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Selection principles in hyperspaces

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Princípios seletivos em hiperespaços

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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To Mr. Manoel Tomás dos Santos
In Memoriam

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Originally I had planned to plot a table of acknowledgements, but I soon realized that LaTeX would not handle the data. So let me do this in the old fashioned way.

First of all I thank to my family, since one needs to exist before doing anything else. Also, they always tried to help me in their own way, and I appreciate that, in my own way[1].

Obviously, I thank my advisor, Prof. Leandro Aurichi, for the nice work we have done since my entrance at ICMC-USP, and I hope this to be just the beginning of a fruitful collaboration. Also, he indirectly introduced me to the life with **crocs**tm.

I cannot imagine a universe where I do not thank to Pryscilla Silva, for everything. I think this is the more precise I can be with just a few words – and without requiring a `Parental Advisory` at the cover of this thesis.

Life is pain, and studying Mathematics does not change that[2]. However, friends have the ability to make life tolerable. At least my friends have this ability, and I have a lot of friends. I am not kidding: friends from Miranda and Aquidauana[3] (including the alligators), friends from Safu (including Safu, *In Memoriam*), friends from São Carlos (including the ones that are not there anymore), friends from São Paulo, friends from GJM, ... If you are in one of the above *sets*, please, feel hugged, but not too much, since I am not a *hugger*.

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Finally, thanks to Capes and CNPq for the financial support.

[1] *Primeiramente, eu agradeço minha família, pois precisa-se existir antes de se fazer qualquer outra coisa. Além disso, eles sempre tentaram me ajudar do modo deles, e eu aprecio isso, do meu modo.*

[2] Actually, the Qualifying Exams contributed a lot in this aspect.

[3] Where I am currently working. By the way, special thanks to Prof. Adriana Wagner, Prof. Cláudio Pupim and Prof. Daniela Philippi, for applying the exams to my students while I was at São Carlos.

“This is your promise that things may be different, Roland
– that there may be *rest*. Even *salvation*.
(...)
Time to get moving.
The man in black fled across the desert, and the gunslinger followed”.

(Stephen King, The Dark Tower)

ABSTRACT

MEZABARBA, R. M. **Selection principles in hyperspaces**. 2018. 102 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

In this work we analyze some selection principles over some classes of hyperspaces. In the first part we consider selective variations of tightness over a class of function spaces whose topologies are determined by bornologies on the space. As results, we extend several well known translations between covering properties and closure properties of the topology of pointwise convergence. In the second part we consider artificial hyperspaces that assist the analysis of productive topological properties. We emphasize the results characterizing productively ccc preorders and the characterization of the Lindelöf property via closed projections.

Keywords: selection principles, function spaces, topological games, hyperspaces.

RESUMO

MEZABARBA, R. M. **Princípios seletivos em hiperespaços**. 2018. 102 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2018.

Neste trabalho analisamos alguns princípios seletivos quando considerados sobre alguns tipos de hiperespaços. Na primeira parte consideramos variações seletivas do *tightness* sobre diversos tipos de espaços de funções, cujas topologias são determinadas por bornologias no espaço. Como resultados, estendemos diversas traduções conhecidas entre propriedades de recobrimento e propriedades de convergência na topologia da convergência pontual. Na segunda parte consideramos hiperespaços artificiais que auxiliam na análise de propriedades topológicas produtivas. Destacamos os resultados que caracterizam as pré-ordens produtivamente ccc e a caracterização da propriedade de Lindelöf em termos de projeções fechadas.

Palavras-chave: princípios seletivos, espaços de funções, jogos topológicos, hiperespaços.

LIST OF SYMBOLS

ω — set of natural numbers

$\prod_{a\in A} B_a$ — set of all functions $f\colon A \to \bigcup_{a\in A} B_a$ such that $f(a) \in B_a$ for all $a \in A$

B^A — set of all functions $f\colon A \to B$

$A^{<\kappa}$ — $\bigcup_{\alpha<\kappa} A^\alpha$

$|B|$ — cardinality of the set B

$[A]^\kappa$ — set of all subsets of A whose cardinality is precisely κ

$[A]^{<\kappa}$ — $\bigcup_{\lambda<\kappa} [A]^\lambda$

$\mathsf{S}_{\mathrm{fin}}(\mathcal{C},\mathcal{D})$ — $\forall (C_n)_{n\in\omega} \in \mathcal{C}^\omega \, \exists (D_n)_{n\in\omega} \in \prod_{n\in\omega} [C_n]^{<\aleph_0}$ s. t. $\bigcup_{n\in\omega} D_n \in \mathcal{D}$

$\mathsf{U}_{\mathrm{fin}}(\mathcal{C},\mathcal{D})$ — $\forall (C_n)_{n\in\omega} \in \mathcal{C}^\omega \, \exists (D_n)_{n\in\omega} \in \prod_{n\in\omega} [C_n]^{<\aleph_0}$ s. t. $\{\bigcup D_n : n \in \omega\} \in \mathcal{D}$

$\mathsf{S}_1(\mathcal{C},\mathcal{D})$ — $\forall (C_n)_{n\in\omega} \in \mathcal{C}^\omega \, \exists (d_n)_{n\in\omega} \in \prod_{n\in\omega} C_n$ s. t. $\{d_n : n \in \omega\} \in \mathcal{D}$

$\mathcal{O}(X)$ — family of all open coverings of X

$\mathrm{diam}_d(\bullet)$ — diameter of a subset of a metric space (M,d)

$\Omega_{x,X}$ — set of those subsets $A \subset X$ such that $x \in \overline{A}$

$\mathsf{G}_{\mathrm{fin}}(\mathcal{C},\mathcal{D})$ — game associated with $\mathsf{S}_{\mathrm{fin}}(\mathcal{C},\mathcal{D})$

$\mathsf{J} \uparrow \mathsf{G}$ — Player J has a winning strategy in the game G

$\mathsf{J} \not\uparrow \mathsf{G}$ — Player J has no winning strategy in the game G

$\mathsf{G}_1(\mathcal{C},\mathcal{D})$ — game associated with $\mathsf{S}_1(\mathcal{C},\mathcal{D})$

$\wp(\mathcal{S})$ — power set of $\mathcal{S}$

$\binom{\mathcal{C}}{\mathcal{D}}_{\mathcal{R}}$ — $\forall C \in \mathcal{C} \, \exists D \in \wp(\mathcal{S})$ s. t. $D \mathcal{R} C$

$\mathsf{L}(\bullet)$ — Lindelöf degree

$\mathsf{K}(X)$ — set of all compact subspaces of X

$(Z)_\kappa$ — κ-modification of Z

$C(X)$ — family of the continuous real functions defined on X

$\mathbb{R}$ — set of all real numbers with the usual topology

$C_p(X)$ — $C(X)$ endowed with the topology of pointwise convergence

$\underline{0}$ — constant zero function $X \to \mathbb{R}$

$\mathrm{t}(\bullet)$ — tightness

$\Omega(X)$ — family of the ω-coverings of X

$C_k(X)$ — $C(X)$ endowed with the compact-open topology

S_ψ — selection principle associated with the function $\psi\colon \omega \to [2,\aleph_0]$

S_k — S_ψ for $\psi \equiv k+1$ and $k \in \omega \setminus \{0\}$

G_ψ — game associated with S_ψ

G_k — G_ψ for $\psi \equiv k+1$, with $k \in \omega \setminus \{0\}$

$\underline{\mathrm{Id}}$ — $\underline{\mathrm{Id}}\colon \omega \to [2,\aleph_0)$, where $\underline{\mathrm{Id}}(n) = n+2$

$\mathcal{U} \wedge \mathcal{V}$ — family of the sets of the form $U \cap V$ such that $U \in \mathcal{U}$ and $V \in \mathcal{V}$

$z^\frown \mathcal{U}$ — concatenation of z with $\mathcal{U}$

$\langle B, \varepsilon\rangle[f]$ — basic open set of $C_{\mathcal{B}}(X)$

$C_{\mathcal{B}}(X)$ — $C(X)$ with the topology of uniform convergence on $\mathcal{B}$

$\widetilde{\mathcal{B}}$ — closure of the family $\mathcal{B}$ under taking finite unions

K_X — the subsets of compact subspaces of X

$\mathcal{O}_{\mathcal{B}}$ — family of all $\mathcal{B}$-coverings of X

$\mathcal{O}^*_{\mathcal{B}}$ — set of the non-trivial $\mathcal{B}$-coverings of X

I_n — open interval $\left(-\frac{1}{n+1}, \frac{1}{n+1}\right) \subset \mathbb{R}$, for $n \in \omega$

$\mathcal{A}(\mathcal{U})$ — set of those functions $f \in C_{\mathcal{B}}(X)$ such that there exists a $U \in \mathcal{U}$ with $f \restriction (X \setminus U) \equiv 1$

$\mathcal{U}_n(A)$ — set of all sets of the form $f^{-1}[I_n]$ for $f \in A \subset C_{\mathcal{B}}(X)$

$\bigoplus_{i \in I} X_i$ — topological sum

$\langle U\rangle$ — basic open set of $\mathcal{B}^+$

$\mathcal{B}^+$ — family $\mathcal{B}$ with the upper semi-finite topology

$\langle \mathcal{U}\rangle$ — family of open sets of $\mathcal{B}^+$ induced by the family $\mathcal{U}$

$\widetilde{\mathcal{W}}$ — set of open subsets of X induced by the family $\mathcal{W}$

$\mathcal{N}_{y,Y}$ — neighborhood filter of $y \in Y$

$\mathcal{G}^\uparrow$ — upwards closure of the family $\mathcal{G}$

$\mathbb{F}(C)$ — family of proper filters on C

$\mathbb{F}_\kappa(C)$ — those proper filters on C having a basis with cardinality $\leq \kappa$

$R[F]$ — image of F by the relation R

$R(\mathcal{F})$ — the upward closure of $\{R[F] : F \in \mathcal{F}\}$

$\mathcal{F} \vee \mathcal{H}$ — upward closure of $\mathcal{F} \wedge \mathcal{H}$

$V(B)$ — family of open sets containing B as a subset

$\Gamma_{\mathcal{B}}(X)$ — upward closure of $\{V(B) : B \in \mathcal{B}\}$

$\Gamma(X)$ — $\Gamma_{[X]^{<\omega}}(X)$

$\mathcal{F} \# \mathcal{G}$ — $F \cap G \neq \emptyset$ for all $(F,G) \in \mathcal{F} \times \mathcal{G}$

Γ_y — set of those subsets D of Y such that $D \setminus U$ is finite for all $U \in \mathcal{N}_{y,Y}$

$\chi(\bullet)$ — character

$\Gamma_{\mathcal{B}}$ — family of $\mathcal{B}$-cofinite open coverings of X

$\bigotimes_{t \in T} \mathcal{B}_t$ — product bornology

$\bigsqcup_{t \in T} \mathcal{B}_t$ — sum bornology

$[G]$ — basic open sets of index topology

$\mathscr{A}^\star$ — the set $\mathscr{A}$ endowed with the index topology

$\mathscr{K}(X)$ — covering number of X

$\mathscr{N}(X)$ — collection of the nice families of antichains

$p \perp q$ — the elements p and q are incompatible

$\mathfrak{F}(\mathscr{A})$ — the set $\bigcup_{\mathcal{A} \in \mathscr{A}} [\mathcal{A}]^{<\aleph_0}$ endowed with the partial order of reverse inclusion

$\mathbb{P} \times \mathbb{Q}$ — product of the preorders $\mathbb{P}$ and $\mathbb{Q}$

$\mathscr{E}_{\mathcal{U}}(X)$ — Escardó hyperspace of X and $\mathcal{U}$

CONTENTS

INTRODUCTION

In this work we treat of two apparently disjoint topics. In Chapter 1 we extend known translations between covering properties of a topological space X and closure properties of $C_p(X)$ to the context of spaces of the form $C_{\mathcal{B}}(X)$, where $\mathcal{B}$ is a *bornology* on X. On the other hand, in Chapter 2 we consider artificial spaces to analyze some topological properties on products. However, both these topics can be thought as the study of *selection principles* in *hyperspaces*. But what does this mean?

Roughly speaking, a *selection principle* is any property like "for each element C of a family $\mathcal{C}$ there exists an element $D \in \mathcal{D}$ satisfying some condition related to C". On the other hand, by a *hyperspace of a space* X, we mean *any* space Y defined in terms of X. At first, it may appears that the range of the present work is larger than it really is, since we intend to analyze a few relations between a topological space and some of its hyperspaces through selection principles. For this reason, we shall discuss the meanings of the previous expressions, in order to provide the reader with a clearer idea of the topics studied along this work.

We assume that the reader is familiar with General Topology and Set Theory. Still, we display a short Glossary after Chapter 2, with the main definitions and results that we assume to be known: the entries of the Glossary are underlined along the text[4]. There is also a List of Symbols preceding the Table of Contents: most of the standard terminologies we use are defined there. Along this work we reserve the word "theorem" for results found in the literature – on the other hand, our results will be called "lemmas", "propositions" and "corollaries".

For introductions to basic concepts, we recommend the books of Willard [101] and Hrbacek and Jech [36], while advanced topics are covered by Engelking [22], Kunen [49] and Jech [38]. It may be helpful to mention that, unlike Engelking, we do not add separation axioms to describe covering properties: for instance, we call "compact Hausdorff" and "regular Lindelöf" the spaces Engelking calls "compact" and "Lindelöf", respectively. Also, unless otherwise stated, we assume that all spaces treated along this work are infinite.

Selection principles and games

Although the framework used in this work to treat selection principles has been developed by Marion Scheepers [74], some of these principles emerged in Mathematics long before the 90's. For families $\mathcal{C}$ and $\mathcal{D}$ of nonempty subsets of a set $\mathcal{S}$, consider the following assertions:

[4] The entries of the Glossary also works as hyperlinks in the electronic version of this thesis.

1. $\mathsf{S}_{\mathrm{fin}}(\mathcal{C},\mathcal{D})$: for each sequence $(C_n)_{n\in\omega}\in\mathcal{C}^\omega$ there exists a sequence $(D_n)_{n\in\omega}$ with $D_n\subset C_n$ and D_n finite for each $n\in\omega$, such that $\bigcup_{n\in\omega}D_n\in\mathcal{D}$;
2. $\mathsf{U}_{\mathrm{fin}}(\mathcal{C},\mathcal{D})$: for each sequence $(C_n)_{n\in\omega}\in\mathcal{C}^\omega$ there exists a sequence $(D_n)_{n\in\omega}$ with $D_n\subset C_n$ and D_n finite for each $n\in\omega$, such that $\{\bigcup D_n : n\in\omega\}\in\mathcal{D}$;
3. $\mathsf{S}_1(\mathcal{C},\mathcal{D})$: for each sequence $(C_n)_{n\in\omega}\in\mathcal{C}^\omega$ there exists a sequence $(d_n)_{n\in\omega}$ with $d_n\in C_n$ for each $n\in\omega$, such that $\{d_n : n\in\omega\}\in\mathcal{D}$.

In this fashion, for each instance of pair $(\mathcal{C},\mathcal{D})$, one obtains a (formally) different instance of a selection principle.

Example I.1 (Menger spaces)**.** For a topological space X, let $\mathcal{O}(X)$ be the family of all open coverings of X, which we will simply write as $\mathcal{O}$ when the space X is clear from the context. Spaces satisfying condition $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ are currently called **Menger spaces**, in reference to Karl Menger. In 1924, Menger [56] conjectured that σ-compact metric spaces are characterized as those spaces with the following property:

for each basis $\mathscr{B}$ of the metric space (M,d), there is a sequence $(B_n)_{n\in\omega}\in\mathscr{B}^\omega$ such that $\mathrm{diam}_d(B_n)\to 0$ and $M=\bigcup_{n\in\omega}B_n$,

where diam_d denotes the diameter of a set with respect to the metric d.

Later, Witold Hurewicz [37] noted[5] that metric spaces with the above property are precisely those spaces satisfying $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. In order to analyze Menger's conjecture, Hurewicz formulated an intermediate property between σ-compactness and $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. In modern terminology, **Hurewicz's property** can be expressed as $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$, where Γ denotes the family of all *point-cofinite*[6] open coverings: an open covering $\mathcal{U}$ of a topological space X is **point-cofinite** if for all $x\in X$, the set $\{U\in\mathcal{U} : x\in U\}$ is cofinite in $\mathcal{U}$.

Although Hurewicz did not solve Menger's conjecture, he observed that

$$\sigma\text{-compactness}\Rightarrow\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)\Rightarrow\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}),\tag{1}$$

then conjecturing that "σ-compactness = $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$ for metric spaces". Of course, a counterexample for this latter conjecture would prove that Menger's conjecture is false. However, even with such a counterexample, one could ask if $\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$ is equal to $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$.

Nowadays, it is known that both conjectures are false and the second implication in (1) cannot be reversed in general. We refer the reader to Tsaban's survey [98], where these topics are deeply discussed. ■

[5] Without proving it. For the proofs we refer the reader to Fremlin and Miller's paper [25] and Scott's answer on mathoverflow [81].

[6] Point-cofinite open coverings are usually called γ-coverings for historical reasons: they were introduced by Gerlits and Nagy [28], related with the third item in a list starting at α. Also, it is usual to suppose that a point-cofinite open covering $\mathcal{U}$ is infinite and such that $X\notin\mathcal{U}$, just to avoid trivial cases.

Example I.2 (Rothberger spaces)**.** In 1919, Émile Borel [15] introduced the following property for metric spaces, now called *strong measure zero*: a subset X of a metric space (M,d) has **strong measure zero** if for each sequence $(\varepsilon_n)_{n\in\omega}$ of positive real numbers there is a sequence $(X_n)_{n\in\omega}$ of subsets of X such that $\mathrm{diam}_d(X_n) < \varepsilon_n$ for each $n \in \omega$ and $X = \bigcup_{n\in\omega} X_n$. Borel then conjectured that the only subsets of the real line with this property are the countable ones.

In his study of Borel's conjecture, Fritz Rothberger [70] introduced a topological analogy of strong measure zero, that can be stated as $\mathsf{S}_1(\mathcal{O},\mathcal{O})$, and for this reason, topological spaces with this property are now called **Rothberger spaces**. What about the Borel's conjecture? The answer may be surprising: it is independent of ZFC. For further results in this topic, we refer the reader to Sakai and Scheepers' survey [80]. ■

Example I.3 (Variations of tightness)**.** Recall that a topological space X has countable tightness at $x \in X$ if for all $A \subset X$ such that $x \in \overline{A}$ there exists a countable subset $B \subset A$ with $x \in \overline{B}$; if the same happens to all points in X, then we simply say that X has countable tightness. In general, first countable spaces have countable tightness, but the converse is not true. By taking the family

$$\Omega_{x,X} := \{A \subset X : x \in \overline{A}\} \tag{2}$$

in the selection principles $\mathsf{S}_{\mathrm{fin}}$ and S_1, one obtains intermediate properties between first countability and countable tightness: clearly,

$$\text{first countable at } x \Rightarrow \mathsf{S}_1(\Omega_{x,X},\Omega_{x,X}) \Rightarrow \mathsf{S}_{\mathrm{fin}}(\Omega_{x,X},\Omega_{x,X}) \Rightarrow \text{countable tightness at } x. \tag{3}$$

Along this work, for simplicity of notation, we write Ω_x instead of $\Omega_{x,X}$. These selective variations of tightness were introduced in the analysis of local properties of function spaces: property $\mathsf{S}_{\mathrm{fin}}(\Omega_x,\Omega_x)$ was called **countable fan tightness** by Arhangel'skii [2], while $\mathsf{S}_1(\Omega_x,\Omega_x)$ is known as **countable strong fan tightness**, due to Sakai [71]. ■

Although Menger's conjecture was settled as false, it is important to point out that, in some sense, it was close to a true *selective characterization*. To discuss this, we need to introduce a *selective game*. Following Scheeper's notation, the game $\mathsf{G}_{\mathrm{fin}}(\mathcal{C},\mathcal{D})$ – associated with $\mathsf{S}_{\mathrm{fin}}$ – is played by two players, Player I and Player II, according to the rules below:

- a play P of $\mathsf{G}_{\mathrm{fin}}(\mathcal{C},\mathcal{D})$ has ω innings;
- for each $n \in \omega$, Player I starts the n-th inning of the play P by choosing an element $C_n \in \mathcal{C}$ and then Player II responds with a finite subset $D_n \subset C_n$;
- Player II wins the play P if and only if $\bigcup_{n\in\omega} D_n \in \mathcal{D}$.

In this framework, the interest lies in asking about the existence of *winning strategies* for the players. Here, a **strategy** for a player in a game G is a rule that "tells" the player how to respond to each legal move of its opponent at every inning of a play of the game G – particularly, we assume the strategy "remembers" all the previous moves of the opponent[7]. For instance, in the game $\mathsf{G}_{\text{fin}}(\mathcal{C},\mathcal{D})$, a strategy for Player II is a function

$$\sigma\colon \left(\mathcal{C}^{<\omega}\setminus\{\emptyset\}\right)\to[\mathcal{D}]^{<\omega}$$

such that for each sequence $(C_0,\dots,C_n)\in\mathcal{C}^{<\omega}$, the finite subset $\sigma((C_0,\dots,C_n))$ is contained in C_n, while a strategy for Player I is a function[8] $\mu\colon ([\mathcal{D}]^{<\omega})^{<\omega}\to\mathcal{C}$ – note that the first move of Player I is $\mu(\emptyset)$. Naturally, a **winning strategy** for a player is a strategy such that there is no way to his opponent to defeat it in every legal play.

To add some topological flavor to this discussion, let us check that if X is a σ-compact space, then Player II has a winning strategy in the game $\mathsf{G}_{\text{fin}}(\mathcal{O},\mathcal{O})$: indeed, if $X=\bigcup_{n\in\omega}K_n$, where each subspace K_n is compact, and $\mathcal{U}_m$ is the open covering chosen by Player I at the m-th inning of a play in $\mathsf{G}_{\text{fin}}(\mathcal{O},\mathcal{O})$, we make Player II respond with a finite subset $\mathcal{V}_m\subset\mathcal{U}_m$ such that $K_m\subset\bigcup\mathcal{V}_m$; since $X=\bigcup_{n\in\omega}K_n$, this procedure gives a winning strategy for Player II.

A nontrivial theorem, due to Telgársky, tells us that the converse of this example is true in the realm of metric spaces.

Theorem I.4 (Telgársky [92])**.** A metric space X is σ-compact if and only if Player II has a winning strategy in the game $\mathsf{G}_{\text{fin}}(\mathcal{O},\mathcal{O})$.

To see why this is related to Menger's conjecture, we introduce Telgársky's notation [89]: for $\mathrm{J}\in\{\mathrm{I},\ \mathrm{II}\}$ we write $\mathrm{J}\uparrow\mathsf{G}_{\text{fin}}(\mathcal{C},\mathcal{D})$ to express the sentence "Player J has a winning strategy in the game $\mathsf{G}_{\text{fin}}(\mathcal{C},\mathcal{D})$", whose negation we denote by $\mathrm{J}\not\uparrow\mathsf{G}_{\text{fin}}(\mathcal{C},\mathcal{D})$. Now, it is not hard to see that

$$\mathrm{II}\uparrow\mathsf{G}_{\text{fin}}(\mathcal{C},\mathcal{D})\Rightarrow\mathrm{I}\not\uparrow\mathsf{G}_{\text{fin}}(\mathcal{C},\mathcal{D})\Rightarrow\mathsf{S}_{\text{fin}}(\mathcal{C},\mathcal{D}).$$

There is no guarantee that any of the above implications can be reversed in general. However, for $\mathcal{C}=\mathcal{D}=\mathcal{O}$, the principles $\mathsf{S}_{\text{fin}}(\mathcal{O},\mathcal{O})$ and $\mathrm{I}\not\uparrow\mathsf{G}_{\text{fin}}(\mathcal{O},\mathcal{O})$ are the *same*, justifying the fact that the game $\mathsf{G}_{\text{fin}}(\mathcal{O},\mathcal{O})$ is also called the **Menger game**.

Theorem I.5 (Hurewicz[9], 1926)**.** For a topological space X, the selection principle $\mathsf{S}_{\text{fin}}(\mathcal{O},\mathcal{O})$ holds if and only if $\mathrm{I}\not\uparrow\mathsf{G}_{\text{fin}}(\mathcal{O},\mathcal{O})$.

Of course, similar games can be related to the selection principles U_{fin} and S_1 as well. In particular, the game associated with $\mathsf{S}_1(\mathcal{C},\mathcal{D})$ is denoted by $\mathsf{G}_1(\mathcal{C},\mathcal{D})$, and it is called the

[7] In game theory terminology, G_{fin} is a perfect information game [30].

[8] Actually, the domain of μ could be smaller than $([\mathcal{D}]^{<\omega})^{<\omega}$: indeed, a strategy for Player I just need to respond to sequences of finite sets of the form $(D_0,\dots,D_n)$ such that $D_{j+1}\subset\mu((D_0,\dots,D_j))$ for each $j<n$. But in this case one can easily extend the domain of μ to the whole set $([\mathcal{D}]^{<\omega})^{<\omega}$.

[9] Although the game $\mathsf{G}_{\text{fin}}(\mathcal{O},\mathcal{O})$ was only defined by Telgársky in 1984, the basic ideas of Theorem I.5 were presented by Hurewicz back in 1926, up to terminology, as mentioned by Scheepers [78]. The reader may find a conceptual proof of this theorem in [86].

Rothberger game when $\mathcal{C} = \mathcal{D} = \mathcal{O}$: it is essentially the same as the game G_{fin}, but now Player II chooses only one element per inning. For more examples and historical background about topological games, we refer the reader to Scheepers' [78] and Telgársky's [89] papers. Variations of these games and their corresponding selection principles are treated in Chapter 1, in connection with (hyper)spaces of the form $C_{\mathcal{B}}(X)$, where $\mathcal{B}$ is a *bornology on* X.

We also deal with *classical* topological properties that can be seen as instances of another type of selection principle. For families $\mathcal{C}$ and $\mathcal{D}$ of nonempty subsets of a set $\mathcal{S}$ and a binary relation $\mathcal{R} \subset \wp(\mathcal{S}) \times \wp(\mathcal{S})$, we denote by $\binom{\mathcal{C}}{\mathcal{D}}_{\mathcal{R}}$ the following assertion:

$$\text{for all } C \in \mathcal{C} \text{ there exists a } D \in \wp(\mathcal{S}) \text{ such that } D\mathcal{R}C \text{ and } D \in \mathcal{D}. \tag{4}$$

This is a variation of the **Bar-Ilan** selection principle, as called by Scheepers [78]. Several topological properties can be described in this fashion, for suitable sets $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{R}$.

Example I.6. For a topological space (X, τ), let $\mathcal{O}_{\leq\kappa} := \{\mathcal{U} \in \mathcal{O} : |\mathcal{U}| \leq \kappa\}$. Then the selection principle $\binom{\mathcal{O}}{\mathcal{O}_{\leq\kappa}}_{\subset}$ describes precisely the property $\mathrm{L}(X) \leq \kappa$, where $\mathrm{L}(X)$ denotes the Lindelöf degree of X. ■

Example I.7. For a topological space (X, τ) and for open coverings $\mathcal{U}, \mathcal{V} \in \mathcal{O}$, let us write $\mathcal{U} \preceq \mathcal{V}$ to indicate that $\mathcal{U}$ refines $\mathcal{V}$. Then, paracompactness of X can be described as $\binom{\mathcal{O}}{\mathcal{O}_{\text{lf}}}_{\preceq}$, where $\mathcal{O}_{\text{lf}}$ denotes the family of locally finite open coverings of X. ■

In this sense, Chapter 2 is intended to analyze the behavior of some topological properties of this form on product spaces by using different types of (hyper)spaces. The second appearance of the expression "(hyper)" in the last paragraph indicates that it is time to give a meaning to the word *hyperspace*.

Hyperspaces

Accordingly to Mizokami and Shimane [63], a topological space (H, τ) is a *hyperspace of* X if τ is a topology on H, in which H is one of the following families:

- $\mathrm{CL}(X) := \{F \subset X : F \text{ is closed}\}$;
- $\mathsf{K}(X) := \{F \subset X : F \text{ is compact}\}$;
- $\mathrm{CL}(X)^* := \{F \subset X : \emptyset \neq F \text{ is closed}\}$;
- $[X]^{<\aleph_0} \setminus \{\emptyset\} = \{F \subset X : |F| < \aleph_0\} \setminus \{\emptyset\}$.

However, we do not restrict the use of the term "hyperspace" to the above cases. Instead of presenting a formal definition broad enough to fulfill our needs[10], we shall illustrate what we understand by *hyperspaces* in the following discussion.

[10] A possible way to define hyperspaces as they are used in this work could be done by using *functors*. For instance, for a *category* $\mathcal{C}$ and a *functor* $H\colon \mathrm{Top} \to \mathcal{C}$, we could call $H(X)$ a hyperspace of X. Since the discussion of this definition would demand the introduction of (notions of) Category Theory that would not be used in most of the work, we chose to keep the definition of hyperspace informal.

Recall that a continuous function $f\colon X \to Y$ between topological spaces X and Y is **proper** if the map

$$\begin{array}{rccc} f\times id_Z\colon & X\times Z & \to & Y\times Z \\ & (x,z) & \mapsto & (f(x),z) \end{array}$$

is closed for every topological space Z (see Bourbaki [16, p. 97]).

For a compact space X, it is easy to see that the (unique) function $X \to \{\emptyset\}$ is proper, since this is equivalent to asking for the projection

$$\begin{array}{rccc} \pi_Z\colon & X\times Z & \to & Z \\ & (x,z) & \mapsto & z \end{array}$$

to be closed for each topological space Z. As mentioned by Engelking [22], this result is due to Kuratowski (1931) and Bourbaki (1940) for the metric and topological cases, respectively. Up to terminology, it is due to Mrówka the observation that the converse also holds.

Theorem I.8 (Mrówka [64])**.** A topological space X is compact if and only if the function $X \to \{\emptyset\}$ is proper.

We mention this result because one possible way to prove its relevant part ($\Leftarrow$) makes use of a space Z, considered by Escardó [23]. For a fixed open covering $\mathcal{U}$ closed under finite unions, the space Z is the topology of X endowed with the following topology: $\mathcal{W} \subset Z$ is open if and only if $\mathcal{W} = \emptyset$ or $\mathcal{W}\cap\mathcal{U} \neq \emptyset$ and $V \in \mathcal{W}$ for all $V \in Z$ such that there exists a $U \in \mathcal{W}$ with $U \subset V$.

Remark I.9. Back in 2015, Professor C. H. Grossi personally asked me about a characterization for Lindelöfness in the fashion of Theorem I.8. Indeed, by taking an open covering closed under unions of κ many elements, $\kappa \geq \aleph_0$, we obtain a space similar to Z that allows us to generalize Theorem I.8 in the following way[11]:

$$\mathrm{L}(X) \leq \kappa \Leftrightarrow \text{ for each topological space } Z, \pi_Z\colon X\times Z \to (Z)_\kappa \text{ is closed,} \tag{5}$$

where $(Z)_\kappa$ denotes the **κ-modification of** Z – the topology over Z whose basic open sets are the intersections of κ many open sets, also known as **G_κ-sets**. The equivalence (5) is the core of Section 2.3. ■

In the above example, we regard Z as a hyperspace of X, because Z is defined *in terms of* X: its points are *the open sets of* X, and its topology is induced from families of *subsets of* X. Similarly, we also say that the space $(Z)_\kappa$ in Remark I.9 is a hyperspace of Z. So, roughly speaking, we consider any space whose *construction* is related to X as a **hyperspace of** X.

Thus, according to this *rule*, we may say that $(C(X),\mathcal{T})$ is a hyperspace of X, where $C(X)$ denotes the set of the continuous real functions defined on a topological space X and $\mathcal{T}$ is

[11] A word of caution: unlike Engelking, we do not suppose that closed mappings are continuous.

a topology on $C(X)$. This was done before, by Scheepers [77], what also motivated our choice of terminology.

Chapter 2 is dedicated to the analysis of some topological properties of products through hyperspaces: Sections 2.1 and 2.2 adapt the hyperspace defined by Aurichi and Zdomskyy [12] to other contexts different from Lindelöfness, while Section 2.3 discusses possible applications of equivalence (5).

A bait section for Functional Analysts

Although the aim of the present work *does not* include the discussion of applications *outside* General Topology and Set Theory, we dedicate this last part of the Introduction to glimpse possible applications for mathematicians living in the complement of both fields. We do this with an illustrative and naive visit to Functional Analysis[12], but we hope that this can be appealing for non-analysts as well.

