet 8 \quad \bullet 9 \\ \diagdown \quad \diagup \\ \bullet 5 \\ | \\ \bullet 4 \\ | \\ \bullet 2 \quad \bullet 3 \\ \diagup \quad \diagdown \\ \bullet 10 \quad \bullet 6 \\ \diagup \quad \diagdown \\ \bullet 7 \end{array} \right] - \left[\begin{array}{c} \bullet 8 \quad \bullet 9 \\ \diagdown \quad \diagup \\ \bullet 5 \\ | \\ \bullet 4 \\ | \\ \bullet 3 \quad \bullet 2 \\ \diagup \quad \diagdown \\ \bullet 10 \quad \bullet 6 \\ \diagup \quad \diagdown \\ \bullet 7 \end{array} \right] \right) \\
& \quad \theta(5)=4
\end{aligned}$$

Proposition 5. The products $\circ_i: \langle B(n) \rangle \times \langle B(m) \rangle \rightarrow \mathcal{FT}(n+m-1)$ satisfy the equivariance, unit, parallel and sequential axioms

Proof. The unit of $\circ_i$ is $\bullet 1$. Indeed, if $s \in L(t, [m])(t \in \mathbb{T}(m))$:

$$s \circ_i \bullet 1 = s' \circ_{\phi} \bullet i \text{ (by Definition 29, } f(x) = x)$$

$$\begin{aligned}
&= s \circ_{\phi} \bullet i \text{ (we replace the node } i \text{ of } s' = s \text{ by the root } \bullet i) \\
&= s \quad (\text{the unique functions } \phi \text{ are the empty function or } \phi(x) = i)
\end{aligned}$$

$$\begin{aligned}
\bullet 1 \circ_1 s &= \bullet 1 \circ_1 s'' \text{ (by Definition 29, } g(x) = x) \\
&= \bullet 1 \circ_1 s \\
&= s
\end{aligned}$$

If $u = [u_1, u_2] \in B(n)$, then $u \circ_i \bullet 1 = u$ is analogous, once it is almost the same operation with trees. Lastly, $\bullet 1 \circ_1 [u_1, u_2] = [u_1, u_2]$, since $\bullet 1$ does not have subtrees.

The operadic axioms also say that it does not matter which composition is calculated first. Formally, for the maps $\circ_i: \langle B(n) \rangle \times \langle B(m) \rangle \rightarrow \mathcal{FT}(n+m-1)$ verify the parallel and sequential axioms, if $s \in B(n), v \in B(n_1), w \in B(n_2)$, we need to have $\forall 1 \leq i < j \leq n$:

$$(s \circ_j w) \circ_i v = (s \circ_i v) \circ_{j+n_1-1} w \text{ (parallel axiom)}$$

$$s \circ_i (v \circ_j w) = (s \circ_i v) \circ_{j+i-1} w \text{ (sequential axiom)}$$

Strictly speaking, there are nine cases both for $(s \circ_j w) \circ_i v$ and for $s \circ_i (v \circ_j w)$. However, the arguments used below are the same in all the cases. So, we prove the parallel axiom just for 2.10 (the proof of sequential axiom is analogous). Considering

$$(s \circ_j w) \circ_i v = \sum_{\phi_1, \phi_2} (s' \circ_{\phi_1}^j w'')' \circ_{\phi_2}^i v'',$$

$$(s \circ_i v) \circ_{j+n_1-1} w = \sum_{\phi_2, \phi_1} (s' \circ_{\phi_2}^i v'')' \circ_{\phi_1}^{j+n_1-1} w'',$$

the aim is to show that $(s' \circ_{\phi_1}^j w'')' \circ_{\phi_2}^i v'' = (s' \circ_{\phi_2}^i v'')' \circ_{\phi_1}^{j+n_1-1} w''$.

In $s \circ_j w$, the operadic composition $\circ_j$ depends on the node $\bullet j$. For each summand of $s \circ_j w$, no nodes labeled by $i \neq j$ disappear, they are just relabeled (with i in s or in w). We also have the subtrees preserved, i.e., if the node labeled by i has subtrees $t_1, \dots, t_n$, then each summand of $s \circ_j w$ keeps this subtrees, as we can see in the next example:

The same arguments are repeated for $s \circ_i v$:

Since $i < j$, due to the bijections f and g , the node labeled by j in s is the same node labeled by $j + n_1 - 1$ in $s \circ_i v$. Thus, if we prove that the the label changes determined by f and g are the same in $(s \circ_j w) \circ_i v$ and $(s \circ_i v) \circ_{j+n_1-1} w$, no matter which operation is calculated first - $(s' \circ_{\phi_1}^j w'')' \circ_{\phi_2}^i v'' = (s' \circ_{\phi_2}^i v'')' \circ_{\phi_1}^{j+n_1-1} w''$ - and this guarantees the parallel axiom holds.

Now, let $x \in [n]$ be a label of s , $y \in [n_2]$ be a label of w and $z \in [n_1]$ be a label of v . The table below give us the new labels of each node in $(s \circ_j w) \circ_i v$ and $(s \circ_i v) \circ_{j+n_1-1} w$ by using the definition of f and g . For instance, a node labeled by x in s , with $i + 1 \leq x \leq n$, in $s \circ_i v$ is labeled by $f(x) = x + n_1 - 1$. In $(s \circ_i v) \circ_{j+n_1-1} w$, the label of the same node depends on to compare $x + n_1 - 1$ with $j + n_1 - 1$. In fact, if $i + 1 \leq x \leq j$, then $x + n_1 - 1 \leq j + n_1 - 1 \Rightarrow f(f(x)) = x + n_1 - 1$. If $j + 1 \leq x \leq n \Rightarrow j + n_1 \leq x + n_1 - 1 \leq n + n_1 - 1 \Rightarrow f(f(x)) = x + n_1 + n_2 - 2$.

	S		W		V	
$s \circ_j w$	x	$1 \leq x \leq j$	$y + j - 1$		z	
	$x + n_2 - 1$	$j + 1 \leq x \leq n$				
$(s \circ_j w) \circ_i v$	$\mathbf{x}$	$1 \leq \mathbf{x} \leq \mathbf{i}$	$\mathbf{y} + \mathbf{n}_1 + \mathbf{j} - 2$	$i < j \Rightarrow$ $i + 1 \leq y + j - 1$	$\mathbf{z} + \mathbf{i} - 1$	
	$\mathbf{x} + \mathbf{n}_1 - 1$	$\mathbf{i} + 1 \leq \mathbf{x} \leq \mathbf{j}$				
	$\mathbf{x} + \mathbf{n}_2 + \mathbf{n}_1 - 2$	$\mathbf{j} + 1 \leq \mathbf{x} \leq \mathbf{n}$				
$s \circ_i v$	x	$1 \leq x \leq i$	y		$z + i - 1$	
	$x + n_1 - 1$	$i + 1 \leq x \leq n$				
$(s \circ_i v) \circ_{j+n_1-1} w$	$\mathbf{x}$	$1 \leq \mathbf{x} \leq \mathbf{i}$	$\mathbf{y} + \mathbf{j} + \mathbf{n}_1 - 2$		$\mathbf{z} + \mathbf{i} - 1$	$i < j,$ $1 \leq z \leq n_1 \Rightarrow$ $z + i - 1$ is less than $j + n_1 - 1$
	$\mathbf{x} + \mathbf{n}_1 - 1$	$\mathbf{i} + 1 \leq \mathbf{x} \leq \mathbf{j}$				
	$\mathbf{x} + \mathbf{n}_1 + \mathbf{n}_2 - 2$	$\mathbf{j} + 1 \leq \mathbf{x} \leq \mathbf{n}$				

By comparing the results, the change of labels coincide.

At least, we have to verify the equivariance axiom. The equivariance is the property

responsible for guaranteeing the compatibility of the $\mathbb{S}$ -module structure with the operadic compositions $\circ_i$. Again, we just prove for 2.10. We need to show that:

$$s^\sigma \circ_j v^\beta = (s \circ_{\sigma(j)} v)^\sigma \circ_j \beta, \quad s \in B(n), \quad v \in B(m), \quad \sigma \in \mathbb{S}_n, \quad \beta \in \mathbb{S}_m.$$

Given $s = l_{t_1, n}(id_n)$ and a node labeled by j in s^σ , since $s^\sigma = l_{t_1, n}([\sigma^{-1}(1)\sigma^{-1}(2)\cdots\sigma^{-1}(n)])$, there is $i \in [n]$ such that $\sigma^{-1}(i) = j$. It follows from Definition 27 that the position of the element in the sequence determines the node. So, the node labeled j in s^σ is the same node labeled $i = \sigma(j)$ in s .

If $v = l_{t_2, m}(id_m)$, we have:

$$s^\sigma \circ_j v^\beta = \sum_{\phi} (s^\sigma)' \circ_{\phi}^j (v^\beta)'' , (s \circ_{\sigma(j)} v)^\sigma \circ_j \beta = \left(\sum s' \circ_{\phi}^{\sigma(j)} v'' \right)^\sigma \circ_j \beta.$$

The maps $\circ_j, \circ_{\sigma(j)}$ are determined by the nodes. However, in this case, the nodes are the same. Thus, the proof is finished if the label of $(s^\sigma)' \circ_{\phi}^j (v^\beta)''$ is equal to the label of $(s' \circ_{\phi}^{\sigma(j)} v'')^\sigma \circ_j \beta$, for each ϕ .

The action of $\sigma \circ_j \beta$ on $s' \circ_{\phi}^{\sigma(j)} v''$ does not change the rooted tree t , such that $l_{t, m+n-1}(\gamma) = s' \circ_{\phi}^i v''$ for some $\gamma \in \mathbb{S}_{n+m-1}$.

In this example, $\phi(4) = 6$, $\sigma = [1325467]$, $\beta = id_2$ and $\alpha = \sigma \circ_4 \beta = [13256478]$.

So, it does not matter if the action happens before or after of we make the compositions, i.e., for each ϕ ,

$$\begin{aligned} (s' \circ_{\phi}^{\sigma(j)} v'')^{\sigma \circ_j \beta} &= (l_{t_1}(f(1), \dots, f(n)) \circ_{\phi}^{\sigma(j)} l_{t_2}(g(1), \dots, g(m)))^{\sigma \circ_j \beta} \\ &= (l_{t_1}(1, 2, i, i+m, \dots, n+m-1)) \circ_{\phi}^{\sigma(j)} l_{t_2}(i, \dots, i+m-1))^{\sigma \circ_j \beta} \\ &= l_{t_1}((\sigma \circ_j \beta)^{-1}(1), \dots, i, (\sigma \circ_j \beta)^{-1}(i+m), \dots, (\sigma \circ_j \beta)^{-1}(n+m-1)) \circ_{\phi}^{\sigma(j)} \\ &\quad l_{t_2}((\sigma \circ_j \beta)^{-1}(i), \dots, (\sigma \circ_j \beta)^{-1}(i+m-1)). \end{aligned}$$

Apart from this, $(\sigma \circ_j \beta)^{-1}$ is the block permutation $\theta = \sigma(id_1 \oplus \cdots \oplus id_1 \oplus \underbrace{\beta^{-1}}_{jth} \oplus id_1 \oplus \cdots \oplus id_1)$ and, by definition:

Returning to the example, since the composition depends on the node, the maps $\phi(4) = 6$ and $\phi(\alpha^{-1}(4)) = \alpha^{-1}(6)$ determine the same rooted tree.

$$\theta(x) = \begin{cases} \sigma^{-1}(x), 1 \leq x \leq i-1 \\ \beta^{-1}(x-i+1)+j-1, i \leq x \leq m+i-1 \\ \sigma^{-1}(x)+m-1, m+i \leq x \leq m+n-1 \end{cases} \Rightarrow$$

$$\begin{aligned} (s' \circ_{\phi}^{\sigma(j)} v'')^{\sigma \circ_j \beta} &= \iota_1((\sigma \circ_j \beta)^{-1}(1), \dots, i, (\sigma \circ_j \beta)^{-1}(i+m), \dots, (\sigma \circ_j \beta)^{-1}(n+m-1)) \circ_{\phi}^{\sigma(j)} \\ &\quad \iota_2((\sigma \circ_j \beta)^{-1}(i), \dots, (\sigma \circ_j \beta)^{-1}(i+m-1)) \\ &= \iota_1(f(\sigma^{-1}(1)), \dots, f(\sigma^{-1}(n))) \circ_{\phi}^j \iota_2(g(\beta^{-1}(1)), \dots, g(\beta^{-1}(m))) \\ &= (s^{\sigma})' \circ_{\phi}^j (v^{\beta})''. \end{aligned}$$

It is important to highlight the node labeled by j in s^{σ} is the same node labeled by $i = \sigma(j)$ in s .

Thus, all axioms hold for the maps $\circ_i: \langle B(n) \rangle \times \langle B(m) \rangle \rightarrow \mathcal{FT}(n+m-1)$ which is ending the proof (see Note 7). $\square$

Definition 32. Let $u \in \mathcal{L}(n)$, we say that u is a **composition** of $v_1 \in B(n_1), \dots, v_k \in B(n_k)$ if there are $i_1, \dots, i_{k-1} \in \mathbb{N}$ and $\sigma \in \mathbb{S}_n$ such that:

$$u = (\dots(\dots((v_1 \circ_{i_1} v_2) \circ_{i_2} v_3) \dots v_{k-1}) \circ_{i_{k-1}} v_k)^{\sigma}.$$

Example 40. If $u = \left[\left[\begin{array}{c} \bullet_3 \\ \bullet_4 \end{array}, \bullet_1 \right], \begin{array}{c} \bullet_6 \bullet_5 \\ \bullet_2 \end{array} \right]$, then u is a composition of $\left[\begin{array}{c} \bullet_1 \\ \bullet_2 \end{array}, \bullet_3 \right]$ and $\left[\bullet_1, \begin{array}{c} \bullet_2 \bullet_3 \\ \bullet_4 \end{array} \right]$. In fact,

$$\left[\bullet_1, \begin{array}{c} \bullet_2 \bullet_3 \\ \bullet_4 \end{array} \right] \circ_1 \left[\begin{array}{c} \bullet_1 \\ \bullet_2 \end{array}, \bullet_3 \right] = \left[\left[\begin{array}{c} \bullet_1 \\ \bullet_2 \end{array}, \bullet_3 \right], \begin{array}{c} \bullet_4 \bullet_5 \\ \bullet_6 \end{array} \right]$$

and for $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 2 & 5 & 4 \end{pmatrix}$, we have $\left[\left[\begin{array}{c} \bullet_1 \\ \bullet_2 \end{array}, \bullet_3 \right], \begin{array}{c} \bullet_4 \bullet_5 \\ \bullet_6 \end{array} \right]^{\sigma} = u$.

