Integral equations in the sense of Kurzweil integral and applications

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Equações integrais no sentido da integral de Kurzweil e aplicações

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Abstract

Being part of a research group on functional differential equations (FDEs, for short), due to my formation in non-absolute integration theory and because certain kinds of FDEs can be expressed as integral equations, I was motivated to investigate the latter. The purpose of this work, therefore, is to develop the theory of integral equations, when the integrals involved are in the sense of Kurzweil-Henstock or Kurzweil-Henstock-Stieltjes, through the correspondence between solutions of integral equations and solutions of generalized ordinary differential equations (we write generalized ODEs, for short).

In order to be able to obtain results for integral equations, we propose extensions of both the Kurzweil integral and the generalized ODEs (found in [36]). We develop the fundamental properties of this new generalized ODE, such as existence and uniqueness of solutions results, and we propose stability concepts for the solutions of our new class of equations. We, then, apply these results to a class of nonlinear Volterra integral equations of the second kind. Finally, we consider a model of population growth (found in [4]) that can be expressed as an integral equation that belongs to this class of nonlinear Volterra integral equations.

Keywords: integral equations, nonabsolute integration, generalized ordinary differential equations, Kurzweil-Henstock integral.

Resumo

Sendo parte de um grupo de pesquisa em equações diferenciais funcionais (escrevemos EDFs), por causa de minha formação em teoria de integração não absoluta e porque certos tipos de EDFs podem ser escritas como equações integrais, decidi estudar esse último tipo de equações. O objetivo desse trabalho, portanto, é desenvolver a teoria de equações integrais, quando as integrais envolvidas são no sentido de Kurzweil-Henstock ou Kurzweil-Henstock-Stieltjes, através da correspondência entre soluções de equações integrais e soluções de equações diferenciais ordinárias generalizadas (ou EDOs generalizadas).

A fim de obter resultados para estas equações integrais, propomos extensões de ambas a integral de Kurzweil e as EDOs generalizadas (encontradas em [36]). Desenvolvemos propriedades fundamentais dessa nova EDO generalizada, como resultados de existência e unicidade de solução, e propomos conceitos de estabilidade para as soluções de nossa nova classe de equações. Nós, então, aplicamos esses resultados a uma classe de equações integrais de Volterra não lineares de segunda espécie. Finalmente, consideramos um modelo de crescimento de populações (encontrado em [4]) que pode ser escrito como uma equação integral pertencente a essa classe de equações integrais de Volterra não lineares.

Palavras-chave: equações integrais, integração não absoluta, equações diferenciais ordinárias generalizadas, integral de Kurzweil-Henstock.

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Introduction

Many problems which appear in Physics, Engineering, Biology, Economy and other sciences can be described as integral equations.

On science's realm some laws are written as differential equations as, for instance, on Newton's dynamics theory. Such differential equations denote an instantaneous balance or local equilibrium, whose solutions are associated to adequate initial or boundary conditions obtained from experiments or observations. Thus natural phenomena can be understood when compared with experimental data and a process in science can be used in order to "coincide" with calculated data.

However, when natural laws are not sufficiently established, as an instantaneous balance or local equilibrium and, therefore, cannot be regarded as a differential equation, or when the experimental methods or observation conditions cannot be analysed separately from the object being studied, so that it cannot be expressed with a initial or boundary condition, then the problem is usually modelled as an integral equation. Moreover, problems with singular boundary conditions and some types of differential equations can be more easily studied when written as integral equations. In short, problems formulated as integral equations satisfy, automatically, initial or boundary conditions within its integral formulation. Furthermore, presenting the solution as an integral equation is often more useful for approximations.

Integral equations have been studied in Kurzweil-Henstock's sense, which coincide with the Perron integral and which integrates highly oscillating functions (i.e. of unbounded

variation) as well as functions with many discontinuities. The papers [1], [10], [11], [12], [13], [14], [15], [16], [17], [27] and [31] are some references in which the authors investigate different types of differential equations using their integral forms, where the integral is considered in the Kurzweil-Henstock's sense, and obtain results through the correspondence between the solutions of these differential equations and generalized ODEs.

As an example, we can cite a nonlinear Volterra integral equation, which arises in the study of population growth, considered in [4]. We assume the growth rate, $g : [0, \infty) \rightarrow \mathbb{R}$, depends only on the size of the population and the probability of death, $P : [0, \infty) \rightarrow [0, 1]$, depends only on the age of the individuals. These assumptions lead to a Volterra integral equation of the form

$$x(t) = f(t) + \int_0^t g(x(t-s))P(s)ds, \quad 0 \leq t < \infty, \quad (0.1)$$

where $f : [0, \infty) \rightarrow \mathbb{R}$ represents the number of survivors at time t from the original population $f(0)$.

This model is useful to studying populations afflicted with a disease, provided people who recover from the disease are not immune to it (thus, the spread of the disease depends only on the number of people with the disease) and there is no incubation period (thus, no delay terms appear in the integral). The papers [7] and [8] derive and use this model to study the spread of gonorrhea.

Finally, this model is also used to describe economic models, where the solution x describes the total value of capital in the instant t , the production of new capital within the economy depends only on $x(t)$ and the rate of production is $g(x(t))$, $P(t)$ is the probability of survival of a unit of equipment to age at least t (considering this equipment retains its full value until breakdown) and $f(t)$ means the amount of capital left from the original investment $f(0)$.

Note that the model (0.1) is a particular case of the nonlinear Volterra integral equation of the second kind

$$x(t) = f(t) + \int_0^t K(t, x(s), s) du(s), \quad 0 \leq t < \infty, \quad (0.2)$$

where the integral is considered in the Kurzweil-Henstock-Stieltjes' sense. Thus, in order to apply the theory of generalized ordinary differential equations (generalized ODEs, for short) to integral equations of the form (0.2), it was necessary to create an extension of the Kurzweil integral and of the standard generalized ODE (both found in [36]).

The first chapter of this work is dedicated to defining this new integral and the larger class of generalized ODE, proving fundamental properties expected from these mathematical objects, including a local existence and uniqueness of solutions result.

The second chapter proposes concepts of stability for the solutions of our generalized ODE and we prove Lyapunov-type theorems for these stability concepts. The paper [15] was the reference for the main results of this chapter.

In Chapter 3, we investigate integral equations of the form (0.2) and show a correspondence result between the solutions of these integral equations and the solutions of the new generalized ODE. We obtain a global existence and uniqueness of solution result to nonlinear integral equations based on ideas from the theory of linear Stieltjes integral equations, found in the paper [37]. Finally, we propose stability concepts for Volterra integral equations using the stability concepts defined for generalized ODEs in Chapter 2.

In the last chapter, we apply the results obtained to the population model (0.1) and we also replicate some of the results from [4] assuming different hypotheses. We also include two appendices: Appendix A contains the fundamental properties of the Kurzweil integral, considered in [36], that we use in this work, and Appendix B contains information about generalized ODEs and concepts of stability of their solutions, both found in [36].

Generalized Ordinary Differential Equations

In the present text, we propose an extension of the integral introduced by J. Kurzweil in 1957, see [27]. The aim of this extension is to propose a better framework to deal with integral equations. This framework is in the sense of the theory of generalized ordinary differential equations (generalized ODEs, for short) proposed by J. Kurzweil. As a matter of fact, our extension of the Kurzweil integral generates a larger class of generalized ODEs.

This chapter introduces this new integral (we call it the K-integral) and its main properties. We then use the K-integral to define a larger class of generalized ODEs and we close the chapter with a result on the existence and uniqueness of a solution of the new class of generalized ODEs.

1.1 The Kurzweil integral

At first, let us provide some definitions and notation.

A **gauge** on $[a, b] \subset \mathbb{R}$ is any function $\delta : [a, b] \rightarrow (0, \infty)$.

A **division** of $[a, b]$ is any finite collection of points $D = \{s_0, s_1, \dots, s_{|D|}\}$ such that $a = s_0 < s_1 < \dots < s_{|D|-1} < s_{|D|} = b$, where $|D|$ denotes the number of subintervals into which $[a, b]$ is divided. For each integer i , the numbers s_i are called **nodes** of the division D .

A **tagged division** of $[a, b]$ is any finite collection of point-interval pairs $D = \{(\tau_i, [s_{i-1}, s_i]) \mid i = 1, \dots, |D|\}$ such that $a = s_0 < s_1 < \dots < s_{|D|-1} < s_{|D|} = b$ and $\tau_i \in [s_{i-1}, s_i]$ for every integer i between 1 and $|D|$. For every integer i , the number τ_i is called **tag** of the interval $[s_{i-1}, s_i]$.

A tagged division $D = \{(\tau_i, [s_{i-1}, s_i])\}_{i=1}^{|D|}$ of $[a, b]$ is **δ -fine**, whenever $[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$ for every integer i between 1 and $|D|$.

Definition 1.1. Let X be a Banach space. A function $F : [a, b] \times [a, b] \rightarrow X$ is **Kurzweil integrable** over the interval $[a, b]$, if there is an element $I \in X$, denoted by $\int_a^b DF(\tau, s)$, such that for every $\varepsilon > 0$, there is a gauge δ on $[a, b]$ such that

$$\left\| \sum_{i=1}^{|D|} [F(\tau_i, s_i) - F(\tau_i, s_{i-1})] - I \right\| < \varepsilon$$

for all δ -fine tagged divisions $D = \{(\tau_i, [s_{i-1}, s_i])\}_{i=1}^{|D|}$ of $[a, b]$. We denote the set of all Kurzweil integrable functions over $[a, b]$ by $K([a, b])$.

The reader may want to consult the Appendix A in the end of this text for the basic results regarding the Kurzweil integral and used in this work.

It is important to discuss whether the Kurzweil integral is well-defined or not, which means verifying that for any gauge δ on $[a, b]$, there is at least one δ -fine tagged division. This is the content of the following lemma, known as Cousin Lemma, whose proof can be found in [21], Theorem 4.1.

Lemma 1.2. Given a gauge δ on $[a, b]$, there is a δ -fine tagged division D of $[a, b]$.

By $L(X)$ we mean the Banach space of all linear operators $T : X \rightarrow X$ with norm $\|T\| = \sup_x \|T(x)\|$, where the supremum is taken over all $x \in X$ with $\|x\| \leq 1$. When

$F : [a, b] \times [a, b] \rightarrow X$ in Definition 1.1 is such that $F(\tau, t) = f(\tau)g(t)$, with $f : [a, b] \rightarrow L(X)$ and $g : [a, b] \rightarrow X$, we obtain the well-known Perron-Stieltjes integral, which is equivalent to the Kurzweil-Henstock-Stieltjes integral, when the functions f and g are real valued. In particular, if $g(t) = t$ for all $t \in [a, b]$, then we obtain the Perron integral, also known as generalized Riemann integral. The book [19] is a great reference for the generalized Riemann integral when $X = \mathbb{R}$.

Reference [36] is a detailed treatise of the theory of Generalized Ordinary Differential Equations, which we will deal with later. The first chapter of such book is dedicated to studying the Kurzweil integral, as presented in Definition 1.1, but when $X = \mathbb{R}^n$. We also include the recent book of J. Kurzweil on the subject, see [28].

A subset H of $K([a, b])$ is said to be **equi-integrable**, if for every $\varepsilon > 0$, there is a gauge δ on $[a, b]$ such that Definition 1.1 is satisfied for every function F in H . Next we use the concept of equi-integrability to define our extension of the Kurzweil integral.

Definition 1.3. *A function $F : [a, b] \times [a, b] \times [a, b] \rightarrow X$ is **K-integrable** over $[a, b]$, if for each $t \in [a, b]$, the function $F_t : [a, b] \times [a, b] \rightarrow X$ given by $F_t(\tau, s) = F(t, \tau, s)$ is Kurzweil integrable over $[a, b]$ and, moreover, the family of functions $\{F_t; t \in [a, b]\}$ is equi-integrable. We denote the set of all K-integrable functions over $[a, b]$ by $\mathcal{K}([a, b])$.*

Each $F \in K([a, b])$ can be identified with an element $\overline{F} \in \mathcal{K}([a, b])$ by setting $\overline{F}(t, \tau, s) = F(\tau, s)$. Hence the set of Kurzweil integrable functions is contained in the set of K-integrable functions, that is, $K([a, b]) \subset \mathcal{K}([a, b])$. However the other inclusion is not valid, because the Kurzweil integral of K-integrable functions is not well-defined. Indeed, the Kurzweil integral is defined for functions whose domain is contained in $\mathbb{R}^2$, while the K-integral is defined for functions whose domain is a subset of $\mathbb{R}^3$.

The definition above can be extended to unbounded intervals, that is, intervals where at least one of its endpoints is $-\infty$ or ∞ . The usual approach in elementary calculus is to define these types of integrals as limits of integrals over bounded intervals. Instead of dealing

with a limit, we impose

$$F(t, \tau, s_2) - F(t, \tau, s_1) = 0, \quad t \notin \{-\infty, \infty\},$$

whenever $s_1 = -\infty$, $s_2 = \infty$ or both. When $t \in \{-\infty, \infty\}$, we set $F(t, \tau, s) = 0$ for all τ, s where F is defined. Thus, it does not matter the value of $F(t, \tau, s)$ when $\tau \in \{-\infty, \infty\}$, because only an unbounded interval can have $-\infty$ or ∞ as a tag. The reference [30] contains more details about integration over unbounded intervals for the case where $F(t, \tau, s) = f(\tau)s$ and f is real valued. Also, [24] is a great reference, in Portuguese, for the Henstock-Kurzweil integral over bounded and unbounded intervals.

In order to simplify the notation, $S(F, t, D)$ is going to represent the Riemann sum corresponding to a function $F : [a, b] \times [a, b] \times [a, b] \rightarrow X$, a point $t \in [a, b]$ and a tagged division $D = \{\tau_i, [s_{i-1}, s_i]\}_{i=1}^{|D|}$ of $[a, b]$:

$$S(F, t, D) = \sum_{i=1}^{|D|} [F(t, \tau_i, s_i) - F(t, \tau_i, s_{i-1})].$$

Likewise, if $J = [t_1, t_2]$ is a subset of $[a, b]$, then

$$\begin{aligned} S(F, J, D) &= S(F, t_2, D) - S(F, t_1, D) = \\ &= \sum_{i=1}^{|D|} [F(t_2, \tau_i, s_i) - F(t_2, \tau_i, s_{i-1}) - F(t_1, \tau_i, s_i) + F(t_1, \tau_i, s_{i-1})]. \end{aligned}$$

As expected, linearity of the integral is valid because of the following identity:

$$S(c_1 F + c_2 G, t, D) = c_1 S(F, t, D) + c_2 S(G, t, D)$$

for any Riemann sum of functions $F, G : [a, b] \times [a, b] \times [a, b] \rightarrow X$, constants $c_1, c_2 \in \mathbb{R}$, a tagged division D of $[a, b]$ and $t \in [a, b]$.

Now, we discuss integrability on subintervals for the K-integral.

Theorem 1.4. *If $F \in \mathcal{K}([a, b])$, then for every $[c, d] \subset [a, b]$, we have $F \in \mathcal{K}([c, d])$.*

Proof. A function $F : [a, b] \times [a, b] \times [a, b] \rightarrow X$ belongs to $\mathcal{K}([c, d])$, if the Kurzweil integral over $[c, d]$ of $F_t : [a, b] \times [a, b] \rightarrow X$, given by $F_t(\tau, s) = F(t, \tau, s)$, exists for every $t \in [c, d]$ and, moreover, the set $\{F_t; t \in [c, d]\}$ is equi-integrable.

Since $F \in \mathcal{K}([a, b])$, the Kurzweil integral of F_t over $[a, b]$ exists for every $t \in [a, b]$ and, thus, we can apply Theorem A.3 (see Appendix A) to conclude F_t is Kurzweil integrable over $[c, d]$ for all $t \in [a, b]$. Hence, F_t is Kurzweil integrable over $[c, d]$ for all $t \in [c, d]$, because $[c, d] \subset [a, b]$. The equi-integrability of $\{F_t; t \in [c, d]\}$ follows from the definition of the K-integral and because $F \in \mathcal{K}([a, b])$. Therefore $F \in \mathcal{K}([c, d])$. $\square$

Most of the proofs of the results of this section follow the same line of thinking: we need to prove a property for the K-integral, we use its analogue available for the Kurzweil integral and finish with an argument similar to the one used in the end of the proof of Theorem 1.4.

Theorem 1.5 (Theorem A.4). *Let $c \in (a, b)$. If $F \in \mathcal{K}([a, c])$ and $F \in \mathcal{K}([c, b])$, then $F \in \mathcal{K}([a, b])$.*

The next result, known as Saks-Henstock Lemma, is a very useful tool, though technical in nature.

Lemma 1.6 (Theorem A.5). *Let $F : [a, b] \times [a, b] \times [a, b] \rightarrow X$ be K-integrable over $[a, b]$. Given $\epsilon > 0$, assume δ is the gauge on $[a, b]$ such that*

$$\left\| S(F, t, D) - \int_a^b D_s F(t, \tau, s) \right\| < \epsilon$$

for every δ -fine tagged division D of $[a, b]$ and $t \in [a, b]$. If

$$a \leq \beta_1 \leq \xi_1 \leq \gamma_1 \leq \beta_2 \leq \xi_2 \leq \gamma_2 \leq \cdots \leq \beta_m \leq \xi_m \leq \gamma_m \leq b$$

satisfies

$$\xi_j \in [\beta_j, \gamma_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$$

for every integer j between 1 and m , then

$$\left\| \sum_{j=1}^m [F(t, \xi_j, \gamma_j) - F(t, \xi_j, \beta_j) - \int_{\beta_j}^{\gamma_j} D_s F(t, \tau, s)] \right\| < \varepsilon \quad (1.1)$$

$t \in [a, b]$.

The K-integral contains all its Cauchy extensions, meaning that it contains its improper integrals. This property is known as Hake-type theorem.

Theorem 1.7 (Theorem A.6). *Let $F : [a, b] \times [a, b] \times [a, b] \rightarrow X$ belong to $\mathcal{K}([a, c])$ for all $c \in [a, b]$ and suppose there is $I = I(t) \in X$, with $t \in [a, b]$, such that*

$$\lim_{c \rightarrow b-} \left[\int_a^c D_s F(t, \tau, s) - F(t, b, c) + F(t, b, b) \right] = I(t).$$

Then $F \in \mathcal{K}([a, b])$ and $\int_a^b D_s F(t, \tau, s) = I(t)$.

The reader may want to consult the Appendix A for more information about the “left endpoint” analogue of Theorem 1.7. Such result can be carried out with a similar proof.

Let Ω be an open subset of a Banach space X and $F : [a, b] \times \Omega \times [a, b] \rightarrow X$ be a function. As a consequence of Theorem 1.7, we can prove that the K-integral $\int_a^b D_s F(t, x(\tau), s)$ exists, whenever $x : [a, b] \rightarrow \Omega$ is a step function.

We recall a function $f : [a, b] \rightarrow X$ is called **regulated**, if its one-sided limits exist at every point of the domain. The space of regulated functions from $[a, b]$ to X is denoted by $G([a, b], X)$. The book [25] and the paper [18] are great references for the theory of regulated functions.

Corollary 1.8. *Let the function $F : [a, b] \times \Omega \times [a, b] \rightarrow X$ be regulated on its third variable, i.e. $F(t, \tau, \cdot)$ is regulated for every $t \in [a, b]$ and $\tau \in \Omega$, and $x : [a, b] \rightarrow \Omega$ be a step function. Then the integral $\int_a^b D_s F(t, x(\tau), s)$ exists.*

Proof. Let $t \in [a, b]$. Then we can apply Corollary B.5 (see Appendix B) and conclude the Kurzweil integral $\int_a^b DF(t, x(\tau), s)$ exists and

$$\begin{aligned} \int_a^b DF(t, x(\tau), s) &= \sum_{j=1}^k [F(t, c_j, s_j-) - F(t, c_j, s_{j-1}+)] + \\ &+ \sum_{j=1}^k [F(t, x(s_{j-1}), s_{j-1}+) - F(t, x(s_{j-1}), s_{j-1}) - F(t, x(s_j), s_j-) + F(t, x(s_j), s_j)], \end{aligned} \quad (1.2)$$

where $a = s_0 < s_1 < \dots < s_k = b$ is a division of $[a, b]$ such that $x(s) = c_j \in X$ for all $s \in (s_{j-1}, s_j)$ and the division depends only on the step function x . Note that the above limits are well-defined, because we assumed the function $F(t, \tau, \cdot)$ is regulated for every $t \in [a, b]$ and $\tau \in \Omega$.