The object that will do the transition between the fields is the space $C_p(X)$. Given a topological space X, let $C(X) := \{f\colon X \to \mathbb{R} \,|\, f \text{ is continuous}\}$, where $\mathbb{R}$ denotes the real numbers with their usual topology. We may consider the hyperspace $C_p(X)$, defined as the set $C(X)$ endowed with the **topology of pointwise convergence**: this is the topology on $C(X)$ as a subspace of $\mathbb{R}^X$, where the latter is endowed with the product topology. As a consequence, $C_p(X)$ is a Tychonoff space.

Now, in the Analysis' side of the corner, for a normed $\mathbb{R}$-vector space $(E, \|\cdot\|)$, the *strong* topology induced by the norm $\|\cdot\|$ determines the set $E^\star$ of all continuous linear functionals, that is, linear continuous functions of the form $E \to \mathbb{R}$. It is usual to call $\sigma(E, E^\star)$ the smallest topology on E such that each $\phi \in E^\star$ remains continuous when considered as a function of the form $(E, \sigma(E, E^\star)) \to \mathbb{R}$: this is the so called *weak topology* on E. In this context, it can be showed that the following assertions are equivalent:

- $\dim_{\mathbb{R}} E < \aleph_0$;
- the weak topology on E is metrizable;
- $(E, \sigma(E, E^\star))$ is first countable.

So, in the most interesting cases occurring in Functional Analysis, the space $(E, \sigma(E, E^\star))$ is not first countable, which raises a natural question: in the sense of (3), how far from being first countable is the space $(E, \sigma(E, E^\star))$?

[12] Based on a talk of Lyubomyr Zdomskyy at the (Conference) Frontiers of Selection Principles [102], held in Warsaw in August 2017.

To answer this with our tools, we need to consider a different topology on $E^\star$. Indeed, the set $E^\star$ also admits a stronger topology induced by the *operator norm*, namely

$$\|\phi\| := \sup\{|\phi(x)| : x \in E \text{ and } \|x\| \leq 1\}.$$

Of course, one can take the weak topology on $E^\star$, the smallest one such that all elements $\Phi \in E^{\star\star}$ remain continuous, but there is yet another alternative. For each $x \in E$, the function $J_E(x) \colon E^\star \to \mathbb{R}$ given by $J_E(x)(\phi) := \phi(x)$ determines an inhabitant of $E^{\star\star}$. Thus, one can consider over $E^\star$ the smallest topology such that $J_E(x)$ remains continuous for all $x \in E$, which is usually called the *weak** topology on $E^\star$, denoted by $\sigma(E^\star, E)$. For brevity, we shall denote $E_w := (E, \sigma(E, E^\star))$ and $E^\star_{w^\star} := (E^\star, \sigma(E^\star, E))$.

Theorem I.10 (Banach-Alaoglu)**.** Let $(E, \|\cdot\|)$ be a normed vector space. Then the closed ball $\{\phi \in E^\star : \|\phi\| \leq 1\}$ is compact as a subspace of $E^\star_{w^\star}$.

It follows at once from the above theorem that $E^\star_{w^\star}$ is σ-compact, hence it is a Menger space. Actually, since finite powers of σ-compact spaces are σ-compact, it follows that each finite power of $E^\star_{w^\star}$ is again a Menger space. This brings us to the next theorem, where we denote by $\underline{0} \colon X \to \mathbb{R}$ the constant zero function, and $\Omega_{\underline{0}}$ is the family of those subsets A of $C_p(X)$ such that $\underline{0} \in \overline{A}$, as defined in (2).

Theorem I.11 (Arhangel'skii [3])**.** Let X be a Tychonoff space. Then $C_p(X)$ satisfies $\mathsf{S}_{\text{fin}}(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ if and only if X^n is Menger for all $n \in \omega$.

By the previous observations about $E^\star_{w^\star}$, the next corollary is immediate.

Corollary I.12. For a normed space $(E, \|\cdot\|)$, the space $C_p(E^\star_{w^\star})$ satisfies $\mathsf{S}_{\text{fin}}(\Omega_{\underline{0}}, \Omega_{\underline{0}})$.

Actually, since $C_p(X)$ is a homogeneous space for every X, it follows that $C_p(E^\star_{w^\star})$ satisfies $\mathsf{S}_{\text{fin}}(\Omega_\Phi, \Omega_\Phi)$ for all $\Phi \in C_p(E^\star_{w^\star})$. Finally, since the function J_E determines an embedding of E_w into $C_p(E^\star_{w^\star})$, we may assume $E_w \subset C_p(E^\star_{w^\star})$, hence E_w satisfies $\mathsf{S}_{\text{fin}}(\Omega_x, \Omega_x)$ for all $x \in E$. In particular, it follows from (3) that E_w has countable tightness, a result attributed to Kaplansky[13].

[13] See [45], for instance.

Chapter

1

BORNOLOGIES AND FILTERS APPLIED TO SELECTION PRINCIPLES AND FUNCTION SPACES

This chapter is an extended version of [10]. In the previous section, Theorem I.11 illustrates an important situation we shall be paying attention to in this chapter: when does a property $\mathfrak{P}$ of $C_p(X)$ translate itself as a topological property $\mathfrak{Q}$ of X? In this sense, the next theorem is another instance of this kind of duality, where $\mathrm{t}(Y)$ stands for the tightness of a space Y.

Theorem 1.1 (Arhangel'skii-Pytkeev[1])**.** For a Tychonoff space X, $\sup_{n\in\omega} \mathrm{L}(X^n) = \mathrm{t}(C_p(X))$.

The cardinal number $\sup_{n\in\omega} \mathrm{L}(X^n)$ also describes a covering property of X, a result due to Gerlits and Nagy [28]. We say that an open covering $\mathcal{U}$ of X is an **ω-covering** if for each finite subset $F \subset X$ there exists a $U \in \mathcal{U}$ such that $F \subset U$. It is straightforward to adapt the definition of Lindelöf degree for ω-coverings is straightforward. The **ω-Lindelöf degree** of X is the following cardinal number

$$\mathrm{L}_\omega(X) := \min\left\{\kappa : \forall \mathcal{U} \in \Omega(X) \left([\mathcal{U}]^{\leq\kappa} \cap \Omega(X) \neq \emptyset\right)\right\} + \aleph_0, \tag{1.1}$$

where $\Omega(X)$ denotes the family of all ω-coverings of X.

Theorem 1.2 (Gerlits and Nagy [28])**.** For every topological space X, $\mathrm{L}_\omega(X) = \sup_{n\in\omega} \mathrm{L}(X^n)$.

The comparison of Theorems I.11, 1.1 and 1.2 suggests that covering properties of X regarding ω-coverings translate as closure properties of $C_p(X)$. To reinforce this suggestion, we present the next two results. To avoid confusion, we emphasize that in the following, we write Ω instead of $\Omega(X)$ to denote the set of ω-coverings of X.

[1] More precisely, as mentioned by Arhangel'skii [3], the inequalities $\sup_{n\in\omega} \mathrm{L}(X^n) \leq \mathrm{t}(C_p(X))$ and $\mathrm{t}(C_p(X)) \leq \sup_{n\in\omega} \mathrm{L}(X^n)$ are due to Arhangel'skii (1976) and Pytkeev (1982), respectively.

Theorem 1.3 (Several authors[2]). Let X be a Tychonoff space and $\bullet \in \{1, \text{fin}\}$. The following are equivalent:

1. Player I does not have a winning strategy in the game $\mathsf{G}_\bullet(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ on $C_p(X)$;
2. $C_p(X)$ has property $\mathsf{S}_\bullet(\Omega_{\underline{0}}, \Omega_{\underline{0}})$;
3. X has property $\mathsf{S}_\bullet(\Omega, \Omega)$;
4. Player I does not have a winning strategy in the game $\mathsf{G}_\bullet(\Omega, \Omega)$ on X.

Theorem 1.4 (Scheepers [79]). Let X be a Tychonoff space and $\bullet \in \{1, \text{fin}\}$. The following are equivalent:

1. Player II has a winning strategy in the game $\mathsf{G}_\bullet(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ on $C_p(X)$;
2. Player II has a winning strategy in the game $\mathsf{G}_\bullet(\Omega, \Omega)$ on X.

On the other hand, since $C(X)$ admits possibly many different topologies, it is natural to ask what happens with the above "dualities" if we change the topology of $C(X)$. For instance, by considering $C_k(X)$, defined as the set $C(X)$ endowed with the compact-open topology, one has the following result, where by $\mathcal{K}$ we mean the family of all **K-coverings** of X, i.e., an open covering $\mathcal{U}$ of X such that for each compact subset $C \subset X$ there exists a $U \in \mathcal{U}$ with $C \subset U$.

Theorem 1.5. Let X be a Tychonoff space X.

1. (McCoy [54]). $C_k(X)$ has countable tightness if and only if every K-covering of X has a countable K-subcovering.
2. (Lin, Liu and Teng [52]). $C_k(X)$ satisfies $\mathsf{S}_{\text{fin}}(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ if and only if X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{K}, \mathcal{K})$.
3. (Kočinac [48]). $C_k(X)$ satisfies $\mathsf{S}_1(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ if and only if X satisfies $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$.

In fact, these dualities are preserved for a larger class of topologies on $C(X)$, defined in terms of *bornologies* on X. The precise definitions are given in Section 1.2, where we also extend our work started in [9], generalizing the equivalence 3 $\Leftrightarrow$ 4 in Theorem 1.3 through the introduction of a "bornological hyperspace". In the subsequent sections (Sections 1.3 and 1.4) we deal with filters and their applications in $C_{\mathcal{B}}$-theory, by adapting the C_p-theory results of Jordan [39]. But first, in the next section we discuss generalizations of the selection principles S_1 and S_{fin} as well as of the games G_1 and G_{fin}, motivated by the work of Aurichi, Bella and Dias [6].

[2] The equivalence 2 $\Leftrightarrow$ 3 is due to Sakai [71] in case $\bullet = 1$, while for $\bullet =$fin it follows from Theorem I.11 (due to Arhangel'skii) together with a result of Just et al. [44]. In both cases, $\bullet = 1$ and $\bullet = $ fin, the equivalences 3 $\Leftrightarrow$ 4 $\Leftrightarrow$ 1 are proved by Scheepers [75].

1.1 In between $\mathsf{S}_1/\mathsf{G}_1$ and $\mathsf{S}_{\text{fin}}/\mathsf{G}_{\text{fin}}$

We begin this section with a brief comparison between the selection principles S_{fin} and S_1. Recall that for collections $\mathcal{C}$ and $\mathcal{D}$ of nonempty subsets of an infinite set $\mathcal{S}$, we define:

$$\mathsf{S}_{\text{fin}}(\mathcal{C},\mathcal{D}) := \ \forall (C_n)_{n\in\omega} \in \mathcal{C}^\omega \ \exists (D_n)_{n\in\omega} \in \prod_{n\in\omega} [C_n]^{<\aleph_0} \text{ such that } \bigcup_{n\in\omega} D_n \in \mathcal{D}, \tag{1.2}$$

$$\mathsf{S}_1(\mathcal{C},\mathcal{D}) := \ \forall (C_n)_{n\in\omega} \in \mathcal{C}^\omega \ \exists (d_n)_{n\in\omega} \in \prod_{n\in\omega} C_n \text{ such that } \{d_n : n\in\omega\} \in \mathcal{D}. \tag{1.3}$$

However, since $d_n \in C_n$ is equivalent to $\{d_n\} \in [C_n]^{<2}$, at first one could state the following sentence as a possible equivalent definition of (1.3):

$$\mathsf{S}_{\underline{1}}(\mathcal{C},\mathcal{D}) := \ \forall (C_n)_{n\in\omega} \in \mathcal{C}^\omega \ \exists (D_n)_{n\in\omega} \in \prod_{n\in\omega} [C_n]^{<2} \text{ such that } \bigcup_{n\in\omega} D_n \in \mathcal{D}. \tag{1.4}$$

Nevertheless, there is a subtle difference between (1.3) and (1.4): the latter allows the possibility of having $D_n = \emptyset$, because $[C_n]^{<2} = \{\{d\} : d \in C_n\} \cup \{\emptyset\}$, but the former excludes this situation. Since the equality $\mathsf{S}_{\underline{1}}(\mathcal{C},\mathcal{D}) = \mathsf{S}_1(\mathcal{C},\mathcal{D})$ holds for all pairs $(\mathcal{C},\mathcal{D})$ considered along this chapter, we choose to adopt a *pragmatic posture* and, from now on, we assume this equality as an additional condition for the families $\mathcal{C}$ and $\mathcal{D}$. With this in mind, the following definition is even more natural.

Let $\psi\colon \omega \to [2,\aleph_0]$ be a function and let $\mathcal{C}$ and $\mathcal{D}$ be families as before. We denote as $\mathsf{S}_\psi(\mathcal{C},\mathcal{D})$ the following assertion:

$$\forall (C_n)_{n\in\omega} \in \mathcal{C}^\omega \ \exists (D_n)_{n\in\omega} \in \prod_{n\in\omega} [C_n]^{<\psi(n)} \text{ such that } \bigcup_{n\in\omega} D_n \in \mathcal{D}. \tag{1.5}$$

If one takes $\psi \equiv 2$, the constant function with value 2, then $\mathsf{S}_\psi = \mathsf{S}_1$. Similarly, if $\psi \equiv \aleph_0$, then $\mathsf{S}_\psi = \mathsf{S}_{\text{fin}}$. More generally, for a constant function $\psi\colon \omega \to [2,\aleph_0]$ with value $k+1 \in \omega$, we write S_k instead of S_ψ.

García-Ferreira and Tamariz-Mascarúa [27] presented the prototype of the above definitions. They analyzed selection variations of tightness in a point y of a Tychonoff space Y, by using the family Ω_y in $\mathsf{S}_f(\Omega_y,\Omega_y)$, for $f\colon \omega \to [2,\aleph_0)$. More recently, Aurichi, Bella and Dias [6] defined game variations of the selection principle S_ψ, but still in the tightness context. We shall now present our definition of the game G_ψ – it is slightly different from the one presented in [6], in order to include both the games G_1 and G_{fin}.

Let $\mathcal{C}$ and $\mathcal{D}$ be families of nonempty subsets of an infinite set $\mathcal{S}$ and let $\psi\colon \omega \to [2,\aleph_0]$ be a function. The game $\mathsf{G}_\psi(\mathcal{C},\mathcal{D})$, played between two players, Player I and Player II, has the same structure of the game $\mathsf{G}_{\text{fin}}(\mathcal{C},\mathcal{D})$, except that now, at the n-th inning, Player II has to choose a subset whose cardinality is strictly less than $\psi(n)$.

As we did in the definition of S_ψ, we assume a similar pragmatic posture concerning G_1, in such a way that the notation $\mathsf{G}_k := \mathsf{G}_\psi$, for $\psi \equiv k+1$, makes sense for all $k \in \omega \setminus \{0\}$. The basic relations between S_ψ and G_ψ are summarized in the following proposition.

Proposition 1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be families of nonempty subsets of an infinite set $\mathcal{S}$. Let $\mathcal{D}_{\aleph_0}$ be the collection of countable elements of $\mathcal{D}$ and take ψ, φ functions of the form $\omega \to [2, \aleph_0]$ such that $\psi(n) \leq \varphi(n)$ for all $n \in \omega$. Then the following implications hold:

$$\begin{array}{ccccccc} \text{II} \uparrow \mathsf{G}_\psi(\mathcal{C},\mathcal{D}) & \Longrightarrow & \text{I} \not\uparrow \mathsf{G}_\psi(\mathcal{C},\mathcal{D}) & \Longrightarrow & \mathsf{S}_\psi(\mathcal{C},\mathcal{D}) & & \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ \text{II} \uparrow \mathsf{G}_\varphi(\mathcal{C},\mathcal{D}) & \Longrightarrow & \text{I} \not\uparrow \mathsf{G}_\varphi(\mathcal{C},\mathcal{D}) & \Longrightarrow & \mathsf{S}_\varphi(\mathcal{C},\mathcal{D}) & \Longrightarrow & \binom{\mathcal{C}}{\mathcal{D}_{\aleph_0}}_{\subset} \end{array} \tag{1.6}$$

Proof. The winning criterion of the game G_ψ implies $\text{II} \uparrow \mathsf{G}_\psi(\mathcal{C},\mathcal{D}) \Rightarrow \text{I} \not\uparrow \mathsf{G}_\psi(\mathcal{C},\mathcal{D})$.

On the other hand, if $\mathsf{S}_\psi(\mathcal{C},\mathcal{D})$ does not hold, then there is a sequence $(C_n)_{n\in\omega} \in \mathcal{C}^\omega$ such that for each sequence $(D_n)_{n\in\omega} \in \prod_{n\in\omega}[C_n]^{<\psi(n)}$ one has $\bigcup_{n\in\omega} D_n \notin \mathcal{D}$. We can easily define a winning strategy for Player I in the game G_ψ with this information. Thus, arguing by contraposition we obtain $\text{I} \not\uparrow \mathsf{G}_\psi(\mathcal{C},\mathcal{D}) \Rightarrow \mathsf{S}_\psi(\mathcal{C},\mathcal{D})$.

Now, if $\mathsf{S}_\psi(\mathcal{C},\mathcal{D})$ holds, then we take $(C)_{n\in\omega} \in \mathcal{C}^\omega$ a constant sequence, and we obtain $(D_n)_{n\in\omega} \in \prod_{n\in\omega}[C]^{<\psi(n)}$ such that $D := \bigcup_{n\in\omega} D_n \in \mathcal{D}$. Since $D_n \subset C$ and $|D_n| < \psi(n) \leq \aleph_0$ for all $n \in \omega$, it follows that $D \subset C$ with $D \in \mathcal{D}_{\aleph_0}$, i.e., $\binom{\mathcal{C}}{\mathcal{D}_{\aleph_0}}_{\subset}$ holds.

Since $\psi\colon \omega \to [2, \aleph_0]$ is an arbitrary function, the previous arguing proves all the horizontal implications in (1.6). Finally, the vertical implications follow essentially because $\psi \leq \varphi$. For instance, if μ is a winning strategy for Player II in $\mathsf{G}_\psi(\mathcal{C},\mathcal{D})$, then the choices of Player II accordingly to μ are valid answers in the game $\mathsf{G}_\varphi(\mathcal{C},\mathcal{D})$, from which it follows that μ is also a winning strategy for Player II in $\mathsf{G}_\varphi(\mathcal{C},\mathcal{D})$. The remaining implications can be proved with similar arguments. □

With these intermediate selection principles defined, it is reasonable to ask whether they are indeed new. In the tightness context for instance, most of them turn out to be equivalent. By **equivalent games** we mean two games G and H as before, such that for all J $\in$ {I, II}, Player J has a winning strategy in the game G if and only if Player J has a winning strategy in the game H.

Theorem 1.7. Let Y be a topological space and let $y \in Y$. Let f and $\underline{\mathrm{Id}}$ be functions of the form $\omega \to [2, \aleph_0)$, where $\underline{\mathrm{Id}}$ is defined by $\underline{\mathrm{Id}}(n) = n+2$.

1. If f is bounded, then

   a) (García-Ferreira and Tamariz-Mascarúa [27]) the selection principle $\mathsf{S}_f(\Omega_y, \Omega_y)$ is equivalent to $\mathsf{S}_1(\Omega_y, \Omega_y)$;

   b) (Aurichi, Bella and Dias [6]) the games $\mathsf{G}_f(\Omega_y, \Omega_y)$ and $\mathsf{G}_{k-1}(\Omega_y, \Omega_y)$ are equivalent, where $k = \limsup_{n\in\omega} f(n) \in \omega$.[3]

[3] A word of caution: in [6], the game G_f imposes choices with cardinality $\leq f(n)$ for Player II, implying a slightly different statement from the one we wrote.

2. If f is unbounded, then

    a) (García-Ferreira and Tamariz-Mascarúa [27]) the selection principle $\mathsf{S}_f(\Omega_y,\Omega_y)$ is equivalent to $\mathsf{S}_{\underline{\mathrm{Id}}}(\Omega_y,\Omega_y)$;

    b) (Aurichi, Bella and Dias [6]) the games $\mathsf{G}_f(\Omega_y,\Omega_y)$ and $\mathsf{G}_{\underline{\mathrm{Id}}}(\Omega_y,\Omega_y)$ are equivalent.

In fact, García-Ferreira and Tamariz-Mascarúa [27] provide examples showing that the principles $\mathsf{S}_1(\Omega_y,\Omega_y)$, $\mathsf{S}_{\underline{\mathrm{Id}}}(\Omega_y,\Omega_y)$ and $\mathsf{S}_{\mathrm{fin}}(\Omega_y,\Omega_y)$ are not equivalent. Also, Aurichi, Bella and Dias [6] showed that for each $k\in\omega\setminus\{0\}$, the games $\mathsf{G}_k(\Omega_y,\Omega_y)$, $\mathsf{G}_{k+1}(\Omega_y,\Omega_y)$, $\mathsf{G}_{\underline{\mathrm{Id}}}(\Omega_y,\Omega_y)$ and $\mathsf{G}_{\mathrm{fin}}(\Omega_y,\Omega_y)$ are indeed different.

It is then natural to investigate the behavior of these intermediate selection principles and games when one replaces the family Ω_y with $\mathcal{O}$, or some other set of open coverings. We deal with the family $\mathcal{O}$ of open coverings for the rest of this section, and we particularize this analysis for different classes of open coverings in the next section.

Let us first observe that for an arbitrary function $f\colon\omega\to[2,\aleph_0)$, the selection principle $\mathsf{S}_f(\mathcal{O},\mathcal{O})$ does not *say* anything new, because of the following theorem.

Theorem 1.8 (García-Ferreira and Tamariz-Mascarúa [27])**.** The selection principles $\mathsf{S}_f(\mathcal{O},\mathcal{O})$ and $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ are equivalent for every topological space Y and every function $f\colon\omega\to[2,\aleph_0)$.

Proof. We present a proof, for the convenience of the reader, adapted from [27]. First, note that for $\mathcal{U},\mathcal{V}\in\mathcal{O}$, one has $\mathcal{U}\wedge\mathcal{V}\in\mathcal{O}$, where

$$\mathcal{U}\wedge\mathcal{V}:=\{U\cap V:(U,V)\in\mathcal{U}\times\mathcal{V}\}. \tag{1.7}$$

Suppose that $\mathsf{S}_f(\mathcal{O},\mathcal{O})$ holds and take a sequence $(\mathcal{U}_n)_{n\in\omega}$ of open coverings for Y. We will show that for each $n\in\omega$ there exists a $U_n\in\mathcal{U}_n$ such that $\{U_n:n\in\omega\}\in\mathcal{O}$. Consider the function $g\colon\{-1\}\cup\omega\to\omega$ defined by $g(-1)=0$ and $g(n)=\left(\sum_{i\leq n}f(i)\right)-1$ for all $n\in\omega$. We define the sequence of open coverings $(\mathcal{V}_n)_{n\in\omega}$ in which

$$\mathcal{V}_n:=\mathcal{U}_{g(n-1)}\wedge\mathcal{U}_{g(n-1)+1}\wedge\ldots\wedge\mathcal{U}_{g(n-1)+f(n)-2},$$

for each $n\in\omega$. Since S_f holds, for each $n\in\omega$ there exists a $\mathcal{W}_n\subset\mathcal{V}_n$ such that $|\mathcal{W}_n|<f(n)$ and $\bigcup_{n\in\omega}\mathcal{W}_n\in\mathcal{O}$. By writing $\mathcal{W}_n=\{W_{n,j}:0\leq j<f(n)-1\}\subset\mathcal{V}_n$, it follows that for each $j\in\{0,1,\ldots,f(n)-2\}$ we may take a $U_{g(n-1)+j}\in\mathcal{U}_{g(n-1)+j}$ such that $W_{n,j}\subset U_{g(n-1)+j}$. Hence

$$X=\bigcup_{n\in\omega}\bigcup\mathcal{W}_n=\bigcup_{n\in\omega}\bigcup_{j=0}^{f(n)-2}W_{n,j}\subset\bigcup_{n\in\omega}\bigcup_{j=0}^{f(n)-2}U_{g(n-1)+j}=\bigcup_{k\in\omega}U_k,$$

which shows that $\{U_n:n\in\omega\}\in\mathcal{O}$, as desired. $\square$

Even more can be said in general, because of the "S_1-version" of Theorem I.5.

Theorem 1.9 (Pawlikowski [68]). For a topological space Y, the selection principle $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ holds if and only if $\mathrm{I} \not\uparrow \mathsf{G}_1(\mathcal{O},\mathcal{O})$.

Corollary 1.10. Let Y be a topological space and let $f\colon \omega \to [2, \aleph_0)$ be a function. The following are equivalent:

1. $\mathrm{I} \not\uparrow \mathsf{G}_1(\mathcal{O},\mathcal{O})$;
2. $\mathrm{I} \not\uparrow \mathsf{G}_f(\mathcal{O},\mathcal{O})$;
3. $\mathsf{S}_f(\mathcal{O},\mathcal{O})$;
4. $\mathsf{S}_1(\mathcal{O},\mathcal{O})$.

Proof. Proposition 1.6 and Theorem 1.8 give $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$, while $(4) \Rightarrow (1)$ follows from the previous theorem. □

Thus, to distinguish the games $\mathsf{G}_f(\mathcal{O},\mathcal{O})$ and $\mathsf{G}_g(\mathcal{O},\mathcal{O})$ for functions $f,g\colon \omega \to [2, \aleph_0)$, it is necessary to consider the situation of Player II. In a recent paper by Nathaniel Hiers et al. [33], it is showed that if the space Y is Hausdorff, then the games $\mathsf{G}_f(\mathcal{O},\mathcal{O})$ and $\mathsf{G}_1(\mathcal{O},\mathcal{O})$ are equivalent (Corollary 2.3 in [33]). However, this is done in a very technical way. Alternatively, by adapting the proof of Scheepers [73] for Theorem I.4, we can show that condition $\mathrm{II} \uparrow \mathsf{G}_f(\mathcal{O},\mathcal{O})$ is very restrictive on a large class of spaces in a reasonable simpler way.

Lemma 1.11. Let Y be a topological space and let $f\colon \omega \to [2, \aleph_0)$ be a function. If σ is a winning strategy for Player II in the game $\mathsf{G}_f(\mathcal{O},\mathcal{O})$, then Y admits a countable covering by sets of the form

$$L_{\mathcal{A},z} := \bigcap_{\mathcal{U}\in\mathcal{A}} \bigcup \sigma(z^\frown \mathcal{U}) \tag{1.8}$$

and

$$\widetilde{L_{\mathcal{A},z}} := \bigcap_{\mathcal{U}\in\mathcal{A}} \overline{\bigcup \sigma(z^\frown \mathcal{U})}, \tag{1.9}$$

where $z = (\mathcal{U}_0,\ldots,\mathcal{U}_n) \in \mathcal{O}^{<\omega}$, $\mathcal{A} \subset \mathcal{O}$ and $z^\frown\mathcal{U}$ denotes the concatenation $(\mathcal{U}_0,\ldots,\mathcal{U}_n,\mathcal{U})$.

Proof. Indeed, if for each $z \in \mathcal{O}^{<\omega}$ we have a countable subset $\mathcal{O}_z \subset \mathcal{O}$ fixed, then we may recursively select countably many finite sequences z in such a way that the corresponding sets $L_{\mathcal{O}_z,z}$ (and $\widetilde{L_{\mathcal{O}_z,z}}$) cover X. More precisely, by induction we may define a function $\mathcal{U}\colon \omega^{<\omega} \to \mathcal{O}$ such that $\mathcal{O}_{\mathcal{U}_s} = \{\mathcal{U}_{s^\frown n} : n \in \omega\}$ for all $s \in \omega^{<\omega}$.

Now, to simplify notations, for a sequence $s = (m_1,\ldots,m_n) \in \omega^{<\omega}$ we write $\mathcal{O}_s$ instead of $\mathcal{O}_{\mathcal{U}_s}$ and $\underline{s} := (\mathcal{U}_{m_0},\mathcal{U}_{m_0,m_1},\ldots,\mathcal{U}_{m_0,m_1,\ldots,m_n})$. Since σ is a winning strategy for Player II in the game $\mathsf{G}_f(\mathcal{O},\mathcal{O})$, we have

$$X = \bigcup_{s\in\omega^{<\omega}} L_{\mathcal{O}_s,\underline{s}} = \bigcup_{s\in\omega^{<\omega}} \widetilde{L_{\mathcal{O}_s,\underline{s}}}. \tag{1.10}$$

Suppose, contrary to our claim, that we could find a $p \in X$ such that $p \notin L_{\mathcal{O}_{s},\underline{s}}$ for all $s \in \omega^{<\omega}$. In particular, for $s = \emptyset$, we would have $p \notin L_{\mathcal{O}_{\emptyset},\emptyset}$, hence we could find a $\mathcal{U}_{n_0} \in \mathcal{O}_{\emptyset}$ such that $p \notin \bigcup \sigma(\mathcal{U}_{n_0})$. Again, $p \notin L_{\mathcal{O}_{(n_0)},\underline{(n_0)}}$, then there would be a $\mathcal{U}_{n_0,n_1} \in \mathcal{O}_{(n_0)}$ such that $p \notin \bigcup \sigma(\mathcal{U}_{n_0}, \mathcal{U}_{n_0,n_1})$. We could thus continue, obtaining a play in the game $\mathsf{G}_f(\mathcal{O},\mathcal{O})$ such that the strategy σ is defeated, a contradiction. Therefore, equality (1.10) holds. $\square$

Proposition 1.12. Let Y be a topological space and let $f\colon \omega \to [2, \aleph_0)$ be a function. If Player II has a winning strategy in the game $\mathsf{G}_f(\mathcal{O},\mathcal{O})$, then Y is countable, provided at least one of the following holds:

1. the space Y is a T_1-space and has a countable basis;

2. the space Y is Hausdorff and all its points are G_δ-sets (in particular if Y is first countable).

Moreover, in this case $\mathrm{II} \uparrow \mathsf{G}_1(\mathcal{O},\mathcal{O})$.

Proof. Because of the previous lemma, it is enough to show that for each $z = (\mathcal{U}_0, \dots, \mathcal{U}_n) \in \mathcal{O}^{<\omega}$ there exists a countable subset $\mathcal{O}_z \subset \mathcal{O}$ such that $L_{\mathcal{O}_z,z}$ (or $\widetilde{L_{\mathcal{O}_z,z}}$) is countable. Fix $m \in \omega \setminus \{0\}$ such that $f(n+1) = m+1$.

1. Assuming the space Y is T_1 with a countable basis $\mathscr{B}$. Let $\mathcal{O}' := \mathcal{O} \cap \wp(\mathscr{B})$.

   - $|L_{\mathcal{O}',z}| \le m$. Indeed, for an $F \subset Y$ such that $|F| = m+1$, let $F_y := F \setminus \{y\}$ for each $y \in F$ and take $\mathcal{V} := \{Y \setminus F_y : y \in F\}$. Since Y is a T_1 space, it follows that each member of $\mathcal{V}$ is an open set of Y and $\mathcal{V} \in \mathcal{O}$. We claim that for every $\mathcal{U} \in \mathcal{O}'$ refining $\mathcal{V}$, $F \not\subset \bigcup \sigma(z^\frown \mathcal{U})$ holds. Indeed, for each $U \in \sigma(z^\frown \mathcal{U})$ there exists an $y(U) \in F$ such that $U \subset Y \setminus F_{y(U)}$, hence
   $$\bigcup \sigma(z^\frown \mathcal{U}) \subset \bigcup_{U \in \sigma(z^\frown \mathcal{U})} Y \setminus F_{y(U)} = Y \setminus \bigcap_{U \in \sigma(z^\frown \mathcal{U})} F_{y(U)};$$
   since $|\{y(U) : U \in \sigma(z^\frown \mathcal{U})\}| \le m < |F|$, there exists an $y \in F$ such that $y \neq y(U)$ for all $U \in \sigma(z^\frown \mathcal{U})$, which implies that $y \in \bigcap_{U \in \sigma(z^\frown \mathcal{U})} F_{y(U)}$.
   - We have $\{\sigma(z^\frown \mathcal{U}) : \mathcal{U} \in \mathcal{O}'\} \subset [\mathscr{B}]^{<\aleph_0}$, hence there exists a countable subfamily $\mathcal{O}_z \subset \mathcal{O}' \subset \mathcal{O}$ such that $\{\sigma(z^\frown \mathcal{U}) : \mathcal{U} \in \mathcal{O}'\} = \{\sigma(z^\frown \mathcal{U}) : \mathcal{U} \in \mathcal{O}_z\}$, as desired.

