A unifying approach to isotropic and radial positive definite kernels

Jean Carlo Guella

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Jean Carlo Guella

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RESUMO

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Neste trabalho, nós generalizamos três resultados famosos obtidos por Schoenberg:

- I) a caracterização dos núcleos contínuos isotrópicos positivos definidos em esferas reais;
- II) a caracterização dos núcleos contínuos radiais positivos definidos em espaços Euclidianos;
- III) a caracterização dos núcleos contínuos radiais condicionalmente negativos definidos em espaços Euclidianos.

A partir destas novas abordagens, reobtemos vários resultados da literatura assim como obtemos novos. Com a exceção de S^1 e $\mathbb{R}$, obtemos condições necessárias e suficientes para que estes núcleos sejam estritamente positivos definidos e estritamente condicionalmente negativos definidos.

Palavras-chave: Núcleos Isotrópicos em esferas, Núcleos Radiais em espaços Euclidianos, Núcleos positivos definidos, Núcleos condicionalmente negativos definidos, Núcleos estritamente positivos definidos.

ABSTRACT

GUELLA, J. C. **A unifying approach to isotropic and radial positive definite kernels.** 2019. 117 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2019.

In this work, we generalize three famous results obtained by Schoenberg:

- I) the characterization of the continuous positive definite isotropic kernels defined on a real sphere;
- II) the characterization of the continuous positive definite radial kernels defined on an Euclidean space;
- III) the characterization of the continuous conditionally negative radial kernels defined on an Euclidean space.

From this new approach, we reobtain several results in the literature and obtain some new ones as well. With the exception of S^1 and $\mathbb{R}$, we obtain necessary and sufficient conditions in order that these kernels be strictly positive definite and strictly conditionally negative definite.

Keywords: Isotropic kernel on spheres, Radial kernels on Euclidean spaces, Positive definite kernels, Conditionally negative definite kernels, Strictly positive definite kernels.

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INTRODUCTION

Positive definite kernels and their various analogues and generalizations have arisen in different parts of mathematics since the beginning of the past century. Although, the first major result on the subject dates back to nineteenth century, the Gram representation Theorem for positive semidefinite matrices.

One of the most prolific period in the development of the subject was during 1930's and 1940's. In 1933 Bochner proved a characterization for the space of continuous and positive definite kernels on Euclidean spaces that are invariant by translation. His result is of great importance, once it provides a strong connection between positive definite kernels on Euclidean spaces and the Fourier transform. Between 1938 to 1942, Schoenberg published three key papers on the subject:

([SCHOENBERG, 1938a](#)), where it was proved a characterization for the continuous and positive definite radial kernels on Euclidean spaces. A characterization of the continuous conditionally negative definite radial kernels on Euclidean spaces was partially proved.

([VON-NEUMANN; SCHOENBERG, 1941](#)), where the proof of the continuous conditionally negative definite radial kernels on Euclidean spaces was fully proved.

([SCHOENBERG, 1942](#)), where it was proved a characterization for the continuous and positive definite isotropic kernels on real spheres.

The articles ([SCHOENBERG, 1937](#)) and ([SCHOENBERG, 1938b](#)) also contain some relevant aspects of the theory, but the focus in these papers was almost entirely on the problem of isometric embeddings of metric spaces. In ([ARONSZAJN, 1950](#)) was proved a characterization for an arbitrary positive definite kernel, similar to the Gram representation Theorem for positive semidefinite matrices.

For more information about the history of the subject see ([STEWART, 1976](#)) and ([FASSHAEUER, 2011](#)).

Schoenberg's results and positive definite kernels appear in many applications, such as coding theory and semidefinite programming (BACHOC, 2009); kernel methods in machine learning (SCHOLKOPF; SMOLA, 2001), (STEINWART; CHRISTMANN, 2008); probability (SRIPERUMBUDUR *et al.*, 2010); geostatistics (DE-IACO *et al.*, 2018), (PORCU; BEVILACQUA; GENTON, 2016); approximation theory (WENDLAND, 2005), (CHENEY; LIGHT, 2009), (FASSHAEUER, 2007).

The contributions of this dissertation are:

In Chapter 2 we provide a connection between matrix valued (strictly) positive definite kernels and (strictly) positive definite kernels on Cartesian products of spaces. A similar connection is proved for operator valued (strictly) positive definite kernels.

In Chapter 3 we provide a generalization of Schoenberg's characterization for continuous and positive definite isotropic kernels on the real unit sphere S^d of $\mathbb{R}^{d+1}$. Precisely, if X is a non empty set, with no topology in principle, we characterize those functions $f : X \times X \times [-1, 1] \rightarrow \mathbb{C}$, for which $f(u, v, \cdot) : [-1, 1] \rightarrow \mathbb{C}$ is a continuous function for each fixed pair $(u, v) \in X^2$, and the kernel

$$(X \times S^d) \times (X \times S^d) \rightarrow f(u, v, x \cdot y) \in \mathbb{C},$$

is positive definite. When $d \geq 2$, we also provide necessary and sufficient conditions for those kernels being strictly positive definite. We analyze this characterization for some specific choices of the set X and reobtain some results found in the literature via this unifying setting.

In Chapter 4 we provide a generalization of Schoenberg's characterization for the continuous and positive definite radial kernels on Euclidean spaces. Precisely, if X is a non-empty set, with no topology in principle, we characterize those functions $f : X \times X \times [0, \infty) \rightarrow \mathbb{C}$, for which $f(u, v, \cdot) : [0, \infty) \rightarrow \mathbb{C}$ is a continuous function for each fixed pair $(u, v) \in X^2$, and the kernel

$$(X \times \mathbb{R}^d) \times (X \times \mathbb{R}^d) \rightarrow f(u, v, \|x - y\|) \in \mathbb{C},$$

is positive definite. When $d \geq 2$, we also provide necessary and sufficient conditions in order that the kernels be strictly positive definite. We analyze this characterization for some specific choices of the set X and reobtain some results found in the literature via this unifying setting.

In Chapter 5 we extend Schoenberg's characterization for the continuous and conditionally negative definite radial kernels on Euclidean spaces to a matrix valued context. When $d \geq 2$, we also provide a characterization for those matrix valued kernels being strictly conditionally negative definite.

POSITIVE DEFINITE KERNELS

In this introductory chapter, Section 2.1 will be focused on the concepts of positive semidefinite and positive definite matrices to be used in the study of positive definite and strictly positive definite scalar valued kernels in Section 2.2. In Section 2.3 we extrapolate this concepts to the matrix valued kernels and discuss some of their properties, mainly those to be used in subsequent chapters. We finish the chapter in Section 2.4, by providing a connection between operator valued (strictly) positive definite kernels and scalar valued (strictly) positive definite kernels on product of spaces.

The bridge theorems in Section 2.4 were obtained by the author. So far, they are unpublished.

2.1 Matrix analysis results

Throughout the text, we will write $M_n(\mathbb{R})$ to denote the set of all square matrices of order n possessing real entries while $M_n(\mathbb{C})$ will denote the set of all square matrices of order n with complex entries. The main reference for this section is ([HORN; JOHNSON, 2012](#)).

Definition 2.1.1. A symmetric matrix $A = [a_{\mu,\nu}]_{\mu,\nu=1}^n \in M_n(\mathbb{R})$ is called **positive semidefinite** if for every real scalars $c_1, c_2, \dots, c_n \in \mathbb{R}$

$$cAc^t = \sum_{\mu,\nu=1}^n c_\mu c_\nu a_{\mu,\nu} \geq 0,$$

where $c = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$. In addition, if $cAc^t > 0$ for all $c \in \mathbb{R}^n \setminus \{0\}$, then we say that the matrix A is **positive definite**.

A very similar notion exists for complex valued matrices.

Definition 2.1.2. A matrix $A = [a_{\mu,\nu}]_{\mu,\nu=1}^n \in M_n(\mathbb{C})$ is called **positive semidefinite** if for every complex scalars $c_1, c_2, \dots, c_n \in \mathbb{C}$

$$cAc^* = \sum_{\mu,\nu=1}^n c_\mu \overline{c_\nu} a_{\mu,\nu} \geq 0,$$

where $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^n$. In addition, if $cAc^* > 0$ for all $c \in \mathbb{C}^n \setminus \{0\}$, then we say that the matrix A is **positive definite**.

It is clear from these definitions that a positive definite matrix is automatically a positive semidefinite matrix.

Below, we show how the definitions above are related:

Proposition 2.1.3. Let $A = [a_{\mu,\nu}]_{\mu,\nu=1}^n \in M_n(\mathbb{C})$.

- (i) If A is positive semidefinite according to Definition 2.1.2, then the matrix A is Hermitian, that is, $A = A^*$.
- (ii) If A has real entries and is symmetric, then it is positive semidefinite (definite) according to Definition 2.1.1 if and only if A is positive semidefinite (definite) according to Definition 2.1.2.

Proof. Let $\mu, \nu \in \{1, 2, \dots, n\}$ with $\mu \neq \nu$. By setting $\{e_1, e_2, \dots, e_n\}$ as the canonical basis of $\mathbb{R}^n$ and applying specific vectors $c \in \mathbb{C}^n$ in Definition 2.1.2, yield the inequalities

$$\begin{aligned} a_{\mu,\mu} &\geq 0, \quad c = e_\mu; \quad a_{\nu,\nu} \geq 0, \quad c = e_\nu \\ a_{\mu,\mu} + a_{\nu,\nu} + a_{\mu,\nu} + a_{\nu,\mu} &\geq 0, \quad c = e_\mu + e_\nu \\ a_{\mu,\mu} + a_{\nu,\nu} - ia_{\mu,\nu} + ia_{\nu,\mu} &\geq 0, \quad c = e_\mu + ie_\nu. \end{aligned}$$

So $a_{\mu,\nu} + a_{\nu,\mu}$ and $-ia_{\mu,\nu} + ia_{\nu,\mu}$ are real numbers, and that can only happen if $a_{\mu,\nu} = \overline{a_{\nu,\mu}}$. This settles (i).

In order to prove (ii), notice that for a real valued and symmetric matrix $A = [a_{\mu,\nu}]_{\mu,\nu=1}^n$,

$$\begin{aligned} \sum_{\mu,\nu=1}^n c_\mu \overline{c_\nu} a_{\mu,\nu} &= \sum_{\mu,\nu=1}^n (c_\mu^1 + ic_\mu^2)(c_\nu^1 - ic_\nu^2) a_{\mu,\nu} \\ &= \sum_{\mu,\nu=1}^n c_\mu^1 c_\nu^1 a_{\mu,\nu} + \sum_{\mu,\nu=1}^n c_\mu^2 c_\nu^2 a_{\mu,\nu} + i \left(- \sum_{\mu,\nu=1}^n c_\mu^1 c_\nu^2 a_{\mu,\nu} + \sum_{\mu,\nu=1}^n c_\mu^2 c_\nu^1 a_{\mu,\nu} \right) \\ &= \sum_{\mu,\nu=1}^n c_\mu^1 c_\nu^1 a_{\mu,\nu} + \sum_{\mu,\nu=1}^n c_\mu^2 c_\nu^2 a_{\mu,\nu}, \end{aligned}$$

where $c_\mu = c_\mu^1 + ic_\mu^2$. The equivalence follows directly from this relation. $\square$

It is worth mentioning that, in the real valued case, we require that the matrix A is symmetric not only because of the consequences of the previous proposition, but also because for every anti-symmetric real valued matrix $B = [b_{i,j}]_{i,j=1}^m$, the double sum $\sum_{i,j=1}^m d_i d_j b_{i,j} = 0$

for whichever $d_1, d_2, \dots, d_m \in \mathbb{R}$. From now on, a positive semidefinite (definite) matrix will always refer to the Definition 2.1.2 and if the matrix has real entries, either its symmetry will be imposed, or will be obtained from the context.

Below, we prove the Gram representation theorem for a positive definite matrix. Although being a simple result, it is a crucial property for the development of the subject. The following classical theorem about diagonalization of matrices will be needed (HORN; JOHNSON, 2012), page 135.

Theorem 2.1.4. Let A be a Hermitian matrix from $M_n(\mathbb{C})$. The following assertions hold:

- (i) A has n real eigenvalues;
- (ii) There exists a unitary matrix U in $M_n(\mathbb{C})$ and a diagonal matrix D in $M_n(\mathbb{R})$, whose entries of which are the eigenvalues of A (counting multiplicities), for which $A = UDU^*$;
- (iii) If the matrix A in (ii) belongs to $M_n(\mathbb{R})$, then the matrix U in the equality $A = UDU^*$ can be taken from $M_n(\mathbb{R})$.

Throughout, we will write “ $\cdot$ ” to denote the usual inner product in $\mathbb{C}^n$.

Theorem 2.1.5. Let $A = [a_{\mu, \nu}]_{\mu, \nu=1}^n \in M_n(\mathbb{C})$ be a positive semidefinite matrix. There exist $x_1, x_2, \dots, x_n \in \mathbb{C}^n$ such that

$$a_{\mu, \nu} = x_\mu \cdot x_\nu,$$

for all $1 \leq \mu, \nu \leq n$. Conversely, for every $m \in \mathbb{N}$ and $x_1, x_2, \dots, x_n \in \mathbb{C}^m$, the matrix $[x_\mu \cdot x_\nu]_{\mu, \nu=1}^n \in M_n(\mathbb{C})$ is positive semidefinite.

In addition, if the matrix $A = [a_{\mu, \nu}]_{\mu, \nu=1}^n$ has real entries, then we can take the vectors $x_1, x_2, \dots, x_n$ in $\mathbb{R}^n$, and the converse holds true for vectors in $\mathbb{R}^m$.

Proof. By Proposition 2.1.3, if A is positive semidefinite then the matrix A is Hermitian, so we can use Theorem 2.1.4 to deduce that $A = UDU^*$, where D is a diagonal matrix and U is unitary. Notice that if λ is an eigenvalue of A , then $\lambda \geq 0$, because if $Av^t = \lambda v^t$ then $\bar{\lambda} \|v\|^2 = v\bar{A}v^* = vA^t v^* \geq 0$, since A^t is positive semidefinite as well.

If $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, then $D = BB$, where $B = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}) \in M_n(\mathbb{R})$. So

$$A = UDU^* = UBBU^* = [UB][UB]^*$$

However, if we write $x_\mu \in \mathbb{C}^n$ to denote the μ -th line of UB , then $[UB]^* = [x_1^*, x_2^*, \dots, x_n^*]$ and we finally have that

$$A = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} [x_1^*, x_2^*, \dots, x_n^*] = [x_\mu x_\nu^*]_{\mu, \nu=1}^n = [x_\mu \cdot x_\nu]_{\mu, \nu=1}^n.$$

This takes care of (i). Conversely, if $A = [x_\mu \cdot x_\nu]_{\mu,\nu=1}^n \in M_n(\mathbb{C})$, where $x_1, x_2, \dots, x_n \in \mathbb{C}^m$, the matrix A is positive semidefinite because

$$\sum_{\mu,\nu=1}^n c_\mu \overline{c_\nu} a_{\mu,\nu} = \left(\sum_{\mu=1}^n c_\mu x_\mu \right) \cdot \left(\sum_{\mu=1}^n c_\mu x_\mu \right) \geq 0,$$

In case the matrix A has real entries, Theorem 2.1.4 implies that we can take U as a real matrix. But then the matrix UB also is a real matrix, implying that its lines, which are the vectors $x_1, x_2, \dots, x_n$, are in $\mathbb{R}^n$. The converse follows from the same arguments of the complex case. $\square$

A **principal minor** of a matrix $A = [a_{\mu,\nu}]_{\mu,\nu=1}^n \in M_n(\mathbb{C})$, is a matrix $D = [d_{\alpha,\beta}]_{\alpha,\beta=1}^l \in M_l(\mathbb{C})$, where $l \leq n$ and $d_{\alpha,\beta} = a_{\sigma(\alpha),\sigma(\beta)}$ for all $\alpha, \beta \in \{1, 2, \dots, l\}$, in which $1 \leq \sigma(1) < \sigma(2) < \dots < \sigma(l) \leq n$. A consequence of the previous theorem is the following. The proof is omitted.

Corollary 2.1.6. Let $A = [a_{\mu,\nu}]_{\mu,\nu=1}^n \in M_n(\mathbb{C})$ be a positive semidefinite matrix. The following assertions hold:

- (i) $a_{\mu,\mu} \geq 0$ for all $\mu \in \{1, 2, \dots, n\}$.
- (ii) $a_{\mu,\nu} = \overline{a_{\nu,\mu}}$ for all $\mu, \nu \in \{1, 2, \dots, n\}$.
- (iii) $|a_{\mu,\nu}|^2 \leq a_{\mu,\mu} a_{\nu,\nu}$ for all $\mu, \nu \in \{1, 2, \dots, n\}$.
- (iv) $|a_{\mu,\nu}|^2 < a_{\mu,\mu} a_{\nu,\nu}$ for all $\mu \neq \nu$ if A is positive definite.
- (v) The matrix A is positive definite if and only if the vectors in its Gram representation are linearly independent in $\mathbb{C}^n$.
- (vi) A is positive definite if and only if it is invertible and positive semidefinite.
- (vii) Every principal minor A is a positive semidefinite.
- (viii) If A is positive definite, then every principal minor of the matrix A is positive definite and conversely.

Now, we introduce two operations between matrices, the Kronecker and the Schur products.

Definition 2.1.7. Let $A = [a_{\mu,\nu}]_{\mu,\nu=1}^n \in M_n(\mathbb{C})$ and $B = [b_{i,j}]_{i,j=1}^m \in M_m(\mathbb{C})$.

- (i) The **Kronecker product** of A and B is the matrix $A \otimes B$ in $M_{nm}(\mathbb{C})$, defined by

$$A \otimes B = [a_{\mu,\nu} B]_{\mu,\nu=1}^n = [[a_{\mu,\nu} b_{i,j}]_{i,j=1}^m]_{\mu,\nu=1}^n.$$

- (ii) If $m = n$, the **Schur product** of A and B is the matrix $A \bullet B$ from $M_n(\mathbb{C})$, defined by

$$A \bullet B = [a_{\mu,\nu} b_{\mu,\nu}]_{\mu,\nu=1}^n.$$

Proposition 2.1.8. Let $A = [a_{\mu,\nu}]_{\mu,\nu=1}^n \in M_n(\mathbb{C})$ and $B = [b_{i,j}]_{i,j=1}^m \in M_m(\mathbb{C})$ be positive semidefinite (definite) matrices. Then $A \otimes B$ is a positive semidefinite (definite) matrix.

If $m = n$, $A \bullet B$ and $aA + bB$ are positive semidefinite (definite) matrices for every positive real numbers a and b with $ab > 0$.

Proof. Being A and B positive semidefinite matrices, Theorem 2.1.5 implies that there exist $x_1, x_2, \dots, x_n \in \mathbb{C}^n$ and $y_1, y_2, \dots, y_m \in \mathbb{C}^m$, such that

$$A = [a_{\mu, \nu}]_{\mu, \nu=1}^n = [x_\mu \cdot x_\nu]_{\mu, \nu=1}^n, \quad B = [b_{i, j}]_{i, j=1}^m = [y_i \cdot y_j]_{i, j=1}^m$$

Define vectors $z_r \in (\mathbb{C}^n)^m = \mathbb{C}^{nm}$, by

$$z_r = (x_\mu(y_i)_1, x_\mu(y_i)_2, \dots, x_\mu(y_i)_m),$$

for $1 \leq r \leq nm$, where $\mu \in \{1, 2, \dots, n\}$ and $i \in \{1, 2, \dots, m\}$ are uniquely determined by the relation $r = (\mu - 1)m + i$. Now, if $s = (\nu - 1)m + j$ and $r = (\mu - 1)m + i$ are arbitrary numbers in $\{1, 2, \dots, nm\}$, then

$$z_r \cdot z_s = \sum_{l=1}^m (y_i)_l (y_j)_l x_\mu \cdot x_\nu = (y_i \cdot y_j)(x_\mu \cdot x_\nu) = [A \otimes B]_{r, s}.$$

Hence, $A \otimes B$ is positive semidefinite. If the matrices A and B are positive definite, the proof that the matrix $A \otimes B$ is positive definite will be postponed to the Lemma 2.1.9 (with $W = \mathbb{C}^n$ in that setting), which will be proved in a more general context.

Finally, if $m = n$, the properties of $aA + bB$ are checked directly through the definition of a positive semidefinite (definite) matrix. As for the Schur product $A \bullet B$, just notice that $A \bullet B$ is a principal minor of $A \otimes B$, in this case, $\sigma(\mu) = (\mu - 1)m + \mu$, $1 \leq \mu \leq m$. $\square$

As a matter of fact, it can be proved that if A and B are nonzero matrices and $A \otimes B$ is a positive semidefinite (definite) matrix, then there exists a unique $\theta \in [0, 2\pi)$ such that $e^{i\theta}A$ and $e^{-i\theta}B$ are positive semidefinite (definite) matrices. On the other hand, if $A \bullet B$ is positive semidefinite (definite), not necessarily A or B will be both positive semidefinite (definite).

The final theorem of this section will play a key role when obtaining sufficient conditions for strictly positive definite kernels on the subsequent chapters. It will be stated in a more generic version than we have dealt so far, a new point of view on positive definiteness to be made clear in Chapter 5.

Lemma 2.1.9. Let $B = [b_{i, j}]_{i, j=1}^m \in M_m(\mathbb{C})$ be a positive semidefinite matrix, $A = [a_{\mu, \nu}]_{\mu, \nu=1}^n \in M_n(\mathbb{C})$ and W a linear subspace of $\mathbb{C}^n$. If A is positive semidefinite in W , that is,

$$\sum_{\mu, \nu=1}^n u_\mu \overline{u_\nu} a_{\mu, \nu} \geq 0,$$

for all $u = (u_1, u_2, \dots, u_n) \in W \setminus \{0\}$, then

$$\sum_{i, j=1}^m \sum_{\mu, \nu=1}^n c_{\mu, i} \overline{c_{\nu, j}} a_{\mu, \nu} b_{i, j} \geq 0,$$

whenever $c_i = (c_{1, i}, c_{2, i}, \dots, c_{n, i}) \in W$, $i = 1, 2, \dots, m$. Further, if A is positive definite in W , then the inequality above is an equality if and only if

$$\sum_{i, j=1}^m c_{\mu, i} \overline{c_{\mu, j}} b_{i, j} = 0,$$

for all $1 \leq \mu \leq n$.

Proof. By the Gram representation in Theorem 2.1.5, $b_{i,j} = z_i \cdot z_j$, where $z_1, z_2, \dots, z_m \in \mathbb{C}^m$. The double double sum in the statement of the lemma takes the form

$$\begin{aligned} \sum_{i,j=1}^m \sum_{\mu,\nu=1}^n c_{\mu,i} \overline{c_{\nu,j}} a_{\mu,\nu} b_{i,j} &= \sum_{i,j=1}^m \sum_{\mu,\nu=1}^n c_{\mu,i} \overline{c_{\nu,j}} a_{\mu,\nu} \sum_{l=1}^m (z_i)_l \overline{(z_j)_l} \\ &= \sum_{l=1}^m \sum_{i,j=1}^m \sum_{\mu,\nu=1}^n c_{\mu,i} \overline{c_{\nu,j}} a_{\mu,\nu} (z_i)_l \overline{(z_j)_l} \\ &= \sum_{l=1}^m \sum_{\mu,\nu=1}^n \left(\sum_{i=1}^m c_{\mu,i} (z_i)_l \right) \overline{\left(\sum_{j=1}^m c_{\nu,j} (z_j)_l \right)} a_{\mu,\nu}. \end{aligned}$$

Since the vectors

$$\left(\sum_{i=1}^m c_{1,i} (z_i)_l, \sum_{i=1}^m c_{2,i} (z_i)_l, \dots, \sum_{i=1}^m c_{n,i} (z_i)_l \right) = \sum_{i=1}^n (z_i)_l (c_{1,i}, c_{2,i}, \dots, c_{n,i})$$

belong to W for all $l = 1, 2, \dots, m$ and A is positive semidefinite in W , then

$$\sum_{i,j=1}^m \sum_{\mu,\nu=1}^n c_{\mu,i} \overline{c_{\nu,j}} a_{\mu,\nu} b_{i,j} \geq 0.$$

As for the final assertion of the lemma, if A is positive definite in W , then

$$0 = \sum_{i,j=1}^m \sum_{\mu,\nu=1}^n c_{\mu,i} \overline{c_{\nu,j}} a_{\mu,\nu} b_{i,j} = \sum_{l=1}^m \sum_{\mu,\nu=1}^n \left(\sum_{i=1}^m c_{\mu,i} (z_i)_l \right) \overline{\left(\sum_{j=1}^m c_{\nu,j} (z_j)_l \right)} a_{\mu,\nu}$$

implies that $\sum_{i=1}^m c_{\mu,i} (z_i)_l = 0$ for all $\mu \in \{1, 2, \dots, n\}$ and $l \in \{1, 2, \dots, m\}$. However

$$\sum_{i,j=1}^m c_{\mu,i} \overline{c_{\mu,j}} b_{i,j} = \sum_{l=1}^m \left(\sum_{i=1}^m c_{\mu,i} (z_i)_l \right) \overline{\left(\sum_{j=1}^m c_{\mu,j} (z_j)_l \right)} = 0,$$

for all $\mu \in \{1, 2, \dots, n\}$. Conversely, if $\sum_{i,j=1}^m c_{\mu,i} \overline{c_{\mu,j}} b_{i,j} = 0$, from the previous equation we obtain that $\sum_{i=1}^m c_{\mu,i} (z_i)_l = 0$ for all $\mu \in \{1, 2, \dots, n\}$ and $l \in \{1, 2, \dots, m\}$, and then

$$\sum_{i,j=1}^m \sum_{\mu,\nu=1}^n c_{\mu,i} \overline{c_{\nu,j}} a_{\mu,\nu} b_{i,j} = 0.$$

The proof is complete. $\square$

2.2 Positive definite kernels

Let X be a nonempty set. A function K mapping a product space $X \times X$ into $\mathbb{C}$ is called a **kernel**. If its values are matrices $M_n(C)$ or elements of $L(\mathcal{H})$ (the space of continuous linear operators from the Hilbert space $\mathcal{H}$ to itself), then K is called a **matrix valued kernel** and an **operator valued kernel**, respectively.

Definition 2.2.1. A kernel $K : X \times X \rightarrow \mathbb{C}$ is **positive definite** if for every finite quantity of distinct points $x_1, x_2, \dots, x_n \in X$, the matrix

$$[K(x_\mu, x_\nu)]_{\mu, \nu=1}^n$$

is positive semidefinite. In addition, if the above matrices are always positive definite, the kernel K is **strictly positive definite**.

There is a small conflict between this terminology and the one in matrix analysis, but they are well established in the literature.

The matrix $[K(x_\mu, x_\nu)]_{\mu, \nu=1}^n$ is usually called the **interpolation matrix** at the points $x_1, x_2, \dots, x_n$ through the kernel K . When the points $x_1, x_2, \dots, x_n \in X$ and the kernel $K : X \times X \rightarrow \mathbb{C}$ are clear from the context, we may simplify the notation by calling $[K(x_\mu, x_\nu)]_{\mu, \nu=1}^n$ an interpolation matrix.

A direct consequence of this definition and Corollary 2.1.6 is:

Corollary 2.2.2. Let $K : X \times X \rightarrow \mathbb{C}$ be a positive definite kernel. The following assertions hold:

- (i) $K(x, x) \geq 0$ for all $x \in X$.
- (ii) $K(x, y) = \overline{K(y, x)}$ for all $x, y \in X$.
- (iii) $|K(x, y)|^2 \leq K(x, x)K(y, y)$ for all $x \neq y \in X$.
- (iv) $2|K(x, y)| \leq K(x, x) + K(y, y)$ for all $x \neq y \in X$.

In addition, if K is strictly positive definite, all the inequalities above are strict.

A natural generalization of the Kronecker and Schur products of matrices are defined below.

Definition 2.2.3. Let $K_1 : X \times X \rightarrow \mathbb{C}$ and $K_2 : Y \times Y \rightarrow \mathbb{C}$ be kernels.

(i) The **Kronecker product** of K_1 and K_2 is the kernel $K_1 \otimes K_2 : (X \times Y) \times (X \times Y) \rightarrow \mathbb{C}$ defined by

$$[K_1 \otimes K_2]((x_1, y_1), (x_2, y_2)) = K_1(x_1, x_2)K_2(y_1, y_2).$$

(ii) If $X = Y$, the **Schur product** of K_1 and K_2 is the kernel $K_1 \bullet K_2 : X \times X \rightarrow \mathbb{C}$ defined by

$$[K_1 \bullet K_2](x_1, x_2) = K_1(x_1, x_2)K_2(x_1, x_2).$$

Similar conclusions as those in Proposition 2.1.8 are valid. The proof is omitted.

Proposition 2.2.4. Let $K_1 : X \times X \rightarrow \mathbb{C}$ and $K_2 : Y \times Y \rightarrow \mathbb{C}$ be kernels. If K_1 and K_2 are (strictly) positive definite kernels then the following assertions hold:

- (i) The kernel $K_1 \otimes K_2$ is a (strictly) positive definite kernel.
- (ii) If $X = Y$, the kernel $K_1 \bullet K_2$ is a (strictly) positive definite kernel, and for every positive real numbers a and b , the kernel $aK_1 + bK_2$ is also a (strictly) positive definite kernel.

Again, it can be proved that if K_1 and K_2 are nonzero kernels and $K_1 \otimes K_2$ is a (strictly) positive definite kernel, then there exists a unique $\theta \in [0, 2\pi)$ such that $e^{i\theta}K_1$ and $e^{-i\theta}K_2$ are (strictly) positive definite kernels. On the other hand, if $K_1 \bullet K_2$ is a (strictly) positive definite kernel, not necessarily K_1 or K_2 is (strictly) positive definite.

Below, we prove a simple property that will be useful when dealing with kernels on a product of spaces.

Lemma 2.2.5. Let $K : (X \times Y) \times (X \times Y) \rightarrow \mathbb{C}$ be a kernel. Then K is (strictly) positive definite if and only if for every finite quantity of distinct points $x_1, x_2, \dots, x_n \in X$ and distinct points $y_1, y_2, \dots, y_m \in Y$, the following matrix in $M_{mn}(\mathbb{C})$

$$[[K((x_\mu, y_i), (x_\nu, y_j))]_{i,j=1}^m]_{\mu,\nu=1}^n$$

is positive semidefinite (definite).

Proof. Notice that the matrix in the statement of the theorem is the interpolation matrix at the mn distinct points

$$(x_1, y_1), (x_1, y_2), \dots, (x_1, y_m), (x_2, y_1), (x_2, y_2), \dots, (x_2, y_m), \dots, (x_n, y_1), (x_n, y_2), \dots, (x_n, y_m)$$

on $X \times Y$ through the kernel K . So if K is a (strictly) positive definite kernel, this matrix has the desirable property.

Conversely, if $(z_1, w_1), (z_2, w_2), \dots, (z_p, w_p)$ are distinct points in $X \times Y$, define $x_1, x_2, \dots, x_n \in X$, as the points that encompasses the distinct elements among $z_1, z_2, \dots, z_p \in X$. Similarly, define $y_1, y_2, \dots, y_m \in Y$, as the points that encompasses the distinct elements among $w_1, w_2, \dots, w_p \in Y$. In particular, $1 \leq n, m \leq p$, and the interpolation matrix $[K((z_k, w_k), (z_l, w_l))]_{k,l=1}^p$ is a principal minor of $[[K((x_\mu, y_i), (x_\nu, y_j))]_{i,j=1}^m]_{\mu,\nu=1}^n$ (up to equal changes in lines and columns). Thus, it is a positive semidefinite (definite) matrix. $\square$

Corollary 2.2.6. Let $K_1 : X \times X \rightarrow \mathbb{C}$ and $K_2 : Y \times Y \rightarrow \mathbb{C}$ be positive definite kernels. The kernel $K : (X \times Y) \times (X \times Y) \rightarrow \mathbb{C}$, defined by

$$K((x_1, y_1), (x_2, y_2)) = K_1(x_1, x_2) + K_2(y_1, y_2)$$

is positive definite. Further, the kernel K is not strictly positive definite when both sets X and Y contain more than one point.

Proof. We will use the notation 1_n to define the matrix in $M_n(\mathbb{C})$, $n \in \mathbb{N}$ where all entries have value 1. By Lemma 2.2.5,

$$\begin{aligned} [[K((x_\mu, y_i), (x_\nu, y_j))]_{i,j=1}^m]_{\mu,\nu=1}^n &= [[K_1(x_\mu, x_\nu) + K_2(y_i, y_j)]_{i,j=1}^m]_{\mu,\nu=1}^n \\ &= [[K_1(x_\mu, x_\nu)]_{i,j=1}^m]_{\mu,\nu=1}^n + [[K_2(y_i, y_j)]_{i,j=1}^m]_{\mu,\nu=1}^n \\ &= 1_m \otimes [K_1(x_\mu, x_\nu)]_{\mu,\nu=1}^n + [K_2(y_i, y_j)]_{i,j=1}^m \otimes 1_n. \end{aligned}$$

Since the matrices $1_m, 1_n, [K_1(x_\mu, x_\nu)]_{\mu, \nu=1}^n$ and $[K_2(y_i, y_j)]_{i, j=1}^m$ are positive semidefinite, Proposition 2.1.8 implies that the interpolation matrix $[[K((x_\mu, y_i), (x_\nu, y_j))]_{i, j=1}^m]_{\mu, \nu=1}^n$ is positive semidefinite. Thus K is a positive definite kernel.

If $x_1 \neq x_2 \in X$ and $y_1 \neq y_2 \in Y$, defining $c_{i,j} = (-1)^{i+j} = (-1)^i(-1)^j$, $i, j \in \{1, 2\}$, then

$$\begin{aligned} \sum_{i,j=1}^2 \sum_{\mu, \nu=1}^2 c_{\mu,i} c_{\nu,j} [K_1(x_\mu, x_\nu) + K_2(y_i, y_j)] &= \sum_{\mu, \nu=1}^2 (c_{\mu,1} + c_{\mu,2})(c_{\nu,1} + c_{\nu,2}) K_1(x_\mu, x_\nu) \\ &+ \sum_{i,j=1}^2 (c_{1,i} + c_{2,i})(c_{1,j} + c_{2,j}) K_2(y_i, y_j) = 0. \end{aligned}$$

Hence, the kernel K is not strictly positive definite. $\square$

The final theorem of this section states that the set of positive definite kernels is closed under pointwise convergence. The proof is omitted.

Proposition 2.2.7. Let $K_n : X \times X \rightarrow \mathbb{C}$, $n \in \mathbb{N}$, be a sequence of positive definite kernels for which the limit

$$K(x, y) := \lim_{n \rightarrow \infty} K_n(x, y),$$

exists for every $x, y \in X$. Then the kernel K is positive definite.

Theorem 2.1.5 has a generalization to positive definite kernels. If $K : X \times X \rightarrow \mathbb{C}$ is a positive definite kernel, there exists a complex Hilbert space $\mathcal{H}$, that is a subspace of the space of complex valued functions defined in X , such that

$$K(x, y) = \langle \phi_x, \phi_y \rangle,$$

for all $x, y \in X$ and, if $f \in \mathcal{H}$

$$f(x) = \langle f, K(\cdot, x) \rangle,$$

for all $x \in X$. To be more precise, $\phi_x(z) = K(z, x)$ for all $z \in X$. This outstanding result, proved in (ARONSZAJN, 1950) is widely used in applications (SCHOLKOPF; SMOLA, 2001), (FASSHAEUER, 2007).

2.3 Matrix valued positive definite kernels

Definition 2.3.1. A matrix valued kernel $K : X \times X \rightarrow M_l(\mathbb{C})$ is called a **positive definite** if for every finite quantity of distinct points $x_1, x_2, \dots, x_n \in X$, the matrix

$$[K(x_\mu, x_\nu)]_{\mu, \nu=1}^n \in M_{nl}(\mathbb{C})$$

is positive semidefinite. If in addition, the above matrices are always positive definite, the kernel K is called **strictly positive definite**.

Now, we connect the definition of a matrix valued kernel with kernels in a product of spaces.

Proposition 2.3.2. If the matrix valued kernel $K : X \times X \rightarrow M_l(\mathbb{C})$ is (strictly) positive definite, then the kernel $H : (X \times \{1, 2, \dots, l\}) \times (X \times \{1, 2, \dots, l\}) \rightarrow \mathbb{C}$ defined by

$$H((x, i), (y, j)) := K_{i,j}(x, y), \quad x, y \in X, \quad i, j \in \{1, 2, \dots, l\}$$

is (strictly) positive definite.

Proof. Given $x_1, x_2, \dots, x_n$ distinct points in X and $y_1, y_2, \dots, y_m$ distinct points in $\{1, 2, \dots, l\}$ (in particular, $m \leq l$), the matrix $[[H((x_\mu, y_i), (x_\nu, y_j))]_{i,j=1}^m]_{\mu,\nu=1}^n$ is a principal minor of the matrix $[K(x_\mu, x_\nu)]_{\mu,\nu=1}^n$ (up to equal changes in lines and columns). If the kernel K is (strictly) positive definite, then the interpolation matrix $[[H((x_\mu, y_i), (x_\nu, y_j))]_{i,j=1}^m]_{\mu,\nu=1}^n$ is positive semidefinite (definite). Hence by Lemma 2.2.5, the kernel H is (strictly) positive definite. $\square$

The converse is also true.

Proposition 2.3.3. If the kernel $H : (X \times \{1, 2, \dots, l\}) \times (X \times \{1, 2, \dots, l\}) \rightarrow \mathbb{C}$ is (strictly) positive definite, then the matrix valued kernel $K : X \times X \rightarrow M_l(\mathbb{C})$, defined by

$$K_{i,j}(x, y) := H((x, i), (y, j)),$$

for all $x, y \in X$ and $i, j \in \{1, 2, \dots, l\}$ is (strictly) positive definite.

Proof. If $x_1, x_2, \dots, x_n$ are distinct points in X , the matrix $[K(x_\mu, x_\nu)]_{\mu,\nu=1}^n$ is identical to the matrix $[[H((x_\mu, i), (x_\nu, j))]_{i,j=1}^l]_{\mu,\nu=1}^n$. So, if the kernel H is (strictly) positive definite the kernel, K is (strictly) positive definite. $\square$

Propositions 2.3.2 and 2.3.3 will be referred to as the **matrix bridge**. The name bridge is used, because they establish a correspondence between (strictly) positive definite scalar valued kernels in $X \times \{1, 2, \dots, l\}$ and $M_l(\mathbb{C})$ valued (strictly) positive definite kernels.

Next, we present a property of matrix valued kernels to be used in the main characterizations for positive definite and strictly positive definite kernels in Chapter 3 (isotropic kernels on real spheres) and Chapter 4 (radial kernels on Euclidean spaces).

Proposition 2.3.4. Let $K : X \times X \rightarrow M_l(\mathbb{C})$ be a (strictly) positive definite kernel. Then for every $u \in \mathbb{C}^l \setminus \{0\}$, the scalar valued kernel $K_u : X \times X \rightarrow \mathbb{C}$, defined by

$$K_u(x, y) := uK(x, y)u^*,$$

for all $x, y \in X$, is (strictly) positive definite.

Proof. Given $u \in \mathbb{C}^l \setminus \{0\}$, distinct points $x_1, x_2, \dots, x_n$ in X , complex numbers $c_1, c_2, \dots, c_n$, we have that

$$\sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} K_u(x_\mu, x_\nu) = \sum_{\mu, \nu=1}^n (c_\mu u) K(x_\mu, x_\nu) (c_\nu u)^* \geq 0,$$

then the kernel K_u is positive definite. If at least one of the complex numbers c_μ is nonzero, then at least one of the vectors $c_\mu u$ is nonzero, which implies that if K is strictly positive definite, K_u also is strictly positive definite for all $u \in \mathbb{C}^l \setminus \{0\}$. $\square$

The converse of Proposition 2.3.4 is not true in general.

Example 2.3.5. Let X be a nonempty set with at least 2 elements. Given two functions $f : X \rightarrow \mathbb{R}$ and $h : X \rightarrow \mathbb{R}$, define the kernel $K : X \times X \rightarrow M_2(\mathbb{C})$ by

$$K(x, y) = \begin{pmatrix} 2f(x)f(y) & f(x)h(y) + h(x)f(y) \\ f(x)h(y) + h(x)f(y) & 2h(x)h(y) \end{pmatrix}$$

For every $u = (c_1, c_2) \in \mathbb{C}^2 \setminus \{0\}$, we have that

$$\begin{aligned} K_u(x, y) &= 2|c_1|^2 f(x)f(y) + (c_1 \overline{c_2} + c_2 \overline{c_1})(f(x)h(y) + h(x)f(y)) + 2|c_2|^2 h(x)h(y) \\ &= [c_1 f(x) + c_2 h(x)] \overline{[c_1 f(y) + c_2 h(y)]} + [c_1 f(y) + c_2 h(y)] \overline{[c_1 f(x) + c_2 h(x)]}. \end{aligned}$$

Thus, the kernel K_u is positive definite. If $x_1, x_2 \in X$ are distinct points, write $a = f(x_1)$, $b = f(x_2)$, $c = h(x_1)$ and $d = h(x_2)$. Clearly,

$$\begin{pmatrix} K(x_1, x_1) & K(x_1, x_2) \\ K(x_2, x_1) & K(x_2, x_2) \end{pmatrix} = \begin{pmatrix} 2a^2 & 2ac & 2ab & ad + bc \\ 2ac & 2c^2 & ad + bc & 2cd \\ 2ab & ad + bc & 2b^2 & 2bd \\ ad + bc & 2cd & 2bd & 2d^2 \end{pmatrix}.$$

Choosing the functions f and h and the points x_1 and x_2 in such a way that $b = c = d = 1$ and $a = 2$, the principal minor

$$\begin{pmatrix} 2c^2 & ad + bc \\ ad + bc & 2b^2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

is not a positive semidefinite matrix. Hence, the matrix valued kernel K is not positive definite, while the scalar valued kernel K_u is positive definite for all $u \in \mathbb{C}^2 \setminus \{0\}$. A similar example works on $M_l(\mathbb{C})$, $l \geq 3$, by placing 0 in all remaining positions.