Proposition 6. If $u \in L(p, [n])$ ($p \in M_m(\mathbb{T})$, $m \geq 3$), then there are $v_1 \in B(n_1), \dots, v_k \in B(n_k)$, such that u is a composition of $\{v_1, \dots, v_k\}$.

Proof. The proof is made by induction over the numbers of rooted trees in the parenthesizing p and it is made for $u = l_{p,n}(id_n)$ (Note 7). If $p \in M_3(\mathbb{T})$ is a parenthesizing of $t_1 \in \mathbb{T}(n_1), t_2 \in \mathbb{T}(n_2), t_3 \in \mathbb{T}(n_3)$, we have: $u = [u_1, [u_2, u_3]]$ or $u = [[u_1, u_2], u_3]$, in which u_i represents the planar tree t_i with a label. Thus,

$$[u_1, [u_2, u_3]] = [u_1, \bullet_{n_1+1}] \circ_{n_1+1} [v_1, v_2], \text{ with } v_1 = l_{t_2, n_2}(id_{n_2}) \text{ and } v_2 = l_{t_3, n_3}(id_{n_3}),$$

$$[[u_1, u_2], u_3] = [\bullet_1, u_3] \circ_1 [u_1, u_2].$$

Now, suppose that the proposition is true for all $u \in L(p, [n])$ such that p is a parenthesizing of $t_1, \dots, t_j$ for all $3 \leq j \leq k$. If $u \in L(p, [n])$, in which p a parenthesizing of $t_1, \dots, t_{k+1}$, then $u = [w_1, w_2]$, in which w_i is either planar tree $t_i \in \mathbb{T}(m_i)$ or a parenthesizing of planar trees $t_i \in \mathbb{M}(m_i)$ ($m_1 + m_2 - 2 = n$) with labels. Besides that,

$$[w_1, w_2] = ([\bullet_1, \bullet_2] \circ_2 v_2) \circ_1 w_1, \text{ (} v_2 = l_{t_2, n_2}(id_{n_2}) \text{)}$$

Since, in the worst hypothesis, w_1 and w_2 are parenthesizings with less planar trees than in $[w_1, w_2]$, w_1 and w_2 are a composition of elements in $\cup_{n \geq 2} B(n)$ due to the induction hypothesis. Thus, if necessary, by using the parallel or the sequential axioms, $[w_1, w_2]$ is also a composition of elements in $\cup_{n \geq 2} B(n)$. $\square$

Definition 33. For u or $s \in \mathcal{L}(n) \setminus B(n)$, $u \circ_i s$ is defined by using the operadic properties of $\circ_i$ to, step by step, reduce $u \circ_i s$ to a product between two elements.

Thus, $\{\mathcal{FT}, \circ_i\}$ is an operad, in which $\circ_i$ is the extension of $\circ_i: \langle B(n) \rangle \times \langle B(m) \rangle \rightarrow \mathcal{FT}(n+m-1)$ to $\mathcal{FT}(n+m-1)$ to $\mathcal{FT}(n) \times \mathcal{FT}(m)$.

Example 41. First, we give an example in which block permutations are not necessary, since by Definition 27 the elements are image of the identity.

$$\begin{aligned} & \begin{array}{c} 1 \\ \bullet \\ 2 \end{array} \circ_1 \left[\left[\begin{array}{c} \bullet \\ 1 \\ \bullet \\ 3 \end{array}, \begin{array}{c} \bullet \\ 2 \\ \bullet \\ 4 \end{array} \right], \begin{array}{c} \bullet \\ 5 \\ \bullet \\ 6 \end{array} \right] = \begin{array}{c} 1 \\ \bullet \\ 2 \end{array} \circ_1 \left(\left[\begin{array}{c} \bullet \\ 1 \end{array}, \begin{array}{c} \bullet \\ 2 \\ \bullet \\ 3 \end{array} \right] \circ_1 \left[\begin{array}{c} \bullet \\ 1 \\ \bullet \\ 3 \end{array}, \begin{array}{c} \bullet \\ 2 \\ \bullet \\ 4 \end{array} \right] \right) \\ &= \left(\begin{array}{c} 1 \\ \bullet \\ 2 \end{array} \circ_1 \left[\begin{array}{c} \bullet \\ 1 \end{array}, \begin{array}{c} \bullet \\ 2 \\ \bullet \\ 3 \end{array} \right] \right) \circ_1 \left[\begin{array}{c} \bullet \\ 1 \\ \bullet \\ 3 \end{array}, \begin{array}{c} \bullet \\ 2 \\ \bullet \\ 4 \end{array} \right] \\ &= \left(\begin{array}{c} 1 \quad 2 \\ \bullet \quad \bullet \\ 4 \quad 3 \end{array} - \begin{array}{c} 2 \quad 1 \\ \bullet \quad \bullet \\ 4 \quad 3 \end{array} \right) \circ_1 \left[\begin{array}{c} \bullet \\ 1 \\ \bullet \\ 3 \end{array}, \begin{array}{c} \bullet \\ 2 \\ \bullet \\ 4 \end{array} \right] \\ &= \begin{array}{c} 1 \quad 2 \quad 5 \\ \bullet \quad \bullet \quad \bullet \\ 3 \quad 4 \quad 6 \end{array} - \begin{array}{c} 1 \quad 2 \quad 5 \\ \bullet \quad \bullet \quad \bullet \\ 4 \quad 3 \quad 6 \end{array} - \begin{array}{c} 5 \quad 1 \quad 2 \\ \bullet \quad \bullet \quad \bullet \\ 6 \quad 3 \quad 4 \end{array} \end{aligned}$$

[illegible]

$$\begin{aligned}
& + \left[\left[\begin{array}{c} \bullet 3 \\ | \\ \bullet 4, \bullet 5 \end{array}, \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \\ | \\ \bullet 6 \bullet 7 \\ \diagdown \diagup \\ \bullet 8 \end{array} \right] + \left[\left[\begin{array}{c} \bullet 3 \\ | \\ \bullet 4, \bullet 5 \end{array}, \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \\ | \\ \bullet 6 \bullet 7 \\ \diagdown \diagup \\ \bullet 8 \end{array} \right] \right) id_3 \circ_3 \sigma \\
& = \left[\left[\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \\ | \\ \bullet 5 \\ | \\ \bullet 6, \bullet 3 \end{array}, \begin{array}{c} \bullet 8 \bullet 7 \\ \diagdown \diagup \\ \bullet 4 \end{array} \right] + \left[\left[\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \\ | \\ \bullet 5 \\ | \\ \bullet 6, \bullet 3 \end{array}, \begin{array}{c} \bullet 8 \bullet 7 \\ \diagdown \diagup \\ \bullet 4 \end{array} \right] \right] \\
& + \left[\left[\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \\ | \\ \bullet 5 \\ | \\ \bullet 6, \bullet 3 \end{array}, \begin{array}{c} \bullet 8 \bullet 7 \\ \diagdown \diagup \\ \bullet 4 \end{array} \right] + \left[\left[\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \\ | \\ \bullet 5 \\ | \\ \bullet 6, \bullet 3 \end{array}, \begin{array}{c} \bullet 8 \bullet 7 \\ \diagdown \diagup \\ \bullet 4 \end{array} \right] \right] \\
& + \left[\left[\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \\ | \\ \bullet 5 \\ | \\ \bullet 6, \bullet 3 \end{array}, \begin{array}{c} \bullet 8 \bullet 7 \\ \diagdown \diagup \\ \bullet 4 \end{array} \right] + \left[\left[\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \\ | \\ \bullet 5 \\ | \\ \bullet 6, \bullet 3 \end{array}, \begin{array}{c} \bullet 8 \bullet 7 \\ \diagdown \diagup \\ \bullet 4 \end{array} \right] \right] .
\end{aligned}$$

2.4 The operad ∇

The operad ∇ is by definition the quotient operad (Definition 11) $\mathcal{FT}/\mathcal{I}$, in which $\mathcal{I}$ is the operadic ideal generated by the set

$$\{[\bullet 1, \bullet 2] + [\bullet 2, \bullet 1], [\bullet 1, [\bullet 2, \bullet 3]] + [\bullet 3, [\bullet 1, \bullet 2]] + [[\bullet 1, \bullet 3], \bullet 2]\}.$$

The operadic products in ∇ are also denoted by $\circ_i$ and are induced by the products in $\mathcal{FT}$ (see Definitions 10, 11 and 15).

Remark 8. Roughly, ∇ is almost the operad $\mathcal{FT}$. The difference is that now all brackets are antisymmetric and Jacobi holds. For instance, it follows from the definition of the operadic compositions that, if $l_{b,n}(id_n) = [u_1, u_2] \in L(b, [n])$ ($b = [t_1, t_2] \in M_2(\mathbb{T})$), then $u = ([\bullet 1, \bullet 2] \circ_2 w_2) \circ_1 w_1$ ($w_i = l_{t_i, n_i}(id_{n_i})$, $n_1 + n_2 = n$). However, in ∇ , $[\bullet 1, \bullet 2] = -[\bullet 2, \bullet 1]$, thus:

$$\begin{aligned}
[u_1, u_2] &= ([\bullet 1, \bullet 2] \circ_2 w_2) \circ_1 w_1 \\
&= -([\bullet 2, \bullet 1] \circ_2 w_2) \circ_1 w_1 \\
&= -[u_2, u_1].
\end{aligned}$$

2.5 Post-Lie algebras and the operad ∇

In this section, it is proved the $\mathcal{PostLie}$ operad is isomorphic to the operad ∇ . By using Definition 18, we present the vector space $\nabla(V)$ as a way of to describe the free post-Lie algebra over a vector space V . We also show, with more details, how the operad ∇ extends the operad $\mathcal{RT}$.

Lemma 3. If $s \in \nabla$, then there are $w_1, \dots, w_n \in \nabla$ such that $s = \sum_{i=1}^n k_i w_i$ ($k_i \in \mathbb{K}$) and w_i is a composition of elements in $\nabla(2)$ (Definition 32).

Proof. Proposition 6 reduces the proof to labeled rooted trees or to brackets of two labeled rooted trees. By definition, every tree $t \in \mathbb{T}(n)$ ($n \geq 3$) is a corolla or is built from corollas (see Proposition 3). The product $s \circ_i v$, whose s and v are labeled trees and i is an external node, is the same one built by using labeled rooted trees (see 2.7). So, every labeled rooted tree is a corolla or a composition of corollas. If $u \in L(b, [n])$ ($b \in M_2(\mathbb{T})$), then there are v_1, v_2 labeled rooted trees and $\sigma \in \mathbb{S}_n$, such that:

$$u = (([\bullet 1, \bullet 2] \circ_2 v_2) \circ_1 v_1)^\sigma \text{ (see Definition 31).} \quad (2.13)$$

Thus, the proof of the lemma is reduced to labeled corollas.

The operadic composition 2.10 provides:

$$\begin{array}{c} 1 \\ \bullet \\ \vdots \\ 2 \end{array} \circ_2 \begin{array}{c} 1 \quad 2 \quad \dots \quad n-1 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \vdots \\ n \end{array} = \begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \vdots \\ n+1 \end{array} + \sum_{1 \leq i \leq n-1} \left(\begin{array}{c} 1 \quad 2 \quad \dots \quad n-1 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \vdots \\ n \end{array} \circ_i \begin{array}{c} 1 \\ \bullet \\ \vdots \\ 2 \end{array} \right)^{\sigma_i},$$

in which $\sigma_i \in \mathbb{S}_{n+1}$, such that $\sigma_i(1) = i$, $\sigma_i(i-k+1) = i-k$, $\sigma_i(i+m) = i+m$, $1 \leq k \leq i-1$, $1 \leq m \leq n+1-i$. Thus,

$$\begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \vdots \\ n+1 \end{array} = \begin{array}{c} 1 \\ \bullet \\ \vdots \\ 2 \end{array} \circ_2 \begin{array}{c} 1 \quad 2 \quad \dots \quad n-1 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \vdots \\ n \end{array} - \sum_{1 \leq i \leq n-1} \left(\begin{array}{c} 1 \quad 2 \quad \dots \quad n-1 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \vdots \\ n \end{array} \circ_i \begin{array}{c} 1 \\ \bullet \\ \vdots \\ 2 \end{array} \right)^{\sigma_i}. \quad (2.14)$$

Since

$$\begin{array}{c} 1 \quad 2 \\ \bullet \quad \bullet \\ \vdots \\ 3 \end{array} = \begin{array}{c} 1 \\ \bullet \\ \vdots \\ 2 \end{array} \circ_2 \begin{array}{c} 1 \\ \bullet \\ \vdots \\ 2 \end{array} - \begin{array}{c} 1 \\ \bullet \\ \vdots \\ 2 \end{array} \circ_1 \begin{array}{c} 1 \\ \bullet \\ \vdots \\ 2 \end{array},$$

by recurrence, for labeled corollas, the lemma holds. □

Theorem 1. The operad ∇ is isomorphic to the operad $\mathcal{PostLie}$.

Proof. The operad $\mathcal{PostLie} = \mathcal{F}(\mathcal{S})/(R)$ (Example 17) is the free operad generated by the with the $\mathbb{S}$ -module $\mathcal{S} = \{S(n)\}_{n \in \mathbb{N}}$ given by

$$S(2) = \mathbb{K}[\mathbb{S}_2] \langle (\bullet \triangleright \bullet), [\bullet, \bullet] \rangle \text{ and } S(m) = 0 \text{ for all } m \neq 2, R = \{r_1, r_2, r_3, r_4\}.$$

The linear map given by

$$[\bullet, \bullet] \mapsto [\bullet 1, \bullet 2]$$

$$(\bullet \triangleright \bullet) \mapsto \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array}$$

defines an $\mathbb{S}$ -module morphism $h: \mathcal{S} \rightarrow \nabla$. By the universal property (Definition 12), this $\mathbb{S}$ -module morphism is extended to an operadic morphism $\tilde{h}: \mathcal{F}(\mathcal{S}) \rightarrow \nabla$.