2. Assuming the space Y is Hausdorff and its points are G_δ-sets of Y. In particular, note that each finite subset H of Y is a G_δ-set, say $H = \bigcap_{n\in\omega} G_{H,n}$, where each $G_{H,n} \subset Y$ is open.

   - $|\widetilde{L_{\mathcal{O},z}}| \le m$. Indeed, for an $F \subset Y$ such that $|F| = m+1$, let $F_y := F \setminus \{y\}$ for each $y \in F$. Since Y is Hausdorff, for each $w \in Y \setminus F_y$ there exists an open set $B_w \subset Y$ such that $w \in B_w \subset \overline{B_w} \subset Y \setminus F_y$, which allows us to take $\mathcal{U} := \bigcup_{y\in F} \mathcal{U}_y$, where $\mathcal{U}_y := \{B_w : w \in Y \setminus F_y\}$. As in the first case, one can easily show that $F \not\subset \overline{\bigcup \sigma(z^\frown \mathcal{U})}$.

- Note that the family $\mathcal{H} := \left\{Y \setminus \overline{\bigcup \sigma(z^\frown \mathcal{U})} : \mathcal{U} \in \mathcal{O}\right\}$ is an open covering for the subspace $Y \setminus H$, where $H := \widetilde{L_{\mathcal{O},z}}$. Hence $\mathcal{H}_n := \mathcal{H} \cup \{G_{H,n}\} \in \mathcal{O}$. Since Y is a Lindelöf space (by Proposition 1.6), there exists a countable family $\mathcal{O}_{z,n} \subset \mathcal{O}$ such that
$$\left\{Y \setminus \overline{\bigcup \sigma(z^\frown \mathcal{U})} : \mathcal{U} \in \mathcal{O}_{z,n}\right\} \cup \{G_{H,n}\} \in \mathcal{O}.$$
By letting $\mathcal{O}_z := \bigcup_{n\in\omega} \mathcal{O}_{z,n}$, it follows that
$$H = \widetilde{L_{\mathcal{O},z}} \subset \widetilde{L_{\mathcal{O}_z,z}} = \bigcap_{n\in\omega} \bigcap_{\mathcal{U}\in\mathcal{O}_{z,n}} \overline{\bigcup \sigma(z^\frown \mathcal{U})} \subset \bigcap_{n\in\omega} G_{H,n} = H,$$
as desired. □

Since closed subsets of metric spaces are G_δ-sets, the next corollary is immediate.

Corollary 1.13. The following are equivalent for every metric space M:

1. the space M is countable;
2. $\mathrm{II} \uparrow \mathsf{G}_1(\mathcal{O}, \mathcal{O})$;
3. $\mathrm{II} \uparrow \mathsf{G}_f(\mathcal{O}, \mathcal{O})$ for every function $f \colon \omega \to [2, \aleph_0)$;
4. $\mathrm{II} \uparrow \mathsf{G}_f(\mathcal{O}, \mathcal{O})$ for some function $f \colon \omega \to [2, \aleph_0)$.

We say that a game G with two players is **undetermined** if both $\mathrm{I} \not\uparrow \mathsf{G}$ and $\mathrm{II} \not\uparrow \mathsf{G}$ hold.

Corollary 1.14. If M is an uncountable metric space satisfying $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$, then the game $\mathsf{G}_f(\mathcal{O}, \mathcal{O})$ is undetermined for every function $f \colon \omega \to [2, \aleph_0)$.

By the above results, the next two (consistent) examples present spaces in which the game $\mathsf{G}_f(\mathcal{O}, \mathcal{O})$ is undetermined for each function $f \colon \omega \to [2, \aleph_0)$.

Example 1.15. Recall that an uncountable subspace $L \subset \mathbb{R}$ is called a **Lusin set** if $|L \cap N| \leq \aleph_0$ for all nowhere dense subsets $N \subset \mathbb{R}$ (see [76]). Although the existence of Lusin sets is independent of ZFC, one can prove that if $L \subset \mathbb{R}$ is a Lusin set, then L satisfies $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$.

Indeed, first we fix a countable dense set $\{d_n : n \in \omega\} \subset L$. Now, for a sequence $(\mathcal{U}_n)_{n\in\omega}$ of open coverings for X, for each $n \in \omega$ we take a $U_{2n} \in \mathcal{U}_{2n}$ such that $d_n \in U_{2n}$. Since the set $N := L \setminus \bigcup_{n\in\omega} U_{2n}$ is nowhere dense[4], it follows that $|L \cap N| \leq \aleph_0$, thus we may cover the complement of N with the (countably many) remaining coverings of the sequence[5]. ■

Example 1.16. In [5], Aurichi shows that with a suitable topology, a *Suslin tree* is an uncountable Hausdorff first countable and Lindelöf space, but with a bit more effort, it can be shown that such space is Rothberger [8].

[4] Since N is disjoint from the dense open set $(\mathbb{R} \setminus \overline{L}) \cup \bigcup_{n\in\omega} U_{2n}$, because $\{d_n : n \in \omega\} \subset \bigcup_{n\in\omega} U_{2n}$.

[5] This argument is adapted from [8], where it was attributed to Rothberger (1938).

Turning our attention back to Menger's game, we mention that Telgársky originally obtained Theorem I.4 as a corollary of the following result, implicitly stated in his works.

Theorem 1.17 (Telgársky [90, 92])**.** Let X be a Tychonoff space such that each compact subset of X is a G_δ-set. Then $\mathrm{II} \uparrow \mathsf{G}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ if and only if X is σ-compact.

Indeed, Telgársky [92, Corollary 3] observed that for a Tychonoff space X, the condition $\mathrm{II} \uparrow \mathsf{G}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ is equivalent to Player I having a winning strategy in a game K, defined in terms of compact sets of X.[6] On the other hand, he [90, Corollary 6.4] showed that if X is a Tychonoff space in which compact sets are G_δ-sets, then the condition $\mathrm{I} \uparrow \mathsf{K}$ is equivalent to σ-compactness of X.

Curiously, our proof for Proposition 1.12, which is an adaptation of Scheepers [73] direct proof for Theorem I.4, yields a direct proof for Theorem 1.17.

Proposition 1.18. Let X be a regular space such that each compact subset of X is contained in a compact G_δ-set of X. Then the space X is σ-compact if and only if $\mathrm{II} \uparrow \mathsf{G}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$.

Proof. As in Proposition 1.12, it is enough to show that for each $z = (\mathcal{U}_0, \dots, \mathcal{U}_n) \in \mathcal{O}^{<\omega}$ there exists a countable family $\mathcal{O}_z \subset \mathcal{O}$ such that $\widetilde{L_{\mathcal{O}_z,z}}$ is (contained in) a compact subset of X.

Let us first observe that $\widetilde{L_{\mathcal{O},z}}$ is compact. Indeed, let $\mathcal{U}$ be an open covering for $\widetilde{L_{\mathcal{O},z}}$, and for each $x \in \widetilde{L_{\mathcal{O},z}}$ let $U_x \in \mathcal{U}$ such that $x \in U_x$. Since X is regular, for each $x \in \widetilde{L_{\mathcal{O},z}}$ we may take an open set $V_x \subset X$ such that $x \in V_x \subset \overline{V_x} \subset U_x$, and for an $x \in X \setminus \widetilde{L_{\mathcal{O},z}}$ we may take an open set $V_x \subset X$ such that $x \in V_x \subset \overline{V_x} \subset X \setminus \widetilde{L_{\mathcal{O},z}}$, because $\widetilde{L_{\mathcal{O},z}}$ is closed, which implies that $\mathcal{V} := \{V_x : x \in X\} \in \mathcal{O}$. Hence there exists a finite subset $F \subset X$ such that $\sigma(z^\frown\mathcal{V}) = \{V_x : x \in F\}$. By the way we defined $\mathcal{V}$, it follows that $\{U_x : x \in F \cap \widetilde{L_{\mathcal{O},z}}\}$ is a finite subcovering of $\mathcal{U}$.

Now, let $K = \bigcap_{n\in\omega} G_{K,n}$ be a compact G_δ-set of X such that $\widetilde{L_{\mathcal{O},z}} \subset K$. It is clear that the family $\mathcal{H} := \left\{X \setminus \overline{\bigcup \sigma(z^\frown\mathcal{U})} : \mathcal{U} \in \mathcal{O}\right\}$ is an open covering of $X \setminus \widetilde{L_{\mathcal{O},z}}$, from which it follows that $\mathcal{H}_n := \mathcal{H} \cup \{G_{K,n}\} \in \mathcal{O}$. Since X is a Lindelöf space (by Proposition 1.6), there exists a countable subfamily $\mathcal{O}_{z,n} \subset \mathcal{O}$ such that $\left\{X \setminus \overline{\bigcup \sigma(z^\frown\mathcal{U})} : \mathcal{U} \in \mathcal{O}_{z,n}\right\} \cup \{G_{K,n}\} \in \mathcal{O}$. Finally, for $\mathcal{O}_z = \bigcup_{n\in\omega} \mathcal{O}_{z,n}$ we have

$$\widetilde{L_{\mathcal{O}_z,z}} := \bigcap_{\mathcal{U}\in\mathcal{O}_z} \overline{\bigcup \sigma(z^\frown\mathcal{U})} = \bigcap_{n\in\omega} \bigcap_{\mathcal{U}\in\mathcal{O}_{z,n}} \overline{\bigcup \sigma(z^\frown\mathcal{U})} \subset \bigcap_{n\in\omega} G_{K,n} = K,$$

as desired. $\square$

[6] Actually, Telgársky denotes the game K as $G(C,X)$, but it would be confusing with our terminology. It is played like this: Player I starts a play of the game K by choosing a compact subset $K_0 \subset X$, and Player II answers with a closed subset $C_1 \subset X \setminus K_0$; in the next inning Player I chooses a compact subset $K_1 \subset C_1$, and then Player II chooses a closed subset $C_2 \subset C_1 \setminus K_1$, and so on; Player I wins the play if $\bigcap_{n\in\omega} C_n = \emptyset$.

In the following diagram we summarize the implications proved so far, for a fixed topological space X and a function $f\colon \omega \to [2, \aleph_0)$.

$$
\begin{tikzcd}
 & {|X| \leq \aleph_0} \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & \mathrm{II} \uparrow \mathsf{G}_1(\mathcal{O},\mathcal{O}) \arrow[r, Rightarrow, "(b)"] \arrow[d, Rightarrow] & \mathrm{I} \not\uparrow \mathsf{G}_1(\mathcal{O},\mathcal{O}) \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\text{Corollary 1.10}"] & \mathsf{S}_1(\mathcal{O},\mathcal{O}) \arrow[d, equal] \\
 & & \mathrm{II} \uparrow \mathsf{G}_f(\mathcal{O},\mathcal{O}) \arrow[r, Rightarrow, "(b)"] \arrow[d, Rightarrow, "(a)"] \arrow[ul, dashed, Rightarrow, "\dagger"'] & \mathrm{I} \not\uparrow \mathsf{G}_f(\mathcal{O},\mathcal{O}) \arrow[r, equal] \arrow[d, Rightarrow, "(d)"] & \mathsf{S}_f(\mathcal{O},\mathcal{O}) \arrow[d, Rightarrow, "(d)"] \\
X \text{ is compact} \arrow[r, Rightarrow] & X \text{ is } \sigma\text{-compact} \arrow[r, Rightarrow] & \mathrm{II} \uparrow \mathsf{G}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) \arrow[r, Rightarrow, "(c)"] \arrow[l, dashed, Rightarrow, bend left, "\star"] & \mathrm{I} \not\uparrow \mathsf{G}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) \arrow[r, equal, "\text{Thm I.5}"] & \mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) \arrow[d, Rightarrow, "(e)"] \\
 & & & & \mathrm{L}(X) \leq \aleph_0
\end{tikzcd}
\tag{1.11}
$$

The double arrows indicate implications that hold in general, while the double lines mean equivalences. The $\dagger$ dashed arrow holds with the hypotheses of Proposition 1.12, while the $\star$ dashed arrow is true with the hypotheses of Proposition 1.18. The converses of the other arrows are not true in general. We present a short list of counterexamples for the converses of the corresponding tagged arrows.

(a) Any uncountable σ-compact metric space works.

(b) Any Lusin set works (consistently). If we do not restrict ourselves to consider subsets of the real line, then we may find counterexamples in ZFC: in [7], Aurichi and Dias obtain a Lindelöf regular space Y such that Y_δ (the $\aleph_0$-modification of Y)[7] is Lindelöf while $\mathrm{II} \not\uparrow \mathsf{G}_{\mathrm{fin}}(\mathcal{O}(Y), \mathcal{O}(Y))$, and it is not hard to see that $\mathrm{L}(Y_\delta) \leq \aleph_0 \Rightarrow \mathsf{S}_1(\mathcal{O}(Y), \mathcal{O}(Y))$.

(c) Fremlin and Miller [25] present non-σ-compact subspaces of $\mathbb{R}$ satisfying $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$.

(d) The space 2^ω is compact, hence $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ holds, but it does not satisfy $\mathsf{S}_1(\mathcal{O},\mathcal{O})$. This is simple to verify: for each $n \in \omega$ let $\mathcal{U}_n := \{\pi_n^{-1}[\{0\}], \pi_n^{-1}[\{1\}]\}$, where $\pi_n\colon 2^\omega \to \{0,1\}$ is the projection onto the n-th coordinate, and note that $(\mathcal{U}_n)_{n\in\omega}$ witnesses the failure of the property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$.

(e) The space ω^ω is a Lindelöf space that does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. In this case, the sequence of open coverings $(\mathcal{U}_m)_{m\in\omega}$, where $\mathcal{U}_m := \{C_{m,n} : n \in \omega\}$ and $C_{m,n} := \{f \in \omega^\omega : f(m) = n\}$, witnesses the failure of the property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$.

In the next section we obtain a way to reduce the games of the form $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ played on certain function spaces to games of the form $\mathsf{G}_\psi(\mathcal{O},\mathcal{O})$ played on certain topological spaces. However, even in the best scenarios these translation spaces are at most T_0-spaces, what prevents us from using the previous results. This suggests the following questions.

[7] Recall the definition at the Remark I.9.

Question 1.19. Let X be a T_0-space. What can be said about the space X if $\mathrm{II} \uparrow \mathsf{G}_f(\mathcal{O}, \mathcal{O})$ holds for a function $f\colon \omega \to [2, \aleph_0)$?

Question 1.20. Let X be a T_0-space and let $f\colon \omega \to [2, \aleph_0)$ be a function. Are the games $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$ and $\mathsf{G}_f(\mathcal{O}, \mathcal{O})$ equivalent in general? What if $f \equiv k+1$ for $k \in \omega \setminus \{0, 1\}$?[8]

Remark 1.21. One could ask about similar variations for the selection principle $\mathsf{U}_{\mathrm{fin}}(\mathcal{C}, \mathcal{D})$, i.e., for a function $\varphi\colon \omega \to [2, \aleph_0]$, consider the assertion

$$\mathsf{U}_\varphi(\mathcal{C}, \mathcal{D}) := \forall (C_n)_{n\in\omega} \in \mathcal{C}^\omega \, \exists (D_n)_{n\in\omega} \in \prod_{n\in\omega} [C_n]^{<\varphi(n)} \text{ such that } \left\{ \bigcup D_n : n \in \omega \right\} \in \mathcal{D}. \tag{1.12}$$

Not surprisingly[9], this was already done by Tsaban [98]. However, we are not aware of similar variations concerning the *Hurewicz* game. ■

[8] As pointed out by Hiers et al. [33], "the equivalence of the above games for arbitrary spaces is no stronger than the equivalence for T_0 spaces (by considering the T_0 quotient of an arbitrary space)".

[9] See Remark 1.44.

1.2 Bornologies and spaces of the form $C_{\mathcal{B}}(X)$

We start this section with a closer look at the hyperspace $C_p(X)$, for a fixed topological space X. With the definition we gave so far, a basic open neighborhood of a function $f \in C_p(X)$ is a set of the form

$$C(X) \cap \prod_{x \in X} V_x,$$

where $V_x \subset \mathbb{R}$ is open for each $x \in X$ and $F := \{x \in X : V_x \neq \mathbb{R}\}$ is finite. Note then that for each $x \in F$ there is an $\varepsilon_x > 0$ such that $(f(x) - \varepsilon_x, f(x) + \varepsilon_x) \subset V_x$. By taking $\varepsilon := \min\{\varepsilon_x : x \in F\}$, it follows that

$$\langle F, \varepsilon \rangle [f] := \left\{ g \in C_p(X) : \forall x \in F \left(|g(x) - f(x)| < \varepsilon\right) \right\} \subset C(X) \cap \prod_{x \in X} V_x.$$

Since $\langle F, \varepsilon \rangle [f]$ itself is a basic open set in $C_p(X)$, it is easy to conclude that the family $\{\langle F, \varepsilon \rangle [f] : F \in [X]^{<\aleph_0}, \varepsilon > 0\}$ is a local basis at f. This framework suggests a natural generalization, by replacing the family $[X]^{<\aleph_0}$ with an arbitrary family of subsets of X.

For a subset $B \subset X$, a real number $\varepsilon > 0$ and a function $f \in C(X)$, consider the sets of the form

$$\langle B, \varepsilon \rangle [f] := \{g \in C(X) : \forall x \in B \left(|g(x) - f(x)| < \varepsilon\right)\}, \tag{1.13}$$

and for a family $\mathcal{B}$ of subsets of X, call $C_{\mathcal{B}}(X)$ the set $C(X)$ with the topology having as a local subbasis at each f the family

$$\mathscr{U}_{\mathcal{B},f} := \{\langle B, \varepsilon \rangle [f] : B \in \mathcal{B} \text{ and } \varepsilon > 0\}. \tag{1.14}$$

Since a sequence[10] $(f_n)_{n \in \omega}$ in $C_{\mathcal{B}}(X)$ converges to a function f if and only if $f_n \to f$ uniformly on B for all $B \in \mathcal{B}$, the topology generated by $\{\mathscr{U}_{\mathcal{B},f}\}_{f \in C(X)}$ is usually called the (topology) **of uniform convergence on** $\mathcal{B}$.

In particular, note that for each pair of subsets $A, B \subset X$ and a positive real number $\varepsilon > 0$, the equality

$$\langle A, \varepsilon \rangle [f] \cap \langle B, \varepsilon \rangle [f] = \langle A \cup B, \varepsilon \rangle [f] \tag{1.15}$$

shows that we may always suppose the family $\mathcal{B}$ to be closed under taking finite unions: more precisely, by letting

$$\widetilde{\mathcal{B}} := \left\{ B \subset X : \exists \mathcal{F} \in [\mathcal{B}]^{<\aleph_0} \text{ such that } B = \bigcup \mathcal{F} \right\}, \tag{1.16}$$

one has $C_{\mathcal{B}}(X) = C_{\widetilde{\mathcal{B}}}(X)$. Moreover, in this case the family $\mathscr{U}_{\widetilde{\mathcal{B}},f}$ is in fact a local basis at $f \in C_{\mathcal{B}}(X)$. Finally, since the correspondence $A \mapsto \langle A, \varepsilon \rangle [f]$ is *contravariant*, it follows that we may also assume that $\mathcal{B}$ is closed for inclusions. Thus, the topology of $C_{\mathcal{B}}(X)$ remains the same if we ask for the family $\mathcal{B}$ to be an ideal of subsets of X.

[10] Obviously, it also holds for nets as well.

Our first contact with spaces of the form $C_{\mathcal{B}}(X)$ was made with the stronger assumption that $\mathcal{B}$ was a *bornology* on X. For a given set X, a family $\mathcal{B}$ of subsets of X is a **bornology** on X if $\mathcal{B}$ is an ideal such that $X = \bigcup \mathcal{B}$. Also, a subset $\mathcal{B}'$ of a bornology $\mathcal{B}$ is a (closed, **compact**, etc.) **basis** for $\mathcal{B}$ if $\mathcal{B}'$ is cofinal in $(\mathcal{B}, \subset)$ (such that all its elements are closed, compact, etc.). Intuitively, a bornology gives a notion of boundedness in X. Indeed, if we call as *bounded sets* the elements of $\mathcal{B}$, note that

- a finite union of bounded sets is bounded,
- a subset of a bounded set is bounded,
- every point is bounded,

which agrees with the behavior of *usual* bounded sets of metric spaces. Although these structures have arisen in Functional Analysis, where they are used in the context of topological vector spaces[11], here we are mainly interested in the fact that bornologies are *nice* generalizations of the following families of a topological space X:

- the family $[X]^{<\aleph_0}$ of the finite subsets of X;
- the family K_X of all subsets of compact subspaces of X – if X is a Hausdorff space, then $\mathsf{K}_X = \{A \subset X : \overline{A} \text{ is compact}\}$.

In the works of Beer and Levi [14] and of Caserta, di Maio and Kočinac [17], a variation of the topology of uniform convergence on $\mathcal{B}$ is studied for the case where X is a metric space and the bornology $\mathcal{B}$ has a closed basis. The results in [17] concerning dualities between covering properties of X and local properties of $C(X)$ in the framework of selection principles triggered our investigation, which started at [9], in the realm of Tychonoff spaces. In this sense, our approach resembles the work of McCoy and Ntantu [55].

For a family $\mathcal{B}$ of subsets of X, we call $\mathcal{O}_{\mathcal{B}}$ the family of all **$\mathcal{B}$-coverings** of X: those open coverings $\mathcal{U}$ of X such that for each $B \in \mathcal{B}$ there exists a $U \in \mathcal{U}$ with $B \subset U$. We also say that the open covering $\mathcal{U}$ is **non-trivial** if $X \notin \mathcal{U}$, and we denote by $\mathcal{O}^*_{\mathcal{B}}$ the collection of the non-trivial $\mathcal{B}$-coverings.

Example 1.22 ($C_p(X)$ and ω-coverings)**.** As our starting point for the definition of $C_{\mathcal{B}}(X)$, it is clear that for $\mathcal{B} = [X]^{<\aleph_0}$ one has $C_{[X]^{<\aleph_0}}(X) = C_p(X)$. Also, $\Omega = \mathcal{O}_{[X]^{<\aleph_0}}$. ■

Example 1.23 ($C_k(X)$ and K-coverings)**.** If X is a Hausdorff space, it is straightforward to show that $C_k(X) = C_{\mathsf{K}_X}(X)$, or in words: the compact open topology coincides with the topology of uniform convergence on compact sets[12]. Similarly, one readily verifies $\mathcal{K} = \mathcal{O}_{\mathsf{K}_X}$. ■

[11] For the reader interested in such applications, we suggest the work of Hogbe-Nlend [34].

[12] In [67], Nokhrin and Osipov discuss this kind of coincidence in a general setting.

As an example of the advantage of this approach, note that the next theorem generalizes some of the previous results about $C_p(X)$ and $C_k(X)$ in connection with ω-coverings and K-coverings, respectively.

Theorem 1.24 (McCoy and Ntantu [55]). Let X be a Tychonoff space and let $\mathcal{B}$ be a bornology on X with a compact basis. Then $\mathrm{t}(C_{\mathcal{B}}(X)) \leq \kappa$ if and only if every $\mathcal{B}$-covering of X has a $\mathcal{B}$-subcovering with cardinality $\leq \kappa$.

If one asks for a normal space instead of a Tychonoff space in the above theorem, then the condition about compactness of the basis may be relaxed to closedness. As we shall see along this chapter, in this kind of duality, for a given element $B \in \mathcal{B}$ and an open set $U \subset X$ such that $\overline{B} \subset U$, one needs to find a continuous function $g\colon X \to \mathbb{R}$ such that $g \restriction B \equiv 0$ and $g \restriction (X \setminus U) \equiv 1$. For normal spaces, this function is provided by Urysohn's Lemma. On the other hand, in the absence of Urysohn's Lemma, we may use the following.

Theorem 1.25 (Engelking [22, Theorem 3.1.7]). If K is a compact subspace of a Tychonoff space X, then for every closed set $C \subset X$ disjoint from K there exists a continuous function $g\colon X \to [0,1]$ such that $g \restriction K \equiv 0$ and $g \restriction C \equiv 1$.

One of the main reasons for considering bornologies instead of arbitrary families of subsets of X is hidden in the next simple lemma.

Lemma 1.26 (Scheepers [74], for ω-coverings). Let $\mathcal{B}$ be a bornology on a topological space X and fix a $\mathcal{U} \in \mathcal{O}^*_{\mathcal{B}}$.

1. If $\{\mathcal{U}_i\}_{i \in I}$ is a finite partition of $\mathcal{U}$, then $\mathcal{U}_i \in \mathcal{O}_{\mathcal{B}}$ for some $i \in I$.
2. If $F \subset \mathcal{U}$ is a finite subset, then $\mathcal{U} \setminus F \in \mathcal{O}_{\mathcal{B}}$.[13]

Proof. We prove 1 by the contrapositive: if each $\mathcal{U}_i$ fails to be a $\mathcal{B}$-covering, then for each $i \in I$ there exists a $B_i \in \mathcal{B}$ such that $B_i \not\subset U$ for all $U \in \mathcal{U}_i$, hence $B := \bigcup_{i \in I} B_i \in \mathcal{B}$ witnesses that $\mathcal{U} \notin \mathcal{O}_{\mathcal{B}}$. The item 2 follows from the previous one because $\{F, \mathcal{U} \setminus F\}$ is a partition of $\mathcal{U}$ and $F \notin \mathcal{O}_{\mathcal{B}}$: if $F = \{U_0, \ldots, U_n\}$, then for each $i \leq n$ there exists an $x_i \in X \setminus U_i$, but then $\{x_0, \ldots, x_n\} \in \mathcal{B}$ is not covered by the elements of F. □

Remark 1.27. Note that to prove item 2 in the above lemma, it would be enough that $\mathcal{B}$ was a *directed covering* for X, i.e., $X = \bigcup \mathcal{B}$ and for each $A, B \in \mathcal{B}$ there exists a $C \in \mathcal{B}$ such that $A \cup B \subset C$. That is, it would be enough that $\mathcal{B}$ was a basis for *some* bornology on X. ■

Both Lemma 1.26 and Theorem 1.25 are essential in the following lemma, adapted from Caserta, di Maio and Kočinac [17], which works as a bridge between covering properties of X

[13] In some sense, this is related to the following fact: if Y is a T_1 space and $y \in \overline{A} \setminus A$ for some $A \subset Y$, then $y \in \overline{A \setminus F}$ for each finite subset $F \subset A$.

concerning $\mathcal{B}$-coverings and local properties of $C_{\mathcal{B}}(X)$. Once again, to avoid confusion and to simplify the notation, we emphasize that in the statement below, $\Omega_{\underline{0}} = \left\{A \subset C_{\mathcal{B}}(X) : \underline{0} \in \overline{A}\right\}$ and $I_n := \left(-\frac{1}{n+1}, \frac{1}{n+1}\right) \subset \mathbb{R}$ for each $n \in \omega$.

Lemma 1.28. Let X be a Tychonoff space and let $\mathcal{B}$ be a bornology on X with a compact basis. For a family $\mathcal{U}$ of open sets of X let

$$\mathcal{A}(\mathcal{U}) := \{f \in C_{\mathcal{B}}(X) : \exists U \in \mathcal{U}(f \restriction (X \setminus U) \equiv 1)\}.$$

Also, for $n \in \omega$ and $A \subset C_{\mathcal{B}}(X)$, let $\mathcal{U}_n(A) := \left\{f^{-1}[I_n] : f \in A\right\}$.

1. If $X \notin \mathcal{U}$, then $\mathcal{U} \in \mathcal{O}_{\mathcal{B}}$ if and only if $\mathcal{A}(\mathcal{U}) \in \Omega_{\underline{0}}$.
2. The function $\underline{0}$ belongs to $\overline{A}$ if and only if $\mathcal{U}_n(A) \in \mathcal{O}_{\mathcal{B}}$ for all $n \in \omega$.
3. If $(A_n)_{n\in\omega}$ is a sequence of finite subsets of $C_{\mathcal{B}}(X)$ such that $\bigcup_{n\in\omega}\mathcal{U}_n(A_n) \in \mathcal{O}^*_{\mathcal{B}}$, then $\bigcup_{n\in\omega} A_n \in \Omega_{\underline{0}}$.
4. If $(A_n)_{n\in\omega}$ is a sequence of finite subsets of $C_{\mathcal{B}}(X)$ such that $\bigcup_{n\in\omega} A_n \in \Omega_{\underline{0}}$ and for all $n \in \omega$ and $g \in A_n$ there exists a $U_g \subsetneq X$ such that $g \restriction (X \setminus U_g) \equiv 1$, then $\bigcup_{n\in\omega}\{U_g : g \in A_n\} \in \mathcal{O}_{\mathcal{B}}$.

Proof.

1. Suppose that $\mathcal{U} \in \mathcal{O}^*_{\mathcal{B}}$ and let $\langle B, \varepsilon\rangle[\underline{0}]$ be a neighborhood of $\underline{0}$. We will obtain a function $g \in \mathcal{A}(\mathcal{U}) \cap \langle B, \varepsilon\rangle[\underline{0}]$: note that $\overline{B} \in \mathcal{B}$ is a compact subspace of X, hence there exists a proper[14] open subset $U \in \mathcal{U}$ such that $\overline{B} \subset U$; since X is a Tychonoff space, Theorem 1.25 gives the desired function. Conversely, for a $B \in \mathcal{B}$ we take an $f \in \mathcal{A}(\mathcal{U}) \cap \langle B, 1\rangle[\underline{0}]$, from which we obtain an open set $U \in \mathcal{U}$ such that $B \subset U$.
2. It follows because $f \in \langle B, \frac{1}{n+1}\rangle[\underline{0}]$ if and only if $B \subset f^{-1}[I_n]$, for every function $f \in C_{\mathcal{B}}(X)$, $B \in \mathcal{B}$ and $n \in \omega$.
3. Because of item 2 from Lemma 1.26, for all $m \in \omega$ one has $\bigcup_{n\geq m}\mathcal{U}_n(A_n) \in \mathcal{O}^*_{\mathcal{B}}$. So, if $\langle B, \varepsilon\rangle[\underline{0}]$ is a neighborhood of $\underline{0}$, then for an $m \in \omega$ such that $\frac{1}{m+1} < \varepsilon$ there are an $n \geq m$ and an $f \in A_n$ such that $B \subset f^{-1}[I_n]$. Therefore $\bigcup_{n\in\omega} A_n \in \Omega_{\underline{0}}$.
4. For a $B \in \mathcal{B}$, we take a $g \in \bigcup_{n\in\omega} A_n \cap \langle B, 1\rangle[\underline{0}]$, from which it follows that $B \subset U_g$, because $g \restriction (X \setminus U_g) \equiv 1$. $\square$

In order to prove Theorem 1.24, one just needs to use the two first items in the above lemma. However, its full strength is required to settle the next proposition, which is a generalization of one of the main results presented in [9].

[14] Since we are supposing the bornology $\mathcal{B}$ with a compact basis, we may only have $X \in \mathcal{B}$ when the space X itself is compact. Everything becomes trivial in this case.

Proposition 1.29. Let X be a Tychonoff space and let $\mathcal{B}$ be a bornology on X with a compact basis. Let $\psi\colon \omega \to [2, \aleph_0]$ be a function.

1. $\mathsf{S}_\psi(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$ holds in X if and only if $\mathsf{S}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ holds in $C_\mathcal{B}(X)$.

2. If ψ is increasing, then the games $\mathsf{G}_\psi(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$ and $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ are equivalent.

Proof. It is essentially an adaptation of the arguments used in [9]. So we postpone its presentation to the end of this section. □

Corollary 1.30. Let X be a Tychonoff space and let $\psi\colon \omega \to [2, \aleph_0]$ be a constant function.

1. X has property $\mathsf{S}_\psi(\Omega, \Omega)$ if and only if $C_p(X)$ has property $\mathsf{S}_\psi(\Omega_{\underline{0}}, \Omega_0)$.

2. X has property $\mathsf{S}_\psi(\mathcal{K}, \mathcal{K})$ if and only if $C_k(X)$ has property $\mathsf{S}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$.

3. The games $\mathsf{G}_\psi(\Omega, \Omega)$ on X and $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ on $C_p(X)$ are equivalent.

4. The games $\mathsf{G}_\psi(\mathcal{K}, \mathcal{K})$ on X and $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ on $C_k(X)$ are equivalent.

If we take functions of the form $f\colon \omega \to [2, \aleph_0)$, then we may drop the monotonicity condition in Proposition 1.29. Indeed, Theorem 1.7 implies that the game $\mathsf{G}_f(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ on $C_\mathcal{B}(X)$ is always equivalent to the game $\mathsf{G}_g(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ for some increasing function $g\colon \omega \to [2, \aleph_0)$. So, we only need a counterpart of this result for the corresponding game $\mathsf{G}_f(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$ on X.