If the set X is endowed with a topology and the space of continuous functions from X to $\mathbb{R}$ includes a function that is not constant, small adjustments in the previous example yields an example where the kernel K is continuous (by that we mean each coordinate function is continuous). If the topology in X is such that the space of continuous functions from X to $\mathbb{R}$ contains only the constant functions, then any continuous matrix valued kernel is necessarily a constant function. In this case the converse of Proposition 2.3.4 is valid.

2.4 Operator valued positive definite kernels

Definition 2.4.1. Let $\mathcal{H}$ be a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. An operator valued kernel $K : X \times X \rightarrow L(\mathcal{H})$ is called **positive definite** if for every finite quantity of distinct points $x_1, x_2, \dots, x_n \in X$, and every vectors $u_1, u_2, \dots, u_n \in \mathcal{H}$

$$\sum_{\mu, \nu=1}^n \langle K(x_\mu, x_\nu) u_\mu, u_\nu \rangle \geq 0.$$

If, in addition, the inequality above is an equality only when the vectors are all zero, the kernel K is called **strictly positive definite**.

We intend to connect the definition of an operator valued kernel with kernels on products of spaces. In order to do that, fix a **Hamel basis** $(v_\xi)_{\xi \in \Lambda}$ of $\mathcal{H}$. Hence, every vector u in $\mathcal{H}$ is uniquely representable as a finite linear combination of the vectors v_ξ , $\xi \in \Lambda$. Next, define $P : \Lambda \times \Lambda \rightarrow \mathbb{C}$ by the formula

$$P(\xi, \tau) = \langle v_\tau, v_\xi \rangle, \quad \xi, \tau \in \Lambda.$$

Observe that this kernel is strictly positive definite. Although the theory of reproducing kernel Hilbert spaces is not the focus of this text, an interesting property of the kernel P is that its reproducing kernel is isomorphically isometric to the space $\mathcal{H}$.

Theorem 2.4.2. Let $K : X \times X \rightarrow L(\mathcal{H})$ be a (strictly) positive definite operator valued kernel. Define the scalar valued kernel H on $X \times \Lambda$ by

$$H((x, \xi), (y, \tau)) := \langle K(x, y) v_\tau, v_\xi \rangle,$$

for every $x, y \in X$ and $\xi, \tau \in \Lambda$. Then the kernel H is (strictly) positive definite. Further, for every $x \in X$, there exists a constant $M_x \geq 0$ such that the kernels

$$(\xi, \tau) \in \Lambda \times \Lambda \rightarrow M_x P(\xi, \tau) - H((x, \xi), (x, \tau)) \in \mathbb{C}$$

$$(\xi, \tau) \in \Lambda \times \Lambda \rightarrow M_x P(\xi, \tau) + H((x, \xi), (x, \tau)) \in \mathbb{C}$$

are positive definite.

Proof. Let $x_1, x_2, \dots, x_n$ be distinct points in X , $\xi_1, \xi_2, \dots, \xi_m$ be distinct points in Λ and nm scalars $c_{\mu, i} \in \mathbb{C}$, $1 \leq \mu \leq n$, $1 \leq i \leq m$. Define the vectors $u_\mu = \sum_{i=1}^m c_{\mu, i} v_{\xi_i}$, $1 \leq \mu \leq n$. By the hypothesis, it follows that

$$\sum_{\mu, \nu=1}^n \sum_{i, j=1}^m c_{\mu, i} \overline{c_{\nu, j}} H((x_\mu, \xi_i), (x_\nu, \xi_j)) = \sum_{\mu, \nu=1}^n \langle K(x_\mu, x_\nu) u_\mu, u_\nu \rangle \geq 0.$$

Thus, H is positive definite by Lemma 2.2.5. If at least one of the scalars $c_{\mu, i}$ is nonzero, then at least one of the vectors u_μ is nonzero. Thus, if K is strictly positive definite so is H by Lemma

2.2.5.

Since $K(x, x) \in L(\mathcal{H})$ for every $x \in X$, there exists a positive constant M_x such that $|\langle K(x, x)u, u \rangle| \leq M_x \langle u, u \rangle$ for every $u \in \mathcal{H}$. But $\langle K(x, x)u, u \rangle$ is a real number for all $u \in \mathcal{H}$, whence the inequalities

$$M_x \langle v, v \rangle - \langle K(x, x)v, v \rangle \geq 0$$

and

$$M_x \langle v, v \rangle + \langle K(x, x)v, v \rangle \geq 0$$

for all $v \in \mathcal{H}$ hold, but this corresponds to the positive definiteness of the two kernels described in the statement of the theorem. $\square$

The converse is also true.

Theorem 2.4.3. Let $H : (X \times \Lambda) \times (X \times \Lambda) \rightarrow \mathbb{C}$ be a positive definite kernel. Assume that for every $x \in X$ there exists a positive constant M_x , such that the kernels

$$(\xi, \tau) \in \Lambda \times \Lambda \rightarrow M_x P(\xi, \tau) - H((x, \xi), (x, \tau)) \in \mathbb{C}$$

$$(\xi, \tau) \in \Lambda \times \Lambda \rightarrow M_x P(\xi, \tau) + H((x, \xi), (x, \tau)) \in \mathbb{C}$$

are positive definite. Thus, the operator valued kernel $K : X \times X \rightarrow L(\mathcal{H})$ defined by the property that if $u = \sum_{i=1}^n a_i v_{\xi_i}$ and $v = \sum_{j=1}^m b_j v_{\tau_j}$ are vectors in $\mathcal{H}$ then

$$\langle K(x, y)u, v \rangle := \sum_{i=1}^n \sum_{j=1}^m a_i \overline{b_j} H((x, \xi_i), (y, \tau_j)),$$

is well defined and positive definite. If H is strictly positive definite, then so is K .

Proof. First, notice that $K(x, y)$ can always be defined, even without these kernels assumptions, but it will only be a sesquilinear form on $\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$, not necessarily continuous. It is well defined because all elements in $\mathcal{H}$ are described uniquely in terms of the Hamel basis $(v_\xi)_{\xi \in \Lambda}$. If $x_1, x_2, \dots, x_n$ are distinct points in X and $u_1, u_2, \dots, u_n$ are vectors in $\mathcal{H}$, then $u_\mu = \sum_{i=1}^m c_{\mu, i} v_{\xi_i}$, $1 \leq \mu \leq n$, for some distinct points $\xi_1, \xi_2, \dots, \xi_n$ in Λ and scalars $c_{\mu, i} \in \mathbb{C}$. So

$$\sum_{\mu, \nu=1}^n \langle K(x_\mu, x_\nu)u_\mu, u_\nu \rangle = \sum_{\mu, \nu=1}^n \sum_{i, j=1}^m c_{\mu, i} \overline{c_{\nu, j}} H((x_\mu, \xi_i), (x_\nu, \xi_j)) \geq 0,$$

because H is positive definite.

If H is strictly positive definite and at least one of the vectors u_μ is nonzero, then at least one of the complex numbers $c_{\mu, i}$ is nonzero. This leads to a strict inequality in the previous equation, so that K would be strictly positive definite too.

We now prove that the image of K lies, indeed, in $L(\mathcal{H})$. If $x, y \in X$, our assumptions along with the positive definiteness of K , imply that the matrix

$$\begin{bmatrix} \langle K(x, x)u, u \rangle & \langle K(x, y)u, v \rangle \\ \langle K(y, x)v, u \rangle & \langle K(y, y)v, v \rangle \end{bmatrix}$$

is positive semidefinite. In particular, since

$$|\langle K(x, y)u, v \rangle|^2 \leq \langle K(x, x)u, u \rangle \langle K(y, y)v, v \rangle \leq M_x M_y \langle u, u \rangle \langle v, v \rangle$$

$$K(x, y) \in L(\mathcal{H}) \text{ and } \|K(x, y)\| \leq \sqrt{M_x M_y}.$$

□

Theorems 2.4.2 and 2.4.3 will be referred to as the **operator bridge**.

Remark 2.4.4. The use of a Hamel basis in order to know whether a given kernel is (strictly) positive definite will usually lead to nowhere, since the Hamel basis is ultimately an “abstract” concept (it is a standard exercise in functional analysis books like (BREZIS, 2010), to show that such basis is uncountable when the dimension of $\mathcal{H}$ is infinite). From this point of view, a standard orthonormal basis would be preferable. If we employ this type of basis instead, Theorem 2.4.2 is true modulus some few changes in its proof. The same is true for Theorem 2.4.3, except for the strict positive definiteness assertion.

The next example will show that Theorem 2.4.3, under the setting of orthonormal basis rather than Hamel basis, the kernel H can be strictly positive definite while the kernel K is not.

Example 2.4.5. Let $x_0 = (1, 1/2, 1/3, 1/4, \dots) \in l^2(\mathbb{N})$, and $x_n = e_n$, the canonical orthonormal basis for $l^2(\mathbb{N})$. The kernel $H : \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \mathbb{R}$ given by

$$H(\mu, \nu) = \langle x_\mu, x_\nu \rangle, \quad \mu, \nu \in \mathbb{Z}_+,$$

is a kernel that satisfies Theorem 2.4.3. It is strictly positive definite (X in this case is a set with only one element, that is the reason why the kernel H is not explicitly defined in a product of spaces). In particular, $M_x = \langle x_0, x_0 \rangle$, $\{x\} = X$, but if $u = (1, -1, -1/2, -1/3, -1/4, \dots)$

$$\langle K(x, x)u, u \rangle = \langle x_0, x_0 \rangle + 2 \sum_{\mu=1}^{\infty} \frac{-1}{\mu} \langle x_0, x_\mu \rangle + \sum_{\mu, \nu=1}^{\infty} \frac{-1}{\mu} \frac{-1}{\nu} \langle x_\mu, x_\nu \rangle = 0.$$

POSITIVE DEFINITE ISOTROPIC KERNELS

In this chapter, we specialize the theory of positive definite kernels to the continuous positive definite isotropic kernels on real spheres. In Section 3.1, we provide two different methods for characterizing such kernels, stating the main arguments of the proofs. In Section 3.2, we characterize the continuous strictly positive definite isotropic kernels on real spheres. In Section 3.3, we generalize the characterization in Section 3.1 to kernels defined on the product space $X \times S^d$, where X is a nonempty set with no topology in principle, that are isotropic in the sphere variable and have a certain type of continuity. In Section 3.4, we present a characterization for the strictly positive definite kernels among those presented in Section 3.3, except for the case S^1 , which remains open.

The results obtained by the author presented in this chapter were published in (GUELLA; MENEGATTO, 2018c), (GUELLA; MENEGATTO; PERON, 2016b), (GUELLA; MENEGATTO; PORCU, 2018), (GUELLA; MENEGATTO; PERON, 2016a), (GUELLA; MENEGATTO, 2016). The main reference is the first article.

3.1 Schoenberg characterization of positive definite isotropic kernels

In this section, we present the Schoenberg characterization for the continuous positive definite isotropic kernels on real spheres. Two different methods for the proof of such characterization are given. The main references for this section are (DAI; XU, 2013) and (SCHOENBERG, 1942).

Definition 3.1.1. Let $d \in \mathbb{N}$ and $K : S^d \times S^d \rightarrow \mathbb{C}$ be a kernel. The kernel K is **isotropic**, or more precisely, rotationally invariant if for every orthogonal matrix $Q \in O(d+1)$ the following equality is satisfied:

$$K(Qx, Qy) = K(x, y),$$

for all $x, y \in S^d$.

It is possible to describe an isotropic kernel on a real sphere in a simpler way.

Lemma 3.1.2. If $d \in \mathbb{N}$ and $K : S^d \times S^d \rightarrow \mathbb{C}$ is an isotropic kernel, then there exists a function $f : [-1, 1] \rightarrow \mathbb{C}$ such that $K(x, y) = f(x \cdot y)$ for all $x, y \in S^d$. Moreover, the kernel K is continuous if and only if the function f is continuous.

Proof. Let $x, y, z, w \in S^d$ where $x \cdot y = z \cdot w$. From the properties of orthogonal matrices there exists a matrix $Q \in O(d+1)$, with $Qx = z$ and $Qy = w$. Take then $f : [-1, 1] \rightarrow \mathbb{C}$, where

$$f(t) := K((1, 0, \dots, 0), (t, \sqrt{1-t^2}, 0, \dots, 0)).$$

The Cauchy-Schwarz inequality yields that $|x \cdot y| \leq 1$ for all $x, y \in S^d$. For $x, y \in S^d$, letting $x \cdot y = s \in [-1, 1]$ and taking $Q \in O(d+1)$ where $Qx = (1, 0, \dots, 0)$ and $Qy = (s, \sqrt{1-s^2}, 0, \dots, 0)$, then

$$f(x \cdot y) = K((1, 0, \dots, 0), (x \cdot y, \sqrt{1-(x \cdot y)^2}, 0, \dots, 0)) = K(Qx, Qy) = K(x, y).$$

If f is continuous, the last equation yields the continuity on K . Now if K is continuous, the definition of f induces its continuity, because $t \in [-1, 1] \rightarrow ((1, 0, \dots, 0), (t, \sqrt{1-t^2}, 0, \dots, 0)) \in S^d \times S^d$ is continuous. $\square$

Corollary 2.2.2 implies that a positive definite isotropic kernel on S^d is necessarily real valued. Indeed

$$K(x, y) = f(x \cdot y) = f(y \cdot x) = K(y, x) = \overline{K(x, y)},$$

for all $x, y \in S^d$.

Lemma 3.1.2 implies that to characterize the continuous and (strictly) positive definite isotropic kernels on a sphere S^d is equivalent to characterize the set of continuous functions $f : [-1, 1] \rightarrow \mathbb{R}$ that define a (strictly) positive definite isotropic kernel on S^d by $(x, y) \in S^d \times S^d \rightarrow f(x \cdot y) \in \mathbb{R}$. In 1942, Schoenberg in his celebrated paper (SCHOENBERG, 1942), proved a characterization of these functions.

Theorem 3.1.3. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function and $d \in \mathbb{N}$. The kernel

$$(x, y) \in S^d \times S^d \rightarrow f(x \cdot y) \in \mathbb{R}$$

is positive definite if and only if

$$f(t) = \sum_{k=0}^{\infty} a_k^d(f) P_k^d(t),$$

for all $t \in [-1, 1]$, where P_k^d is the usual Gegenbauer polynomial of degree k attached to the number $(d-1)/2$, the coefficients a_k^d are nonnegative, and the series converges at $t = 1$. Moreover, the representation is unique and the coefficients $a_k^d(f)$ are described by the equation

$$a_k^d(f) = C(k, d) \int_{-1}^1 f(s) P_k^d(s) (1-s^2)^{(d-2)/2} ds,$$

for all $k \in \mathbb{Z}_+$, where

$$C(k, d) := \left[\int_{-1}^1 [P_k^d(s)]^2 (1-s^2)^{(d-2)/2} ds \right]^{-1}.$$

The proof of Theorem 3.1.3 can be found in (DAI; XU, 2013) and (SCHOENBERG, 1942). The main references for Gegenbauer polynomials (sometimes called ultraspherical polynomials) in this dissertation are (SZEGŐ, 1975) and (DAI; XU, 2013). It is worth mentioning that the Gegenbauer polynomials in (SZEGŐ, 1975) are denoted as P_k^λ , which is exactly the same polynomial C_k^λ used in (DAI; XU, 2013), but very different from our notation P_k^d , which satisfies $C_k^{(d-1)/2} = P_k^d$. The notation $P_k^d(t)$ in this dissertation is being used in order to simplify the writing.

By the invariance of the orthogonal group $O(d+1)$ with respect to the surface measure ρ of S^d , it is possible to prove, (DAI; XU, 2013) page 412, that for every continuous function $g : [-1, 1] \rightarrow \mathbb{R}$ and $y \in S^d$, we have

$$\frac{1}{\text{vol}(S^d)} \int_{S^d} g(x \cdot y) d\rho(x) = C(0, d) \int_{-1}^1 g(t) (1-t^2)^{(d-2)/2} dt, \quad (3.1)$$

where the constant $\text{vol}(S^d) := 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$ is the usual total mass of the surface measure ρ in S^d . On the other hand, the polynomial P_k^d has degree k and they are orthogonal with respect to the weight function $(1-t^2)^{(d-2)/2}$ in $[-1, 1]$, meaning,

$$\begin{aligned} \int_{-1}^1 P_k^d(t) P_l^d(t) (1-t^2)^{(d-2)/2} dt &= 0, \quad k \neq l \\ \int_{-1}^1 [P_k^d(t)]^2 (1-t^2)^{(d-2)/2} dt &> 0, \quad k \in \mathbb{Z}_+. \end{aligned} \quad (3.2)$$

They also satisfy $P_0^d = 1$ for all $d \in \mathbb{N}$, $P_1^d(t) = (d-1)t$ for $d \geq 2$, $P_1^1(t) = t$, and the 3-term recurrence relation

$$\begin{aligned} P_{k+2}^d(t) &= \frac{1}{k+2} (2k+d+1) t P_{k+1}^d(t) - (k+d-1) P_k^d(t), \quad d \geq 2, \\ P_{k+2}^1(t) &= 2t P_{k+1}^1(t) - P_k^1(t). \end{aligned} \quad (3.3)$$

With the help of Equation (3.2) and Equation (3.3), it is possible to prove the generating function formula, (SZEGŐ, 1975) page 83

$$\begin{aligned} \frac{1}{(1-2zt+z^2)^{(d-1)/2}} &= \sum_{k=0}^{\infty} P_k^d(t) z^k, \quad d \geq 2, \\ \frac{1-tz}{1-2zt+z^2} &= \sum_{k=0}^{\infty} P_k^1(t) z^k, \end{aligned}$$

for all $t \in [-1, 1]$ and $z \in (-1, 1)$. From the generating function, we obtain that

$$\begin{aligned} (2k+d+1)P_{k+2}^d(t) &= (d-1)(P_{k+2}^{d+2}(t) - P_k^{d+2}(t)), \quad d \geq 2, \\ 2P_{k+2}^1(t) &= P_{k+2}^3(t) - P_k^3(t), \end{aligned}$$

for all $t \in [-1, 1]$ and $k \in \mathbb{Z}_+$. And then a very important generating function appears

$$\begin{aligned} \frac{1-z^2}{(1-2zt+z^2)^{(d+1)/2}} &= \sum_{k=0}^{\infty} \frac{2k+d-1}{d-1} P_k^d(t) z^k, \quad d \geq 2 \\ \frac{1-z^2}{1-2zt+z^2} &= 1 + \sum_{k=1}^{\infty} 2P_k^1(t) z^k, \end{aligned} \tag{3.4}$$

for all $t \in [-1, 1]$ and $k \in \mathbb{Z}_+$. Below we state the main steps towards the proof of Theorem 3.1.3, without proving them.

The converse of the Schoenberg's characterization follows from the fact that the polynomial P_k^d defines a positive definite isotropic kernel on the sphere S^d . There are two main methods to prove this result. One approach relies on the addition formula for spherical harmonics, (DAI; XU, 2013) page 10:

$$\frac{2k+d-1}{d-1} P_k^d(x \cdot y) = \sum_{i=1}^{a(k,d)} Y_{k,i}^{d+1}(x) Y_{k,i}^{d+1}(y), \quad d \geq 2$$

for all $x, y \in S^d$, where $a(k, d) = \binom{k+d-1}{k} - \binom{k+d-3}{k-2}$ if $k \geq 2$ and $a(1, d) = d$. For $d = 1$ and $k \geq 1$, it reads

$$P_k^1(x \cdot y) = \frac{1}{2} [Y_{k,1}^1(x) Y_{k,1}^1(y) + Y_{k,2}^1(x) Y_{k,2}^1(y)],$$

for all $x, y \in S^1$. The other one is based on an induction argument on the addition formula for Gegenbauer polynomials (this is the method Schoenberg actually used). The addition formula is, (ASKEY, 1975) page 30,

$$\begin{aligned} P_k^d(\cos \theta \cos \phi + \sin \theta \sin \phi \cos \psi) &= P_k^d(\cos \theta) P_k^d(\cos \phi) \\ &+ \sum_{n=1}^k a_{n,k}^{(d-1)/2} (\sin \theta)^n P_{k-n}^{d+2n}(\cos \theta) (\sin \phi)^n P_{k-n}^{d+2n}(\cos \phi) P_n^{d-1}(\cos \psi), \end{aligned} \tag{3.5}$$

where $a_{n,k}^{(d-1)/2} \geq 0$, $\theta, \phi, \psi \in [0, 2\pi]$ and $k \in \mathbb{Z}_+$. The polynomial P_k^1 defines a positive definite isotropic kernel on the sphere S^1 , because $P_k^1(\cos \theta) = \cos(k\theta)$. The general case is proved using a decomposition for points of S^{d+1} , $x = (\cos(\theta), \sin(\theta)x')$ for some $\theta \in [0, 2\pi]$ and $x' \in S^d$, along with Equation (3.5).

To conclude the converse of Schoenberg's Theorem, notice that since P_k^d defines a positive definite isotropic kernel on S^d , Corollary 2.2.2–(iii) implies the inequality $|P_k^d(t)| \leq P_k^d(1)$ for $t \in [-1, 1]$. If $(a_k)_{k \in \mathbb{Z}_+}$ is a sequence of nonnegative numbers such that $\sum_{k=0}^{\infty} a_k P_k^d(1) < \infty$, the function

$$t \in [-1, 1] \rightarrow \sum_{k=0}^{\infty} a_k P_k^d(t) \in \mathbb{R}$$

is continuous and defines a positive definite isotropic kernel on S^d by Proposition 2.2.7.

Now, to prove that a continuous function $f : [-1, 1] \rightarrow \mathbb{R}$ that defines a positive definite isotropic kernel on S^d has such a characterization, we recall the Dirichlet problem on the sphere, (AKHIEZER, 1988) page 28.

Theorem 3.1.4. Let $F : S^d \rightarrow \mathbb{R}$ be a continuous function. There exists only one continuous function $u : B_{d+1} \rightarrow \mathbb{R}$, that is harmonic on the open ball such that $u = F$ on S^d . This function can be represented by the Poisson integral

$$u(rx) = \frac{1}{\text{vol}(S^d)} \int_{S^d} \frac{1-r^2}{(1-2rx \cdot \xi + r^2)^{(d+1)/2}} F(\xi) d\rho(\xi), \quad 0 \leq r < 1, \quad x \in S^d.$$

The Gegenbauer polynomials satisfy a reproducing formula, that is also a generalization of Equation (3.1): for every continuous function $g : [-1, 1] \rightarrow \mathbb{R}$ and all $x, y \in S^d$, $d \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{\text{vol}(S^d)} \int_{S^d} \frac{2k+d-1}{d-1} P_k^d(x \cdot \xi) g(\xi \cdot y) d\rho(\xi) &= a_k^d(g) P_k^d(x \cdot y) \\ \frac{1}{\text{vol}(S^1)} \int_{S^1} 2P_k^1(x \cdot \xi) g(\xi \cdot y) d\rho(\xi) &= a_k^1(g) P_k^1(x \cdot y), \end{aligned} \quad (3.6)$$

for all $k \in \mathbb{N}$, where

$$a_k^d(g) := C(k, d) \int_{-1}^1 g(s) P_k^d(s) (1-s^2)^{(d-2)/2} ds, \quad k \in \mathbb{N}, \quad d \in \mathbb{N}.$$

In (DAI; XU, 2013), this is proved by using the so called Funk-Hecke formula for spherical harmonics, while Schoenberg proved that by using Equation (3.5). Both ways rely heavily on the properties of the measure ρ in S^d .

The proof of the Schoenberg's characterization starts by proving that $a_k^d(f) \geq 0$ for all $k \in \mathbb{Z}_+$, where the hypothesis that f defines a positive definite isotropic kernel on S^d and the fact that P_k^d also defines a positive definite isotropic kernel on S^d are essential. Define the continuous function $F : S^d \rightarrow \mathbb{R}$ given by $F(x) = f(x \cdot e_1)$. The solution of the Dirichlet problem in Theorem 3.1.4 states that for every fixed $r \in [0, 1)$ and $d \geq 2$,

$$\begin{aligned} u(rx) &= \frac{1}{\text{vol}(S^d)} \int_{S^d} \frac{1-r^2}{(1-2rx \cdot \xi + r^2)^{(d+1)/2}} F(\xi) d\rho(\xi) \\ &= \frac{1}{\text{vol}(S^d)} \int_{S^d} \left[\sum_{k=0}^{\infty} \frac{2k+d-1}{d-1} P_k^d(x \cdot \xi) r^k \right] f(\xi \cdot e_1) d\rho(\xi) \\ &= \sum_{k=0}^{\infty} \left[\frac{1}{\text{vol}(S^d)} \int_{S^d} \frac{2k+d-1}{d-1} P_k^d(x \cdot \xi) f(\xi \cdot e_1) d\rho(\xi) \right] r^k = \sum_{k=0}^{\infty} a_k^d(f) P_k^d(x \cdot e_1) r^k. \end{aligned}$$

The change of order between the integral and the infinite sum is possible because $|P_k^d(x \cdot \xi)| \leq P_k^d(1)$. The case $d = 1$ follows by the same argument. The Dirichlet problem also implies that

$$\lim_{r \rightarrow 1^-} u(rx) = F(x) = f(x \cdot e_1).$$

Since all the coefficients $a_k^d(f)$ are nonnegative the power series in the variable r is Abel summable, and that implies

$$f(x \cdot e_1) = \sum_{k=0}^{\infty} a_k^d(f) P_k^d(x \cdot e_1), \quad \text{for all } x \in S^d.$$

The proof ends by picking $x = (t, \sqrt{1-t^2}, 0, \dots, 0)$, for all $t \in [-1, 1]$. The representation is unique because the Dirichlet problem has only one solution. Another argument for this property is reached by the uniform convergence of the series along with the orthogonality properties of the Gegenbauer polynomials in Equation (3.2).

Note that if $f : [-1, 1] \rightarrow \mathbb{R}$ defines a continuous (strictly) positive definite isotropic kernel on S^d then it also defines a continuous (strictly) positive definite isotropic kernel on S^m for all $m \leq d$. Hence, it makes sense to ask which continuous functions $f : [-1, 1] \rightarrow \mathbb{R}$ define a (strictly) positive definite isotropic kernel on S^d for all $d \in \mathbb{N}$. This class of functions defines the so called **positive definite isotropic kernel on S^∞** .

As an application of the Gram Representation Theorem 2.1.5, a function $f : [-1, 1] \rightarrow \mathbb{R}$ defines a (strictly) positive definite isotropic kernel on S^∞ if and only if the kernel $K : S \times S \rightarrow \mathbb{R}$ given by $K(x, y) = f(\langle x, y \rangle)$ is (strictly) positive definite, where $S := \{x \in \mathcal{H}, \langle x, x \rangle = 1\}$ and $\mathcal{H}$ is an infinite dimensional real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Thus, a (strictly) positive definite isotropic kernel on the sphere of $l^2(\mathbb{N})$ can be used as a synonym for a (strictly) positive definite isotropic kernel on S^∞ .

In the same article (SCHOENBERG, 1942), Schoenberg proved a characterization for the continuous functions that define a positive definite isotropic kernel on S^∞ .

Theorem 3.1.5. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function. The kernel

$$(x, y) \in S^d \times S^d \rightarrow f(x \cdot y) \in \mathbb{R}$$

is positive definite for all $d \in \mathbb{N}$ if and only if

$$f(t) = \sum_{k=0}^{\infty} a_k^\infty(f) P_k^\infty(t),$$

for all $t \in [-1, 1]$, where $P_k^\infty(t) := t^k$, the coefficients a_k^∞ are nonnegative, and the series converges at $t = 1$. Moreover, the representation is unique.

The aforementioned fact that the representation is unique is a direct consequence of standard properties of analytic functions.

The converse of Theorem 3.1.5 follows because, by Proposition 2.2.5–(ii), the kernel $(x, y) \in S^d \times S^d \rightarrow a_k(x \cdot y)^k \in \mathbb{R}$ is positive definite for all $k \in \mathbb{Z}_+$, and the conclusion follows from Proposition 2.2.7.

There are two different methods to prove the representation of a continuous positive definite isotropic kernel on S^∞ given in Theorem 3.1.5. The proof Schoenberg gave proves that for every fixed $t \in [-1, 1]$

$$\lim_{d \rightarrow \infty} \frac{P_k^d(t)}{P_k^d(1)} \rightarrow t^k \quad (3.7)$$

uniformly in the variable $k \in \mathbb{Z}_+$, which follows from the integral equation 4.10.3 in page 94 of (SZEGŐ, 1975). Since $|a_k^d P_k^d(1)| \leq f(1)$ for all $k \in \mathbb{Z}_+$ and $d \in \mathbb{N}$, a Cantor diagonal type of argument provides a sequence $(a_k^\infty(f))_{k \in \mathbb{Z}_+}$ and a subsequence $(d_l)_{l \in \mathbb{N}}$, such that

$$\lim_{l \rightarrow \infty} a_k^{d_l} P_k^{d_l}(1) = a_k^\infty(f)$$

for every $k \in \mathbb{N}$. The representation is proved using these two properties.

The other method is rather different and is based on Choquet theory. For further details on that alternative see (BERG; CHRISTENSEN; RESSEL, 1984) and (CHRISTENSEN; RESSEL, 1978). It is relevant to say that in (CHRISTENSEN; RESSEL, 1978) the hypothesis that the function that defines the kernel is continuous is removed, and in this context, only two more functions appear in the series representation: the simple functions $\chi_{\{1\}} - \chi_{\{-1\}}$ and $\chi_{\{-1,1\}}$.

In some papers, including (SCHOENBERG, 1942), but mostly those with a statistics point of view, an isotropic positive definite function on S^d is a continuous function $\psi : [0, \pi] \rightarrow \mathbb{R}$ such that the kernel $(x, y) \in S^d \times S^d \rightarrow \psi(d(x, y))$, is positive definite, where $d(x, y)$ is the geodesic distance between x and y . The connection between the definition we presented and this new one is made by the formula $d(x, y) = \arccos(x \cdot y)$, as explained in chapter 17 of (CHENEY; LIGHT, 2009). So we can pass from one definition to the other by using the cos and arccos functions.

On the subsequent chapters some basic properties of Gegenbauer polynomials will be required. Their proofs are omitted and can be found at chapter 4 in (SZEGŐ, 1975).

Proposition 3.1.6. The Gegenbauer polynomials P_k^d satisfy the following properties:

- (i) $P_k^d(1) = \binom{d-2+k}{k}$, for $d \geq 2$.
- (ii) $P_k^1(\cos \theta) = \cos k\theta$.
- (iii) $P_k^d(-t) = (-1)^k P_k^d(t)$.
- (iv) $|P_k^d(t)| \leq P_k^d(1)$.
- (v) $\frac{d}{dt}[P_k^d(t)] = (d-1)P_{k-1}^{d+2}(t)$, for $d \geq 2$ and $k \geq 1$.

It is possible to generalize the concept of isotropic kernels to a complex sphere $\Omega_q := \{x \in \mathbb{C}^q; \|x\| = 1\}$, $q \in \mathbb{N}$. A kernel $K : \Omega_q \times \Omega_q \rightarrow \mathbb{C}$ is isotropic, or more precisely invariant through the unitary group $U(q+1)$, if for all $Q \in U(q+1)$ and $x, y \in \Omega_q$

$$K(x, y) = K(Qx, Qy).$$

An isotropic kernel on Ω_q can be described as a function $g : \bar{D} \rightarrow \mathbb{C}$, where $\bar{D} := \{x \in C, \|x\| \leq 1\}$ and $K(x, y) = g(x \cdot y)$. Similar to the real sphere, the kernel K is continuous if and only if g is continuous.

The characterization of the continuous functions $g : \bar{D} \rightarrow \mathbb{C}$ that define a positive definite isotropic kernel on Ω_q , for $q \in \mathbb{N}$ was achieved in (MENEGATTO; PERON, 2001). The characterization of the continuous functions $g : \bar{D} \rightarrow \mathbb{C}$ that define a positive definite isotropic kernel on Ω_q for every $q \in \mathbb{N}$ was achieved in (CHRISTENSEN; RESSEL, 1982).

A conclusive characterization of the continuous functions $g : \bar{D} \rightarrow \mathbb{C}$ such that

$$(x, y) \in \Omega_q \times \Omega_q \rightarrow g(x \cdot y) \in \mathbb{C}$$

is strictly positive definite, with $q \in \mathbb{N} \cup \{\infty\}$, was recently achieved in (GUELLA; MENEGATTO, 2018d).

3.2 Strictly positive definite isotropic kernels

In this section, we characterize the set of the continuous functions that define a strictly positive definite isotropic kernel on S^d , when $d \in \mathbb{N} \cup \{\infty\}$. The proof for the case $d \in \{2, 3, \dots, \infty\}$, is based in (CHEN; MENEGATTO; SUN, 2003), with the difference that the one provided here is based on the asymptotics for the Gegenbauer polynomials to be obtained in Lemma 3.2.2, while the proof in (CHEN; MENEGATTO; SUN, 2003) is based on asymptotics obtained by the addition formula in Equation (3.5). The case $d = 1$ is based on (MENEGATTO; OLIVEIRA; PERON, 2006). The main references for this section are (CHEN; MENEGATTO; SUN, 2003) and (MENEGATTO; OLIVEIRA; PERON, 2006).

Let $x = (\cos \theta, \sin \theta)$ and $y = (\cos \lambda, \sin \lambda)$ be points on S^1 . A simple relation that will help us achieve the characterization for the strict case is

$$\begin{aligned} P_k^1(x \cdot y) &= P_k^1(\cos \theta \cos \lambda + \sin \theta \sin \lambda) = P_k^1(\cos(\theta - \lambda)) \\ &= \cos(k(\theta - \lambda)) = (\cos k\theta, \sin k\theta) \cdot (\cos k\lambda, \sin k\lambda) \end{aligned} \quad (3.8)$$

Recall the following simple fact about roots of unity: if $m \in \mathbb{N}$ and $z \in \mathbb{C}$ is such that, $z^m - 1 = 0$, but $z \neq 1$, then

$$\sum_{\alpha=1}^m z^\alpha = 0. \quad (3.9)$$

Theorem 3.2.1. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function that defines a positive definite isotropic kernel on S^d , $d \in \{2, 3, \dots, \infty\}$, and consider its series expansion

$$f(t) = \sum_{k=0}^{\infty} a_k^d(f) P_k^d(t).$$

If the isotropic kernel defined by f is strictly positive definite, then the set $\{k \in \mathbb{Z}_+ : a_k^d(f) > 0\}$ contains infinitely many even and infinitely many odd integers.

Proof. For all $d \in \{2, 3, \dots, \infty\}$ and $k \in \mathbb{Z}_+$, the function P_k^d is a polynomial, so

$$P_k^d(t) = \sum_{l=0}^k a_{k,l}^d P_l^1(t),$$

with $a_{k,k}^d \neq 0$. But the function $P_k^d(t)$ also defines a positive definite isotropic kernel on S^1 , which implies that all coefficients $a_{k,l}^d \geq 0$, and in particular that $a_{k,k}^d > 0$. Proposition 3.1.6–(iii) shows that P_k^d is even when k is even and odd when k is odd. So, we may also deduce that $a_{k,l}^d = 0$, when $k - l$ is an odd number.

If $\{k \in \mathbb{Z}_+ : a_{2k}^d > 0\} \subset \{0, 1, \dots, q-1\}$ for some number $q \in \mathbb{N}$, define the $4q$ distinct points $x_\alpha = (\cos(2\pi\alpha/4q), \sin(2\pi\alpha/4q), 0, \dots, 0) \in S^d$ and $4q$ scalars $c_\alpha = (-1)^\alpha = e^{i\pi\alpha}$, $1 \leq \alpha \leq 4q$. Note that $x_\alpha = -x_{\alpha+2q}$ and $c_\alpha = c_{\alpha+2q}$, when $1 \leq \alpha \leq 2q$. Clearly,

$$\sum_{\alpha, \beta=1}^{4q} c_\alpha c_\beta f(x_\alpha \cdot x_\beta) = \sum_{k=0}^{\infty} a_k^d(f) \sum_{\alpha, \beta=1}^{4q} c_\alpha c_\beta P_k^d(x_\alpha \cdot x_\beta).$$

But, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned} \sum_{\alpha, \beta=1}^{4q} c_\alpha c_\beta P_{2k+1}^d(x_\alpha \cdot x_\beta) &= \sum_{\alpha, \beta=1}^{2q} c_\alpha c_\beta P_{2k+1}^d(x_\alpha \cdot x_\beta) + \sum_{\alpha=1}^{2q} \sum_{\beta=2q+1}^{4q} c_\alpha c_\beta P_{2k+1}^d(x_\alpha \cdot x_\beta) \\ &\quad + \sum_{\beta=1}^{2q} \sum_{\alpha=2q+1}^{4q} c_\alpha c_\beta P_{2k+1}^d(x_\alpha \cdot x_\beta) + \sum_{\alpha, \beta=2q+1}^{4q} c_\alpha c_\beta P_{2k+1}^d(x_\alpha \cdot x_\beta), \end{aligned}$$

and then

$$\begin{aligned} \sum_{\alpha, \beta=1}^{4q} c_\alpha c_\beta P_{2k+1}^d(x_\alpha \cdot x_\beta) &= \sum_{\alpha, \beta=1}^{2q} c_\alpha c_\beta P_{2k+1}^d(x_\alpha \cdot x_\beta) + \sum_{\alpha, \beta=1}^{2q} c_\alpha c_\beta P_{2k+1}^d(x_\alpha \cdot -x_\beta) \\ &\quad + \sum_{\alpha, \beta=1}^{2q} c_\alpha c_\beta P_{2k+1}^d(-x_\alpha \cdot x_\beta) + \sum_{\alpha, \beta=1}^{2q} c_\alpha c_\beta P_{2k+1}^d(-x_\alpha \cdot -x_\beta) \\ &= \sum_{\alpha, \beta=1}^{2q} [1 + (-1)^{2k+1} + (-1)^{2k+1} + 1] c_\alpha c_\beta P_{2k+1}^d(x_\alpha \cdot x_\beta) = 0. \end{aligned}$$

On the other hand, for $k \in \{0, 1, \dots, q-1\}$,

$$\begin{aligned} \sum_{\alpha, \beta=1}^{4q} c_\alpha c_\beta P_{2k}^d(x_\alpha \cdot x_\beta) &= \sum_{\alpha, \beta=1}^{2q} [1 + (-1)^{2k} + (-1)^{2k} + 1] c_\alpha c_\beta P_{2k}^d(x_\alpha \cdot x_\beta) \\ &= 4 \sum_{\alpha, \beta=1}^{2q} c_\alpha c_\beta \sum_{l=0}^k a_{2k, 2l}^d P_{2l}^1(x_\alpha \cdot x_\beta) \\ &= 4 \sum_{l=0}^k a_{2k, 2l}^d \sum_{\alpha, \beta=1}^{2q} c_\alpha c_\beta (\cos l\pi\alpha/q, \sin l\pi\alpha/q) \cdot (\cos l\pi\beta/q, \sin l\pi\beta/q) \\ &= 4 \sum_{l=0}^k a_{2k, 2l}^d \left\| \sum_{\alpha=1}^{2q} (-1)^\alpha (\cos l\pi\alpha/q, \sin l\pi\alpha/q) \right\|^2 = 0. \end{aligned}$$

The last equality is a consequence of Equation (3.9). Indeed, if $l \in \{0, 1, \dots, q-1\}$, then $(e^{i2\pi(q+l)/2q})^{2q} - 1 = 0$ and $e^{i2\pi(q+l)/2q} \neq 1$, so

$$0 = \sum_{\alpha=1}^{2q} e^{i2\alpha\pi(q+l)/2q} = \sum_{\alpha=1}^{2q} e^{i\alpha\pi} e^{i\pi\alpha l/q} = \sum_{\alpha=1}^{2q} (-1)^\alpha [\cos l\pi\alpha/q + i \sin l\pi\alpha/q].$$

Thus, f does not define a strictly positive definite isotropic kernel on S^d .

Now, suppose that the set $\{k \in \mathbb{Z}_+ : a_{2k+1}^d > 0\}$ is finite. The continuous function $g(t) := tf(t)$ defines a positive definite isotropic kernel on S^d by the Schur product result in Proposition 2.2.4. Proposition 3.1.6–(iii) reveals that the even part of the function f is $\sum_{k=0}^{\infty} a_{2k}^d(f) P_{2k}^d(t)$, while its odd part is $\sum_{k=0}^{\infty} a_{2k+1}^d(f) P_{2k+1}^d(t)$. A similar remark applies to g . In particular,

$$t \sum_{k=0}^{\infty} a_{2k}^d(f) P_{2k}^d(t) = \sum_{k=0}^{\infty} a_{2k+1}^d(g) P_{2k+1}^d(t),$$

and

$$t \sum_{k=0}^{\infty} a_{2k+1}^d(f) P_{2k+1}^d(t) = \sum_{k=0}^{\infty} a_{2k}^d(g) P_{2k}^d(t).$$

This implies that the set $\{k \in \mathbb{Z}_+ : a_{2k}^d(g) > 0\}$ is finite. By the previous part of the proof, the function g does not define a strictly positive definite isotropic kernel on S^d . So, there exist distinct points $x_1, x_2, \dots, x_n \in S^d$, scalars $c_1, c_2, \dots, c_n \in \mathbb{R}$, not all zero, such that

$$\sum_{\mu, \nu=1}^n c_\mu c_\nu g(x_\mu \cdot x_\nu) = \sum_{\mu, \nu=1}^n c_\mu c_\nu (x_\mu \cdot x_\nu) f(x_\mu \cdot x_\nu) = 0.$$

Introducing coordinates to the points x_μ , we deduce that

$$0 = \sum_{\mu, \nu=1}^n c_\mu c_\nu \sum_{l=1}^{d+1} (x_\mu)_l (x_\nu)_l f(x_\mu \cdot x_\nu) = \sum_{l=1}^{d+1} \sum_{\mu, \nu=1}^n [c_\mu (x_\mu)_l] [c_\nu (x_\nu)_l] f(x_\mu \cdot x_\nu),$$

that is,

$$\sum_{\mu, \nu=1}^n [c_\mu (x_\mu)_l] [c_\nu (x_\nu)_l] f(x_\mu \cdot x_\nu) = 0$$

for all $1 \leq l \leq d+1$. Since at least one of the $n(d+1)$ scalars $c_\mu (x_\mu)_l$, $1 \leq l \leq d+1$, $1 \leq \mu \leq n$, is nonzero, the function f does not define a strictly positive definite isotropic kernel on S^d . $\square$

Now we state and prove a key lemma to obtain the converse of Theorem 3.2.1.