The following identities guarantee $\tilde{h}(r_i) = 0$, which induce an operadic morphism (Definition 11)

$$f: \text{PostLie} \rightarrow \nabla \quad (2.15)$$

$$\begin{aligned} \tilde{h}_2([\bullet, \bullet] + [\bullet, \bullet]^{[21]}) &= [\bullet 1, \bullet 2] + [\bullet 2, \bullet 1] \\ &= 0, \end{aligned}$$

$$\begin{aligned} \tilde{h}_3([\bullet, \bullet], \bullet - [\bullet, [\bullet, \bullet]] - [[\bullet, \bullet], \bullet]^{[132]}) &= \tilde{h}_3([\bullet, \bullet] \circ_1 [\bullet, \bullet] - [\bullet, \bullet] \circ_2 [\bullet, \bullet] - ([\bullet, \bullet] \circ_1 [\bullet, \bullet])^{[132]}) \\ &= \tilde{h}_2([\bullet, \bullet]) \circ_1 \tilde{h}_2([\bullet, \bullet]) - \tilde{h}_2([\bullet, \bullet]) \circ_2 \tilde{h}_2([\bullet, \bullet]) \\ &\quad - (\tilde{h}_2([\bullet, \bullet]) \circ_1 \tilde{h}_2([\bullet, \bullet]))^{[132]} \\ &= [\bullet 1, \bullet 2] \circ_2 [\bullet 1, \bullet 2] - ([\bullet 1, \bullet 2] \circ_2 [\bullet 1, \bullet 2]) \\ &\quad - ([\bullet 1, \bullet 2] \circ_1 [\bullet 1, \bullet 2])^{[132]} \\ &= [\bullet 1, [\bullet 2, \bullet 3]] - [[\bullet 1, \bullet 2], \bullet 3] - [[\bullet 1, \bullet 3], \bullet 2] \\ &= 0, \end{aligned}$$

$$\begin{aligned} \tilde{h}_3((\bullet \triangleright [\bullet, \bullet]) - [(\bullet \triangleright \bullet), \bullet] - [\bullet, (\bullet \triangleright \bullet)]^{[213]}) &= \tilde{h}_2((\bullet \triangleright \bullet)) \circ_2 \tilde{h}_2([\bullet, \bullet]) - \tilde{h}_2([\bullet, \bullet]) \circ_1 \tilde{h}_2((\bullet \triangleright \bullet)) \\ &\quad - (\tilde{h}_2([\bullet, \bullet]) \circ_2 \tilde{h}_2((\bullet \triangleright \bullet)))^{[213]} \\ &= \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \circ_2 [\bullet 1, \bullet 2] - \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array}, \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array}, \bullet 3 - \begin{array}{c} 2 \bullet \\ \downarrow \\ \bullet 1, 3 \end{array} \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \\ &= \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \circ_2 [\bullet 1, \bullet 2] - \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array}, \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array}, \bullet 3 - \begin{array}{c} 1 \bullet \\ \downarrow \\ \bullet 2, 3 \end{array} \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \\ &= 0, \end{aligned}$$

$$\begin{aligned} \tilde{h}_3([\bullet, \bullet] \triangleright \bullet - a_{\triangleright} + a_{\triangleright}^{[213]}) &= \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \circ_1 [\bullet 1, \bullet 2] - \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \circ_2 \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} + \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \circ_1 \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} + \left(\begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \circ_2 \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \right)^{[213]} \\ &\quad - \left(\begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \circ_1 \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \right)^{[213]} \\ &= \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \circ_1 [\bullet 1, \bullet 2] - \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} \begin{array}{c} 2 \bullet \\ \downarrow \\ 3 \bullet \end{array} + \begin{array}{c} 2 \bullet \\ \downarrow \\ 3 \bullet \end{array} \begin{array}{c} 1 \bullet \\ \downarrow \\ 2 \bullet \end{array} = 0. \end{aligned}$$

By Lemma 3, f_n is onto for all $n \in \mathbb{N}^*$. The same lemma is used to define a family of $\mathbb{S}$ -modules morphisms $g_n: \nabla(n) \rightarrow \mathcal{PostLie}(n)$ (Definition 8):

$$g_1: \bullet 1 \mapsto \bullet, \quad g_2: \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \mapsto [\bullet, \bullet], \quad \dots, \quad g_n: t \in \mathcal{L}(n) \mapsto c_t.$$

In which c_t is obtained by induction, in a similar way we prove the Lema 3. In fact, for

$$t = \begin{array}{c} \bullet 1 \\ \swarrow \searrow \\ \bullet 2 \quad \bullet 3 \end{array} :$$

$$\begin{aligned} g_3 \left(\begin{array}{c} \bullet 2 \quad \bullet 1 \\ \swarrow \searrow \\ \bullet 3 \end{array} \right) &= g_3 \left(\begin{array}{c} \bullet 1 \quad \bullet 2 \\ \swarrow \searrow \\ \bullet 3 \end{array} \right)^{[213]} \\ &= g_3 \left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} - \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_1 \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \right)^{[213]} \right) \\ &:= \left(g_2 \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \right) \circ_2 g_2 \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \right) \right)^{[213]} - \left(g_2 \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \right) \circ_1 g_2 \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \right) \right)^{[213]} \\ &= a_{\triangleright}^{[213]}. \end{aligned}$$

The linear maps g_n are well defined, since

$$g_2([\bullet 1, \bullet 2] + [\bullet 2, \bullet 1]) = [\bullet, \bullet] + [\bullet, \bullet]^{[21]} = 0$$

$$g_3([\bullet 1, [\bullet 2, \bullet 3]] - [[\bullet 1, \bullet 2], \bullet 3] - [[\bullet 1, \bullet 3], \bullet 2]) = [[\bullet, \bullet], \bullet] - [\bullet, [\bullet, \bullet]] - [[\bullet, \bullet], \bullet]^{[132]} = 0.$$

It follows from the definition of g that g_n is onto $\forall n \in \mathbb{N}^*$. Thus, $\forall n \in \mathbb{N}^*$, $f_n \circ g_n$ is onto, and the Rank–nullity theorem says $f_n \circ g_n$ is an isomorphism. However, $f_n \circ g_n$ isomorphism $\Rightarrow g_n$ is one-to-one and, therefore, an isomorphism. So, $f_n = (f_n \circ g_n) \circ g_n^{-1}$ is an isomorphism too. Then, $f = \{f_n\}_{n \in \mathbb{N}^*}$ is an operadic morphism such that f_n is an isomorphism of vector spaces $\Rightarrow \mathcal{PostLie}$ is isomorphic to ∇ .⁴ $\square$

The idea used to construct the operad ∇ comes from Chapoton and Livernet's operad $\mathcal{RT}$ (Definition 28), so it is natural to ask what relations hold between the two operads. For instance, the product on $\mathcal{RT}$ is product (2.10). However, in $\mathcal{RT}$ there is no rule about the graphing side, since the trees are non-planar.

⁴ The argument used in this proof is an adaptation of a similar argument found in the proof of Chapoton's Proposition 2 [4].

Consider the operadic ideal $\mathcal{J} = ([\bullet 1, \bullet 2]) \subset \nabla$. In the operad $\nabla/\mathcal{J}$,

$$s \circ_i [\bullet 1, \bullet 2] = [\bullet 1, \bullet 2] \circ_j s = 0, \forall s + \mathcal{J}(n) \in (\nabla/\mathcal{J})(n), i \in [n], j \in [2].$$

Thus, all parenthesizings are identically zero (see 2.13) and for all $i \in [n-1]$:

$$0 = \begin{array}{c} 1 \quad 2 \quad \dots \quad n-1 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ n \end{array} \quad \circ_i [\bullet 1, \bullet 2] = \begin{array}{c} 1 \quad 2 \quad \dots \quad i \quad i+1 \quad \dots \quad n \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ n+1 \end{array} - \begin{array}{c} 1 \quad 2 \quad \dots \quad i+1 \quad i \quad \dots \quad n \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ n+1 \end{array}.$$

So, every corolla is non-planar:

$$\begin{array}{c} 1 \quad 2 \quad \dots \quad i \quad i+1 \quad \dots \quad n \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ n+1 \end{array} = \begin{array}{c} 1 \quad 2 \quad \dots \quad i+1 \quad i \quad \dots \quad n \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ n+1 \end{array}.$$

Every rooted tree that is not a corolla is a composition of corollas (see proof of Lemma 3). Thus, the labeled rooted trees in $\nabla/\mathcal{J}$ are non-planar, which proves the following proposition:

Proposition 7. The map defined by:

$$s + \mathcal{J}(n) \mapsto s,$$

is an isomorphism from $\nabla/\mathcal{J}$ to $\mathcal{RT}$.

All pre-Lie algebras $(V, *)$ are post-Lie algebras with the products $*$ and $[\cdot, \cdot] \equiv 0$ (see (1.25)). The isomorphism defined before translates this result into an operadic context.

We also use Proposition 7 to prove Theorem 1.9 in [5] as a corollary of Theorem 1.

Corollary 1 ([5, Theorem 1.9]). The operad $\mathcal{RT}$ is isomorphic to the operad $\mathcal{PL}$ that encodes the pre-Lie algebras.

Proof. Let $([\bullet, \bullet])$ be the operadic ideal generated by $\{[\bullet, \bullet]\}$ in $\mathcal{PostLie}$ (Example 17). Note that $\frac{\mathcal{PostLie}}{([\bullet, \bullet])} \simeq \mathcal{PL}$. In fact, the $\mathbb{S}$ -module morphism given by $(\bullet \triangleright \bullet) \mapsto (\bullet * \bullet)$ determines this isomorphism, since

$$0 = (\bullet \triangleright \bullet) \circ_1 [\bullet, \bullet] = a_{\triangleright} - a_{\triangleright}^{[213]}.$$

By the isomorphism f , we have $f_2^{-1}([\bullet 1, \bullet 2]) = [\bullet, \bullet] \Rightarrow f^{-1}(\mathcal{J}) = \{f_n^{-1}(\mathcal{J}(n))\} = ([\bullet, \bullet])$.

With $\frac{\mathcal{PostLie}}{f^{-1}(\mathcal{J})} \simeq \mathcal{PL}$ and Theorem 1, the diagram below describes how we obtain an operadic isomorphism from $\mathcal{RT}$ to $\mathcal{PL}$:

$$\begin{array}{ccc} \nabla(n) & \xleftrightarrow{f_n^{-1}} & \mathcal{PostLie}(n) \\ \downarrow \text{proj} & & \downarrow \text{proj} \\ \frac{\nabla(n)}{\mathcal{J}(n)} & \xleftrightarrow{\quad} & \frac{\mathcal{PostLie}(n)}{f_n^{-1}(\mathcal{J}(n))} \\ \updownarrow & & \updownarrow \\ \mathcal{RT}(n) & \xleftrightarrow{\quad} & \mathcal{PL}(n). \end{array}$$

□

Definition 34 ([22]). Consider $T_{\mathcal{C}}$ the set of planar trees labeled by a set $\mathcal{C}$ and $\tau_1, \tau_2 \in T_{\mathcal{C}}$. A **left grafting** product $\tau_1 \triangleright \tau_2$ given by the sum of all the planar trees resulting from attaching the root of τ_1 to all the nodes of the tree τ_2 from the left.

Example 42. For $\mathcal{C} = \{a, b\}$:

$$\begin{array}{c} a \\ | \\ a \end{array} \triangleright \begin{array}{c} a \\ | \\ b \end{array} = \begin{array}{c} a \\ | \\ a \\ | \\ b \end{array} + \begin{array}{c} a \\ | \\ a \\ / \backslash \\ b \quad a \end{array} .$$

Proposition 8 ([22, Proposition 3.1]). Let the free Lie algebra $\mathfrak{g} = \text{Lie}(T_{\mathcal{C}})$ be equipped with a product $\triangleright : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, extended from the **left grafting** defined on $T_{\mathcal{C}}$ as:

$$\begin{aligned}
 x \triangleright [y, z] &= [x \triangleright y, z] + [y, x \triangleright z] \\
 [x, y] \triangleright z &= a_{\triangleright}(x, y, z) - a_{\triangleright}(y, x, z), \quad \forall x, y, z \in \mathfrak{g}.
 \end{aligned}$$

Then $\{\text{Lie}(T_{\mathcal{C}}), \triangleright, [\cdot, \cdot]\}$ is post-Lie.

Proposition 8 shows that the free Lie algebra of planar trees has the structure of a post-Lie algebra $\text{postLie}(\mathcal{C}) := \{\text{Lie}(T_{\mathcal{C}}, \triangleright, [\cdot, \cdot])\}$. Munthe-Kaas and Lundervold call this the free post-Lie algebra over the set $\mathcal{C}$ for the following reason:

Corollary 2 ([22, Theorem 3.2]). For any post-Lie algebra $\{\mathcal{A}, \triangleright, [\cdot, \cdot]\}$ and any function $f : \mathcal{C} \rightarrow \mathcal{A}$, there exists a unique morphism of post-Lie algebras $F : \text{postLie}(\mathcal{C}) \rightarrow \mathcal{A}$ such that $F(c) = f(c)$ for all $c \in \mathcal{C}$.

We proof Corollary 2 by using an operadic approach given by Lemmas 1 and 2.

Proof. Up to isomorphism, $\text{PostLie}(V) = \sum_{n \in \mathbb{N}} (\text{PostLie}(n) \otimes V^{\otimes n})_{\mathbb{S}_n}$ is the free post-Lie algebra (see (1.24)). Since $\text{PostLie} \simeq \nabla$, $\nabla(V)$ is also the free post-Lie algebra.

Due to 1.22, the correspondence below is one-to-one:

$$l_{t,n}([\sigma(1) \cdots \sigma(n)]) \otimes x_1 \cdots x_n \in \nabla(V) \text{ to } l_t(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

For example :

$$\begin{array}{c} 6 \\ | \\ 5 \\ / \backslash \\ 4 \quad 2 \\ | \quad | \\ 1 \quad 7 \end{array} \quad \otimes x_1 x_2 x_3 x_4 x_5 x_6 x_7 \mapsto \begin{array}{c} x_6 \\ | \\ x_5 \\ / \backslash \\ x_4 \quad x_2 \\ | \quad | \\ x_1 \quad x_7 \end{array} .$$
(2.16)

So, if $x_i \in X$, where X is a base of V , $\nabla(V)$ is the vector space generated by planar trees or by parenthesizings of planar trees labeled by X .