Proposition 1.31. Let X be a topological space with a bornology $\mathcal{B}$ and let $f\colon \omega \to [2, \aleph_0)$ be a function.

1. If $f\colon \omega \to [2, \aleph_0)$ is bounded, then the games $\mathsf{G}_f(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$ and $\mathsf{G}_{k-1}(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$ are equivalent, where $k = \limsup_{n\in\omega} f(n)$.

2. If $f\colon \omega \to [2, \aleph_0)$ is unbounded and $g\colon \omega \to [2, \aleph_0)$ is another unbounded function, then the games $\mathsf{G}_f(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$ and $\mathsf{G}_g(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$ are equivalent.

Proof. We postpone this proof to the end of this section. □

Corollary 1.32. Let X be a Tychonoff space and suppose that the bornology $\mathcal{B}$ has a compact basis. Let $f\colon \omega \to [2, \aleph_0)$ be a function. Then the games $\mathsf{G}_f(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$ on X and $\mathsf{G}_f(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ on $C_\mathcal{B}(X)$ are equivalent.

Remark 1.33. We emphasize again that the above results remain true if one works in a normal space X with a bornology $\mathcal{B}$ having a closed basis. ■

Of course, the same equivalences in Theorem 1.7 concerning the selection principle $\mathsf{S}_f(\Omega_y,\Omega_y)$ hold for $\mathsf{S}_f(\mathcal{O}_{\mathcal{B}},\mathcal{O}_{\mathcal{B}})$ when one takes the bornology $\mathcal{B}$ with a compact basis: it follows immediately from Proposition 1.29 and Theorem 1.7. However, as we shall see soon, these principles behave like $\mathsf{S}_f(\mathcal{O},\mathcal{O})$ in Corollary 1.10 – and without any requirements over the family $\mathcal{B}$. This will follow from a more general approach that we begin to discuss now.

Comparing Theorems 1.3 and 1.4 with the results obtained so far, it is natural to ask[15] the following questions.

Question 1.34. Is it true that $\mathsf{S}_1(\mathcal{O}_{\mathcal{B}},\mathcal{O}_{\mathcal{B}})$ holds if and only if $\mathrm{I} \not\uparrow \mathsf{G}_1(\mathcal{O}_{\mathcal{B}},\mathcal{O}_{\mathcal{B}})$ holds?

Question 1.35. Is it true that $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{\mathcal{B}},\mathcal{O}_{\mathcal{B}})$ holds if and only if $\mathrm{I} \not\uparrow \mathsf{G}_{\mathrm{fin}}(\mathcal{O}_{\mathcal{B}},\mathcal{O}_{\mathcal{B}})$ holds?

A reasonable way to begin this investigation would be to analyze Scheepers' original proofs in the context of ω-coverings [75]. In a nutshell, in order to prove

$$\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega) \Rightarrow \mathrm{I} \not\uparrow \mathsf{G}_{\mathrm{fin}}(\Omega,\Omega) \tag{1.17}$$

Scheepers follows four natural steps:

(S-i) obtain a hyperspace $Y = Y(X)$ such that property $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ of X translates itself as $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ of Y;

(S-ii) fix a strategy σ for Player I in the game $\mathsf{G}_{\mathrm{fin}}(\Omega,\Omega)$ and define a related strategy σ' for Player I in the game $\mathsf{G}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$;

(S-iii) with Theorem I.5, take a play P' in the game $\mathsf{G}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ such that Player I uses the strategy σ' and loses the play;

(S-iv) obtain a play P in the game $\mathsf{G}_{\mathrm{fin}}(\Omega,\Omega)$ from the play P', such that Player I uses the strategy σ and loses the play.

A similar sketch is used for proving $\mathsf{S}_1(\Omega,\Omega) \Rightarrow \mathrm{I} \not\uparrow \mathsf{G}_1(\Omega,\Omega)$, except that in this case Theorem I.5 is replaced by Theorem 1.9.

The problem in following this sketch when trying to generalize it for $\mathcal{B}$-coverings consists in finding an appropriate hyperspace $Y(X)$. Scheepers originally uses $Y(X) = \bigoplus_{n\in\omega} X^n$, the topological sum of the family $\{X^n : n \in \omega\}$, but we were not able to relate this construction to the bornology $[X]^{<\aleph_0}$ in a reasonable way: this would be the natural first step to generalize it to arbitrary $\mathcal{B}$-coverings.

However, since any space $Y(X)$ satisfying the conditions in the *four steps* above would fit the proof, we tried to topologize the bornology $\mathcal{B}$ itself as a *naive* try – and it worked.

15 These questions were addressed to us by Angelo Bella and by the referee of our paper [9]. Unfortunately we were not able to provide a proof at the time.

More generally, let X be a topological space and let $\mathcal{B}$ be a family of nonempty subsets of X, that we fix for the rest of this section. We consider the sets of the form

$$\langle U \rangle := \{B \in \mathcal{B} : B \subset U\}, \tag{1.18}$$

in which U ranges over the open subsets of X. Since the equality $\langle U \rangle \cap \langle V \rangle = \langle U \cap V \rangle$ holds for every pair $U, V \subset X$ of open sets, it follows that the family

$$\mathcal{T}_{\mathcal{B}} := \{\langle U \rangle : U \subset X \text{ is open}\} \tag{1.19}$$

is a basis for a topology on $\mathcal{B}$. This type of hyperspace has been studied already in the literature[16]. In [57], Ernest Michael considers over $\mathcal{A}(X) := \{A \subset X : A \neq \emptyset\}$ the topology generated by sets of the form

$$U^+ := \{A \in \mathcal{A}(X) : A \subset U\}, \tag{1.20}$$

with U ranging over the open sets of X, and he calls it the **upper semi-finite topology** on $\mathcal{A}(X)$; by restricting this construction to the the family of all nonempty closed subsets of X, one obtains the so called *upper Vietoris* topology [35]. Since the topology on $\mathcal{B}$ generated by $\mathcal{T}_{\mathcal{B}}$ is the topology of $\mathcal{B}$ as a subspace of $\mathcal{A}(X)$, we shall write $\mathcal{B}^+$ to denote the family $\mathcal{B}$ endowed with the topology induced by $\mathcal{T}_{\mathcal{B}}$.

The main problem with the hyperspace $\mathcal{B}^+$ is its poor separation properties: one readily sees that if there are $A, B \in \mathcal{B}$ such that $A \subset B$, then they cannot be separated as points of $\mathcal{B}^+$, showing that $\mathcal{B}^+$ is not T_1. However, in our context this lack of separation properties will be harmless.

To fix notations, for a family $\mathcal{U}$ of open sets of X, let

$$\langle \mathcal{U} \rangle := \{\langle U \rangle : U \in \mathcal{U}\} \tag{1.21}$$

be the corresponding family of basic open sets of $\mathcal{B}^+$. Also, for a family $\mathcal{W}$ of basic open sets of $\mathcal{B}^+$, for each $W \in \mathcal{W}$ we choose an open set $U_W \subset X$ such that $W = \langle U_W \rangle$, allowing us to define

$$\widetilde{\mathcal{W}} := \{U_W : W \in \mathcal{W}\}. \tag{1.22}$$

The next innocuous lemma allows us to translate properties of X concerning $\mathcal{B}$-coverings as properties of (general) open coverings of $\mathcal{B}^+$, and vice versa.

Lemma 1.36. With the notations of (1.21) and (1.22):

1. if $\mathcal{U} \in \mathcal{O}_{\mathcal{B}}$, then $\langle \mathcal{U} \rangle \in \mathcal{O}(\mathcal{B}^+)$;
2. if $\mathcal{W} \in \mathcal{O}(\mathcal{B}^+)$ is an open covering by basic open sets, then $\widetilde{\mathcal{W}} \in \mathcal{O}_{\mathcal{B}}$.

[16] I would like to thank Professor Valentin Gutev for pointing this out after my talk at the Conference Frontiers of Selection Principles.

Proof. Trivial. □

Proposition 1.37. Let $\psi\colon \omega \to [2, \aleph_0]$ be a function.

1. The space X has property $\mathsf{S}_\psi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$ if and only if $\mathcal{B}^+$ has property $\mathsf{S}_\psi(\mathcal{O}(\mathcal{B}^+), \mathcal{O}(\mathcal{B}^+))$.
2. The games $\mathsf{G}_\psi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$ and $\mathsf{G}_\psi(\mathcal{O}(\mathcal{B}^+), \mathcal{O}(\mathcal{B}^+))$ are equivalent.

Proof. Note that the implications

- $\mathsf{S}_\psi(\mathcal{O}(\mathcal{B}^+), \mathcal{O}(\mathcal{B}^+)) \Rightarrow \mathsf{S}_\psi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$,
- $\mathrm{I} \uparrow \mathsf{G}_\psi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}}) \Rightarrow \mathrm{I} \uparrow \mathsf{G}_\psi(\mathcal{O}(\mathcal{B}^+), \mathcal{O}(\mathcal{B}^+))$ and
- $\mathrm{II} \uparrow \mathsf{G}_\psi(\mathcal{O}(\mathcal{B}^+), \mathcal{O}(\mathcal{B}^+)) \Rightarrow \mathrm{II} \uparrow \mathsf{G}_\psi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$

follow directly from item 1 in Lemma 1.36. Their converses follow from item 2 in Lemma 1.36, because of the lemma below. □

Lemma 1.38. Let Y be a topological space and take $\mathcal{G}$ a basis for the topology of Y. Consider the family $\mathcal{O}|_{\mathcal{G}} := \{\mathcal{U} \in \mathcal{O} : \mathcal{U} \subset \mathcal{G}\}$ and let $\psi\colon \omega \to [2, \aleph_0]$ be a function. Then:

1. the selection principles $\mathsf{S}_\psi(\mathcal{O}, \mathcal{O})$ and $\mathsf{S}_\psi(\mathcal{O}|_{\mathcal{G}}, \mathcal{O}|_{\mathcal{G}})$ are equivalent.
2. the games $\mathsf{G}_\psi(\mathcal{O}, \mathcal{O})$ and $\mathsf{G}_\psi(\mathcal{O}|_{\mathcal{G}}, \mathcal{O}|_{\mathcal{G}})$ are equivalent.

Proof. It follows by refining open coverings with basic open sets of $\mathcal{G}$. □

Finally, the next corollary gives affirmative answers for both Questions 1.34 and 1.35.

Corollary 1.39. Let $\alpha \in \{2, \aleph_0\}$ and let $\psi \equiv \alpha$. If $\mathsf{S}_\psi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$ holds, then $\mathrm{I} \not\uparrow \mathsf{G}_\psi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$ holds.

Proof. Just take $Y(X) = \mathcal{B}^+$ in the four steps ((S-i)–(S-iv)) of Scheepers. □

Corollary 1.40. Let $f\colon \omega \to [2, \aleph_0)$ be a function. The following conditions are equivalent:

1. $\mathrm{I} \not\uparrow \mathsf{G}_1(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$;
2. $\mathrm{I} \not\uparrow \mathsf{G}_f(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$;
3. $\mathsf{S}_f(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$;
4. $\mathsf{S}_1(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$.

Proof. Follows from Corollary 1.10 and Proposition 1.37. □

Corollary 1.41. Suppose that X is a Tychonoff space and assume that $\mathcal{B}$ is a bornology on X with a compact basis. Let $f\colon \omega \to [2, \aleph_0)$ be a function. The following conditions are equivalent:

1. $\mathsf{S}_1(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ on X;
2. $\mathsf{S}_f(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ on X;
3. $\mathrm{I} \not\uparrow \mathsf{G}_1(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ on X;
4. $\mathrm{I} \not\uparrow \mathsf{G}_f(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ on X;
5. $\mathsf{S}_1(\Omega_{\underline{0}},\Omega_{\underline{0}})$ on $C_\mathcal{B}(X)$;
6. $\mathsf{S}_f(\Omega_{\underline{0}},\Omega_{\underline{0}})$ on $C_\mathcal{B}(X)$;
7. $\mathrm{I} \not\uparrow \mathsf{G}_1(\Omega_{\underline{0}},\Omega_{\underline{0}})$ on $C_\mathcal{B}(X)$;
8. $\mathrm{I} \not\uparrow \mathsf{G}_f(\Omega_{\underline{0}},\Omega_{\underline{0}})$ on $C_\mathcal{B}(X)$.

Proof. Follows from Propositions 1.6, 1.29 and Corollary 1.40. □

García-Ferreira and Tamariz-Mascarúa [27] proved equivalence 6 ⇔ 5 above for the case $\mathcal{B} = [X]^{<\aleph_0}$. Since $C_\mathcal{B}(X)$ is a topological group, the last corollary suggests the next question.

Question 1.42. Let G be a topological group. Is it true that $\mathsf{S}_1(\Omega_g,\Omega_g) \Rightarrow \mathrm{I} \not\uparrow \mathsf{G}_1(\Omega_g,\Omega_g)$ for every $g \in G$? What if G is just a homogeneous space?

Also, as a consequence of the results we proved so far, note that for every pair of functions $f,g\colon \omega \to [2,\aleph_0)$, the only possible differences between the games $\mathsf{G}_f(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ and $\mathsf{G}_g(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ concern the situation of Player II.

Question 1.43. Are there a topological space X and a family $\mathcal{B}$ of subsets of X (maybe a bornology) such that the game $\mathsf{G}_1(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ is undetermined but $\mathrm{II} \uparrow \mathsf{G}_2(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$?

Remark 1.44 (A "Newton-Leibniz" situation)**.** Following the announcement of [10], Boaz Tsaban brought to our attention that a result similar to Corollary 1.39 also appears in the (thus far, unpublished)[17] MSc thesis of his student Nadav Samet [72].

Instead of considering a topology over a family of subsets of X, they take a family $\mathbb{P}$ of filters of open sets and observe that sets of the form $O_U := \{p \in \mathbb{P} : U \in p\}$ define a basis for a topology over $\mathbb{P}$ when U ranges over the open sets of X.

A possible conclusion after this surprising coincidence is that this (upper semi-finite) approach is the natural way to deal with these problems. ■

We shall finish this section with a few more comments concerning the hyperspace $\mathcal{B}^+$. First of all, we justify our interest in T_0-spaces back in Questions 1.19 and 1.20.

Proposition 1.45. If X is a regular space and $\mathcal{B}$ is a family of closed subsets of X, then $\mathcal{B}^+$ is a T_0-space.

Proof. Let $A \neq B$ be two distinct *points* of $\mathcal{B}^+$. If there is an $x \in A \setminus B$, then there are open sets $U, V \subset X$ such that $x \in U$, $B \subset V$ and $U \cap V = \emptyset$. Hence $B \in \langle V \rangle$ and $A \notin \langle V \rangle$. □

[17] Arguably, a more adequate title for this remark could be *a "Gaussian" situation*.

Now, let us reverse the general setting of dualities studied along this section. So far, we have been concerned with relations like

$$\text{“global” property of } X \text{ vs. “local” property of } C_{\mathcal{B}}(X).$$

Could there be something in the opposite direction, i.e., “global” property of $C_{\mathcal{B}}(X)$ vs. “local” property of X? As a motivation, consider the following theorem.

Theorem 1.46 (Asanov[18], 1979)**.** For a Tychonoff space X, $\sup_{n\in\omega} \mathrm{t}(X^n) \leq \mathrm{L}(C_p(X))$.

Now, recall that our definition of $\mathcal{B}^+$ was triggered by the topological sum $Y := \bigoplus_{n\in\omega} X^n$. Since in this situation one has $\mathrm{t}(Y) \leq \sup_{n\in\omega} \mathrm{t}(X^n)$, it follows from Theorem 1.46 that

$$\mathrm{t}(Y) \leq \mathrm{L}(C_p(X)).$$

It is then reasonable to wonder if a similar inequality holds when one replaces the pair $(Y, C_p(X))$ with $(\mathcal{B}^+, C_{\mathcal{B}}(X))$. The next proposition tell us what happens in this case[19].

Proposition 1.47. Let X be a Tychonoff space and let $\mathcal{B}$ be a family of compact subsets of X. Then $\mathrm{t}(\mathcal{B}^+) \leq \mathrm{L}(C_{\mathcal{B}}(X))$.

Proof. Let $B \in \mathcal{B}^+$ and let $\mathcal{C} \subset \mathcal{B}^+$ be a subset such that $B \in \overline{\mathcal{C}}$. We shall exhibit an open covering $\{U_C\}_{C\in\mathcal{C}}$ for a fixed closed subset $A \subset C_{\mathcal{B}}(X)$, and then we will show that every subcovering $\mathcal{V}$ from $\{U_C\}_{C\in\mathcal{C}}$ induces a subset $\mathcal{D} \subset \mathcal{C}$ such that $B \in \overline{\mathcal{D}}$ and $|\mathcal{D}| \leq |\mathcal{V}|$. This establishes the desired inequality, since $\mathrm{L}(A) \leq \mathrm{L}(C_{\mathcal{B}}(X))$ holds whenever A is closed.

We claim that $A := \{g \in C_{\mathcal{B}}(X) : g \upharpoonright B \equiv 1\} \subset C_{\mathcal{B}}(X)$ is closed. Indeed, for if $f \in \overline{A}$, then for each $\varepsilon > 0$ the set $\langle B, \varepsilon\rangle[f]$ is an open set containing f, so there exists a $g_\varepsilon \in A \cap \langle B, \varepsilon\rangle[f]$, hence $|g_\varepsilon - f| < \varepsilon$ holds in B. Thus $|1 - f(x)| < \varepsilon$ for all $x \in B$. Therefore $f \in A$.

Note that $U_C := \{f \in C_{\mathcal{B}}(X) : \forall x \in C(|f(x)| > 0)\} \subset C_{\mathcal{B}}(X)$ is open, for each $C \in \mathcal{B}$. Indeed, for each $f \in U_C$, let $c_f := \min\{|f(x)| : x \in C\} > 0$ and note that for every $\varepsilon > 0$ such that $\varepsilon < c_f$ we have $f \in \langle C, \varepsilon\rangle[f] \subset U_C$: if $g \in \langle C, \varepsilon\rangle[f]$, then $|f(x) - g(x)| < \varepsilon$ for all $x \in C$, hence

$$|f(x)| \leq |f(x) - g(x)| + |g(x)| \Rightarrow |g(x)| \geq |f(x)| - |f(x) - g(x)| \geq c_f - \varepsilon > 0,$$

showing that $g \in U_C$.

We also have $A \subset \bigcup_{C\in\mathcal{C}} U_C$. Indeed, for $f \in A$ the set $U := f^{-1}[(0, +\infty)] \subset X$ is an open set containing B. Hence $B \in \langle U\rangle$, from which it follows that $\langle U\rangle \cap \mathcal{C} \neq \emptyset$, i.e., there exists a $C \in \mathcal{C} \cap \langle U\rangle$. Thus $C \subset U$, which shows that $f(x) > 0$ for all $x \in C$, i.e., $f \in U_C$.

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[18] According to Tkachuk [94].

[19] Our arguments are adapted from the proof for Theorem 1.46 presented by Tkachuk [94, S.189].

Finally, let $\mathcal{D} := \{C_\alpha : \alpha < \kappa\} \subset \mathcal{C}$ be a family such that $A \subset \bigcup_{\alpha<\kappa} U_{C_\alpha}$. Since B is compact, for an open set $U \subset X$ with $B \in \langle U \rangle$, Theorem 1.25 gives a continuous function $g\colon X \to \mathbb{R}$ such that $g \restriction B \equiv 1$ and $g \restriction (X \setminus U) \equiv 0$. So, $g \in A$ and there exists an $\alpha < \kappa$ such that $g \in U_{C_\alpha}$, which shows that $g(x) > 0$ for all $x \in C_\alpha$. Consequently, $C_\alpha \in \langle U \rangle$, showing that $\{C_\alpha : \alpha < \kappa\} \cap \langle U \rangle \neq \emptyset$. $\square$

Recall that by $\mathsf{K}(X)$ we mean the family of compact subsets of the space X.

Corollary 1.48. Let X be a Tychonoff space.

1. $\mathrm{t}\left(([X]^{<\aleph_0})^+\right) \leq \mathrm{L}\left(C_p(X)\right)$.
2. $\mathrm{t}\left((\mathsf{K}(X))^+\right) \leq \mathrm{L}\left(C_k(X)\right)$.

In [93, 94], Tkachuk mentions that a reasonable characterization for $\mathrm{L}\left(C_p(X)\right)$ in terms of X is still unknown, suggesting the following question.

Question 1.49. Does the reverse inequality in Proposition 1.47 hold?

Postponed proofs

Now we present the postponed proofs of Propositions 1.29 and 1.31. In the following, X always denotes a Tychonoff space endowed with a bornology $\mathcal{B}$. Most of the work on Proposition 1.29 follows from the next lemma.

Lemma 1.50. Suppose that $\mathcal{B}$ has a compact basis and let $\psi\colon \omega \to [2, \aleph_0]$ be a function.

- Let $\mathsf{S}^*_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ be the assertion: for each sequence $(A_n)_{n\in\omega} \in \Omega_{\underline{0}}^\omega$ such that $X \notin \mathcal{U}_n(A_n)$ for all n, there exists a sequence $(C_n)_{n\in\omega} \in \prod_{n\in\omega}[A_n]^{<\psi(n)}$ such that $\bigcup_{n\in\omega} C_n \in \Omega_{\underline{0}}$.
- Let $\mathsf{G}^*_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ be the game played as $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$, with the additional rule that for each $n \in \omega$, at the n-inning, Player I chooses $A_n \in \Omega_{\underline{0}}$ such that $X \notin \mathcal{U}_n(A_n)$.
- For a strategy σ for Player I in the game $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$, call a sequence $P = ((A_n, B_n) : n < m)$ of length $m \in \omega$ a *game start* in σ if $|B_n| < \psi(n)$ and $B_n \subset A_n = \sigma((B_j : j < n))$ for all $n < m$.

Then:

1. the selection principles $\mathsf{S}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ and $\mathsf{S}^*_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ are equivalent;
2. $\mathrm{II} \uparrow \mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ holds if and only if $\mathrm{II} \uparrow \mathsf{G}^*_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ holds;
3. if σ is a winning strategy for Player I in the game $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ and P is a game start in σ of length m, then there are a $p \geq m$ and a game start P' in σ with length $p+1$ extending P such that $X \notin \mathcal{U}_p(A_p)$.

Proof. Suppose that $\mathsf{S}^*_\psi(\Omega_{\underline{0}},\Omega_{\underline{0}})$ holds and let $(A_n)_{n\in\omega}\in\Omega_{\underline{0}}^\omega$ be an arbitrary sequence. If the family $T:=\{n\in\omega : X\in\mathcal{U}_n(A_n)\}$ is infinite, then we may choose witnesses $f_n\in A_n$ with $f_n^{-1}[I_n]=X$ for each $n\in T$, in such a way that $\{f_n : n\in T\}\in\Omega_{\underline{0}}$. On the other hand, if there exists an $m\in\omega$ such that $X\notin\mathcal{U}_n(A_n)$ occurs for all $n\geq m$, then for each $n<m$ we replace the term A_n with $A'_n:=\mathcal{A}(\widetilde{\mathcal{U}})$, where

$$\widetilde{\mathcal{U}}:=\{X\setminus\{x\}:x\in X\}\in\mathcal{O}^*_{\mathcal{B}}$$

and $\mathcal{A}(\widetilde{\mathcal{U}})\in\Omega_{\underline{0}}$ is taken as in Lemma 1.28. Since $X\notin\mathcal{U}_n(\mathcal{A}(\widetilde{\mathcal{U}}))$ for all $n\in\omega$, the hypothesis $\mathsf{S}^*_\psi(\Omega_{\underline{0}},\Omega_{\underline{0}})$ works for the sequence $(A'_0,\dots,A'_{m-1})^\frown(A_n)_{n\geq m}$. Now the result follows because $C_{\mathcal{B}}(X)$ is a T_1-space (see footnote 13, page 42). The converse is obviously true.

For the game part concerning Player II, note that

$$\mathrm{II}\uparrow\mathsf{G}_\psi(\Omega_{\underline{0}},\Omega_{\underline{0}})\Rightarrow\mathrm{II}\uparrow\mathsf{G}^*_\psi(\Omega_{\underline{0}},\Omega_{\underline{0}})\tag{1.23}$$

is trivial. Thus we only need to care about the converse of (1.23). If σ is a winning strategy for Player II in the game $\mathsf{G}^*_\psi(\Omega_{\underline{0}},\Omega_{\underline{0}})$, one may define a winning strategy μ for Player II in the game $\mathsf{G}_\psi(\Omega_{\underline{0}},\Omega_{\underline{0}})$ by simply replacing the sets A_n such that $X\in\mathcal{U}_n(A_n)$ with $\mathcal{A}(\widetilde{\mathcal{U}})$ and taking the corresponding witnesses, as we did in the last paragraph.

For the last part, suppose the contrary. If every game start in σ extending

$$P=((A_0,B_0),\dots,(A_m,B_m))$$

does not satisfy the desired condition, then there exists a function $f_{m+1}\in A_{m+1}=\sigma((B_0,\dots,B_m))$ such that $f_{m+1}^{-1}[I_{m+1}]=X$, hence there exists an $f_{m+2}\in A_{m+2}=\sigma((B_0,\dots,B_m,\{f_{m+1}\}))$ such that $f_{m+2}^{-1}[I_{m+2}]=X$, and so on. It is easy to see that $\{f_n:n\geq m\}\in\Omega_{\underline{0}}$, which shows that σ is not a winning strategy. □

Proposition 1.29. Suppose that $\mathcal{B}$ has a compact basis and let $\psi\colon\omega\to[2,\aleph_0]$ be a function.

1. $\mathsf{S}_\psi(\mathcal{O}_{\mathcal{B}},\mathcal{O}_{\mathcal{B}})$ holds in X if and only if $\mathsf{S}_\psi(\Omega_{\underline{0}},\Omega_{\underline{0}})$ holds in $C_{\mathcal{B}}(X)$.
2. If ψ is increasing, then the games $\mathsf{G}_\psi(\mathcal{O}_{\mathcal{B}},\mathcal{O}_{\mathcal{B}})$ and $\mathsf{G}_\psi(\Omega_{\underline{0}},\Omega_{\underline{0}})$ are equivalent.

Proof. The idea is to use items 1 and 2 from Lemma 1.28 to "transfer" informations from $\mathcal{O}_{\mathcal{B}}$ to $\Omega_{\underline{0}}$, and vice versa. On the other hand, items 3 and 4 in the same lemma are used to validate this "transfer process". In this validation step, there are situations in which we need to avoid trivial $\mathcal{B}$-coverings, and we use Lemma 1.50 in order to do this.

The proofs of $\mathsf{S}_\psi(\mathcal{O}_{\mathcal{B}},\mathcal{O}_{\mathcal{B}})\Leftrightarrow\mathsf{S}_\psi(\Omega_{\underline{0}},\Omega_{\underline{0}})$ and $\mathrm{II}\uparrow\mathsf{G}_\psi(\mathcal{O}_{\mathcal{B}},\mathcal{O}_{\mathcal{B}})\Leftrightarrow\mathrm{II}\uparrow\mathsf{G}_\psi(\Omega_{\underline{0}},\Omega_{\underline{0}})$ are similar and straightforward, so we leave them to the reader. We only present the proof for the equivalence $\mathrm{I}\uparrow\mathsf{G}_\psi(\mathcal{O}_{\mathcal{B}},\mathcal{O}_{\mathcal{B}})\Leftrightarrow\mathrm{I}\uparrow\mathsf{G}_\psi(\Omega_{\underline{0}},\Omega_{\underline{0}})$, which falls naturally in two parts.

1) $\mathrm{I} \uparrow \mathsf{G}_\psi(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B}) \Rightarrow \mathrm{I} \uparrow \mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$.

Given a winning strategy ρ for Player I in the game $\mathsf{G}_\psi(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$, we will obtain a winning strategy η for Player I in the game $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$. First of all, note that since ρ is a winning strategy, its choices are non-trivial $\mathcal{B}$-coverings for X. This allows us to use item 1 from Lemma 1.28. For brevity, for each $\mathcal{B}$-covering $\mathcal{U} \in \mathcal{O}^*_\mathcal{B}$ and each $g \in \mathcal{A}(\mathcal{U})$ we choose a $U_g \in \mathcal{U}$ such that $g \restriction (X \setminus U_g) \equiv 1$. Now, for $G \subset \mathcal{A}(\mathcal{U})$ let

$$\mathcal{U}[G] := \{U_g : g \in G\}.$$

We define η in the following way:

1. $\eta(\emptyset) := \mathcal{A}(\mathcal{U}_0)$, in which $\mathcal{U}_0 := \rho(\emptyset)$;
2. if Player II answers with $G_0 \in [\eta(\emptyset)]^{<\psi(0)}$, then $\eta((G_0)) := \mathcal{A}(\rho((\mathcal{U}_0[G_0])))$;
3. for $n > 1$, if Player II chooses $G_n \in [\eta((G_0, \dots, G_{n-1}))]^{<\psi(n)}$, then
$$\eta((G_0, \dots, G_n)) := \mathcal{A}(\rho((\mathcal{U}_0[G_0], \dots, \mathcal{U}_n[G_n]))),$$
where $\mathcal{U}_0 := \rho(\emptyset), \mathcal{U}_1 := \rho((\mathcal{U}_0[G_0])), \dots, \mathcal{U}_n := \rho((\mathcal{U}_0[G_0], \dots, \mathcal{U}_{n-1}[G_{n-1}]))$.

If $((A_n, G_n))_{n\in\omega}$ is a play in the game $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ in which Player I follows the strategy η, then the corresponding sequence $((\mathcal{U}_n, \mathcal{U}_n[G_n]))_{n\in\omega}$, as in the description above, is a play in the game $\mathsf{G}_\psi(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$ in which Player I follows the winning strategy ρ. Thus $\bigcup_{n\in\omega} \mathcal{U}_n[G_n] \notin \mathcal{O}_\mathcal{B}$, from which the result follows from the contrapositive of item 4 from Lemma 1.28.

2) $\mathrm{I} \uparrow \mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}}) \Rightarrow \mathrm{I} \uparrow \mathsf{G}_\psi(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$. This one is tricky.

Let σ be a winning strategy for Player I in the game $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$. By applying item 3 from Lemma 1.50, we define a strategy ν for Player I in the game $\mathsf{G}_\psi(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$. First, for $P = \emptyset$, we obtain a game start P_0 in σ with lenght $p_0 + 1 \geq 1$, say

$$P_0 = ((A_{0,0}, B_{0,0}), \dots, (A_{p_0,0}, B_{p_0,0})),$$

such that $X \notin \mathcal{U}_{p_0}(A_{p_0,0})$. Let the first move of Player I be $\nu(\emptyset) := \mathcal{U}_{p_0}(A_{p_0,0})$.

If $W_0 \in [\nu(\emptyset)]^{<\psi(0)}$ is the answer of Player II, then there is no loss of generality in taking $B_{p_0,0} \in [A_{p_0,0}]^{<\psi(0)}$ such that $\mathcal{U}_{p_0}(B_{p_0,0}) = W_0$, since $|W_0| < \psi(0) \leq \psi(p_0)$ – because ψ is increasing. Applying item 3 from Lemma 1.50 again, but this time to the game start P_0, yields a game start P_1 in σ, say

$$P_1 = ((A_{0,1}, B_{0,1}), \dots, (A_{p_1,1}, B_{p_1,1})),$$

extending P_0 and such that $p_0 + 1 \leq p_1$ with $X \notin \mathcal{U}_{p_1}(A_{p_1,1})$. We define the answer of Player I to W_0 as $\nu((W_0)) := \mathcal{U}_{p_1}(A_{p_1,1})$.

Thus, item 3 from Lemma 1.50 allows us to proceed in this fashion for each $n \in \omega$, from which it follows that we obtain a strategy ν for Player I in the game $\mathsf{G}_\psi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$. Note that in this way, the strategy σ does not answer trivial $\mathcal{B}$-coverings. We will see that ν is a winning strategy for Player I in the game $\mathsf{G}_\psi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$.

Let $R = ((\mathcal{V}_n, W_n))_{n\in\omega}$ be a play in the game $\mathsf{G}_\psi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$ in which Player I follows the strategy ν. By the way we defined the strategy ν, there are a strictly increasing sequence $(p_n)_{n\in\omega}$ of natural numbers and a sequence $(P_n)_{n\in\omega}$ of game starts in σ, such that for all $m \in \omega$:

1. P_m has lenght $p_m + 1$, say $P_m = ((A_{0,m}, B_{0,m}), \dots, (A_{p_m,m}, B_{p_m,m}))$;
2. $P_{m+1} \restriction (p_m + 1) = P_m$;
3. $X \notin W_m = \mathcal{U}_{p_m}(B_{p_m,m})$;
4. $\mathcal{V}_m = \mathcal{U}_{p_m}(A_{p_m,m})$.