Lemma 3.2.2. The Gegenbauer polynomials P_k^d satisfy

- (i) $(1-t^2) \frac{P_k^{d+2}(t)}{P_k^{d+2}(1)} = \frac{d}{2k+d+1} \left[\frac{P_k^d(t)}{P_k^d(1)} - \frac{P_{k+2}^d(t)}{P_{k+2}^d(1)} \right], \quad t \in [-1, 1], \quad d \in \mathbb{N}.$
- (ii) $P_k^2(\cos \theta) = \frac{1}{\pi} \int_0^\pi (\cos \theta + i \sin \theta \cos \lambda)^k d\lambda, \quad 0 < \theta < \pi.$

Proof. For all $d \in \mathbb{N}$ and $k \in \mathbb{Z}_+$, the function $(1-t^2)P_k^{d+2}$ is a polynomial of degree $k+2$, so we can write

$$(1-t^2) \frac{P_k^{d+2}(t)}{P_k^{d+2}(1)} = \sum_{\mu=1}^{k+2} a_{\mu}^{k,d} \frac{P_{\mu}^d(t)}{P_{\mu}^d(1)},$$

with $a_{k+2}^{k,d} \neq 0$. The orthogonality relations in Equation (3.2) imply that if, $0 \leq v \leq k-1$, then

$$\begin{aligned} 0 &= \int_{-1}^1 P_k^{d+2}(s) P_v^d(s) (1-s^2)^{d/2} ds = \int_{-1}^1 (1-s^2) P_k^{d+2}(s) P_v^d(s) (1-s^2)^{(d-2)/2} ds \\ &= P_k^{d+2}(1) \int_{-1}^1 \sum_{\mu=1}^{k+2} a_{\mu}^{k,d} \frac{P_{\mu}^d(s)}{P_{\mu}^d(1)} P_v^d(s) (1-s^2)^{(d-2)/2} ds \\ &= P_k^{d+2}(1) \sum_{\mu=1}^{k+2} a_{\mu}^{k,d} \frac{1}{P_{\mu}^d(1)} \int_{-1}^1 P_{\mu}^d(s) P_v^d(s) (1-s^2)^{(d-2)/2} ds \\ &= P_k^{d+2}(1) a_v^{k,d} \frac{1}{P_v^d(1)} \int_{-1}^1 P_v^d(s) P_v^d(s) (1-s^2)^{(d-2)/2} ds. \end{aligned}$$

So, $a_{\mu}^{k,d} = 0$ for all $1 \leq \mu \leq k-1$, and then we can write

$$(1-t^2) \frac{P_k^{d+2}(t)}{P_k^{d+2}(1)} = a_k^{k,d} \frac{P_k^d(t)}{P_k^d(1)} + a_{k+1}^{k,d} \frac{P_{k+1}^d(t)}{P_{k+1}^d(1)} + a_{k+2}^{k,d} \frac{P_{k+2}^d(t)}{P_{k+2}^d(1)}.$$

The number $a_{k+1}^{k,d}$ is zero, because, in the last equation, the function on the left side is even when k is even, and is odd when k is odd. On the other hand, $a_k^{k,d} = -a_{k+2}^{k,d}$, because the function on the left side at $t = 1$ is zero. Finally, if the coefficient of t^k in $P_k^{d+2}(t)$ is $a \in \mathbb{R}$ and the coefficient of t^{k+2} in $P_{k+2}^d(t)$ is $b \in \mathbb{R}$, then $-a/P_k^{d+2}(1) = a_{k+2}^{k,d} b/P_{k+2}^d(1)$. The numbers a and b can be calculated by differentiating $P_k^{d+2}(t)$ and $P_{k+2}^d(t)$, k and $k+2$ times respectively and using Proposition 3.1.6–(i) and (v). This settles (i).

As for the proof of (ii), first we need the Rodriguez formula, page 82 equation 4.7.12 (SZEGŐ, 1975),

$$P_k^2(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k} (t^2 - 1)^k.$$

The cases $k = 0$ and $k = 1$ are immediate, while the general case is proved using the 3-term recurrence relation in Equation (3.3). For every fixed $\theta \in (0, \pi)$, consider the circle $C_{\theta} = \cos \theta + \sin \theta e^{i\lambda}$, $\lambda \in [0, 2\pi]$. By the Cauchy integral formula, page 47 (STEIN; SHAKARCHI, 2003)

$$\begin{aligned} P_k^2(\cos \theta) &= \frac{1}{2^k 2i\pi} \int_{C_{\theta}} \frac{(\xi^2 - 1)^k}{(\xi - \cos \theta)^{k+1}} d\xi \\ &= \frac{1}{2^k 2\pi} \int_0^{2\pi} \frac{(\cos^2 \theta + \sin^2 \theta e^{2i\lambda} + 2 \cos \theta \sin \theta e^{i\lambda} - 1)^k}{(\sin \theta e^{i\lambda})^k} d\lambda \\ &= \frac{1}{\pi} \int_0^{\pi} (\cos \theta + i \sin \theta \cos \lambda)^k d\lambda. \end{aligned}$$

The result follows. □

If $d \in \mathbb{N} \cup \{\infty\}$, a set $D \subset S^d$ is **antipodal free** if $D \cap (-D) = \emptyset$, where $-D := \{-z : z \in D\}$. Thus, if D is antipodal free and $x \in D$, its antipodal $-x$ does not belong to D . The next Lemma is the crucial step towards the characterization of the strictly positive definite isotropic kernels on S^d

Lemma 3.2.3. Let $\{x_1, x_2, \dots, x_q\}$ be an antipodal free set of S^d , $d \in \{2, 3, \dots, \infty\}$. Then there exists a natural number k_0 , that depends upon the subset, such that the matrices

$$[P_k^d(x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q$$

are positive definite for all $k \geq k_0$.

Proof. Since the set $\{x_1, x_2, \dots, x_q\}$ contains only distinct points and is antipodal free, we get that $-1 < x_\alpha \cdot x_\beta < 1$ when $\alpha \neq \beta$. If $d \in \{2, 3, \dots, \infty\}$, the Gegenbauer polynomials, satisfy

$$\lim_{k \rightarrow \infty} \frac{P_k^d(t)}{P_k^d(1)} = 0,$$

for all $-1 < t < 1$. This is immediate for $d = \infty$, as for $d < \infty$ we separate in 2 cases. When $d \geq 3$, it follows from Lemma 3.2.2–(i) and Proposition 3.1.6–(v). If $d = 2$, since

$$\lim_{k \rightarrow \infty} (\cos \theta + i \sin \theta \cos \lambda)^k = 0$$

for all $\theta, \lambda \in (0, \pi)$, it follows from the Lebesgue Dominated Convergence Theorem and Lemma 3.2.2–(ii).

Thus, we have concluded that

$$\lim_{k \rightarrow \infty} \left[\frac{P_k^d(x_\alpha \cdot x_\beta)}{P_k^d(1)} \right]_{\alpha, \beta=1}^q = I_q \in M_q(\mathbb{R}).$$

The determinant is a continuous function on the space of square matrices, and the determinant of I_q is 1. So, for k_0 large enough, the determinant of the matrices $[P_k^d(x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q$, $k \geq k_0$, are not zero. Since these matrices are already positive semidefinite, the result is valid. $\square$

Note that if $q \geq 3$ and $x_1, x_2, \dots, x_q$ are distinct points in S^1 , the matrix $[P_k^1(x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q$ is not positive definite, because, by Equation (3.8), the vectors in its Gram representation belong to a 2 dimensional space.

In (MENEGATTO, 1999), it was proved that for $d = \infty$, every subset having q distinct points that are antipodal free, the minimum value for k_0 in Lemma 3.2.3 is at most $q - 1$. Only partial results are known for $d < \infty$.

Before proving the sufficiency, we state a lemma that will simplify its proof.

Lemma 3.2.4. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function that defines a positive definite isotropic kernel on S^d , $d \in \{2, 3, \dots, \infty\}$, and consider its series expansion

$$f(t) = \sum_{k=0}^{\infty} a_k^d(f) P_k^d(t).$$

The isotropic kernel defined by f is strictly positive definite if and only if for every finite antipodal free set $\{x_1, x_2, \dots, x_q\}$ of S^d , the only $2q$ scalars $c_\alpha^1, c_\alpha^2 \in \mathbb{R}$, $1 \leq \alpha \leq q$, such that

$$\sum_{\alpha, \beta=1}^q c_\alpha^1 c_\beta^1 P_k^d(x_\alpha \cdot x_\beta) = 0, \quad k \in (2\mathbb{Z}_+ + 1) \cap \{k \in \mathbb{Z}_+ : a_k^d(f) > 0\}$$

and

$$\sum_{\alpha, \beta=1}^q c_\alpha^2 c_\beta^2 P_k^d(x_\alpha \cdot x_\beta) = 0, \quad k \in 2\mathbb{Z}_+ \cap \{k \in \mathbb{Z}_+ : a_k^d(f) > 0\}.$$

are zero.

Proof. Suppose that there exists a finite antipodal free set $\{x_1, x_2, \dots, x_q\}$ of S^d and q scalars $c_\alpha \in \mathbb{R}$, $1 \leq \alpha \leq q$, not all zero, such that

$$\sum_{\alpha, \beta=1}^q c_\alpha c_\beta P_k^d(x_\alpha \cdot x_\beta) = 0, \quad k \in (2\mathbb{Z}_+ + 1) \cap \{k \in \mathbb{Z}_+ : a_k^d(f) > 0\}$$

Define $v = (-c_1, -c_2, \dots, -c_q, c_1, c_2, \dots, c_q) \in \mathbb{R}^{2q} \setminus \{0\}$. The interpolation matrix at the points $x_1, x_2, \dots, x_q, -x_1, -x_2, \dots, -x_q$ through the isotropic kernel defined by f is:

$$\begin{pmatrix} f(x_1 \cdot x_1) & f(x_1 \cdot x_2) & \dots & f(x_1 \cdot x_q) & f(x_1 \cdot -x_1) & f(x_1 \cdot -x_2) & \dots & f(x_1 \cdot -x_q) \\ f(x_2 \cdot x_1) & f(x_2 \cdot x_2) & \dots & f(x_2 \cdot x_q) & f(x_2 \cdot -x_1) & f(x_2 \cdot -x_2) & \dots & f(x_2 \cdot -x_q) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f(x_q \cdot x_1) & f(x_q \cdot x_2) & \dots & f(x_q \cdot x_q) & f(x_q \cdot -x_1) & f(x_q \cdot -x_2) & \dots & f(x_q \cdot -x_q) \\ f(-x_1 \cdot x_1) & f(-x_1 \cdot x_2) & \dots & f(-x_1 \cdot x_q) & f(-x_1 \cdot -x_1) & f(-x_1 \cdot -x_2) & \dots & f(-x_1 \cdot -x_q) \\ f(-x_2 \cdot x_1) & f(-x_2 \cdot x_2) & \dots & f(-x_2 \cdot x_q) & f(-x_2 \cdot -x_1) & f(-x_2 \cdot -x_2) & \dots & f(-x_2 \cdot -x_q) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f(-x_q \cdot x_1) & f(-x_q \cdot x_2) & \dots & f(-x_q \cdot x_q) & f(-x_q \cdot -x_1) & f(-x_q \cdot -x_2) & \dots & f(-x_q \cdot -x_q) \end{pmatrix}$$

Denote the above matrix by $A \in M_{2q}(\mathbb{R})$. Then

$$\begin{aligned} vAv^t &= 2 \sum_{\alpha, \beta=1}^q c_\alpha c_\beta f(x_\alpha \cdot x_\beta) - 2 \sum_{\alpha, \beta=1}^q c_\alpha c_\beta f(-x_\alpha \cdot x_\beta) = 2 \sum_{\alpha, \beta=1}^q c_\alpha c_\beta [f(x_\alpha \cdot x_\beta) - f(-x_\alpha \cdot x_\beta)] \\ &= 4 \sum_{\alpha, \beta=1}^q c_\alpha c_\beta \sum_{k=0}^{\infty} a_{2k+1}^d(f) P_{2k+1}^d(x_\alpha \cdot x_\beta) = 4 \sum_{k=0}^{\infty} a_{2k+1}^d(f) \sum_{\alpha, \beta=1}^q c_\alpha c_\beta P_{2k+1}^d(x_\alpha \cdot x_\beta) = 0. \end{aligned}$$

Thus, f does not define a strictly positive definite isotropic kernel on S^d . If there exists a finite antipodal free set $\{x_1, x_2, \dots, x_q\}$ of S^d and q scalars $c_\alpha \in \mathbb{R}$, $1 \leq \alpha \leq q$, not all zero, such that

$$\sum_{\alpha, \beta=1}^q c_\alpha c_\beta P_k^d(x_\alpha \cdot x_\beta) = 0, \quad k \in (2\mathbb{Z}_+) \cap \{k \in \mathbb{Z}_+ : a_k^d(f) > 0\},$$

a similar calculation with $v = (c_1, c_2, \dots, c_q, c_1, c_2, \dots, c_q) \in \mathbb{R}^{2q} \setminus \{0\}$ implies the same conclusion.

Conversely, if the isotropic kernel defined by f is not strictly positive definite, there exist distinct points $z_1, z_2, \dots, z_n$ on S^d , such that the interpolation matrix $[f(z_\mu \cdot z_\nu)]_{\mu, \nu=1}^n$ is not positive definite. Pick distinct points $x_1, x_2, \dots, x_q \in S^d$, that are antipodal free set and such that $z_\mu \in \{\pm x_1, \pm x_2, \dots, \pm x_q\}$ for all $1 \leq \mu \leq n$. In particular, the interpolation matrix $[f(z_\mu \cdot z_\nu)]_{\mu, \nu=1}^n$ is a principal minor (up to equal changes in lines and columns) of the matrix $B \in M_{2q}(\mathbb{R})$, the interpolation matrix at the points $x_1, x_2, \dots, x_q, -x_1, -x_2, \dots, -x_q$ through the isotropic kernel defined by f . By the hypothesis on the matrix $[f(z_\mu \cdot z_\nu)]_{\mu, \nu=1}^n$, B is not positive definite, so there is a vector $v = (v_1, v_2) \in (\mathbb{R}^q)^2$ such that

$$vBv^t = v \begin{pmatrix} [f(x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q & [f(x_\alpha \cdot -x_\beta)]_{\alpha, \beta=1}^q \\ [f(-x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q & [f(x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q \end{pmatrix} v^t = 0.$$

But, if

$$O := \left[\sum_{k=0}^{\infty} a_{2k+1}(f) P_{2k+1}^d(x_\alpha \cdot x_\beta) \right]_{\alpha, \beta=1}^q \in M_q(\mathbb{R})$$

$$E := \left[\sum_{k=0}^{\infty} a_{2k+1}(f) P_{2k}^d(x_\alpha \cdot x_\beta) \right]_{\alpha, \beta=1}^q \in M_q(\mathbb{R}),$$

then

$$[f(x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q = E + O, \quad [f(-x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q = E - O.$$

So,

$$(v_1 + v_2)E(v_1 + v_2)^t + (v_1 - v_2)O(v_1 - v_2)^t = v_1[f(x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q v_1^t + v_2[f(x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q v_2^t \\ + v_1[f(-x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q v_2^t + v_2[f(-x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q v_1^t = vBv^t = 0.$$

In particular,

$$\sum_{\alpha, \beta=1}^q (v_1 + v_2)_\alpha (v_1 + v_2)_\beta P_k^d(x_\alpha \cdot x_\beta) = 0, \quad k \in (2\mathbb{Z}_+ + 1) \cap \{k \in \mathbb{Z}_+ : a_k^d(f) > 0\}$$

and

$$\sum_{\alpha, \beta=1}^q (v_1 - v_2)_\alpha (v_1 - v_2)_\beta P_k^d(x_\alpha \cdot x_\beta) = 0, \quad k \in 2\mathbb{Z}_+ \cap \{k \in \mathbb{Z}_+ : a_k^d(f) > 0\}.$$

Since either $v_1 + v_2$ or $v_1 - v_2$ is a nonzero vector in $\mathbb{R}^q$, the lemma follows. $\square$

Theorem 3.2.5. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function that defines a positive definite isotropic kernel on S^d , $d \in \{2, 3, \dots, \infty\}$, and consider its series expansion

$$f(t) = \sum_{k=0}^{\infty} a_k^d(f) P_k^d(t).$$

The isotropic kernel defined by f in S^d is strictly positive definite if and only if the set $\{k \in \mathbb{Z}_+ : a_k^d(f) > 0\}$ contains infinitely many even and infinitely many odd integers.

Proof. If the kernel is strictly positive definite, Theorem 3.2.1 implies that the set $\{k \in \mathbb{Z}_+ : a_k^d(f) > 0\}$ contains infinitely many even and infinitely many odd integers.

Conversely, assume that the set $\{k \in \mathbb{Z}_+ : a_k^d(f) > 0\}$ contains infinitely many even and infinitely many odd integers. Let $\{x_1, x_2, \dots, x_q\}$ be a finite antipodal free set in S^d and $c_\alpha^1, c_\alpha^2 \in \mathbb{R}$, $1 \leq \alpha \leq q$, such that

$$\sum_{\alpha, \beta=1}^q c_\alpha^1 c_\beta^1 P_k^d(x_\alpha \cdot x_\beta) = 0, \quad k \in (2\mathbb{Z}_+ + 1) \cap \{k \in \mathbb{Z}_+ : a_k^d(f) > 0\},$$

and

$$\sum_{\alpha, \beta=1}^q c_\alpha^2 c_\beta^2 P_k^d(x_\alpha \cdot x_\beta) = 0, \quad k \in 2\mathbb{Z}_+ \cap \{k \in \mathbb{Z}_+ : a_k^d(f) > 0\}.$$

We can pick an even and an odd integer sufficiently large in $\{k \in \mathbb{Z}_+ : a_k^d(f) > 0\}$, so that we can apply Lemma 3.2.3 at the set $\{x_1, x_2, \dots, x_q\}$, but then $c_\alpha^1 = c_\alpha^2 = 0$, for all $1 \leq \alpha \leq q$, which imply that the isotropic kernel defined by f is strictly positive definite by Lemma 3.2.4. $\square$

The proof for the characterization of the continuous strictly positive definite isotropic kernels on S^1 , given in (MENEGATTO; OLIVEIRA; PERON, 2006) is rather different. It has subtle connections with the kernels studied in (PINKUS, 2004b) and (PINKUS, 2004a).

Lemma 3.2.6. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function that defines a positive definite isotropic kernel on S^1 , and consider its series representation

$$f(t) = \sum_{k=0}^{\infty} a_k^1(f) P_k^1(t).$$

The function f defines a strictly positive definite kernel on S^1 if and only if for every distinct points $\theta_1, \theta_2, \dots, \theta_n \in [0, 2\pi)$, the following subset of $\mathbb{C}^q$

$$\{(c_1, c_2, \dots, c_n) \in \mathbb{C}^q : \sum_{\mu=1}^n c_\mu e^{-il\theta_\mu} = 0, \text{ for all } l \in \{k \in \mathbb{Z} : a_{|k|}^1(f) > 0\}\} = \{0\}.$$

Proof. First, note that if $f : [-1, 1] \rightarrow \mathbb{R}$ is a continuous function that defines a positive definite isotropic kernel on S^1 and has a series expansion

$$f(t) = \sum_{k=0}^{\infty} a_k^1(f) P_k^1(t),$$

then

$$f(\cos \theta) = \sum_{k=0}^{\infty} a_k^1(f) P_k^1(\cos \theta) = \sum_{k=0}^{\infty} a_k^1(f) \cos k\theta = \sum_{k \in \mathbb{Z}} b_k^1(f) e^{ik\theta}$$

where $2b_k^1(f) = a_k^1(f)$, if $k \neq 0$, and $b_0^1(f) = a_0^1(f)$. Recalling Equation (3.8), it follows that the isotropic kernel defined by f is strictly positive definite if and only if the kernel $F : [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{C}$, defined by

$$(\theta, \lambda) \in [0, 2\pi) \times [0, 2\pi) \rightarrow \sum_{k \in \mathbb{Z}_+} b_k^1(f) e^{ik(\theta - \lambda)}$$

is strictly positive definite. By the definition of the kernel F , we have that the kernel F is strictly positive definite if and only if given distinct points $\theta_1, \theta_2, \dots, \theta_n$ in $[0, 2\pi)$, it holds

$$\{(c_1, c_2, \dots, c_n) \in \mathbb{C}^n : \sum_{\mu=1}^n c_{\mu} e^{-il\theta_{\mu}} = 0, \text{ for all } l \in \{k \in \mathbb{Z} : b_k^1(f) > 0\}\} = \{0\}.$$

The proof is complete. $\square$

A full arithmetic progression in $\mathbb{Z}$ is a set $N\mathbb{Z} + j := \{Nk + j \in \mathbb{Z} : k \in \mathbb{Z}\}$, where $N \in \{2, 3, \dots\}$ and $j \in \{0, 1, \dots, N-1\}$.

Theorem 3.2.7. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function that defines a positive definite isotropic kernel on S^1 and consider its series expansion

$$f(t) = \sum_{k=0}^{\infty} a_k^1(f) P_k^1(t).$$

If the function f defines a strictly positive definite isotropic kernel on S^1 , then the set $\{|k| \in \mathbb{Z} : a_k^1(f) > 0\}$ intersects every full arithmetic progression in $\mathbb{Z}$.

Proof. Suppose that there exists $N \in \{2, 3, \dots\}$ and $j \in \{0, 1, \dots, N-1\}$, such that $(N\mathbb{Z} + j) \cap \{|k| \in \mathbb{Z} : a_k^1(f) > 0\} = \emptyset$. Define $\theta_{\mu} := 2\pi\mu/N \in [0, 2\pi)$, $0 \leq \mu \leq N-1$. Since the Vandermonde matrix $[e^{i\pi\alpha\beta/N}]_{\alpha, \beta=0}^{N-1}$ is invertible, there exists a vector $v \in \mathbb{C}^N \setminus \{0\}$ such that $[e^{i\pi\alpha\beta/N}]_{\alpha, \beta=0}^{N-1} v = e_j$, but then

$$v \in \{(c_1, c_2, \dots, c_n) \in \mathbb{C}^n : \sum_{\mu=1}^n c_{\mu} e^{-il\theta_{\mu}} = 0, \text{ for all } l \in \{|k| \in \mathbb{Z} : a_k^1(f) > 0\}\}.$$

Thus, f does not define a strictly positive definite isotropic kernel on S^1 . $\square$

As for the converse of Theorem 3.2.7, we need a result from number theory, the famous Skolem-Mahler-Lech Theorem. Recall that a **linear recurrence sequence** is a sequence $(a_k)_{k \in \mathbb{Z}}$ in $\mathbb{C}$ satisfying a homogeneous linear equation

$$a_{k+n} = s_1 a_{k-1+n} + \dots + s_{n-1} a_{k+1} + s_n a_k$$

for all $k \in \mathbb{Z}$, with constants $s_1, s_2, \dots, s_n \in \mathbb{C}$. For instance, the Fibonacci sequence $a_{k+2} = a_{k+1} + a_k$, is an example of a linear recurrence sequence. The Skolem-Mahler-Lech Theorem reads as follows, (EVEREST *et al.*, 2003) page 25.

Theorem 3.2.8. Let $(a_k)_{k \in \mathbb{Z}}$ be a linear recurrence sequence in $\mathbb{C}$. The set

$$\{k \in \mathbb{Z} : a_k = 0\}$$

is a finite union of disjoint full arithmetic progressions in $\mathbb{Z}$ and a finite set.

The Skolem-Mahler-Lech Theorem is important for our purposes because given distinct points $\theta_1, \theta_2, \dots, \theta_n$ in $[0, 2\pi)$ and scalars $c_1, c_2, \dots, c_n \in \mathbb{C}$, the sequence

$$a_k := \sum_{\mu=1}^n c_\mu e^{-ik\theta_\mu} \in \mathbb{C}$$

is a linear recurrence sequence. Indeed, consider the polynomial of degree n

$$p(x) := \prod_{\mu=1}^n (1 - e^{i\theta_\mu} x) = 1 - s_1 x - s_2 x^2 - \dots - s_n x^n.$$

Although the actual value of the coefficients $s_1, s_2, \dots, s_n \in \mathbb{C}$ can be computed, we will not need this information. Then

$$\begin{aligned} a_{k+n} &= \sum_{\mu=1}^n c_\mu e^{-ik\theta_\mu} e^{-in\theta_\mu} = \sum_{\mu=1}^n c_\mu e^{-ik\theta_\mu} e^{-in\theta_\mu} [p(e^{i\theta_\mu}) + s_1 e^{i\theta_\mu} + s_2 e^{2i\theta_\mu} + \dots + s_n e^{in\theta_\mu}] \\ &= \sum_{\mu=1}^n c_\mu e^{-ik\theta_\mu} [s_1 e^{-i(n-1)\theta_\mu} + s_2 e^{-(n-2)i\theta_\mu} + \dots + s_n] \\ &= s_1 \sum_{\mu=1}^n c_\mu e^{-i(k+n-1)\theta_\mu} + \dots + s_n \sum_{\mu=1}^n c_\mu e^{-ik\theta_\mu} = s_1 a_{k-1+n} + \dots + s_{n-1} a_{k+1} + s_n a_k. \end{aligned}$$

Lemma 3.2.9. Let $(a_k)_{k \in \mathbb{Z}}$ be a linear recurrence sequence in $\mathbb{C}$. If the set $\{k \in \mathbb{Z} : a_k = 0\}$ intersects every full arithmetic progression in $\mathbb{Z}$, then $a_k = 0$ for all $k \in \mathbb{Z}$.

Proof. If $(a_k)_{k \in \mathbb{Z}}$ is a linear recurrence sequence in $\mathbb{C}$, Theorem 3.2.8 implies that

$$\{k \in \mathbb{Z} : a_k = 0\} = \bigcup_{\alpha=1}^m (N_\alpha \mathbb{Z} + j_\alpha) \cup F$$

where $N_\alpha \in \{2, 3, \dots\}$, $j_\alpha \in \{0, 1, \dots, N_\alpha - 1\}$, for all $1 \leq \alpha \leq m$, the set $F \subset \mathbb{Z}$ is finite and $(N_\alpha \mathbb{Z} + j_\alpha) \cap (N_\beta \mathbb{Z} + j_\beta) = \emptyset$ when $\alpha \neq \beta$. Define $N := \prod_{\alpha=1}^m N_\alpha \in \{2, 3, \dots\}$. The full arithmetic progression $N_\alpha \mathbb{Z} + j_\alpha$ is the disjoint union of full arithmetic progressions $N\mathbb{Z} + \tilde{j}_{\alpha,l}$, where $\tilde{j}_{\alpha,l} = j_\alpha + lN_\alpha$, $1 \leq l \leq N/N_\alpha$, for all $1 \leq \alpha \leq m$. In other words, we can write

$$\{k \in \mathbb{Z} : a_k = 0\} = \bigcup_{\xi=1}^M (N\mathbb{Z} + j_\xi) \cup F$$

where $N \in \{2, 3, \dots\}$, $j_\xi \in \{0, 1, \dots, N - 1\}$, for all $1 \leq \xi \leq M \leq N - 1$, the set $F \subset \mathbb{Z}$ is finite and $j_\xi \neq j_\rho$, when $\xi \neq \rho$.

Note that if the set $\{k \in \mathbb{Z} : a_k = 0\}$ intersects every full arithmetic progression in $\mathbb{Z}$, then it does infinitely many times, and then, since F is finite, for every $j \in \{0, 1, \dots, N - 1\}$, there exists an integer in $\{k \in \mathbb{Z} : a_k = 0\} \cap (N\mathbb{Z} + j)$ but not in F . But this implies that $j = j_\xi$ for some $1 \leq \xi \leq M$, and then $\{k \in \mathbb{Z} : a_k = 0\} = \mathbb{Z}$. $\square$

Theorem 3.2.10. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function that defines a positive definite isotropic kernel on S^1 and consider its series expansion

$$f(t) = \sum_{k=0}^{\infty} a_k^1(f) P_k^1(t).$$

The function f defines a strictly positive definite isotropic kernel on S^1 if and only if the set $\{|k| \in \mathbb{Z} : a_k^1(f) > 0\}$ intersects every full arithmetic progression in $\mathbb{Z}$.

Proof. If the isotropic kernel defined by f is strictly positive definite, Theorem 3.2.7 implies that the set $\{|k| \in \mathbb{Z} : a_k^1(f) > 0\}$ intersects every full arithmetic progression in $\mathbb{Z}$.

Conversely, suppose that the set $\{|k| \in \mathbb{Z} : a_k^1(f) > 0\}$ intersects every full arithmetic progression in $\mathbb{Z}$. Given distinct points $\theta_1, \theta_2, \dots, \theta_n \in [0, 2\pi)$, if the scalars $c_1, c_2, \dots, c_n \in \mathbb{C}$ are such that

$$\sum_{\mu=1}^n c_{\mu} e^{-ik\theta_{\mu}} = 0$$

for all $k \in \{|l| \in \mathbb{Z} : a_l^1(f) > 0\}$, then the linear recurrence sequence $(\sum_{\mu=1}^n c_{\mu} e^{-ik\theta_{\mu}})_{k \in \mathbb{Z}}$ intersects every full arithmetic progression in $\mathbb{Z}$. Lemma 3.2.9 implies that $\sum_{\mu=1}^n c_{\mu} e^{-ik\theta_{\mu}} = 0$ for all $k \in \mathbb{Z}$. In particular, since the Vandermonde matrix $[e^{i\pi^v \theta_{\mu}}]_{\mu, v=1}^n$ is invertible, the scalars $c_{\mu} = 0$, for all $1 \leq \mu \leq n$, and then Lemma 3.2.6 implies that the isotropic kernel defined by f is strictly positive definite. $\square$

Much of the harmonic analysis on real spheres, can be generalized to a larger class of spaces, the **compact two-point homogeneous spaces**. These spaces were classified in (WANG, 1952) as:

- (i) The real spheres S^d , $d \in \mathbb{N}$,
- (ii) The real projective spaces $P^d(\mathbb{R})$, $d \in \{2, 3, \dots\}$,
- (iii) The complex projective spaces $P^d(\mathbb{C})$, $d \in \{4, 6, 8, \dots\}$,
- (iv) The quaternionic projective spaces $P^d(\mathbb{H})$, $d \in \{4, 8, 12, \dots\}$,
- (v) The Cayley projective plane $P^d(\text{Cay})$, $d = 16$.

The number d refers to dimension of the corresponding real manifold. For more information on these spaces and two point homogeneous spaces see Chapter 4 (HELGASON, 1984). A full characterization of the continuous functions $f : [0, \pi] \rightarrow \mathbb{R}$, such that the kernel

$$(x, y) \in M^d \times M^d \rightarrow f(|xy|)$$

is positive definite, where M^d is a fixed compact two-point homogeneous space and $|xy|$ stands for the geodesic distance between x and y , was given in (GANGOLLI, 1967). A full characterization for the above kernels that are strictly positive definite was recently obtained in (BARBOSA; MENEGATTO, 2016).

A similar characterization of Theorem 3.1.5 to these new spaces was proved in (GUELLA; MENEGATTO, 2018b), a characterization of the continuous functions $f : [0, \pi] \rightarrow \mathbb{R}$, for which

the kernel

$$(x, y) \in M^d \times M^d \rightarrow f(|xy|)$$

is positive definite for all real projective spaces, for all complex projective spaces and for all quaternionic projective spaces. In particular, these 3 families of functions are the same. It was also proved a characterization for the strictly positive definite case for these 3 families of functions.

3.3 A unifying perspective. Part I

In this section we provide a generalization of the Schoenberg results on continuous positive definite isotropic kernels on real spheres, that is, a characterization of the kernels $K : (X \times S^d) \times (X \times S^d) \rightarrow \mathbb{C}$, $d \in \mathbb{N} \cup \{\infty\}$, that are positive definite, isotropic on the sphere variable, meaning

$$K((u, x), (v, y)) = K((u, Qx), (v, Qy)),$$

for all $u, v \in X$, every points $x, y \in S^d$ and $Q \in O(d+1)$, along with the assumption that the kernels

$$(x, y) \in S^d \times S^d \rightarrow K((u, x), (v, y)) \in \mathbb{C}$$

are continuous for every $u, v \in X$.

Similar conclusions from Lemma 3.1.1 and the comments above it, imply that our task is equivalent to characterize the set of functions $f : X \times X \times [-1, 1] \rightarrow \mathbb{C}$ such that the kernel

$$(X \times S^d) \times (X \times S^d) \rightarrow f(u, v, x \cdot y) \in \mathbb{C}$$

is positive definite, under the assumption that $f(u, v, \cdot)$ is a continuous function on $[-1, 1]$, for each fixed $u, v \in X$.

This section is based in (GUELLA; MENEGATTO, 2018c). With this characterization, we were able to reobtain the main results of (GUELLA; MENEGATTO; PERON, 2016a) and (BERG; PORCU, 2017), the latter being the main source of inspiration for (GUELLA; MENEGATTO, 2018c). We also reobtain the main results in (WANG; DU; MA, 2014), using the bridge propositions on matrix valued kernels in Section 2.3.

The following lemma plays a key role in order to obtain information about positive definite kernels on $X \times S^d$.

Lemma 3.3.1. Let $((u, x), (v, y)) \in (X \times S^d) \times (X \times S^d) \rightarrow f(u, v, x \cdot y)$ be a positive definite kernel, $d \in \mathbb{N} \cup \{\infty\}$, and assume that each function $f(u, v, \cdot)$ is continuous on $[-1, 1]$. For distinct points $u_1, u_2, \dots, u_p$ in X and complex numbers $c_1, c_2, \dots, c_p$, define

$$g(t) = \sum_{\mu, \nu=1}^p c_\mu \overline{c_\nu} f(u_\mu, u_\nu, t), \quad t \in [-1, 1].$$

The following assertions hold:

- (i) The function g is real and continuous.
- (ii) The function g defines a positive definite isotropic kernel on S^d .
- (iii) If the c_μ are not all zero and the kernel is strictly positive definite, then g defines a strictly positive definite isotropic kernel on S^d .

Proof. The continuity of g on $[-1, 1]$ is obvious. Since

$$\overline{g(t)} = \sum_{\mu, \nu=1}^p \overline{c_\mu} c_\nu \overline{f(u_\mu, u_\nu, t)} = \sum_{\mu, \nu=1}^p \overline{c_\mu} c_\nu f(u_\nu, u_\mu, t) = g(t), \quad t \in [-1, 1],$$

g is clearly a real function. In order to verify (ii) holds, observe that if $x_1, x_2, \dots, x_q$ are distinct points on S^d and $b = (b_1, b_2, \dots, b_q)$ belongs to $\mathbb{R}^q$, then Lemma 2.2.5 leads to

$$\begin{aligned} b [g(x_\alpha \cdot x_\beta)]_{\alpha, \beta=1}^q b^t &= \sum_{\alpha, \beta=1}^q \sum_{\mu, \nu=1}^p b_\alpha c_\mu \overline{c_\nu} b_\beta f(u_\mu, u_\nu, x_\alpha \cdot x_\beta) \\ &= \sum_{\alpha, \beta=1}^q \sum_{\mu, \nu=1}^p (b_\alpha c_\mu) (\overline{b_\beta c_\nu}) f(u_\mu, u_\nu, x_\alpha \cdot x_\beta) \geq 0. \end{aligned}$$

Finally, assume that at least one c_μ is nonzero. If at least one b_α is nonzero, then one of the pq numbers $b_\alpha c_\mu$ is likewise nonzero. Hence, if the kernel is strictly positive definite, then the previous inequality is actually strict. $\square$

The next lemma emphasizes that under some assumptions, a continuous complex valued function defined on $[-1, 1]$ is uniquely described in terms of its Gegenbauer polynomials coefficients.

Lemma 3.3.2. Let $(a_k)_{k \in \mathbb{Z}_+}$ be a sequence in $\mathbb{C}$ and $d \in \mathbb{N} \cup \{\infty\}$. Suppose that the series $\sum_{k=0}^\infty a_k P_k^d(1)$ converges absolutely. If

$$\sum_{k=0}^\infty a_k P_k^d(t) = 0,$$

for all $t \in [-1, 1]$, then $a_k = 0$ for all $k \in \mathbb{Z}_+$.

Proof. Since the series $\sum_{k=0}^\infty a_k P_k^d(1)$ converges absolutely, Proposition 3.1.6–(iv) along with the Weierstrass M-test implies that the series

$$\sum_{k=0}^\infty a_k P_k^d(t)$$

converges uniformly on $[-1, 1]$. If $d = \infty$, then $a_k = 0$ for all $k \in \mathbb{Z}_+$ by standard properties of analytic functions. If $d \in \mathbb{N}$, for all $k \in \mathbb{Z}_+$ we have that

$$\begin{aligned} a_k &= C(k, d) \sum_{l=0}^\infty a_l \int_{-1}^1 P_k^d(s) P_l^d(s) (1-s^2)^{(d-2)/2} ds \\ &= C(k, d) \int_{-1}^1 \left[\sum_{l=0}^\infty a_l P_l^d(s) \right] P_k^d(s) (1-s^2)^{(d-2)/2} ds = 0. \end{aligned}$$

□

The main theorem in this section is as follows.

Theorem 3.3.3. Let $f : X \times X \times [-1, 1] \rightarrow \mathbb{C}$ be a function such that $f(u, v, \cdot)$ is a continuous on $[-1, 1]$, for each fixed $u, v \in X$ and $d \in \mathbb{N} \cup \{\infty\}$. The following properties are equivalent:

- (i) The kernel $((u, x), (v, y)) \in (X \times S^d) \times (X \times S^d) \rightarrow f(u, v, x \cdot y)$ is positive definite.
- (ii) Each function $f(u, v, \cdot)$ has a series representation in the form

$$f(u, v, t) = \sum_{k=0}^{\infty} a_k^d(u, v) P_k^d(t),$$

for all $t \in [-1, 1]$, where $a_k^d(u, v) \in \mathbb{C}$ for all $k \in \mathbb{Z}_+$, each kernel $(u, v) \in X \times X \rightarrow a_k^d(u, v)$ is positive definite, and $\sum_{k=0}^{\infty} a_k^d(u, u) P_k^d(1) < \infty$, for all $u \in X$. This representation is unique.

Moreover, if f satisfies (i) and $d \in \mathbb{N}$, the following integral formula holds for $a_k^d(u, v)$, for all $u, v \in X$:

$$a_k^d(u, v) = C(k, d) \int_{-1}^1 f(u, v, s) P_k^d(s) (1 - s^2)^{(d-2)/2} ds.$$

Proof. If (i) holds, the last assertion in Lemma 3.3.1 implies that each function $f(u, u, \cdot)$ is continuous and, in addition, defines a positive definite kernel on S^d . Theorem 3.1.3 and Theorem 3.1.5 provide a unique series representation for each $f(u, u, \cdot)$ according to (ii). As a matter of fact, we have that $a_k^d(u, u) \geq 0$ and $\sum_{k=0}^{\infty} a_k^d(u, u) P_k^d(1) < \infty$ for all $u \in X$. If $u, v \in X$ and $u \neq v$, Lemma 3.3.1 implies that all three functions

$$f(u, u, \cdot) + f(v, v, \cdot) + f(u, v, \cdot) + f(v, u, \cdot),$$

$$f(u, u, \cdot) + f(v, v, \cdot) - f(u, v, \cdot) - f(v, u, \cdot),$$

$$f(u, u, \cdot) + f(v, v, \cdot) - if(u, v, \cdot) + if(v, u, \cdot)$$

are continuous and, also, define positive definite isotropic kernels on S^d . Given that $f(u, v, \cdot)$ is a linear combination of them (with complex coefficients), it is promptly seen, via Theorem 3.1.3 and Theorem 3.1.5 once again, that $f(u, v, \cdot)$ has a series representation as in the statement of the theorem and that the series $\sum_{k=0}^{\infty} a_k^d(u, v) P_k^d(1)$ converges absolutely for all $u, v \in X$.

Next, we verify that the kernels $(u, v) \in X \times X \rightarrow a_k^d(u, v)$ are positive definite for all $k \in \mathbb{Z}_+$. If $u_1, u_2, \dots, u_p$ are distinct points on X and $c = (c_1, c_2, \dots, c_p) \in \mathbb{C}^p$, Lemma 3.3.1 reveals that

$$g(t) = \sum_{\mu, \nu=1}^p c_{\mu} \overline{c_{\nu}} f(u_{\mu}, u_{\nu}, t), \quad t \in [-1, 1],$$

is real, continuous and defines a positive definite isotropic kernel on S^d . Since

$$g(t) = \sum_{k=0}^{\infty} \left(c \left[a_k^d(u_{\mu}, u_{\nu}) \right]_{\mu, \nu=1}^p c^* \right) P_k^d(t), \quad t \in [-1, 1],$$

Lemma 3.3.2 implies that

$$c \left[a_k^d(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p c^* = a_k^d(g) \geq 0.$$

The coefficients $a_k^d(u, v) \in \mathbb{C}$, $k \in \mathbb{Z}_+$, are unique for every $u, v \in X$ in this type of representation, because since each kernel $(u, v) \in X \times X \rightarrow a_k^d(u, v)$ is positive definite, and $\sum_{k=0}^{\infty} a_k^d(u, u) P_k^d(1) < \infty$, for all $u \in X$, we get that

$$\sum_{k=0}^{\infty} |a_k^d(u, v)| P_k^d(1) \leq \sum_{k=0}^{\infty} a_k^d(u, u) P_k^d(1) + \sum_{k=0}^{\infty} a_k^d(v, v) P_k^d(1) < \infty,$$

and the conclusion follows from Lemma 3.3.2. This shows that (i) implies (ii). Conversely, assume that each $f(u, v, \cdot)$ has a series representation satisfying the properties given in (ii). Since each kernel $(u, v) \in X \times X \rightarrow a_k^d(u, v)$ is positive definite, we may infer that

$$\begin{aligned} \sum_{k=0}^{\infty} |a_k^d(u, v) P_k^d(t)| &= \sum_{k=0}^{\infty} |a_k^d(u, v)| |P_k^d(t)| \\ &\leq \frac{1}{2} \sum_{k=0}^{\infty} \left[a_k^d(u, u) + a_k^d(v, v) \right] P_k^d(1) < \infty, \quad u, v \in X. \end{aligned}$$

Then, the convergence is uniform for each fixed pair $u, v \in X$. To conclude the proof, since $(u, v) \in X \times X \rightarrow a_k^d(u, v)$ and $(x, y) \in S^d \times S^d \rightarrow P_k^d(x \cdot y)$ are positive definite kernels for each $k \in \mathbb{Z}_+$, Proposition 2.2.4 implies that the kernels

$$((u, x), (v, y)) \in (X \times S^d) \times (X \times S^d) \rightarrow a_k^d(u, v) P_k^d(x \cdot y)$$

are positive definite for all $k \in \mathbb{Z}_+$, while Proposition 2.2.7 concludes that

$$((u, x), (v, y)) \in (X \times S^d) \times (X \times S^d) \rightarrow \sum_{k=0}^{\infty} a_k^d(u, v) P_k^d(x \cdot y) = f(u, v, x \cdot y)$$

is a positive definite kernel. The integral representation for a_k^d when $d \in \mathbb{N}$ is a directly consequence of the integral representation in Theorem 3.1.3 and Lemma 3.3.2. $\square$

We now illustrate the previous theorem for some relevant choices of the set X and obtain some additional properties.

