Fractional differential equations:
a novel study of local and global solutions
in Banach spaces

Paulo Mendes de Carvalho Neto

Equações diferenciais fracionárias:
um novo estudo de soluções locais e globais
em espaços de Banach

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***Orientador:* Prof. Dr. Alexandre Nolasco de Carvalho**

***Co-Orientador:* Prof. Dr. Pedro Marín-Rubio**

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" It is said that offering someone a dedicatory, is the most primordial demonstration of love that anyone could do. Personally, I think it is much more meaningful to dedicate all this work to you, without saying your name..."

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“Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry.”

Bertrand Russell (1919)

Mathematics can be seen as a different way of understand the life itself. The patterns are hidden in plain sight, we just have to know where to search for them. Things most people see as chaos, or as nonsense, actually follow subtle laws of behavior, a mathematical behavior. Galaxies, ocean, plants, seashells, even things that we are used to treat as common stuff, mobiles, television, computers. The truth of mathematics never lies, but only some of us can see how the pieces fit together to somehow organize this chaotic information and “decode it”, understand and in some way create a new knowledge.

The years studying mathematics during the Master Degree and the PhD still remains one of the happiest times in my life. Also I’m very grateful to all the mathematicians I met during this chapter of my formation, their contagious enthusiasm lead me to continue inspired during all this time. I came to São Carlos many years ago without a clear

objective, but now I proudly say that I'm leaving as a researcher.

First of all, I would like to thank both of my advisors and an important friend and researcher, for the opportunity to study and learn with them. They are models as teachers and mathematicians to all the researchers that are beginning, as myself.

– In Brazil, **Prof. Dr. Alexandre Nolasco de Carvalho**, was my mentor in many ways, teaching in courses, in life and on the research itself. He is responsible for most of all my math training, and for that, I have an immense gratitude. I will never forget the puzzled look he gave me when I introduced myself and, in the same breath, asked him to be my thesis advisor. Also, I can't forget the day that he introduce me to my thesis final objective. At the beginning, the fractional differential equations seemed so odd, but with time and dedication, once again I noticed that Alexandre was right, when he told me at that first day what a wonderful puzzle-world I was being introduced.

–And during my research time in Spain, **Prof. Dr. Pedro Marín Rubio**, was a father and a friend in the life on a new country, but with him besides of learning mathematics, he taught me how to understand and observe the details on the math-world. I clearly remember the first day in July of 2011 when we first met and began to study together. His support, insight and enthusiasm have fueled this process from beginning to end. In addition, he showed great patience and understanding in dealing with a foreign student that barely spoke his language.

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Next I want to thank all my friends. I wouldn't have been able to write this thesis without their support, but I have so many names to remember that it seems an almost impossible task. Even if I tried, I'm afraid this acknowledges would grow longer than the body of my thesis... So, as I have just this small piece of paper to write everything I want,

it looks better to be subtle. In few words, I can say that apart of being distant from some of them during different periods, the way that lead me to this final text was certainly guided in parts for their friendship, companionship and for the importance that each one of us has in the life of the other. Be known that once we've met, we became hard-wired with the impulses to share our ideas, to be connected in many ways, to understand each other... this is the family I was able to choose each of its members, becoming my brothers and sisters, the ones I choose to share happiness, sorrows, victories and defeats, hands to lift me up and shoulders to lean on. Thus, the only words I can image to write here are: **Thanks for everything, for each second, for each word... I'm very happy to have the opportunity to call them friends.**

Now I want to thank my family for everything they've done for me, their support and comprehension during my studies. I owe my family an immeasurable debt of gratitude for their unwavering support of my ever endeavor, despite my repeated selection of distant universities. My first and greatest teachers were my parents, **Armando e Odete**. I can't tell just in words how much I'm happy to have them as my first teachers. My Father was a beacon in my life, teaching me and understanding my choice (*to him so odd*) of the academic and research work. For that I consider myself a lucky guy. And my Mom, that was always there to help me and to stand by me whenever was needed (*even when she couldn't understand the real problem*). Her ability to find kind words at the most difficult times meant more that mere words can convey. She isn't here today to see the result of all the effort, but I know that if she were, she would be very proud. *There wasn't a single day, that I haven't thought in you.*

And in special, I want to say some words to someone that just started to have a very important role in my life. Sometimes we just don't get what wonderful gifts life gave us, but when we notice and understand that even in a gray and bitter time, just a word or a look is sufficient to make everything get colored and sweet again, you can consider yourself special. That spark sunk deeply in your eyes guided me for a long time and still guide... So I wish to thank you so much, **Mariana**, that just the word "thanks" isn't enough to mean it. I love you so much.

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Resumo

Motivados pelo êxito das aplicações nas equações abstratas em muitas áreas da ciência e da engenharia, e pelas perguntas ainda abertas, neste trabalho estudamos questões relativas aos problemas fracionários abstratos de Cauchy de ordem $\alpha \in (0, 1)$. Buscamos responder algumas perguntas: por exemplo, analisamos a existência de soluções locais fracas do problema e sua possível continuação em um intervalo maximal de existência. O caso da não-linearidade crítica e sua correspondente solução regular fraca também é abordado. Por último, mediante o estabelecimento de alguns resultados gerais de comparação, chegamos a conclusão de que as soluções de uma equação diferencial parcial fracionária, proveniente da teoria de condução de calor, existe globalmente.

Resumen

Motivados por el éxito de las aplicaciones de las ecuaciones abstractas fraccionarias en muchas áreas de la ciencia y la ingeniería, y por las preguntas abiertas en esta teoría, en este trabajo se estudian varias cuestiones relativas a los problemas abstractos de Cauchy fraccionarios de orden $\alpha \in (0, 1)$. Buscamos responder a algunas preguntas que estaban abiertas: por ejemplo, se analiza la existencia de soluciones locales débiles del problema, y su posible continuación a un intervalo maximal de existencia. En el caso de la no linealidad crítica, también se estudia la existencia de la correspondiente solución regular débil. Por último, mediante el establecimiento de algunos resultados generales de comparación, llegamos a la conclusión del buen planteamiento de una solución global de una ecuación diferencial parcial fraccionaria, procedente de la teoría de la conducción del calor.

Abstract

Motivated by the huge success of the applications of the abstract fractional equations in many areas of science and engineering, and by the unsolved question in this theory, in this work we study several matters related to abstract fractional Cauchy problems of order $\alpha \in (0, 1)$. We search to answer some questions that were open: for instance, we analyze the existence of local mild solutions for the problem, and its possible continuation to a maximal interval of existence. The case of critical nonlinearities and corresponding regular mild solutions is also studied. Finally, by establishing some general comparison results, we apply them to conclude the global well-posedness of a fractional partial differential equation coming from heat conduction theory.

Contents

	Page
Introduction	1
1 Preliminary Knowledges	11
1.1 <i>General results</i>	12
1.2 <i>Linear operators, semigroups and evolution equations</i>	14
1.3 <i>Fractional powers</i>	18
2 Fractional Calculus	21
2.1 <i>Tools and special functions</i>	21
2.2 <i>Fractional integration and derivation</i>	37
2.3 <i>Fractional differential equations - bounded operators</i>	44
2.4 <i>The Mittag-Leffler operators</i>	47
3 Abstract Fractional Equations	59
3.1 <i>Existence, uniqueness and the fractional limit</i>	61
3.2 <i>Comparison and global existence of solutions</i>	80
3.3 <i>The critical case</i>	91
4 Some Comments on Open Problems and Remaining Questions	109
Bibliography	115
Subject Index	122

Introduction

At a first glance, when we begin to study the fractional derivative and fractional integral, we can be a little unwilling, thinking how odd is the definition and how difficult it can be to be manipulated. But for instance, when we return to our primary school knowledge, and remember that we used to learn that exponents provide a short notation for what is essentially a repeated multiplication of a numerical value, the physical definition can clearly become confused when considering exponents of non integer value. Almost anyone can relate that $x^4 = x \cdot x \cdot x \cdot x$, but how can an ordinary person describe the physical meaning of $x^{5,4}$, or moreover the transcendental exponent in x^π . One cannot conceive what it might be like to multiply a number or quantity by itself 5,4 times, or π times, and yet these expressions have a definite value, verifiable by infinite series expansion. Now in the same way, consider the integral and the derivative. Although they are indeed concepts of a higher complexity by nature, it is still fairly easy to physically represent their meaning and their relation with the positive integer numbers. Our question arrives when we try to think as done before and change the integer number of derivation or integration exponent to a fractional number. In this direction, we will study other operators, sometimes called fractional differentiation or integration operators. In this text, one notices that this definition follows naturally.

BACKGROUNDS

The fractional calculus can be considered in many ways, a novel topic, once it is only during the last thirty years that it has been the subject of specialized conferences and treatises. Everything has begun with the important applications discovered in numerous diverse and widespread fields in science, engineering and finance. More specifically, we can easily find a direct application of fractional calculus in the study of Fluids Flow,

Porous Structures, Control Theory of Dynamical Systems, Rheology Theory, Viscoelasticity, Chemical Physics, Optics, Signal Processing and in many other problems, see [13, 58, 59, 60, 61, 75] among others. One can also read on many specialized texts, that the fractional derivatives and integrals are very suitable for modeling the memory properties of various materials and processes that are governed by anomalous diffusion (*term used to describe a diffusion process with a non-linear relationship to time*).

It is important to understand that the concept of fractional calculus is not a new one, even we already said that it is a novel topic. The history is believed to have emerged from a question raised in the year 1695 by Marquis de L'Hôpital to Gottfried Wilhelm Leibniz. In his letter, L'Hôpital asked about a particular notation Leibniz had used on his publications for the derivative. He used to write

$$\frac{d^n}{dx^n}f(x)$$

to symbolize the n -th derivative of a function f , to $n \in \mathbb{N}^* := \{1, 2, \dots\}$. L'Hôpital posed his question, arguing about the possibility of taking $n = 1/2$. In his reply, dated 30 September of 1695, Leibniz wrote back saying

"... This is an apparent paradox from which, one day, useful consequences will be draw. ..."

Following this line, in 1730, Euler mentioned interpolating between integral orders of a derivative, in 1812 Laplace defined a kind of fractional derivative by means of an integral, and in 1819 there appeared the first discussion of fractional derivative on a calculus text written by S. F. Lacroix. In his 700-page book entitled "*Traité du Calcul Différentiel et du Calcul Intégral*", he answered the question, giving as result, that

$$\frac{d^{1/2}}{dx^{1/2}}x = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

With this answer, formally the fractional calculus was born [42, 47].

Even Fourier mention the fractional derivatives but did not gave applications or at least examples. So been the first to make applications, N. H. Abel in 1823 [1] studied the fractional calculus in the solution of an integral equation which arises in the formulation of the tautochrone problem: it consist in finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire slides to the lowest point of the wire in the same time regardless of where the bead is placed. The brachistochrone problem deals with the shortest time of slide.

Abel's solution was so elegant that in the guess of many mathematicians, was what attracted the attention of Liouville [45] who made the first major attempt to give a logical definition of the nowadays fractional calculus structure. Liouville's first definition involved

Figure 1: Tautochrone Curve

an infinite series and as we know, the notion of convergence ever would interfere in the definition itself, what was his first obstacle. It was Liouville's second definition that solved the last problem. The notion adopted for the fractional differential of an integrable function $f : [t_0, t_1] \subset \mathbb{R} \rightarrow \mathbb{R}$ was

$${}_t D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d^m}{dt^m} \right) \int_{t_0}^t (t-s)^{m-\alpha-1} f(s) ds, \quad t \in [t_0, t_1]$$

where $\alpha > 0$ and m is the first integer greater or equal than α .

As the last advance, in order to present a notion more compatible with the usual theory of differential equations, comes the notion introduced by M. Caputo in 1967 in his celebrated paper [18]. In contrast to the Riemann-Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to deal with the singularity on $t = 0$ and define the fractional order initial conditions, which eventually could be unpleasant to the physics theory. Caputo's definition is illustrated for regular functions $f : [t_0, t_1] \subset \mathbb{R} \rightarrow \mathbb{R}$ as follows

$${}_t D_t^\alpha f(t) = {}_t D_t^\alpha \left[f(t) - \sum_{k=0}^{m-1} \left(\frac{d^k f(t)}{dt^k} \right) \Big|_{t=t_0} \frac{(t-t_0)^k}{k!} \right], \quad t \in [t_0, t_1]$$

where we also have that $\alpha > 0$ and m is the first integer greater or equal than α .

Even now that we have a rich literature on analytical methods for solving differential equations of fractional order, see for instance [42, 48, 51, 57], it is to be remarked

that solutions in closed form have been found only for such equations with constant coefficients and for a rather small class of equations with particular variable coefficients. In general, numerical solution techniques are required to obtain more precise answers. We cannot forget to relate the fundamental and basic role of Mittag-Leffler type functions to “represent” the solutions in this theory. The most simple function of Mittag-Leffler type, $E_\alpha(z)$ for example, depends on two variables: the complex argument z and the real parameter α . Experience in the computation of special functions of mathematical physics teach us that these functions behave in huge different way in distinct parts of the complex plane which make us think in how strongly this information could affect the abstract computations and proofs.

A practical example to the study of the fractional differential equations comes when one tries to understand a **Rheological Constitutive Equations** on the basis of well known mechanics models, i.e., the study of the flow of matter, primarily in the liquid state, but also as “soft solids” or solids under conditions in which they respond with plastic flow rather than deforming elastically in response to an applied force. With some physics manipulations and assumptions, we deduce the following equations:

$$\begin{cases} \sigma_L(t) + C_1 {}^C D_t^{\alpha-\beta} \sigma_L(t) = C_2 {}^C D_t^\alpha \epsilon(t), \\ \sigma_R(t) = C_3 {}^C D_t^\gamma \epsilon(t) \end{cases}$$

where $\gamma \in (0, 1)$, $0 < \beta < \alpha < 1$, C_i for $i \in \{1, 2, 3\}$ are positive constants, $\sigma_L(t)$ is the stresses in the left, $\sigma_R(t)$ the stresses in the right and $\epsilon(t)$ is the strain. This model is the so-called **Zener Model** or the standard solid model. This is a method of modeling the behavior of a viscoelastic material using a linear combination of springs and dashpots to represent elastic and viscous components, respectively (*More details in [60]*).

THESIS OBJECTIVES AND OUTLINES

The study of the fractional differential equations found place in several different topics, already discussed and solved for the usual differential equations. Among all these topics, one that stands out is the study of the abstract Cauchy problems with the Caputo fractional derivative, when we consider the fractional exponent $\alpha \in (0, 1)$.

Exhaustively, autonomous and nonautonomous evolution equations and related topics were been studied with the hope to settle down a huge and strong theory. But in other hand, the difficulties and the problems that eventually appears when one tries to adapt or create a new proof to a result valid in the usual case, made this effort much more difficult than it looks. There are a bunch of recent articles studying these kind of problems, for

instance [11, 26, 30, 36, 41, 53, 55, 66, 67]. Our objective is to study some new properties and relations between the ordinary theory and the fractional theory.

So motivated by this we consider the abstract fractional Cauchy problem

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f(t, u(t)), & t > 0 \\ u(0) = u_0 \in X, \end{cases} \quad (1)$$

where X is a Banach space, $\alpha \in (0, 1)$, $A : D(A) \subset X \rightarrow X$ is a positive sectorial operator, cD_t^α is the Caputo fractional derivative and $f : [0, \infty) \times X \rightarrow X$ is a suitable function.

In order to start our discussion, let us recall some preliminaries. We understand as the Caputo fractional derivative (see Section 2.2 to a better definition) the operator

$$cD_t^\alpha u(t) = D_t^\alpha (u(t) - u(0)),$$

where D_t^α is the Riemann-Liouville fractional derivative, that is given by

$$D_t^\alpha u(t) = D_t \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \right\}.$$

Consider also $\{E_\alpha(-t^\alpha A) : t \geq 0\}$ and $\{E_{\alpha,\alpha}(-t^\alpha A) : t \geq 0\}$ the Mittag-Leffler families associated to $-A$ (see Section 2.4 for details). We adopt the following concepts for solutions to problem (1).

- i) A function $u : [0, \infty) \rightarrow X$ is said to be a global mild solution to problem (1) in $[0, \infty)$ if u is continuous and

$$u(t) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s)) ds, \quad t \geq 0. \quad (2)$$

- ii) Let $\tau > 0$. A function $u : [0, \tau] \rightarrow X$ is said to be a local mild solution to problem (1) in $[0, \tau]$ if $u \in C([0, \tau]; X)$ and

$$u(t) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s)) ds, \quad t \in [0, \tau].$$

Our main purpose in this work is to ensure sufficient conditions for existence and uniqueness of solution to (1) and to establish some regularity and comparison results for this solution on Banach spaces.

In **Chapter 1**, we begin presenting the base of the theory that we want to study. We discuss the vectorial functions in Banach spaces, the theory of unbounded operators, semigroups and finally the theory of fractional powers of sectorial operators. This chapter

is fundamental to the basic estimates and constructions that will be recurrently used during all this text.

In **Chapter 2**, the emphases are the fractional calculus and the Mittag-Leffler functions, that plays an important role in this theory. Among other things, we study some properties of the gamma function, the beta function and other special functions. In particular, we study the behavior of the families $\{E_\alpha(-t^\alpha A) : t \geq 0\}$ and $\{E_{\alpha,\alpha}(-t^\alpha A) : t \geq 0\}$ on the fractional power spaces associated to the positive sectorial operator A and we establish expressions for these families, similar to the second fundamental limit for semigroups.

In **Chapter 3**, we finally consider the new results. We devote the **Section 3.1** to study existence, uniqueness and continuation results to (1) when the nonlinear term is a locally Lipschitz continuous function $f : [0, \infty) \times X \rightarrow X$. In this direction we want to obtain an argument to guarantee the existence of global solution or the “Blow-Up” of maximal local mild solutions. The first important theorem of this section is the following:

Theorem *Let $f : [0, \infty) \times X \rightarrow X$ be a continuous function, locally Lipschitz in the second variable, uniformly with respect to the first variable, and bounded (i.e. it maps bounded sets onto bounded sets). Then the problem (1) has a global mild solution in $[0, \infty)$ or there exists $\omega \in (0, \infty)$ such that $u : [0, \omega) \rightarrow X$ is a maximal local mild solution, and in such a case, $\limsup_{t \rightarrow \omega^-} \|u(t)\| = \infty$.*

Finally, we consider for $\gamma \in (0, 1]$ the problem

$$\begin{cases} {}^c D_t^\gamma u(t) = -Au(t) + f(t, u(t)), & t \geq 0 \\ u(0) = u_0 \in X, \end{cases} \quad (P_\gamma)$$

where ${}^c D_t^\gamma$ is the Caputo fractional derivative, $A : \mathcal{D}(A) \subset X \rightarrow X$ is a positive sectorial operator and the function $f : [0, \infty) \times X \rightarrow X$ is globally Lipschitz and prove the latest important theorem in this section.

Theorem *Consider the problem (P_γ) , for $\gamma \in (0, 1]$ and suppose that $u_\gamma(t)$ is the maximum local mild solution of (P_γ) defined over $[0, \omega_\gamma)$. Then there exists $t^* > 0$ such that*

$$[0, t^*] \subset \bigcap_{\gamma \in [1/2, 1]} [0, \omega_\gamma)$$

and for each fixed $t \in [0, t^*]$

$$\lim_{\gamma \rightarrow 1^-} \|u_\gamma(t) - u_1(t)\| = 0.$$

In **Section 3.2** we consider the problem of comparison and positivity of solutions. We start with the linear version of (1). Particularly, we establish the equivalence between the

positivity of the families $\{E_\alpha(-t^\alpha A) : t \geq 0\}$ and $\{E_{\alpha,\alpha}(-t^\alpha A) : t \geq 0\}$ with the positivity of the function $\lambda \mapsto (\lambda^\alpha + A)^{-1}$. With respect to the semilinear problem (1), we establish results on the positivity of the solutions to conclude the following comparison result.

Theorem *Let $(X, \leq_X)$ be an ordered Banach space and suppose that the families $\{E_\alpha(-t^\alpha A) : t \geq 0\}$ and $\{E_{\alpha,\alpha}(-t^\alpha A) : t \geq 0\}$ are increasing.*

- i) *Let $u_0, u_1 \in X$, and assume that there exist $t_0, t_1 \in [0, \infty)$ such that $\{u_f(t, u_i)\}_{i=0,1}$ are local mild solutions in $[0, t_i]$, $i \in \{0, 1\}$, to*

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f(t, u(t)), & t > 0, \\ u(0) = u_i \in X. \end{cases}$$

Then, if $t^ = \min\{t_0, t_1\}$, $f(t, \cdot)$ is increasing a.e. $t \in [0, t^*]$, and $u_1 \leq_X u_0$, it holds that $u_f(t, u_1) \leq_X u_f(t, u_0)$ for all $t \in [0, t^*]$.*

- ii) *Consider functions f_0 and f_1 , and $u_0 \in X$, and assume that there exist $t_0, t_1 \in [0, \infty)$ such that $\{u_{f_i}(t, u_0)\}_{i=0,1}$ are local mild solutions in $[0, t_i]$, $i \in \{0, 1\}$, to*

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f_i(t, u(t)), & t > 0, \\ u(0) = u_0 \in X. \end{cases}$$

Then, if $t^ = \min\{t_0, t_1\}$ and $f_0(t, x) \leq_X f_1(t, x)$ a.e. $t \in [0, t^*]$ and for all $x \in X$, it holds that $u_{f_0}(t, u_0) \leq_X u_{f_1}(t, u_0)$ for all $t \in [0, t^*]$.*

- iii) *Consider functions f_0 and f_1 , and $u_0, u_1 \in X$, and assume that there exist $t_0, t_1 \in [0, \infty)$ such that $\{u_{f_i}(t, u_i)\}_{i=0,1}$ are local mild solutions in $[0, t_i]$, $i \in \{0, 1\}$, to*

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f_i(t, u(t)), & t > 0, \\ u(0) = u_i \in X. \end{cases}$$

Then, if $t^ = \min\{t_0, t_1\}$, and $x \leq_X y$ imply $f_0(t, x) \leq_X f_1(t, y)$ a.e. $t \in [0, t^*]$, and $u_0 \leq_X u_1$, it holds that $u_{f_0}(t, u_0) \leq_X u_{f_1}(t, u_1)$ for all $t \in [0, t^*]$.*

Finally, as an application of this last result we ensure existence of global mild solution for a fractional partial differential equation coming from heat conduction theory.

In **Section 3.3** we observe that the assumption on the nonlinear term f is rather general but it does not allow to treat problems where the nonlinearity has critical growth. Hence we treat the critical case, proving existence of local ϵ -regular mild solution to (1).

In order to state a brief resume to this section, let us introduce some notation. For $\beta \geq 0$, let $X^\beta = \mathcal{D}(A^\beta)$ be the fractional power spaces associated to the positive sectorial operator A . In this section we use the following concepts.

Definition A continuous function $u : [0, \tau] \rightarrow X^1$ is called a local ϵ -regular mild solution to (1) if $u \in C((0, \tau]; X^{1+\epsilon})$ and verifies (2) in $[0, \tau]$.

Definition For $\epsilon > 0$ we say that a map g is an ϵ -regular map relative to the pair (X^1, X^0) if there exist $\rho > 1$, $\gamma(\epsilon)$ with $\rho\epsilon \leq \gamma(\epsilon) < 1$, and a positive constant c , such that $g : X^{1+\epsilon} \rightarrow X^{\gamma(\epsilon)}$ and

$$\|g(x) - g(y)\|_{X^{\gamma(\epsilon)}} \leq c \left(1 + \|x\|_{X^{1+\epsilon}}^{\rho-1} + \|y\|_{X^{1+\epsilon}}^{\rho-1} \right) \|x - y\|_{X^{1+\epsilon}},$$

for all $x, y \in X^{1+\epsilon}$.

Next theorem is the main result of Section 3.3 (where details of the suitable class of functions $\mathcal{F}(\epsilon, \rho, \gamma(\epsilon), c, v(\cdot), \xi)$ are given).

Theorem Let $\alpha \in (0, 1)$ and $f \in \mathcal{F}(\epsilon, \rho, \gamma(\epsilon), c, v(\cdot), \xi)$. If $v_0 \in X^1$, there exist positive numbers r and τ_0 such that for any $u_0 \in B_{X^1}(v_0, r)$ there exists a continuous function $u(\cdot, u_0) : [0, \tau_0] \rightarrow X^1$ with $u(0, u_0) = u_0$, which is a local ϵ -regular mild solution to the problem

$$\begin{cases} cD_t^\alpha u = -Au + f(t, u(t)), & t > 0 \\ u(0) = u_0. \end{cases}$$

This solution satisfies

$$u \in C((0, \tau_0]; X^{1+\theta}), \quad 0 \leq \theta < \gamma(\epsilon),$$

$$\lim_{t \rightarrow 0^+} t^{\alpha\theta} \|u(t, u_0)\|_{X^{1+\theta}} = 0, \quad 0 < \theta < \gamma(\epsilon).$$

Moreover, for each $\theta_0 < \gamma(\epsilon)$ there exists a constant $C = C(B_{X^1}(v_0, r)) > 0$ such that for any $u_0, w_0 \in B_{X^1}(v_0, r)$,

$$t^{\alpha\theta} \|u(t, u_0) - u(t, w_0)\|_{X^{1+\theta}} \leq C \|u_0 - w_0\|_{X^1} \quad \forall t \in [0, \tau_0], \quad 0 \leq \theta \leq \theta_0.$$

In particular we obtain an existence theorem in X^1 without the nonlinearity being defined on X^1 . Finally, as an application of this last result we ensure existence of a local mild solution for the fractional partial differential equation coming from heat conduction theory.

We end this manuscript, with **Chapter 4**. It is a quite common knowledge between the researchers that the open problems in this area are not difficult to understand, but

difficult to study. In this chapter we show some general open problems that were studied, however have not yet been completely solved. We also discuss some interesting questions concerning about the basic theory of the fractional calculus.

1

Preliminary Knowledges

This first chapter contains some concepts and knowledge used throughout the course of the thesis and has the objective to be a library of basic results. In short, we shall discuss some basic and classical results of operators in Banach spaces and their implications. We begin fixing the notation and establishing important definitions. Then we study linear operators and the existence of C_0 -semigroups associated to sectorial operators. Finally, we study the fractional power of sectorial operators, in order to obtain estimations and interesting properties. We assume that the reader has prior knowledge of the contents in this chapter, which will not be proved.

Unless otherwise noted, throughout this manuscript we consider that $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are notations to set of the Natural, Integer, Real and Complex numbers. Also, we always consider $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ as notation to Banach spaces (*i.e.* a complete normed vector space) over $\mathbb{C}$ and $\mathcal{L}(X, Y)$ as the space of all linear and continuous maps from X to Y , with the norm

$$\|B\|_{\mathcal{L}(X, Y)} := \sup_{\|x\|_X \leq 1} \|Bx\|_Y.$$

We shall use $\mathcal{L}(X)$ to symbolize the space $\mathcal{L}(X, X)$. Furthermore, let $B : D \subset X \rightarrow Y$ be a linear map.

- i) The map B is said to be a **closed map** if the graph set of B

$$G(B) = \{(x, Bx) \in X \times Y : x \in D\}$$

is a closed subset of $X \times Y$.

- ii) The map B is said to be a **densely defined map**, if the domain of B is a dense subset of X .
- iii) When $Y = X$, we will say that B is a **linear operator**.

Remark 1.1 To simplify this reading, we emphasize:

- i) for any linear map $B : D \subset X \rightarrow Y$, we will use $\mathcal{D}(B)(= D)$ to denote its domain and $\mathcal{R}(B)$ to denote its range;
- ii) and whenever there is no possibility of confusion, we shall discard the notation of the norm with a subscript as defined above, writing instead just $\|\cdot\|$.

Theorem 1.2 (Closed graph theorem) *Let $B : X \rightarrow Y$ be a linear map. If B is a closed map, then B is continuous.*

§ 1.1 GENERAL RESULTS

We begin studying the overall results of Functional Analysis in Banach spaces. More explicit considerations and proofs of the following results can be found in [15, 28, 43, 63].

As usual in Complex Analysis, we introduce some tools to deal with vector functions $f : \mathcal{D}(f) \subset \mathbb{C} \rightarrow X$ and explain how the classical results behave in this new frame.

Remark 1.3 Let $[a, b] \subset \mathbb{R}$. If $\gamma : [a, b] \rightarrow \mathbb{C}$ is a continuous function, then:

- i) It is called a **simple path** (or just **path**) in the Complex plane.
- ii) If it is a differentiable function, we call it a **smooth path**.
- iii) If $\gamma(a) = \gamma(b)$, we call it a **closed path**.
- iv) And if there exists a constant $M \geq 0$ such that for any partition

$$P := a = t_0 < t_1 < \dots < t_{n_p} = b$$

of $[a, b]$ the sum

$$v(\gamma, P) := \sum_{i=1}^{n_p} |\gamma(t_i) - \gamma(t_{i-1})| \leq M,$$

we say that γ have **bounded variation**.

Theorem 1.4 (Riemann-Stieltjes integral) Consider a path $\gamma : [a, b] \subset \mathbb{R} \rightarrow \mathbb{C}$ with bounded variation and a vectorial function $f : [a, b] \subset \mathbb{R} \rightarrow X$. Then there exists a vector $I \in X$ with the following property: Given $\epsilon > 0$, there exist $\delta > 0$ such that, if

$$P := a = t_0 < t_1 < \dots < t_{n_p} = b$$

is a partition of $[a, b]$ with $\|P\| := \max\{t_i - t_{i-1} : 1 \leq i \leq n_p\} < \delta$, then

$$\left\| I - \sum_{i=1}^{n_p} f(\tau_i) [\gamma(t_i) - \gamma(t_{i-1})] \right\| < \epsilon,$$

for any choice of $\tau_i \in [t_{i-1}, t_i]$, $1 \leq i \leq n_p$. We denote this vector for $\int_a^b f d\gamma$ and call it the **Riemann-Stieltjes integral** of f over the path γ .

Now we consider the contour integral when the domain of a vectorial function is the complex plane.

Definition 1.5 (Contour integral) Let $\gamma : [a, b] \subset \mathbb{R} \rightarrow \mathbb{C}$ be a path with bounded variation and $f : \{\gamma\} \subset \mathbb{C} \rightarrow X$ a continuous vectorial function. The **countour integral** of f over γ is defined as

$$\int_a^b (f \circ \gamma) d\gamma$$

where the above integral is considered on Riemann-Stieltjes sense. We shall denote it by

$$\int_{\gamma} f(z) dz.$$

Proposition 1.6 Let $\gamma : [a, b] \subset \mathbb{R} \rightarrow \mathbb{C}$ be a smooth path and $f : \{\gamma\} \subset \mathbb{C} \rightarrow X$ a continuous vectorial function. Then

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

Definition 1.7 (Holomorphic functions) Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \rightarrow X$ a vectorial function. If for all $\lambda \in \Omega$ there exist $f'(\lambda) \in X$ such that

$$\lim_{z \rightarrow \lambda} \left\| \frac{f(z) - f(\lambda)}{z - \lambda} - f'(\lambda) \right\| = 0,$$

we say that f is **holomorphic** and we call $f' : \Omega \rightarrow X$ the **derivative** of f .

Definition 1.8 Let $\Omega \subset \mathbb{C}$. We say that Ω is a **Cauchy domain** if it has a finite number of connected components and its boundary $\partial\Omega$ is composed of a finite number of closed

paths, positively oriented $(+\partial\Omega)$.

Theorem 1.9 (Cauchy theorem) *Let $\Omega \subset \mathbb{C}$ be a Cauchy domain and $f : \overline{\Omega} \rightarrow X$ be a continuous function that is holomorphic in Ω . Then*

$$\int_{+\partial\Omega} f(z) dz = 0.$$

Moreover, for any $\lambda \in \Omega$,

$$f(\lambda) = \frac{1}{2\pi i} \int_{+\partial\Omega} \frac{f(z)}{z - \lambda} dz.$$

Theorem 1.10 (Cauchy general theorem) *If $\Omega \subset \mathbb{C}$ is a Cauchy domain and $f : \overline{\Omega} \rightarrow X$ is a continuous function that is holomorphic in Ω , then f is infinitely differentiable in Ω and for any $\lambda \in \Omega$,*

$$f^{(n)}(\lambda) = \frac{n!}{2\pi i} \int_{+\partial\Omega} \frac{f(z)}{(z - \lambda)^{n+1}} dz$$

for any $n \geq 0$.

§ 1.2 LINEAR OPERATORS, SEMIGROUPS AND EVOLUTION EQUATIONS

In this section we discuss spectral properties of linear operators and semigroups. More details can be found in [15, 33, 63, 76].

Definition 1.11 Let $B : \mathcal{D}(B) \subset X \rightarrow X$ be a linear operator. The set

$$\rho(B) = \{\lambda \in \mathbb{C} : (\lambda I - B)^{-1} : \mathcal{R}(\lambda I - B) \subset X \rightarrow X \text{ is injective, bounded and } \overline{\mathcal{R}(\lambda I - B)} = X\}$$

is called **resolvent set** of B . The complementary of this set, $\sigma(B) = \mathbb{C} \setminus \rho(B)$, is called the **spectrum** of B . For $\lambda \in \rho(B)$ the operator $(\lambda I - B)^{-1}$ is called the **resolvent**.

From this point on, to simplify the notation, we will omit the identity operator when writing $(\lambda I - B)$, writing just $(\lambda - B)$.

Theorem 1.12 (Resolvent equality) *Let $B : \mathcal{D}(B) \subset X \rightarrow X$ be any linear operator. Then, for $\lambda, \mu \in \rho(B)$, we obtain*

$$(\lambda - B)^{-1} - (\mu - B)^{-1} = (\mu - \lambda)(\mu - B)^{-1}(\lambda - B)^{-1}.$$

Remark 1.13 If $B : \mathcal{D}(B) \subset X \rightarrow X$ is a closed linear operator, then by the Closed Graph theorem, for each $\lambda \in \rho(B)$ the application $(\lambda - B)^{-1}$ is an everywhere defined continuous linear operator.