Since t was chosen arbitrarily in $[a, b]$ and the above division does not rely on the point t , we can conclude the K-integral $\int_a^b D_s F(t, x(\tau), s)$ exists. $\square$

Given $F \in \mathcal{K}([a, b])$, for each $t \in [a, b]$, the function $F_t(\tau, s) = F(t, \tau, s)$ admits an indefinite integral, in Kurzweil's sense, given by

$$\xi \in [a, b] \mapsto \int_a^\xi DF_t(\tau, s) = \int_a^\xi DF(t, \tau, s), \quad (1.3)$$

whose continuity is discussed in [36], Theorem 1.16 (see Theorem A.8 in Appendix A in the sequel) and Remark 1.17 from [36].

Lemma 1.9. *The indefinite K-integral, given by*

$$\xi \in [a, b] \mapsto \int_a^\xi D_s F(t, \tau, s),$$

is continuous at $\xi \in [a, b]$, if and only if, the function $F_{t\xi} : [a, b] \rightarrow X$ given by $F_{t\xi}(s) = F(t, \xi, s)$ is continuous at ξ , where $t \in [a, b]$.

Proof. Let $\varepsilon > 0$ be given and let δ be a gauge on $[a, b]$ which corresponds to ε in the definition of the K-integrability of F (Definition 1.3). Then, for each $t \in [a, b]$ and for each

$\sigma \in [\xi - \delta(\xi), \xi + \delta(\xi)] \cap [a, b]$, we can apply Saks-Henstock Lemma (Lemma 1.6) to obtain

$$\left\| F(t, \xi, \sigma) - F(t, \xi, \xi) - \int_{\xi}^{\sigma} D_s F(t, \tau, s) \right\| < \varepsilon, \quad (1.4)$$

that is,

$$\lim_{\sigma \rightarrow \xi} \int_{\xi}^{\sigma} D_s F(t, \tau, s) = \lim_{\sigma \rightarrow \xi} [F(t, \xi, \sigma) - F(t, \xi, \xi)]. \quad (1.5)$$

The above limit yields the result. $\square$

We are also interested in determining when the following limit is valid:

$$\lim_{\sigma \rightarrow t} \int_a^b D_s(\sigma, \tau, s) = \int_a^b D_s F(t, \tau, s).$$

At first, we need a version of a convergence theorem for the K-integral.

Lemma 1.10 (Theorem A.9). *Let functions $F, F_n : [a, b] \times [a, b] \times [a, b] \rightarrow X$ be given, where $F_n \in \mathcal{K}([a, b])$ for $n \in \mathbb{N}$. Assume there is a gauge ω on $[a, b]$ such that*

$$\lim_{n \rightarrow \infty} [F_n(t, \tau, s_2) - F_n(t, \tau, s_1)] = F(t, \tau, s_2) - F(t, \tau, s_1)$$

for $t \in [a, b]$ and all $s_1, s_2 \in [a, b]$ such that $s_1 \leq \tau \leq s_2$, $[s_1, s_2] \subset (\tau - \omega(\tau), \tau + \omega(\tau))$.

Assume, further, that the sequence $\{F_n\}_{n \in \mathbb{N}}$ is equi-integrable. Then $F \in \mathcal{K}([a, b])$ and

$$\lim_{n \rightarrow \infty} \int_a^b D_s F_n(t, \tau, s) = \int_a^b D_s F(t, \tau, s).$$

Corollary 1.11. *Let $F : [a, b] \times [a, b] \times [a, b] \rightarrow X$ be a function. If, for every $\tau, s \in [a, b]$, $F(\cdot, \tau, s)$ is continuous and $F \in \mathcal{K}([a, b])$, then*

$$\lim_{\sigma \rightarrow t} \int_a^b D_s F(\sigma, \tau, s) = \int_a^b D_s F(t, \tau, s).$$

Proof. We will prove the above limit is true when $\sigma \rightarrow t+$. The case $\sigma \rightarrow t-$ follows analogously.

For each $n \in \mathbb{N}$, let us define $F_n : [a, b] \times [a, b] \times [a, b] \rightarrow X$ by

$$F_n(t, \tau, s) = F\left(t + \frac{1}{n}, \tau, s\right).$$

Then, for every $t, \tau, s_1, s_2 \in [a, b]$ such that $\tau \in [s_1, s_2] \subset [a, b]$, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} [F_n(t, \tau, s_2) - F_n(t, \tau, s_1)] = \\ & = \lim_{n \rightarrow \infty} \left[F\left(t + \frac{1}{n}, \tau, s_2\right) - F\left(t + \frac{1}{n}, \tau, s_1\right) \right] = F(t, \tau, s_2) - F(t, \tau, s_1) \end{aligned}$$

since F is continuous on the first variable. From the definition of the K-integral (Definition 1.3), it is clear the sequence $\{F_n\}_{n \in \mathbb{N}}$ is equi-integrable. Hence we can apply Lemma 1.10 to conclude

$$\lim_{\sigma \rightarrow t+} \int_a^b D_s F(\sigma, \tau, s) = \lim_{n \rightarrow \infty} \int_a^b D_s F_n(t, \tau, s) = \int_a^b D_s F(t, \tau, s)$$

and the proof is complete. $\square$

1.2 Basic concepts about generalized ODEs

This section introduces a new kind of generalized ODE using the K-integral. We recall our main reference for generalized ODEs, as introduced by J. Kurzweil, is [36] and the results we used from this book are collected in Appendix B.

Let Ω be an open set of an arbitrary Banach space X . We are going to consider functions $F : G \rightarrow X$, where $G = [a, b] \times \Omega \times [a, b]$, for some $a, b \in \mathbb{R}$.

Definition 1.12. *A function $x : [\alpha, \beta] \rightarrow \Omega$ is called a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = D_s F(t, x, s) \tag{1.6}$$

on the interval $[\alpha, \beta] \subset [a, b]$, if

$$x(t) = \int_{\alpha}^{\beta} D_s F(t, x(\tau), s)$$

holds for every $t \in [\alpha, \beta]$.

Our first example points out (1.6) is a notation only, whose purpose is to describe a formal equation-like object defined by its solutions. Moreover, note that the symbol $\frac{dx}{d\tau}$ does not stand for differentiability.

Example 1.13. Let $r : [0, 1] \rightarrow \mathbb{R}$ be a continuous function for which its derivative does not exist at any point of the interval $[0, 1]$ and $F(t, x, s) = r(s)\chi_{[0,t]}(s)$, where $\chi_{[0,t]} : [0, 1] \rightarrow \mathbb{R}$ is the characteristic function of $[0, t]$ on $[0, 1]$, that is,

$$\chi_{[0,t]}(s) = \begin{cases} 1 & s \in [0, t] \\ 0 & s \in (t, 1]. \end{cases}$$

Then $x : [0, 1] \rightarrow \mathbb{R}$, defined by $x(t) = r(t) - r(0)$, is a solution of the generalized differential equation

$$\frac{dx}{d\tau} = D_s F(t, x, s)$$

on $[0, 1]$ because, by the definition of the integral, we have

$$\int_0^1 D_s F(t, x, s) = \int_0^t D_s r(s) = r(t) - r(0) = x(t)$$

though the function x has no derivative at any point of $[0, 1]$.

Now, we would like to investigate how the standard generalized ODE, introduced by J. Kurzweil, fits in this new definition.

As a consequence of Proposition B.2, any function $x : [\alpha, \beta] \rightarrow \Omega$ is a solution of the generalized ODE

$$\frac{dx}{d\tau} = DU(x, t)$$

on $[\alpha, \beta]$, where $U : \Omega \times [a, b] \rightarrow X$, $\Omega \subset X$ is open and $[\alpha, \beta] \subset [a, b]$, provided

$$x(t) = x(\alpha) + \int_{\alpha}^t DU(x(\tau), s) \quad t \in [\alpha, \beta].$$

Let us define $F : [\alpha, \beta] \times \Omega \times [\alpha, \beta] \rightarrow X$ by

$$F(t, \tau, s) = U(\tau, s)\chi_{[\alpha, t]}(s) + \frac{x(\alpha)}{\beta - \alpha}s.$$

Then

$$\int_{\alpha}^{\beta} D_s F(t, x(\tau), s) = \int_{\alpha}^t DU(x(\tau), s) + x(\alpha) = x(t) - x(\alpha) + x(\alpha) = x(t)$$

which means that $x : [\alpha, \beta] \rightarrow \Omega$ is a solution of

$$\frac{dx}{d\tau} = D_s F(t, x, s)$$

on $[\alpha, \beta]$. Thus, we proved that any solution of a generalized ODE, in the classical sense, is a solution of a generalized ODE as presented in Definition 1.12.

In order to obtain qualitative properties of the solutions of our generalized ODE, we are going to consider a special class of functions given by the definition below.

Definition 1.14. *Let $h : [a, b] \rightarrow \mathbb{R}$ be a nondecreasing function. A function $F : G \rightarrow X$ belongs to the class $\mathcal{F}(G, h)$, if the following inequalities*

$$\left\| \int_a^b D_s [F(t_2, x(\tau), s) - F(t_1, x(\tau), s)] \right\| \leq |h(t_2) - h(t_1)| \quad (1.7)$$

and

$$\left\| \int_a^b D_s [F(t_2, x(\tau), s) - F(t_1, x(\tau), s) - F(t_2, y(\tau), s) + F(t_1, y(\tau), s)] \right\| \leq \|x - y\| |h(t_2) - h(t_1)| \quad (1.8)$$

hold for all $t_1, t_2 \in [a, b]$ and $x, y \in E$, where E is the set of all step functions $x : [a, b] \rightarrow \Omega$.

For the remainder of this section, we are going to present some basic properties expected for the solutions of a generalized ODE whose right-hand side F belongs to $\mathcal{F}(G, h)$ for some nondecreasing real function h .

Now, we would like to describe the discontinuities of the solutions of a generalized ODE. If $x : [\alpha, \beta] \rightarrow X$ is a solution of the generalized ODE (1.6) and F is continuous on the first variable, then we can apply Corollary 1.11 and conclude every solution of (1.6) is continuous.

Lemma 1.15. *Assume $F : \Omega \rightarrow X$ satisfies condition (1.7). If $[\alpha, \beta] \subset [a, b]$ and $x : [\alpha, \beta] \rightarrow X$ is a solution of (1.6), then for every $t_1, t_2 \in [\alpha, \beta]$ the inequality*

$$\|x(t_2) - x(t_1)\| \leq |h(t_2) - h(t_1)|$$

holds.

Proof. From Definition 1.12 and condition (1.7), we have

$$\|x(t_2) - x(t_1)\| = \left\| \int_{\alpha}^{\beta} D_s [F(t_2, x(\tau), s) - F(t_1, x(\tau), s)] \right\| \leq |h(t_2) - h(t_1)|$$

and this concludes our proof. $\square$

Hence, if $x : [\alpha, \beta] \rightarrow X$ is a solution of (1.6), where F satisfies condition (1.7), then x is continuous on $t \in [a, b]$, whenever h is continuous on t .

There is also a particular case of Definition 1.12 which is worth considering. Let $F, \tilde{F} : G \rightarrow X$ be such that

$$\tilde{F}(t, \tau, s) = F(t, \tau, s)\chi_{[a, t]}(s). \quad (1.9)$$

If $x : [\alpha, \beta] \rightarrow \Omega$, $[\alpha, \beta] \subset [a, b]$ is a solution of

$$\frac{dx}{d\tau} = D_s \tilde{F}(t, x, s) \quad (1.10)$$

then

$$x(t) = \int_{\alpha}^{\beta} D_s \tilde{F}(t, x(\tau), s) = \int_{\alpha}^t D_s F(t, x(\tau), s).$$

In the next result, we assume the functions $\tilde{F}$ and F satisfy (1.9).

Lemma 1.16. *If $x : [\alpha, \beta] \rightarrow \Omega$ is a solution of the generalized ODE (1.10) and $\tilde{F} : G \rightarrow X$, given by (1.9), satisfies condition (1.7), then*

$$\begin{aligned} x(r+) - x(r) &= \lim_{\sigma \rightarrow r+} x(\sigma) - x(r) = \\ &= \lim_{\sigma \rightarrow r+} \left(\int_{\alpha}^r D_s [F(\sigma, x(\tau), s) - F(r, x(\tau), s)] \right) + F(r+, x(r), r+) - F(r+, x(r), r) \end{aligned} \quad (1.11)$$

for $r \in [\alpha, \beta)$ and

$$\begin{aligned} x(r) - x(r-) &= x(r) - \lim_{\sigma \rightarrow r-} x(\sigma) = \\ &= \lim_{\sigma \rightarrow r-} \left(\int_{\alpha}^{\sigma} D_s [F(r, x(\tau), s) - F(\sigma, x(\tau), s)] \right) + F(r, x(r), r) - F(r, x(r), r-) \end{aligned} \quad (1.12)$$

for $r \in (\alpha, \beta]$, where $F(r+, x(r), r+) - F(r+, x(r), r) = \lim_{\sigma \rightarrow r+} (F(\sigma, x(r), \sigma) - F(\sigma, x(r), r))$ for $r \in [\alpha, \beta)$ and $F(r, x(r), r-) = \lim_{\sigma \rightarrow r-} F(r, x(r), \sigma)$ for $r \in (\alpha, \beta]$.

Proof. At first, note that all the above limits involving the function F exist because of the estimate in condition (1.7).

If σ is sufficiently close to r and $\sigma > r$, then for any $\varepsilon > 0$, we can write

$$\begin{aligned} \|x(\sigma) - x(r)\| &= \left\| \int_{\alpha}^{\beta} D_s \tilde{F}(\sigma, x(\tau), s) - \int_{\alpha}^{\beta} D_s \tilde{F}(r, x(\tau), s) \right\| = \\ &= \left\| \int_{\alpha}^r D_s [F(\sigma, x(\tau), s) - F(r, x(\tau), s)] + \int_r^{\sigma} D_s F(\sigma, x(\tau), s) \right\| \leq \\ &\leq \left\| \int_{\alpha}^r D_s [F(\sigma, x(\tau), s) - F(r, x(\tau), s)] \right\| + \varepsilon + \|F(\sigma, x(r), \sigma) - F(\sigma, x(r), r)\| \end{aligned}$$

where the third inequality follows from (1.4) in Lemma 1.9. Since ε was taken arbitrarily, the equality (1.11) holds.

The relation (1.12) can be proved analogously. □

Corollary 1.17. *If $x : [\alpha, \beta] \rightarrow \Omega$ is a solution of (1.10), $F : G \rightarrow X$ satisfies condition (1.7) and F is continuous on the first variable, then*

$$x(r+) - x(r) = F(r+, x(r), r+) - F(r+, x(r), r) = F(r, x(r), r+) - F(r, x(r), r)$$

for $r \in [\alpha, \beta)$ and

$$x(r) - x(r-) = F(r, x(r), r) - F(r, x(r), r-)$$

for $r \in (\alpha, \beta]$.

Proof. It follows from Lemma 1.16 and Corollary 1.11. □

Note that Lemma 1.15 implies that any solution of the generalized ODE (1.6) is of bounded variation in its domain, whenever F satisfies condition (1.7). It is well-known that every function of bounded variation is regulated (for a proof, check [37], Proposition 1.5 on page 437). Hence it is reasonable to restrict our discussion to functions F whose integrals $\int_a^b D_s F(t, x(\tau), s)$ exist for regulated functions x .

Theorem 1.18. *Given $F \in \mathcal{F}(G, h)$, let $x : [a, b] \rightarrow \Omega$ be the pointwise limit of functions $x_k : [a, b] \rightarrow \Omega$, $k \in \mathbb{N}$, and assume the integral $\int_a^b D_s F(t, x_k(\tau), s)$ exists for every $k \in \mathbb{N}$ and $t \in [a, b]$. Then the integral $\int_a^b D_s F(t, x(\tau), s)$ exists and*

$$\int_a^b D_s F(t, x(\tau), s) = \lim_{k \rightarrow \infty} \int_a^b D_s F(t, x_k(\tau), s)$$

uniformly in $t \in [a, b]$.

Proof. Let $\mathcal{X}$ be the set of all functions $x : [a, b] \rightarrow \Omega$. Then there is a subset S of $\mathcal{X}$ containing all the functions x such that the K-integral $\int_a^b D_s F(t, x(\tau), s)$ exists. Note that $E \subset S$, where E is the set of all step functions from $[a, b]$ to Ω (Corollary 1.8).

We are going to consider a sequence of functions $\xi_k : [a, b] \rightarrow S$ given by $\xi_k(\tau) = x_k$, where x_k is the sequence we have assumed pointwise convergent. Then ξ_k converges pointwisely to $\xi : [a, b] \rightarrow \mathcal{X}$ given by $\xi(\tau) = x$.

Let us set $\Phi : S \times [a, b] \rightarrow X$ by

$$\Phi(x, t) = \int_a^b D_s F(t, x(\tau), s). \quad (1.13)$$

Then

$$\|\Phi(x, t_2) - \Phi(x, t_1)\| = \left\| \int_a^b D_s [F(t_2, x(\tau), s) - F(t_1, x(\tau), s)] \right\| \leq |h(t_2) - h(t_1)|$$

and

$$\begin{aligned} & \|\Phi(x, t_2) - \Phi(x, t_1) - \Phi(y, t_2) + \Phi(y, t_1)\| = \\ & = \left\| \int_a^b D_s [F(t_2, x(\tau), s) - F(t_1, x(\tau), s) - F(t_2, y(\tau), s) + F(t_1, y(\tau), s)] \right\| \leq \\ & \leq \|x - y\| |h(t_2) - h(t_1)| \end{aligned}$$

for all $t_1, t_2 \in [a, b]$ and $x, y \in S$. Hence, the function Φ belongs to $\mathcal{F}_0(C, h)$. The class of functions $\mathcal{F}_0$ is introduced in Definition B.3.

Let us fix $t \in [a, b]$. Then, for any tagged division $D = \{(\sigma_j, [t_{j-1}, t_j]) \mid j = 1, \dots, |D|\}$ of $[a, t]$, we have

$$\sum_{j=1}^{|D|} (\Phi(\xi_k(\sigma_j), t_j) - \Phi(\xi_k(\sigma_j), t_{j-1})) = \Phi(x_k, t) - \Phi(x_k, a).$$

Hence the Kurzweil integral $\int_a^t D\Phi(\xi_k(\tau), s)$ exists, for every $k \in \mathbb{N}$ and $t \in [a, b]$, and it is given by

$$\int_a^t D\Phi(\xi_k(\tau), s) = \int_a^b D_s [F(t, x_k(\tau), s) - F(a, x_k(\tau), s)]$$

which allows us to apply Theorem B.4 to conclude that the Kurzweil integral $\int_a^t D\Phi(\xi(\tau), s)$ exists and, hence,

$$\int_a^b D_s [F(t, x(\tau), s) - F(a, x(\tau), s)] = \lim_{k \rightarrow \infty} \int_a^b D_s [F(t, x_k(\tau), s) - F(a, x_k(\tau), s)]. \quad (1.14)$$

If we fix $t = a$ in (1.13), then it is clear we can apply Theorem B.4 to get the existence of the Kurzweil integral $\Phi(x, a) = \int_a^b DF(a, x(\tau), s)$ and

$$\int_a^b DF(a, x(\tau), s) = \lim_{k \rightarrow \infty} \int_a^b DF(a, x_k(\tau), s).$$

Therefore, it follows from (1.14) that the integral $\int_a^b D_s F(t, x(\tau), s)$ exists and

$$\int_a^b D_s F(t, x(\tau), s) = \lim_{k \rightarrow \infty} \int_a^b D_s F(t, x_k(\tau), s).$$

This concludes our proof. □

Since the K-integral $\int_a^b D_s F(t, x(\tau), s)$ exists, whenever $x : [a, b] \rightarrow X$ is a step function, we can apply the above result and conclude the K-integral $\int_a^b D_s F(t, x(\tau), s)$ exists, whenever $x : [a, b] \rightarrow X$ is a regulated function, because every regulated function is the uniform limit of step functions (check [25], Theorem 3.1 on page 16).

1.3 Local existence and uniqueness of solution result

This section is devoted to establishing a local existence and uniqueness result for the generalized ODE

$$\frac{dx}{d\tau} = D_s F(t, x, s),$$

when F belongs to $\mathcal{F}(G, h)$ and it satisfies

$$F(t, \tau, s) = \tilde{F}(t, \tau, s) \chi_{[a, t]}(s)$$

for some $\tilde{F} : G \rightarrow X$. We are also going to assume the nondecreasing real function h is continuous from the left on $(a, b]$, because it simplifies the proof and the general case, where h is only a nondecreasing real function, can be treated with the same reasoning.

From the results of the previous section, we know that the candidates for a solution of (1.6) must be of bounded variation, continuous from the left and its discontinuities from the right satisfy (1.11). Thus, given an initial condition $\tilde{x} \in \Omega$, we need to assume $\tilde{x}+ \in \Omega$ in order to be possible to continue the solution for $t > \alpha$. Applying Lemma 1.16 for $r = \alpha$, we obtain

$$\tilde{x}+ = \tilde{x} + F(\alpha+, \tilde{x}, \alpha+) - F(\alpha+, \tilde{x}, \alpha). \quad (1.15)$$

The local existence and uniqueness result we are going to present follows the ideas of the proof of Theorem 2.15 from [16], which relies on the Banach Fixed Point Theorem (check [26], for a proof of the latter).