Theorem 3.3.4. Let X be endowed with a topology and $f : X \times X \times [-1, 1] \rightarrow \mathbb{C}$ a continuous function such that $((u, x), (v, y)) \in (X \times S^d) \times (X \times S^d) \rightarrow f(u, v, x \cdot y)$ is a positive definite kernel, $d \in \mathbb{N} \cup \{\infty\}$. Then the coefficient functions $a_k^d : X \times X \rightarrow \mathbb{C}$, provided by Theorem 3.3.3 are continuous.

Proof. We prove the case $d \in \mathbb{N}$ first. For each pair $(u, v) \in X^2$, the set $\{(u, v, t) : t \in [-1, 1]\}$ is compact in $X \times X \times [-1, 1]$ and then, we can conclude that for any given $\varepsilon > 0$, there are open sets U and V in X such that $|f(a, b, t) - f(u, v, t)| < \varepsilon$, whenever $a, x \in U$ and $b, y \in V$. The

integral representation of a_k^d in Theorem 3.3.3 provides the means to prove its continuity. As for the continuity of a_k^∞ , it is a little bit trickier. First of all, we need to observe that for $u, v \in X$ fixed,

$$\lim_{d \rightarrow \infty} a_k^d(u, v) P_k^d(1) = a_k^\infty(u, v).$$

This property is a consequence of the results proved in (SCHOENBERG, 1942) and the uniqueness representation in Theorem 4.1.8. We also have the recurrence relation (which is derived from Proposition 3.2.2–(i) and the integral formula for $a_k^d(u, v)$ in Theorem 3.3.3)

$$\begin{aligned} a_k^{d+2}(u, v) P_k^{d+2}(1) &= \frac{(k+d-1)(k+d)}{d(2k+d-1)} a_k^d(u, v) P_k^d(1) \\ &\quad - \frac{(k+1)(k+2)}{d(2k+d+3)} a_{k+2}^d(u, v) P_{k+2}^d(1), \quad u, v \in X. \end{aligned}$$

But then

$$\left| a_k^\infty(u, v) - a_k^d(u, v) P_k^d(1) \right| \leq \frac{C}{d} [f(u, u, 1) + f(v, v, 1)], \quad u, v \in X,$$

where C is a positive constant depending on k . Hence, we have local uniform convergence in

$$\lim_{d \rightarrow \infty} a_k^d(u, v) P_k^d(1) = a_k^\infty(u, v)$$

so, the continuity of a_k^∞ follows from the continuity of a_k^d , when $d < \infty$. $\square$

For each $u, v \in X$ fixed, the convergence of the series representation for $f(u, v, \cdot)$ in Theorem 3.3.3 is uniform in $[-1, 1]$. We complement Theorem 3.3.3, if improved convergence is wanted for the series representation of the kernel.

Lemma 3.3.5. Let $((u, x), (v, y)) \in (X \times S^d)^2 \rightarrow f(u, v, x \cdot y)$ be a kernel satisfying the conditions in Theorem 3.3.3–(ii). Assume that there exists a sequence $(b_k)_{k \in \mathbb{Z}_+}$ so that

$$a_k^d(u, u) \leq b_k,$$

for all $u \in X$ and $\sum_{k=0}^\infty b_k P_k^d(1) < \infty$. Then the series representation provided by Theorem 3.3.3 is uniformly convergent for $(u, v, t) \in X^2 \times [-1, 1]$.

Proof. Let $\varepsilon > 0$ be given and select $k_0 \in \mathbb{Z}_+$ so that $\sum_{k=k_0+1}^\infty b_k P_k^d(1) < \varepsilon$. Now observe that

$$\left| f(u, v, t) - \sum_{k=0}^{k_0} a_k^d(u, v) P_k^d(t) \right| \leq \sum_{k=k_0+1}^\infty |a_k^d(u, v)| |P_k^d(t)| \leq \sum_{k=k_0+1}^\infty |a_k^d(u, v)| P_k^d(1).$$

Since the kernel $(u, v) \in X^2 \rightarrow a_k^d(u, v)$ is positive definite, this implies that

$$\sum_{k=k_0+1}^\infty |a_k^d(u, v)| P_k^d(1) \leq \frac{1}{2} \sum_{k=k_0+1}^\infty [a_k^d(u, u) + a_k^d(v, v)] P_k^d(1) \leq \sum_{k=k_0+1}^\infty b_k P_k^d(1) < \varepsilon.$$

The result follows. $\square$

Lemma 3.3.6. Let X be a group G with operation $*$ and $g : G \times [-1, 1] \rightarrow \mathbb{C}$ a function such that $g(u, \cdot) : [-1, 1] \rightarrow \mathbb{C}$ is continuous for all $u \in G$. The kernel $((u, x), (v, y)) \in (G \times S^d)^2 \rightarrow g(u * v^{-1}, x \cdot y)$ is positive definite if and only if there exist functions $b_k^d : G \rightarrow \mathbb{C}$, $k \in \mathbb{Z}_+$, such that

$$g(u, t) = \sum_{k=0}^{\infty} b_k^d(u) P_k^d(t), \quad u \in G, \quad t \in [-1, 1].$$

The kernels $(u, v) \in G \times G \rightarrow b_k^d(u * v^{-1})$ are positive definite for all $k \in \mathbb{Z}_+$ and $\sum_{k=0}^{\infty} b_k^d(e) P_k^d(1) < \infty$, where e is the neutral element in G . Further, the convergence of the series is uniform.

Proof. By Theorem 3.3.3, the kernel $((u, x), (v, y)) \in (G \times S^d)^2 \rightarrow g(u * v^{-1}, x \cdot y)$ is positive definite if and only if

$$g(u * v^{-1}, t) = \sum_{k=0}^{\infty} a_k^d(u, v) P_k^d(t), \quad u, v \in G, \quad t \in [-1, 1],$$

where each kernel $(u, v) \in G \times G \rightarrow a_k^d(u, v)$ is positive definite and $\sum_{k=0}^{\infty} a_k^d(u, u) P_k^d(1) < \infty$, for all $u \in G$. But, for all $t \in [-1, 1]$,

$$g(u * v^{-1}, t) = g((u * v^{-1}) * e, t) = \sum_{k=0}^{\infty} a_k^d(u * v^{-1}, e) P_k^d(t).$$

Since the representation is unique, we have that $a_k^d(u, v) = a_k^d(u * v^{-1}, e)$, for all $u, v \in G$. Defining $b_k^d(w) := a_k^d(w, e)$, $w \in G$, we obtain that the kernel a_k^d is positive definite if and only if the kernel $(u, v) \in G \times G \rightarrow b_k^d(u * v^{-1})$ is positive definite, while the convergence $\sum_{k=0}^{\infty} a_k^d(u, u) P_k^d(1) < \infty$, for all $u \in G$ is equivalent to $\sum_{k=0}^{\infty} b_k^d(e) P_k^d(1) < \infty$. The convergence is uniform by Lemma 3.3.5. $\square$

If X is a locally compact group, then Theorem 3.3.4 and Lemma 3.3.6 combined is all that is needed in order to recover the main results proved in (BERG; PORCU, 2017).

A result found in (YAGLOM, 1987) characterizes the continuous matrix valued positive definite isotropic kernels on S^d , $d \in \mathbb{N} \cup \{\infty\}$. We may obtain the very same theorem from the bridge Propositions 2.3.2 and 2.3.3 and Theorem 3.3.3. The proof is omitted.

Theorem 3.3.7. Let $f : [-1, 1] \rightarrow M_l(\mathbb{C})$ be a continuous function and $d \in \mathbb{N} \cup \{\infty\}$. The kernel

$$(x, y) \in S^d \times S^d \rightarrow f(x \cdot y) \in M_l(\mathbb{C})$$

is positive definite if and only if f has a series representation

$$f(t) = \sum_{k=0}^{\infty} A_k^d P_k^d(t)$$

for all $t \in [-1, 1]$, where $A_k^d \in M_l(\mathbb{C})$ is a positive semidefinite matrix for all $k \in \mathbb{Z}_+$ and the series converges at $t = 1$.

Next, we prove a characterization for the continuous and positive definite isotropic kernels on a product of spheres. The proof of this theorem in (GUELLA; MENEGATTO; PERON, 2016a) is constructive, and the major steps are essentially reformulations of the properties that appear in (SCHOENBERG, 1942) to this new context. Contrary to the method we are using in this section, the knowledge of Schoenberg characterization does not play the same role in (GUELLA; MENEGATTO; PERON, 2016a).

Theorem 3.3.8. Let $f : [-1, 1] \times [-1, 1] \rightarrow \mathbb{C}$ be a continuous function and $d, m \in \mathbb{N} \cup \{\infty\}$. The kernel

$$((x, w), (y, z)) \in (S^d \times S^m)^2 \rightarrow f(x \cdot y, w \cdot z)$$

is positive definite if and only if f has a series representation

$$f(t, s) = \sum_{k, l=0}^{\infty} a_{k, l} P_k^d(t) P_l^m(s)$$

where $a_{k, l} \geq 0$ for all $k, l \in \mathbb{Z}_+$ and the series converges at $t = s = 1$. The representation is unique.

Proof. If the kernel is positive definite, Theorem 3.3.3 and Theorem 3.3.4 imply that

$$f(x \cdot y, s) = \sum_{l=0}^{\infty} a_l^m(x, y) P_l^m(s), \quad x, y \in S^d, \quad s \in [-1, 1],$$

where a_l^m is a continuous positive definite kernel in S^d and the series converges at $t = 1$ for all pair of points $x = y \in S^d$. Since $f(Qx \cdot Qy, s) = f(x \cdot y, s)$ for all orthogonal matrices $Q \in O(d+1)$, all points $x, y \in S^d$ and $s \in [-1, 1]$, the uniqueness in the representation on Theorem 3.3.3 leads to $a_l^m(Qx, Qy) = a_l^m(x, y)$ for all $x, y \in S^d$ and $l \in \mathbb{Z}_+$. Then, there exists a continuous function $g_l : [-1, 1] \rightarrow \mathbb{C}$, that defines a positive definite isotropic kernel on S^d such that $a_l^m(x, y) = g_l(x \cdot y)$, for all $x, y \in S^d$ and $l \in \mathbb{Z}_+$. In particular, each function g_l has a unique series representation

$$g_l(t) = \sum_{k=0}^{\infty} a_{k, l} P_k^d(t), \quad t \in [-1, 1]$$

with $a_{k, l} \geq 0$ for all $k, l \in \mathbb{Z}_+$ and the series converges at $t = 1$. Finally,

$$f(x \cdot y, s) = \sum_{l=0}^{\infty} a_l^m(x, y) P_l^m(s) = \sum_{l=0}^{\infty} g_l(x \cdot y) P_l^m(s) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} a_{k, l} P_k^d(x \cdot y) P_l^m(s)$$

and $\sum_{l=0}^{\infty} \sum_{k=0}^{\infty} a_{k, l} P_k^d(x \cdot x) P_l^m(1) = \sum_{k, l=0}^{\infty} a_{k, l} P_k^d(1) P_l^m(1) < \infty$ for all $x \in S^d$. The representation is unique, because all previous representations were unique, but this fact can also be checked using the orthogonality relations for Gegenbauer polynomials in Equation (3.2) and standard properties of analytic functions.

Note that the kernels

$$((x, w), (y, z)) \in (S^d \times S^m)^2 \rightarrow P_k^d(x \cdot y) P_l^m(w \cdot z)$$

are positive definite for all $k, l \in \mathbb{Z}_+$ by Proposition 2.2.4. The conclusion comes from Proposition 2.2.7, since the limit is well defined because $\sum_{k, l=0}^{\infty} a_{k, l} P_k^d(1) P_l^m(1) < \infty$. $\square$

In (BERG; PERON; PORCU, 2017), the authors generalized the results obtained in (BERG; PORCU, 2017) and also the main results in this section, by replacing the sphere with a compact Gelfand pair. For information on compact Gelfand pairs see (WOLF, 2007) and (DIJK, 2009).

3.4 The strictly positive definite case

In this section, we present necessary and sufficient conditions on a positive definite kernel representable as in Theorem 3.3.3, in order that it be strictly positive definite, when $d \in \{2, 3, \dots, \infty\}$. With this characterization, we were able to reobtain the main results in (GUELLA; MENEGATTO; PERON, 2016b) and (GUELLA; MENEGATTO, 2016). We also provide a characterization of the continuous matrix valued strictly positive definite isotropic kernels on S^d , when $d \in \{2, 3, \dots, \infty\}$. This section is based in (GUELLA; MENEGATTO, 2018c).

In order to simplify the proof for the characterization, we will work with sets defined by the concept below.

Definition 3.4.1. For a subset $\{u_1, u_2, \dots, u_p\}$ of X and a antipodal free set $\{x_1, x_2, \dots, x_q\}$ of S^d , $d \in \{2, 3, \dots, \infty\}$, the **enhanced subset** of $X \times S^d$ generated by them is the ordered set

$$\begin{aligned} & \{(u_1, x_1), (u_2, x_1), \dots, (u_p, x_1), (u_1, x_2), (u_2, x_2), \dots, (u_p, x_2), \dots, \\ & (u_1, x_q), (u_2, x_q), \dots, (u_p, x_q), (u_1, -x_1), (u_2, -x_1), \dots, (u_p, -x_1), \\ & (u_1, -x_2), (u_2, -x_2), \dots, (u_p, -x_2), \dots, (u_1, -x_q), (u_2, -x_q), \dots, (u_p, -x_q)\}. \end{aligned}$$

The positive numbers p and q in the previous definition are independent of each other. We are fixing an order among the elements of the set above so that the writing of the upcoming results can be made easier. In order to simplify the notation, we will mention an enhanced set, without referring to the initial subset of X or the antipodal free set of S^d .

It is not hard to see that if $B = \{(u'_1, x'_1), (u'_2, x'_2), \dots, (u'_n, x'_n)\}$ is a subset of $X \times S^d$, then there exists an enhanced set A of $X \times S^d$ that contains B . Indeed, if $\{u_1, u_2, \dots, u_p\}$ encompasses the distinct elements among the u'_μ , the subset $\{x_1, x_2, \dots, x_q\}$ of S^d contains no pairs of antipodal points of S^d and satisfies $x'_\beta \in \{\pm x_1, \pm x_2, \dots, \pm x_q\}$, $\beta = 1, 2, \dots, n$, then the enhanced subset of $X \times S^d$ generated by $\{u_1, u_2, \dots, u_p\}$ and $\{x_1, x_2, \dots, x_q\}$ provides an example. With this in mind, a similar conclusion of Lemma 2.2.5 is possible, we omit the proof due to its similarities.

Lemma 3.4.2. Let $((u, x), (v, y)) \in (X \times S^d)^2 \rightarrow f(u, v, x \cdot y)$ be a positive definite kernel. The kernel is strictly positive definite if and only if for every enhanced subset of $X \times S^d$, its interpolation matrix through this kernel is positive definite.

For a kernel $((u, x), (v, y)) \in (X \times S^d)^2 \rightarrow f(u, v, x \cdot y)$ satisfying Theorem 3.3.3-(ii), we describe what happens with the interpolation matrix at an enhanced set A of $X \times S^d$ through the this kernel. First

$$\mathcal{J}(f, A) = \sum_{k=0}^{\infty} \mathcal{J}(f, A, k)$$

where $\mathcal{J}(f, A, k) \in M_{2pq}(\mathbb{C})$ is the interpolation matrix through the kernel

$$((u, x), (v, y)) \in (X \times S^d)^2 \rightarrow a_k^d(u, v) P_k^d(x \cdot y)$$

The order we have fixed for the elements in the enhanced set A of $X \times S^d$ determines a special block representation for each matrix $\mathcal{J}(f, A, k)$, that is,

$$\mathcal{J}(f, A, k) = [M_{\rho\sigma}(k)]_{\rho, \sigma=1}^2,$$

where each block $M_{\rho\sigma}(k) \in M_{pq}(\mathbb{C})$. Note that the order of the elements imply that $M_{11}(k)$ is the interpolation matrix on the first half of the elements of A through the kernel $a_k^d(u, v) P_k^d(x \cdot y)$ while $M_{22}(k)$ is the interpolation matrix on the second half of the elements of A through the kernel $a_k^d(u, v) P_k^d(x \cdot y)$.

Each matrix $M_{\rho\sigma}(k)$ has its own block structure

$$M_{\rho\sigma}(k) = [M_{\rho\sigma}^{\alpha\beta}(k)]_{\alpha, \beta=1}^q, \quad \rho, \sigma = 1, 2,$$

where $M_{\rho\sigma}^{\alpha\beta}(k) \in M_p(\mathbb{C})$ can be described as

$$M_{\rho\sigma}^{\alpha\beta}(k) = \begin{cases} [a_k^d(u_\mu, u_\nu)]_{\mu, \nu=1}^p P_k^d(x_\alpha \cdot x_\beta), & \rho = \sigma = 1 \\ [a_k^d(u_\mu, u_\nu)]_{\mu, \nu=1}^p P_k^d(x_\alpha \cdot -x_\beta), & \rho = 1, \sigma = 2 \\ [a_k^d(u_\mu, u_\nu)]_{\mu, \nu=1}^p P_k^d(-x_\alpha \cdot x_\beta), & \rho = 2, \sigma = 1 \\ [a_k^d(u_\mu, u_\nu)]_{\mu, \nu=1}^p P_k^d(-x_\alpha \cdot -x_\beta), & \rho = \sigma = 2 \end{cases}$$

For instance, if A is generated by $\{u_1, u_2\} \subset X$ and $\{x_1, x_2\} \subset S^d$, then $\mathcal{J}(f, A, k)$ is the 8×8 matrix

$$\begin{pmatrix} a_{1,1}^k b_{1,1}^k & a_{1,2}^k b_{1,1}^k & a_{1,1}^k b_{1,2}^k & a_{1,2}^k b_{1,2}^k & a_{1,1}^k c_{1,1}^k & a_{1,2}^k c_{1,1}^k & a_{1,1}^k c_{1,2}^k & a_{1,2}^k c_{1,2}^k \\ a_{2,1}^k b_{1,1}^k & a_{2,2}^k b_{1,1}^k & a_{2,1}^k b_{1,2}^k & a_{2,2}^k b_{1,2}^k & a_{2,1}^k c_{1,1}^k & a_{2,2}^k c_{1,1}^k & a_{2,1}^k c_{1,2}^k & a_{2,2}^k c_{1,2}^k \\ a_{1,1}^k b_{2,1}^k & a_{1,2}^k b_{2,1}^k & a_{1,1}^k b_{2,2}^k & a_{1,2}^k b_{2,2}^k & a_{1,1}^k c_{2,1}^k & a_{1,2}^k c_{2,1}^k & a_{1,1}^k c_{2,2}^k & a_{1,2}^k c_{2,2}^k \\ a_{2,1}^k b_{2,1}^k & a_{2,2}^k b_{2,1}^k & a_{2,1}^k b_{2,2}^k & a_{2,2}^k b_{2,2}^k & a_{2,1}^k c_{2,1}^k & a_{2,2}^k c_{2,1}^k & a_{2,1}^k c_{2,2}^k & a_{2,2}^k c_{2,2}^k \\ \\ a_{1,1}^k c_{1,1}^k & a_{1,2}^k c_{1,1}^k & a_{1,1}^k c_{1,2}^k & a_{1,2}^k c_{1,2}^k & a_{1,1}^k b_{1,1}^k & a_{1,2}^k b_{1,1}^k & a_{1,1}^k b_{1,2}^k & a_{1,2}^k b_{1,2}^k \\ a_{2,1}^k c_{1,1}^k & a_{2,2}^k c_{1,1}^k & a_{2,1}^k c_{1,2}^k & a_{2,2}^k c_{1,2}^k & a_{2,1}^k b_{1,1}^k & a_{2,2}^k b_{1,1}^k & a_{2,1}^k b_{1,2}^k & a_{2,2}^k b_{1,2}^k \\ a_{1,1}^k c_{2,1}^k & a_{1,2}^k c_{2,1}^k & a_{1,1}^k c_{2,2}^k & a_{1,2}^k c_{2,2}^k & a_{1,1}^k b_{2,1}^k & a_{1,2}^k b_{2,1}^k & a_{1,1}^k b_{2,2}^k & a_{1,2}^k b_{2,2}^k \\ a_{2,1}^k c_{2,1}^k & a_{2,2}^k c_{2,1}^k & a_{2,1}^k c_{2,2}^k & a_{2,2}^k c_{2,2}^k & a_{2,1}^k b_{2,1}^k & a_{2,2}^k b_{2,1}^k & a_{2,1}^k b_{2,2}^k & a_{2,2}^k b_{2,2}^k \end{pmatrix}$$

where $a_{\mu, \nu}^k = a_k^d(u_\mu, u_\nu)$, $b_{\alpha, \beta}^k = P_k^d(x_\alpha \cdot x_\beta)$ and $c_{\alpha, \beta}^k = P_k^d(-x_\alpha \cdot x_\beta)$.

The facts described above and the previous results lead to the following characterization for strict positive definiteness based on the interpolation matrices at enhanced sets.

Lemma 3.4.3. Let $((u, x), (v, y)) \in (X \times S^d)^2 \rightarrow f(u, v, x \cdot y)$ be a kernel as in Theorem 3.3.3-(ii). The following assertions are equivalent;

- (i) The kernel is strictly positive definite;
- (ii) If A is an enhanced subset of $X \times S^d$ generated by $\{u_1, u_2, \dots, u_p\} \subset X$ and the antipodal free set $\{x_1, x_2, \dots, x_q\}$ of S^d , then the only solution $(v_1, v_2) \in (\mathbb{C}^{pq})^2$ to the system

$$[v_1 + (-1)^k v_2] M_{11}(k) [v_1 + (-1)^k v_2]^* = 0,$$

for $k \in \mathbb{Z}_+$, is the trivial solution $v_1 = v_2 = 0$.

- (iii) If A is an enhanced subset of $X \times S^d$ generated by $\{u_1, u_2, \dots, u_p\} \subset X$ and the antipodal free set $\{x_1, x_2, \dots, x_q\}$ of S^d , then the only vector $(v_1^1, v_1^2, \dots, v_1^q, v_2^1, v_2^2, \dots, v_2^q)$ of $(\mathbb{C}^p)^{2q}$ satisfying

$$\sum_{\alpha, \beta=1}^q \left\{ \left(v_1^\alpha + (-1)^k v_2^\alpha \right) \left[a_k^d(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p \left(v_1^\beta + (-1)^k v_2^\beta \right)^* \right\} P_k^d(x_\alpha \cdot x_\beta) = 0,$$

for all $k \in \mathbb{Z}_+$, is the zero vector.

Proof. In order to prove that (i) implies (ii), observe that if $v = (v_1, v_2) \in (\mathbb{C}^{pq})^2$ is such that $v \mathcal{J}(f, A) v^* = 0$, then $\sum_{k=0}^\infty v \mathcal{J}(f, A, k) v^* = 0$. Each matrix $\mathcal{J}(f, A, k)$ is positive semidefinite, and this along with the previous equality implies that $v \mathcal{J}(f, A, k) v^* = 0$ for all $k \in \mathbb{Z}_+$. To conclude, notice that $P_k^d((-1)^\rho x_\alpha \cdot (-1)^\sigma x_\beta) = (-1)^{k(\rho+\sigma)} P_k^d(x_\alpha \cdot x_\beta)$ by Proposition 3.1.6-(iii), so $\mathcal{J}(f, A, k) = [M_{\rho\sigma}(k)]_{\rho, \sigma=1}^2 = [(-1)^{k(\rho+\sigma)} M_{11}(k)]_{\rho, \sigma=1}^2$, and then the equations in (ii) are just the fact that $v \mathcal{J}(f, A, k) v^* = 0$. The converse is true, since if $v \mathcal{J}(f, A, k) v^* = 0$ for all $k \in \mathbb{Z}_+$, by summing all this equations we get that $v \mathcal{J}(f, A) v^* = 0$.

The equivalence between (ii) and (iii) follows after we introduce components in the vectors v_1 and v_2 appearing in (ii). $\square$

In the next result, we break the main equation in Lemma 3.4.3-(iii) according to the parity of the indices k involved.

Proposition 3.4.4. Let $((u, x), (v, y)) \in (X \times S^d)^2 \rightarrow f(u, v, x \cdot y)$ be a kernel as in Theorem 3.3.3-(ii). The following assertions are equivalent:

- (i) The kernel is strictly positive definite;
- (ii) If p is a positive integer, $u_1, u_2, \dots, u_p$ are distinct points in X and $\{x_1, x_2, \dots, x_q\}$ is a antipodal free set of S^d , then the only solution $(d_1^1, d_1^2, \dots, d_1^q, d_2^1, d_2^2, \dots, d_2^q)$ in $(\mathbb{C}^p)^{2q}$ to the system

$$\left| \begin{array}{l} \sum_{\alpha, \beta=1}^q \left\{ d_1^\alpha \left[a_k^d(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p (d_1^\beta)^* \right\} P_k^d(x_\alpha \cdot x_\beta) = 0, \quad k \in 2\mathbb{Z}_+ + 1 \\ \sum_{\alpha, \beta=1}^q \left\{ d_2^\alpha \left[a_k^d(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p (d_2^\beta)^* \right\} P_k^d(x_\alpha \cdot x_\beta) = 0, \quad k \in 2\mathbb{Z}_+ \end{array} \right.$$

is the trivial one.

Proof. For distinct points $u_1, u_2, \dots, u_p$ in X and an antipodal free set $\{x_1, x_2, \dots, x_q\}$ of S^d , assume that the system in (ii) has a nontrivial solution $(d_1^1, d_1^2, \dots, d_1^q, d_2^1, \dots, d_2^q)$. Defining $v_1^\alpha = -d_1^\alpha + d_2^\alpha$ and $v_2^\alpha = d_1^\alpha + d_2^\alpha$, $\alpha = 1, 2, \dots, q$, it is promptly seen that $(v_1^1, v_1^2, \dots, v_1^q, v_2^1, v_2^2, \dots, v_2^q)$ belongs to $(\mathbb{C}^p)^{2q} \setminus \{0\}$ and that

$$\sum_{\alpha, \beta=1}^q \left[(v_1^\alpha + (-1)^k v_2^\alpha) [a_k(u_\mu, u_\nu)]_{\mu, \nu=1}^p (\overline{v_1^\beta} + (-1)^k \overline{v_2^\beta}) \right] P_k^d(x_\alpha \cdot x_\beta) = 0,$$

for all $k \in \mathbb{Z}_+$, that is, Lemma 3.4.3-(iii) does not hold. In particular, the kernel in the statement of the theorem is not strictly positive definite. This shows that (i) implies (ii).

The converse follows along the same lines. If (i) does not hold, Lemma 3.4.3 guarantees the existence of distinct points $u_1, u_2, \dots, u_p$ in X , an antipodal free set $\{x_1, x_2, \dots, x_q\}$ of S^d and a vector $(v_1^1, v_1^2, \dots, v_1^q, v_2^1, v_2^2, \dots, v_2^q)$ in $(\mathbb{C}^p)^{2q} \setminus \{0\}$ so that Lemma 3.4.3-(iii) does not hold. The vector $(v_1^1 - v_2^1, v_1^2 - v_2^2, \dots, v_1^q - v_2^q, v_1^1 + v_2^1, v_1^2 + v_2^2, \dots, v_1^q + v_2^q)$ of $(\mathbb{C}^p)^{2q}$ then provides a non-trivial solution to the system in (ii). $\square$

Our next theorem provides a characterization of the strictly positive definite kernels on $X \times S^d$ as in Theorem 3.3.3-(ii), when $d \in \{2, 3, \dots, \infty\}$.

Theorem 3.4.5. Let $d \in \{2, 3, \dots, \infty\}$ and $((u, x), (v, y)) \in (X \times S^d)^2 \rightarrow f(u, v, x \cdot y)$ be a kernel as in Theorem 3.3.3-(ii). The following assertions are equivalent:

- (i) The kernel is strictly positive definite;
- (ii) If p is a positive integer, $u_1, u_2, \dots, u_p$ are distinct points in X and $c = (c_1, c_2, \dots, c_p)$ is a nonzero vector in $\mathbb{C}^p$, then

$$\{k \in \mathbb{Z}_+ : c \left[a_k^d(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p c^* > 0\}$$

contains infinitely many even and infinitely many odd integers.

Proof. If the kernel in the statement of the theorem is strictly positive definite, then for distinct points $u_1, u_2, \dots, u_p$ in X and a nonzero vector c in $\mathbb{C}^p$, Lemma 3.3.1-(iii) reveals that

$$s \in [-1, 1] \rightarrow \sum_{\mu, \nu=1}^p c_\mu \overline{c_\nu} f(u_\mu, u_\nu, s) = \sum_{k=0}^{\infty} c \left[a_k^d(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p c^* P_k^d(s)$$

defines a strictly positive definite isotropic kernel on S^d , and then, by Theorem 3.2.5, the set

$$\{k \in \mathbb{Z}_+ : c \left[a_k^d(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p c^* > 0\}$$

contains infinitely many even and infinitely many odd integers. Conversely, assume that (ii) holds. Let $u_1, u_2, \dots, u_p$ be distinct points in X , an antipodal free set $\{y_1, y_2, \dots, y_q\}$ of S^d and a vector $(d_1^1, d_1^2, \dots, d_1^q, d_2^1, d_2^2, \dots, d_2^q)$ in $(\mathbb{C}^p)^{2q}$ so that the two equations in Proposition 3.4.4-(ii) hold. We will focus on the even case

$$\sum_{\alpha, \beta=1}^q \left\{ (d_2^\alpha) \left[a_k^d(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p (d_2^\beta)^* \right\} P_k^d(y_\alpha \cdot y_\beta) = 0, \quad k \in 2\mathbb{Z}_+, \quad (3.10)$$

once the odd case can be handled similarly. By Lemma 3.2.3, we can pick $k_0 \in \mathbb{Z}_+$ such that the matrices

$$[P_k^d(y_\alpha \cdot y_\beta)]_{\alpha, \beta=1}^q$$

are positive definite for all $k \geq k_0$. Applying Lemma 2.1.9 in Equation (3.10), we obtain

$$d_2^\alpha \left[a_k^d(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p (d_2^\alpha)^* = 0,$$

for all $\alpha \in \{1, 2, \dots, p\}$ and $k \in 2\mathbb{N}$ with $k \geq k_0$. But then the sets

$$\{k \in \mathbb{Z}_+ : d_2^\alpha \left[a_k^d(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p (d_2^\alpha)^* > 0\}$$

contain finitely many even numbers. However, by the hypothesis, this can only happen if $d_2^\alpha = 0$ for all α , and then the kernel is strictly positive definite by Proposition 3.4.4. $\square$

A characterization similar to Theorem 3.4.5 for $d = 1$ remains open. The easy part is reported below without proof.

Theorem 3.4.6. Let $((u, x), (v, y)) \in (X \times S^1)^2 \rightarrow f(u, v, x \cdot y)$ be a kernel as in Theorem 3.3.3-(ii). If it is strictly positive definite, then for any distinct points $u_1, u_2, \dots, u_p$ in X and a nonzero vector c in $\mathbb{C}^p$, the set

$$\left\{ k \in \mathbb{Z} : c \left[a_{|k|}^1(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p c^* > 0 \right\}$$

intersects every full arithmetic progression in $\mathbb{Z}$.

The next result depends upon some new notation. Given a natural number γ and a positive definite kernel $((u, x), (v, y)) \in (X \times S^d)^2 \rightarrow f(u, v, x \cdot y)$ as in Theorem 3.3.3-(ii), we introduce the truncated functions

$$a_\gamma^o(u, v) = \sum_{k \geq \gamma} a_{2k+1}^d(u, v), \quad u, v \in X$$

and

$$a_\gamma^e(u, v) = \sum_{k \geq \gamma} a_{2k}^d(u, v), \quad u, v \in X.$$

Since $P_k^d(1) \geq 1$ for all $d \in \{2, 3, \dots, \infty\}$, $k \in \mathbb{Z}_+$, and each kernel $(u, v) \in X \times X \rightarrow a_k^d(u, v)$ is positive definite, we have that

$$\begin{aligned} \left| \sum_{k=k_1}^{k_2} a_{2k}^d(u, v) \right| &\leq \sum_{k=k_1}^{k_2} |a_{2k}^d(u, v)| P_k^d(1) \\ &\leq \sum_{k=k_1}^{k_2} [a_{2k}^d(u, u) + a_{2k}^d(v, v)] P_k^d(1), \quad u, v \in X, \quad k_2 \geq k_1. \end{aligned}$$

But $\sum_{k=0}^\infty a_k^d(u, u) P_k^d(1) < \infty$, $u \in X$, so each function a_γ^e is well defined. A similar argument leads to the same conclusion for the functions a_γ^o .

Another characterization for strictly positive definite kernels as in Theorem 3.3.3-(ii) is based upon the two functions introduced above.

Theorem 3.4.7. Let $d \in \{2, 3, \dots, \infty\}$ and let $((u, x), (v, y)) \in (X \times S^d)^2 \rightarrow f(u, v, x \cdot y)$ be a kernel as in Theorem 3.3.3-(ii). The following assertions are equivalent:

- (i) The kernel is strictly positive definite;
- (ii) For each $\gamma \in \mathbb{N}$, the kernels $(u, v) \in X^2 \rightarrow a_\gamma^o(u, v)$ and $(u, v) \in X^2 \rightarrow a_\gamma^e(u, v)$ are strictly positive definite.

Proof. If $u_1, u_2, \dots, u_p$ are distinct points in X and $c \in \mathbb{C}^p$, we have that

$$c \left[a_\gamma^o(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p c^* = \sum_{k \geq \gamma} c \left[a_{2k+1}(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p c^*.$$

Hence, if the kernel is strictly positive definite and $c \neq 0$, then Theorem 3.4.5 implies that

$$c \left[a_\gamma^o(u_\mu, u_\nu) \right]_{\mu, \nu=1}^p c^* > 0.$$

In particular, $(u, v) \in X^2 \rightarrow a_\gamma^o(u, v)$ is strictly positive definite. A similar argument leads to the same conclusion for $(u, v) \in X^2 \rightarrow a_\gamma^e(u, v)$. Conversely, if both kernels in (ii) are strictly positive definite, it is promptly seen that Condition (ii) in Theorem 3.4.5 holds. Therefore, the kernel is strictly positive definite. $\square$

As an application of the previous result, we provide the characterization of the kernels in Theorem 3.3.8 that are strictly positive definite when $d, m \in \mathbb{N} \cup \{\infty\}$, except the case $d = m = 1$, which is proved using other methods in (GUELLA; MENEGATTO; PERON, 2017). First, we deal with the case $d, m \in \{2, 3, \dots, \infty\}$ that appeared in (GUELLA; MENEGATTO, 2016).

Theorem 3.4.8. Let $f : [-1, 1] \times [-1, 1] \rightarrow \mathbb{C}$ be a continuous function and $d, m \in \{2, 3, \dots, \infty\}$. The kernel $((x, w), (y, z)) \in (S^d \times S^m)^2 \rightarrow f(x \cdot y, w \cdot z)$ is strictly positive definite if and only if the set

$$K_f := \{(k, l) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : a_{k,l}^{d,m} > 0\}$$

contains four sequences $(k_n^{i,j}, l_n^{i,j})_{n \in \mathbb{N}} \subset K_f$, $i, j \in \{0, 1\}$, with the property that

$$(k_n^{i,j}, l_n^{i,j}) \in (2\mathbb{N} + i) \times (2\mathbb{N} + j)$$

for all $n \in \mathbb{N}$ and $i, j \in \{0, 1\}$, and

$$\lim_{n \rightarrow \infty} \min\{k_n^{i,j}, l_n^{i,j}\} = \infty$$

for all $i, j \in \{0, 1\}$.

Proof. Using the same notation of Theorem 3.3.8 and Theorem 3.4.7, the kernel is strictly positive definite if and only if

$$a_\gamma^o(x, y) = \sum_{l \geq \gamma} a_{2l+1}^m(x, y) = \sum_{k=0}^{\infty} \sum_{l \geq \gamma} a_{2l+1,k}^{d,m} P_k^d(x \cdot y)$$

and

$$a_\gamma^o(x, y) = \sum_{l \geq \gamma} a_{2l}^m(x, y) = \sum_{k=0}^{\infty} \sum_{l \geq \gamma} a_{2l, k}^{d, m} P_k^d(x \cdot y)$$

are strictly positive definite kernels for all $\gamma \in \mathbb{N}$. Note that the left side of both equations imply that the kernels a_γ^o and a_γ^e are continuous and isotropic. From that, the kernel is strictly positive definite if and only if the sets

$$K_f^{\gamma, 1} := \{l \in \mathbb{Z}_+ : k \geq \gamma, \quad k \in 2\mathbb{Z}_+ + 1, \quad a_{k, l}^{d, m} > 0\}$$

$$K_f^{\gamma, 0} := \{l \in \mathbb{Z}_+ : k \geq \gamma, \quad k \in 2\mathbb{Z}_+, \quad a_{k, l}^{d, m} > 0\}$$

contain infinitely many even and infinitely many odd integers, for all $\gamma \in \mathbb{N}$. Thus, if the kernel is strictly positive definite, for every $\gamma \in \mathbb{N}$ we can pick $(k_\gamma^{i, j}, l_\gamma^{i, j}) \in K_f$, $i, j \in \{0, 1\}$, with the property that $k_\gamma^{i, j} \in 2\mathbb{N} + i$, $l_\gamma^{i, j} \in 2\mathbb{N} + j$, and $k_\gamma^{i, j} \geq \gamma$, $l_\gamma^{i, j} \geq \gamma$, which are the desired four sequences. For the converse, having the four sequences at our disposal, its properties assert that the sets $K_f^{\gamma, 1}$ and $K_f^{\gamma, 0}$ contain infinitely many even and infinitely many odd integers for all $\gamma \in \mathbb{N}$. $\square$

The case $d \in \{2, 3, \dots, \infty\}$ and $m = 1$ or $m \in \{2, 3, \dots, \infty\}$ and $d = 1$, can be proved using similar arguments, taking into account Theorem 3.2.10. This case appeared in (GUELLA; MENEGATTO; PERON, 2016b). We omit the proof.

Theorem 3.4.9. Let $f : [-1, 1] \times [-1, 1] \rightarrow \mathbb{C}$ be a continuous function and $d \in \{2, 3, \dots, \infty\}$. The kernel $((x, w), (y, z)) \in (S^d \times S^1)^2 \rightarrow f(x \cdot y, w \cdot z)$ is positive definite if and only if for each $\gamma \in \mathbb{N}$, the sets

$$K_f^{\gamma, 1} = \{l \in \mathbb{Z} : k \geq \gamma, \quad k \in 2\mathbb{Z}_+ + 1, \quad a_{k, |l|}^{d, 1} > 0\}$$

$$K_f^{\gamma, 0} = \{l \in \mathbb{Z} : k \geq \gamma, \quad k \in 2\mathbb{Z}_+, \quad a_{k, |l|}^{d, 1} > 0\}$$

intersect every arithmetic progression in $\mathbb{Z}$.

At last, a useful sufficient condition for a kernel as in Theorem 3.3.3-(ii) being strictly positive definite is presented. Note that it includes the case S^1 .

Theorem 3.4.10. Let $((u, x), (v, y)) \in (X \times S^d)^2 \rightarrow f(u, v, x \cdot y)$ be a kernel as in Theorem 3.3.3-(ii). In order that it be strictly positive definite it is sufficient that:

- (i) ($d \in \{2, 3, \dots, \infty\}$) The kernels $(u, v) \in X^2 \rightarrow a_k^d(u, v)$ are strictly positive definite for infinitely many even k and infinitely many odd k .
- (ii) ($d = 1$) The set $\{k \in \mathbb{Z} : (u, v) \in X \times X \rightarrow a_{|k|}^1(u, v) \text{ is strictly positive definite}\}$ intersects every full arithmetic progression in $\mathbb{Z}$.

