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Bi-Lipschitz invariant geometry

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To Jesus, the Prince of Peace.

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“Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. You go into the first room and it’s dark, completely dark. You stumble around, bumping into the furniture. Gradually, you learn where each piece of furniture is. And finally, after six months or so, you find the light switch and turn it on. Suddenly, it’s all illuminated and you can see exactly where you were. Then you enter the next dark room. . . ”

(Andrew Wiles)

ABSTRACT

DA SILVA, T. F.. **Bi-Lipschitz invariant geometry**. 2018. 143 f. Doctoral dissertation (Doctorate Candidate Program in Mathematics) – Instituto de Ciências Matemáticas e de Computação (ICMC/USP), São Carlos – SP.

The study about bi-Lipschitz equisingularity has been a very important subject in Singularity Theory in last decades. Many different approach have cooperated for a better understanding about. One can see that the bi-Lipschitz geometry is able to detect large local changes in curvature more accurately than other kinds of equisingularity. The aim of this thesis is to investigate the bi-Lipschitz geometry in an algebraic viewpoint. We define some algebraic tools developing classical properties. From these tools, we obtain algebraic criterions for the bi-Lipschitz equisingularity of some families of analytic varieties. We present a categorical and homological viewpoints of these algebraic structure developed before. Finally, we approach algebraically the bi-Lipschitz equisingularity of a family of Essentially Isolated Determinantal Singularities.

Key-words: Double of a module, Lipschitz saturation of a module, bi-Lipschitz equisingularity, Determinantal varieties, Integral closure of ideals and modules.

RESUMO

DA SILVA, T. E. **Geometria bi-Lipschitz Invariante**. 2017. 143 p. Tese de doutorado (Candidato ao Programa de Doutorado em Matemática) – Instituto de Ciências Matemáticas e de Computação (ICMC/USP), São Carlos – SP.

O estudo da equisingularidade bi-Lipschitz tem sido amplamente investigado nas últimas décadas. Diversas abordagens têm contribuído para uma melhor compreensão a respeito. Observa-se que a geometria bi-Lipschitz é capaz de detectar grandes alterações locais de curvatura com maior precisão quando comparada a outros padrões de equisingularidade. O objetivo desta tese é investigar a geometria bi-Lipschitz do ponto de vista algébrico. Definimos algumas estruturas algébricas desenvolvendo algumas propriedades clássicas. A partir de tais estruturas obtemos critérios algébricos para a equisingularidade bi-Lipschitz de algumas classes de famílias de variedades analíticas. Apresentamos uma visão categórica e homológica dos elementos desenvolvidos. Finalmente abordamos algebricamente a equisingularidade de famílias de Singularidades Determinantais Essencialmente Isoladas.

Palavras-chave: O double de um módulo, Saturação Lipschitz de um módulo, Equisingularidade bi-Lipschitz, Variedades Determinantais, Fecho Integral de ideais e módulos.

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LIST OF SYMBOLS

$\mathbb{Z}$ — Integer numbers

$\mathbb{C}$ — Complex numbers

$\mathbb{R}$ — Real numbers

$\mathbb{K}$ — The field $\mathbb{R}$ or $\mathbb{C}$

$\mathbb{P}^N$ — N -dimensional complex projective space

$\mathcal{O}_X$ — Analytic sheaf of rings of the analytic variety X

$\Delta(X)$ — Diagonal of the set X in $X \times X$

$\ell(M)$ — The length of a module M

$\overline{M}$ — Integral closure of M

$\mathcal{R}(M)$ — Rees algebra of the module M

I_S — Lipschitz saturation of the ideal I

$B_I(X)$ — Blow-up of X with respect to the ideal I

M_D — Double of the module M

$\text{Proj}(G)$ — The projective spectrum of a graded ring G

$e(M)$ — The Buchsbaum-Rim multiplicity of a module M

$e(M, N)$ — The multiplicity of the pair of modules (M, N)

$\mathcal{F}_x$ — The stalk of the sheaf $\mathcal{F}$ at the point x

$\text{supp}(\mathcal{F})$ — The support of the sheaf $\mathcal{F}$

$\text{cosupp}(\mathcal{F})$ — The cosupport of the sheaf $\mathcal{F}$

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INTRODUCTION

The study about bi-Lipschitz equisingularity was started by Pham and Teissier [84, 85] at 1969. In these papers they defined the notion of the Lipschitz saturation of a local ring on the context of analytic complex algebras. After, in 1971 Zariski [111] developed the Lipschitz saturation of a complete local domain of dimension one. In the same year, [112] he defined the saturation of a local ring of dimension one, and in 1975 [113] he generalized this notion for local rings with arbitrary dimension. For more about Lipschitz saturation of local rings one can see the important contributions of Lipman [65, 66].

As reported by Trotman in [101], while Thom [99] was describing Whitney's work about stratifications of analytic varieties [109, 110] in 1964 at the Bourbaki seminar, he suggested that the vector fields yielding local topological triviality of Whitney stratifications (conjectured by Whitney in his contribution to the Morse Jubilee volume [109], although Whitney's conjecture, still unproven, required a much stronger notion of local triviality) might be made Lipschitz, hence the local bi-Lipschitz triviality would follow as a corollary. In his 1985 dissertation, Mostowski [72] proved the Thom's conjecture. Few years before in 1969, Pham and Teissier [84, 85] obtained a positive result for the special case of families of plane curves.

Using a set of equisingularity conditions, Mostowski [72] defined a notion of Lipschitz stratification (see the section 1.6) stronger than Verdier's (w) -regularity (see the section 1.3), and he proved that every complex analytic variety admits a Lipschitz stratification. Further, Mostowski proved that these Lipschitz stratifications are locally bi-Lipschitz trivial along strata by integrating stratified Lipschitz vector fields. Then, Parusiński generalized these theorems for every analytic real variety [78], after for all semianalytic sets [79], and finally in 1994 for every subanalytic set [82].

There are canonical Whitney (equivalently Verdier) stratifications of complex analytic varieties by the theory of Teissier [98] (see also [54]) which shows identifies canonical Whitney strata as precisely the loci of equimultiplicity of the different polar varieties. However, in general there is no canonical Lipschitz stratification attached to a given subanalytic set, even in the special cases of real algebraic varieties or complex algebraic varieties. For details, see Mostowski's habilitation dissertation [72]. Nevertheless, Mostowski [71] gave equivalent algebraic criteria for Lipschitz regularity in the case of complex surfaces and in [70] he also explained the close relation of the polar varieties of a complex analytic variety X and the stratifications of X with locally trivializing Lipschitz vector fields.

Many mathematicians worked or are still working on bi-Lipschitz geometry. We can

cite, for instance, Birbrair [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], Comte [20], Costa [22, 23, 24], Fernandes [6, 7, 8, 9, 10, 11, 12, 13, 32, 33, 34, 35], Gaffney [39, 40], Grandjean [8, 9], Grulha [26], Henry [55], Juniati [57, 58, 59], Kovalev [60], Lê [10], Mostowski [70, 71, 72, 73], Neumann [11, 12, 14, 74], O'Shea [9, 77], Parusiński [55, 78, 79, 80, 81, 82], Paunescu [61], Pham [84, 85], Pichon [74], Ruas [6, 33, 34, 91], Saia [23, 24], Sampaio [10, 35, 92], Teissier [85], Trotman [58, 59, 101], Valette [57, 59, 91, 102, 103, 104, 105, 106] and Zariski [111, 112, 113].

In [41], Gaffney used the integral closure of modules to describe the Whitney equisingularity. If $X \subseteq \mathbb{C}^n \times \mathbb{C}^k$ is a family of complex analytic varieties defined by an analytic map $F : (\mathbb{C}^n \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}^p, 0)$, $Y = 0 \times \mathbb{C}^k \equiv \mathbb{C}^k \subseteq X$ the parameter space and the singular locus of a small enough representative of X , Gaffney showed that $(X - Y, Y)$ is Whitney equisingular if and only if all the partial derivatives of F with respect to the parameter space are in the integral closure of the submodule generated by $\left\{ z_i \frac{\partial F}{\partial z_j} \right\}_{i,j=1}^n$, where $z_1, \dots, z_n$ are the coordinate functions on $\mathbb{C}^n$. We denote this inclusion as

$$JM(X)_Y \subseteq \overline{m_Y JM_z(X)}.$$

Since there is a close relation between the double structure and Lipschitz behavior, it is natural to hope that the above condition may describe the bi-Lipschitz equisingularity adding the double structure, i.e., $(JM(X)_Y)_D \subseteq \overline{(m_Y JM_z(X))_D}$, or even a weaker condition as $(JM(X)_Y)_D \subseteq \overline{(JM_z(X))_D}$. In the hypersurface case, Gaffney [39] called these the infinitesimal Lipschitz conditions m_Y and A, respectively.

This thesis is divided into five chapters.

In the first chapter we first remember the main definitions and results about the integral closure of ideals and modules. Then we recall the Whitney and Verdier regularity conditions and its algebraic characterization obtained by Gaffney in [41] using the integral closure of modules. We see the definition of the Lipschitz saturation of an ideal, following the ideas of Pham, Teissier [85] and Zariski [111, 112, 113]. We also remember the main about the double of an ideal and the infinitesimal Lipschitz conditions for hypersurfaces given by Gaffney in [39, 40]. After we recall the definition of Determinantal Varieties, which will be used on the fourth chapter. We end the first chapter presenting the classical approach about bi-Lipschitz geometry due to Mostowski and Parusiński.

In the second chapter we start generalizing the concept of the double for modules, and we obtain several properties and results. Then we use the double structure to rephrase the infinitesimal Lipschitz conditions in any codimension, and we generalize some results of [39].

In the third chapter we define some different notions of Lipschitz saturations of a module, each one generalizing the original concept for ideals, and we get relations among them. At the end we apply the algebraic machinery developed to obtain conditions for bi-Lipschitz equisingularity of some families of analytic varieties.

In the fourth chapter we investigate the bi-Lipschitz equisingularity for families of determinantal varieties. First we remember the classification of simple isolated singularities of codimension 2 in $\mathbb{C}^4$ and $\mathbb{C}^5$, given by Frühbis-Krüger and Neumer in [37]. Then we get a canonical stratified vector field defined on a semi-universal 1-unfolding of a determinantal variety which is Lipschitz when the double of the partial derivative of the unfolding with respect to the parameter belongs to the integral closure of the ideal generated by the double of the another partial derivatives. We also see that for the special case that the matrix of deformation is constant, the last above condition always is true and then the canonical vector field is Lipschitz.

Finally, in the fifth chapter we develop a categorical viewpoint of the double structure. We start defining the double of a homomorphism in a quite natural way. Then we look for homological properties of the double structure and we compare algebraic properties of the double homomorphism with the "single" homomorphism. We define the double category and we end the chapter generalizing the double homomorphism for two analytic varieties linked by an analytic map germ.

Have a good reading!

PRELIMINARIES

1.1 The integral closure of ideals

In this section we present some classical results and definitions about the integral closure of an ideal following [95].

For more about this subject one can see the works of Bivià-Ausina [15, 16].

Definition 1.1.1. *Let I be an ideal in a ring R . An element $h \in R$ is said to be **integral over I** if there exists a positive integer n and elements $a_i \in I^i$, with $i \in \{1, \dots, n\}$, such that*

$$h^n + a_1 h^{n-1} + \dots + a_{n-1} h + a_n = 0.$$

Such an equation is called an **equation of integral dependence of h over I** .

The set of all elements that are integral over I is called **the integral closure of I** , and is denoted by $\bar{I}$. If $\bar{I} = I$, then I is called **integrally closed**. If $I \subseteq J$ are ideals, we say that J is integral over I if $J \subseteq \bar{I}$.

A basic example is the following:

Example 1.1.2. *For arbitrary elements $x, y \in R$, $xy \in \overline{(x^2, y^2)}$. In fact, take $n = 2$, $a_1 = 0 \in (x^2, y^2)$ and $a_2 = -x^2 y^2 \in (x^2, y^2)^2$, notice that*

$$(xy)^2 + a_1(xy) + a_2 = 0$$

is an equation of integral dependence of xy over (x^2, y^2) . Therefore, $(x, y)^2$ is integral over (x^2, y^2) .

Definition 1.1.3. *Let $R \subseteq S$ be an extension of rings. An element $s \in S$ is called **integral over R** if $f(s) = 0$ for some monic polynomial $f \in R[T]$.*

The next lemma is a link between the notions of integral closure of extensions of rings and ideals, and it is an important tool to prove that the integral closure of an ideal is also an ideal.

Lemma 1.1.4. ([64] Lemma 1.3) *Let R be a ring and denote by $R[T]$ the ring of polynomials of one variable over R . Let $h \in R$ and I an ideal of R . Then: h is integral over I if and only if hT is integral over the subring $\bigoplus_{n \in \mathbb{Z}^+} I^n T^n$ of $R[T]$.*

Now, using the previous lemma and the fact that the set of the elements which are integral over a subring is also a subring, we have the next

Proposition 1.1.5. ([64] Corollary 1.4) *If I is an ideal of R then $\bar{I}$ is an integrally closed ideal of R , i.e., $\bar{\bar{I}} = \bar{I}$.*

Remark 1.1.6. ([95] Remark 1.1.3) *Basic properties of the integral closure of ideals:*

- $I \subseteq \bar{I}$, since for each $r \in I$, taking $n = 1$ and $a_1 = -r$ give an equation of integral dependence over I .
- If $I \subseteq J$ are ideals then $\bar{I} \subseteq \bar{J}$, since every equation of integral dependence of r over I is also an equation of integral dependence of r over J .
- $\bar{I} \subseteq \sqrt{I}$, since from the equation of integral dependence of some degree n for $h \in \bar{I}$, we conclude that $h^n \in (a_1, \dots, a_n) \subseteq I$.
- Radical, hence prime and maximal, ideals are integrally closed.
- The nilradical $\sqrt{0}$ of the ring is contained in $\bar{I}$ for every ideal I , because for each nilpotent element r there exists an integer n such that $r^n = 0$ and this is an equation of integral dependence over I .
- Intersections of integrally closed ideals are integrally closed.
- The following property is called **persistence**: if $\varphi : R \longrightarrow S$ is a ring homomorphism, then $\varphi(\bar{I}) \subseteq \overline{\varphi(I)S}$. This follows as by applying φ to an equation of integral dependence of an element r over I to obtain an equation of integral dependence of $\varphi(r)$ over $\varphi(I)S$.
- Another important property is contraction: if $\varphi : R \longrightarrow S$ is a ring homomorphism and I is an integrally closed ideal of S , then $\varphi^{-1}(I)$ is integrally closed in R . In fact, if r is integral over $\varphi^{-1}(I)$, then applying φ to an equation of integral dependence of r over $\varphi^{-1}(I)$ gives an equation of integral dependence of $\varphi(r)$ over I , hence $\varphi(r) \in I$ and $r \in \varphi^{-1}(I)$.
- In particular, if R is a subring of S , and I is an integrally closed ideal of S , then $I \cap R$ is an integrally closed ideal in R .

Integral closure behaves well under localization:

Proposition 1.1.7. ([95] Proposition 1.1.4) *Let R be a ring and I an ideal in R . For any multiplicatively closed subset W of R ,*

$$W^{-1}\bar{I} = \overline{W^{-1}I}.$$

Furthermore, the following are equivalent:

- a) $\bar{I} = I$;
- b) $\overline{W^{-1}I} = W^{-1}I$, for all multiplicatively closed subsets W of R ;
- c) $\bar{I}_P = I_P$, for all prime ideals P of R ;
- d) $\bar{I}_M = I_M$, for all maximal ideals M of R .

The following proposition reduces questions about the integral closure to questions about integral closure in integral domains.

Proposition 1.1.8. ([95] Proposition 1.1.5) *Let R be a ring and I an ideal of R .*

- a) *The image of the integral closure of I in R_{red} is the integral closure of the image of I in R_{red} : $\bar{I}R_{red} = \overline{IR_{red}}$. Thus $\bar{I}$ equals the natural lift to R of the integral closure of I in the reduced ring R_{red} .*
- b) *An element $r \in R$ is in the integral closure of I if and only if for every minimal prime ideal P of R , the image of r in $\frac{R}{P}$ is in the integral closure of $\frac{I+P}{P}$.*

We next rephrase integral closure with ideal equalities:

Proposition 1.1.9. ([95] Proposition 1.1.7) *Let R be a ring, I an ideal of R and $r \in R$. Then: $r \in \bar{I}$ if and only if there exists an integer n such that $(I + (r))^n = I(I + (r))^{n-1}$.*

One can also use modules to express integral dependence, usually known as *determinantal trick*.

Corollary 1.1.10. ([95] Corollary 1.1.8) *Let I be an ideal of R and $r \in R$. Then, the following are equivalent:*

- (a) r is integral over I ;
- (b) *There exists a finitely generated R -module M such that $rM \subseteq IM$ and such that whenever $aM = 0$ for some $a \in R$, then $ar \in \sqrt{0}$.*

Moreover, if I is finitely generated and contains a non-zerodivisor, r is integral over I if and only if there exists a finitely generated faithful R -module M such that $IM = (I + (r))M$.

We introduce reductions, which are an extremely useful tool for integral closure in general.

Definition 1.1.11. *Let $J \subseteq I$ be ideals in R . The ideal J is said to be a **reduction** of I if there exists a non-negative integer n such that $I^{n+1} = JI^n$.*

As a consequence of Proposition 1.1.9 we have the following result.

Corollary 1.1.12. *([95] Corollary 1.2.2) An element $r \in R$ is integral over J if and only if J is a reduction of $J + (r)$.*

Remark 1.1.13. *Note that if $JI^n = I^{n+1}$, then for all positive integers m , $I^{m+n} = JI^{m+n-1} = \dots = J^m I^n$. In particular, if $J \subseteq I$ is a reduction, there exists an integer n such that for all $m \geq 1$, $I^{m+n} \subseteq J^m$.*

The reduction property is transitive:

Proposition 1.1.14. *([95] Proposition 1.2.4) Let $K \subseteq J \subseteq I$ be ideals in R .*

- a) *If K is a reduction of J and J is a reduction of I , then K is a reduction of I ;*
- b) *If K is a reduction of I then J is a reduction of I ;*
- c) *If I is finitely generated, $J = K + (r_1, \dots, r_k)$, and K is a reduction of I , then K is a reduction of J .*

The next corollary gives a relation between the integral closure and reductions.

Corollary 1.1.15. *([95] Corollary 1.2.5) Let $J \subseteq I$ be ideals in R . Suppose that I is finitely generated. Then J is a reduction of I if and only if $I \subseteq \bar{J}$.*

In particular, if R is a noetherian ring, then reduction property in R is equivalent to the integral dependence.

Notice that in Example 1.1.2 we have seen that (x^2, y^2) is a reduction of $(x, y)^2$.

The next result says about the behaviour of the integral closure with powers of an ideal.

Proposition 1.1.16. *([64] Corollary 1.8) Let I be an ideal of R .*

- a) $(\bar{I})^n \subseteq \overline{I^n}$;
- b) $I \cdot \bar{I}^n \subseteq \overline{I^{n+1}}$.

As a corollary of the Corollary 1.1.10, we have the

Corollary 1.1.17. ([64] Corollary 1.11) *If I and J are ideals in R then*

$$\bar{I} \cdot \bar{J} \subseteq \overline{I \cdot J}.$$

Proposition 1.1.18. ([64] Proposition 1.14) *Let R be an excellent reduced ring and I an ideal in R containing a non-zerodivisor element of R . Then, there exists an integer N such that, if $n \geq N$:*

- a) $I \cdot \bar{I}^n = \overline{I^{n+1}};$
- b) $I \cdot (\bar{I})^n = (\bar{I})^{n+1}.$

For the definition of excellent rings, see [52]. Every complete local ring is an excellent ring. For example, the local rings of the analytic geometry.

Notation 1.1.19. *Let I be an ideal of R . We denote by $e(I)$ the Hilbert-Samuel multiplicity of I .*

Proposition 1.1.20. ([64] Proposition 1.18) *Let (R, m) be a local noetherian ring. If I and J are m -primary ideals of R with the same integral closure, i.e., $\bar{I} = \bar{J}$ then $e(I) = e(J)$.*

For the definition of m -primary ideals and Hilbert-Samuel multiplicity see [31].

Remark 1.1.21. *If (R, m) is an analytic equidimensional local algebra, David Rees [86] showed a reciprocal: if I and J are m -primary ideals of R such that $e(I) = e(J)$ and $I \subseteq J$ then $\bar{I} = \bar{J}$.*

Let us recall the definition of *nowhere dense* in a topological space.

Definition 1.1.22. *Let X be a topological space. A subset $A \subseteq X$ is called **nowhere dense** if $\bar{A}$ has empty interior. Equivalently, a nowhere dense set is a set that is not dense in any non-empty open subset of X .*

The next theorem is one of the most important result to our work. It gives several points of view of the integral closure of an ideal when we are working in analytic geometry.

Theorem 1.1.23. ([64] Theorem 2.1) *Let $(X, \mathcal{O}_X)$ be a reduced complex analytic space, $x \in X$, and let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{O}_X$. Denote $I = \mathcal{I}_x$ the stalk of $\mathcal{I}$ on x , which is an ideal of $\mathcal{O}_{X,x}$. Let $h \in \mathcal{O}_{X,x}$. Suppose that $\mathcal{I}$ defines a nowhere dense closed subset of X . The following are equivalent:*

- (a) h is integral over I ;
- (b) *There exist a neighborhood U of x , a positive real number C , representatives of the space germ X , the function germ h , and generators $g_1, \dots, g_m$ of I on U , which we identify with the corresponding germs, so that*

$$\|h(z)\| \leq C \max \{ \|g_1(z)\|, \dots, \|g_m(z)\| \},$$

for all $z \in U$;

- (c) For all analytic path germs $\phi : (\mathbb{C}, 0) \rightarrow (X, x)$, the pullback $\phi^*(h)$ is contained in the ideal generated by $\phi^*(I)$ in the local ring $\mathcal{O}_{\mathbb{C}, 0}$;
- (d) Let NB denote the normalization of the blowup of X by I , $\bar{D}$ the pullback of the exceptional divisor of the blowup of X by I . Then, for any irreducible component C of the underlying set of $\bar{D}$, the order of vanishing of the pullback of h to NB along C is greater than or equal to the order of the divisor $\bar{D}$ along C .

We end this section with a theorem that allows us work with the integral closure of a sheaf of ideals.

Theorem 1.1.24. ([64] Theorem 2.6) Let $(X, \mathcal{O}_X)$ be a reduced complex analytic space, and let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{O}_X$. Suppose that $\mathcal{I}$ defines a nowhere dense closed subset of X . Then there exists a coherent sheaf of ideals of $\mathcal{O}_X$, denoted $\bar{\mathcal{I}}$, such that

$$(\bar{\mathcal{I}})_x = \overline{\mathcal{I}_x},$$

for all $x \in X$.

1.2 The integral closure of modules

In this section we develop the notion of the integral closure of a module and we develop analogous properties for this object. In the contexts considered in this work, it seems most convenient to define the integral closure of a module using the valuative criterion. We follow the Gaffney's language for the integral closure of a module following [41].

Here we denote $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n, 0}$ and m_n its maximal ideal.

Definition 1.2.1. Suppose (X, x) is a complex analytic germ and that M is a submodule of $\mathcal{O}_{X, x}^p$. Let $h \in \mathcal{O}_{X, x}^p$.

- We say that **h is integral over M** if $h \circ \phi \in (\phi^*(M))\mathcal{O}_1$, for all analytic path $\phi : (\mathbb{C}, 0) \rightarrow (X, x)$.
- We say that **h is strictly integral over M** if $h \circ \phi \in m_1(\phi^*(M))\mathcal{O}_1$, for all analytic path $\phi : (\mathbb{C}, 0) \rightarrow (X, x)$.

The **integral closure of M** is defined as $\bar{M} := \{h \in \mathcal{O}_{X, x}^p \mid h \text{ is integral over } M\}$ and the **strict integral closure of M** is defined as $M^\dagger := \{h \in \mathcal{O}_{X, x}^p \mid h \text{ is strictly integral over } M\}$. Using the definition it is clear that $\bar{M}$ and $M^\dagger$ are submodules of $\mathcal{O}_{X, x}^p$, $M \subseteq \bar{M}$ and $M^\dagger \subseteq \bar{M}$.

Example 1.2.2. ([41] Example 1.4) Suppose $X = \mathbb{C}^2$ and that $M \subseteq \mathcal{O}_2^2$ is generated by $\{(x, 0), (0, y), (y, x)\}$. Then, $\bar{M} = m_2\mathcal{O}_2^2$. Indeed, let $\phi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$. Then, $(\phi^*(M))\mathcal{O}_1$ is generated by $\{(t^n, 0), (0, t^n)\}$, where $n = \min\{o(\phi_1), o(\phi_2)\}$ and $o(\phi_i)$ denotes the order of vanishing of ϕ_i , for $i \in \{1, 2\}$. But these are the generators of $(\phi^*(m_2)\mathcal{O}_2^2)\mathcal{O}_1$ as well.

If (X, x) has several components, then it is clear that M induces a submodule M_V in $\mathcal{O}_{V,x}^p$, V any component of (X, x) . It is also clear from the definition that $h \in \overline{M}$ if and only if $h_V \in \overline{M_V}$ for all components V of (X, x) , where h_V is the element of $\mathcal{O}_{V,x}^p$ induced by h . It is clear from the definition that $\overline{\overline{M}} = \overline{M}$.

The following generalization of Nakayama's Lemma is often useful.

Proposition 1.2.3. ([41] Proposition 1.5) Suppose N, M are submodules of $\mathcal{O}_{X,x}^p$, and $m = m_{X,x}$.

- a) If $\overline{(m\overline{M} + N)} = \overline{M}$ then $\overline{N} = \overline{M}$;
- b) If $M \subseteq m\overline{M} + N \subseteq \overline{M}$ then $\overline{N} = \overline{M}$.

The link between the integral closure of ideals and modules is very strong. If M is a submodule of $\mathcal{O}_{X,x}^p$, and $[M]$ is a matrix of generators of M , let $J_k(M)$ denote the ideal generated by $k \times k$ minors of $[M]$. This is the same as the $(p - k)$ -th Fitting ideal of $\frac{\mathcal{O}_{X,x}^p}{M}$, hence is independent of the choice of generators of M (see [36] and [68]). If $h \in \mathcal{O}_{X,x}^p$, let (h, M) denote the submodule generated by h and M .

The following lemma, which is a generalization of Cramer's rule, is helpful in establishing the connection between $\overline{M}$ and $\overline{J_k(M)}$.

Lemma 1.2.4. ([41] Lemma 1.6) Suppose $h \in \mathcal{O}_{X,x}^p$, M is an $\mathcal{O}_{X,x}$ -submodule of $\mathcal{O}_{X,x}^p$, $J_{k+1}(h, M) = 0$ and no element of $J_k(M)$ is a zero divisor on $\mathcal{O}_{X,x}$. Then

$$J_k(M) \cdot h \subseteq J_k(h, M) \cdot M.$$

The link between $\overline{M}$ and $\overline{J_k(M)}$ is established by

Proposition 1.2.5. ([41] Proposition 1.7) Suppose M is a submodule of $\mathcal{O}_{X,x}^p$, X irreducible, $h \in \mathcal{O}_{X,x}^p$. Then $h \in \overline{M}$ if and only if $J_k(h, M) \subseteq \overline{J_k(M)}$, where k is the largest integer such that $J_k(h, M) \neq 0$.

The proof of the above proposition in [41] shows that $h \in \overline{M}$ implies $J_k(h, M) \subseteq \overline{J_k(M)}$ for all k with no assumptions on X or the generic rank of M .

If X is not irreducible, one obtains:

Corollary 1.2.6. ([41] Corollary 1.8) Suppose (X, x) is a complex analytic germ with irreducible components (V_i) , and M is a submodule of $\mathcal{O}_{X,x}^p$. Then $h \in \overline{M}$ if and only if $J_{k_i}(h, M_i) \subseteq \overline{J_{k_i}(M_i)}$, for all i , where M_i is the submodule of $\mathcal{O}_{V_i,x}^p$ induced from M and k_i is the generic rank of (h, M_i) on V_i .

In [87] Rees defined the notion of the integral closure of M in $K \otimes_R M$, where R is a noetherian domain, K its field of fractions and M is a finitely generated torsion-free R -module.

His definition is based on the theory of discrete valuations. However, the previous proposition and the Theorem 1.2 of [87] show that $\overline{M}$ in our language is exactly the set of elements of $\mathcal{O}_{X,x}^p$ which are integral over M in Rees's language.

Using Proposition 1.2.5 and Corollary 1.2.6, Gaffney proved the next result.

Proposition 1.2.7. ([41] Proposition 1.9) *Suppose M is a submodule of $\mathcal{O}_{X,x}^p$ and $h \in \mathcal{O}_{X,x}^p$. Then $h \in \overline{M}$ if and only if on each component V of (X, x) , there exists an ideal I of $\mathcal{O}_{V,x}$, $I \neq 0$ such that $I \cdot h \subseteq I \cdot M$ in $\mathcal{O}_{V,x}^p$.*

In the particular case when the generic rank of M is 0 or maximal, Gaffney obtained a description of the integral closure of M stronger than the Proposition 1.2.7.

Proposition 1.2.8. ([41] Proposition 1.10) *Suppose M is a submodule of $\mathcal{O}_{X,x}^p$ of generic rank p or 0 on each component of X , and let $h \in \mathcal{O}_{X,x}^p$. Then $h \in \overline{M}$ if and only if there exists a faithful submodule I of $\mathcal{O}_{X,x}$ such that $I \cdot h \subseteq I \cdot M$.*

The next proposition contains the growth condition for integral closure in the module case. The version we work here was suggested by Looijenga. In what follows, we let $\Gamma(E)$ denote sections of a vector bundle.

Proposition 1.2.9. ([41] Proposition 1.11) *Suppose $h \in \mathcal{O}_{X,x}^p$ and that M is a submodule of $\mathcal{O}_{X,x}^p$. Then $h \in \overline{M}$ if and only if for each choice of generators $\{s_i\}$ of M there exists a neighborhood U of x such that for all $\varphi \in \Gamma(\text{Hom}(\mathbb{C}^p, \mathbb{C}))$,*

$$\|\varphi(z) \cdot h(z)\| \leq C \sup_i \|\varphi(z) \cdot s_i(z)\|$$

for all $z \in U$.

We consider the "sheafification" of the construction of the integral closure of modules. The connection between M and $J_k(M)$ allows us to show that the integral closure gives rise to a coherent sheaf. The proof uses a description of $\overline{M}$ in terms of blowing up.

Proposition 1.2.10. ([41] Proposition 1.12) *Suppose $\mathcal{M}$ is a coherent sheaf of submodules of $\mathcal{O}_X^p$. Then, there exists a unique coherent sheaf $\overline{\mathcal{M}}$ on X such that for each $x \in X$, $(\overline{\mathcal{M}})_x = \overline{\mathcal{M}_x}$.*

There is another way to make a connection between the integral closure of modules and ideals.

Let us define the *ideal sheaf* $\rho(M)$ on $X \times \mathbb{P}^{p-1}$ associated to a submodule sheaf M of $\mathcal{O}_X^p$ (see [48]): Given $h = (h_1, \dots, h_p) \in \mathcal{O}_X^p$ and $(x, [t_1, \dots, t_p]) \in X \times \mathbb{P}^{p-1}$, with $t_i \neq 0$, we define $\rho(h)$ as the germ of the analytic map given by $\sum_{j=1}^p h_j(z) \frac{T_j}{T_i}$ which is well-defined on a Zariski open subset of $X \times \mathbb{P}^{p-1}$ that contains the point $(x, [t_1, \dots, t_p])$. We define $\rho(M)$ as the ideal generated by $\{\rho(h) \mid h \in M\}$.

The next result gives a relation between the integral closure of modules and ideals.

Proposition 1.2.11. ([48] Proposition 3.4) *Let $h \in \mathcal{O}_{X,x}^p$. Then $h \in \overline{M}$ at x if and only if $\rho(h) \in \overline{\rho(M)}$ at all point $(x, [t_1, \dots, t_p]) \in V(\rho(M))$.*

The Multiplicity of a Pair of Modules

We end this section defining the multiplicity of a pair of modules following the lines of [42].