The parenthesizings are responsible for the structure of the free Lie algebra in $\nabla(V)$ (see Note 8). To conclude this proof, let's determine the post-Lie product.

Consider the labeled rooted trees $s_1, s_2 \in \nabla$, the special composition $\triangleright$ is defined by:

$$s_1 \triangleright s_2 := \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 s_2 \right) \circ_1 s_1. \quad (2.17)$$

By definition, each summand of $s_1 \triangleright s_2$ is obtained first by connecting $\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array}$ in all the nodes of s_2'' and after the node $\bullet 1$ is replaced by the tree s_1'' , by attaching the root of s_1'' to the outgoing edge of $\bullet 1$. For example:

$$\begin{aligned} \begin{array}{c} \bullet 1 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 3 \end{array} \triangleright \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} &= \left(\begin{array}{c} \bullet 1 \quad \bullet 1 \\ | \quad | \\ \bullet 2 \quad \bullet 2 \end{array} \right) \circ_1 \begin{array}{c} \bullet 1 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 3 \end{array} \\ &= \left(\begin{array}{c} \bullet 1 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 3 \end{array} + \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \\ | \\ \bullet 3 \end{array} \right) \circ_1 \begin{array}{c} \bullet 1 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 3 \end{array} \\ &= \begin{array}{c} \bullet 1 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 3 \quad \bullet 4 \\ \diagdown \quad \diagup \\ \bullet 5 \end{array} + \begin{array}{c} \bullet 1 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 3 \\ | \\ \bullet 4 \\ | \\ \bullet 5 \end{array}. \end{aligned}$$

Thus, up to label change, $s_1 \triangleright s_2$ is the sum of all the trees that result from attaching the root of s_1 to all the nodes of the tree s_2 on the left. With this composition and 1.22, we define a product $\triangleright$ in $\nabla(V)$:

$$(s_1 \otimes x_1 \cdots x_n, s_2 \otimes x_1 \cdots x_m) \mapsto (s_1 \triangleright s_2 \otimes x_1 \cdots x_{n+m}).$$

For instance, if $x, y, z \in X$, then

$$\begin{array}{c} \bullet z \quad \bullet y \\ \diagdown \quad \diagup \\ \bullet x \end{array} \triangleright \begin{array}{c} \bullet y \\ | \\ \bullet x \end{array} = \begin{array}{c} \bullet z \quad \bullet y \\ \diagdown \quad \diagup \\ \bullet x \quad \bullet y \\ \diagdown \quad \diagup \\ \bullet x \end{array} + \begin{array}{c} \bullet z \quad \bullet y \\ \diagdown \quad \diagup \\ \bullet x \\ | \\ \bullet y \\ | \\ \bullet x \end{array}.$$

This product is the **left grafting** (Definition 34) and is a post-Lie product in $\nabla(V)$.

Indeed, it is enough to verify that 2.17 is compatible with the brackets:

$$\begin{aligned} \bullet 1 \triangleright [\bullet 1, \bullet 2] &= \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 [\bullet 1, \bullet 2] \right) \circ_1 \bullet 1 \\ &= \left[\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array}, \bullet 3 \right] + [\bullet 2, \begin{array}{c} \bullet 1 \\ | \\ \bullet 3 \end{array}], \end{aligned}$$

$$\begin{aligned}
[\bullet 1, \bullet 2] \triangleright \bullet 1 &= \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 \bullet 1 \right) \circ_1 [\bullet 1, \bullet 2] \\
&= \begin{array}{c} \bullet 1 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 3 \end{array} - \begin{array}{c} \bullet 2 \quad \bullet 1 \\ \diagdown \quad \diagup \\ \bullet 3 \end{array}.
\end{aligned}$$

Thus, for the labeled rooted trees $s_i \in \nabla(k_i)$, $i \in [3]$:

$$\begin{aligned}
s_1 \triangleright [s_2, s_3] &= \left(\left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 [\bullet 1, \bullet 2] \right) \circ_1 \bullet 1 \right) \circ_3 s_3 \right) \circ_2 s_2 \circ_1 s_1 \\
&= \left(\left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array}, \bullet 3 \right] + \left[\bullet 2, \begin{array}{c} \bullet 1 \\ | \\ \bullet 3 \end{array} \right] \right) \circ_3 s_3 \right) \circ_2 s_2 \circ_1 s_1 \\
&= \left(\left(\left[\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array}, \bullet 3 \right] \circ_3 s_3 \right) \circ_2 s_2 \right) \circ_1 s_1 + \left(\left(\left[\bullet 1, \begin{array}{c} \bullet 2 \\ | \\ \bullet 3 \end{array} \right] \right)^{[213]} \circ_3 s_3 \right) \circ_2 s_2 \circ_1 s_1 \\
&= \left(\left(\left[\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array}, \bullet 3 \right] \circ_3 s_3 \right) \circ_2 s_2 \right) \circ_1 s_1 + \left(\left(\left(\left[\bullet 1, \begin{array}{c} \bullet 2 \\ | \\ \bullet 3 \end{array} \right] \circ_3 s_3 \right) \circ_1 s_2 \right) \circ_2 s_1 \right)^\theta \\
&= [s_1 \triangleright s_2, s_3] + [s_2, s_1 \triangleright s_3], \\
[s_1, s_2] \triangleright s_3 &= \left(\left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 \bullet 1 \right) \circ_1 [\bullet 1, \bullet 2] \right) \circ_3 s_3 \right) \circ_2 s_2 \circ_1 s_1 \\
&= \left(\left(\begin{array}{c} \bullet 1 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 3 \end{array} - \begin{array}{c} \bullet 2 \quad \bullet 1 \\ \diagdown \quad \diagup \\ \bullet 3 \end{array} \circ_3 s_3 \right) \circ_2 s_2 \right) \circ_1 s_1 \\
&= \left(\left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} - \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_1 \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} - \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} - \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_1 \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \right)^{[213]} \right) \circ_3 s_3 \right) \circ_2 s_2 \right) \circ_1 s_1 \\
&= \left(\left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \right) \circ_3 s_3 \right) \circ_2 s_2 \right) \circ_1 s_1 - \left(\left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_1 \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \right) \circ_3 s_3 \right) \circ_2 s_2 \right) \circ_1 s_1 \\
&\quad - \left(\left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \right) \circ_3 s_3 \right) \circ_1 s_2 \right) \circ_2 s_1 \right)^\theta + \left(\left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_1 \begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \right) \circ_3 s_3 \right) \circ_1 s_2 \right) \circ_2 s_1 \right)^\theta \\
&= \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 \left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 s_3 \right) \circ_1 s_2 \right) \right) \circ_1 s_1 - \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_1 \left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_1 s_1 \right) \circ_2 s_2 \right) \right) \circ_2 s_3 \\
&\quad - \left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 \left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 s_3 \right) \circ_1 s_1 \right) \right) \circ_1 s_2 \right)^\theta + \left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_1 \left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_1 s_2 \right) \circ_2 s_1 \right) \right) \circ_2 s_3 \right)^\theta \\
&= \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 (s_3 \triangleright s_2) \right) \circ_1 s_1 - \left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_1 (s_1 \triangleright s_2) \right) \circ_2 s_3 \\
&\quad - \left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_2 (s_1 \triangleright s_3) \right) \circ_1 s_2 \right)^\theta + \left(\left(\begin{array}{c} \bullet 1 \\ | \\ \bullet 2 \end{array} \circ_1 (s_2 \triangleright s_1) \right) \circ_2 s_3 \right)^\theta \\
&= s_1 \triangleright (s_2 \triangleright s_3) - (s_1 \triangleright s_2) \triangleright s_3 - (s_2 \triangleright (s_1 \triangleright s_3) - (s_2 \triangleright s_1) \triangleright s_3),
\end{aligned}$$

in which θ is the inverse of the bloch permutation $[213](id_{k_1} \oplus id_{k_2} \oplus id_{k_3})$ and the label changes necessary to describe the operadic compositions are omitted to simplify the notation. $\square$

∇-ALGEBRAS

In Sections 3.2 and 3.3, some properties of the ∇ -algebras are used to alternatively prove Propositions 3.1 and 3.2 of Ebrahimi-Fard et al. [8] and Proposition 5.3 of Guin and Oudom in [16]. Section 3.1 is dedicated to present some prerequisites of Sections 3.2 and 3.3 and give an introduction about these sections. We also explain, in Section 3.1, the problem that gave rise to this thesis.

3.1 A bit more context

Considering an operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n \in \mathbb{N}}$, we denote

$$((\cdots((x \circ_n y_n) \circ_{n-1} y_{n-1}) \cdots) \circ_1 y_1), \quad x \in \mathcal{P}(n), \quad y_i \in \mathcal{P}(m_i), \quad 1 \leq i \leq n \text{ by } \gamma_{n;m_1, \dots, m_n}(x; y_1, \dots, y_n).$$

The collection of multilinear operations called **braces** on $\oplus_{n \in \mathbb{N}} \mathcal{P}(n)$ is given by:

$$\{x; y_1, \dots, y_n\} = \sum \gamma_{m; k_1, \dots, k_m}(x; 1, \dots, 1, y_1, 1, \dots, 1, y_n, 1, \dots, 1), \quad (3.1)$$

for $x \in \mathcal{P}(m)$, $m \geq n$, $y_i \in \mathcal{P}(m_i)$, $1 \leq i \leq n$, $1 \in \mathcal{P}(1)$ the unit (see 1.5), in which the sum runs over all the possible substitutions of $y_1, \dots, y_n$ into $\gamma_{m; k_1, \dots, k_m}$, in the same order and completing with the unit if necessary.¹ To represent the use of this unit, the convention $\{x; \emptyset\} := x$ is used.

Example 43. For $x \in \mathcal{P}(3)$, $y_1 \in \mathcal{P}(m_1)$, $y_2 \in \mathcal{P}(m_2)$:

$$\{x; y_1, y_2\} = \gamma_{3; 1, m_1, m_2}(x; 1, y_1, y_2) + \gamma_{3; m_1, 1, m_2}(x; y_1, 1, y_2) + \gamma_{3; m_1, m_2, 1}(x; y_1, y_2, 1).$$

Due to the unit, parallel and sequential axioms, the following properties hold:

$$\{\{x; y_1, \dots, y_n\}; z_1, \dots, z_m\} = \sum \{x; z_1, \dots, z_{i_1}, \{y_1; z_{i_1+1}, \dots, z_{j_1}\}, z_{j_1+1}, \dots, z_{i_n}, \{y_n; z_{i_n+1}, \dots, z_{j_n}\}, z_{j_n+1}, \dots, z_m\}. \quad (3.2)$$

¹ In [15], in the original definition a sign is used. However, in our context, it is not relevant and the sign is omitted.

Roughly, on the right side of the equality, the elements $\{z_1, \dots, z_m\}$ are written between the y_i 's in all the possible ways, in the same order and considering the possible products between y_i and the z_j 's with the convention $\{y_i; \emptyset\} := y_i$.

Example 44. For $x \in \mathcal{P}(3)$, $y_1 \in \mathcal{P}(m_1)$, $y_2 \in \mathcal{P}(m_2)$, $z \in \mathcal{P}(n)$:

$$\{\{x; y_1, y_2\}; z\} = \{x; z, y_1, y_2\} + \{x; y_1, z, y_2\} + \{x; \{y_1; z\}, y_2\} + \{x; y_1, y_2, z\} + \{x; y_1, \{y_2; z\}\}.$$

Definition 35. A (right) **brace algebra** is a vector space V with a family of linear maps $\{\{-; -, \dots, -\}: V^{\otimes n} \rightarrow V\}_{n \in \mathbb{N}, n \geq 2}$ such that relations (3.2) and the convention $\{y_i; \emptyset\} := y_i$ hold.

Proposition 9 ([20][Proposition 13.11.4]). Given an operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n \in \mathbb{N}}$, the vector space $\mathcal{P} = \bigoplus_n \mathcal{P}(n)$ is a brace algebra with the operations $B_{1,n}: \mathcal{P} \otimes \mathcal{P}^n \rightarrow \mathcal{P}$ defined as follows:

- $B_{1,n}(x; y_1, \dots, y_n) = 0$, if $m < n$;
- $B_{1,n}(x; y_1, \dots, y_n) = \{x; y_1, \dots, y_n\}$, if $m \geq n$.

For $x \in \mathcal{P}(m)$, $y_i \in \mathcal{P}(n_i)$.

Every brace algebra is a pre-Lie algebra. Indeed, the product $\{; \}: V^{\otimes 2} \rightarrow V$ is pre-Lie:

$$\begin{aligned} \{\{x; y\}; z\} - \{x; \{y; z\}\} &= \{x; z, y\} + \{x; y, z\} \\ &= \{\{x; z\}; y\} - \{x; \{z; y\}\}, \quad \forall x, y, z \in V. \end{aligned}$$

The problem that gave rise to this thesis: Since a pre-Lie algebra with a bracket identically zero is a post-Lie algebra, it is natural to think if there is an algebra similar to brace algebras in the post-Lie case.

Unfortunately, we do not have an answer to this question. However, as we see next, if instead of brace algebras, we consider symmetric brace algebras, the ∇ -algebras provide an answer.

Symmetric brace algebras are defined by Lada and Daily in [7]. They also prove in [6] that the symmetrization of a brace algebra is a symmetric brace algebra. To define symmetric brace algebras, it is interesting to define **shuffles** beforehand.

Definition 36. A (p, q) -**shuffle** is a sequence of natural numbers $[i_1 \dots i_p \mid j_1 \dots j_q]$ that is a permutation of $[p+q]$ and $i_1 < \dots < i_p$, $j_1 < \dots < j_q$.