Definition 1.14 A **strongly continuous semigroup of linear operators** on X is a family $\{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ such that:

i) $T(0) = I_X$, where $I = I_X$ is the identity operator in X .

ii) For any $t, s \geq 0$

$$T(t)T(s) = T(t+s).$$

iii) The map $\mathbb{R}^+ \times X \ni (t, x) \rightarrow T(t)x \in X$ is continuous.

An immediate conclusion of the above properties is

$$T(t)T(s) = T(t+s) = T(s)T(t),$$

that is, the family $\{T(t) : t \geq 0\}$ is commutative in respect to the composition.

Remark 1.15 Note that whenever there is no possibility of confusion, from this point on, we will call the family $\{T(t) : t \geq 0\}$ described in the last definition, just by **C_0 -semigroup**.

Definition 1.16 Let $\{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ be a C_0 -semigroup on X . Its **infinitesimal generator** is the linear operator $B : \mathcal{D}(B) \subset X \rightarrow X$, where

$$\mathcal{D}(B) := \left\{ x \in X : \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Bx := \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}, \text{ for all } x \in \mathcal{D}(B).$$

Theorem 1.17 Suppose that $\{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ is a C_0 -semigroup on X . Then:

i) If $B : \mathcal{D}(B) \subset X \rightarrow X$ is the infinitesimal generator of $\{T(t) : t \geq 0\}$, then B is a closed and densely defined linear operator. Also, for any $x \in \mathcal{D}(B)$ the application $[0, \infty) \ni t \rightarrow T(t)x \in X$ is continuously differentiable and

$$\frac{d}{dt}T(t)x = BT(t)x = T(t)Bx, \quad t > 0.$$

ii) $\bigcap_{m \geq 1} \mathcal{D}(B^m)$ is dense in X .

iii) There exist $\sigma > 0$ such that if $\operatorname{Re}(\lambda) > \sigma$, then $\lambda \in \rho(B)$ and

$$(\lambda - B)^{-1}x = \int_0^\infty e^{-\lambda t} T(t)x \, dt, \quad \text{for all } x \in X.$$

Theorem 1.18 (Second fundamental limit theorem) *Let $\{T(t) : t \geq 0\}$ be a C_0 -semigroup on X . If $B : \mathcal{D}(B) \subset X \rightarrow X$ is its infinitesimal generator, then*

$$T(t)x = \lim_{n \rightarrow +\infty} \left(I - \frac{t}{n} B \right)^{-n} x = \lim_{n \rightarrow +\infty} \left[\frac{n}{t} \left(\frac{n}{t} - B \right)^{-1} B \right]^n x.$$

We now describe an important curve in the complex plane, that will be used throughout the text. This path was first used to study the gamma function in the complex plane. Today it is an important path as tool in one of the methods of complex contour integration.

Figure 1.1: Hankel's path

Definition 1.19 We say that Ha is a **Hankel's path** if there exist $\epsilon > 0$ and $\theta \in (\pi/2, \pi)$ where $Ha = Ha_1 + Ha_2 - Ha_3$ and the paths Ha_i are given

$$Ha = \begin{cases} Ha_1 := \{te^{i\theta} : t \in [\epsilon, \infty)\} \\ Ha_2 := \{\epsilon e^{it} : t \in [-\theta, \theta]\} \\ Ha_3 := \{te^{-i\theta} : t \in [\epsilon, \infty)\} \end{cases}.$$

We also write $Ha = Ha(\epsilon, \theta)$ to show the angle and the radius dependence (see Figure 1.1)

above).

Finally, the following constructions provides a brief description of the basic results of the theory of analytic semigroups which forms a functional analytic background for the study done in this manuscript. We start discussing the sectorial operators, using the notation of Henry at [33] (for more detailed information see also [39, 40]).

Figure 1.2: Sector $S_{\phi,a}$

Definition 1.20 Let $A : \mathcal{D}(A) \subset X \rightarrow X$ be a closed and densely defined operator. The operator A is said to be a **sectorial operator** if there exist constants $a \in \mathbb{R}$, $N \geq 1$ and $\phi \in (0, \pi/2)$ such that

$$S_{\phi,a} := \{\lambda \in \mathbb{C} : \phi \leq |\arg(\lambda - a)| \leq \pi\} \subset \rho(A)$$

(see Figure 1.2 above) and

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{N}{|\lambda - a|}, \quad \forall \lambda \in S_{\phi,a} \setminus \{a\}.$$

Remark 1.21 If $a = 0$, on the last definition, we say that A is a **positive sectorial operator**

and write just S_ϕ to represent its sector.

Theorem 1.22 *If $A: \mathcal{D}(A) \subset X \rightarrow X$ is a sectorial operator, then $-A$ generates an C_0 -semigroup $\{T(t) : t \geq 0\}$,*

$$T(t) = \int_{-a+Ha} e^{\lambda t} (\lambda + A)^{-1} d\lambda$$

where $-a + Ha$ is the shift of any Hankel's path (as in Definition 1.19) contained in $\rho(-A)$. Moreover, there exist $C > 0$ such that

$$\|T(t)\|_{\mathcal{L}(X)} \leq C e^{-at}, \quad \|AT(t)\|_{\mathcal{L}(X)} \leq (C/t) e^{-at}$$

for all $t > 0$. Finally, for any $x \in X$ the function $[0, \infty) \ni t \mapsto T(t)x \in X$ is analytic and

$$\frac{d}{dt} T(t)x = -AT(t)x$$

for $t > 0$.

§ 1.3 FRACTIONAL POWERS

The study of fractional powers of operators has a long history, which may go back to Abel's work on the tautochrone, the Riemann-Liouville integral, and its generalizations by M. Riesz. The problem of finding a suitable representation for a fractional power of an unbounded operator defined on a Banach space X has attracted much attention and in this section, we study this theory when $A: \mathcal{D}(A) \subset X \rightarrow X$ is a positive sectorial operator. Properties associated with these fractional powers will then be established in a natural manner as a framework to the study that follows.

The fractional powers of sectorial operators play a fundamental role in the theory of existence of solutions to non-linear partial differential equations of parabolic type and to analysis of the asymptotic behavior of solutions to these problems. For more information see [5, 6, 7, 33].

Definition 1.23 Let A be a positive sectorial operator on X and $\beta > 0$. Then we define

$$A^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} T(s) ds$$

where $\{T(t) : t \geq 0\}$ is the C_0 -semigroup generated by $-A$.

Remark 1.24 We consider, to the completeness of the above definition, that $A^0 = Id_X$.

Proposition 1.25 *Let A be a positive sectorial operator on X . Then for any $\beta \geq 0$ the operator $A^{-\beta} \in \mathcal{L}(X)$ and it is injective. Moreover, if β and δ are non negative real numbers, then*

$$A^{-\beta}A^{-\delta} = A^{-(\beta+\delta)}.$$

The above construction guarantees the existence of a bijective operator into its range, for any positive sectorial operator. Now we want to define an operator that play the same role that the fractional positive powers of bounded operators. Supported by Proposition 1.25 we can define the fractional power of A , for any $\beta \geq 0$, as

$$A^\beta : \mathcal{D}(A^\beta) \subset X \rightarrow X,$$

where $\mathcal{D}(A^\beta) := \mathcal{R}(A^{-\beta})$ and $A^\beta := (A^{-\beta})^{-1}$.

Proposition 1.26 *Let A be a positive sectorial operator on X . Then for any $\alpha, \beta \geq 0$ we observe that*

- i) *If $\beta > 0$, then A^β is a closed and densely defined operator.*
- ii) *If $\alpha \geq \beta$, then $\mathcal{D}(A^\alpha) \subset \mathcal{D}(A^\beta)$.*
- iii) *$A^\alpha A^\beta = A^\beta A^\alpha = A^{\alpha+\beta}$ on $\mathcal{D}(A^\gamma)$, where $\gamma = \max\{\alpha, \beta, \alpha + \beta\}$.*
- iv) *$A^\alpha T(t) = T(t)A^\alpha$ on $\mathcal{D}(A^\alpha)$, for $t > 0$, where $\{T(t) : t \geq 0\}$ is the C_0 -semigroup generated by $-A$.*

Theorem 1.27 *Let A be a positive sectorial operator in X and $\beta \in (0, 1)$, then*

$$\|A^\beta (\lambda + A)^{-1} x\|_X \leq C |\lambda|^{\beta-1} \|x\|_X,$$

for any $x \in X$ and any λ in the sector of the operator $-A$.

Now we start to define important fractional spaces that will be used on the chapters that follows.

Definition 1.28 *Let A be a positive sectorial operator on X . Consider for each $\beta \geq 0$*

$$X^\beta = \mathcal{D}(A^\beta)$$

with the graph norm

$$\|x\|_{X^\beta} = \|A^\beta x\|_X$$

for any $x \in X^\beta$.

These spaces defined above will provide the topology used on Section 2.4 and on Section 3.3, so we just cite a last theorem that collect certain properties, merely reformulating the theorems proved until now.

Theorem 1.29 *Let A be a positive sectorial operator on X and $\beta \geq 0$. Then $(X^\beta, \|\cdot\|_{X^\beta})$ is a Banach space. Moreover, if $\sigma \geq \beta \geq 0$, then X^σ is a dense subset of X^β with continuous inclusion.*

2

Fractional Calculus

One of the essential knowledge for the study that follows, is the notion of fractional calculus. Therefore this chapter is concerned with the study of some concepts and results in Banach spaces to familiarize the reader with this kind of calculations. We start studying the usual notations and the basic definitions. Then we focus on the study of the fractional operators (*of integration and differentiation*).

It is also important to understand that in this chapter we will study two fractional differential operators (*the Riemann-Liouville and the Caputo*). However, our objective is to consider the Caputo fractional differentiation operator on the abstract Cauchy problems that we will study on Chapter 3.

§ 2.1 TOOLS AND SPECIAL FUNCTIONS

In this section, we study some preliminary concepts related to the fractional calculus (*for more details, see, for instance, [12, 27, 42]*).

To begin this section, we recall the study of the locally Bochner integrable functions and the L^p -spaces ($1 \leq p < \infty$) for the Dunford-Schwartz integral with respect to a Banach space [25, Chapter 3]. This theory of integration over vector valued functions was constructed by Bochner in [14]. Consider $S \subset \mathbb{R}$ and the following Banach spaces, in the sense of Bochner.

- $(L^p(S, X), \|\cdot\|_{L^p(S, X)})$, for $1 \leq p < \infty$, denotes the set of all measurable functions on

S to X such that $\|u(t)\|_X^p$ is integrable and its norm is given by

$$\|u(t)\|_{L^p(S,X)} := \left(\int_S \|u(t)\|_X^p dt \right)^{1/p}.$$

- $L_{loc}^p(S,X)$, for $1 \leq p < \infty$, denotes the space of functions that belongs to $L^p(\tilde{S},X)$ for all $\tilde{S} \subset\subset S$ (*compactly contained*). These functions are called locally integrable functions.
- $(W^{p,k}(S,X), \|\cdot\|_{W^{p,k}(S,X)})$, denotes the space of functions $f \in L^p(S,X)$ which have weak derivative of order less or equal then k , all being on $L^p(S,X)$, with norm

$$\|u(t)\|_{W^{p,k}(S,X)} := \left(\sum_{n=0}^k \int_S \|D^n u(t)\|_X^p dt \right)^{1/p}, \text{ if } 1 \leq p < \infty.$$

- $C(S,X)$ denotes the space of the continuous functions on S to X and when S is compact we define its norm as

$$\|u(t)\|_{C(S,X)} := \sup_{t \in S} \|u(t)\|_X.$$

Theorem 2.1 (Dominated Convergence) *Let X be a Banach space and $1 \leq p < \infty$. Consider $g \in L^p(S,X)$ and $\{f_\eta\} \subset L^p(S,X)$ a sequence of elements such that $\|f_\eta\|_X \leq \|g\|_X$ a.e. (almost everywhere) for each η . Then if $f_\eta \rightarrow f$ a.e. (almost everywhere), we conclude that the function $f \in L^p(S,X)$ and*

$$\lim_{\eta \rightarrow \infty} \int_S \|f_\eta(t) - f(t)\|^p dt = 0.$$

Proof: The proof can be found on [25, Theorem III.3.7]. □

Remark 2.2 We recall from the literature, that in measure theory a property holds **almost everywhere** if the set of elements for which the property does not hold is a set of measure zero. Therefore, from this point on, we avoid the complete sentence “almost everywhere”, writing just “a.e.”.

✂ THE LAPLACE TRANSFORM – Now let us consider the Laplace transform, which is a very powerful tool for solving differential equations. Its discovery is attributed to the French mathematician *Pierre-Simon Laplace (1749-1827)*.

The following definitions were based on the the classical sufficient condition for the existence of the Laplace transform operator, based on the work of Vignaux in [65]. Vignaux studied a basic theory of asymptotic Laplace transforms applicable to locally Bochner integrable functions, defined on the half line taking values in a Banach space. See also [17, 24, 46, 77, 78], for more details.

Definition 2.3 We say that a function $f : [0, \infty) \rightarrow X$ is of **exponential type** , if there exist $t_0, M > 0$ and a $\gamma \in \mathbb{R}$ such that

$$\|f(t)\| \leq M e^{\gamma t}, \quad \text{for all } t \geq t_0.$$

In other words, the function $f(t)$ must not grow faster than a certain exponential function when $t \rightarrow \infty$.

Proposition 2.4 Let $f : [0, \infty) \rightarrow X$ be a locally integrable function of exponential type. Then there exist $\gamma > 0$ such that

$$\int_0^\infty e^{-\lambda t} f(t) dt \quad (2.1)$$

is convergent for $\operatorname{Re}(\lambda) > \gamma$.

Using the last proposition, if we consider the function $\hat{f} : \mathcal{D}(\hat{f}) \subset \mathbb{C} \rightarrow X$ given by

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt, \quad (2.2)$$

we assure at least that $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > \gamma\} \subset \mathcal{D}(\hat{f})$. Hence, $\hat{f}(\lambda)$ will be called the **Laplace transform** of $f(t)$.

In other words, we can define a linear map $\mathcal{L} : \mathcal{D}(\mathcal{L}) \rightarrow \mathcal{F}(\mathbb{C}, X)$, where

$$\mathcal{D}(\mathcal{L}) := \{f \in L_{loc}^1([0, \infty), X) : f \text{ is of exponential type}\}$$

and

$$\mathcal{F}(\mathbb{C}, X) := \{ \text{the set of the functions defined on a subset of } \mathbb{C} \text{ with Range contained in } X \}.$$

We call it the **Laplace transform operator** and denote it as

$$\mathcal{L}\{f(t)\}(\lambda) := \hat{f}(\lambda).$$

It can be shown that the Laplace transform ($\mathcal{L}$) is a bijective and continuous operator (see [57, 70] for instance); hence, we can define the inverse Laplace transform and

conclude that it is uniquely defined. In general, the computation of inverse Laplace transform require techniques from complex analysis. The simplest inversion formula is given by the so-called **Bromwich integral**.

Definition 2.5 Let $\hat{f} : \mathcal{D}(\hat{f}) \subset \mathbb{C} \rightarrow \mathbb{C}$ be an integrable function. Then we introduce

$$f(t) = \mathcal{L}^{-1}\{\hat{f}(\lambda)\}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} \hat{f}(\lambda) d\lambda, \quad \text{where } c > c_0,$$

where c_0 lies in the right half plane of the absolute convergence of the Laplace integral (2.2). To denote the **inverse Laplace transform** we write $\mathcal{L}^{-1}$. The direct evaluation of the inverse Laplace transform by the above formula is usually complicated, but sometimes it gives useful information of the original function.

Example 2.6 Let $\gamma \in \mathbb{C}$ be such that $\operatorname{Re}(\gamma) > -1$, and $f : [0, \infty) \rightarrow [0, \infty)$ be given by

$$f(t) = t^\gamma.$$

Then, for $\operatorname{Re}(\lambda) > 0$, we obtain

$$\mathcal{L}\{t^\gamma\}(\lambda) = \frac{\Gamma(\gamma+1)}{\lambda^{(\gamma+1)}}.$$

Since it has appeared in a natural way in the above example, it is an appropriate moment to recall the definition and some properties of the gamma function.

✦ THE GAMMA FUNCTION – This transcendental function, represented by $\Gamma(z)$, has caught the interest of some of the most prominent mathematicians of all time. Its history, notably documented by *Philip J. Davis (1923 - nowadays)* in an article, see [23], that won him the 1963 Chauvenet Prize, reflects many of the major developments within mathematics since the 18th century. In the words of Davis, “*Each generation has found something of interest to say about the gamma function. Perhaps the next generation will also*”.

Historically, during the 18th century it was studied by the Swiss mathematician and physicist *Leonhard Euler (1707-1783)* and by the Scottish mathematician *James Stirling (1692-1770)*, but it was *Carl Friedrich Gauss (1777-1855)*, on the 19th century, that rewrote Euler’s results as an infinite product, that allowed him to discover new properties of the gamma function, been the first to consider complex variables. His formulation was

$$\Gamma(z) := \lim_{n \rightarrow \infty} \frac{n! n^z}{z(1+z)(2+z) \dots (n+z)}$$

for all $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$. More details can be found on [52, Chapter 2].

Definition 2.7 The **gamma function**, defined in $\mathcal{D}(\Gamma) := \mathbb{C} \setminus \{0, -1, -2, \dots\}$, have the following properties.

i) If $\operatorname{Re}(z) > 0$,

$$\Gamma(z) = \int_0^\infty s^{z-1} e^{-s} ds.$$

ii) When $n \in \mathbb{N}$,

$$\Gamma(n+1) = n!.$$

iii) And finally, if $z \in \mathcal{D}(\Gamma)$,

$$z\Gamma(z) = \Gamma(z+1).$$

A fundamental result, used all through this text, was obtained by H. Hankel in [35]. Its intention is to represent the gamma function as an integral over a infinite Hankel's path.

Theorem 2.8 Given $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$,

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{Ha} \mu^{-z} e^\mu d\mu,$$

where Ha is any Hankel's path (see Definition 1.19).

Proof: First fix $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$. Let $Ha = Ha(\epsilon, \theta)$, with $\epsilon > 0$ and $\theta \in (\pi/2, \pi)$. Observe that this integral is finite. Indeed,

$$\left\| \int_{Ha} \mu^{-z} e^\mu d\mu \right\| \leq \int_{Ha} |\mu|^{-\operatorname{Re}(z)} e^{\operatorname{Re}(\mu) + \operatorname{Im}(z) \arg(\mu)} d|\mu| \leq M_z \epsilon^{-\operatorname{Re}(z)} \int_{Ha} e^{\operatorname{Re}(\mu)} d|\mu| < \infty.$$

Moreover, it is independent of the Hankel's path chosen. To check this, we start proving that the integral over Ha is equivalent to the integral over a parallel line to the imaginary axis, which the elements have the real value bigger than ϵ . To this end, we first fix $\gamma, R > \epsilon$ and consider the path $\eta_R = \beta_{1,R} - \beta_{2,R} - \beta_{3,R} + \beta_{4,R}$, where

$$\eta_R = \begin{cases} \beta_{1,R} := \{\gamma + it : t \in [-R, R]\} \\ \beta_{2,R} := \{t + iR : t \in [R/\tan \theta, \gamma]\} \\ \beta_{3,R} := \{z \in Ha : \|z\| \leq R/\sin \theta\} \\ \beta_{4,R} := \{t - iR : t \in [R/\tan \theta, \gamma]\} \end{cases}$$

as on the Figure 2.1. By the Cauchy's theorem we have that

Figure 2.1: Finite Path η_R

$$\int_{\eta_R} \mu^{-z} e^{\mu} d\mu = 0$$

and rewriting, we obtain

$$\int_{\beta_{3,R}} \mu^{-z} e^{\mu} d\mu = \int_{\beta_{1,R}-\beta_{2,R}+\beta_{4,R}} \mu^{-z} e^{\mu} d\mu.$$

Doing a limit process over R , the expression above can be written as

$$\int_{Ha} \mu^{-z} e^{\mu} d\mu = \lim_{R \rightarrow \infty} \int_{\beta_{3,R}} \mu^{-z} e^{\mu} d\mu = \lim_{R \rightarrow \infty} \int_{\beta_{1,R}-\beta_{2,R}+\beta_{4,R}} \mu^{-z} e^{\mu} d\mu.$$

Therefore, if we show that over $\beta_{2,R}$ and over $\beta_{4,R}$ the integrals tends to zero we would obtain the equality

$$\int_{Ha} \mu^{-z} e^{\mu} d\mu = \int_{\gamma-i\infty}^{\gamma+i\infty} \mu^{-z} e^{\mu} d\mu.$$

Indeed, observe that

$$\begin{aligned} \left\| \int_{\beta_{2,R}} \mu^{-z} e^{\mu} d\mu \right\| &= \left\| \int_{R/\tan \theta}^{\gamma} e^{-z \log(t+iR) + (t+iR)} dt \right\| \\ &\leq \int_{R/\tan \theta}^{\gamma} e^t (t^2 + R^2)^{-\operatorname{Re}(z)/2} e^{\operatorname{Im}(z) \arg(t+iR)} dt \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \left\| \int_{\beta_{2,R}} \mu^{-z} e^{\mu} d\mu \right\| &\leq M \int_{R/\tan \theta}^{\gamma} e^t R^{-\operatorname{Re}(z)} dt \\ &= MR^{-\operatorname{Re}(z)} (e^{\gamma} - e^{R/\tan \theta}) \rightarrow 0, R \rightarrow \infty. \end{aligned}$$

So the last computations also allow to prove that the definition of the integral is independent of the Hankel's path chosen.

Now if we consider $t > 0$ fixed by making the substitution $\mu = t\omega$ in the integral over the curve Ha , we obtain

$$\frac{1}{2\pi i} \int_{Ha} \mu^{-z} e^{\mu} d\mu = \frac{t^{1-z}}{2\pi i} \int_{Ha} \mu^{-z} e^{\mu t} d\mu = \frac{t^{1-z}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mu^{-z} e^{\mu t} d\mu.$$

But we know by the Example 2.6 that if $F(t) = t^{z-1}$ then its Laplace transform is

$$\mathcal{L}\{F(t)\}(\mu) = \Gamma(z) \mu^{-z}$$

and therefore, using the inverse Laplace operator,

$$\frac{1}{2\pi i} \int_{Ha} \mu^{-z} e^{\mu} d\mu = \frac{t^{1-z}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mu^{-z} e^{\mu t} d\mu = t^{1-z} \mathcal{L}^{-1}\{\mu^{-z}\}(t) = \frac{1}{\Gamma(z)}.$$

□

✦ THE BETA FUNCTION – We remark that another useful mathematical function in fractional calculus is the Beta function. It was discovered and studied by the Swiss mathematician and physicist *Leonard Euler* (1707-1783) and by the French mathematician *Adrien-Marie Legendre* (1752-1833) more than 200 years ago and was given its name by the French mathematician, physicist and astronomer *Jacques Philippe Marie Binet* (1786-1856). The importance of this function lies on the fact that it shares a form that is characteristically

similar to the fractional integral of many functions, particularly polynomials of the form t^α .

Definition 2.9 The **beta function** $(B(z_1, z_2))$, defined in

$$\mathcal{D}(B) := \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re}(z_1) > 0, \operatorname{Re}(z_2) > 0\},$$

is given by the integral formula

$$B(z_1, z_2) = \int_0^1 s^{z_1-1} (1-s)^{z_2-1} ds.$$

The connection between the beta function and the gamma function is given by the following

Proposition 2.10 Let $z_1, z_2 \in \mathbb{C}$ such that $\operatorname{Re}(z_1) > 0$ and $\operatorname{Re}(z_2) > 0$. Then

$$\Gamma(z_1)\Gamma(z_2) = \Gamma(z_1 + z_2)B(z_1, z_2).$$

Proof: First observe that

$$\Gamma(z_1)\Gamma(z_2) = \int_0^\infty \int_0^\infty e^{-(t+s)} t^{z_1-1} s^{z_2-1} dt ds.$$

Now, changing the variables $t = x^2$ and $s = y^2$ on the right side of the above equality, we obtain

$$\Gamma(z_1)\Gamma(z_2) = 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2z_1-1} y^{2z_2-1} dx dy.$$

Then, taking $x = r \cos \theta$ and $y = r \sin \theta$, we conclude that

$$\Gamma(z_1)\Gamma(z_2) = 4 \int_0^\infty e^{-r^2} r^{2(z_1+z_2)-1} dr \int_0^{\pi/2} (\cos \theta)^{2z_2-1} (\sin \theta)^{2z_1-1} d\theta$$

which with the change of variables $r^2 = w_1$ and $(\cos \theta)^2 = w_2$, guarantees that

$$\Gamma(z_1)\Gamma(z_2) = \int_0^\infty e^{-w_1} (w_1)^{z_1+z_2-1} dw_1 \int_0^1 (w_2)^{z_2-1} (1-w_2)^{z_1-1} dw_2 = \Gamma(z_1 + z_2)B(z_1, z_2),$$

what completes the proof. □

✦ THE CONVOLUTION – One of the important tools that we're focusing at this point, is the convolution between functions defined in finite intervals $I = [0, \tau] \subset [0, \infty)$, which is a particular case of composition products considered by Italian mathematician *Vito Volterra* (1860-1940). This is a kind of convolution quite common in the theory of Volterra integral equations of first kind (see for instance [9]).

Definition 2.11 Let $\tau \in (0, \infty)$ and consider two measurable functions $f : [0, \tau] \rightarrow \mathbb{R}$ and $g : [0, \tau] \rightarrow X$. We define the **convolution** in $[0, \tau]$ between f and g as

$$f * g(t) := \int_0^t f(t-s)g(s) ds, \quad \text{for all } t \in [0, \tau],$$

whenever the above integral exist.

Remark 2.12 Let $\tau \in (0, \infty)$ and consider the measurable functions $f, g : [0, \tau] \rightarrow \mathbb{R}$ and $h : [0, \tau] \rightarrow X$. Assuming that all the integrals exist, we can prove that

$$\text{i) } f * g = g * f;$$

$$\text{ii) } (f * g) * h = g * (f * h).$$

Lemma 2.13 Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ and $g : \mathbb{R}^+ \rightarrow X$ be functions of exponential type. Then the function $f * g : \mathbb{R}^+ \rightarrow X$ is of exponential type.

Proof : Since f, g are of exponential type, there exist $t_1, t_2, M_1, M_2 > 0$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\|f(t)\| \leq M_1 e^{\gamma_1 t}, \quad t \geq t_1 \quad \text{and} \quad \|g(t)\| \leq M_2 e^{\gamma_2 t} \quad t \geq t_2.$$

Without loss of generality, we may assume that $t^* := t_1 = t_2$ and that $\gamma_2 \geq \gamma_1$.

Hence, observe that if $t \geq 2t^*$ we obtain

$$\int_0^t f(t-s)g(s) ds = \underbrace{\int_0^{t^*} f(t-s)g(s) ds}_{\mathcal{J}_1(t)} + \underbrace{\int_{t^*}^{t-t^*} f(t-s)g(s) ds}_{\mathcal{J}_2(t)} + \underbrace{\int_{t-t^*}^t f(t-s)g(s) ds}_{\mathcal{J}_3(t)}.$$

Now we estimate each part of the right side of the last equality.

i) First, observe that for $0 \leq s \leq t^*$ we conclude that $t^* \leq (t-s) \leq t$ and therefore

$$\begin{aligned} \|\mathcal{J}_1(t)\| &\leq M_1 \int_0^{t^*} e^{\gamma_1(t-s)} \|g(s)\| ds \\ &\leq M_1 \left(\sup_{s \in [0, t^*]} e^{-\gamma_1 s} \right) \|g\|_{L^1(0, t^*, X)} e^{\gamma_1 t} \end{aligned}$$

and therefore

$$\|\mathcal{J}_1(t)\| \leq \tilde{M}_1 e^{\gamma_1 t} \leq \tilde{M}_1 e^{\gamma_2 t}.$$

ii) Second, for $t^* \leq s \leq t - t^*$ we conclude that $t^* \leq (t - s) \leq t - t^*$ and therefore

$$\begin{aligned} \|\mathcal{J}_2(t)\| &\leq M_1 M_2 \int_{t^*}^{t-t^*} e^{\gamma_1(t-s)} e^{\gamma_2 s} ds \\ &\leq \frac{M_1 M_2 e^{(\gamma_1 - \gamma_2)t^*}}{\gamma_2 - \gamma_1} e^{\gamma_2 t} = \tilde{M}_2 e^{\gamma_2 t}. \end{aligned}$$

iii) Finally, if $t - t^* \leq s \leq t$ (i.e. $t^* \leq s \leq t$) then $0 \leq (t - s) \leq t^*$ and therefore

$$\begin{aligned} \|\mathcal{J}_3(t)\| &\leq M_2 \int_{t-t^*}^t f(t-s) e^{\gamma_2 s} ds \\ &= M_2 \left(\max\{1, e^{-\gamma_2 t^*}\} \right) \|f\|_{L^1(0, t^*, X)} e^{\gamma_2 t} = \tilde{M}_3 e^{\gamma_2 t}. \end{aligned}$$

Joining all the conclusions above and taking $\tilde{M} = \max\{\tilde{M}_1, \tilde{M}_2, \tilde{M}_3\}$, we obtain that

$$\|f * g(t)\| \leq \tilde{M} e^{\gamma_2 t}, \quad t \geq 2t^*$$

what conclude the proof. □

Theorem 2.14 Let $f : [0, \infty) \rightarrow \mathbb{R}$ and $g : [0, \infty) \rightarrow X$ be locally integrable functions of exponential type and suppose that $\mathcal{L}$ is the Laplace transform. Then the function $f * g : [0, \infty) \rightarrow X$ is locally integrable of exponential type and

$$\mathcal{L}\{f * g(t)\}(\lambda) = \mathcal{L}\{f(t)\}(\lambda) \mathcal{L}\{g(t)\}(\lambda),$$

where λ is on the suitable convergence region of both functions.

Proof : In the literature, we can easily check that $f * g$ is locally integrable (see for instance [28]), and by Lemma 2.13 we conclude that $f * g$ is of exponential type.

Now, observe that

$$\begin{aligned}
 \mathcal{L} \left\{ \int_0^t f(t-s)g(s) ds \right\}(\lambda) &= \int_0^\infty e^{-\lambda t} \int_0^t f(t-s)g(s) ds dt \\
 &= \int_0^\infty \int_s^\infty e^{-\lambda t} f(t-s)g(s) dt ds \\
 &= \int_0^\infty g(s) \int_s^\infty e^{-\lambda t} f(t-s) dt ds
 \end{aligned}$$

and changing the variables $t-s=\tau$, we obtain

$$\begin{aligned}
 \mathcal{L} \left\{ \int_0^t f(t-s)g(s) ds \right\}(\lambda) &= \int_0^\infty g(s) \int_0^\infty e^{-\lambda(s+\tau)} f(\tau) d\tau ds \\
 &= \left(\int_0^\infty e^{-\lambda s} g(s) ds \right) \left(\int_0^\infty e^{-\lambda \tau} f(\tau) d\tau \right) \\
 &= \mathcal{L}\{f(t)\}(\lambda) \mathcal{L}\{g(t)\}(\lambda).
 \end{aligned}$$

□

Remark 2.15 Using the same notations of the last proposition, we can rewrite the equality obtained above in other way. Suppose that $\mathcal{L}\{f(t)\}(\lambda) = F(\lambda)$, $\mathcal{L}\{g(t)\}(\lambda) = G(\lambda)$, then

$$\int_0^t \mathcal{L}^{-1}\{F(\lambda)\}(t-s) \mathcal{L}^{-1}\{G(\lambda)\}(s) ds = \mathcal{L}^{-1}\{F(\lambda)G(\lambda)\}(t).$$

✦ THE GENERAL MITTAG-LEFFLER FUNCTION – Motivated essentially by the success of the applications of the Mittag-Leffler functions in many areas of science and engineering, we discuss this subject in a brief survey of their interesting and useful properties. During the last two decades this function has come into prominence after about nine decades of its discovery by the Swedish mathematician *Magnus Gustaf(Gösta) Mittag-Leffler (1846-1927)*.

Definition 2.16 Let α and β be strictly positive real numbers. Then $E_{\alpha,\beta} : \mathbb{C} \rightarrow \mathbb{C}$ is the **general Mittag-Leffler function**, given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

Remark 2.17 There are many important functions related to the general Mittag-Leffler

function.

- i) If $\beta = 1$, we obtain the more studied of those functions, which we will call as the **Mittag-Leffler function**, denoting

$$E_\alpha(z) = E_{\alpha,1}(z).$$

- ii) If $\alpha = \beta = 1$ we obtain that

$$E_{\alpha,\beta}(z) = E_{1,1}(z) = e^z.$$

- iii) Another behavior to observe is that for $\alpha = 2$ and $\beta = 1$, we obtain

$$E_{\alpha,\beta}(-z^2) = E_{2,1}(-z^2) = \sum_{k=0}^{\infty} \frac{(-z^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos z.$$

- iv) A last important observation, for $|z| < 1$, we can force the above definition and consider $\alpha = 0$ and $\beta = 1$, to conclude that

$$E_{0,1}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(1)} = \frac{1}{1-z}.$$

- v) With the notion given on the last items, it is natural to ask about the properties of this general function. Since it can represent either the cosine function, the exponential function and eventually a rational function, we should expect as a partial answer that it behaves in a “different way”. Indeed, it can be proven, for $\alpha \in (0, 1)$, that when $|z| \rightarrow \infty$ (see Figure 2.2 below)

$$E_\alpha(z) = \begin{cases} \frac{1}{\alpha} e^{z^{1/\alpha}} + O(|z|^{-1}), & \text{for } |\arg(z)| \leq \pi\alpha/2, \\ -\frac{1}{z\Gamma(1-\alpha)} + O(|z|^{-1}), & \text{for } \pi\alpha/2 < |\arg(z)| \leq \pi. \end{cases}$$

So we cannot expect a uniform asymptotic behavior of these functions, what can be a big difficult. For more information, see the literature [12, 36, 37, 54].