The multiplicity of an ideal or module or pair of modules is one of the most important invariants we can associate to an m -primary module. It is intimately connected with integral closure.

Let (X, x) be a germ of an analytic space and assume that (X, x) has dimension d . Let $M \subseteq N$ be sheaves of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$, where M has finite colength in N . Denote $\mathcal{M}$ the associated ideal sheaf $\rho(M)$. Let $\mathcal{R}(M)$ and $\mathcal{R}(N)$ the Rees algebra of M and N , respectively.

Then, we have canonical maps which give us the following diagram:

$$\begin{array}{ccc} B_{\mathcal{M}}(\text{Proj}(\mathcal{R}(N))) & \xrightarrow{\pi_N} & \text{Proj}(\mathcal{R}(N)) \\ \pi_M \downarrow & & \downarrow \pi_{XN} \\ \text{Proj}(\mathcal{R}(M)) & \xrightarrow{\pi_{XM}} & X \end{array}$$

On the blow up $B_{\mathcal{M}}(\text{Proj}(\mathcal{R}(N)))$ we have two tautological bundles. One is the pullback of the bundle on $\text{Proj}(\mathcal{R}(N))$. The other comes from $\text{Proj}(\mathcal{R}(M))$. Denote the corresponding Chern classes by c_M and c_N , and denote the exceptional divisor by $D_{M,N}$. Suppose the generic rank of N (and hence M) is g .

Then, the multiplicity of a pair of modules (M, N) at x is:

$$e(M, N) := \sum_{j=0}^{d+g-2} \int D_{M,N} \cdot c_M^{d+g-2-j} \cdot c_N^j.$$

Kleiman and Thorup show that this multiplicity is well defined at $x \in X$ as long as $\overline{M} = \overline{N}$ on a deleted neighborhood of x . This condition implies that $D_{M,N}$ lies in the fiber over x , hence is compact. It also possible to conclude that, when $N = \mathcal{O}_{X,x}^p$, and M has finite colength in $\mathcal{O}_{X,x}^p$, then $e(M, \mathcal{O}_{X,x}^p)$ is the Buchsbaum-Rim multiplicity $e(M)$ defined in [19]. There is a fundamental result due to Kleiman and Thorup, the principle of additivity [62], which states that given a sequence of $\mathcal{O}_{X,x}$ -modules $M \subseteq N \subseteq P$ such that the multiplicity of the pairs is well defined, then

$$e(M, P) = e(M, N) + e(N, P).$$

Also if $\overline{M} = \overline{N}$ at x then $e(M, N) = 0$, and the converse also holds if X is equidimensional. These results will be used in this work.

1.3 Whitney equisingularity and the integral closure of modules

In this section, we follow the Gaffney's approach about Whitney equisingularity and integral closure of modules presented in [42].

The way how Gaffney got algebraic conditions that characterizes the Whitney equisingularity using the integral closure of modules have inspired us to develop some algebraic tools in this work that will be useful in order to describe the bi-Lipchitz Equisingularity.

We start with some notation to describe a family of analytic sets.

Setup 1.3.1. Let $(X, 0) \subseteq (\mathbb{C}^{n+k}, 0)$ be the germ of the analytic space defined by an analytic map $F : \mathbb{C}^n \times \mathbb{C}^k \rightarrow \mathbb{C}^p$, where X is a sufficient small representative such that $Y = \mathbb{C}^k = 0 \times \mathbb{C}^k \subseteq X$ is the singular set of X . Let $F_1, \dots, F_p : \mathbb{C}^n \times \mathbb{C}^k \rightarrow \mathbb{C}$ be the coordinates functions of F , for each $y \in Y$ let $f_y : \mathbb{C}^n \rightarrow \mathbb{C}^p$ given by $f_y(z) := F(z, y)$ and let $X_y := f_y^{-1}(0)$. Assume that X_y has an isolated singularity in 0, for all $y \in Y$ and $n \geq p$. Let $z_1, \dots, z_n, y_1, \dots, y_k$ be the coordinates on $\mathbb{C}^{n+k}$, let m_Y be the ideal of $\mathcal{O}_X$ generated by $\{z_1, \dots, z_n\}$, let $JM(X)$ be the jacobian module of X , let $JM(X)_Y$ be the module generated by $\{\frac{\partial F}{\partial y_1}, \dots, \frac{\partial F}{\partial y_k}\}$ and let $JM_z(X)$ be the module generated by $\{\frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_n}\}$. In this setup we assume that F defines X with reduced structure and X is equidimensional. Usually Y is called the **parameter space** of the family X .

Given a family as above, we say the family is **holomorphically trivial** if there exists a holomorphic family of origin preserving bi-holomorphic germs r_y such that $r_y(X_0) = X_y$. If the map-germs are only homeomorphisms, we say the family is C^0 trivial.

Example 1.3.2. ([42] Example 1.1) Let $X \subseteq \mathbb{C}^3$ be the family of four moving lines with equation $F(x, y, z) = xz(z+x)(z-(1+y)x) = 0$. Here y is the parameter, the x and z axis are fixed as is the line $z+x=0$, while the line $z=(1+y)x$ moves with y .

For each $y \in \mathbb{C}$ let $f_y : (\mathbb{C}^2, 0) \rightarrow \mathbb{C}$ given by $f_y(z) = F(x, y, z)$.

It is easy to see that if $\theta : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a $\mathbb{C}$ -linear map which preserves the lines $x=0$, $z=0$ and $z=-x$ on $\mathbb{C}^2$ then θ is a multiple of the identity on $\mathbb{C}^2$.

Suppose by contradiction that X is a holomorphically trivial family. So we have a family $r_y : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$, $y \in \mathbb{C}$, of bi-holomorphic germs such that $r_y(X_0) = X_y$, for all $y \in \mathbb{C}$. Take $y \in \mathbb{C}$, $y \neq 0$. Since $r_y(X_0) = X_y$ then the differential $Dr_y(0) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ takes the tangent lines of X_0 into the tangent lines of X_y , i.e:

$$\{z = -x\} \subseteq X_0 \text{ maps to } \{z = -x\} \subseteq X_y \text{ through } Dr_y(0)$$

Figure 1 – Four moving lines

$$\begin{aligned} \{z = 0\} \subseteq X_0 &\text{ maps to } \{z = 0\} \subseteq X_y \text{ through } Dr_y(0) \\ \{x = 0\} \subseteq X_0 &\text{ maps to } \{x = 0\} \subseteq X_y \text{ through } Dr_y(0) \\ \{z = x\} \subseteq X_0 &\text{ maps to } \{z = x\} \subseteq X_y \text{ through } Dr_y(0). \end{aligned}$$

In particular $Dr_y(0)$ is a $\mathbb{C}$ -linear map which preserves the lines $x = 0$, $z = 0$ and $z = -x$ on $\mathbb{C}^2$. Thus, $Dr_y(0)$ is a multiple of the identity, i.e, there exists $a \in \mathbb{C}$ such that $Dr_y(0) = a \cdot Id_{\mathbb{C}^2}$. Taking $x \neq 0$ sufficiently nearby 0, once we apply $Dr_y(0)$ at the point (x, x) , we conclude that $a = 1$ and $a = 1 + y$, which implies that $y = 0$, contradiction.

Therefore, the family X is not holomorphically trivial. However it should be equisingular for any reasonable definition of equisingularity.

This example shows that we need a notion of equisingularity that is less restrictive than holomorphic equivalence.

The Whitney conditions imply C^0 -triviality but also imply the family is well-behaved at the infinitesimal level.

Here we denote $\mathbb{K}$ as the field $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.3.3. (Whitney's conditions) Suppose X is a subset of $\mathbb{K}^n$, X_0 is the set of smooth points on X and that Y is a smooth subset of X .

- The pair (X_0, Y) satisfies **Whitney's condition A** at $y \in Y$ if for all sequences (x_i) of points of X_0 such that

$$x_i \rightarrow y$$

$$T_{x_i}X \rightarrow T$$

implies $T_y Y \subseteq T$;

- The pair (X_0, Y) satisfies **Whitney's condition B** at $y \in Y$ if for all sequences (x_i) of points of X_0 and (y_i) of points of Y such that

$$x_i \rightarrow y$$

$$y_i \rightarrow y$$

$$T_{x_i}X \rightarrow T$$

$$\sec(x_i, y_i) \rightarrow L$$

implies $L \subseteq T$.

John Mather first pointed out that Whitney's condition B implies Whitney's condition A in the notes of his lectures at Harvard in 1970 [69].

Example 1.3.4. Let us go back to the Example 1.3.2, and to prove that the family of four moving lines satisfies the Whitney's conditions. As pointed out by Mather, we only have to check the condition B.

Let Y be a smooth subspace of X , $u \in Y$, $(x_i)_{i \in \mathbb{N}}$ a sequence of points of X_0 and $(y_i)_{i \in \mathbb{N}}$ a sequence of points of Y such that

$$x_i \rightarrow u$$

$$y_i \rightarrow u$$

$$T_{x_i}X \rightarrow T$$

$$\sec(x_i, y_i) \rightarrow L.$$

Let M, N and P the planes $x = 0, z = 0$ and $z = -x$, respectively, and let R the surface $z = (1 + y)x$. So $M \cap N \cap P \cap R$ is the y -axis and $X = M \cup N \cup P \cup R$. Consider

$$\tilde{M} := \{i \in \mathbb{N} \mid x_i \in M\}$$

$$\tilde{N} := \{i \in \mathbb{N} \mid x_i \in N\}$$

$$\tilde{P} := \{i \in \mathbb{N} \mid x_i \in P\}$$

$$\tilde{R} := \{i \in \mathbb{N} \mid x_i \in R\}.$$

Clearly $\mathbb{N} = \tilde{M} \cup \tilde{N} \cup \tilde{P} \cup \tilde{R}$ and one of these sets has to be not bounded.

(i) Suppose that $\tilde{M}$ is not bounded. So we have a subsequence $(T_{x_i}X)_{i \in \tilde{M}} \rightarrow T$, hence $T = M$. We also have that $u = \lim_{i \rightarrow \infty} x_i = \lim_{i \in \tilde{M}} x_i \in \overline{M} = M$ which implies that $T_u M = M$. Since Y is a smooth subset of X and $u \in M$ there exists an open subset U of Y such that $u \in U$ and U is a submanifold of M (this occurs because M, N, P and R intersects transversely pairwise). Since $(x_i), (y_i) \rightarrow u$ then $x_i, y_i \in U \subseteq M, \forall i \in \mathbb{N}$ sufficiently large. Therefore, the set $\hat{M} := \{i \in \mathbb{N} \mid x_i, y_i \in M\}$ is not bounded. Since M is a plane then $\sec(x_i, y_i) \subseteq M, \forall i \in \hat{M}$ and since this subsequence also converges to L then $L \subseteq M = T$.

The same argument can be applied when $\tilde{N}$ or $\tilde{P}$ once these subsets of X are planes as well.

(ii) Suppose that $\tilde{R}$ is not bounded. Analogously to the previous case, we have that $u \in R$, there exists an open subset U of Y such that $u \in U$ such that U is a submanifold of R and the set $\hat{R} := \{i \in \mathbb{N} \mid x_i, y_i \in R\}$. Since R and U is a submanifold of R then the pair (R, U) satisfies the Whitney's conditions. Since

$$\begin{aligned} (x_i)_{i \in \hat{R}} &\rightarrow u \\ (y_i)_{i \in \hat{R}} &\rightarrow u \\ (T_{x_i} R)_{i \in \hat{R}} &\rightarrow T \\ (\sec(x_i, y_i))_{i \in \hat{R}} &\rightarrow L. \end{aligned}$$

then $L \subseteq T$.

Therefore, the family of four moving lines satisfies the Whitney's conditions.

Example 1.3.5. ([42] Example 1.4) Consider $X \subseteq \mathbb{R}^3$ the family defined by $F(x, y, z) = z^3 + x^2 - y^2 z^2 = 0$. The members of the family X_y consist of node singularities where the loop is pulled smaller and smaller as y tends to zero becoming a cusp at $y = 0$ (see the picture below). The singular locus is the y -axis. Whitney A holds because every limiting tangent plane contains the y -axis. But Whitney B fails. Notice that the parabola $z = y^2$ is in the surface, and letting $x_i = (0, t_i, t_i^2)$ and $y_i = (0, t_i, 0)$, (t_i) any sequence tending to 0, we see that the limiting secant line is the z -axis, while the limiting tangent plane along this curve is the xy -plane.

We see that the dimension of the limiting tangent planes at the origin is 1, while it is zero elsewhere on the y -axis. This kind of drastic change at the infinitesimal level is prevented by the Whitney conditions.

Reading: There are many places that we can read about the Whitney conditions. An important reference is the Chapter III of [98], which is more in the spirit of the way that Gaffney develops the subject. Surely one can see the works of Whitney [109, 110] as well.

Verdier's Condition W (or (w)-regularity)

The next condition, while is equivalent to the Whitney conditions in the complex analytic case (proved by Teissier in [98]), is easier to work using algebra.

The (w)-regularity says that the distance between the tangent space to X at a point x_i of X_0 and the tangent space to Y at y goes to zero as fast as the distance between x_i and Y . We first need to define what we mean by the distance between two linear spaces.

Suppose A, B are linear subspaces of $\mathbb{K}^N$. We define the distance from A to B as

$$\text{dist}(A, B) = \sup \left\{ \frac{\|\langle u, v \rangle\|}{\|u\| \|v\|} \mid u \in B^\perp - \{0\} \text{ and } v \in A - \{0\} \right\}.$$

Figure 2 – $X: y^2 z^2 = z^3 + x^2$

In the applications, B is the "big space" and A is the "small" space. The inner product is Hermitian inner product when we work over $\mathbb{C}$.

Example 1.3.6. ([42] Example 1.5) Work in $\mathbb{R}^3$. Let $A = x$ -axis, B a plane with unit normal u_0 , then the distance from A to B is $\cos \theta$, where θ is the small angle between u_0 and the x -axis, in the plane they determine. So when the distance is 0, B contains the x -axis.

We recall Verdier's condition W , also called (w) -regularity.

Definition 1.3.7. Suppose $Y \subseteq \overline{X}$, where X, Y are strata in a stratification of an analytic space. Then the pair (X, Y) satisfies the **Verdier's condition W** at $0 \in Y$ if there exists a positive real number C such that $\text{dist}(TY_0, TX_x) \leq C \text{dist}(x, Y)$ for all x close to Y .

The next theorem was proved first by Teissier in the case of a family of analytic hypersurfaces, and generalized by Gaffney, for any codimension.

Theorem 1.3.8. ([41] Theorem 2.5) With the setup 1.3.1, the Verdier's condition W holds for $(X - Y, Y)$ at $(0, 0)$ if and only if $(X - Y, Y)$ satisfies the Whitney's conditions at $(0, 0)$ if and only if

$$JM(X)_Y \subseteq \overline{m_Y JM_z(X)}.$$

1.4 The Lipschitz saturation and the double of an ideal

Following the approach of Pham-Teissier in [85], let A be a commutative local ring over $\mathbb{C}$, and $\overline{A}$ its normalization (we can assume that A is the local ring of an analytic space X at the

origin in $\mathbb{C}^n$). Let I be the kernel of the inclusion

$$\overline{A} \otimes_{\mathbb{C}} \overline{A} \rightarrow \overline{A} \otimes_A \overline{A}$$

In this construction, the tensor product is the analytic tensor product which has the right universal property for the category of analytic algebras, and which gives the analytic algebra for the analytic fiber product.

Pham and Teissier then defined the Lipschitz saturation of A , denoted $\tilde{A}$, to consist of all elements $h \in \overline{A}$ such that $h \otimes 1 - 1 \otimes h \in \overline{A} \otimes_{\mathbb{C}} \overline{A}$ is in the integral closure of I (for related results see [66]). The connection between this notion and that of Lipschitz functions is as follows. If we pick generators $(z_1, \dots, z_n)$ of the maximal ideal of the local ring A , then $z_i \otimes 1 - 1 \otimes z_i \in \overline{A} \otimes_{\mathbb{C}} \overline{A}$ give a set of generators of I . Choosing z_i so that they are the restriction of coordinates on the ambient space, using the supremum criterion given by Lejeune-Jalabert and Teissier in [64], we see the integral closure condition is equivalent to

$$|h(z_1, \dots, z_n) - h(z'_1, \dots, z'_n)| \leq C \sup_i \{|z_i - z'_i|\}$$

holding on some neighborhood U of $(0,0)$ on $X \times X$. This last inequality is what is meant by the meromorphic function h being Lipschitz at the origin on X . Note that the integral closure condition is equivalent to the inequality holding on a neighborhood U for some C for any set of generators of the maximal ideal of the local ring. The constant C and the neighborhood U will depend on the choice.

If (X, x) is normal then passing to the Lipschitz saturation does not add any functions.

Definition 1.4.1. Let I be an ideal of $\mathcal{O}_{X,x}$, $SB_I(X)$ the Lipschitz saturation of the blow-up, which is the space whose analytic sheaf of rings $\mathcal{O}_{SB_I(X)}$ is the Lipschitz saturation of the analytic sheaf of rings $\mathcal{O}_{B_I(X)}$, i.e.,

$$\mathcal{O}_{SB_I(X)} = \tilde{\mathcal{O}}_{B_I(X)}.$$

Let $\pi_S : SB_I(X) \rightarrow X$ be the projection map. The **Lipschitz saturation** of the ideal I is denoted I_S , and is the ideal $I_S := \{h \in \mathcal{O}_{X,x} \mid \pi_S^*(h) \in \pi_S^*(I)\}$.

Since the normalization of a local ring A contains the Lipschitz Saturation of A then $I \subseteq I_S \subseteq \bar{I}$. In particular, if I is integrally closed then $I_S = \bar{I}$.

Here is a viewpoint on the Lipschitz saturation of an ideal I , which will be useful later. Given an ideal I , and an element h that we want to check for inclusion in I_S , we can consider $(B_I(X), \pi)$, $\pi^*(I)$ and $h \circ \pi$. Since $\pi^*(I)$ is locally principal, working at a point z on the exceptional divisor E , we have a local generator $f \circ \pi$ of $\pi^*(I)$. Consider the quotient $(h/f) \circ \pi$. Then $h \in I_S$ if and only if at the generic point of any component of E , $(h/f) \circ \pi$ is Lipschitz with respect to a system of local coordinates. If this holds we say $h \circ \pi \in (\pi^*(I))_S$.

We can also work on the normalized blow-up $(NB_I(X), \pi_N)$. Then we say $h \circ \pi_N \in (\pi_N^*(I))_S$ if $(h/f) \circ \pi_N$ satisfies a Lipschitz condition at the generic point of each component of the exceptional divisor of $(NB_I(X), \pi_N)$ with respect to the pullback to $(NB_I(X), \pi_N)$ of a system of local coordinates $B_I(X)$ at the corresponding points of $B_I(X)$. As usual, the inequalities at the level of $NB_I(X)$ can be pushed down and are equivalent to inequalities on a suitable collection of open sets on X .

This definition can be given in an equivalent statement using the theory of integral closure of modules. Since Lipschitz conditions depend on controlling functions at two different points as the points come together, we should look for a sheaf defined on $X \times X$. We describe a way of moving from a sheaf of ideals on X to a sheaf on $X \times X$.

Let $\pi_1, \pi_2 : X \times X \rightarrow X$ be the projections to the i -th factor, $i \in \{1, 2\}$, and let $h \in \mathcal{O}_{X,x}$. Define $h_D \in \mathcal{O}_{X \times X, (x,x)}^2$ as $(h \circ \pi_1, h \circ \pi_2)$, called **the double of h** . We define the double of the ideal I , denoted I_D , as the submodule of $\mathcal{O}_{X \times X, (x,x)}^2$ generated by h_D , where h is an element of I .

If I is an ideal sheaf on a space X then intuitively, $h \in \bar{I}$ if h tends to zero as fast as the elements of I do as you approach a zero of I . If h_D is in $\bar{I}_D$ then the element defined by $(1, -1) \cdot (h \circ \pi_1, h \circ \pi_2) = h \circ \pi_1 - h \circ \pi_2$ should be in the integral closure of the ideal generated by applying $(1, -1)$ to the generators of I_D , namely the ideal generated by $g \circ \pi_1 - g \circ \pi_2$, g any element of I . This implies the difference of h at two points goes to zero as fast as the difference of elements of I at the two points go to zero as the points approach each other. It is reasonable that elements in I_S should have this property. In fact, as we can see in [40], the following result gives a nice link between Lipschitz saturation and integral closure of modules.

Theorem 1.4.2. ([40] Theorem 2.3) *Suppose (X, x) is a complex analytic set germ, $I \subseteq \mathcal{O}_{X,x}$ is an ideal and $h \in \mathcal{O}_{X,x}$. Then $h \in I_S$ if and only if $h_D \in \bar{I}_D$.*

Corollary 1.4.3. ([39] Corollaries 3.4 and 3.5) *Let $I \subseteq J \subseteq \bar{I}$ be ideals of $\mathcal{O}_{X,x}$, with X equidimensional. Suppose that I_D has finite colength in J_D and J_D has finite colength in $(\bar{I})_D$. Then*

$$e(I_D, (\bar{I})_D) = e(J_D, (\bar{I})_D) \text{ if and only if } \bar{I}_D = \bar{J}_D$$

if and only if

$$I_S = J_S.$$

Here is an example showing the difference between the integral closure of the jacobian ideal and its saturation, in particular, the difference between the integral closure and saturation.

Example 1.4.4. ([39]) Consider $F(x, y) = x^2 + y^p$, $p \geq 3$ odd. Denote by X the plane curve defined by F . Then X has a normalization given by $\phi(t) = (t^p, t^2)$. The elements in the integral closure of the jacobian ideal are just those ring elements h such that $h \circ \phi \in \phi^*(J(F)) = (t^p)$. Now, $y^q \circ \phi = t^{2q}$, so $y^q \in \overline{J(F)}$, for $q > \frac{p}{2}$. Denote a matrix of generators for $J(F)_D$ by $[J(F)_D]$. Consider the curve mapping into $X \times X$ given by $\Phi(t) = (t^p, t^2, t^p, ct^2)$, where c is a p^{th} root of unity different from 1. Now consider the ideal generated by the entries of the vector

$$(1, -1)[J(F)_D] \circ \Phi(t).$$

This ideal is generated by $(y^{p-1} - y'^{p-1}, (y - y')(x, x', y^{p-1}, y'^{p-1})) \circ \Phi(t) = (t^{p+2})$. Meanwhile the order in t of $(1, -1)(y^q, y'^q) \circ \Phi(t)$ is $2q$. If $p < 2q < p + 2$, i.e., $q = \frac{p+1}{2}$, then (y^q, y'^q) cannot be in $\overline{J(F)_D}$, hence $y^q \notin J(F)_S$, but $y^q \in \overline{J(F)}$.

Using the Lipschitz saturation of ideals (and doubles), in [39] we have set the infinitesimal Lipschitz conditions for hypersurfaces.

Setup: Let $X^{n+k}, 0 \subseteq \mathbb{C}^{n+1+k}, 0$ be a hypersurface, containing a smooth subset Y embedded in $\mathbb{C}^{n+1+k}$ as $0 \times \mathbb{C}^k$, with p_Y the projection to Y . Assume $Y = S(X)$, the singular set of X . Suppose F is the defining equation of X , (z, y) coordinates on $\mathbb{C}^{n+1+k}$. Denote by $f_y(z) = F(z, y)$ the family of functions defined by F and $X_y := f_y^{-1}(0)$. Assume that f_y has an isolated singularity at the origin. Let m_Y denote the ideal defining Y , and $J(F)_Y$, the ideal generated by the partial derivatives with respect to the y coordinates, $J_z(F)$, those with respect to the z coordinates. Here we work with the double relative to Y , which means that we work with the projections π_1 and π_2 defined on the fibered product $X \times_Y X$.

Definition 1.4.5. We say the pair (X, Y) satisfies the *infinitesimal Lipschitz condition* $\mathbf{m}_Y$ ($\mathbf{iL}_{\mathbf{m}_Y}$) at the origin if either of the two equivalent conditions hold:

1. $J(F)_Y \subseteq (m_Y J_z(F))_S$;
2. $(J(F)_Y)_D \subseteq \overline{(m_Y J_z(F))_D}$.

An analogous condition for iL_{m_Y} is $J(F)_Y \subseteq \overline{m_Y J_z(F)}$. This is the equivalent to the Verdier's condition W or the Whitney conditions. Next we give the definition of iL_A .

Definition 1.4.6. We say the pair (X, Y) satisfies the *infinitesimal Lipschitz condition A* ($\mathbf{iL}_A$) at the origin if either of the two equivalent conditions hold:

1. $J(F)_Y \subseteq (J_z(F))_S$;
2. $(J(F)_Y)_D \subseteq \overline{(J_z(F))_D}$.

The analogous condition is $J(F)_Y \subseteq \overline{J_z(F)}$. If one works on the ambient space, then this is equivalent to the A_F condition.

Since there are different ways in which the total space X^{n+k} can be made into a family of spaces, it is natural to ask if the conditions we have defined depend on the projection to Y which defines the family. The following result says the iL_{m_Y} does not depend on the projection to Y .

Proposition 1.4.7. ([39] Proposition 3.8) *In the above setup the following conditions are equivalent:*

$$(a) (J(F)_Y)_D \subseteq \overline{(m_Y J_z(F))_D};$$

$$(b) (J(F)_Y)_D \subseteq \overline{(m_Y J(F))_D}.$$

While a similar result for iL_A does not make sense, if we ask that $(J(F)_Y)_D \subseteq (J_z(F))_D^\dagger$ then an analogous result holds.

In Proposition 4.1 of [39] is proved that the cosupport of $(m_Y J_z(F))_D$ and $(J_z(F))_D$ on $X \times_Y X$ are equal, and consist of

$$\Delta(X) \cup (X \times_Y 0) \cup (0 \times_Y X).$$

This set is where makes sense to ask about the infinitesimal Lipschitz conditions. The next result says us that both iL conditions hold on the above set off $(0,0) \times Y$.

Proposition 1.4.8. ([39] Proposition 4.2) *iL_A and iL_{m_Y} hold at all points of $\Delta(X) - ((0,0) \times Y)$, and both conditions hold at all point of $(0 \times_Y X) \cup (X \times_Y 0) - ((0,0) \times Y)$ if W condition holds at all point $(0,y)$, $y \in Y$.*

In the chapter 3 we generalize this results for an arbitrary codimension of X .

Finally, in $(0,0) \times Y$, Gaffney proved that the iL_A condition holds generically.

Theorem 1.4.9. ([39] 4.3) *In the setup of this section, there exists a Zariski open subset U of Y such that iL_A holds for the pair $(X - Y, U \cap Y)$ along Y .*

1.5 Determinantal Varieties

We first recall the definition of determinantal varieties. Let $\Sigma_t \subseteq \text{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ be the subset consisting of the maps that have rank less than t , with $1 \leq t \leq \min(n, p)$. It is possible to show that Σ_t is an irreducible singular algebraic variety of codimension $(n-t+1)(p-t+1)$ (see [18]). Moreover the singular set of Σ_t is exactly Σ_{t-1} . The set Σ_t is called a *generic determinantal variety* of size (n, p) obtained from $t \times t$ minors.

The representation of the variety Σ_t as the union $\Sigma_i \setminus \Sigma_{i-1}$, $i = 1, \dots, t$ is a stratification of Σ_t , which is locally holomorphically trivial, and it is called the *rank stratification* of Σ_t .

Definition 1.5.1. Let $U \subseteq \mathbb{C}^r$ be an open domain, $F = (m_{ij}(x))$ be an $n \times p$ matrix whose entries are complex analytic functions on U , $0 \in U$ and f the function defined by the $t \times t$ minors of M . We say that $X = V(f)$ is a determinantal variety if it has codimension $(n - t + 1)(p - t + 1)$.

Currently, determinantal varieties have been an important object of study in Singularity Theory. For example, we can refer to the works of Damon [29], Frühbis-Krüger [37, 38], Gaffney [39, 47], Grulha [47], Nuño-Ballesteros [2, 76], Oréface-Okamoto [2, 76], Pereira [83, 89], Pike [29], Ruas [47, 89], Tomazella [2, 76], Zhang [114] and others.

In the case where X is a codimension two determinantal variety, we can use the Hilbert-Burch theorem to obtain a good description of X and its deformations in terms of its presentation matrix. In fact, if X is a codimension two Cohen-Macaulay variety, then X can be defined by the maximal minors of a $n \times (n + 1)$ matrix. Moreover, any perturbation of a $n \times (n + 1)$ matrix gives rise to a deformation of X and any deformation of X can be obtained through a perturbation of the presentation matrix (see [93]). We can use this correspondence to study properties of codimension two Cohen-Macaulay varieties through their presentation matrix. This is the approach of Frühbis-Krüger, Pereira and Ruas.

In order to introduce the notion of EIDS and relate it with the classical approach of singularity theory, let us recall some concepts in this field.

Let $\mathcal{R}$ be the group of coordinate changes (on the source) in $(\mathbb{C}^r, 0)$. We denote $GL_i(\mathcal{O}_r)$ the group of invertible matrices of size $i \times i$ with entries in the local ring $\mathcal{O}_r$. Consider the group $\mathcal{H} := GL_p(\mathcal{O}_r) \times GL_n(\mathcal{O}_r)$.

Given two matrices, we are interested in studying these germs according to the following equivalence relation.

Definition 1.5.2. Let $\mathcal{G} := \mathcal{R} \ltimes \mathcal{H}$ be the semi-direct product of $\mathcal{R}$ and $\mathcal{H}$. We say that two germs $F_1, F_2 \in Mat_{(n,p)}(\mathcal{O}_r)$ are $\mathcal{G}$ -equivalent if there exist $(\phi, R, L) \in \mathcal{G}$ such that $F_1 = L^{-1}(\phi^* F_2)R$.

It is not difficult to see that $\mathcal{G}$ is one of Damon's geometric subgroups of $\mathcal{K}$ (see [83]), hence as a consequence of Damon's result ([27]) we can use the techniques of singularity theory, for instance, those concerning finite determinacy. The notions of $\mathcal{G}$ -equivalence and $\mathcal{K}_\Delta$ -equivalence, where Δ consists of the subvariety of matrices of rank less than the maximal rank [27], coincide for finitely determined germs (see [17]).

The next result is a Geometric Criterion of Finite Determinacy for families of $n \times p$ matrices and was proved by Pereira in her Ph. D. thesis.

Theorem 1.5.3. (Geometric Criterion of Finite Determinacy, [83] Theorem 2.4.1) A representative of a germ $F : \mathbb{C}^r, 0 \rightarrow Mat_{(n,p)}(\mathbb{C})$ is $\mathcal{G}$ -finitely determined if and only if F is transverse to the strata of the rank stratification of $Mat_{(n,p)}(\mathbb{C})$ outside the origin.

It follows that if F is a $n \times p$ matrix with entries in the maximal ideal of $\mathcal{O}_r$, defining an isolated singularity, then F is $\mathcal{G}$ -finitely determined. Moreover if F is $\mathcal{G}$ -finitely determined, then the germ of X at a singular point is holomorphic to either the product of Σ_t with an affine space or a transverse slice of Σ_t . This motivates the following definition ([30]):

Definition 1.5.4. *A point $x \in X = F^{-1}(\Sigma_t)$ is called essentially non-singular if, at the point x , the map F is transversal to the corresponding stratum of the variety Σ_t . A germ $(X, 0) \subseteq (\mathbb{C}^r, 0)$ of a determinantal variety has an essentially isolated singular point at the origin (or is an essentially isolated determinantal singularity: EIDS) if it has only essentially non-singular points in a punctured neighborhood of the origin in X .*

If $X = F^{-1}(\Sigma_t)$ then a perturbation of X is obtained by perturbing the entries of F . This yields an unfolding of F , and if X is an EIDS then happens to also give a deformation of X which is transverse to the strata of $\text{Hom}(\mathbb{C}^n, \mathbb{C}^p)$.