The associated permutation $\sigma \in \mathbb{S}_{p+q}$ given by

$$\sigma(1) = i_1, \dots, \sigma(p) = i_p, \sigma(p+1) = j_1, \dots, \sigma(p+q) = j_q$$

is also called (p, q) -**shuffle** by abuse of terminology. The subset of $\mathbb{S}_{p+q}$ of all (p, q) -shuffles is denoted by $Sh(p, q)$. We also consider $\{id_p\}$ as shuffle, i. e. $q = 0$, in this case we denote $Sh(p, q) = Sh(p) = \{id_p\}$.

Example 45. The unique $(1, 2)$ -shuffles are $[1|23]$, $[2|13]$, $[3|12]$.

The idea of $(i_1, \dots, i_k)$ -shuffle ($Sh(i_1, \dots, i_k)$) is extended naturally.

Definition 37. A (left) **symmetric brace algebra** is a vector space V with a family $\{\langle - \dots -; - \rangle : V^{\otimes n} \rightarrow V\}_{n \in \mathbb{N}, n \geq 2}$ of linear maps that are symmetric on the left, such that for all $x, y_i, z_j \in V$:

$$\langle \emptyset; x \rangle := x, \quad (3.3)$$

$$\langle \emptyset, y_1, \dots, y_n; x \rangle := \langle y_1, \dots, y_n; x \rangle, \quad (3.4)$$

$$\langle z_1, \dots, z_m; \langle y_1, \dots, y_n; x \rangle \rangle = \sum_{\sigma} \langle \dots, \langle \dots; y_1 \rangle, \langle \dots; y_2 \rangle, \langle \dots; y_3 \rangle, \dots, \langle \dots; y_n \rangle; x \rangle. \quad (3.5)$$

On the left-hand side, the dots are filled with the variables z_i 's, indexed by all the possible shuffle sequences $\sigma \in Sh(t_1, \dots, t_j)$ of $[m]$, by using $\emptyset$ if necessary.

Example 46. For $\langle z_1; \langle y_1; x \rangle \rangle$, only $\{[1]\} = Sh(1)$ is considered:

$$\begin{aligned} \langle z_1; \langle y_1; x \rangle \rangle &= \langle \emptyset, \langle z_1; y_1 \rangle; x \rangle + \langle z_1, \langle \emptyset; y_1 \rangle; x \rangle \\ &= \langle \langle z_1; y_1 \rangle; x \rangle + \langle z_1, y_1; x \rangle. \end{aligned}$$

However, for $\langle z_1, z_2; \langle y_1, y_2; x \rangle \rangle$, $Sh(2) = \{id_2\}$, $Sh(1, 1) = \{[1|2], [2|1]\}$ are considered:

$$\begin{aligned} \langle z_1, z_2; \langle y_1, y_2; x \rangle \rangle &= \langle \emptyset, \langle z_1, z_2; y_1 \rangle, \langle \emptyset; y_2 \rangle; x \rangle + \langle \emptyset, \langle \emptyset; y_1 \rangle, \langle z_1, z_2; y_2 \rangle; x \rangle \\ &\quad + \langle z_1, z_2, \langle \emptyset; y_1 \rangle, \langle \emptyset; y_2 \rangle; x \rangle + \langle \emptyset, \langle z_1; y_1 \rangle, \langle z_2; y_2 \rangle; x \rangle + \langle \emptyset, \langle z_2; y_1 \rangle, \langle z_1; y_2 \rangle; x \rangle \\ &\quad + \langle z_1, \langle z_2; y_1 \rangle, \langle \emptyset; y_2 \rangle; x \rangle + \langle z_2, \langle z_1; y_1 \rangle, \langle \emptyset; y_2 \rangle; x \rangle + \langle z_1, \langle \emptyset; y_1 \rangle, \langle z_2; y_2 \rangle; x \rangle \\ &\quad + \langle z_2, \langle \emptyset; y_1 \rangle, \langle z_1; y_2 \rangle; x \rangle \\ &= \langle \langle z_1, z_2; y_1 \rangle, y_2; x \rangle + \langle y_1, \langle z_1, z_2; y_2 \rangle; x \rangle + \langle z_1, z_2, y_1, y_2; x \rangle \\ &\quad + \langle \langle z_1; y_1 \rangle, \langle z_2; y_2 \rangle; x \rangle + \langle \langle z_2; y_1 \rangle, \langle z_1; y_2 \rangle; x \rangle + \langle z_1, \langle z_2; y_1 \rangle, y_2; x \rangle \\ &\quad + \langle z_2, \langle z_1; y_1 \rangle, y_2; x \rangle + \langle z_1, y_1, \langle z_2; y_2 \rangle; x \rangle + \langle z_2, y_1, \langle z_1; y_2 \rangle; x \rangle. \end{aligned}$$

Lastly, for $\langle z_1, z_2, z_3; \langle y_1; x \rangle \rangle$, $Sh(3) = \{id_3\}$, $Sh(1, 2) = \{[1|23], [2|13], [3|12]\}$, $Sh(2, 1) = \{[12|3], [13|2], [23|1]\}$ are considered :

$$\begin{aligned} \langle z_1, z_2, z_3; \langle y_1; x \rangle \rangle &= \langle \emptyset, \langle z_1, z_2, z_3; y_1 \rangle; x \rangle + \langle z_1, z_2, z_3, \langle \emptyset; y_1 \rangle; x \rangle + \langle z_1, \langle z_2, z_3; y_1 \rangle; x \rangle \\ &\quad + \langle z_2, \langle z_1, z_3; y_1 \rangle; x \rangle + \langle z_3, \langle z_1, z_2; y_1 \rangle; x \rangle + \langle z_1, z_2, \langle z_3; y_1 \rangle; x \rangle \\ &\quad + \langle z_1, z_3, \langle z_2; y_1 \rangle; x \rangle + \langle z_2, z_3, \langle z_1; y_1 \rangle; x \rangle \\ &= \langle \langle z_1, z_2, z_3; y_1 \rangle; x \rangle + \langle z_1, z_2, z_3, y_1; x \rangle + \langle z_1, \langle z_2, z_3; y_1 \rangle; x \rangle \\ &\quad + \langle z_2, \langle z_1, z_3; y_1 \rangle; x \rangle + \langle z_3, \langle z_1, z_2; y_1 \rangle; x \rangle + \langle z_1, z_2, \langle z_3; y_1 \rangle; x \rangle \\ &\quad + \langle z_1, z_3, \langle z_2; y_1 \rangle; x \rangle + \langle z_2, z_3, \langle z_1; y_1 \rangle; x \rangle. \end{aligned}$$

Besides Lada and Daily's papers, the reader can find more information on symmetric brace algebras in the Ph.D theses of Capaldi [3] and Hillman [17]. We also suggest our main reference [20].

As well as brace algebras, every (left) symmetric brace algebra is a (left) pre-Lie algebra. Indeed, the map $\langle -; - \rangle : V^{\otimes 2} \rightarrow V$ is a pre-Lie product:

$$\langle z; \langle y; x \rangle \rangle - \langle \langle z; y \rangle; x \rangle = \langle z, y; x \rangle = \langle y, z; x \rangle = \langle y; \langle z; x \rangle \rangle - \langle \langle y; z \rangle; x \rangle, \forall x, y, z \in V.$$

However, the relation between pre-Lie algebras and symmetric brace algebras is stronger. In fact, the categories of pre-Lie algebras and symmetric brace algebras are isomorphic, i. e., given a pre-Lie product, it is possible to obtain symmetric brace products. This result is due to Proposition 5.3 of Guin and Oudom in [16]. Daily and Lada in [7] also prove the same result with a different approach. The operadic approach used in this work avoid us to obtain this isomorphism of categories by using the equivalence between the categories of ∇ -algebras and post-Lie algebras (see Propositions 11 and 12).

Definition 38. A **unital associative algebra** (H, m, u) is an associative algebra (H, m) with a linear map $u : \mathbb{K} \rightarrow H$ (unit) in which the following diagram commutes:

$$\begin{array}{ccccc} H \otimes \mathbb{K} & \xrightarrow{\quad} & H & \xleftarrow{\quad} & \mathbb{K} \otimes H \\ \downarrow id \otimes u & & \downarrow id & & \downarrow u \otimes id \\ H \otimes H & \xrightarrow{m} & H & \xleftarrow{m} & H \otimes H. \end{array}$$

Definition 39. Let (H_1, m_1, u_1) and (H_2, m_2, u_2) be unital associative algebras. A **unital associative algebra morphism** or just algebra morphism, is a linear map $\phi : H_1 \rightarrow H_2$ such that the following diagrams commute:

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi} & H_2 \\ m_1 \uparrow & & \uparrow m_2 \\ H_1 \otimes H_1 & \xrightarrow{\phi \otimes \phi} & H_2 \otimes H_2 \end{array} \quad \begin{array}{ccc} H_1 & \xrightarrow{\phi} & H_2 \\ u_1 \uparrow & \nearrow u_2 & \\ \mathbb{K} & & \end{array}$$

Definition 40. A **counital coassociative coalgebra** (H, Δ, ε) is a vector space H with two linear maps $H \xrightarrow{\Delta} H \otimes H$ (coproduct) and $H \xrightarrow{\varepsilon} \mathbb{K}$ (counit) such that the following diagrams commute:

$$\begin{array}{ccc} & H \otimes H \otimes H & \\ \Delta \otimes id \nearrow & & \searrow id \otimes \Delta \\ H \otimes H & & H \otimes H \\ \Delta \nwarrow & & \nearrow \Delta \\ & H & \end{array} \quad \begin{array}{ccccc} H \otimes \mathbb{K} & \xleftarrow{\quad} & H & \xrightarrow{\quad} & \mathbb{K} \otimes H \\ \uparrow id \otimes \varepsilon & & \uparrow id & & \uparrow \varepsilon \otimes id \\ H \otimes H & \xleftarrow{\Delta} & H & \xrightarrow{\Delta} & H \otimes H \end{array}$$

Remark 9 (Sweedler's notation). Let H be a coalgebra. Given $c \in H$, we denote

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}.$$

To indicate that we apply Δ on $c_{(j)}$ we will write $\Delta(c_{(j)}) = c_{(j:1)} \otimes c_{(j:2)}$.

Definition 41. Let $(H_1, \Delta_1, \varepsilon_1)$ and $(H_2, \Delta_2, \varepsilon_2)$ be counital coassociative coalgebras. A **counital coassociative coalgebra morphism** or just coalgebra morphism, is a linear map $\psi: H_1 \rightarrow H_2$ such that the following diagrams commute:

$$\begin{array}{ccc} H_2 & \xleftarrow{\psi} & H_1 \\ \Delta_2 \downarrow & & \downarrow \Delta_1 \\ H_2 \otimes H_2 & \xleftarrow{\psi \otimes \psi} & H_1 \otimes H_1 \end{array} \qquad \begin{array}{ccc} H_2 & \xleftarrow{\psi} & H_1 \\ \varepsilon_2 \downarrow & \swarrow \varepsilon_1 & \\ \mathbb{K} & & \end{array}$$

Definition 42. The vector space H is a **bialgebra** if (H, m, u) is a unital associative algebra, (H, Δ, ε) is an counital coassociative coalgebra, m, u are algebra morphisms and Δ, ε are coalgebra morphisms.

Definition 43. A **Hopf algebra** $(H, m, u, \Delta, \varepsilon)$ is a bialgebra with a linear map $S: H \rightarrow H$, called **antipode** map, such that the following diagram commutes:

$$\begin{array}{ccccc} & & H \otimes H & \xrightarrow{S \otimes id} & H \otimes H \\ & \nearrow \Delta & & & \downarrow m \\ H & \xrightarrow{\varepsilon} & \mathbb{K} & \xrightarrow{u} & H \\ & \searrow \Delta & & & \uparrow m \\ & & H \otimes H & \xrightarrow{id \otimes S} & H \otimes H \end{array}$$

We indicate the following references, in case the reader is not familiar with the definition of Hopf algebras: Underwood [23], Abe [1], Loday and Vallette [20].

Example 47. Given a vector space $\mathfrak{g}$, the tensor algebra $T(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N}} \mathfrak{g}^{\otimes n}$ is a Hopf algebra with the associative product $m: T(\mathfrak{g}) \otimes T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$ given by $m(x_1 \cdots x_n \otimes x_1 \cdots x_m) = x_1 \cdots x_{n+m}$ and the coassociative coproduct $\Delta: T(\mathfrak{g}) \rightarrow T(\mathfrak{g}) \otimes T(\mathfrak{g})$, given by

$$\Delta(1) = 1 \otimes 1,$$

$$\Delta(x_1 \cdots x_n) = x_1 \cdots x_n \otimes 1 + 1 \otimes x_1 \cdots x_n + \sum_{\substack{p+q=n \\ \sigma \in Sh(p,q)}} x_{\sigma(1)} \cdots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \cdots x_{\sigma(p+q)}.$$

The coproduct Δ is also known as **shuffle coproduct**.

If we consider the ideals $\mathcal{I}_1, \mathcal{I}_2$ generated by

$$\{x_1 x_2 - x_2 x_1; x_1, x_2 \in \mathfrak{g}\},$$

$$\{[x_1, x_2] - x_1 x_2 + x_2 x_1; x_1, x_2 \in \mathfrak{g}, \mathfrak{g} \text{ is a Lie algebra with the Lie bracket } [\cdot, \cdot]\}$$

respectively, then we have the symmetric algebra

$$S(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I}_1$$

and the enveloping algebra

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I}_2,$$

which are also Hopf algebras with the maps induced by m and Δ .

Remark 10. A coproduct is cocommutative if $\tau \circ \Delta = \Delta$, with $\tau: H \otimes H \rightarrow H \otimes H$ the isomorphism given by $\tau(x \otimes y) = y \otimes x$. For example, the shuffle coproduct on $T(\mathfrak{g})$ is cocommutative. In fact, given $z = x_1 \cdots x_n \in \mathfrak{g}^{\otimes n}$,

$$\Delta(x_1 \cdots x_n) = x_1 \cdots x_n \otimes 1 + 1 \otimes x_1 \cdots x_n + \sum_{\substack{p+q=n \\ \sigma \in Sh(p,q)}} x_{\sigma(1)} \cdots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \cdots x_{\sigma(p+q)}.$$

Note that, for each

$$[\sigma(1) \dots \sigma(p) | \sigma(p+1) \dots \sigma(p+q)] \in Sh(p, q)$$

we have $\beta \in Sh(q, p)$ with

$$\beta(1) = \sigma(p+1), \dots, \beta(q) = \sigma(p+q), \beta(q+1) = \sigma(1), \dots, \beta(q+p) = \sigma(p).$$

Thus, the cocommutativity holds for the shuffle coproduct:

$$\Delta(z) = z_{(1)} \otimes z_{(2)} = z_{(2)} \otimes z_{(1)} = \tau \circ \Delta(z).$$

However, the shuffle coproduct has a property stronger than the cocommutativity. For

$$\Delta \otimes \Delta(\Delta(z)) = z_{(1:1)} \otimes z_{(1:2)} \otimes z_{(2:1)} \otimes z_{(2:2)},$$

we have that all $z_{(i:j)}$ commutes, since each summand is indexed by $\sigma \in Sh(p_1, p_2, p_3, p_4)$ with $p_1 + p_2 + p_3 + p_4 = n$.