Proof. Assertion (i) follows from Theorem 3.4.7. As for (ii), assume that

$$K := \{k \in \mathbb{Z} : (u, v) \in X \times X \rightarrow a_{|k|}^1(u, v) \text{ is strictly positive definite}\}$$

intersects every full arithmetic progression in $\mathbb{Z}$. Given distinct points $u_1, u_2, \dots, u_p$ in X and distinct points $x_1, x_2, \dots, x_q$ in S^1 , if $c_{\alpha, \mu} \in \mathbb{C}$, $1 \leq \alpha \leq q$, $1 \leq \mu \leq p$ are such that

$$0 = \sum_{\mu, \nu=1}^p \sum_{\alpha, \beta=1}^q c_{\mu, \alpha} \overline{c_{\nu, \beta}} f(u_\mu, u_\nu, x_\alpha \cdot x_\beta) = \sum_{k=0}^{\infty} \sum_{\mu, \nu=1}^p \sum_{\alpha, \beta=1}^q c_{\mu, \alpha} \overline{c_{\nu, \beta}} a_k^1(u_\mu, u_\nu) P_k^1(x_\alpha \cdot x_\beta),$$

then

$$0 = \sum_{\mu, \nu=1}^p \sum_{\alpha, \beta=1}^q c_{\mu, \alpha} \overline{c_{\nu, \beta}} a_{|k|}^1(u_\mu, u_\nu) P_{|k|}^1(x_\alpha \cdot x_\beta), \quad k \in K.$$

By the hypothesis, the matrices $[a_{|k|}^1(u_\mu, u_\nu)]_{\mu, \nu=1}^p$ are positive definite for all $k \in K$. Applying Lemma 2.1.9, we obtain that

$$0 = \sum_{\alpha, \beta=1}^q c_{\mu, \alpha} \overline{c_{\mu, \beta}} P_{|k|}^1(x_\alpha \cdot x_\beta), \quad k \in K, \quad 1 \leq \mu \leq p.$$

Finally, by Theorem 3.2.10, the continuous function $g(t) = \sum_{k \in K} 2^{-|k|} P_{|k|}^1(t)$ defines a strictly positive definite isotropic kernel on S^1 , so the equality

$$\sum_{\alpha, \beta=1}^q c_{\mu, \alpha} \overline{c_{\mu, \beta}} g(x_\alpha \cdot x_\beta) = \sum_{k \in K} \frac{1}{2^{|k|}} \sum_{\alpha, \beta=1}^q c_{\mu, \alpha} \overline{c_{\mu, \beta}} P_{|k|}^1(x_\alpha \cdot x_\beta) = 0,$$

implies that the scalars $c_{\mu, \alpha}$ are all zero. $\square$

By the Schur product result in Proposition 2.2.4, if $f : X \times X \times [-1, 1] \rightarrow \mathbb{C}$ and $g : X \times X \times [-1, 1] \rightarrow \mathbb{C}$ are two functions that satisfies Theorem 3.3.3–(i) for the same real sphere S^d , $d \in \mathbb{N} \cup \{\infty\}$, then the function $h : X \times X \times [-1, 1] \rightarrow \mathbb{C}$, defined by

$$h(u, v, t) := f(u, v, t)g(u, v, t),$$

for all $u, v \in X$ and $t \in [-1, 1]$ also satisfies Theorem 3.3.3–(i) for the same real sphere S^d , $d \in \mathbb{N} \cup \{\infty\}$. In (BONFIM; GUELLA; MENEGATTO, 2018), it was presented a characterization for the strict positive definiteness of the isotropic kernel defined by h on $X \times S^d$, for $d \in \{2, 3, \dots, \infty\}$. It was also presented a characterization for some specific choices of the set X , related questions for matrix valued positive definite isotropic kernels and similar characterizations involving two point compact homogeneous spaces and their continuous isotropic kernels.

POSITIVE DEFINITE RADIAL KERNELS

In this chapter, we move to the continuous positive definite radial kernels on Euclidean spaces. In Section 4.1, we obtain a characterization for the continuous positive definite radial kernels from the famous Bochner's theorem. In Section 4.2, we characterize the continuous strictly positive definite radial kernels, based in ([CASTELL; FILBIR; SZWARC, 2005](#)), except for the case $\mathbb{R}$. In Section 4.3, we generalize the characterization given in Section 4.1 to kernels defined on the product space $X \times \mathbb{R}^d$, where X is a nonempty set with no topology in principle, that is radial in the Euclidean variable and has a certain type of continuity. In Section 4.4, we present a characterization for the strictly positive definite kernels among those presented in Section 4.3, except for the case $\mathbb{R}$.

Some results obtained by the author presented in this chapter were included in the submitted paper ([GUELLA; MENEGATTO, 2018a](#)).

4.1 Schoenberg characterization of positive definite radial kernels

In this introductory section, we develop the concept of radial kernels on $\mathbb{R}^d$ and characterize the continuous positive definite radial kernels on $\mathbb{R}^d$. The main references are ([SCHOENBERG, 1938a](#)), ([WELLS; WILLIAMS, 1975](#)) and ([WENDLAND, 2005](#)).

Definition 4.1.1. Let $d \in \mathbb{N}$ and $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ be a kernel. The kernel K is **radial**, or more precisely, invariant through rotations and translations, if for every orthogonal matrix $Q \in O(d)$ and every vector $z \in \mathbb{R}^d$, the following equality is satisfied:

$$K(z + Qx, z + Qy) = K(x, y),$$

for all $x, y \in \mathbb{R}^d$.

As in the case of isotropic kernels on real spheres, a radial kernel on $\mathbb{R}^d$ can be described in a simpler way.

Lemma 4.1.2. If $d \in \mathbb{N}$ and $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ is a radial kernel, then there exists a function $f : [0, \infty) \rightarrow \mathbb{C}$ such that $K(x, y) = f(\|x - y\|)$ for all $x, y \in \mathbb{R}^d$. Moreover, the kernel K is continuous if and only if the function f is continuous.

Proof. Let $x, y, u, w \in \mathbb{R}^d$ such that $\|x - y\| = \|u - w\|$. From the properties of orthogonal matrices, there exist a matrix $Q \in O(d)$, with $Q(x - y) = u - w$. Define $f : [0, \infty) \rightarrow \mathbb{C}$, by

$$f(t) := K((t, 0, \dots, 0), 0).$$

If $\|x - y\| = s \in [0, \infty)$, we can take $Q \in O(d)$ where $Q(x - y) = (s, 0, \dots, 0)$, to obtain

$$f(\|x - y\|) = K((\|x - y\|, 0, \dots, 0), 0) = K(Q(x - y), 0) = K(Qx, Qy) = K(x, y).$$

If f is continuous, the last equation yields the continuity on K . Conversely, if K is continuous, the definition of f induces its continuity, because

$$t \in [0, \infty) \rightarrow ((t, 0, \dots, 0), 0) \in \mathbb{R}^d \times \mathbb{R}^d$$

is continuous. □

Corollary 2.2.2 implies that a positive definite radial kernel on $\mathbb{R}^d$ is necessarily real valued. Indeed,

$$K(x, y) = f(\|x - y\|) = f(\|y - x\|) = K(y, x) = \overline{K(x, y)},$$

for all $x, y \in \mathbb{R}^d$.

Lemma 4.1.2 implies that to characterize the continuous and (strictly) positive definite radial kernels on $\mathbb{R}^d$, is equivalent to characterize the set of continuous functions $f : [0, \infty) \rightarrow \mathbb{R}$ that define (strictly) positive definite radial kernels on $\mathbb{R}^d$ by $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(\|x - y\|) \in \mathbb{R}$. In 1938, Schoenberg in (SCHOENBERG, 1938a), presented the following characterization:

Theorem 4.1.3. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function and $d \in \mathbb{N}$. The kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(\|x - y\|) \in \mathbb{R}$$

is positive definite if and only if

$$f(t) = \int_0^\infty \Omega_d(rt) d\sigma(r),$$

for all $t \geq 0$, where σ is a finite nonnegative Borel measure on $[0, \infty)$, and

$$\Omega_d(\|x\|) := \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} e^{-ix \cdot \omega} d\rho(\omega),$$

for all $x \in \mathbb{R}^d$. Moreover,

$$\Omega_d(s) := 1 + \sum_{l=1}^{\infty} \frac{(-1)^l s^{2l}}{2^l l! \prod_{q=0}^{l-1} (d+2q)},$$

for all $s \in [0, \infty)$. The finite Borel measure σ is unique.

It is worth mentioning that a nonnegative measure is what is called a measure on (FOLLAND, 2013) page 24, while a positive measure is a measure for which there exists a measurable set whose measure lies in $(0, \infty)$.

Theorem 4.1.3 is a direct consequence of the famous Bochner's Theorem, (BOCHNER, 1933).

Theorem 4.1.4. Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$ be a continuous function and $d \in \mathbb{N}$. The kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(x - y) \in \mathbb{C}$$

is positive definite if and only if

$$f(x) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} d\sigma(r),$$

for all $x \in \mathbb{R}^d$, where σ is a finite nonnegative Borel measure on $\mathbb{R}^d$. The finite Borel measure σ is unique.

The kernels studied by Bochner are the continuous kernels $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ that are translational invariant, meaning that the following equality holds

$$K(x + z, y + z) = K(x, y),$$

for all $x, y, z \in \mathbb{R}^d$. This result is well established in the literature, see (WENDLAND, 2005), (GELFAND; VILENKIN, 1964), and was generalized to various contexts, such as locally compact Abelian Hausdorff groups (RUDIN, 1962), Gelfand pairs (WOLF, 2007), among others. By making use of Bochner's Theorem, we prove Schoenberg's characterization for the continuous positive definite radial kernels on $\mathbb{R}^d$, $d \in \mathbb{N}$. The proof of the series representation for Ω_d will be postponed.

Before moving to the proof itself, we need a concept from measure theory, adapted to our needs, that will appear often in the text.

Let X_1 and X_2 , be two topological spaces and $g : X_1 \rightarrow X_2$ a continuous function. If σ_1 is a Borel measure on X_1 , there exists a Borel measure σ_g in X_2 , called the **image measure** (or **push forward measure**) of the measure σ_1 through the continuous function g , such that

$$\int_{X_2} h(x) d\sigma_g(x) = \int_{X_1} h(g(y)) d\sigma_1(y),$$

for all functions $h : X_2 \rightarrow \mathbb{C}$ such that one side of the previous equality is well defined. To be precise, if $E \subset X_2$ is a Borel set, then $\sigma_g(E) = \sigma_1(g^{-1}(E))$. For further details and generalizations of this concept see (FOLLAND, 2013), page 314.

Below, we prove Theorem 4.1.3.

Proof. First note that the kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} e^{-i(x-y) \cdot \omega} d\rho(\omega) \in \mathbb{R}$$

is continuous. Moreover, since ρ is invariant by $O(d)$, the kernel is radial. Lemma 4.1.2 provides the existence of a continuous function $\Omega_d : [0, \infty) \rightarrow \mathbb{R}$ where

$$\Omega_d(\|x\|) = \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} e^{-ix \cdot \omega} d\rho(\omega),$$

for all $x \in \mathbb{R}^d$. Recall that when $d = 1$, $S^0 := \{-1, 1\}$ and $\text{vol}(S^0) = 2$, and then $\Omega_1(t) = \cos(t)$. If $f : [0, \infty) \rightarrow \mathbb{R}$ is continuous and the kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(\|x - y\|) \in \mathbb{R}$ is positive definite, we can apply Theorem 4.1.4 on the continuous function $g(x) := f(\|x\|)$, $x \in \mathbb{R}^d$. It implies that there exists a unique finite Borel measure σ such that

$$g(x) = f(\|x\|) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} d\sigma(\xi),$$

for all $x \in \mathbb{R}^d$. In particular,

$$\begin{aligned} f(\|x\|) &= \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} g(\|x\|\omega) d\rho(\omega) = \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} \int_{\mathbb{R}^d} e^{-i(\|x\|\omega) \cdot \xi} d\sigma(\xi) d\rho(\omega) \\ &= \int_{\mathbb{R}^d} \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} e^{-i(\|x\|\xi) \cdot \omega} d\rho(\omega) d\sigma(\xi) = \int_{\mathbb{R}^d} \Omega_d(\|x\|\|\xi\|) d\sigma(\xi). \end{aligned}$$

The change on the order of integration is possible due to Fubini-Tonelli Theorem, since σ and ρ are finite measures and $|e^{-i(\|x\|\omega) \cdot \xi}| \leq 1$. Thus

$$f(\|x\|) = \int_{\mathbb{R}^d} \Omega_d(\|x\|\|\xi\|) d\sigma(\xi) = \int_0^\infty \Omega_d(\|x\|r) d\sigma_{\|\cdot\|}(r),$$

where the finite and nonnegative measure $\sigma_{\|\cdot\|}$ is the push forward measure of the measure σ through the continuous function $\xi \in \mathbb{R}^d \rightarrow \|\xi\| \in [0, \infty)$.

For the converse, given distinct points $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ and $c_1, c_2, \dots, c_n \in \mathbb{R}$

$$\sum_{\mu, \nu=1}^n c_\mu c_\nu f(\|x_\mu - x_\nu\|) = \int_0^\infty \sum_{\mu, \nu=1}^n c_\mu c_\nu \Omega_d(\|x_\mu - x_\nu\|r) d\sigma_{\|\cdot\|}(r).$$

But, for every $r \geq 0$,

$$\sum_{\mu, \nu=1}^n c_\mu c_\nu \Omega_d(\|x_\mu - x_\nu\|r) = \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} \left| \sum_{\mu=1}^n c_\mu e^{-irx_\mu \cdot \omega} \right|^2 d\rho(\omega) \geq 0.$$

In order to prove that the finite Borel measure $\sigma_{\|\cdot\|}$ in $[0, \infty)$ is unique, let γ be a finite Borel measure on $[0, \infty)$ such that

$$f(\|x\|) = \int_0^\infty \Omega_d(\|x\|r) d\gamma(r),$$

for all $x \in \mathbb{R}^d$. Define the finite nonnegative Borel measure μ on $\mathbb{R}^d$ by

$$\mu := (\gamma \times \tilde{\rho})_{\mathbb{R}^d},$$

where $\tilde{\rho} = \frac{1}{\text{vol}(S^{d-1})} \rho$. The measure $(\gamma \times \tilde{\rho})_{\mathbb{R}^d}$ is the push forward measure of the finite nonnegative Borel measure $\gamma \times \tilde{\rho}$ in $[0, \infty) \times S^{d-1}$ through the continuous function $(r, \omega) \in [0, \infty) \times S^{d-1} \rightarrow r\omega \in \mathbb{R}^d$. The properties of the push forward measure give us that for all $x \in \mathbb{R}^d$,

$$\begin{aligned} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} d\mu(\xi) &= \int_{\mathbb{R}^d} e^{-ix \cdot \xi} d((\gamma \times \tilde{\rho})_{\mathbb{R}^d})(\xi) = \int_{[0, \infty)} \int_{S^{d-1}} e^{-ix \cdot (r\omega)} d\tilde{\rho}(\omega) d\gamma(r) \\ &= \int_{[0, \infty)} \Omega_d(r\|x\|) d\gamma(r) = f(\|x\|). \end{aligned}$$

This equation implies that the Fourier transform of the nonnegative and finite Borel measures μ and σ are equal to the function g , then $\mu = \sigma$ by Theorem 4.1.4. If $C \subset [0, \infty)$ is a Borel set, consider $B := \{x\omega : x \in C, \omega \in S^{d-1}\} \subset \mathbb{R}^d$. The properties of the push forward measure imply that

$$\gamma(C) = \mu(B) = \sigma(B) = \sigma_{\|\cdot\|}(C).$$

This proves that the finite Borel measure is uniquely defined. $\square$

Note that if $f : [0, \infty) \rightarrow \mathbb{R}$ defines a (strictly) positive definite radial kernel on $\mathbb{R}^d$ then it also defines a (strictly) positive definite radial kernel on $\mathbb{R}^m$ for all $m \leq d$. As for isotropic kernels on real spheres, it makes sense to ask which continuous functions $f : [0, \infty) \rightarrow \mathbb{R}$ define a (strictly) positive definite radial kernel on $\mathbb{R}^d$ for all $d \in \mathbb{N}$. This class of functions define the so called **(strictly) positive definite radial kernels on $\mathbb{R}^\infty$** .

There exists a geometric interpretation for the continuous functions that define a (strictly) positive definite radial kernel on $\mathbb{R}^\infty$. There is a correspondence between them and the continuous functions $f : [0, \infty) \rightarrow \mathbb{R}$ for which the kernel $(x, y) \in \mathcal{H} \times \mathcal{H} \rightarrow f(\sqrt{\langle x - y, x - y \rangle})$ is (strictly) positive definite, where $\mathcal{H}$ is an infinite dimensional real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, which is settled by an application of the Gram-Schmidt orthonormalization process.

In the same article (SCHOENBERG, 1938a), Schoenberg proved a characterization for the continuous functions that define a positive definite radial kernel on $\mathbb{R}^\infty$. Before proving that characterization, we need two technical lemmas, one concerning the functions Ω_d , where the series representation for Ω_d is proved, and the other related to weak* convergence of measures.

Lemma 4.1.5. The following properties hold:

$$(i) \quad \Omega_d(t) = 1 + \sum_{l=1}^{\infty} \frac{(-1)^l t^{2l}}{2^l l! \prod_{q=0}^{l-1} (d + 2q)}, \text{ for all } t \geq 0.$$

- (ii) $\lim_{d \rightarrow \infty} (1 - r^2/2d)_+^{(d-3)/2} = e^{-r^2/4}$, being the convergence in $L^1(\mathbb{R})$.
 (iii) $\lim_{d \rightarrow \infty} \Omega_d(t\sqrt{2d}) = e^{-t^2}$ uniformly in $t \in [0, \infty)$.
 (iv) $\lim_{t \rightarrow \infty} \Omega_d(t) = 0$, when $d \geq 2$.

Proof. First, we prove (i) in the case $d \geq 2$. Using the reproducing formula for the Gegenbauer polynomial P_0^{d-1} , in Equation (3.1), with $g_x(t) = e^{-it\|x\|}$, for $x \neq 0$,

$$\begin{aligned} \Omega_d(\|x\|) &= \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} e^{-ix \cdot \xi} d\xi = \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} g_x(\xi \cdot x/\|x\|) d\xi \\ &= C(0, d-1) \int_{-1}^1 g_x(r) (1-r^2)^{(d-3)/2} dr, \end{aligned}$$

and then, for all $t \geq 0$,

$$\Omega_d(t) = C(0, d-1) \int_{-1}^1 e^{-itr} (1-r^2)^{(d-3)/2} dr = C(0, d-1) \sum_{l=0}^{\infty} \frac{(-i)^l t^l}{l!} \int_{-1}^1 r^l (1-r^2)^{(d-3)/2} dr.$$

But, as a consequence of Euler's relation, (ASKEY, 1975) page 19 equation 3.2, the moments of the weight function $(1-r^2)^{(d-3)/2}$ in $[-1, 1]$ are

$$\int_{-1}^1 r^l (1-r^2)^{(d-3)/2} dr = \begin{cases} 0, & l = 2k+1 \\ \frac{\Gamma(k+1/2)\Gamma((d-1)/2)}{\Gamma(k+d/2)}, & l = 2k \end{cases}$$

The conclusion follows by using standard properties of the gamma function. The case $d = 1$ is immediate from the definition of S^0 .

In order to prove (ii), note that

$$\lim_{d \rightarrow \infty} (1 - r^2/2d)_+^{(d-3)/2} = e^{-r^2/4},$$

for all $r \in \mathbb{R}$. This is a direct consequence of the limit formula $\lim_{\lambda \rightarrow \infty} (1 + x/\lambda)^\lambda = e^x$, for all $x \in \mathbb{R}$. But, if $d \geq 4$,

$$0 \leq (1 - r^2/2d)_+^{(d-3)/2} \leq C \min\{1, 1/r^2\},$$

for some $C > 0$ that does not depend on d . Since the function $r(1 - r/2d)_+^{(d-3)/2}$ attains its maximum at the point $4d/(d-1)$, then

$$0 \leq r(1 - r/2d)_+^{(d-3)/2} \leq \frac{4d}{d-1} \left(1 - \frac{2}{d-1}\right)_+^{(d-3)/2},$$

for all $r \in [0, \infty)$ and $\lim_{d \rightarrow \infty} [4d/(d-1)](1 - 2/(d-1))_+^{(d-3)/2} = 4e^{-1}$. On the other hand, $\min\{1, 1/r^2\} \in L_1(\mathbb{R})$, so the Lebesgue Dominated Convergence Theorem can be applied and

the result is valid.

In order to prove (iii), note that if $d \geq 3$,

$$\begin{aligned} \frac{\sqrt{2d}}{C(0, d-1)} \Omega(t\sqrt{2d}) &= \sqrt{2d} \int_{-1}^1 e^{-it\sqrt{2d}r} (1-r^2)^{(d-3)/2} dr \\ &= \sqrt{2d} \frac{1}{\sqrt{2d}} \int_{-\sqrt{2d}}^{\sqrt{2d}} e^{-itw} \left(1 - \frac{w^2}{2d}\right)^{(d-3)/2} dw = \int_{\mathbb{R}} e^{-itw} \left(1 - \frac{w^2}{2d}\right)_+^{(d-3)/2} dw, \end{aligned}$$

for all $t \in [0, \infty)$. Recall the Fourier transform of the Gaussian function is

$$e^{-r^2} = \frac{1}{2\pi^{1/2}} \int_{\mathbb{R}} e^{-w^2/4} e^{-iwr} dw.$$

If

$$b_d := C(0, d-1) \frac{1}{\sqrt{2d}} 2\pi^{1/2} = \frac{\Gamma(d/2)}{\Gamma(1/2)\Gamma((d-1)/2)} \frac{1}{\sqrt{2d}} 2\pi^{1/2},$$

then

$$\lim_{d \rightarrow \infty} b_d = 1$$

by Stirling's formula. Finally,

$$2\pi^{1/2} |\Omega_d(t\sqrt{2d}) - \Omega_\infty(t)| = \left| \int_{\mathbb{R}} e^{-iwt} \left[b_d \left(1 - \frac{w^2}{2d}\right)_+^{(d-3)/2} - e^{-w^2/4} \right] dw \right|.$$

The uniform convergence is then a consequence of (ii).

As for (iv), it is true because

$$\Omega_d(t) = C(0, d-1) \int_{\mathbb{R}} e^{-itr} (1-r^2)_+^{(d-3)/2} dr$$

and the conclusion is a consequence of the Riemann-Lebesgue Lemma, (FOLLAND, 2013) page 249. $\square$

Next, we will write $C_0([0, \infty))$ to denote the subspace of $C([0, \infty))$ of functions that converge to 0 at infinity.

Definition 4.1.6. Let $(\gamma_k)_{k \in \mathbb{N}}$ be a sequence of finite Borel measures in $[0, \infty)$. The sequence $(\gamma_k)_{k \in \mathbb{N}}$ is **vaguely convergent** (more precisely, convergent in the weak* topology of $C_0([0, \infty))$) to a finite Borel measure γ if

$$\lim_{k \rightarrow \infty} \int_0^\infty h(r) d\gamma_k(r) = \int_0^\infty h(r) d\gamma(r),$$

for all $h \in C_0([0, \infty))$.

In order to obtain the result we are aiming at, we will use the **one point compactification** (or **Alexandroff compactification**) of the half interval $[0, \infty)$, which will be denoted by $[0, \infty]$. The “point” ∞ is the compactification point. The topology in $[0, \infty]$ is such that a subset $A \subset [0, \infty]$ is open if it is an open subset of the interval $[0, \infty)$ with its standard topology or it contains the

point ∞ and $[0, \infty] - A$ is a compact set of the interval $[0, \infty)$ with its standard topology. A function $F : [0, \infty] \rightarrow \mathbb{C}$ is continuous if and only if F , restricted to the interval $[0, \infty)$, is continuous in the usual sense and $F(\infty) = \lim_{\lambda \rightarrow \infty} F(\lambda)$. These facts can be found in (FOLLAND, 2013) chapter 4.

Proposition 4.1.7. Let $(\gamma_k)_{k \in \mathbb{N}}$ be a bounded sequence of nonnegative Borel measures in $[0, \infty)$. There exists a subsequence $(\gamma_{k_l})_{l \in \mathbb{N}}$ which is vaguely convergent to a finite nonnegative Borel measure γ defined in $[0, \infty]$. In addition,

$$\lim_{l \rightarrow \infty} \gamma_{k_l}([0, \infty)) = \gamma([0, \infty)).$$

Proof. Since $[0, \infty]$ is compact and metrizable (the induced metric by the stereographic projection), then the space $C([0, \infty])$ is separable, (CONWAY, 2013) page 140, Theorem 6.6.

The dual of $C([0, \infty])$ is the space $M([0, \infty])$ of all finite Borel measures in $[0, \infty]$, (FOLLAND, 2013) page 223, Corollary 7.18.

In particular, being $C([0, \infty])$ a separable space, every closed ball with center at 0 in $M([0, \infty])$ in the strong topology is metrizable in the weak* topology, (BREZIS, 2010) page 74, Theorem 3.28.

By the Banach-Alaoglu Theorem, every closed ball with center at 0 in $M([0, \infty])$ in the strong topology is compact in the weak* topology.

From that, we obtain that every closed ball with center at 0 in $M([0, \infty])$ at the strong topology is metrizable in the weak* topology and is a compact set, so, every bounded sequence in $M([0, \infty])$ in the strong topology has a subsequence that is convergent in the weak* topology.

This proves the existence of γ and the vague convergence. In order to prove that γ is nonnegative, notice that since all the γ_k are nonnegative measures, for every nonnegative function $h \in C([0, \infty])$, $\int_{[0, \infty]} h(r) d\gamma_k(r) \geq 0$. But then

$$\int_{[0, \infty]} h(r) d\gamma(r) \geq 0,$$

for all nonnegative functions $h \in C([0, \infty])$. The Riesz Representation Theorem, (FOLLAND, 2013) page 212, implies that γ is nonnegative.

Finally,

$$\lim_{l \rightarrow \infty} \gamma_{k_l}([0, \infty)) = \gamma([0, \infty)),$$

because the function $t \in [0, \infty] \rightarrow 1 \in \mathbb{R}$ belongs to the space $C([0, \infty])$ and

$$\lim_{l \rightarrow \infty} \gamma_{k_l}([0, \infty)) = \lim_{l \rightarrow \infty} \int_{[0, \infty]} 1 d\gamma_{k_l} = \int_{[0, \infty]} 1 d\gamma = \gamma([0, \infty)).$$

The proof is complete. □

Theorem 4.1.8. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function. The kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(\|x - y\|) \in \mathbb{R}$$

is positive definite for all $d \in \mathbb{N}$ if and only if

$$f(t) = \int_0^\infty \Omega_\infty(rt) d\sigma(r),$$

for all $t \geq 0$, where σ is a finite nonnegative Borel measure on $[0, \infty)$ and $\Omega_\infty(r) := e^{-r^2}$, $r \in [0, \infty)$. Moreover, the finite Borel measure σ is unique.

Proof. If f defines a radial positive definite function on $\mathbb{R}^d$ for every d , then

$$f(t) = \int_0^\infty \Omega_d(rt) d\sigma_d(r),$$

for all $t \geq 0$, in which σ_d is a finite nonnegative Borel measure on $[0, \infty)$. In particular

$$f(t) = \int_0^\infty \Omega_d(rt\sqrt{2d}/\sqrt{2d}) d\sigma_d(r) = \int_0^\infty \Omega_d(rt\sqrt{2d}) d\gamma_d(r),$$

where γ_d is the push forward measure of the measure σ_d through the continuous function $s \in [0, \infty) \rightarrow s/\sqrt{2d} \in [0, \infty)$. Since $(\gamma_d)_{d \in \mathbb{N}}$ is a bounded sequence of measures ($\int_0^\infty d\gamma_d(r) = f(0)$ for all $d \in \mathbb{N}$) and, by Lemma 4.1.5–(iii),

$$\lim_{d \rightarrow \infty} \Omega_d(rt\sqrt{2d}) \rightarrow \Omega_\infty(rt)$$

uniformly as $d \rightarrow \infty$, we obtain that

$$f(t) = \lim_{d \rightarrow \infty} \int_0^\infty \Omega_\infty(rt) d\gamma_d(r),$$

for all $t \geq 0$. Being the sequence of measures $(\gamma_d)_{d \in \mathbb{N}}$ defined on the set $[0, \infty)$, Proposition 4.1.7 implies the existence of a subsequence $(\gamma_{d_k})_{k \in \mathbb{N}}$ that is vaguely convergent to a finite nonnegative Borel measure σ defined in $[0, \infty]$. Since $\Omega_\infty(\cdot t) \in C_0([0, \infty))$ for all $t > 0$, then the integral representation for f in the statement of the theorem is true for all $t > 0$. For $t = 0$, the sequence of nonnegative functions $(\Omega_\infty(\cdot 1/n))_{n \in \mathbb{N}}$ is bounded by the constant function 1 and converge pointwise to the same function. The continuity of f and the Lebesgue Dominated Convergence Theorem imply that

$$f(0) = \lim_{n \rightarrow \infty} f(1/n) = \lim_{n \rightarrow \infty} \int_0^\infty \Omega_\infty(r/n) d\sigma(r) = \int_0^\infty \Omega_\infty(0) d\sigma(r).$$

Conversely, given $d \in \mathbb{N}$, distinct points $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ and $c_1, c_2, \dots, c_n \in \mathbb{R}$,

$$\begin{aligned} \sum_{\mu, \nu=1}^n c_\mu c_\nu \Omega_\infty(s\|x_\mu - x_\nu\|) &= \sum_{\mu, \nu=1}^n c_\mu c_\nu e^{-s^2\|x_\mu - x_\nu\|^2} \\ &= \sum_{\mu, \nu=1}^n [c_\mu e^{-s^2\|x_\mu\|^2}] [c_\nu e^{-s^2\|x_\nu\|^2}] e^{2s^2 x_\mu \cdot x_\nu} \geq 0, \end{aligned}$$

because the kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow e^{2s^2 x \cdot y} \in \mathbb{R}$ is positive definite for all $d \in \mathbb{N}$ and $s \in \mathbb{R}$. This is easily proved noticing that

$$e^{2s^2 x \cdot y} = \sum_{l=0}^{\infty} \frac{(2s^2)^l}{l!} (x \cdot y)^l$$

and for each $l \in \mathbb{N}$ the kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow (x \cdot y)^l$ is positive definite by the Schur Product result in Proposition 2.2.4, the conclusion comes from Proposition 2.2.7. Being σ a nonnegative measure, we obtain

$$\sum_{\mu, \nu=1}^n c_\mu c_\nu f(\|x_\mu - x_\nu\|) = \int_0^\infty \sum_{\mu, \nu=1}^n c_\mu c_\nu \Omega_\infty(r\|x_\mu - x_\nu\|) d\sigma(r) \geq 0.$$

Our final claim, is that the finite Borel measure is unique. Indeed, if σ_1 and σ_2 are two finite nonnegative Borel measures in $[0, \infty)$ such that

$$\int_0^\infty \Omega_\infty(rt) d\sigma_1(r) = \int_0^\infty \Omega_\infty(rt) d\sigma_2(r),$$

for all $t \geq 0$, then $\int_0^\infty \Omega_\infty(rt) d\gamma(r) = 0$, where $\gamma = \sigma_1 - \sigma_2$. If α is the finite Borel measure defined by the push forward measure of γ through the continuous function $r \in [0, \infty) \rightarrow e^{-r^2} \in (0, 1]$, then

$$0 = \int_0^\infty e^{-r^2 t^2} d\gamma(r) = \int_{(0,1]} u^2 d\alpha(u) = \int_{[0,1]} u^2 d\alpha(u).$$

Here, we are imposing that $\alpha(\{0\}) = 0$. In particular, $\int_{[0,1]} u^n d\alpha(u) = 0$, for all $n \in \mathbb{Z}_+$. The Stone–Weierstrass Theorem implies that $\int_{[0,1]} g(u) d\alpha(u) = 0$, for all continuous functions $g : [0, 1] \rightarrow \mathbb{R}$. The Riesz Representation Theorem, (FOLLAND, 2013) page 223, implies that α is the zero measure, and then $\sigma_1 = \sigma_2$. $\square$

A consequence of (PINKUS, 2004b) is that the kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow e^{x \cdot y} \in \mathbb{R}$ is strictly positive definite for all $d \in \mathbb{N}$, and this information implies that the kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow \Omega_\infty(s\|x - y\|)$ is strictly positive definite for all $s > 0$ and $d \in \mathbb{N}$. We reach the same conclusion by a different argument in Section 4.2.

It is worth mentioning that there exists an alternative characterization for the positive definite radial kernels on $\mathbb{R}^\infty$ based on an equivalence between completely monotone functions and the Laplace transform of measures.

A continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ is called **completely monotone** if f is infinitely many times differentiable in the interval $(0, \infty)$ ($f \in C^\infty((0, \infty))$) and $(-1)^n f^{(n)}(r) \geq 0$, for all $n \in \mathbb{Z}_+$ and $r > 0$. The following theorem connects those different properties, (WENDLAND, 2005) chapter 7.

Theorem 4.1.9. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function. The following properties are equivalent:

- (i) The kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(\|x - y\|) \in \mathbb{R}$ is positive definite for all $d \in \mathbb{N}$.
- (ii) The function f has an integral representation

$$f(t) = \int_0^\infty e^{-rt^2} d\gamma(r),$$

for all $t \geq 0$, in which σ is a finite nonnegative Borel measure on $[0, \infty)$.

- (iii) The function $t \in (0, \infty) \rightarrow f(\sqrt{t})$ is completely monotone.

The difference between the term “ r^2 ” in Theorem 4.1.8 and the term “ r ” in the previous theorem is just a renormalization of the measures σ and γ .

4.2 Strictly positive definite radial kernels

In this section, we characterize the set of continuous functions that define a strictly positive definite radial kernel on $\mathbb{R}^d$, for $d \in \{2, 3, \dots, \infty\}$. This characterization was proved in (SUN, 1993). We also characterize the set of continuous functions $f : [0, \infty) \times [-1, 1] \rightarrow \mathbb{R}$, for which the kernel

$$((u, x), (v, y)) \in (\mathbb{R}^m \times S^d) \times (\mathbb{R}^m \times S^d) \rightarrow f(\|u - v\|, x \cdot y) \in \mathbb{R}$$

is positive definite when $m, d \in \mathbb{N} \cup \{\infty\}$ and strictly positive definite when $d \in \{1, 2, \dots, \infty\}$ and $m \in \{2, \dots, \infty\}$. This result was published in (GUELLA; MENEGATTO, 2018c).

We begin with a surprising result based on the article (CASTELL; FILBIR; SZWARC, 2005).

Theorem 4.2.1. Let $s > 0$ and $d \in \{2, 3, \dots\}$. The kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow \Omega_d(s\|x - y\|) \in \mathbb{R}$$

is strictly positive definite.

Proof. Notice that if the result is proved for $s = 1$ then it will follow for all $s > 0$. Indeed, the interpolation matrix $[\Omega_d(s\|x_\mu - x_\nu\|)]_{\mu, \nu=1}^n$ is also the interpolation matrix $[\Omega_d(\|sx_\mu - sx_\nu\|)]_{\mu, \nu=1}^n$. Let $x_1, x_2, \dots, x_n$ be distinct points in $\mathbb{R}^d$ and $c_1, c_2, \dots, c_n \in \mathbb{R}$ such that

$$0 = \sum_{\mu, \nu=1}^n c_\mu c_\nu \Omega_d(\|x_\mu - x_\nu\|) = \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} \left| \sum_{\mu=1}^n c_\mu e^{-ix_\mu \cdot \omega} \right|^2 d\rho(\omega).$$

The continuity of the function $\omega \in S^{d-1} \rightarrow \sum_{\mu=1}^n c_\mu e^{-ix_\mu \cdot \omega} \in \mathbb{R}$ implies that

$$\sum_{\mu=1}^n c_\mu e^{-ix_\mu \cdot \omega} = 0, \tag{4.1}$$

for all $\omega \in S^{d-1}$. Pick a vector $\omega_1 \in S^{d-1}$ with the property

$$x_\mu \cdot \omega_1 \neq x_\nu \cdot \omega_1, \quad \mu \neq \nu.$$

Such a vector exists once each set

$$A_{\mu, \nu} = \{\omega \in S^{d-1} : (x_\mu - x_\nu) \cdot \omega = 0\}, \quad \mu \neq \nu,$$

has ρ measure zero. Without loss of generality, assume that $x_1 \cdot \omega_1 < x_2 \cdot \omega_1 < \dots < x_n \cdot \omega_1$. Pick $\omega_2 \in S^{d-1}$, orthogonal to ω_1 and consider the entire function

$$z \in \mathbb{C} \rightarrow \sum_{\mu=1}^n c_{\mu} e^{-ix_{\mu} \cdot [\omega_1 \sin(z) + \omega_2 \cos(z)]}.$$

Since $\|\omega_1 \sin(z) + \omega_2 \cos(z)\| = 1$ for all $z \in \mathbb{R}$, Equation 4.1 implies that this entire function is zero on the real line. The separation of zeros for entire functions implies that, this entire function must be zero on all complex numbers. Using the identities $\cos(it) = \cosh(t)$ and $\sin(it) = i \sinh(t)$, $t \in \mathbb{R}$, we get

$$\sum_{\mu=1}^n c_{\mu} e^{\sinh(t)x_{\mu} \cdot \omega_1 - i \cosh(t)x_{\mu} \cdot \omega_2} = 0,$$

for all $t \in \mathbb{R}$. Dividing this last equation by $e^{\sinh(t)x_n \cdot \omega_1 - i \cosh(t)x_n \cdot \omega_2}$, we obtain

$$\sum_{\mu=1}^n c_{\mu} e^{\sinh(t)(x_{\mu} - x_n) \cdot \omega_1 - i \cosh(t)(x_{\mu} - x_n) \cdot \omega_2} = 0,$$

for all $t \in \mathbb{R}$. Finally, letting $t \rightarrow \infty$, we obtain that $c_n = 0$. The same procedure induces the other equalities $c_{\mu} = 0$, for $\mu \in \{1, 2, \dots, n-1\}$. $\square$

Radiality for continuous positive definite kernels on $\mathbb{R}$ does not add much information compared to the Bochner setting. This happens because a positive definite kernel on $\mathbb{R}$ according to Bochner satisfies $f(t) = \overline{f(-t)}$, for all $t \in \mathbb{R}$. So, if f is in fact a real valued function, then it has the radiality property, and conversely. In terms of the measure on Bochner's Theorem 4.1.4, this is equivalent at the measure σ in the integral representation satisfies $\sigma(A) = \sigma(-A)$, for all Borel sets $A \subset \mathbb{R}$.

There is no complete characterization of the continuous strictly positive definite kernels in Bochner's Theorem. The article (DERRIEN, 2010) contains information about this problem on $\mathbb{R}$. The additional assumption of integrability leads to interesting results, (WENDLAND, 2005) page 76.

A direct consequence of Theorem 4.2.1 is the characterization of the strictly positive definite radial kernels on $\mathbb{R}^d$, for $d \in \{2, 3, \dots\}$. We write δ_0 to denote the Borel measure on $[0, \infty)$ that has value 1 for all sets that contains the element 0, and zero for all the other sets.

Theorem 4.2.2. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function and $d \in \{2, 3, \dots\}$. The kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(\|x - y\|) \in \mathbb{R}$$

is strictly positive definite if and only if f defines a positive definite radial kernel on $\mathbb{R}^d$ and f is not a constant function.

Moreover, if f defines a positive definite radial kernel on $\mathbb{R}^d$, the function f is constant if and only if the measure σ in Theorem 4.1.3 is a multiple of δ_0 .

Proof. If the kernel is strictly positive definite, is immediate that f defines a radial positive definite function on $\mathbb{R}^d$ and that it cannot be the constant function.

Conversely, if f defines a positive definite radial kernel on $\mathbb{R}^d$, then

$$f(t) = \int_0^\infty \Omega_d(rt) d\sigma_d(r),$$

for all $t \geq 0$, in which σ_d is a finite nonnegative Borel measure on $[0, \infty)$. Let $x_1, x_2, \dots, x_n$ be distinct points in $\mathbb{R}^d$ and $c_1, c_2, \dots, c_n \in \mathbb{R}$, not all zero. Then

$$\sum_{\mu, \nu=1}^n c_\mu c_\nu f(\|x_\mu - x_\nu\|) = \int_0^\infty \sum_{\mu, \nu=1}^n c_\mu c_\nu \Omega_d(r\|x_\mu - x_\nu\|) d\sigma(r).$$

But, Theorem 4.2.1 implies that $\sum_{\mu, \nu=1}^n c_\mu c_\nu \Omega_d(r\|x_\mu - x_\nu\|) > 0$ for all $r > 0$, so if $\sigma((0, \infty)) > 0$, the above integral is positive. This implies that the kernel is strictly positive definite. If $\sigma((0, \infty)) = 0$, then $\sigma = \sigma([0, \infty))\delta_0$, but that would imply that $f(t) = \sigma([0, \infty))$, which is a constant function which is not a strictly positive definite radial kernel on $\mathbb{R}^d$. $\square$

With this result, it is easy to obtain an analogue of Theorem 4.2.1 for $\mathbb{R}^\infty$.

Corollary 4.2.3. For every $s > 0$ the kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow \Omega_\infty(s\|x - y\|) \in \mathbb{R}$$

is strictly positive definite.

Proof. The function $r \in [0, \infty) \rightarrow \Omega_\infty(sr)$ defines a positive definite radial kernel on $\mathbb{R}^d$ for every $d \in \mathbb{N}$ and is not a constant function. Theorem 4.2.2 implies that it defines a strictly positive definite radial kernel on $\mathbb{R}^d$ for every $d \in \mathbb{N}$. $\square$

Arguments similar to those in Theorem 4.2.2, lead to the following additional result.

Theorem 4.2.4. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function. The kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(\|x - y\|) \in \mathbb{R}$$

is strictly positive definite for every $d \in \mathbb{N}$ if and only if f defines a positive definite radial kernel on $\mathbb{R}^\infty$ and is not a constant function.

Moreover, if f defines a positive definite radial kernel on $\mathbb{R}^\infty$, the function f is constant if and only if the measure σ is a multiple of δ_0 .

Theorem 3.3.3 and Theorem 3.4.10 along with the results we proved so far in this chapter enable us to obtain the characterization of the continuous functions $f : [0, \infty) \times [-1, 1] \rightarrow \mathbb{R}$, for which the kernel

$$((u, x), (v, y)) \in (\mathbb{R}^m \times S^d) \times (\mathbb{R}^m \times S^d) \rightarrow f(\|u - v\|, x \cdot y) \in \mathbb{R}$$

is positive definite for $d, m \in \mathbb{N} \cup \{\infty\}$ and strictly positive definite, when $d \in \{1, 2, \dots, \infty\}$ and $m \in \{2, \dots, \infty\}$. These results were proved in (GUELLA; MENEGATTO, 2018c). Note that, as in the case of radial and isotropic kernels, the assumption that the previous kernel is real valued is not a restriction.