Theorem 1.19. *Let $F : G \rightarrow X$ be a function belonging to the class $\mathcal{F}(G, h)$ and $(\alpha, \tilde{x}+, \alpha) \in G$, where $\tilde{x}+$ is given by (1.15). Then there is $\Delta > 0$ such that the generalized ODE (1.6) admits a unique solution $x : [\alpha, \alpha + \Delta] \rightarrow X$ with initial condition $x(\alpha) = \tilde{x}$.*

Proof. At first, we assume the function h is continuous at α .

Let $\Delta > 0$ be such that $[\alpha, \alpha + \Delta] \subset [a, b]$, $h(\alpha + \Delta) - h(\alpha) < \frac{1}{2}$ and $\|x - \tilde{x}\| \leq |h(t) - h(\alpha)|$ implies $x \in \Omega$ for all $t \in [\alpha, \alpha + \Delta]$.

We denote by $BV([\alpha, \alpha + \Delta], X)$ the space of all functions $z : [\alpha, \alpha + \Delta] \rightarrow X$ of bounded variation. When equipped with the variation norm given by

$$\|z\|_{BV} = \|z(\alpha)\| + \text{var}_\alpha^{\alpha+\Delta} z,$$

where

$$\text{var}_\alpha^{\alpha+\Delta} z = \sup \left\{ \sum_{j=1}^{|D|} \|z(\sigma_j) - z(\sigma_{j-1})\| \right\}$$

and the supremum is taken over all finite divisions D of $[\alpha, \alpha + \Delta]$, with $\alpha = \sigma_0 < \sigma_1 < \dots < \sigma_{|D|} = \alpha + \Delta$. The set $BV([\alpha, \alpha + \Delta], X)$ is a Banach space (check [23], page 187).

Note that the supremum norm and the bounded variation norm are related by the inequality

$$\|z\|_\infty := \sup_{t \in [\alpha, \alpha + \Delta]} \|z(t)\| \leq \|z\|_{BV}. \quad (1.16)$$

Let $Q \subset BV([\alpha, \alpha + \Delta], X)$ be the set of functions $z : [\alpha, \alpha + \Delta] \rightarrow X$ such that $\|z(t) - \tilde{x}\| \leq |h(t) - h(\alpha)|$ for all $t \in [\alpha, \alpha + \Delta]$. We are going to show that Q is closed. Indeed, let $\{z_n\}_{n \in \mathbb{N}}$ be a convergent sequence in Q and let $z^* \in BV([\alpha, \alpha + \Delta], X)$ be its limit. Then $z_n \rightarrow z^*$ uniformly because of (1.16). Hence, given $\varepsilon > 0$, take $n_0 \in \mathbb{N}$ from the uniform convergence of $\{z_n\}_{n \in \mathbb{N}}$. Then for all $n > n_0$, we have

$$\|z^*(t) - \tilde{x}\| \leq \|z^*(t) - z_n(t)\| + \|z_n(t) - \tilde{x}\| \leq \varepsilon + |h(t) - h(\alpha)|$$

for all $t \in [\alpha, \alpha + \Delta]$ and we can conclude that $z^* \in Q$. Consequently, Q equipped with the variation norm is also a Banach space.

Now, we are ready to define the operator which we will use as the contraction for the Banach Fixed Point Theorem.

Let $T : Q \rightarrow BV([\alpha, \alpha + \Delta], X)$ be given by

$$Tz(t) := \tilde{x} + \int_{\alpha}^t D_s F(t, z(\tau), s)$$

for all $t \in [\alpha, \alpha + \Delta]$. This operator is well-defined because the integral above always exists when z is regulated (every function of bounded variation is regulated) and $F \in \mathcal{F}(\Omega, h)$. Furthermore, from condition (1.7) of functions in $\mathcal{F}(\Omega, h)$, we have

$$\|Tz(t) - \tilde{x}\| = \left\| \int_{\alpha}^t D_s F(t, z(\tau), s) \right\| \leq |h(t) - h(\alpha)|.$$

Therefore $Tz \in Q$ and, hence, $T : Q \rightarrow Q$.

Now, we will prove T is a contraction.

Let $z_1, z_2 \in Q$, $t_1, t_2 \in [\alpha, \alpha + \Delta]$ with $t_1 < t_2$ and $\varepsilon > 0$. Then

$$\begin{aligned} & \|Tz_2(t_2) - Tz_1(t_2) - Tz_2(t_1) + Tz_1(t_1)\| = \\ & = \left\| \int_{\alpha}^{t_2} D_s[F(t_2, z_2(\tau), s) - F(t_2, z_1(\tau), s)] - \int_{\alpha}^{t_1} D_s[F(t_1, z_2(\tau), s) - F(t_1, z_1(\tau), s)] \right\| \leq \\ & \leq \|z_2 - z_1\| [h(t_2) - h(t_1)]. \end{aligned}$$

Note that we used the definition of the operator T and condition (1.8) from the class $\mathcal{F}(G, h)$. Also, it is clear that all $z \in Q$ satisfy $z(\alpha) = \tilde{x}$. Since $t_1, t_2 \in [\alpha, \alpha + \Delta]$ are arbitrary and (1.16) is valid, we have

$$\|Tz_2 - Tz_1\|_{BV} \leq \|z_2 - z_1\|_{BV} [h(\alpha + \Delta) - h(\alpha)] < \frac{1}{2} \|z_2 - z_1\|_{BV}.$$

Therefore, by the Banach Fixed Point Theorem, the operator T admits a unique fixed point and we can conclude Theorem 1.19 is valid when h is continuous at α .

Now, we assume h is not continuous at α . Consider a function $\tilde{h} : [a, b] \rightarrow \mathbb{R}$ given by

$$\tilde{h}(t) = \begin{cases} h(t), & \text{if } t \leq \alpha, \\ h(t) - (h(\alpha+) - h(\alpha)), & \text{if } t \geq \alpha. \end{cases}$$

Then $\tilde{h}$ is a non-decreasing function continuous at α .

Moreover, define $\hat{F} : G \rightarrow X$ by

$$\hat{F}(t, x, s) = \begin{cases} F(t, x, s), & \text{if } t \leq \alpha, \\ F(t, x, s) - (F(\alpha+, \tilde{x}, s) - F(\alpha, \tilde{x}, \alpha)), & \text{if } t > \alpha. \end{cases}$$

Then $\hat{F} \in \mathcal{F}(G, \tilde{h})$. Therefore we can follow the steps of the previous case and conclude the generalized ODE

$$\frac{dx}{d\tau} = D_s \hat{F}(t, x(\tau), s),$$

with initial condition $x(\alpha) = \tilde{x}+$, admits a unique solution z . Hence $x : [\alpha, \alpha + \Delta] \rightarrow X$ given by

$$x(t) = \begin{cases} \tilde{x}, & \text{if } t = \alpha, \\ z(t), & \text{if } t > \alpha, \end{cases}$$

is the solution of $\frac{dx}{d\tau} = D_s F(t, x(\tau), s)$ with $x(\alpha) = \tilde{x}$. Thus the proof is complete. $\square$

Qualitative Properties of Solutions of Generalized ODEs

Let $G = [0, \infty) \times B_c \times [0, \infty)$, where $B_c = \{x \in X; \|x\| < c, c > 0\}$. Throughout this chapter we assume $F : G \rightarrow X$ belongs to the class $\mathcal{F}(G, h)$, satisfies $F(t, 0, s_2) - F(t, 0, s_1) = 0$ for $s_1, s_2, t \in [0, \infty)$ and

$$F(t, \tau, s) = \tilde{F}(t, \tau, s)\chi_{[0,t]}(s)$$

for some $\tilde{F} : G \rightarrow X$. Thus $x \equiv 0$ is a solution on $[0, \infty)$ of the generalized ODE

$$\frac{dx}{d\tau} = D_s F(t, x, s), \quad (2.1)$$

because

$$\int_0^t D_s F(t, 0, s) ds = F(t, 0, t) - F(t, 0, 0) = 0$$

for every $t \in [0, \infty)$.

This chapter proposes stability concepts for the solutions of generalized ODEs and applies the method of Lyapunov functionals for these stability concepts. We use the paper [15] as

the main reference and make the necessary adaptations in order to apply its results to the generalized ODE we defined in Chapter 1.

2.1 Stability concepts for generalized ODEs

We start this section by defining all the stability concepts we are going to use.

Definition 2.1. *The trivial solution $x \equiv 0$ of (2.1) is called regularly stable, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\bar{x} : [\alpha, \beta] \rightarrow B_c$, with $0 \leq \alpha < \beta < \infty$, is a regulated function which satisfies $\|\bar{x}(\alpha)\| < \delta$ and*

$$\sup_{t \in [\alpha, \beta]} \left\| \bar{x}(t) - \bar{x}(\alpha) - \int_{\alpha}^t D_s F(t, \bar{x}(\tau), s) \right\| < \delta,$$

then

$$\|\bar{x}(t)\| < \varepsilon, \quad \text{for } t \in [\alpha, \beta].$$

Definition 2.2. *The trivial solution $x \equiv 0$ of (2.1) is called regularly attracting, if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that if $\bar{x} : [\alpha, \beta] \rightarrow B_c$, with $0 \leq \alpha < \beta < \infty$, is a regulated function satisfying $\|\bar{x}(\alpha)\| < \delta_0$ and*

$$\sup_{t \in [\alpha, \beta]} \left\| \bar{x}(t) - \bar{x}(\alpha) - \int_{\alpha}^t D_s F(t, \bar{x}(\tau), s) \right\| < \rho,$$

then

$$\|\bar{x}(t)\| < \varepsilon, \quad \text{for } t \in [\alpha, \beta] \cap [\alpha + T, \infty).$$

Definition 2.3. *The trivial solution $x \equiv 0$ of (2.1) is called regularly asymptotically stable, if it is regularly stable and regularly attracting.*

Besides the generalized ODE (2.1), we consider the perturbed generalized ODE

$$\frac{dx}{d\tau} = D_s[F(t, x, s) + P(s)\chi_{[0,t]}(s)], \quad (2.2)$$

where $P : [0, \infty) \rightarrow X$. Let $G^-([\alpha, \beta], X)$ denote the space of left-continuous regulated functions from $[\alpha, \beta]$ to X .

Definition 2.4. *The trivial solution $x \equiv 0$ of (2.1) is called regularly stable with respect to perturbations, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|x_0\| < \delta$ and $P \in G^-([\alpha, \beta], X)$ with*

$$\sup_{t \in [\alpha, \beta]} \|P(t) - P(\alpha)\| < \delta,$$

then

$$\|\bar{x}(t, \alpha, x_0)\| < \varepsilon, \quad \text{for } t \in [\alpha, \beta],$$

where $\bar{x}(t, \alpha, x_0)$ is a solution of the perturbed generalized ODE (2.2), with initial condition $\bar{x}(\alpha, \alpha, x_0) = x_0$ and $[\alpha, \beta] \subset [0, \infty)$.

Definition 2.5. *The trivial solution $x \equiv 0$ of (2.1) is called regularly attracting with respect to perturbations, if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that if $\|x_0\| < \delta_0$ and*

$$\sup_{t \in [\alpha, \beta]} \|P(t) - P(\alpha)\| < \rho,$$

with $P \in G^-([\alpha, \beta], X)$, then

$$\|\bar{x}(t, \alpha, x_0)\| < \varepsilon, \quad \text{for all } t \geq \alpha + T, \quad t \in [\alpha, \beta],$$

where $\bar{x}(t, \alpha, x_0)$ is a solution of the perturbed generalized ODE (2.2), with initial condition $\bar{x}(\alpha, \alpha, x_0) = x_0$ and $[\alpha, \beta] \subset [0, \infty)$.

Definition 2.6. *The trivial solution $x \equiv 0$ of (2.1) is called regularly asymptotically stable with respect to perturbations, if it is both stable and attracting with respect to perturbations.*

Before proceeding, we would like to make some comments about the nature of these stability concepts. Regular stability means that if $\bar{x}$, defined on $[\alpha, \beta] \subset [0, \infty)$, has initial

value $\bar{x}(\alpha)$ sufficiently small and is such that

$$\bar{x}(t) - \bar{x}(\alpha) - \int_{\alpha}^t D_s F(t, \bar{x}(\tau), s), \quad (2.3)$$

where $t \in [\alpha, \beta]$, is small enough when measured with the supremum norm, then $\bar{x}$ on $[\alpha, \beta]$ is as close to zero as we wish. Note that when $\bar{x}$ is a solution of the generalized ODE (2.1), then (2.3) equals zero.

On the other hand, regular stability with respect to perturbations means that any solution of the perturbed generalized ODE (2.2), defined on a certain $[\alpha, \beta]$, is close to zero whenever the initial value $\bar{x}(\alpha)$ and the perturbing term,

$$\sup_{t \in [\alpha, \beta]} \|P(t) - P(\alpha)\|,$$

are small enough. In particular, when $X = \mathbb{R}^n$ and $P(t) = \int_{\alpha}^t p(s)ds$, $t \geq \alpha$, with $p \in L_1([\alpha, \beta], \mathbb{R}^n)$, then P is absolutely continuous and the derivative $\dot{P}(s) = \frac{dP}{ds}$ exists almost everywhere in $[\alpha, \beta]$ and

$$\sup_{t \in [\alpha, \beta]} \|P(t) - P(\alpha)\| \leq \text{var}_{\alpha}^{\beta} P = \int_{\alpha}^{\beta} \|\dot{P}(s)\|ds = \int_{\alpha}^{\beta} \|p(s)\|ds.$$

Thus we can easily see that these regular stability concepts of the trivial solution of (2.1) are more general than Halanay's concepts of integral stability (see [20] and [17]).

In the book [36], the author proposes similar stability concepts for generalized ODEs which we have included in Appendix B. Instead of using the supremum norm, he uses the variation of the functions

$$\bar{x}(t) - \bar{x}(\alpha) - \int_{\alpha}^t D_s F(t, \bar{x}(\tau), s)$$

and

$$P(t) - P(\alpha).$$

This seems a natural choice, since the solutions of generalized ODEs are functions of bounded variation (check [36], Lemmas 3.9 and 3.10). However, we choose regular stability concepts, because they are more general than the ones found in [36] and made it possible to connect stability theory for generalized ODEs with integral stability for measure functional differential equations, as described in [15].

The next result shows us how all the previous concepts of regular stability can be related and it follows the line of thinking of Theorem 4.7 from [15].

Theorem 2.7. *The following statements hold.*

- (i) *The trivial solution $x \equiv 0$ of (2.1) is regularly stable if, and only if, it is regularly stable with respect to perturbations.*
- (ii) *The trivial solution $x \equiv 0$ of (2.1) is regularly attracting if, and only if, it is regularly attracting with respect to perturbations.*
- (iii) *The trivial solution $x \equiv 0$ of (2.1) is regularly asymptotically stable if, and only if, it is regularly asymptotically stable with respect to perturbations.*

Sketch of the proof. The last item in the statement above clearly follows from (i) and (ii); the proofs of (i) and (ii) are very similar. In order to illustrate the techniques, we are going to write down the main ideas of the proof of (i).

If $x \equiv 0$ is regularly stable, then given $\varepsilon > 0$, we pick up the δ given by the definition of regular stability (Definition 2.1). Any solution x on $[\alpha, \beta]$ of the perturbed generalized ODE (2.2) is regulated and satisfies

$$x(t) - x(\alpha) - \int_{\alpha}^t D_s F(t, x(\tau), s) = P(t) - P(\alpha).$$

Hence, assuming $\|x(\alpha)\| < \delta$ and $\sup_{t \in [\alpha, \beta]} \|P(t) - P(\alpha)\| < \delta$, we obtain

$$\sup_{t \in [\alpha, \beta]} \left\| x(t) - x(\alpha) - \int_{\alpha}^t D_s F(t, x(\tau), s) \right\| < \delta.$$

This means $\|x(t)\| < \varepsilon$, for $t \in [\alpha, \beta]$, because $x \equiv 0$ is regularly stable. Therefore, the trivial solution of (2.1) is regularly stable with respect to perturbations.

Conversely, given $\varepsilon > 0$, we pick up the δ from the definition of regular stability with respect to perturbations (Definition 2.4). Any regulated function $\bar{x} : [\alpha, \beta] \rightarrow \Omega$ is the solution of a perturbed generalized ODE (2.2), where

$$P(t) = P(\alpha) + \bar{x}(t) - \bar{x}(\alpha) - \int_{\alpha}^t D_s F(t, \bar{x}(\tau), s) \quad (2.4)$$

and $t \in [\alpha, \beta]$.

Hence, assuming $\|x(\alpha)\| < \delta$ and

$$\sup_{t \in [\alpha, \beta]} \left\| \bar{x}(t) - \bar{x}(\alpha) - \int_{\alpha}^t D_s F(t, \bar{x}(\tau), s) \right\| < \delta,$$

we obtain

$$\sup_{t \in [\alpha, \beta]} \|P(t) - P(\alpha)\| < \delta,$$

because of (2.4). This means $\|x(t)\| < \varepsilon$ for $t \in [\alpha, \beta]$, because $x \equiv 0$ is regularly stable with respect to perturbations. Therefore the trivial solution of (2.1) is regularly stable.

2.2 Lyapunov-type theorems for generalized ODEs

Before presenting a direct method of Lyapunov functionals for the stability concepts described in the previous section, we need some auxiliary results.

Lemma 2.8 (Proposition 10.11 from [36]). *Assume that $-\infty < a < b < \infty$ and $f, g : [a, b] \rightarrow \mathbb{R}$ are left-continuous functions on $(a, b]$. If for every $\sigma \in [a, b]$, there is $\delta(\sigma) > 0$ such that for every $\eta \in (0, \delta(\sigma))$ the inequality*

$$f(\sigma + \eta) - f(\sigma) \leq g(\sigma + \eta) - g(\sigma)$$

holds, then

$$f(t) - f(a) \leq g(t) - g(a)$$

for all $t \in [a, b]$.

Lemma 2.8 is a simple real analysis result that relies on the properties of supremum and continuity.

The next lemma, though very technical, is going to be proved in detail, because many features of the theory of generalized ODEs discussed in the first chapter are employed here and there are some technical adaptations we make from the proof of Lemma 6.1 from [15].

We remind the reader $G = [0, \infty) \times B_c \times [0, \infty)$, where $B_c = \{x \in X; \|x\| < c\}$.

Lemma 2.9. *Let $F \in \mathcal{F}(G, h)$. Suppose $V : [0, \infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$, where $\rho < c$, is continuous from the left on $(0, \infty)$ for $x \in \overline{B}_\rho$ and satisfies*

$$|V(t, z) - V(t, y)| \leq K\|z - y\| \quad z, y \in \overline{B}_\rho, \quad t \in [0, \infty) \quad (2.5)$$

where K is a positive constant. Furthermore, suppose there exists a function $\Phi : \overline{B}_\rho \rightarrow \mathbb{R}$ such that for every solution $x : [\alpha, \beta] \rightarrow \overline{B}_\rho$, $[\alpha, \beta] \subset [0, \infty)$, of the generalized ODE (2.1), we have

$$\dot{V}(t, x(t)) = \limsup_{\eta \rightarrow 0+} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leq \Phi(x(t)), \quad t \in [\alpha, \beta]. \quad (2.6)$$

If $\bar{x} : [\gamma, v] \rightarrow \overline{B}_\rho$, $0 \leq \gamma < v < \infty$, is left-continuous on $(\gamma, v]$ and regulated on $[\gamma, v]$, then

$$V(v, \bar{x}(v)) - V(\gamma, \bar{x}(\gamma)) \leq K \sup_{t \in [\gamma, v]} \left\| \bar{x}(t) - \bar{x}(\gamma) - \int_\gamma^t D_s F(t, \bar{x}(\tau), s) \right\| + M(v - \gamma) \quad (2.7)$$

where $M = \sup_{t \in [\gamma, v]} \Phi(\bar{x}(t))$.

Proof. Given a function $\bar{x} : [\gamma, v] \rightarrow \overline{B}_\rho$ which is left-continuous on $(\gamma, v]$ and regulated on $[\gamma, v] \subset [0, \infty)$, we can apply Theorem 1.18 to prove the integral $\int_\gamma^v D_s F(t, \bar{x}(\tau), s)$ exists.

Take $\sigma \in [\gamma, v]$. From our local existence and uniqueness theorem (Theorem 1.19), we know (2.1) has a local unique solution $x : [\sigma, \sigma + \eta_1(\sigma)] \rightarrow \overline{B}_\rho$ with initial condition $x(\sigma) = \overline{x}(\sigma)$.