Since each P_m is a game start in σ, condition 2 implies that $P = \bigcup_{n\in\omega} P_n$ is a play in the game $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ in which Player I follows the winning strategy σ. So, if Player II wins the play R, then $\bigcup_{m\in\omega} B_{p_m,m} \in \Omega_{\underline{0}}$ (by item 3 from Lemma 1.28), but it implies that Player II also wins the play P in the game $\mathsf{G}_\psi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$, a contradiction. □

The second postponed proof concerns the monotonicity hypothesis in the previous proposition: if we take functions of the form $f\colon \omega \to [2, \aleph_0)$, then no requirements of monotonicity are needed at all.

Lemma 1.51. Let $f\colon \omega \to [2, \aleph_0)$ be a bounded function and let $k = \limsup_{n\in\omega} f(n)$.

1. The set $M = \{n \in \omega : f(n) > k\}$ is finite.
2. The set $N = \{n \in \omega : f(n) = k\}$ is infinite.

Proof. Since f is bounded, we have

$$k = \limsup_{n\in\omega} f(n) := \inf_{m\geq 0} \left(\sup_{n\geq m} f(n) \right) = \min \left\{ \sup_{n\geq m} f(n) : m \in \omega \right\} \in \omega.$$

Thus, there exists an $m_0 \in \omega$ such that $k = \sup_{n\geq m_0} f(n)$, so if $m > m_0$, then

$$k := \limsup_{n\in\omega} f(n) \leq \sup_{n\geq m} f(n) \leq \sup_{n\geq m_0} f(n) = k,$$

which shows that $f(m) \leq k$, thereafter $|M| \leq m_0 + 1 < \aleph_0$. If it were true that the set N is finite, there would be a maximal element for the set $M \cup N$, say m_1. By choosing $m' > m_1$, we obtain $\sup_{n\geq m'} f(n) < k$, contradicting the minimality of k. □

Proposition 1.31. Let $f\colon \omega \to [2,\aleph_0)$ be a function.

1. If $f\colon \omega \to [2,\aleph_0)$ is bounded, then the games $\mathsf{G}_f(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ and $\mathsf{G}_{k-1}(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ are equivalent, where $k = \limsup_{n\in\omega} f(n)$.

2. If $f\colon \omega \to [2,\aleph_0)$ is unbounded and $g\colon \omega \to [2,\aleph_0)$ is another unbounded function, then the games $\mathsf{G}_f(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ and $\mathsf{G}_g(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ are equivalent.

Proof. Since we have Corollary 1.40, we only need to worry about Player II.

1. If μ is a winning strategy for Player II in the game $\mathsf{G}_f(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$, then μ defines a winning strategy for Player II in the game $\mathsf{G}_{k-1}(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$. In fact, from item 1 in Lemma 1.51, there is an $m_0 \in \omega$ such that $f(n) \le k$ for all $n \ge m_0$. Hence, for $n \ge m_0$, the moves of Player II in the n-th inning following the strategy μ are legal choices in the game $\mathsf{G}_{k-1}(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$. Thus we may define a strategy μ' for Player II in the game $\mathsf{G}_{k-1}(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ in the following way: if $\mathcal{U}_0,\dots,\mathcal{U}_n \in \mathcal{O}_\mathcal{B}$ are $\mathcal{B}$-coverings for X, then

$$\mu'((\mathcal{U}_0,\dots,\mathcal{U}_n)) = \mu((\underbrace{\mathcal{U}_0,\dots,\mathcal{U}_0}_{m_0\text{ times}},\mathcal{U}_0,\mathcal{U}_1,\dots,\mathcal{U}_n)) \in [\mathcal{U}_n]^{<f(m_0+n)} \subset [\mathcal{U}_n]^{<k}.$$

Since there is no loss of generality in supposing that Player I does not play trivial $\mathcal{B}$-coverings, item 2 from Lemma 1.26 yields that μ' is a winning strategy for Player II in the game $\mathsf{G}_{k-1}(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$.

Conversely, let η be a winning strategy for Player II in the game $\mathsf{G}_{k-1}(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$. Since the set $N = \{n\in\omega : f(n) = k\}$ is infinite by the previous lemma, it is enough for Player II to use the strategy η in the game $\mathsf{G}_f(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ at those n-th innings such that $n \in N$ – ignoring[20] Player I in all the other innings.

2. By symmetry, it is enough to prove $\mathrm{II}\uparrow \mathsf{G}_f(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B}) \Rightarrow \mathrm{II}\uparrow \mathsf{G}_g(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$

Let μ be a winning strategy for Player II in the game $\mathsf{G}_f(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ and let $(n_i)_{i\in\omega}$ be an increasing sequence such that $g(n_i) \ge f(i)$ for all $i\in\omega$. We may define a winning strategy for Player II in the game $\mathsf{G}_g(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$ by simply ignoring the innings n such that $n \in \omega\setminus\{n_i : i\in\omega\}$ and using the strategy μ in the other cases. More precisely, if in the n-th inning the play was $\mathcal{U}_0 \supset V_0,\mathcal{U}_1 \supset V_1,\dots,\mathcal{U}_n$, then Player II chooses

- an arbitrary $V_n \subset \mathcal{U}_n$ such that $|V_n| < g(n)$ if $n \ne n_i$ for all i;
- $V_{n_i} = \mu((\mathcal{U}_{n_0},\dots,\mathcal{U}_{n_i}))$ if $n = n_i$, where $|V_{n_i}| < f(i) \le g(n_i)$.

It is clear that in this fashion we obtain a winning strategy for Player II in the game $\mathsf{G}_g(\mathcal{O}_\mathcal{B},\mathcal{O}_\mathcal{B})$. □

[20] Choosing an arbitrary but legal subset of the cover played by Player I. In chess language, we are putting Player I in a *zugzwang*-like situation [85], possibly infinite many times.

1.3 Applications of filters in $C_{\mathcal{B}}$-theory

Recall that a **filter** $\mathcal{F}$ on a set C is a family of subsets of C, upward closed and closed for taking finite intersections – it is called a proper filter if $\emptyset \notin \mathcal{F}$. Filters have arisen [21] as a way to generalize the notion of convergence of sequences to arbitrary topological spaces.

Indeed, for a topological space (Y,τ) and a point $y \in Y$, we say that a filter $\mathcal{F}$ on Y **converges to** y if $\mathcal{N}_{y,Y} \subset \mathcal{F}$, where

$$\mathcal{N}_{y,Y} := \{N \subset Y : \exists V \in \tau\,(y \in V \subset N)\} \tag{1.24}$$

is the (proper) **neighborhood filter of** $y \in Y$. It is easy to see that this definition generalizes convergence of sequences. Indeed, a sequence $(y_n)_{n\in\omega} \in Y^{\omega}$ converges to y if and only if for all $N \in \mathcal{N}_{y,Y}$ there exists an $m_N \in \omega$ such that $\{y_n : n \geq m_N\} \subset N$, which is equivalent to asking for the filter $\{\{y_n : n \geq m\} : m \in \omega\}^{\uparrow}$ to contain the filter $\mathcal{N}_{y,Y}$ – where $\mathcal{G}^{\uparrow}$ denotes the **upwards closure** of a family $\mathcal{G}$, i.e.,

$$\mathcal{G}^{\uparrow} := \{F \subset Y : \exists G \in \mathcal{G}\,(G \subset F)\}. \tag{1.25}$$

Actually, local properties far more delicate than convergence of sequences are captured by filters. We present two simple examples below, where $\mathcal{F}$ is a filter on a set C.

Example 1.52 (Character)**.** Call a subset $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{G}^{\uparrow} = \mathcal{F}$ a **basis** of $\mathcal{F}$. We say that a filter $\mathcal{F}$ is **countably based** if $\mathcal{F}$ has a countable basis. Then, note that a point $y \in Y$ has a countable local basis if and only if the filter $\mathcal{N}_{y,Y}$ is countably based. Naturally this can be generalized in order to define the *character* of a filter. ■

Example 1.53 (Tightness)**.** Define the **tightness** of $\mathcal{F}$ as the least cardinal $\kappa \geq \aleph_0$ with the following property: for all $A \subset C$ such that $A \cap F \neq \emptyset$ holds for every element of $\mathcal{F}$, there exists a $B \in [A]^{\leq\kappa}$ such that $B \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. If we call $\mathrm{t}(\mathcal{F})$ the tightness of $\mathcal{F}$, then it is easy to see that $\mathrm{t}(\mathcal{N}_{y,Y}) = \mathrm{t}(y,Y)$ for every point y in a topological space Y. ■

The above examples were adapted from the work of Jordan and Mynard [42], in which the authors describe many local properties via filters. Particularly, Jordan [39] uses this approach to obtain general dualities between local properties of $C_p(X)$ and properties concerning ω-coverings of X. In this section we extend his results for the context of $C_{\mathcal{B}}$-theory.

We begin with a few definitions and notations, adapted from [39, 42]. For a set C, denote by $\mathbb{F}(C)$ the family of proper filters on C. For a cardinal $\kappa \geq \aleph_0$, let $\mathbb{F}_{\kappa}(C)$ be the family of proper filters on C having a basis with cardinality $\leq \kappa$ – particularly, $\mathbb{F}_1(C)$ and $\mathbb{F}_{\aleph_0}(C)$ are the families of the **principal filters**[22] and countably based filters on C, respectively.

[21] Due to Henri Cartan, according to Bourbaki [16, p. 166].
[22] Do not confuse with principal ultrafilters. Our definition agrees with the *canon* [38].

Remark 1.54. For a family $\mathcal{G} \in \wp(\wp(C))$, the correspondence $\mathcal{G} \mapsto \mathcal{G}^{\uparrow}$ *does not* define a function from $\wp(\wp(C))$ to $\mathbb{F}(C)$. In fact, $\mathcal{G}^{\uparrow}$ is a proper filter on C if and only if $\emptyset \notin \mathcal{G}$ and for every pair $A, B \in \mathcal{G}$ there exists a $G \in \mathcal{G}$ such that $G \subset A \cap B$. ■

For a pair of sets C, D, a subset $F \subset C$ and a binary relation $R \subset C \times D$, it is usual to write $R[F] := \{d \in D : \exists c \in F\ ((c,d) \in R)\}$. For a family $\mathcal{F} \subset \wp(C)$, let

$$R(\mathcal{F}) := \{R[F] : F \in \mathcal{F}\}^{\uparrow}. \tag{1.26}$$

We say that $\mathbb{K}$ is a **class of filters** if $\mathbb{K}$ denotes a property about general filters, and we write $\mathcal{F} \in \mathbb{K}$ to indicate that $\mathcal{F}$ has property $\mathbb{K}$. We shall call a topological space Y a $\mathbb{K}$**-space** if $\mathcal{N}_{y,Y} \in \mathbb{K}$ for all $y \in Y$. For instance:

- the class $\mathbb{T}_\kappa$ of those filters $\mathcal{F}$ such that $\mathrm{t}(\mathcal{F}) \leq \kappa$;
- the class $\mathbb{F}_{\aleph_0}$ of all countably based filters.

Remark 1.55. Note that $\mathbb{T}_\kappa$-spaces are precisely those topological spaces Y such that $\mathrm{t}(Y) \leq \kappa$, while $\mathbb{F}_{\aleph_0}$-spaces are precisely the first countable spaces. ■

A class of filters $\mathbb{K}$ is said to be $\mathbb{F}_1$**-composable** if for each pair of sets C, D and each relation $R \subset C \times D$, the following holds:

$$\mathcal{F} \in \mathbb{F}(C) \cap \mathbb{K} \text{ and } R(\mathcal{F}) \in \mathbb{F}(D) \Rightarrow R(\mathcal{F}) \in \mathbb{K}. \tag{1.27}$$

Note that the condition "$R(\mathcal{F}) \in \mathbb{F}(D)$" above is necessary, because generally we cannot be sure that $R(\mathcal{F})$ is a proper filter on D. Indeed, if $R \subset C \times D$ is such that $R[F] = \emptyset$ for some $F \in \mathcal{F}$, then $R(\mathcal{F})$ is not proper. However, we do not need to worry about this when the relation R is a function.

Theorem 1.56 (Folklore)**.** Let C and D be sets and let $f\colon C \to D$ be a function.

1. If $\mathcal{F} \in \mathbb{F}(C)$, then $f(\mathcal{F}) \in \mathbb{F}(D)$.
2. If $\mathcal{H} \in \mathbb{F}(D)$, then $f^{-1}(\mathcal{H}) \in \mathbb{F}(C)$ if and only if $H \cap f[C] \neq \emptyset$ for all $H \in \mathcal{H}$.

Finally, a class of filters $\mathbb{K}$ is $\mathbb{F}_{\aleph_0}$**-steady** if for each set C and for all filters $\mathcal{F} \in \mathbb{F}(C) \cap \mathbb{K}$ and $\mathcal{H} \in \mathbb{F}_{\aleph_0}(C)$ such that $F \cap H \neq \emptyset$ for all $(F, H) \in \mathcal{F} \times \mathcal{H}$, the following holds:

$$\mathcal{F} \vee \mathcal{H} := (\mathcal{F} \wedge \mathcal{H})^{\uparrow} = \{F \cap H : (F,H) \in \mathcal{F} \times \mathcal{H}\}^{\uparrow} \in \mathbb{K}. \tag{1.28}$$

For the rest of this chapter, we fix a topological space (X, τ) endowed with a bornology $\mathcal{B}$. For each $B \in \mathcal{B}$, let $V(B) := \{U \in \tau : B \subset U\}$ and consider the following filter on τ:

$$\Gamma_{\mathcal{B}}(X) := \{V(B) : B \in \mathcal{B}\}^{\uparrow}, \tag{1.29}$$

which is a proper filter, since $V(B) \cap V(B') = V(B \cup B') \neq \emptyset$ for every $B, B' \in \mathcal{B}$. In particular, if $\mathcal{B}' \subset \mathcal{B}$ is a basis for the bornology $\mathcal{B}$, then $\Gamma_{\mathcal{B}}(X) = \{V(B) : B \in \mathcal{B}'\}^{\uparrow}$. By taking $\mathcal{B} = [X]^{<\aleph_0}$ in (1.29), one obtains the filter defined by Jordan [39], denoted simply as $\Gamma(X)$.

Proposition 1.57. Let (X, τ) be a Tychonoff space and suppose that the bornology $\mathcal{B}$ has a compact basis. Suppose that $\mathbb{K}$ is a $\mathbb{F}_1$-composable class of filters. If $C_{\mathcal{B}}(X)$ is a $\mathbb{K}$-space, then $\Gamma_{\mathcal{B}}(X) \in \mathbb{K}$.

Proof. We shall take a neighborhood $\mathcal{F}$ on $C_{\mathcal{B}}(X)$, a set Y together with functions $\pi\colon Y \to \tau$ and $\Phi\colon Y \to C_{\mathcal{B}}(X)$ such that $\Phi^{-1}(\mathcal{F})$ is a proper filter and $\Gamma_{\mathcal{B}}(X) = \pi\left(\Phi^{-1}(\mathcal{F})\right)$. Note that this proves the desired result:

- $\mathcal{F} \in \mathbb{K}$, because $C_{\mathcal{B}}(X)$ is a $\mathbb{K}$-space;
- since $\Phi^{-1}(\mathcal{F})$ is a proper filter and $\mathbb{K}$ is $\mathbb{F}_1$-composable, it follows that $\Phi^{-1}(\mathcal{F}) \in \mathbb{K}$;
- for the filter $\Phi^{-1}(\mathcal{F})$ and the function π, the above argument yields $\pi\left(\Phi^{-1}(\mathcal{F})\right) \in \mathbb{K}$.

Now we fix a compact basis $\mathcal{B}' \subset \mathcal{B}$ for the bornology $\mathcal{B}$.

- Let $\mathcal{F} = \mathcal{N}_{\underline{0}, C_{\mathcal{B}}(X)}$ and note that $\mathcal{F} = \mathfrak{B}^{\uparrow}$, in which $\mathfrak{B}$ is a local basis at $\underline{0} \in C_{\mathcal{B}}(X)$ defined as $\mathfrak{B} := \{\langle B, \varepsilon\rangle[\underline{0}] : B \in \mathcal{B}', \varepsilon \in (0,1)\}$.
- Let $Y := \{(B,U) \in \mathcal{B}' \times \tau : B \subset U\}$, and take $\pi\colon Y \to \tau$ the projection onto τ. Since X is a Tychonoff space, for each $(B,U) \in Y$, Theorem 1.25 gives a function $f_{(B,U)} \in C_{\mathcal{B}}(X)$ such that $f_{(B,U)} \restriction B \equiv 0$ and $f_{(B,U)} \restriction (X \setminus U) \equiv 1$. Define $\Phi\colon Y \to C_{\mathcal{B}}(X)$ by $\Phi((B,U)) := f_{(B,U)}$.
- Item 2 from Theorem 1.56 implies $\Phi^{-1}(\mathcal{F}) \in \mathbb{F}(Y)$.
- Since $f_{(B,U)} \in \langle B, \varepsilon\rangle[\underline{0}]$ if and only if $B \subset U$, for each $B \in \mathcal{B}'$ and $\varepsilon \in (0,1)$ we get the equality $V(B) = \pi\left[\Phi^{-1}\left[\langle B, \varepsilon\rangle[\underline{0}]\right]\right]$.
- By the theorem below, the item above shows that $\Gamma_{\mathcal{B}}(X)$ and $\pi\left(\Phi^{-1}(\mathcal{F})\right)$ have the same basis, from which the result follows. □

Theorem 1.58 (Folklore)**.** Let C and D be sets, let $f\colon C \to D$ be a function and consider the filters $\mathcal{F} \in \mathbb{F}(C)$ and $\mathcal{H} \in \mathbb{F}(D)$.

1. If $\mathfrak{B}$ is a basis of $\mathcal{F}$, then $\{f[B] : B \in \mathfrak{B}\}$ is a basis of $f(\mathcal{F})$.
2. If $f^{-1}(\mathcal{H}) \in \mathbb{F}(C)$ and $\mathfrak{G}$ is a basis of $\mathcal{H}$, then $\{f^{-1}[G] : G \in \mathfrak{G}\}$ is a basis of $f^{-1}(\mathcal{H})$.

Proof. It follows by the *monotonicity* of taking images and inverse images. □

Proposition 1.59. Let $\mathbb{K}$ be a class of filters that is $\mathbb{F}_1$-composable and $\mathbb{F}_{\aleph_0}$-steady. If $\Gamma_{\mathcal{B}}(X) \in \mathbb{K}$, then $C_{\mathcal{B}}(X)$ is a $\mathbb{K}$-space.

Proof. We shall obtain a set Y, a proper filter $\mathcal{H} \in \mathbb{F}_{\aleph_0}(Y)$, functions $\pi\colon Y \to C_{\mathcal{B}}(X)$ and $\Phi\colon Y \to \tau$ such that $\Phi^{-1}(\Gamma_{\mathcal{B}}(X))$ and $\Phi^{-1}(\Gamma_{\mathcal{B}}(X)) \vee \mathcal{H}$ are proper filters on Y, satisfying

$$\pi\left(\Phi^{-1}(\Gamma_{\mathcal{B}}(X)) \vee \mathcal{H}\right) = \mathcal{N}_{\underline{0},C_{\mathcal{B}}(X)}.$$

The hypotheses about the class of filters $\mathbb{K}$ imply $\mathcal{N}_{\underline{0},C_{\mathcal{B}}(X)} \in \mathbb{K}$, and the desired result thus follows, because $C_{\mathcal{B}}(X)$ is a homogeneous space.

- Let $Y := \{(f,B,n) \in C_{\mathcal{B}}(X) \times \mathcal{B} \times \omega : f[B] \subset I_n\}$ and $\mathcal{H} := \{M_n : n \in \omega\}^{\uparrow}$, where for each $n \in \omega$ we put $M_n := \{(f,B,m) : m \geq n\}$. The function $\pi\colon Y \to C_{\mathcal{B}}(X)$ is the projection onto $C_{\mathcal{B}}(X)$ and $\Phi\colon Y \to \tau$ is defined by $\Phi((f,B,n)) = f^{-1}[I_n]$.
- Item 2 from Theorem 1.56 yields that $\Phi^{-1}(\Gamma_{\mathcal{B}}(X))$ is a proper filter on Y – since $\mathbb{K}$ is $\mathbb{F}_1$-composable, it follows that $\Phi^{-1}(\Gamma_{\mathcal{B}}(X)) \in \mathbb{K}$.
- For each $n \in \omega$ and $B \in \mathcal{B}$ we have $M_n \cap \Phi^{-1}[V(B)] \neq \emptyset$, thus $\Phi^{-1}(\Gamma_{\mathcal{B}}(X)) \vee \mathcal{H} \in \mathbb{K}$, because $\mathbb{K}$ is $\mathbb{F}_{\aleph_0}$-steady.
- Item 1 from Theorem 1.56 gives $\pi\left(\Phi^{-1}(\Gamma_{\mathcal{B}}(X)) \vee \mathcal{H}\right) \in \mathbb{K}$.
- As in the previous proposition, the desired equality falls by observing that both filters $\pi\left(\Phi^{-1}(\Gamma_{\mathcal{B}}(X)) \vee \mathcal{H}\right)$ and $\mathcal{N}_{\underline{0},C_{\mathcal{B}}(X)}$ have the same basis. This follows from Theorem 1.58 by noting that for each $B \in \mathcal{B}$ and $n \in \omega$, one has

$$\left\langle B, \frac{1}{n+1} \right\rangle [\underline{0}] = \pi\left[\Phi^{-1}[V(B)] \cap M_n\right]. \qquad \square$$

Corollary 1.60. Let X be a Tychonoff space and suppose that the bornology $\mathcal{B}$ has a compact basis. If $\mathbb{K}$ is a class of filters $\mathbb{F}_1$-composable and $\mathbb{F}_{\aleph_0}$-steady, then $\Gamma_{\mathcal{B}}(X) \in \mathbb{K}$ if and only if $C_{\mathcal{B}}(X)$ is a $\mathbb{K}$-space.

The original result of Jordan [39, Theorem 3] follows from the previous corollary by simply putting $\mathcal{B} = [X]^{<\aleph_0}$. Depending on the class of filters $\mathbb{K}$ considered, we obtain a different instance of duality between local properties of $C_{\mathcal{B}}(X)$ and properties of $\mathcal{B}$-coverings of X. We shall see below a few illustrative examples.

For brevity, for families $\mathcal{F}$ and $\mathcal{H}$, following Jordan's notation [39], we write

$$\mathcal{F} \# \mathcal{G}, \tag{1.30}$$

and we say that $\mathcal{F}$ **and** $\mathcal{G}$ **mesh**, to abbreviate the sentence "$F \cap G \neq \emptyset$ for all $(F,G) \in \mathcal{F} \times \mathcal{G}$". Particularly, for $\mathcal{F} = \{A\}$ we simply write $A \# \mathcal{G}$ instead of $\{A\} \# \mathcal{G}$, and for another subset B, $A \# B$ means $A \cap B \neq \emptyset$.

Example 1.61 (Revisiting Theorem 1.24)**.** The class of filters $\mathbb{T}_\kappa$ is $\mathbb{F}_1$-composable and $\mathbb{F}_{\aleph_0}$-steady.

- $\mathbb{T}_\kappa$ is $\mathbb{F}_1$-composable: if $\mathcal{F} \in \mathbb{T}_\kappa(C)$ and $R \subset C \times D$ is a relation such that $R(\mathcal{F}) \in \mathbb{F}(D)$, we shall prove that $R(\mathcal{F}) \in \mathbb{T}_\kappa(D)$. Indeed, for if $A \subset D$ is such that $A \# R(\mathcal{F})$, then $R^{-1}[A] \# \mathcal{F}$, and the condition about $\mathcal{F}$ thus implies that there exists a $B \in [R^{-1}[A]]^{\leq \kappa}$ with $B \# \mathcal{F}$. For each $b \in B$ we choose an $a_b \in A$ such that $(b, a_b) \in R$, which shows that $B' = \{a_b : b \in B\} \in [A]^{\leq \kappa}$ satisfies $B' \# R(\mathcal{F})$. From this we obtain $R(\mathcal{F}) \in \mathbb{T}_\kappa(D)$.
- $\mathbb{T}_\kappa$ is $\mathbb{F}_{\aleph_0}$-steady: if $\mathcal{F} \in \mathbb{T}_\kappa(C)$ and $\mathcal{H} = \{H_n : n \in \omega\}^{\uparrow} \in \mathbb{F}_{\aleph_0}(C)$ is such that $\mathcal{H} \# \mathcal{F}$, then $\mathcal{F} \vee \mathcal{H} \in \mathbb{T}_\kappa(C)$. Indeed, if $A \subset C$ is such that $A \# (\mathcal{F} \vee \mathcal{H})$, then $(A \cap H_n) \# \mathcal{F}$ holds for each $n \in \omega$, from which it follows that there exists a $B_n \subset A \cap H_n$ such that $|B_n| \leq \kappa$ and $B_n \# \mathcal{F}$. Thus, $B = \bigcup_{n \in \omega} B_n \subset A$ satisfies $B \# (\mathcal{F} \vee \mathcal{H})$ and $|B| \leq \kappa$.

Corollary 1.62 (From Corollary 1.60)**.** Let X be a Tychonoff space and suppose that the bornology $\mathcal{B}$ has a compact basis. Then $\mathrm{t}(C_{\mathcal{B}}(X)) \leq \kappa$ if and only if $\Gamma_{\mathcal{B}}(X) \in \mathbb{T}_\kappa$.

Since $\Gamma_{\mathcal{B}}(X) \in \mathbb{T}_\kappa$ holds if and only if $\mathrm{L}(\mathcal{B}^+) \leq \kappa$, we obtain an alternative proof for Theorem 1.24. ■

Example 1.63 (Revisiting Proposition 1.29)**.** Let $\mathcal{F}$ be a proper filter on a set C and consider the family $\mathcal{M}_{\mathcal{F}} := \{A \subset C : \forall F \in \mathcal{F}(A \cap F \neq \emptyset)\}$.

For a function $\varphi\colon \omega \to [2, \aleph_0]$, let $\mathbb{S}_\varphi$ and $\mathbb{G}_\varphi$ be the classes of those proper filters $\mathcal{F}$ satisfying $\mathsf{S}_\varphi(\mathcal{M}_{\mathcal{F}}, \mathcal{M}_{\mathcal{F}})$ and $\mathrm{II} \uparrow \mathsf{G}_\varphi(\mathcal{M}_{\mathcal{F}}, \mathcal{M}_{\mathcal{F}})$, respectively. Note that both classes of filters $\mathbb{S}_\varphi$ and $\mathbb{G}_\varphi$ are $\mathbb{F}_1$-composable. Thus, with the hypotheses of Theorem 1.57, it follows that

- if $\mathsf{S}_\varphi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ holds in $C_{\mathcal{B}}(X)$, then $\Gamma_{\mathcal{B}}(X) \in \mathbb{S}_\varphi$, and
- if $\mathrm{II} \uparrow \mathsf{G}_\varphi(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ holds in $C_{\mathcal{B}}(X)$, then $\Gamma_{\mathcal{B}}(X) \in \mathbb{G}_\varphi$.

On the other hand, Proposition 1.59 yields the converses of the above implications provided the function φ is constant, because in this case we are able to prove that the classes $\mathbb{S}_\varphi$ and $\mathbb{G}_\varphi$ are $\mathbb{F}_{\aleph_0}$-steady. Since $\Gamma_{\mathcal{B}}(X) \in \mathbb{S}_\varphi \Leftrightarrow \mathsf{S}_\varphi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$ and $\Gamma_{\mathcal{B}}(X) \in \mathbb{G}_\varphi \Leftrightarrow \mathrm{II} \uparrow \mathsf{G}_\varphi(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$, we note that Corollary 1.60 generalizes part of Proposition 1.29. We were not able to apply the same reasoning to obtain the results about Player I in Proposition 1.29. ■

Example 1.64 (Hiding point-cofinite open coverings)**.** Let $\mathbb{S}(C)$ be the class of those proper filters $\mathcal{F}$ on C satisfying the following condition: for each sequence $(H_n)_{n \in \omega}$ of subsets of C such that $H_n \# \mathcal{F}$ for all $n \in \omega$, there exists a decreasing family $\mathcal{G} = \{G_n : n \in \omega\}$ of subsets of C such that $G_n \# H_n$ for all $n \in \omega$ and $\mathcal{G}$ refines $\mathcal{F}$. For a topological space Y and a point $y \in Y$, it is easy to see that $\mathcal{N}_{y,Y} \in \mathbb{S}(Y)$ if and only if Y satisfies the selection principle $\mathsf{S}_1(\Omega_y, \Gamma_y)$, where

$$\Gamma_y := \{D \subset Y : \forall U \in \mathcal{N}_{y,Y}\, (|D \setminus U| < \aleph_0)\}. \tag{1.31}$$

Since the class of filters $\mathbb{S}$ is $\mathbb{F}_1$-composable and $\mathbb{F}_{\aleph_0}$-steady, Corollary 1.60 implies that $C_{\mathcal{B}}(X)$ satisfies $\mathsf{S}_1(\Omega_{\underline{0}},\Gamma_{\underline{0}})$ if and only if $\Gamma_{\mathcal{B}}(X) \in \mathbb{S}$. ∎

The last example is related with *Fréchet condition*[23]. Recall that for a topological space Y and a point $y \in Y$, we say that Y is a **Fréchet space at** $y \in Y$ if $\binom{\Omega_y}{\Gamma_y}_{\subset}$ holds, i.e., for all $A \subset Y$ such that $y \in \overline{A}$ there exists a sequence $\{y_n : n \in \omega\} \subset A$ such that $y_n \to y$. Naturally, when $\binom{\Omega_y}{\Gamma_y}_{\subset}$ holds for all $y \in Y$, we simply say that Y is a **Fréchet space**. On the other hand, a space Y satisfying the selective variation $\mathsf{S}_1(\Omega_y,\Gamma_y)$ is said to be a **strictly Fréchet space** at $y \in Y$. Of course, one has the following implications,

$$\chi(y,Y) \leq \aleph_0 \Rightarrow \mathsf{S}_1(\Omega_y,\Gamma_y) \Rightarrow \binom{\Omega_y}{\Gamma_y}_{\subset} \Rightarrow \mathrm{t}(y,Y) \leq \aleph_0,$$

where χ denotes the character of Y at y.

Strictly Fréchet spaces were introduced by Gerlits and Nagy [28] together with the notion of point-cofinite open coverings (see Example I.1), in order to fully characterize the Fréchet property of $C_p(X)$.

Theorem 1.65 (Gerlits and Nagy [28])**.** For a Tychonoff space X, the following are equivalent:

1. X satisfies $\binom{\Omega}{\Gamma}_{\subset}$;
2. X has property $\mathsf{S}_1(\Omega,\Gamma)$;
3. $C_p(X)$ is strictly Fréchet;
4. $C_p(X)$ is Fréchet.

Spaces satisfying $\binom{\Omega}{\Gamma}_{\subset}$ became known as **γ-spaces** (or γ sets if they are subsets of the real line), because this property was stated as the third item in a list starting at α. In the above theorem, while the proof of (3)$\Rightarrow$(4)$\Rightarrow$(1) is quite standard, (1)$\Rightarrow$(2) has a trick and (2) $\Leftrightarrow$ (3) is hidden in Example 1.64, for $\mathcal{B} = [X]^{<\aleph_0}$. Indeed, taking C as the topology of X, note that

- for $\mathcal{U} = H$, condition $\mathcal{U}\#\Gamma_{[X]^{<\aleph_0}}(X)$ means that $\mathcal{U}$ is an ω-covering of X;
- if $\mathcal{G} = \{G_n : n \in \omega\}$ is a decreasing family of subsets of τ such that $\mathcal{G}$ refines $\Gamma_{[X]^{<\aleph_0}}(X)$, then every selection $(U_n)_{n\in\omega} \in \prod_{n\in\omega} G_n$ yields a point-cofinite open covering.

Actually, in the last bullet we can see that point-cofinite open coverings are cofinite in a *stronger* sense: each finite subset of the space is contained in all but finitely many elements of the covering. This suggests a natural adaptation for $\mathcal{B}$-coverings: a collection $\mathcal{U}$ of open sets of X is **$\mathcal{B}$-cofinite** if $\{U \in \mathcal{U} : B \subset U\}$ is cofinite in $\mathcal{U}$ for all $B \in \mathcal{B}$. We shall call $\Gamma_{\mathcal{B}}$ the family of all $\mathcal{B}$-cofinite open coverings of X, and a space satisfying $\binom{\mathcal{O}_{\mathcal{B}}}{\Gamma_{\mathcal{B}}}_{\subset}$ will be called a **$\gamma_{\mathcal{B}}$-space**.