Definition 44. A unital associative algebra V is a **filtered algebra** if $V = \bigoplus V_i$ such that, each V_i is a vector space satisfying $\mathbb{K} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \cdots$ and $V_i V_j \subseteq V_{i+j}$.

In Abe [1], the reader finds more details on filtered algebras and the enveloping algebra of a Lie algebra.

Example 48. The enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a filtered algebra. Indeed, the filtration in $\mathcal{U}(\mathfrak{g})$ is induced by the filtration in $T(\mathfrak{g})$ given by:

$$T_k(\mathfrak{g}) = \mathbb{K} \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \cdots \oplus \mathfrak{g}^{\otimes k}.$$

If $(\mathfrak{g}, *)$ is a pre-Lie algebra, Guin and Oudom in [16, Lemma 3.11] define an associative product that extends the pre-Lie product from $\mathfrak{g}$ to $S(\mathfrak{g})$. This new product and the shuffle coproduct also determine a Hopf algebra structure on $S(\mathfrak{g})$.

Since a symmetric algebra with a bracket $[\cdot, \cdot] \equiv 0$ is an enveloping algebra of a Lie algebra, the question is if it is possible to define a similar product on enveloping algebras. The answer is yes. Indeed, Ebrahimi-Fard et al. [8, Proposition 3.3] define an associative product for the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a post-Lie algebra $(\mathfrak{g}, \triangleright, [\cdot, \cdot])$, similar to the product defined by Gui and Oudom.

The aim of Section 3.3 is to use properties of the ∇ -algebras, displayed in Section 3.2, to prove Propositions 3.1 and 3.2 of Ebrahimi-Fard et al. [8].

3.2 Some properties of ∇ -algebras

According to Definition 16, a ∇ -algebra is a vector space V , with an operadic morphism $E: \nabla \rightarrow \mathcal{E}nd(V)$. The elements and operadic compositions of ∇ determine in $\mathcal{E}nd(V)$ operations on V (Definition 7). For instance, $E([\bullet 1, \bullet 2]) = [\cdot, \cdot]: V \otimes V \rightarrow V$, such that $[\cdot, \cdot]$ is a Lie bracket:

$$\begin{aligned} 0 &= E([\bullet 1, \bullet 2] + [\bullet 2, \bullet 1]) \\ &= E([\bullet 1, \bullet 2]) + E([\bullet 1, \bullet 2])^{[21]} \\ &= [\cdot, \cdot] + [\cdot, \cdot]^{[21]} \end{aligned} \quad (3.6)$$

$$\begin{aligned} 0 &= E([\bullet 1, [\bullet 2, \bullet 3]] + [\bullet 3, [\bullet 1, \bullet 2]] + [[\bullet 1, \bullet 3], \bullet 2]) \\ &= E([\bullet 1, [\bullet 2, \bullet 3]] + E([\bullet 1, [\bullet 2, \bullet 3]])^{[231]} + E([\bullet 1, \bullet 2], \bullet 3)]^{[132]} \\ &= E([\bullet 1, \bullet 2] \circ_2 [\bullet 1, \bullet 2]) + E([\bullet 1, \bullet 2] \circ_2 [\bullet 1, \bullet 2])^{[231]} + E([\bullet 1, \bullet 2] \circ_1 [\bullet 1, \bullet 2])^{[132]} \\ &= [\cdot, \cdot] \circ_2 [\cdot, \cdot] + ([\cdot, \cdot] \circ_2 [\cdot, \cdot])^{[231]} + ([\cdot, \cdot] \circ_1 [\cdot, \cdot])^{[132]} \end{aligned} \quad (3.7)$$

Due to 3.6 $[x_1, x_2] = -[x_2, x_1]$, $\forall x_1, x_2 \in V$. Due to 3.7 and 3.6 $[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + [x_2, [x_1, x_3]]$, $\forall x_1, x_2, x_3 \in V$, i.e., Jacobi holds.

The ∇ -algebras verify other relations associated to compositions 2.10 and 2.11.

Proposition 10. Let $E \left(\begin{array}{c} \bullet 1 \quad \bullet 2 \quad \dots \quad \bullet n \\ \diagdown \quad \diagup \quad \diagup \quad \diagdown \\ \bullet n+1 \end{array} \right) := T_{n,1}(-, \dots, -; -)$. For all $n, m \in \mathbb{N}^*$, $1 \leq i \leq n, x_1, \dots, x_{n+2} \in V$:

1. $(T_{n,1} \circ_i [\cdot, \cdot])(x_1, \dots, x_{n+2}) = T_{n,1}(x_1, \dots, [x_i, x_{i+1}], \dots; x_{n+2}) = T_{n+1,1}(x_1, \dots, x_i, x_{i+1}, \dots; x_{n+2}) - T_{n+1,1}(x_1, \dots, x_{i+1}, x_i, \dots; x_{n+2})$;
2. $T_{n,1} \circ_{n+1} T_{m,1}$ satisfy relations 3.5 with conventions 3.3 and 3.4.
3. $(T_{n,1} \circ_{n+1} [\cdot, \cdot])(x_1, \dots, x_{n+2}) = T_{n,1}(x_1, x_2, \dots, x_n; [x_{n+1}, x_{n+2}]) = [T_{n,1}(x_1, x_2, \dots, x_n; x_{n+1}), x_{n+2}] + \sum_{\substack{p+q=n \\ \sigma \in Sh(p,q)}} [T_{p,1}(x_{\sigma(1)}, \dots, x_{\sigma(p)}; x_{n+1}), T_{q,1}(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)}; x_{n+2})] + [x_{n+1}, T_{n,1}(x_1, \dots, x_n; x_{n+2})]$;

Proof. In fact,

$$\begin{aligned} T_{n,1} \circ_i [\cdot, \cdot] &= E \left(\begin{array}{c} \bullet 1 \quad \bullet 2 \quad \dots \quad \bullet n \\ \diagdown \quad \diagup \quad \diagup \quad \diagdown \\ \bullet n+1 \end{array} \circ_i [\bullet 1, \bullet 2] \right), \\ T_{n,1} \circ_{n+1} T_{m,1} &= E \left(\begin{array}{c} \bullet 1 \quad \bullet 2 \quad \dots \quad \bullet n \\ \diagdown \quad \diagup \quad \diagup \quad \diagdown \\ \bullet n+1 \end{array} \circ_{n+1} \begin{array}{c} \bullet 1 \quad \bullet 2 \quad \dots \quad \bullet m \\ \diagdown \quad \diagup \quad \diagup \quad \diagdown \\ \bullet m+1 \end{array} \right), \\ T_{n,1} \circ_{n+1} [\cdot, \cdot] &= E \left(\begin{array}{c} \bullet 1 \quad \bullet 2 \quad \dots \quad \bullet n \\ \diagdown \quad \diagup \quad \diagup \quad \diagdown \\ \bullet n+1 \end{array} \circ_{n+1} [\bullet 1, \bullet 2] \right). \end{aligned}$$

The proposition follows from operadic compositions 2.10 and 2.11. $\square$

A further discussion on the proof above may be enlightening, in case the reader feels comfortable about it may skip to page 82.

To enlighten the argument used to prove Proposition 10, we can use the label defined for rooted trees and brackets of rooted trees. Since E is an operadic morphism,

$$E \left(\begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \diagdown \quad \diagup \quad \diagup \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \end{array} \right) = E \left(\begin{array}{c} \sigma^{-1}(1) \quad \sigma^{-1}(2) \quad \dots \quad \sigma^{-1}(n) \\ \diagdown \quad \diagup \quad \diagup \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \end{array} \right) = T_{n,1}^\sigma, \text{ with}$$

$$T_{n,1}^\sigma(x_1, \dots, x_n; x_{n+1}) = T_{n,1}(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}; x_{\sigma^{-1}(n+1)}).$$

Therefore, the variable positions in $T_{n,1}$ (from left to right) correspond to the labels of the rooted tree (see 2.4). In a similar approach to the one we use in 2.16, we can describe the elements $T_{n,1}(x_1, \dots, x_n; x_{n+1})$ by using the following correspondence:

$$T_{n,1}(x_1, \dots, x_n; x_{n+1}) \mapsto \begin{array}{c} x_1 \quad x_2 \quad \dots \quad x_n \\ \diagdown \quad \diagup \quad \diagup \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \end{array} \quad (3.8)$$

The idea is to use 3.8 to obtain relations of ∇ -algebras hold due to operadic compositions of ∇ . By the proof of Lemma 3:

- Every labeled rooted tree that is not a corolla is a composition of corollas;
- Every bracket of two labeled rooted trees is a composition of labeled rooted trees with $[\bullet 1, \bullet 2]$.

With proper adjustments, it does not matter if we use the label $l_{t,n}(\sigma)$ or $l_t(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ ($\forall t \in M(\mathbb{T}), n \in \mathbb{N}^*, \sigma \in \mathbb{S}_n$) to calculate the operadic compositions and the action of $\mathbb{S}_n$.

Example 49. For $\sigma = [1324]$:

$$\begin{array}{ccc} \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \circ_2 \begin{array}{c} 1 \\ \bullet \\ 2 \end{array} & = & \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \circ_2 \begin{array}{c} 2 \\ \bullet \\ 3 \end{array} \\ & = & \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ & & \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} x_1 \quad x_2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \circ_{x_2} \begin{array}{c} x_1 \\ \bullet \\ x_2 \end{array} & = & \begin{array}{c} x_1 \quad x_2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \circ_{x_2} \begin{array}{c} x_2 \\ \bullet \\ x_3 \end{array} \\ & = & \begin{array}{c} x_1 \quad x_2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \end{array}$$

$$\begin{array}{ccc} \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} [1324] & = & \begin{array}{c} \sigma^{-1}(1) \quad \sigma^{-1}(2) \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ & = & \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} x_1 \quad x_2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} [1324] & = & \begin{array}{c} x_{\sigma^{-1}(1)} \quad x_{\sigma^{-1}(2)} \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ & = & \begin{array}{c} x_1 \quad x_2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \end{array}$$

Thus, the map E guarantees the translation given by 3.8 can be used to describe a composition of maps $T_{n,1}$ or maps $T_{n,1}$ with $[\cdot, \cdot]$. For instance,

$$\begin{aligned}
 E \left(\begin{array}{c} \bullet 1 \quad \bullet 3 \\ \diagdown \quad \diagup \\ \bullet 4 \quad \bullet 2 \end{array} \right) &= E \left(\begin{array}{c} \bullet 1 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 3 \end{array} \circ_2 \begin{array}{c} \bullet 1 \\ \bullet 2 \end{array} \right)^{[1324]} \\
 &= \left(E \left(\begin{array}{c} \bullet 1 \quad \bullet 2 \\ \diagdown \quad \diagup \\ \bullet 3 \end{array} \right) \circ_2 E \left(\begin{array}{c} \bullet 1 \\ \bullet 2 \end{array} \right) \right)^{[1324]} \\
 &= (T_{2,1} \circ_2 T_{1,1})^{[1324]}.
 \end{aligned}$$

By using 3.8:

$$\begin{array}{c} \bullet x_1 \quad \bullet x_3 \\ \diagdown \quad \diagup \\ \bullet x_4 \quad \bullet x_2 \end{array} = \left(\begin{array}{c} \bullet x_1 \quad \bullet x_2 \\ \diagdown \quad \diagup \\ \bullet x_3 \end{array} \circ_{x_2} \begin{array}{c} \bullet x_1 \\ \bullet x_2 \end{array} \right)^{[1324]} \\
 T_{2,1}(x_1, T_{1,1}(x_3, x_2); x_4) \quad (T_{2,1} \circ_2 T_{1,1})^{[1324]}(x_1, x_2, x_3; x_4)$$

Returning to Proposition 10, by Definition 2.11:

$$\begin{aligned}
 T_{n,1} \circ_i [\cdot, \cdot] &= E \left(\begin{array}{c} \bullet 1 \quad \bullet 2 \quad \dots \quad \bullet n \\ \diagdown \quad \diagup \quad \quad \diagup \\ \bullet n+1 \end{array} \circ_i [\bullet 1, \bullet 2] \right) \\
 &= E \left(\begin{array}{c} \bullet 1 \quad \bullet 2 \quad \dots \quad \bullet i \quad \bullet i+1 \quad \dots \quad \bullet n+1 \\ \diagdown \quad \diagup \quad \quad \diagup \quad \diagdown \quad \diagup \\ \bullet n+2 \end{array} \right) - E \left(\begin{array}{c} \bullet 1 \quad \bullet 2 \quad \dots \quad \bullet i+1 \quad \bullet i \quad \dots \quad \bullet n+1 \\ \diagdown \quad \diagup \quad \quad \diagup \quad \diagdown \quad \diagup \\ \bullet n+2 \end{array} \right)
 \end{aligned}$$

The labels of the rooted trees determine the order of the variables in $T_{n+1,1}$, thus 1 is immediate.