From the integrability on subintervals, there is $\eta_2 > 0$ such that $\eta_2 \leq \eta_1(\sigma)$, $[\sigma, \sigma + \eta_2] \subset [\sigma, v]$ and the integral $\int_\sigma^{\sigma+\eta_2} D_s[F(t, \overline{x}(\tau), s) - F(t, x(\tau), s)]$ exists. Thus, given $\varepsilon > 0$, there is a gauge δ on $[\sigma, \sigma + \eta_2]$ corresponding to ε in the definition of the last integral. We can take a smaller η_2 satisfying $\eta_2 < \delta(\sigma)$. By (2.6), there is $0 < \eta \leq \eta_2$ such that

$$V(\sigma + \eta, x(\sigma + \eta)) - V(\sigma, x(\sigma)) \leq \eta \Phi(x(\sigma)) \leq \eta M. \quad (2.8)$$

From the Saks-Henstock Lemma (Lemma 1.6), we can get inequalities similar to (1.4):

$$\left\| F(t, \overline{x}(\sigma), \sigma + \eta) - F(t, \overline{x}(\sigma), \sigma) - \int_\sigma^{\sigma+\eta} D_s F(t, \overline{x}(\tau), s) \right\| < \frac{\eta \varepsilon}{2K} \quad (2.9)$$

and

$$\left\| F(t, x(\sigma), \sigma + \eta) - F(t, x(\sigma), \sigma) - \int_\sigma^{\sigma+\eta} D_s F(t, x(\tau), s) \right\| < \frac{\eta \varepsilon}{2K}. \quad (2.10)$$

Since $\overline{x}(\sigma) = x(\sigma)$, from (2.9) and (2.10), we obtain

$$\left\| \int_\sigma^{\sigma+\eta} D_s [F(t, \overline{x}(\tau), s) - F(t, x(\tau), s)] \right\| \leq \frac{\eta \varepsilon}{K}. \quad (2.11)$$

Indeed, we have

$$\begin{aligned} \left\| \int_\sigma^{\sigma+\eta} D_s [F(t, \overline{x}(\tau), s) - F(t, x(\tau), s)] \right\| &= \left\| \int_\sigma^{\sigma+\eta} D_s [F(t, \overline{x}(\tau), s) - F(t, x(\tau), s)] + \right. \\ &\quad \left. + F(t, \overline{x}(\sigma), \sigma + \eta) - F(t, x(\tau), \sigma + \eta) + F(t, x(\sigma), \sigma) - F(t, \overline{x}(\tau), \sigma) \right\| \leq \\ &\leq \frac{\eta \varepsilon}{2K} + \frac{\eta \varepsilon}{2K}. \end{aligned}$$

Moreover, (2.5) implies

$$V(\sigma + \eta, \overline{x}(\sigma + \eta)) - V(\sigma + \eta, x(\sigma + \eta)) \leq$$

$$\begin{aligned}
&\leq K\|\bar{x}(\sigma + \eta) - x(\sigma + \eta)\| = K\|\bar{x}(\sigma + \eta) - \bar{x}(\sigma) + x(\sigma) - x(\sigma + \eta)\| = \\
&= K\left\|\bar{x}(\sigma + \eta) - \bar{x}(\sigma) - \int_{\sigma}^{\sigma+\eta} D_s F(\sigma + \eta, x(\tau), s) d\tau\right\|
\end{aligned}$$

Then the inequality above, (2.11) and (2.8) yield

$$\begin{aligned}
&V(\sigma + \eta, \bar{x}(\sigma + \eta)) - V(\sigma, \bar{x}(\sigma)) = \\
&= V(\sigma + \eta, \bar{x}(\sigma + \eta)) - V(\sigma + \eta, x(\sigma + \eta)) + V(\sigma + \eta, x(\sigma + \eta)) - V(\sigma, x(\sigma)) \leq \\
&\leq K\left\|\bar{x}(\sigma + \eta) - \bar{x}(\sigma) - \int_{\sigma}^{\sigma+\eta} D_s F(\sigma + \eta, \bar{x}(\tau), s) d\tau\right\| + \eta M \leq \\
&\leq K\left\|\bar{x}(\sigma + \eta) - \bar{x}(\sigma) - \int_{\sigma}^{\sigma+\eta} D_s F(\sigma + \eta, \bar{x}(\tau), s) d\tau\right\| + \\
&+ K\left\|\int_{\sigma}^{\sigma+\eta} D_s [F(\sigma + \eta, \bar{x}(\tau), s) - F(\sigma + \eta, x(\tau), s)] d\tau\right\| + \eta M \leq \\
&\leq K\left\|\bar{x}(\sigma + \eta) - \bar{x}(\sigma) - \int_{\sigma}^{\sigma+\eta} D_s F(\sigma + \eta, \bar{x}(\tau), s) d\tau\right\| + \eta(\varepsilon + M).
\end{aligned}$$

Given $t, r \in [\gamma, v]$, we define

$$P(t, r) = \bar{x}(r) - \int_{\gamma}^r D_s F(t, \bar{x}(\tau), s) d\tau.$$

The function $P(t, \cdot)$ is left-continuous on $(\gamma, v]$, since $\bar{x}$ is left-continuous on $(\gamma, v]$. Thus

$$\begin{aligned}
P(t, \sigma + \eta) - P(t, \sigma) &= \bar{x}(\sigma + \eta) - \bar{x}(\sigma) - \int_{\gamma}^{\sigma+\eta} D_s F(t, \bar{x}(\tau), s) d\tau + \int_{\gamma}^{\sigma} D_s F(t, \bar{x}(\tau), s) d\tau = \\
&= \bar{x}(\sigma + \eta) - \bar{x}(\sigma) - \int_{\sigma}^{\sigma+\eta} D_s F(t, \bar{x}(\tau), s) d\tau.
\end{aligned}$$

Now, we define

$$f(t, r) = K\|P(t, r) - P(t, \sigma)\| + (\varepsilon + M)t.$$

Then

$$\begin{aligned} V(\sigma + \eta, \bar{x}(\sigma + \eta)) - V(\sigma, \bar{x}(\sigma)) &\leq K \|P(\sigma + \eta, \sigma + \eta) - P(\sigma + \eta, \sigma)\| + (\varepsilon + M)\eta = \\ &= f(\sigma + \eta, \sigma + \eta) - f(\sigma + \eta, \sigma) \leq \sup_{t \in [\gamma, v]} (f(t, \sigma + \eta) - f(t, \sigma)). \end{aligned}$$

This means we can apply Lemma 2.8 in order to obtain

$$\begin{aligned} V(v, \bar{x}(v)) - V(\gamma, \bar{x}(\gamma)) &\leq \sup_{t \in [\gamma, v]} \{f(t, v) - f(t, \gamma)\} = \\ &= K \sup_{t \in [\gamma, v]} (\|P(t, v) - P(t, \sigma)\| - \|P(t, \gamma) - P(t, \sigma)\|) + (\varepsilon + M)(v - \gamma) \leq \\ &\leq K \sup_{t \in [\gamma, v]} \|P(t, v) - P(t, \gamma)\| + (\varepsilon + M)(v - \gamma) = \\ &= K \sup_{t \in [\gamma, v]} \left\| \bar{x}(v) - \bar{x}(\gamma) - \int_{\gamma}^v D_s F(t, \bar{x}(\tau), s) \right\| + (\varepsilon + M)(v - \gamma) \leq \\ &\leq K \sup_{t \in [\gamma, v]} \left\| \bar{x}(t) - \bar{x}(\gamma) - \int_{\gamma}^t D_s F(t, \bar{x}(\tau), s) \right\| + (\varepsilon + M)(v - \gamma). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the result follows. $\square$

Now, we present a definition of a Lyapunov functional for generalized ODEs. It is borrowed from [15].

Definition 2.10. *We say that $V : [0, \infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$ is a Lyapunov functional with respect to the generalized ODE (2.1), if the following conditions are satisfied:*

- (i) $V(\cdot, x) : [0, \infty) \rightarrow \mathbb{R}$ is continuous from the left on $(0, \infty)$, for every $x \in \overline{B}_\rho$;
- (ii) There is a continuous and increasing function $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $b(0) = 0$ (we say that such function is of Hahn class) and

$$V(t, x) \geq b(\|x\|),$$

for every $t \in [0, \infty)$ and $x \in \overline{B}_\rho$;

(iii) For every solution $\bar{x} : [\gamma, v] \rightarrow \overline{B}_\rho$ of (2.1) with $[\gamma, v] \subset [0, \infty)$, we have

$$\dot{V}(t, \bar{x}(t)) = \limsup_{\eta \rightarrow 0+} \frac{V(t + \eta, \bar{x}(t + \eta)) - V(t, \bar{x}(t))}{\eta} \leq 0, \quad t \in [\gamma, v],$$

which means the right-derivative of V is non-positive along the solutions of (2.1).

The next result presents sufficient conditions on a Lyapunov functional for the regular stability of the trivial solution of (2.1). We borrowed the main ideas from the proof of Theorem 6.3 in [15] and made simple adaptations.

Theorem 2.11. *Let $V : [0, \infty) \times \overline{B}_\rho \rightarrow \mathbb{R}$ be a Lyapunov functional where $0 < \rho < c$. Suppose V satisfies the following conditions:*

(i) $V(t, 0) = 0$, $t \in [0, \infty)$;

(ii) There is a constant $K > 0$ such that

$$|V(t, z) - V(t, y)| \leq K \|z - y\|, \quad t \in [0, \infty) \quad z, y \in \overline{B}_\rho.$$

Then the trivial solution $x \equiv 0$ of (2.1) is regularly stable.

Proof. Let $\bar{x} : [\gamma, v] \rightarrow X$ be a regulated function on $[\gamma, v] \subset [0, \infty)$. By condition (iii) from Definition 2.10 and by Lemma 2.9, we may write

$$V(t, \bar{x}(t)) \leq V(\gamma, \bar{x}(\gamma)) + K \sup_{r \in [\gamma, t]} \left\| \bar{x}(r) - \bar{x}(\gamma) - \int_\gamma^r D_s F(r, \bar{x}(\tau), s) \right\|$$

for every $t \in [\gamma, v]$.

Since V is a Lyapunov functional, there is a function $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ of Hahn class satisfying

$$V(t, x) \geq b(\|x\|), \tag{2.12}$$

for every $(t, x) \in [\gamma, v] \times \overline{B}_\rho$.

Given $\varepsilon > 0$ and $b(\varepsilon) > 0$, let $\delta(\varepsilon) > 0$ be such that $2K\delta(\varepsilon) < b(\varepsilon)$. If

$$\sup_{r \in [\gamma, t]} \left\| \bar{x}(r) - \bar{x}(\gamma) - \int_{\gamma}^r D_s F(r, \bar{x}(\tau), s) \right\| < \delta(\varepsilon),$$

then

$$\begin{aligned} V(t, \bar{x}(t)) &\leq V(\gamma, \bar{x}(\gamma)) + K \sup_{r \in [\gamma, t]} \left\| \bar{x}(r) - \bar{x}(\gamma) - \int_{\gamma}^r D_s F(r, \bar{x}(\tau), s) \right\| < \\ &< |V(\gamma, \bar{x}(\gamma))| + K\delta(\varepsilon) \leq 2K\delta(\varepsilon) < b(\varepsilon), \end{aligned}$$

for every $t \in [\gamma, v]$.

On the other hand, if there is $u \in [\gamma, v]$ such that $\|\bar{x}(u)\| \geq \varepsilon$, then, by (2.12), we have

$$V(u, \bar{x}(u)) \geq b(\|\bar{x}(u)\|) \geq b(\varepsilon),$$

which contradicts the previous inequality. Therefore $\|\bar{x}(t)\| < \varepsilon$ for $t \in [\gamma, v]$ and the result follows. $\square$

The next result adds one hypothesis on the Lyapunov functional in order to get regular asymptotic stability for the trivial solution of (2.1). Here, we are only going to present a sketch of the proof.

Theorem 2.12 (Theorem 6.4 from [15]). *Let $V : [0, \infty) \times \overline{B}_{\rho} \rightarrow \mathbb{R}$ be a Lyapunov functional where $0 < \rho < c$. Suppose V satisfies conditions (i) and (ii) from Theorem (2.11). Moreover, assume there is a continuous function $\Phi : X \rightarrow \mathbb{R}$ satisfying $\Phi(0) = 0$ and $\Phi(x) > 0$ for $x \neq 0$, such that for every solution $x : [\gamma, v] \rightarrow B_{\rho}$ of (2.1) with $[\gamma, v] \subset [0, \infty)$, we have*

$$\dot{V}(t, x(t)) \leq -\Phi(x(t)), \quad t \in [\gamma, v].$$

Then the trivial solution $x \equiv 0$ of (2.1) is regularly asymptotically stable.

Sketch of the proof. Given $\varepsilon > 0$ and a regulated function $\bar{x} : [\gamma, v] \rightarrow X$ on $[\gamma, v] \subset [0, \infty)$, from the regular stability of $x \equiv 0$, which is clear, we define $T = T(\varepsilon)$ necessary for the trivial solution of (2.1) to be regular attracting.

Next we make a similar argument used in Theorem 2.11 involving an estimate for $V(t, \bar{x}(t))$ and a function b of Hahn class to conclude $x \equiv 0$ is regularly attracting.

Volterra Integral Equations

In this chapter, we introduce a class of Volterra integral equations and obtain some qualitative properties of their solutions through the theory of generalized ODEs. Most of the results regarding Volterra integral equations found in literature restrain the functions involved in order to get a global unique continuous solution. A basic reference for Volterra integral equations is [29], for more advanced numerical and theoretical results the reader may want to consult [5] and [32].

Here, we admit solutions of bounded variation and obtain a global existence and uniqueness result using ideas from the theory of linear generalized ODEs. Furthermore, we propose stability concepts for Volterra integral equations based on the existing ones for generalized ODEs mentioned in the previous chapter.

3.1 Integral equations as generalized ODEs

In this section, we show a correspondence result between solutions of generalized ODEs and solutions of a class of Volterra integral equations of the second kind.

Let $t \in [a, b] \subset \mathbb{R}$. An equation of the form

$$x(t) = \int_a^t K(t, x(s), s) du(s) + P(t), \quad t \in [a, b], \quad (3.1)$$

is a Volterra integral equation of the second kind. Here, $K : [a, b] \times \Omega \times [a, b] \rightarrow X$ is a function, where Ω is an open subset of the arbitrary Banach space X , $u : [a, b] \rightarrow \mathbb{R}$ and $P : [a, b] \rightarrow X$ are both functions of bounded variation in $[a, b]$. A function $x : [a, b] \rightarrow \Omega$ is a solution of this equation, whenever it satisfies (3.1).

If the integral in (3.1) exists, when considered in the Kurzweil-Henstock-Stieltjes' sense, then it can be approximated by the following Riemann sum

$$\sum_{i=1}^{|D|} K(t, x(\tau_i), \tau_i)(u(s_i) - u(s_{i-1})).$$

In other words, for sufficiently fine divisions $D = \{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, |D|\}$ of $[a, t]$, where $t \in [a, b]$, the sum above converges to the integral term given by equation (3.1).

In order to express the integral equation (3.1) as a generalized ODE, we define

$$F(t, x(\tau), s) = K(t, x(\tau), \tau) \chi_{[a, t]}(s) u(s). \quad (3.2)$$

Thus, given any tagged division $\tilde{D} = \{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, |\tilde{D}|\}$ of $[a, b]$ where $t = s_m$ for some positive integer m , with $0 \leq m \leq |\tilde{D}|$, we have

$$\sum_{i=1}^{|\tilde{D}|} (F(t, x(\tau_i), s_i) - F(t, x(\tau_i), s_{i-1})) = \sum_{i=1}^m K(t, x(\tau_i), \tau_i)(u(s_i) - u(s_{i-1})).$$

Therefore, for sufficiently fine divisions of $[a, b]$ containing t as a node, the K-integral

$$\int_a^b D_s F(t, x(\tau), s)$$

and the Kurzweil-Henstock-Stieltjes integral

$$\int_a^t K(t, x(s), s) du(s)$$

coincide. This means equation (3.1) is equivalent to the perturbed generalized ODE

$$\frac{dx}{d\tau} = D_s[F(t, x, s) + P(s)\chi_{[a,t]}(s)] \quad (3.3)$$

with initial condition $x(a) = P(a)$.

Thus, we have proved the following result.

Theorem 3.1. *Let $K : [a, b] \times \Omega \times [a, b] \rightarrow X$ be a function, where Ω is an open subset of the Banach space X . Assume the functions $u : [a, b] \rightarrow \mathbb{R}$ and $P : [a, b] \rightarrow X$ are of bounded variation in $[a, b]$. A function $x : [a, b] \rightarrow X$ is a solution of*

$$x(t) = \int_a^t K(t, x(s), s) du(s) + P(t), \quad t \in [a, b],$$

if, and only if, x is a solution of the generalized ODE

$$\frac{dx}{d\tau} = D_s[F(t, x, s) + P(s)\chi_{[a,t]}(s)]$$

with initial condition $x(a) = P(a)$, where $F : [a, b] \times \Omega \times [a, b] \rightarrow X$ is given by

$$F(t, x(\tau), s) = K(t, x(\tau), \tau)\chi_{[a,t]}(s)u(s).$$

Before proceeding, we would like to make clear that considering arbitrary tagged divisions of $[a, b]$ or the ones containing a fixed $t \in [a, b]$ as a node result in the same integral. In other words, given any tagged division of $[a, b]$, we can replace $(\tau_\xi, [s_{\xi-1}, s_\xi])$, where $t \in [s_{\xi-1}, s_\xi]$, by $(t, [s_{\xi-1}, t])$ and $(t, [t, s_\xi])$. Indeed, given $\varepsilon > 0$, let δ be the gauge on $[a, b]$ that makes F K-integrable. If $\delta(t) \geq \delta(\tau_\xi)$, then we can assume $(\tau_\xi, [s_{\xi-1}, s_\xi])$, $(t, [s_{\xi-1}, t])$ and $(t, [t, s_\xi])$

are δ -fine tagged intervals and, by the Saks-Henstock Lemma (Lemma 1.6), we have

$$\left\| F(t, \tau_\xi, s_\xi) - F(t, \tau_\xi, s_{\xi-1}) - \int_{\xi-1}^{\xi} D_s F(t, x(\tau), s) \right\| < \varepsilon$$

and

$$\begin{aligned} & \left\| F(t, t, s_\xi) - F(t, t, t) + F(t, t, t) - F(t, t, s_{\xi-1}) - \right. \\ & \quad \left. - \left(\int_{\xi-1}^t D_s F(t, x(\tau), s) + \int_t^{\xi} D_s F(t, x(\tau), s) \right) \right\| = \\ & = \left\| F(t, t, s_\xi) - F(t, t, t) + F(t, t, t) - F(t, t, s_{\xi-1}) - \int_{\xi-1}^{\xi} D_s F(t, x(\tau), s) \right\| < \varepsilon, \end{aligned}$$

which means that the integral does not change.

If $\delta(t) < \delta(\tau_\xi)$, $t > \tau_\xi$ and $t + \delta(t) \geq s_\xi$, then we can replace $(\tau_\xi, [s_{\xi-1}, s_\xi])$ by $(\tau_\xi, [s_{\xi-1}, t])$ and $(t, [t, s_\xi])$. The interval $[t, s_\xi]$ requires further splitting when $t + \delta(t) < s_\xi$. But in both cases, the reasoning follows as above. The case $\delta(t) < \delta(\tau_\xi)$ and $t < \tau_\xi$ is analogous to the previous ones.