To better study this function, we present an integral formula for the general Mittag-Leffler function (See [12, 54]).

Figure 2.2: Asymptotic behavior of $E_\alpha(z)$

Proposition 2.18 Let $\alpha, \beta > 0$ and $E_{\alpha,\beta}(z)$ be as defined above. Then for $z \in \mathbb{C}$, we choose $Ha_z = Ha(\epsilon_z, \theta)$ with $\epsilon_z > |z|^{1/\alpha}$ and $\theta \in (\pi/2, \pi)$ to reformulate

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{Ha_z} \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu.$$

Proof: Fix $z \in \mathbb{C}$. Observe that if Ha is any Hankel's path, Theorem 2.8 guarantees that

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} = \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i} \int_{Ha} \mu^{-\alpha k - 1} e^\mu d\mu.$$

Since the inequality above remains true for any Ha , if we choose a particular Ha as in the theorem hypothesis, by the uniform convergence of the sum, we conclude

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{Ha} \mu^{-1} e^\mu \sum_{k=0}^{\infty} \left(\frac{z}{\mu^\alpha} \right)^k d\mu$$

and therefore that

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{Ha} \mu^{-1} e^\mu \frac{1}{1 - z/\mu^\alpha} d\mu = \frac{1}{2\pi i} \int_{Ha} \frac{\mu^{\alpha-1} e^\mu}{\mu^\alpha - z} d\mu.$$

□

Finally, since we will deal with integrals that have singularities, it follows a fractional singular version of the Gronwall's theorem (see for instance [33]).

Theorem 2.19 (Fractional Gronwall inequality theorem) *Given $b \geq 0$, $\alpha > 0$ and $\ell : [0, \infty) \rightarrow \mathbb{R}$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq \infty$). Assume that $u : [0, \infty) \rightarrow \mathbb{R}$ is a nonnegative function locally integrable on $[0, T)$ with*

$$u(t) \leq \ell(t) + b \int_0^t (t-s)^{\alpha-1} u(s) ds$$

on this interval. Then

$$u(t) \leq \ell(t) + \theta \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\theta(t-s)^\alpha) \ell(s) ds, \quad t \in [0, T],$$

where $\theta = b\Gamma(\alpha)$.

Proof: We start defining the operator

$$T(\phi(t)) = b \int_0^t (t-s)^{\alpha-1} \phi(s) ds, \quad t \geq 0,$$

to nonnegative functions ϕ , that are locally integrable on $[0, T]$. Observe that this operator is well defined, since Theorem 2.14 guarantees the locally integrability of this convolution.

Then we make some remarks:

i) Observe that

$$T^2(\phi(t)) = b \int_0^t (t-s)^{\alpha-1} T(\phi(s)) ds = b \int_0^t (t-s)^{\alpha-1} \left\{ b \int_0^s (s-\tau)^{\alpha-1} \phi(\tau) d\tau \right\} ds$$

for $t \geq 0$. Changing the order of integration, we obtain

$$T^2(\phi(t)) = b^2 \int_0^t \int_\tau^t (t-s)^{\alpha-1} (s-\tau)^{\alpha-1} \phi(\tau) ds d\tau$$

and changing of variables $s = (t-\tau)r + \tau$,

$$T^2(\phi(t)) = b^2 \int_0^t (t-\tau)^{2\alpha-1} \left\{ \int_0^1 (1-r)^{\alpha-1} (r)^{\alpha-1} dr \right\} \phi(\tau) d\tau.$$

Finally, by Proposition 2.10, we conclude that

$$T^2(\phi(t)) = b^2 \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)} \int_0^t (t-\tau)^{2\alpha-1} \phi(\tau) d\tau.$$

By a recursive process and by induction, we can prove that

$$T^k(\phi(t)) = \int_0^t [b\Gamma(\alpha)]^k (t-s)^{k\alpha-1} \phi(s) \frac{ds}{\Gamma(k\alpha)}.$$

ii) Also, observe that for each fixed $t \in (0, T)$, if we choose $(n+1) > 1/\alpha$,

$$\begin{aligned} T^{n+1}(\phi(t)) &\leq \int_0^t [b\Gamma(\alpha)]^{n+1} t^{(n+1)\alpha-1} \phi(s) \frac{ds}{\Gamma((n+1)\alpha)} \\ &\leq \left[\frac{b\Gamma(\alpha)}{t^{1-\alpha}} \int_0^t \phi(s) ds \right] \frac{(b\Gamma(\alpha)t^\alpha)^n}{\Gamma(n\alpha + \alpha)}. \end{aligned}$$

Therefore, since the above inequality involves the term of the sum that defines the function $E_{\alpha,\alpha}$, we conclude that

$$\lim_{n \rightarrow \infty} T^{n+1}(\phi(t)) = C(t) \lim_{n \rightarrow \infty} \frac{(b\Gamma(\alpha)t^\alpha)^n}{\Gamma(n\alpha + \alpha)} = 0.$$

Now by hypothesis, $u(t) \leq \ell(t) + T(u(t))$. Using this recursive inequality, we conclude that for each $n \in \mathbb{N}$

$$u(t) \leq \ell(t) + \sum_{k=1}^{n-1} T^k(\ell(t)) + T^n(u(t))$$

and using the items above, when $n \rightarrow \infty$, we obtain for each $t \in [0, T)$

$$u(t) \leq \ell(t) + \sum_{k=1}^{\infty} T^k(\ell(t)) = \ell(t) + \sum_{k=1}^{\infty} \int_0^t [b\Gamma(\alpha)]^k (t-s)^{k\alpha-1} \ell(s) \frac{ds}{\Gamma(k\alpha)},$$

and using the Dominated Convergence theorem together with the uniform convergence of the series, we finally obtain

$$\sum_{k=1}^{\infty} T^k(\ell(t)) = \int_0^t \sum_{k=1}^{\infty} [b\Gamma(\alpha)]^k (t-s)^{k\alpha-1} \ell(s) \frac{ds}{\Gamma(k\alpha)}$$

what guarantees that

$$\begin{aligned} \sum_{k=1}^{\infty} T^k(\ell(t)) &= \int_0^t b\Gamma(\alpha)(t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{[b\Gamma(\alpha)]^k (t-s)^{k\alpha}}{\Gamma(k\alpha + \alpha)} \ell(s) ds \\ &= b\Gamma(\alpha) \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}([b\Gamma(\alpha)](t-s)^\alpha) \ell(s) ds. \end{aligned}$$

Now the last computations implies that

$$u(t) \leq \ell(t) + \theta \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\theta(t-s)^\alpha) \ell(s) ds, \quad t \in [0, T],$$

where $\theta = b\Gamma(\alpha)$. □

✂ THE WRIGHT-TYPE FUNCTION – This function was named in honor of *Sir Edward Maitland Wright (1906 - 2005)*, the eminent British mathematician, who introduced and investigated this function in a series of notes starting from 1933 in the framework of the theory of partitions, see for instance [72, 73, 74].

Definition 2.20 Let $\lambda > -1$ and $\mu \in \mathbb{C}$. The **Wright function** ($W_{\lambda,\mu}(z)$) is given by the complex series representation, convergent in the whole complex plane, as follows

$$W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)}.$$

We devote more attention to one particular function of the Wright type, in virtue of their role in applications of fractional calculus. It is called Mainardi function. More results and applications can be seen in [48].

Definition 2.21 Let $\alpha \in (0, 1)$. The **Mainardi function** ($M_\alpha(z)$) is given by :

$$M_\alpha(z) := W_{-\alpha, 1-\alpha}(-z).$$

Finally we aim to study some properties of the Mainardi function and link it with the Mittag-Leffler function. The motivation to this link was based on the inversion of certain Laplace transforms in order to obtain the fundamental solutions of the fractional diffusion-wave equation in space-time domain. The following result can also be found in [66].

Proposition 2.22 Let $\alpha \in (0, 1)$, $-1 < r < \infty$, $\lambda > 0$ and $z \in \mathbb{C}$. The Mainardi function M_α has the following properties:

i) $M_\alpha(t) \geq 0$, for all $t \geq 0$;

$$\text{ii) } \int_0^\infty t^r M_\alpha(t) dt = \frac{\Gamma(r+1)}{\Gamma(\alpha r+1)};$$

$$\text{iii) } \mathcal{L}\{\alpha t M_\alpha(t)\}(z) = E_{\alpha,\alpha}(-z);$$

$$\text{iv) } \mathcal{L}\{M_\alpha(t)\}(z) = E_\alpha(-z);$$

$$\text{v) } \mathcal{L}\{\alpha t^{-(1+\alpha)} M_\alpha(t^{-\alpha})\}(\lambda) = e^{-\lambda^\alpha}.$$

Proof: The proof of this theorem can be found in [32, 48]. □

§ 2.2 FRACTIONAL INTEGRATION AND DERIVATION

Our initial goal in this section is to introduce an extension of the operations of integration and differentiation to the case of fractional powers. The main definition used on studies in this area, was given by Caputo-Riemann-Liouville. We start this study with the motivation that inspired this definition.

The main structure of this section is partially based in [57, Chapter 2].

✂ AN INFORMAL INTRODUCTION – If X is a Banach space, $b \in (0, \infty)$ and we consider the application

$$\begin{aligned} I : L^1(0, b; X) &\rightarrow C([0, b], X) \\ f(t) &\mapsto \int_0^t f(s) ds \end{aligned}$$

we observe that

$$(I^2 f)(t) = \int_0^t \int_0^s f(r) dr ds = \int_0^t \int_r^t f(r) ds dr = \int_0^t \frac{(t-r)}{1!} f(r) dr.$$

Again if we repeat the above operation,

$$(I^3 f)(t) = \int_0^t \int_0^r \int_0^s f(\tau) d\tau ds dr = \int_0^t \frac{(t-r)^2}{2!} f(r) dr.$$

Inductively we conclude that $(I^n f)$, i.e., the n -th time we apply the integral to the function f , is given by

$$(I^n f)f(t) = \frac{1}{(n-1)!} \int_0^t (t-r)^{n-1} f(r) dr.$$

Once we realize the formula involved to deduct the n -th integration, to introduce a concept that generalize the deduction made above, we rewrite the last equality in another way

$$(I^n f)f(t) = \frac{1}{\Gamma(n)} \int_0^t (t-r)^{n-1} f(r) dr.$$

This is a clue to understand a way to define the generalization. So, informally we may say that the fractional integration of order $\alpha > 0$ of f is given by

$$(I^\alpha f)f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f(r) dr. \quad (2.3)$$

✂ THE FRACTIONAL CALCULUS THEORY – In order to obtain a formal definition for such calculations, let us recall some additional definitions and results.

Definition 2.23 Let $\alpha > 0$. Consider the function $g_\alpha : \mathbb{R} \rightarrow [0, \infty)$ given by

$$g_\alpha(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

where $\Gamma(\alpha)$ is the gamma function.

Proposition 2.24 Let $\alpha, \beta > 0$ and consider $g_\alpha : \mathbb{R} \rightarrow [0, \infty)$ given in Definition 2.23. Then

$$g_\alpha * g_\beta = g_{\alpha+\beta}, \quad t > 0.$$

Proof: Indeed, consider $t > 0$ and observe that

$$\begin{aligned} g_\alpha * g_\beta(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} s^{\beta-1} ds \\ &= \frac{1}{\Gamma(\alpha+\beta)B(\alpha,\beta)} t^{\alpha+\beta-1} \int_0^1 (1-s)^{\alpha-1} s^{\beta-1} ds \\ &= \frac{1}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} = g_{\alpha+\beta}(t). \end{aligned}$$

□

Remark 2.25 Note that for $\alpha \in (0, 1)$ and $f \in L^1(0, b; X)$, we can represent the relation (2.3) as the convolution

$$f * g_\alpha(t) = \int_0^t g_\alpha(t-s)f(s) ds, \quad \text{a.e. in } [0, b].$$

We are now ready to define the concepts of integral and fractional derivative.

Definition 2.26 Let $\alpha \in (0, 1)$, $b > 0$ and $f \in L^1(0, b; X)$. The **Riemann-Liouville fractional integral of order α** , is denoted by $J_t^\alpha f(t)$, and is given by

$$J_t^\alpha f(t) := (g_\alpha * f)(t), \quad \text{a.e. in } [0, b].$$

Remark 2.27 We define, for completeness of the Riemann-Liouville fractional integral operator, that

$$J_t^0 f(t) := f(t).$$

Theorem 2.28 Given $\alpha \in (0, 1)$ and $b > 0$, the Riemann-Liouville fractional integral operator

$$\begin{aligned} J_t^\alpha : L^1(0, b; X) &\rightarrow L^1(0, b; X) \\ f(t) &\mapsto \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \end{aligned}$$

is a bounded operator.

Proof : A general version of this theorem was proved by Samko–Marichev–Kilbas in Theorem 2.6 on Section 2.7 on the book [58]. $\square$

Definition 2.29 Let $\alpha \in (0, 1)$, $b > 0$ and $f \in L^1(0, b; X)$ with $f * g_{1-\alpha} \in W^{1,1}(0, b; X)$. We define the **Riemann-Liouville fractional derivative of order α** , which is denoted as $D_t^\alpha f(t)$, by

$$D_t^\alpha f(t) := D_t^1 J_t^{1-\alpha} f(t) = D_t^1 (g_{1-\alpha} * f)(t), \quad \text{a.e. in } [0, b],$$

where $D_t^1 = \left(\frac{d}{dt} \right)$.

Remark 2.30 Notice that.

i) Explicitly, we obtain

$$D_t^\alpha f(t) = D_t \left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s) ds \right\}, \quad \text{a.e. in } [0, b].$$

- ii) Let $AC([0, b], X)$ be the absolutely continuous functions on $[0, b]$. If $\alpha \in (0, 1)$ and $f \in AC([0, b], X)$, then $D_t^\alpha f(t) \in L^r(0, b; X)$ for $1 \leq r < 1/\alpha$. See [58, Lemma 2.2].

Example 2.31 Let $\alpha \in (0, 1)$, $\beta \in (-1, \infty)$ and consider the function $f(t) = ct^\beta$. Then

$$D_t^\alpha f(t) = c \frac{\Gamma(\beta + 1)}{\Gamma(1 - \alpha + \beta)} t^{\beta - \alpha}.$$

Proof: If we compute the fractional Riemann-Liouville derivative, we obtain

$$\begin{aligned} D_t^\alpha f(t) &= D_t \left\{ \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} c s^\beta ds \right\} \\ &= D_t \left\{ \frac{c t^{1 + \beta - \alpha}}{\Gamma(1 - \alpha)} \int_0^1 (1 - s)^{-\alpha} s^\beta ds \right\} \\ &= (1 + \beta - \alpha) \frac{c t^{\beta - \alpha}}{\Gamma(1 - \alpha)} B(1 - \alpha, \beta + 1) \\ &= c \frac{\Gamma(\beta + 1)}{\Gamma(1 - \alpha + \beta)} t^{\beta - \alpha}. \end{aligned}$$

□

Remark 2.32 Observe on Example 2.31 that the Riemann-Liouville fractional derivative of a constant function is given by

$$D_t^\alpha c = c \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}.$$

Also, when calculating the Riemann-Liouville fractional derivative of a general function, we cannot expect a non-singular behavior at zero. So to avoid this difficulties, we adopt the concept of Caputo fractional derivative, that represents a sort of regularization in the time origin for the Riemann-Liouville fractional derivative and satisfies the relevant property of being zero when applied to a constant.

The last remark and other mainly physical and practical reasons led to the definition of the Caputo fractional derivative.

Definition 2.33 Let $\alpha \in (0, 1)$, $b > 0$ and $f \in C([0, b], X)$ with $f * g_{1-\alpha} \in W^{1,1}(0, b; X)$.

We define the **Caputo fractional derivative of order α** , which is denoted as ${}_t cD_t^\alpha f(t)$, by

$${}_t cD_t^\alpha f(t) := D_t^\alpha(f(t) - f(0)), \quad \text{a.e. in } [0, b].$$

Proposition 2.34 *Let $\alpha \in (0, 1)$ and suppose that $f \in C^1([0, b], X)$. then ${}_t cD_t^\alpha f(t) \in C([0, b], X)$ and*

$${}_t cD_t^\alpha f(t) = J_t^{1-\alpha} f'(t), \quad \text{for all } t \in [0, b].$$

Proof: First observe by [42, Theorem 2.2] that ${}_t cD_t^\alpha f(t) \in C([0, b], X)$. Then, to verify the statement, observe that

$${}_t cD_t^\alpha f(t) = D_t^\alpha(f(t) - f(0)) = D_t^1 \left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} (f(s) - f(0)) ds \right\}.$$

Since f is differentiable in the usual sense, integrating by parts we obtain

$$\begin{aligned} {}_t cD_t^\alpha f(t) &= D_t^1 \left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^t \left[\frac{1}{\alpha-1} D_s^1 (t-s)^{1-\alpha} \right] (f(s) - f(0)) ds \right\} \\ &= D_t^1 \left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{\alpha-1} (t-s)^{1-\alpha} f'(s) ds \right\}. \end{aligned}$$

Finally, by Leibniz Rule we conclude that

$${}_t cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds = J_t^{1-\alpha} f'(t).$$

□

The next result is a classical computation done in the books that study the fractional calculus, for instance [42, 51, 57, 58]. Nevertheless, we believe it is important that we study its proof.

Proposition 2.35 *The following properties are valid in respect of the fractional integral and derivative definitions: given $\alpha_1, \alpha_2, b \geq 0$, with $f \in L^1(0, b; X)$ and $h \in C([0, b], X)$:*

- i) $J_t^{\alpha_1} J_t^{\alpha_2} f(t) = J_t^{\alpha_1 + \alpha_2} f(t);$
- ii) $D_t^{\alpha_1} J_t^{\alpha_1} f(t) = f(t);$
- iii) If $g_{1-\alpha_1} * f \in W^{1,1}(0, b; X)$, then

$$J_t^{\alpha_1} D_t^{\alpha_1} f(t) = f(t) - \frac{1}{\Gamma(\alpha_1)} t^{\alpha_1-1} \{J_s^{1-\alpha_1} f(s)\} \big|_{s=0}.$$

Moreover, if there exists an integrable function ϕ such that $f = J_t^{\alpha_1} \phi(t)$, then

$$J_t^{\alpha_1} D_t^{\alpha_1} f(t) = f(t);$$

$$\text{iv) } {}^c D_t^{\alpha_1} J_t^{\alpha_1} h(t) = h(t);$$

$$\text{v) If } g_{1-\alpha_1} * h \in W^{1,1}(0, b; X), \text{ then } J_t^{\alpha_1} {}^c D_t^{\alpha_1} h(t) = h(t) - h(0).$$

Proof: We prove each item separately.

i) We just observe that by Remark 2.12 and Remark 2.24

$$J_t^{\alpha_1} J_t^{\alpha_2} f(t) = g_{\alpha_1} * (g_{\alpha_2} * f)(t) = (g_{\alpha_1} * g_{\alpha_2}) * f(t) = g_{\alpha_1 + \alpha_2} * f(t) = J_t^{\alpha_1 + \alpha_2} f(t).$$

ii) Using item i) proved above, we compute

$$D_t^{\alpha_1} J_t^{\alpha_1} f(t) = D_t J_t^{1-\alpha_1} J_t^{\alpha_1} f(t) = D_t J_t^1 f(t).$$

By the fundamental theorem of calculus, we have that

$$D_t^1 J_t^1 f(t) = \frac{d}{dt} \left(\int_0^t f(s) ds \right) = f(t).$$

iii) This item demands a more carefully computation. Observe that $D_t^{\alpha_1} f(t) \in L^1(0, b; X)$, therefore

$$\begin{aligned} J_t^{\alpha_1} D_t^{\alpha_1} f(t) &= \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} D_s^{\alpha_1} f(s) ds \\ &= \frac{1}{\alpha_1 \Gamma(\alpha_1)} \int_0^t D_t^1 (t-s)^{\alpha_1} D_s^{\alpha_1} f(s) ds. \end{aligned}$$

Then by the Leibniz Rule

$$\begin{aligned} J_t^{\alpha_1} D_t^{\alpha_1} f(t) &= D_t^1 \left\{ \frac{1}{\alpha_1 \Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1} D_s^{\alpha_1} f(s) ds \right\} \\ &= D_t^1 \left\{ \frac{1}{\alpha_1 \Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1} D_s^1 J_s^{1-\alpha_1} f(s) ds \right\} \end{aligned}$$

and finally integrating by parts

$$\begin{aligned}
J_t^{\alpha_1} D_t^{\alpha_1} f(t) &= D_t^1 \left\{ \frac{1}{\alpha_1 \Gamma(\alpha_1)} \left\{ (t-s)^{\alpha_1} J_s^{1-\alpha_1} f(s) \right\} \Big|_0^t \right. \\
&\quad \left. - \frac{1}{\alpha_1 \Gamma(\alpha_1)} \int_0^t D_s^1 (t-s)^{\alpha_1} J_s^{1-\alpha_1} f(s) ds \right\} \\
&= D_t^1 \left\{ - \frac{1}{\alpha_1 \Gamma(\alpha_1)} t^{\alpha_1} \left\{ J_s^{1-\alpha_1} f(s) \right\} \Big|_{s=0} \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} J_s^{1-\alpha_1} f(s) ds \right\} \\
&= - \frac{1}{\Gamma(\alpha_1)} t^{\alpha_1-1} \left\{ J_s^{1-\alpha_1} f(s) \right\} \Big|_{s=0} + D_t^1 J_t^{\alpha_1} J_t^{1-\alpha_1} f(t) \\
&= f(t) - \frac{1}{\Gamma(\alpha_1)} t^{\alpha_1-1} \left\{ J_s^{1-\alpha_1} f(s) \right\} \Big|_{s=0}.
\end{aligned}$$

Now, if $f(t) = J_t^{\alpha_1} \phi(t)$, then by item *ii*), we conclude that

$$J_t^{\alpha_1} D_t^{\alpha_1} f(t) = J_t^{\alpha_1} D_t^{\alpha_1} J_t^{\alpha_1} \phi(t) = J_t^{\alpha_1} \phi(t) = f(t).$$

iv) Using the last item and the fact that

$$\|J_s^{\alpha_1} h(s)\|_{s=0} \leq \|h\|_{C([0,b],X)} \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\sigma)^{\alpha_1-1} d\sigma \Big|_{s=0} = 0,$$

we conclude that

$$cD_t^{\alpha_1} J_t^{\alpha_1} h(t) = D_t^{\alpha_1} (J_t^{\alpha_1} h(t) - J_s^{\alpha_1} h(s)\Big|_{s=0}) = D_t^{\alpha_1} J_t^{\alpha_1} h(t) = h(t).$$

v) Finally, observe that if $H(t) = h(t) - h(0)$, by item *iii*) and by the fact observed on item *iv*), we obtain

$$J_t^{\alpha_1} cD_t^{\alpha_1} h(t) = J_t^{\alpha_1} D_t^{\alpha_1} H(t) = H(t) - \frac{1}{\Gamma(\alpha_1)} t^{\alpha_1-1} \left\{ J_s^{1-\alpha_1} H(s) \right\} \Big|_{s=0} = h(t) - h(0).$$

□

§ 2.3 FRACTIONAL DIFFERENTIAL EQUATIONS - BOUNDED OPERATORS

This section is devoted to prove the existence and uniqueness of solutions to Cauchy problems for linear differential equations

$$\begin{cases} cD_t^\alpha u(t) = Bu(t), & t > 0 \\ u(0) = u_0 \in X \end{cases} \quad (2.4)$$

where X is a Banach space, $\alpha \in (0, 1)$, cD_t^α is the Caputo fractional derivative and $B \in \mathcal{L}(X)$. The basic ideas of the proofs done in this section are taken from [22, 42, 57]. Let $0 < \tau < \infty$. We start considering the vector space $(C^\alpha([0, \tau], X), \|\cdot\|_{C([0, \tau], X)})$, where

$$C^\alpha([0, \tau], X) := \{u \in C([0, \tau], X) : cD_t^\alpha u \in C([0, \tau], X)\}.$$

Definition 2.36 We say that a continuous function $u : [0, \infty) \rightarrow X$ is a global solution of (2.4), if $u \in C^\alpha([0, \tau], X)$ for all $\tau > 0$ and satisfies the equations of (2.4).

Lemma 2.37 Let $u : [0, \infty) \rightarrow X$ be a continuous function such that $u \in C([0, \tau], X)$ for all $\tau > 0$. Then u is a global solution of (2.4) if, and only if, u satisfies the integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bu(s) ds, \quad t \geq 0.$$

Proof: ($\Rightarrow$) Let $\tau > 0$. Since u is a global solution of (2.4), then $u \in C([0, \tau], X)$, $cD_t^\alpha u \in C([0, \tau], X)$ and

$$cD_t^\alpha u(t) = Bu(t), \quad t \in (0, \tau].$$

Thus, by applying J_t^α in both sides of the equality (since $cD_t^\alpha u \in L^1(0, \tau; X)$) we obtain

$$u(t) = u(0) + J_t^\alpha Bu(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bu(s) ds, \quad t \in [0, \tau].$$

Since $\tau > 0$ was an arbitrary choice, u satisfies the integral equation for all $t \geq 0$, as we wish.

($\Leftarrow$) On the other hand, choose $\tau > 0$ (but arbitrary). By hypothesis, $u \in C([0, \tau], X)$ and satisfies the integral equation,

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bu(s) ds, \quad t \in [0, \tau].$$

Observing also that $u(0) = u_0$ and rewriting the equality above, we obtain

$$u(t) = u_0 + J_t^\alpha Bu(s), \quad t \in [0, \tau].$$

Since $Bu(s) \in C([0, \tau], X)$, we conclude, by item *ii*) of Proposition 2.35, that we can apply cD_t^α in both sides of the integral equation, obtaining

$$cD_t^\alpha u(t) = Bu(t), \quad t \in [0, \tau],$$

what lead us to verify that $cD_t^\alpha u \in C([0, \tau], X)$. Since $\tau > 0$ was an arbitrary choice, we conclude that the function u is a global solution to (2.4). $\square$

Theorem 2.38 *Let $\alpha \in (0, 1)$, $B \in \mathcal{L}(X)$ and $u_0 \in X$. Then the problem (2.4) have a unique global solution.*

Proof : Choose $\tau > 0$. Then consider $K_\tau = \{u \in C([0, \tau], X) : u(0) = u_0\}$ and the operator $T : K_\tau \rightarrow K_\tau$ given by

$$T(u(t)) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bu(s) ds.$$

We show that a power (*with respect to the composition*) of this operator is a contraction, and therefore, by Banach's Fixed Point Theorem, T have a unique fixed point in K_τ . To this end, observe that for any $u, v \in K_\tau$:

$$\begin{aligned} \|T(u(t)) - T(v(t))\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\|B\|_{\mathcal{L}(X)} \|u(s) - v(s)\|}{(t-s)^{1-\alpha}} ds \\ &\leq \frac{t^\alpha}{\alpha \Gamma(\alpha)} \|B\|_{\mathcal{L}(X)} \sup_{s \in [0, \tau]} \|u(s) - v(s)\| \\ &= \frac{t^\alpha}{\Gamma(\alpha + 1)} \|B\|_{\mathcal{L}(X)} \sup_{s \in [0, \tau]} \|u(s) - v(s)\|. \end{aligned}$$

By iterating this relation, we find that

$$\begin{aligned} \|T^n(u(t)) - T^n(v(t))\| &\leq \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \|B\|_{\mathcal{L}(X)}^n \sup_{s \in [0, \tau]} \|u(s) - v(s)\| \\ &\leq \frac{\tau^{n\alpha}}{\Gamma(n\alpha + 1)} \|B\|_{\mathcal{L}(X)}^n \sup_{s \in [0, \tau]} \|u(s) - v(s)\|, \end{aligned}$$

and for an sufficiently large n , the constant in question is less than 1, i.e., there exists a fixed point $u \in K_\tau$. Observe now that $\tau > 0$ was an arbitrary choice, so we conclude that the fixed point $u \in C([0, \tau], X)$ for all $\tau > 0$ and by Lemma 2.37, we obtain the existence and uniqueness of a global solution to the problem (2.4). $\square$

Now we aim to understand how the solution to the problem (2.4) looks like. To this, we need to construct its Taylor series, using the Piccard Theorem, as done in the usual case of ordinary differential equation.

Corollary 2.39 *Consider the same hypothesis of Theorem 2.38.*

i) Let $\{U_n(t)\}_{n=0}^\infty$ be a sequence of continuous functions

$$U_n : [0, \infty) \rightarrow X$$

given by

$$U_0(t) = u_0, \quad U_n = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B U_{n-1}(s) ds, \quad n \in \{1, 2, \dots\}.$$

Then there exists a continuous function $U : [0, \infty) \rightarrow X$, such that for any $\tau > 0$ we conclude that $U_n \rightarrow U$ in $C([0, \tau], X)$. Moreover, $U(t)$ is the unique global solution of (2.4).

ii) It holds that

$$U(t) = \sum_{k=0}^{\infty} \frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k + 1)}.$$

Proof:

i) It follows directly from proof of Theorem 2.38.

ii) It is trivial that $U_0(t) = u_0$. So we compute, using the gamma function properties, that

$$U_1(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B u_0 ds = u_0 + \frac{t^\alpha B u_0}{\alpha \Gamma(\alpha)} = u_0 + \frac{t^\alpha B u_0}{\Gamma(\alpha + 1)}.$$

By a simple induction process, we conclude that

$$U_n(t) = \sum_{k=0}^n \frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k + 1)}$$

and therefore

$$U(t) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k + 1)} = \sum_{k=0}^{\infty} \frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k + 1)} := E_\alpha(t^\alpha B)u_0.$$

□

Remark 2.40 A last important observation about the result obtained on Corollary 2.39, is that even if the theory studied until now guarantees that the solution of the problem (2.4) follows “the same” construction properties as the usual ordinary case, we can verify in the literature, for instance [55, 56], that is impossible to suppose that the “semigroup property” remain valid to this new solution, i.e., for any $\alpha \in (0, 1)$ there will always exist $t, s \in [0, \infty)$, such that

$$E_\alpha((t+s)^\alpha B) \neq E_\alpha(t^\alpha B)E_\alpha(s^\alpha B).$$

§ 2.4 THE MITTAG-LEFFLER OPERATORS

In this section we aim to define and study the central operators involved in this thesis. In Section 2.3 we proved that if $B \in \mathcal{L}(X)$, then $E_\alpha(t^\alpha B)$ is the solution operator of the fractional differential equation

$$\begin{cases} cD_t^\alpha u(t) = Bu(t), & t > 0 \\ u(0) = u_0 \in X, \end{cases}$$

where $\alpha \in (0, 1)$ and cD_t^α is the Caputo fractional derivative. The study of this linear problem was the first step to start the approach of more complicated problems. Since our objective in this section is to make this study more general, we begin considering the problem

$$\begin{cases} cD_t^\alpha u(t) = -Au(t), & t > 0 \\ u(0) = u_0 \in X, \end{cases}$$

where $A : \mathcal{D}(A) \subset X \rightarrow X$ is a positive sectorial operator. Our concern is to understand the problem in this new frame and study the solution associated and another important Mittag-Leffler operator.

We start with a generalization of the Cauchy representation for semigroups associated to positive sectorial operators. To this end we recall that given $\epsilon > 0$ and $\theta \in (\pi/2, \pi)$, the Hankel’s path $Ha = Ha(\epsilon, \theta)$ is the path given by $Ha = Ha_1 + Ha_2 - Ha_3$, where Ha_i

are such that

$$Ha_1 := \{te^{i\theta} : t \in [\epsilon, \infty)\}, Ha_2 := \{\epsilon e^{it} : t \in [-\theta, \theta]\}, Ha_3 := \{te^{-i\theta} : t \in [\epsilon, \infty)\}. \quad (2.5)$$

Inspired by the construction done on the last sections, we consider the following. (See, for instance, [11, 21, 62, 66]).

Theorem 2.41 *Let $\alpha \in (0, 1)$ and suppose that $A : \mathcal{D}(A) \subset X \rightarrow X$ is a positive sectorial operator. Then, the operators*

$$E_\alpha(-t^\alpha A) := \frac{1}{2\pi i} \int_{Ha} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} d\lambda, \quad t \geq 0,$$

and

$$E_{\alpha,\alpha}(-t^\alpha A) := \frac{t^{1-\alpha}}{2\pi i} \int_{Ha} e^{\lambda t} (\lambda^\alpha + A)^{-1} d\lambda, \quad t \geq 0$$

(where $Ha \subset \rho(-A)$ and is given by (2.5)) are well defined and $E_\alpha(-t^\alpha A)$ is strongly continuous, i.e., for each $x \in X$

$$\lim_{t \rightarrow 0^+} \|E_\alpha(-t^\alpha A)x - x\| = 0.$$

Furthermore, there exists a constant $M > 0$ (uniform on α) such that

$$\sup_{t \geq 0} \|E_\alpha(-t^\alpha A)\|_{\mathcal{L}(X)} \leq M \quad \text{and} \quad \sup_{t \geq 0} \|E_{\alpha,\alpha}(-t^\alpha A)\|_{\mathcal{L}(X)} \leq M.$$

Proof : Let $\phi \in (0, \pi/2)$, S_ϕ be the sector associated with the positive sectorial operator A and $N \geq 1$ such that

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{N}{|\lambda|}, \quad \forall \lambda \in S_\phi \setminus \{0\}.$$

For each $t > 0$, assume that $\epsilon = 1/t$, $\theta \in (\pi/2, \pi - \phi)$ and consider the Hankel's path $Ha = Ha(\epsilon, \theta)$ (see the Figure 2.3 below).