Example 50. Considering the composition $T_{3,1} \circ_4 T_{1,1}(x_1, x_2, x_3, x_4; x_5) = T_{3,1}(x_1, x_2, x_3; T_{1,1}(x_4; x_5))$, an example of property 2 is:

$$\begin{array}{c} \bullet x_1 \quad \bullet x_2 \quad \bullet x_3 \\ \diagdown \quad \diagup \quad \diagup \\ \bullet x_4 \end{array} \circ_{x_4} \begin{array}{c} \bullet x_1 \\ \bullet x_2 \end{array} = \begin{array}{c} \bullet x_1 \quad \bullet x_2 \quad \bullet x_3 \quad \bullet x_4 \\ \diagdown \quad \diagup \quad \diagup \quad \diagup \\ \bullet x_5 \end{array} + \begin{array}{c} \bullet x_1 \quad \bullet x_2 \\ \diagdown \quad \diagup \\ \bullet x_3 \end{array} \circ_{x_5} \begin{array}{c} \bullet x_4 \end{array} \\
 T_{3,1} \circ_4 T_{1,1}(x_1, x_2, x_3, x_4; x_5) \quad T_{4,1}(x_1, x_2, x_3, x_4; x_5) \quad T_{2,1}(x_3, T_{2,1}(x_1, x_2; x_4); x_5)
 \end{array} \quad (3.9)$$

$$\begin{aligned}
 &+ \begin{array}{c} \bullet x_1 \quad \bullet x_3 \\ \diagdown \quad \diagup \\ \bullet x_4 \end{array} \circ_{x_5} \begin{array}{c} \bullet x_2 \end{array} + \begin{array}{c} \bullet x_2 \quad \bullet x_3 \\ \diagdown \quad \diagup \\ \bullet x_4 \end{array} \circ_{x_5} \begin{array}{c} \bullet x_1 \end{array} \\
 &+ \begin{array}{c} \bullet x_1 \quad \bullet x_2 \quad \bullet x_3 \\ \diagdown \quad \diagup \quad \diagup \\ \bullet x_5 \end{array} + \begin{array}{c} \bullet x_1 \quad \bullet x_3 \quad \bullet x_2 \\ \diagdown \quad \diagup \quad \diagup \\ \bullet x_5 \end{array} \\
 &+ \begin{array}{c} \bullet x_1 \quad \bullet x_2 \quad \bullet x_3 \\ \diagdown \quad \diagup \quad \diagup \\ \bullet x_5 \end{array} + \begin{array}{c} \bullet x_1 \quad \bullet x_2 \quad \bullet 3 \\ \diagdown \quad \diagup \quad \diagup \\ \bullet x_4 \end{array} \\
 &T_{2,1}(x_2, T_{2,1}(x_1, x_3; x_4); x_5) \quad T_{2,1}(x_1, T_{2,1}(x_2, x_3; x_4); x_5) \\
 &T_{3,1}(x_1, x_2, T_{1,1}(x_3; x_4); x_5) \quad T_{3,1}(x_1, x_3, T_{1,1}(x_2; x_4); x_5) \\
 &T_{3,1}(x_2, x_3, T_{1,1}(x_1; x_4); x_5) \quad T_{1,1}(T_{3,1}(x_1, x_2, x_3; x_4); x_5)
 \end{aligned} \quad (3.10)$$

In general, by Definition 2.10:

$$T_{n,1} \circ_{n+1} T_{m,1} = \sum_{\phi} E \left(\begin{array}{c} 1 \quad 2 \quad \cdots \quad n \\ \searrow \quad \swarrow \quad \searrow \quad \swarrow \\ n+1 \end{array} \circ_{n+1}^{\phi} \begin{array}{c} n+1 \quad n+2 \quad \cdots \quad n+m-2 \\ \searrow \quad \swarrow \quad \searrow \quad \swarrow \\ n+m-1 \end{array} \right). \quad (3.11)$$

Each summand is the tree obtained by graphing the incoming edges of $\bullet n+1$ on the nodes of at left and keeping the original

order (crescent) that they appear in . Thus, 3.11 is the sum of the map T_{n+m-1} with the partial compositions between $T_{l,1}$ ($l \geq m$) and all possible $T_{j,1}$ at left. The index of the variables at left in $T_{j,1}$ s are in a crescent order, i.e., labeled by shuffle sequences. The use of $\emptyset$ is associated with the nodes labeled $i \in \{n+1, \dots, n+m-1\}$ that do not belong to the image of the function ϕ , i.e., nodes in which connections do not occur. For instance, in 3.9

$$T_{5,1}(x_1, x_2, x_3, x_4; x_5) = T_{5,1}(x_1, x_2, x_3, T_{1,1}(\emptyset; x_4); x_5)$$

and in 3.10

$$T_{1,1}(T_{3,1}(x_1, x_2, x_3; x_4); x_5) = T_{2,1}(\emptyset, T_{3,1}(x_1, x_2, x_3; x_4); x_5).$$

Property 3 follows similar arguments to those used in property 2. Since

$$T_{n,1} \circ_{n+1} [\cdot, \cdot] = \sum_{\phi} E \left(\begin{array}{c} 1 \quad 2 \quad \cdots \quad n \\ \searrow \quad \swarrow \quad \searrow \quad \swarrow \\ n+1 \end{array} \circ_{\phi}^{n+1} [\bullet n+1, \bullet n+2] \right),$$

we have by definition $\circ_{\phi}^{n+1} [\bullet n+1, \bullet n+2]$ is the bracket of two labeled rooted trees obtained by graphing the incoming edges of $\bullet n+1$ on $\bullet n+1$ or on $\bullet n+2$, at left

and preserving the original order they appear in (from left to right). These rules show that the index of the variables at left in $T_{p,1}$ and in $T_{q,1}$ are in a increasing order for each possible composition, i. e., the variables are indexed by a (p, q) -shuffle sequence, what ends our discussion on the proof of Proposition 10.

Remark 11. With Proposition 10, we can define ∇ -algebras as Lie-algebras $(\mathfrak{g}, [\cdot, \cdot])$ with a family of linear maps $\{T_{n,1}\}_{n \in \mathbb{N}^*}$ that satisfy the relations 1, 2, 3.

Due to Proposition 2, given the operads $\mathcal{P}$ and $\mathcal{Q}$, we have $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic iff the category of $\mathcal{P}$ -algebras is equivalent to the category of $\mathcal{Q}$ -algebras. However, it is interesting to see how it works in a particular case. Below, we present the equivalence of categories considering the operadic isomorphism between $\mathcal{PostLie}$ (Example 17) and the operad ∇ .

Proposition 11. The categories of post-Lie algebras and ∇ -algebras are isomorphic.

Proof. If $(W, [\cdot, \cdot], \triangleright)$ is a post-Lie algebra, then there is an operadic morphism $h: \mathcal{PostLie} \rightarrow \mathcal{End}(W)$. Since $\mathcal{PostLie} \xrightarrow{f} \nabla$ (see (2.15) in the proof of Theorem 1), the composition $h \circ f$ gives to W an ∇ -algebra structure. In particular, $h \circ f$ determines the following maps by recurrence (see 2.14 in the proof of Lemma 3):

$$\triangleright = T_{1,1}$$

$$T_{n,1}(x_1, \dots, x_n; x_{n+1}) = T_{1,1}(x_1; T_{n-1,1}(x_2, \dots, x_{n+1})) - \sum_{2 \leq i \leq n} T_{n-1,1}(x_2, \dots, T_{1,1}(x_1; x_i), \dots, x_{n+1}), \quad (3.12)$$

$\forall x_1, \dots, x_{n+1} \in \mathfrak{g}$. Which give us a functor F from the category of post-Lie algebras to the category of ∇ -algebras. The converse also holds, every ∇ -algebra is a post-Lie algebra, since $T_{1,1}$ is a post-Lie product. This relation give us a functor G from the category of ∇ -algebras to the category of post-Lie algebras. Note that, $F^{-1} = G$. Thus, we have the isomorphism intended. $\square$

In particular, if $(\mathfrak{g}, \triangleright, [\cdot, \cdot])$ is a post-Lie algebra with $[\cdot, \cdot] \equiv 0$, then $(\mathfrak{g}, \triangleright)$ is a pre-Lie algebra. Indeed, by Proposition 10:

$$0 = T_{1,1}([x_1, x_2]; x_3) = T_{2,1}(x_1, x_2; x_3) - T_{2,1}(x_2, x_1; x_3) \quad (3.13)$$

By using 3.12 we have:

$$T_{2,1}(x_1, x_2; x_3) = a_{\triangleright}(x_1, x_2, x_3)$$

So, by 3.13:

$$a_{\triangleright}(x_1, x_2, x_3) = T_{2,1}(x_1, x_2; x_3) = T_{2,1}(x_2, x_1; x_3) = a_{\triangleright}(x_2, x_1, x_3).$$

The argument used in 3.13 applied to $T_{n,1} \circ_i [\cdot, \cdot]$ ($\forall 1 \leq i \leq n$) guarantees that $T_{n,1}$ is symmetric at left. So, when $[\cdot, \cdot] \equiv 0$, 3.12 determines a symmetric brace structure on $\mathfrak{g}$ (Proposition 10, item 2), which proves the following result due to Guin and Oudom:

Proposition 12. The category of symmetric brace algebras and pre-Lie algebras are isomorphic.

Proposition 12 in an algebraic context is equivalent to Corollary 1 in an operadic context. Moreover, the reader can prove the proposition above just by using Chapoton and Livernet's operad. The use of ∇ -algebras is related to showing ∇ -algebras as an extension of symmetric brace algebras, i. e., ∇ -algebras like a “brace algebra” in a post-Lie case, as mentioned in Section 3.1 page 74. In [20][p. 517], Loday and Vallette describe explicitly how we obtain a symmetric brace algebra from a pre-Lie product. The ∇ -algebra is the structure obtained when we consider the same construction with a post-Lie product.

3.3 ∇ -algebras and the enveloping algebra of a post-Lie algebra

In the previous section, we obtain products $T_{n,1} : \mathfrak{g}^n \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ from the post-Lie product $\triangleright$ of a post-Lie algebra $(\mathfrak{g}, \triangleright, [\cdot, \cdot])$ (see (3.12)). With these products and the properties given by Proposition 10, in this section we prove the existence of an extension $\triangleright$ from $\mathfrak{g}$ to $\mathcal{U}(\mathfrak{g})$ of the post-Lie product $\triangleright$. This extension is defined by Ebrahimi-Fard et al. [8].

Proposition 13 (Ebrahimi-Fard et al. [8][Proposition 3.1]). Let $a, b, c \in \mathcal{U}(\mathfrak{g})$ and $u, v \in \mathfrak{g}$. There is a unique extension $\triangleright$ of the post-Lie product from $\mathfrak{g}$ to $\mathcal{U}(\mathfrak{g})$ given by:

$$1 \triangleright a = a, \quad (3.14)$$

$$ua \triangleright v = u \triangleright (a \triangleright v) - (u \triangleright a) \triangleright v, \quad (3.15)$$

$$a \triangleright bc = (a_{(1)} \triangleright b)(a_{(2)} \triangleright c). \quad (3.16)$$

This product interacts with the Hopf algebra structure of $\mathcal{U}(\mathfrak{g})$ (Example 47).

Proposition 14 (Ebrahimi-Fard et al. [8][Proposition 3.2]). Let $a, b, c \in \mathcal{U}(\mathfrak{g})$. We have:

$$a \triangleright 1 = \varepsilon(a) \quad (3.17)$$

$$\varepsilon(b \triangleright a) = \varepsilon(b)\varepsilon(a) \quad (3.18)$$

$$\Delta(b \triangleright a) = (b_{(1)} \triangleright a_{(1)}) \otimes (b_{(2)} \triangleright a_{(2)}) \quad (3.19)$$

$$c \triangleright (b \triangleright a) = (c_{(1)}(c_{(2)} \triangleright b)) \triangleright a, \quad (3.20)$$

being ε the counit of $(\mathcal{U}(\mathfrak{g}), \Delta)$.

With Proposition 14, Ebrahimi-Fard et al. prove the product $*$ defined below also determines a Hopf algebra structure on $(\mathcal{U}(\mathfrak{g}), \Delta)$ ([8][Proposition 3.3]):

$$a * b = a_{(1)}(a_{(2)} \triangleright b).$$

As expected, the proofs of Proposition 3.1 and Proposition 3.1 in [8] are, roughly, analogous to the one in [16, Propositions 3.7 and 3.9], slightly modified to match to the structure of $(\mathfrak{g}, \triangleright, [\cdot, \cdot])$. We also prove most of Propositions 13 and 14 by adapting the arguments used in [16, Propositions 3.7 and 3.9]. Our advantage lies in the fact that we can use the relations given by Proposition 10. In particular, (3.20) is proved in a different way to the one in [8][Proposition 3.2].

Remark 12. The proof of Proposition 13 involves the construction of the map $\triangleright : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$. During this construction, some abuse of notation occur. Indeed, at the end of the construction, the reader can notice that even if 3.21 and 3.22 describe different maps, the use of the same notation is justified since both are particular cases of the same product.

Remark 13. Along the text, we use the definition of shuffle coproduct (Example 47), the Sweedler's notation (Definition 9) and the filtered structure on $\mathcal{U}(\mathfrak{g})$ (Example 48) without explicit mention.

Now, we can start the proofs.

Proof of Proposition 13. To prove this proposition, we define an action of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$ given by:

$$x \triangleright a := \sum_{1 \leq i \leq n} x_1 x_2 \cdots x \triangleright x_i \cdots x_n \quad x \triangleright 1 := 0, \quad (3.21)$$

with $x \in \mathfrak{g}$, $a = x_1 \cdots x_n \in \mathcal{U}(\mathfrak{g})$.

This action is well defined. Indeed, it is enough to prove that

$$x \triangleright [x_1, x_2] = x \triangleright x_1 x_2 - x \triangleright x_2 x_1.$$

$$\begin{aligned} x \triangleright [x_1, x_2] &= [x \triangleright x_1, x_2] + [x_1, x \triangleright x_2] \quad (\text{post-Lie property}) \\ &= (x \triangleright x_1) x_2 - x_2 (x \triangleright x_1) + x_1 x \triangleright x_2 - x \triangleright x_2 x_1 \quad (\text{in } \mathcal{U}(\mathfrak{g}), [x_1, x_2] = x_1 x_2 - x_2 x_1) \\ &= x \triangleright x_1 x_2 - x \triangleright x_2 x_1. \end{aligned}$$

Now, consider the map $\triangleright : \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ given by:

$$\begin{aligned} (x_1 \cdots x_n) \otimes u &\mapsto (x_1 \cdots x_n) \triangleright u := T_{n,1}(x_1 \cdots x_n; u) \\ 1 \otimes u &\mapsto 1 \triangleright u := u \end{aligned} \quad (3.22)$$

By Proposition 10 item 1, we have:

$$\begin{aligned} T_{n,1}(x_1 \cdots [x_i, x_{i+1}] \cdots; u) &= T_{n+1,1}(x_1 \cdots x_i x_{i+1} \cdots; u) - T_{n+1,1}(x_1 \cdots x_{i+1} x_i \cdots; u) \\ &= T_{n+1,1}(x_1 \cdots x_i x_{i+1} - x_{i+1} x_i \cdots; u), \quad \forall 1 \leq i \leq n. \end{aligned}$$

Thus, $\triangleright : \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ is well defined.