Lemma 3.2. Assume $P : [a, b] \rightarrow X$ and $u : [a, b] \rightarrow \mathbb{R}$ are functions of bounded variation in $[a, b]$ and $K : [a, b] \times \Omega \times [a, b] \rightarrow X$ satisfies the following conditions:

- (i) For all $x \in G([a, b], \Omega)$, the function $K(t, x(s), s)$ is K -integrable with respect to u .
- (ii) There is a Lebesgue-Stieltjes integrable function $V : [a, b] \rightarrow \mathbb{R}$ with respect to u such that, for every $x \in G([a, b], \Omega)$ and $t_1, t_2 \in [a, b]$, we have

$$\left\| \int_a^{t_2} K(t_2, x(s), s) du(s) - \int_a^{t_1} K(t_1, x(s), s) du(s) \right\| \leq \int_{t_1}^{t_2} V(s) du(s).$$

- (iii) There is a Lebesgue-Stieltjes integrable function $W : [a, b] \rightarrow \mathbb{R}$ with respect to u such that, for every $x, y \in G([a, b], \Omega)$ and $t_1, t_2 \in [a, b]$, we have

$$\left\| \int_a^{t_2} (K(t_2, x(s), s) - K(t_2, y(s), s)) du(s) - \int_a^{t_1} (K(t_1, x(s), s) - K(t_1, y(s), s)) du(s) \right\| \leq$$

$$\leq \int_{t_1}^{t_2} W(s) \|x - y\|_{\infty} du(s).$$

Then $F : G \rightarrow X$ given by (3.2) is such that $F(t, x(\tau), s) + P(s)\chi_{[a,t]}(s) \in \mathcal{F}(G, h)$, where $G = [a, b] \times \Omega \times [a, b]$, and $h : [a, b] \rightarrow \mathbb{R}$ is a nondecreasing function given by

$$h(t) = \int_a^t (V(s) + W(s)) du(s) + \text{var}_a^t P(s).$$

Proof. Condition (i) and the proof of Theorem 3.1 ensure that the integral $\int_a^b D_s F(t, x(\tau), s)$ exists. Given $x \in G([a, b], X)$ and $t_1, t_2 \in [a, b]$, it follows by condition (ii) that

$$\begin{aligned} & \left\| \int_a^b D_s [F(t_2, x(\tau), s) + P(s)\chi_{[a,t_2]}(s) - F(t_1, x(\tau), s) - P(s)\chi_{[a,t_1]}(s)] \right\| \leq \\ & \leq \left\| \int_a^{t_2} K(t_2, s, x(s)) du(s) - \int_a^{t_1} K(t_1, s, x(s)) du(s) \right\| + \|P(t_2) - P(a) - P(t_1) + P(a)\| \leq \\ & \leq \int_{t_1}^{t_2} V(s) du(s) + \text{var}_{t_1}^{t_2} P(s) = |h(t_2) - h(t_1)|. \end{aligned}$$

Similarly, by condition (iii),

$$\begin{aligned} & \left\| \int_a^b D_s [F(t_2, x(\tau), s) + P(s)\chi_{[a,t_2]}(s) - F(t_2, y(\tau), s) - P(s)\chi_{[a,t_2]}(s) - \right. \\ & \quad \left. - F(t_1, x(\tau), s) - P(s)\chi_{[a,t_1]}(s) + F(t_1, y(\tau), s) + P(s)\chi_{[a,t_1]}(s)] \right\| = \\ & = \left\| \int_a^{t_2} (K(t_2, s, x(s)) - K(t_2, s, y(s))) du(s) - \int_a^{t_1} (K(t_1, s, x(s)) - K(t_1, s, y(s))) du(s) \right\| \leq \\ & \leq \int_{t_1}^{t_2} W(s) \|x - y\|_{\infty} du(s) \leq \|x - y\|_{\infty} |h(t_2) - h(t_1)|, \end{aligned}$$

for all $x, y \in G([a, b], X)$ and $t_1, t_2 \in [a, b]$. □

The conditions (i), (ii) and (iii) play an important role for obtaining a correspondence of solutions result between generalized ODEs and several other types of differential equations such as measure functional differential equations, functional dynamic equations on time

scales, retarded functional differential equations with variable impulses, among others. The papers [1], [13] and [15] deal with such equations via generalized ODEs.

Now, we can apply our local existence and uniqueness result for generalized ODEs, namely Theorem 1.19, in order to obtain a new existence-uniqueness result for Volterra integral equations of the second kind.

Theorem 3.3. *Let $K : [a, b] \times \Omega \times [a, b] \rightarrow X$ be a function that satisfies conditions (i), (ii), (iii) from Lemma 3.2. Let the functions $u : [a, b] \rightarrow \mathbb{R}$ and $P : [a, b] \rightarrow X$ be of bounded variation in $[a, b]$ and assume that, for each $\sigma \in [a, b]$, we have*

$$P(\sigma+) + K(\sigma+, P(\sigma), \sigma)(u(\sigma+) - u(\sigma)) \in \Omega. \quad (3.4)$$

Then, for each $\sigma \in [a, b]$, there exist $\Delta = \Delta(\sigma) > 0$ and a unique function $x : [\sigma, \sigma + \Delta] \rightarrow \Omega$ such that x satisfies

$$x(t) = \int_{\sigma}^t K(t, x(s), s) du(s) + P(t), \quad \text{for } t \in [\sigma, \sigma + \Delta]. \quad (3.5)$$

Proof. We know, from Theorem 3.1, that

$$x(t) = \int_{\sigma}^t K(t, x(s), s) du(s) + P(t), \quad t \in [0, \infty),$$

is equivalent to the perturbed generalized ODE

$$\frac{dx}{d\tau} = D_s[F(t, x, s) + P(s)\chi_{[\sigma, t]}(s)] \quad (3.6)$$

with initial condition $x(\sigma) = P(\sigma)$, where $F : [a, b] \times \Omega \times [a, b] \rightarrow X$ is given by

$$F(t, x(\tau), s) = K(t, x(\tau), \tau)\chi_{[a, t]}(s)u(s).$$

Since all the assumptions of Lemma 3.2 are satisfied, we can conclude the right-hand side of (3.6) belongs to $\mathcal{F}(G, h)$, where $G = [a, b] \times \Omega \times [a, b]$ and $h : [a, b] \rightarrow \mathbb{R}$ is given by $h(t) = \int_a^t (V(s) + W(s)) du(s) + \text{var}_a^t P(s)$.

We need to verify whether it is possible to continue the solution of (3.5) to the right. We have

$$\begin{aligned} x(\sigma+) &= x(\sigma) + F(\sigma+, x(\sigma), \sigma) - F(\sigma, x(\sigma), \sigma) = \\ &= P(\sigma) + K(\sigma, P(\sigma), \sigma)u(\sigma) + P(\sigma) - K(\sigma, P(\sigma), \sigma)u(\sigma) - P(\sigma) = \\ &= P(\sigma) + K(\sigma, P(\sigma), \sigma)(u(\sigma) - u(\sigma)), \end{aligned}$$

which means that $x(\sigma+) \in \Omega$. Therefore we can apply Theorem 1.19 to get the result. $\square$

Example 3.4. Let $K : [a, b] \times [a, b] \rightarrow L(X)$, $P : [a, b] \rightarrow X$ and $u : [a, b] \rightarrow \mathbb{R}$ be functions. Given $\sigma \in [a, b]$, we are going to apply Theorem 3.3 to the linear Volterra integral equation

$$x(t) = P(t) + \int_{\sigma}^t K(t, s)x(s)du(s), \quad (3.7)$$

where t belongs to some interval containing $\sigma \in [a, b]$.

At first, note that the solution x takes values in X , that is, there is no open Ω where we need to verify whether the solution can be continued to the right. Hence we take $\Omega = X$ and (3.4) is always satisfied.

We assume $u : [a, b] \rightarrow \mathbb{R}$ and $P : [a, b] \rightarrow \mathbb{R}$ are functions of bounded variation in $[a, b]$ and the integral term in (3.7) is well-defined as a K -integral (Definition 1.3) for every $x \in G([a, b], X)$. Further, assume there exists a Lebesgue-Stieltjes integrable function $W : [a, b] \rightarrow \mathbb{R}$ with respect to u such that

$$\|K(t, s)\|_{\infty} \leq W(s),$$

for every $t \in [a, b]$. In order to simplify the discussion, we restrict the candidates for a solution of (3.7) to the functions x in $G([a, b], X)$ with a fixed bound $\|x\|_\infty \leq A$, for some $A > 0$. This subset of $G([a, b], X)$ is clearly closed, hence it is also a Banach space.

The integrability condition in Lemma 3.2 (condition (i)) is clearly satisfied. For condition (ii), we have

$$\begin{aligned}
& \left\| \int_a^{t_2} K(t_2, s)x(s)du(s) - \int_a^{t_1} K(t_1, s)x(s)ds \right\| \leq \\
& \leq \varepsilon + \sum_{i=1}^{|D_1|} \|K(t_2, \tau_i)x(\tau_i)\| |u(s_i) - u(s_{i-1})| + \varepsilon + \sum_{j=1}^{|D_2|} \|K(t_1, \xi_j)x(\xi_j)\| |u(r_j) - u(r_{j-1})| \leq \\
& \leq 2\varepsilon + A \sum_{i=1}^{|D_1|} W(\tau_i) |u(s_i) - u(s_{i-1})| + A \sum_{j=1}^{D_2} W(\xi_j) |u(r_j) - u(r_{j-1})| \leq \\
& \leq 4\varepsilon + A \int_a^{t_2} W(s)du(s) + A \int_a^{t_1} W(s)du(s) = 4\varepsilon + \int_{t_1}^{t_2} AW(s)du(s),
\end{aligned}$$

where $\varepsilon > 0$ is arbitrary, $t_1, t_2 \in [a, b]$ and the tagged divisions $D_1 = \{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, |D_1|\}$ of $[a, t_2]$ and $D_2 = \{(\xi_j, [r_{j-1}, r_j]), j = 1, \dots, |D_2|\}$ of $[a, t_1]$ are sufficiently fine.

Similarly, for condition (iii), we have

$$\begin{aligned}
& \left\| \int_a^{t_2} K(t_2, s)(x(s) - y(s))du(s) - \int_a^{t_1} K(t_1, s)(x(s) - y(s))du(s) \right\| \leq \\
& \leq \varepsilon + \sum_{i=1}^{|D_1|} \|K(t_2, \tau_i)(x(\tau_i) - y(\tau_i))\| |u(s_i) - u(s_{i-1})| + \\
& + \varepsilon + \sum_{j=1}^{|D_2|} \|K(t_1, \xi_j)(x(\xi_j) - y(\xi_j))\| |u(r_j) - u(r_{j-1})| \leq \\
& \leq 2\varepsilon + \sum_{i=1}^{|D_1|} W(\tau_i) \|x - y\|_\infty |u(s_i) - u(s_{i-1})| + \sum_{j=1}^{|D_2|} W(\xi_j) \|x - y\|_\infty |u(r_j) - u(r_{j-1})| \leq \\
& \leq 4\varepsilon + \int_{t_1}^{t_2} W(s) \|x - y\|_\infty du(s),
\end{aligned}$$

where $\varepsilon > 0$ is arbitrary, $t_1, t_2 \in [a, b]$ and the tagged divisions $D_1 = \{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, |D_1|\}$ of $[a, t_2]$ and $D_2 = \{(\xi_j, [r_{j-1}, r_j]), j = 1, \dots, |D_2|\}$ of $[a, t_1]$ are sufficiently fine.

Therefore Theorem 3.3 yields that the integral equation (3.7) admits a unique solution $x : [\sigma, \sigma + \Delta] \rightarrow X$ for some $\Delta > 0$ satisfying $\|x\|_\infty < A$.

Now, we make a few comments about the K-integrability of $K(t, x(s), s)$ with respect to $u : [a, b] \rightarrow \mathbb{R}$. If $x : [a, b] \rightarrow \Omega$ is a step function, then we can apply Lemma 1.8 to conclude the integral $\int_a^t D_s K(t, x(s), s) du(s)$ exists, with $t \in [a, b]$ and F given by (3.2). Thus, if $K : G \rightarrow X$ satisfies conditions (ii) and (iii) from Lemma 3.2, whenever x is a step function, then Theorem 1.18 ensures $K(t, x(s), s)$ is integrable with respect to u for every $x \in G([a, b], \Omega)$.

3.2 A global existence and uniqueness result

In this section, we use some ideas from the theory of linear Stieltjes integral equations, found in the paper [37], to create a global existence and uniqueness of solutions result for Volterra integral equations. At first, we need to define a new class of functions.

We remind the reader X is a Banach space, Ω is an open subset of X and $G = [a, b] \times \Omega \times [a, b]$.

Definition 3.5. Let the function $u : [a, b] \rightarrow \mathbb{R}$ be of bounded variation and $W : [a, b] \rightarrow \mathbb{R}$ be a Lebesgue-Stieltjes integrable function with respect to u . Assume $K : G \rightarrow X$ belongs to the class $\mathcal{A}(G, u)$ if, and only if, the function $K(t, x(s), s)$ is K-integrable with respect to u and, further, the following inequality

$$\left\| \int_{r_1}^{r_2} (K(t, x(s), s) - K(t, y(s), s)) du(s) \right\| \leq \int_{r_1}^{r_2} W(s) \|x(s) - y(s)\| du(s) \quad (3.8)$$

holds for all $r_1, r_2, t \in [a, b]$ and $x, y \in G([a, b], X)$.

This new class of functions and a boundedness condition under the above function W lead to a local existence and uniqueness of solutions result for the Volterra integral equation

$$x(t) = \int_a^t K(t, x(s), s) du(s) + P(t), \quad (3.9)$$

where $t \in [a, b]$ and $P : [a, b] \rightarrow X$ is a regulated function. Its proof follows the line of thinking of Proposition 2.4 from [37].

Theorem 3.6. *Assume that $K : G \rightarrow X$ belongs to $\mathcal{A}(G, u)$ and the functions $u : [a, b] \rightarrow \mathbb{R}$ and $W : [a, b] \rightarrow \mathbb{R}$ are such that, for every $\sigma \in [a, b]$, there exist $\rho = \rho(\sigma) \in (0, 1)$ and $\Delta = \Delta(\sigma) > 0$ satisfying*

$$(\text{var}_{\sigma-\Delta}^{\sigma+\Delta} u) \sup_{s \in [\sigma-\Delta, \sigma+\Delta]} (W(s)) < \rho. \quad (3.10)$$

Then, for every $\sigma \in [a, b]$ and $P \in G([a, b], X)$, there is a unique function $x \in G(J_\sigma, X)$, defined on the interval $J_\sigma = [\sigma - \Delta, \sigma + \Delta] \cap [a, b]$, such that

$$x(t) = \int_\sigma^t K(t, x(s), s) du(s) + P(t) \quad t \in J_\sigma.$$

Proof. At first, we remind the reader that the set $G(J_\sigma, X)$, endowed with the supremum norm, is a Banach space. Let $Q = \{x \in G(J_\sigma, X) : x(\sigma) = P(\sigma)\}$. This set is closed because, given a sequence $\{x_k\}_{k \in \mathbb{N}}$ in Q that converges to x , we have

$$\|x(\sigma) - P(\sigma)\| = \|x(\sigma) - x_k(\sigma)\| \leq \|x - x_k\|_\infty \leq \varepsilon,$$

where $\varepsilon > 0$ is arbitrary and the last inequality is valid for sufficiently large positive integers k .

Let $\Delta = \Delta(\sigma) > 0$ be the value given by the assumption (3.10). Define the operator

$$Tx(t) = \int_\sigma^t K(t, x(s), s) du(s) + P(t), \quad t \in J_\sigma,$$

on Q . Then,

$$Tx(t) - Ty(t) = \int_{\sigma}^t (K(t, x(s), s) - K(t, y(s), s)) du(s), \quad t \in J_{\sigma}.$$

Since $K \in \mathcal{A}(G, u)$, we can write

$$\|Tx(t) - Ty(t)\| = \left\| \int_{\sigma}^t (K(t, x(s), s) - K(t, y(s), s)) du(s) \right\| \leq \int_{\sigma}^t W(s) \|x(s) - y(s)\| du(s)$$

for some locally Lebesgue-Stieltjes integrable real function W and $t \in J_{\sigma}$. If also $t < \sigma$, we can apply Theorem 1.7 (Hake-type Theorem) to obtain

$$\begin{aligned} \|Tx(t) - Ty(t)\| &\leq \\ &\leq \lim_{\delta \rightarrow 0+} \left[\int_t^{\sigma-\delta} W(s) \|y(s) - x(s)\| du(s) + W(\sigma) \|y(\sigma) - x(\sigma)\| (u(\sigma) - u(\sigma - \delta)) \right] = \\ &= \lim_{\delta \rightarrow 0+} \left[\int_t^{\sigma-\delta} W(s) \|y(s) - x(s)\| du(s) \right]. \end{aligned}$$

Note that $x(\sigma) = y(\sigma) = P(\sigma)$, because $x, y \in Q$. From the definition of the Kurzweil integral, we obtain the following estimate

$$\begin{aligned} &\left| \int_t^{\sigma-\delta} W(s) \|y(s) - x(s)\| du(s) \right| \leq \\ &\leq \left| \int_t^{\sigma-\delta} W(s) \|y(s) - x(s)\| du(s) - \sum_{i=1}^{|D|} W(\tau_i) \|y(\tau_i) - x(\tau_i)\| (u(s_i) - u(s_{i-1})) \right| + \\ &\quad + \left| \sum_{i=1}^{|D|} W(\tau_i) \|y(\tau_i) - x(\tau_i)\| (u(s_i) - u(s_{i-1})) \right| \leq \\ &\leq \varepsilon + \left| \sum_{i=1}^{|D|} W(\tau_i) \|y(\tau_i) - x(\tau_i)\| (u(s_i) - u(s_{i-1})) \right| \end{aligned}$$

where $\varepsilon > 0$ comes from the definition of the Kurzweil integral and $D = \{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, |D|\}$ is a sufficiently fine tagged division of $[t, \sigma - \delta]$. Since ε is arbitrary, we obtain

$$\begin{aligned} \left| \int_t^{\sigma-\delta} W(s) \|y(s) - x(s)\| du(s) \right| &\leq \sup_{s \in [t, \sigma-\delta]} W(s) \|x - y\|_\infty |u(\sigma - \delta) - u(t)| \leq \\ &\leq \sup_{s \in [t, \sigma-\delta]} W(s) \|x - y\|_\infty (var_{\sigma-\delta}^t u). \end{aligned}$$

Thus, from (3.10), we conclude

$$\begin{aligned} \|Tx(t) - Ty(t)\| &\leq \lim_{\delta \rightarrow 0^+} \left| \int_t^{\sigma-\delta} W(s) \|y(s) - x(s)\| du(s) \right| \leq \\ &\leq \lim_{\delta \rightarrow 0^+} \sup_{s \in [t, \sigma-\delta]} W(s) \|x - y\|_\infty (var_{\sigma-\delta}^t u) \leq \\ &\leq (var_{\sigma-\Delta}^{\sigma+\Delta} u) \sup_{s \in [\sigma-\Delta, \sigma+\Delta]} W(s) \|x - y\|_\infty \leq \rho \|x - y\|_\infty. \end{aligned}$$

For $t \in J_\sigma$ and $\sigma < t$, we can show the same inequality similarly. Hence, passing to the supremum over $t \in J_\sigma$, we get

$$\|Tx - Ty\|_\infty \leq \rho \|x - y\|_\infty.$$

Therefore, the operator T acting on Q is a contraction and, by the Banach Fixed Point Theorem (see [26]), it has a unique fixed point, i.e. there is $x \in G(J_\sigma, X)$ such that

$$x(t) = \int_\sigma^t K(t, x(s), s) du(s) + P(t), \quad t \in J_\sigma,$$

and, thus, we conclude the proof. $\square$

In order to prove a global existence and uniqueness result for the solution of (3.9), we need to impose more restrictions on the discontinuities of the function u and the operator K . At first, we remind the reader that, given $t \in [a, b]$, every function of bounded variation

$u : [a, b] \rightarrow \mathbb{R}$ is regulated on its domain (check [37]). Moreover, for every $\varepsilon > 0$, the sets

$$\{s \in [a, b]; \Delta^+ u(s) = u(s+) - u(s) \geq \varepsilon\} \quad (3.11)$$

and

$$\{s \in (a, b]; \Delta^- u(s) = u(s) - u(s-) \geq \varepsilon\} \quad (3.12)$$

are finite, otherwise $u(b) - u(a)$ would not be finite and, consequently, u would not be of bounded variation. Also, the set of discontinuity points of u is at most countable (see [33]).

Let $a \leq \sigma < t \leq b$. The next step is to define a function $L : [\sigma, t] \rightarrow \mathbb{R}$ involving the discontinuities of u ,

$$L(\sigma) = 0; \quad L(s) = \Delta^- u(s), \quad \text{for } s \in (\sigma, t]. \quad (3.13)$$

The result below is based on Lemma 2.5 and Corollary 2.6 from [37]. Its proof can be carried out with simple adaptations.

Lemma 3.7. *Let $0 \leq \sigma < t < \infty$. The function $L : [\sigma, t] \rightarrow \mathbb{R}$, given by (3.13), belongs to $BV([\sigma, t], \mathbb{R})$.*

Lemma 3.8 (Proposition 2.7 from [37]). *Let $0 \leq \sigma < t < \infty$. If L is given by (3.13), then, for every $x \in G_0([\sigma, \infty), \Omega)$ and $K : G \rightarrow X$, with $K \in \mathcal{A}(G, u)$, the Perron-Stieltjes integral $\int_{\sigma}^t K(t, x(s), s) dL(s)$ exists and*

$$\int_{\sigma}^t K(t, x(s), s) dL(s) = K(t, x(t), t) L(t) = K(t, x(t), t) \Delta^- u(t),$$

for $t \in (\sigma, \infty)$.

Sketch of the proof. The existence of the integral, because $K \in \mathcal{A}(G, u)$. Next, we use the fact that the set, where u is continuous, is dense in $[a, b]$ to build a special gauge δ . This gauge is used to show the integral above only depends on the right endpoint t . After applying Theorem 1.7 (Hake-type Theorem), we get the result.