In the particular case where X is Cohen-Macaulay of codimension 2, it is a consequence of the Auslander-Buchsbaum formula and the Hilbert-Burch Theorem that any deformation of X can be given as a perturbation of the presentation matrix (see [38]). Therefore we can study these varieties and their deformations using their representation matrices. We can express the normal module $N(X)$, and the space of the first order deformations T_X^1 , in terms of matrices, hence we can treat the base of the semi-universal deformation using matrix representation and we can express the normal module $N(X)$ in terms of matrices.

1.6 Bi-Lipschitz geometry and Lipschitz stratifications

In this section we follow the lines of [70] for an understanding of the bi-Lipschitz geometry.

Let M and N be metric spaces.

Definition 1.6.1. *A map $f : M \rightarrow N$ is called **Lipschitz** if there exists a constant $C > 0$ such that*

$$d(f(x), f(y)) \leq Cd(x, y),$$

for all $x, y \in M$.

*A homeomorphism $f : M \rightarrow N$ is called **bi-Lipschitz** if f is Lipschitz and the inverse $f^{-1} : N \rightarrow M$ is Lipschitz as well.*

*In this case we say that f is a **bi-Lipschitz equivalence** and that M and N are **bi-Lipschitz equivalent**.*

In order to understand the bi-Lipschitz equisingularity, we recall here some important notions developed by Mostowski and Parusiński.

We denote $\mathbb{K}$ the field $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.6.2. A **filtration** of a subset $X \subseteq \mathbb{K}^n$ is a sequence of subsets of X

$$X = X^{d_1} \supset X^{d_2} \supset \dots \supset X^{d_l} \neq \emptyset$$

with $d_1 > d_2 > \dots > d_l$, and $X_{\text{sing}}^{d_i} \subseteq X^{d_{i+1}}$, $\forall i \in \{1, \dots, l\}$. The superscripts d_i denote dimensions, the X^{d_i} are called **skeletons**, the manifolds $\mathring{X}^{d_i} := X^{d_i} - X^{d_{i+1}}$ **strata** and the collection $\{\mathring{X}^{d_i}\}$ is called a **stratification** of X .

The stratification $\{\mathring{X}^{d_i}\}$ is called **weakly Lipschitz** if for every i , for all point $p \in \mathring{X}^{d_i}$ and vector $v \in T_p \mathring{X}^{d_i}$, there is a Lipschitz vector field $\tilde{v}$ defined in a neighborhood of p in $\mathbb{K}^n$ such that $\tilde{v}(p) = v$ and the local flow v_t generated by v satisfies $v_t(X) \subseteq X$.

The weakly Lipschitz stratifications have the following property: if v is a Lipschitz vector field defined in a neighborhood of a point $p \in \mathring{X}^{d_i}$ in $\mathring{X}^{d_i}$, which is tangent to $\mathring{X}^{d_i}$, then v extends to a Lipschitz vector field defined in a neighborhood of p in $\mathbb{K}^n$, whose local flow preserves X .

Definition 1.6.3. Under the above notation, we say that X is **locally Lipschitz equisingular along a stratum** $\mathring{X}^{d_i}$ if for every $p \in \mathring{X}^{d_i}$ there is a neighborhood U_0 of p in $\mathring{X}^{d_i}$, a tubular neighborhood U of U_0 in $\mathbb{K}^n$ with a projection $\pi : U \rightarrow U_0$, and a bi-Lipschitz homeomorphism

$$h : U \rightarrow U_0 \times H_0,$$

where H is the complement of $T_p U_0$ in $T_p \mathbb{K}^n = \mathbb{K}^n$, i.e.,

$$T_p \mathbb{K}^n = T_p U_0 \oplus H,$$

H_0 is a neighborhood of p in H , such that

1. $pr \circ h = \pi$;
2. $h(U \cap X) = U_0 \times (H_0 \cap X)$,

where $pr : U_0 \times H_0 \rightarrow U_0$ is the projection.

It is clear that X is locally Lipschitz equisingular along every stratum of a weakly Lipschitz stratification. The existence of weakly Lipschitz stratifications is proved in [72].

By $\text{dist}(\star, X)$ we mean the function of distance to X . If $X = \emptyset$ we set $\text{dist}(\star, \emptyset) \equiv 1$. For any linear operator A defined on $\mathbb{K}^n$, we define the *norm* of A as

$$\|A\| = \sup\{\|A(v)\| \mid v \in \mathbb{K}^n, \|v\| = 1\}$$

For $q \in \mathring{X}^{d_j}$, let $P_q : \mathbb{K}^n \rightarrow T_q \mathring{X}^{d_j}$ be the orthogonal projection onto the tangent space of $\mathring{X}^{d_j}$ at q , and let $P_q^\perp = Id_{\mathbb{K}^n} - P_q : \mathbb{K}^n \rightarrow T_q^\perp \mathring{X}^{d_j}$ be the orthogonal projection onto the normal space.

For technical reasons we replace the distance functions $\text{dist}(q, X^{d_j})$ by semianalytic functions (i.e, continuous functions with semianalytic graphs) such that

$$\frac{1}{2n} \text{dist}(q, X^{d_j}) \leq \rho_j(q) \leq 2n \text{dist}(q, X^{d_j}).$$

For a proof of the existence of these functions ρ_j 's see [67].

If X^{d_j} 's are algebraic then the distance functions themselves can be used [28].

Definition 1.6.4. (*Chain*) Let c_0 be a fixed constant, $c_0 \geq 2n$. A **c_0 -chain** (or **chain**) for a point $q \in \mathring{X}^{d_j}$ is a strictly increasing sequence of indices $j = j_1, j_2, \dots, j_r = l$ and a sequence of points $q_{j_s} \in \mathring{X}^{d_{j_s}}$ such that $j_1 = j$, $q_{j_1} = q$ and each j_s ($s \geq 2$) is the smallest integer greater than j_{s-1} for which $\rho_{j_{s-1}}(q) \geq 2c_0^2 \rho_{j_s}(q)$, and $\|q - q_{j_s}\| \leq c_0 \rho_{j_s}(q)$.

Now we are able to recall the following definition (see [80]).

Definition 1.6.5. (*Mostowski's conditions for Lipschitz stratifications*) We call a stratification $\{\mathring{X}^{d_i}\}$ a **Lipschitz stratification** if for some constant $C > 0$, every c_0 -chain $q = q_{j_1}, q_{j_2}, \dots, q_{j_r}$ and every $k \in \{1, \dots, r\}$ we have:

1. $\left\| P_q^\perp \circ P_{q_{j_2}} \circ \dots \circ P_{q_{j_k}} \right\| \leq C \frac{\|q - q_{j_2}\|}{\text{dist}(q, X^{d_{j_{k-1}}})}.$
- If $q' \in \mathring{X}^{d_j}$ such that $\|q - q'\| \leq \frac{\text{dist}(q, X^{d_{j-1}})}{2c_0}$ then
2. $\left\| (P_q - P_{q'}) \circ P_{q_{j_2}} \circ \dots \circ P_{q_{j_k}} \right\| \leq C \frac{\|q - q'\|}{\text{dist}(q, X^{d_{j_{k-1}}})};$
3. $\|P_q - P_{q'}\| \leq C \frac{\|q - q'\|}{\text{dist}(q, X^{d_{j-1}})}.$

Example 1.6.6. Consider the cone X given by the equation $z^2 = x^2 + y^2$. If we take $X^2 := X$ and $X^0 := (0, 0, 0)$ then $\{X^0, X^2 - X^0\}$ is a stratification of the cone X which satisfies the three above Mostowski conditions.

Figure 3 – Cone

An alternate definition of Lipschitz stratification is given in [70].

Definition 1.6.7. *We call a stratification $\{\mathring{X}^{d_i}\}$ a **Lipschitz stratification** if for some constant $C > 0$, for every compact set F such that*

$$X^{d_{i+1}} \subseteq F \subseteq X^{d_i}$$

and every Lipschitz vector field v , defined on F and tangent to strata (i.e $v(x) \in T_x \mathring{X}^{d_j}$ for all $x \in \mathring{X}^{d_j}$, $j > i$, or $x \in \mathring{X}^{d_i} \cap F$), there exists a Lipschitz vector field $\tilde{v}$, defined on $\mathbb{K}^n$, tangent to strata and extending v , such that the best Lipschitz constant for $\tilde{v}$ is less than or equal to C .

We state the main result of this section, due to Mostowski.

Theorem 1.6.8. ([72] Proposition 1.2) *Every complex analytic set admits a complex analytic Lipschitz stratification. Moreover such Lipschitz stratification are locally bi-Lipschitz trivial along strata*

In [78], Parusiński obtained the above theorem for analytic real varieties. After, he generalized this result for subanalytic sets.

Theorem 1.6.9. ([82] Theorem 1.4) *Every subanalytic set admits a subanalytic Lipschitz stratification. Moreover such Lipschitz stratification are locally bi-Lipschitz trivial along strata*

There is a construction of a Lipschitz stratification of an analytic hypersurface in [72], but it was not made canonically. Hence, one important thing to investigate is when it is possible to construct a canonical vector field satisfying the previous Lipschitz conditions.

Example 1.6.10. ([101] Example 5.1) *Consider $\mathbb{R}^3$ with coordinates x, y and z . For each z consider $X_z = \{y = 0\} \cup \{(x, x^z, z)\}$. Then the bi-Lipschitz type of X_z and $X_{z'}$ are different if $z \neq z'$, both greater than 1. Let us to verify this for natural numbers $n \neq m$. For each n consider the map-germ $f_n : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ given by $f_n(x) = x^n$. Thus f_n is finitely C^0 - $\mathcal{K}$ -determined map germs. Since $\deg f_n \neq \deg f_m$ whenever $n \neq m$ then by Theorem 3.1 of [21] f_n and f_m are not topologically $\mathcal{K}$ -equivalent if $n \neq m$. In particular, f_n and f_m are not bi-Lipschitz $\mathcal{K}$ -equivalent for $n \neq m$, and therefore the fibers X_n and X_m have not the same bi-Lipschitz type. Hence, there is no bi-Lipschitz trivial stratification in this case.*

For more results toward the contact equivalence, see [50, 96].

Example 1.6.11. ([59] Example 1.5) *This is a semialgebraic example due to Satoshi Koike, showing that (w) -regularity does not imply local bi-Lipschitz triviality. Let $X = \{y^2 = z^2 x^2 + x^3, x \geq 0\}$ in $\mathbb{R}^3$. Let Y be the z -axis. Then the pair $(X - Y, Y)$ is (w) -regular, but the bi-Lipschitz type of the germ of X at point $(0, 0, 0) \in Y$ is different from the bi-Lipschitz type of the germ of X at points $(0, 0, z) \in Y$ if $z \neq 0$.*

Figure 4 – $\{y = 0\} \cup \{y = x^2\}, z \geq 0$

In fact, suppose that $X_0 = \{y^2 = x^3\}$ and $X_z = \{y^2 = z^2x^2 + x^3\}$ have the same bi-Lipschitz type, with $z \neq 0$. In particular X_0 and X_z are homeomorphic, hence $X_0 - \{0\}$ and $X_z - \{0\}$ as well, which is a contradiction because these sets have 2 and 3 connected components, respectively.

Figure 5 – Cusp and Node

THE DOUBLE OF A MODULE

In order to deal with bi-Lipschitz equisingularity in a family of hypersurfaces and to get algebraic infinitesimal Lipschitz conditions, Gaffney defined in [39, 40] the double of an ideal in a quite natural way. In this chapter we extend this notion for modules and we generalize several results of [39].

2.1 The double of a module and basic properties

Let $X \subseteq \mathbb{C}^n$ be an analytic space. Consider the projection maps

$$\begin{array}{ccc} & X \times X & \\ \swarrow \pi_1 & & \searrow \pi_2 \\ X & & X \end{array}$$

where π_i is the projection onto the i^{th} -factor, for every $i \in \{1, 2\}$.

Definition 2.1.1. Let $h \in \mathcal{O}_{X,x}^p$. The **double** of h is defined as the element

$$h_D := (h \circ \pi_1, h \circ \pi_2) \in \mathcal{O}_{X \times X, (x,x)}^{2p}.$$

The double of a submodule $M \subseteq \mathcal{O}_{X,x}^p$ is denoted by M_D , and is defined as the $\mathcal{O}_{X \times X, (x,x)}$ -submodule of $\mathcal{O}_{X \times X, (x,x)}^{2p}$ generated by $\{h_D \mid h \in M\}$.

Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$. We can assume that $\mathcal{M}$ is finitely generated by global sections. If $h \in \mathcal{O}_X^p(X)$ is a global section, given an arbitrary point $(x, x') \in X \times X$, we can consider $h_D \in \mathcal{O}_{X \times X, (x,x')}^{2p}$, the **double** of h , as the germ of $(h \circ \pi_1, h \circ \pi_2) \in \mathcal{O}_{X \times X}^{2p}(X \times X)$ at the point (x, x') . Thus, we define the **double** of the sheaf $\mathcal{M}$, denoted $\mathcal{M}_D$, as the $\mathcal{O}_{X \times X}$ -submodule of $\mathcal{O}_{X \times X}^{2p}$ generated by $\{h_D \mid h \in \mathcal{M}(X) \text{ is a global section}\}$. Unless we say the

opposite, we denote by M the stalk of $\mathcal{M}$ at an arbitrary point of X and M_D the stalk of $\mathcal{M}_D$ in an arbitrary point of $X \times X$. If we take the germ of h at a point, we use the same notation, if there is no confusion. For any sheaf of $\mathcal{O}_X$ -modules $\mathcal{F}$, the notation $h \in \mathcal{F}$ means that h is a global section of $\mathcal{F}$.

We want to recover some results which are true in the ideal case, i.e, when $p = 1$ and $M = I$ is an ideal. We start by obtain a set of generators for M_D from a set of generators of M .

Consider $z_1, \dots, z_n$ the coordinates on $\mathbb{C}^n$.

Lemma 2.1.2. *Under the above notation, we have the following properties:*

- a) $(\alpha h)_D = -(0_{\mathcal{O}_{X \times X}^p}, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h \circ \pi_2)) + (\alpha \circ \pi_1)h_D$, for all $\alpha \in \mathcal{O}_X$ and $h \in \mathcal{O}_X^p$;
- b) $(0_{\mathcal{O}_{X \times X}^p}, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h \circ \pi_2)) \in M_D$, for all $h \in M$ and $\alpha \in \mathcal{O}_X$;
- c) $\alpha \circ \pi_1 - \alpha \circ \pi_2 \in I(\Delta(X)) = (z_1 \circ \pi_1 - z_1 \circ \pi_2, \dots, z_n \circ \pi_1 - z_n \circ \pi_2)$, for all $\alpha \in \mathcal{O}_X$;
- d) $(g + h)_D = g_D + h_D$, for all $g, h \in \mathcal{O}_X^p$;
- e) $(0_{\mathcal{O}_{X \times X}^p}, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h \circ \pi_2)) \in \overline{\mathcal{M}_D}$ at (x, x') , for all $h \in \overline{\mathcal{M}_{x'}}$ and $\alpha \in \mathcal{O}_X$. The analogous property holds if we look at the first p coordinates instead the last.

Proof. (a) We have: $(\alpha h)_D = ((\alpha h) \circ \pi_1, (\alpha h) \circ \pi_2) = ((\alpha \circ \pi_1)(h \circ \pi_1), (\alpha \circ \pi_2)(h \circ \pi_2))$
 $= -(0_{\mathcal{O}_{X \times X}^p}, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h \circ \pi_2)) + ((\alpha \circ \pi_1)(h \circ \pi_1), (\alpha \circ \pi_1)(h \circ \pi_2))$
 $= -(0_{\mathcal{O}_{X \times X}^p}, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h \circ \pi_2)) + (\alpha \circ \pi_1)h_D.$

(b) Since $h \in M$ then $\alpha h \in M$, so $h_D \in M_D$ and $(\alpha h)_D \in M_D$. Thus, by (a) we have that $(0_{\mathcal{O}_{X \times X}^p}, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h \circ \pi_2)) = (\alpha \circ \pi_1)h_D - (\alpha h)_D \in M_D$.

(c) Obviously $\alpha \circ \pi_1 - \alpha \circ \pi_2$ vanishes on the diagonal of X , which finishes the proof of (c).

(d) Notice that: $(g + h)_D = ((g + h) \circ \pi_1, (g + h) \circ \pi_2) = (g \circ \pi_1 + h \circ \pi_1, g \circ \pi_2 + h \circ \pi_2) = (g \circ \pi_1, g \circ \pi_2) + (h \circ \pi_1, h \circ \pi_2) = g_D + h_D$.

(e) Let $\phi : (\mathbb{C}, 0) \rightarrow (X \times X, (x, x'))$ be an arbitrary analytic curve, $\phi = (\phi_1, \phi_2)$. Since $h \in \overline{\mathcal{M}_{x'}}$ then we can write

$$h \circ \phi_2 = \sum_l \alpha_l (v_l \circ \phi_2)$$

with $\alpha_l \in \mathcal{O}_{\mathbb{C}, 0}$ and $v_l \in \mathcal{M}_{x'}$, for all l . So we have

$$\begin{aligned} (0, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h \circ \pi_2)) \circ \phi &= (0, (\alpha \circ \phi_1 - \alpha \circ \phi_2)(h \circ \phi_2)) \\ &= \sum_l \alpha_l (0, (\alpha \circ \phi_1 - \alpha \circ \phi_2)(v_l \circ \phi_2)) = \sum_l \alpha_l (\underbrace{(0, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(v_l \circ \pi_2))}_{\in \mathcal{M}_D}) \circ \phi \in \phi^*(\mathcal{M}_D). \end{aligned}$$

Hence, $(0, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h \circ \pi_2)) \in \overline{\mathcal{M}_D}$. □

The next proposition gives a set of generators of M_D , from a known set of generators of M , which will be useful in the proof of many results in this section and in the section 2.2.

Proposition 2.1.3. *Suppose that M is generated by $\{h_1, \dots, h_r\}$. Then, the following sets are generators of M_D :*

- a) $\mathcal{B} = \{(h_1)_D, \dots, (h_r)_D\} \cup \{(0_{\mathcal{O}_{X \times X}^p}, (z_i \circ \pi_1 - z_i \circ \pi_2)(h_j \circ \pi_2)) \mid i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, r\}\}$;
- b) $\mathcal{B}' = \{(h_1)_D, \dots, (h_r)_D\} \cup \{((z_i \circ \pi_1 - z_i \circ \pi_2)(h_j \circ \pi_1), 0_{\mathcal{O}_{X \times X}^p}) \mid i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, r\}\}$;
- c) $\mathcal{B}'' = \{(h_1)_D, \dots, (h_r)_D\} \cup \{(z_i h_j)_D \mid i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, r\}\}$.

Proof. (a) Let N be the submodule of $\mathcal{O}_{X \times X}^{2p}$ generated by $\mathcal{B}$. By Lemma 2.1.2 (b) we have that $N \subseteq M_D$. Now, in order to verify that $M_D \subseteq N$, it is enough to check that $h_D \in N, \forall h \in M$. Indeed, if $h \in M$ we can write $h = \sum_{j=1}^r \alpha_j h_j$, for some $\alpha_j \in \mathcal{O}_X$. By Lemma 2.1.2 (a) and (d) we have that

$$\begin{aligned} h_D &= \left(\sum_{j=1}^r \alpha_j h_j \right)_D = \sum_{j=1}^r (\alpha_j h_j)_D \\ &= \left(\sum_{j=1}^r (\alpha_j \circ \pi_1)(h_j)_D \right) - \left(\sum_{j=1}^r (0_{\mathcal{O}_{X \times X}^p}, (\alpha_j \circ \pi_1 - \alpha_j \circ \pi_2)(h_j \circ \pi_2)) \right). \end{aligned}$$

Clearly the first sum is in N . By Lemma 2.1.2 (c) we have that each $\alpha_j \circ \pi_1 - \alpha_j \circ \pi_2$ belongs to the ideal $I(\Delta(X))$, so the second sum is in N , which finishes the proof.

(b) This is completely analogous to the item (a).

(c) We use (a). Let N be the submodule of $\mathcal{O}_{X \times X}^{2p}$ generated by $\mathcal{B}''$. For all $j \in \{1, \dots, r\}$ and $i \in \{1, \dots, n\}$ we have

$$(z_i h_j)_D = (z_i \circ \pi_1)(h_j)_D - (0_{\mathcal{O}_{X \times X}^p}, (z_i \circ \pi_1 - z_i \circ \pi_2)(h_j \circ \pi_2)) \in M_D,$$

by the previous lemma. Hence, $N \subseteq M_D$. Now, in order to check that $M_D \subseteq N$, it suffices to verify that all the generators of M_D given in (a) are in N . We already have $(h_j)_D \in N$, for all $j \in \{1, \dots, r\}$. Also, for all j and i we have

$$(0_{\mathcal{O}_{X \times X}^p}, (z_i \circ \pi_1 - z_i \circ \pi_2)(h_j \circ \pi_2)) = (z_i \circ \pi_1)(h_j)_D - (z_i h_j)_D \in N$$

which finishes the proof. $\square$

We can develop the same notion for the double in the family case. Suppose that $X \subseteq \mathbb{C}^{n+k}$ is an analytic space and let $Y = 0 \times \mathbb{C}^k \subseteq X$. Identifying $Y = 0 \times \mathbb{C}^k = \mathbb{C}^k$ we have that $X \subseteq \mathbb{C}^n \times Y$. Let $\mathbf{p} : X \subseteq \mathbb{C}^n \times Y \rightarrow Y$ be the projection, $X \times_X X$ the fibered product relative to the projection onto Y , with the projections maps $\pi_1, \pi_2 : X \times_X X \rightarrow X$.

$$\begin{array}{ccc}
X \times_Y X & & X \times_Y X \\
\pi_1 \searrow & & \swarrow \pi_2 \\
& X & \\
& \downarrow \mathbf{p} & \\
& Y &
\end{array}$$

Let $h \in \mathcal{O}_X^p$. The **double of h relative to Y** is defined by

$$h_{D,Y} := h_D := (h \circ \pi_1, h \circ \pi_2) \in \mathcal{O}_{X \times_Y X}^{2p}.$$

In the same way, the double of a sheaf of $\mathcal{O}_X$ -submodules $\mathcal{M}$ of $\mathcal{O}_X^p$ relative to Y is defined as the $\mathcal{O}_{X \times_Y X}$ -submodule of $\mathcal{O}_{X \times_Y X}^{2p}$ generated by $\{h_{D,Y} \mid h \in \mathcal{M}(X) \text{ is a global section}\}$, and is denoted by $\mathcal{M}_D$ (or $\mathcal{M}_{D,Y}$).

Let $z_1, \dots, z_n, y_1, \dots, y_k$ be the coordinates on $\mathbb{C}^{n+k}$. It is easy to see that the Lemma 2.1.2 still holds when we are working with the projections restricted to the fibered product $X \times_Y X$ and, since each $y_l \circ \pi_1 - y_l \circ \pi_2$ vanishes on the fibered product, then we get the following proposition, whose proof is completely analogous to the proof of the Proposition 2.1.3.

Proposition 2.1.4. *Suppose that M is generated by $\{h_1, \dots, h_r\}$. Then, the following sets are generators of M_D relative to Y :*

- a) $\mathcal{B} = \{(h_1)_D, \dots, (h_r)_D\} \cup \{(0_{\mathcal{O}_{X \times X}^p}, (z_i \circ \pi_1 - z_i \circ \pi_2)(h_j \circ \pi_2)) \mid i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, r\}\}$;
- b) $\mathcal{B}' = \{(h_1)_D, \dots, (h_r)_D\} \cup \{((z_i \circ \pi_1 - z_i \circ \pi_2)(h_j \circ \pi_1), 0_{\mathcal{O}_{X \times X}^p}) \mid i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, r\}\}$;
- c) $\mathcal{B}'' = \{(h_1)_D, \dots, (h_r)_D\} \cup \{(z_i h_j)_D \mid i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, r\}\}$.

Example 2.1.5. Consider the ideal $I = (x^2, y^2)$ of $\mathcal{O}_{\mathbb{C}^2, 0}$. Notice that I_D has generators further the doubles of x^2 and y^2 . In fact, by Proposition 2.1.3 the double I_D is generated by $\mathcal{B} := \{(x^2)_D, (y^2)_D, (0, (x - x')x'), (0, (x - x')y'), (0, (y - y')x'), (0, (y - y')y')\}$. It is easy to see that no element of $\mathcal{B}$ can be generated by the another elements of $\mathcal{B}$, i.e, $\mathcal{B}$ is a minimal set of generators of I_D . Hence, we need further elements than the doubles of x^2 and y^2 to generate the double of I .

Remember that for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$ -modules, the **support** of $\mathcal{F}$ is defined as the subset of X given by

$$\text{supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

If $\mathcal{G}$ is a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{F}$, the **cosupport** of $\mathcal{G}$ in $\mathcal{F}$ is defined as

$$\text{cosupp}(\mathcal{G}) := \text{supp} \left(\frac{\mathcal{F}}{\mathcal{G}} \right)$$

Let us to fix some notations. Let us assume that $\mathcal{M}$ is generated by global sections $\{h_1, \dots, h_r\}$ of $\mathcal{O}_X^p$, let $[\mathcal{M}]$ be the $p \times r$ matrix whose columns are the generators of $\mathcal{M}$, and for each $x \in X$ let $[\mathcal{M}(x)]$ be the $p \times r$ matrix whose columns are $h_1(x), \dots, h_r(x)$.

Let $\Sigma(\mathcal{M}) := \{x \in X \mid \text{rank}[\mathcal{M}(x)] < p\}$. This set is called **the singular set of $\mathcal{M}$** .

In the next theorem we compute the cosupport of the double $\mathcal{M}_D$ in $\mathcal{O}_{X \times X}^{2p}$.

Theorem 2.1.6. *Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$. Then*

$$\text{cosupp}(\mathcal{M}_D) = \Delta(X) \cup (X \times \Sigma(\mathcal{M})) \cup (\Sigma(\mathcal{M}) \times X).$$

Proof. Let $(x, x') \in X \times X$ and suppose that (x, x') is not in the $\text{cosupp}(\mathcal{M}_D)$.

(1) In a neighborhood of this point, by Proposition 2.1.3 we can write

$$(1, 0, \dots, 0, 0_{\mathcal{O}_{X \times X}^p}) = \left(\sum \alpha_i (h_i \circ \pi_1, h_i \circ \pi_2) \right) + \left(\sum \beta_{ij} ((z_i \circ \pi_1 - z_i \circ \pi_2)(h_j \circ \pi_1), 0_{\mathcal{O}_{X \times X}^p}) \right)$$

for some $\alpha_i, \beta_{ij} \in \mathcal{O}_{X \times X}$.

If $(x, x') \in \Delta(X)$, applying the above equation in this point, the second sum vanishes and we get

$$(1, 0, \dots, 0, 0_{\mathbb{C}^p}) = \sum \alpha_i(x, x) (h_i(x), h_i(x)).$$

Notice that the right hand side of the above equation is in the diagonal of $\mathbb{C}^p$ and the left hand side is not, which is a contradiction. So, $(x, x') \notin \Delta(X)$.

(2) Suppose by contradiction that $(x, x') \in \Sigma(\mathcal{M}) \times X$. Then $\text{rank}[\mathcal{M}(x)] < p$, so there exists $v \in \mathbb{C}^p$ such that v is not a linear combination of $h_1(x), \dots, h_r(x)$. Consider the element $(v, 0_{\mathcal{O}_{X \times X}^p}) \in \mathcal{O}_{X \times X}^{2p}$ (thinking v as a constant function). Since (x, x') is not in the cosupport of $\mathcal{M}_D$ then in a neighborhood of this point, by Proposition 2.1.3 we can write

$$(v, 0_{\mathcal{O}_{X \times X}^p}) = \left(\sum \alpha_i (h_i \circ \pi_1, h_i \circ \pi_2) \right) + \left(\sum \beta_{ij} (0_{\mathcal{O}_{X \times X}^p}, (z_i \circ \pi_1 - z_i \circ \pi_2)(h_j \circ \pi_2)) \right)$$

for some $\alpha_i, \beta_{ij} \in \mathcal{O}_{X \times X}$. Now, comparing the first p coordinates in above equation and applying in (x, x') we have that $v = \sum \alpha_i(x, x') h_i(x)$, so v is a linear combination of $h_1(x), \dots, h_r(x)$, which is a contradiction. Hence, $(x, x') \notin \Sigma(\mathcal{M}) \times X$.

(3) Analogously we can conclude that $(x, x') \notin X \times \Sigma(\mathcal{M})$ by using again the Proposition 2.1.3 with the natural changes.

The three above steps prove the inclusion

$$\text{cosupp}(\mathcal{M}_D) \supseteq \Delta(X) \cup (X \times \Sigma(\mathcal{M})) \cup (\Sigma(\mathcal{M}) \times X).$$

Now, let $(x, x') \in X \times X$ such that $x \neq x'$, $[\mathcal{M}(x)]$ and $[\mathcal{M}(x')]$ have rank $\geq p$. For each $z \in X$ denote by $[\mathcal{M}(z)]_p$ the $p \times p$ submatrix given by the first p columns of $[\mathcal{M}(z)]$. Since $[\mathcal{M}(x)]$ and $[\mathcal{M}(x')]$ have rank $\geq p$, without loss of generality we can assume that $[\mathcal{M}(x)]_p$ and $[\mathcal{M}(x')]_p$ are invertible. Thus, there exist U and U' open subsets of X containing x and x' , respectively, such that $[\mathcal{M}(z)]_p$ and $[\mathcal{M}(z')]_p$ are invertible, $\forall (z, z') \in U \times U'$. Since $x \neq x'$ then there exists $k \in \{1, \dots, n\}$ such that $(z_k \circ \pi_1 - z_k \circ \pi_2)(x, x') \neq 0$, so $z_k \circ \pi_1 - z_k \circ \pi_2$ is invertible in $\mathcal{O}_{X \times X, (x, x')}$.

Let us to prove that $(e_i, 0_{\mathcal{O}_{X \times X}^p}) \in \mathcal{M}_D$ at (x, x') , $\forall i \in \{1, \dots, p\}$. In fact, given a such i , consider the analytic map $\alpha : U \times U' \rightarrow \mathbb{C}^p$ given by $\alpha(z, z') := [\mathcal{M}(z)]_p^{-1} \cdot e_i$, with α_j its coordinates functions. So, it is easy to see that $\sum_{j=1}^p \alpha_j(z, z') h_j(z) = e_i$, $\forall (z, z') \in U \times U'$. Thus, in the open subset $U \times U'$ (which contains the point (x, x')) we have $e_i = \sum_{j=1}^p \alpha_j(h_j \circ \pi_1)$, which implies that

$$(e_i, 0_{\mathcal{O}_{X \times X}^p}) = \sum_{j=1}^p \alpha_j (z_k \circ \pi_1 - z_k \circ \pi_2)^{-1} \underbrace{\left((z_k \circ \pi_1 - z_k \circ \pi_2)(h_j \circ \pi_1), 0_{\mathcal{O}_{X \times X}^p} \right)}_{\in \mathcal{M}_D}$$

$$\Rightarrow (e_i, 0_{\mathcal{O}_{X \times X}^p}) \in \mathcal{M}_D.$$

Analogously, $(0_{\mathcal{O}_{X \times X}^p}, e_i) \in \mathcal{M}_D$ at (x, x') , $\forall i \in \{1, \dots, p\}$, hence $\mathcal{M}_D = \mathcal{O}_{X \times X, (x, x')}^{2p}$ and $(x, x') \notin \text{cosupp}(\mathcal{M}_D)$. $\square$

It is easy to see this theorem still holds in the family case, replacing the corresponding sets by $X \times_Y \Sigma(\mathcal{M})$ and $\Sigma(\mathcal{M}) \times_Y X$.

Remark 2.1.7. Notice that if the singular set of $\mathcal{M}$ is empty then $\text{cosupp}(\mathcal{M}_D) = \Delta(X)$ the diagonal of X and if the singular set of $\mathcal{M}$ is the whole X then $\text{cosupp}(\mathcal{M}_D) = X \times X$.