Theorem 4.2.5. Let $d, m \in \mathbb{N} \cup \{\infty\}$ and $f : [0, \infty) \times [-1, 1] \rightarrow \mathbb{R}$ be a continuous function.

(i) The kernel

$$((u, x), (v, y)) \in (\mathbb{R}^m \times S^d) \times (\mathbb{R}^m \times S^d) \rightarrow f(\|u - v\|, x \cdot y) \in \mathbb{R}$$

is positive definite if and only if the function f has a series representation

$$f(t, s) = \sum_{k=0}^{\infty} b_k^m(t) P_k^d(s), \quad s \in [-1, 1], \quad t \in [0, \infty),$$

where

$$b_k^m(t) = \int_0^{\infty} \Omega_m(rt) \sigma_k(r),$$

for all $t \geq 0$. The Borel measure σ_k in $[0, \infty)$ is finite, nonnegative and $\sum_{k=0}^{\infty} b_k^m(0) P_k^d(1) < \infty$. This representation is unique.

(ii) ($m \neq 1$) The kernel is strictly positive definite if and only if it satisfies (i) and the set

$$K_f := \{k \in \mathbb{Z}_+ : b_k^m(t) \text{ is not constant} \}$$

contains infinitely many even and infinitely many odd integers when $d \geq 2$ and $K_f \cup (-K_f)$ intersects every full arithmetic progression in $\mathbb{Z}$ when $d = 1$.

Proof. By Theorem 3.3.4 and Theorem 3.3.3, if the kernel in (i) is positive definite, the function f has a unique series representation

$$f(\|u - v\|, s) = \sum_{k=0}^{\infty} a_k^d(u, v) P_k^d(s),$$

where each kernel $a_k^d : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous, positive definite and $\sum_{k=0}^{\infty} a_k^d(u, u) P_k^d(1) < \infty$ for all $u \in \mathbb{R}^m$. Observe that $a_k^d(z + Qu, z + Qv) = a_k^d(u, v)$ for all $u, v, z \in \mathbb{R}^m$ and $Q \in O(m)$, because

$$f(\|u - v\|, s) = f(\|(z + Qu) - (z - Qv)\|, s),$$

for all $s \in [-1, 1]$ and the uniqueness of the representation. Lemma 4.1.2 and the characterization for positive definite radial kernels provided by Theorems 4.1.8 and 4.1.3 induces the series representation for f and also that $a_k^d(u, u) = b_k^m(\|u - u\|) = b_k^m(\|0\|)$ for all $u \in \mathbb{R}^m$. Thus the desired convergence of $\sum_{k=0}^{\infty} b_k^m(0) P_k^d(1)$ holds.

Conversely, if the function f admits this type of series representation, all the functions b_k^m define positive definite radial kernels on $\mathbb{R}^m$, and for all $u \in \mathbb{R}^m$, $\sum_{k=0}^{\infty} b_k^d(\|u - u\|) P_k^d(1) < \infty$. Now, the converse of Theorem 3.3.3 implies that the kernel

$$((u, x), (v, y)) \in (\mathbb{R}^m \times S^d) \times (\mathbb{R}^m \times S^d) \rightarrow f(\|u - v\|, x \cdot y) \in \mathbb{R}$$

is positive definite. The representation is unique, because the series representation in Theorem 3.3.3 is unique. This settles (i).

In order to prove (ii), if the kernel is strictly positive definite, pick arbitrary distinct points $u, v \in \mathbb{R}^m$. By Lemma 3.3.1, the function

$$s \in [-1, 1] \rightarrow 2f(0, s) - 2f(\|u - v\|, s) = 2 \sum_{k=0}^{\infty} [b_k^m(0) - b_k^m(\|u - v\|)] P_k^d(s) \in \mathbb{R}$$

defines a continuous strictly positive definite isotropic kernel on S^d . In particular, the set $K_{u,v} := \{k \in \mathbb{Z}_+ : b_k^m(0) - b_k^m(\|u - v\|) \neq 0\}$ contains infinitely many even and infinitely many odd integers when $d \geq 2$ and $K_{u,v} \cup (-K_{u,v})$ intersects every full arithmetic progression on $\mathbb{Z}$ for $d = 1$. Since $K_{u,v} \subset K_f$, the same holds for K_f .

Conversely, since each nonconstant continuous function that defines a positive definite radial kernel on $\mathbb{R}^m$, actually defines a strictly positive definite radial kernel on $\mathbb{R}^m$, with K_f as in (ii), it is possible to apply Lemma 3.4.10 in order to obtain that the kernel

$$((u, x), (v, y)) \in (\mathbb{R}^m \times S^d) \times (\mathbb{R}^m \times S^d) \rightarrow f(\|u - v\|, x \cdot y) \in \mathbb{R}$$

is strictly positive definite. □

In (GUELLA; MENEGATTO, 2017), based on the ideas used to obtain the characterization for the continuous strictly positive definite isotropic kernels on S^1 in Section 3.2, a characterization of the continuous strictly positive definite kernels on the torus $T_m := \{x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : -\pi \leq x_j \leq \pi, 1 \leq j \leq m\}$, $m \in \mathbb{N}$, that are invariant through translations was achieved. A proof for the characterization of the continuous positive definite kernels on T_m that are invariant through translations can be found in Chapter 4 in (SHAPIRO, 2011). The characterization of the positive definite case can be seen as the characterization of the continuous functions in $\mathbb{R}^m$, that define a positive definite kernel in Bochner's setting, and are periodic with period 2π in each variable of $\mathbb{R}^m$.

4.3 A unifying perspective. Part II

In this section, we provide a generalization of Schoenberg's results on continuous positive definite radial kernels on Euclidean spaces, that is, a characterization of the kernels $K : (X \times \mathbb{R}^d) \times (X \times \mathbb{R}^d) \rightarrow \mathbb{C}$, $d \in \mathbb{N} \cup \{\infty\}$, that are positive definite, radial on the Euclidean variable, meaning

$$K((u, x), (v, y)) = K((u, z + Qx), (v, z + Qy)),$$

for all $u, v \in X$, every points $x, y, z \in \mathbb{R}^d$ and $Q \in O(d)$, along with the assumption that the kernels

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow K((u, x), (v, y)) \in \mathbb{C}$$

are continuous for every $u, v \in X$.

Similar conclusions from Lemma 4.1.2 and the comments above it, imply that our task is equivalent to characterize the set of functions $f : X \times X \times [0, \infty) \rightarrow \mathbb{C}$ such that the kernel

$$(X \times \mathbb{R}^d) \times (X \times \mathbb{R}^d) \rightarrow f(u, v, \|x - y\|) \in \mathbb{C}$$

is positive definite, under the assumption that $f(u, v, \cdot)$ is a continuous function on $[0, \infty)$, for each fixed $u, v \in X$.

With this characterization and the bridge propositions on matrix valued kernels in Section 2.3, we are able to reobtain the results of (WANG; DU; MA, 2014). We also reobtain the characterization of the continuous functions $f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$, such that the kernel

$$((x, z), (y, w)) \in (\mathbb{R}^d \times \mathbb{R}^{d'}) \times (\mathbb{R}^d \times \mathbb{R}^{d'}) \rightarrow f(\|x - y\|_d, \|z - w\|_{d'})$$

is positive definite, where $d, d' \in \mathbb{N} \cup \{\infty\}$, proved in (ALONSO-MALAYER; PORCU; GIRALDO, 2015). Unless stated otherwise, all theorems in this section were obtained by the author, and so far left unpublished.

The following lemma will play a key role in this section. The proof is similar to that of Lemma 3.3.1. We omit the proof.

Lemma 4.3.1. Let $((u, x), (v, y)) \in (X \times \mathbb{R}^d) \times (X \times \mathbb{R}^d) \rightarrow f(u, v, \|x - y\|)$ be a positive definite kernel, $d \in \mathbb{N} \cup \{\infty\}$, and assume that each function $f(u, v, \cdot)$ is continuous on $[0, \infty)$. For distinct points $u_1, u_2, \dots, u_p$ in X and complex numbers $c_1, c_2, \dots, c_p$, define $g : [0, \infty) \rightarrow \mathbb{C}$ by

$$g(t) = \sum_{\mu, \nu=1}^p c_\mu \overline{c_\nu} f(u_\mu, u_\nu, t), \quad t \in [0, \infty)$$

The following assertions hold:

- (i) The function g is real and continuous.
- (ii) The function g defines a positive definite radial kernel on $\mathbb{R}^d$.
- (iii) If the c_μ are not all zero and the kernel is strictly positive definite, then g defines a strictly positive definite radial kernel on $\mathbb{R}^d$.

The arguments here will demand the use of the Lebesgue-Radon-Nykodim Theorem, (FOLLAND, 2013) page 93. A version of this theorem adapted to our purposes is presented:

Proposition 4.3.2. Let σ be a finite nonnegative Borel measure on $[0, \infty)$ and ϑ a complex valued measure on $[0, \infty)$. Assume that $\vartheta \ll \sigma$, in the sense that if E is a Borel set of $[0, \infty)$ with $\sigma(E) = 0$, then $\vartheta(E) = 0$. Then there exists a function g in $L^1([0, \infty), \sigma)$ such that

$$\vartheta(A) = \int_A g(s) d\sigma(s)$$

for all Borel sets A of $[0, \infty)$.

The next lemma is concerned with representations for certain families of measures.

Lemma 4.3.3. Let $\sigma_{\alpha,\beta}$, $\alpha, \beta = 1, 2, \dots, l$, be finite complex Borel measures on $[0, \infty)$ fulfilling the following property: if $c_1, c_2, \dots, c_l$ are complex numbers, the measure $\sum_{\alpha,\beta=1}^l c_\alpha \overline{c_\beta} \sigma_{\alpha,\beta}$ is nonnegative. If

$$\sigma_D := \sum_{\alpha=1}^l \sigma_{\alpha,\alpha},$$

then there exist functions $g_{\alpha,\beta}$, $\alpha, \beta = 1, 2, \dots, l$ in $L^1([0, \infty), \sigma_D)$ so that

$$d\sigma_{\alpha,\beta} = g_{\alpha,\beta} d\sigma_D.$$

Further, for each $r \in [0, \infty)$, the matrix $[g_{\alpha,\beta}(r)]_{\alpha,\beta=1}^l$ is positive semidefinite.

Proof. Our assumption on the measures $\sigma_{\alpha,\beta}$ asserts that the matrix $[\sigma_{\alpha,\beta}(A)]_{\alpha,\beta=1}^l$ is positive semidefinite, for all Borel sets A of $[0, \infty)$. In particular,

$$\sigma_{\alpha,\beta}(A) = \overline{\sigma_{\beta,\alpha}(A)},$$

for each such A and $\alpha, \beta = 1, 2, \dots, l$. Thus, $\sigma_{\alpha,\beta} = \overline{\sigma_{\beta,\alpha}}$, for all $\alpha, \beta = 1, 2, \dots, l$. Also,

$$2|\sigma_{\alpha,\beta}(A)| \leq \sigma_{\alpha,\alpha}(A) + \sigma_{\beta,\beta}(A) \leq \sigma_D(A),$$

for all $\alpha, \beta = 1, 2, \dots, l$. Since σ_D is a finite nonnegative Borel measure on $[0, \infty)$, an application of Theorem 4.3.2 provides the existence of functions $h_{\alpha,\beta}$, $\alpha, \beta = 1, 2, \dots, l$, in the space $L^1([0, \infty), \sigma_D)$, so that

$$\sigma_{\alpha,\beta}(A) = \int_A h_{\alpha,\beta}(s) d\sigma_D(s),$$

for all $\alpha, \beta = 1, 2, \dots, l$ and each Borel set A of $[0, \infty)$. In other words,

$$d\sigma_{\alpha,\beta} = h_{\alpha,\beta} d\sigma_D.$$

Since $\sigma_{\alpha,\beta} = \overline{\sigma_{\beta,\alpha}}$, we can assume that $h_{\alpha,\beta}(r) = \overline{h_{\beta,\alpha}(r)}$ for all $r \geq 0$. In order to prove the last statement of the theorem, first observe that if v is a nonzero vector on $\mathbb{C}^l$ and A is an arbitrary Borel subset of $[0, \infty)$, then

$$\int_A v[h_{\alpha,\beta}(s)]_{\alpha,\beta=1}^l v^* d\sigma_D(s) = v[\sigma_{\alpha,\beta}(A)]_{\alpha,\beta=1}^l v^* \geq 0.$$

Hence, since the matrix $[h_{\alpha,\beta}(r)]_{\alpha,\beta=1}^l$ is Hermitian, for each such v ,

$$B_v := \{r \in [0, \infty) : v[h_{\alpha,\beta}(r)]_{\alpha,\beta=1}^l v^* < 0\}$$

is a σ_D -null subset of $[0, \infty)$. We now infer that

$$B := \bigcup_{v \in \mathbb{C}^l \setminus \{0\}} B_v$$

is also a σ_D -null subset of $[0, \infty)$. Indeed, if $s \in B$, then $s \in B_v$ for some $v \in \mathbb{C}^l \setminus \{0\}$. Since $v[h_{\alpha,\beta}(s)]_{\alpha,\beta=1}^l v^* < 0$ and the function $w \in \mathbb{C}^l \rightarrow w[h_{\alpha,\beta}(s)]_{\alpha,\beta=1}^l w^*$ is continuous, we can select a neighborhood U_v of v on $\mathbb{C}^l$ so that

$$u[h_{\alpha,\beta}(s)]_{\alpha,\beta=1}^l u^* < 0, \quad u \in U_v.$$

Now, pick a dense subset $\{v_1, v_2, \dots\}$ of $\mathbb{C}^l \setminus \{0\}$. Since $v_m \in U_v$ for some m , it follows that $v_m[h_{\alpha,\beta}(s)]_{\alpha,\beta=1}^l v_m^* < 0$ for at least one m . In other words, $s \in B_{v_m}$, for at least one m . Therefore, $B \subset \bigcup_{m \in \mathbb{N}} B_{v_m}$ and, consequently, $\sigma_D(B) = 0$. The arguments above reveal that $[h_{\alpha,\beta}(r)]_{\alpha,\beta=1}^l$ is positive semidefinite, except for r in a σ_D -null subset B of $[0, \infty)$. To circumvent this last minute inconvenience, we finally define the functions

$$g_{\alpha,\beta}(r) := \begin{cases} h_{\alpha,\beta}(r), & r \in [0, \infty) - B \\ 0, & r \in B, \end{cases}$$

for all $\alpha, \beta = 1, 2, \dots, l$ and observe that the matrices $[g_{\alpha,\beta}(r)]_{\alpha,\beta=1}^l$, $r \in [0, \infty)$, are all positive semidefinite. Since $g_{\alpha,\beta} = h_{\alpha,\beta}$, except for a σ_D -null subset of $[0, \infty)$, we obtain

$$d\sigma_{\alpha,\beta} = h_{\alpha,\beta}d\sigma_D = g_{\alpha,\beta}d\sigma_D,$$

for all $\alpha, \beta = 1, 2, \dots, l$. The proof is complete. $\square$

Lemma 4.3.4. Let γ be a finite complex valued Borel measure on $[0, \infty)$ and $d \in \mathbb{N} \cup \{\infty\}$. Suppose that

$$\int_0^\infty \Omega_d(rt) d\gamma(r) = 0,$$

for all $t \in [0, \infty)$. Then the measure γ is the zero measure.

Proof. Since the measure γ is finite and complex valued, then there exist finite and real valued Borel measures γ_{Re} and γ_{Im} , such that $\gamma = \gamma_{Re} + i\gamma_{Im}$. Since the function Ω_d is a real valued function, we have that

$$0 = \int_0^\infty \Omega_d(rt) d\gamma(r) = \int_0^\infty \Omega_d(rt) d\gamma_{Re}(r) + i \int_0^\infty \Omega_d(rt) d\gamma_{Im}(r),$$

for all $t \in [0, \infty)$, that is

$$\int_0^\infty \Omega_d(rt) d\gamma_{Re}(r) = \int_0^\infty \Omega_d(rt) d\gamma_{Im}(r) = 0,$$

for all $t \in [0, \infty)$. In order to prove γ_{Re} is the zero measure write, via the Jordan Decomposition Theorem, (FOLLAND, 2013) page 87, $\gamma_{Re} = \gamma_{Re}^+ - \gamma_{Re}^-$, where γ_{Re}^+ and γ_{Re}^- are finite nonnegative Borel measures on $[0, \infty)$. Defining $f : [0, \infty) \rightarrow \mathbb{R}$ by

$$f(t) := \int_0^\infty \Omega_d(rt) d\gamma_{Re}^+(r) = \int_0^\infty \Omega_d(rt) d\gamma_{Re}^-(r),$$

we see that f defines a continuous positive definite radial kernel on $\mathbb{R}^d$. Theorem 4.1.3 and Theorem 4.1.8 imply that $\gamma_{Re}^+ = \gamma_{Re}^-$, so that γ_{Re} is the zero measure. The same proof implies that γ_{Im} is the zero measure. $\square$

Now we are able to prove the main result of this section.

Theorem 4.3.5. Let $f : X \times X \times [0, \infty) \rightarrow \mathbb{C}$ be a function such that $f(u, v, \cdot)$ is continuous on $[0, \infty)$, for each fixed $u, v \in X$. The following properties are equivalent:

- (i) The kernel $((u, x), (v, y)) \in (X \times \mathbb{R}^d)^2 \rightarrow f(u, v, \|x - y\|)$ is positive definite.
- (ii) Each function $f(u, v, \cdot)$ has an integral representation in the form

$$f(u, v, s) = \int_0^\infty \Omega_d(rs) d\sigma_{u,v}(s),$$

where $\sigma_{u,v}$ is a finite complex valued Borel measure on $[0, \infty)$, and for every Borel measurable set $A \subset [0, \infty)$, the kernel $(u, v) \in X \times X \rightarrow \sigma_{u,v}(A) \in \mathbb{C}$ is positive definite. The finite complex valued Borel measures $\sigma_{u,v}$ are unique for each pair $(u, v) \in X^2$.

Proof. If (i) holds, the last assertion in Lemma 4.3.1 implies that each function $f(u, u, \cdot)$ is continuous and, in addition, defines a positive definite radial kernel on $\mathbb{R}^d$. Theorem 4.1.3 and Theorem 4.1.8 provide a unique integral representation for each $f(u, u, \cdot)$ according to (ii). If $u, v \in X$ and $u \neq v$, Lemma 4.3.1 implies that all three functions

$$f(u, u, \cdot) + f(v, v, \cdot) + f(u, v, \cdot) + f(v, u, \cdot),$$

$$f(u, u, \cdot) + f(v, v, \cdot) - f(u, v, \cdot) - f(v, u, \cdot),$$

$$f(u, u, \cdot) + f(v, v, \cdot) - if(u, v, \cdot) + if(v, u, \cdot),$$

are continuous and, also, define positive definite radial kernels on $\mathbb{R}^d$. Given that $f(u, v, \cdot)$ is a linear combination of them (with complex coefficients), it is promptly seen, via Theorem 4.1.3 and Theorem 4.1.8 once again, that $f(u, v, \cdot)$ has the integral representation as in the statement of the theorem. The finite complex valued Borel measure $\sigma_{u,v}$ is unique by Lemma 4.3.4 for each fixed pair $(u, v) \in X^2$.

Next, we verify that the kernels $(u, v) \in X \times X \rightarrow \sigma_{u,v}(A)$ are positive definite for every Borel set $A \subset [0, \infty)$. If $u_1, u_2, \dots, u_p$ are distinct points on X and $c_1, c_2, \dots, c_p \in \mathbb{C}$, Lemma 4.3.1 reveals that

$$g(s) = \sum_{\mu, \nu=1}^p c_\mu \overline{c_\nu} f(u_\mu, u_\nu, s) = \int_0^\infty \Omega_d(rs) d\sigma_g(s),$$

for some finite nonnegative Borel measure σ_g in $[0, \infty)$. Since

$$g(s) = \int_0^\infty \Omega_d(rs) d \left(\sum_{\mu, \nu=1}^p c_\mu \overline{c_\nu} \sigma_{u_\mu, u_\nu}(s) \right),$$

Lemma 4.3.4 implies that

$$\sum_{\mu, \nu=1}^p c_\mu \overline{c_\nu} \sigma_{u_\mu, u_\nu} = \sigma_g.$$

In particular,

$$\sum_{\mu, \nu=1}^p c_\mu \overline{c_\nu} \sigma_{\mu, \nu}(A) = \sigma_g(A) \geq 0.$$

This shows that (i) implies (ii).

Conversely, assume that each $f(u, v, \cdot)$ has an integral representation satisfying the properties given in (ii). Since each kernel $(u, v) \in X \times X \rightarrow \sigma_{u, v}(A)$ is positive definite for all Borel sets $A \subset [0, \infty)$, given distinct points $u_1, u_2, \dots, u_p$ in X and $c_1, c_2, \dots, c_p \in \mathbb{C}$, the finite Borel measure $\sum_{\mu, \nu=1}^p c_\mu \overline{c_\nu} \sigma_{\mu, \nu}$ is nonnegative. Lemma 4.3.3 implies that

$$f(u_\alpha, u_\beta, t) = \int_0^\infty \Omega_d(rt) d\sigma_{\alpha, \beta}(r) = \int_0^\infty \Omega_d(r\|x_\mu - x_\nu\|) g_{\alpha, \beta}(r) d\sigma_D(r),$$

for all $t \in [0, \infty)$. Given $d_{\mu, \alpha} \in \mathbb{C}$, $1 \leq \mu \leq n$, $1 \leq \alpha \leq p$, and distinct points $x_1, x_2, \dots, x_n \in \mathbb{R}^d$

$$\begin{aligned} \sum_{\alpha, \beta=1}^p \sum_{\mu, \nu=1}^n d_{\mu, \alpha} \overline{d_{\nu, \beta}} f(u_\alpha, u_\beta, \|x_\mu - x_\nu\|) \\ = \int_0^\infty \sum_{\alpha, \beta=1}^p \sum_{\mu, \nu=1}^n d_{\mu, \alpha} \overline{d_{\nu, \beta}} \Omega_d(r\|x_\mu - x_\nu\|) g_{\alpha, \beta}(r) d\sigma_D(r). \end{aligned}$$

The integrand above is nonnegative for all $r \geq 0$, because it is the Hadamard product of two positive semidefinite matrices, one being the interpolation matrix $[\Omega_d(r\|x_\mu - x_\nu\|)]_{\mu, \nu=1}^n$ and the other $[g_{\alpha, \beta}(r)]_{\alpha, \beta=1}^p$. The proof is complete. $\square$

Now we prove a version of Theorem 3.3.4 in this new context.

Theorem 4.3.6. Let X be endowed with a topology and $f : X \times X \times [0, \infty) \rightarrow \mathbb{C}$ be a continuous function such that the kernel

$$((u, x), (v, y)) \in (X \times \mathbb{R}^d)^2 \rightarrow f(u, v, \|x - y\|)$$

is positive definite. Then, for every $h \in C_0([0, \infty))$, the kernel $T_h : X \times X \rightarrow \mathbb{C}$ defined by

$$T_h(u, v) = \int_0^\infty h(r) \sigma_{u, v}(r)$$

is continuous. Further, if h is a nonnegative function, then the kernel T_h is positive definite.

Proof. The last assertion of the theorem is a direct consequence of the fact that for every Borel measurable set $A \subset [0, \infty)$, the kernel $(u, v) \in X \times X \rightarrow \sigma_{u, v}(A) \in \mathbb{C}$ is positive definite.

As for the continuity, we assume that the topology in X is first countable. This additional hypothesis will suffice to our applications, and the general case can be proved by similar arguments with nets. So, in order to prove that T_h is continuous, it is sufficient to prove that if a sequence $(u_n, v_n) \rightarrow (u, v)$ then $T_h(u_n, v_n) \rightarrow T_h(u, v)$.

Let $(u_n, v_n)_{n \in \mathbb{N}} \in [0, \infty)^2$ be a sequence converging to $(u, v) \in [0, \infty)^2$. We will prove that $(\sigma_{u_n, v_n})_{n \in \mathbb{N}}$ converges vaguely to $\sigma_{u, v}$ and that $\sigma_{u_n, v_n}([0, \infty)) \rightarrow \sigma_{u, v}([0, \infty))$. Indeed, since $u_n \rightarrow$

u and $v_n \rightarrow v$, the numerical sequences $(f(u_n, u_n, 0))_{n \in \mathbb{N}}$ and $(f(v_n, v_n, 0))_{n \in \mathbb{N}}$ are bounded, so, the sequences of measures $(\sigma_{u_n, u_n})_{n \in \mathbb{N}}$ and $(\sigma_{v_n, v_n})_{n \in \mathbb{N}}$ are bounded. But, $|\sigma_{u_n, v_n}(A)|^2 \leq \sigma_{u_n, u_n}(A) \sigma_{v_n, v_n}(A)$ for all Borel sets $A \subset [0, \infty)$, whence the sequence of finite Borel measures $(\sigma_{u_n, v_n})_{n \in \mathbb{N}}$ is also bounded. Proposition 4.1.7 substantiates the existence of a subsequence $(\sigma_{u_{n_k}, v_{n_k}})_{k \in \mathbb{N}}$ and a finite Borel measure γ in $[0, \infty]$ such that $(\sigma_{u_{n_k}, v_{n_k}})_{k \in \mathbb{N}}$ converges vaguely to γ and $\sigma_{u_{n_k}, v_{n_k}}([0, \infty)) \rightarrow \gamma([0, \infty))$. But, for all $t > 0$, the function $\Omega_d(\cdot, t) \in C_0([0, \infty))$, and then

$$\int_0^\infty \Omega_d(rt) \sigma_{u, v}(r) = f(u, v, t) = \lim_{k \rightarrow \infty} f(u_{n_k}, v_{n_k}, t) = \lim_{k \rightarrow \infty} \int_0^\infty \Omega_d(rt) \sigma_{u_{n_k}, v_{n_k}}(r) = \int_0^\infty \Omega_d(rt) \gamma(r).$$

Since the sequence of bounded functions $\Omega_d(\cdot, 1/n)$ converges pointwise to the constant function 1, the Lebesgue Dominated Convergence Theorem implies that $\sigma_{u, v}([0, \infty)) = \gamma([0, \infty))$. All this information can be summarized as

$$\int_0^\infty \Omega_d(rt) \sigma_{u, v}(r) = \int_0^\infty \Omega_d(rt) \gamma(r),$$

for all $t \in [0, \infty)$. Finally, Lemma 4.3.4 implies that $\sigma_{u, v} = \gamma$. Thus, we have proved that if $(u_n, v_n) \rightarrow (u, v)$ then a subsequence of $T_h(u_n, v_n)$ converges to $T_h(u, v)$. This implies the continuity of T_h . $\square$

Lemma 4.3.7. Let X be a group G with operation $*$ and $g : G \times [0, \infty) \rightarrow \mathbb{C}$ a function such that $g(u, \cdot) : [0, \infty) \rightarrow \mathbb{C}$ is continuous for all $u \in G$. The kernel $((u, x), (v, y)) \in (G \times \mathbb{R}^d)^2 \rightarrow g(u^* v^{-1}, \|x - y\|)$ is positive definite if and only if the function g has a unique integral representation

$$g(u, t) = \int_0^\infty \Omega_d(rt) d\sigma_u(r),$$

for all $u \in G$ and $t \in [0, \infty)$, where σ_u is a finite complex valued Borel measure on $[0, \infty)$ such that for every Borel set $A \subset [0, \infty)$, the kernel $(u, v) \in G \times G \rightarrow \sigma_{u^* v^{-1}}(A) \in \mathbb{C}$ is positive definite.

Proof. Theorem 4.3.5 implies that the kernel $((u, x), (v, y)) \in (G \times \mathbb{R}^d)^2 \rightarrow g(u^* v^{-1}, \|x - y\|)$ is positive definite if and only if

$$g(u^* v^{-1}, t) = \int_0^\infty \Omega_d(rt) d\sigma_{u, v}(r),$$

where $\sigma_{u, v}$ is a finite complex valued Borel measure on $[0, \infty)$, and for every Borel set $A \subset [0, \infty)$, the kernel $(u, v) \in G \times G \rightarrow \sigma_{u, v}(A) \in \mathbb{C}$ is positive definite. But, if e is the neutral element in G ,

$$g(u^* v^{-1}, t) = g((u^* v^{-1}) * e^{-1}, t) = \int_0^\infty \Omega_d(rt) d\sigma_{u^* v^{-1}, e}(r).$$

Since the integral representation is unique, we have that $\sigma_{u, v} = \sigma_{u^* v^{-1}, e}$. Defining $\sigma_u := \sigma_{u, e}$, $u \in G$, given a Borel set $A \subset [0, \infty)$, we obtain that kernel $(u, v) \in G \times G \rightarrow \sigma_{u, v}(A) \in \mathbb{C}$ is positive definite if and only if the kernel $(u, v) \in G \times G \rightarrow \sigma_{u^* v^{-1}}(A) \in \mathbb{C}$ is positive definite. $\square$

Using Theorem 4.3.5 and the bridge propositions on matrix valued kernels in Section 2.3, we can reobtain the main theorem in (WANG; DU; MA, 2014). The proof is omitted.

Theorem 4.3.8. Let $F : [0, \infty) \rightarrow M_l(\mathbb{C})$ be a continuous function. The following assertions are equivalent:

- (i) The kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow F(\|x - y\|)$ is positive definite.
- (ii) The function F has a unique integral representation

$$[F(t)]_{\alpha, \beta} = \int_0^\infty \Omega_d(rt) d\sigma_{\alpha, \beta}(r),$$

for all $1 \leq \alpha, \beta \leq l$, where $\sigma_{\alpha, \beta}$ is a finite complex valued Borel measure on $[0, \infty)$, and for every $c_1, c_2, \dots, c_l \in \mathbb{C}$, the measure $\sum_{\alpha, \beta=1}^l c_\alpha \overline{c_\beta} \sigma_{\alpha, \beta}$ is nonnegative.

- (iii) For every $c \in \mathbb{C}^l$, the function $F_c(t) = cF(t)c^*$, $t \in [0, \infty)$ defines a positive definite radial kernel on $\mathbb{R}^d$.

The generalization of Theorem 4.3.8 to operator valued kernels is as follows.

Theorem 4.3.9. Let $\mathcal{H}$ be a complex Hilbert space and $F : [0, \infty) \rightarrow L(\mathcal{H})$ a continuous function in the weak operator topology. The following properties are equivalent:

- (i) The kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow F(\|x - y\|) \in L(\mathcal{H})$ is positive definite.
- (ii) For every $v \in \mathcal{H}$, the function $F_v(t) = \langle F(t)v, v \rangle$, $t \in [0, \infty)$ defines a continuous positive definite radial kernel on $\mathbb{R}^d$.

Proof. If the operator valued kernel is positive definite, arguments similar to those in the proof of Proposition 4.3.1, reveal that F_v defines a positive definite radial kernel, for every $v \in \mathcal{H}$. Since F is a continuous function in the weak operator topology, the scalar valued function F_v is also continuous. Hence (i) implies (ii).

Conversely, given distinct points $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ and vectors $v_1, v_2, \dots, v_n \in \mathcal{H}$, if $(u_\lambda)_{\lambda \in \Lambda}$ is a Hamel basis for $\mathcal{H}$, then $v_\mu = \sum_{\alpha=1}^l c_{\mu, \alpha} u_{\lambda_\alpha}$, $1 \leq \mu \leq n$, for some complex numbers $c_{\mu, \alpha}$ and finitely many indices λ_α in Λ . But then

$$\begin{aligned} \sum_{\mu, v=1}^n \langle F(\|x_\mu - x_v\|) v_\mu, v_v \rangle &= \sum_{\mu, v=1}^n \sum_{\alpha, \beta=1}^l c_{\mu, \alpha} \overline{c_{v, \beta}} \langle F(\|x_\mu - x_v\|) u_{\lambda_\alpha}, u_{\lambda_\beta} \rangle \\ &= \sum_{\mu, v=1}^n c_\mu G(\|x_\mu - x_v\|) c_v^*, \end{aligned}$$

where $G(t) = [\langle F(t) u_{\lambda_\alpha}, u_{\lambda_\beta} \rangle]_{\alpha, \beta=1}^l \in M_l(\mathbb{C})$, $t \in [0, \infty)$, and $c_\mu = (c_{\mu, 1}, c_{\mu, 2}, \dots, c_{\mu, l}) \in \mathbb{C}^l$. The matrix valued function G is continuous, because F is continuous in the weak operator topology. For every $c = (c_1, c_2, \dots, c_l) \in \mathbb{C}^l$, the function $G_c(t) = cG(t)c^*$, $t \in [0, \infty)$ defines a continuous positive definite radial kernel on $\mathbb{R}^d$ because $cG(t)c^* = \langle F(t)v, v \rangle$, where $v = \sum_{\alpha=1}^l c_\alpha u_{\lambda_\alpha}$. By Theorem 4.3.8, the function G defines a matrix valued positive definite radial kernel on $\mathbb{R}^d$. Thus, F defines an operator valued positive definite radial kernel on $\mathbb{R}^d$. $\square$

The article (NEEB, 1998) contains a characterization of the ultraweakly continuous functions $F : \mathbb{R}^d \rightarrow L(\mathcal{H})$ that are positive definite via an integral representation similar to that in Theorem 4.1.4, in terms of operator valued measures.

We close this section by providing two different methods for characterizing the continuous functions $f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$, such that

$$((x, z), (y, w)) \in (\mathbb{R}^d \times \mathbb{R}^{d'}) \times (\mathbb{R}^d \times \mathbb{R}^{d'}) \rightarrow f(\|x - y\|_d, \|z - w\|_{d'})$$

is positive definite, where $d, d' \in \mathbb{N} \cup \{\infty\}$. First, we prove a direct approach using similar arguments as those in Theorem 4.1.3 and Theorem 4.1.8. This proof is similar to the one given in (ALONSO-MALAVER; PORCU; GIRALDO, 2015).

Theorem 4.3.10. Let $f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be a continuous function and $d, d' \in \mathbb{N} \cup \{\infty\}$. The kernel

$$((x, z), (y, w)) \in (\mathbb{R}^d \times \mathbb{R}^{d'}) \times (\mathbb{R}^d \times \mathbb{R}^{d'}) \rightarrow f(\|x - y\|_d, \|z - w\|_{d'})$$

is positive definite if and only if f has an integral representation

$$f(t, s) = \int_{[0, \infty)^2} \Omega_d(rt) \Omega_{d'}(us) d\sigma(r, u),$$

for all $t, s \geq 0$, where σ is a finite nonnegative Borel measure on $[0, \infty)^2$. The finite and nonnegative Borel measure σ is unique.

Proof. If $f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is continuous and the kernel

$$((x, z), (y, w)) \in (\mathbb{R}^d \times \mathbb{R}^{d'}) \times (\mathbb{R}^d \times \mathbb{R}^{d'}) \rightarrow f(\|x - y\|_d, \|z - w\|_{d'}) \in \mathbb{R}$$

is positive definite, we can apply Bochner's Theorem 4.1.4 on the continuous function $g(x, z) := f(\|x\|_d, \|z\|_{d'})$, $x \in \mathbb{R}^d$, $z \in \mathbb{R}^{d'}$. Then,

$$g(x, z) = f(\|x\|_d, \|z\|_{d'}) = \int_{\mathbb{R}^{d+d'}} e^{-i(x, z) \cdot (\xi_1, \xi_2)} d\sigma((\xi_1, \xi_2)),$$

for all $(x, z) \in \mathbb{R}^{d+d'}$. However, that yields

$$\begin{aligned} f(\|x\|_d, \|z\|_{d'}) &= g(x, z) = \int_{S^{d-1}} \int_{S^{d'-1}} g(\|x\|_d \omega, \|z\|_{d'} \omega') d\tilde{\rho}_d(\omega) d\tilde{\rho}_{d'}(\omega') \\ &= \int_{S^{d-1}} \int_{S^{d'-1}} \int_{\mathbb{R}^{d+d'}} e^{-i(\|x\|_d \omega, \|z\|_{d'} \omega') \cdot \xi} d\sigma(\xi) d\tilde{\rho}_d(\omega) d\tilde{\rho}_{d'}(\omega') \\ &= \int_{\mathbb{R}^{d+d'}} \int_{S^{d-1}} e^{-i(\|x\|_d \xi_1) \cdot \omega} d\tilde{\rho}_d(\omega) \int_{S^{d'-1}} e^{-i(\|z\|_{d'} \xi_2) \cdot \omega'} d\tilde{\rho}_{d'}(\omega') d\sigma((\xi_1, \xi_2)) \\ &= \int_{\mathbb{R}^{d+d'}} \Omega_d(\|x\|_d \|\xi_1\|) \Omega_{d'}(\|z\|_{d'} \|\xi_2\|) d\sigma((\xi_1, \xi_2)) \\ &= \int_{[0, \infty)^2} \Omega_d(\|x\|_d r) \Omega_{d'}(\|z\|_{d'} u) d\sigma_{\|\cdot\|_d, \|\cdot\|_{d'}}(r, u), \end{aligned}$$

where $\tilde{\rho}_d = \frac{1}{\text{vol}(S^{d-1})} \rho_d$ and $\tilde{\rho}_{d'} = \frac{1}{\text{vol}(S^{d'-1})} \rho_{d'}$. The finite and nonnegative Borel measure $\sigma_{\|\cdot\|_d, \|\cdot\|_{d'}}$ is the push forward measure of the measure σ through the continuous function $(\xi_1, \xi_2) \in \mathbb{R}^{d+d'} \rightarrow (\|\xi_1\|_d, \|\xi_2\|_{d'}) \in [0, \infty) \times [0, \infty)$.

For the converse, given distinct points $x_1, x_2, \dots, x_n \in \mathbb{R}^d$, distinct points $z_1, z_2, \dots, z_m \in \mathbb{R}^{d'}$ and $c_{\mu,i} \in \mathbb{R}$, $1 \leq \mu \leq n$, $1 \leq i \leq m$,

$$\begin{aligned} & \sum_{\mu,v=1}^n \sum_{i,j=1}^m c_{\mu,i} c_{v,j} f(\|x_\mu - x_v\|_d, \|z_i - z_j\|_{d'}) \\ &= \int_0^\infty \int_0^\infty \sum_{\mu,v=1}^n \sum_{i,j=1}^m c_{\mu,i} c_{v,j} \Omega_d(\|x_\mu - x_v\|_d r) \Omega_{d'}(\|z_i - z_j\|_{d'} u) d\sigma_{\|\cdot\|_d, \|\cdot\|_{d'}}(r, u). \end{aligned}$$

For every $r, s \geq 0$, the matrices $[\Omega_d(\|x_\mu - x_v\|_d r)]_{\mu,v=1}^n$ and $[\Omega_{d'}(\|z_i - z_j\|_{d'} u)]_{i,j=1}^m$ are positive semidefinite, and then

$$\sum_{\mu,v=1}^n \sum_{i,j=1}^m c_{\mu,i} c_{v,j} \Omega_d(\|x_\mu - x_v\|_d r) \Omega_{d'}(\|z_i - z_j\|_{d'} u) \geq 0.$$

Thus, the kernel is positive definite by Lemma 2.2.5.

In order to prove that the finite Borel measure $\sigma_{\|\cdot\|_d, \|\cdot\|_{d'}}$ is unique, let γ be a finite nonnegative Borel measure on $[0, \infty)^2$, such that

$$f(t, s) = \int_{[0, \infty)^2} \Omega_d(rt) \Omega_{d'}(us) d\gamma(r, u),$$

for all $t, s \geq 0$. Define the finite nonnegative Borel measure μ on $\mathbb{R}^{d+d'}$ by

$$\mu := (\gamma \times \tilde{\rho}_d \times \tilde{\rho}_{d'})_{\mathbb{R}^{d+d'}},$$

where the measure $(\gamma \times \tilde{\rho}_d \times \tilde{\rho}_{d'})_{\mathbb{R}^{d+d'}}$ is the push forward measure of the finite nonnegative Borel measure $\gamma \times \tilde{\rho}_d \times \tilde{\rho}_{d'}$ in $[0, \infty)^2 \times S^{d-1} \times S^{d'-1}$, through the continuous function $(r, u, \omega, \omega') \in [0, \infty)^2 \times S^{d-1} \times S^{d'-1} \rightarrow (r\omega, u\omega') \in \mathbb{R}^{d+d'}$. The properties of the push forward measure give us that for all $(x, z) \in \mathbb{R}^{d+d'}$

$$\begin{aligned} \int_{\mathbb{R}^{d+d'}} e^{-i(x,z) \cdot (\xi_1, \xi_2)} d\mu((\xi_1, \xi_2)) &= \int_{[0, \infty)^2} \int_{S^{d-1}} \int_{S^{d'-1}} e^{-i(x,z) \cdot (r\omega, u\omega')} d\gamma(r, u) d\rho_d(\omega) d\rho_{d'}(\omega') \\ &= \int_{[0, \infty)^2} \Omega_d(r\|x\|) \Omega_{d'}(u\|z\|) d\gamma(r, u) = f(\|x\|_d, \|z\|_{d'}). \end{aligned}$$

This equation implies that the Fourier transform of the nonnegative and finite Borel measures μ and σ are equal to the function g , then $\mu = \sigma$ by Theorem 4.1.4. If $C, D \subset [0, \infty)$ are Borel sets, consider $A := \{r\omega, r \in C, \omega \in S^{d-1}\} \subset \mathbb{R}^d$ and $B := \{u\omega', u \in D, \omega' \in S^{d'-1}\} \subset \mathbb{R}^{d'}$. Using again the properties of push forward measure, we have that

$$\mu(A \times B) = \gamma(C \times D),$$

and the uniqueness follows. $\square$

Now we prove the same characterization using Theorem 4.3.5. A technical result concerning measures on products of spaces will be needed. A version of this result adapted to our purposes is presented, (BERG; CHRISTENSEN; RESSEL, 1984) page 24.