Let $a \leq \sigma < t \leq b$. Now, we consider a function $B : [\sigma, t] \rightarrow \mathbb{R}$ given by

$$B(\sigma) = u(\sigma); \quad B(t) = u(t-), \quad \text{for } t \in (\sigma, t]. \quad (3.14)$$

Since $B(s) = u(s) - L(s)$ for every $s \in [\sigma, t]$, we can use the previous lemma to obtain

$$\int_{\sigma}^t K(t, x(s), s) du(s) = \int_{\sigma}^t K(t, x(s), s) dB(s) + K(t, x(t), t) \Delta^- u(t). \quad (3.15)$$

Note that B is left-continuous on $(\sigma, t]$, hence we can apply Theorem 1.7 (Hake-type Theorem) to write

$$\lim_{r \rightarrow t-} \int_{\sigma}^r K(t, x(s), s) dB(s) = \int_{\sigma}^t K(t, x(s), s) dB(s),$$

which means that the integral $\int_{\sigma}^t K(t, x(s), s) dB(s)$ does not depend on the value $K(t, x(t), t)$.

Before proceeding, we introduce some new notation. For each $t \in [a, b]$, let us define a new subset of X , namely

$$G_t = \{y \in X; y = x(t) \text{ for some } x \in G([a, b], \Omega)\}$$

and a function $\tilde{K}_t : G_t \rightarrow X$ given by

$$\tilde{K}_t(y) = K(t, y, t) = K(t, x(t), t), \quad \text{for some } x \in G([a, b], \Omega). \quad (3.16)$$

Theorem 3.9 (Proposition 2.8 from [37]). *Let $\sigma \in [a, b]$. Assume that $P \in G([a, b], X)$, the function $u : [a, b] \rightarrow \mathbb{R}$ is of bounded variation in $[a, b]$, $K : G \rightarrow X$ belongs to the class $\mathcal{A}(G, u)$ and*

$$(I - \Delta^- u(t) \tilde{K}_t) : G_t \rightarrow X,$$

where $\tilde{K}_t$ is given by (3.16) and I is the identity function, admits an inverse for every $t \in (a, b]$. Further, assume the functions u and $W : [a, b] \rightarrow \mathbb{R}$ satisfy condition (3.10). Then

there is $x \in G([\sigma, b], \Omega)$ such that

$$x(t) = \int_{\sigma}^t K(t, x(s), s) du(s) + P(t) \quad (3.17)$$

for every $r \in [\sigma, b]$.

Proof. By the local existence result, given in Theorem 3.6, there is $\Delta > 0$ such that there is a function $x \in G([\sigma, \sigma + \Delta], \Omega)$ satisfying (3.17) for $r \in [\sigma, \sigma + \Delta]$. For each $t \in [\sigma, b]$, define

$$t^* = \sup_{T \in (\sigma, b]} \{ \exists x \in G([\sigma, T], \Omega) \text{ such that (3.17) holds for } t \in [\sigma, T] \}. \quad (3.18)$$

If $t^* = b$, then there is nothing left to prove. If $\sigma < t^* < b$, then there is a function $x : [\sigma, t^*) \rightarrow X$ such that (3.17) holds for every $t \in [\sigma, t^*)$.

Consider

$$x(t^*) = [I - \Delta^- u(t^*) \tilde{K}_{t^*}]^{-1} \left(\int_{\sigma}^{t^*} K(t^*, x(s), s) dB(s) + P(t^*) \right), \quad (3.19)$$

where B is given by (3.14) and $\tilde{K}_{t^*}$, by (3.16). Since $\int_{\sigma}^{t^*} K(t, x(s), s) dB(s)$ depends only on the values of $K(s, x(s), s) = \tilde{K}_s(x)$ at $s \in [\sigma, t^*)$ and $\tilde{K}_{t^*}(x)$ admits an inverse, the value $x(t^*) \in X$ is well-defined and we can write

$$[I - \Delta^- u(t^*) \tilde{K}_{t^*}] x(t^*) = \int_{\sigma}^{t^*} K(t^*, x(s), s) dB(s) + P(t^*).$$

Then, by (3.15), we get

$$x(t^*) = \int_{\sigma}^{t^*} K(t^*, x(s), s) du(s) + P(t^*),$$

where t^* is given by (3.18). Hence, the function x , with $x(t^*)$ given by (3.19), is a solution of the integral equation (3.17) on $[\sigma, t^*]$.

If we set $\sigma = t^*$, then the local existence result (Theorem 3.6) can be used again to show there exists $\Delta(t^*) > 0$ such that there exists a solution $y \in G([t^*, t^* + \Delta(t^*)], X)$ of

$$y(t) = \int_{t^*}^t K(t, s, x(s)) du(s) + P(t), \quad t \in [t^*, t^* + \Delta(t^*)] \cap [\sigma, b].$$

We can assume, without loss of generality, that $t^* + \Delta(t^*) < b$. Putting $z(t) = x(t)$ for $t \in [\sigma, t^*]$ and $z(t) = y(t)$ for $t \in [t^*, t^* + \Delta(t^*)]$, we obtain a solution of (3.17) on $[\sigma, t^* + \Delta(t^*)]$, which contradicts the properties of the supremum and, therefore, $t^* = t$. $\square$

In a completely analogous way, the following statement can be proved.

Theorem 3.10. *Let $\sigma \in [a, b]$. Assume that $P \in G([a, b], X)$, the function $u : [a, b] \rightarrow \mathbb{R}$ is of bounded variation in $[a, b]$, $K : G \rightarrow X$ belongs to the class $\mathcal{A}(G, u)$ and*

$$(I - \Delta^+ u(t) \tilde{K}_t) : G_t \rightarrow X,$$

where $\tilde{K}_t$ is given by (3.16) and I is the identity function, admits an inverse for every $t \in [a, b]$. Further, assume the functions u and $W : [a, b] \rightarrow \mathbb{R}$ satisfy condition (3.10). Then there is $x \in G([a, \sigma], \Omega)$ such that

$$x(t) = \int_{\sigma}^t K(t, x(s), s) du(s) + P(t)$$

for every $t \in [a, \sigma]$.

Before proceeding, we present a situation where the operators $I - \Delta^+ u(t) \tilde{K}_t$ and $I - \Delta^- u(t) \tilde{K}_t$ always admit an inverse in their domains. At first, note that if u is a continuous function, then these operators are equal to the identity function and, therefore, always admit an inverse, regardless of the function K . The next example deals with a different case.

Example 3.11. *Let $K : [a, b] \times \Omega \times [a, b] \rightarrow X$ and $\hat{K} : \Omega \rightarrow X$ be functions such that*

$$K(t, x(s), s) = \hat{K}(x(t - s)),$$

where $x \in G([a, b], \Omega)$ and $0 \in [a, b]$. These convolution-type functions are important for applications. More information about these functions can be found in next chapter and in the books [5] and [29].

Let $t \in [a, b]$. Then, for every $y \in G_t$, $y = x(t)$ for some $x \in G([a, b], \Omega)$ and

$$\begin{aligned} (I - \Delta^- u(t) \tilde{K}_t)(y) &= (I - \Delta^- u(t) \tilde{K}_t)(x(t)) = x(t) - \Delta^- u(t) K(t, x(t), t) = \\ &= x(t) - \Delta^- u(t) \hat{K}(x(0)) = y - \Delta^- u(t) \hat{K}(x(0)). \end{aligned}$$

Hence the operator $I - \Delta^- u(t) \tilde{K}_t$ is actually the identity plus a constant and, therefore, this operator always admit an inverse. The operator $I - \Delta^+ u(t) \tilde{K}_t$ can be treated similarly.

The next result puts together the forward solution given by Theorem 3.9 and the backward solution given by Theorem 3.10. Moreover, it describes the unique solution of (3.9).

Theorem 3.12. Assume that $P \in G([a, b], X)$, the function $u : [a, b] \rightarrow \mathbb{R}$ is of bounded variation, $K : G \rightarrow X$ belongs to the class $\mathcal{A}(G, u)$, where $(I - \Delta^- u(t) \tilde{K}_t) : G_t \rightarrow X$ admits an inverse for every $t \in (a, b]$ and $(I - \Delta^+ u(t) \tilde{K}_t) : G_t \rightarrow X$ admits an inverse for every $t \in [a, b)$. Further, assume the functions u and $W : [a, b] \rightarrow \mathbb{R}$ satisfy condition (3.10). Then, for every choice of $\sigma \in [a, b]$, there is $x \in G([a, b], \Omega)$ such that

$$x(t) = \int_{\sigma}^t K(t, x(s), s) du(s) + P(t) \quad (3.20)$$

for every $t \in [a, b]$.

Proof. Given $\sigma \in [a, b]$, Theorem 3.9 can be used for proving the existence of a forward solution $y \in G([\sigma, b], X)$ and Theorem 3.10 can be used for proving the existence of a backward solution $z \in G([a, \sigma], X)$ of the equation (3.20). Taking

$$x(t) = \begin{cases} y(t), & t \in [\sigma, b], \\ z(t), & t \in [a, \sigma], \end{cases}$$

we get $x \in G([a, b], X)$, which satisfies (3.20) for all $t \in [a, b]$. $\square$

In order to prove uniqueness of a solution x in the above result, we need to strengthen the assumptions on the function u . We assume $u : [a, b] \rightarrow \mathbb{R}$ is a continuous nondecreasing function and prove that the solution x of the integral equation (3.20) is uniquely determined.

Let $x_1, x_2 \in G([a, b], X)$ be two solutions satisfying (3.20) for $t \in [a, b]$. Then the difference $z(t) = x_1(t) - x_2(t)$ satisfies

$$z(t) = \int_{\sigma}^t (K(t, x_1(s), s) - K(t, x_2(s), s)) du(s), \quad t \in [a, b].$$

Since $K \in \mathcal{A}(G, u)$, we obtain

$$\begin{aligned} \|z(t)\| &= \left\| \int_{\sigma}^t (K(t, x_1(s), s) - K(t, x_2(s), s)) du(s) \right\| \leq \\ &\leq \int_{\sigma}^t W(s) \|x_1(s) - x_2(s)\| du(s) = \int_{\sigma}^t W(s) \|z(s)\| du(s), \quad t \in [a, b]. \end{aligned}$$

On the other hand, the function u is continuous. Hence, for every $\varepsilon > 0$, there is $\rho > 0$ such that $|u(t) - u(\sigma)| < \varepsilon$, whenever $|\sigma - t| < \rho$. Thus,

$$\begin{aligned} \|z(t)\| &\leq \int_{\sigma}^t W(s) \|z(s)\| du(s) \leq \\ &\leq \varepsilon + \sum_{i=1}^{|D|} W(\tau_i) \|z(\tau_i)\| |u(s_i) - u(s_{i-1})|, \quad t \in (\sigma - \rho, \sigma + \rho), \end{aligned}$$

where D is a sufficiently fine tagged division of $[\sigma, t]$ (assuming $t > \sigma$, otherwise D is a tagged division of $[t, \sigma]$). Since u is nondecreasing and we can assume, without loss of generality, that W is bounded in $(\sigma - \rho, \sigma + \rho)$ (we are using condition (3.10) and taking a smaller ρ , if necessary), we obtain

$$\|z(t)\| \leq \varepsilon + M \|z\|_{\infty} |u(t) - u(\sigma)| \leq \varepsilon (1 + M \|z\|_{\infty}),$$

where $|W(s)| < M$ for $s \in (\sigma - \rho, \sigma + \rho)$. Since $\sigma \in [a, b]$ is arbitrary, we get $z(t) = 0$ for every $t \in [a, b]$, which yields uniqueness of a solution of the integral equation (3.20).

3.3 Stability concepts for Volterra integral equations

Now, we are going to discuss stability concepts for Volterra integral equations. Let $G = [a, b] \times \Omega \times [a, b]$, where $\Omega = B_c = \{x \in X; \|x\| < c, c > 0\}$. We remind the reader some notation from the previous section. For each $t \in [a, b]$, let us define

$$G_t = \{y \in X; y = x(t), \text{ for some } x \in G([a, b], \Omega)\}$$

and $\tilde{K}_t : G_t \rightarrow X$ given by

$$\tilde{K}_t(y) = K(t, y, t) = K(t, x(t), t), \text{ for some } x \in G([a, b], \Omega). \quad (3.21)$$

Assume the operators $(I - \Delta^- u(t) \tilde{K}_t) : G_t \rightarrow X$ and $(I - \Delta^+ u(t) \tilde{K}_t) : G_t \rightarrow X$ admit an inverse for every $t \in (a, b]$ and $t \in [a, b]$ respectively. Further, assume the functions $u : [a, b] \rightarrow \mathbb{R}$ and $W : [a, b] \rightarrow \mathbb{R}$ are such that, for every $\sigma \in [a, b]$, there exist $\rho = \rho(\sigma) \in (0, 1)$ and $\Delta = \Delta(\sigma) > 0$ satisfying

$$(var_{\sigma-\Delta}^{\sigma+\Delta} u) \sup_{s \in [\sigma-\Delta, \sigma+\Delta]} (W(s)) < \rho.$$

Moreover, assume $K : G \rightarrow X$ belongs to the class $\mathcal{A}(G, u)$ and satisfies $K(t, 0, s) = 0$ for all $t, s \in [a, b]$. Then $x \equiv 0$ is a solution of the Volterra integral equation

$$x(t) = \int_a^t K(t, x(s), s) du(s), \quad t \in [a, b], \quad (3.22)$$

where the function $u : [a, b] \rightarrow \mathbb{R}$ is left-continuous and nondecreasing.

Note that, for every regulated function $P : [a, b] \rightarrow X$, the following integral equation

$$y(t) = \int_a^t K(t, y(s), s) du(s) + P(t), \quad t \in [a, b], \quad (3.23)$$

admits at least one global solution, because of Theorem 3.12.

In the sequel, $G^-([a, b], X)$ represents the Banach space of regulated and left-continuous functions $x : [a, b] \rightarrow X$.

Definition 3.13. *The trivial solution $x \equiv 0$ of (3.22) is called stable with respect to perturbations, if, for every $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that if $P \in G^-([a, b], X)$ with*

$$\sup_{s \in [a, b]} \|P(s) - P(a)\| < \delta,$$

then $\bar{x} : [a, b] \rightarrow \Omega$ satisfies $\|\bar{x}(t)\| < \varepsilon$ on $t \in [a, b]$, where $\bar{x}$ is the solution of the integral equation (3.23) on $t \in [a, b]$.

Definition 3.14. *The trivial solution $x \equiv 0$ of (3.22) is called attracting with respect to perturbations, if, for every $\varepsilon > 0$, there is $T = T(\varepsilon) > 0$ and $\rho = \rho(\varepsilon) > 0$ such that if $P \in G^-([a, b], X)$ with*

$$\sup_{s \in [a, b]} \|P(s) - P(a)\| < \rho,$$

then $\bar{x} : [a, b] \rightarrow \Omega$ satisfies $\|\bar{x}(t)\| < \varepsilon$ on $t \in [T, b]$, where $\bar{x}$ is the solution of the integral equation (3.23) on $t \in [a, b]$.

Definition 3.15. *The trivial solution $x \equiv 0$ of (3.22) is called asymptotically stable with respect to perturbations, if it is both stable and attracting with respect to perturbations.*

We remind the reader that if $F : G \rightarrow X$ is given by

$$F(t, x(\tau), s) = K(t, x(\tau), \tau)u(s)\chi_{[a, t]}(s), \quad (3.24)$$

then the generalized ODE

$$\frac{dx}{d\tau} = D[F(t, x, s) + P(s)\chi_{[a,t]}(s)], \quad (3.25)$$

with initial condition $x(a) = P(a)$, is equivalent to the integral equation (3.23), because of Theorem 3.1. Moreover, F satisfies $F(t, 0, s) = 0$ for every $t, s \in [a, b]$. Hence $x \equiv 0$ is a solution of

$$\frac{dx}{d\tau} = DF(t, x, s). \quad (3.26)$$

The next result describes how these stability concepts are related to those introduced in the last chapter for generalized ODEs. The main ideas of the proof are borrowed from Theorem 5.1 of [15].

Theorem 3.16. *Suppose the function $K : G \rightarrow X$ satisfies all the hypotheses from the beginning of this section, $P : [a, b] \rightarrow X$ belongs to $G^-([a, b], X)$ and $u : [a, b] \rightarrow \mathbb{R}$ is nondecreasing and left-continuous. Let $F : G \rightarrow X$ be the function given by (3.24). Then the following statements hold:*

- (i) *The trivial solution $x \equiv 0$ of (3.22) is stable with respect to perturbations if, and only if, the trivial solution $x \equiv 0$ of (3.26) is regularly stable.*
- (ii) *The trivial solution $x \equiv 0$ of (3.22) is attracting with respect to perturbations if, and only if, the trivial solution $x \equiv 0$ of (3.26) is regularly attracting.*
- (iii) *The trivial solution $x \equiv 0$ of (3.22) is asymptotically stable with respect to perturbations if, and only if, the trivial solution $x \equiv 0$ of (3.26) is regularly asymptotically stable.*

Proof. At first, we prove (i). We assume the trivial solution of the integral equation (3.22) is stable with respect to perturbations. Then we prove that the trivial solution of the generalized ODE (3.26) is regularly stable with respect to perturbations. The result follows from Theorem 2.7, item (i).

Let $x : [a, b] \rightarrow \Omega$ be a solution of the perturbed generalized ODE (3.25) with initial condition $x(a) = P(a)$. Let $\varepsilon > 0$ and take $\delta = \delta(\varepsilon) > 0$ given by Definition 3.13. Assume

$P : [a, b] \rightarrow X$ belongs to $G^-([a, b], X)$, $P(a) < \delta$ and

$$\sup_{s \in [a, b]} \|P(s) - P(a)\| < \delta.$$

Since (3.25) is equivalent to (3.23), for F given by (3.24), and since we assume the trivial solution of (3.22) is stable with respect to perturbations, it follows that $\|x(t)\| < \varepsilon$ whenever $t \in [a, b]$. Hence, the trivial solution of (3.26) is regularly stable with respect to perturbations.

Conversely, suppose the trivial solution of (3.26) is regularly stable. Theorem 2.7, item (i), yields that $x \equiv 0$ is regularly stable with respect to perturbations.

Let $x : [0, \infty) \rightarrow \Omega$ be a solution of the integral equation (3.22). Let $\varepsilon > 0$ and take $\delta = \delta(\varepsilon) > 0$ given by Definition 2.4. Assume $P : [a, b] \rightarrow X$ belongs to $G^-([a, b], X)$, $P(a) < \delta$ and

$$\sup_{s \in [a, b]} \|P(s) - P(a)\| < \delta.$$

Since (3.23) is equivalent to (3.25), for F given by (3.24), and since we assume the trivial solution of (3.26) is stable with respect to perturbations, it follows that $\|x(t)\| < \varepsilon$ whenever $t \in [a, b]$. Hence the trivial solution of (3.22) is stable with respect to perturbations.

Item (ii) can be proved with similar reasoning. Item (iii) follows from items (i) and (ii). □

Now we discuss Lyapunov-type results for Volterra integral equations.

Definition 3.17. *We say that $U : [a, b] \times \overline{B}_\rho \rightarrow \mathbb{R}$ is a Lyapunov functional with respect to the Volterra integral equation (3.22), if the following conditions are satisfied:*

- (i) $U(\cdot, x) : [a, b] \rightarrow \mathbb{R}$ is continuous from the left on $(a, b]$, for every $x \in \overline{B}_\rho$;
- (ii) There is a continuous and strictly increasing function $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $b(0) = 0$ (we say that such function is of Hahn class) and

$$U(t, x) \geq b(\|x\|),$$

for every $t \in [a, b]$ and $x \in \overline{B}_\rho$;

(iii) For every solution $\bar{x} : [\gamma, v] \rightarrow \overline{B}_\rho$ of (3.22) with $[\gamma, v] \subset [a, b]$, we have

$$\dot{U}(t, \bar{x}(t)) = \limsup_{\eta \rightarrow 0^+} \frac{U(t + \eta, \bar{x}(t + \eta)) - U(t, \bar{x}(t))}{\eta} \leq 0, \quad t \in [\gamma, v],$$

which means the right-derivative of U is non-positive along the solutions of (3.22).

Note this is the same definition of a Lyapunov functional for generalized ODEs, namely Definition 2.10. According to Theorem 3.1, the correspondence of solutions function between Volterra integral equations and generalized ODEs is the identity function, whenever the hypothesis from this result are satisfied, this is why the definition of Lyapunov function and Lyapunov type results do not change.

Theorem 3.18. *Let $U : [a, b] \times \overline{B}_\rho \rightarrow \mathbb{R}$ be a Lyapunov functional with respect to the Volterra integral equation (3.22), where $0 < \rho < c$. Suppose U satisfies the following conditions:*

(i) $U(t, 0) = 0$, $t \in [a, b]$;

(ii) There is a constant $K > 0$ such that

$$|U(t, z) - U(t, y)| \leq K \|z - y\|, \quad t \in [a, b] \text{ } z, y \in \overline{B}_\rho.$$

Then the trivial solution $x \equiv 0$ of (3.22) is stable with respect to perturbations.