Example 2.1.8. Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^p$ be an analytic map with $n < p$ and $X = F^{-1}(0)$. Let $JM(X)$ be the jacobian module of X . In this case, the rank of $JM(X)$ at any point of X is less than or equal to $\min\{n, p\} = n < p$, then $\Sigma(JM(X)) = X$ and $\text{cosupp}((JM(X))_D) = X \times X$.

Example 2.1.9. Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^p$ be an analytic map and suppose that F is a submersion. Thus $X = F^{-1}(0)$ is a smooth submanifold of $\mathbb{C}^n$ of dimension $n - p$ and the singular set of the jacobian module $JM(X)$ is empty. Therefore, $\text{cosupp}((JM(X))_D) = \Delta(X)$.

Example 2.1.10. Let us consider the cusp deformation $X = F^{-1}(0) \subseteq \mathbb{C}^3$ given by the equation $F(x, y, z) = x^3 + z^2 x^2 - y^2 = 0$. Thus the gradient of F at a point (x, y, z) is $(3x^2 + 2z^2 x, -2y, 2zx^2)$ and it is easy to see that the singular set of the jacobian module of X is the z -axis. Therefore, $\text{cosupp}((JM(X))_D) = \Delta(X) \cup (X \times (z\text{-axis})) \cup ((z\text{-axis}) \times X)$.

The next proposition generalizes the Corollary 3.4 of [39] for modules.

Proposition 2.1.11. *Let $\mathcal{M} \subseteq \mathcal{N} \subseteq \overline{\mathcal{M}}$ be $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$, with X equidimensional. Suppose that M_D has finite colength in N_D and N_D has finite colength in $(\overline{M})_D$. Then*

$$e(M_D, (\overline{M})_D) = e(N_D, (\overline{M})_D) \text{ if and only if } \overline{M}_D = \overline{N}_D.$$

Proof. By the principle of additivity (see Theorem 6.7 of [62]), we have that

$$e(M_D, (\overline{M})_D) = e(M_D, N_D) + e(N_D, (\overline{M})_D).$$

Notice that all these multiplicities are well-defined by hypothesis. So, $e(M_D, (\overline{M})_D) = e(N_D, (\overline{M})_D)$ if and only if $e(M_D, N_D) = 0$, which is equivalent to the equality $\overline{M}_D = \overline{N}_D$, since X is equidimensional. $\square$

The following proposition and corollary will be useful later to make a relation between the saturation and the double of a module, and to work with the infinitesimal Lipschitz conditions.

Proposition 2.1.12. *Let $h \in \mathcal{O}_X^p$.*

- a) If $h_D \in \overline{\mathcal{M}}_D$ at (x, x') then $h \in \overline{\mathcal{M}}$ at x and x' ;*
- b) If $h_D \in (\mathcal{M}_D)^\dagger$ at (x, x') then $h \in \mathcal{M}^\dagger$ at x and x' .*

The same result still holds in the family case.

Proof. (a) Let us to prove that $h \in \overline{\mathcal{M}}$ at x (the case at x' is completely analogous). Let $\phi : (\mathbb{C}, 0) \rightarrow (X, x)$ be an arbitrary analytic curve. Define $\gamma : (\mathbb{C}, 0) \rightarrow (X \times X, (x, x'))$ given by $\gamma(t) = (\phi(t), x')$. Since $h_D \in \overline{\mathcal{M}}_D$ then $h_D \circ \gamma \in \mathcal{M}_D \circ \gamma$, so we can write

$$h_D \circ \gamma = \sum_j \alpha_j ((g_j)_D \circ \gamma)$$

with $g_j \in \mathcal{M}$ and $\alpha_j \in \mathcal{O}_{\mathbb{C}, 0}$. Since $\pi_1 \circ \gamma = \phi$, comparing the first p coordinates of the above equation, we get $h \circ \phi = \sum_j \alpha_j (g_j \circ \phi) \in \mathcal{M} \circ \phi$. Therefore, $h \in \overline{\mathcal{M}}$ at x .

(b) Let t be the coordinate on $\mathbb{C}$. Let us to prove that $h \in \mathcal{M}^\dagger$ at x (the case at x' is completely analogous). Let $\phi : (\mathbb{C}, 0) \rightarrow (X, x)$ be an arbitrary analytic curve. Define $\gamma : (\mathbb{C}, 0) \rightarrow (X \times X, (x, x'))$ given by $\gamma(t) = (\phi(t), x')$. Since $h_D \in (\mathcal{M}_D)^\dagger$ then $h_D \circ \gamma \in m_1(\mathcal{M}_D \circ \gamma)$, so we can write

$$h_D \circ \gamma = \sum_j t \alpha_j ((g_j)_D \circ \gamma)$$

with $g_j \in \mathcal{M}$ and $\alpha_j \in \mathcal{O}_{\mathbb{C}, 0}$. Since $\pi_1 \circ \gamma = \phi$, comparing the first p coordinates of the above equation, we get $h \circ \phi = \sum_j t \alpha_j (g_j \circ \phi) \in m_1(\mathcal{M} \circ \phi)$. Therefore, $h \in \mathcal{M}^\dagger$ at x . $\square$

The proof in the family case is quite analogous, working on the fibered product $X \times_Y X$.

Example 2.1.13. As in Example 1.4.4, consider $F(x, y) = x^2 + y^p$, $p \geq 3$ odd. Denote by X the plane curve defined by F . We have seen that if we take $q = \frac{p+1}{2}$ then $y^q \in \overline{J(f)}$ and $(y^q)_D \notin \overline{(J(f))_D}$. Hence, the converse of the Proposition 2.1.12 (a) is not true in general.

Despite the converse of the Proposition 2.1.12 (a) is not true in general, the following proposition gives a condition such that the converse holds.

Proposition 2.1.14. Let M be an $\mathcal{O}_{x,x}$ -submodule of $\mathcal{O}_{X,x}^p$ and suppose that $I_\Delta^{\oplus p} \oplus I_\Delta^{\oplus p} \subseteq \overline{M_D}$.

If $h \in \overline{M}$ then:

- a) $(h \circ \pi_1, h \circ \pi_1) \in \overline{M_D}$;
- b) $(h \circ \pi_2, h \circ \pi_2) \in \overline{M_D}$;
- c) $h_D = (h \circ \pi_1, h \circ \pi_2) \in \overline{M_D}$;
- d) $(h \circ \pi_2, h \circ \pi_1) \in \overline{M_D}$.

The same result still holds in the family case.

Proof. (a) Assume first that $h \in M$. We can write $(h \circ \pi_1, h \circ \pi_1) = h_D + (0, h \circ \pi_1 - h \circ \pi_2)$. In this case we have that $h_D \in M_D \subseteq \overline{M_D}$ and by hypothesis $(0, h \circ \pi_1 - h \circ \pi_2) \in 0 \oplus I_\Delta^{\oplus p} \subseteq \overline{M_D}$. Hence, $(h \circ \pi_1, h \circ \pi_1) \in \overline{M_D}$.

Now, let us go back to the general case. Suppose that $h \in \overline{M}$ and take an arbitrary analytic curve $\phi : (\mathbb{C}, 0) \rightarrow (X \times X, (x, x))$ written as $\phi = (\phi_1, \phi_2)$. Since $h \in \overline{M}$ then we can write $h \circ \phi_1 = \sum_i \alpha_i (g_i \circ \phi_1)$ with $g_i \in M$ and $\alpha_i \in \mathcal{O}_{\mathbb{C}, 0}$, for all i . By what we have proved before, we know that $(g_i \circ \pi_1, g_i \circ \pi_1) \in \overline{M_D}$ for all i . Thus, $(h \circ \pi_1, h \circ \pi_1) \circ \phi = (h \circ \phi_1, h \circ \phi_1)$

$$= (\sum_i \alpha_i (g_i \circ \phi_1), \sum_i \alpha_i (g_i \circ \phi_1)) = \sum_i \alpha_i \left(\underbrace{(g_i \circ \pi_1, g_i \circ \pi_1)}_{\in \overline{M_D}} \circ \phi \right) \in \phi^*(\overline{M_D}). \text{ Hence,}$$

$$(h \circ \pi_1, h \circ \pi_1) \in \overline{\overline{M_D}} = \overline{M_D}.$$

(b) The proof is analogous to the item (a) using the fact that $I_\Delta^{\oplus p} \oplus 0 \subseteq \overline{M_D}$.

(c) We can write $h_D = (h \circ \pi_1, h \circ \pi_1) + (0, h \circ \pi_2 - h \circ \pi_1)$. By the item (a) we already know that $(h \circ \pi_1, h \circ \pi_1) \in \overline{M_D}$ and since $(0, h \circ \pi_2 - h \circ \pi_1) \in 0 \oplus I_\Delta^{\oplus p} \subseteq \overline{M_D}$ then $h_D \in \overline{M_D}$.

(d) Notice that $(h \circ \pi_2, h \circ \pi_1) = (h \circ \pi_2 - h \circ \pi_1, 0) + (h \circ \pi_1, h \circ \pi_1)$. By the item (a) we already know that $(h \circ \pi_1, h \circ \pi_1) \in \overline{M_D}$ and since $(h \circ \pi_2 - h \circ \pi_1, 0) \in I_\Delta^{\oplus p} \oplus 0 \subseteq \overline{M_D}$ then $(h \circ \pi_2, h \circ \pi_1) \in \overline{M_D}$. $\square$

Corollary 2.1.15. *Let $\mathcal{M}$ and $\mathcal{N}$ be sheaves of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$.*

- a) *If $\mathcal{M}_D \subseteq \overline{\mathcal{N}_D}$ at (x, x') then $\mathcal{M} \subseteq \overline{\mathcal{N}}$ at x and x' ;*
- b) *If $\mathcal{M}_D \subseteq (\mathcal{N}_D)^\dagger$ at (x, x') then $\mathcal{M} \subseteq \mathcal{N}^\dagger$ at x and x' .*

The same result still holds in the family case.

Proof. (a) Suppose that $h \in \mathcal{M}$ at x and x' . So, $h_D \in \mathcal{M}_D$ at (x, x') , and since $\mathcal{M}_D \subseteq \overline{\mathcal{N}_D}$ at (x, x') then $h_D \in \overline{\mathcal{N}_D}$ at (x, x') . By Proposition 2.1.12 (a) we conclude that $h \in \overline{\mathcal{N}}$ at x and x' .

(b) Suppose that $h \in \mathcal{M}$ at x and x' . So, $h_D \in \mathcal{M}_D$ at (x, x') , and since $\mathcal{M}_D \subseteq (\mathcal{N}_D)^\dagger$ at (x, x') then $h_D \in (\mathcal{N}_D)^\dagger$ at (x, x') . By Proposition 2.1.12 (b) we conclude that $h \in \mathcal{N}^\dagger$ at x and x' . $\square$

Looking to the Example 2.1.13, we conclude that the converse of the Corollary 2.1.15 (a) also is not true in general.

In the next corollary, we prove that the double of a module is contained in the integral closure of the double of another module, provided the doubles of the generators are.

Corollary 2.1.16. *Let M and N be $\mathcal{O}_{X,x}$ -submodules of $\mathcal{O}_{X,x}^p$. Suppose that M is generated by $\{g_1, \dots, g_r\}$ and let M'_D be the submodule of $\mathcal{O}_{X \times X, (x,x)}^{2p}$ generated by $\{(g_1)_D, \dots, (g_r)_D\}$. If $M'_D \subseteq \overline{N_D}$ then $M_D \subseteq \overline{N_D}$. The same result still holds in the family case.*

Proof. Let us prove that $(0, (z_i \circ \pi_1 - z_i \circ \pi_2)(g_j \circ \pi_2)) \in \overline{N_D}$, for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, r\}$. In fact, let i and j be arbitrary. Since $(g_j)_D \in M'_D \subseteq \overline{N_D}$ then by Proposition 2.1.12 (a) we have that $g_j \in \overline{N}$. By Lemma 2.1.2 (e) we conclude that $(0, (z_i \circ \pi_1 - z_i \circ \pi_2)(g_j \circ \pi_2)) \in \overline{N_D}$.

Therefore, all the generators of M_D belong to $\overline{N_D}$ and so $M_D \subseteq \overline{N_D}$. $\square$

In the next theorem we compute the generic rank of the double of a module.

Theorem 2.1.17. *Let (X, x) be an irreducible analytic complex germ of dimension $d \geq 1$ and $M \subseteq \mathcal{O}_{X,x}^p$ a submodule of generic rank k . Then M_D has generic rank $2k$ at (x, x) .*

The same result still holds in the family case.

Proof. Let $\{g_1, \dots, g_r\}$ be a set of generators of M . We can write $[M] = \begin{bmatrix} | & & | \\ g_1 & \dots & g_r \\ | & & | \end{bmatrix} =$

$$\begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_p & - \end{bmatrix}, \text{ where } v_1, \dots, v_p \in \mathcal{O}_{X,x}^r \text{ are the rows of } [M]. \text{ Since } M \text{ has generic rank } k \text{ then}$$

we can choose k rows of $[M]$ which are $\mathcal{O}_{X,x}$ -linearly independent. Without loss of generality, we can suppose that $v_1, \dots, v_k$ are linearly independent.

For each $l \in \{1, \dots, p\}$, let

$$w_l := (v_l \circ \pi_1, 0_{\mathcal{O}_{X,x}^{n,r}})$$

and

$$\bar{w}_l := (v_l \circ \pi_2, (z_1 \circ \pi_1 - z_1 \circ \pi_2)(v_l \circ \pi_2), \dots, (z_n \circ \pi_1 - z_n \circ \pi_2)(v_l \circ \pi_2)).$$

Thus, $w_1, \dots, w_p, \bar{w}_1, \dots, \bar{w}_p$ are the rows of the matrix $[M_D]$ given by the generators of M_D induced by the generators of M .

Claim 1: $\{w_1, \dots, w_k, \bar{w}_1, \dots, \bar{w}_k\}$ is $\mathcal{O}_{X \times X, (x,x)}$ -linearly independent.

In fact, let $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathcal{O}_{X \times X, (x,x)}$ be arbitrary such that

$$\sum_{l=1}^k \alpha_l w_l + \sum_{l=1}^k \beta_l \bar{w}_l = 0_{\mathcal{O}_{X \times X, (x,x)}^{r+n,r}}.$$

Then, we get two equations:

$$\sum_{l=1}^k \alpha_l (v_l \circ \pi_1) + \sum_{l=1}^k \beta_l (v_l \circ \pi_2) = 0_{\mathcal{O}_{X \times X, (x,x)}^r} \quad (1)$$

$$\sum_{l=1}^k \beta_l ((z_1 \circ \pi_1 - z_1 \circ \pi_2)(v_l \circ \pi_2), \dots, (z_n \circ \pi_1 - z_n \circ \pi_2)(v_l \circ \pi_2)) = 0_{\mathcal{O}_{X \times X, (x,x)}^{n,r}} \quad (2)$$

The equality (2) implies that

$$(z_i \circ \pi_1 - z_i \circ \pi_2) \sum_{l=1}^k \beta_l (v_l \circ \pi_2) = 0_{\mathcal{O}_{X \times X, (x,x)}^r} \quad (3)$$

for all $i \in \{1, \dots, n\}$. Since $d \geq 1$ then X is not a point. If $z_i \circ \pi_1 - z_i \circ \pi_2 = 0_{\mathcal{O}_{X \times X, (x,x)}}$ for all i then $X \times X \subseteq V(I(\Delta)) = \Delta$ which implies $X \times X = \Delta$, so X is a point, contradiction. So $z_i \circ \pi_1 - z_i \circ \pi_2 \neq 0_{\mathcal{O}_{X \times X, (x,x)}}$, for some i . Since (X, x) is irreducible then $(X \times X, (x, x))$ is irreducible, so $\mathcal{O}_{X \times X, (x,x)}$ is a domain, and by the equation (3) we have

$$\sum_{l=1}^k \beta_l (v_l \circ \pi_2) = 0_{\mathcal{O}_{X \times X, (x,x)}^r}. \quad (4)$$

We can find an open subset U of X , with $x \in U$, such that all $v_1, \dots, v_k$ are defined on U , and all $\beta_1, \dots, \beta_k$ are defined on $U \times U$. For each $z \in U$ define

$$\begin{aligned} \beta_l^z : U &\longrightarrow \mathbb{C} \\ w &\longmapsto \beta_l(z, w) \end{aligned}$$

Then, the germ $\beta_l^z \in \mathcal{O}_{X,x}$, $\forall z \in U$ and $l \in \{1, \dots, k\}$.

Let us prove that $\beta_l^z = 0_{\mathcal{O}_{X,x}}$, $\forall z \in U$ and $l \in \{1, \dots, k\}$. In fact, given z and l , by the equation (4), for all $w \in U$ we have

$$0_{\mathbb{C}^r} = \left(\sum_{l=1}^k \beta_l(v_l \circ \pi_2) \right)(z, w) = \sum_{l=1}^k \beta_l(z, w) v_l(w) = \left(\sum_{l=1}^k \beta_l^z v_l \right)(w)$$

which implies $\sum_{l=1}^k \beta_l^z v_l = 0_{\mathcal{O}_{X,x}^r}$. Since $v_1, \dots, v_k$ are $\mathcal{O}_{X,x}$ -linearly independent then $\beta_l^z = 0_{\mathcal{O}_{X,x}}$.

Hence, $\beta_l = 0_{\mathcal{O}_{X \times X, (x,x)}}$, for all $l \in \{1, \dots, k\}$. So, the equation (1) becomes

$$\sum_{l=1}^k \alpha_l(v_l \circ \pi_1) = 0_{\mathcal{O}_{X \times X, (x,x)}^r}$$

which is similar to the equation (4). Using the same argument, we conclude that $\alpha_l = 0_{\mathcal{O}_{X \times X, (x,x)}}$, for all $l \in \{1, \dots, k\}$. Thus, the Claim 1 is proved.

Hence, the generic rank of M_D is $\geq 2k$.

Claim 2: It does not exist more than k rows $\mathcal{O}_{X \times X, (x,x)}$ -linearly independent on the first p rows of $[M_D]$.

In fact, suppose that there exist. So, we have at least $k+1$ rows linearly independent on the first p rows of $[M_D]$. Without loss of generality, we can assume that the first $k+1$ rows satisfy it. Let us prove that $\{v_1, \dots, v_{k+1}\}$ is $\mathcal{O}_{X,x}$ -linearly independent. Indeed, let $\alpha_1, \dots, \alpha_{k+1} \in \mathcal{O}_{X,x}$ such that $\sum_{l=1}^{k+1} \alpha_l v_l = 0_{\mathcal{O}_{X,x}^r}$. Take $\lambda_l := \alpha_l \circ \pi_1$, $\forall l \in \{1, \dots, k+1\}$. For all (z, w) in a small enough neighborhood of (x, x) we have

$$\left(\sum_{l=1}^{k+1} \lambda_l w_l \right)(z, w) = \left(\left(\sum_{l=1}^{k+1} \alpha_l v_l \right)(z), 0_{\mathbb{C}^{n.r}} \right) = 0_{\mathbb{C}^{r+n.r}}$$

which implies that $\sum_{l=1}^{k+1} \lambda_l w_l = 0_{\mathcal{O}_{X \times X, (x,x)}^{r+n.r}}$. Since $\{w_l\}_{l=1}^{k+1}$ is linearly independent then $\lambda_l = 0_{\mathcal{O}_{X \times X, (x,x)}}$, for all $l \in \{1, \dots, k+1\}$. Thus, for all z in a small neighborhood of x we have $0 = \lambda_l(z, z) = \alpha_l \circ \pi_1(z, z) = \alpha_l(z)$, hence $\alpha_l = 0_{\mathcal{O}_{X,x}}$, $\forall l \in \{1, \dots, k+1\}$. Then, $v_1, \dots, v_{k+1}$ is linearly independent and the generic rank of M is $\geq k+1$, contradiction. Therefore, the Claim 2 is proved.

Claim 3: It does not exist more than k rows $\mathcal{O}_{X \times X, (x,x)}$ -linearly independent on the last p rows of $[M_D]$.

In fact, suppose that there exist. So, we have at least $k+1$ rows linearly independent on the last p rows of $[M_D]$. Without loss of generality, we can assume that $\bar{w}_1, \dots, \bar{w}_{k+1}$ are linearly independent. Let us prove that $\{v_1, \dots, v_{k+1}\}$ is $\mathcal{O}_{X,x}$ -linearly independent. Indeed, let $\alpha_1, \dots, \alpha_{k+1} \in \mathcal{O}_{X,x}$ such that $\sum_{l=1}^{k+1} \alpha_l v_l = 0_{\mathcal{O}_{X,x}^r}$. Take $\lambda_l := \alpha_l \circ \pi_2$, $\forall l \in \{1, \dots, k+1\}$. For all (z, w) in a small enough neighborhood of (x, x) we have

$$\left(\sum_{l=1}^{k+1} \lambda_l \bar{w}_l \right)(z, w) = \left(\left(\sum_{l=1}^{k+1} \alpha_l v_l \right)(w), (z_1 - w_1) \left(\sum_{l=1}^{k+1} \alpha_l v_l \right)(w), \dots, (z_n - w_n) \left(\sum_{l=1}^{k+1} \alpha_l v_l \right)(w) \right) = 0_{\mathbb{C}^{r+n.r}}.$$

Since $\{\bar{w}_l\}_{l=1}^{k+1}$ is linearly independent then $\lambda_l = 0_{\mathcal{O}_{X \times X, (x,x)}}$, for all $l \in \{1, \dots, k+1\}$, and $\alpha_l = 0_{\mathcal{O}_{X,x}}$, $\forall l \in \{1, \dots, k+1\}$. Then, $v_1, \dots, v_{k+1}$ is linearly independent and the generic rank of M is $\geq k+1$, contradiction. Therefore, the Claim 3 is proved.

Finally, suppose by contradiction that the generic rank of M_D is $\geq 2k+1$. Then we have $2k+1$ rows of $[M_D]$ which are $\mathcal{O}_{X \times X, (x,x)}$ -linearly independent. By the Claim 2, we have at most k of these rows on the first p rows of $[M_D]$. Then, we have at least $k+1$ of these rows on the last p rows, which contradicts the Claim 3. Therefore, the generic rank of M_D is $2k$.

The proof for the family case is completely analogous working on the fibered product $X \times_Y X$. $\square$

Corollary 2.1.18. *Let $\{V_i\}$ be the irreducible components of (X, x) . For each i , if M has generic rank k_i on V_i then M_D has generic rank $2k_i$ on $V_i \times V_i$.*

Remark 2.1.19. *Let M be an $\mathcal{O}_{X,x}$ -submodule of $\mathcal{O}_{X,x}^p$ of generic rank k , and suppose that $\dim X = 0$. Then X is a point and $\mathcal{O}_{X,x} \cong \mathbb{C}$. In this case, M is a k -dimensional $\mathbb{C}$ -vector subspace of $\mathbb{C}^p$. Notice that, since $X \times X$ is also a point, then $\pi_1 = \pi_2$. Hence, the map*

$$\begin{aligned} M \subseteq \mathbb{C}^p &\longrightarrow M_D \subseteq \mathbb{C}^{2p} \\ h &\longmapsto (h, h) \end{aligned}$$

is an isomorphism of $\mathbb{C}$ -vector spaces which implies that $\dim_{\mathbb{C}} M_D = k$, i.e, M_D has generic rank k .

In next proposition we prove that the integral closure of modules commutes with finite direct sum of modules.

Proposition 2.1.20. *Let $M \subseteq \mathcal{O}_{X,x}^p$ and $N \subseteq \mathcal{O}_{X,x}^q$ be $\mathcal{O}_{X,x}$ -submodules. Then*

$$\overline{M \oplus N} = \overline{M} \oplus \overline{N}.$$

Proof. First we prove that $\overline{M \oplus 0} \subseteq \overline{M \oplus N}$. Let $v \in \overline{M \oplus 0}$. Then we can write $v = (g, 0)$ with $g \in \overline{M}$. Let $\phi : (\mathbb{C}, 0) \rightarrow (X, x)$ be an arbitrary analytic curve. Since $g \in \overline{M}$ then we can write $g \circ \phi = \sum_i \alpha_i (g_i \circ \phi)$, with $g_i \in M$ and $\alpha_i \in \mathcal{O}_{\mathbb{C}, 0}$, for all i . Thus, $v \circ \phi = (g \circ \phi, 0)$
 $= (\sum_i \alpha_i (g_i \circ \phi), 0) = \sum_i \alpha_i (g_i \circ \phi, 0) = \sum_i \alpha_i \phi^*(g_i, 0) \in \phi^*(M \oplus N)$. Hence, $v \in \overline{M \oplus N}$ and $\overline{M \oplus 0} \subseteq \overline{M \oplus N}$. Analogously, $0 \oplus \overline{N} \subseteq \overline{M \oplus N}$. Therefore, $\overline{M \oplus N} \subseteq \overline{M \oplus N}$.

Conversely, let $v \in \overline{M \oplus N}$. We can write $v = (g, h)$ with $g \in \mathcal{O}_{X,x}^p$ and $h \in \mathcal{O}_{X,x}^q$. Let $\phi : (\mathbb{C}, 0) \rightarrow (X, x)$ be an arbitrary analytic curve. Then we can write $\phi^*(v) = \sum_i \alpha_i \phi^*(g_i, h_i)$ where $g_i \in M$, $h_i \in N$ and $\alpha_i \in \mathcal{O}_{\mathbb{C}, 0}$, for all i . Comparing the coordinates of the last equation we conclude that $\phi^*(g) = \sum_i \alpha_i \phi^*(g_i)$ and $\phi^*(h) = \sum_i \alpha_i \phi^*(h_i)$. Thus, $g \in \overline{M}$ and $h \in \overline{N}$. Therefore, $v \in \overline{M \oplus N}$. $\square$

Corollary 2.1.21. *Let $M_i \subseteq \mathcal{O}_{X,x}^{p_i}$ be $\mathcal{O}_{X,x}$ -submodules, $i \in \{1, \dots, r\}$. Then*

$$\overline{\bigoplus_{i=1}^r M_i} = \bigoplus_{i=1}^r \overline{M_i}.$$

Proof. Induction on r and apply the Proposition 2.1.20. □

Let us recall some basic results in commutative algebra.

Remark 2.1.22. *Let (R, m) be a local ring and consider the residue field $k := \frac{R}{m}$.*

- a) If M is a free R -module of rank p then $\frac{M}{mM}$ is a k -vector space of dimension p ;*
- b) If M is a free R -submodule of R^p of rank p then $M = R^p$.*

Proof. (a) By hypothesis there exists an R -basis $\{v_1, \dots, v_p\}$ of M . Clearly the images of these elements in $\frac{M}{mM}$ form a generator set of $\frac{M}{mM}$. So, it remains to prove that these elements are R -linearly independent. Suppose we have an equation

$$\sum_{i=1}^p (\alpha_i + m)(v_i + mM) = 0 + mM$$

with $\alpha_i \in R$, for all $i \in \{1, \dots, p\}$. Then $\sum_{i=1}^p \alpha_i v_i \in mM$ which implies that we can write $\sum_{i=1}^p \alpha_i v_i = \sum_{i=1}^p \beta_i v_i$, for some $\beta_1, \dots, \beta_p \in m$. Since $\{v_1, \dots, v_p\}$ is R -linearly independent then $\alpha_i = \beta_i \in m$, for all $i \in \{1, \dots, p\}$. Hence, $\alpha_i + m = 0 + m$, for all $i \in \{1, \dots, p\}$.

(b) By hypothesis the submodule M has an R -basis $\{v_1, \dots, v_p\}$. Thus, $\{v_1, \dots, v_p\}$ is an R -linearly independent subset of R^p . By the proof of the item (a) we have that $\{v_1 + mR^p, \dots, v_p + mR^p\}$ is a k -linearly independent subset of $\frac{R^p}{mR^p}$. By the item (a) we have $\dim_k \frac{R^p}{mR^p} = p$, hence $\{v_1 + mR^p, \dots, v_p + mR^p\}$ is a k -basis of $\frac{R^p}{mR^p}$. By Nakayama's Lemma $\{v_1, \dots, v_p\}$ is a generator set of R^p and we conclude that $M = R^p$. □

Remark 2.1.23. *Let M be an $\mathcal{O}_{X,x}$ -submodule of $\mathcal{O}_{X,x}^p$. If $\overline{M} = \mathcal{O}_{X,x}^p$ then $M = \mathcal{O}_{X,x}^p$.*

In fact, let s be the generic rank of M . Take an analytic curve $\phi : (\mathbb{C}, 0) \rightarrow (X, x)$ such that the rank of $\phi^(M)$ is generically s . Since $\overline{M} = \mathcal{O}_{X,x}^p$ then $\phi^*(M) = \phi^*(\mathcal{O}_{X,x}^p)$ which has rank p , hence $s = p$. Thus, M has generic rank p and there exists a free $\mathcal{O}_{X,x}$ -submodule M_p of M of rank p . By Remark 2.1.22 (b) we have that $M_p = \mathcal{O}_{X,x}^p$. Since $M_p \subseteq M$ then $M = \mathcal{O}_{X,x}^p$.*

Therefore, in order to verify the equation $M = \mathcal{O}_{X,x}^p$ it suffices to check that $\overline{M} = \mathcal{O}_{X,x}^p$, which may be easier sometimes.

In the next proposition we calculate the stalk of the double sheaf $\mathcal{M}_D$ of a sheaf of modules $\mathcal{M}$ in a pair of points of X such that one of them is not a singular point of $\mathcal{M}$, i.e, the matrix $[\mathcal{M}]$ in one of these points has maximal rank p .

Proposition 2.1.24. *Let $\mathcal{M} \subseteq \mathcal{O}_X^p$ be a sheaf of submodules. Let $(x, x') \in X \times X$ with $x \neq x'$.*

a) *If $\mathcal{M}_x = \mathcal{O}_{X,x}^p$ then $\overline{\mathcal{M}}_D = \mathcal{O}_{X \times X, (x, x')}^p \oplus \overline{(\mathcal{M}_{x'} \circ \pi_2)}$ at (x, x') ;*

b) *If $\mathcal{M}_{x'} = \mathcal{O}_{X, x'}^p$ then $\overline{\mathcal{M}}_D = \overline{(\mathcal{M}_x \circ \pi_1)} \oplus \mathcal{O}_{X \times X, (x, x')}^p$ at (x, x') .*

The same result still holds in the family case, working on the fibered product $X \times_Y X$.

Proof. We may assume that $\mathcal{M}$ is generated by global sections $\{g_1, \dots, g_r\}$. Since $x \neq x'$ then $(z_l \circ \pi_1 - z_l \circ \pi_2)(x, x') \neq 0$, for some $l \in \{1, \dots, n\}$, thus $z_l \circ \pi_1 - z_l \circ \pi_2$ is an invertible element of $\mathcal{O}_{X \times X, (x, x')}$.

(a) Let us prove that $(0, h \circ \pi_2) \in \overline{\mathcal{M}}_D$ at (x, x') , $\forall h \in \overline{\mathcal{M}_{x'}}$. In fact, let $\phi : (\mathbb{C}, 0) \rightarrow (X \times X, (x, x'))$ be an arbitrary analytic curve. Write $\phi = (\phi_1, \phi_2)$. Since $h \in \overline{\mathcal{M}_{x'}}$ then we can write $h \circ \phi_2 = \sum_i \alpha_i (g_i \circ \phi_2)$ with $\alpha_i \in \mathcal{O}_{\mathbb{C}, 0}$, for all i . Since $(0, (z_l \circ \pi_1 - z_l \circ \pi_2)(g_i \circ \phi_2)) \in \mathcal{M}_D$ at (x, x') , for all i , and $z_l \circ \pi_1 - z_l \circ \pi_2$ is invertible then $(0, g_i \circ \pi_2) \in \mathcal{M}_D$ at (x, x') , for all i . Hence, $(0, h \circ \pi_2) \circ \phi = (0, h \circ \phi_2) = \sum_i \alpha_i (0, g_i \circ \phi_2) = \sum_i \alpha_i ((0, g_i \circ \pi_2) \circ \phi) \in \phi^*(\mathcal{M}_D)$. Therefore, $(0, h \circ \pi_2) \in \overline{\mathcal{M}}_D$ at (x, x') and we conclude that $0 \oplus (\overline{\mathcal{M}_{x'} \circ \pi_2}) \subseteq \overline{\mathcal{M}}_D$ at (x, x') .