Lemma 4.3.11. Let $\mathcal{B}([0, \infty))$ be the set of all Borel measurable sets in $[0, \infty)$. If $\gamma: \mathcal{B}([0, \infty)) \times \mathcal{B}([0, \infty)) \rightarrow [0, \infty)$ is a bounded function such that for all $A, B \in \mathcal{B}([0, \infty))$, $\gamma(A, \cdot)$ and $\gamma(\cdot, B)$ are nonnegative Borel measures in $[0, \infty)$, then there exists a unique nonnegative Borel measure on $[0, \infty)^2$ such that

$$\sigma(A \times B) = \gamma(A, B)$$

for all $A, B \in \mathcal{B}([0, \infty))$.

Theorem 4.3.12. Let $f: [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be a continuous function and $d, d' \in \mathbb{N} \cup \{\infty\}$. The kernel

$$((x, z), (y, w)) \in (\mathbb{R}^d \times \mathbb{R}^{d'}) \times (\mathbb{R}^d \times \mathbb{R}^{d'}) \rightarrow f(\|x - y\|_d, \|z - w\|_{d'})$$

is positive definite if and only if f has an integral representation

$$f(t, s) = \int_{[0, \infty)^2} \Omega_d(rt) \Omega_{d'}(us) d\sigma(r, u),$$

for all $t, s \in [0, \infty)$, where σ is a finite nonnegative Borel measure on $[0, \infty)^2$. The finite and nonnegative Borel measure σ is unique.

Proof. If the kernel is positive definite, Theorem 4.3.5 implies that the function $((x, y), t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty) \rightarrow f(\|x - y\|, t) \in \mathbb{R}$ has a unique integral representation

$$f(\|x - y\|, t) = \int_0^\infty \Omega_{d'}(ut) d\sigma_{x,y}(u),$$

where $\sigma_{x,y}$, $x, y \in \mathbb{R}^d$, is a finite complex valued Borel measure on $[0, \infty)$ and for every Borel set $A \subset [0, \infty)$, the kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow \sigma_{x,y}(A)$ is positive definite. In particular, $\sigma_{x,y} = \sigma_{x',y'}$, if $\|x - y\|_d = \|x' - y'\|_d$. Theorem 4.3.6 implies that for every $h \in C_0([0, \infty))$, the kernel $T_h: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ defined by

$$T_h(x, y) = \int_0^\infty h(r) d\sigma_{x,y}(r)$$

is continuous and positive definite when h nonnegative. Note that T_h is a radial kernel on $\mathbb{R}^d$, so there is a unique integral representation

$$T_h(x, y) = \int_0^\infty \Omega_d(r\|x - y\|) d\sigma_h(r),$$

where σ_h , $h \in C_0([0, \infty))$, is a finite complex valued Borel measure on $[0, \infty)$, which is nonnegative whenever h is nonnegative. In particular, for every fixed $h \in C_0([0, \infty))$, if $(C_n)_{n \in \mathbb{N}}$ is a disjoint sequence of Borel sets in $[0, \infty)$, we have that

$$\sigma_h(\cup_{n=1}^\infty C_n) = \sum_{n=1}^\infty \sigma_h(C_n).$$

Note that for every Borel set $B \subset [0, \infty)$, the linear functional

$$h \in C_0([0, \infty)) \rightarrow \sigma_h(B)$$

is positive. The Riesz Representation Theorem, (FOLLAND, 2013) page 223, implies the existence of a unique finite nonnegative Borel measure σ_B in $[0, \infty)$ such that

$$\sigma_h(B) = \int_0^\infty h(r) \sigma_B(r).$$

Define $\gamma: \mathcal{B}([0, \infty)) \times \mathcal{B}([0, \infty)) \rightarrow [0, \infty)$, by $\gamma(C, D) = \sigma_C(D)$. The function γ is nonnegative. For every $C, D \in \mathcal{B}([0, \infty))$

$$\sigma_C(D) \leq \sigma_C([0, \infty)) \leq \sigma_{[0, \infty)}([0, \infty)).$$

The first inequality is true because σ_C is a nonnegative measure, while the second one is a consequence of the relation $\sigma_h(A) + \sigma_h([0, \infty) - A) = \sigma_h([0, \infty))$, for all $A \in \mathcal{B}([0, \infty))$. Thus γ is a bounded function. If $(C)_{n \in \mathbb{N}}$ is a disjoint sequence of Borel sets in $[0, \infty)$, we have that

$$\begin{aligned} \int_0^\infty h(r) d\sigma_{(\cup_{n=1}^\infty C_n)}(r) &= \sigma_h(\cup_{n=1}^\infty C_n) = \sum_{n=1}^\infty \sigma_h(C_n) \\ &= \sum_{n=1}^\infty \int_0^\infty h(r) d\sigma_{C_n}(r) = \int_0^\infty h(r) d\left(\sum_{n=1}^\infty \sigma_{C_n}(r)\right) \end{aligned}$$

for all $h \in C_0([0, \infty))$, and then, $\gamma(\cup_{n=1}^\infty C_n, D) = \sum_{n=1}^\infty \gamma(C_n, D)$ for all Borel sets $D \subset [0, \infty)$. On the other hand, if $(D)_{n \in \mathbb{N}}$ is a disjoint sequence of Borel sets in $[0, \infty)$, we have $\gamma(C, \cup_{n=1}^\infty D_n) = \sum_{n=1}^\infty \gamma(C, D_n)$, simply because σ_C is a finite nonnegative Borel measure on $[0, \infty)$ for all Borel sets $C \subset [0, \infty)$.

Lemma 4.3.11 implies the existence of a unique finite nonnegative Borel measure σ in $[0, \infty)^2$, such that

$$\sigma(C \times D) = \gamma(C, D) = \sigma_C(D)$$

In order to prove the integral representation, let $s \neq 0$ and $x, y \in \mathbb{R}^d$ be distinct points. Pick a sequence of bounded simple functions $(\varphi_n)_{n \in \mathbb{N}}$ defined on $[0, \infty)$ that converges to $\Omega_d(\|x - y\| \cdot)$. Then

$$\begin{aligned} f(\|x - y\|, s) &= \int_0^\infty \Omega_{d'}(us) d\sigma_{x,y}(u) = T_{\Omega_{d'}(s \cdot)}(x, y) \\ &= \int_0^\infty \Omega_d(r\|x - y\|) d\sigma_{\Omega_{d'}(s \cdot)}(r) = \lim_{n \rightarrow \infty} \int_0^\infty \varphi_n(r) d\sigma_{\Omega_{d'}(s \cdot)}(r). \end{aligned}$$

But, since φ_n is a simple function, after adding its components, we can prove that

$$\int_0^\infty \varphi_n(r) d\sigma_{\Omega_{d'}(s \cdot)}(r) = \int_{[0, \infty)} \varphi_n(r) \Omega_{d'}(su) d\sigma(r, u),$$

and then

$$f(\|x - y\|, s) = \int_{[0, \infty)} \Omega_d(\|x - y\|r) \Omega_{d'}(su) d\sigma(r, u).$$

The integral representation at the points $(0, t)$ and $(t, 0)$, $t \in [0, \infty)$, is proved using the continuity of f and the Lebesgue Dominated Convergence Theorem.

The converse is true by the same arguments in the proof of Theorem 4.3.10.

The proof is complete. $\square$

The set of continuous functions $f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ that define positive definite radial kernels on $\mathbb{R}^\infty \times \mathbb{R}^\infty$, has a similar structure as the set of functions in Theorem 4.1.9. They are related to completely monotone functions in several variables. This relation is justified by the integral representation obtained in Theorem 4.3.10 and the representation for completely monotone functions in several variables given in (BOCHNER, 2005).

4.4 The strictly positive definite case

In this section, we present necessary and sufficient conditions on a positive definite kernel representable as in Theorem 4.3.5 in order that it be strictly positive definite, when $d \in \{2, 3, \dots, \infty\}$. With this result and the bridge propositions on matrix valued kernels in Section 2.3, we are able to characterize the continuous matrix valued strictly positive definite radial kernels on $\mathbb{R}^d$, $d \in \{2, 3, \dots, \infty\}$.

We also characterize the kernels representable as in Theorem 4.3.10 that are strictly positive definite, when $d, d' \in \mathbb{N} \cup \{\infty\}$, except when $d = d' = 1$. Unless stated otherwise, all the results in this section were obtained by the author. Some of them were included in (GUELLA; MENEGATTO, 2018a).

Theorem 4.4.1. Let $f : X \times X \times [0, \infty) \rightarrow \mathbb{C}$ be a function such that $f(u, v, \cdot)$ is continuous on $[0, \infty)$, for each fixed $u, v \in X$ and $d \in \{2, 3, \dots, \infty\}$. If the kernel $((u, x), (v, y)) \in (X \times \mathbb{R}^d)^2 \rightarrow f(u, v, \|x - y\|)$ is positive definite, the following assertions are equivalent:

- (i) The kernel $((u, x), (v, y)) \in (X \times \mathbb{R}^d)^2 \rightarrow f(u, v, \|x - y\|)$ is strictly positive definite.
- (ii) For every finite quantity of distinct points $u_1, u_2, \dots, u_p \in X$ and whichever scalars $c_1, c_2, \dots, c_p \in \mathbb{C}$, not all zero, the function $t \in [0, \infty) \rightarrow \sum_{\alpha, \beta=1}^p c_\alpha \overline{c_\beta} f(u_\alpha, u_\beta, t) \in \mathbb{R}$ defines a strictly positive definite radial kernel on $\mathbb{R}^d$.

Proof. If the kernel is strictly positive definite, assertion (ii) is valid by Lemma 4.3.1-(iii).

If (ii) holds, given distinct points $u_1, u_2, \dots, u_p \in X$, distinct points $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ and scalars $c_{\mu, \alpha} \in \mathbb{C}$, $1 \leq \mu \leq n$, $1 \leq \alpha \leq p$, such that

$$\sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^p c_{\mu, \alpha} \overline{c_{\nu, \beta}} f(u_\alpha, u_\beta, \|x_\mu - x_\nu\|) = 0,$$

by making use of Lemma 4.3.3, we obtain

$$\begin{aligned} 0 &= \sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^p c_{\mu, \alpha} \overline{c_{\nu, \beta}} f(u_\alpha, u_\beta, \|x_\mu - x_\nu\|) \\ &= \sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^p c_{\mu, \alpha} \overline{c_{\nu, \beta}} \int_0^\infty \Omega_d(r \|x_\mu - x_\nu\|) d\sigma_{u_\alpha, u_\beta}(r) \\ &= \int_0^\infty \sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^p c_{\mu, \alpha} \overline{c_{\nu, \beta}} \Omega_d(r \|x_\mu - x_\nu\|) g_{\alpha, \beta}(r) d\sigma_D(r). \end{aligned}$$

Since the matrices $[\Omega_d(r\|x_\mu - x_\nu\|)]_{\mu,\nu=1}^q$ and $[g_{\alpha,\beta}(r)]_{\alpha,\beta=1}^p$ are positive semidefinite for all $r \geq 0$, so is their Kroenecker product by Lemma 2.1.8. Hence,

$$\sum_{\mu,\nu=1}^q \sum_{\alpha,\beta=1}^p c_{\mu,\alpha} \overline{c_{\nu,\beta}} \Omega_d(r\|x_\mu - x_\nu\|) g_{\alpha,\beta}(r) \geq 0,$$

for all $r \geq 0$. Since the function $t \in [0, \infty) \rightarrow \sum_{\alpha=1}^l f(u_\alpha, u_\alpha, t) \in \mathbb{R}$ defines a strictly positive definite radial kernel on $\mathbb{R}^d$, Theorem 4.2.2 and Theorem 4.2.4 imply that σ_D is a positive measure that is not a multiple of δ_0 . It is now seen that if

$$B := \left\{ s \in (0, \infty) : \sum_{\mu,\nu=1}^q \sum_{\alpha,\beta=1}^p c_{\mu,\alpha} \overline{c_{\nu,\beta}} \Omega_d(s\|x_\mu - x_\nu\|) g_{\alpha,\beta}(s) = 0 \right\},$$

then $(0, \infty) - B$ is a σ_D -null set. Recalling the second part of Lemma 2.1.9 and the fact that the matrices $[\Omega_d(s\|x_\mu - x_\nu\|)]_{\mu,\nu=1}^q$ are positive definite for all $s > 0$ by Theorem 4.2.1 and Theorem 4.2.3, we have that

$$B = \left\{ s \in (0, \infty) : \sum_{\alpha,\beta=1}^p c_{\mu,\alpha} \overline{c_{\mu,\beta}} g_{\alpha,\beta}(s) = 0, \text{ for all } \mu = 1, 2, \dots, n \right\}.$$

In particular, for every $\mu = 1, 2, \dots, n$,

$$\begin{aligned} \sum_{\alpha,\beta=1}^q c_{\mu,\alpha} \overline{c_{\mu,\beta}} f(u_\alpha, u_\beta, t) &= \int_0^\infty \Omega_d(rt) \sum_{\alpha,\beta=1}^p c_{\mu,\alpha} \overline{c_{\mu,\beta}} g_{\alpha,\beta}(r) d\sigma_D(r) \\ &= \left(\int_{\{0\}} + \int_B + \int_{(0,\infty)-B} \right) \Omega_d(rt) \sum_{\alpha,\beta=1}^p c_{\mu,\alpha} \overline{c_{\mu,\beta}} g_{\alpha,\beta}(r) d\sigma_D(r) \\ &= \sum_{\alpha,\beta=1}^q c_{\mu,\alpha} \overline{c_{\mu,\beta}} \sigma_{u_\alpha, u_\beta}(\{0\}) + 0 + 0 \end{aligned}$$

for all $t \in [0, \infty)$. But then $\sum_{\alpha,\beta=1}^q c_{\mu,\alpha} \overline{c_{\mu,\beta}} f(u_\alpha, u_\beta, t)$ is a constant function. However, by Theorems 4.2.2 and 4.2.4, that can only occur if $c_{\mu,\alpha} = 0$ for all $1 \leq \mu \leq p$, $1 \leq \alpha \leq q$. $\square$

As a direct consequence of Theorem 4.4.1 and the bridge propositions on matrix valued kernels in Section 2.3, we can reobtain one of the main theorems in (GUELLA; MENEGATTO, 2018a). The proof is omitted.

Theorem 4.4.2. Let $F : [0, \infty) \rightarrow M_l(\mathbb{C})$ be a continuous function. The following assertions are equivalent:

- (i) The kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow F(\|x - y\|) \in M_l(\mathbb{C})$ is strictly positive definite.
- (ii) F has an integral representation

$$[F(t)]_{\alpha,\beta} = \int_0^\infty \Omega_d(rt) d\sigma_{\alpha,\beta}(r), \quad 1 \leq \alpha, \beta \leq l,$$

where $\sigma_{\alpha,\beta}$ is a finite complex valued Borel measure on $[0, \infty)$, and for every $c_1, c_2, \dots, c_l \in \mathbb{C}$, not all zero, the measure $\sum_{\alpha,\beta=1}^l c_\alpha \overline{c_\beta} \sigma_{\alpha,\beta}$ is positive and not a multiple of δ_0 .

- (iii) For every $v \in \mathbb{C}^l \setminus \{0\}$, the continuous function $f_v(t) := vF(t)v^* \in \mathbb{R}$, $t \in [0, \infty)$, defines a strictly positive definite radial kernel on $\mathbb{R}^d$.

The generalization of Theorem 4.4.2 to operator valued kernels is presented. The proof is omitted because it is analogous to that of Theorem 4.3.9.

Theorem 4.4.3. Let $\mathcal{H}$ be a complex Hilbert space and $F : [0, \infty) \rightarrow L(\mathcal{H})$ a continuous function in the weak operator topology. The following properties are equivalent:

- (i) The kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow F(\|x - y\|) \in L(\mathcal{H})$ is strictly positive definite.
- (ii) For every $v \in \mathcal{H} \setminus \{0\}$, the continuous function $F_v(t) = \langle F(t)v, v \rangle, t \in [0, \infty)$, defines a strictly positive definite radial kernel on $\mathbb{R}^d$.

Now, we move to the characterization of the strictly positive definite kernels described in Theorem 4.3.10, except when $d = d' = 1$. We split it in two cases. The first one is a direct application of Theorem 4.4.1, and the proof of which is omitted.

Theorem 4.4.4. Let $f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be a continuous function and $d \in \{2, 3, \dots, \infty\}$. If the kernel

$$((x, z), (y, w)) \in (\mathbb{R}^d \times \mathbb{R}) \times (\mathbb{R}^d \times \mathbb{R}) \rightarrow f(\|x - y\|_d, |z - w|)$$

is positive definite, the following assertions are equivalent:

- (i) The kernel is strictly positive definite.
- (ii) For every distinct points $t_1, t_2, \dots, t_q \in \mathbb{R}$ and whichever scalars $c_1, c_2, \dots, c_q \in \mathbb{R}$, not all zero, the function

$$s \in [0, \infty) \rightarrow \sum_{\alpha, \beta=1}^q c_\alpha c_\beta f(s, |t_\alpha - t_\beta|) \in \mathbb{R}$$

is not a constant function.

Theorem 4.4.5. Let $f : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be a continuous function and $d, d' \in \{2, 3, \dots, \infty\}$. If the kernel

$$((x, z), (y, w)) \in (\mathbb{R}^d \times \mathbb{R}^{d'}) \times (\mathbb{R}^d \times \mathbb{R}^{d'}) \rightarrow f(\|x - y\|_d, |z - w|_{d'})$$

is positive definite, the following assertions are equivalent:

- (i) The kernel is strictly positive definite.
- (ii) The function f is not of the type $f(t, s) = f_1(t) + f_2(s)$, where f_1 defines a positive definite radial kernel on $\mathbb{R}^d$ and f_2 defines a positive definite radial kernel on $\mathbb{R}^{d'}$.
- (iii) The measure σ described in Theorem 4.3.10 does not satisfy $\sigma((0, \infty) \times (0, \infty)) = 0$.

Proof. If the kernel is strictly positive definite, the function f is not of the type $f(t, s) = f_1(t) + f_2(s)$ by Corollary 2.2.6. This shows that (i) implies (ii).

If $\sigma((0, \infty) \times (0, \infty)) = 0$, then

$$\begin{aligned} f(t, s) &= \int_{[0, \infty)^2} \Omega_d(rt) \Omega_{d'}(us) d\sigma(r, s) \\ &= \int_{[0, \infty) \times \{0\}} \Omega_d(rt) \Omega_{d'}(us) d\sigma(r, s) + \int_{\{0\} \times (0, \infty)} \Omega_d(rt) \Omega_{d'}(us) d\sigma(r, s) = f_1(t) + f_2(s) \end{aligned}$$

where f_1 defines a positive definite radial kernel on $\mathbb{R}^d$ and f_2 defines a positive definite radial kernel on $\mathbb{R}^{d'}$. This settles that (ii) implies (iii).

Finally, if (iii) is valid, given distinct points $x_1, x_2, \dots, x_n$ on $\mathbb{R}^d$, distinct points $z_1, z_2, \dots, z_m$ on $\mathbb{R}^{d'}$, scalars $c_{\mu, \alpha} \in \mathbb{R}$, not all zero, $1 \leq \mu \leq n$, $1 \leq \alpha \leq m$, then

$$\begin{aligned} & \sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^m c_{\mu, \alpha} c_{\nu, \beta} f(\|x_\mu - x_\nu\|_d, \|z_\alpha - z_\beta\|_{d'}) \\ &= \int_0^\infty \int_0^\infty \sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^m c_{\mu, \alpha} c_{\nu, \beta} \Omega_d(\|x_\mu - x_\nu\|_d r) \Omega_{d'}(\|z_\alpha - z_\beta\|_{d'} u) d\sigma(r, u). \end{aligned}$$

Theorem 4.2.1 implies that for every r and $u > 0$ the matrices $[\Omega_d(\|x_\mu - x_\nu\|_d r)]_{\mu, \nu=1}^n$ and $[\Omega_{d'}(\|z_\alpha - z_\beta\|_{d'} u)]_{\alpha, \beta=1}^m$ are positive definite, and then

$$\sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^m c_{\mu, \alpha} c_{\nu, \beta} \Omega_d(\|x_\mu - x_\nu\|_d r) \Omega_{d'}(\|z_\alpha - z_\beta\|_{d'} u) > 0,$$

for all $r, u > 0$. Since $\sigma((0, \infty) \times (0, \infty)) > 0$, the above integral is positive. Thus, the kernel is strictly positive definite by Lemma 2.2.5. $\square$

CONDITIONALLY NEGATIVE DEFINITE RADIAL KERNELS

In Section 5.1, we provide the necessary background on conditionally negative definite kernels, taking values in $\mathbb{C}$ or $M_l(\mathbb{C})$ and obtain an interesting result concerning the existence of the inverse of interpolation matrices defined by a Hermitian kernel that is strictly conditionally negative definite and nonnegative in the diagonal. In Section 5.2, we obtain a characterization for the continuous conditionally negative definite radial kernels, based on the Lévy-Khinchin formula, along with a characterization for the continuous strictly conditionally negative definite radial kernels. In Section 5.3, we present a characterization for the continuous matrix valued (strictly) conditionally negative definite radial kernels.

The matrix valued results in Section 5.3 and the existence results on the inverse of an interpolation matrix in Section 5.1 were included in the submitted paper ([GUELLA; MENEGATTO, 2018a](#)).

5.1 Conditionally negative definite kernels

In this section, we introduce the so called (strictly) conditionally negative kernels, and present some interesting and important relations between them and the positive definite kernels. We also define matrix valued (strictly) conditionally negative kernels, and present some interesting and important relations between them and the positive definite kernels.

Definition 5.1.1. A matrix $A = [a_{\mu,v}]_{\mu,v=1}^n \in M_n(\mathbb{C})$ is called **conditionally negative definite** if for every complex scalars $c_1, c_2, \dots, c_n \in \mathbb{C}$, restricted to the hyperplane $\sum_{\mu=1}^n c_\mu = 0$, the inequality below is satisfied

$$\sum_{\mu,v=1}^n c_\mu \overline{c_v} a_{\mu,v} \leq 0.$$

In addition, if all the inequalities above are an equality only when all the scalars c_μ are zero (but still restricted to the hyperplane $\sum_{\mu=1}^n c_\mu = 0$), the matrix A is called **strictly conditionally negative definite**.

A kernel $K : X \times X \rightarrow \mathbb{C}$ is called **conditionally negative definite** if for every finite quantity of distinct points $x_1, x_2, \dots, x_n \in X$ the interpolation matrix

$$[K(x_\mu, x_\nu)]_{\mu, \nu=1}^n$$

is conditionally negative semidefinite. In addition, if all the interpolation matrices above are strictly conditionally negative definite, the kernel K is called **strictly conditionally negative definite**.

The terminology conflict that appeared in Section 2.2 does not show up in this context.

Unlike positive definite kernels, a conditionally negative definite kernel (matrix) is not necessarily Hermitian. For instance, the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

is conditionally negative definite. From now on, for a conditionally negative definite kernel (matrix), its Hermitian property will either be imposed, or will be obtained from the context.

We present a connection between Hermitian conditionally negative definite kernels and positive definite kernels. For further results, see chapter 3 in (BERG; CHRISTENSEN; RESSEL, 1984).

Lemma 5.1.2. Let $\psi : X \times X \rightarrow \mathbb{C}$ be a Hermitian kernel and $x_0 \in X$. Define the kernel $\phi : X \times X \rightarrow \mathbb{C}$ by

$$\phi(x, y) := \psi(x, x_0) + \psi(x_0, y) - \psi(x, y) - \psi(x_0, x_0).$$

Then ψ is a conditionally negative definite if and only if ϕ is positive definite.

Proof. Given distinct points $x_1, x_2, \dots, x_n \in X$, and scalars $c_1, c_2, \dots, c_n \in \mathbb{C}$ such that $\sum_{\mu=1}^n c_\mu = 0$, we have

$$\begin{aligned} \sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} \phi(x_\mu, x_\nu) &= \sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} [\psi(x_\mu, x_0) + \psi(x_0, x_\nu) - \psi(x_\mu, x_\nu) - \psi(x_0, x_0)] \\ &= - \sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} \psi(x_\mu, x_\nu) \end{aligned}$$

Hence, if ϕ is positive definite, then the previous equalities imply that ψ is conditionally negative definite.

Conversely, given distinct points $y_1, y_2, \dots, y_m \in X$ and scalars $d_1, d_2, \dots, d_m \in \mathbb{C}$, define $d_0 = -\sum_{i=1}^m d_i$ and $y_0 = x_0$, and notice that

$$\begin{aligned} \sum_{i,j=0}^m d_i \bar{d}_j \psi(y_i, y_j) &= \sum_{i,j=1}^m d_i \bar{d}_j \psi(y_i, y_j) + \sum_{i=1}^m d_i \bar{d}_0 \psi(y_i, y_0) + \sum_{j=1}^m d_0 \bar{d}_j \psi(y_0, y_j) + |d_0|^2 \psi(y_0, y_0) \\ &= \sum_{i,j=1}^m d_i \bar{d}_j [\psi(y_i, y_j) - \psi(y_i, y_0) - \psi(y_0, y_j) + \psi(y_0, y_0)] \\ &= - \sum_{i,j=1}^m d_i \bar{d}_j \phi(y_i, y_j) \end{aligned}$$

Hence, if ψ is conditionally negative definite, then the previous equality implies that ϕ is positive definite. $\square$

Corollary 5.1.3. Let $A = [a_{\mu,\nu}]_{\mu,\nu=1}^l \in M_l(R)$ be a symmetric matrix. If $a_{\mu,\mu} = 0$ for all $\mu \in \{1, 2, \dots, l\}$, then A is conditionally negative definite if and only if there exists $z_1, z_2, \dots, z_l \in \mathbb{R}^l$, such that

$$a_{\mu,\nu} = (z_\mu - z_\nu) \cdot (z_\mu - z_\nu) = \|z_\mu - z_\nu\|^2$$

Proof. If $\sum_{\mu=1}^l c_\mu = 0$, then

$$\sum_{\mu,\nu=1}^l c_\mu \bar{c}_\nu a_{\mu,\nu} = -2 \left(\sum_{\mu=1}^l c_\mu z_\mu \right) \cdot \left(\sum_{\mu=1}^l c_\mu z_\mu \right) \leq 0,$$

and one implication follows.

On the other hand, by Lemma 5.1.2, if the matrix A is conditionally negative definite, then the matrix $B = [b_{\mu,\nu}]_{\mu,\nu=1}^l \in M_l(R)$, given by

$$b_{\mu,\nu} := a_{\mu,1} + a_{1,\nu} - a_{\mu,\nu} - a_{1,1} = a_{\mu,1} + a_{1,\nu} - a_{\mu,\nu}$$

for all $\mu, \nu \in \{1, 2, \dots, l\}$, is positive definite. Hence, by the Gram Representation Theorem 2.1.5, $[b_{\mu,\nu}]_{\mu,\nu=1}^l = [x_\mu \cdot x_\nu]_{\mu,\nu=1}^l$, where $x_1, x_2, \dots, x_l \in \mathbb{R}^l$. Since

$$\begin{aligned} (x_\mu - x_\nu) \cdot (x_\mu - x_\nu) &= x_\mu \cdot x_\mu + x_\nu \cdot x_\nu - x_\mu \cdot x_\nu - x_\nu \cdot x_\mu \\ &= [a_{\mu,1} + a_{1,\mu} - a_{\mu,\mu}] + [a_{\nu,1} + a_{1,\nu} - a_{\nu,\nu}] \\ &\quad - [a_{\mu,1} + a_{1,\nu} - a_{\mu,\nu}] - [a_{\nu,1} + a_{1,\mu} - a_{\nu,\mu}] = 2a_{\mu,\nu}, \end{aligned}$$

the proof is complete after we define $z_\mu = x_\mu / \sqrt{2}$. $\square$

An interesting consequence of Corollary 5.1.3 is that a metric space (X, d) , where X is a finite set, can be isometrically embedded into some real Hilbert space $\mathcal{H}$, more precisely, there exists a real Hilbert space $\mathcal{H}$ and a function $F : X \rightarrow \mathcal{H}$, such that $d(x, y) = \|F(x) - F(y)\|_{\mathcal{H}}$, for all $x, y \in X$ if and only if the kernel

$$(x, y) \in X \times X \rightarrow d^2(x, y) \in \mathbb{R}$$

is conditionally negative definite. The same is true even if X is not a finite set, (WELLS; WILLIAMS, 1975) page 3 (this proof relies on Zorn's Lemma), (BERG; CHRISTENSEN; RESSEL, 1984) page 82 (this proof is essentially the same of Corollary 5.1.3, and relies on the theory of reproducing kernels of (ARONSZAJN, 1950)).

We prove another relation between Hermitian conditionally negative definite kernels and positive definite kernels due to Schoenberg.

Theorem 5.1.4. Let $K : X \times X \rightarrow \mathbb{C}$ be a Hermitian kernel. It is conditionally negative definite if and only if for every $\lambda > 0$ the kernel

$$(x, y) \in X \times X \rightarrow e^{-\lambda K(x, y)} \in \mathbb{C}$$

is positive definite.

Proof. If the kernel $(x, y) \in X \times X \rightarrow e^{-\lambda K(x, y)}$ is positive definite for all $\lambda > 0$, an immediate inspection of the Definition 5.1.1 implies that the kernel $(x, y) \in X \times X \rightarrow 1 - e^{-\lambda K(x, y)}$ is conditionally negative definite for all $\lambda > 0$. Similar to Proposition 2.2.7, the set of Hermitian conditionally negative definite kernels on a set X is closed under pointwise convergence. Thus, the kernel

$$(x, y) \in X \times X \rightarrow K(x, y) = \lim_{\lambda \rightarrow 0^+} \frac{1 - e^{-\lambda K(x, y)}}{\lambda}$$

is conditionally negative definite.

Conversely, suppose that K is conditionally negative definite and pick a point $x_0 \in X$. Lemma 5.1.2 implies that for all $\lambda > 0$, the kernel

$$(x, y) \in X \times X \rightarrow \phi_\lambda(x, y) := \lambda K(x, x_0) + \lambda K(x_0, y) - \lambda K(x, y) - \lambda K(x_0, x_0) \in \mathbb{C}$$

is positive definite. But then, given distinct points $x_1, x_2, \dots, x_n \in X$ and complex scalars $c_1, c_2, \dots, c_n \in \mathbb{C}$,

$$\begin{aligned} \sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} e^{-\lambda K(x_\mu, x_\nu)} &= \sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} e^{-\lambda K(x_\mu, x_0) - \lambda K(x_0, x_\nu) + \phi_\lambda(x_\mu, x_\nu) + \lambda K(x_0, x_0)} \\ &= \sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} e^{\lambda K(x_0, x_0)} e^{-\lambda K(x_\mu, x_0)} e^{\phi_\lambda(x_\mu, x_\nu)} e^{-\lambda K(x_0, x_\nu)} \\ &= e^{\lambda K(x_0, x_0)} \sum_{\mu, \nu=1}^n \left[c_\mu e^{-\lambda K(x_\mu, x_0)} \right] \left[\overline{c_\nu e^{-\lambda K(x_\nu, x_0)}} \right] e^{\phi_\lambda(x_\mu, x_\nu)}. \end{aligned}$$

Now, $e^{\lambda K(x_0, x_0)} > 0$, because K is Hermitian. The kernel $(x, y) \in X \times X \rightarrow e^{\phi_\lambda(x, y)} = \sum_{k=0}^{\infty} (\phi_\lambda(x, y))^k / k!$ is positive definite by the Schur Product result in Proposition 2.2.4 and Proposition 2.2.7. Thus, the double sum above is positive if at least one of the scalars $c_1, c_2, \dots, c_n$ is nonzero. Therefore, the kernel $(x, y) \in X \times X \rightarrow e^{-\lambda K(x, y)} \in \mathbb{C}$ is positive definite. $\square$

Now, we move to matrix valued conditionally negative definite kernels.

Definition 5.1.5. A Hermitian matrix valued kernel $K : X \times X \rightarrow M_l(\mathbb{C})$ is called **conditionally negative definite** if for every finite quantity of distinct points $x_1, x_2, \dots, x_n \in X$ and whichever vectors $u_1, u_2, \dots, u_n \in \mathbb{C}^l$, restricted to $\sum_{\mu=1}^n u_\mu = 0$, the inequality below is satisfied

$$\sum_{\mu, \nu=1}^n u_\mu K(x_\mu, x_\nu) u_\nu^* \leq 0.$$

In addition, if all the inequalities above are an equality only when all the vectors v_μ are zero (but still restricted to $\sum_{\mu=1}^n u_\mu = 0$), the kernel K is called **strictly conditionally negative definite**.

The exact same proof in Lemma 5.1.2 shows a connection between conditionally negative definite matrix valued kernels and positive definite matrix valued kernels. But, a property similar to Theorem 5.1.4 is still elusive.

One of the most interesting properties of the strictly positive definite kernels is that the interpolation problem is well posed. Precisely, if $K : X \times X \rightarrow \mathbb{C}$ is a strictly positive definite kernel, given distinct points $x_1, x_2, \dots, x_n \in X$, the interpolation matrix $[K(x_\mu, x_\nu)]_{\mu, \nu=1}^n$ is an invertible matrix, and has n positive eigenvalues. In the matrix valued setting, the same happens: if $L : X \times X \rightarrow M_l(\mathbb{C})$ is a strictly positive definite kernel, the interpolation matrix $[L(x_\mu, x_\nu)]_{\mu, \nu=1}^n \in M_{nl}(\mathbb{C})$ is an invertible matrix, and has nl positive eigenvalues.

Some Hermitian strictly conditionally negative definite kernels share a similar property. In order to justify that, first, we recall the min-max Courant-Fischer Theorem, (HORN; JOHNSON, 2012) page 179. In order to simplify the notation, we will write $S \leq \mathbb{C}^m$ to indicate that S is a linear subspace of $\mathbb{C}^m$.

Theorem 5.1.6. Let $A \in M_m(\mathbb{C})$ be a Hermitian matrix having eigenvalues $\lambda_1 \leq \lambda_1 \leq \dots \leq \lambda_m$, including multiplicities. Then, for every k in $\{1, \dots, m\}$, it holds

$$\begin{aligned} \lambda_k &= \min_{S \leq \mathbb{C}^m} \left\{ \max_{0 \neq u \in S} \frac{uAu^*}{\|u\|^2} : \dim(S) = k \right\} \\ &= \max_{S \leq \mathbb{C}^m} \left\{ \min_{0 \neq u \in S} \frac{uAu^*}{\|u\|^2} : \dim(S) = m - k + 1 \right\}. \end{aligned}$$

A direct consequence is ((WENDLAND, 2005) page 99 for the case $l = 1$).

Corollary 5.1.7. Let $K : X \times X \rightarrow M_l(\mathbb{C})$ be a Hermitian strictly conditionally negative definite kernel such that $K(x, x)$ is a positive semidefinite matrix for all $x \in X$. If $n \geq 2$ and $x_1, x_2, \dots, x_n$ are distinct points in X , then the interpolation matrix $[K(x_\mu, x_\nu)]_{\mu, \nu=1}^n$ is invertible and has $(n - 1)l$ negative eigenvalues and l positive eigenvalues.

Proof. First, since the kernel K is Hermitian, all interpolation matrices defined through K are Hermitian. Write A to denote the interpolation matrix in the statement of the theorem. The matrix

A has nl real eigenvalues by Theorem 2.1.4, and they are, in order, $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{nl}$, including multiplicities. The sets

$$P := \{(v_1, v_2, \dots, v_n) \in (\mathbb{C}^l)^n : \sum_{\mu=1}^n v_\mu = 0\}$$

and

$$Q := \{(v, v, \dots, v) \in (\mathbb{C}^l)^n : v \in \mathbb{C}^l\}$$

are $(n-1)l$ -dimensional and l -dimensional subspaces of $\mathbb{C}^{ln}$, respectively. As a matter of fact, $\mathbb{C}^{ln}$ is a direct sum of them. If $S \leq \mathbb{C}^{ln}$ and $\dim(S) = l+1$, then $\dim(S \cap P) \geq 1$. Hence, since K is a strictly conditionally negative definite kernel, then

$$\min_{0 \neq u \in S} \frac{uAu^*}{\|u\|^2} < 0,$$

and, consequently,

$$\lambda_{(n-1)l} = \max_{S \leq \mathbb{C}^{ln}} \left\{ \min_{0 \neq u \in S} \frac{uAu^*}{\|u\|^2} : \dim(S) = l+1 \right\} < 0.$$

In particular, since the eigenvalues of A are ordered, we conclude that A has at least $(n-1)l$ negative eigenvalues. Next, if $x \neq y$, then $-K(x, x) - K(y, y) + K(x, y) + K(y, x)$ is a positive definite matrix. Indeed, since K is strictly conditionally negative definite, given $v \in \mathbb{C}^l \setminus \{0\}$,

$$-v[-K(x, x) - K(y, y) + K(x, y) + K(y, x)]v^* = (v, -v) \begin{pmatrix} K(x, x) & K(x, y) \\ K(y, x) & K(y, y) \end{pmatrix} (v, -v)^* < 0.$$

Thus, for all $v \in \mathbb{C}^l \setminus \{0\}$,

$$\begin{aligned} \sum_{\mu, v=1}^n vK(x_\mu, x_v)v^* &= n \sum_{\mu=1}^n vK(x_\mu, x_\mu)v^* \\ &+ \sum_{1 \leq \mu < v \leq n} v[-K(x_\mu, x_\mu) - K(x_v, x_v) + K(x_\mu, x_v) + K(x_v, x_\mu)]v^* > 0. \end{aligned}$$

If $S \leq \mathbb{C}^{ln}$ and $\dim(S) = (n-1)l+1$, then $\dim(S \cap Q) \geq 1$. Hence,

$$\max_{0 \neq x \in S} \frac{xAx^*}{\|x\|^2} > 0,$$

and, consequently,

$$\lambda_{(n-1)l+1} = \min_{S \leq \mathbb{C}^{ln}} \left\{ \max_{0 \neq x \in S} \frac{xAx^*}{\|x\|^2} : \dim(S) = (n-1)l+1 \right\} > 0.$$

The proof is complete. □

Corollary 5.1.7 was proved in (GUELLA; MENEGATTO, 2018a) for a continuous strictly conditionally negative definite radial kernel K on $\mathbb{R}^d$.

5.2 Conditionally negative definite radial kernels

In this section, we characterize the set of continuous functions $f : [0, \infty) \rightarrow \mathbb{C}$ that define a conditionally negative definite radial kernel on $\mathbb{R}^d$, for $d \in \mathbb{N} \cup \{\infty\}$, via the famous Lévy-Khinchin formula. We also characterize the set of continuous functions that define a strictly conditionally negative definite radial kernel on $\mathbb{R}^d$, for $d \in \{2, 3, \dots, \infty\}$, based on results of (CASTELL; FILBIR; SZWARC, 2005). The main references for this section are (BERG; FORST, 1975), (WELLS; WILLIAMS, 1975), (SCHOENBERG, 1938a) and (VON-NEUMANN; SCHOENBERG, 1941).

The characterization of the continuous conditionally negative definite radial kernels first appeared in (VON-NEUMANN; SCHOENBERG, 1941), although it was conjectured and partially proved in (SCHOENBERG, 1938a). The interest in conditionally negative definite radial kernels on $\mathbb{R}^d$ came from the following question in metric geometry proposed by Schoenberg:

Which metrics D in $\mathbb{R}^d$ can be embedded into some real Hilbert space $\mathcal{H}$? In other words, when there exists an isometry from $\mathbb{R}^d$ to $\mathcal{H}$?

Schoenberg proved in (SCHOENBERG, 1938a) that this is possible if and only if the kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow D^2(x, y)$$

is conditionally negative definite.

In particular, suppose that the metric D is radial, that is

$$D(z + Qy, z + Qz) = D(x, y)$$

for all $x, y, z \in \mathbb{R}^d$ and $Q \in O(d)$, and also that

$$D : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$$

is a continuous function in the standard topology of $\mathbb{R}^d$. Lemma 4.1.2 implies that $D(x, y) = F(\|x - y\|)$ for all $x, y \in \mathbb{R}^d$, where $F : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function. (VON-NEUMANN; SCHOENBERG, 1941) proved a characterization of those metrics by characterizing the set of continuous functions $F : [0, \infty) \rightarrow \mathbb{R}$, for which $F(0) = 0$ and F^2 define a conditionally negative definite radial kernel on $\mathbb{R}^d$.

In order to characterize the conditionally negative definite radial kernels on $\mathbb{R}^d$, for $d \in \mathbb{N}$, we recall the so called Lévy-Khinchin formula, which is a characterization, in the spirit of Bochner's Theorem 4.1.4, of the continuous translation invariant conditionally negative definite kernels on $\mathbb{R}^d$.

Theorem 5.2.1. Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$ be a continuous function. The kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(x - y) \in \mathbb{C}$$

is conditionally negative definite if and only if

$$f(x) = f(0) + b \cdot x + xAx^t + \int_{\mathbb{R}^d \setminus \{0\}} \left[1 - e^{-ix \cdot \xi} - \frac{ix \cdot \xi}{1 + \|\xi\|^2} \right] \frac{1 + \|\xi\|^2}{\|\xi\|^2} d\sigma(\xi),$$

for all $x \in \mathbb{R}^d$, where $b \in \mathbb{R}^d$, $A \in M_d(\mathbb{R})$ is a symmetric conditionally negative definite matrix, the Borel measure σ is nonnegative and finite. Moreover, the vector b , the symmetric matrix A and the finite nonnegative Borel measure σ are unique.

For a proof of this theorem see (BERG; FORST, 1975), where it is proved for an arbitrary locally compact Abelian Hausdorff group.

Theorem 5.2.2. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function and $d \in \mathbb{N}$. The kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(\|x - y\|) \in \mathbb{R}$$

is conditionally negative definite if and only if

$$f(t) = f(0) + at^2 + \int_{(0, \infty)} (1 - \Omega_d(rt)) \frac{1 + r^2}{r^2} d\gamma(r),$$

for all $t \geq 0$, where a is a nonnegative real number, γ is a nonnegative finite Borel measure on $(0, \infty)$. The number a and the finite measure γ are unique.