Theorem 3.19. *Let $U : [a, b] \times \overline{B}_\rho \rightarrow \mathbb{R}$ be a Lyapunov functional with respect to the Volterra integral equation (3.22), where $0 < \rho < c$. Suppose U satisfies conditions (i) and (ii) from Theorem 2.11. Moreover, assume there is a continuous function $\Phi : X \rightarrow \mathbb{R}$ satisfying $\Phi(0) = 0$ and $\Phi(x) > 0$ for $x \neq 0$, such that for every solution $x : [\gamma, v] \rightarrow B_\rho$ of (2.1) with $[\gamma, v] \subset [a, b]$, we have*

$$\dot{U}(t, x(t)) \leq -\Phi(x(t)), \quad t \in [\gamma, v].$$

Then the trivial solution $x \equiv 0$ of (2.1) is asymptotically stable with respect to perturbations.

Applications

In this chapter, we investigate a nonlinear Volterra integral equation, which arises in the study of population growth, considered in [4]. We assume the growth rate, $g : [0, \infty) \rightarrow \mathbb{R}^+$, depends only on the size of the population and the probability of death, $P : [0, \infty) \rightarrow [0, 1]$, depends only on age. Here, $\mathbb{R}^+$ is the set of nonnegative numbers. These assumptions lead to a Volterra integral equation of the form

$$x(t) = f(t) + \int_0^t g(x(t-s))P(s)ds, \quad 0 \leq t < \infty, \quad (4.1)$$

where $f : [0, \infty) \rightarrow \mathbb{R}^+$ represents the number of survivors at time t from the original population $f(0)$.

This model is also useful to studying populations afflicted with a disease, provided people who recover from the disease are not immune to it (thus, the spread of the disease depends only on the number of people with the disease) and there is no incubation period (thus, no delay terms appear in the integral). The papers [7] and [8] derive and use this model to study the spread of gonorrhea.

The results in [4] deal with aspects of boundedness of solutions. Here, we adapt some of the results from [4] and apply the results from Chapter 3 to the model (4.1).

4.1 Existence and uniqueness of solutions

In [4], the author is interested in questions about the behavior of a solution of the integral equation

$$x(t) = f(t) + \int_0^t g(x(t-s))P(s)ds, \quad 0 \leq t < \infty. \quad (4.2)$$

as t tends to infinity. Since the integral equation (4.2) is a model for population growth, we assume that all the functions involved in this equation are nonnegative.

At first, we describe the meaning of these functions and the fundamental hypotheses the author considers in [4]. The function $f : [0, \infty) \rightarrow \mathbb{R}^+$ is continuous and of bounded variation in $[0, \infty)$, $f(t)$ is the number of survivors, at time t , of the original population $f(0)$, hence it is an initial condition of equation (4.2). The function $P : [0, \infty) \rightarrow [0, 1]$ represents the probability a member of the population survives to age t , hence this function only takes values in the interval $[0, 1]$ and it can be normalized so that $P(0) = 1$. Furthermore, $P(t)$ is monotone nonincreasing and differentiable on $[0, \infty)$. Finally, $g : [0, \infty) \rightarrow \mathbb{R}^+$ is defined so that $g(x)$ is the number of members added to the population in unit time when the population size is x and, moreover, $g(x)$ is continuously differentiable on $[0, \infty)$ and $g(0) = 0$.

The purpose of this section is to adapt some of the results of [4] to a more general setting of functions. At first, note that a simple change of variables yields that (4.2) is equivalent to

$$x(t) = f(t) + \int_0^t g(x(s))P(t-s)ds, \quad 0 \leq t < \infty. \quad (4.3)$$

Before proving that the integral equation (4.2) admits a solution, we prove that if it admits a nonnegative solution, then this solution is bounded. The proof is borrowed from [4], Theorem 1, but the hypotheses are weaker.

Lemma 4.1. *Let $f : [0, \infty) \rightarrow \mathbb{R}^+$ be a bounded function, $P : [0, \infty) \rightarrow [0, 1]$ be Perron integrable and satisfy $\int_0^\infty P(s)ds < \infty$. Also, let the function $g : [0, \infty) \rightarrow \mathbb{R}$ be regulated such that*

$$\left(\int_0^\infty P(s)ds \right) \limsup_{x \rightarrow \infty} \frac{g(x)}{x} < 1. \quad (4.4)$$

Then every nonnegative solution of (4.2) is bounded in $[0, \infty)$.

Proof. Given $0 < \rho < 1$, there is a sufficiently large $K > 0$ such that

$$\left(\int_0^\infty P(s)ds \right) g(x) < \rho x, \quad (4.5)$$

for $x \geq K$, because of (4.4). Let us consider

$$I_1 = \{s; x(s) > K\}, \quad I_2 = \{s; x(s) \leq K\}.$$

Since f is bounded, there is $R > 0$ such that $f(t) \leq R$ on $0 \leq t < \infty$. Since g is continuous, there is $L > 0$ such that $g(x(s)) \leq L$ for $x(s) \leq K$. Thus, using (4.5), we obtain

$$\begin{aligned} x(t) &= f(t) + \int_0^t g(x(s))P(t-s)ds = \\ &= f(t) + \int_{I_1} g(x(s))P(t-s)ds + \int_{I_2} g(x(s))P(t-s)ds \leq \\ &\leq R + \frac{\rho}{\int_0^\infty P(s)ds} \int_{I_1} x(s)P(t-s)ds + L \int_{I_2} P(t-s)ds \leq \\ &\leq R + \rho \sup_{s \in [0, t]} x(s) + L \int_0^\infty P(s)ds. \end{aligned}$$

Since this last inequality is valid for every $t \geq 0$ and we have $\rho < 1$, we obtain

$$\sup_{0 \leq t \leq T} x(t) \leq \frac{R + L \int_0^\infty P(s)ds}{1 - \rho},$$

for every $T > 0$. Finally, since this bound is independent of T , we have

$$\sup_{0 \leq t \leq \infty} x(t) \leq \frac{R + L \int_0^\infty P(s) ds}{1 - \rho}$$

and the theorem is proved. $\square$

Corollary 4.2. *Let $f : [0, \infty) \rightarrow \mathbb{R}^+$ be a bounded function, $P : [0, \infty) \rightarrow [0, 1]$ be Perron integrable and satisfy $\int_0^\infty P(s) ds < \infty$. Also, let the function $g : [0, \infty) \rightarrow \mathbb{R}$ be regulated such that $\limsup_{x \rightarrow \infty} \frac{g(x)}{x} = 0$. Then every nonnegative solution of (4.2) is bounded in $[0, \infty)$.*

Proof. It is immediate, because $\limsup_{x \rightarrow \infty} \frac{g(x)}{x} = 0$ implies condition (4.4) is valid. $\square$

Denote by $G_0([0, \infty), \mathbb{R})$ the Banach space of all regulated and bounded functions $x : [0, \infty) \rightarrow \mathbb{R}$. Our aim is to use Theorem 3.12 and Lemma 4.1 to prove the integral equation (4.2) admits a unique global solution in $G_0([0, \infty), \mathbb{R})$. Since we consider $u(s) = s$, the function $K : [0, \infty) \times [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ given by $K(t, x(s), s) = g(x(t - s))P(s)$ is such that $\tilde{K}_t(y) = K(t, x(t), t) = g(x(0))P(t)$, where $\tilde{K}_t$ is given by (3.16). This means that $I - \Delta^- u(t) \tilde{K}_t = I$ always admits an inverse on $(0, \infty)$, because it is the identity function. Likewise, $I - \Delta^+ u(t) \tilde{K}_t$ admits an inverse on $[0, \infty)$.

Theorem 4.3. *Let $\sigma \in [0, \infty)$, $f : [0, \infty) \rightarrow \mathbb{R}^+$ be a regulated and bounded function. Let $P : [0, \infty) \rightarrow [0, 1]$ be Perron integrable and $\int_0^\infty P(s) ds < \infty$. Further, assume $g : [0, \infty) \rightarrow \mathbb{R}^+$ is regulated, the K -integral*

$$\int_0^t g(x(t - s))P(s) ds$$

exists for every $t \in [0, \infty)$, there exists a locally Lebesgue integrable and locally bounded function $W : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\left| \int_{r_1}^{r_2} g(x(t - s))P(s) ds \right| \leq \int_{r_1}^{r_2} W(s) |x(s) - y(s)| ds,$$

for every $r_1, r_2, t \in [0, \infty)$, and every $x, y \in G_0([0, \infty), X)$, and

$$\left(\int_0^\infty P(s) ds \right) \limsup_{x \rightarrow \infty} \frac{g(x)}{x} < 1.$$

Then there is a unique $x \in G_0([0, \infty), \Omega)$ such that

$$x(t) = f(t) + \int_{\sigma}^t g(x(t-s))P(s)ds. \quad (4.6)$$

Proof. Let us define $K : [0, \infty) \times [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ by $K(t, x(s), s) = g(x(t-s))P(s)$ for every $x \in G_0([0, \infty), \mathbb{R})$. Then it is clear that $K \in \mathcal{A}(G, u)$, where $G = [0, \infty) \times [0, \infty) \times [0, \infty)$ and $u(s) = s$ for $s \in [0, \infty)$. Since u is continuous and nondecreasing and W is locally bounded, Theorem 3.12 yields there is a unique solution of the integral equation (4.6) in $G([0, \infty), \mathbb{R})$. Furthermore, Lemma 4.1 yields that this solution belongs to $G_0([0, \infty), \mathbb{R})$. $\square$

Theorem 3.1 states that if $F : [0, \infty) \times [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is given by

$$F(t, x(\tau), s) = g(x(t-\tau))P(\tau)\chi_{[\sigma, t]}(\tau)s,$$

where $x \in G([0, \infty), \mathbb{R})$, then the generalized ODE

$$\frac{dx}{d\tau} = D_s[F(t, x, s) + f(s)\chi_{[\sigma, t]}(s)], \quad (4.7)$$

with initial condition $x(\sigma) = P(\sigma)$, is equivalent to the integral equation (4.2). Moreover, the generalized ODE (4.7) admits a unique global solution on $[\sigma, \infty)$, whenever the hypotheses from Theorem 4.3 are satisfied.

Even though the existence and uniqueness of a global continuous solution of integral equations, obtained through strong smoothness hypotheses on the functions involved (as in [4]), is useful for the numerical treatment of integral equations (see [5] and [29]), it is important to mention that some real world problems present more complex behavior, which requires models that admit solutions in a broader sense.

For example, assume the life span of a certain species is one unit of time and just before the species dies out, eggs are laid which are hatched one unit of time later. This type of behavior is observed in nature. A couple of examples are the common mayfly, *stenonema canadense*, which lives as a larva for a year and as an adult for less than a day, and the 17

year cicada, *magicicada septendecim*, which lives as a larva for 17 years and as an adult for perhaps a week. These examples and others can be found in [6].

This property of the population (to “vanish” for a season, come back to lay the eggs for the next generation and then die) is not well captured for neither continuous nor discrete models. On the other hand, the theory of dynamic equations on time scales, whose primary purpose was to unify the continuous and discrete theories, can be applied for this kind of example. The reader may want to consult [2] for more information about time scales, [13] and [14] to see how a dynamic equation on time scales can be treated as generalized ODEs.

4.2 Stability results

Now, we are going to discuss stability concepts for the population model given in [4]. Let $G = [0, \infty) \times [0, c) \times [0, \infty)$, where $c > 0$. We assume $f : [0, \infty) \rightarrow \mathbb{R}^+$ is a regulated and bounded function. Let $P : [0, \infty) \rightarrow [0, 1]$ be Perron integrable and $\int_0^\infty P(s)ds < \infty$. Finally, assume $g : [0, \infty) \rightarrow \mathbb{R}^+$ is regulated, the K-integral

$$\int_0^t g(x(t-s))P(s)ds$$

exists for every $t \in [0, \infty)$, there exists a locally Lebesgue integrable and locally bounded function $W : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\left| \int_{r_1}^{r_2} g(x(t-s))P(s)ds \right| \leq \int_{r_1}^{r_2} W(s)|x(s) - y(s)|ds,$$

for every $r_1, r_2, t \in [0, \infty)$, and every $x, y \in G_0([0, \infty), X)$, and

$$\left(\int_0^\infty P(s)ds \right) \limsup_{x \rightarrow \infty} \frac{g(x)}{x} < 1.$$

Since the growth rate function g only depends on the population size, it is reasonable to assume that $g(0) = 0$. Then $K : G \rightarrow \mathbb{R}$ given by $K(t, x(s), s) = g(x(t-s))P(s)$ belongs to

the the class $\mathcal{A}(G, u)$, where $u(s) = s$ for every $s \in [0, \infty)$, and satisfies $K(t, 0, s) = 0$ for all $t, s \in [0, \infty)$. Furthermore, $x \equiv 0$ is a solution of the Volterra integral equation

$$x(t) = \int_0^t K(t, x(s), s) du(s) = \int_0^t g(x(t-s))P(s)ds, \quad 0 \leq t < \infty. \quad (4.8)$$

Note that, for every regulated and bounded function $f : [0, \infty) \rightarrow \mathbb{R}^+$, the equation

$$x(t) = f(t) + \int_0^t K(t, x(s), s) du(s) = f(t) + \int_0^t g(x(t-s))P(s)ds, \quad 0 \leq t < \infty \quad (4.9)$$

admits a unique global solution, because of Theorem 4.3.

In order to discuss stability of the trivial solution of (4.8), it is necessary to differentiate both sides of equation (4.8). We start with some preliminary results on this subject.

Lemma 4.4 (Corollary 1.46 from [36]). *Let $[\alpha, \beta], [a, b] \subset \mathbb{R}$, $-\infty < \alpha < \beta < \infty$, $-\infty < a < b < \infty$ be given intervals. Assume that $u : [\alpha, \beta] \rightarrow \mathbb{R}$, $v : [a, b] \rightarrow \mathbb{R}$ are functions of bounded variation in $[\alpha, \beta]$ and in $[a, b]$ respectively. Assume, further, that $\phi : [\alpha, \beta] \times [a, b] \rightarrow \mathbb{R}$ is a given function such that for every $y \in [a, b]$ the Perron-Stieltjes integral $\int_\alpha^\beta \phi(x, y) du(x)$ exists and the K -integral $\int_a^b D_{\widehat{y}} \widehat{\phi}(x, y, \widehat{y})$ exists, where $\widehat{\phi} : [\alpha, \beta] \times [a, b] \times [a, b] \rightarrow \mathbb{R}$ is given by $\widehat{\phi}(x, y, \widehat{y}) = \phi(x, y)v(\widehat{y})$. Then both Perron-Stieltjes integrals*

$$\int_\alpha^\beta \left(\int_a^b \phi(x, y) dv(y) \right) du(x)$$

and

$$\int_a^b \left(\int_\alpha^\beta \phi(x, y) du(x) \right) dv(y)$$

exist and have the same value, i.e.

$$\int_\alpha^\beta \left(\int_a^b \phi(x, y) dv(y) \right) du(x) = \int_a^b \left(\int_\alpha^\beta \phi(x, y) du(x) \right) dv(y).$$

Lemma 4.5 (Corollary 3 from [39]). *Let $\Phi : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable almost everywhere on (a, b) . Then Φ' is Perron integrable over $[a, b]$ and $\int_a^t \Phi'(t)dt = \Phi(t) - \Phi(a)$ for all $t \in [a, b]$.*

Lemma 4.6 (Corollary 8, item (i), from [39]). *Let $\phi : [\alpha, \beta] \times [a, b] \rightarrow \mathbb{R}$. Suppose $\phi(\cdot, y)$ is continuous on $[\alpha, \beta]$ for almost all $y \in (a, b)$ and is differentiable nearly everywhere on (α, β) for almost all $y \in (a, b)$. If the following equality*

$$\int_s^t \int_a^b \frac{\partial \phi}{\partial x}(x, y) dy dx = \int_a^b \int_s^t \frac{\partial \phi}{\partial x}(x, y) dx dy$$

holds for all $[s, t] \subset [\alpha, \beta]$, then $\Phi : [\alpha, \beta] \rightarrow \mathbb{R}$, given by

$$\Phi(x) = \int_a^b \phi(x, y) dy,$$

satisfies

$$\Phi'(x) = \int_a^b \frac{\partial \phi}{\partial x}(x, y) dy$$

for almost all $x \in (\alpha, \beta)$.

We use Lemmas 4.4, 4.5 and 4.6 to obtain sufficient conditions to differentiate equation (4.8).

Proposition 4.7. *Assume the function $g : [0, \infty) \rightarrow \mathbb{R}^+$ is regulated, the K -integral*

$$\int_0^t g(x(t-s))P(s)ds \tag{4.10}$$

exists for every $t \in [0, \infty)$, whenever $x \in G_0([0, \infty), \mathbb{R})$, and P is Perron integrable with $\int_0^\infty P(s)ds < \infty$, $g(0) = 0$, there exists a locally Lebesgue integrable and locally bounded function $W : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\left| \int_{r_1}^{r_2} g(x(t-s))P(s)ds \right| \leq \int_{r_1}^{r_2} W(s)|x(s) - y(s)|ds,$$

for every $r_1, r_2, t \in [0, \infty)$, $x, y \in G_0([0, \infty), X)$ and

$$\left(\int_0^\infty P(s) ds \right) \limsup_{x \rightarrow \infty} \frac{g(x)}{x} < 1.$$

Further, assume the function $P : [0, \infty) \rightarrow [0, 1]$ is continuous and differentiable almost everywhere on $[0, \infty)$. Then $\xi : [0, \infty) \rightarrow \mathbb{R}$ given by

$$\xi(t) = \int_0^t g(x(t-s))P(s)ds$$

is differentiable almost everywhere on $[0, \infty)$ and

$$\xi'(t) = \int_0^t g(x(s))P'(t-s)ds.$$

Proof. At first, we recall that

$$\int_0^t g(x(t-s))P(s)ds = \int_0^t g(x(s))P(t-s)ds, \quad 0 \leq t < \infty.$$

We consider $\phi : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ given by $\phi(t, s) = g(x(s))P(t-s)$, where $x \in G_0([0, \infty)\mathbb{R})$. The function $\phi(\cdot, s)$ is continuous on $[0, \infty)$ and differentiable almost everywhere on $[0, \infty)$, because we assumed the probability function P is continuous on $[0, \infty)$ and differentiable almost everywhere on $[0, \infty)$.

For each $t \in [0, \infty)$, we pick up an arbitrary $T > t$. The functions $u, v : [0, T] \rightarrow \mathbb{R}$ given by $u(s) = v(s) = s$ are clearly of bounded variation in $[0, T]$. We consider a function $\tilde{\phi} : [0, T] \times [0, T] \rightarrow \mathbb{R}$ given by

$$\tilde{\phi}(t, s) = \frac{\partial \phi}{\partial t}(t, s)\chi_{[0, t]}(s) = g(x(s))P'(t-s)\chi_{[0, t]}(s).$$

For each $s \in [0, T]$, the function $\tilde{\phi}(\cdot, s) : [0, T] \rightarrow \mathbb{R}$ is given by

$$\tilde{\phi}(t) = \begin{cases} g(x(s))P'(t-s), & s \in [0, t], \\ 0, & s \in (t, T]. \end{cases}$$

Note that $\phi(t, s) = g(x(s))P(t - s)$, where $s \in [0, T]$ is fixed, is continuous at $t \in [s, T]$ and differentiable almost everywhere at $t \in (s, T)$. Furthermore, the function $\phi_0(t) = 0$, where $t \in [0, s]$, is continuously differentiable on $[0, s]$. Hence, for each $s \in [0, T]$, Lemma 4.5 yields

$$\int_0^T \tilde{\phi}(t, s) dt = \int_s^T g(x(s))P'(t - s) dt = g(x(s))(P(T - s) - P(0)).$$

Note that Lemma 4.5 yields that P' is Perron integrable and

$$\left| \int_0^\infty P'(s) ds \right| = \left| \lim_{r \rightarrow \infty} \int_0^r P'(s) ds \right| = \left| \lim_{r \rightarrow \infty} P(r) - P(0) \right| = P(0) < \infty,$$

hence the K-integral

$$\int_0^T D_s \hat{\phi}(t, s, \hat{s}) = \int_0^t g(x(s))P'(t - s) ds$$

exists because of (4.10), where $\hat{\phi}(t, s, \hat{s}) = \tilde{\phi}(t, s)\hat{s}$ and $t \in [0, T]$. Hence, we can apply Lemma 4.4 to conclude that

$$\int_0^T \left(\int_0^T \tilde{\phi}(t, s) ds \right) dt = \int_0^T \left(\int_0^T \tilde{\phi}(t, s) dt \right) ds.$$

Note that

$$\frac{\partial}{\partial t} (\phi(t, s)\chi_{[0, t]}(s)) = g(x(s))P'(t - s)\chi_{[0, t]}(s) = \tilde{\phi}(t, s), \quad t, s \in [0, T].$$

Moreover,

$$\xi(t) = \int_0^T \phi(t, s)\chi_{[0, t]}(s) ds,$$

which, by Lemma 4.6, yields

$$\xi'(t) = \int_0^T \frac{\partial}{\partial t} (\phi(t, s)\chi_{[0, t]}(s)) ds = \int_0^t g(x(s))P'(t - s) ds$$

and the proposition is proved. □

Now, we are ready to give sufficient conditions for the trivial solution of (4.8) to be stable with respect to perturbations (Definition 3.13). Let $U : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be given by

$$U(t, x) = \frac{|x|^3}{3}. \quad (4.11)$$

At first, we prove U is a Lyapunov functional.