Let $e_1, \dots, e_p$ be the canonical basis elements of $\mathcal{O}_X^p$. Since all these elements belong to $\mathcal{M}$ at x then $((z_l \circ \pi_1 - z_l \circ \pi_2)(e_j \circ \pi_1), 0) \in \mathcal{M}_D$ at (x, x') , $\forall j \in \{1, \dots, p\}$. The element $z_l \circ \pi_1 - z_l \circ \pi_2$ is invertible, hence $(e_j \circ \pi_1, 0) \in \mathcal{M}_D$ at (x, x') , $\forall j \in \{1, \dots, p\}$. Thus, $\mathcal{O}_{X \times X, (x, x')}^p \oplus 0 \subseteq \mathcal{M}_D \subseteq \overline{\mathcal{M}}_D$ at (x, x') , and $\mathcal{O}_{X \times X, (x, x')}^p \oplus (\overline{\mathcal{M}_{x'} \circ \pi_2}) \subseteq \overline{\mathcal{M}}_D$ at (x, x') .

Clearly $\mathcal{M}_D \subseteq \mathcal{O}_{X \times X, (x, x')}^p \oplus (\mathcal{M}_{x'} \circ \pi_2) \subseteq \mathcal{O}_{X \times X, (x, x')}^p \oplus (\overline{\mathcal{M}_{x'} \circ \pi_2})$ at (x, x') . Therefore,

$$\mathcal{M}_D \subseteq \mathcal{O}_{X \times X, (x, x')}^p \oplus (\overline{\mathcal{M}_{x'} \circ \pi_2}) \subseteq \overline{\mathcal{M}}_D$$

at (x, x') and using the Proposition 2.1.20 we conclude that $\overline{\mathcal{M}}_D = \overline{\mathcal{O}_{X \times X, (x, x')}^p \oplus (\overline{\mathcal{M}_{x'} \circ \pi_2})} = \overline{\mathcal{O}_{X \times X, (x, x')}^p} \oplus \overline{(\overline{\mathcal{M}_{x'} \circ \pi_2})} = \mathcal{O}_{X \times X, (x, x')}^p \oplus (\overline{\mathcal{M}_{x'} \circ \pi_2})$ at (x, x') .

(b) It is quite analogous to the item (a). □

Corollary 2.1.25. *Let $\mathcal{M} \subseteq \mathcal{O}_X^p$ be a sheaf of submodules. Let $(x, x') \in X \times X$ with $x \neq x'$. If $\mathcal{M}_x = \mathcal{O}_{X,x}^p$ or $\mathcal{M}_{x'} = \mathcal{O}_{X, x'}^p$ then*

$$\overline{(\mathcal{M})}_D = \overline{\mathcal{M}}_D \text{ at } (x, x').$$

The same result still holds in the family case, working on the fibered product $X \times_Y X$.

Proof. Assume that $\mathcal{M}_x = \mathcal{O}_{X,x}^p$ (the another case is analogous). By Proposition 2.1.24 (a) we have that

$$\overline{\mathcal{M}}_D = \mathcal{O}_{X \times X, (x, x')}^p \oplus \overline{(\mathcal{M}_{x'} \circ \pi_2)}$$

at (x, x') . Since $\overline{\mathcal{M}}_x = \mathcal{O}_{X,x}^p$ then applying again the Proposition 2.1.24 (a) we get

$$\overline{(\overline{\mathcal{M}})}_D = \mathcal{O}_{X \times X, (x, x')}^p \oplus \overline{(\overline{\mathcal{M}_{x'}} \circ \pi_2)}$$

at (x, x') . We already know that $\overline{\mathcal{M}_{x'}} = \overline{\mathcal{M}_{x'}}$. Therefore $\overline{(\overline{\mathcal{M}})}_D = \overline{\mathcal{M}}_D$ at (x, x') . $\square$

Now we calculate the stalk of the double sheaf $\mathcal{M}_D$ at a point of the diagonal of X whose corresponding point in X is a non-singular point of $\mathcal{M}$.

Proposition 2.1.26. *Let $\mathcal{M} \subseteq \mathcal{O}_X^p$ be a sheaf of submodules, $x \in X$ such that $\mathcal{M}_x = \mathcal{O}_{X,x}^p$. Then:*

- a) $\mathcal{M}_D = I_{\Delta}^{\oplus p} \oplus I_{\Delta}^{\oplus p} + E$ at (x, x) , where E is the submodule generated by $\{(e_i)_D \mid i \in \{1, \dots, p\}\}$, where $e_1, \dots, e_p$ are the canonical basis elements of $\mathcal{O}_X^p$;
- b) $\overline{(\overline{\mathcal{M}})}_D = \overline{\mathcal{M}}_D$ at (x, x) .

The same result still holds in the family case, working on the fibered product $X \times_Y X$.

Proof. For each $i \in \{1, \dots, p\}$ we have that $e_i \in \mathcal{M}_x$, which implies that

$((z_j \circ \pi_1 - z_j \circ \pi_2)(e_i \circ \pi_1), 0), (0, (z_j \circ \pi_1 - z_j \circ \pi_2)(e_i \circ \pi_2)) \in \mathcal{M}_D$ at (x, x) , for all $j \in \{1, \dots, n\}$ and $i \in \{1, \dots, p\}$. Hence, $I_{\Delta}^{\oplus p} \oplus I_{\Delta}^{\oplus p} \subseteq \mathcal{M}_D$ at (x, x) . Since $E \subseteq \mathcal{M}_D$ at (x, x) then $I_{\Delta}^{\oplus p} \oplus I_{\Delta}^{\oplus p} + E \subseteq \mathcal{M}_D$ at (x, x) .

Notice that $(0, (z_j \circ \pi_1 - z_j \circ \pi_2)(e_i \circ \pi_2)) \in I_{\Delta}^{\oplus p} \oplus I_{\Delta}^{\oplus p}$, for all $j \in \{1, \dots, n\}$ and $i \in \{1, \dots, p\}$. Thus, all the generators of $\mathcal{M}_D$ at (x, x) belong to $I_{\Delta}^{\oplus p} \oplus I_{\Delta}^{\oplus p} + E$, hence $\mathcal{M}_D \subseteq I_{\Delta}^{\oplus p} \oplus I_{\Delta}^{\oplus p} + E$ at (x, x) . Therefore,

$$\mathcal{M}_D = I_{\Delta}^{\oplus p} \oplus I_{\Delta}^{\oplus p} + E$$

at (x, x) .

(b) Since $\mathcal{M}_x = \mathcal{O}_{X,x}^p$ then $\overline{\mathcal{M}}_x = \mathcal{M}_x$ at x , which implies that $\overline{(\overline{\mathcal{M}})}_D = \mathcal{M}_D$ at (x, x) , and taking the integral closure we conclude that $\overline{(\overline{\mathcal{M}})}_D = \overline{\mathcal{M}}_D$ at (x, x) . $\square$

We use the previous results to obtain an algebraic invariant associated to an analytic variety with isolated singularity.

Let $F : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$ be an analytic map, $n \geq p$ such that $X = F^{-1}(0)$ has an isolated singularity at the origin, $d = \dim X$ and $JM(F)$ the jacobian module defined on X . In general the submodule $(JM(F))_D$ does not have finite colength in $\mathcal{O}_{X \times X}^{2p}$, even if X is an isolated hypersurface singularity. Nevertheless, the multiplicity of the pair offers a way around this. The module $(JM(F))_D$ has a simple description, as we will see, off the origin in its cosupport, and any integral closure condition we wish to use is easily checked because of this structure. In the notation of [43], let $H_{2d-1}((JM(F))_D)$ be the largest sheaf of modules whose integral closure

agrees with $(JM(F))_D$ off the origin. This is the integral hull of $(JM(F))_D$ of codimension $2d - 1$, which means the integral closure of $(JM(F))_D$ and $H_{2d-1}((JM(F))_D)$ agree off a set of codimension $2d$, i.e, off $(0, 0)$ in $X^d \times X^d$.

The next result identifies $H_{2d-1}((JM(F))_D)$ and is a generalization of the Lemma 3.3 of [39].

Corollary 2.1.27. *Let $F : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$ be an analytic map with $n \geq p$ and suppose that $X = F^{-1}(0)$ has isolated singularity at the origin. Let $JM(F)$ be the jacobian module defined in X . Then the multiplicity of the pair of modules $e((JM(F))_D, \overline{(JM(F))_D})$ is well defined at the origin and if X is an ICIS then*

$$H_{2d-1}((JM(F))_D) = \left(\overline{JM(F)} \right)_D.$$

Proof. By Corollary 2.1.25 and Proposition 2.1.26 (b) we have that $\overline{(JM(F))_D} = \overline{(JM(F))_D}$ off the origin. $\square$

As a corollary of the proof of the last result, we have that

$$H_{2d-1}(\mathcal{M}_D) = (\overline{\mathcal{M}})_D$$

for any sheaf of submodules $\mathcal{M} \subseteq \mathcal{O}_X^p$ of finite colength in $\mathcal{O}_X^p$.

2.2 The infinitesimal Lipschitz conditions iL_A and iL_{m_Y}

Now we use some of the results presented in last section to recover some properties about the infinitesimal Lipschitz conditions for the following more general setup.

Setup 2.2.1. *Let $(X, 0) \subseteq (\mathbb{C}^{n+k}, 0)$ be the germ of the analytic space defined by an analytic map $F : \mathbb{C}^n \times \mathbb{C}^k \rightarrow \mathbb{C}^p$, where X is a sufficient small representative such that $Y = \mathbb{C}^k = 0 \times \mathbb{C}^k \subseteq X$ is the singular set of X . Let $F_1, \dots, F_p : \mathbb{C}^n \times \mathbb{C}^k \rightarrow \mathbb{C}$ be the coordinates functions of F , for each $y \in Y$ let $f_y : \mathbb{C}^n \rightarrow \mathbb{C}^p$ given by $f_y(z) := F(z, y)$ and let $X_y := f_y^{-1}(0)$. Assume that X_y has an isolated singularity in 0, for all $y \in Y$ and $n \geq p$. Let $z_1, \dots, z_n, y_1, \dots, y_k$ be the coordinates on $\mathbb{C}^{n+k}$, let m_Y be the ideal of $\mathcal{O}_X$ generated by $\{z_1, \dots, z_n\}$, let $JM(X)$ be the jacobian module of X , let $JM(X)_Y$ be the module generated by $\{\frac{\partial F}{\partial y_1}, \dots, \frac{\partial F}{\partial y_k}\}$ and let $JM_z(X)$ be the module generated by $\{\frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_n}\}$.*

In this section we work with the doubles relative to Y and with the projections

$$\begin{array}{ccc} & X \times X & \\ & \downarrow Y & \\ X & \xleftarrow{\pi_1} & \xrightarrow{\pi_2} X \end{array}$$

Since X_Y has an isolated singularity in 0, for all $y \in Y$, then $\Sigma(JM_z(X)) = \{(0, y) \mid y \in Y\} = Y$, so by Theorem 2.1.6 we have that

$$\text{cosupp}((JM_z(X))_D) = \Delta(X) \cup (X \times_Y 0) \cup (0 \times_Y X).$$

Definition 2.2.2. Under the above setup, we introduce the infinitesimal Lipschitz conditions.

- The pair (X, Y) satisfies the **infinitesimal Lipschitz condition m_Y (iL_{m_Y})** at $(x, x') \in X \times_Y X$ if $(JM(X)_Y)_D \subseteq \overline{(m_Y JM_z(X))_D}$ at (x, x') ;
- The pair (X, Y) satisfies the **infinitesimal Lipschitz condition A (iL_A)** at $(x, x') \in X \times_Y X$ if $(JM(X)_Y)_D \subseteq \overline{(JM_z(X))_D}$ at (x, x') .

Notice that iL_{m_Y} implies iL_A .

By Theorem 2.5 of [41], the pair (X, Y) satisfies the Verdier (w) -regularity at $x \in X$ if and only if $JM(X)_Y \subseteq \overline{m_Y JM_z(X)}$ at x . Thus, the iL_{m_Y} condition is the Lipschitz version of the W condition. The iL_A condition is the Lipschitz version of the A_F condition, which is $JM(X)_Y \subseteq \overline{JM_z(X)}$, if this holds on the ambient space.

Example 2.2.3. (Family of cusps) Consider $X \subseteq \mathbb{C}^3$ given by the equation $F(x, y, z) = z^2 - x^3 = 0$. Since the gradient of F at a point (x, y, z) is $(-3x^2, 2z, 0)$ then the singular locus of X is the y -axis. In this case, it is easy to see that $JM(X)_Y = \{0\}$ and therefore both infinitesimal Lipschitz conditions are trivially satisfied at any point of $X \times_Y X$.

Figure 6 – Family of Cusps

In general, if X is defined by an analytic map which does not depend of the parameter space then $JM(X)_Y = \{0\}$ and both infinitesimal Lipschitz conditions are trivially satisfied.

Example 2.2.4. Consider the analytic map $f(x, y) = x^2 + y^3$. Taking $q := \frac{3+1}{2} = 2$ we have seen in Example 1.4.4 that $y^2 \in \overline{J(f)}$ but $(y^2)_D \notin \overline{(J(f))_D}$. In this case, $J(f)_Y$ is generated by y^2 . Then, $(J(f)_Y)_D \not\subseteq \overline{(J(f))_D}$.

Hence, neither of the infinitesimal Lipschitz conditions are satisfied.

We want to see if the condition iL_{m_Y} at a point of $(0,0) \times Y$ depends of the projection $\mathbf{p} : X \rightarrow Y$. The next lemma will be useful to answer this question.

Lemma 2.2.5. *Consider the setup 2.2.1. For any parameter $y \in Y$ we have:*

a) *If $(JM(X)_Y)_D \subseteq \overline{(m_Y JM(X))_D}$ at $(0,y,0,y) \in (0,0) \times Y$ then*

$$JM(X)_Y \subseteq \overline{m_Y JM_z(X)}$$

at $(0,y)$, i.e, the W condition holds at $(0,y)$;

b) *If $(JM(X)_Y)_D \subseteq ((m_Y JM(X))_D)^\dagger$ at $(z,y,z,y) \in X \times_Y X$ then*

$$JM(X)_Y \subseteq (m_Y JM_z(X))^\dagger$$

at (z,y) .

Proof. (a) Let $\phi : (\mathbb{C}, 0) \rightarrow (X, (0,y))$ be an arbitrary analytic curve. By the hypothesis and Corollary 2.1.15 (a) we have that $JM(X)_Y \subseteq \overline{m_Y JM(X)}$, hence $\phi^*(JM(X)_Y) \subseteq \phi^*(m_Y JM(X))$.

Now, since $\phi^*(m_Y) \subseteq m_1$, $JM(X) = JM(X)_Y + JM_z(X)$ and ϕ^* preserves the module operations then:

$$\begin{aligned} \phi^*(JM(X)_Y) &\subseteq \phi^*(m_Y JM(X)) = \phi^*(m_Y JM(X)_Y + m_Y JM_z(X)) \\ &\subseteq m_1 \phi^*(JM(X)_Y) + \phi^*(m_Y JM_z(X)). \end{aligned}$$

By Nakayama's Lemma we conclude that $\phi^*(JM(X)_Y) \subseteq \phi^*(m_Y JM_z(X))$. Therefore, (a) is proved by the curve criterion.

(b) Let $\phi : (\mathbb{C}, 0) \rightarrow (X, (z,y))$ be an arbitrary analytic curve. By the hypothesis and Corollary 2.1.15 (b) we have that $JM(X)_Y \subseteq (m_Y JM(X))^\dagger$, hence $\phi^*(JM(X)_Y) \subseteq m_1 \phi^*(m_Y JM(X))$, and furthermore, $\phi^*(JM(X)_Y) \subseteq m_1 \phi^*(m_Y JM(X)) = m_1 \phi^*(m_Y JM(X)_Y) + m_1 \phi^*(m_Y JM_z(X)) \subseteq m_1 \phi^*(JM(X)_Y) + (m_1 \phi^*(m_Y JM_z(X)))$. By Nakayama's Lemma we have that $\phi^*(JM(X)_Y) \subseteq m_1 \phi^*(m_Y JM_z(X))$. Therefore, (b) is proved. $\square$

The next result gives us a kind of “independence of the projection onto Y ” of the iL_{m_Y} condition at the origin .

Proposition 2.2.6. *Consider the setup 2.2.1. At any point of $(0,0) \times Y$, we have that $(JM(X)_Y)_D \subseteq \overline{(m_Y JM(X))_D}$ if and only if $(JM(X)_Y)_D \subseteq \overline{(m_Y JM_z(X))_D}$.*

Proof. The implication $(\Leftarrow)$ is obvious. Let us to prove $(\Rightarrow)$.

Let $\phi : (\mathbb{C}, 0) \rightarrow (X \times_Y X, (0,y,0,y))$ be an arbitrary analytic curve. Write $\phi = (\phi_1, \phi_2)$, with $\phi_1, \phi_2 : (\mathbb{C}, 0) \rightarrow (X, (0,y))$.

Claim 1: $((z_i \circ \phi_1 - z_i \circ \phi_2)(\frac{\partial F}{\partial y_l} \circ \phi_1), 0) \in \phi^*((m_Y JM_z(X))_D)$, for all $i \in \{1, \dots, n\}$ and $l \in \{1, \dots, k\}$.

In fact, by Lemma 2.2.5 (a) we have $(JM(X))_Y \subseteq \overline{(m_Y JM_z(X))}$, so $\frac{\partial F}{\partial y_l} \in \overline{(m_Y JM_z(X))} \implies \frac{\partial F}{\partial y_l} \circ \phi_1 \in \phi_1^*(m_Y JM_z(X))$ and we can write $\frac{\partial F}{\partial y_l} \circ \phi_1 = \sum_{r,j} \beta_{rj} ((z_r \frac{\partial F}{\partial z_j}) \circ \phi_1)$, with $\beta_{rj} \in \mathcal{O}_{\mathbb{C},0}$. Then,

$$((z_i \circ \phi_1 - z_i \circ \phi_2) (\frac{\partial F}{\partial y_l} \circ \phi_1), 0) = \sum_{r,j} \beta_{rj} ((z_i \circ \phi_1 - z_i \circ \phi_2) ((z_r \frac{\partial F}{\partial z_j}) \circ \phi_1), 0)$$

$$= \sum_{r,j} \beta_{rj} \underbrace{\phi^*(((z_i \circ \pi_1 - z_i \circ \pi_2) (\frac{\partial F}{\partial z_j}) \circ \pi_1), 0))}_{\in \phi^*((m_Y JM_z(X))_D)} \in \phi^*((m_Y JM_z(X))_D). \text{ So the Claim 1 is proved.}$$

Claim 2: $\phi^*((m_Y JM(X))_Y)_D \subseteq m_1 \phi^*((m_Y JM(X))_D) + \phi^*((m_Y JM_z(X))_D)$.

In fact, it is enough to look to the images of the generators of $(m_Y JM(X))_Y$. For all $i, j \in \{1, \dots, n\}$ and $l \in \{1, \dots, k\}$ we have

$$\phi^*((z_i \frac{\partial F}{\partial y_l})_D) = (z_i \circ \phi_2) (\frac{\partial F}{\partial y_l} \circ \phi_1, \frac{\partial F}{\partial y_l} \circ \phi_2) + ((z_i \circ \phi_1 - z_i \circ \phi_2) (\frac{\partial F}{\partial y_l} \circ \phi_1), 0).$$

We have that $z_i \circ \phi_2 \in m_1$. By hypothesis we have $\phi^*((JM(X))_Y)_D \subseteq \phi^*((m_Y JM(X))_D)$, so $\phi^*((\frac{\partial F}{\partial y_l})_D) \in \phi^*((m_Y JM(X))_D)$. Thus, the first term on the right hand side of the above sum is in $m_1 \phi^*((m_Y JM(X))_D)$. The second term on the right hand side of the above sum is in $\phi^*((m_Y JM_z(X))_D)$ by Claim 1.

Hence, $\phi^*((z_i \frac{\partial F}{\partial y_l})_D) \in m_1 \phi^*((m_Y JM(X))_D) + \phi^*((m_Y JM_z(X))_D)$. Also, we have

$$\phi^*((z_i \circ \pi_1 - z_i \circ \pi_2) ((z_j \frac{\partial F}{\partial y_l}) \circ \pi_1), 0) = (z_j \circ \phi_1) \underbrace{((z_i \circ \phi_1 - z_i \circ \phi_2) (\frac{\partial F}{\partial y_l} \circ \phi_1), 0)}_{\in \phi^*((m_Y JM_z(X))_D)}, \text{ so the Claim 2}$$

is proved.

Thus, $\phi^*((m_Y JM(X))_D) = \phi^*((m_Y JM(X))_Y)_D + \phi^*((m_Y JM_z(X))_D)$
 $\subseteq (m_1 \phi^*((m_Y JM(X))_D) + \phi^*((m_Y JM_z(X))_D)) + \phi^*((m_Y JM_z(X))_D)$
 $= m_1 \phi^*((m_Y JM(X))_D) + \phi^*((m_Y JM_z(X))_D)$. By Nakayama's Lemma we conclude that $\phi^*((m_Y JM(X))_D) \subseteq \phi^*((m_Y JM_z(X))_D)$.

Since $\phi^*((JM(X))_Y)_D \subseteq \phi^*((m_Y JM(X))_D)$ then $\phi^*((JM(X))_Y)_D \subseteq \phi^*((m_Y JM_z(X))_D)$. By the curve criterion we conclude that $(JM(X))_Y)_D \subseteq \overline{(m_Y JM_z(X))_D}$. $\square$

While a similar result for iL_A does not make sense, if we work on the strict iL_A , then we get an analogous result, as we can see in the following proposition.

Proposition 2.2.7. Consider the setup 2.2.1. At any point $(x, x') \in X \times_Y X$ we have:

$$(JM(X)_Y)_D \subseteq (JM(X)_D)^\dagger \text{ if and only if } (JM(X)_Y)_D \subseteq (JM_z(X)_D)^\dagger.$$

Proof. The implication $(\Leftarrow)$ is obvious. Now, suppose that $(JM(X)_Y)_D \subseteq (JM(X)_D)^\dagger$, and let $\phi : (\mathbb{C}, 0) \rightarrow (X \times_Y X, (x, x'))$ be an arbitrary curve. Then,
 $\phi^*((JM(X)_Y)_D) \subseteq m_1 \phi^*(JM(X)_D) = m_1 \phi^*((JM(X)_Y)_D) + m_1 \phi^*((JM_z(X))_D)$. By Nakayama's Lemma we conclude that $\phi^*((JM(X)_Y)_D) \subseteq m_1 \phi^*(JM_z(X)_D)$, which finishes the proof. $\square$

Lemma 2.2.8. *Consider the setup 2.2.1. If $x = (z, y) \in X$ with $z \neq 0$ then $m_Y JM_z(X) = JM_z(X) = \mathcal{O}_X^p$ at x .*

Proof. Since $z \neq 0$ then the jacobian matrix $[J(f_y)(z)] = [JM_z(X)(x)]$ has maximal rank p . Without loss of generality we can assume that the $p \times p$ submatrix formed by the first p columns of $[JM_z(X)(x)]$ is invertible. Denote this matrix by $[JM_z(X)(x)]_p$. Then, there exists an open subset U of X containing x such that $[JM_z(X)(w)]_p$ is invertible, $\forall w \in U$. Let $e_1, \dots, e_p$ be the canonical generators of $\mathcal{O}_X^p$.

Let us to prove that $e_j \in JM_z(X)$ at x , $\forall j \in \{1, \dots, p\}$. In fact, given such j , consider the analytic map $\alpha : U \rightarrow \mathbb{C}^p$ given by $\alpha(w) := [JM_z(X)(w)]_p^{-1} \cdot e_j$, with coordinates function $\alpha_1, \dots, \alpha_p$. So, we have $e_j = \sum_{i=1}^p \alpha_i \frac{\partial F}{\partial z_i}$ on U , hence $e_j \in JM_z(X)$ at x . Therefore, $JM_z(X) = \mathcal{O}_X^p$ at x .

Now, since $z \neq 0$ then $z_k(x) \neq 0$ for some k , so z_k is invertible in $\mathcal{O}_{X,x}$. Since $\frac{\partial F}{\partial z_i} = z_k^{-1} (z_k \frac{\partial F}{\partial z_i})$, for all $i \in \{1, \dots, n\}$ then $JM_z(X) = m_Y JM_z(X)$ at x . $\square$

The next result generalizes the Proposition 4.1 of [39].

Proposition 2.2.9. *Consider the setup 2.2.1. Then*

$$\text{cosupp}((m_Y JM_z(X))_D) = \text{cosupp}((JM_z(X))_D) = \Delta(X) \cup (X \times_Y 0) \cup (0 \times_Y X).$$

Proof. Since $(m_Y JM_z(X))_D \subseteq (JM_z(X))_D$ at any point, then $\text{cosupp}((JM_z(X))_D) \subseteq \text{cosupp}((m_Y JM_z(X))_D)$. We already know that

$$\text{cosupp}((JM_z(X))_D) = \Delta(X) \cup (X \times_Y 0) \cup (0 \times_Y X).$$

Let us to prove that

$$X \times_Y X - (\text{cosupp}((JM_z(X))_D)) \subseteq X \times_Y X - (\text{cosupp}((m_Y JM_z(X))_D)).$$

In fact, if (x, x') is not in the cosupport of $(JM_z(X))_D$ then $(JM_z(X))_D = \mathcal{O}_{X \times_Y X}^{2p}$ at (x, x') , and we can write $x = (z, y)$ and $x' = (z', y)$, with $z \neq 0$, $z' \neq 0$ and $z \neq z'$. So there exist $t, k, r \in \{1, \dots, n\}$ such that $z_t \circ \pi_1$, $z_k \circ \pi_2$ and $z_r \circ \pi_1 - z_r \circ \pi_2$ are invertibles in $\mathcal{O}_{X \times_Y X, (x, x')}$.

Notice that: for all $j \in \{1, \dots, n\}$ we have:

$$\begin{aligned} \left(\frac{\partial F}{\partial z_j}\right)_D &= (z_t \circ \pi_1)^{-1} \underbrace{\left((z_t \frac{\partial F}{\partial z_j}) \circ \pi_1, (z_t \frac{\partial F}{\partial z_j}) \circ \pi_2 \right)}_{\in (m_Y JM_z(X))_D} \\ &\quad - (z_t \circ \pi_1)^{-1} (z_r \circ \pi_1 - z_r \circ \pi_2)^{-1} \underbrace{\left(0, (z_r \circ \pi_1 - z_r \circ \pi_2) \left((z_t \frac{\partial F}{\partial z_j}) \circ \pi_2 \right) \right)}_{\in (m_Y JM_z(X))_D} \\ &\quad + (z_k \circ \pi_2)^{-1} (z_r \circ \pi_1 - z_r \circ \pi_2)^{-1} \underbrace{\left(0, (z_r \circ \pi_1 - z_r \circ \pi_2) \left((z_k \frac{\partial F}{\partial z_j}) \circ \pi_2 \right) \right)}_{\in (m_Y JM_z(X))_D} \in (m_Y JM_z(X))_D. \end{aligned}$$

Furthermore, for all $i, j \in \{1, \dots, n\}$ we have $(0, (z_i \circ \pi_1 - z_i \circ \pi_2)(\frac{\partial F}{\partial z_j} \circ \pi_2)) = (z_k \circ \pi_2)^{-1}(0, (z_i \circ \pi_1 - z_i \circ \pi_2)((z_k \frac{\partial F}{\partial z_j}) \circ \pi_2)) \in (m_Y JM_z(X))_D$.

Thus, we conclude that $(m_Y JM_z(X))_D = (JM_z(X))_D = \mathcal{O}_{X \times_Y X}^{2p}$ at (x, x')
 $\Rightarrow (x, x') \notin \text{cosupp}((m_Y JM_z(X))_D)$.

Hence, $\text{cosupp}(m_Y JM_z(X))_D \subseteq \text{cosupp}(JM_z(X))_D$, which finishes the proof. $\square$

The next result generalizes the Proposition 4.2 of [39], and states that both infinitesimal Lipschitz conditions hold in any point of $X \times_Y X$ off $(0, 0) \times Y$.

Proposition 2.2.10. *Consider the setup 2.2.1.*

- a) *If $(x, x) \in \Delta(X) - ((0, 0) \times Y)$ then $(JM(X)_Y)_D \subseteq (JM_z(X))_D$ and $(JM(X)_Y)_D \subseteq (m_Y JM_z(X))_D$ at (x, x) . In particular, iL_A and iL_{m_Y} hold at (x, x) ;*
- b) *If $(x, x') \in (X \times_Y 0) \cup (0 \times_Y X) - ((0, 0) \times Y)$ and the W condition holds in any point $(0, y)$, $y \in Y$, then iL_A and iL_{m_Y} conditions hold at (x, x') .*

Proof. (a) We can write $x = (z, y)$ with $z \neq 0$. By Lemma 2.2.8 we have that $JM(X)_Y \subseteq \mathcal{O}_X^p = JM_z(X) = m_Y JM_z(X)$ at x , which implies that $(JM(X)_Y)_D \subseteq (JM_z(X))_D$ and $(JM(X)_Y)_D \subseteq (m_Y JM_z(X))_D$ at (x, x) .

(b) We can assume that $(x, x') \in (X \times_Y 0) - ((0, 0) \times Y)$ (the other case is completely analogous). So, we can write $x = (z, y)$ and $x' = (0, y)$, with $z \neq 0$. So $z_k(x) \neq 0$ for some k , and follows that $(z_k \circ \pi_1 - z_k \circ \pi_2)(x, x') = z_k(x) - 0 = z_k(x) \neq 0 \Rightarrow z_k \circ \pi_1$ and $z_k \circ \pi_1 - z_k \circ \pi_2$ are invertibles in $\mathcal{O}_{X \times_Y X(x, x')}$. Let $e_1, \dots, e_p$ be the canonical generators of $\mathcal{O}_{X \times_Y X}^p$ and $e'_1, \dots, e'_p$ be the canonical generators of $\mathcal{O}_X^p$. Thus $e_j = e'_j \circ \pi_1, \forall j \in \{1, \dots, p\}$. Since $z \neq 0$ then by Lemma 2.2.8 we have $JM_z(X) = \mathcal{O}_X^p$ at x .

Let us to prove that $(e_j, 0) \in (m_Y JM_z(X))_D, \forall j \in \{1, \dots, p\}$ at (x, x') . In fact, given a such j , we have $e'_j \in \mathcal{O}_X^p = JM_z(X)$ at x , so we can write $e'_j = \sum_{i=1}^n \alpha_i \frac{\partial F}{\partial z_i} \Rightarrow (e_j, 0) = \sum_{i=1}^n (\alpha_i \circ \pi_1)(z_k \circ \pi_1 - z_k \circ \pi_2)^{-1}(z_k \circ \pi_1)^{-1} \left((z_k \circ \pi_1 - z_k \circ \pi_2)(z_k \frac{\partial F}{\partial z_i}) \circ \pi_1, 0 \right)$, hence $(e_j, 0) \in (m_Y JM_z(X))_D$.

$\underbrace{\hspace{15em}}_{\in (m_Y JM_z(X))_D}$

Let us to prove that $(0, \frac{\partial F}{\partial y_l} \circ \pi_2) \in \overline{(m_Y JM_z(X))_D}$ at (x, x') , $\forall l \in \{1, \dots, k\}$. In fact, given a such l , we will use the curve criterion. Let $\phi : (\mathbb{C}, 0) \rightarrow (X \times_Y X, (x, x'))$ be an arbitrary analytic curve. Write $\phi = (\varphi, \gamma)$. Since $\gamma : (\mathbb{C}, 0) \rightarrow (X, x')$ is an analytic curve and $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM_z(X)}$ at $x' = (0, y)$ then $\frac{\partial F}{\partial y_l} \circ \gamma = g \circ \gamma$, for some $g \in m_Y JM_z(X)$ at x' . So, we can write $g = \sum_{r,j=1}^n \beta_{rj} z_r \frac{\partial F}{\partial z_j}$.