We will estimate the function $\|E_\alpha(-t^\alpha A)\|_{\mathcal{L}(X)}$ on each Ha_i , $i \in \{1, 2, 3\}$. Just observe that

i) On Ha_1 , it holds that

$$\left\| \frac{1}{2\pi i} \int_{Ha_1} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} d\lambda \right\|_{\mathcal{L}(X)} \leq \frac{1}{2\pi} \left\| \int_\epsilon^\infty e^{tse^{i\theta}} (se^{i\theta})^{\alpha-1} ((se^{i\theta})^\alpha + A)^{-1} e^{i\theta} ds \right\|_{\mathcal{L}(X)}$$

and using that if $\lambda = se^{i\theta} \in Ha(\epsilon, \theta) \subset -S_\phi$, then $\lambda^\alpha \in -S_\phi$, we obtain by the

Figure 2.3: Sector $-S_\phi$ and Hankel's path

sectorial property that

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_{Ha_1} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} d\lambda \right\|_{\mathcal{L}(X)} &\leq \frac{N}{2\pi} \int_\epsilon^\infty e^{ts \cos(\theta)} |(se^{i\theta})|^{-1} ds \\ &\leq \frac{N}{2\pi\epsilon} \int_\epsilon^\infty e^{ts \cos(\theta)} ds = \frac{Ne^{\cos(\theta)}}{-2\pi \cos(\theta)}. \end{aligned}$$

ii) On Ha_2 , we have that

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_{Ha_2} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} d\lambda \right\|_{\mathcal{L}(X)} &\leq \frac{1}{2\pi} \left\| \int_{-\theta}^\theta e^{te^{is}} (\epsilon e^{is})^{\alpha-1} ((\epsilon e^{is})^\alpha + A)^{-1} i \epsilon e^{is} ds \right\|_{\mathcal{L}(X)} \\ &\leq \frac{N}{2\pi} \int_{-\theta}^\theta e^{t\epsilon \cos(s)} ds \leq \frac{\theta N e}{\pi}. \end{aligned}$$

iii) On Ha_3 we proceed in the same way as in Ha_1 .

Taking M as the maximum over all the bounds obtained above, we deduce that the norm $\|E_\alpha(-t^\alpha A)\|_{\mathcal{L}(X)}$ is well defined for each $t > 0$ and bounded over the specific Hankel's path, chosen before.

Now we shall seek a uniform bound for the function $\|E_\alpha(-t^\alpha A)\|_{\mathcal{L}(X)}$. To conclude, it is

enough to justify that the value of the integral considered is independent of the Hankel's path chosen, once the estimates does not depends on $t > 0$. For this, observe that if $0 < \epsilon < \epsilon'$ and $\pi/2 < \theta' < \theta < \pi - \phi$, we take $Ha = Ha(\epsilon, \theta)$ and $Ha' = Ha(\epsilon', \theta')$ to obtain the equality (Notice, by the choices made on the last line, that $Ha, Ha' \subset \rho(-A)$)

$$\frac{1}{2\pi i} \int_{Ha} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} d\lambda = \frac{1}{2\pi i} \int_{Ha'} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} d\lambda \quad \forall t > 0,$$

doing just some computations. Indeed, first consider the path $\xi_R = Ha'_R - \xi_{1,R} - Ha_R + \xi_{2,R}$, where

$$\xi_R = \begin{cases} Ha'_R := \{z \in Ha' : |z| \leq R/\sin \theta'\} \\ \xi_{1,R} := \{t + iR : t \in [R/\tan \theta, R/\tan \theta']\} \\ Ha_R := \{z \in Ha : |z| \leq R/\sin \theta\} \\ \xi_{2,R} := \{t - iR : t \in [R/\tan \theta, R/\tan \theta']\} \end{cases}$$

as on the Figure 2.4.

Figure 2.4: ξ_R path

Then by Cauchy's Theorem, we conclude that for any $R > \epsilon'$,

$$\frac{1}{2\pi i} \int_{\xi_R} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} d\lambda = 0, \quad \forall t > 0,$$

which guarantees that

$$\frac{1}{2\pi i} \int_{Ha_R} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} d\lambda = \frac{1}{2\pi i} \int_{Ha'_R - \xi_{1,R} + \xi_{2,R}} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} d\lambda, \quad \forall t > 0.$$

Now if we just do the same computations as before in Proposition 2.8, when $R \rightarrow \infty$, we obtain the desired result.

To verify the boundedness for $t = 0$, we observe that making a change of variables

$$E_\alpha(-t^\alpha A) = \frac{1}{2\pi i} \int_{Ha} e^{\lambda} \lambda^{\alpha-1} (\lambda^\alpha + A t^\alpha)^{-1} d\lambda, \quad t \geq 0,$$

and therefore, for $t = 0$, we obtain

$$E_\alpha(-0^\alpha A) = \left[\frac{1}{2\pi i} \int_{Ha} e^{\lambda} \lambda^{-1} d\lambda \right] Id,$$

where Id is the identity operator. By standard Complex Analysis arguments using the Cauchy's theorem, we conclude that

$$E_\alpha(-0^\alpha A) = Id$$

Finally, let us prove that $E_\alpha(-t^\alpha A)$ is strongly continuous. Observe that it is sufficient to prove for $x \in \mathcal{D}(A)$, once $\|E_\alpha(-t^\alpha A)\|_{\mathcal{L}(X)} \leq M$ for all $t \geq 0$. Let $x \in \mathcal{D}(A)$ and observe for $t > 0$, that

$$\begin{aligned} E_\alpha(-t^\alpha A)x - x &= \frac{1}{2\pi i} \int_{Ha} e^{\lambda t} \left[\lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} - \lambda^{-1} \right] x d\lambda \\ &= \frac{1}{2\pi i} \int_{Ha} e^{\lambda t} \left[\lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} - \lambda^{-1} (\lambda^\alpha + A)^{-1} (\lambda^\alpha + A) \right] x d\lambda \\ &= \frac{1}{2\pi i} \int_{Ha} e^{\lambda t} (\lambda^\alpha + A)^{-1} \left[-\lambda^{-1} A \right] x d\lambda \\ &= -\frac{1}{2\pi i} \int_{Ha} e^{\mu} \left[\left(\frac{\mu}{t} \right)^\alpha + A \right]^{-1} \left[\left(\frac{\mu}{t} \right)^{-1} A \right] x \frac{d\mu}{t} \end{aligned}$$

therefore

$$\|E_\alpha(-t^\alpha A)x - x\| \leq \frac{1}{2\pi} \int_{Ha} e^{\operatorname{Re}(\mu)} N \frac{t^\alpha}{|\mu|^{\alpha+1}} \|Ax\| d|\mu| \leq C \|Ax\| t^\alpha$$

which conclude the proof of this property.

A similar procedure proves that

$$\sup_{t \in [0, \infty)} \|E_{\alpha, \alpha}(-t^\alpha A)\|_{\mathcal{L}(X)} \leq M$$

independently of the Hankel's path chosen. $\square$

Now using the Mainardi function defined on Section 2.1, we construct another integral formula for this operator. This new formula is quite used as a tool in fractional framework to prove some useful properties (see [62, 66, 79]).

Theorem 2.42 *Consider the same hypothesis of Theorem 2.41. Then, we can rewrite the operators $\{E_\alpha(-t^\alpha A) : t \geq 0\}$ and $\{E_{\alpha, \alpha}(-t^\alpha A) : t \geq 0\}$ as follows*

$$E_\alpha(-t^\alpha A) = \int_0^\infty M_\alpha(s) T(st^\alpha) ds, \quad t \geq 0,$$

and

$$E_{\alpha, \alpha}(-t^\alpha A) = \int_0^\infty \alpha s M_\alpha(s) T(st^\alpha) ds, \quad t \geq 0,$$

where $\{T(t) : t \geq 0\}$ is the C_0 -semigroup generated by $-A$.

Proof: This proof can be found in [62, Lemma 9]. $\square$

The following proposition is an important result and can be found on [11].

Proposition 2.43 *Consider $\alpha \in (0, 1)$ and suppose that $A : \mathcal{D}(A) \subset X \rightarrow X$ is a positive sectorial operator. Then for any $x \in X$, it holds that*

$$\mathcal{L}\{E_\alpha(-t^\alpha A)x\}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha + A)^{-1}x$$

and

$$\mathcal{L}\{t^{\alpha-1}E_{\alpha, \alpha}(-t^\alpha A)x\}(\lambda) = (\lambda^\alpha + A)^{-1}x.$$

Proof: Both equalities may be proved analogously, so we will only check the second

one. For any $x \in X$, observe that by Theorem 2.42

$$\begin{aligned}\mathcal{L}\{t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha A)x\}(\lambda) &= \int_0^\infty e^{-\lambda t} t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha A)x \, dt \\ &= \int_0^\infty e^{-\lambda t} t^{\alpha-1} \left(\int_0^\infty \alpha s M_\alpha(s) T(st^\alpha)x \, ds \right) dt\end{aligned}$$

and with a change of variables ($s = \omega t^{-\alpha}$), we conclude

$$\begin{aligned}\mathcal{L}\{t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha A)x\}(\lambda) &= \int_0^\infty e^{-\lambda t} t^{\alpha-1} \left(\int_0^\infty \alpha(\omega t^{-\alpha}) M_\alpha(\omega t^{-\alpha}) T(\omega)x t^{-\alpha} d\omega \right) dt \\ &= \int_0^\infty \omega \underbrace{\left(\int_0^\infty \alpha t^{-(1+\alpha)} M_\alpha(\omega t^{-\alpha}) e^{-\lambda t} dt \right)}_{\mathcal{H}} T(\omega)x \, d\omega.\end{aligned}$$

So taking $t = \tau \omega^{1/\alpha}$ in $\mathcal{H}$, we obtain by item v) of Proposition 2.22, that

$$\begin{aligned}\mathcal{H} &= \int_0^\infty \alpha(\tau \omega^{1/\alpha})^{-(1+\alpha)} M_\alpha(\omega(\tau \omega^{1/\alpha})^{-\alpha}) e^{-\lambda(\tau \omega^{1/\alpha})} \omega^{1/\alpha} d\tau \\ &= \omega^{-1} \int_0^\infty \alpha \tau^{-(1+\alpha)} M_\alpha(\tau^{-\alpha}) e^{-(\lambda \omega^{1/\alpha})\tau} d\tau \\ &= \omega^{-1} e^{-\lambda^\alpha \omega}.\end{aligned}$$

Therefore by Theorem 1.17, we conclude that

$$\mathcal{L}\{t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha A)x\}(\lambda) = \int_0^\infty e^{-\lambda^\alpha \omega} T(\omega)x \, d\omega = (\lambda^\alpha + A)^{-1}x.$$

□

At this point, we turn our attention to a fundamental result about the solution of the linear fractional abstract problem in Banach spaces. Observe that together with Theorem 2.41, it forms a fractional version of Theorem 1.22, to positive sectorial operators.

Theorem 2.44 *Consider the same hypothesis of Theorem 2.41. Then, for each $x \in X$, the function $E_\alpha(t^\alpha A)x$ is analytic for $t \geq 0$ and it is the unique solution of*

$$cD_t^\alpha E_\alpha(-t^\alpha A)x = -AE_\alpha(-t^\alpha A)x, \quad t > 0.$$

Proof: Consider $x \in X$ and define $v(t) = E_\alpha(-t^\alpha A)x$. The analyticity follows from a theorem proved in [11, Chapter 2]. Therefore, by Proposition 2.34 we compute

$$cD_t^\alpha v(t) = J_t^{1-\alpha} v'(t) = g_{1-\alpha} * v'(t).$$

Observe that the analyticity and Theorem 2.41 guarantees the remaining conditions to $v'(t)$ be in the domain of the Laplace operator. Hence, we obtain

$$\mathcal{L}\{cD_t^\alpha v(t)\}(\lambda) = \mathcal{L}\{g_{1-\alpha} * v'(t)\}(\lambda) = \mathcal{L}\{g_{1-\alpha}(t)\}(\lambda) \mathcal{L}\{v'(t)\}(\lambda)$$

for $\operatorname{Re}(\lambda) > 0$. Using the Laplace Transform properties and Example 2.6, we rewrite the later equality as

$$\mathcal{L}\{cD_t^\alpha v(t)\}(\lambda) = [\lambda^{\alpha-1}] [\lambda \mathcal{L}\{v(t)\}(\lambda) - v(0)]$$

and since Proposition 2.43 guarantees a formula to $\hat{v}(\lambda)$, we conclude that

$$\mathcal{L}\{cD_t^\alpha v(t)\}(\lambda) = \lambda^{2\alpha-1}(\lambda^\alpha + A)^{-1}x - \lambda^{\alpha-1}x,$$

for $\operatorname{Re}(\lambda) > 0$.

On the other hand, since A is a closed operator, by Proposition 2.43

$$\mathcal{L}\{-Av(t)\}(\lambda) = -A\hat{v}(\lambda) = -A[\lambda^{\alpha-1}(\lambda^\alpha + A)^{-1}x],$$

for $\operatorname{Re}(\lambda) > 0$. Using the identity $A(\lambda^\alpha + A)^{-1}x = x - \lambda^\alpha(\lambda^\alpha + A)^{-1}x$, we conclude that

$$\mathcal{L}\{-Av(t)\}(\lambda) = \lambda^{2\alpha-1}(\lambda^\alpha + A)^{-1}x - \lambda^{\alpha-1}x,$$

for $\operatorname{Re}(\lambda) > 0$. By the uniqueness of the inverse Laplace transform we complete the proof of the theorem. The uniqueness of solution to the equation is guarantee by the standard literature. (See, for instance, [11, 66]). $\square$

Our goal now is to establish an expression for Mittag-Leffler functions associated to positive sectorial operators similar to the second fundamental limit for semigroups (see Theorem 1.18). For this, we recall the Post-Widder inversion formula, which can be found in [76]. This expression will be very useful.

Lemma 2.45 (Post-Widder) *Let $u : [0, \infty) \rightarrow X$ be a continuous function such that $u(t) = O(\exp(\omega t))$ as $t \rightarrow \infty$ for some $\omega \in \mathbb{R}$, and let $\hat{u}$ be the Laplace transform of u .*

Then

$$u(t) = \lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \left(\frac{d^n}{d\lambda^n} \hat{u}\right)(n/t),$$

uniformly on compact sets of $(0, \infty)$.

The following result proves some interesting properties of the Mittag-Leffler operators. We shall observe that only the first part of this theorem was proved on the literature (see for instance [11, Section 2.1]). The second part is one of the new results obtained on this thesis.

Proposition 2.46 Consider $A : \mathcal{D}(A) \subset X \rightarrow X$ a positive sectorial operator and $\alpha \in (0, 1)$. Let $\{E_\alpha(-t^\alpha A) : t \geq 0\}$ and $\{E_{\alpha,\alpha}(-t^\alpha A) : t \geq 0\}$ be the Mittag-Leffler families associated to $-A$. Then, for each $x \in X$ we have

$$E_\alpha(-t^\alpha A)x = \lim_{n \rightarrow \infty} \frac{1}{n!} \sum_{k=1}^{n+1} b_{k,n+1}^\alpha \left(\left(\frac{n}{t}\right)^\alpha \left(\left(\frac{n}{t}\right)^\alpha + A \right)^{-1} \right)^k x, \quad (2.6)$$

and

$$t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha A)x = \lim_{n \rightarrow \infty} \frac{1}{n!} \sum_{k=1}^n \alpha k b_{k,n}^\alpha \left(\frac{n}{t}\right)^{\alpha k+1} \left(\left(\frac{n}{t}\right)^\alpha + A \right)^{-(k+1)} x, \quad (2.7)$$

uniformly on compact sets of $(0, \infty)$, where the positive real numbers $b_{k,n}^\alpha$ are defined by

$$\begin{cases} b_{1,1}^\alpha = 1, \\ b_{k,n}^\alpha = (n-1-\alpha k)b_{k,n-1}^\alpha + \alpha(k-1)b_{k-1,n-1}^\alpha, & \text{if } 1 \leq k \leq n \text{ and } n = 2, 3, \dots \\ b_{k,n}^\alpha = 0, & \text{if } k > n \text{ and } n = 1, 2, \dots \end{cases} \quad (2.8)$$

Proof: By Theorem 2.41 there exist $M \geq 0$ such that

$$\max \left\{ \sup_{t \geq 0} \|E_\alpha(-t^\alpha A)\|_{\mathcal{L}(X)}, \sup_{t \geq 0} \|E_{\alpha,\alpha}(-t^\alpha A)\|_{\mathcal{L}(X)} \right\} \leq M.$$

Let us begin proving (2.6). Observe that the function $\lambda^{\alpha-1}(\lambda^\alpha + A)^{-1}$ is analytic over $\rho(-A)$. Therefore, for $\operatorname{Re} \lambda > 0$ we obtain

$$\begin{aligned} \frac{d}{d\lambda} (\lambda^{\alpha-1}(\lambda^\alpha + A)^{-1})x &= (\alpha-1)\lambda^{\alpha-2}(\lambda^\alpha + A)^{-1}x - \alpha\lambda^{2(\alpha-1)}(\lambda^\alpha + A)^{-2}x \\ &= -\lambda^{-2} \sum_{k=1}^2 b_{k,2}^\alpha [\lambda^\alpha (\lambda^\alpha + A)^{-1}]^k x. \end{aligned}$$

By induction in n , we conclude that

$$\frac{d^n}{d\lambda^n}(\lambda^{\alpha-1}(\lambda^\alpha + A)^{-1})x = (-1)^n \lambda^{-(n+1)} \sum_{k=1}^{n+1} b_{k,n+1}^\alpha [\lambda^\alpha (\lambda^\alpha + A)^{-1}]^k x, \quad n = 1, 2, \dots$$

for all $x \in X$. Furthermore, by Proposition 2.43, for each $x \in X$ the continuous function $v(t) = E_\alpha(-t^\alpha A)x$ is such that

$$\mathcal{L}\{v(t)\}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha + A)^{-1}x.$$

Hence, (2.6) follows by Lemma 2.45.

On the other hand, to prove (2.7), observe that

$$\frac{d^n}{d\lambda^n}(\lambda^\alpha + A)^{-1}x = (-1)^n \lambda^{-n} \sum_{k=1}^n \alpha k b_{k,n}^\alpha [\lambda^{\alpha k} (\lambda^\alpha + A)^{-(k+1)}]x, \quad n = 1, 2, \dots$$

for all $x \in X$. Again by Proposition 2.43, for each $x \in X$ the function $v(t) = t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha A)x$ verifies

$$\mathcal{L}\{v(t)\}(\lambda) = (\lambda^\alpha + A)^{-1}x.$$

Using again Lemma 2.45, the proof is complete. $\square$

The next result give us information on the Mittag-Leffler families in the fractional power spaces X^β (see Section 1.3), $\beta \geq 0$, associated to the positive sectorial operator A . This formulation is an implementation of fractional spaces in the fractional calculus, to study the existence of solutions in such spaces (see Section 3.3 for more information). A similar result can be found on [66].

Theorem 2.47 Consider $\alpha \in (0, 1)$, $0 \leq \beta \leq 1$, and suppose that $A : \mathcal{D}(A) \subset X \rightarrow X$ is a positive sectorial operator. Then, there exists a constant $M > 0$ such that

$$\|E_\alpha(-t^\alpha A)x\|_{X^\beta} \leq M t^{-\alpha\beta} \|x\|_X \quad \text{and} \quad \|E_{\alpha,\alpha}(-t^\alpha A)x\|_{X^\beta} \leq M t^{-\alpha\beta} \|x\|_X$$

for all $t > 0$.

Proof: We have that

$$\|E_\alpha(-t^\alpha A)x\|_{X^\beta} = \left\| \frac{A^\beta}{2\pi i} \int_{Ha} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} x d\lambda \right\|_X.$$

Then, since A is a closed operator, we conclude that

$$\begin{aligned} \|E_\alpha(-t^\alpha A)x\|_{X^\beta} &= \left\| \frac{1}{2\pi i} \int_{Ha} e^{\lambda t} \lambda^{\alpha-1} A^\beta (\lambda^\alpha + A)^{-1} x \, d\lambda \right\|_X \\ &= \frac{t^{-\alpha}}{2\pi} \left\| \int_{Ha} e^\mu \mu^{\alpha-1} A^\beta ((\mu/t)^\alpha + A)^{-1} x \, d\mu \right\|_X. \end{aligned}$$

Hence, estimating the last equation we obtain

$$\|E_\alpha(-t^\alpha A)x\|_{X^\beta} \leq \frac{t^{-\alpha}}{2\pi} \int_{Ha} e^{\operatorname{Re}(\mu)} |\mu|^{\alpha-1} \|A^\beta ((\mu/t)^\alpha + A)^{-1} x\|_X \, d\mu$$

and by Theorem 1.27

$$\|E_\alpha(-t^\alpha A)x\|_{X^\beta} \leq t^{-\alpha\beta} \left(\frac{C}{2\pi} \int_{Ha} e^{\operatorname{Re}(\mu)} |\mu|^{\alpha(\beta-1)} \, d|\mu| \right) \|x\|_X.$$

Observe now that it is sufficient to choose $M > 0$ such that

$$\frac{C}{2\pi} \int_{Ha} e^{\operatorname{Re}(\lambda)} |\lambda|^{\alpha(\beta-1)} \, d|\lambda| \leq M.$$

By a similar procedure one may prove the second estimate. □

3

Abstract Fractional Equations

The abstract fractional Cauchy problem has been studied for some time and although recently many relevant results were obtained in this area, even the very basic theory of the fractional differential equations is incomplete and there is much that needs to be done (*see for instance [11]*). In this chapter we discuss issues that seek to answer some questions outstanding in this area. To that end, we consider the fractional Cauchy problem

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f(t, u(t)), & t \geq 0 \\ u(0) = u_0 \in X, \end{cases} \quad (3.1)$$

where X is a Banach space over $\mathbb{C}$, $\alpha \in (0, 1)$, $A : \mathcal{D}(A) \subset X \rightarrow X$ is a positive sectorial operator, cD_t^α is the Caputo fractional derivative and $f : [0, \infty) \times X \rightarrow X$ is a continuous function.

Consider $\{E_\alpha(-t^\alpha A) : t \geq 0\}$ and $\{E_{\alpha, \alpha}(-t^\alpha A) : t \geq 0\}$ the Mittag-Leffler families associated to $-A$, discussed at Section 2.4. At this point, even formally, we want to find an appropriate definition for the concept of solution to the problem (3.1).

Its quite a common knowledge, that one of the most difficult points in this kind of new approach to obtain existence and uniqueness of solutions to differential equations, is to give a reasonable concept of solutions (*see the work of Hernandez-O'Regan-Balachandran [34]*). Therefore, we should discuss the two different notions of solution to (3.1), used in the literature.

One of the approaches follows the idea that: since A is a positive sectorial operator, we already know that there exists a C_0 -semigroup $\{T(t) : t \geq 0\}$ associated to $-A$ and inspired by the usual case, some research was done in the study of the solution given by

the integral representation

$$u(t) = T(t)u_0 + \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, u(s)) ds, \quad t \geq 0,$$

as can be seen in [38, 49, 50]. But, following the same observations done in [66], by Wang-Chen-Xiao, we understand the need of a warning. This definition does not fit some basic properties, for instance:

- i) If $f \equiv 0$, then we obtain that $u(t) = T(t)u_0$, which contradicts what was proved on Section 2.4 (see Theorem 2.44), since $\{T(t) : t \geq 0\}$ is not the Mittag-Leffler operator.
- ii) Moreover, it does not matter how smooth is the initial data u_0 , there is no proof in the literature to ensure that the function

$$u(t) = T(t)u_0 + \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, u(s)) ds, \quad t \geq 0,$$

satisfies the equations of the problem (3.1).

For this we will construct a more acceptable notion of solution, according to the theory itself. Suppose for a moment that $u : [0, \infty) \rightarrow X$ is a continuous function that satisfies (3.1). Then, operating with J_t^α at both sides of the fractional differential equation, observe that

$$\begin{aligned} u(t) &= u(0) + J_t^\alpha(-A)u(t) + J_t^\alpha f(t, u(t)) \\ &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (-A)u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds, \quad t \geq 0. \end{aligned}$$

Now assuming that this function is of exponential type and is locally integrable, we apply the Laplace transform ($\mathcal{L}$) on both sides and by Theorem 2.14 we obtain

$$\hat{u}(\lambda) = \frac{u_0}{\lambda} + \frac{1}{\lambda^\alpha} (-A)\hat{u}(\lambda) + \frac{1}{\lambda^\alpha} (\widehat{f(u)})(\lambda) \Rightarrow \lambda^\alpha \hat{u}(\lambda) = u_0 \lambda^{\alpha-1} + (-A)\hat{u}(\lambda) + (\widehat{f(u)})(\lambda),$$

where $\hat{u}(\lambda) = \mathcal{L}\{u(t)\}(\lambda)$ and $\widehat{f(u)}(\lambda) = \mathcal{L}\{f(t, u(t))\}(\lambda)$. Suppose that $\lambda^\alpha \in \rho(-A)$. Following with the algebraic manipulation, we can conclude that

$$\hat{u}(\lambda) = \lambda^{\alpha-1} (\lambda^\alpha + A)^{-1} u_0 + (\lambda^\alpha + A)^{-1} (\widehat{f(u)})(\lambda).$$

Applying the inverse of the Laplace transform in both sides of the equality, we deduce

(formally) the expression

$$u(t) = \mathcal{L}^{-1}\{\lambda^{\alpha-1}(\lambda^\alpha + A)^{-1}u_0\}(t) + \mathcal{L}^{-1}\{(\lambda^\alpha + A)^{-1}(\widehat{f(u)})(\lambda)\}(t), \quad t \geq 0$$

and using the Remark 2.15,

$$\begin{aligned} u(t) &= \mathcal{L}^{-1}\{\lambda^{\alpha-1}(\lambda^\alpha + A)^{-1}u_0\}(t) + \int_0^t \mathcal{L}^{-1}\{(\lambda^\alpha + A)^{-1}\}(t-s) \mathcal{L}^{-1}\{(\widehat{f(u)})(\lambda)\}(s) ds \\ &= \mathcal{L}^{-1}\{\lambda^{\alpha-1}(\lambda^\alpha + A)^{-1}u_0\}(t) + \int_0^t \mathcal{L}^{-1}\{(\lambda^\alpha + A)^{-1}\}(t-s) f(s, u(s)) ds. \end{aligned}$$

Finally, by Proposition 2.43, we end up with the formula

$$u(t) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s)) ds, \quad t \geq 0.$$

Inspired by the above discussion and the literature (see [44, 49, 67]), we adopt the following concept of global mild solution.

Definition 3.1 Let $u : [0, \infty) \rightarrow X$ be a continuous function. We say that u is a global mild solution to (3.1) if

$$u(t) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s)) ds, \quad t \geq 0.$$

Observe that this definition does not discuss the notion of local mild solution, since the Laplace transform is applied only to functions defined over the positive real line. Thus, motivated by this definition and by previous literature of this theory, we consider the following (see, for instance, [67]).

Definition 3.2 Let $\tau > 0$.

- i) A function $u : [0, \tau] \rightarrow X$ is called a local mild solution of (3.1) in $[0, \tau]$ if $u \in C([0, \tau]; X)$ and

$$u(t) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s)) ds, \quad t \in [0, \tau]. \quad (3.2)$$

§ 3.1 EXISTENCE, UNIQUENESS AND THE FRACTIONAL LIMIT

Our main purpose in this section is to ensure sufficient conditions for existence and uniqueness of mild solution to (3.1). Then we consider a solution $u^\alpha(t)$ of (3.1) and

compare the limit when $\alpha \rightarrow 1^-$ with the solution of the usual problem ($\alpha = 1$). Notice that the notion of mild solution to (3.1) is intimately connected to the Mittag-Leffler functions, therefore we use extensively the results obtained in Chapter 2. We start with the following local result.

Theorem 3.3 *Let $f : [0, \infty) \times X \rightarrow X$ be a continuous function, and locally Lipschitz in the second variable, uniformly with respect to the first variable, that is, for each fixed $x \in X$, there exists an open ball B_x and a constant $L = L(B_x) \geq 0$ such that*

$$\|f(t, z) - f(t, y)\| \leq L\|z - y\|$$

for all $z, y \in B_x$ and $t \in [0, \infty)$. Then, there exist $t_0 > 0$ such that (3.1) has a unique local mild solution in $[0, t_0]$.

Proof: Given $u_0 \in X$, let $B_{u_0}(r)$ be the open ball with center u_0 and radius $r > 0$ and $L = L(B_{u_0}(r))$ be the Lipschitz constant of f associated to $B_{u_0}(r)$. Fix $\beta \in (0, r)$ and using Theorem 2.41, choose $t_0 > 0$ such that

$$(M/\alpha)(L\beta + C)t_0^\alpha \leq \beta/2 \quad \text{and} \quad \|E_\alpha(-t^\alpha A)u_0 - u_0\| \leq \beta/2 \quad \forall t \in [0, t_0],$$

where $M = \sup_{t \in [0, \infty)} \|E_{\alpha, \alpha}(-t^\alpha A)\|_{\mathcal{L}(X)}$ and $C = \sup_{t \in [0, t_0]} \|f(s, u_0)\|$. Consider

$$K := \{u \in C([0, t_0]; X) : u(0) = u_0 \quad \text{and} \quad \|u(t) - u_0\| \leq \beta \quad \forall t \in [0, t_0]\}$$

with the norm

$$\|u\|_K := \sup_{s \in [0, t_0]} \|u(s)\|.$$

Then define the operator T on K by

$$T(u(t)) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A) f(s, u(s)) ds.$$

If $u \in K$, then $T(u(0)) = u_0$ and $T(u(t)) \in C([0, t_0]; X)$. Furthermore, we have that

$$\begin{aligned} \|T(u(t)) - u_0\| &\leq \|E_\alpha(-t^\alpha A)u_0 - u_0\| \\ &\quad + \int_0^t (t-s)^{\alpha-1} M (\|f(s, u(s)) - f(s, u_0)\| + \|f(s, u_0)\|) ds \\ &\leq \|E_\alpha(-t^\alpha A)u_0 - u_0\| + (M/\alpha)(L\beta + C)t_0^\alpha \leq \beta/2 + \beta/2 = \beta \end{aligned}$$

that is, $T(K) \subset K$. Now, if $u, v \in K$, we see that

$$\begin{aligned} \|T(u(t)) - T(v(t))\| &\leq \int_0^t (t-s)^{\alpha-1} M \|f(s, u(s)) - f(s, v(s))\| ds \\ &\leq \frac{LMt_0^\alpha}{\alpha} \sup_{s \in [0, t_0]} \|u(s) - v(s)\| \\ &\leq \frac{1}{2} \sup_{s \in [0, t_0]} \|u(s) - v(s)\|. \end{aligned}$$

So, by the Banach contraction principle we have that T has a unique fixed point in K . This proves the theorem. □

The remainder of this section will be devoted to the problem of continuation of local mild solutions and existence of global mild solutions to (3.1).

Definition 3.4 Let $u : [0, \tau] \rightarrow X$ be the unique local mild solution in $[0, \tau]$ to (3.1). If $\tau^* > \tau$ and $u^* : [0, \tau^*] \rightarrow X$ is a local mild solution to (3.1) in $[0, \tau^*]$, then we say that u^* is a continuation of u over $[0, \tau^*]$.

Definition 3.5 If $u : [0, \tau^*) \rightarrow X$ is the unique local mild solution to (3.1) in $[0, \tau]$ for all $\tau \in (0, \tau^*)$ and does not have a continuation, then we call it a maximal local mild solution.

Now we prove a theorem related to the existence of a continuation to a local mild solution, as defined before. First, we highlight the importance of such theorem. When we deal with the case $\alpha = 1$, we already know that different solutions of autonomous equations does not intersect. However, the nonlinearity carried by the definition of the Riemann-Liouville fractional integral operator, does not allow us to have this property for autonomous fractional differential equations.

Even though the theorem that follows has a very similar statement to the case $\alpha = 1$, its proof follows subtle distinct lines.

Theorem 3.6 Let $f : [0, \infty) \times X \rightarrow X$ be as in Theorem 3.3. If $u : [0, t_0] \rightarrow X$ is the unique local mild solution to (3.1) in $[0, t_0]$, then there exists a unique continuation u^* of u in some interval $[0, t_0 + \tau]$ with $\tau > 0$.

Proof: Let $u : [0, t_0] \rightarrow X$ be the unique local mild solution to (3.1) in $[0, t_0]$. Since f is locally Lipschitz, there exist $r > 0$, an open ball $B = B_{u(t_0)}(r)$ with center in $u(t_0)$ and radius r , and $L = L_{u(t_0)}$ the Lipschitz constant of f associated to B . Fix $\beta \in (0, r)$ and

choose $\tau > 0$ such that the following conditions are satisfied

- $\|E_\alpha(-t^\alpha A)u_0 - E_\alpha(-t_0^\alpha A)u_0\| \leq \beta/4,$
- $(M/\alpha)(L\beta + C)\tau^\alpha \leq \beta/4, \quad (MD/\alpha)[t^\alpha - (t - t_0)^\alpha - t_0^\alpha] \leq \beta/4,$
- $\int_0^{t_0} (t_0 - s)^{\alpha-1} \| [E_{\alpha,\alpha}(-(t-s)^\alpha A) - E_{\alpha,\alpha}(-(t_0-s)^\alpha A)] f(s, u(s)) \| ds \leq \beta/4$

for all $t \in [t_0, t_0 + \tau]$, where

$$C = \sup_{s \in [0, t_0]} \|f(s, u(t_0))\|,$$

$$D = \sup_{s \in [0, t_0]} \|f(s, u(s))\|,$$

$$M = \max \left\{ \sup_{t \in [0, \infty)} \|E_{\alpha,\alpha}(-t^\alpha A)\|_{\mathcal{L}(X)}, \sup_{t \in [0, \infty)} \|E_\alpha(-t^\alpha A)\|_{\mathcal{L}(X)} \right\}.$$

Consider

$$K := \left\{ w \in C([0, t_0 + \tau]; X) : \begin{array}{l} w(t) = u(t) \text{ for all } t \in [0, t_0] \text{ and} \\ \|w(t) - u(t_0)\| \leq \beta \text{ for all } t \in [t_0, t_0 + \tau] \end{array} \right\}$$

and $T : K \rightarrow C([0, t_0 + \tau]; X)$ given by

$$T(w(t)) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, w(s)) ds.$$

We check that $T(K) \subset K$.

i) If $w \in K$, then $w(t) = u(t)$ in $[0, t_0]$ with u the local mild solution to (3.1) in $[0, t_0]$.