Proposition 4.8. *Let us assume the function $g : [0, \infty) \rightarrow \mathbb{R}^+$ is regulated, the K -integral*

$$\int_0^t g(x(t-s))P(s)ds$$

exists for every $t \in [0, \infty)$, whenever $x \in G_0([0, \infty), \mathbb{R})$ and P is Perron integrable with $\int_0^\infty P(s)ds < \infty$, $g(0) = 0$, there exists a locally Lebesgue integrable and bounded function $W : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\left| \int_{r_1}^{r_2} g(x(t-s))P(s)ds \right| \leq \int_{r_1}^{r_2} W(s)|x(s) - y(s)|ds,$$

for every $r_1, r_2, t \in [0, \infty)$, $x, y \in G_0([0, \infty), X)$ and

$$\left(\int_0^\infty P(s)ds \right) \limsup_{x \rightarrow \infty} \frac{g(x)}{x} < 1.$$

Further, assume the function $P : [0, \infty) \rightarrow [0, 1]$ is nonincreasing, continuous and differentiable almost everywhere on $[0, \infty)$. Then the function $U : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ given by (4.11) is a Lyapunov functional with respect to the Volterra integral equation (4.8).

Proof. Note that, for each $x \in [0, \infty)$, the function $U(\cdot, x) : [0, \infty) \rightarrow \mathbb{R}$ is constant. Hence, in particular, it is left-continuous on $(0, \infty)$.

The function $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, given by $b(x) = \frac{x^3}{3}$, is clearly a function of Hahn class (strictly increasing, continuous and such that $b(0) = 0$). Furthermore,

$$U(t, x) = \frac{|x|^3}{3} \geq b(|x|),$$

where the last inequality is actually an equality and the norm in $b(|x|)$ is unnecessary, because $x \in [0, \infty)$.

Now, let us consider $\bar{x} : [0, \infty) \rightarrow [0, \rho]$ a solution of (4.8), then

$$\dot{U}(t, \bar{x}(t)) = (\bar{x}(t))^2 \bar{x}'(t).$$

By Proposition 4.7, we have

$$\dot{U}(t, \bar{x}(t)) = (\bar{x}(t))^2 \int_0^t g(\bar{x}(s)) P'(t-s) ds \leq 0,$$

because g and $(\bar{x}(t))^2$ are nonnegative functions and the probability function P is non-increasing. Therefore, according to Definition 3.17, U is a Lyapunov functional with respect to the Volterra integral equation (4.8). $\square$

Theorem 4.9. *Assume the function $g : [0, \infty) \rightarrow \mathbb{R}^+$ is regulated, the K -integral*

$$\int_0^t g(x(t-s)) P(s) ds$$

exists for every $t \in [0, \infty)$, whenever $x \in G_0([0, \infty), \mathbb{R})$, and P is Perron integrable with $\int_0^\infty P(s) ds < \infty$, $g(0) = 0$, there exists a locally Lebesgue integrable and locally bounded function $W : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\left| \int_{r_1}^{r_2} g(x(t-s)) P(s) ds \right| \leq \int_{r_1}^{r_2} W(s) |x(s) - y(s)| ds,$$

for every $r_1, r_2, t \in [0, \infty)$, and every $x, y \in G_0([0, \infty), X)$, and

$$\left(\int_0^\infty P(s) ds \right) \limsup_{x \rightarrow \infty} \frac{g(x)}{x} < 1.$$

Further, assume the function $P : [0, \infty) \rightarrow [0, 1]$ is nonincreasing, continuous and differentiable almost everywhere on $[0, \infty)$. Then the trivial solution of (4.8) is stable with respect to perturbations.

Proof. According to Proposition 4.8, the function $U : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ given by (4.11) is a Lyapunov functional with respect to the Volterra integral equation (4.8).

Note that the function U clearly satisfies $U(t, 0) = 0$ for every $t \in [0, \infty)$. Moreover, let $x, y \in [0, \rho]$. Then

$$\begin{aligned} |U(t, x) - U(t, y)| &= \frac{1}{3}|x^3 - y^3| = \\ &= \frac{1}{3}|(x - y)(x^2 + xy + y^2)| \leq \frac{1}{3}3\rho^2|x - y| = \rho^2|x - y|. \end{aligned}$$

Hence, by Theorem 3.18, we conclude the trivial solution of (4.8) is stable with respect to perturbations. $\square$

Note that the need to strengthen the hypotheses on the function P , in order to obtain stability with respect to perturbations of the trivial solution of the integral equation (4.8), came from Proposition 4.7, where we differentiate under the integral sign. For integrodifferential equations such as

$$x'(t) = - \int_{t-r}^t p(t-s)g(x(s))ds, \quad 0 \leq t < \infty, \quad (4.12)$$

it would be possible to avoid dealing with differentiation under the integral sign. The equation (4.12) arises in the theory of a circulating fuel nuclear reactor (see [9]). In [1], the authors investigated the equation (4.12) under impulsive behavior and obtained results on boundedness of the solution.

Appendices

The Kurzweil Integral

The Appendix A collects results from the Kurzweil integral and generalized ODEs that we used throughout the text. Most of them can be found in the book [36], where all the results are specified for functions taking values in the euclidean space $\mathbb{R}^n$. The corresponding versions for functions taking values in a Banach space can be obtained through simple adaptations.

At first, we remind the reader the definition of the Kurzweil integral.

A **gauge** on $[a, b] \subset \mathbb{R}$ is any function $\delta : [a, b] \rightarrow (0, \infty)$.

A **division** of $[a, b]$ is any finite collection of points $D = \{s_0, s_1, \dots, s_{|D|}\}$ such that $a = s_0 < s_1 < \dots < s_{|D|-1} < s_{|D|} = b$, where $|D|$ denotes the number of subintervals in which $[a, b]$ is divided.

A **tagged division** of $[a, b]$ is any finite collection of point-interval pairs $D = \{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, |D|\}$ such that $a = s_0 < s_1 < \dots < s_{|D|-1} < s_{|D|} = b$ and $\tau_i \in [s_{i-1}, s_i]$ for every integer i between 1 and $|D|$. For each integer i , the number τ_i is called the **tag** of the interval $[s_{i-1}, s_i]$.

A tagged division $D = \{\tau_i, [s_{i-1}, s_i]\}_{i=1}^{|D|}$ of $[a, b]$ is δ -**fine**, whenever $[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$ for every integer i between 1 and $|D|$.

Definition A.1. Let X be a Banach space. A function $F : [a, b] \times [a, b] \rightarrow X$ is **Kurzweil integrable** over the interval $[a, b]$, if there is an element $I \in X$, denoted by $\int_a^b DF(\tau, s)$, such that for every $\varepsilon > 0$, there is a gauge δ on $[a, b]$ such that

$$\left\| \sum_{i=1}^{|D|} [F(\tau_i, s_i) - F(\tau_i, s_{i-1})] - I \right\| < \varepsilon$$

for all δ -fine tagged divisions $D = \{\tau_i, [s_{i-1}, s_i]\}_{i=1}^{|D|}$ of $[a, b]$. We denote the set of all Kurzweil integrable functions over $[a, b]$ as $K([a, b])$.

Now we present some basic properties about the Kurzweil integral, starting by the linearity of the integral.

Theorem A.2 (Theorem 1.9 from [36]). If $U, V \in K([a, b])$ and $c_1, c_2 \in \mathbb{R}$, then $c_1U + c_2V \in K([a, b])$ and

$$\int_a^b D[c_1U(\tau, t) + c_2V(\tau, t)] = c_1 \int_a^b DU(\tau, t) + c_2 \int_a^b DV(\tau, t).$$

Integrability on subintervals is also valid.

Theorem A.3 (Theorem 1.10 from [36]). If $U \in K([a, b])$, then for every $[c, d] \subset [a, b]$ we have $U \in K([c, d])$.

Moreover, we have additivity on adjacent intervals.

Theorem A.4 (Theorem 1.11 from [36]). If $c \in (a, b)$ and $U : [a, b] \times [a, b] \rightarrow X$ is such that $U \in K([a, c])$ and $U \in K([c, d])$, then $U \in K([a, b])$ and

$$\int_a^b DU(\tau, t) = \int_a^c DU(\tau, t) + \int_c^b DU(\tau, t).$$

Next, we present the Saks-Henstock Lemma, whose original version belongs to S. Saks, [34], adapted later for generalized integrals using Riemann-like sums by R. Henstock, see [22].

Lemma A.5 (Lemma 1.13 from [36]). *Let $U : [a, b] \times [a, b] \rightarrow X$ be Kurzweil integrable over $[a, b]$. Given $\epsilon > 0$, assume the gauge δ on $[a, b]$ satisfies*

$$\left\| \sum_{j=1}^{|D|} [U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})] - \int_a^b DU(\tau, t) \right\| < \epsilon$$

for every δ -fine tagged partition $D = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{|D|-1}, \tau_{|D|}, \alpha_{|D|}\}$ of $[a, b]$. If

$$a \leq \beta_1 \leq \xi_1 \leq \gamma_1 \leq \beta_2 \leq \xi_2 \leq \gamma_2 \leq \dots \leq \beta_m \leq \xi_m \leq \gamma_m \leq b$$

satisfies

$$\xi_j \in [\beta_j, \gamma_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$$

for every integer j between 1 and m , then

$$\left\| \sum_{j=1}^m [U(\xi_j, \gamma_j) - U(\xi_j, \beta_j) - \int_{\beta_j}^{\gamma_j} DU(\tau, s)] \right\| < \epsilon.$$

The Saks-Henstock Lemma is used, for instance, to prove a Hake-type result as the following theorem.

Theorem A.6 (Theorem 1.14 from [36]). *Let $U : [a, b] \times [a, b] \rightarrow X$ belong to $K([a, c])$ for all $c \in [a, b)$ and suppose there is a finite limit*

$$\lim_{c \rightarrow b-} \left[\int_a^c DU(\tau, s) - U(b, c) + U(b, b) \right] = I.$$

Then $U \in K([a, b])$ and $\int_a^b DU(\tau, s) = I$.

The left endpoint analogue of Theorem A.6 can be proved in a similar manner.

Theorem A.7. *Let $U : [a, b] \times [a, b] \rightarrow X$ belong to $K([c, b])$ for all $c \in (a, b]$ and suppose there is a finite limit*

$$\lim_{c \rightarrow a+} \left[\int_c^b DU(\tau, s) - U(a, c) + U(a, a) \right] = I.$$

Then $U \in K([a, b])$ and $\int_a^b DU(\tau, s) = I$.

The next result regards the continuity of the indefinite Kurzweil integral.

Theorem A.8 (Theorem 1.16 from [36]). *Let $U : [a, b] \times [a, b] \rightarrow X$ be such that $U \in K([a, b])$ and $c \in [a, b]$. Then*

$$\lim_{s \rightarrow c} \left[\int_a^s DU(\tau, t) - U(c, s) + U(c, c) \right] = \int_a^c DU(\tau, t).$$

Theorem A.8 implies the indefinite Kurzweil integral is continuous at a point $c \in [a, b]$ if, and only if, the function $U(c, \cdot) : [a, b] \rightarrow X$ is continuous at the point c .

Theorem A.9 (Theorem 1.25 from [36]). *Let functions $U, U_n : [a, b] \times [a, b] \rightarrow X$ be given, where $U_n \in K([a, b])$ and $n \in \mathbb{N}$. Assume there is a gauge ω on $[a, b]$ such that*

$$\lim_{n \rightarrow \infty} [U_n(\tau, t_2) - U_n(\tau, t_1)] = U(\tau, t_2) - U(\tau, t_1)$$

for $\tau \in [a, b]$ and all $t_1, t_2 \in [a, b]$ such that $t_1 \leq \tau \leq t_2$, $[t_1, t_2] \subset (\tau - \omega(\tau), \tau + \omega(\tau))$.

Assume, further, that the sequence $\{U_n\}_{n \in \mathbb{N}}$ is equi-integrable. Then $U \in K([a, b])$ and

$$\lim_{n \rightarrow \infty} \int_a^b DU_n(\tau, s) = \int_a^b DU(\tau, s).$$

Generalized Ordinary Differential Equations

This second appendix deals with basic properties and stability concepts of solutions of generalized ODEs, see [36] and [38]. Let Ω be an open subset of X and $G = \Omega \times [a, b]$ for some $a, b \in \mathbb{R}$.

Definition B.1. *A function $x : [\alpha, \beta] \rightarrow X$ is called a solution of the generalized ODE*

$$\frac{dx}{d\tau} = DU(x, t) \tag{B.1}$$

on the interval $[\alpha, \beta] \subset \mathbb{R}$, if $(x(t), t) \in G$ for all $t \in [\alpha, \beta]$ and if

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DU(x(\tau), t)$$

holds for every $s_1, s_2 \in [\alpha, \beta]$.

An equivalent way of formulating Definition B.1 is given by the following proposition.

Proposition B.2 (Proposition 3.5 from [36]). *If a function $x : [\alpha, \beta] \rightarrow X$ is a solution of the generalized ODE (B.1) on $[\alpha, \beta]$, then for every fixed $\gamma \in [\alpha, \beta]$, we have*

$$x(s) = x(\gamma) + \int_{\gamma}^s DU(x(\tau), s), \quad s \in [\alpha, \beta]. \quad (\text{B.2})$$

If a function $x : [\alpha, \beta] \rightarrow X$ satisfies the integral equation (B.2) for some $\gamma \in [\alpha, \beta]$ and all $s \in [\alpha, \beta]$, then x is a solution of the generalized ODE (B.1).

The next definition introduces a special class of functions considered as right-hand sides of the generalized ODE (B.1).

Definition B.3. *Let $h : [a, b] \rightarrow \mathbb{R}$ be a nondecreasing function. A function $U : G \rightarrow X$ belongs to the class $\mathcal{F}_0(G, h)$, if the following inequalities*

$$\|U(x, t_2) - U(x, t_1)\| \leq |h(t_2) - h(t_1)| \quad (\text{B.3})$$

and

$$\|U(x, t_2) - U(x, t_1) - U(y, t_2) + U(y, t_1)\| \leq \|x - y\| |h(t_2) - h(t_1)| \quad (\text{B.4})$$

hold for all $(x, t_2), (x, t_1), (y, t_2), (y, t_1) \in G$.

When $U \in \mathcal{F}_0(G, h)$, the solutions of the generalized ODE (B.1) are functions of bounded variation in their domains (see Lemmas 3.9 and 3.10 from [36]) and their discontinuities are described by Lemma 3.12 from [36]. Next, we have a convergence result.

Theorem B.4 (Theorem 3.14 from [36]). *Given $U \in \mathcal{F}_0(G, h)$, let $x : [a, b] \rightarrow \Omega$ be the pointwise limit of functions $x_k : [a, b] \rightarrow \Omega$, $k \in \mathbb{N}$ and assume the integral $\int_a^b DU(x_k(\tau), s)$ exists for every $k \in \mathbb{N}$. Then the integral $\int_a^b DU(x(\tau), s)$ exists and*

$$\int_a^b DU(x(\tau), s) = \lim_{k \rightarrow \infty} \int_a^b DU(x_k(\tau), s).$$

Corollary B.5 (Corollary 3.15 from [36]). *Given $U \in \mathcal{F}_0(G, h)$, let $x : [a, b] \rightarrow \Omega$ be the pointwise limit of a sequence of finite step functions such that $(x(s), s) \in G$ for every $s \in [a, b]$. Then the integral $\int_a^b DU(x(\tau), t)$ exists.*

Note that, in order to prove the integral $\int_a^b DU(x(\tau), t)$ exists, whenever x is a step function, it is only necessary to use basic properties of the Kurzweil integral, given in Appendix A.

We close this appendix with the variational stability concepts of the trivial solution of generalized ODEs found in Chapter X of [36]. We include the definitions of these stability concepts so that the reader can compare them with the ones presented in Chapter 2. These variational stability concepts admit similar results to those presented in Chapter 2. The reader may want to consult [36] and [38].

Let $G = B_c \times [0, \infty)$, where $B_c = \{x \in X; \|x\| < c, c > 0\}$, and $h : [0, \infty) \rightarrow \mathbb{R}$ be a nondecreasing function. We now assume $U : G \rightarrow X$ belongs to the class $\mathcal{F}_0(G, h)$ and it satisfies

$$U(0, t_2) - U(0, t_1) = 0 \quad t_1, t_2 \geq 0.$$

This condition yields $x \equiv 0$ is a solution on $[0, \infty)$ of the generalized ODE

$$\frac{dx}{d\tau} = DU(x, s), \tag{B.5}$$

because

$$\int_{s_1}^{s_2} DU(0, t) = U(0, s_2) - U(0, s_1) = 0, \quad s_1, s_2 \in [0, \infty).$$

Definition B.6. *The trivial solution $x \equiv 0$ of (B.5) is called variationally stable, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\bar{x} : [t_0, t_1] \rightarrow B_c$, with $0 \leq t_0 < t_1 < \infty$, is a function of bounded variation in $[t_0, t_1]$, continuous from the left on $(t_0, t_1]$ which satisfies $\|\bar{x}(t_0)\| < \delta$ and*

$$\text{var}_{t_0}^{t_1} \left(\bar{x}(s) - \int_{t_0}^s DU(\bar{x}(\tau), t) \right) < \delta,$$

then

$$\|\bar{x}(t)\| < \varepsilon, \quad \text{for } t \in [t_0, t_1].$$

Definition B.7. *The trivial solution $x \equiv 0$ of (B.5) is called variationally attracting, if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that if $\bar{x} : [t_0, t_1] \rightarrow B_c$, with $0 \leq t_0 < t_1 < \infty$, is a function of bounded variation in $[t_0, t_1]$, continuous from the left on $(t_0, t_1]$ satisfying $\|\bar{x}(t_0)\| < \delta_0$ and*

$$\text{var}_{t_0}^{t_1} \left(\bar{x}(s) - \int_{t_0}^s DU(\bar{x}(\tau), t) \right) < \rho,$$

then

$$\|\bar{x}(t)\| < \varepsilon, \quad \text{for } t \in [t_0, t_1] \cap [t_0 + T, \infty).$$

Definition B.8. *The trivial solution $x \equiv 0$ of (B.5) is called variationally asymptotically stable, if it is variationally stable and variationally attracting.*

Besides the generalized ODE (B.5), we consider the perturbed generalized ODE

$$\frac{dx}{d\tau} = D[U(t, x, s) + P(t)], \tag{B.6}$$

where $P : [0, \infty) \rightarrow X$ is a function which is locally of bounded variation and continuous from the left on $(0, \infty)$. Clearly $U(x, t) + P(t)$ belongs to the class $\mathcal{F}_0(G, \tilde{h})$, with $\tilde{h}(t) = h(t) + \text{var}_0^t P$.

Definition B.9. *The trivial solution $x \equiv 0$ of (B.5) is called variationally stable with respect to perturbations, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|x_0\| < \delta$ and P is of bounded variation in $[t_0, t_1]$, continuous from the left on $(t_0, t_1]$ with*

$$\text{var}_{t_0}^{t_1} P < \delta,$$

then

$$\|\bar{x}(t, t_0, x_0)\| < \varepsilon, \quad \text{for } t \in [t_0, t_1],$$

where $\bar{x}(t, t_0, x_0)$ is a solution of the perturbed generalized ODE (B.6), with initial condition $\bar{x}(t_0, t_0, x_0) = x_0$ and $[t_0, t_1] \subset [0, \infty)$.

Definition B.10. The trivial solution $x \equiv 0$ of (B.5) is called *attractive* with respect to perturbations, if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that if $\|x_0\| < \delta_0$ and

$$\text{var}_{t_0}^{t_1} P < \rho,$$

where P is of bounded variation in $[t_0, t_1]$ and continuous from the left on $(t_0, t_1]$, then

$$\|\bar{x}(t, t_0, x_0)\| < \varepsilon, \quad \text{for all } t \geq t_0 + T,$$

where $\bar{x}(t, t_0, x_0)$ is a solution of the perturbed generalized ODE (B.6), with initial condition $\bar{x}(t_0, t_0, x_0) = x_0$ and $[t_0, t_1] \subset [0, \infty)$.

Definition B.11. The trivial solution $x \equiv 0$ of (B.5) is called *asymptotically stable* with respect to perturbations, if it is both *stable* and *attractive* with respect to perturbations.

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