Thus, we have that $(0, g \circ \pi_2) = \sum (\beta_{rj} \circ \pi_2)(z_k \circ \pi_1 - z_k \circ \pi_2)^{-1} \left(0, (z_k \circ \pi_1 - z_k \circ \pi_2)(z_r \frac{\partial F}{\partial z_j}) \circ \pi_2 \right)$

$\underbrace{\hspace{15em}}_{\in (m_Y JM_z(X))_D}$

$\Rightarrow (0, g \circ \pi_2) \in (m_Y JM_z(X))_D$. Then, $(0, \frac{\partial F}{\partial y_l} \circ \pi_2) \circ \phi$
 $= (0, g \circ \pi_2) \circ \phi \in (m_Y JM_z(X))_D \circ \phi$. By curve criterion we conclude that
 $(0, \frac{\partial F}{\partial y_l} \circ \pi_2) \in \overline{(m_Y JM_z(X))_D}$ at (x, x') .

Finally, let us to prove that $(JM(X)_Y)_D \subseteq \overline{(m_Y JM_z(X))_D}$ at (x, x') . In fact, for all
 $i \in \{1, \dots, n\}$ and $l \in \{1, \dots, k\}$ we have that $(0, (z_i \circ \pi_1 - z_i \circ \pi_2)(\frac{\partial F}{\partial y_l} \circ \pi_2)) =$
 $(z_i \circ \pi_1 - z_i \circ \pi_2)(0, \frac{\partial F}{\partial y_l} \circ \pi_2) \in \overline{(m_Y JM_z(X))_D}$ and

$$\left(\frac{\partial F}{\partial y_l} \right)_D = \left(\sum_{j=1}^p \left(\frac{\partial F_j}{\partial y_l} \circ \pi_1 \right) \underbrace{(e_j, 0)}_{\in (m_Y JM_z(X))_D} \right) + \underbrace{(0, \frac{\partial F}{\partial y_l} \circ \pi_2)}_{\in \overline{(m_Y JM_z(X))_D}} \in \overline{(m_Y JM_z(X))_D}.$$

Therefore, $(JM(X)_Y)_D \subseteq \overline{(m_Y JM_z(X))_D}$ at (x, x') and the iL_{m_Y} and iL_A condition are satisfied in (x, x') . $\square$

The next result generalizes the Theorem 4.3 of [39] and states that the infinitesimal Lipschitz condition A holds generically along the parameter space Y .

Theorem 2.2.11. *Consider the setup 2.2.1. Then there exists a dense Zariski open subset U of Y such that the infinitesimal Lipschitz condition A holds for the pair $(X - Y, U \cap Y)$ along Y .*

Proof. We can write a matrix of generators of $(JM_z(X))_D$ as

$$[(JM_z(X))_D] = \begin{bmatrix} JM_z(x) \circ \pi_1 & 0 \\ JM_z(X) \circ \pi_2 & (0, (z_i \circ \pi_1 - z_i \circ \pi_2)(\frac{\partial F}{\partial z_s} \circ \pi_2))_{i,s=1}^n \end{bmatrix}$$

whose entries are in $\mathcal{O}_{X \times_Y X}$. Since $(JM_z(X))_D$ is a sheaf of submodules of $\mathcal{O}_{X \times_Y X}^{2p}$ then, choosing $S_1, \dots, S_{2p}$ as the homogeneous coordinates on $\mathbb{P}^{2p-1}$, we can consider the sheaf of ideals of $\mathcal{O}_{X \times_Y X \times \mathbb{P}^{2p-1}}$ induced by $(JM_z(X))_D$, namely $\rho((JM_z(X))_D)$, which is generated by the entries of the vector

$$[1 \quad \frac{S_2}{S_1} \quad \dots \quad \frac{S_{2p}}{S_1}] \cdot [(JM_z(X))_D]$$

on the chart $U_1 := \{[S_1, \dots, S_{2p}] \in \mathbb{P}^{2p-1} \mid S_1 \neq 0\}$ which is a dense Zariski open subset of $\mathbb{P}^{2p-1}$.

Denote by $N := NB_{\rho((JM_z(X))_D)}(X \times_Y X \times \mathbb{P}^{2p-1})$ the normalized blow-up of $X \times_Y X \times \mathbb{P}^{2p-1}$ with respect to the sheaf of ideals $\rho((JM_z(X))_D)$ of $\mathcal{O}_{X \times_Y X \times \mathbb{P}^{2p-1}}$. Consider the projection map $\pi : N \rightarrow X \times_Y X \times \mathbb{P}^{2p-1}$ and let $E \subseteq N$ be the normalized exceptional divisor. To prove this theorem we use the Corollary 2.1.16 and the module criterion (see Proposition 3.5 in [48]), i.e, in order to verify the condition $(JM(X)_Y)_D \subseteq \overline{(JM_z(X))_D}$ in a dense Zariski open subset U of Y , it suffices to check that on each component of the exceptional divisor, the pullback of the element

$\rho((\frac{\partial F}{\partial y})_D)$ to the normalized blow-up is in the pullback of $\rho((JM_z(X))_D)$, for every coordinate y in the parameter space.

Let $\mathbf{p} : X \subseteq \mathbb{C}^n \times Y \rightarrow Y$ be the projection onto Y . For each $\ell \in \{1, 2\}$ consider the projection map $p_\ell : X \times_Y X \times \mathbb{P}^{2p-1} \rightarrow X$ on the ℓ^{th} factor and $\bar{\pi}_\ell : N \rightarrow X$ given $\bar{\pi}_\ell := p_\ell \circ \pi$.

$$\begin{array}{ccccccc}
 & & N & & & & \\
 & \swarrow \bar{\pi}_1 & \downarrow \pi & \searrow \bar{\pi}_2 & & & \\
 Y & \xleftarrow{\mathbf{p}} & X & \xleftarrow{p_1} & X \times_Y X \times \mathbb{P}^{2p-1} & \xrightarrow{p_2} & X \xrightarrow{\mathbf{p}} Y \\
 & & \nwarrow \pi_1 & & \downarrow \bar{p} & & \nearrow \pi_2 \\
 & & & & X \times_Y X & &
 \end{array}$$

By Proposition 2.2.10 we need only consider those components of the exceptional divisor which projects to Y under the map to $X \times_Y X$. Since we are working over a dense Zariski open subset of Y we may assume that every such components maps surjectively onto Y . Since N is a normal space and E has codimension 1 in N then we can work at a point q of the normalized exceptional divisor E such that E is smooth at q , N is smooth at q and the projection to Y is a submersion at q . Thus we can choose coordinates (y', u', x') such that $y' = y \circ \mathbf{p}$, u' defines E locally with reduced structure and $\frac{\partial u'}{\partial y'} = 0$, i.e, u' and y' are independent coordinates. Working on the subset $U_1 \subset \mathbb{P}^{2p-1}$, since X is defined by F then the germ of

$$\begin{bmatrix} 1 & \frac{s_2}{s_1} & \dots & \frac{s_{2p}}{s_1} \end{bmatrix} \cdot \begin{bmatrix} F_1 \circ p_1 \\ \vdots \\ F_p \circ p_1 \\ F_1 \circ p_2 \\ \vdots \\ F_p \circ p_2 \end{bmatrix} = 0$$

is identically zero on $X \times_Y X \times \mathbb{P}^{2p-1}$. Pull this back to N by π and take the partial derivative with respect to y' at q . We get by the chain rule:

$$[1 \quad \frac{S_2}{S_1} \quad \dots \quad \frac{S_{2p}}{S_1}] \cdot \begin{bmatrix} \frac{\partial F_1}{\partial y} \circ \bar{\pi}_1 + \sum_{i=1}^n \left(\frac{\partial F_1}{\partial z_i} \circ \bar{\pi}_1 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} \right) \\ \vdots \\ \frac{\partial F_p}{\partial y} \circ \bar{\pi}_1 + \sum_{i=1}^n \left(\frac{\partial F_p}{\partial z_i} \circ \bar{\pi}_1 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} \right) \\ \frac{\partial F_1}{\partial y} \circ \bar{\pi}_2 + \sum_{i=1}^n \left(\frac{\partial F_1}{\partial z_i} \circ \bar{\pi}_2 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_2)}{\partial y'} \right) \\ \vdots \\ \frac{\partial F_p}{\partial y} \circ \bar{\pi}_2 + \sum_{i=1}^n \left(\frac{\partial F_p}{\partial z_i} \circ \bar{\pi}_2 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_2)}{\partial y'} \right) \end{bmatrix} = 0 \quad (\star)$$

Since $F_j \circ \bar{\pi}_1 = F_j \circ \bar{\pi}_2 = 0$ for all $j \in \{1, \dots, p\}$ then there is no term involving the derivatives of the homogeneous coordinates with respect to y' . Notice that all z_i vanish along Y and $z_i \circ \bar{\pi}_1$ and $z_i \circ \bar{\pi}_2$ vanish along E at q then we can assume that the order of vanishing of $z_1 \circ \bar{\pi}_\ell$ is minimal among $\{z_i \circ \bar{\pi}_\ell\}$ and that the strict transform of $z_1 \circ \bar{\pi}_\ell$ do not pass through q , $\forall \ell \in \{1, 2\}$.

$$\text{By the equation } (\star) \text{ we have that } \rho((\frac{\partial F}{\partial y})_D) \circ \pi = [1 \quad \frac{S_2}{S_1} \quad \dots \quad \frac{S_{2p}}{S_1}] \cdot \begin{bmatrix} \frac{\partial F_1}{\partial y} \circ \bar{\pi}_1 \\ \vdots \\ \frac{\partial F_p}{\partial y} \circ \bar{\pi}_1 \\ \frac{\partial F_1}{\partial y} \circ \bar{\pi}_2 \\ \vdots \\ \frac{\partial F_p}{\partial y} \circ \bar{\pi}_2 \end{bmatrix} = -v,$$

where

$$v := [1 \quad \frac{S_2}{S_1} \quad \dots \quad \frac{S_{2p}}{S_1}] \cdot \begin{bmatrix} \sum_{i=1}^n \left(\frac{\partial F_1}{\partial z_i} \circ \bar{\pi}_1 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} \right) \\ \vdots \\ \sum_{i=1}^n \left(\frac{\partial F_p}{\partial z_i} \circ \bar{\pi}_1 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} \right) \\ \sum_{i=1}^n \left(\frac{\partial F_1}{\partial z_i} \circ \bar{\pi}_2 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_2)}{\partial y'} \right) \\ \vdots \\ \sum_{i=1}^n \left(\frac{\partial F_p}{\partial z_i} \circ \bar{\pi}_2 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_2)}{\partial y'} \right) \end{bmatrix}.$$

In order to simplify the notation, for each $i \in \{1, \dots, n\}$ define

$$w_i := \begin{bmatrix} 1 & \frac{S_2}{S_1} & \dots & \frac{S_{2p}}{S_1} \end{bmatrix} \cdot \begin{bmatrix} \left(\frac{\partial F_1}{\partial z_i} \circ \bar{\pi}_1 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} \right) \\ \vdots \\ \left(\frac{\partial F_p}{\partial z_i} \circ \bar{\pi}_1 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} \right) \\ \left(\frac{\partial F_1}{\partial z_i} \circ \bar{\pi}_2 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} \right) \\ \vdots \\ \left(\frac{\partial F_p}{\partial z_i} \circ \bar{\pi}_2 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} \right) \end{bmatrix}$$

and

$$\tilde{w}_i := - \begin{bmatrix} 1 & \frac{S_2}{S_1} & \dots & \frac{S_{2p}}{S_1} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \left(\frac{\partial F_1}{\partial z_i} \circ \bar{\pi}_2 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} - \frac{\partial(z_i \circ \bar{\pi}_2)}{\partial y'} \right) \\ \vdots \\ \left(\frac{\partial F_p}{\partial z_i} \circ \bar{\pi}_2 \right) \left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} - \frac{\partial(z_i \circ \bar{\pi}_2)}{\partial y'} \right) \end{bmatrix}.$$

Clearly $v = \sum_{i=1}^n (w_i + \tilde{w}_i)$. For every $i \in \{1, \dots, n\}$ we have that

$$w_i = \frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} \pi^*(\rho((\frac{\partial F}{\partial z_i})_D)) \in \pi^*(\rho((JM_z(X))_D)).$$

Now it suffices to check that $\tilde{w}_i \in \pi^*(\rho((JM_z(X))_D))$, $\forall i \in \{1, \dots, n\}$. Since the pullback of the ideal $\rho((JM_z(X))_D)$ is locally principal then we can work at a point q such that $\pi^*(\rho((JM_z(X))_D))$ is generated by u'^r , a power of u' . Since $\mathcal{O}_{N,q}$ is a normal ring then the Lemma 1.12 of [64] implies that the ideal $\pi^*(\rho((JM_z(X))_D))$ is integrally closed, i.e., $\overline{\pi^*(\rho((JM_z(X))_D))} = \pi^*(\rho((JM_z(X))_D))$. So, it is enough to prove that $\tilde{w}_i \in \overline{\pi^*(\rho((JM_z(X))_D))}$, for all $i \in \{1, \dots, n\}$. Let $i \in \{1, \dots, n\}$ be arbitrary. We use the curve criterion. Let $\tilde{\phi} : (\mathbb{C}, 0) \rightarrow (N, q)$ be an analytic curve. We can choose $\tilde{\phi}$ such that $\phi : (\mathbb{C}, 0) \rightarrow (X \times_Y X \times \mathbb{P}^{2p-1}, \pi(q))$ given by $\phi := \pi \circ \tilde{\phi}$ meets the dense Zariski open subset U_1 , $\phi = (\phi_1, \phi_2, \psi)$ and $\psi = \left[1, \frac{\psi_2}{\psi_1}, \dots, \frac{\psi_{2p}}{\psi_1} \right]$. Further, $\tilde{\phi}$ can be chosen such that $\tilde{\phi}$ is transverse to the component so that $u' \circ \tilde{\phi} = t$, where t is the generator of the maximal ideal of $\mathcal{O}_{\mathbb{C},0}$. Hence, the pullback of the ideal $\pi^*(\rho((JM_z(X))_D))$ is generated by t^r . Consider the element

$$\hat{w}_i := - \begin{bmatrix} 1 & \frac{S_2}{S_1} & \dots & \frac{S_{2p}}{S_1} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \left(\frac{\partial F_1}{\partial z_i} \circ \bar{\pi}_2 \right) (z_i \circ \bar{\pi}_1 - z_i \circ \bar{\pi}_2) \\ \vdots \\ \left(\frac{\partial F_p}{\partial z_i} \circ \bar{\pi}_2 \right) (z_i \circ \bar{\pi}_1 - z_i \circ \bar{\pi}_2) \end{bmatrix}.$$

Notice that $\hat{w}_i = -\pi^*(\rho(\underbrace{(0, (z_i \circ \pi_1 - z_i \circ \pi_2)(\frac{\partial F}{\partial z_i} \circ \pi_2))}_{\in (JM_z(X)_D)})) \in \pi^*(\rho((JM_z(X))_D))$. Since y' and u'

are independent coordinates then the order of $\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} - \frac{\partial(z_i \circ \bar{\pi}_2)}{\partial y'}$ in u' is the same as the order of $z_i \circ \bar{\pi}_1 - z_i \circ \bar{\pi}_2$ in u' . Then the pullback of both have the same order in t , so there exists an invertible element $\alpha_i \in \mathcal{O}_{\mathbb{C},0}$ such that

$$\tilde{\phi}^*\left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} - \frac{\partial(z_i \circ \bar{\pi}_2)}{\partial y'}\right) = \alpha_i(\tilde{\phi}^*(z_i \circ \bar{\pi}_1 - z_i \circ \bar{\pi}_2)).$$

Hence, $\tilde{\phi}^*(\tilde{w}_i) = -\sum_{j=1}^p \left(\frac{\partial F_j}{\partial z_i} \circ \phi_2\right) \left(\tilde{\phi}^*\left(\frac{\partial(z_i \circ \bar{\pi}_1)}{\partial y'} - \frac{\partial(z_i \circ \bar{\pi}_2)}{\partial y'}\right)\right) \frac{\psi_{p+j}}{\psi_1}$
 $= \alpha_i \tilde{\phi}^*(\hat{w}_i) \in \tilde{\phi}^*(\pi^*(\rho((JM_z(X))_D)))$.

Therefore, $\tilde{w}_i \in \overline{\pi^*(\rho((JM_z(X))_D))}$, for all $i \in \{1, \dots, n\}$ which finishes the proof. $\square$

2.3 The genericity theorem applied in a family of hyperplane sections

Given X an analytic variety with isolated singularity at the origin, we can consider the sections of X by hyperplanes. One natural question is if there exists a generic set of hyperplanes for which the family of hyperplanes sections satisfies the infinitesimal Lipschitz condition A. We will show this is true. First, we recall some important notions in order to make precise statements. For more details see [44].

Let us work on the Grassmanian modification of $X = f^{-1}(0)$, defined by an analytic map $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$, X with isolated singularity at the origin, $n \geq p$.

For each $y = [y_1, \dots, y_n] \in \mathbb{P}^{n-1}$, consider the hyperplane on $\mathbb{C}^n$ given by

$$H_y := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid z \cdot y := \sum_{i=1}^n z_i y_i = 0\}.$$

Let E_{n-1} be the canonical bundle over $\mathbb{P}^{n-1}$, i.e.,

$$E_{n-1} := \{(z, y) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid z \in H_y\}.$$

Consider the projection map $\beta : E_{n-1} \rightarrow \mathbb{C}^n$. We call $\tilde{X} := \beta^{-1}(X)$ the **(n-1)-Grassmanian modification of X**. Here we will simply refer to the $(n-1)$ -modification as the Grassmanian modification of X . We can see $\mathbb{P}^{n-1}$ embedded into E_{n-1} as the zero section of the bundle E_{n-1} , which allows us to think of $0 \times \mathbb{P}^{n-1}$ as a stratum of $\tilde{X}$. Note that the projection to $0 \times \mathbb{P}^{n-1}$ makes $\tilde{X}$ a family of analytic sets with $0 \times \mathbb{P}^{n-1}$ as the parameter space, which we denote by Y . The members of this family are just $\{H_y \cap X\}$ as y varies in $\mathbb{P}^{n-1}$.

$$\begin{array}{ccc}
 & \tilde{X} & \\
 \beta \downarrow \tilde{X} & & \\
 X & & Y
 \end{array}
 \quad \hookrightarrow \quad
 \begin{array}{ccc}
 & E_{n-1} & \\
 \beta \downarrow & & \\
 \mathbb{C}^n & & Y
 \end{array}$$

Consider the chart

$$U_n := \{[y_1, \dots, y_n] \in \mathbb{P}^{n-1} \mid y_n \neq 0\} = \{[y_1, \dots, y_{n-1}, -1] \mid (y_1, \dots, y_{n-1}) \in \mathbb{C}^{n-1}\} \cong \mathbb{C}^{n-1}$$

which is a dense Zariski open subset of $\mathbb{P}^{n-1}$. Working on the dense Zariski open subset $E_{n-1} \cap (\mathbb{C}^n \times U_n)$ of E_{n-1} , we have local coordinates given by $(z_1, \dots, z_n, y_1, \dots, y_{n-1})$. In these coordinates, the projection map β satisfies the equation

$$\beta(z_1, \dots, z_n, y_1, \dots, y_{n-1}) = (z_1, \dots, z_{n-1}, \sum_{i=1}^{n-1} y_i z_i).$$

Consider the analytic map

$$\begin{aligned}
 F : E_{n-1} \cap (\mathbb{C}^n \times U_n) &\rightarrow \mathbb{C}^p \\
 (z, y) &\mapsto f \circ \beta(z, y)
 \end{aligned}$$

Thus, $F^{-1}(0) = \beta^{-1}(f^{-1}(0)) = \beta^{-1}(X) = \tilde{X}$, hence $\tilde{X}$ is defined by F . For each $y = (y_1, \dots, y_{n-1}) \equiv [y_1, \dots, y_{n-1}, -1] \in U_n$, let $F_y : \mathbb{C}^n \rightarrow \mathbb{C}^p$ given by $F_y(z) := F(z, y)$ and let $\tilde{X}_y := F^{-1}(0)$. In these coordinates, clearly $\tilde{X}_y = (f^{-1}(0)) \cap H_y = X \cap H_y$. Therefore, F defines the family of sections of X by the hyperplanes H_y , as y varies on the dense Zariski open subset U_n of $\mathbb{P}^{n-1}$.

The next result generalizes the Theorem 4.4 of [39].

Theorem 2.3.1. *Suppose $(X, 0)$ is a germ of an isolated singularity analytic variety defined by an analytic map-germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$, $n \geq p$, and consider the Grassmanian modification $\tilde{X}$ of X . Then, there exists a non-empty Zariski open subset U of $\mathbb{P}^{n-1}$, such that the iL_A condition holds for the pair $(\tilde{X} - U, U)$ along U .*

Proof. As we have seen, $\tilde{X}$ is a family defined by the above analytic map F .

Let us prove that $\tilde{X}_y$ has isolated singularity at $(0, y)$ for all y varying in a non-empty Zariski open subset U' of U_n . In fact, we already know that the set of limiting tangent hyperplanes of X at the origin is a Zariski proper closed subset of $\mathbb{P}^{n-1}$. Call this set W . Let $U' := U_n - (W \cap U_n)$. Since $\mathbb{P}^{n-1}$ is irreducible then U_n is irreducible. Since U_n is a dense subset of $\mathbb{P}^{n-1}$ then $W \cap U_n$ also is a proper Zariski closed subset of U_n , hence U' is a dense Zariski open subset of U_n . Let $y \in U'$. We want to show that $(0, y)$ is an isolated singularity of $\tilde{X}_y$. By hypothesis, H_y is not a limiting tangent hyperplane of X at the origin, and by Lemma 4.1 (a) of [48] we have that $\overline{JM(X)_{H_y}} = \overline{JM(X)}$ at the origin, where $JM(X)_{H_y} := \{\frac{\partial f}{\partial v} \mid v \in H_y\}$. Thus, in a neighborhood of the origin, the generic rank of $JM(X)$ and $JM(X)_{H_y}$ is the same. Thus, if we take

z in this neighborhood, such that $z \in H_y$, $z \neq 0$ then the generic rank of $JM(\tilde{X}_y) = JM(X \cap H_y)$ at z is the generic rank of $JM(X)_{H_y}$ at z , which is the generic rank of $JM(X)$ at z . Since $z \neq 0$ and X has isolated singularity at the origin then we can choose this neighborhood so that z is a non-singular point of X , which implies that z is not a singular point of $\tilde{X}_y$. Therefore, $\tilde{X}_y$ has isolated singularity at the origin, for all $y \in U'$.

Now, the existence of U follows from the Theorem 2.2.11. $\square$

Let us go back to the discussion before the last theorem. We have seen that $\tilde{X}$ is defined by the map $F : E_{n-1} \cap (\mathbb{C}^n \times U_n) \rightarrow \mathbb{C}^p$ given by $F(z, y) = f \circ \beta(z, y)$. From the chain rule we have that

$$[dF_{(z,y)}] = [df_{\beta(z,y)}] \cdot \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ y_1 & y_2 & \dots & y_{n-1} & 0 & z_1 & \dots & z_{n-1} \end{bmatrix}$$

which implies that $\frac{\partial F}{\partial y_i} = z_i \left(\frac{\partial f}{\partial z_n} \circ \beta \right)$ and $\frac{\partial F}{\partial z_i} = \frac{\partial f}{\partial z_i} \circ \beta + \sum_{j=1}^{n-1} y_j \left(\frac{\partial f}{\partial z_n} \circ \beta \right)$, for all $i \in \{1, \dots, n-1\}$, and $\frac{\partial F}{\partial z_n} = 0$. Thus, we have immediately the next result, which is a generalization of the Corollary 4.5 of [39].

Corollary 2.3.2. *The point $(0, P) \in E_{n-1} \cap (\mathbb{C}^n \times U_n)$ belongs to the Zariski open subset of the last theorem if and only if $\left(z_i \left(\frac{\partial f}{\partial z_n} \circ \beta \right) \right)_D \in \overline{(JM_z(\tilde{X}))_D}$ at $(0, P)$, for all $i \in \{1, \dots, n-1\}$.*

The corollary tell us that in order to check if a hyperplane is generic, it is enough to verify that for all curves ϕ_1, ϕ_2 , tangent to P at the origin with lifts $\tilde{\phi}_1, \tilde{\phi}_2$, respectively, with $\phi := (\phi_1, \phi_2)$ and $\tilde{\phi} := (\tilde{\phi}_1, \tilde{\phi}_2)$, we have that $\left(z_i \left(\frac{\partial f}{\partial z_n} \circ \beta \right) \right) \circ \phi$ is an $\mathcal{O}_{\mathbb{C},0}$ -linear combination of $\left\{ \left(\frac{\partial f}{\partial z_s} \circ \phi \right) + \left(\sum_{j=1}^{n-1} y_j \left(\frac{\partial f}{\partial z_n} \circ \beta \right) \right) \circ \tilde{\phi} \right\}_{s=1}^{n-1}$, for all $i \in \{1, \dots, n-1\}$.

In [39] Gaffney gave a description of these generic hyperplanes using analytic invariants in the jacobian ideal case. Now we generalize this description for the jacobian module case. For the rest of this section we assume that the hyperplanes H_y are not limiting tangent hyperplanes of $(X, 0)$. As we have seen, this implies that $\tilde{X}_y = X \cap H_y$ has isolated singularity at the origin and $\overline{JM(X)_{H_y}} = \overline{JM(X)}$ at the origin.

The invariants that we use here appeared earlier in section 2.1. Since $\tilde{X}_y$ has isolated singularity at the origin then by Corollary 2.1.27 the multiplicity of the pair $e((JM(\tilde{X}_y))_D, \overline{(JM(\tilde{X}_y))_D})$ is well defined.

These invariants have been used in this context before. As pointed out by Gaffney in [39], for ICIS singularities, and more generally in [43], to check for whether or not a hyperplane is in

the generic set of hyperplanes for which the hyperplane sections form a Whitney equisingular family, you can use the multiplicity of the pair $(JM(\tilde{X}_y), \mathcal{O}_{\tilde{X}_y}^p)$, which is the Buchsbaum-Rim multiplicity $e(JM(\tilde{X}_y))$.

The hyperplane is generic if this multiplicity is minimal, and the minimal number is the sum of the Milnor number of $X \cap H_y$ and $X \cap H_y \cap H_{y'}$, where H_y and $H_{y'}$ are generic hyperplanes.

The proof that the minimal value of $e((JM(\tilde{X}_y))_D, \overline{(JM(\tilde{X}_y))}_D)$ identifies generic hyperplanes will be done using the Multiplicity Polar Theorem (see Corollary 1.4 [45]). Now we identify the modules we will use.

We will work on the fibered product $\tilde{X} \times_{\mathbb{P}^{n-1}} \tilde{X} \subseteq X \times \mathbb{P}^{n-1} \times X$. Let $N := (\beta^*(\overline{JM(X)}))_D$ and $M := (JM_z(\tilde{X}))_D$, considering $\tilde{X}$ defined by the analytic map $F : E_{n-1} \cap (\mathbb{C}^n \times U_n) \rightarrow \mathbb{C}^p$, given by $F(z, y) = f \circ \beta(z, y)$. Clearly M restricted to the fiber of the family $\tilde{X}$ over the hyperplane H_y is just $(JM(X \cap H_y))_D$ and N restricted to H_y is $(\overline{JM(X)}|_{H_y})_D$. Further, since we are assuming that H_y is not a limiting tangent hyperplane of $(X, 0)$ then $\overline{JM(X)}|_{H_y} = \overline{JM(X)_{H_y}}$, hence N restricted to H_y is $(\overline{JM(\tilde{X}_y)})_D$. Therefore, the multiplicity of the pair $(M|_{H_y}, N|_{H_y})$ is the same as $e((JM(\tilde{X}_y))_D, \overline{(JM(\tilde{X}_y))}_D)$.

The next result is a generalization of the Theorem 4.6 of [39].

Theorem 2.3.3. *Under the above notations, let U be the set of hyperplanes which are not limiting tangent hyperplanes of $(X, 0)$. Suppose that N has no polar variety with the same codimension of U . Then:*

a) *The map*

$$\begin{aligned} U &\rightarrow \mathbb{Z} \\ H_y &\mapsto e((JM(\tilde{X}_y))_D, \overline{(JM(\tilde{X}_y))}_D) \end{aligned}$$

is upper semicontinuous on U ;

b) *The iL_A condition holds along U at a hyperplane H_y for which the value of $e((JM(\tilde{X}_y))_D, \overline{(JM(\tilde{X}_y))}_D)$ is minimal.*

Proof. (a) By the definition of U , $(\overline{JM(\tilde{X}_y)})_D$ is the restriction of N to the fiber $\tilde{X}_y = X \cap H_y$. Since N has no polar variety with the same codimension of U then the multiplicity polar theorem implies that $e((JM(\tilde{X}_y))_D, \overline{(JM(\tilde{X}_y))}_D)$ is upper semicontinuous.

(b) Suppose $H_y \in U$ gives the minimal value of the multiplicity. Since this value already is minimal then it cannot go down, hence it must be constant. This implies that the polar variety of M of the same codimension as U is empty, which puts restrictions on the size of the fiber of $Proj(\mathcal{R}(M))$. We already know that $\{\frac{\partial F}{\partial y_i}\}_{i=1}^{n-1}$ are in $\overline{M}$ generically. Since the dimension of the fiber of $Proj(\mathcal{R}(M))$ is bounded then by Theorem A1 of [62] we have that $\{\frac{\partial F}{\partial y_i}\}_{i=1}^{n-1}$ are in $\overline{M}$ at H_y , which finishes the proof. $\square$

In order to check the hypothesis over N , we have to check if the fiber dimension of $\text{Proj}(\mathcal{R}(N))$ is small enough so that N has no polar variety of the same codimension of U .

THE LIPSCHITZ SATURATION OF A MODULE

Motivated by some properties that the Lipschitz saturation has on the ideal case, we define some kinds of Lipschitz saturation for modules.

3.1 The Lipschitz saturation of a module and basic properties

Let $X \subseteq \mathbb{C}^n$ be an analytic set and let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$ generated by global sections $\{g_1, \dots, g_r\}$. Consider the ideal sheaf $\rho(\mathcal{M})$ on $X \times \mathbb{P}^{p-1}$ induced by $\mathcal{M}$. Then $\rho(\mathcal{M})$ is generated by $\{\rho(g_1), \dots, \rho(g_r)\}$, so we can consider the blowup $B_{\rho(\mathcal{M})}(X \times \mathbb{P}^{p-1}) \subseteq (X \times \mathbb{P}^{p-1}) \times \mathbb{P}^{r-1}$ and the blowup map

$$\pi : B_{\rho(\mathcal{M})}(X \times \mathbb{P}^{p-1}) \rightarrow X \times \mathbb{P}^{p-1}.$$

Let $T_1, \dots, T_p$ and $S_1, \dots, S_r$ be the homogeneous coordinates of $\mathbb{P}^{p-1}$ and $\mathbb{P}^{r-1}$, respectively.

Definition 3.1.1. *Let $x \in X$. The **1-Lipschitz saturation** of $\mathcal{M}$ at $\mathbf{x}$ is denoted by $(\mathcal{M}_{S_1})_x$ and is defined by*

$$(\mathcal{M}_{S_1})_x := \{h \in \mathcal{O}_{X,x}^p \mid \rho(h) \in (\rho(\mathcal{M}))_S \text{ at all } (x, [t_1, \dots, t_p]) \in V(\rho(\mathcal{M}))\}.$$

*In the family case, the **1-Lipschitz saturation** of $\mathcal{M}$ at $\mathbf{x} \in \mathbf{X}$ relative to $\mathbf{Y}$ is defined as above taking the Lipschitz saturation of $\rho(\mathcal{M})$ relative to Y .*

Unless we say the opposite, we denote by M_{S_1} the stalk of the sheaf $\mathcal{M}_{S_1}$ at an arbitrary point of X .