So, if $t \in [0, t_0]$,

$$\begin{aligned} T(w(t)) &= E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, w(s)) ds \\ &= E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s)) ds = u(t). \end{aligned}$$

ii) If $t \in [t_0, t_0 + \tau]$, then

$$\begin{aligned}
\|T(w(t)) - u(t_0)\| &\leq \|E_\alpha(-t^\alpha A)u_0 - E_\alpha(-t_0^\alpha A)u_0\| \\
&\quad + \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, w(s)) ds \right. \\
&\quad \left. - \int_0^{t_0} (t_0-s)^{\alpha-1} E_{\alpha,\alpha}(-(t_0-s)^\alpha A) f(s, w(s)) ds \right\| \\
&\leq \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 \\
&\leq \beta,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{J}_1 &= \|E_\alpha(-t^\alpha A)u_0 - E_\alpha(-t_0^\alpha A)u_0\| \leq \beta/4, \\
\mathcal{J}_2 &= \left\| \int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, w(s)) ds \right\| \leq \beta/4, \\
\mathcal{J}_3 &= \left\| \int_0^{t_0} [(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}] E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, w(s)) ds \right\| \leq \beta/4, \\
\mathcal{J}_4 &= \left\| \int_0^{t_0} (t_0-s)^{\alpha-1} [E_{\alpha,\alpha}(-(t-s)^\alpha A) - E_{\alpha,\alpha}(-(t_0-s)^\alpha A)] f(s, w(s)) ds \right\| \leq \beta/4.
\end{aligned}$$

By similar computations, we conclude that for every $t \in [0, t_0 + \tau]$

$$\begin{aligned}
\|T(\omega(t)) - T(v(t))\| &\leq \frac{LM\tau^\alpha}{\alpha} \sup_{s \in [0, t_0 + \tau]} \|\omega(s) - v(s)\| \\
&\leq \frac{1}{2} \sup_{s \in [0, t_0 + \tau]} \|\omega(s) - v(s)\| \quad \forall u, v \in K.
\end{aligned}$$

Therefore, by the Banach contraction principle, we conclude that there exists a unique fixed point $u^* \in K$ of the integral equation. $\square$

Now we present a fundamental result to the proof of the main theorem of this section.

Lemma 3.7 Consider $\omega \in (0, \infty)$, $u : [0, \omega) \rightarrow X$ a bounded continuous function, and $f : [0, \infty) \times X \rightarrow X$ a continuous function that maps bounded sets onto bounded sets. If $\{t_n\} \subset [0, \omega)$ satisfies $\lim_{n \rightarrow \infty} t_n = \omega$, then

$$\lim_{n \rightarrow \infty} \int_0^{t_n} (t_n - r)^{\alpha-1} \left\| [E_{\alpha, \alpha}(-(t_n - r)^\alpha A) - E_{\alpha, \alpha}(-(w - r)^\alpha A)] f(r, u(r)) \right\| dr = 0.$$

Proof: Let $M = \sup_{t \in [0, \infty)} \|E_{\alpha, \alpha}(-t^\alpha A)\|_{\mathcal{L}(X)}$ and $K = \sup_{s \in [0, \omega)} \|f(s, u(s))\|$. Given $\epsilon > 0$, fix $\gamma \in (0, \omega)$ such that

$$\frac{(\omega - \gamma)^\alpha}{\alpha} MK < \epsilon/4.$$

Now, choose $N \in \mathbb{N}$ such that

$$\text{i) } t_n > \gamma, \forall n \geq N;$$

$$\text{ii) } \int_0^\gamma (t_n - r)^{\alpha-1} \left\| [E_{\alpha, \alpha}(-(t_n - r)^\alpha A) - E_{\alpha, \alpha}(-(w - r)^\alpha A)] f(r, u(r)) \right\| dr < \epsilon/2, \forall n \geq N.$$

Hence, we conclude that for $n \geq N$,

$$\begin{aligned} \int_0^{t_n} (t_n - r)^{\alpha-1} \left\| [E_{\alpha, \alpha}(-(t_n - r)^\alpha A) - E_{\alpha, \alpha}(-(w - r)^\alpha A)] f(r, u(r)) \right\| dr \\ \leq \epsilon/2 + \int_\gamma^{t_n} (t_n - r)^{\alpha-1} 2MK dr \\ \leq \epsilon/2 + 2 \frac{(\omega - \gamma)^\alpha}{\alpha} MK < \epsilon. \end{aligned}$$

This finishes the proof. $\square$

At last we study a result about the existence of global mild solutions or the “Blow Up” of maximal local mild solutions.

Theorem 3.8 Let $f : [0, \infty) \times X \rightarrow X$ be a continuous function, locally Lipschitz in the second variable, uniformly with respect to the first variable, and bounded (i.e. it maps bounded sets onto bounded sets). Then the problem (1) has a global mild solution in $[0, \infty)$ or there exist $\omega \in (0, \infty)$ such that $u : [0, \omega) \rightarrow X$ is a maximal local mild solution, and in such a case, $\limsup_{t \rightarrow \omega^-} \|u(t)\| = \infty$.

Proof: Consider

$$H := \{ \tau \in [0, \infty) : \exists u_\tau : [0, \tau] \rightarrow X \text{ unique local mild solution to (3.1) in } [0, \tau] \}.$$

If we denote $w = \sup H$, we can consider a continuous function $u : [0, \omega) \rightarrow X$ that is a local mild solution to (3.1) in $[0, \omega)$. If $\omega = \infty$, then u is a global mild solution in $[0, \infty)$. Otherwise, if $\omega < \infty$ we will prove that $\limsup_{t \rightarrow \omega^-} \|u(t)\| = \infty$. By contradiction, suppose that there exist $\tilde{K} < \infty$ such that $\|u(t)\| \leq \tilde{K}$ for all $t \in [0, \omega)$. Therefore, it follows from Lemma 3.7 that if $\{t_n\} \subset [0, \omega)$ is a sequence that converges to ω , given $\epsilon > 0$, there exist $N \in \mathbb{N}$, such that, if $m, n \geq N$, we have

$$\|E_\alpha(-t_n^\alpha A)u_0 - E_\alpha(-t_m^\alpha A)u_0\| < \epsilon/5, \quad |t_n - t_m|^\alpha \frac{MK}{\alpha} < \epsilon/5, \quad |t_n^\alpha - (t_n - t_m)^\alpha - t_m^\alpha| \frac{MK}{\alpha} < \epsilon/5,$$

$$\int_0^{t_n} (t_n - r)^{\alpha-1} \left\| [E_{\alpha,\alpha}(-(t_n - r)^\alpha A) - E_{\alpha,\alpha}(-(w - r)^\alpha A)] f(r, u(r)) \right\| dr < \epsilon/5,$$

$$\int_0^{t_m} (t_m - r)^{\alpha-1} \left\| [E_{\alpha,\alpha}(-(t_m - r)^\alpha A) - E_{\alpha,\alpha}(-(w - r)^\alpha A)] f(r, u(r)) \right\| dr < \epsilon/5,$$

where

$$K = \sup_{t \in [0, \omega)} \|f(t, u(t))\|, \quad M = \max \left\{ \sup_{t \in [0, \infty)} \|E_{\alpha,\alpha}(-t^\alpha A)\|_{\mathcal{L}(X)}, \sup_{t \in [0, \infty)} \|E_\alpha(-t^\alpha A)\|_{\mathcal{L}(X)} \right\}.$$

Hence, for $n, m \geq N$ and assuming, without loss of generality, that $t_n > t_m$, it follows from the estimate

$$\|u(t_n) - u(t_m)\| \leq \|E_\alpha(-t_n^\alpha A)u_0 - E_\alpha(-t_m^\alpha A)u_0\| + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,$$

where

$$\begin{aligned} \mathcal{J}_1 &= \int_{t_m}^{t_n} (t_n - r)^{\alpha-1} \left\| E_{\alpha,\alpha}(-(t_n - r)^\alpha A) f(r, u(r)) \right\| dr \leq |t_n - t_m|^\alpha \frac{MK}{\alpha}; \\ \mathcal{J}_2 &= \int_0^{t_m} (t_n - r)^{\alpha-1} \left\| [E_{\alpha,\alpha}(-(t_n - r)^\alpha A) - E_{\alpha,\alpha}(-(t_m - r)^\alpha A)] f(r, u(r)) \right\| dr \\ &\leq \int_0^{t_m} (t_n - r)^{\alpha-1} \left\| [E_{\alpha,\alpha}(-(t_n - r)^\alpha A) - E_{\alpha,\alpha}(-(\omega - r)^\alpha A)] f(r, u(r)) \right\| dr \\ &\quad + \int_0^{t_m} (t_n - r)^{\alpha-1} \left\| [E_{\alpha,\alpha}(-(t_m - r)^\alpha A) - E_{\alpha,\alpha}(-(\omega - r)^\alpha A)] f(r, u(r)) \right\| dr \end{aligned}$$

which can be estimated by

$$\begin{aligned} \mathcal{J}_2 &\leq \int_0^{t_n} (t_n - r)^{\alpha-1} \left\| [E_{\alpha,\alpha}(-(t_n - r)^\alpha A) - E_{\alpha,\alpha}(-(\omega - r)^\alpha A)] f(r, u(r)) \right\| dr \\ &\quad + \int_0^{t_m} (t_m - r)^{\alpha-1} \left\| [E_{\alpha,\alpha}(-(t_m - r)^\alpha A) - E_{\alpha,\alpha}(-(\omega - r)^\alpha A)] f(r, u(r)) \right\| dr, \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_3 &= \int_0^{t_m} \left| [(t_n - s)^{\alpha-1} - (t_m - s)^{\alpha-1}] \right| \left\| E_{\alpha,\alpha}(-(t_m - s)^\alpha A) f(s, u(s)) \right\| ds \\ &\leq |t_n^\alpha - (t_n - t_m)^\alpha - t_m^\alpha| \frac{MK}{\alpha}, \end{aligned}$$

that

$$\|u(t_n) - u(t_m)\| < \epsilon.$$

This computation shows that $\{u(t_n)\}_{n=0}^\infty$ is a Cauchy sequence and therefore it has a limit, $u_t \in X$. Then, we may extend u over $[0, \omega]$ obtaining the equality

$$u(t) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s)) ds$$

for all $t \in [0, \omega]$. With this, by Theorem 3.6, we can extend the solution to some bigger interval, which is a contradiction with the definition of ω . So if $\omega < \infty$, by the contradiction above, $\limsup_{t \rightarrow \omega^-} \|u(t)\| = \infty$. This concludes the proof. $\square$

Corollary 3.9 *Let $f : [0, \infty) \times X \rightarrow X$ be as in Theorem 3.8 and suppose that there exists a constant $K > 0$ such that if $\|u_0\| \leq K$, then the solutions to (3.1), while exist, are bounded by K . Then (3.1) possesses a unique global mild solution to each $u_0 \in X$ such that $\|u_0\| \leq K$.*

Proof: It is a trivial consequence of the last theorem. $\square$

Proposition 3.10 *Consider $A : \mathcal{D}(A) \subset X \rightarrow X$ a positive sectorial operator and $f : [0, \infty) \times X \rightarrow X$ a continuous function as in Theorem 3.8. Suppose that there exist $\beta \in [0, \infty)$ and constants $c_0, c_1 \geq 0$ such that*

$$\sup_{x \in X} \|f(t, x)\| \leq c_0 \|x\| + c_1 t^\beta, \quad t \geq 0.$$

Then the problem

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f(t, x), & t > 0 \\ u(0) = u_0 \in X, \end{cases}$$

has a global mild solution.

Proof: Indeed, observe by Theorem 3.3 and Theorem 3.8 (since f maps bounded sets into bounded sets), that there exists a maximal mild solution u in $[0, \omega)$, given by

$$u(t) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s)) ds, \quad t \in [0, \omega). \quad (3.3)$$

Now if

$$M = \max \left\{ \sup_{t \in [0, \infty)} \|E_{\alpha,\alpha}(-At^\alpha)\|_{\mathcal{L}(X)}, \sup_{t \in [0, \infty)} \|E_\alpha(-At^\alpha)\|_{\mathcal{L}(X)} \right\},$$

and we suppose for a moment that $\omega < \infty$, we would obtain from the equality (3.3) for all $t \in [0, \omega)$, that

$$\begin{aligned} \|u(t)\| &\leq M\|u_0\| + M \int_0^t (t-s)^{\alpha-1} [c_0\|u(s)\| + c_1 s^\beta] ds \\ &\leq M \left[\|u_0\| + c_1 \int_0^t (t-s)^{\alpha-1} s^\beta ds \right] + M c_0 \int_0^t (t-s)^{\alpha-1} \|u(s)\| ds \\ &\leq \underbrace{M \left[\|u_0\| + c_1 \omega^{\alpha+\beta} B(\alpha, \beta+1) \right]}_{\tilde{M}} + M c_0 \int_0^t (t-s)^{\alpha-1} \|u(s)\| ds \end{aligned}$$

and by the singular Grownwall inequality (Theorem 2.19)

$$\|u(t)\| \leq \tilde{M} + \theta \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\theta(t-s)^\alpha) \tilde{M} ds$$

where $\theta = M c_0 \Gamma(\alpha)$. Since $\sup_{s \in [0, \omega)} |E_{\alpha,\alpha}(\theta s^\alpha)| \leq \infty$, we conclude that

$$\sup_{t \in [0, \omega)} \|u(t)\| < \infty,$$

what contradicts Theorem 3.8. So, u is a global mild solution. $\square$

Corollary 3.11 Consider $A : \mathcal{D}(A) \subset X \rightarrow X$ a positive sectorial operator and $C \in X$. Then

the problem

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + C, & t > 0 \\ u(0) = u_0 \in X, \end{cases}$$

has a global mild solution.

We end this section with the study of the fractional limit depending on the fractional order of the equation. This question is slightly studied in [4, 44] and among other authors. This study is sometimes called the study of approximation of the fractional solutions. For instance, Almeida-Ferreira at [4] studied this kind of problems involving the semilinear fractional partial differential equation,

$$\begin{cases} cD_t^{1+\alpha} u(t, x) = \Delta_x u(t, x) + |u(t, x)|^\rho u(t, x), & (t, x) \in [0, \infty) \times \mathbb{R}^n \\ u(0, x) = u_0 \text{ and } \partial_t u(0, x) = 0, & x \in \mathbb{R}^n. \end{cases}$$

for $\alpha \in (0, 1)$, which interpolates the semilinear heat and wave equations. They studied the existence, uniqueness and regularity of solution to this problem in Morrey spaces and then showed that when $\alpha \rightarrow 1^-$, the solution of the problem loses its native regularity. Our intention here is a little different.

To better understand the problem at focus now, consider the fractional abstract differential equation

$$\begin{cases} cD_t^\gamma u(t) = -Au(t) + f(t, u(t)), & t \geq 0 \\ u(0) = u \in X, \end{cases} \quad (P_\gamma)$$

and the standard abstract differential equation

$$\begin{cases} \frac{d}{dt} u(t) = -Au(t) + f(t, u(t)), & t \geq 0 \\ u(0) = u \in X, \end{cases} \quad (P_1)$$

where $\gamma \in (0, 1)$, cD_t^γ is the Caputo fractional derivative, $A : \mathcal{D}(A) \subset X \rightarrow X$ is a positive sectorial operator and the function $f : [0, \infty) \times X \rightarrow X$ is globally Lipschitz on the second variable, uniformly with respect to the first variable, that is, there exist $L > 0$ such that

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for all $x, y \in X$ and all $t \in [0, \infty)$.

The optimum result we wish to obtain is that, if $u_\gamma(t)$ is a solution of (P_γ) defined over

the interval $[0, t_\gamma]$, for any $\gamma \in (0, 1]$, then for suitable $t > 0$

$$\lim_{\gamma \rightarrow 1^-} u_\gamma(t) = u_1(t).$$

Observe that we are not ready to answer this inquiry yet. There are many other questions to answer before we begin to study the veracity of the statement made above. We start proving a result about the linear part of the problem $(P_\gamma)(f \equiv 0)$. It is important to notice that the solutions of the linear part are always defined over all the positive real axis.

Lemma 3.12 *Let $\gamma \in (0, 1)$. Consider the families $\{E_\gamma(-t^\gamma A) : t \geq 0\}$, $\{E_{\gamma,\gamma}(-t^\gamma A) : t \geq 0\}$ and the C_0 -semigroup $\{T(t) : t \geq 0\}$ associated to $-A$. Then for each $x \in X$ and $t \geq 0$,*

$$\lim_{\gamma \rightarrow 1^-} E_\gamma(-t^\gamma A)x = T(t)x$$

and

$$\lim_{\gamma \rightarrow 1^-} E_{\gamma,\gamma}(-t^\gamma A)x = T(t)x.$$

Moreover, the convergence is uniform on bounded subsets of X and on intervals $[a, b] \subset \mathbb{R}^+$, for $a > 0$.

Proof: Fix $x \in X$ and $t > 0$. Let $\phi \in (0, \pi/2)$, S_ϕ be the sector associated with the positive sectorial operator A and $N \geq 0$ such that

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{N}{|\lambda|}, \quad \forall \lambda \in S_\phi \setminus \{0\}.$$

Choose $\epsilon > 1$, $\theta \in (\pi/2, \pi - \phi)$ and consider the Hankel's path $Ha = Ha(\epsilon, \theta)$, given by $Ha = Ha_1 + Ha_2 - Ha_3$, where

$$Ha_1 := \{te^{i\theta} : t \in [\epsilon, \infty)\}, \quad Ha_2 := \{\epsilon e^{it} : t \in [-\theta, \theta]\}, \quad Ha_3 := \{te^{-i\theta} : t \in [\epsilon, \infty)\}.$$

Since by the choices made above $Ha \subset \rho(-A)$, we just need to observe by Theorem 1.22 and Theorem 2.41, that

$$\begin{aligned} \|E_\gamma(-t^\gamma A)x - T(t)x\| &= \left\| \frac{1}{2\pi i} \int_{Ha} e^{\lambda t} \lambda^{\gamma-1} (\lambda^\gamma + A)^{-1} x - e^{\lambda t} (\lambda + A)^{-1} x \, d\lambda \right\| \\ &\leq \frac{1}{2\pi} \int_{Ha} e^{\operatorname{Re}(\lambda)t} \left\| \lambda^{\gamma-1} (\lambda^\gamma + A)^{-1} x - (\lambda + A)^{-1} x \right\| d|\lambda| \end{aligned}$$

what finally implies that

$$\begin{aligned} \|E_\gamma(-t^\gamma A)x - T(t)x\| &\leq \frac{1}{2\pi} \int_{Ha} e^{\operatorname{Re}(\lambda)t} \left\| \lambda^{\gamma-1}(\lambda^\gamma + A)^{-1}x - \lambda^{\gamma-1}(\lambda + A)^{-1}x \right\| d|\lambda| \\ &\quad + \frac{1}{2\pi} \int_{Ha} e^{\operatorname{Re}(\lambda)t} \left\| \lambda^{\gamma-1}(\lambda + A)^{-1}x - (\lambda + A)^{-1}x \right\| d|\lambda|. \end{aligned}$$

Using the resolvent equality and the sectorial operator property, we conclude that

$$\begin{aligned} \|E_\gamma(-t^\gamma A)x - T(t)x\| &\leq \frac{1}{2\pi} \int_{Ha} e^{\operatorname{Re}(\lambda)t} |\lambda|^{\gamma-1} |\lambda - \lambda^\gamma| \frac{M}{|\lambda|} \frac{M}{|\lambda|^\gamma} \|x\| d|\lambda| \\ &\quad + \frac{1}{2\pi} \int_{Ha} e^{\operatorname{Re}(\lambda)t} |\lambda|^{\gamma-1} - 1 \frac{M}{|\lambda|} \|x\| d|\lambda| \end{aligned}$$

and therefore

$$\|E_\gamma(-t^\gamma A)x - T(t)x\| \leq \frac{(M + M^2)\|x\|}{2\pi} \int_{Ha} e^{\operatorname{Re}(\lambda)t} \frac{|\lambda - \lambda^\gamma|}{|\lambda|^2} d|\lambda|.$$

Now, since $\epsilon > 1$, observe that

i) Over Ha_1 , we obtain that:

$$\int_{Ha_1} e^{\operatorname{Re}(\lambda)t} \frac{|\lambda - \lambda^\gamma|}{|\lambda|^2} d|\lambda| = \int_\epsilon^\infty e^{ts \cos(\theta)} \frac{|se^{i\theta} - s^\gamma e^{i\theta\gamma}|}{s^2} ds \leq \int_\epsilon^\infty e^{ts \cos(\theta)} |se^{i\theta} - s^\gamma e^{i\theta\gamma}| ds$$

and since $e^{ts \cos(\theta)} |se^{i\theta} - s^\gamma e^{i\theta\gamma}| \leq 2se^{ts \cos(\theta)}$, i.e., it is integrable on $[\epsilon, \infty)$ and

$$|se^{i\theta} - s^\gamma e^{i\theta\gamma}| \xrightarrow{(\gamma \rightarrow 1^-)} 0$$

for each fixed $s \in [\epsilon, \infty)$, by the Dominated Convergence theorem, we conclude that

$$\lim_{\gamma \rightarrow 1^-} \int_{Ha_1} e^{\operatorname{Re}(\lambda)t} \frac{|\lambda - \lambda^\gamma|}{|\lambda|^2} d|\lambda| = 0.$$

ii) Over Ha_2 , following similar computations, observe that

$$\int_{Ha_2} e^{\operatorname{Re}(\lambda)t} \frac{|\lambda - \lambda^\gamma|}{|\lambda|^2} d|\lambda| \leq e^{t\epsilon} \int_{-\theta}^\theta |\epsilon e^{is} - \epsilon^\gamma e^{is\gamma}| \epsilon ds$$

and that $|\epsilon e^{is} - \epsilon^\gamma e^{is\gamma}| \leq 2\epsilon^2$. Since the function on the right side of the last

inequality is integrable on $[-\theta, \theta]$ and

$$|\epsilon e^{is} - \epsilon^\gamma e^{is\gamma}| \epsilon^{\gamma \rightarrow 1^-} 0$$

for each fixed $s \in [-\theta, \theta]$, by the Dominated Convergence theorem, we conclude that

$$\lim_{\gamma \rightarrow 1^-} \int_{Ha_2} e^{\operatorname{Re}(\lambda)t} \frac{|\lambda - \lambda^\gamma|}{|\lambda|^2} d|\lambda| = 0.$$

iii) The computations over Ha_3 are analogous to this in item i).

Therefore

$$\lim_{\gamma \rightarrow 1^-} \|E_\gamma(-t^\gamma A)x - T(t)x\| \leq \frac{(M + M^2)\|x\|}{2\pi} \lim_{\gamma \rightarrow 1^-} \int_{Ha} e^{\operatorname{Re}(\lambda)t} \frac{|\lambda - \lambda^\gamma|}{|\lambda|^2} d|\lambda| = 0.$$

Moreover, it is not difficult to notice in the last inequality that we could have supposed $t \in [a, b]$ for any $a > 0$ and x on any bounded subset of X , proving an uniform convergence on these sets. The proof for the other family of operators is analogous. $\square$

Now we start to deal with the nonlinearity f , introducing new computations to the results obtained so far. An important question that appears when we deal with the nonlinearity of (P_γ) , arises from the fact that Theorem 3.3 does not guarantee any kind of uniform (on the fractional exponent γ) existence of solutions. Indeed, we know that for any $\gamma \in (0, 1]$ there exist $t_\gamma > 0$ and a unique continuous function $u : [0, t_\gamma] \rightarrow X$, but eventually

$$\bigcap_{\gamma \in (0, 1]} [0, t_\gamma] = \{0\}, \quad (3.4)$$

which would lead us to face the possibility of calculate the fractional limit just for the trivial $t = 0$. To avoid this complication, we prove the following lemmas.

Lemma 3.13 *If $f : [0, \infty) \times X \rightarrow X$ is globally Lipschitz on the second variable, uniformly with respect to the first variable, that is, there exist $L > 0$ such that*

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for all $x, y \in X$ and all $t \in [0, \infty)$, then f maps bounded sets into bounded sets.

Proof: Just observe that for any $t \geq 0$ and $x \in X$ we have the inequality

$$\|f(t, x)\| \leq \|f(t, x) - f(t, 0)\| + \|f(t, 0)\| \leq C\|x\| + \|f(t, 0)\|.$$

□

Lemma 3.14 Consider the problem (P_γ) , for $\gamma \in (0, 1]$. If $u_\gamma(t)$ is the maximal local mild solution of (P_γ) defined over the interval $[0, \omega_\gamma)$, then there exist $t^* > 0$ such that

$$[0, t^*] \subset \bigcap_{\gamma \in [1/2, 1]} [0, \omega_\gamma).$$

Proof: Observe that for each $\gamma \in (0, 1]$, by Lemma 3.13 and Theorem 3.8 we know that the maximal interval of existence for the solution $u_\gamma(t)$ can be $[0, \infty)$ or a finite interval $[0, \omega_\gamma)$ such that

$$\limsup_{t \rightarrow \omega_\gamma^-} \|u_\gamma(t)\| = \infty.$$

Therefore, if we suppose for an instant that

$$\bigcap_{\gamma \in [1/2, 1]} [0, \omega_\gamma) = \{0\},$$

there will exist a sequence $\{\gamma_n\}_{n=1}^\infty \subset [1/2, 1]$ and a related decreasing sequence of positive real numbers $\{\omega_{\gamma_n}\}_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} \omega_{\gamma_n} = 0$. By the property stated above and choosing a subsequence of $\{\omega_{\gamma_n}\}_{n=1}^\infty$, if necessary, we suppose that there exist $\gamma \in [1/2, 1]$ such that γ_n converges to γ and we can construct a sequence $\{t_n\}_{n=1}^\infty$ with the following properties:

$$\text{i) } 0 < \omega_{\gamma_{n+1}} < t_n < \omega_{\gamma_n}, \text{ for } n \in \mathbb{N},$$

$$\text{ii) } \|u_{\gamma_n}(t)\| < \|u_0\| + 1 \text{ for } t \in [0, t_n) \text{ and } \|u_{\gamma_n}(t_n)\| = \|u_0\| + 1, \text{ for } n \in \mathbb{N}.$$

Then observe that

$$\|u_{\gamma_n}(t_n) - u_0\| \leq \|E_{\gamma_n}(-(t_n)^{\gamma_n} A)u_0 - u_0\| + \left\| \int_0^{t_n} (t_n - s)^{\gamma_n - 1} E_{\gamma_n, \gamma_n}(-(t_n - s)^{\gamma_n} A) f(s, u_{\gamma_n}(s)) ds \right\|$$

and since

$$\begin{aligned} & \left\| \int_0^{t_n} (t_n - s)^{\gamma_n - 1} E_{\gamma_n, \gamma_n}(-(t_n - s)^{\gamma_n} A) f(s, u_{\gamma_n}(s)) ds \right\| \\ & \leq M \int_0^{t_n} (t_n - s)^{\gamma_n - 1} \left[\|f(s, u_{\gamma_n}(s)) - f(s, 0)\| + \|f(s, 0)\| \right] ds \\ & \leq ML [\|u_0\| + 1 + \bar{M}] \int_0^{t_n} (t_n - s)^{\gamma_n - 1} ds \end{aligned}$$

where $M > 1$ is given uniformly on $\gamma \in [1/2, 1]$ by Theorem 2.41 and

$$\bar{M} = \frac{\sup_{s \in [0, \omega_{\gamma_1}]} \|f(s, 0)\|}{L},$$

we conclude that

$$\begin{aligned} \|u_{\gamma_n}(t_n) - u_0\| &\leq \|E_{\gamma_n}(-(t_n)^{\gamma_n} A)u_0 - u_0\| + ML \left[\|u_0\| + 1 + \bar{M} \right] \int_0^{t_n} (t_n - s)^{\gamma_n - 1} ds \\ &= \|E_{\gamma_n}(-(t_n)^{\gamma_n} A)u_0 - u_0\| + ML \left[\|u_0\| + 1 + \bar{M} \right] \frac{(t_n)^{\gamma_n}}{\gamma_n} \end{aligned}$$

and therefore

$$\lim_{n \rightarrow \infty} \|u_{\gamma_n}(t_n) - u_0\| = 0,$$

what contradicts item *ii*) which claims that $\|u_{\gamma_n}(t_n)\| = \|u_0\| + 1$, for $n \in \mathbb{N}$. This conclude the proof of the lemma. □

Now we study some important technical lemmas, that will help to prove the final main theorem of this section.

Lemma 3.15 Consider $\tau > 0$ and a family $\{v_\gamma(t)\}_{\gamma \in (0,1]} \subset C([0, \tau], [0, \infty))$. If for each $t \in [0, \tau]$,

$$\lim_{\gamma \rightarrow 1^-} v_\gamma(t) = 0,$$

and if there exist $\bar{M} > 0$ such that

$$\sup_{\gamma \in (0,1]} \left\{ \sup_{s \in [0, \tau]} |v_\gamma(s)| \right\} \leq \bar{M},$$

then, for each $t \in [0, \tau]$

$$\lim_{\gamma \rightarrow 1^-} \int_0^t (t-s)^{\gamma-1} v_\gamma(s) ds = 0.$$

Proof: Indeed, choose and fix $p, q \in (1, \infty)$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

By the Young's inequality with ϵ , for any $\epsilon > 0$, we have that

$$\int_0^t (t-s)^{\gamma-1} v_\gamma(s) ds \leq \epsilon \int_0^t (t-s)^{p(\gamma-1)} ds + C(\epsilon) \int_0^t [v_\gamma(s)]^q ds$$

where $C(\epsilon) = (1/q)(1/\epsilon p)^{q/p}$. If we suppose $\gamma \in ((p-1)/p, 1]$, then by integration of the above first term we obtain

$$\int_0^t (t-s)^{\gamma-1} v_\gamma(s) ds \leq \epsilon \frac{t^{p(\gamma-1)+1}}{p(\gamma-1)+1} + C(\epsilon) \int_0^t [v_\gamma(s)]^q ds.$$

Using the Dominated Convergence theorem on the above second term, we conclude that

$$\begin{aligned} \lim_{\gamma \rightarrow 1^-} \int_0^t (t-s)^{\gamma-1} v_\gamma(s) ds &\leq \epsilon t + C(\epsilon) \lim_{\gamma \rightarrow 1^-} \int_0^t [v_\gamma(s)]^q ds \\ &= \epsilon t. \end{aligned}$$

Since $\epsilon > 0$ was an arbitrary choice, we finish the proof of the lemma. $\square$

Lemma 3.16 *Let be $\gamma \in (0, 1)$, $f : [0, \infty) \times X \rightarrow X$ a globally Lipschitz function, $\tau > 0$ and $v : [0, \tau] \rightarrow X$ a continuous function. Consider also $\{E_{\gamma, \gamma}(-t^\gamma A) : t \geq 0\}$, the family of operators associated to $-A$. Then for each $t \in [0, \tau]$, it holds that*

$$\lim_{\gamma \rightarrow 1^-} \left\| \int_0^t [E_{\gamma, \gamma}(-(t-s)^\gamma A) - T(t-s)] f(s, v(s)) ds \right\| = 0.$$

Proof: By Theorem 1.22 and Theorem 2.41, let $M, \bar{M} > 0$ be such that

$$\max \left\{ \sup_{s \geq 0} \|T(s)\|_{\mathcal{L}(X)}, \sup_{s \geq 0} \|E_{\gamma, \gamma}(-s^\gamma A)\|_{\mathcal{L}(X)} \right\} \leq M, \quad \text{for all } \gamma \in (0, 1)$$

and

$$\max \left\{ \sup_{s \in [0, \tau]} \|f(s, 0)\|, \sup_{s \in [0, \tau]} \|v(s)\| \right\} \leq \bar{M}.$$

Fix $t \in [0, \tau]$, and observe that for all $\gamma \in (0, 1)$,

$$\left\| [E_{\gamma, \gamma}(-(t-s)^\gamma A) - T(t-s)] f(s, v(s)) \right\| \leq 2M\bar{M} [L+1]$$

which is integrable in the variable s over $[0, t]$. Since Lemma 3.12 guarantees that for each fixed $s \in [0, t]$

$$\lim_{\gamma \rightarrow 1^-} \left\| [E_{\gamma, \gamma}(-(t-s)^\gamma A) - T(t-s)] f(s, v(s)) \right\| = 0,$$

using the Dominated Convergence theorem we conclude that

$$\lim_{\gamma \rightarrow 1^-} \left\| \int_0^t \left[E_{\gamma, \gamma}(-(t-s)^\gamma A) - T(t-s) \right] f(s, v(s)) ds \right\| = 0$$

what finishes the proof of the lemma. $\square$

Lemma 3.17 *Let $\gamma \in (0, 1)$, $f : [0, \infty) \times X \rightarrow X$ a globally Lipschitz function, $\tau > 0$ and $v : [0, \tau] \rightarrow X$ a continuous function. Consider the family $\{E_{\gamma, \gamma}(-t^\gamma A) : t \geq 0\}$ and the C_0 -semigroup $\{T(t) : t \geq 0\}$ both associated to the operator $-A$. Given $\tilde{M} > 0$ and $x \in X$, define the function*

$$\begin{aligned} L_\gamma[v(t), x, \tilde{M}](t) &= \|E_\gamma(-t^\gamma A)x - T(t)x\| + \tilde{M} \left| (t^\gamma / \gamma) - t \right| \\ &\quad + \left\| \int_0^t \left[E_{\gamma, \gamma}(-(t-s)^\gamma A) - T(t-s) \right] f(s, v(s)) ds \right\| \end{aligned}$$

for $t \in [0, \tau]$. Then

$$\lim_{\gamma \rightarrow 1^-} L_\gamma[v(t), x, \tilde{M}](t) = 0,$$

for each fixed $t \in [0, \tau]$.