[23] Also known as Fréchet-Urysohn condition.

McCoy and Ntantu [55] have introduced $\mathcal{B}$-cofinite open coverings in their generalization of Theorem 1.65, calling them *$\mathcal{B}$-sequences*. Although they just stated items (1) and (4) of the theorem below, their arguments, which are adapted from Gerlits and Nagy [28], can be used to prove the following.

Theorem 1.66 (McCoy and Ntantu [55])**.** Let X be a Tychonoff space and suppose that the bornology $\mathcal{B}$ has a compact basis. The following are equivalent:

1. X is a $\gamma_{\mathcal{B}}$-space;
2. X has property $\mathsf{S}_1(\mathcal{O}_{\mathcal{B}}, \Gamma_{\mathcal{B}})$;
3. $C_{\mathcal{B}}(X)$ is strictly Fréchet;
4. $C_{\mathcal{B}}(X)$ is Fréchet.

It can be shown that the above theorem follows mostly from Corollary 1.60. In order to do this, we need to write the conditions above in the *filter language*. We already know the class $\mathbb{S}$ of strictly Fréchet filters from Example 1.64: those proper filters $\mathcal{F}$ on C such that for each sequence $(H_n)_{n\in\omega}$ of subsets of C meshing with $\mathcal{F}$ there is a decreasing family $\mathcal{G} = \{G_n : n \in \omega\}$ refining $\mathcal{F}$ such that $G_n \# H_n$ for all n.

Calling τ the topology of X, it is easy to see that $\Gamma_{\mathcal{B}}(X) \in \mathbb{S}(\tau)$ if and only if $\mathsf{S}_1(\mathcal{O}_{\mathcal{B}}, \Gamma_{\mathcal{B}})$ holds[24]. So, if one proves that the class of filters $\mathbb{S}$ is $\mathbb{F}_1$-composable and $\mathbb{F}_{\aleph_0}$-steady, then equivalence 2$\Leftrightarrow$3 follows from Corollary 1.60.

Lemma 1.67. The class of filters $\mathbb{S}$ is $\mathbb{F}_1$-composable and $\mathbb{F}_{\aleph_0}$-steady.

Proof. The class $\mathbb{S}$ is $\mathbb{F}_1$-composable: for $\mathcal{F} \in \mathbb{S}(C)$ and $R \subset C \times D$ satisfying $R(\mathcal{F}) \in \mathbb{F}(D)$, we shall show that $R(\mathcal{F}) \in \mathbb{S}$. Indeed, for a sequence $(H_n)_{n\in\omega}$ such that $H_n \# R(\mathcal{F})$ for all $n \in \omega$, we consider the sequence $(R^{-1}[H_n])_{n\in\omega}$, which satisfies $R^{-1}[H_n] \# \mathcal{F}$ for all n. Hence, there is a decreasing family $\{G_n : n \in \omega\}$ refining $\mathcal{F}$ such that $G_n \# R^{-1}[H_n]$ for all n. Now, for each $n \in \omega$, let $G'_n := \{h \in H_n : \exists g \in G_n\, (g,h) \in R\}$ and note that $G'_n \# H_n$. Finally, let $G''_m := \bigcup_{n\geq m} G'_n$ for each $m \in \omega$ and note $\mathcal{G}'' := \{G''_m : m \in \omega\}$ is a decreasing refinement of $\mathcal{R}(\mathcal{F})$.

The class $\mathbb{S}$ is $\mathbb{F}_{\aleph_0}$-steady: for $\mathcal{F} \in \mathbb{S}(C)$ and $\mathcal{H} \in \mathbb{F}_{\aleph_0}(C)$ such that $\mathcal{H} \# \mathcal{F}$, we show that $\mathcal{F} \vee \mathcal{H} \in \mathbb{S}(C)$. Let $(A_n)_{n\in\omega}$ be a sequence of subsets of C such that $A_n \# \mathcal{F} \vee \mathcal{H}$ for all $n \in \omega$. Fixing a decreasing basis $\{H_n : n \in \omega\}$ for $\mathcal{H}$, note that $(A_n \cap H_n)_{n\in\omega}$ is again a sequence of subsets of C such that $A_n \cap H_n \# \mathcal{F}$. Therefore, there is a decreasing family $\mathcal{G} := \{G_n : n \in \omega\}$ refining $\mathcal{F}$ such that $G_n \# A_n \cap H_n$ for all $n \in \omega$. Finally, take $\mathcal{G}' := \{G_n \cap H_n : n \in \omega\}$ and note that $\mathcal{G}'$ has the desired properties. □

It remains to define Fréchet filters. Following Jordan's definitions [39, 42], Fréchet filters are those proper filters $\mathcal{F}$ on C such that for each $H \subset C$ meshing with $\mathcal{F}$ there is a countably based filter $\mathcal{G}$ refining $\mathcal{F} \vee H$. Let us denote by $\mathbb{F}^r(C)$ the class of Fréchet filters on C. To settle any doubts about this definition, we prove the next lemma.

[24] Caution: $\Gamma_{\mathcal{B}}$ denotes the family of $\mathcal{B}$-cofinite coverings of X, while $\Gamma_{\mathcal{B}}(X)$ denotes a filter on τ.

Lemma 1.68. Let Y be a topological space and let $y \in Y$. Then Y is Fréchet at y if and only if $\mathcal{N}_{y,Y} \in \mathbb{F}^r(Y)$.

Proof. A subset $H \subset X$ satisfies $H \# \mathcal{N}_{y,Y}$ if and only if $y \in \overline{H}$. Thus, assuming Y to be Fréchet at y, we may take $\mathcal{G} = \{\{h_n : n \geq m\} : m \in \omega\}^{\uparrow}$, where $\{h_n : n \in \omega\} \subset H$ is such that $h_n \to y$. Conversely, if $\mathcal{G} \in \mathbb{F}_{\aleph_0}(X)$ refines $\mathcal{N}_{y,Y} \vee H$, we take a decreasing countable basis $\{G_n : n \in \omega\}$ for $\mathcal{G}$ and for each n we pick an $h_n \in G_n$. Clearly one has $\{h_n : n \in \omega\} \cap H \in \Gamma_y$. □

Particularly, one can readily verify that $\Gamma_{\mathcal{B}}(X) \in \mathbb{F}^r(\tau)$ if and only if X satisfies $\binom{\mathcal{O}_{\mathcal{B}}}{\Gamma_{\mathcal{B}}}_{\subset}$, hence it is enough to prove that $\mathbb{F}^r$ is an $\mathbb{F}_1$-composable class of filters to obtain $4 \Rightarrow 1$ as an instance of Proposition 1.57.

Lemma 1.69. The class of filters $\mathbb{F}^r$ is $\mathbb{F}_1$-composable.

Proof. For $\mathcal{F} \in \mathbb{F}^r(C)$ and $R \subset C \times D$ satisfying $R(\mathcal{F}) \in \mathbb{F}(D)$, we shall show that $R(\mathcal{F}) \in \mathbb{F}^r$. For a set $H \subset D$ such that $H \# R(\mathcal{F})$, we consider the set $R^{-1}[H]$, which satisfies $R^{-1}[H] \# \mathcal{F}$. Hence, there is a countably based filter $\mathcal{G} = \{G_n : n \in \omega\}^{\uparrow}$ refining $R^{-1}[H] \vee \mathcal{F}$. Suppose that $\{G_n : n \in \omega\}$ is decreasing, for each $n \in \omega$ let $G'_n := \{h \in H : \exists g \in G_n\, (g,h) \in R\}$ and note that $\mathcal{G}' := \{G'_n : n \in \omega\}^{\uparrow}$ is a countably based filter refining $\mathcal{R}(\mathcal{F}) \vee H$. □

Now, to finish this indirect proof for Theorem 1.66, we only need to deal with the nontrivial implication $\binom{\mathcal{O}_{\mathcal{B}}}{\Gamma_{\mathcal{B}}}_{\subset} \Rightarrow \mathsf{S}_1(\mathcal{O}_{\mathcal{B}}, \Gamma_{\mathcal{B}})$: this is the only part that falls outside the scope of Corollary 1.60.

Theorem 1.70 (Gerlits-Nagy trick)**.** Let X be a topological space and let $\mathcal{B}$ be a bornology on X. Then $\binom{\mathcal{O}_{\mathcal{B}}}{\Gamma_{\mathcal{B}}}_{\subset}$ implies $\mathsf{S}_1(\mathcal{O}_{\mathcal{B}}, \Gamma_{\mathcal{B}})$.

Proof. Without loss of generality we take a sequence $(\mathcal{U}_n)_{n \in \omega}$ of $\mathcal{B}$-coverings such that $X \notin \mathcal{U}_n$ for all $n \in \omega$, which obviously adds the assumption $X \notin \mathcal{B}$. Now, we let $\mathcal{V}_0 := \mathcal{U}_0$ and for $n > 0$, let $\mathcal{V}_{n+1} := \mathcal{V}_n \wedge \mathcal{U}_{n+1}$ as in the proof of Theorem 1.8. Note that $(\mathcal{V}_n)_{n \in \omega}$ is a sequence of $\mathcal{B}$-coverings such that $\mathcal{V}_n$ refines both $\mathcal{U}_n$ and $\mathcal{V}_m$ for all $m \leq n < \omega$.

Since $X \notin \mathcal{B}$, the set $\mathcal{G} := \{X \setminus \{x\} : x \in X\}$ is a $\mathcal{B}$-covering for X, and then $\binom{\mathcal{O}_{\mathcal{B}}}{\Gamma_{\mathcal{B}}}_{\subset}$ gives a $\mathcal{B}$-cofinite subcovering $\mathcal{G}'$, which we may assume to be countably infinite[25], say $\mathcal{G}' = \{X \setminus \{x_n\} : n \in \omega\}$, where $x_n \neq x_m$ if $n \neq m$. For each $n \in \omega$, let $\mathcal{W}'_n := \{V \setminus \{x_n\} : V \in \mathcal{U}_n\}$ and set $\mathcal{W} := \bigcup_{n \in \omega} \mathcal{W}'_n$. Clearly $\mathcal{W} \in \mathcal{O}_{\mathcal{B}}$, from which we get a $\mathcal{B}$-cofinite subcovering, say $\mathcal{W}' := \{V_k \setminus \{x_{n_k}\} : k \in \omega\}$, with $V_k \in \mathcal{V}_{n_k}$ for each $k \in \omega$, with $(n_k)_{k \in \omega}$ strictly increasing. Finally, for each $i \in \omega$ such that $n_k \leq i < n_{k+1}$, we may take a $U_i \in \mathcal{U}_i$ such that $V_k \subset U_i$, from which it follows that $\{U_i : i \in \omega\} \in \Gamma_{\mathcal{B}}$. □

[25] Every infinite subset of a $\mathcal{B}$-cofinite nontrivial covering is again a $\mathcal{B}$-cofinite covering.

In the next section we shall analyze the class of *productively γ spaces* under the perspective of Corollary 1.60.

Remark 1.71 (Is there any?)**.** It can be shown that a Tychonoff space X is countable if and only if $C_p(X)$ is first countable[26], hence in this case $C_p(X)$ is a Fréchet space and X satisfies $\binom{\Omega}{\Gamma}_{\subset}$. This raises a natural and interesting question: is there a space X such that $C_p(X)$ is Fréchet but it is not first countable? Or, equivalently: is there an uncountable space X such that $\binom{\Omega}{\Gamma}_{\subset}$ holds?

Restricting the range of X to subspaces of the real line, this simple question turns out to be connected with several problems in General Topology and Set Theory[27]. Although the discussion of these connections falls outside the scope of our work, it is worthwhile to mention a couple of them, for the (consistent) convenience of the reader.

1. Still in [28], Gerlits and Nagy shows that a subset of the real line satisfying $\binom{\Omega}{\Gamma}_{\subset}$ necessarily has Rothberger property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$, which in turns implies strong measure zero. But it is consistent that each strong measure zero subset of the real line is countable [51].

2. On the other hand, it can be shown that every subspace X of the real line such that $|X| < \mathfrak{p}$ is necessarily a γ-space. Since $\aleph_1 < \mathfrak{p}$ is also consistent, it follows that (consistently) *there exists* some uncountable γ-space. This was outlined in one of Tsaban's talks at the Frontiers of Selection Principles [99]. ■

[26] More generally, if X is a Tychonoff space and $\mathcal{B}$ is a bornology on X with a closed basis, then $\chi(C_{\mathcal{B}}(X))$ is the *cofinality* of $\mathcal{B}$, i.e., the smallest cardinality of a (closed) basis of $\mathcal{B}$. This is Theorem 4.4.1 in [55].

[27] If we drop this restriction, then we may find examples in ZFC. For instance, Gerlits and Nagy [28] show that every Lindelöf *P-space* is a γ-space, and there are examples of uncountable Lindelöf P-spaces in ZFC: see section 7 in Telgársky's paper [91]. We discuss P-spaces in Section 2.3.

1.4 γ-productive spaces

We begin the last section of this chapter by introducing two classes of filters, following Jordan and Mynard's [41] definitions. We say that a filter $\mathcal{F}$ on a set C is:

1. **strongly Fréchet** if for each filter $\mathcal{H} \in \mathbb{F}_{\aleph_0}(C)$ such that $\mathcal{H}\#\mathcal{F}$, there is a filter $\mathcal{G} \in \mathbb{F}_{\aleph_0}(C)$ refining the filter $\mathcal{F} \vee \mathcal{H}$;
2. **productively Fréchet** if for each strongly Fréchet filter $\mathcal{H}$ such that $\mathcal{H}\#\mathcal{F}$, there is a filter $\mathcal{G} \in \mathbb{F}_{\aleph_0}(C)$ refining $\in \mathcal{F} \vee \mathcal{H}$.

It can be shown that both these classes are $\mathbb{F}_1$-composable and $\mathbb{F}_{\aleph_0}$-steady.

In the topological context, a space Y is **strongly Fréchet** in $y \in Y$ if and only if the filter $\mathcal{N}_{y,Y}$ is strongly Fréchet, which is equivalent to the following[28]: for every *decreasing* sequence $(A_n)_{n\in\omega}$ of subsets of Y such that $y \in \bigcap_{n\in\omega} \overline{A_n}$, there exists a sequence $(y_n)_{n\in\omega} \in \prod_{n\in\omega} A_n$ such that $y_n \to y$. It is then easy to see that in the following list, each item implies the next one, while all of them are equivalent in spaces of the form $C_{\mathcal{B}}(X)$, due to Theorem 1.66:

1. the space Y is strictly Fréchet at y;
2. the space Y is strongly Fréchet at y;
3. the space Y is Fréchet at y.

However, the topological translation of productively Fréchet filters does not correspond with the grammatical appeal of its name. In fact, calling a space Y **productively Fréchet** if its neighborhood filters are productively Fréchet, the following characterization is known[29].

Theorem 1.72 (Jordan and Mynard [41])**.** A space Y is productively Fréchet if and only if $Y \times Z$ is a Fréchet (equivalently, strongly Fréchet) space for each strongly Fréchet space Z.

Following Jordan [39], we say that a Tychonoff space X is **γ-productive** if the filter $\Gamma(X) := \Gamma_{[X]^{<\aleph_0}}$ is productively Fréchet. Now, the name of this latter class of spaces is natural, at least in one direction.

Theorem 1.73 (Jordan [39])**.** If the space X is γ-productive, then $X \times Y$ is a γ-space for all γ-spaces Y.

With the terminology of the next chapter, the above theorem says that *γ-productive spaces are productively γ*. The result we prove at the end of this section suggests that the converse does not hold.

[28] Strongly Fréchet spaces were independently introduced by Michael [59] and Siwiec [84].
[29] In the terminology adopted in Chapter 2, a more suitable name for the above class of spaces would be *productively strongly Fréchet spaces*.

More recently, Miller, Tsaban and Zdomskyy [61] proved the above theorem with a short detour via C_p-theory, by using the characterizations of Theorems 1.65 and 1.72. By adapting their approach, we shall extend Theorem 1.73 with a similar detour throughout $C_{\mathcal{B}}$-theory, by using Theorems 1.66 and 1.72.

First of all, since we shall deal with products of spaces with different bornologies, we have to discuss their behavior with respect to products and sums of spaces.

Lemma 1.74. Let $\{X_t : t \in T\}$ be a family of pairwise disjoint topological spaces and suppose that for each $t \in T$ there exists a family $\mathcal{B}_t$ of subsets of X_t. If for each $t \in T$ the family $\mathcal{B}_t$ is a (compact) basis for a bornology on X_t, then

1. the family
$$\bigotimes_{t\in T} \mathcal{B}_t := \left\{ \prod_{t\in T} B_t : (B_t)_{t\in T} \in \prod_{t\in T} \mathcal{B}_t \right\} \tag{1.32}$$
is a (compact) basis for a bornology on $\prod_{t\in T} X_t$;

2. the family
$$\bigsqcup_{t\in T} \mathcal{B}_t := \left\{ \bigcup_{t\in T} B_t : (B_t)_{t\in T} \in \prod_{t\in T} \mathcal{B}_t \text{ and } |\{t\in T : B_t \neq \emptyset\}| < \aleph_0 \right\} \tag{1.33}$$
is a (compact) basis for a bornology on $\bigoplus_{t\in T} X_t$.

Proof. It is straightforward and left to the reader. □

Lemma 1.75. Let $\{X_t : t \in T\}$ be a family of pairwise disjoint topological spaces and for each $t \in T$ let $\mathcal{B}_t$ be a bornology on X_t. Then $C_{\bigsqcup_{t\in T}\mathcal{B}_t}(\bigoplus_{t\in T} X_t)$ is linearly homeomorphic to $\prod_{t\in T} C_{\mathcal{B}_t}(X_t)$.

Proof. It is enough to note that the correspondence $f \mapsto (f \restriction X_t)_{t\in T}$ defines a homeomorphism from $C_{\bigsqcup_{t\in T}\mathcal{B}_t}(\bigoplus_{t\in T} X_t)$ to $\prod_{t\in T} C_{\mathcal{B}_t}(X_t)$. □

Remark 1.76. In particular, if $\mathcal{B}_t = [X_t]^{<\aleph_0}$ for each $t \in T$, then $\bigsqcup_{t\in T}[X_t]^{<\aleph_0} = [\bigoplus_{t\in T} X_t]^{<\aleph_0}$, hence
$$C_p\left(\bigoplus_{t\in T} X_t\right) = C_{[\bigoplus_{t\in T} X_t]^{<\aleph_0}}\left(\bigoplus_{t\in T} X_t\right) = C_{\bigsqcup_{t\in T}[X_t]^{<\aleph_0}}\left(\bigoplus_{t\in T} X_t\right) \cong \prod_{t\in T} C_{[X_t]^{<\aleph_0}}(X_t) = \prod_{t\in T} C_p(X_t), \tag{1.34}$$
where the symbol "$\cong$" indicates the existence of a homeomorphism. Similarly, if for each $t \in T$ the space X_t is Hausdorff, then
$$C_k\left(\bigoplus_{t\in T} X_t\right) \cong \prod_{t\in T} C_k(X_t). \tag{1.35}$$
For brevity, when $X_t = X$ and $\mathcal{B}_t = \mathcal{B}$ for all $t \in T$, we denote $\mathcal{B}^{|T|} := \bigotimes_{t\in T} \mathcal{B}$. ■

The next theorem is as essential in the alternative proof for Theorem 1.73 presented in [61] as it is in our own generalization of it. Since its proof is reasonably short, we present it here for the convenience of the reader.

Theorem 1.77 (Miller, Tsaban and Zdomskyy [61])**.** Let $\mathfrak{P}$ be a topological property hereditary for closed subspaces and preserved under finite powers. Then for each pair of topological spaces X and Y, $X \times Y$ has property $\mathfrak{P}$ whenever $X \oplus Y$ has property $\mathfrak{P}$.

Proof. Since property $\mathfrak{P}$ is preserved under taking finite powers, it follows that $(X \oplus Y)^2$ has property $\mathfrak{P}$. On the other hand, since property $\mathfrak{P}$ is hereditary for closed subsets, it follows that $X \times Y$ has property $\mathfrak{P}$, because $X \times Y$ is a closed subset of

$$(X \oplus Y)^2 = X^2 \oplus (X \times Y) \oplus (Y \times X) \oplus Y^2. \qquad \square$$

Our next proposition is meant to show that the property "X has a bornology $\mathcal{B}$ with a compact basis such that X is a $\gamma_{\mathcal{B}}$-space" satisfies the conditions of Theorem 1.77 – for brevity, we shall call this property as $\mathsf{CB}\gamma$. In order to do so, we first prove the following lemma, adapted from Di Maio, Kočinac and Meccariello [19].

Lemma 1.78. Suppose that the bornology $\mathcal{B}$ has a compact basis $\mathcal{B}'$ and let $n \in \omega$. If $\mathcal{U}$ is a non-trivial $\mathcal{B}^n$-covering of X^n, then there exists a $\mathcal{B}$-covering $\mathcal{V}$ of X such that $\{V^n : V \in \mathcal{V}\}$ is a $\mathcal{B}^n$-covering of X^n refining $\mathcal{U}$.

Proof. For a compact subset $B \in \mathcal{B}'$, there exists a $U_B \in \mathcal{U}$ such that $B^n \subset U_B$. It follows from the Tube Lemma that there are open sets $V_{0,B}, \dots, V_{n-1,B} \subset X$ such that

$$B^n \subset \prod_{0 \leq j < n} V_{j,B} \subset U_B.$$

Then, let $V_B := \bigcap_{0 \leq j < n} V_{j,B}$ and note that $\mathcal{V} := \{V_B : B \in \mathcal{B}'\}$ satisfies the desired conditions. $\square$

Proposition 1.79. Property CB_γ satisfies the conditions of Theorem 1.77. More precisely, the following holds.

1. If $Y \subset X$, then $\mathcal{B}_Y := \{B \cap Y : B \in \mathcal{B}\}$ is a bornology on Y. Also, if Y is closed and $\mathcal{B}$ has a compact basis (on X), then $\mathcal{B}_Y$ has a compact basis (on Y).

2. If X is a $\gamma_{\mathcal{B}}$-space and $Y \subset X$ is closed, then Y is a $\gamma_{\mathcal{B}_Y}$-space.

3. Let $n \in \omega$. If the bornology $\mathcal{B}$ has a compact basis, then X is a $\gamma_{\mathcal{B}}$-space if and only if X^n is a $\gamma_{\mathcal{B}^n}$-space.

Proof.

1. Clearly the family $\mathcal{B}_Y$ is a bornology on Y. Also, if $\mathcal{B}' \subset \mathcal{B}$ is a basis for $\mathcal{B}$, then $\mathcal{B}'_Y$ is a basis for $\mathcal{B}_Y$. Particularly, if Y is closed, then $B \cap Y$ is compact whenever B is compact, from which the desired result follows.

2. If $Y \in \mathcal{B}$, then Y has no non-trivial $\mathcal{B}_Y$-coverings, from which it follows that Y is vacuosly a $\gamma_{\mathcal{B}_Y}$-space. If $Y \notin \mathcal{B}$ and $\mathcal{U} = \{U \cap Y : U \in \mathcal{V}\}$ is a non-trivial $\mathcal{B}_Y$-covering of Y for some family $\mathcal{V}$ of open sets of X, then $\mathcal{W} = \mathcal{V} \cup \{X \setminus Y\} \cup \{U \cup (X \setminus Y) : U \in \mathcal{V}\}$ is a non-trivial $\mathcal{B}$-covering of X. Finally, note that every $\mathcal{B}$-cofinite subcovering of $\mathcal{W}$ induces a $\mathcal{B}_Y$-cofinite subcovering of $\mathcal{V}$.

3. By using Lemma 1.78, it is easy to see that X^n is a $\gamma_{\mathcal{B}^n}$-space whenever X is a $\gamma_{\mathcal{B}}$-space. The converse holds by item (2) and the next lemma, since X is a closed subspace of $\bigoplus_{0 \leq j < n} X$. □

Lemma 1.80. If X is a $\gamma_{\mathcal{B}}$-space and Z is a $\gamma_{\mathcal{L}}$-space for a bornology $\mathcal{L}$ on Z, such that $X \times Z$ is a $\gamma_{\mathcal{B}\otimes\mathcal{L}}$-space, then $X \oplus Z$ is a $\gamma_{\mathcal{B}\sqcup\mathcal{L}}$-space.

Proof. If $\mathcal{U}$ is a non-trivial $\mathcal{B} \sqcup \mathcal{L}$-covering for $X \oplus Z$, then $\mathcal{U}^2 := \{U^2 : U \in \mathcal{U}\}$ is a non-trivial $\mathcal{B} \otimes \mathcal{L}$-covering for $X \times Z$. Then note that a $\mathcal{B} \otimes \mathcal{L}$-cofinite subcovering of $\mathcal{U}^2$ induces a $\mathcal{B} \sqcup \mathcal{L}$-cofinite subcovering of $\mathcal{U}$. □

Corollary 1.81. Let X and Y be topological spaces endowed with bornologies $\mathcal{B}$ and $\mathcal{L}$, respectively, both of them with compact bases. Then $X \times Y$ is a $\gamma_{\mathcal{B}\otimes\mathcal{L}}$-space if and only if $X \oplus Y$ is a $\gamma_{\mathcal{B}\sqcup\mathcal{L}}$-space.

Proof. The last proposition showed that property CB_γ satisfies the conditions of Theorem 1.77, which proves one direction. The converse is Lemma 1.80. □

Corollary 1.82. Let X be a Tychonoff space and suppose that the bornology $\mathcal{B}$ has a compact basis. If $C_{\mathcal{B}}(X)$ is productively Fréchet, then $X \times Y$ is a $\gamma_{\mathcal{B}\otimes\mathcal{L}}$-space for each Tychonoff space Y endowed with a bornology $\mathcal{L}$ with a compact basis such that Y is a $\gamma_{\mathcal{L}}$-space.

Proof. Since Y is a $\gamma_{\mathcal{L}}$-space, Theorem 1.66 gives that $C_{\mathcal{L}}(Y)$ is strictly Fréchet, hence strongly Fréchet. On the other hand, since $C_{\mathcal{B}}(X)$ is productively Fréchet, Theorem 1.72 yields that the space $C_{\mathcal{B}}(X) \times C_{\mathcal{L}}(Y)$ is Fréchet. However, one has $C_{\mathcal{B}}(X) \times C_{\mathcal{L}}(Y) \cong C_{\mathcal{B}\sqcup\mathcal{L}}(X \oplus Y)$, and Theorem 1.66 implies again that $X \oplus Y$ is a $\gamma_{\mathcal{B}\sqcup\mathcal{L}}$-space, hence the result follows from the previous corollary. □

Now, it makes sense to define *$\gamma_{\mathcal{B}}$-productive spaces* in the following way. For a topological space X endowed with a bornology $\mathcal{B}$, we say that X is **$\gamma_{\mathcal{B}}$-productive** if the filter $\Gamma_{\mathcal{B}}(X)$ is productively Fréchet. Thus, we may rewrite the last corollary.

Corollary 1.83. Let X be a Tychonoff space and suppose that the bornology $\mathcal{B}$ has a compact basis. If X is $\gamma_{\mathcal{B}}$-productive, then $X \times Y$ is a $\gamma_{\mathcal{B}\otimes\mathcal{L}}$-space for each Tychonoff space Y endowed with a bornology $\mathcal{L}$ with a compact basis such that Y is a $\gamma_{\mathcal{L}}$-space.

In particular, whenever a Tychonoff space X is γ-productive, the space $X \times Y$ is a $\gamma_{[X]^{<\aleph_0}\otimes\mathcal{L}}$-space for each $\gamma_{\mathcal{L}}$-space Y, in which Y is a Tychonoff space and $\mathcal{L}$ is a bornology with a compact basis on Y. In some sense, this is stronger than the result stated in Theorem 1.73, and also suggests that its converse may be false: possibly, the class of γ-spaces such that its product with every γ-space is again a γ-space, i.e., the class of *productively γ-spaces*, is strictelly bigger than the class of γ-productive spaces. This observation may be useful in the following question.

Question 1.84 (Jordan [40])**.** If $X \times Y$ is a γ-space for each γ-space Y, then is X γ-productive?

Chapter 2

HYPERSPACES AND SELECTIVE PROPERTIES ON PRODUCTS

Given a topological property, it is natural to ask about its behavior with respect to the product operation. Compactness and the separation axioms T_i (for $i \in \{0,1,2,3,3\frac{1}{2}\}$) are classical examples of *productive properties*, in the sense that a nonempty product of topological spaces has one of these properties if and only if each one of the factors also has the same property[1]. On the other hand, normality illustrates the opposite behavior: the Sorgenfrey line is a usual example of a normal space whose square is not normal.

Whenever a topological property $\mathfrak{P}$ is not productive, it does make sense to analyze which class of spaces are *productive* with respect to the property $\mathfrak{P}$. Explicitly, we shall say that a topological space X is **productively** $\mathfrak{P}$ if $X \times Y$ has property $\mathfrak{P}$ for all spaces Y having property $\mathfrak{P}$. For instance, Theorem 1.73 may be restated in the following way.

Theorem 1.73' (Jordan [39])**.** If X is a γ-productive space, then X is productively γ.

Productively normal spaces have become a fertile field of research in General Topology[2]. Particularly, two problems have arisen from this topic: to find intrinsic characterizations of *productively paracompact* spaces and *productively Lindelöf* spaces, both of them posed by Przymusiński [69], although the latter is attributed to Tamano.

More recently, Aurichi and Zdomskyy [12] presented an internal characterization of regular productively Lindelöf spaces. Roughly speaking, they prove that a regular space X is productively Lindelöf if and only if $X \times \mathscr{L}$ is a Lindelöf space for all Lindelöf spaces $\mathscr{L}$ in a *restrict* class of hyperspaces of X. Since their work triggered most of the investigations done along this chapter, we shall outline their method below.

[1] The productivity of compactness is the Tychonoff Theorem, and the productivity of the separation axioms are *folklore* – a concise proof for the latter is presented by Engelking [22, p. 80], where the counterexample of the Sorgenfrey line is also discussed.

[2] An interesting survey in this topic is provided by Atsuji [4].

For the rest of this section, let X be a topological space. Now, let $\mathcal{T}$ be the topology on the set $\mathcal{O}(X) := \mathcal{O}$ whose basic open sets are of the form

$$[A_0,\dots,A_n] := \{\mathcal{U} \in \mathcal{O} : A_0,\dots,A_n \in \mathcal{U}\}, \tag{2.1}$$

where $n \in \omega$ and $A_0,\dots,A_n \subset X$ are open subsets. The intuition behind this topology is quite simple: the more open sets two coverings have in common, the closer they are to each other. This is enough for us to (at least) state the anticipated characterization, in which we consider $\mathcal{O}$ endowed with the topology $\mathcal{T}$.

Theorem 2.1 (Aurichi and Zdomskyy [12])**.** A regular space X is productively Lindelöf if and only if $X \times \mathscr{L}$ is Lindelöf for every Lindelöf subspace $\mathscr{L} \subset \mathcal{O}$.

Sketch of the proof. One direction is trivial. For the converse, let Y be a Lindelöf space and let $\mathcal{W}$ be an open covering for $X \times Y$, of which we want to obtain a countable subcovering. Let $\mathscr{L} := \{\mathcal{U}_y : y \in Y\}$, where $\mathcal{U}_y := \{A \subset X : \exists B \subset Y$ such that $A \times B \in \mathcal{W}$ and $y \in B\} \in \mathcal{O}$.

Although one can prove directly that $\mathscr{L} \subset \mathcal{O}$ is a Lindelöf subspace, it is easier to observe that the correspondence $y \mapsto \mathcal{U}_y$ defines a continuous onto function $Y \to \mathscr{L}$. Note then that the collection

$$\mathscr{D}_{\mathscr{L}} := \left\{A \times [A] : A \in \bigcup \mathscr{L}\right\}$$

is an open covering for $X \times \mathscr{L}$. Since the hypothesis says that $X \times \mathscr{L}$ is a Lindelöf space, it follows that $\mathscr{D}_{\mathscr{L}}$ has a countable subcovering, from which we obtain *in some way* a countable subcovering for $\mathcal{W}$. $\square$

The tricky part of the above argument, in which the regularity hypothesis is used, is hidden in some way in the sentence above. Indeed, suppose that $\{A_n \times [A_n] : n \in \omega\}$ is a countable subcovering of $\mathscr{D}_{\mathscr{L}}$. It is natural to choose for each $n \in \omega$ an open set $B_n \subset Y$ such that $A_n \times B_n \in \mathcal{W}$, and it is tempting to assume that the corresponding family $\{A_n \times B_n : n \in \omega\}$ has to be a subcovering of $\mathcal{W}$. Surely, for a pair $(x,y) \in X \times Y$ there is an $n \in \omega$ such that $(x,\mathcal{U}_y) \in A_n \times [A_n]$ or, equivalently, $x \in A_n$ and $A_n \in \mathcal{U}_y$. Here is the problem: $A_n \in \mathcal{U}_y$ means that there exists a $B \subset Y$ such that $A_n \times B \in \mathcal{W}$ and $y \in B$, but possibly $B \neq B_n$.