Write $h = (h_1, \dots, h_p)$ and each generator $g_j = (g_{1j}, \dots, g_{pj})$. Suppose $t_1 \neq 0$ and take $(x, [t_1, \dots, t_p], [s_1, \dots, s_r]) \in \pi^{-1}((x, [t_1, \dots, t_p]))$, with $s_1 \neq 0$. Then, $\pi^*(\rho(\mathcal{M}))$ is locally generated by $\rho(g_1) \circ \pi$, and $\rho(h) \in (\rho(\mathcal{M}))_S$ at $(x, [t_1, \dots, t_p])$ if and only if $\frac{\rho(h) \circ \pi}{\rho(g_1) \circ \pi}$ is Lipschitz with respect to the system of coordinates $z_1, \dots, z_n, \frac{T_1}{T_1}, \dots, \frac{T_p}{T_1}, \frac{S_1}{S_1}, \dots, \frac{S_r}{S_1}$.

Remark 3.1.2. $\mathcal{M}_{S_1}$ is the Lipschitz saturation of an ideal if $p = 1$. In fact, in this case $\rho(h) = h$, $\forall h \in \mathcal{O}_{X,x}^p$ and $\rho(\mathcal{M}) = \mathcal{M}$, and the condition $\rho(h) \in (\rho(\mathcal{M}))_S$ at all $(x, [1]) \in V(\rho(\mathcal{M}))$ is equivalent to $h \in \mathcal{M}_S$ at x . Hence, $\mathcal{M}_{S_1} = \mathcal{M}_S$ at x if $\mathcal{M}$ is an ideal sheaf of $\mathcal{O}_X$.

Proposition 3.1.3. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$. Then:

- a) $\mathcal{M}_{S_1}$ is an $\mathcal{O}_X$ -submodule of $\mathcal{O}_X^p$;
- b) $M \subseteq M_{S_1} \subseteq \overline{M}$. In particular, M is a reduction of M_{S_1} and, if M has finite colength in M_{S_1} then $e(M, M_{S_1}) = 0$.

Proof. Let $x \in X$ be an arbitrary point.

(a) Let $h, h' \in \mathcal{M}_{S_1}$ at x and $\alpha \in \mathcal{O}_{X,x}$. So, in every point $(x, [t_1, \dots, t_p]) \in V(\rho(\mathcal{M}))$ we have $\rho(\alpha h + h') = \alpha \rho(h) + \rho(h') \in (\rho(\mathcal{M}))_S$. So, $\alpha h + h' \in \mathcal{M}_{S_1}$ at x , which finishes the proof of (a).

(b) It is pretty obvious that $M \subseteq M_{S_1}$. Let us to prove that $M_{S_1} \subseteq \overline{M}$.

Let $h \in M_{S_1}$ at x . So $\rho(h) \in (\rho(M))_S$ in every point $(x, [t_1, \dots, t_p]) \in V(\rho(M))$. By Proposition 2.1 of [40] we have $(\rho(M))_S \subseteq \overline{\rho(M)}$, hence $\rho(h) \in \overline{\rho(M)}$ in every point $(x, [t_1, \dots, t_p]) \in V(\rho(M))$. By Proposition 3.4 of [48] we conclude that $h \in \overline{M}$ at x . Therefore, $M_{S_1} \subseteq \overline{M}$. $\square$

Definition 3.1.4. Consider the Setup 2.2.1. We define the infinitesimal Lipschitz conditions with respect to the 1-Lipschitz saturation of a module.

- The pair (X, Y) satisfies the **1-infinitesimal Lipschitz condition $\mathbf{m}_Y$ ($1\text{-iL}_{\mathbf{m}_Y}$)** at $\mathbf{x} \in \mathbf{X}$ if $JM(X)_Y \subseteq (m_Y JM_z(X))_{S_1}$ at x ;
- The pair (X, Y) satisfies the **1-infinitesimal Lipschitz condition A (1-iL_A)** at $\mathbf{x} \in \mathbf{X}$ if $JM(X)_Y \subseteq (JM_z(X))_{S_1}$ at x .

Here we consider the 1-Lipschitz saturation relative to the parameter space Y . Notice that $1\text{-iL}_{\mathbf{m}_Y}$ implies 1-iL_A .

Corollary 3.1.5. Consider the Setup 2.2.1. If $x = (z, y) \in X$ with $z \neq 0$ then $1\text{-iL}_{\mathbf{m}_Y}$ and 1-iL_A hold at x .

Proof. By Lemma 2.2.8 we have that $JM(X)_Y \subseteq m_Y JM_z(X)$ at x . By Proposition 3.1.3 the inclusion $m_Y JM_z(X) \subseteq (m_Y JM_z(X))_{S_1}$ holds and therefore the condition $1\text{-iL}_{\mathbf{m}_Y}$ holds at x . In particular, 1-iL_A holds at x . $\square$

Thus, it remains to investigate the 1-infinitesimal Lipschitz conditions along the parameter space $Y \equiv 0 \times Y$.

Definition 3.1.6. *The 2-Lipschitz saturation of the submodule $M \subseteq \mathcal{O}_{X,x}^p$ is denoted by M_{S_2} , and is defined by*

$$M_{S_2} := \{h \in \mathcal{O}_{X,x}^p \mid h_D \in \overline{M_D} \text{ at } (x, x)\}.$$

In the family case, the 2-Lipschitz saturation of M at $x \in X$ relative to Y is defined as above taking the double relative to Y .

If $\mathcal{M}$ is an $\mathcal{O}_X$ -submodule of $\mathcal{O}_X^p$ then we can consider the sheaf of $\mathcal{O}_X$ -modules $\mathcal{M}_{S_2}$ where each stalk $(\mathcal{M}_{S_2})_x$ is $(\mathcal{M}_x)_{S_2}$.

Unless we say the opposite, we denote by M_{S_2} the stalk of the sheaf $\mathcal{M}_{S_2}$ at an arbitrary point of X .

Remark 3.1.7. M_{S_2} is the Lipschitz saturation of an ideal if $p = 1$. In fact, by Theorem 1.4.2 we have that $h \in M_{S_2}$ if and only if $h_D \in \overline{M_D}$ if and only if $h \in M_S$. Therefore, $M_{S_2} = M_S$ if M is an ideal of $\mathcal{O}_{X,x}^p$.

Proposition 3.1.8. *Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$. Then:*

- a) $\mathcal{M}_{S_2}$ is an $\mathcal{O}_X$ -submodule of $\mathcal{O}_X^p$;
- b) $M \subseteq M_{S_2} \subseteq \overline{M}$. In particular, M is a reduction of M_{S_2} and, if M has finite colength in M_{S_2} then $e(M, M_{S_2}) = 0$.

The same result holds in the family case.

Proof. Let $x \in X$ be an arbitrary point.

(a) Let $\alpha \in \mathcal{O}_{X,x}$ and $h, h' \in \mathcal{M}_{S_2}$ at x .

Claim: $(0, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h \circ \pi_2)) \in \overline{M_D}$ at (x, x) . In fact, let $\phi : (\mathbb{C}, 0) \rightarrow (X \times X, (x, x))$ be an arbitrary curve. Since $h_D \in \overline{M_D}$ then by Proposition 2.1.12 (a) we have that $h \in \overline{M}$ at x . If we write $\phi = (\phi_1, \phi_2)$, we have that $h \circ \phi_2 \in M \circ \phi_2$, so we can write $h \circ \phi_2 = \sum \alpha_j (h_j \circ \phi_2)$, with $\alpha_j \in \mathcal{O}_{\mathbb{C},0}$ and $h_j \in M$. By the Lemma 2.1.2 (b) we have that $(0, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h_j \circ \pi_2)) \in M_D$, for all j . Thus, we get: $(0, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h \circ \pi_2)) \circ \phi = (0, (\alpha \circ \phi_1 - \alpha \circ \phi_2)(h \circ \phi_2)) = (0, (\alpha \circ \phi_1 - \alpha \circ \phi_2)(\sum \alpha_j (h_j \circ \phi_2))) = \sum \alpha_j \underbrace{(0, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h_j \circ \pi_2)) \circ \phi}_{\in M_D} \in M_D \circ \phi$, which proves the claim.

Finally, we have that $h_D, h'_D \in \overline{M_D}$ at (x, x) , hence: $(\alpha h + h')_D = (\alpha \circ \pi_1)h_D + (0, (\alpha \circ \pi_1 - \alpha \circ \pi_2)(h \circ \pi_2)) + h'_D$, and each factor on the right hand side belongs to $\overline{M_D}$ at (x, x) , so the proof of (a) is done.

(b) If $h \in \mathcal{M}$ at x then $h_D \in \mathcal{M}_D \subseteq \overline{\mathcal{M}_D}$ at (x, x) , so $h \in \mathcal{M}_{S_2}$ at x . Therefore, $M \subseteq M_{S_2}$.

Furthermore, if $h \in \mathcal{M}_{S_2}$ at x then $h_D \in \overline{\mathcal{M}_D}$ at (x, x) , and by Proposition 2.1.12 (a) we have that $h \in \overline{\mathcal{M}}$ at x . Therefore, $M_{S_2} \subseteq \overline{M}$. $\square$

Definition 3.1.9. Consider the Setup 2.2.1. We define the infinitesimal Lipschitz conditions with respect to the 2-Lipschitz saturation of a module.

- The pair (X, Y) satisfies the **2-infinitesimal Lipschitz condition $\mathbf{m}_Y$ ($2\text{-iL}_{\mathbf{m}_Y}$)** at $\mathbf{x} \in \mathbf{X}$ if $JM(X)_Y \subseteq (m_Y JM_z(X))_{S_2}$ at x ;
- The pair (X, Y) satisfies the **2-infinitesimal Lipschitz condition A (2-iL_A)** at $\mathbf{x} \in \mathbf{X}$ if $JM(X)_Y \subseteq (JM_z(X))_{S_2}$ at x .

Here we consider the 2-Lipschitz saturation relative to the parameter space Y . Notice that $2\text{-iL}_{\mathbf{m}_Y}$ implies 2-iL_A .

Clearly the above definition is equivalent to the Definition 2.2.2.

Corollary 3.1.10. Consider the Setup 2.2.1. If $x = (z, y) \in X$ with $z \neq 0$ then $2\text{-iL}_{\mathbf{m}_Y}$ and 2-iL_A hold at x . Furthermore, there exists a dense Zariski open subset U of Y such the 2-iL_A holds in $(0, y)$, for all $y \in U$.

Proof. By Lemma 2.2.8 we have that $JM(X)_Y \subseteq m_Y JM_z(X)$ at x . By Proposition 3.1.8 the inclusion $m_Y JM_z(X) \subseteq (m_Y JM_z(X))_{S_2}$ holds and therefore the condition $2\text{-iL}_{\mathbf{m}_Y}$ holds at x . In particular, 2-iL_A holds at x .

The second statement is a consequence of the Theorem 2.2.11. $\square$

Before to define the third Lipschitz saturation, Let us fix some notations.

For each $\psi : X \rightarrow \text{Hom}(\mathbb{C}^p, \mathbb{C})$ analytic map, $\psi = (\psi_1, \dots, \psi_p)$ and $h = (h_1, \dots, h_p) \in \mathcal{O}_X^p$, we define $\psi \cdot h \in \mathcal{O}_X$ given by $(\psi \cdot h)(z) := \sum_{i=1}^p \psi_i(z) h_i(z)$. We define $\psi \cdot M$ as the ideal of $\mathcal{O}_X$ generated $\{\psi \cdot h \mid h \in M\}$.

Lemma 3.1.11. Under the above notation, we have the following properties:

- $\psi \cdot (\alpha g + h) = \alpha(\psi \cdot g) + (\psi \cdot h)$, $\forall g, h \in \mathcal{O}_X^p$ and $\alpha \in \mathcal{O}_X$.
- If M is generated by $\{h_1, \dots, h_r\}$ then $\psi \cdot M$ is generated by $\{\psi \cdot h_1, \dots, \psi \cdot h_r\}$.

Proof. It is easy to see that (b) is a straightforward consequence of (a). Now, write $g = (g_1, \dots, g_p)$ and $h = (h_1, \dots, h_p)$. Then, for every z we have $(\psi \cdot (\alpha g + h))(z) = \sum_{i=1}^p \psi_i(z) (\alpha(z) g_i(z) + h_i(z)) = \alpha(z) \sum_{i=1}^p \psi_i(z) g_i(z) + \sum_{i=1}^p \psi_i(z) h_i(z) = (\alpha(\psi \cdot g) + (\psi \cdot h))(z)$, so (a) is proved. $\square$

Lemma 3.1.12. *Suppose the submodule $M \subseteq \mathcal{O}_{X,x}^p$ has generic rank k on each component of X . If $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ are indexes with $j_1 = 1$ then there exists $\psi : X \rightarrow \text{Hom}(\mathbb{C}^p, \mathbb{C})$ such that:*

$$a) \quad \psi \cdot h = J_{IJ}(h, M), \quad \forall h \in \mathcal{O}_{X,x}^p;$$

$$b) \quad \psi \cdot M \subseteq J_k(M).$$

Proof. Let us fix a matrix of generators of M , $[M] = \begin{bmatrix} | & & | \\ g_1 & \dots & g_r \\ | & & | \end{bmatrix}$. We have $J_{IJ}(h, M) = \det[h, M]_{IJ}$, where $[h, M]_{IJ} = \begin{bmatrix} h_{i_1} & g_{i_1, j_2-1} & \dots & g_{i_1, j_k-1} \\ \vdots & \vdots & & \vdots \\ h_{i_k} & g_{i_k, j_2-1} & \dots & g_{i_k, j_k-1} \end{bmatrix}$, for all $h = (h_1, \dots, h_p)$. Let G_{i_l, j_s-1} be the (l, s) -cofactor of $[h, M]_{IJ}$, for all $l, s \in \{1, \dots, k\}$. Notice that the $(l, 1)$ -cofactors $G_{i_l, 0}$ does not depend of h . Then, $J_{IJ}(h, M) = \det[h, M]_{IJ} = \sum_{l=1}^k G_{i_l, 0} \cdot h_{i_l}$, for all h (taking the cofactor expansion at the first column, since $j_1 - 1 = 0$). Take $\psi : X \rightarrow \text{Hom}(\mathbb{C}^p, \mathbb{C})$ given by $(\psi_1, \dots, \psi_p)$ where $\psi_{i_l} = G_{i_l, 0}$, for all $l \in \{1, \dots, k\}$, and $\psi_j = 0$, for every index j off I . Thus, for all $h \in \mathcal{O}_{X,x}^p$ we get $J_{IJ}(h, M) = \psi_I \cdot h_I = \psi \cdot h$. So, (a) is proved.

Now, let $g \in M$ arbitrary. Then $(g, M) = M$. By (a) we have $\psi \cdot g = J_{I,J}(g, M) \in J_k((g, M)) = J_k(M)$, hence $\psi \cdot M \subseteq J_k(M)$. $\square$

The Propositions 1.2.5 and 1.2.11 are characterizations for the integral closure of modules using integral closure of ideals. Notice the next proposition is another characterization of the same type.

Proposition 3.1.13. *Let $h \in \mathcal{O}_{X,x}^p$ and suppose that the submodule $M \subseteq \mathcal{O}_{X,x}^p$ has generic rank k on each component of X . Then, $h \in \overline{M}$ at x if and only if $\psi \cdot h \in \overline{\psi \cdot M}$ at x , for every $\psi : X \rightarrow \text{Hom}(\mathbb{C}^p, \mathbb{C})$ analytic map.*

Proof. ($\implies$) Let $\phi : (\mathbb{C}, 0) \rightarrow (X, x)$ be an arbitrary curve. Since $h \in \overline{M}$ then $h \circ \phi \in \phi^*(M)$, so $(\psi \circ \phi)(h \circ \phi) \in (\psi \circ \phi)\phi^*(M) \Rightarrow (\psi \cdot h) \circ \phi \in \phi^*(\psi \cdot M)$. Hence, $\psi \cdot h \in \overline{\psi \cdot M}$.

($\impliedby$) The proof use 1.2.5 and 1.2.6.

By these results it is enough to check that $J_{IJ}(h, M) \in \overline{J_k(M)}$, for all indexes I and J .

Write $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$. Let $[M] = \begin{bmatrix} | & & | \\ g_1 & \dots & g_r \\ | & & | \end{bmatrix}$ be a matrix of generators of

M . Write $h = (h_1, \dots, h_p)$. Then, $[h, M] = \begin{bmatrix} | & | & & | \\ h & g_1 & \dots & g_r \\ | & | & & | \end{bmatrix}$. If $j_1 > 1$ then $J_{IJ}(h, M)$ is a $k \times k$ minor taken only among the generators of M , hence $J_{IJ}(h, M) \in J_k(M) \subseteq \overline{J_k(M)}$.

Now suppose that $j_1 = 1$. By the Lemma 3.1.12 there exists $\psi : X \rightarrow \text{Hom}(\mathbb{C}^p, \mathbb{C})$ such that $\psi \cdot h = J_{I,J}(h, M)$ and $\psi \cdot M \subseteq J_k(M)$. By the hypothesis we have $\psi \cdot h \in \overline{\psi \cdot M}$, hence $J_{I,J}(h, M) = \psi \cdot h \in \overline{\psi \cdot M} \subseteq \overline{J_k(M)}$, which finishes the proof. $\square$

The last proposition will be useful to prove that the 3-Lipschitz Saturation satisfies the same statement of the Propositions 3.1.3 and 3.1.8.

Definition 3.1.14. The **3-Lipschitz saturation** of the submodule $M \subseteq \mathcal{O}_{X,x}^p$ is denoted by M_{S_3} , and is defined by

$$M_{S_3} := \{h \in \mathcal{O}_{X,x}^p \mid \psi \cdot h \in (\psi \cdot M)_S, \forall \psi : X \rightarrow \text{Hom}(\mathbb{C}^p, \mathbb{C}) \text{ analytic map}\}.$$

In the family case, the **3-Lipschitz saturation of M at $x \in X$ relative to Y** is defined as above taking the Lipschitz saturation of $\psi \cdot M$ relative to Y .

If $\mathcal{M}$ is an $\mathcal{O}_X$ -submodule of $\mathcal{O}_X^p$ then we can consider the sheaf of $\mathcal{O}_X$ -modules $\mathcal{M}_{S_3}$ where each stalk $(\mathcal{M}_{S_3})_x$ is $(\mathcal{M}_x)_{S_3}$.

Unless we say the opposite, we denote by M_{S_3} the stalk of the sheaf $\mathcal{M}_{S_3}$ at an arbitrary point of X .

Remark 3.1.15. M_{S_3} is the Lipschitz saturation of an ideal if $p = 1$.

First, let us prove that $\alpha \cdot M_S \subseteq (\alpha \cdot M)_S$, $\forall \alpha \in \mathcal{O}_{X,x}$. In fact consider the canonical projection $\pi : SB_M(X) \rightarrow X$. We have a canonical projection $B_{\alpha \cdot M}(X) = B_{(\alpha)}(B_M(X)) \rightarrow B_M(X)$ which gives rise a canonical projection $\pi_\alpha : SB_{\alpha \cdot M}(X) \rightarrow SB_M(X)$ and $\pi \circ \pi_\alpha : SB_{\alpha \cdot M}(X) \rightarrow X$ is the canonical projection for the Lipschitz saturation of the blowup of X with respect to the ideal $\alpha \cdot M$. We already know that

$$M_S = \{h \in \mathcal{O}_{X,x} \mid \pi^*(h) \in \pi^*(M)\} \text{ and} \\ (\alpha \cdot M)_S = \{h \in \mathcal{O}_{X,x} \mid (\pi \circ \pi_\alpha)^*(h) \in (\pi \circ \pi_\alpha)^*(\alpha \cdot M)\}.$$

Thus, given $h \in M_S$ we have that $\pi^*(h) \in \pi^*(M)$ which implies that $(\pi \circ \pi_\alpha)^*(\alpha h) = \pi_\alpha^* \circ \pi^*(\alpha h) = \pi_\alpha^*(\pi^*(\alpha)) \cdot \pi_\alpha^*(\pi^*(h)) \in (\pi \circ \pi_\alpha)^*(\alpha) \cdot (\pi \circ \pi_\alpha)^*(M) \subseteq (\pi \circ \pi_\alpha)^*(\alpha \cdot M)$. Hence $\alpha h \in (\alpha M)_S$ and therefore $\alpha \cdot M_S \subseteq (\alpha \cdot M)_S$.

Finally, let us prove that $M_{S_3} = M_S$. Indeed, if $h \in M_{S_3}$, taking $\psi_0 : X \rightarrow \text{Hom}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$, $\psi_0 \equiv 1$ we have that $h = \psi_0 \cdot h \in (\psi_0 \cdot M)_S = M_S$. Conversely, let $h \in M_S$. For every $\psi : X \rightarrow \text{Hom}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$ we can look to the germ $\psi \in \mathcal{O}_{X,x}$ and $\psi \cdot h \in \psi \cdot M_S \subseteq (\psi \cdot M)_S$, i.e., $\psi \cdot h \in (\psi \cdot M)_S$. Hence, $h \in M_{S_3}$.

Proposition 3.1.16. *Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$. Then:*

- a) $\mathcal{M}_{S_3}$ is an $\mathcal{O}_X$ -submodule of $\mathcal{O}_X^p$;
- b) $M \subseteq M_{S_3} \subseteq \overline{M}$. In particular, M is a reduction of M_{S_3} and, if M has finite colength in M_{S_3} then $e(M, M_{S_3}) = 0$.

The same result holds in the family case.

Proof. Let $x \in X$ be an arbitrary point.

(a) Let $h, h' \in \mathcal{M}_{S_3}$ and $\alpha \in \mathcal{O}_X$ at x , and let $\psi : X \rightarrow \text{Hom}(\mathbb{C}^p, \mathbb{C})$ arbitrary. Then, $\psi \cdot h, \psi \cdot h' \in (\psi \cdot M)_S$ at x , and by Lemma 3.1.11 (a) we have $\psi \cdot (\alpha h + h') = \alpha(\psi \cdot h) + \psi \cdot h' \in (\psi \cdot M)_S$ at x . Therefore, $\alpha h + h' \in \mathcal{M}_{S_3}$ at x .

(b) If $h \in M$ then $\psi \cdot h \in \psi \cdot M \subseteq (\psi \cdot M)_S$, so $h \in M_{S_3}$ and $M \subseteq M_{S_3}$.

Now, let $h \in M_{S_3}$. Then, $\psi \cdot h \in (\psi \cdot M)_S \subseteq \overline{\psi \cdot M}, \forall \psi : X \rightarrow \text{Hom}(\mathbb{C}^p, \mathbb{C})$. By Proposition 3.1.13 we conclude that $h \in \overline{M}$. $\square$

Definition 3.1.17. *Consider the Setup 2.2.1. We define the infinitesimal Lipschitz conditions with respect to the 3-Lipschitz saturation of a module.*

- The pair (X, Y) satisfies the **3-infinitesimal Lipschitz condition $\mathbf{m}_Y$ ($3\text{-iL}_{\mathbf{m}_Y}$)** at $\mathbf{x} \in \mathbf{X}$ if $JM(X)_Y \subseteq (m_Y JM_z(X))_{S_3}$ at x ;
- The pair (X, Y) satisfies the **3-infinitesimal Lipschitz condition $\mathbf{A}$ ($3\text{-iL}_{\mathbf{A}}$)** at $\mathbf{x} \in \mathbf{X}$ if $JM(X)_Y \subseteq (JM_z(X))_{S_3}$ at x .

Here we consider the 3-Lipschitz saturation relative to the parameter space Y . Notice that 3-iL_{m_Y} implies $3\text{-iL}_{\mathbf{A}}$.

Corollary 3.1.18. *Consider the Setup 2.2.1. If $x = (z, y) \in X$ with $z \neq 0$ then 3-iL_{m_Y} and $3\text{-iL}_{\mathbf{A}}$ hold at x . Furthermore, there exists a dense Zariski open subset U of Y such the $3\text{-iL}_{\mathbf{A}}$ holds in $(0, y)$, for all $y \in U$.*

Proof. By Lemma 2.2.8 we have that $JM(X)_Y \subseteq m_Y JM_z(X)$ at x . By Proposition 3.1.16 the inclusion $m_Y JM_z(X) \subseteq (m_Y JM_z(X))_{S_3}$ holds and therefore the condition 3-iL_{m_Y} holds at x . In particular, $3\text{-iL}_{\mathbf{A}}$ holds at x .

The second statement is a consequence of the Theorem 2.2.11 and the Proposition 3.1.22. $\square$

Proposition 3.1.19. *Suppose the submodule $M \subseteq \mathcal{O}_{X,x}^p$ has generic rank k on every component of X . Let $\pi_S : SB_{J_k(M)}(X) \rightarrow B_{J_k(M)}(X)$, $\mathbf{p} : B_{J_k(M)}(X) \rightarrow X$ and $\pi = \mathbf{p} \circ \pi_S$ be the projections maps.*

$$\begin{array}{ccc} SB_{J_k(M)}(X) & \xrightarrow{\pi_S} & B_{J_k(M)}(X) \\ & \searrow \pi \quad \swarrow \mathbf{p} & \\ & X & \end{array}$$

Let $h \in \mathcal{O}_{X,x}^p$. Then:

$$J_k(h, M) \subseteq (J_k(M))_S \text{ if and only if } \pi^*(h) \in \pi^*(M).$$

Proof. Fix a set of generators $\{g_1, \dots, g_r\}$ of M . We work at $x' \in E$, $E = \pi_S^{-1}(E_B)$, E_B the exceptional divisor of $B_{J_k(M)}(X)$.

Suppose first that $J_k(h, M) \subseteq (J_k(M))_S$. Let $\pi^*(J_{I,J}(M))$ be a local generator of the principal ideal $\pi^*(J_k(M))$. Then, by Cramer's rule we can write

$$(1) \quad (J_{I,J}(M) \circ \pi)(h_I \circ \pi) = \sum_{j \in J} (J_{I,j}(h_I, M_I) \circ \pi)(m_{I,j} \circ \pi)$$

with $m_j \in M$. We claim in fact that

$$(J_{I,J}(M) \circ \pi)(h \circ \pi) = \sum_{j \in J} (J_{I,j}(h_I, M_I) \circ \pi)(m_j \circ \pi).$$

To see this pick a curve $\phi : (\mathbb{C}, 0) \rightarrow (B_{J_k(M)}(X)_S, x')$ and choose ϕ so that the rank of $\pi^*(M)|_\phi$ is generically k . Since by hypothesis $h \in \overline{M}$ then $h \circ \pi \circ \phi \in \phi^*(\pi^*(M))$. So, the element

$$\star = h \circ \pi \circ \phi - \sum_{j \in J} \left(\frac{J_{I,j}(h_I, M_I) \circ \pi}{J_{I,J}(M) \circ \pi} \circ \phi \right) (m_j \circ \pi \circ \phi) \in \phi^*(\pi^*(M))$$

because the above quotients by hypothesis are regular functions on $SB_{J_k(M)}(X)$. By equation (1), the above element has 0 for the entries indexed by I . So, if $\star$ is not zero then $\phi^*(\pi^*(M))$ has rank at least $k+1$, which is a contradiction. Since the images of ϕ fill up a $\mathbb{Z}$ -open set it follows that (see Lemma 3.3 of [48]) $\star$ is locally zero. Therefore,

$$h \circ \pi = \sum_{j \in J} \left(\frac{J_{I,j}(h_I, M_I) \circ \pi}{J_{I,J}(M) \circ \pi} \right) (m_j \circ \pi) \in \pi^*(M).$$

Conversely, suppose that $\pi^*(h) \in \pi^*(M)$. Then we can write

$$(2) \quad h \circ \pi = \sum_{j=1}^r \alpha_j (g_j \circ \pi)$$

where $\alpha_j \in \widetilde{\mathcal{O}}_{B_{J_k(M)}(X), x'}$, $\forall j \in \{1, \dots, r\}$.

Let c be an arbitrary generator of $J_k(h, M)$. Then we can write $c = \det[h, M]_{IJ}$ for some k -indexes I and J . If the k -index J does not pick the first column of $[h, M]$, then c is a $k \times k$ minor of the matrix $[M]$, hence $c \in J_k(M) \subset (J_k(M))_S$ and we are done. Thus we may assume that J pick the first column of $[h, M]$. Then, we can write

$$c = \det \begin{bmatrix} h_I & (g_{j_1})_I & \dots & (g_{j_{k-1}})_I \end{bmatrix}.$$

Applying the pullback of the projection map we have

$$\pi^*(c) = \det \begin{bmatrix} h_I \circ \pi & (g_{j_1})_I \circ \pi & \dots & (g_{j_{k-1}})_I \circ \pi \end{bmatrix}.$$

By the equation (2), if we look only to the entries of the k -index I , we get

$$h_I \circ \pi = \sum_{j=1}^r \alpha_j ((g_j)_I \circ \pi).$$

Using the linearity of the determinant in the first column, we conclude that $\pi^*(c) = \sum_{j=1}^r \alpha_j (v_j \circ \pi)$, where

$$v_j := \det \begin{bmatrix} (g_j)_I & (g_{j_1})_I & \dots & (g_{j_{k-1}})_I \end{bmatrix},$$

for all $j \in \{1, \dots, r\}$. Clearly, $v_j \in J_k(M)$, $\forall j \in \{1, \dots, r\}$. Hence, $\pi^*(c) \in \pi^*(J_k(M))$ and by the definition of the Lipschitz saturation of an ideal, we conclude that $c \in (J_k(M))_S$. Therefore, $J_k(h, M) \subseteq (J_k(M))_S$. $\square$

Corollary 3.1.20. *Suppose the submodule $M \subseteq \mathcal{O}_{X,x}^p$ has generic rank k on every component of X . If $J_k(h, M) \subseteq (J_k(M))_S$ then there exists an open covering of X such that h can be written on each U of the covering as an element of M by using Lipschitz functions (Lipschitz with respect to (z_i) and $(\frac{J_{I',J'}(M)}{J_{I,J}(M)})$, where $J_{I,J}(M)$ is the minor which gives the local generator on the preimage of U in $SB_{J_k(M)}(X)$).*

Proof. It is a straightforward consequence of the proof of the last proposition, more precisely, the equation (2) above. $\square$

Proposition 3.1.21. *Suppose the submodule $M \subseteq \mathcal{O}_{X,x}^p$ has generic rank k on each component of X . If $h \in M_{S_3}$ then $J_k(h, M) \subseteq (J_k(M))_S$.*

Proof. By Theorem 2.3 of [40] it is enough to prove that $(J_k(h, M))_D \subseteq \overline{(J_k(M))_D}$ at (x, x) . So, we have to prove that all the generators $(J_{IJ}(h, M))_D$ and $(0, (z_i \circ \pi_1 - z_i \circ \pi_2)(J_{IJ}(h, M)) \circ \pi_2)$ are in $\overline{(J_k(M))_D}$, for all $i \in \{1, \dots, n\}$ and for all indexes I and J . Write $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$. We have two cases.