Proof: It is a consequence of Lemma 3.12 and Lemma 3.16. $\square$

Theorem 3.18 *Consider the problem (P_γ) for $\gamma \in (0, 1]$, and suppose that $u_\gamma(t)$ is the maximum local mild solution of (P_γ) defined over $[0, \omega_\gamma)$. Then there exists $t^* > 0$ such that*

$$[0, t^*] \subset \bigcap_{\gamma \in [1/2, 1]} [0, \omega_\gamma)$$

and for each fixed $t \in [0, t^*]$

$$\lim_{\gamma \rightarrow 1^-} \|u_\gamma(t) - u_1(t)\| = 0.$$

Proof: Observe that Theorem 3.8 guarantees the existence of maximal mild solutions $u_\gamma(t)$ defined over a maximal interval of existence $[0, \omega_\gamma)$ that for $\gamma \in (0, 1)$ satisfies

$$u_\gamma(t) = E_\gamma(-t^\gamma A)u_0 + \int_0^t (t-s)^{\gamma-1} E_{\gamma, \gamma}(-(t-s)^\gamma A) f(s, u_\gamma(s)) ds, \quad t \in [0, \omega_\gamma).$$

and analogously, for the problem (P_1) we have that the corresponding maximal mild

solution u_1 in the associated maximal interval of existence, satisfies

$$u_1(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u_\gamma(s))ds, \quad t \in [0, \omega_1).$$

Applying Lemma 3.12, we conclude that there exist $t^* > 0$ such that

$$[0, t^*] \subset \bigcap_{\gamma \in [1/2, 1]} [0, \omega_\gamma).$$

Consider $M, C^* > 0$ such that

$$\max \left\{ \sup_{s \geq 0} \|E_{\gamma, \gamma}(-s^\gamma A)\|_{\mathcal{L}(X)}, \sup_{s \geq 0} \|T(s)\|_{\mathcal{L}(X)} \right\} \leq M$$

uniformly on $\gamma \in [1/2, 1)$ and that

$$\sup_{s \in [0, t^*]} \|u_1(s)\| \leq C^*.$$

Fix $t \in (0, t^*]$ and observe that for $\gamma \in [1/2, 1]$

$$\begin{aligned} \|u_\gamma(t) - u_1(t)\| &\leq \|E_\gamma(-t^\gamma A)u_0 - T(t)u_0\| \\ &\quad + \left\| \int_0^t (t-s)^{\gamma-1} E_{\gamma, \gamma}(-(t-s)^\gamma A) f(s, u_\gamma(s)) ds \right. \\ &\quad \left. - \int_0^t T(t-s)f(s, u_1(s))ds \right\|. \end{aligned}$$

After some algebraic manipulations, we obtain

$$\|u_\gamma(t) - u_1(t)\| \leq \|E_\gamma(-t^\gamma A)u_0 - T(t)u_0\| + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3$$

where

$$\mathcal{J}_1 = \left\| \int_0^t (t-s)^{\gamma-1} E_{\gamma, \gamma}(-(t-s)^\gamma A) [f(s, u_\gamma(s)) - f(s, u_1(s))] ds \right\|,$$

$$\mathcal{J}_2 = \left\| \int_0^t [(t-s)^{\gamma-1} - 1] E_{\gamma, \gamma}(-(t-s)^\gamma A) f(s, u_1(s)) ds \right\|,$$

$$\mathcal{J}_3 = \left\| \int_0^t [E_{\gamma, \gamma}(-(t-s)^\gamma A) - T(t-s)] f(s, u_1(s)) ds \right\|.$$

Estimating $\mathcal{J}_i$, for $i \in \{1, 2, 3\}$, we obtain

$$\begin{aligned} \|u_\gamma(t) - u_1(t)\| &\leq \|E_\gamma(-t^\gamma A)u_0 - T(t)u_0\| \\ &\quad + ML \int_0^t (t-s)^{\gamma-1} \|u_\gamma(s) - u_1(s)\| ds \\ &\quad + MLC^* \left| (t^\gamma/\gamma) - t \right| \\ &\quad + \left\| \int_0^t \left[E_{\gamma,\gamma}(-(t-s)^\gamma A) - T(t-s) \right] f(s, u_1(s)) ds \right\|. \end{aligned}$$

Using the notation of Lemma 3.17, we rewrite the above inequality as

$$\|u_\gamma(t) - u_1(t)\| \leq L_\gamma[u_1(t), u_0, MLC^*](t) + ML \int_0^t (t-s)^{\gamma-1} \|u_\gamma(s) - u_1(s)\| ds.$$

Also, observe that for any $s \in [0, t]$,

$$\|L_\gamma[u_1(s), u_0, MLC^*](s)\| \leq 2M\|u_0\| + 2MLC^* \max \left\{ (t^*)^\gamma/\gamma, t^*, 1 \right\}$$

i.e., $L_\gamma[u_1(s), u_0, MLC^*](s)$ is integrable on $[0, t]$. Therefore, using the singular Gronwall inequality (*Theorem 2.19*) we conclude that

$$\begin{aligned} \|u_\gamma(t) - u_1(t)\| &\leq L_\gamma[u_1(t), u_0, MLC^*](t) \\ &\quad + \theta_\gamma \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(\theta_\gamma(t-s)^\gamma) L_\gamma[u_1(s), u_0, MLC^*](s) ds. \end{aligned}$$

where $\theta_\gamma = ML\Gamma(\gamma)$. Since Lemma 3.17 guarantees, for each fixed $t \in [0, t^*]$, that

$$\lim_{\gamma \rightarrow 1^-} L_\gamma[u_1(t), u_0, MLC^*](t) = 0,$$

to conclude this proof, it suffices to prove that

$$\lim_{\gamma \rightarrow 1^-} \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(\theta_\gamma(t-s)^\gamma) L_\gamma[u_1(s), u_0, MLC^*](s) ds = 0.$$

But we already know that there exist $\bar{M} > 0$ such that $E_{\gamma,\gamma}(\theta_\gamma s^\gamma) \leq \bar{M}$ for all $s \in [0, t^*]$ and

all $\gamma \in [1/2, 1]$ (see for details [12, 36, 37, 54]), i.e.,

$$\begin{aligned} & \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(\theta_\gamma(t-s)^\gamma) L_\gamma[u_1(s), u_0, MLC^*](s) ds \\ & \leq \bar{M} \int_0^t (t-s)^{\gamma-1} L_\gamma[u_1(s), u_0, MLC^*](s) ds. \end{aligned}$$

Finally, by Lemma 3.15 we have that

$$\lim_{\gamma \rightarrow 1^-} \int_0^t (t-s)^{\gamma-1} L_\gamma[u_1(s), u_0, MLC^*](s) ds = 0.$$

Thus, the theorem is proved. $\square$

Remark 3.19 All the computations done in the end of this section, about the theory of the fractional limit, could be done in more general way. Indeed, if instead of considering the fractional limit as 1, we could have consider an arbitrary $\alpha_0 \in (0, 1]$ and studied the limit

$$\lim_{\gamma \rightarrow \alpha_0} u_\gamma(t) = u_{\alpha_0}(t)$$

following analogous computations as above.

§ 3.2 COMPARISON AND GLOBAL EXISTENCE OF SOLUTIONS

In this section we will study some results about comparison and positivity of solutions to problems like (3.1). We intend to prove abstract monotonicity results for the Mittag-Leffler operators discussed on Section 2.4. The objective is to obtain, with the comparison results, a tool to guarantee the existence and uniqueness of global mild solutions.

We start introducing the concepts suitable for the study that follows.

Definition 3.20 Let S be a set. Then if $\mathcal{R}$ is a relation on S , we say that $\mathcal{R}$ is a **partial order relation** if it is reflexive, antisymmetric, and transitive, i.e., for s_1, s_2 and s_3 elements of S :

- i) $s_1 \mathcal{R} s_1$;
- ii) if $s_1 \mathcal{R} s_2$ and $s_2 \mathcal{R} s_1$, then $s_1 = s_2$;
- iii) if $s_1 \mathcal{R} s_2$ and $s_2 \mathcal{R} s_3$, then $s_1 \mathcal{R} s_3$.

Definition 3.21 Let X be a real Banach space. A subset C of X is called a cone if the following conditions are satisfied:

- i) C is nonempty and nontrivial (i.e., C contains a nonzero point);
- ii) $\lambda C \subset C$ for any nonnegative λ ;
- iii) C is convex;
- iv) C is closed; and
- v) $C \cap (-C) = \{0\}$.

In resume, we can say that these conditions have fairly intuitive meanings. The first one is a condition to guarantee the existence of the cone, the second one shows that a cone is a collection of rays emanating from the origin, the third requirement ensures that the cone contains no holes, and the fourth guarantees that the cone contains its boundaries. Finally the last condition makes sure that the cone is not too big: it must not take up more than half the space.

Example 3.22 Examples of cones are easy to come by:

- i) In $\mathbb{R}^3$ the definition of a cone precisely matches our geometrical intuition, with the stipulation that the vertex must coincide with the origin. In $\mathbb{R}^2$, any wedge which extends to infinity from the origin is a cone. And finally, in $\mathbb{R}$ the nonnegative numbers forms a cone.

Now we can begin to define a partial order relation on the Banach space X . For this, we consider a cone C and define the relation $\leq_C$ as follows.

Definition 3.23 We say that $x \leq_C y$ if and only if $y - x \in C$.

Proposition 3.24 The relation defined on Definition 3.23 is a partial order relation on X .

Proof: Observe that

Reflexive: Since 0 belongs to C , we see that $x - x = 0 \in C$ and therefore $x \leq_C x$.

Antisymmetric: If $x \leq_C y$ and $y \leq_C x$, then both $y - x$ and $-(y - x)$ belongs to C . It follows that $y - x = 0$ and therefore that $x = y$.

Transitive: If $x \leq_C y$ and $y \leq_C z$, then both $y - x$ and $z - y$ belongs to C . By the convexity of C we know that

$$\frac{1}{2}(y - x) + \frac{1}{2}(z - y) \in C$$

so the scalar multiplication $2(\frac{1}{2}(y-x) + \frac{1}{2}(z-y))$ also belongs to C , i.e., $(z-x) \in C$ and therefore $x \leq_C z$.

□

Finally we introduce the concept of order in a Banach space which will be the main framework for the study of comparison results:

Definition 3.25 Let X be a Banach space and $\leq_C$ a fixed partial order relation (for a cone) as above. Then the couple $(X, \leq_C)$ is a **ordered Banach space**. Some most useful properties of this couple are the following:

- i) $x \leq_C y$ implies $x + z \leq_C y + z$, for all $z \in X$;
- ii) for any $\lambda \geq 0$, if $x \leq_C y$ then $\lambda x \leq_C \lambda y$;
- iii) The ordering preserves limits. If $y_n \rightarrow y$ and $y_n \leq_C x$ for each n , then $y \leq_C x$.
Reciprocally, if $x \leq_C y_n$ for each n , then $x \leq_C y$.

Example 3.26 Let $X = L^p(\Omega)$, $p \in [1, \infty]$ and $C := \{f \in X : f(x) \geq 0, \text{ a.e.}\}$. Then we define a partial order “ $f \leq_C g$ if and only if $f(x) \leq g(x)$ a.e.”. Its easy to see that $(X, \leq_C)$ is an ordered Banach space. We also could consider $X = (L^p(\Omega))^n$, for $p \in [1, \infty]$ and $n > 1$. Using the partial order induced in each entry by the partial order from $L^p(\Omega)$, we again obtain an ordered Banach space. The same can be done with $C(\overline{\Omega})$, with the cone $C := \{f \in X : f(x) \geq 0\}$.

From this point, we consider that a cone and respectively a partial order is always established to a given Banach space X and we will ever denote this partial order as $\leq_X$.

Definition 3.27 Suppose that $(X, \leq_X)$ and $(Y, \leq_Y)$ are ordered Banach spaces. A function $T : \mathcal{D}(T) \subset X \rightarrow Y$ is called **increasing** if $x \leq_X y$ implies $Tx \leq_Y Ty$ for all $x, y \in \mathcal{D}(T)$. Also, if we have only that $0 \leq_X x$ implies $0 \leq_Y Tx$, for all $x \in \mathcal{D}(T)$, then T is called **positive**.

Lemma 3.28 Let $(X, \leq_X)$ be an ordered Banach space and consider $f \in L^1(a, b; X)$ such that $0 \leq_X f(t)$ a.e. in (a, b) . Then

$$0 \leq_X \int_a^b f(s) ds.$$

Proof: It is sufficient to prove this result in a dense subset of $L^1(a, b; X)$. Suppose $f \in C([a, b]; X)$ and remember that the integration above can be written as the limit

$$\int_a^b f(s) ds = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^{n-1} \sup_{\tilde{t} \in [t_{i+1}-t_i]} f(\tilde{t}) [t_{i+1} - t_i]$$

where $P = \{t_1 = a, t_2, \dots, t_n = b\}$ is a partition of the interval $[a, b]$ and

$$\|P\| := \sup_{i \in \{1, \dots, n-1\}} \{t_{i+1} - t_i\}.$$

Now the proof follows by the property *iii)* stated on Definition 3.25. $\square$

The following result establishes the equivalence between the positivity of the resolvent operator of A and the positivity of the Mittag-Leffler families.

Proposition 3.29 *Let $(X, \leq_X)$ be an ordered Banach space and consider $\alpha \in (0, 1)$ and $A : \mathcal{D}(A) \subset X \rightarrow X$ a positive sectorial operator. Suppose that there exist $\lambda_0 > 0$ such that $(\lambda^\alpha + A)^{-1}$ is increasing for any $\lambda > \lambda_0$. Then, $E_\alpha(-t^\alpha A)$ is increasing for all $t \geq 0$. Conversely, if $X \ni u_0 \mapsto E_\alpha(-t^\alpha A)u_0 \in C([0, \infty); X)$ is increasing for all $t \geq 0$, then $(\lambda^\alpha + A)^{-1}$ is increasing for any $\lambda > 0$.*

Proof: The case $t = 0$ is easily verified. To prove the case $t > 0$, consider $u_0 \in X$ such that $0 \leq_X u_0$. From Proposition 2.46, write

$$E_\alpha(-t^\alpha A)u_0 = \lim_{n \rightarrow \infty} \frac{1}{n!} \sum_{k=1}^{n+1} b_{k,n+1}^\alpha \left(\left(\frac{n}{t} \right)^\alpha \left(\left(\frac{n}{t} \right)^\alpha + A \right)^{-1} \right)^k u_0, \quad t > 0. \quad (3.5)$$

If $n \in \mathbb{N}$ is such that $n/t > \lambda_0$, then

$$0 \leq \left(\left(\frac{n}{t} \right)^\alpha + A \right)^{-1} u_0.$$

Since the values $b_{k,n+1}^\alpha$, given by (2.8), are positive real numbers and the positive cone is closed, it follows from (3.5) that $0 \leq_X E_\alpha(-t^\alpha A)u_0$.

Conversely, since the integral is a positive operator and

$$\lambda^{\alpha-1}(\lambda^\alpha + A)^{-1} = \int_0^\infty e^{-\lambda t} E_\alpha(-t^\alpha A) dt, \quad \forall \lambda > 0,$$

we conclude that $(\lambda^\alpha + A)^{-1}$ is a positive operator whenever $\lambda > 0$. $\square$

Remark 3.30 Consider the following observations.

- i) Proposition 3.29 is equivalent to the following result: if $0 \leq_X u_0$, then the solution

of the linear homogeneous problem

$$\begin{cases} cD_t^\alpha u(t) = -Au(t), \\ u(0) = u_0 \in X, \end{cases}$$

verifies $0 \leq_X u(t)$ for all $t \geq 0$. Equivalently, if $u_0 \leq_X u_1$, then $u_0(t) \leq_X u_1(t)$ for all $t \geq 0$, where u_0 and u_1 are the corresponding solutions to the problem with initial conditions $u_0(0) = u_0$ and $u_1(0) = u_1$, respectively.

- ii) The assumption on the operator A is equivalent to the following positivity result for elliptic problems: for any $\lambda > \lambda_0$ and for any f with $0 \leq_X f$, the solution to the problem $\lambda^\alpha u + Au = f$ satisfies $0 \leq_X u$.

A similar result may be stated in terms of the family $\{E_{\alpha,\alpha}(-t^\alpha A) : t \geq 0\}$.

Proposition 3.31 *Let $(X, \leq_X)$ be an ordered Banach space and consider $\alpha \in (0, 1)$ and $A : \mathcal{D}(A) \subset X \rightarrow X$ a positive sectorial operator. Suppose that there exist $\lambda_0 > 0$ such that $(\lambda^\alpha + A)^{-1}$ is increasing for any $\lambda > \lambda_0$. Then, $E_{\alpha,\alpha}(-t^\alpha A)$ is increasing for all $t \geq 0$. Conversely, if $X \ni u_0 \mapsto E_{\alpha,\alpha}(-t^\alpha A)u_0 \in C([0, \infty); X)$ is increasing for all $t \geq 0$, then $(\lambda^\alpha + A)^{-1}$ is increasing for any $\lambda > 0$.*

Proof: It suffices to observe that if $u_0 \in X$, then from Proposition 2.46,

$$t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha A)u_0 = \lim_{n \rightarrow \infty} \frac{1}{n!} \sum_{k=1}^n \alpha k b_{k,n}^\alpha \left(\frac{n}{t}\right)^{\alpha k+1} \left(\left(\frac{n}{t}\right)^\alpha + A\right)^{-(k+1)} u_0, \quad t \geq 0.$$

Now, we may proceed as in the proof of Proposition 3.29. Conversely, we have that

$$(\lambda^\alpha + A)^{-1} = \int_0^\infty e^{-\lambda t} t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha A) dt,$$

for any $\lambda > 0$, which concludes the proof. $\square$

Corollary 3.32 *Let $(X, \leq_X)$ be an ordered Banach space and consider $\alpha \in (0, 1)$ and $A : \mathcal{D}(A) \subset X \rightarrow X$ a positive sectorial operator. Suppose that there exist $\lambda_0 > 0$ such that $(\lambda^\alpha + A)^{-1}$ is increasing for any $\lambda > \lambda_0$. Let $u_f(t, u_0)$ be the local mild solution in $[0, \tau]$, for some $\tau > 0$, to the problem*

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f(t), & t > 0, \\ u(0) = u_0 \in X, \end{cases}$$

and suppose that $f \in L^1(0, \tau; X)$. Assume that $u_1 \leq_X u_0$ and $f_1(t) \leq_X f_0(t)$ a.e. $t \in [0, \tau]$. Then $u_{f_1}(t, u_1) \leq_X u_{f_0}(t, u_0)$ for all $t \in [0, \tau]$. In particular, if $0 \leq_X u_0$ and $0 \leq_X f(t)$ a.e. $t \in [0, \tau]$, then $0 \leq_X u_f(t, u_0)$ for all $t \in [0, \tau]$.

Proof: Observe that the corresponding solutions ($i = 0, 1$) are given by

$$u_{f_i}(t) = E_\alpha(-t^\alpha A)u_i + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f_i(s) ds, \quad t \in [0, \tau].$$

Since $E_\alpha(-t^\alpha A)u_1 \leq_X E_\alpha(-t^\alpha A)u_0$ and for all $0 < s < t < \tau$,

$$E_{\alpha,\alpha}(-(t-s)^{\alpha-1} A) f_1(s) \leq_X E_{\alpha,\alpha}(-(t-s)^{\alpha-1} A) f_0(s),$$

the result follows from the fact that the integral is a positive operator. $\square$

Corollary 3.33 *Let $(X, \leq_X)$ be an ordered Banach space and suppose that the families*

$$\{E_\alpha(-t^\alpha A) : t \geq 0\} \quad \text{and} \quad \{E_{\alpha,\alpha}(-t^\alpha A) : t \geq 0\}$$

are increasing. If $0 \leq_X f(t, x)$ a.e. $t \in [0, t_0]$ and for all $x \in X$ with $0 \leq_X x$, then $0 \leq_X u_0$ implies that the local mild solution $u_f(t, u_0)$ of the problem

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f(t), & t > 0, \\ u(0) = u_0 \in X, \end{cases}$$

is positive, i.e., $0 \leq_X u_f(t, u_0)$ for all $t \in [0, t_0]$.

Proof: Let T be the operator defined by

$$T(u)(t) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s)) ds, \quad t \in [0, t_0].$$

We proved in Theorem 3.3 that if $\tau \in [0, t_0]$ and $\beta > 0$ are small enough, then T is a contraction in $K = \{u \in C([0, \tau]; X) : \|u(t) - u_0\| \leq \beta\}$ and it has a unique fixed point. Consider the function $y_0(s) = u_0$, to $s \in [0, \tau]$. Consequently, $0 \leq_X y_0(s)$ and $0 \leq_X f(s, y_0(s))$ a.e. $[0, \tau]$. Hence, $y_1(t) = T(y_0(t))$, $t \in [0, \tau]$ satisfies $0 \leq_X y_1(t)$. Iterating, we obtain

$$0 \leq_X y_n(t) = T(y_{n-1}(t)), \quad t \in [0, \tau].$$

Since $\{y_n\}_{n \in \mathbb{N}}$ converges to u in $C([0, \tau]; X)$ we have that $0 \leq_X u(t)$ for all $t \in [0, \tau]$. Now, combining this with some continuation arguments from Section 3.1, we may conclude that

$0 \leq_X u(t)$ for all $t \in [0, t_0]$. □

This leads us to the following comparison result.

Theorem 3.34 *Let $(X, \leq_X)$ be an ordered Banach space and suppose that the families*

$$\{E_\alpha(-t^\alpha A) : t \geq 0\} \quad \text{and} \quad \{E_{\alpha,\alpha}(-t^\alpha A) : t \geq 0\}$$

are increasing.

i) *Let $u_0, u_1 \in X$ be given and assume that there exist $t_0, t_1 \in [0, \infty)$ such that $\{u_f(t, u_i)\}_{i=0,1}$ are local mild solutions in $[0, t_i]$, $i \in \{0, 1\}$, to*

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f(t, u(t)), & t > 0, \\ u(0) = u_i \in X. \end{cases}$$

Then, if $t^ = \min\{t_0, t_1\}$, $f(t, \cdot)$ is increasing a.e. $t \in [0, t^*)$ and $u_1 \leq_X u_0$, it holds that*

$$u_f(t, u_1) \leq_X u_f(t, u_0)$$

for all $t \in [0, t^]$.*

ii) *Consider functions f_0 and f_1 , and $u_0 \in X$, and assume that there exist $t_0, t_1 \in [0, \infty)$ such that $\{u_{f_i}(t, u_0)\}_{i=0,1}$ are local mild solutions in $[0, t_i]$, $i \in \{0, 1\}$, to*

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f_i(t, u(t)), & t > 0, \\ u(0) = u_0 \in X. \end{cases}$$

Then, if $t^ = \min\{t_0, t_1\}$ and $f_0(t, x) \leq_X f_1(t, x)$ a.e. $t \in [0, t^*]$ and for all $x \in X$, it holds that $u_{f_0}(t, u_0) \leq_X u_{f_1}(t, u_0)$ for all $t \in [0, t^*]$.*

iii) *Consider functions f_0 and f_1 , and $u_0, u_1 \in X$, and assume that there exist $t_0, t_1 \in [0, \infty)$ such that $\{u_{f_i}(t, u_i)\}_{i=0,1}$ are local mild solutions in $[0, t_i]$, $i \in \{0, 1\}$, to*

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f_i(t, u(t)), & t > 0, \\ u(0) = u_i \in X. \end{cases}$$

Then, if $t^ = \min\{t_0, t_1\}$, and $x \leq_X y$ imply $f_0(t, x) \leq_X f_1(t, y)$ a.e. $t \in [0, t^*]$, and $u_0 \leq_X u_1$, it holds that $u_{f_0}(t, u_0) \leq_X u_{f_1}(t, u_1)$ for all $t \in [0, t^*]$.*

Proof: i) For $i = 0, 1$, we know that $u_i(t) = u_f(t, u_i)$ is a fixed point of the operator

$$T(u)(t) = E_\alpha(-t^\alpha A)u_i + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s)) ds$$

for $t \in [0, t_i]$.

Consider initially the function $y_i^0(t) = u_i$, $t \in [0, t_i]$, $i = 0, 1$. Iterating, we have $y_i^n(t) = T(y_i^{n-1})(t)$ and this sequence converges to $u_i(t)$ in $C([0, t_i]; X)$. Furthermore, we have $y_1^0(t) \leq_X y_0^0(t)$ a.e. $t \in (0, t^*)$ and hence

$$f(t, y_1^0(t)) \leq_X f(t, y_0^0(t)), \quad \text{a.e. } t \in (0, t^*).$$

Then, $y_1^1(t) \leq_X y_0^1(t)$ in $[0, t^*]$. Iterating, we obtain $y_1^n(t) \leq_X y_0^n(t)$ and consequently $u_1(t) \leq_X u_0(t)$ in $[0, t^*]$.

Statements ii) and iii) can be proved analogously. □

Next, we use the comparison of solutions to establish the existence of a unique mild solution in a certain interval. As a consequence, when the solutions of the auxiliary problems are globally defined in time, from the continuation results we obtained previously, we conclude the existence of a unique global mild solution.

Corollary 3.35 *Let $(X, \leq_X)$ be an ordered Banach space and $C > 0$ such that*

$$x \leq_X y \leq_X z \Rightarrow \|y\|_X \leq C [\|x\|_X + \|z\|_X] \quad \forall x, y, z \in X.$$

Suppose that the families $\{E_\alpha(-t^\alpha A) : t \geq 0\}$ and $\{E_{\alpha,\alpha}(-t^\alpha A) : t \geq 0\}$ are increasing. Let $u_0 \in X$ and functions $f_1, f_2 : [0, \infty) \times X \rightarrow X$. Let $g : [0, \infty) \times X \rightarrow X$ be as in Theorem 3.3, and suppose that g maps bounded sets into bounded sets. Moreover, assume that $f_1(t, x) \leq_X g(t, x) \leq_X f_2(t, x)$ a.e. $t \geq 0$ and for all $x \in X$. Then if the problem

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f_i(t, u(t)), & t > 0, \\ u(0) = u_0 \in X. \end{cases}$$

for each $i = \{1, 2\}$, has a unique global mild solution $\{u_{f_i}\}$, then the problem

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + g(t, u(t)), & t > 0, \\ u(0) = u_0 \in X. \end{cases}$$

has a unique global mild solution.

Proof: Observe that by Theorem 3.3 and Theorem 3.8, we already know about the existence and uniqueness of a maximal local mild solution u_g on some interval $[0, \omega)$. Let us prove that $\omega = \infty$. Indeed, if we suppose that $\omega < \infty$, we already know that

$$\limsup_{t \rightarrow \omega^-} \|u_g(t)\|_X = \infty. \quad (3.6)$$

By hypothesis and by Theorem 3.34

$$\|u_g(t)\|_X \leq C [\|u_{f_1}(t)\|_X + \|u_{f_2}(t)\|_X]$$

for all $t \in [0, \omega)$. But above inequality guarantees

$$\sup_{t \in [0, \omega)} \|u_g(t)\|_X < \infty$$

what contradicts (3.6). □

Finally, as an application of this last result (*and some of Section 3.1*) we ensure existence of global mild solution for the following fractional partial differential equation coming from heat conduction theory.

Example 3.36 Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be a bounded open set with smooth boundary $\partial\Omega$. Consider the problem

$$\begin{cases} c\partial_t^\alpha u(t, x) = D\Delta u(t, x) + F(u(t, x)), & t > 0, \quad x \in \Omega, \\ \partial u / \partial \vec{v}(t, x) = 0, & t > 0, \quad x \in \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega. \end{cases} \quad (3.7)$$

where $c\partial_t^\alpha$ is Caputo fractional derivative, $u = (u_1, \dots, u_n)^\top$, $n \geq 1$, $\frac{\partial u}{\partial \vec{v}} = (\langle \nabla u_1, \vec{v} \rangle, \dots, \langle \nabla u_n, \vec{v} \rangle)^\top$, where $\vec{v}$ is the outward normal vector, and D is the diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}_{n \times n}$$

with $d_i > 0$, for all $i \in \{0, 1, \dots, n\}$. The nonlinearity

$$F = (F_1, \dots, F_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is assumed to be locally Lipschitz.

To treat this problem we set $X = (L^q(\Omega))^n$, for $1 < q < \infty$, with the usual order $\leq_{(L^q(\Omega))^n}$ (see Example 3.26) and we consider $A_D = \text{diag}(A_1, \dots, A_n)$, where for each $i \in \{0, \dots, n\}$ we define $A_i : D(A_i) \subset L^q(\Omega) \rightarrow L^q(\Omega)$ by

$$\begin{aligned} D(A_i) &= \{\phi \in W^{2,q}(\Omega); \partial \phi / \partial \vec{\nu} = 0, \text{ on } \partial \Omega\} =: W_{\mathcal{N}}^{2,q}(\Omega); \\ A_i \phi &= -d_i \Delta \phi \end{aligned}$$

and suppose that

$$\inf\{\text{Re}(\sigma(A_D))\} = \tilde{d}_1 > 0.$$

In [19, Proposition 2.8] it was shown that this operator is sectorial and has positive resolvent. Then, rewriting (3.7) in the abstract form

$$\begin{cases} cD_t^\alpha u(t) = -A_D u(t) + F(u(t)), & t > 0, \\ u(0) = u_0 \in (L^q(\Omega))^n, \end{cases}$$

we may ensure that F has all the properties asked to the force function in Theorem 3.3. Moreover, if each F_i is an increasing function and there exist constants $c_0 \in [0, \tilde{d}_1)$ and $c_1 \in [0, \infty)$ such that

$$-c_0|u_i| - c_1 \leq F_i(u) \leq c_0|u_i| + c_1 \quad \forall i \in \{1, \dots, n\},$$

and if u_0 belongs to the positive cone of X , the problem (3.7) has a global mild solution. Indeed, we start the proof of this last statement making important observations.

i) For any $u \in (L^q(\Omega))^n$

$$-c_0|u| - C_1 \leq_{(L^q(\Omega))^n} F(u) \leq_{(L^q(\Omega))^n} c_0|u| + C_1,$$

where $|u| := (|u_1|, |u_2|, \dots, |u_n|)$ and $C_1 = (c_1, c_1, \dots, c_1)$.

ii) The function $H : (L^q(\Omega))^n \rightarrow (L^q(\Omega))^n$ given by

$$H(u) = c_0|u| + C_1 := (c_0|u_1| + c_1, c_0|u_2| + c_1, \dots, c_0|u_n| + c_1)$$

is globally Lipschitz, since for $w, v \in (L^q(\Omega))^n$ we have

$$\|H(w) - H(v)\|_{(L^q(\Omega))^n} = \sqrt{\sum_{k=1}^n \|c_0|w_i| + c_1 - c_0|v_i| - c_1\|_{L^q(\Omega)}^2} = c_0 \sqrt{\sum_{k=1}^n \| |w_i| - |v_i| \|_{L^q(\Omega)}^2}$$

and therefore,

$$\|H(w) - H(v)\|_{(L^q(\Omega))^n} \leq c_0 \sqrt{\sum_{k=1}^n \|w_i - v_i\|_{L^q(\Omega)}^2} = c_0 \|w - v\|_{(L^q(\Omega))^n}$$

(The same computations can be done for the other function).

iii) Since by item ii), those functions maps bounded sets into bounded sets, observe that Proposition 3.10 guarantees that the problems

$$\begin{cases} cD_t^\alpha u(t) = -A_D u(t) - c_0|u| - C_1, & t > 0, \\ u(0) = u_0 \in (L^q(\Omega))^n, \end{cases}$$

and

$$\begin{cases} cD_t^\alpha u(t) = -A_D u(t) + c_0|u| + C_1, & t > 0, \\ u(0) = u_0 \in (L^q(\Omega))^n, \end{cases}$$

have a unique global mild solutions $u_-(t)$ and $u_+(t)$ respectively.

iv) If $F_1, F_2, F_3 \in (L^q(\Omega))^n$ and $F_1 \leq_{(L^q(\Omega))^n} F_2 \leq_{(L^q(\Omega))^n} F_3$, then for any $i \in \{1, 2, \dots, n\}$

$$(F_1)_i(x) \leq (F_2)_i(x) \leq (F_3)_i(x)$$

a.e. $x \in \Omega$ (see Example 3.26). Hence, observe that

a) If $x \in \Omega^+ := \{x \in \Omega : (F_2)_i(x) > 0\}$, then $|(F_2)_i(x)| \leq |(F_3)_i(x)|$ and

$$|(F_2)_i(x)| \leq |(F_1)_i(x)| + |(F_3)_i(x)|.$$

b) Also, if $x \in \Omega^- := \{x \in \Omega : (F_2)_i(x) \leq 0\}$, then $|(F_2)_i(x)| \leq |(F_1)_i(x)|$ and

$$|(F_2)_i(x)| \leq |(F_1)_i(x)| + |(F_3)_i(x)|.$$

c) Therefore, for a.e. $x \in \Omega$ we conclude that

$$|(F_2)_i(x)|^q \leq \left[|(F_1)_i(x)| + |(F_3)_i(x)| \right]^q \leq C(q) \left[|(F_1)_i(x)|^q + |(F_3)_i(x)|^q \right].$$

Hence, we compute that

$$\begin{aligned} \|(F_2)_i\|_{L^q(\Omega)}^2 &\leq (C(q))^{2/q} [\|(F_1)_i\|_{L^q(\Omega)} + \|(F_3)_i\|_{L^q(\Omega)}]^2 \\ &\leq 2(C(q))^{2/q} [\|(F_1)_i\|_{L^q(\Omega)}^2 + \|(F_3)_i\|_{L^q(\Omega)}^2] \end{aligned}$$

what guarantees that

$$\|F_2\|_{(L^q(\Omega))^n} \leq \sqrt{2}(C(q))^{1/q} [\|F_1\|_{(L^q(\Omega))^n} + \|F_3\|_{(L^q(\Omega))^n}].$$

Now by items *i)* and *iv)*, we conclude that

$$\|F(u)\|_{(L^q(\Omega))^n} \leq 2^{3/2}(C(q))^{1/q} [c_0\|u\|_{(L^q(\Omega))^n} + \|C_1\|_{(L^q(\Omega))^n}]$$

in other words, F maps bounded sets into bounded sets. Finally item *iii)* and the last observation guarantees, by Corollary 3.35, that (3.7) possesses a unique global mild solution.