Proof. If the function f has such a representation, the integrand is nonnegative, because $|\Omega_d(r)| \leq \Omega_d(1) = 1$ for all $r > 0$. For $t = 0$, the integrand is the zero function. By Lemma 4.1.5–(i), (iv), if $d \geq 2$ and $t > 0$,

$$\lim_{r \rightarrow 0^+} (1 - \Omega_d(rt)) \frac{1 + r^2}{r^2} = \frac{t^2}{2d} \quad \text{and} \quad \lim_{r \rightarrow \infty} (1 - \Omega_d(rt)) \frac{1 + r^2}{r^2} = 1.$$

It follows that the function $(1 - \Omega_d(rt))(1 + r^2)/r^2$ is bounded and continuous. So, the integral is well defined. If $d = 1$, the function $(1 - \Omega_1(rt))(1 + r^2)/r^2$ is also bounded and continuous, but it is not continuous at infinity.

Given distinct points $x_1, x_2, \dots, x_n \in \mathbb{R}^d$, scalars $c_1, c_2, \dots, c_n \in \mathbb{C}$, restricted to $\sum_{\mu=1}^n c_\mu = 0$, we have that

$$\begin{aligned} & \sum_{\mu, v=1}^n c_\mu \bar{c}_v f(\|x_\mu - x_v\|) \\ &= \sum_{\mu, v=1}^n c_\mu \bar{c}_v \left[f(0) + a\|x_\mu - x_v\|^2 + \int_{(0, \infty)} (1 - \Omega_d(r\|x_\mu - x_v\|)) \frac{1 + r^2}{r^2} d\gamma(r) \right], \end{aligned}$$

and then

$$\begin{aligned} \sum_{\mu, v=1}^n c_\mu \bar{c}_v f(\|x_\mu - x_v\|) &= -a \left(\sum_{\mu=1}^n c_\mu x_\mu \right) \cdot \left(\sum_{\mu=1}^n c_\mu x_\mu \right) \\ &\quad - \int_{(0, \infty)} \sum_{\mu, v=1}^n c_\mu \bar{c}_v \Omega_d(r\|x_\mu - x_v\|) \frac{1 + r^2}{r^2} d\gamma(r) \leq 0, \end{aligned}$$

because $\Omega_d(rt)$ defines a positive definite radial kernel on $\mathbb{R}^d$ for all $r > 0$.

Conversely, assume that the kernel is conditionally negative definite. Without loss of generality, we can assume that $f(0) = 0$. Indeed, every constant function in $[0, \infty)$ defines a conditionally negative definite radial kernel on $\mathbb{R}^d$, for every $d \in \mathbb{N}$. We can apply Theorem 5.2.1 on the continuous function $g(x) := f(\|x\|)$, $x \in \mathbb{R}^d$, to write

$$g(x) = f(\|x\|) = b \cdot x + xAx^t + \int_{\mathbb{R}^d \setminus \{0\}} \left[1 - e^{-ix \cdot \xi} - \frac{ix \cdot \xi}{1 + \|\xi\|^2} \right] \frac{1 + \|\xi\|^2}{\|\xi\|^2} d\sigma(\xi),$$

for all $x \in \mathbb{R}^d$, where σ is a finite nonnegative Borel measure on $\mathbb{R}^d \setminus \{0\}$, $b \in \mathbb{R}^d$ and $A \in M_d(\mathbb{R})$ is a symmetric conditionally negative definite matrix. This implies that for every $Q \in O(d)$ and $x \in \mathbb{R}^d$,

$$g(xQ) = (bQ^t) \cdot x + xQAQ^t x^t + \int_{\mathbb{R}^d \setminus \{0\}} \left[1 - e^{-ix \cdot \xi} - \frac{ix \cdot \xi}{1 + \|\xi\|^2} \right] \frac{1 + \|\xi\|^2}{\|\xi\|^2} d\sigma_Q(\xi),$$

where σ_Q is the push forward measure of the finite nonnegative Borel measure σ through the continuous function $\xi \in \mathbb{R}^d \setminus \{0\} \rightarrow \xi Q \in \mathbb{R}^d \setminus \{0\}$. Since the representation in Theorem 5.2.1 is unique, we have that

$$b = bQ^t, \quad A = QAQ^t, \quad \sigma = \sigma_Q$$

for all orthogonal matrices $Q \in O(d)$. In particular, the vector b is zero. The matrix A is a multiple $a \in \mathbb{R}$ of the identity, since A is real symmetric, and if a is an eigenvalue of A with eigenvector v , then for every $Q \in O(d)$, a is an eigenvalue of A with eigenvector Qv . The number a is nonnegative, because the kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow (x - y)A(x - y)^t \in \mathbb{R}$ is conditionally negative definite. We also have that for every $x \in \mathbb{R}^d$,

$$\begin{aligned} g(x) - a\|x\|^2 &= \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} g(\omega\|x\|) - a(\|\omega\|x\|)^2 d\rho(\omega) \\ &= \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} \int_{\mathbb{R}^d \setminus \{0\}} \left[1 - e^{-i\|x\|\omega \cdot \xi} - \frac{i\|x\|\omega \cdot \xi}{1 + \|\xi\|^2} \right] \frac{1 + \|\xi\|^2}{\|\xi\|^2} d\sigma(\xi) d\rho(\omega) \\ &= \int_{\mathbb{R}^d \setminus \{0\}} \frac{1}{\text{vol}(S^{d-1})} \int_{S^{d-1}} \left[1 - e^{-i\omega \cdot \xi\|x\|} - \|x\| \frac{i\omega \cdot \xi}{1 + \|\xi\|^2} \right] \frac{1 + \|\xi\|^2}{\|\xi\|^2} d\rho(\omega) d\sigma(\xi) \\ &= \int_{\mathbb{R}^d \setminus \{0\}} (1 - \Omega_d(\|x\|\|\xi\|)) \frac{1 + \|\xi\|^2}{\|\xi\|^2} d\sigma(\xi). \end{aligned}$$

The change in the order of integration is possible due to Fubini-Tonelli Theorem, once σ and ρ are finite measures and the function

$$\left[1 - e^{-iy \cdot \xi} - \frac{iy \cdot \xi}{1 + \|\xi\|^2} \right] \frac{1 + \|\xi\|^2}{\|\xi\|^2} = (1 - \cos(y \cdot \xi)) \frac{1 + \|\xi\|^2}{\|\xi\|^2} + i \left(\sin(y \cdot \xi) + \frac{\sin(y \cdot \xi) - y \cdot \xi}{\|\xi\|^2} \right)$$

is bounded on the variable $\xi \in \mathbb{R}^d \setminus \{0\}$ for every fixed $y \in \mathbb{R}^d$. Thus,

$$g(x) - a\|x\|^2 = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \Omega_d(\|x\|\|\xi\|)) \frac{1 + \|\xi\|^2}{\|\xi\|^2} d\sigma(\xi) = \int_{(0, \infty)} (1 - \Omega_d(\|x\|r)) \frac{1 + r^2}{r^2} d\sigma_{\|\cdot\|}(r),$$

where the finite and nonnegative Borel measure $\sigma_{\|\cdot\|}$ is the image measure of the measure σ through the continuous function $\xi \in \mathbb{R}^d \setminus \{0\} \rightarrow \|\xi\| \in (0, \infty)$.

In order to prove that the finite nonnegative Borel measure $\sigma_{\|\cdot\|}$ in $(0, \infty)$ and the nonnegative number a obtained are unique, let γ be a finite Borel measure on $(0, \infty)$ and a nonnegative number c , such that

$$f(\|x\|) = f(0) + c\|x\|^2 + \int_{(0, \infty)} (1 - \Omega_d(\|x\|r) \frac{1+r^2}{r^2}) d\gamma(r),$$

for all $x \in \mathbb{R}^d$. Define the finite nonnegative Borel measure μ on $\mathbb{R}^d \setminus \{0\}$ by

$$\mu := (\gamma \times \tilde{\rho})_{\mathbb{R}^d \setminus \{0\}},$$

where $\tilde{\rho} = \frac{1}{\text{vol}(S^{d-1})} \rho$. The measure $(\gamma \times \tilde{\rho})_{\mathbb{R}^d \setminus \{0\}}$ is the push forward measure of the finite nonnegative Borel measure $\gamma \times \tilde{\rho}$ on $(0, \infty) \times S^{d-1}$ through the continuous function $(r, \omega) \in (0, \infty) \times S^{d-1} \rightarrow r\omega \in \mathbb{R}^d \setminus \{0\}$. The properties of the push forward measure give us that, for all $x \in \mathbb{R}^d$,

$$\begin{aligned} & \int_{\mathbb{R}^d \setminus \{0\}} \left[1 - e^{-ix \cdot \xi} - \frac{ix \cdot \xi}{1 + \|\xi\|^2} \right] \frac{1 + \|\xi\|^2}{\|\xi\|^2} d\mu(\xi) \\ &= \int_{(0, \infty)} \int_{S^{d-1}} \left[1 - e^{-ix \cdot r\omega} - \frac{ix \cdot r\omega}{1 + \|r\omega\|^2} \right] \frac{1 + \|r\omega\|^2}{\|r\omega\|^2} d\tilde{\rho}(\omega) d\gamma(r) \\ &= \int_{(0, \infty)} (1 - \Omega_d(\|x\|r) \frac{1+r^2}{r^2}) d\gamma(r) = f(\|x\|) - a\|x\|^2 = f(\|x\|) - x(cI_d)x^t. \end{aligned}$$

This equation and Theorem 5.2.1 imply that $A = aI_d = cI_d$. They also imply that μ and σ are equal. Thus, if $C \subset (0, \infty)$ is a Borel set, consider $B := \{x\omega : x \in C, \omega \in S^{d-1}\} \subset \mathbb{R}^d \setminus \{0\}$. The properties of the push forward measure imply that

$$\gamma(C) = \mu(B) = \sigma(B) = \sigma_{\|\cdot\|}(B),$$

which proves that $\gamma = \sigma_{\|\cdot\|}$. □

In (GUO; HU; SUN, 1993), the integral representation for the continuous functions that define conditionally negative definite radial kernels on $\mathbb{R}^d$ is of the form

$$f(t) = f(0) + at^2 + \int_{(0, \infty)} \frac{1 - \Omega_d(rt)}{r^2} d\sigma(r) \quad (5.1)$$

for all $t \geq 0$, where σ is a nonnegative Borel measure on $(0, \infty)$ such that $\min\{1, r^{-2}\}$ is integrable and a is a nonnegative number. The representation is also unique. The measure σ in Equation (5.1) does not need to be finite. The proof is based on results found in (GELFAND; VILENKIN, 1964), and does not rely on the Lévy-Khinchin formula. In the same paper, they also prove a characterization for the conditionally negative definite radial kernels of order k , $k \in \mathbb{Z}_+$, being $k = 0$ the positive definite case, and $k = 1$ the “conditionally positive definite” case. For more information on conditionally negative definite kernels of order k see (WENDLAND, 2005) chapter 8 and references therein.

Our characterization in Theorem 5.2.2 will simplify the arguments when dealing with matrix valued conditionally negative definite radial kernels in Section 5.3, the main reason for this is that the representation in Theorem 5.2.2 only involves finite measures.

Note that if $f : [0, \infty) \rightarrow \mathbb{R}$ defines a (strictly) conditionally negative definite radial kernel on $\mathbb{R}^d$, then it also defines a (strictly) conditionally negative definite radial kernel on $\mathbb{R}^m$ for all $m \leq d$. Similar to the positive definite radial kernels, it makes sense to ask which continuous functions $f : [0, \infty) \rightarrow \mathbb{R}$ define a (strictly) conditionally negative definite radial kernel on $\mathbb{R}^d$ for all $d \in \mathbb{N}$. This class of functions define the so called **(strictly) conditionally negative definite radial kernel on $\mathbb{R}^\infty$** .

Theorem 5.2.3. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function. The kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(\|x - y\|) \in \mathbb{R}$$

is conditionally negative definite for all $d \in \mathbb{N}$ if and only if

$$f(t) = f(0) + at^2 + \int_{(0, \infty)} (1 - \Omega_\infty(rt)) \frac{1 + r^2}{r^2} d\gamma(r),$$

for all $t \geq 0$, where γ is a nonnegative finite Borel measure on $(0, \infty)$ and a is a nonnegative real number. The nonnegative number a and the finite nonnegative Borel measure γ are unique.

Proof. If the function f has such a representation, the integrand is nonnegative, since $|\Omega_\infty(r)| < \Omega_\infty(0) = 1$ for all $r > 0$. When $t = 0$, the integrand is the zero function. If $t > 0$

$$\lim_{r \rightarrow 0^+} (1 - \Omega_\infty(rt)) \frac{1 + r^2}{r^2} = t^2 \quad \text{and} \quad \lim_{r \rightarrow \infty} (1 - \Omega_\infty(rt)) \frac{1 + r^2}{r^2} = 1.$$

So, the function $(1 - \Omega_\infty(rt))(1 + r^2)/r^2$ is a bounded continuous function and the integral is well defined. Given $d \in \mathbb{N}$, distinct points $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ and scalars $c_1, c_2, \dots, c_n \in \mathbb{C}$, restricted to $\sum_{\mu=1}^n c_\mu = 0$, we have that

$$\begin{aligned} & \sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} f(\|x_\mu - x_\nu\|) \\ &= -a \left(\sum_{\mu=1}^n c_\mu x_\mu \right) \cdot \left(\sum_{\mu=1}^n c_\mu x_\mu \right) - \int_{(0, \infty)} \sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} \Omega_\infty(r\|x_\mu - x_\nu\|) \frac{1 + r^2}{r^2} d\gamma(r) \leq 0, \end{aligned}$$

because $\Omega_\infty(r \cdot)$ defines a positive definite radial kernel on $\mathbb{R}^\infty$ for all $r \geq 0$.

Conversely, assume that the kernel is conditionally negative definite for all $d \in \mathbb{N}$. Without loss of generality, we can assume that $f(0) = 0$, because every constant function in $[0, \infty)$ defines a conditionally negative definite radial kernel on $\mathbb{R}^\infty$. For every $k \in \mathbb{N}$, the function $t \in [0, \infty) \rightarrow e^{-f(t)/k}$ defines a positive definite radial kernel on $\mathbb{R}^\infty$ by Theorem 5.1.4. Theorem 4.1.8 implies that it has a unique integral representation a in the form

$$e^{-f(t)/k} = \int_0^\infty \Omega_\infty(rt) d\sigma_k(r),$$

for all $t \geq 0$, where σ_k is a nonnegative Borel measure on $[0, \infty)$, with $\sigma_k([0, \infty)) = e^{-f(0)k} = 1$, for all $k \in \mathbb{N}$. Hence, it follows that

$$\begin{aligned} f(t) &= \lim_{k \rightarrow \infty} k[1 - e^{-f(t)/k}] = \lim_{k \rightarrow \infty} k \int_0^\infty 1 - \Omega_\infty(rt) d\sigma_k(r) \\ &= \lim_{k \rightarrow \infty} \int_{(0, \infty)} (1 - \Omega_\infty(rt)) \frac{1+r^2}{r^2} \left[k \frac{r^2}{1+r^2} d\sigma_k(r) \right]. \end{aligned}$$

The measure $\gamma_k(r) := k(r^2/(1+r^2))d\sigma_k(r)$ on $(0, \infty)$ is nonnegative. It is finite because there are positive numbers $D, E \in \mathbb{R}$, such that

$$D \frac{r^2}{1+r^2} \leq 1 - \Omega_\infty(r) \leq E \frac{r^2}{1+r^2},$$

for all $r \geq 0$. Defining $\gamma_k(\{0\}) = 0$, for all $k \in \mathbb{N}$, Proposition 4.1.7 implies that there exists a nonnegative Borel measure γ on $[0, \infty]$ and a subsequence $(\gamma_{k_l})_{l \in \mathbb{N}}$, vaguely convergent to γ , such that $\lim_{l \rightarrow \infty} \gamma_{k_l}((0, \infty)) = \gamma([0, \infty])$. Taking $h_t(r) = -1 + (1 - \Omega_\infty(rt))(1+r^2)/r^2 \in C_0([0, \infty))$, $t > 0$, we obtain that

$$\begin{aligned} f(t) &= \lim_{l \rightarrow \infty} \int_0^\infty (1 - \Omega_\infty(rt)) \frac{1+r^2}{r^2} d\gamma_{k_l}(r) = \lim_{l \rightarrow \infty} \int_0^\infty h_t(r) d\gamma_{k_l}(r) + \lim_{l \rightarrow \infty} \gamma_{k_l}([0, \infty)) \\ &= \int_0^\infty h_t(r) d\gamma(r) + \gamma([0, \infty)) = \gamma(\{0\})t^2 + \int_{(0, \infty)} (1 - \Omega_\infty(rt)) \frac{1+r^2}{r^2} d\gamma(r) + \gamma(\{\infty\}). \end{aligned}$$

Since $f(t) \geq \gamma(\{\infty\}) \geq 0$, for all $t \in (0, \infty)$, the continuity of f implies that $\gamma(\{\infty\}) = 0$.

Our final claim, is that the nonnegative finite Borel measure γ and the nonnegative real number a in the representation are unique. Indeed, $a = \lim_{t \rightarrow \infty} f(t)/t^2$. If γ_1 and γ_2 are two finite nonnegative Borel measures in $(0, \infty)$, extend both to $[0, \infty)$, by defining $\gamma_1(\{0\}) = \gamma_2(\{0\}) = 0$. Suppose that

$$\int_0^\infty (1 - \Omega_\infty(rt)) \frac{1+r^2}{r^2} d\gamma_1(r) = \int_0^\infty (1 - \Omega_\infty(rt)) \frac{1+r^2}{r^2} d\gamma_2(r),$$

for all $t \geq 0$, that is

$$\int_0^\infty (1 - e^{-r^2 t}) \frac{1+r^2}{r^2} d\gamma_1(r) = \int_0^\infty (1 - e^{-r^2 t}) \frac{1+r^2}{r^2} d\gamma_2(r),$$

for all $t > 0$. Differentiating both sides with respect to t , leads to

$$\int_0^\infty e^{-r^2 t} (1+r^2) d\gamma_1(r) = \int_0^\infty e^{-r^2 t} (1+r^2) d\gamma_2(r),$$

for all $t > 0$. In particular, we obtain that

$$\int_0^\infty \Omega_\infty(rt) \left[e^{-r^2} (1+r^2) d\gamma_1 \right] (r) = \int_0^\infty \Omega_\infty(rt) \left[e^{-r^2} (1+r^2) d\gamma_2 \right] (r),$$

for all $t \geq 0$. Theorem 4.1.8 implies that the finite nonnegative Borel measures $e^{-r^2} (1+r^2) \gamma_1$ and $e^{-r^2} (1+r^2) \gamma_2$ are equal, and then $\gamma_1 = \gamma_2$. $\square$

The proof of Theorem 5.2.3 is based in (WELLS; WILLIAMS, 1975) page 39. A very similar proof works for continuous conditionally negative definite radial kernels on $\mathbb{R}^d$, for $d \in \{2, 3, \dots\}$, but we were unable to find a proof for the uniqueness of the representation without relying on Theorem 5.2.1.

We already know that a constant function in $[0, \infty)$ defines a conditionally negative radial kernel on $\mathbb{R}^d$, $d \in \mathbb{N} \cup \{\infty\}$, that is not strictly conditionally negative definite. The function $f(t) = t^2$ also defines a conditionally negative radial kernel on $\mathbb{R}^d$, $d \in \mathbb{N} \cup \{\infty\}$. Likewise, it does not define a strictly conditionally negative radial kernel. Indeed, pick distinct points $x_1, x_2, x_3 \in \mathbb{R}^1$ such that $x_1 + x_2 - 2x_3 = 0$, and notice that

$$\sum_{\mu, \nu=1}^3 c_\mu \overline{c_\nu} \|x_\mu - x_\nu\|^2 = 0,$$

where $2c_1 = 2c_2 = -c_3 = 2$. Essentially, these are the only radial kernels with this property. The characterization of the continuous functions that define a strictly conditionally negative radial kernel on $\mathbb{R}^d$ relies on arguments similar to those in Theorem 4.2.2. This characterization first appeared in (SUN, 1993).

Theorem 5.2.4. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function and $d \in \{2, 3, \dots, \infty\}$. The kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(\|x - y\|) \in \mathbb{R}$$

is strictly conditionally negative definite if and only if f defines a conditionally negative definite radial kernel on $\mathbb{R}^d$ and is not a polynomial of the type $at^2 + b$, $a, b \in \mathbb{R}$.

Moreover, if f defines a conditionally negative radial kernel on $\mathbb{R}^d$, the function f being a polynomial of the type $at^2 + b$ is equivalent to the measure γ in Theorem 5.2.2 or 5.2.3 being zero.

Proof. If the kernel is strictly conditionally negative definite, it is immediate that f defines a conditionally negative definite radial kernel on $\mathbb{R}^d$ and that it cannot be a polynomial of the type $at^2 + b$.

Conversely, if f defines a conditionally negative definite radial kernel on $\mathbb{R}^d$, then

$$f(t) = f(0) + at^2 + \int_{(0, \infty)} (1 - \Omega_d(rt)) \frac{1 + r^2}{r^2} d\gamma(r),$$

for all $t \geq 0$, where γ is a finite nonnegative Borel measure on $(0, \infty)$ and a is a nonnegative real number. Let $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ be distinct points and $c_1, c_2, \dots, c_n \in \mathbb{R}$, not all zero and restricted to $\sum_{\mu=1}^n c_\mu = 0$. It follows that

$$\begin{aligned} & \sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} f(\|x_\mu - x_\nu\|) \\ &= -a \left(\sum_{\mu=1}^n c_\mu x_\mu \right) \cdot \left(\sum_{\mu=1}^n c_\mu x_\mu \right) - \int_{(0, \infty)} \sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} \Omega_d(r\|x_\mu - x_\nu\|) \frac{1 + r^2}{r^2} d\gamma(r). \end{aligned}$$

But Theorem 4.2.1 and Corollary 4.2.3 imply that $\sum_{\mu, \nu=1}^n c_\mu \overline{c_\nu} \Omega_d(r \|x_\mu - x_\nu\|) > 0$ for all $r > 0$, so, if $\gamma((0, \infty)) > 0$, then the above integral is negative. This implies that the kernel is strictly conditionally negative definite. If $\gamma((0, \infty)) = 0$, then $f(t) = at^2 + f(0)$. $\square$

Similar to the continuous positive definite radial kernels on $\mathbb{R}^\infty$, there exists another characterization for the continuous conditionally negative definite radial kernels on $\mathbb{R}^\infty$, also based on the concept of completely monotone functions.

Theorem 5.2.5. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function. The following assertions are equivalent

- (i) The kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow f(\|x - y\|) \in \mathbb{R}$ is conditionally negative definite for all $d \in \mathbb{N}$.
- (ii) The function f is infinitely many times differentiable in $(0, \infty)$, and the function $g(t) := f(\sqrt{t})$, $t \in (0, \infty)$ is such that g' is a completely monotone function on $(0, \infty)$.

A continuous function $g : [0, \infty) \rightarrow \mathbb{R}$ such that g' exists and is a completely monotone function on $(0, \infty)$ is usually called a **Bernstein function**. For more information and examples of Bernstein functions see (SCHILLING; SONG; VONDRACEK, 2012).

The continuous function $f(t) = t$, $t \in [0, \infty)$, defines a strictly conditionally negative definite radial kernel on $\mathbb{R}^\infty$. Indeed, it defines a conditionally negative definite radial kernel on $\mathbb{R}^\infty$, because the function f is infinitely many times differentiable on $(0, \infty)$ and the function $g(t) := \sqrt{t}$, $t \in [0, \infty)$, is such that $g'(t) = t^{-1/2}/2$, which is a completely monotone function on $(0, \infty)$. The kernel is strictly conditionally negative definite, because it is not of the type $at^2 + b$. An amazing consequence of this fact is that for every $d \in \mathbb{N}$ and distinct points $x_1, x_2, \dots, x_n \in \mathbb{R}^d$, $n \geq 2$, the distance matrix $[\|x_\mu - x_\nu\|]_{\mu, \nu=1}^n$ is invertible and has $n - 1$ negative eigenvalues and 1 positive eigenvalue.

5.3 Matrix valued (strictly) conditionally negative definite radial kernels

In this section we prove a characterization for the matrix valued conditionally negative definite radial kernels on $\mathbb{R}^d$, for $d \in \mathbb{N} \cup \{\infty\}$ and a characterization for the strictly conditionally negative definite radial kernels on $\mathbb{R}^d$, for $d \in \{2, 3, \dots, \infty\}$. These results were proved in (GUELLA; MENEGATTO, 2018a).

Matrix valued conditionally negative definite radial kernels are discussed in (XIE, 1994), under strong assertions on the differentiability of the function. An integral representation for them was not proved.

Before proving the main results of this section, we state a technical lemma that will be useful. The proof is omitted because it follows by the exact same arguments as in Lemma 4.3.4.

Lemma 5.3.1. Let σ be a finite complex valued Borel measure on $(0, \infty)$, $a \in \mathbb{C}$ and $d \in \mathbb{N} \cup \{\infty\}$. Suppose that

$$at^2 + \int_{(0, \infty)} (1 - \Omega_d(rt)) \frac{1+r^2}{r^2} d\sigma(r) = 0,$$

for all $t \in [0, \infty)$. Then, the measure σ is the zero measure and the complex number a is zero.

Theorem 5.3.2. Let $F : [0, \infty) \rightarrow M_l(\mathbb{C})$ be a continuous function and $d \in \mathbb{N} \cup \{\infty\}$. The following assertions are equivalent:

- (i) The kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow F(\|x - y\|) \in M_l(\mathbb{C})$ is conditionally negative definite.
- (ii) The function F has a representation

$$[F(t)]_{\alpha, \beta} := f_{\alpha, \beta}(t) = f_{\alpha, \beta}(0) + a_{\alpha, \beta}t^2 + \int_{(0, \infty)} (1 - \Omega_d(rt)) \frac{1+r^2}{r^2} d\gamma_{\alpha, \beta}(r),$$

for all $1 \leq \alpha, \beta \leq l$. The Borel measures $\gamma_{\alpha, \beta}$ on $(0, \infty)$ are finite and complex valued. For every $c_1, c_2, \dots, c_l \in \mathbb{C}$, the finite Borel measure $\sum_{\alpha, \beta=1}^l c_\alpha \overline{c_\beta} \gamma_{\alpha, \beta}$ is nonnegative and the matrix $[a_{\alpha, \beta}]_{\alpha, \beta=1}^l \in M_l(\mathbb{C})$ is positive semidefinite. The representation is unique.

- (iii) For every $v \in \mathbb{C}^l$, the continuous function $t \in [0, \infty) \rightarrow F_v(t) := vF(t)v^* \in \mathbb{C}$, defines a conditionally negative definite radial kernel on $\mathbb{R}^d$.

Proof. If the matrix valued kernel is conditionally negative definite, given $v \in \mathbb{C}^l$, distinct points $x_1, x_2, \dots, x_n$ on $\mathbb{R}^d$ and scalars $c_1, c_2, \dots, c_l \in \mathbb{C}$, such that $\sum_{\mu=1}^n c_\mu = 0$, then $\sum_{\mu=1}^n c_\mu v = 0$, and

$$\sum_{\mu, v=1}^n c_\mu \overline{c_v} v F(\|x_\mu - x_v\|) v^* = \sum_{\mu, v=1}^n (c_\mu v) F(\|x_\mu - x_v\|) (c_\mu v)^* \leq 0.$$

Thus (i) implies (iii). If (iii) is valid, the continuous functions $f_{\alpha, \alpha}$ define conditionally negative definite radial kernels on $\mathbb{R}^d$, $1 \leq \alpha \leq l$. If $\alpha \neq \beta$, all three functions

$$f_{\alpha, \alpha} + f_{\beta, \beta} + f_{\alpha, \beta} + f_{\beta, \alpha},$$

$$f_{\alpha, \alpha} + f_{\beta, \beta} - f_{\alpha, \beta} - f_{\beta, \alpha}$$

$$f_{\alpha, \alpha} + f_{\beta, \beta} - if_{\alpha, \beta} + if_{\beta, \alpha}$$

are continuous and also define conditionally negative definite radial kernels on $\mathbb{R}^d$. Given that $f_{\alpha, \beta}$ is a linear combination of them (with complex coefficients), it is promptly seen, via Theorem 5.2.2, that $f_{\alpha, \beta}$ can be represented as in the statement of the theorem. The representation is unique by Lemma 5.3.1.

Next, we verify that the matrix $[a_{\alpha, \beta}]_{\alpha, \beta=1}^l \in M_l(\mathbb{C})$ is positive semidefinite and for every $c_1, c_2, \dots, c_l \in \mathbb{C}$, the measure $\sum_{\alpha, \beta=1}^l c_\alpha \overline{c_\beta} \sigma_{\alpha, \beta}$ is nonnegative. Indeed,

$$h(t) := \sum_{\alpha, \beta=1}^l c_\alpha \overline{c_\beta} f_{\alpha, \beta}(t) = h(0) + a_h t^2 + \int_{(0, \infty)} (1 - \Omega_d(rt)) \frac{1+r^2}{r^2} d\gamma_h(r)$$

for all $t \in [0, \infty)$, where $a_h \geq 0$ and the Borel measure γ_h on $(0, \infty)$ is finite and nonnegative. Since

$$\begin{aligned} h(t) &= \left(\sum_{\alpha, \beta=1}^l c_{\alpha} \overline{c_{\beta}} f_{\alpha, \beta}(0) \right) + \left(\sum_{\alpha, \beta=1}^l c_{\alpha} \overline{c_{\beta}} a_{\alpha, \beta} \right) t^2 \\ &\quad + \int_{(0, \infty)} (1 - \Omega_d(rt)) \frac{1+r^2}{r^2} d \left(\sum_{\alpha, \beta=1}^l c_{\alpha} \overline{c_{\beta}} \gamma_{\alpha, \beta} \right) (r) \end{aligned}$$

and the representation in Theorem 5.2.2 is unique, the equalities below are satisfied

$$\sum_{\alpha, \beta=1}^l c_{\alpha} \overline{c_{\beta}} \sigma_{\alpha, \beta} = \sigma_h, \quad \sum_{\alpha, \beta=1}^l c_{\alpha} \overline{c_{\beta}} a_{\alpha, \beta} = a_g.$$

This shows that (iii) implies (ii).

Finally, if (ii) holds, by Lemma 4.3.3, we can write

$$\begin{aligned} f_{\alpha, \beta}(t) - f_{\alpha, \beta}(0) - a_{\alpha, \beta} t^2 &= \int_{(0, \infty)} (1 - \Omega_d(rt)) \frac{1+r^2}{r^2} d\gamma_{\alpha, \beta}(r) \\ &= \int_{(0, \infty)} (1 - \Omega_d(rt)) \frac{1+r^2}{r^2} g_{\alpha, \beta}(r) d\gamma_D(r), \end{aligned}$$

for all $t \geq 0$, $1 \leq \alpha, \beta \leq l$, where $\gamma_D := \sum_{\alpha=1}^l \gamma_{\alpha, \alpha}$, the functions $g_{\alpha, \beta} \in L^1((0, \infty), \gamma_D)$ and for each $r \in (0, \infty)$, the matrix $[g_{\alpha, \beta}(r)]_{\alpha, \beta=1}^l$ is positive semidefinite. But then, given distinct points $x_1, x_2, \dots, x_n$ on $\mathbb{R}^d$ and $v_1, v_2, \dots, v_n \in \mathbb{C}^l$, such that $\sum_{\mu=1}^n v_{\mu} = 0$, we have that

$$\begin{aligned} &\sum_{\mu, \nu=1}^n v_{\mu} F(\|x_{\mu} - x_{\nu}\|) v_{\nu}^* \\ &= \sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\nu, \beta}} \left[f_{\alpha, \beta}(0) + a_{\alpha, \beta} \|x_{\mu} - x_{\nu}\|^2 + \int_{(0, \infty)} (1 - \Omega_d(r\|x_{\mu} - x_{\nu}\|)) \frac{1+r^2}{r^2} d\gamma_{\alpha, \beta}(r) \right]. \end{aligned}$$

Note that

$$\sum_{\alpha, \beta=1}^l \sum_{\mu, \nu=1}^n v_{\mu, \alpha} \overline{v_{\nu, \beta}} f_{\alpha, \beta}(0) = 0,$$

while

$$\sum_{\alpha, \beta=1}^l \sum_{\mu, \nu=1}^n v_{\mu, \alpha} \overline{v_{\nu, \beta}} a_{\alpha, \beta} \|x_{\mu} - x_{\nu}\|^2 = -2 \sum_{\alpha, \beta=1}^l \sum_{\mu, \nu=1}^n v_{\mu, \alpha} \overline{v_{\nu, \beta}} a_{\alpha, \beta} x_{\mu} \cdot x_{\nu} \leq 0.$$

On the other hand,

$$\begin{aligned} &\sum_{\alpha, \beta=1}^l \sum_{\mu, \nu=1}^n v_{\mu, \alpha} \overline{v_{\nu, \beta}} \int_{(0, \infty)} (1 - \Omega_d(r\|x_{\mu} - x_{\nu}\|)) \frac{1+r^2}{r^2} d\gamma_{\alpha, \beta}(r) \\ &= \sum_{\alpha, \beta=1}^l \sum_{\mu, \nu=1}^n v_{\mu, \alpha} \overline{v_{\nu, \beta}} \int_{(0, \infty)} (1 - \Omega_d(r\|x_{\mu} - x_{\nu}\|)) \frac{1+r^2}{r^2} g_{\alpha, \beta}(r) d\gamma_D(r) \\ &= - \int_{(0, \infty)} \sum_{\alpha, \beta=1}^l \sum_{\mu, \nu=1}^n v_{\mu, \alpha} \overline{v_{\nu, \beta}} \Omega_d(r\|x_{\mu} - x_{\nu}\|) g_{\alpha, \beta}(r) \frac{1+r^2}{r^2} d\gamma_D(r) \leq 0. \end{aligned}$$

The last inequality is true, since for every $r \geq 0$, the Kroenecker product of the positive semidefinite matrices $[\Omega_d(r)\|x_\mu - x_\nu\|]_{\mu,\nu=1}^n$ and $[g_{\alpha,\beta}(r)]_{\alpha,\beta=1}^l$ is positive semidefinite by Proposition 2.1.8. Thus

$$\sum_{\mu,\nu=1}^n v_\mu F(\|x_\mu - x_\nu\|) v_\nu^* \leq 0,$$

and (i) holds. $\square$

Now we prove a characterization for the strictly conditionally negative definite radial kernels on $\mathbb{R}^d$, for $d \in \{2, 3, \dots, \infty\}$.

Theorem 5.3.3. Let $F : [0, \infty) \rightarrow M_l(\mathbb{C})$ be a continuous function and $d \in \{2, 3, \dots, \infty\}$. The following assertions are equivalent:

- (i) The kernel $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow F(\|x - y\|) \in M_l(\mathbb{C})$ is strictly conditionally negative definite.
- (ii) The function F has a representation

$$[F(t)]_{\alpha,\beta} = f_{\alpha,\beta}(t) = f_{\alpha,\beta}(0) + a_{\alpha,\beta} t^2 + \int_{(0,\infty)} (1 - \Omega_d(rt)) \frac{1+r^2}{r^2} d\gamma_{\alpha,\beta}(r),$$

for all $1 \leq \alpha, \beta \leq l$. The matrix $[a_{\alpha,\beta}]_{\alpha,\beta=1}^l \in M_l(\mathbb{C})$ is positive semidefinite. The Borel measures $\gamma_{\alpha,\beta}$ on $(0, \infty)$ are finite and complex valued. For every $c_1, c_2, \dots, c_l \in \mathbb{C}$, not all zero, the finite Borel measure $\sum_{\alpha,\beta=1}^l c_\alpha \overline{c_\beta} \gamma_{\alpha,\beta}$ is positive. The representation is unique.

- (iii) For every $v \in \mathbb{C}^l \setminus \{0\}$, the continuous function $t \in [0, \infty) \rightarrow F_v(t) := vF(t)v^* \in \mathbb{C}$, defines a strictly conditionally negative definite radial kernel on $\mathbb{R}^d$.

Proof. If the matrix valued kernel is strictly conditionally negative definite, given $v \in \mathbb{C}^l \setminus \{0\}$, distinct points $x_1, x_2, \dots, x_n$ on $\mathbb{R}^d$ and scalars $c_1, c_2, \dots, c_l \in \mathbb{C}$, not all zero and such that $\sum_{\mu=1}^n c_\mu = 0$, then $\sum_{\mu=1}^n c_\mu v = 0$, and

$$\sum_{\mu,\nu=1}^n c_\mu \overline{c_\nu} v F(\|x_\mu - x_\nu\|) v^* = \sum_{\mu,\nu=1}^n (c_\mu v) F(\|x_\mu - x_\nu\|) (c_\nu v)^* < 0.$$

Indeed, at least one of the vectors $c_1 v, c_2 v, \dots, c_l v \in \mathbb{C}^l$ is nonzero. Thus (i) implies (iii).

Next, we assume that (iii) holds. By Theorem 5.3.2, the continuous functions $f_{\alpha,\beta}$ have a unique integral representation as in its statement and satisfy all the properties in (ii) with the possible exception that for every $c_1, c_2, \dots, c_l \in \mathbb{C}$, not all zero, the measure $\sum_{\alpha,\beta=1}^l c_\alpha \overline{c_\beta} \gamma_{\alpha,\beta}$ is positive. Since F_c , for $c = (c_1, c_2, \dots, c_l)$, defines a strictly conditionally negative definite radial kernel on $\mathbb{R}^d$, we obtain via Theorem 5.2.4 that the measure $\sum_{\alpha,\beta=1}^l c_\alpha \overline{c_\beta} \gamma_{\alpha,\beta}$ is positive, because the representation is unique. Thus, (ii) holds.

Finally, (ii) implies (i), because if $x_1, x_2, \dots, x_n$ are distinct points on $\mathbb{R}^d$ and $v_1, v_2, \dots, v_n \in \mathbb{C}^l$, restricted to $\sum_{\mu=1}^n v_\mu = 0$, are such that

$$\sum_{\mu,\nu=1}^n v_\mu F(\|x_\mu - x_\nu\|) v_\nu^* = 0,$$

then

$$\begin{aligned} 0 &= \sum_{\mu, \nu=1}^n v_{\mu} F(\|x_{\mu} - x_{\nu}\|) v_{\nu}^* = -2 \sum_{\alpha, \beta=1}^l \sum_{\mu, \nu=1}^n v_{\mu, \alpha} \overline{v_{\nu, \beta}} a_{\alpha, \beta} x_{\mu} \cdot x_{\nu} \\ &\quad + \int_{(0, \infty)} \sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\nu, \beta}} \Omega_d(r \|x_{\mu} - x_{\nu}\|) \frac{1+r^2}{r^2} d\gamma_{\alpha, \beta}(r). \end{aligned}$$

In particular

$$\int_{(0, \infty)} \sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\nu, \beta}} \Omega_d(r \|x_{\mu} - x_{\nu}\|) \frac{1+r^2}{r^2} d\gamma_{\alpha, \beta}(r) = 0.$$

By making use of Lemma 4.3.3, we obtain

$$\begin{aligned} 0 &= \int_{(0, \infty)} \sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\nu, \beta}} \Omega_d(r \|x_{\mu} - x_{\nu}\|) \frac{1+r^2}{r^2} d\gamma_{\alpha, \beta}(r) \\ &= \int_{(0, \infty)} \sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\nu, \beta}} \Omega_d(r \|x_{\mu} - x_{\nu}\|) g_{\alpha, \beta}(r) \frac{1+r^2}{r^2} d\gamma_D(r). \end{aligned}$$

The matrix $[\Omega_d(r \|x_{\mu} - x_{\nu}\|)]_{\mu, \nu=1}^n$ is positive definite for all $r > 0$ and the matrix $[g_{\alpha, \beta}(r)]_{\alpha, \beta=1}^l$ is positive semidefinite for all $r > 0$. Thus,

$$\sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\nu, \beta}} \Omega_d(r \|x_{\mu} - x_{\nu}\|) g_{\alpha, \beta}(r) \geq 0,$$

for all $r > 0$, by Lemma 2.1.8. But the finite Borel measure $\gamma_D := \sum_{\alpha=1}^l \gamma_{\alpha, \alpha}$ is positive, so if

$$B := \left\{ s \in (0, \infty) : \sum_{\mu, \nu=1}^n \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\nu, \beta}} \Omega_d(s \|x_{\mu} - x_{\nu}\|) g_{\alpha, \beta}(s) = 0 \right\},$$

then $(0, \infty) - B$ is a γ_D -null set. Recalling the second part of Lemma 2.1.9 and the fact that $[\Omega_d(r \|x_{\mu} - x_{\nu}\|)]_{\mu, \nu=1}^n$ is positive definite for all $r > 0$, we have that

$$B = \left\{ s \in (0, \infty) : \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\mu, \beta}} g_{\alpha, \beta}(s) = 0, \text{ for all } \mu = 1, 2, \dots, n \right\}.$$

In particular, for every $\mu = 1, 2, \dots, n$,

$$\begin{aligned} \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\mu, \beta}} \gamma_{\alpha, \beta}((0, \infty)) &= \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\mu, \beta}} \int_{(0, \infty)} g_{\alpha, \beta}(r) \frac{1+r^2}{r^2} d\gamma_D(r) \\ &= \int_B \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\mu, \beta}} g_{\alpha, \beta}(r) \frac{1+r^2}{r^2} d\gamma_D(r) \\ &\quad + \int_{(0, \infty) - B} \sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\mu, \beta}} g_{\alpha, \beta}(r) \frac{1+r^2}{r^2} d\gamma_D(r) = 0. \end{aligned}$$

But then, the finite Borel measure $\sum_{\alpha, \beta=1}^l v_{\mu, \alpha} \overline{v_{\mu, \beta}} \gamma_{\alpha, \beta}$ is zero for all $1 \leq \mu \leq n$, which can only occur if $v_{\mu, \alpha} = 0$ for all $1 \leq \mu \leq n$, $1 \leq \alpha \leq l$. $\square$

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