However, if the given open covering $\mathcal{W}$ is such that $|\{B \subset Y : A \times B \in \mathcal{W}\}| \leq \aleph_0$ for every open set $A \subset X$, then the above sketch works, *mutatis mutandis*. Following Aurichi at the Toposym-2016[3], we call an open covering $\mathcal{W}$ with this property an **ω-good** covering. The gap in the above sketch is prevented with the following technical result.

Theorem 2.2 (Aurichi and Zdomskyy [12])**.** If X is a regular space and Y is a Lindelöf space, then every open covering of $X \times Y$ has an ω-good refinement.

[3] Held in Prague, as usual.

Remark 2.3. Since it is usual to consider regularity and Lindelöfness together (e.g., Engelking [22]), one could ask about a characterization for productively "regular Lindelöf" spaces, in which case the above construction would not work, because the space $\mathscr{L}$ fails to be regular. However, in this scenario we are able to use the following.

Theorem 2.4 (Duanmu, Tall and Zdomskyy [21])**.** Let X be a Lindelöf space. If there exists a Lindelöf space Z such that $X \times Z$ is not Lindelöf, then there exists a Lindelöf regular space Z' such that $X \times Z'$ is not Lindelöf. ■

A closer look at the sketch of the proof for Theorem 2.1 reveals that we do not need the Lindelöfness of the product space $X \times \mathscr{L}$: we only need that the *diagonal covering* $\mathscr{D}_{\mathscr{L}}$ to have a countable subcovering. So we have an alternative statement of the characterization.

Theorem 2.1$'$ (Aurichi and Zdomskyy [12])**.** A regular space X is productively Lindelöf if and only if the covering $\mathscr{D}_{\mathscr{L}}$ has a countable subcovering for every Lindelöf subfamily $\mathscr{L} \subset \mathcal{O}$.

Since $\mathcal{O}$ is a hyperspace of X, in some sense the above result gives an internal characterization for the class of regular productively Lindelöf spaces, in such a way that it is possible to restate it *explicitly* in terms of X. Let us say that a family $\mathscr{L}$ of open coverings of X is a **Lindelöf family** if $\mathscr{L}$ is a Lindelöf subspace of $\mathcal{O}$. In this way, we have the following.

Theorem 2.1$''$ (Aurichi and Zdomskyy [12])**.** A regular space X is productively Lindelöf if and only if for every Lindelöf family $\mathscr{L} \subset \mathcal{O}$ there exists a sequence $(A_n)_{n\in\omega}$ of open sets of X such that $\mathcal{U} \cap \{A_n : n \in \omega\} \in \mathcal{O}$ for all $\mathcal{U} \in \mathscr{L}$.

Proof. It is enough to notice that $\mathscr{D}_{\mathscr{L}}$ has a countable subcovering if and only if there exists a sequence $(A_n)_{n\in\omega}$ of open sets of X such that $\mathcal{U} \cap \{A_n : n \in \omega\} \in \mathcal{O}$ for all $\mathcal{U} \in \mathscr{L}$. □

It is then natural to ask if the same method, or a variation of it, can be used in order to characterize other classes of productive spaces. This is the main motivation of this chapter.

We first present our *attack* on covering properties. Although we are able to adapt the method for a few covering properties, there are cases in which the characterization obtained is trivial in some sense. Anyway, we found out internal descriptions for productively $\mathsf{S}_\varphi(\mathcal{O},\mathcal{O})$ spaces and for $\leq \kappa$-L-*productive spaces*, as we shall see in Section 2.1.

In an entirely different direction, but with essentially the same ideas of Theorem 2.1, we obtain a ZFC internal characterization for productively ccc spaces, as well as for productively ccc orderings, which is detailed in Section 2.2. Finally, in the last section we return to discuss Lindelöfness, but with a *categorical flavored* approach: by treating Lindelöfness via closed projections (see Remark I.9), we present a new tool to analyze classical open problems regarding products of Lindelöf spaces.

2.1 The wasteland of index sets

As we said before, the main problem on the sketched proof of Theorem 2.1 is the non-injectivity of the correspondence

$$A \mapsto B, \tag{2.2}$$

where A is a subset of X and B is such that $A \times B \in \mathcal{W}$. What if we could *force*[4] the injectivity of the correspondence in (2.2)?

One way to do this is to consider the indexes instead of the open sets of a covering. More precisely, for an arbitrary family $\mathscr{A}$, consider the sets of the form

$$[G] := \{\mathcal{A} \in \mathscr{A} : G \subset \mathcal{A}\}, \tag{2.3}$$

for each $G \in [\bigcup \mathscr{A}]^{<\aleph_0}$. Since $G \cup H \in [\bigcup \mathscr{A}]^{<\aleph_0}$ and $[G] \cap [H] = [G \cup H]$ for all finite subsets $G, H \in [\bigcup \mathscr{A}]^{<\aleph_0}$, it follows that the family

$$\mathscr{B}_{\mathscr{A}} := \left\{[G] : G \in \left[\bigcup \mathscr{A}\right]^{<\aleph_0}\right\} \tag{2.4}$$

is a basis for a topology on $\mathscr{A}$.

We shall call this topology on $\mathscr{A}$ the **index topology**, and we write $\mathscr{A}^\star$ to indicate the set $\mathscr{A}$ endowed with its index topology. For brevity, we simply write $[g]$ instead of $[\{g\}]$ when $G = \{g\}$, and for a topological property $\mathfrak{P}$, we call $\mathscr{A}$ a $\mathfrak{P}$ **family** if $\mathscr{A}^\star$ has property $\mathfrak{P}$.

Remark 2.5. For $\mathscr{A} = \mathcal{O}$, note that $\mathcal{O}^\star$ is precisely the set $\mathcal{O}$ endowed with the topology defined by Aurichi and Zdomskyy. ■

As an illustration, we use the above *index hyperspace* to characterize $\leq \kappa$*-L-productive spaces*[5]. As defined by Duanmu, Tall and Zdomskyy [21], a space X is said to be $\leq \kappa$**-L-productive** if $\mathrm{L}(X \times Y) \leq \mathrm{L}(Y)$ for all spaces Y such that $\mathrm{L}(Y) \leq \kappa$. In what follows, we need to use the **covering number** of X, denoted[6] as $\mathscr{K}(X)$, and defined as the least cardinal of the form $|\mathcal{K}| + \aleph_0$, where $\mathcal{K}$ is a covering for X by compact sets. It is straightforward that the inequality

$$\mathrm{L}(X \times Y) \leq \mathscr{K}(X) \cdot \mathrm{L}(Y) \tag{2.5}$$

holds for every pair of spaces X and Y – still, see Corollary 2.34.

Proposition 2.6. A topological space X is $\leq \kappa$-L-productive if and only if $\mathrm{L}(X \times \mathscr{A}^\star) \leq \kappa$ holds for every subset $\mathscr{A} \subset \wp(\mathscr{K}(X) \cdot \kappa)$ such that $\mathrm{L}(\mathscr{A}^\star) \leq \kappa$.

4 Not in the forcing sense. This is a joke, and this footnote is a forced metajoke.
5 In our terminology, this is the class of productively "Lindelöf degree $\leq \kappa$" spaces.
6 In standard terminology, $\mathscr{K}(X)$ should be denoted as $\mathrm{cov}(\mathrm{K}_X)$.

Proof. Let Y be a topological space such that $\mathrm{L}(Y) \leq \kappa$ and let $\mathcal{W}$ be an open covering for $X \times Y$ by basic open sets. Taking $\lambda = \mathscr{K}(X) \cdot \mathrm{L}(Y) \leq \mathscr{K}(X) \cdot \kappa$, inequality (2.5) shows that there is no loss of generality in considering $\mathcal{W} := \{A_\alpha \times B_\alpha : \alpha < \lambda\}$.

Similarly to what we did in Theorem 2.1, for each $y \in Y$ let

$$\mathcal{A}_y := \{\alpha \in \lambda : y \in B_\alpha\} \subset \lambda,$$

and let $\mathscr{A} := \{\mathcal{A}_y : y \in Y\}$. Again, the surjection $Y \to \mathscr{A}^\star$ that makes $y \mapsto \mathcal{A}_y$ is continuous: indeed, if $\mathcal{A}_y \in [G]$ for a finite subset $G \subset \lambda$, then $y \in B_\alpha$ for all $\alpha \in G$, hence $B = \bigcap_{\alpha \in G} B_\alpha \subset Y$ is an open set such that $y \in B$ and $\mathcal{A}_w \in [G]$ for all $w \in B$. Thus, $\mathrm{L}(\mathscr{A}^\star) \leq \kappa$.

Now, note that

$$\mathscr{D}_{\mathscr{A}} := \{A_\alpha \times [\alpha] : \alpha < \lambda\}$$

is an open covering for $X \times \mathscr{A}^\star$. The hypothesis implies that $\mathrm{L}(X \times \mathscr{A}^\star) \leq \kappa$, from which it follows that there exists a κ-sequence $(\alpha_\gamma)_{\gamma<\kappa}$ such that $\{A_{\alpha_\gamma} \times [\alpha_\gamma] : \gamma < \kappa\}$ covers $X \times \mathscr{A}^\star$. Finally, we will be done if we show that $\{A_{\alpha_\gamma} \times B_{\alpha_\gamma} : \gamma < \kappa\}$ covers $X \times Y$, which follows from the fact that if $(x, \mathcal{A}_y) \in A_{\alpha_\gamma} \times [\alpha_\gamma]$, then $(x, y) \in A_{\alpha_\gamma} \times B_{\alpha_\gamma}$.

The converse is trivial. □

Since the cardinal $\mathscr{K}(X)$ depends only on the space X, the previous proposition indeed gives an internal characterization for κ-L-productive spaces. Particularly, by taking $\kappa = \aleph_0$ one obtains the following separation axiom free version of Theorem 2.1.

Corollary 2.7. A topological space X is productively Lindelöf if and only if $X \times \mathscr{A}^\star$ is Lindelöf for every Lindelöf family $\mathscr{A} \subset \wp(\mathscr{K}(X))$.

Besides solving Tamano's Problem, it would be desirable to use these tools on other open problems concerning Lindelöf spaces, but we have not been able to do this. Still, we list two main problems in the subject. In [58], Michael showed that under the Continuum Hypothesis (CH), there exists a regular Lindelöf space X such that $X \times \omega^\omega$ is not Lindelöf – nowadays such a space is called a **Michael space**.

Question 2.8 (Michael's Problem)**.** Is there a Michael space (in ZFC)?

Following Aurichi and Tall's [11] terminology, we call a space X **powerfully Lindelöf** if X^ω is Lindelöf. The next question, attributed to Michael and raised in [69], remains unsolved.

Question 2.9. Are productively Lindelöf spaces powerfully Lindelöf?

In the last section we shall return to these topics. Now we turn our attention to selective variations of Lindelöfness, like the Menger and Rothberger properties. In a series of recent papers [60, 61, 87, 88], Tsaban an his collaborators obtained several results concerning productively

Menger and productively Rothberger spaces. Also, Dias and Scheepers [20] derived several results relating games and productive selective properties.

Their work motivated us to investigate the applicability of our method to characterize productively Menger/Rothberger spaces. We shall see below that a similar idea works for property $\mathsf{S}_\varphi(\mathcal{O},\mathcal{O})$, where $\varphi\colon \omega \to [2,\aleph_0]$ is a function.

Proposition 2.10. Let $\varphi\colon \omega \to [2,\aleph_0]$ be a function. A topological space X is productively $\mathsf{S}_\varphi(\mathcal{O},\mathcal{O})$ if and only if $\mathsf{S}_\varphi(\mathcal{O},\mathcal{O})$ holds on $X \times \mathscr{A}^\star$ for each $\mathsf{S}_\varphi(\mathcal{O},\mathcal{O})$ family $\mathscr{A} \subset \wp(\mathscr{K}(X)\times\omega)$.

Proof. For brevity we call $\kappa = \mathscr{K}(X)$. Let Y be a topological space satisfying $\mathsf{S}_\varphi(\mathscr{O},\mathscr{O})$ and let $(\mathcal{W}_n)_{n\in\omega}$ be a sequence of open coverings for $X\times Y$. There is no loss of generality in taking $\mathcal{W}_n := \{A_{\alpha,n}\times B_{\alpha,n} : \alpha<\kappa\}$. Now, for each $y\in Y$, let $\mathcal{A}_y := \{(\alpha,n) : y\in B_{\alpha,n}\}$ and consider $\mathscr{A} := \{\mathcal{A}_y : y\in Y\}$. We will show that $\mathscr{A}^\star$ satisfies $\mathsf{S}_\varphi(\mathscr{O},\mathscr{O})$.

Let $(\mathcal{C}_n)_{n\in\omega}$ be a sequence of open coverings for $\mathscr{A}^\star$, which we may suppose that consists of basic open sets, say $\mathcal{C}_n = \{[H_{j,n}] : j\in J\}$ for each $n\in\omega$, where J is some index set and $H_{j,n}\in[\kappa\times\omega]^{<\aleph_0}$. For each $n\in\omega$ and $j\in J$, let

$$V_{j,n} := \bigcap_{(\alpha,m)\in H_{j,n}} B_{\alpha,m}$$

and set $C_n := \{V_{j,n} : j\in J\}$. This gives a sequence $(C_n)_{n\in\omega}$ of open coverings for Y. Since $\mathsf{S}_\varphi(\mathscr{O},\mathscr{O})$ holds on Y, it follows that there exists a sequence $(G_n)_{n\in\omega}$ such that $G_n\in[J]^{<\varphi(n)}$ and $Y=\bigcup_{n\in\omega}\bigcup_{j\in G_n}V_{j,n}$. Then, for each $n\in\omega$ we put $\mathcal{D}_n := \{[H_{j,n}] : j\in G_n\}\in[\mathcal{C}_n]^{<\varphi(n)}$. Note that for $\mathcal{A}_y\in\mathscr{A}^\star$, there exists an $n\in\omega$ such that $y\in V_{j,n}$ for some $j\in G_n$, from which it follows that $\mathcal{A}_y\in[H_{j,n}]$, showing that $\bigcup_{n\in\omega}\bigcup\mathcal{D}_n=\mathscr{A}^\star$, as desired.

Now the hypothesis implies that $X\times\mathscr{A}^\star$ satisfies $\mathsf{S}_\varphi(\mathscr{O},\mathscr{O})$. Clearly, the family

$$\mathscr{W}_n := \{A_{\alpha,n}\times[(\alpha,n)] : \alpha<\kappa\}$$

is an open covering for $X\times\mathscr{A}^\star$ for each $n\in\omega$. Hence, for each $n\in\omega$ we may select a subset $\mathscr{Z}_n\subset\mathscr{W}_n$ such that $|\mathscr{Z}_n|<\varphi(n)$ and $\bigcup_{n\in\omega}\bigcup\mathscr{Z}_n = X\times\mathscr{A}^\star$. Finally, let

$$\mathcal{Z}_n := \{A_{\alpha,n}\times B_{\alpha,n} : A_{\alpha,n}\times[(\alpha,n)]\in\mathscr{Z}_n\}\subset\mathcal{W}_n,$$

and note that the sequence $(\mathcal{Z}_n)_{n\in\omega}$ has the desired properties. □

Corollary 2.11. Let X be a topological space and let $\kappa = \mathscr{K}(X)$.

1. X is productively Rothberger if and only if $X\times\mathscr{A}^\star$ has the Rothberger property for every Rothberger family $\mathscr{A}\subset\wp(\kappa\times\omega)$.

2. X is productively Menger if and only if $X\times\mathscr{A}^\star$ has the Menger property for every Menger family $\mathscr{A}\subset\wp(\kappa\times\omega)$.

However, we emphasize that an extra dose of caution is needed when dealing with this index approach, which we illustrate by discussing the application our attempt to characterize productively paracompact spaces.

At first, our aim is to show that X is productively paracompact if and only if $X \times \mathscr{A}^\star$ is paracompact for every paracompact familiy $\mathscr{A} \subset \mathcal{H}(X)$, where $\mathcal{H}(X)$ is some hyperspace of X. Assuming the latter, we take a paracompact space Y and an open covering $\mathcal{W}$ for $X \times Y$ by basic open sets, say $\mathcal{W} := \{A_i \times B_i : i \in I\}$.

Now, as before, we take $\mathscr{A} := \{\mathcal{A}_y : y \in Y\}$, where $\mathcal{A}_y := \{i \in I : y \in B_i\}$, and we want to show that $\mathscr{A}^\star$ is paracompact. Here our first problem appears: to do this, we needed to add the supposition that the family $\{B_i : i \in I\}$ is a basis for the topology of Y. Since there is no bound for the weight of paracompact spaces, we get no bound for the cardinal $|I|$, so the corresponding hyperspace $\mathcal{H} = \wp(|I|)^\star$ does not depend only on X.

Anyway, with this supposition, one can prove the following result, with the same arguments we used on Proposition 2.6, *mutatis mutandis*.

Proposition 2.12. A topological space X is productively paracompact if and only if $X \times \mathscr{A}^\star$ is paracompact for all $\kappa \geq \aleph_0$ and all paracompact families $\mathscr{A} \subset \wp(\kappa)$.

However, a closer look on the details of our methods reveals a second problem. Indeed, it is a deeper (and annoying) issue: in some sense, the above proposition just says that $0 = 0$.

Lemma 2.13 (*The "nuke" lemma*)**.** Let Y be a topological space and let $\mathcal{W} := \{B_i : i \in I\}$ be an open covering for Y. Consider $\mathscr{A} := \{\mathcal{A}_y : y \in Y\}$, where $\mathcal{A}_y := \{i \in I : y \in B_i\}$ and let $\mathcal{A} : Y \to \mathscr{A}^\star$ be the onto function given by $\mathcal{A}(y) = \mathcal{A}_y$.

1. The function $\mathcal{A}$ is continuous.
2. If $\mathcal{W}$ is a basis for Y, then the function $\mathcal{A}$ is open.
3. In addition to item 2, if Y is also a T_0-space, then $\mathcal{A}$ is an 1-1 function. In particular, it follows that $\mathcal{A}$ is a homeomorphism.

Proof. We have already noted that $\mathcal{A}$ is a continuous function. For the second statement, note that if $U \subset Y$ is an open set and $y \in U$, then for each finite subset $G \subset I$ such that $\mathcal{A}_y \in [G]$ there exists some $i \in I$ with $y \in B_i \subset \bigcap_{g \in G} B_g$, from which it follows that

$$\mathcal{A}_y \in [i] \subset \mathcal{A}[U] = \{\mathcal{A}_u : u \in U\},$$

thus proving that $\mathcal{A}$ is an open function. Finally, for the last statement, observe that for distinct points $y, y' \in Y$, there exists some $i \in I$ such that $y \in B_i$ and $y' \notin B_i$, or $y \notin B_i$ and $y' \in B_i$, hence $i \in \mathcal{A}_y \setminus \mathcal{A}_{y'}$ or $i \in \mathcal{A}_{y'} \setminus \mathcal{A}_y$. In any case, one has $\mathcal{A}_y \neq \mathcal{A}_{y'}$. In particular, a continuous open bijection is a homeomorphism. $\square$

In face of the previous lemma, the requirement "for all $\kappa \geq \aleph_0$" in Proposition 2.12 turns its statement in a merely rewriting of the very definition of productively paracompactness. The same kind of problem appeared when we tried to use this method for (productively) *countably compact spaces*, γ-spaces and *Baire spaces*[7]. On the other hand, on Propositions 2.6 and 2.10, this "triviality situation" is prevented because we bound the cardinality of $\mathscr{A}^\star$ at $2^{\mathscr{K}(X)}$.

[7] Although there is a known characterization for productively countably compact Tychonoff spaces, due to Frolík [26]. In fact, even a characterization for productively paracompact Hausdorff spaces is known: it is due to Katuta [46], but as mentioned by Atsuji [4]: "a simpler characterization is also desired".

2.2 Productively ccc orders and the Knaster property

Although we present this as the second section, the investigations on this chapter started around *productively ccc spaces*. Recall that a topological space satisfy the **c**ountable **c**hain **c**ondition, or is a **ccc** space, if it has no uncountable *antichains* of nonempty open sets, where by **antichain** we mean a family of pairwise disjoint subsets of the space. Because of the peculiar behavior of ccc spaces under products, it is natural to ask about (ZFC) characterizations for productively ccc spaces. Let us explain what we mean by *peculiar behavior*.

Recall that by using the so called Δ-System Lemma, one can reduce the study of arbitrary products of ccc spaces to finite products.

Theorem 2.14 (Possibly Juhász [43])**.** Let $\{X_i : i \in I\}$ be a family of topological spaces. Then $\prod_{i\in I} X_i$ is a ccc space if and only if $\prod_{i\in F} X_i$ is a ccc space for all $F \in [I]^{<\aleph_0}$.

It may sound surprising for the unfamiliar reader that productivity of ccc property is independent of ZFC. Indeed, in the realm of Martin's Axiom (MA) plus the negation of CH, one can prove that every ccc space is productively ccc[8], while a Suslin line turns out to be a ccc space whose square is not ccc[9]. Since each one of these statements ("MA $+\neg$ CH" and "there exists a Suslin line") is independent of ZFC, it follows that productivity of ccc spaces is itself independent of ZFC.

Nevertheless, the class of productively ccc spaces can be completely determined in ZFC, as we show in the next proposition.

Proposition 2.15. A topological space (X,τ) is productively ccc if and only if $X \times \mathscr{A}^\star$ is ccc for each ccc family $\mathscr{A} \subset \wp(\tau)$.

Proof. As before, we only need to worry about one direction. So, let Y be a ccc space and let $\mathcal{W}$ be an uncountable family of nonempty basic open sets of $X \times Y$. We want to show that $\mathcal{W}$ is not pairwise disjoint. As we did in the previous section, for each $y \in Y$ let

$$\mathcal{A}_y := \{A \subset X : \exists B \subset Y, y \in B \text{ and } A \times B \in \mathcal{W}\},$$

let $Y' := \{y \in Y : \mathcal{A}_y \neq \emptyset\}$ and take $\mathscr{A} := \{\mathcal{A}_y : y \in Y'\}$. The natural claim is that $\mathscr{A}^\star$ is ccc.

Indeed, if $\{[G_\gamma] : \gamma < \omega_1\}$ is an uncountable family of basic nonempty open sets of $\mathscr{A}^\star$, then for each $\gamma < \omega_1$ there exists an $y_\gamma \in Y$ such that $\mathcal{A}_{y_\gamma} \in [G_\gamma]$, thus for each $A \in G_\gamma$ there exists some $B_{A,\gamma} \subset Y$ such that $y_\gamma \in B_{A,\gamma}$ and $A \times B_{A,\gamma} \in \mathcal{W}$. Hence, $\mathcal{V} := \{B_\gamma : \gamma < \omega_1\}$ is a

[8] Fremlin [24] refers this result to Hajnal and Juhász [31], while they refer it to F. Rowbottom. Jech [38] also attributes this result to Rowbottom, as well as to Kunen, Solovay "and possibly others". We shall discuss it further by the end of this section.

[9] This result is due to Kurepa (1950). An outline of the proof can be found in the book of Kharazishvili [47, Exercise 9, Appendix 3], or in Kunen's book [49, Lemma III.2.18].

family of nonempty open sets of Y, where $B_\gamma := \bigcap_{A\in G_\gamma} B_{A,\gamma}$. If $|\mathcal{V}| \leq \aleph_0$, then we may obtain[10] distinct $\gamma,\gamma' < \omega_1$ such that $B_\gamma = B_{\gamma'}$, from which it follows that $\mathcal{A}_{y_\gamma} \in [G_\gamma]\cap[G_{\gamma'}]$. On the other hand, if $|\mathcal{V}| > \aleph_0$, then we may use the fact that Y is a ccc space to obtain a pair $\gamma,\gamma' < \omega_1$ such that $\gamma \neq \gamma'$ and $B_\gamma \cap B_{\gamma'} \neq \emptyset$, from which it follows that $[G_\gamma]\cap[G_{\gamma'}] \neq \emptyset$.

Now, we examine the cardinality of the family $\bigcup\mathscr{A} = \bigcup_{y\in Y'} \mathcal{A}_y$. Again, if $|\bigcup\mathscr{A}| \leq \aleph_0$, then by using the Pigeonhole Principle and the fact that Y is ccc, one can easily conclude that the family $\mathcal{W}$ is not pairwise disjoint. On the other hand, if $|\bigcup\mathscr{A}| > \aleph_0$, then the *diagonal* family

$$\mathscr{D}_{\mathscr{A}} := \left\{A\times[A] : A \in \bigcup\mathscr{A}\right\}$$

is also uncountable. Since $X\times\mathscr{A}^\star$ is ccc by our hypothesis, there are $A,A' \in \bigcup\mathscr{A}$ such that $A \neq A'$ and $(A\times[A])\cap(A'\times[A']) \neq \emptyset$, which implies that

$$(x,\mathcal{A}_y) \in (A\times[A])\cap(A'\times[A']),$$

for some $x\in X$ and $y\in Y$. Finally, the above sentence means that $x\in A\cap A'$ and $y\in B\cap B'$ for some $B,B' \subset Y$ such that $A\times B, A'\times B' \in \mathcal{W}$, as desired. □

Remark 2.16. We emphasize some steps in the above proof.

1. *The non-injectivity of the correspondence $A\mapsto B$ (2.2) was a problematic issue in the (sketched) proof for Theorem 2.1. Why the same did not happen with Proposition 2.15?*

   Note that in the previous cases, one has the following implication:

   $$(x,\mathcal{A}_y) \in A\times[A] \Rightarrow \exists B\subset Y \text{ such that } (x,y)\in A\times B \text{ and } A\times B\in\mathcal{W}.$$

   Unlike the Lindelöf case, in the ccc context we just want objects witnessing that the family $\mathcal{W}$ is not an antichain, so the above implication is enough.

2. Similarly as in Theorem 2.1', in Proposition 2.15 it is enough to show that the family $\mathscr{D}_{\mathscr{A}}$ has at least two sets with nonempty intersection.

3. If the subset $\mathcal{W}$ is an antichain of $X\times Y$, then the elements of the family $\mathscr{A}$, as defined in the previous proof, are antichains of X. On the other hand, if the corresponding family $\mathscr{D}_{\mathscr{A}}$ is not an antichain of $X\times\mathscr{A}^\star$, then there exists an $\mathcal{A}\in\mathscr{A}$ that is not an antichain of X.

4. The Pigeonhole Principle allows us to consider just those ccc families $\mathscr{A}\subset\wp(\tau)$ such that $|\bigcup\mathscr{A}| = \aleph_1$.

5. A family $\mathscr{A}\subset\wp(\tau)$ is ccc if and only if for every uncountable family $\mathscr{F}\subset\bigcup_{\mathcal{A}\in\mathscr{A}}[\mathcal{A}]^{<\aleph_0}$ there are $F,G\in\mathscr{F}$ and $\mathcal{A}\in\mathscr{A}$ such that $F\neq G$ and $F\cup G\subset\mathcal{A}$. ■

[10] With the so called **Pigeonhole Principle**. Its classic version states that if n objects (pigeons) must be put into m boxes (pigeonholes), with $m<n$, then at least one box must contain more than one object. However, in this work we use one of its variations for infinite sets: if $\mathcal{A}$ is a countable partition of an uncountable set $\mathcal{S}$, then there is an $A\in\mathcal{A}$ such that A is uncountable.

Putting all these comments together yields the following restatement of Proposition 2.15, where for brevity we call

$$\mathscr{N}(X) := \left\{ \mathscr{A} \subset \wp(\tau) : \left|\bigcup \mathscr{A}\right| = \aleph_1 \text{ and } \forall \mathcal{A} \in \mathscr{A} (\mathcal{A} \text{ is an antichain}) \right\}, \tag{2.6}$$

and we refer to an element $\mathscr{A} \in \mathscr{N}(X)$ as a **nice family of antichains** of X.

Proposition 2.15'. A topological space X is productively ccc if and only if for every nice family of antichains $\mathscr{A}$ there exists an uncountable $\mathscr{F} \subset \bigcup_{\mathcal{A} \in \mathscr{A}} [\mathcal{A}]^{<\aleph_0}$ such that for each pair $F, G \in \mathscr{F}$ with $F \neq G$ there is no $\mathcal{A} \in \mathscr{A}$ such that $F \cup G \subset \mathcal{A}$.

As an example of application, we can prove the following.

Corollary 2.17 (Fremlin [24, Corollary 12J])**.** Every product of separable spaces is productively ccc.

Proof. By using Theorem 2.14, it is enough to verify that separable spaces are productively ccc. So, let D be a countable dense subset of a topological space X, and let $\mathscr{A}$ be a nice family of antichains. Since the set D is countable and dense and $|\bigcup \mathscr{A}| = \aleph_1$, it follows that there are a $d \in D$ and a $\mathcal{C} \subset \bigcup \mathscr{A}$ such that $|\mathcal{C}| = \aleph_1$ and $d \in C$ for all $C \in \mathcal{C}$. Hence, no antichain $\mathcal{A} \in \mathscr{A}$ contains two distinct elements $C, D \in \mathcal{C}$, because $d \in C \cap D$. $\square$

We still may restate Proposition 2.15 in a shorter way, by (explicitly) entering in the realm of *posets*. Recall that a pair $(\mathbb{P}, \leq)$, where $\mathbb{P}$ is a set endowed with a binary relation $\leq$, is said to be a **preordered set**, for short a **poset**, if the relation $\leq$ is *reflexive* and *transitive* – we will write it simply as $\mathbb{P}$ when no confusion can arise. If we have additionally that $\leq$ is also antisymmetric, then $(\mathbb{P}, \leq)$ is called a **partial order**.

In this order context, disjointness of open sets is replaced by incompatibility of elements in the poset. To fix notations, two elements p and q of a poset $\mathbb{P}$ are said to be **compatible** if there exists an $r \in \mathbb{P}$ such that $r \leq p, q$. Naturally, we say that p and q are **incompatible** when they are not compatible, what we abbreviate as $p \perp q$. A subset $\mathcal{A} \subset \mathbb{P}$ of pairwise incompatible elements is said to be an **antichain**.

It is a trivial observation that antichains of nonempty open sets of a topological space (X, τ) are precisely the antichains of the partial ordered set $(\tau \setminus \{\emptyset\}, \subset)$, justifying their coincident names. Thus, it is natural to say that a poset $\mathbb{P}$ is **ccc** if it has no uncountable antichains. This is enough for restating Proposition 2.15 in another short way. In the following we call $\mathfrak{F}(\mathscr{A})$ the partially ordered set

$$\left(\bigcup_{\mathcal{A} \in \mathscr{A}} [\mathcal{A}]^{<\aleph_0}, \leq \right), \tag{2.7}$$

where $\mathscr{A}$ is a nice family of antichains of X and $\leq$ is the reverse inclusion.

Proposition 2.15*. A topological space X is productively ccc if and only if the partial order $\mathfrak{F}(\mathscr{A})$ is not ccc for all $\mathscr{A} \in \mathscr{N}(X)$.

Proof. It is enough to observe that for $p,q \in \mathfrak{F}(\mathscr{A})$,

$$p \perp q \Leftrightarrow \nexists r \in \mathfrak{F}(\mathscr{A})\,(p \cup q \subset r) \Leftrightarrow \nexists \mathcal{A} \in \mathscr{A}\,(p \cup q \subset \mathcal{A}). \qquad \square$$

This last restatement suggests a similar result for *productively ccc orders*. Indeed, for posets $\mathbb{P}$ and $\mathbb{Q}$, the **product preorder** $\mathbb{P} \times \mathbb{Q}$ is defined as the cartesian product of the sets $\mathbb{P}$ and $\mathbb{Q}$ endowed with the preorder

$$(p,q) \leq (p',q') \Leftrightarrow p \leq p' \text{ and } q \leq q'. \tag{2.8}$$

Thus, it makes sense to call a poset $\mathbb{P}$ **productively ccc** if $\mathbb{P} \times \mathbb{Q}$ is a ccc poset for each ccc poset $\mathbb{Q}$. With so many coincident definitions, for a poset $\mathbb{P}$ it is harmless to consider the family $\mathscr{N}(\mathbb{P})$, and for an $\mathscr{A} \in \mathscr{N}(\mathbb{P})$, the partial order $\mathfrak{F}(\mathscr{A})$, both of them defined in the obvious way as the non-topological counterparts of (2.6) and (2.7).