§ 3.3 THE CRITICAL CASE

In this final section we prove a local existence theorem for abstract parabolic problems given by (3.1), when the nonlinearity f satisfies certain critical conditions. Consider during this section that X^β , for $\beta \geq 0$, are the power spaces related to the positive sectorial operator A , i.e. $D(A^\beta)$ (see Section 1.3 for more information).

To start the discussion that this section wants to promote, let us assume for a moment that the map f is time independent and we are dealing with the problem

$$\begin{cases} cD_t^\alpha u(t) = -Au(t) + f(u(t)), & t \geq 0 \\ u(0) = u_0 \in X^1, \end{cases} \quad (3.8)$$

where for some $0 < \beta < 1$, the function $f : X^1 \rightarrow X^\beta$ is locally Lipschitz, that is, for each fixed $x \in X^1$, there exists an open ball $B_x \subset X^1$ centered at x and a constant $L = L(B_x) \geq 0$ such that

$$\|f(z) - f(y)\|_{X^\beta} \leq L\|z - y\|_{X^1}$$

for all $z, y \in B_x$. Then, the problem (3.8) is well posed in X^1 , i.e., for each $u_0 \in X^1$ there

exist $t_0 > 0$ and a continuous function $u : [0, t_0] \rightarrow X^1$ that satisfies

$$u(t) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(u(s)) ds, \quad t \in [0, t_0].$$

Indeed, observe by Theorem 2.47 and some simple computations, that (*a more general inequality is discussed in (3.9)-(3.10)*)

$$\|E_\alpha(-t^\alpha A)x\|_{X^1} \leq M t^{-\alpha(1-\beta)} \|x\|_{X^\beta} \quad \text{and} \quad \|t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha A)x\|_{X^1} \leq M t^{\alpha\beta-1} \|x\|_{X^\beta}$$

for some $M > 0$. So, given $u_0 \in X^1$, let $B_{u_0}(r) \subset X^1$ be the open ball with center u_0 and radius $r > 0$ and $L = L(B_{u_0}(r))$ be the Lipschitz constant of f associated to $B_{u_0}(r)$. Fix $\gamma \in (0, r)$ and choose $t_0 > 0$ such that

$$(M/\alpha\beta)(L\gamma + \|f(t, u_0)\|_{X^\beta}) t_0^{\alpha\beta} \leq \gamma/2 \quad \text{and} \quad \|E_\alpha(-t^\alpha A)u_0 - u_0\| \leq \gamma/2 \quad \forall t \in [0, t_0].$$

Consider $K := \{u \in C([0, t_0]; X^1) : u(0) = u_0 \text{ and } \|u(t) - u_0\|_{X^1} \leq \gamma, \quad \forall t \in [0, t_0]\}$ and define the operator T on K by

$$T(u(t)) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(u(s)) ds.$$

If $u \in K$, then $T(u(0)) = u_0$ and $T(u(t)) \in C([0, t_0]; X^1)$. Furthermore, we have that

$$\|T(u(t)) - u_0\|_{X^1} \leq \|E_\alpha(-t^\alpha A)u_0 - u_0\|_{X^1} + \mathcal{J}_1(t)$$

where

$$\mathcal{J}_1(t) = \int_0^t (t-s)^{\alpha\beta-1} M \left(\|f(s, u(s)) - f(s, u_0)\|_{X^\beta} + \|f(s, u_0)\|_{X^\beta} \right) ds$$

for $t \in [0, t_0]$. Therefore

$$\begin{aligned} \|T(u(t)) - u_0\|_{X^1} &\leq \|E_\alpha(-t^\alpha A)u_0 - u_0\|_{X^1} + (M/\alpha\beta)(L\gamma + \|f(s, u_0)\|_{X^\beta}) t_0^{\alpha\beta} \\ &\leq \gamma/2 + \gamma/2 = \gamma, \end{aligned}$$

that is, $T(K) \subset K$. Now, if $u, v \in K$, we see that

$$\|T(u(t)) - T(v(t))\|_{X^1} \leq \int_0^t (t-s)^{\alpha\beta-1} M \|f(u(s)) - f(v(s))\|_{X^\beta} ds$$

and therefore

$$\begin{aligned} \|T(u(t)) - T(v(t))\|_{X^1} &\leq \frac{LMt_0^{\alpha\beta}}{\alpha\beta} \sup_{s \in [0, t_0]} \|u(s) - v(s)\|_{X^1} \\ &\leq \frac{1}{2} \sup_{s \in [0, t_0]} \|u(s) - v(s)\|_{X^1}. \end{aligned}$$

So, by the Banach contraction principle we have that T has a unique fixed point in K , i.e. the unique solution of the proposed problem.

In the analysis above, the necessity of $\beta > 0$ is essential, since the whole argument is based on the convergence of the integral

$$\int_0^t (t-s)^{\alpha\beta-1} ds$$

for t in some compact interval $[0, t_0]$. In other words, since $A : X^1 \rightarrow X^0$, the fact that $f : X^1 \rightarrow X^\beta$ for $\beta > 0$, means that the solutions of problem (3.8) can be obtained as perturbations of the solutions of the linear problem $cD_t^\alpha u = -Au$. In this section we address the question of local solvability of problem (3.3) when $\beta = 0$ (and much more).

In this respect there are several results in the literature that can give some insight into this problem in the abstract setting when we are dealing with the usual semigroup theory. In resume, we study the fractional version of the study done by Arrieta-Carvalho in [8], keeping their notation.

Let ϵ , $\gamma(\epsilon)$, ξ , ζ , c , and δ be positive constants. Also, let $v : [0, \infty) \rightarrow [0, \delta)$ be a function such that $\lim_{t \rightarrow 0^+} v(t) = 0$.

Definition 3.37 A continuous function $u : [0, \tau] \rightarrow X^1$ is called an ϵ -regular mild solution to (3.1) if $u \in C((0, \tau]; X^{1+\epsilon})$ and verifies (3.2) in $[0, \tau]$ (see page 61).

Definition 3.38 For $\epsilon > 0$ we say that a map g is an ϵ -regular map relative to the pair (X^1, X^0) if there exist $\rho > 1$, $\gamma(\epsilon)$ with $\rho\epsilon \leq \gamma(\epsilon) < 1$, and a positive constant c , such that $g : X^{1+\epsilon} \rightarrow X^{\gamma(\epsilon)}$ and

$$\|g(x) - g(y)\|_{X^{\gamma(\epsilon)}} \leq c \left(1 + \|x\|_{X^{1+\epsilon}}^{\rho-1} + \|y\|_{X^{1+\epsilon}}^{\rho-1} \right) \|x - y\|_{X^{1+\epsilon}},$$

for all $x, y \in X^{1+\epsilon}$.

Remark 3.39 The conditions defined above guarantees that $0 < \epsilon < \gamma(\epsilon) < 1$.

Now we define the class of functions whose the non linearity will belong. The choice

of this space is done to ensure that the problem proposed above is interesting.

Definition 3.40 The set $\mathcal{F}(\epsilon, \rho, \gamma(\epsilon), c, \nu(\cdot), \xi, \zeta)$ is the family of functions f such that, for each $t \geq 0$, the function $f(t, \cdot)$ is an ϵ -regular map relative to the pair (X^1, X^0) , satisfying

$$\|f(t, x) - f(t, y)\|_{X^{\gamma(\epsilon)}} \leq c \left(\|x\|_{X^{1+\epsilon}}^{\rho-1} + \|y\|_{X^{1+\epsilon}}^{\rho-1} + \nu(t)t^{-\xi} \right) \|x - y\|_{X^{1+\epsilon}}$$

and

$$\|f(t, x)\|_{X^{\gamma(\epsilon)}} \leq c \left(\|x\|_{X^{1+\epsilon}}^{\rho} + \nu(t)t^{-\xi} \right)$$

for all $x, y \in X^{1+\epsilon}$.

Without loss of generality we may assume that the function ν is non-decreasing, once $\lim_{t \rightarrow 0^+} \nu(t) = 0$. We also suppose that

$$0 \leq \zeta \leq \alpha(\gamma(\epsilon) - \epsilon), \quad 0 \leq \xi \leq \alpha\gamma(\epsilon)$$

and $\alpha \in (0, 1)$. In most cases in the arguments below we will fix the parameters $\epsilon, \gamma(\epsilon), \rho, \xi, \zeta$ and c , and we will denote the class $\mathcal{F}$ defined above by $\mathcal{F}(\nu(\cdot))$.

We recall from [8] the following classification for a map f time-independent which is ϵ -regular, for ϵ belonging to some (later specified) interval I , relative to the pair (X^1, X^0) .

- If $I = [0, \epsilon_1]$ for some $\epsilon_1 > 0$ and $\gamma(0) > 0$, we say that f is a **subcritical** map relative to (X^1, X^0) .
- If $I = [0, \epsilon_1]$ for some $\epsilon_1 > 0$ with $\gamma(\epsilon) = \rho\epsilon$, $\epsilon \in I$, and if f is not subcritical, then we say that f is a **critical** map relative to (X^1, X^0) .
- If $I = (0, \epsilon_1]$ for some $\epsilon_1 > 0$ with $\gamma(\epsilon) = \rho\epsilon$, $\epsilon \in I$, and f is not subcritical or critical, then we say that f is a **double-critical** map relative to (X^1, X^0) .
- If $I = [\epsilon_0, \epsilon_1]$ for some $\epsilon_1 > \epsilon_0 > 0$ with $\gamma(\epsilon_0) > \rho\epsilon_0$ and f is not subcritical, critical or double critical, then we say that f is an **ultra-subcritical** map relative to (X^1, X^0) .
- If $I = [\epsilon_0, \epsilon_1]$ for some $\epsilon_1 > \epsilon_0 > 0$ with $\gamma(\epsilon) = \rho\epsilon$, $\epsilon \in I$, and if f is not subcritical, critical, double critical or ultra-subcritical, then we say that f is an **ultra-critical** map relative to (X^1, X^0) .

Note that if f is subcritical then $f : X^1 \rightarrow X^{\gamma(0)}$, $\gamma(0) > 0$, which is the usual definition of subcritical map. When f is a critical map it takes X^1 into X^0 but there is no positive constant θ such that f takes X^1 into X^α . When f is double-critical (*this name first appears*

in [16]) it is not defined as a map from X^1 into X^0 but it is ϵ -regular for arbitrarily small positive values of ϵ . When f is ultra-subcritical or ultra-critical, it is not a well defined map in $X^{1+\epsilon}$ for small values of $\epsilon > 0$, and it is only an ϵ -regular map when $\epsilon > \epsilon_0 > 0$, for some ϵ_0 . The main difference between ultra-subcritical and ultra-critical maps is that for the first definition, the time of existence of the solution can be chosen uniformly on bounded sets of X^1 , while for the second definition this is still an unknown property.

An immediate consequence of Theorem 2.47 is that for $0 \leq \theta \leq \beta \leq 1$, and $x \in X^\beta$,

$$t^{\alpha(1+\theta-\beta)} \|E_\alpha(-t^\alpha A)x\|_{X^{1+\theta}} \leq M \|x\|_{X^\beta} \quad (3.9)$$

and

$$t^{\alpha(\theta-\beta)+1} \|t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha A)x\|_{X^{1+\theta}} \leq M \|x\|_{X^\beta}. \quad (3.10)$$

Remark 3.41 This constant $M > 0$ above will be frequently used on the proof of the following theorems.

To prove the main theorem of this section, we need some previous results.

Lemma 3.42 Consider $\alpha \in (0, 1)$, $\theta \in [0, 1]$, and a positive sectorial operator A . The operators $\{t^{\alpha\theta} E_\alpha(-t^\alpha A) : X^1 \rightarrow X^{1+\theta}\}_{t>0}$ are bounded linear operators satisfying

$$\|t^{\alpha\theta} E_\alpha(-t^\alpha A)\|_{\mathcal{L}(X^1, X^{1+\theta})} \leq M,$$

with $M > 0$ independent of t . Moreover, given a compact subset J of X^1 , we have

$$\lim_{t \rightarrow 0^+} \sup_{x \in J} \|t^{\alpha\theta} E_\alpha(-t^\alpha A)x\|_{X^{1+\theta}} = 0.$$

Proof : The fact that $\|t^{\alpha\theta} E_\alpha(-t^\alpha A)\|_{\mathcal{L}(X^1, X^{1+\theta})} \leq M$ follows directly from (3.9). The remainder of the proof will follow in some steps.

Claim: $\lim_{t \rightarrow 0^+} \|t^{\alpha\theta} E_\alpha(-t^\alpha A)x\|_{X^{1+\theta}} = 0$, for each $x \in X^1$.

Fix $x \in X^1$. Then there exists a sequence $\{x_n\}_{n=1}^\infty \subset X^{1+\theta}$ such that $x_n \rightarrow x$ in the topology of X^1 . Given $\epsilon > 0$

i) Choose $N \in \mathbb{N}$ such that

$$M \|x_N - x\|_{X^1} < \epsilon/2.$$

ii) Then, choose $\delta > 0$ such that

$$t^{\alpha\theta} \|x_N\|_{X^{1+\theta}} \sup_{s \geq 0} \|E_\alpha(-s^\alpha A)\|_{\mathcal{L}(X^0)} < \epsilon/2$$

for all $0 < t < \delta$.

Therefore, using (3.9) we conclude for $|t| \leq \delta$, that

$$\begin{aligned} \|t^{\alpha\theta} E_{\alpha}(-t^{\alpha}A)x\|_{X^{1+\theta}} &\leq \|t^{\alpha\theta} E_{\alpha}(-t^{\alpha}A)(x - x_N)\|_{X^{1+\theta}} + \|t^{\alpha\theta} E_{\alpha}(-t^{\alpha}A)x_N\|_{X^{1+\theta}} \\ &\leq M\|x - x_N\|_{X^1} + t^{\alpha\theta}\|x_N\|_{X^{1+\theta}} \sup_{s \geq 0} \|E_{\alpha}(-s^{\alpha}A)\|_{\mathcal{L}(X^0)} \\ &< \epsilon, \end{aligned}$$

what finishes the proof of this claim.

Now let us prove the general result. For any $\epsilon > 0$ there exist $x_{\epsilon} \in J \subset X^1$ such that

$$0 \leq \sup_{x \in J} \|t^{\alpha\theta} E_{\alpha}(-t^{\alpha}A)x\|_{X^{1+\theta}} < \|t^{\alpha\theta} E_{\alpha}(-t^{\alpha}A)x_{\epsilon}\|_{X^{1+\theta}} + \epsilon$$

and the Claim guarantees that $0 \leq \lim_{t \rightarrow 0^+} \sup_{x \in J} \|t^{\alpha\theta} E_{\alpha}(-t^{\alpha}A)x\|_{X^{1+\theta}} \leq \epsilon$. Therefore, since $\epsilon > 0$ was arbitrary, we conclude that

$$0 \leq \lim_{t \rightarrow 0^+} \sup_{x \in J} \|t^{\alpha\theta} E_{\alpha}(-t^{\alpha}A)x\|_{X^{1+\theta}} \leq 0$$

what proves the theorem. □

The following lemma have a more involved notation, so we define a suitable function that will help us during the proof.

Lemma 3.43 *Let $\alpha \in (0, 1)$ and $f \in \mathcal{F}(\nu(\cdot))$ (see Definition 3.40). If we consider, for $0 < \zeta < 1$, the function*

$$B_{\epsilon}^{\theta}(\zeta) = \sup_{0 \leq \eta \leq \theta} \{B(\alpha(\gamma(\epsilon) - \eta), 1 - \zeta), B(\alpha(\gamma(\epsilon) - \eta), 1 - \alpha\rho\epsilon)\},$$

where $B(\cdot, \cdot)$ is the Beta function and $u \in C((0, \tau]; X^{1+\epsilon})$, then for all $0 \leq \theta < \gamma(\epsilon)$ we have that

$$\begin{aligned} t^{\alpha\theta} \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^{\alpha}A) f(s, u(s)) ds \right\|_{X^{1+\theta}} \\ \leq M c B_{\epsilon}^{\theta}(\xi) \left(\nu(t) t^{\alpha\gamma(\epsilon) - \xi} + \lambda(t)^{\rho} t^{\alpha(\gamma(\epsilon) - \rho\epsilon)} \right) \end{aligned}$$

for $0 < t \leq \tau$, where $\lambda(t) = \sup_{s \in (0, t]} \{s^{\alpha\epsilon} \|u(s)\|_{X^{1+\epsilon}}\}$.

Proof: Indeed, using (3.10), we have that

$$\begin{aligned} t^{\alpha\theta} \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s))\|_{X^{1+\theta}} ds \\ \leq M t^{\alpha\theta} \int_0^t (t-s)^{\alpha(\gamma(\epsilon)-\theta)-1} c \left(v(s) s^{-\xi} + \|u(s)\|_{X^{1+\epsilon}}^\rho \right) ds \end{aligned}$$

and since

$$\begin{aligned} M t^{\alpha\theta} \int_0^t (t-s)^{\alpha(\gamma(\epsilon)-\theta)-1} c \left(v(s) s^{-\xi} + \|u(s)\|_{X^{1+\epsilon}}^\rho \right) ds \\ \leq M c v(t) t^{\alpha\theta} \int_0^t (t-s)^{\alpha(\gamma(\epsilon)-\theta)-1} s^{-\xi} ds \\ + M c t^{\alpha\theta} \int_0^t (t-s)^{\alpha(\gamma(\epsilon)-\theta)-1} s^{-\alpha\rho\epsilon} (s^{\alpha\epsilon} \|u(s)\|_{X^{1+\epsilon}})^\rho ds \\ \leq M c v(t) t^{\alpha\theta-\xi+\alpha(\gamma(\epsilon)-\theta)} \int_0^1 (1-s)^{\alpha(\gamma(\epsilon)-\theta)-1} s^{-\xi} ds \\ + M c \lambda(t)^\rho t^{\alpha\theta-\alpha\rho\epsilon+\alpha(\gamma(\epsilon)-\theta)} \int_0^1 (1-s)^{\alpha(\gamma(\epsilon)-\theta)-1} s^{-\alpha\rho\epsilon} ds \\ \leq M c B_\epsilon^\theta(\xi) \left(v(t) t^{\alpha\gamma(\epsilon)-\xi} + \lambda(t)^\rho t^{\alpha(\gamma(\epsilon)-\rho\epsilon)} \right) \end{aligned}$$

we complete the proof. □

Observe that the function

$$B_\epsilon^\theta(\cdot) = \sup_{0 \leq \eta \leq \theta} \{B(\alpha(\gamma(\epsilon) - \eta), 1 - \cdot), B(\alpha(\gamma(\epsilon) - \eta), 1 - \alpha\rho\epsilon)\}$$

defined on the last result will be recursively used during the following theorems. Also, notice that we are not talking about the involved constants purposely, since they are implied by the definition of the space where the non-linearity belongs.

Lemma 3.44 *Considering the above notation, let $f \in \mathcal{F}(v(\cdot))$ (see Definition 3.40) and consider the functions $u, v \in C((0, \tau]; X^{1+\epsilon})$ such that*

$$\sup_{t \in (0, \tau]} t^{\alpha\epsilon} \|u(t)\|_{X^{1+\epsilon}} \leq \mu \quad \text{and} \quad \sup_{t \in (0, \tau]} t^{\alpha\epsilon} \|v(t)\|_{X^{1+\epsilon}} \leq \mu$$

for some $\mu > 0$. Then, for all $0 \leq \theta < \gamma(\epsilon) < 1$, we have

$$t^{\alpha\theta} \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) (f(s, u(s)) - f(s, v(s))) ds \right\|_{X^{1+\theta}} \leq \mathcal{J}_1(t), \quad t \in (0, \tau]$$

where

$$\mathcal{J}_1(t) = M c B_\epsilon^\theta (\zeta + \alpha\epsilon) \left[v(t) t^{\alpha(\gamma(\epsilon)-\epsilon)-\zeta} + 2\mu^{\rho-1} t^{\alpha(\gamma(\epsilon)-\rho\epsilon)} \right] \sup_{0 < s \leq \tau} s^{\alpha\epsilon} \|u(s) - v(s)\|_{X^{1+\epsilon}}.$$

Proof: It follows from the ϵ -regularity property of f that

$$\begin{aligned} & t^{\alpha\theta} \int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) (f(s, u(s)) - f(s, v(s))) \right\|_{X^{1+\theta}} ds \\ & \leq M t^{\alpha\theta} \int_0^t (t-s)^{\alpha(\gamma(\epsilon)-\theta)-1} \|f(s, u(s)) - f(s, v(s))\|_{X^{\gamma(\epsilon)}} ds \\ & \leq M c t^{\alpha\theta} \int_0^t (t-s)^{\alpha(\gamma(\epsilon)-\theta)-1} \|u(s) - v(s)\|_{X^{1+\epsilon}} \left(\|u(s)\|_{X^{1+\epsilon}}^{\rho-1} + \|v(s)\|_{X^{1+\epsilon}}^{\rho-1} + v(s) s^{-\zeta} \right) ds. \end{aligned}$$

Since v is an increasing function and

$$\max \left\{ \sup_{t \in (0, \tau]} t^{\alpha\epsilon} \|u(t)\|_{X^{1+\epsilon}}, \sup_{t \in (0, \tau]} t^{\alpha\epsilon} \|u(t)\|_{X^{1+\epsilon}} \leq \mu \right\} \leq \mu,$$

splitting the right term of the last inequality we obtain

$$\begin{aligned} & t^{\alpha\theta} \int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) (f(s, u(s)) - f(s, v(s))) \right\|_{X^{1+\theta}} ds \\ & \leq M c v(t) t^{\alpha\theta} \int_0^t (t-s)^{\alpha(\gamma(\epsilon)-\theta)-1} s^{-\zeta-\alpha\epsilon} s^{\alpha\epsilon} \|u(s) - v(s)\|_{X^{1+\epsilon}} ds \\ & \quad + M c t^{\alpha\theta} \int_0^t (t-s)^{\alpha(\gamma(\epsilon)-\theta)-1} s^{-\alpha\rho\epsilon} \left[(s^{\alpha\epsilon} \|u(s)\|_{X^{1+\epsilon}})^{\rho-1} \right. \\ & \quad \left. + (s^{\alpha\epsilon} \|v(s)\|_{X^{1+\epsilon}})^{\rho-1} \right] s^{\alpha\epsilon} \|u(s) - v(s)\|_{X^{1+\epsilon}} ds \\ & \leq \left[v(t) t^{\alpha(\gamma(\epsilon)-\epsilon)-\zeta} + 2\mu^{\rho-1} t^{\alpha(\gamma(\epsilon)-\rho\epsilon)} \right] M c B_\epsilon^\theta (\zeta + \alpha\epsilon) \sup_{0 < s \leq \tau} s^{\alpha\epsilon} \|u(s) - v(s)\|_{X^{1+\epsilon}}. \end{aligned}$$

□

Now we may prove the main result of this section.

Theorem 3.45 *Let $\alpha \in (0, 1)$ and $f \in \mathcal{F}(\epsilon, \rho, \gamma(\epsilon), c, \nu(\cdot), \xi)$. If $v_0 \in X^1$, there exist positive values r and τ_0 such that for any $u_0 \in B_{X^1}(v_0, r)$ there exists a continuous function $u(\cdot, u_0) : [0, \tau_0] \rightarrow X^1$ with $u(0) = u_0$, which is an ϵ -regular mild solution to the problem*

$$\begin{cases} cD_t^\alpha u = -Au + f(t, u(t)), & t > 0 \\ u(0) = u_0. \end{cases} \quad (3.11)$$

This solution satisfies

$$u \in C((0, \tau_0]; X^{1+\theta}), \quad 0 \leq \theta < \gamma(\epsilon),$$

$$\lim_{t \rightarrow 0^+} t^{\alpha\theta} \|u(t, u_0)\|_{X^{1+\theta}} = 0, \quad 0 < \theta < \gamma(\epsilon).$$

Moreover, for each $\theta_0 < \gamma(\epsilon)$ there exists a constant $C > 0$ such that if $u_0, w_0 \in B_{X^1}(v_0, r)$, then

$$t^{\alpha\theta} \|u(t, u_0) - u(t, w_0)\|_{X^{1+\theta}} \leq C \|u_0 - w_0\|_{X^1} \quad \forall t \in [0, \tau_0], \quad 0 \leq \theta \leq \theta_0.$$

Proof : Define μ by

$$McB_\epsilon^\epsilon \mu^{\rho-1} = \frac{1}{8},$$

where $B_\epsilon^\theta := \max \{B_\epsilon^\theta(\xi), B_\epsilon^\theta(\zeta + \alpha\epsilon)\}$ and choose $r = r(\mu, M) > 0$ such that

$$r = \frac{\mu}{4M} = \frac{1}{4M(8cMB_\epsilon^\epsilon)^{\frac{1}{\rho-1}}}.$$

For each $v_0 \in X^1$ fixed, choose $\tau_0 \in (0, 1]$ and $\delta > 0$ such that $\nu(t) < \delta$ for all $t \in (0, \tau_0]$,

$$\|t^{\alpha\epsilon} E_\alpha(-t^\alpha A) v_0\|_{X^{1+\epsilon}} \leq \frac{\mu}{2} \quad \text{if } 0 \leq t \leq \tau_0,$$

and

$$Mc\delta B_\epsilon^\epsilon = \min \left\{ \frac{\mu}{8}, \frac{1}{4} \right\}.$$

Consider

$$K(\tau_0) := \left\{ u \in C((0, \tau_0]; X^{1+\epsilon}) : \sup_{t \in (0, \tau_0]} t^{\alpha\epsilon} \|u(t)\|_{X^{1+\epsilon}} \leq \mu \right\}$$

with norm

$$\|u\|_{K(\tau_0)} = \sup_{t \in (0, \tau_0]} t^{\alpha\epsilon} \|u(t)\|_{X^{1+\epsilon}}.$$

Observe that $(K(\tau_0), \|\cdot\|_{K(\tau_0)})$ is a Banach space (see for instance Arrieta-Carvalho in [8]). Suppose that $u_0 \in X^1$ with $\|u_0 - v_0\|_{X^1} < r$ and define on $K(\tau_0)$ the map

$$Tu(t) = E_\alpha(-t^\alpha A)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s)) ds.$$

Our purpose is to show that for any $u_0 \in B_{X^1}(v_0, r)$, $T(K(\tau_0)) \subset K(\tau_0)$ and that T is a contraction. Firstly, let us prove that $T : K(\tau_0) \rightarrow K(\tau_0)$ is well defined.

Claim 1: if $u \in K(\tau_0)$, then $Tu \in C((0, \tau_0]; X^{1+\theta})$ for all $\theta \in [0, \gamma(\epsilon))$.

For this, let $t_0 \in (0, \tau_0]$, and choose $0 < t < t_0$; then, for every $0 \leq \theta < \gamma(\epsilon)$, we have that

$$\|Tu(t_0) - Tu(t)\|_{X^{1+\theta}} \leq \|E_\alpha(-t_0^\alpha A)u_0 - E_\alpha(-t^\alpha A)u_0\|_{X^{1+\theta}} + \mathcal{J}_1(t) + \mathcal{J}_2(t)$$

where

$$\mathcal{J}_1(t) = \left\| \int_0^t ((t_0-s)^{\alpha-1} E_{\alpha,\alpha}(-(t_0-s)^\alpha A) - (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A)) f(s, u(s)) ds \right\|_{X^{1+\theta}}$$

and

$$\mathcal{J}_2(t) = \left\| \int_t^{t_0} (t_0-s)^{\alpha-1} E_{\alpha,\alpha}(-(t_0-s)^\alpha A) f(s, u(s)) ds \right\|_{X^{1+\theta}}.$$

By Theorem 2.41 the first term goes to 0 as $t \rightarrow t_0^-$. A similar procedure to Lemma 3.7 and Theorem 3.8 shows that $\mathcal{J}_1(t)$ also goes to 0 as $t \rightarrow t_0^-$.

Now, let us consider $\mathcal{J}_2(t)$. Observe that, with similar computations, we obtain

$$\begin{aligned} & \left\| \int_t^{t_0} (t_0-s)^{\alpha-1} E_{\alpha,\alpha}(-(t_0-s)^\alpha A) f(s, u(s)) ds \right\|_{X^{1+\theta}} \\ & \leq M \int_t^{t_0} (t_0-s)^{\alpha(\gamma(\epsilon)-\theta)-1} \|f(s, u(s))\|_{X^{\gamma(\epsilon)}} ds \\ & \leq Mc \int_t^{t_0} (t_0-s)^{\alpha(\gamma(\epsilon)-\theta)-1} c(v(s)s^{-\xi} + \|u(s)\|_{X^{1+\epsilon}}^\rho) ds \\ & \leq Mc\delta \int_t^{t_0} (t_0-s)^{\alpha(\gamma(\epsilon)-\theta)-1} s^{-\xi} ds + Mc \int_t^{t_0} (t_0-s)^{\alpha(\gamma(\epsilon)-\theta)-1} \|u(s)\|_{X^{1+\epsilon}}^\rho ds \end{aligned}$$

and therefore

$$\begin{aligned}
& \left\| \int_t^{t_0} (t_0 - s)^{\alpha-1} E_{\alpha,\alpha}(-(t_0 - s)^\alpha A) f(s, u(s)) ds \right\|_{X^{1+\theta}} \\
& \leq Mc\delta t_0^{\alpha(\gamma(\epsilon)-\theta)-\xi} \int_{t/t_0}^1 (1-s)^{\alpha(\gamma(\epsilon)-\theta)-1} s^{-\xi} ds \\
& \quad + Mc\mu^\rho t^{\alpha(\gamma(\epsilon)-\theta-\rho\epsilon)} \int_{t/t_0}^1 (1-s)^{\alpha(\gamma(\epsilon)-\theta)-1} s^{-\alpha\epsilon\rho} ds,
\end{aligned}$$

which goes to 0 as $t \rightarrow t_0^-$, using the Dominated Convergence theorem. The case $t_0 \leq t \leq \tau_0$ is analogous.

Claim 2: $t^{\alpha\epsilon} \|Tu(t)\|_{X^{1+\epsilon}} \leq \mu$ for all $t \in (0, \tau_0]$.

Indeed, we may estimate as follows

$$t^{\alpha\epsilon} \|Tu(t)\|_{X^{1+\epsilon}} \leq \|t^{\alpha\epsilon} E_\alpha(-t^\alpha A)u_0\|_{X^{1+\epsilon}} + Mt^{\alpha\epsilon} \int_0^t (t-s)^{\alpha(\gamma(\epsilon)-\epsilon)-1} \|f(s, u(s))\|_{X^{\gamma(\epsilon)}} ds$$

and since

$$\begin{aligned}
& \|t^{\alpha\epsilon} E_\alpha(-t^\alpha A)u_0\|_{X^{1+\epsilon}} + Mt^{\alpha\epsilon} \int_0^t (t-s)^{\alpha(\gamma(\epsilon)-\epsilon)-1} \|f(s, u(s))\|_{X^{\gamma(\epsilon)}} ds \\
& \leq \|t^{\alpha\epsilon} E_\alpha(-t^\alpha A)(u_0 \pm v_0)\|_{X^{1+\epsilon}} + Mct^{\alpha\epsilon} \int_0^t (t-s)^{\alpha(\gamma(\epsilon)-\epsilon)-1} (v(s)s^{-\xi} + \|u(s)\|_{X^{1+\epsilon}}^\rho) ds \\
& \leq Mr + \|t^{\alpha\epsilon} E_\alpha(-t^\alpha A)v_0\|_{X^{1+\epsilon}} + Mc\delta t^{\alpha\gamma(\epsilon)-\xi} \int_0^1 (1-s)^{\alpha(\gamma(\epsilon)-\epsilon)-1} s^{-\xi} ds \\
& \quad + Mc\mu^\rho t^{\alpha(\gamma(\epsilon)-\rho\epsilon)} \int_0^1 (1-s)^{\alpha(\gamma(\epsilon)-\epsilon)-1} s^{-\alpha\epsilon\rho} ds \\
& \leq Mr + \|t^{\alpha\epsilon} E_\alpha(-t^\alpha A)v_0\|_{X^{1+\epsilon}} + Mc\delta B_\epsilon^\epsilon + Mc\mu^\rho B_\epsilon^\epsilon \leq \mu.
\end{aligned}$$

Taking $\theta = \epsilon$ in Claim 1 and using Claim 2, we guarantees that $T(K(\tau_0)) \subset K(\tau_0)$.