With these two maps, we define a product $a \triangleright b$ ($a, b \in \mathcal{U}(\mathfrak{g})$). If $a \in \mathfrak{g}$, then we use 3.21. If $b \in \mathfrak{g}$, then we use 3.22. In other cases, if $a = y_1 \cdots y_n$ and $b = z_1 \cdots z_m$ ($y_i, z_j \in \mathfrak{g}$), the idea is “to partition” a and b into as many parts as necessary in order to apply 3.21 or 3.22. The problem is that partition must be compatible with the $\mathcal{U}(\mathfrak{g})$ structure. The shuffle coproduct helps us with this problem.

Let $\Delta(a) = a_{(1)} \otimes a_{(2)}$. We partition b into $z_1 | z_2 \cdots z_m$. At best, if $a_{(2)}$ belongs to $\mathfrak{g}$ in all summands of $\Delta(a)$, then

$$a \triangleright b = (a_{(1)} \triangleright z_1)(a_{(2)} \triangleright z_2 \cdots z_m) \quad (3.23)$$

makes sense, because $a \triangleright b$ is the element of $\mathcal{U}(\mathfrak{g})$, whose left term is obtained by 3.22 and the right one by 3.21. In case $a_{(2)} \notin \mathfrak{g}$, if necessary, we continue using the coproduct and partitioning until we can apply 3.21 or 3.22. For instance, if $m > 2$:

$$a \triangleright b = (a_{(1)} \triangleright z_1)(a_{(2:1)} \triangleright z_2)(a_{(2:2)} \triangleright z_3 \cdots z_m), \text{ and so on.}$$

So far, we define the product $\triangleright : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$ considering a specific partition of $z_1 \cdots z_m$. However, due to the coassociativity of Δ (Definition 40), the way we partition $z_1 \cdots z_m$ does not matter. For $m = 3$, since $(a_{(1:1)} \otimes a_{(1:2)}) \otimes a_{(2)} = a_{(1)} \otimes (a_{(2:1)} \otimes a_{(2:2)})$:

$$\begin{aligned} (a_{(1)} \triangleright z_1 z_2)(a_{(2)} \triangleright z_3) &= (a_{(1:1)} \triangleright z_1)(a_{(1:2)} \triangleright z_2)(a_{(2)} \triangleright z_3) \\ &= (a_{(1)} \triangleright z_1)(a_{(2:1)} \triangleright z_2)(a_{(2:2)} \triangleright z_3) \\ &= (a_{(1)} \triangleright z_1)(a_{(2)} \triangleright z_2 z_3). \end{aligned}$$

Thus, by induction, $\triangleright$ does not depend on the partition of $z_1 \cdots z_m$.

We still need to show that:

$$a \triangleright z_1 \cdots [u, v] \cdots z_m = a \triangleright z_1 \cdots (uv - vu) \cdots z_m.$$

The proof has some cases. However, it is enough to prove for $m = 1$, since the other cases are reduced to this case or they follow from (3.21).

On one hand, if $z_1 = [u, v]$ and $a = x_1 \cdots x_n$, $x_i \in \mathfrak{g}$:

$$\begin{aligned} a \triangleright z_1 &= T_{n,1}(x_1 \cdots x_n, [u, v]) \\ &= [T_{n,1}(x_1 x_2 \cdots x_n; u), v] + [u, T_{n,1}(x_1 \cdots x_n; v)] + \\ &\quad \sum_{\substack{p+q=n \\ \sigma \in Sh(p,q)}} [T_{p,1}(x_{\sigma(1)}, \dots, x_{\sigma(p)}; u), T_{q,1}(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)}; v)] \quad (\text{Proposition 10 item 3}) \\ &= T_{n,1}(x_1 x_2 \cdots x_n; u)v - vT_{n,1}(x_1 x_2 \cdots x_n; u) + uT_{n,1}(x_1 \cdots x_n; v) \\ &\quad - T_{n,1}(x_1 \cdots x_n; v)u \\ &\quad + \sum_{\substack{p+q=n \\ \sigma \in Sh(p,q)}} T_{p,1}(x_{\sigma(1)} \cdots x_{\sigma(p)}; u)T_{q,1}(x_{\sigma(p+1)} \cdots x_{\sigma(p+q)}; v) \\ &\quad - \sum_{\substack{p+q=n \\ \sigma \in Sh(p,q)}} T_{q,1}(x_{\sigma(p+1)} \cdots x_{\sigma(p+q)}; v)T_{p,1}(x_{\sigma(1)} \cdots x_{\sigma(p)}; u) \\ &= (in \mathcal{U}(\mathfrak{g}), [x, y] = xy - yx \ \forall x, y \in \mathfrak{g}). \end{aligned}$$

On the other hand, by definition:

$$a \triangleright uv - vu = (a_{(1)} \triangleright u)(a_{(2)} \triangleright v) - (a_{(1)} \triangleright v)(a_{(2)} \triangleright u).$$

Since

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)} = x_1 \cdots x_n \otimes 1 + \sum_{\substack{p+q=n \\ \sigma \in Sh(p,q)}} x_{\sigma(1)} \cdots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \cdots x_{\sigma(p+q)},$$

we have $a \triangleright [u, v] = a \triangleright (uv - vu)$.

When $a, b \in \mathbb{K}$, the product $a \triangleright b$ is missing. We know by 3.22 that $1 \triangleright u = u$ for $u \in \mathfrak{g} \Rightarrow 1 \triangleright 1 = 1$. Since the map $\triangleright$ is linear, we have $a \triangleright b = ab$.

Note that, by using 3.12 and 3.21:

$$ua \triangleright v = u \triangleright (a \triangleright v) - (u \triangleright a) \triangleright v, \forall u, v \in \mathfrak{g} \text{ and } a \in \mathcal{U}(\mathfrak{g}).$$

□

Proof of Proposition 14. To prove 3.17, we use 3.21: $x \triangleright 1 = 0 = \varepsilon(x)$, for $x \in \mathfrak{g}$. Thus, by induction and by definition of $\triangleright$,

$$a \triangleright 1 = 0 \text{ for } a \in \mathcal{U}(\mathfrak{g}) - \mathbb{K}.$$

If $a \in \mathbb{K}$, since $1 \triangleright 1 = 1$, we have

$$1 \triangleright a = a(1 \triangleright 1) = a = \varepsilon(a).$$

Now to prove 3.18, it suffices to note that $a, b \in \mathcal{U}(\mathfrak{g}) - \mathbb{K} \Rightarrow b \triangleright a \in \mathcal{U}(\mathfrak{g}) - \mathbb{K}$. In this case, $\varepsilon(b \triangleright a) = 0 = \varepsilon(b)\varepsilon(a)$. If $a, b \in \mathbb{K}$, then $\varepsilon(b \triangleright a) = \varepsilon(ba(1 \triangleright 1)) = ba = \varepsilon(b)\varepsilon(a)$.

For 3.19, if $a, b \in \mathbb{K}$:

$$\begin{aligned} \Delta(b \triangleright a) &= ba\Delta(1 \triangleright 1) \\ &= ba(1 \otimes 1) \\ &= ba(1 \triangleright 1 \otimes 1 \triangleright 1) \\ &= b \triangleright a \otimes 1 \triangleright 1 \\ &= (b_{(1)} \triangleright a_{(1)}) \otimes (b_{(2)} \triangleright a_{(2)}). \end{aligned}$$

Now, for $b, a \in \mathcal{U}(\mathfrak{g})$, we prove the first case

$$\Delta(b \triangleright x), \text{ with } b = z_1 \cdots z_m \text{ and } x \in \mathfrak{g}.$$

In this case, we know that $b \triangleright x \in \mathfrak{g} \Rightarrow \Delta(b \triangleright x) = (b \triangleright x) \otimes 1 + 1 \otimes (b \triangleright x)$. By definition of Δ :

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$$\Delta(b) = z_1 \cdots z_m \otimes 1 + 1 \otimes z_1 \cdots z_m + \sum_{\substack{\sigma \in Sh(p,q), \\ p+q=m, \\ p,q \neq 0}} Z_{\sigma_1} \otimes Z_{\sigma_2},$$

with $Z_{\sigma_1} \otimes Z_{\sigma_2} = z_{\sigma(1)} \cdots z_{\sigma(p)} \otimes z_{\sigma(p+1)} \cdots z_{\sigma(p+q)}$. Note that,

$$\begin{aligned} (b_{(1)} \triangleright x_{(1)}) \otimes (b_{(2)} \triangleright x_{(2)}) &= (z_1 \cdots z_m \triangleright x) \otimes (1 \triangleright 1) + (1 \triangleright x) \otimes (z_1 \cdots z_m \triangleright 1) \\ &\quad + \sum_{\substack{\sigma \in Sh(p,q), \\ p+q=m, \\ p,q \neq 0}} (Z_{\sigma_1} \triangleright x) \otimes (Z_{\sigma_2} \triangleright 1) + (z_1 \cdots z_m \triangleright 1) \otimes (1 \triangleright x) \\ &\quad + (1 \triangleright 1) \otimes (z_1 \cdots z_m \triangleright x) + \sum_{\substack{\sigma \in Sh(p,q), \\ p+q=m, \\ p,q \neq 0}} (Z_{\sigma_1} \triangleright 1) \otimes (Z_{\sigma_2} \triangleright x) \\ &= (z_1 \cdots z_m \triangleright x) \otimes 1 + 1 \otimes (z_1 \cdots z_m \triangleright x) \\ &= (b \triangleright x) \otimes 1 + 1 \otimes (b \triangleright x), \end{aligned}$$

since $a \triangleright 1 = \varepsilon(a) = 0, \forall a \in \mathcal{U}(\mathfrak{g}) \setminus \mathbb{K}$ and $1 \triangleright 1 = 1$.

By induction hypothesis, suppose that identity 3.19 holds for monomials $a \in \mathcal{U}(\mathfrak{g})$ of weight n . We prove the case $b \triangleright xa$, with $x \in \mathfrak{g}$.

$$\begin{aligned} \Delta(b \triangleright xa) &= \Delta((b_{(1)} \triangleright x)(b_{(2)} \triangleright a)) \text{ (by 3.16)} \\ &= \Delta(m(b_{(1)} \triangleright x \otimes b_{(2)} \triangleright a)) \\ &= (m \otimes m) \circ \tau_{23} \circ (\Delta \otimes \Delta)(b_{(1)} \triangleright x \otimes b_{(2)} \triangleright a) \text{ (}\Delta \text{ is an algebra morphism)} \\ &= (m \otimes m) \circ \tau_{23}(b_{(1:1)} \triangleright x_{(1)} \otimes b_{(1:2)} \triangleright x_{(2)} \otimes b_{(2:1)} \triangleright a_{(1)} \otimes b_{(2:2)} \triangleright a_{(2)}) \text{ (induction hyp.)} \\ &= (m \otimes m) \circ \tau_{23}(b_{(1:1)} \triangleright x_{(1)} \otimes b_{(2:1)} \triangleright x_{(2)} \otimes b_{(1:2)} \triangleright a_{(1)} \otimes b_{(2:2)} \triangleright a_{(2)}) \text{ (Note 10)} \\ &= (b_{(1:1)} \triangleright x_{(1)})(b_{(1:2)} \triangleright a_{(1)}) \otimes (b_{(2:1)} \triangleright x_{(2)})(b_{(2:2)} \triangleright a_{(2)}) \\ &= b_{(1)} \triangleright x_{(1)} a_{(1)} \otimes b_{(2)} \triangleright x_{(2)} a_{(2)} \text{ (by 3.16)} \\ &= b_{(1)} \triangleright (xa)_{(1)} \otimes b_{(2)} \triangleright (xa)_{(2)} \text{ (}\Delta \text{ is an algebra morphism)} \end{aligned}$$

Finally, we use induction on weight of a to prove 3.20. If a weighs 0, then we have $0 = 0$. If a weighs 1, then 3.20 holds due to Proposition 10 item 2.

Thus, suppose by induction that $c \triangleright (b \triangleright a) = (c_{(1)}(c_{(2)} \triangleright b)) \triangleright a$, with a being a monomial of weight n :

$$\begin{aligned} c \triangleright (b \triangleright xa) &= c \triangleright (b_{(1)} \triangleright x)(b_{(2)} \triangleright a) \text{ (by 3.16)} \\ &= (c_{(1)} \triangleright (b_{(1)} \triangleright x))(c_{(2)} \triangleright (b_{(2)} \triangleright a)) \text{ (by 3.16)} \\ &= (c_{(1:1)}(c_{(1:2)} \triangleright b_{(1)}) \triangleright x)(c_{(2:1)}(c_{(2:2)} \triangleright b_{(2)}) \triangleright a) \text{ (induction hyp.)} \end{aligned}$$

$$\begin{aligned}
&= (c_{(1:1)}(c_{(2:1)} \triangleright b_{(1)}) \triangleright x)(c_{(1:2)}(c_{(2:2)} \triangleright b_{(2)}) \triangleright a) \text{ (Note 10)} \\
&= c_{(1)}(c_{(2)} \triangleright b) \triangleright xa \text{ (by 3.16)}.
\end{aligned}$$

□

As we said before, if $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra with $[\cdot, \cdot] \equiv 0$, $S(\mathfrak{g})$ is an enveloping algebra and due to Proposition 10 $T_{n,1}$ is symmetric at left. Thus, 3.21 and 3.22 determine maps in $S(\mathfrak{g})$ that are extended to a product $\triangleright: S(\mathfrak{g}) \otimes S(\mathfrak{g}) \rightarrow S(\mathfrak{g})$, the same product defined by Guin and Oudom [16].

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