Proposition 2.18. A poset $\mathbb{P}$ is productively ccc if and only if the partial order $\mathfrak{F}(\mathscr{A})$ is not ccc for all $\mathscr{A} \in \mathscr{N}(\mathbb{P})$.

Proof. If $\mathbb{P}$ is productively ccc and $\mathfrak{F}(\mathscr{A})$ is ccc, then $\mathbb{P} \times \mathfrak{F}(\mathscr{A})$ is a ccc poset, hence the following uncountable subset

$$\mathscr{D}_{\mathscr{A}} := \left\{(p,\{p\}) : p \in \bigcup \mathscr{A}\right\} \subset \mathbb{P} \times \mathfrak{F}(\mathscr{A})$$

cannot be an antichain. Thus, there are some $p,p' \in \bigcup \mathscr{A}$ and a pair $(r,F) \in \mathbb{P} \times \mathfrak{F}(\mathscr{A})$ such that $(r,F) \leq (p,\{p\}),(p',\{p'\})$, from which it follows that for an $\mathcal{A} \in \mathscr{A}$ one has $p \not\perp p'$ and $\{p,p'\} \subset F \subset \mathcal{A}$, which contradicts the fact that elements of $\mathscr{A}$ are antichains of $\mathbb{P}$.

Conversely, arguing by contraposition, we take a ccc poset $\mathbb{Q}$ and an uncountable antichain $\mathcal{W} = \{(p_\alpha, q_\alpha) : \alpha < \omega_1\}$ of $\mathbb{P} \times \mathbb{Q}$, and we shall obtain a nice family of antichains $\mathscr{A}$ of $\mathbb{P}$ such that the partial order $\mathfrak{F}(\mathscr{A})$ is ccc.

Now, for each $r \in \mathbb{Q}$, let

$$\mathcal{A}_r := \{p \in \mathbb{P} : \exists q \in \mathbb{Q}\,(r \leq q \text{ and } (p,q) \in \mathcal{W})\},$$

and consider the family $\mathscr{A} := \{\mathcal{A}_r : r \in \mathbb{Q}\}$. One readily verifies that each $\mathcal{A}_r$ is an antichain of $\mathbb{P}$, and by using the Pigeonhole Principle we can see that $\mathscr{A} \in \mathscr{N}(\mathbb{P})$. It remains to show that the partial order $\mathfrak{F}(\mathscr{A})$ is ccc.

For an uncountable subset $\mathscr{F} \subset \mathfrak{F}(\mathscr{A})$, for each $F \in \mathscr{F}$ we take an $r_F \in \mathbb{Q}$ such that $F \subset \mathcal{A}_{r_F}$ and then consider the family $\mathcal{F} := \{r_F : F \in \mathscr{F}\}$. If $|\mathcal{F}| \leq \aleph_0$, then for some pairwise distinct $F,G,H \in \mathscr{F}$ one has $F \cup G \subset \mathcal{A}_{r_H}$, which shows that $\mathscr{F}$ is not an antichain. If $|\mathcal{F}| > \aleph_0$ instead, since $\mathbb{Q}$ is a ccc poset, we obtain some $F,G \in \mathscr{F}$ and an $r \in \mathbb{Q}$ such that $F \neq G$ and $r \leq r_F, r_G$, from which it follows that $F \cup G \subset \mathcal{A}_r$, which implies that $\mathscr{F}$ is not an antichain. $\square$

Remark 2.19. Comfort and Negrepontis [18] attribute to Argyros the following problem.

Question 2.20. Is there a (ccc) space X that can serve as a 'test space' for productively ccc spaces in the following sense: if Y is a space and $X \times Y$ is a ccc space, then Y is productively ccc?

Although our construction does not fully address the above question, it is somewhat related to it, but in a *dual* and "local" sense: Y is productively ccc if and only if $\mathfrak{F}(\mathscr{A})$ is *not* ccc for all $\mathscr{A} \in \mathscr{N}(Y)$. ■

To discuss possible applications for the characterization of productively ccc orders/spaces, we shall return to the surroundings of Martin's Axiom. To do so, we introduce a few more terminologies.

For a poset $\mathbb{P}$ and a natural number $n \in \omega \setminus \{0,1\}$, we say that a subset $A \subset \mathbb{P}$ is ***n*-linked** if for all $F \in [A]^n$ there exists a $p_A \in \mathbb{P}$ such that $p_A \leq p$ for each $p \in F$ – the set A is said to be **centered** whenever A is n-linked for all $n \geq 2$. The following definitions are taken from Todorčević and Veličković's article [95]. We say that the poset $\mathbb{P}$ has property $\mathbf{K}_n$ if

$$\forall A \in [\mathbb{P}]^{\aleph_1} \, \exists A' \in [A]^{\aleph_1} \text{ such that } A' \text{ is } n\text{-linked.} \tag{2.9}$$

The corresponding property to centered subsets is usually called $\aleph_1$-*precaliber*, but we shall refer to it as K_σ: the poset $\mathbb{P}$ has **property $\mathbf{K}_\sigma$** if

$$\forall A \in [\mathbb{P}]^{\aleph_1} \, \exists A' \in [A]^{\aleph_1} \text{ such that } A' \text{ is centered.} \tag{2.10}$$

Now we consider the following assertions:

- $\mathscr{K}_\sigma$: every ccc poset has property K_σ;
- $\mathscr{K}_n$: every ccc poset has property K_n;
- $\mathscr{C}^2$: every ccc poset is productively ccc.

For $n = 2$, it is usual to drop the index 2. So, we refer to property K_2 simply as **property K**, what makes reference to Knaster, the first to treat this kind of property, back in 1941, in the *Scottish Book*[11]. One easily sees that for a poset $\mathbb{P}$ and $n \geq 2$,

$$\mathbb{P} \text{ has property } \mathrm{K}_\sigma \Rightarrow \mathbb{P} \text{ has property } \mathrm{K}_n \Rightarrow \mathbb{P} \text{ has property } \mathrm{K}, \tag{2.11}$$

which can be extended a little further.

[11] There is a published version of the Scottish Book, edited by R. D. Mauldin [53], with some of the history about the book as well as with transcriptions of the original problems, some of which with commentaries made by researchers in the respective fields. In particular, the problem concerning the Knaster property (192) is commented by Mary Ellen Rudin.

Theorem 2.21 (Folklore[12])**.** If a poset $\mathbb{P}$ has property K, then $\mathbb{P}$ is productively ccc.

Proof. It is enough to take a nice family $\mathscr{A}$ of antichains of $\mathbb{P}$ and show that $\mathfrak{F}(\mathscr{A})$ is not ccc. But since $A := \bigcup \mathscr{A} \in [\mathbb{P}]^{\aleph_1}$, property K gives a subset $A' \in [A]^{\aleph_1}$ such that A' is linked. Note that the family $\{\{p\} : p \in A'\}$ witnesses that $\mathfrak{F}(\mathscr{A})$ is not ccc. □

The usual steps to prove that (MA $+\neg$ CH) implies $\mathscr{C}^2$ consist in showing the implications

$$\mathrm{MA}_{\aleph_1} \Rightarrow \mathscr{K}_\sigma \Rightarrow \mathscr{K} \Rightarrow \mathscr{C}^2, \tag{2.12}$$

what naturally suggests the following questions, posed by Larson and Todorčević [50], and still unsolved.

Question 2.22. Does $\mathscr{K}$ imply $\mathrm{MA}_{\aleph_1}$?

Question 2.23. Does $\mathscr{C}^2$ imply $\mathscr{K}$? Does it imply $\mathrm{MA}_{\aleph_1}$?

Remark 2.24. It is important to point out that the above questions *are not* equivalent to their specifications to a particular poset $\mathbb{P}$. For instance, the question "Does $\mathscr{C}^2$ imply $\mathscr{K}$?" is not the same as "Does the productively ccc property imply property K?". The assertion $\mathscr{C}^2$ carries informations about all possible ccc posets, while the assertion "$\mathbb{P}$ is productively ccc" concerns just the poset $\mathbb{P}$.[13] ■

Remark 2.25. One could also ask if $\mathscr{K}_\sigma$ implies $\mathrm{MA}_{\aleph_1}$. This implication is true, and it follows from the following theorem.

Theorem 2.26 (Todorčević and Veličković [95])**.** $\mathrm{MA}_{\aleph_1}$ holds if and only if every uncountable ccc poset has an uncountable centered subset.

Thus, if $\mathscr{K}_\sigma$ holds and $\mathbb{P}$ is an uncountable ccc poset, then there exists an uncountable centered subset of $\mathbb{P}$, showing that $\mathrm{MA}_{\aleph_1}$ holds. ■

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[12] In the Scottish Book this is credited to B. Lance and M. Wiszik (1941).

[13] If one assumes CH, it can be proved that there exists a productively ccc poset $\mathbb{P}$ that does not have property K [100].

2.3 Lindelöfness via closed projections

The choice for closing this chapter with the present section may look a bit odd at first, since one of its (undesirable) immediate consequences is the splitting of our considerations concerning Lindelöfness. Although we deal with hyperspaces in this section, the motivating idea behind our next results is considerably far from the set-theoretical perspective that ruled the previous sections.

Indeed, from now on we shall follow a (shy) *categorical approach*, in which relations between objects are more relevant than the internal structure of the objects. However, we do this just enough to obtain useful informations about the objects[14]. As we mentioned earlier in the Introduction, Escardó's proof [23] for Theorem I.8 triggered the results we shall be treating along this section.

First of all, recall that for a topological space Y and a cardinal κ, we called $(Y)_\kappa$ the set Y endowed with the topology generated by its G_κ-sets (see Remark I.9). In what follows, the lemma bellow will be very useful.

Lemma 2.27. Let $f\colon X \times Y \to Z$ be a function, where X, Y and Z are topological spaces. Then f is closed if and only if the set $\{z \in Z : f^{-1}[\{z\}] \subset W\}$ is open for all open sets $W \subset X \times Y$.

Proof. It follows from the equality $\{z \in Z : f^{-1}[\{z\}] \subset W\} = Z \setminus f[(X \times Y) \setminus W]$. □

Proposition 2.28. Let $\kappa \geq \aleph_0$ be a cardinal. For a topological space X, $\mathrm{L}(X) \leq \kappa$ if and only if the projection $\pi_Y\colon X \times Y \to (Y)_\kappa$ is closed for all topological spaces Y.

Proof. Assume $\mathrm{L}(X) \leq \kappa$. Taking a closed set $C \subset X \times Y$, its complement is an open set that can be written as a union of basic open sets of $X \times Y$, say $\bigcup_{i\in I} U_i \times V_i$. Now, proving that

$$\pi_Y[C] = \{y \in Y : \exists x \in X\,((x,y) \in C)\}$$

is a closed subset of $(Y)_\kappa$ is equivalent to show that its complement is open, which can be done as follows. For $y \in (Y)_\kappa \setminus \pi_Y[C]$, we have $(x,y) \in \bigcup_{i\in I} U_i \times V_i$ for all $x \in X$, which allows us to take an $i_x \in I$ such that $(x,y) \in U_{i_x} \times V_{i_x}$, thus obtaining an open covering $\mathcal{U} = \{U_{i_x} : x \in X\}$ for X. By selecting a subset $\{x_\alpha : \alpha < \kappa\} \subset X$ such that the family $\{U_{i_{x_\alpha}} : \alpha < \kappa\}$ still covers X, it follows that $y \in \bigcap_{\alpha<\kappa} V_{i_{x_\alpha}} \subset (Y)_\kappa \setminus \pi_Y[C]$, as desired.

To prove the converse, it is enough to take an open covering $\mathcal{U}$ of X such that

$$\mathcal{V} \in [\mathcal{U}]^{\leq\kappa} \Rightarrow \bigcup \mathcal{V} \in \mathcal{U},$$

and then show that $X \in \mathcal{U}$.

Now, let τ be the topology of X and consider $\mathscr{E}_{\mathcal{U}}(X)$ as the set τ endowed with the following topology: a nonempty family $\mathscr{W} \subset \mathscr{E}_{\mathcal{U}}(X)$ is open if and only if $\mathscr{W} \cap \mathcal{U} \neq \emptyset$ and $\mathscr{W}$

[14] Of course, we do not present any formal terminology of Category Theory.

is upwards closed. Beyond being a topological space, $\mathscr{E}_{\mathcal{U}}(X)$ is such that every G_κ-set is open. Indeed, let $\{\mathscr{W}_\alpha : \alpha < \kappa\}$ be a family of nonempty open sets of $\mathscr{E}_{\mathcal{U}}(X)$.

- If $U \in \bigcap_{\alpha<\kappa} \mathscr{W}_\alpha$ and $U \subset V \in \tau$, then $V \in \mathscr{W}_\alpha$ for all $\alpha < \kappa$, showing that $\bigcap_{\alpha<\kappa} \mathscr{W}_\alpha$ is upwards closed.
- For each $\alpha < \kappa$ take a $U_\alpha \in \mathcal{U} \cap \mathscr{W}_\alpha$, then $\bigcup_{\alpha<\kappa} U_\alpha \in \mathcal{U}$ witnesses that $\mathcal{U} \cap \bigcap_{\alpha<\kappa} \mathscr{W}_\alpha \neq \emptyset$.

Note that sets of the form $V^\uparrow := \{W \in \mathscr{E}_{\mathcal{U}}(X) : V \subset W\}$, where V ranges over the family $\{V \in \tau : \exists U \in \mathcal{U}\,(V \subset U)\}$, are open subsets of $\mathscr{E}_{\mathcal{U}}(X)$. In particular, the set

$$\mathscr{D}_{\mathcal{U}} := \{(x,V) : x \in V \in \tau\} \subset X \times \mathscr{E}_{\mathcal{U}}(X) \tag{2.13}$$

is open in the space $X \times \mathscr{E}_{\mathcal{U}}(X)$: for if $(x,V) \in \mathscr{D}_{\mathcal{U}}$, then there exists a $U \in \mathcal{U}$ such that $x \in U$, hence $(x,V) \in (V \cap U) \times (V \cap U)^\uparrow \subset \mathscr{D}_{\mathcal{U}}$. By putting $Y = \mathscr{E}_{\mathcal{U}}(X)$, $Z = (Y)_\kappa$, $f = \pi_Y$ and $W = \mathscr{D}_{\mathcal{U}}$ on Lemma 2.27, it follows from the assumption about X that the set

$$\mathcal{P}_{\mathcal{U}} := \{U \in (\mathscr{E}_{\mathcal{U}}(X))_\kappa : X \times \{U\} \subset \mathscr{D}_{\mathcal{U}}\} \tag{2.14}$$

is open in $(\mathscr{E}_{\mathcal{U}}(X))_\kappa$, i.e., $\mathcal{P}_{\mathcal{U}}$ is a union of G_κ-sets of $\mathscr{E}_{\mathcal{U}}(X)$. Since G_κ-sets are open in this space, it follows that $\mathcal{P}_{\mathcal{U}}$ is an open subset of $\mathscr{E}_{\mathcal{U}}(X)$. Also, $\mathcal{P}_{\mathcal{U}} \neq \emptyset$ because $X \in \mathcal{P}_{\mathcal{U}}$. By the way we defined the topology of $\mathscr{E}_{\mathcal{U}}(X)$, it follows that there exists a $U \in \mathcal{U} \cap \mathcal{P}_{\mathcal{U}}$, but we can only have $U = X$, as desired. □

Call **$\mathbf{P}_\kappa$-spaces** those topological spaces in which G_κ-sets are open – for $\kappa = \aleph_0$ we simply say **P-space**[15]. We obtain the following cleaner version of the previous proposition.

Corollary 2.29. Let X be a topological space and let $\kappa \geq \aleph_0$ be a cardinal. Then $\mathrm{L}(X) \leq \kappa$ if and only if the projection $\pi_Y : X \times Y \to Y$ is closed for all P_κ-spaces Y.

Remark 2.30. The careful reader may have noticed that with slightly different definitions we could obtain a generalization of Corollary 2.29 with essentially the same arguments used to prove Proposition 2.28. Indeed, let us say that a space Y is a $\mathbf{P}_{<\kappa}$-space if every G_λ-set of Y is open, for each cardinal $\lambda < \kappa$. Then we have the following.

Proposition 2.31. Let $\kappa \geq \aleph_0$ be a cardinal. For a topological space X, the following are equivalent:

1. for every open covering $\mathcal{U}$ of X there is a subcovering $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| < \kappa$;
2. the projection $\pi_Y : X \times Y \to Y$ is closed for all $\mathrm{P}_{<\kappa}$-spaces Y.

[15] For $\kappa = \aleph_0$, Gillman and Henriksen [29] call pseudo-discrete spaces, P-spaces for short, those Tychonoff spaces on which every continuous real function is constant on some neighborhood of each point of the space, which is equivalent to asking for every G_δ-set to be open. Earlier, Sikorski [83] introduced P_κ-spaces under the name of κ-additive spaces.

Note that the statement above generalizes both Proposition 2.28 and Theorem I.8 simultaneously, since every topological space is a $\mathrm{P}_{<\aleph_0}$-space. ■

Remark 2.32. Similarly, but with a different proof, P-spaces are characterized as those spaces X such that the projection $\pi_Y : X \times Y \to Y$ is closed for all Lindelöf spaces Y. This is Theorem 2.1 in Misra's paper [62]. ■

Corollary 2.33. If X is a compact space, then X is $\leq \kappa$-L-productive for every cardinal $\kappa \geq \aleph_0$.

Proof. Since compositions of closed functions are again closed, the result follows from the commutativity of the diagram

$$\begin{tikzcd} X \times Y \times Z \arrow[d] \arrow[dr] & \\ Y \times Z \arrow[r] & Z \end{tikzcd}$$

where the arrows are the obvious projections, $\mathrm{L}(Y) \leq \kappa$ and Z is a P_κ-space. □

Corollary 2.34. For every pair of topological spaces X and Y, one has $\mathrm{L}(X \times Y) \leq \mathscr{K}(X) \cdot \mathrm{L}(Y)$.

Although there are many ways to prove Theorem I.8, as the reader can see at *n*Lab's article [66], Escardó's general approach[16] has shown to be more suitable for the Lindelöf case afterall. In the proofs for Theorem I.8 and Proposition 2.28, the key step in the converse consists in showing that the point

$$\{X\} = \{U \in \tau : X \times \{U\} \subset \mathscr{D}_{\mathcal{U}}\}$$

is isolated in the **Escardó hyperspace** $\mathscr{E}_{\mathcal{U}}(X)$, yielding the desired membership $X \in \mathcal{U}$. However, this internal machinary is not necessary in order to prove the next proposition, whose arguments are adapted from a proof for the Tychonoff Theorem presented in *n*Lab [66].

Proposition 2.35. If $\{X_m : m \in \omega\}$ is a family of Lindelöf P-spaces, then $\prod_{m\in\omega} X_m$ is Lindelöf.

Proof. We will prove that for a P-space Y, the projection $\pi_0 \colon Y \times \prod_{m\in\omega} X_m \to Y$ is closed. First of all, for each $n \in \omega$ we consider the following commutative diagram with the obvious projections

$$\begin{tikzcd} & Y \times \prod_{m\in\omega} X_m \arrow[dl, "\pi_{n+1}"'] \arrow[d, "\pi_n"] \arrow[dr, "\pi_0"] & \\ Y \times \prod_{m\leq n} X_m \arrow[r, "\pi_n^{n+1}"'] & Y \times \prod_{m<n} X_m \arrow[r, "\pi_0^n"'] & Y \end{tikzcd}$$

In particular, note that the product of finitely many P-spaces is again a P-space. Since each X_m is a Lindelöf P-space, it follows that the projections π_n^{n+1} and π_0^n are closed for each $n \in \omega$.

[16] Credited by him to Leopoldo Nachbin [65].

Now, for a closed subset $C \subset Y \times \prod_{m\in\omega} X_m$, we want to show that $\pi_0[C] = \overline{\pi_0[C]}$. In order to prove this, for each $n \in \omega$, let $C_n := \overline{\pi_n[C]}$ and observe that $C_n \subset \pi_n^{n+1}[C_{n+1}]$.

Indeed, since the above diagram is commutative, we have

$$\pi_n[C] = \pi_n^{n+1}[\pi_{n+1}[C]] \subset \pi_n^{n+1}[C_{n+1}] \Rightarrow C_n := \overline{\pi_n[C]} \subset \overline{\pi_n^{n+1}[C_{n+1}]} = \pi_n^{n+1}[C_{n+1}],$$

where the last equality follows from the closedness of the projection π_n^{n+1}.

Thus, if $y_0 \in \overline{\pi_0[C]}$, then for each $n \in \omega$ there exists an $y_n \in C_n$ such that $\pi_0^n(y_n) = y_0$ and $\pi_n^{n+1}(y_{n+1}) = y_n$ – we are extending y_0 to an n-tuple y_n, one step at a time. It then follows that there exists an $y \in Y \times \prod_{m\in\omega} X_m$ such that $\pi_n(y) = y_n$ for each $n \in \omega$. So it is enough to show that $y \in C$, because then we will have $\overline{\pi_0[C]} \subset \pi_0[C]$.

Now, for a finite subset $F \subset \omega$, there exists an $n \in \omega$ such that $m < n$ for all $m \in F$, hence

$$\pi_F(y) = \pi_F^n(\pi_n(y)) = \pi_F^n(y_n) \in \pi_F^n[C_n] = \pi_F^n[\overline{\pi_n[C]}] \subset \overline{\pi_F^n[\pi_n[C]]} = \overline{\pi_F[C]},$$

where the inclusion follows by the continuity of the projection $\pi_F^n\colon Y \times \prod_{m<n} X_m \to Y \times \prod_{m\in F} X_m$. Finally, the theorem below gives $y \in \overline{C} = C$, as desired. □

Theorem 2.36 (Folklore[17])**.** For a family $\{X_i : i \in I\}$ of topological spaces, a point $y \in X = \prod_{i\in I} X_i$ and a subset $A \subset X$, $y \in \overline{A}$ if $\pi_F(y) \in \overline{\pi_F[A]}$ for each finite subset $F \subset I$, where π_F is the obvious projection.

Corollary 2.37. Lindelöf P-spaces are powerfully Lindelöf.

We emphasize that the above corollary is not new. In fact, Alster [1] defined a class of topological spaces that is both productively Lindelöf and powerfully Lindelöf – this class was named after him by Barr, Kennison and Raphael [13]: a space X is said to be **Alster** if every K-covering of X by G_δ-sets has a countable subcovering.

Theorem 2.38 (Alster [1])**.** Alster spaces are productively Lindelöf and powerfully Lindelöf.

Proof. We provide an alternative proof for the first statement using our characterization of Lindelöfness. It is quite similar to the classical one.

Let A, L and P be an Alster space, a Lindelöf space and a P-space, respectively. We want to show that the projection $\pi_P\colon A \times L \times P \to P$ is closed. For brevity, we call $X = A \times L \times P$. Let $F \subset X$ be closed and take a $p \notin \pi_P[F]$. For a compact subset $K \subset A$ and a point $l \in L$, one has

$$K \times \{l\} \times \{p\} \subset X \setminus F.$$

It follows from the Tube Lemma that there are open subsets $U_{K,l} \subset A$, $V_{K,l} \subset L$ and $W_{K,l} \subset P$ such that $K \times \{l\} \times \{p\} \subset U_{K,l} \times V_{K,l} \times W_{K,l} \subset X \setminus F$.

[17] I learned this theorem in nLab's article [66], as a technical step of the proof for Tychonoff Theorem with the closed projection characterization of compactness.

Making the point l range over L gives an open covering $\{V_{K,l} : l \in L\}$ for L, from which we obtain a countable subset $L_K \subset L$ such that $\{V_{K,l} : l \in L_K\} \in \mathcal{O}_L$. When K ranges over the compact subsets of A, the G_δ-sets of the form $U_K = \bigcap\{U_{K,l} : l \in L_K\}$ define a K-covering for A by G_δ-sets, from which we obtain a countable family $\mathcal{K}$ of compact subsets of A such that $\{U_K : K \in \mathcal{K}\}$ covers A. Finally, $W = \bigcap\{W_{K,l} : (K,l) \in \bigcup_{K\in\mathcal{K}} \mathcal{K} \times L_K\}$ is a G_δ-set of P such that $p \in W \subset P \setminus \pi_P[F]$. □

Since Lindelöf P-spaces are clearly Alster, Corollary 2.37 follows from the above theorem. But comparing all these results, a few natural questions arise.

Question 2.39. Is the product of countably many productively Lindelöf spaces a Lindelöf space?

Question 2.40. Is the product of countably many Alster spaces a Lindelöf space?

Question 2.41. Is there a characterization for Alster spaces in terms of (closed) projections?

One could also ask for a generalization of Proposition 2.35 for uncountable families of P-spaces, but it cannot be done. In fact, the following theorem puts a countable bound in any hope of this kind.

Theorem 2.42 (Folklore[18])**.** For a Hausdorff space X, the following are equivalent:

1. X is compact;
2. X^κ is a Lindelöf space for all cardinals $\kappa \geq \aleph_0$;
3. X^{ω_1} is Lindelöf.

Even if one ignores the above theorem, a generalization of Proposition 2.35 for uncountably many P-spaces would require that products of countably many P-spaces to be P-spaces. But, as pointed out by Misra [62], $2 = \{0,1\}$ is clearly a P-space, but every point in 2^ω is a non-isolated (point) G_δ-set.

[18] A proof for this result may be found in an answer by Hamkins on mathoverflow [32].

GLOSSARY

For the convenience of the reader, we display below a list with the main definitions and results that we assumed to be known along this work. In the electronic version of this thesis, the following entries are linked ("back and forth") with their respective first appearances in the text, if there is one.

Cardinal function: a correspondence ψ between the class of topological spaces and the class of (transfinite) cardinals, such that $\psi(X) = \psi(Y)$ whenever X and Y are homeomorphic topological spaces.

Character: at a point p, is the smallest cardinality of a local base at p; the character of the space is the supremum of the local characters.

Compact space: a space in which every open covering has a finite subcovering.

Compact-open topology: is the topology on $C(X)$ whose basic open sets are finite intersections of sets of the form $[K,V] := \{f \in C(X) : f[K] \subset V\}$, where K and V range over the compact subsets of X and the open sets of X, respectively; see also Example 1.23.

Continuum Hypothesis: in ZFC (see below), is the statement $2^{\aleph_0} = \aleph_1$.

Δ-System Lemma: for every uncountable family $\mathscr{F}$ of finite sets there are an uncountable subfamily $\mathscr{F}' \subset \mathscr{F}$ and a finite set Δ satisfying the equality $F \cap G = \Delta$ for each $F, G \in \mathscr{F}'$ such that $F \neq G$; in Jech's book [38], this result is attributed to Shanin [82].

First countable space: a space in which each of its points has a countable local base.

Homogeneous space: a space X such that for each $x, y \in X$ there exists a homeomorphism $h\colon X \to X$ with $h(x) = y$.

Ideal: of subsets of a set X is a collection $\mathcal{I}$ of subsets of X such that $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and $A \subset B$ with $B \in \mathcal{I}$ implies $A \in \mathcal{I}$. It is the dual notion of a filter (see page 55).

Lindelöf degree: denoted as L, is the smallest cardinal $\kappa \geq \aleph_0$ such that every open covering $\mathcal{U}$ has a subcovering $\mathcal{V}$ such that $|\mathcal{V}| \leq \kappa$.

Lindelöf space: a space in which every open covering has a countable subcovering, i.e., a space with countable Lindelöf degree.

Martin's Axiom: for a poset $\mathbb{P}$, $\mathrm{MA}_\kappa(\mathbb{P})$ denotes the assertion "for each family $\mathcal{D}$ of maximal antichains of $\mathbb{P}$ such that $|\mathcal{D}| \leq \kappa$, there is a filter $\mathcal{F}$ on $\mathbb{P}$ intercepting every member of $\mathcal{D}$". Martin's Axiom is the assertion "for each ccc poset $\mathbb{P}$ and for each $\kappa < 2^{\aleph_0}$, $\mathrm{MA}_\kappa(\mathbb{P})$ holds".

Normal space: a T_1-space that is also T_4.

Nowhere dense set: a set whose closure has empty interior.

Paracompact space: a space in which every open covering has a locally finite open refinement (see below), i.e., a refinement $\mathscr{U}$ such that each point of the space is in a neighborhood intersecting at most finitely many elements of $\mathscr{U}$.

Product topology: for a family $\mathcal{F} = \{X_i : i \in I\}$ of topological spaces, the (standard) product topology on the set $\prod \mathcal{F} = \prod_{i \in I} X_i$ is the coarsest (smallest) topology making the projections $\pi_i \colon \prod \mathcal{F} \to X_i$ continuous for all $i \in I$; equivalently, it is the topology on $\prod \mathcal{F}$ whose sub-basic open sets are of the form $\pi_i^{-1}[V]$ for each open set $V \subset X_i$, for all $i \in I$.

Refinement: if $\mathscr{U}$ and $\mathscr{V}$ are collections of subsets of a set $\mathscr{S}$, we say that $\mathscr{U}$ refines $\mathscr{V}$ if for all $V \in \mathscr{V}$ there exists a $U \in \mathscr{U}$ such that $U \subset V$.

Regular space: a space that is T_1 and T_3.

Separable space: a space with a countable dense subset.

σ-compact space: a countable union of compact spaces.

Sorgenfrey's line: the set $\mathcal{R}$ of all real numbers with the topology induced by sets of the form $[a,b) := \{c \in \mathcal{R} : a \leq c < b\}$.

Suslin line: it is a ccc partial ordered set $\mathbb{P}$ that is *linear* (every pair of elements is comparable in the order), *dense* (for every $p, q \in \mathbb{P}$ such that $p < q$ there exists a $r \in \mathbb{P}$ such that $p < r < q$) but it is not separable in the topological sense, i.e., when $\mathbb{P}$ is endowed with the usual order topology; particularly, the existence of a Suslin line is independent of ZFC.

T_0-space: a space such that for each pair of points there exists an open set that contains one of these points and not the other.

T_1-space: a space in which finite subsets are closed.

T_2-space (or Hausdorff space): a space in which every pair of distinct points may be separated by disjoint neighborhoods.

T_3-space: a space in which points and closed sets are separated by disjoint open sets, i.e., if p is a point of the space and P is a closed subset of the space such that $p \notin P$, then there are disjoint open sets U and V such that $p \in U$ and $P \subset V$.

$T_{3\frac{1}{2}}$-space: a space where points and closed sets are separated by continuous functions, i.e., if F is a closed set not containing a point p, then there is a continuous real function f such that $f \restriction F \equiv 1$ and $f(p) = 0$.

T_4-space: a space where disjoint closed sets are separated by disjoint open sets.

Tightness: for a point $x \in X$, the tightness of X at x, denoted as $\mathrm{t}(x, X)$, is the least cardinal $\kappa \geq \aleph_0$ with the property that for every subset $A \subset X$ such that $x \in \overline{A}$, there exists a $B \subset A$

such that $|B| \leq \kappa$ and $x \in \overline{B}$; the tightness of X, denoted as $\mathrm{t}(X)$, is the cardinal $\sup_{x \in X} \mathrm{t}(x, X)$. Alternatively, see Example 1.53.

Topological group: a group $(G, +)$ with a topology such that the operation $+$ and the function $g \mapsto g^{-1}$ are both continuous.

Topological sum: of a family $\mathcal{F} := \{(X_i, \tau_i) : i \in I\}$ of topological spaces, is the disjoint union $\bigsqcup_{i \in I} X_i$ with the topology generated by $\bigsqcup_{i \in I} \tau_i$.

Tube Lemma: if K and L are compact subsets of topological spaces X and Y, respectively, such that there exists an open set $O \subset X \times Y$ with $K \times L \subset O$, then there are open sets $U \subset X$ and $V \subset Y$ such that $K \times L \subset U \times V \subset O$. For the proof (of a general version) we refer the reader to Engelking [22, Theorem 3.2.10, p. 140].

Tychonoff space (or completely regular): a space simultaneously T_1 and $\mathrm{T}_{3\frac{1}{2}}$.

Urysohn's Lemma: a topological space is T_4 if and only if every pair of disjoint closed subsets can be separated by a continuous real function.

Weight: the smallest cardinality of a base of a topological space.

ZFC: for Zermelo-Fraenkel-Choice, the standard axioms used to formalize Set Theory.

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