Claim 3: The operator T is a strict contraction in $K(\tau_0)$.

Indeed, for any $v, u \in K(\tau_0)$, since

$$t^{\alpha\epsilon} \|Tu(t) - Tv(t)\|_{X^{1+\epsilon}} \leq t^{\alpha\epsilon} \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) [f(s, u(s)) - f(s, v(s))] ds \right\|_{X^{1+\epsilon}}$$

using Lemma 3.44, we conclude that

$$t^{\alpha\epsilon} \|Tu(t) - Tv(t)\|_{X^{1+\epsilon}} \leq McB_\epsilon^\epsilon [\delta + 2\mu^{\rho-1}] \sup_{0 < s \leq \tau} s^{\alpha\epsilon} \|u(s) - v(s)\|_{X^{1+\epsilon}}.$$

Therefore we obtain the inequality

$$\|Tu(t) - Tv(t)\|_{K(\tau_0)} \leq \frac{1}{2} \|u(s) - v(s)\|_{K(\tau_0)},$$

what concludes the proof of Claim 3.

Now since the space $K(\tau_0)$ is a complete metric space, by the Banach contraction principle we have that T has a unique fixed point in $K(\tau_0)$, which will be denoted by $u(\cdot, u_0)$. It is defined for $\|u_0 - v_0\|_{X^1} < r$, $0 \leq t \leq \tau_0$. Moreover, observe that Claim 1 guarantees that the function $u(\cdot, u_0) \in C((0, \tau_0]; X^{1+\theta})$ for all $0 \leq \theta < \gamma(\epsilon)$, since $u(\cdot, u_0) \in K(\tau_0)$ and $Tu(\cdot, u_0) = u(\cdot, u_0)$.

Now let us prove some limit results.

Claim 4: $\sup_{0 < s \leq t} \{s^{\alpha\epsilon} \|u(s, u_0)\|_{X^{1+\epsilon}}\} \rightarrow 0$ as $t \rightarrow 0^+$.

Indeed, fix $t > 0$ and choose $\tilde{t} \in (0, t)$. From Lemma 3.43 we have that

$$\begin{aligned} & \tilde{t}^{\alpha\epsilon} \|u(\tilde{t}, u_0)\|_{X^{1+\epsilon}} \\ & \leq \tilde{t}^{\alpha\epsilon} \|E_\alpha(-\tilde{t}^\alpha A)u_0\|_{X^{1+\epsilon}} + \tilde{t}^{\alpha\epsilon} \left\| \int_0^{\tilde{t}} (\tilde{t}-s)^{\alpha-1} E_{\alpha,\alpha}(-(\tilde{t}-s)^\alpha A) f(s, u(s, u_0)) ds \right\|_{X^{1+\epsilon}} \\ & \leq \tilde{t}^{\alpha\epsilon} \|E_\alpha(-\tilde{t}^\alpha A)u_0\|_{X^{1+\epsilon}} + McB_\epsilon^\epsilon(\xi) v(\tilde{t}) + McB_\epsilon^\epsilon(\xi) \mu^{\rho-1} \sup_{0 < s \leq t} \{s^{\alpha\epsilon} \|u(s, u_0)\|_{X^{1+\epsilon}}\}. \end{aligned}$$

Therefore, we compute that for any $\tilde{t} \in (0, t)$,

$$\tilde{t}^{\alpha\epsilon} \|u(\tilde{t}, u_0)\|_{X^{1+\epsilon}} \leq \sup_{0 < s \leq t} \{s^{\alpha\epsilon} \|E_\alpha(-s^\alpha A)u_0\|_{X^{1+\epsilon}}\} + McB_\epsilon^\epsilon v(t) + \frac{1}{8} \sup_{0 < s \leq t} \{s^{\alpha\epsilon} \|u(s, u_0)\|_{X^{1+\epsilon}}\},$$

from which we obtain

$$\sup_{0 < s \leq t} \{s^{\alpha\epsilon} \|u(s, u_0)\|_{X^{1+\epsilon}}\} \leq \frac{8}{7} \left(\sup_{0 < s \leq t} \{s^{\alpha\epsilon} \|E_\alpha(-s^\alpha A)u_0\|_{X^{1+\epsilon}}\} + McB_\epsilon^\epsilon v(t) \right) \rightarrow 0 \text{ as } t \rightarrow 0^+.$$

Finally, we obtain the first main limit result.

Claim 5: $t^{\alpha\theta} \|u(t, u_0)\|_{X^{1+\theta}} \rightarrow 0$ as $t \rightarrow 0^+$, for all $0 < \theta < \gamma(\epsilon)$.

Again, from Lemma 3.43 we have that

$$\begin{aligned} & t^{\alpha\theta} \|u(t, u_0)\|_{X^{1+\theta}} \\ & \leq t^{\alpha\theta} \|E_\alpha(-t^\alpha A)u_0\|_{X^{1+\theta}} + t^{\alpha\theta} \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s, u_0)) ds \right\|_{X^{1+\theta}} \\ & \leq t^{\alpha\theta} \|E_\alpha(-t^\alpha A)u_0\|_{X^{1+\theta}} + M c B_\epsilon^\theta(\xi) \nu(t) + M c B_\epsilon^\theta(\xi) \mu^{\rho-1} \sup_{0 < s \leq t} \{s^{\alpha\epsilon} \|u(s, u_0)\|_{X^{1+\epsilon}}\}. \end{aligned}$$

Using Claim 4 and doing the limit when $t \rightarrow 0^+$, we conclude that Claim 5 holds.

Claim 6: $\lim_{t \rightarrow 0^+} \|u(t, u_0) - u_0\|_{X^1} = 0$.

For this observe again that from Lemma 3.43 it holds

$$\begin{aligned} & \|u(t, u_0) - u_0\|_{X^1} \\ & \leq \|E_\alpha(-t^\alpha A)u_0 - u_0\|_{X^1} + \int_0^t \|(t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) f(s, u(s, u_0))\|_{X^1} ds \\ & \leq \|E_\alpha(-t^\alpha A)u_0 - u_0\|_{X^1} + M c B_\epsilon^0(\xi) \left(\nu(t) + \left[\sup_{0 < s \leq t} \{s^{\alpha\epsilon} \|u(s, u_0)\|_{X^{1+\epsilon}}\} \right]^\rho \right). \end{aligned}$$

Therefore, the claim follows and we have that $u(t, u_0)$ is an ϵ -regular mild solution starting at u_0 and it is the unique ϵ -regular mild solution, starting at u_0 , in the set $K(\tau_0)$. We will call it the K -solution starting at u_0 .

Claim 7: $\sup_{0 < s \leq \tau_0} \{s^{\alpha\epsilon} \|u(s, u_0) - u(s, w_0)\|_{X^{1+\epsilon}}\} \leq 2M \|u_0 - w_0\|_{X^1}$ for all $u_0, w_0 \in B_{X^1}(\nu_0, r)$.

In fact, if $u_0, w_0 \in B_{X^1}(\nu_0, r)$, it follows from Lemma 3.44 and the choice of τ_0 that

$$\begin{aligned} & t^{\alpha\epsilon} \|u(t, u_0) - u(t, w_0)\|_{X^{1+\epsilon}} \\ & \leq t^{\alpha\epsilon} \|E_\alpha(-t^\alpha A)(u_0 - w_0)\|_{X^{1+\epsilon}} \\ & \quad + t^{\alpha\epsilon} \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) (f(s, u(s, u_0)) - f(s, u(s, w_0))) ds \right\|_{X^{1+\epsilon}} \end{aligned}$$

and therefore

$$\begin{aligned} & t^{\alpha\epsilon} \|u(t, u_0) - u(t, w_0)\|_{X^{1+\epsilon}} \\ & \leq M \|u_0 - w_0\|_{X^1} + M c B_\epsilon^\epsilon (\zeta + \alpha\epsilon) \nu(t) t^{\alpha(\gamma(\epsilon) - \epsilon) - \zeta} \sup_{0 < s \leq \tau_0} \{s^{\alpha\epsilon} \|u(s, u_0) - u(s, w_0)\|_{X^{1+\epsilon}}\} \\ & \quad + M c B_\epsilon^\epsilon (\zeta + \alpha\epsilon) 2\mu^{\rho-1} t^{\alpha(\gamma(\epsilon) - \rho\epsilon)} \sup_{0 < s \leq \tau_0} \{s^{\alpha\epsilon} \|u(s, u_0) - u(s, w_0)\|_{X^{1+\epsilon}}\}. \end{aligned}$$

Then we obtain

$$t^{\alpha\epsilon} \|u(t, u_0) - u(t, w_0)\|_{X^{1+\epsilon}} \leq M \|u_0 - w_0\|_{X^1} + \frac{1}{2} \sup_{0 < s \leq \tau_0} \{s^{\alpha\epsilon} \|u(t, u_0) - u(t, w_0)\|_{X^{1+\epsilon}}\},$$

which implies

$$\sup_{0 < s \leq \tau_0} \{s^{\alpha\epsilon} \|u(t, u_0) - u(t, w_0)\|_{X^{1+\epsilon}}\} \leq 2M \|u_0 - w_0\|_{X^1},$$

what finishes the proof of this Claim.

Claim 8: For each fixed $0 < \theta_0 < \gamma(\epsilon)$, there exist $C(\theta_0)$ such that

$$t^{\alpha\theta} \|u(t, u_0) - u(t, w_0)\|_{X^{1+\theta}} \leq C(\theta_0) \|u_0 - w_0\|_{X^1},$$

for all $\theta \in [0, \theta_0]$ and for all $u_0, w_0 \in B_{X^1}(\nu_0, r)$.

Again, if $u_0, w_0 \in B_{X^1}(\nu_0, r)$ and $0 \leq \theta < \gamma(\epsilon)$, by Lemma 3.44 and the choice of τ_0 we conclude that

$$\begin{aligned} & t^{\alpha\theta} \|u(t, u_0) - u(t, w_0)\|_{X^{1+\theta}} \\ & \leq t^{\alpha\theta} \|E_\alpha(-t^\alpha A)(u_0 - w_0)\|_{X^{1+\theta}} \\ & \quad + t^{\alpha\theta} \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A) (f(s, u(s, u_0)) - f(s, u(s, w_0))) ds \right\|_{X^{1+\theta}} \\ & \leq M \|u_0 - w_0\|_{X^1} + M c B_\epsilon^\theta (\zeta + \alpha\epsilon) \nu(t) t^{\alpha(\gamma(\epsilon) - \epsilon) - \zeta} \sup_{0 < s \leq \tau_0} \{s^{\alpha\epsilon} \|u(s, u_0) - u(s, w_0)\|_{X^{1+\epsilon}}\} \\ & \quad + M c B_\epsilon^\theta (\zeta + \alpha\epsilon) 2\mu^{\rho-1} t^{\alpha(\gamma(\epsilon) - \rho\epsilon)} \sup_{0 < s \leq \tau_0} \{s^{\alpha\epsilon} \|u(s, u_0) - u(s, w_0)\|_{X^{1+\epsilon}}\}. \end{aligned}$$

Then if we choose $0 < \theta_0 < \gamma(\epsilon)$, for any $\theta \in [0, \theta_0]$ we have from Claim 7 that

$$\begin{aligned} & t^{\alpha\theta} \|u(t, u_0) - u(t, w_0)\|_{X^{1+\theta}} \\ & \leq M \|u_0 - w_0\|_{X^1} + 2M^2 cB_\epsilon^\theta (\zeta + \alpha\epsilon) \left[\nu(t) t^{\alpha(\gamma(\epsilon)-\epsilon)-\zeta} + 2\mu^{\rho-1} t^{\alpha(\gamma(\epsilon)-\rho\epsilon)} \right] \|u_0 - w_0\|_{X^1} \\ & \leq C(\theta_0) \|u_0 - w_0\|_{X^1}, \end{aligned}$$

where

$$C(\theta_0) = M \left(1 + 2M \sup_{t \in [0, \tau_0], 0 \leq \theta \leq \theta_0} \left\{ cB_\epsilon^\theta (\zeta + \alpha\epsilon) [\nu(t) t^{\alpha(\gamma(\epsilon)-\epsilon)-\zeta} + 2\mu^{\rho-1} t^{\alpha(\gamma(\epsilon)-\rho\epsilon)}] \right\} \right).$$

This concludes the proof of the theorem. $\square$

We have the following two consequences of Theorem 3.45. The first one is that if f is independent of time, we obtain the same result. Observe that this is not just by an application of the theorem, since now we do not have a time-dependent class $\mathcal{F}$ any more. However, it is not difficult to readapt this proof.

Corollary 3.46 *With the above notation, assume that f is independent of time and it is an ϵ -regular map, for some $\epsilon > 0$, relative to the pair (X^1, X^0) . Then if $v_0 \in X^1$, there exist $r = r(v_0) > 0$ and $\tau_0 = \tau_0(v_0) > 0$ such that for every $u_0 \in B_{X^1}(v_0, r)$ there is a continuous function $u(\cdot, u_0) : [0, \tau_0] \rightarrow X^1$ with $u(0) = u_0$, which is an ϵ -regular mild solution to the problem (3.11) starting at u_0 . Furthermore, this solution satisfies*

$$u \in C((0, \tau_0]; X^{1+\theta}), \quad 0 \leq \theta < \gamma(\epsilon),$$

$$\lim_{t \rightarrow 0^+} t^{\alpha\theta} \|u(t, u_0)\|_{X^{1+\theta}} = 0, \quad 0 < \theta < \gamma(\epsilon).$$

Moreover, there exists a constant $C > 0$ such that if $u_0, w_0 \in B_{X^1}(v_0, r)$, then

$$t^{\alpha\theta} \|u(t, u_0) - u(t, w_0)\|_{X^{1+\theta}} \leq C \|u_0 - w_0\|_{X^1} \quad \forall t \in [0, \tau_0].$$

The second consequence of Theorem 3.45 is the following.

Corollary 3.47 *Let f be as in Theorem 3.45 and K a relatively compact set in X^1 , then there exist $\tau_0 = \tau_0(K)$ such that the ϵ -regular solution starting at u_0 exists in the time interval $[0, \tau_0]$ for any $u_0 \in K$.*

Remark 3.48 Observe that we cannot prove the uniqueness of solution to Theorem 3.45,

based on intrinsic problems. Indeed, If we try to adapt the arguments from [8], we quickly meet the obstacle that guarantees that the Mittag-Leffler functions do not satisfies the concatenation property (see Remark 2.40). The other approach involves the fractional Gronwall theorem, that makes a trickery development, and does not guarantees the convergence of some integrals during the computation.

In particular we obtain an existence theorem in X^1 without the nonlinearity being defined on X^1 . The main motivation to consider situations as in Theorem 3.45 is the fact that if the only requirement on the nonlinear term is that $f : X^1 \rightarrow X^0$ be locally Lipschitz, we cannot ensure that problem (3.1) is well-posed in an ϵ -regular sense. For example, taking $f(u) = 2Au$, which satisfies $f : X^1 \rightarrow X^0$ and is globally Lipschitz, we will have $cD_t^\alpha u = Au$, which is not locally well-posed, in general. Hence, some extra conditions should be imposed on f to guarantee the existence of solutions of the above problem.

As application to the results of this section, we consider our abstract result in the framework of fractional heat equations.

Remark 3.49 In the example that follows, we will constantly use certain well known embeddings that we summarize as:

$$H_{p_1}^{l_1}(\Omega) \hookrightarrow H_{p_2}^{l_2}(\Omega), \quad \frac{l_1}{N} - \frac{1}{p_1} \geq \frac{l_2}{N} - \frac{1}{p_2}, \quad 1 < p_1 \leq p_2 < \infty,$$

$$H_p^l(\Omega) \hookrightarrow C^\eta(\bar{\Omega}), \quad l - \frac{N}{p} > \eta > 0,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and the spaces $H_p^l(\Omega)$ are the Bessel potentials, also called Lebesgue spaces. Notice that $H_p^l(\Omega) = W^{l,p}(\Omega)$, the standard Sobolev-Slobodeckii spaces, whenever $p = 2$ and $l \in \mathbb{R}$, or $p > 1$ and $l \in \mathbb{Z}^+$ (see [2, 71]).

Example 3.50 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. We will treat the fractional equation (with $\alpha \in (0, 1)$)

$$\begin{cases} \partial_t^\alpha u = \Delta u + u|u|^{\rho-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.12)$$

in the L^q theory, for $1 < q < \infty$ and $q = N(\rho - 1)/2$. Observe that this equation for the case $\alpha = 1$ is well-known (e.g. [8, 16, 68, 69]).

Let $A = -\Delta$ with Dirichlet boundary conditions in Ω . Then, A can be seen as an

unbounded operator in $E_q^0 = L^q(\Omega)$ with domain

$$\mathcal{D}(A) := W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega).$$

It is well known that the scale of fractional power spaces $\{E_q^\beta\}_{\beta \in \mathbb{R}}$ associated to A (see [6, 8]) verifies

$$E_q^\beta \hookrightarrow H_q^{2\beta}(\Omega), \quad \beta \geq 0, \quad 1 < q < \infty,$$

$$E_q^{-\beta} \hookrightarrow (E_{q'}^\beta)', \quad \beta \geq 0, \quad 1 < q < \infty, \quad q' = \frac{q}{q-1}.$$

Therefore, using Remark 3.49, we conclude

$$E_q^\beta \hookrightarrow L^r(\Omega), \quad r \leq \frac{Nq}{N-2q\beta}, \quad 0 \leq \beta \leq \frac{N}{2q},$$

$$E_q^0 = L^q(\Omega),$$

$$E_q^\beta \hookleftarrow L^r(\Omega), \quad r \geq \frac{N}{N-2q\beta}, \quad -\frac{N}{2q'} \leq \beta \leq 0.$$

Let A_β be the realization of A in E_q^β . Then we have that A_β is an isometry from $E_q^{\beta+1}$ into E_q^β and

$$-A_\beta : D(A_\beta) = E_q^{\beta+1} \subset E_q^\beta \rightarrow E_q^\beta$$

is a sectorial operator. Denote $X_q^\beta := E_q^{\beta-1}$, $\beta \in \mathbb{R}$. Since we want to discuss the existence of solution in L^q , we observe that the fractional power spaces associated to

$$-A_{-1} : X_q^1 \subset X_q^0 \rightarrow X_q^0$$

satisfy

$$X_q^\beta \hookrightarrow L^r(\Omega) \text{ for } r \leq \frac{Nq}{N+2q-2q\beta}, \quad 1 \leq \beta < \frac{2q+N}{2q},$$

$$X_q^1 = L^q(\Omega), \tag{3.13}$$

$$X_q^\beta \hookleftarrow L^r(\Omega) \text{ for } r \geq \frac{Nq}{N+2q-2q\beta}, \quad \frac{2q'-N}{2q'} < \beta \leq 1.$$

We follow the classification of [8, Lemma 8] and classify the nonlinear term of (3.12)

in the following way.

Theorem 3.51 Consider $f : X_q^{\gamma(\epsilon)} \rightarrow X_q^{1+\epsilon}$ given by $f(u) = u|u|^{\rho-1}$. Then we classify

- i) If $q > \frac{N}{N-2}$, then f is an ϵ -regular map relative to (X_q^1, X_q^0) for $0 \leq \epsilon < \frac{N}{N+2q}$ and $\gamma(\epsilon) = \rho\epsilon$. Therefore f is a critical map.
- ii) If $q = \frac{N}{N-2}$, then f is an ϵ -regular map relative to (X_q^1, X_q^0) for $0 < \epsilon < \frac{N}{N+2q}$ and $\gamma(\epsilon) = \rho\epsilon$. Therefore f is a double-critical map.
- iii) If $1 < q < \frac{N}{N-2}$, then f is an ϵ -regular map relative to (X_q^1, X_q^0) for $0 < \epsilon_0(q) < \epsilon < \frac{N}{N+2q}$, with $\epsilon_0(q) = \frac{N}{N+2q} \left(1 - \frac{N}{2} \left(1 - \frac{1}{q}\right)\right)$ and $\gamma(\epsilon) = \rho\epsilon$. Therefore f is an ultra-critical map.

Proof : i) Observe that (3.13) guarantees that there exist $M_1 > 0$ such that for any $x, y \in X_q^{1+\epsilon}$,

$$\|f(x) - f(y)\|_{X_q^{\gamma(\epsilon)}} \leq M_1 \|f(x) - f(y)\|_{L^{\omega/\rho}(\Omega)} \leq \bar{M}_1 \left\| |x - y| \left(|x|^{\rho-1} + |y|^{\rho-1} \right) \right\|_{L^{\omega/\rho}(\Omega)}$$

for $\omega = Nq\rho/(N + 2q - 2q\gamma(\epsilon))$, since $\omega/\rho = Nq/(N + 2q - 2q\gamma(\epsilon))$. Computing, as in [16], we conclude that

$$\|f(x) - f(y)\|_{X_q^{\gamma(\epsilon)}} \leq \bar{M}_1 \|x - y\|_{L^{\omega}(\Omega)} \left(\|x\|_{L^{\omega}(\Omega)}^{\rho-1} + \|y\|_{L^{\omega}(\Omega)}^{\rho-1} \right).$$

Again using (3.13), there exist $M_2 > 0$ such that

$$\|x - y\|_{L^{\omega}(\Omega)} \left(\|x\|_{L^{\omega}(\Omega)}^{\rho-1} + \|y\|_{L^{\omega}(\Omega)}^{\rho-1} \right) \leq M_2 \|x - y\|_{X_q^{1+\epsilon}} \left(\|x\|_{X_q^{1+\epsilon}}^{\rho-1} + \|y\|_{X_q^{1+\epsilon}}^{\rho-1} \right)$$

since $\omega \leq Nq/(N + 2q - 2q(1 + \epsilon))$. Therefore

$$\|f(x) - f(y)\|_{X_q^{\gamma(\epsilon)}} \leq \bar{M}_1 M_2 \|x - y\|_{X_q^{1+\epsilon}} \left(\|x\|_{X_q^{1+\epsilon}}^{\rho-1} + \|y\|_{X_q^{1+\epsilon}}^{\rho-1} \right),$$

what concludes the proof of the first item. The other items are proved as in [8, 16]. $\square$

Applying Corollary 3.46, for each $u_0 \in L^q(\Omega)$ we ensure the existence of an ϵ -regular solution to the above problem, starting at u_0 , for any $\epsilon \in (\epsilon_0(q), N/(N+2q))$. Furthermore, for any $0 < \theta < \gamma(N/(N+2q)) = 1$ this solution verifies

$$t^{\alpha\theta} \|u(t)\|_{X^{1+\theta}} \rightarrow 0 \text{ as } t \rightarrow 0^+,$$

$$t^{\alpha\theta} \|u(t, u_0) - u(t, v_0)\|_{X^{1+\theta}} \leq C \|u_0 - v_0\|_{L^q}, \quad 0 < t < \tau(u_0, v_0).$$

4

Some Comments on Open Problems and Remaining Questions

“In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation adds a new story to the old structure.”

Hermann Hankel (1839-1873)

Since its born, the fractional calculus has made great strides. Recent progress in the area of fractional derivatives and integrals implies a promising potential for future developments and applications of the theory in various scientific areas. This manuscript presented several studies involving the implementation of fractional equations in a well known frame to the ordinary derivative and the results demonstrate that we can expect much more from it.

In summary, in this text we first introduced the concepts of functional analysis, the notion of sectorial operators and the relation with the abstract equations. Then we start the study of the fractional theory and the relation with the Mittag-Lefflers operators. Finally, in the last chapter we present ideas that would justify existence and uniqueness of

local, global and regular solutions. So, after a little reflection, we find ourselves thinking about the question that remains to be answered:

**And now, what other problems we will
try to solve in this fractional
setting?**

During our study, we encountered many problems whose the treatment was harsh and that ended up not being answered. Some of them will be discussed in what follows.

✠ THE FRACTIONAL-SEMIGROUP PROPERTY – As discussed on Section 2.3, we know that $S_\alpha(t) = E_\alpha(-t^\alpha A)$ is the operator that solves the linear fractional equation. Now Remark 2.40 explains that we cannot expect any kind of semigroup properties, in other words,

$$S(t)S(s) \neq S(t+s),$$

for this fractional operator and until now, we just know that it satisfies (see [56])

$$\begin{aligned} & \int_0^{t+s} (t+s-\tau)^{-\alpha} S_\alpha(\tau) d\tau - \int_0^t (t+s-\tau)^{-\alpha} S_\alpha(\tau) d\tau - \int_0^s (t+s-\tau)^{-\alpha} S_\alpha(\tau) d\tau \\ &= \alpha \int_0^t \int_0^s (t+s-\tau_1-\tau_2)^{-(1+\alpha)} S_\alpha(\tau_1) S_\alpha(\tau_2) d\tau_1 d\tau_2, \quad t, s \geq 0. \end{aligned}$$

On the other hand, it is well known that the dynamical properties and the notion of existence of an attractor are quite dependent of the semigroup property. So an open question until now is to find a “better property” ($S(t)S(s) = ?$ or $S(t+s) = ?$) that should insight us in a way to define these structures and how to deal with the theory that already exist.

✠ THE FRACTIONAL SADDLE-POINT – In dynamical systems, a saddle point is a fixed point whose stable and unstable manifolds have a dimension that is not zero. If the dynamic is given by a differentiable map f then a point is hyperbolic if and only if the differential of f^n (where n is the period of the point) has no eigenvalue on the unit circle (complex) when computed at the point. More explicitly, consider

$$\dot{u}(t) = Bu(t) + f(u(t)). \quad (4.1)$$

where $B \in M^n(\mathbb{R})$ (the space of $n \times n$ matrix with real entries) and $f : \mathbb{C}^n \mapsto \mathbb{C}^n$ is at least a continuous function. Also, let us suppose that the spectrum of B does not intersects the

imaginary axis and that $x_0 = 0$ is a saddle point for the equation (4.1). Consider for each $\lambda \in \mathbb{C}$, the generalized space

$$M_\lambda(B) := \text{Ker}(\lambda - B)$$

and

$$\mathbb{C}_+^n := \bigoplus_{\substack{\lambda \in \sigma(B) \\ \text{Re}(\lambda) > 0}} M_\lambda(B) \quad \text{and} \quad \mathbb{C}_-^n := \bigoplus_{\substack{\lambda \in \sigma(B) \\ \text{Re}(\lambda) < 0}} M_\lambda(B).$$

Definition 4.1 We define the following:

i) The stable manifold is given by:

$$W_f^s(0) := \{x \in \mathbb{R}^n : \exists \text{ a global solution of (4.1) } \phi : \mathbb{R} \mapsto \mathbb{R}^n \text{ such that } \phi(0) = x \text{ and } \lim_{t \rightarrow +\infty} \phi(t) = 0\}.$$

ii) The unstable manifold is given by:

$$W_f^u(0) := \{x \in \mathbb{R}^n : \exists \text{ a global solution of (4.1) } \phi : \mathbb{R} \mapsto \mathbb{R}^n \text{ such that } \phi(0) = x \text{ and } \lim_{t \rightarrow -\infty} \phi(t) = 0\}.$$

We can find in the literature the proof of the following theorem.

Theorem 4.2 Consider the above hypothesis and $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that

$$|f(x) - f(y)| \leq \sigma|x - y|, \forall x, y \text{ such that } |x|, |y| \leq \sigma.$$

Define the projections $\Pi_+ : \mathbb{C}^n \rightarrow \mathbb{C}_+^n$ and $\Pi_- : \mathbb{C}^n \rightarrow \mathbb{C}_-^n$ and suppose that

$$\begin{aligned} |e^{Bt} \Pi_+ x_0| &\leq K e^{\gamma t} |\Pi_+ x_0|, \quad t \leq 0 \\ |e^{Bt} \Pi_- x_0| &\leq K e^{-\gamma t} |\Pi_- x_0|, \quad t \geq 0. \end{aligned}$$

Then, there exist $\delta > 0$ and a set S ,

$$S := W_f^\delta(0) = \{u_0 : |\Pi_- x_0| \leq \delta/(2K) \text{ and } |x(t, x_0)| \leq \delta, \quad t \geq 0\}$$

where for each $u_0 \in S$, the function $u(t, u_0)$ is the solution of (4.1). Moreover, S is homeomorphic to an open ball with radius $\frac{\delta}{2K}$ of $\mathbb{C}_-^n$ and there exist $\tilde{K}, \tilde{\gamma} > 0$ such that

$$|u(t, u_0)| \leq \tilde{K} e^{-\tilde{\gamma} t} |u_0|, \quad t \geq 0.$$

(A similar conclusion can be obtained to the unstable manifold).

Figure 4.1: Saddle Point

Now going back to the fractional differential equations, we want to understand and study the same kind of problem. Note that this is an excellent question, since the answer could give us a tip in a way to define a “good dynamics” behavior and eventually would help us to construct an attractor theory to this Mittag-Leffler operators.

On the other hand, there are many other simpler problems to deal before we begin to study the main problem above. For instance;

- i) We need to define a notion to the fractional unstable manifold, once the solutions of the problem (4.1) are defined only to positive values of time.
- ii) Also, there is a intrinsic need to “decouple” the non-local definition of the fractional derivative and try to resolve and give some meaning to the problem

$$\begin{aligned} {}^c D_t^\alpha u(t) &= Bu(t) + f(u(t)) \\ u(t_0) &= u_0 \in \mathbb{R}^n \end{aligned} \tag{4.2}$$

to some $t_0 > 0$. A partial answer to this question is proposed. Suppose for an instant that the operator $E_\alpha(t^\alpha B)$ have an inverse for all $t \geq 0$ and that we assume:

(P) A function $u : [0, \infty) \rightarrow \mathbb{R}^n$ is said to be a global solution to (4.2), if ${}^c D_t^\alpha u(t) =$

$Bu(t) + f(u(t))$ for all $t > 0$ and $u(t_0) = u_0$.

Then we can deduce that a continuous function satisfies (P) if, and only if it satisfies

$$u(t) = E_\alpha(t^\alpha B)[E_\alpha(t_0^\alpha B)]^{-1} \left[x - \int_0^{t_0} (t_0 - s)^{\alpha-1} E_{\alpha,\alpha}((t_0 - s)^\alpha B) f(u(s)) ds \right] \\ + \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha}((t - s)^\alpha B) f(u(s)) ds, \quad t \geq 0.$$

Yet, the lack of information of the Mittag-Leffler operators is a difficult barrier to go through.

✂ FAEDO-GALERKIN METHOD – This method of solution is very commonly used in Fluid Dynamics [20, 31, 64]. In short, this procedure resume the infinite dimensional equation to the problem of solving a series of finite dimensional problems. Using energy estimates, weak convergence and considering a weak notion of solution we obtain a solution of the original problem. But here we find a slightly problem, when we try to adapt this kind of calculations to the fractional environment.

Now, remember that a fundamental step on this method is ensured by the following theorem (see [64]).

Theorem 4.3 *Let U , V and U' be Hilbert spaces with $U \subset V \subset U'$, where U' is the dual space of U . If a function $u \in L^2(0, T; U)$ with $u' \in L^2(0, T; U')$, then $u \in C^0([0, T]; V)$ and we prove that*

$$\frac{d}{dt} \|u\|_V^2 = 2\langle u', u \rangle.$$

The question in the fractional calculus is much more complex for proving an analogous theorem. Indeed, we first need to better understand the fractional product rule of two continuous functions. We already know [10, 31, 58] that for regular functions ϕ and ψ

$${}_c D_t^\alpha [\phi(t)\psi(t)] = -\phi(0)f(0) \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{k=0}^{\infty} \binom{\alpha}{k} \phi^{(k)}(t) D_t^{\alpha-k} f(t)$$

where $D_t^\gamma = J_t^{-\gamma}$ for $\gamma < 0$.

Our objective now is try to understand a way to give a “non-exotic” formulation to $\langle u, D_t^\alpha u \rangle$. Finally after that, we would aim to study a “fractional Faedo-Galerkin method” to solve fractional fluid dynamics equations.

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Subject Index

- Almost everywhere, 22
- Asymptotic behavior of $E_\alpha(z)$, 32
- Bromwich integral, 24
- Caputo
 - fractional derivative, 40
- Cauchy
 - domain, 13
 - problem, 44, 47, 59, 70, 91
- Cone, 81
- Contour integral, 13
- Convolution, 28
- Fractional power
 - definition of, 19
 - spaces, 19
- Function
 - beta, 27
 - continuous, 22
 - derivative of, 13
 - exponential type, 23
 - gamma, 24
 - general Mittag-Leffler, 31
 - globally Lipschitz, 70, 73
 - holomorphic, 13
 - increasing, 82
 - integrable, 21
 - locally Lipschitz, 62, 91
 - Mainardi, 36
 - Mittag-Leffler, 32
 - of bounded variation, 12
 - Wright-type, 36
- Infinitesimal generator, 15
- Map
 - closed, 11
 - densely defined, 12
 - norm of a continuous linear, 11
- Operator
 - linear, 12
 - Mittag-Leffler, 48
 - positive sectorial, 17
 - sectorial, 17
- Partial order relation, 80
- Path
 - closed, 12
 - Hankel's, 16
 - simple, 12
 - smooth, 12

Resolvent

equality, 14

set, 14

Riemann-Liouville

fractional derivative, 39

fractional integral, 39

Riemann-Stieltjes integral, 13

Semigroup

 C_0 , 15

definition, 15

Solution of differential equation

 ϵ -regular mild solution, 93

global mild solution, 61

global solution, 44

local mild solution, 61

Space

Banach, 11

ordered Banach, 82

Spectrum, 14

Theorem

dominated convergence, 22

Cauchy, 14

Cauchy general, 14

closed graph, 12

fractional Gronwall inequality, 34

Post-Widder, 54

second fundamental limit, 16

Transform

inverse Laplace, 24

Laplace, 23