Suppose $j_1 > 1$. Then, as we have seen, $J_{IJ}(h, M) \in J_k(M)$, so $(J_{IJ}(h, M))_D \in (J_k(M))_D$, in particular, $(J_{IJ}(h, M))_D \in \overline{(J_k(M))_D}$. Furthermore, since $J_{IJ}(h, M) \in J_k(M)$, by Lemma 2.1.2

(b) we have that

$$(0, (z_i \circ \pi_1 - z_i \circ \pi_2)(J_{IJ}(h, M)) \circ \pi_2) \in (J_k(M))_D \subseteq \overline{(J_k(M))_D}.$$

Suppose $j_1 = 1$. By the Lemma 3.1.12 there exists $\psi : X \rightarrow \text{Hom}(\mathbb{C}^p, \mathbb{C})$ such that $\psi \cdot h = J_{IJ}(h, M)$ and $\psi \cdot M \subseteq J_k(M)$. Since h is in the 3-Lipschitz Saturation of M at x then $\psi \cdot h \in (\psi \cdot M)_S$, which is equivalent to $(\psi \cdot h)_D \in \overline{(\psi \cdot M)_D}$. Thus, $(J_{IJ}(h, M))_D = (\psi \cdot h)_D \in \overline{(\psi \cdot M)_D} \subseteq \overline{(J_k(M))_D}$. Let us prove that

$(0, (z_i \circ \pi_1 - z_i \circ \pi_2)(J_{IJ}(h, M) \circ \pi_2)) \in \overline{(J_k(M))_D}$ by using the curve criterion. Let $\phi : (\mathbb{C}, 0) \rightarrow (X \times X, (x, x))$ be an analytic curve. Write $\phi = (\phi_1, \phi_2)$. Since $\psi \cdot h \in (\psi \cdot M)_S \subseteq \overline{\psi \cdot M}$ then we can write $(\psi \cdot h) \circ \phi_2 = \sum_j \beta_j ((\psi \cdot g_j) \circ \phi_2)$, with $\beta_j \in \mathcal{O}_{\mathbb{C}, 0}$ and $g_j \in M$. Thus, we get $(0, (z_i \circ \pi_1 - z_i \circ \pi_2)(J_{IJ}(h, M) \circ \pi_2)) \circ \phi = \sum_j \beta_j ((0, (z_i \circ \pi_1 - z_i \circ \pi_2)(\psi \cdot g_j) \circ \pi_2) \circ \phi) \in (\psi \cdot M)_D \circ \phi \subseteq (J_k(M))_D \circ \phi$, which finishes the proof. $\square$

Proposition 3.1.22. *Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$. Then, $\mathcal{M}_{S_2} \subseteq \mathcal{M}_{S_3}$ in every point $x \in X$.*

Proof. Let $h = (h_1, \dots, h_p) \in \mathcal{M}_{S_2}$ at x . Then $h_D \in \overline{\mathcal{M}_D}$ at (x, x) . It is enough to check that $(\psi \cdot h)_D \in \overline{(\psi \cdot M)_D}$, for all $\psi : X \rightarrow \text{Hom}(\mathbb{C}^p, \mathbb{C})$. Let $\phi = (\phi_1, \phi_2) : (\mathbb{C}, 0) \rightarrow (X \times X, (x, x))$ be an arbitrary analytic curve. Since $h_D \in \overline{\mathcal{M}_D}$ then we can write

$$h_D \circ \phi = \sum_j \alpha_j \phi^*((g_j)_D)$$

with $g_j = (g_{1j}, \dots, g_{pj}) \in M$ and $\alpha_j \in \mathcal{O}_{\mathbb{C}, 0}$ for all j . Looking to the above equation and comparing the $2p$ coordinates we conclude that $h_i \circ \phi_1 = \sum_j \alpha_j (g_{ij} \circ \phi_1)$ and $h_i \circ \phi_2 = \sum_j \alpha_j (g_{ij} \circ \phi_2)$, for all $i \in \{1, \dots, p\}$. Let $\psi_1, \dots, \psi_p$ be the coordinates functions of ψ .

$$\begin{aligned} \text{Thus: } (\psi \cdot h)_D \circ \phi &= ((\psi \cdot h) \circ \phi_1, (\psi \cdot h) \circ \phi_2) \\ &= (\sum_i (\psi_i \circ \phi_1) \cdot (h_i \circ \phi_1), \sum_i (\psi_i \circ \phi_2) \cdot (h_i \circ \phi_2)) \\ &= (\sum_{i,j} (\psi_i \circ \phi_1) \alpha_j (g_{ij} \circ \phi_1), \sum_{i,j} (\psi_i \circ \phi_2) \alpha_j (g_{ij} \circ \phi_2)) \\ &= \sum_j \alpha_j (\sum_i (\psi_i \circ \phi_1) \cdot (g_{ij} \circ \phi_1), \sum_i (\psi_i \circ \phi_2) \cdot (g_{ij} \circ \phi_2)) \\ &= \sum_j \alpha_j ((\psi \cdot g_j) \circ \phi_1, (\psi \cdot g_j) \circ \phi_2) \\ &= \sum_j \alpha_j \underbrace{((\psi \cdot g_j)_D \circ \phi)}_{\in (\psi \cdot M)_D \circ \phi} \in (\psi \cdot M)_D \circ \phi, \text{ and the proof is done.} \end{aligned} \quad \square$$

Definition 3.1.23. *Suppose that $M \subseteq \mathcal{O}_{X,x}^p$ is an $\mathcal{O}_{X,x}$ -submodule of generic rank k on each component of X . The **4-Lipschitz saturation** of M is denoted by M_{S_4} , and is defined by*

$$M_{S_4} := \{h \in \mathcal{O}_{X,x}^p \mid J_k(h, M) \subseteq (J_k(M))_S\}.$$

*In the family case, the **4-Lipschitz saturation of M at $x \in X$ relative to Y** is defined as above taking the Lipschitz saturation of $J_k(M)$ relative to Y .*

If $\mathcal{M}$ is an $\mathcal{O}_X$ -submodule of $\mathcal{O}_X^p$ then we can consider the sheaf of $\mathcal{O}_X$ -modules $\mathcal{M}_{S_4}$ where each stalk $(\mathcal{M}_{S_4})_x$ is $(\mathcal{M}_x)_{S_4}$.

Unless we say the opposite, we denote by M_{S_4} the stalk of the sheaf $\mathcal{M}_{S_4}$ at an arbitrary point of X .

Remark 3.1.24. M_{S_4} is the Lipschitz saturation of an ideal if $p = 1$. In fact, since $p = 1$ then the generic rank of M is 0 or 1. If the generic rank is 0 then $M = M_{S_4} = M_S = (0)$. Suppose the generic rank is 1. Then: $h \in M_{S_4} \Leftrightarrow J_1(h, M) \subseteq (J_1(M))_S \Leftrightarrow (h, M) \subseteq M_S \Leftrightarrow h \in M_S$.

The next proposition allows us to see $\mathcal{M}_{S_4}$ as a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$.

Proposition 3.1.25. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$ of generic rank k on each component of X . Then:

- a) $\mathcal{M}_{S_4}$ is an $\mathcal{O}_X$ -submodule of $\mathcal{O}_X^p$;
- b) $M \subseteq M_{S_4} \subseteq \overline{M}$. In particular, M is a reduction of M_{S_4} and, if M has finite colength in M_{S_4} then $e(M, M_{S_4}) = 0$.

The same result holds in the family case.

Proof. (a) Let $g, h \in \mathcal{M}_{S_4}$ at $x \in X$ and $\alpha \in \mathcal{O}_{X,x}$. By the basic properties of determinants we have that

$$J_k(\alpha g + h, M) \subseteq \alpha J_k(g, M) + J_k(h, M) \subseteq (J_k(M))_S.$$

Hence, $\alpha g + h \in M_{S_4}$ at x .

(b) Since $J_k(M) \subseteq (J_k(M))_S$ then $M \subseteq M_{S_4}$. Now, let $h \in M_{S_4}$. Thus, $J_k(h, M) \subseteq (J_k(M))_S \subseteq \overline{J_k(M)}$ which implies that $h \in \overline{M}$. Therefore, the inclusion $M_{S_4} \subseteq \overline{M}$ is proved. $\square$

Definition 3.1.26. Consider the Setup 2.2.1. We define the infinitesimal Lipschitz conditions with respect to the 4-Lipschitz saturation of a module.

- The pair (X, Y) satisfies the **4-infinitesimal Lipschitz condition $\mathbf{m}_Y$ ($4\text{-iL}_{\mathbf{m}_Y}$)** at $\mathbf{x} \in \mathbf{X}$ if $JM(X)_Y \subseteq (m_Y JM_z(X))_{S_4}$ at x ;
- The pair (X, Y) satisfies the **4-infinitesimal Lipschitz condition $\mathbf{A}$ ($4\text{-iL}_{\mathbf{A}}$)** at $\mathbf{x} \in \mathbf{X}$ if $JM(X)_Y \subseteq (JM_z(X))_{S_4}$ at x .

Here we consider the 4-Lipschitz saturation relative to the parameter space Y . Notice that 4-iL_{m_Y} implies $4\text{-iL}_{\mathbf{A}}$.

Corollary 3.1.27. Consider the Setup 2.2.1. If $x = (z, y) \in X$ with $z \neq 0$ then 4-iL_{m_Y} and $4\text{-iL}_{\mathbf{A}}$ hold at x . Furthermore, there exists a dense Zariski open subset U of Y such the $4\text{-iL}_{\mathbf{A}}$ holds in $(0, y)$, for all $y \in U$.

Proof. By Lemma 2.2.8 we have that $JM(X)_Y \subseteq m_Y JM_z(X)$ at x . By Proposition 3.1.25 the inclusion $m_Y JM_z(X) \subseteq (m_Y JM_z(X))_{S_4}$ holds and therefore the condition $4-iL_{m_Y}$ holds at x . In particular, $4-iL_A$ holds at x .

The second statement is a consequence of the Theorem 2.2.11, Propositions 3.1.21 and 3.1.22. $\square$

Notice that we can rephrase the Proposition 3.1.21 by saying that $\mathcal{M}_{S_3} \subseteq \mathcal{M}_{S_4}$ at every point $x \in X$.

Example 3.1.28. Consider $X = \mathbb{C}$ with coordinate t and $p = 2$. Consider the $\mathcal{O}_{\mathbb{C},0}$ -submodule $M \subseteq \mathcal{O}_{\mathbb{C},0}^2$ generated by $\{(t,0)\}$. Let $h := (0,t) \in \mathcal{O}_{\mathbb{C},0}^2$. Notice that the generic rank of M is 1, $[M] = \begin{bmatrix} t \\ 0 \end{bmatrix}$ and $[h, M] = \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}$. Thus, $J_1(h, M) = (t)$ and $J_1(M) = (t)$. In particular, $J_1(h, M) \subseteq J_1(M) \subseteq (J_1(M))_S$ which implies that $h \in M_{S_4}$.

Now consider $\psi : \mathbb{C} \rightarrow \text{Hom}(\mathbb{C}^2, \mathbb{C})$ given by $\psi(t) := (t, 1)$. Then $\psi \cdot M = (t^2)$ and $\psi \cdot h = t$. Since $t \notin \overline{(t^2)}$ then $\psi \cdot h \notin \overline{\psi \cdot M}$, in particular, $\psi \cdot h \notin (\psi \cdot M)_S$. Hence, $h \notin M_{S_3}$ and we conclude that $M_{S_3} \subsetneq M_{S_4}$.

In [39] Gaffney gave a geometric interpretation of the infinitesimal Lipschitz condition A, presenting vector fields defined in a open covering of $B_{J_z(G)}(X)$, where X is a hypersurface defined by G . Here we extend this interpretation for the jacobian module case.

Consider X a family of analytic varieties as on the setup 2.2.1. Let s be the generic rank of $M := JM_z(X)$. Let g be the number of generators of $J_s(M)$, i.e, $g = \#\{(I, J) \mid I \text{ and } J \text{ are } s\text{-indexes}\}$. Let T_{IJ} be the homogeneous coordinates of $\mathbb{P}^{g-1}$ and M_{IJ} be the generators of $J_s(M)$, (I, J) varying on the set of all pairs of s -indexes. Consider the diagram

$$\begin{array}{ccc} SB_{J_s(M)}(X) & \xrightarrow{\pi_S} & B_{J_s(M)}(X) \quad \subseteq X \times \mathbb{P}^{g-1} \\ & \searrow \pi \quad \swarrow \mathbf{p} & \\ & X & \end{array}$$

as in the Proposition 3.1.19. Let V_{IJ} be the dense Zariski open subset of $\mathbb{P}^{g-1}$ defined by $T_{IJ} \neq 0$ and let $U_{IJ} := B_{J_s(M)}(X) \cap (X \times V_{IJ})$. Clearly the collection $\{U_{IJ}\}_{(I,J)}$ is an open covering of $B_{J_s(M)}(X)$. At each point of U_{IJ} , $M_{IJ} \circ \pi$ is a local generator of the principal ideal sheaf $\pi^*(J_s(M))$.

Suppose that $JM(X)_Y \subseteq (JM_z(X))_{S_t}$ for some $t \in \{2, 3, 4\}$. By Propositions 3.1.21 and 3.1.22 we have $JM(X)_Y \subseteq (JM_z(X))_{S_4}$. By the proof of the Proposition 3.1.19, for each $\ell \in \{1, \dots, k\}$ we can write

$$\frac{\partial F}{\partial y_\ell} \circ \pi = \sum_{j \in J} \frac{M_{IJ,j,\ell} \circ \pi}{M_{IJ} \circ \pi} \left(\frac{\partial F}{\partial z_j} \circ \pi \right)$$

where $M_{IJ,j,\ell}$ is the determinant of the matrix obtained by replacing the j^{th} column of M_{IJ} by $(\frac{\partial F}{\partial y_\ell})_I$, for all $j \in J$.

Thus, for each pair of s -indexes (I, J) , we have $s.k$ vector fields tangent to X

$$\vec{v}_{IJ,j,\ell} := \frac{\partial}{\partial y_\ell} - \frac{M_{IJ,j,\ell}}{M_{IJ}} \frac{\partial}{\partial z_j}$$

defined on $\pi(U_{IJ})$. Since $\frac{M_{IJ,j,\ell} \circ \pi}{M_{IJ} \circ \pi}$ is Lipschitz, for all $j \in J$ and $\ell \in \{1, \dots, k\}$ then $\vec{v}_{IJ,j,\ell}$ is a Lipschitz vector field relative to Y .

3.2 The generic equivalence among the Lipschitz saturations

Now we look for conditions ensuring the above notions of Lipschitz saturations are generically equivalent. The next theorem gives us a such condition.

Theorem 3.2.1. *Let M be an $\mathcal{O}_{X,x}$ -submodule of $\mathcal{O}_{X,x}^p$ with generic rank k on each component of X and $h \in \mathcal{O}_{X,x}^p$. Suppose that there exists an ideal I of $\mathcal{O}_{X \times X, (x,x)}^{2p}$ such that*

$$\begin{aligned} I \cdot J_2((J_k(M))_D) &\subseteq \overline{J_{2k}(M_D)} \\ J_{2k}(h_D, M_D) &\subseteq \overline{I \cdot J_2((J_k(h, M))_D)}. \end{aligned}$$

Then, the following condition are equivalent:

- (a) $h \in M_{S_2}$;
- (b) $h \in M_{S_3}$;
- (c) $h \in M_{S_4}$.

Proof. (a) $\implies$ (b) Follows from the Proposition 3.1.22.

(b) $\implies$ (c) Follows from the Proposition 3.1.21.

(c) $\implies$ (a) Suppose that $J_k(h, M) \subseteq (J_k(M))_S$. Then, $(J_k(h, M))_D \subseteq \overline{(J_k(M))_D}$ and so

$$\overline{(J_k(h, M))_D} = \overline{(J_k(M))_D}.$$

Let us prove that $J_{2k}(h_D, M_D) \subseteq \overline{J_{2k}(M_D)}$. In fact, let $\phi : (\mathbb{C}, 0) \rightarrow (X \times X, (x, x))$ be an arbitrary analytic curve. Since $\overline{(J_k(h, M))_D} = \overline{(J_k(M))_D}$ then $\phi^*((J_k(h, M))_D) = \phi^*((J_k(M))_D)$. Using the inclusions of the hypothesis and the curve criterion for the integral closure of modules we have $\phi^*(J_{2k}(h_D, M_D)) \subseteq \phi^*(I \cdot J_2((J_k(h, M))_D)) = \phi^*(I) \cdot \phi^*(J_2((J_k(h, M))_D))$

$$= \phi^*(I).J_2(\phi^*((J_k(h, M))_D)) = \phi^*(I).J_2(\phi^*((J_k(M))_D)) = \phi^*(I).\phi^*(J_2((J_k(M))_D)) \\ = \phi^*(I.J_2((J_k(M))_D)) \subseteq \phi^*(J_{2k}(M_D)). \text{ Hence, the desired inclusion is proved.}$$

By Theorem 2.1.17 we have that M_D has generic rank $2k$ on each component of $(X \times X, (x, x))$. Therefore, by Corollary 1.2.6 we conclude that $h_D \in \overline{M_D}$. $\square$

Corollary 3.2.2. *Let $M \subseteq \mathcal{O}_{X,x}^p$ be an $\mathcal{O}_{X,x}$ -submodule of generic rank k on each component of X . Suppose that there exists an ideal I of $\mathcal{O}_{X \times X, (x,x)}^{2p}$ such that*

$$I.J_2((J_k(M))_D) \subseteq \overline{J_{2k}(M_D)} \\ J_{2k}(h_D, M_D) \subseteq \overline{I.J_2((J_k(h, M))_D)}, \forall h \in \overline{M}.$$

Then

$$M_{S_2} = M_{S_3} = M_{S_4}$$

In order to use the above criterion, we prove some useful lemmas that allow us to get the desired equivalence.

Notation: For each object A , we denote $A = A \circ \pi_1$ and $A' = A \circ \pi_2$. For k -indexes I, J , we denote by M_{IJ} the submatrix of $[M]$ defined by these k -indexes.

Lemma 3.2.3. *Let $M \subseteq \mathcal{O}_{X,x}^p$ be a submodule and $k \in \mathbb{N}$. Then*

$$(z_{t_1} - z'_{t_1}) \dots (z_{t_k} - z'_{t_k}) \det(M_{IJ}) \det(M'_{KL}) \in J_{2k}(M_D) \text{ at } (x, x)$$

for every $t_1, \dots, t_k \in \{1, \dots, n\}$ and k -indexes I, J, K, L .

Proof. Write $M_{KL} = (g_{rs})$. The $2k \times 2k$ matrix

$$\left[\begin{array}{c|c} M_{IJ} & 0_{k \times k} \\ \hline M'_{IJ} & \begin{pmatrix} (z_{t_1} - z'_{t_1})g'_{11} & \dots & (z_{t_k} - z'_{t_k})g'_{1k} \\ \vdots & & \vdots \\ (z_{t_1} - z'_{t_1})g'_{k1} & \dots & (z_{t_k} - z'_{t_k})g'_{kk} \end{pmatrix} \end{array} \right]$$

is a submatrix of $[M_D]$. So, its determinant, which is $(z_{t_1} - z'_{t_1}) \dots (z_{t_k} - z'_{t_k}) \det(M_{IJ}) \det(M'_{KL})$, belongs to $J_{2k}(M_D)$. $\square$

The next result gives us the first condition of the Theorem 3.2.1 in terms of the ideal coming from the diagonal.

Lemma 3.2.4. *Let $M \subseteq \mathcal{O}_{X,x}^p$ be a submodule and $k \in \mathbb{N}$.*

- a) $I_{\Delta}^k J_2((J_k(M))_D) \subseteq J_{2k}(M_D)$ at (x, x) ;
- b) If $J_k(M)$ is principal then $I_{\Delta}^{k-1} J_2((J_k(M))_D) \subseteq J_{2k}(M_D)$ at (x, x) .

Proof. (a) We have that

$$[(J_k(M))_D] = \begin{bmatrix} \det(M_{IJ}) & \dots & 0 \\ \det(M'_{IJ}) & \dots & (z_i - z'_i) \det(M'_{KL}) \end{bmatrix}$$

varying the k -indexes I, J, K, L and $i \in \{1, \dots, n\}$.

Thus, the desired inclusion is a straightforward consequence of the previous lemma.

(b) Since $J_k(M)$ is principal then there exist k -indexes I, J such that $g = \det(M_{IJ})$ and $J_k(M)$ is generated by $\{g\}$. Thus we can write

$$[(J_k(M))_D] = \begin{bmatrix} g & 0 \\ g' & (z_i - z'_i)g' \end{bmatrix}$$

varying $i \in \{1, \dots, n\}$.

Thus, in this case $J_2((J_k(M))_D)$ is generated by $\{g \cdot g'(z_i - z'_i) \mid i \in \{1, \dots, n\}\}$. So, by previous lemma we have that

$$[(z_{t_1} - z'_{t_1}) \dots (z_{t_{k-1}} - z'_{t_{k-1}})] \cdot [g \cdot g'(z_i - z'_i)] \in J_{2k}(M_D),$$

for all $t_1, \dots, t_{k-1}, i \in \{1, \dots, n\}$.

Therefore, $I_\Delta^{k-1} J_2((J_k(M))_D) \subseteq J_{2k}(M_D)$ at (x, x) . $\square$

Now, we start to get conditions in order to obtain the second condition of the Theorem 3.2.1.

Lemma 3.2.5. *Let M be an $\mathcal{O}_{X,x}$ -submodule of $\mathcal{O}_{X,x}^p$ and $k \in \mathbb{N}$. Denote by $[\tilde{M}]$ the matrix such that*

$$[M_D] = \begin{bmatrix} [M] & 0 \\ [M]' & [\tilde{M}] \end{bmatrix}.$$

Let $\mathcal{J}_{2k}(M_D)$ be the subideal of $J_{2k}(M_D)$ generated by

$$\{\det(M_{IJ}) \det(\tilde{M}_{KL}) \mid I, J, K, L \text{ are } k\text{-indexes}\}.$$

Then,

$$\mathcal{J}_{2k}(M_D) \subseteq I_\Delta^{k-1} J_2((J_k(M))_D) \text{ at } (x, x).$$

Proof. It suffices to prove that each generator of $\mathcal{J}_{2k}(M_D)$ belongs to $I_\Delta^{k-1} J_2((J_k(M))_D)$.

The columns of $\tilde{M}_{KL}$ are columns of $[M]'$ possibly in a different order with terms of type $z_i - z'_i$ multiplied on each column. If the k -indexes K, L pick repeated columns then $\det(\tilde{M}_{KL}) = 0$ and we are done. If this does not occur then

$$\det(\tilde{M}_{KL}) = (z_{i_1} - z'_{i_1}) \dots (z_{i_k} - z'_{i_k}) (\pm \det(M'_{K'L'}))$$

for some reorganization k -indexes K', L' , where $i_1, \dots, i_k \in \{1, \dots, n\}$ are the indexes which $z_{i_1} - z'_{i_1}, \dots, z_{i_k} - z'_{i_k}$ appear on each of the k columns of $\tilde{M}_{KL}$.

Since $\pm(z_{i_1} - z'_{i_1}) \dots (z_{i_{k-1}} - z'_{i_{k-1}}) \in I_{\Delta}^{k-1}$ and $\det(M_{IJ})(z_{i_k} - z'_{i_k}) \det(M'_{KL}) \in J_2((J_k(M))_D)$ then $\det(M_{IJ}) \det(\tilde{M}_{KL}) \in I_{\Delta}^{k-1} J_2((J_k(M))_D)$, which finishes the proof. $\square$

The next result ensures that any free submodule satisfies the second condition of the Theorem 3.2.1 for a suitable power of the ideal coming from the diagonal.

Lemma 3.2.6. *Let M be a free $\mathcal{O}_{X,x}$ -submodule of $\mathcal{O}_{X,x}^p$ of rank k . Then*

$$J_{2k}(M_D) \subseteq I_{\Delta}^{k-1} J_2((J_k(M))_D) \text{ at } (x, x).$$

Proof. We have that

$$[M_D] = \left[\begin{array}{c|c} [M]_{p \times k} & 0 \\ \hline - & - \\ [M]'_{p \times k} & (z_i - z'_i)[M]' \end{array} \right]$$

varying $i \in \{1, \dots, n\}$.

Let $d \in J_{2k}(M_D)$ at (x, x) be an arbitrary generator. Then $d = \det N$ where N is a $2k \times 2k$ submatrix of $[M_D]$.

(i) Suppose first that there are $k+t$ columns of N taken on the part

$$\left[\begin{array}{c} 0 \\ (z_i - z'_i)[M]' \end{array} \right]$$

of $[M_D]$, with $1 \leq t \leq k$. Then we can write

$$\begin{aligned} N &= \left[\begin{array}{c|c} (M_{IJ})_{k \times (k-t)} & 0_{k \times (k+t)} \\ \hline - & - \\ (M_{IJ})'_{k \times (k-t)} & (\tilde{M})_{KL} \end{array} \right] \\ &= \left[\begin{array}{cc|c} M_{IJ} & 0_{k \times t} & 0_{k \times k} \\ \hline - & - & - \\ M'_{IJ} & * & ** \end{array} \right] \end{aligned}$$

for some matrices $*$ and $**$, where $**$ is a square matrix of size $k \times k$. So in this case we have that $d = \det N = \det \begin{bmatrix} M_{IJ} & 0_{k \times t} \end{bmatrix} \cdot \det(**) = 0 \in I_{\Delta}^{k-1} J_2((J_k(M))_D)$.

(ii) Now suppose that we have exactly k columns of N taken on the part

$$\left[\begin{array}{c} 0 \\ (z_i - z'_i)[M]' \end{array} \right]$$

of $[M_D]$. Then we can write

$$N = \left[\begin{array}{c|c} M_{IJ} & 0_{k \times k} \\ \hline M'_{IJ} & \tilde{M}_{KL} \end{array} \right].$$

Thus, $d = \det(M_{IJ}) \det(\tilde{M}_{KL}) \in \mathcal{J}_{2k}(M_D)$ and by previous lemma we conclude that $d \in I_{\Delta}^{k-1} J_2((J_k(M))_D)$. $\square$

As a consequence of the previous lemma, we get another lemma which states that the second condition of the Theorem 3.2.1 holds in a dense Zariski open subset of X in a sheaf of submodules of $\mathcal{O}_X^p$.

Lemma 3.2.7. *Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$ of generic rank k on each component of X . Then there exists a dense Zariski open subset U of X such that $U \cap V$ is a dense Zariski open subset of V , $\forall V$ component of X , $J_k(\mathcal{M})$ is principal at x and*

$$J_{2k}(\mathcal{M}_D) \subseteq I_{\Delta}^{k-1} J_2((J_k(\mathcal{M}))_D) \text{ at } (x, x),$$

for all $x \in U$.

Proof. Consider $[\mathcal{M}]$ a matrix of generators of $\mathcal{M}$. Using the Cramer's rule, we can choose k $\mathcal{O}_X$ -linear independents columns of $[\mathcal{M}]$ such that these columns generates $\mathcal{M}$ in a such dense Zariski open subset U of X . Let $\mathcal{M}_k$ be the $\mathcal{O}_X$ -submodule of $\mathcal{O}_X^p$ generated by the columns chosen above. Thus, given $x \in U$, we have that $\mathcal{M}_x = (\mathcal{M}_k)_x$ is a free $\mathcal{O}_{X,x}$ -submodule of $\mathcal{O}_{X,x}^p$ of rank k and the desired inclusion is a consequence of the previous lemma. The ideal $J_k(\mathcal{M}) = J_k(\mathcal{M}_k)$ is principal at x , because $[\mathcal{M}_k]$ is a square $k \times k$ matrix. $\square$

Lemma 3.2.8. *Let M be an $\mathcal{O}_{X,x}$ -submodule of $\mathcal{O}_{X,x}^p$ of generic rank k on each component of X , where $\mathcal{O}_{X,x}$ is a reduced ring. If $h \in \overline{M}$ then (h, M) also has generic rank k on each component of X .*

Proof. Let V be an arbitrary component of (X, x) and let t be the generic rank of (h, M) at V . Since V is irreducible then there exists a prime ideal I_x of $\mathcal{O}_{X,x}$ such that $\mathcal{O}_{V,x} = \frac{\mathcal{O}_{X,x}}{I_x}$. Since I_x is a prime ideal (hence radical ideal) then $\mathcal{O}_{V,x}$ is a domain (hence reduced ring).

Since $M \subseteq (h, M)$ then $t \geq k$. Suppose by contradiction that $t > k$. Then $J_t(M) = (0)$, once M has generic rank k . Since $h \in \overline{M}$ then (see [41]) $J_t(h, M) \subseteq \overline{J_t(M)} = \overline{(0)} \subseteq \sqrt{(0)} = (0)$ at the component V . Hence, $J_t(h, M) = (0)$ at V , contradiction. Therefore, $t = k$. $\square$

The next theorem gives us the equivalence desired.

Theorem 3.2.9. *Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$ -submodules of $\mathcal{O}_X^p$ of generic rank k on each component of X , and suppose that X has reduced structure. Then there exists a dense Zariski open subset U of X such that $U \cap V$ is a dense Zariski open subset of V , for all V component of X and*

$$\mathcal{M}_{S_2} = \mathcal{M}_{S_3} = \mathcal{M}_{S_4}$$

along U .

In the case when $k = p$ is the maximal rank, the same conclusion holds even if X does not have reduced structure.

Proof. Since $X \subseteq \mathbb{C}^n$ is an analytic complex variety then $\mathcal{O}_X$ is a noetherian sheaf of rings, hence $\mathcal{O}_X^p$ is a noetherian sheaf of $\mathcal{O}_X$ -modules. Since $\mathcal{M}_{S_4}$ is a sheaf of $\mathcal{O}_X$ -modules then $\mathcal{M}_{S_4}$ is finitely generated by global sections $h_1, \dots, h_r$. Since $h_i \in \overline{\mathcal{M}}$, $\forall i \in \{1, \dots, r\}$ then by previous lemma $(h_i, \mathcal{M})$ also has generic rank k on each component of X , $\forall i \in \{1, \dots, r\}$. By Lemma 3.2.7, for each $i \in \{1, \dots, r\}$ there exists a dense Zariski open subset U_i of X such that $U_i \cap V$ is a dense Zariski open subset of V , for all V component of X , and

$$J_{2k}((h_i)_D, \mathcal{M}_D) \subseteq I_{\Delta}^{k-1} J_2((J_k(h_i, \mathcal{M}))_D) \text{ at } (x, x),$$

$\forall x \in U_i$.

Also by Lemma 3.2.7 there exists a dense Zariski open subset U_0 of X such that $U_0 \cap V$ is a dense Zariski open subset of V , for all component V of X , and $J_k(\mathcal{M})$ is principal at x , $\forall x \in U_0$. From the Lemma 3.2.4(b), we conclude that

$$I_{\Delta}^{k-1} J_2((J_k(\mathcal{M}))_D) \subseteq J_{2k}(\mathcal{M}_D) \text{ at } (x, x),$$

$\forall x \in U_0$.

Take $U := \bigcap_{j=0}^r U_j$. Then U is a dense Zariski open subset of X and $U \cap V$ is a dense Zariski open subset of V , for all V component of X .

We already know that $\mathcal{M}_{S_2} \subseteq \mathcal{M}_{S_3} \subseteq \mathcal{M}_{S_4}$ at any point of X , in particular, along U .

Let us prove that $\mathcal{M}_{S_4} \subseteq \mathcal{M}_{S_2}$ along U . In fact, let $x \in U$. Given an arbitrary $i \in \{1, \dots, r\}$, since $x \in U_0$ and $x \in U_i$ then

$$I_{\Delta}^{k-1} J_2((J_k(\mathcal{M}))_D) \subseteq J_{2k}(\mathcal{M}_D) \text{ at } (x, x)$$

and

$$J_{2k}((h_i)_D, \mathcal{M}_D) \subseteq I_{\Delta}^{k-1} J_2((J_k(h_i, \mathcal{M}))_D) \text{ at } (x, x).$$

Since $h_i \in \mathcal{M}_{S_4}$ at x then by Theorem 3.2.1 we have that $h_i \in \mathcal{M}_{S_2}$ at x , $\forall i \in \{1, \dots, r\}$. Since $\mathcal{M}_{S_4}$ is generated by $h_1, \dots, h_r$ at x , then $\mathcal{M}_{S_4} \subseteq \mathcal{M}_{S_2}$ at x , and the theorem is proved. $\square$

3.3 Geometric applications

The next result gives us conditions involving the double and the integral closure of modules so that we can construct Lipschitz vector fields.

Proposition 3.3.1. *Let $X \subseteq \mathbb{C}^n$ be an analytic curve. Let $\mathcal{M}$ be an $\mathcal{O}_X$ -submodule of $\mathcal{O}_X^p$ of generic rank k , with $k+1$ generators. Let $\mathcal{M}_k$ be a reduction of $\mathcal{M}$ generated by the first k columns of $[\mathcal{M}]$, whose rank is k . Let $h \in \mathcal{O}_X^p$. Let $c(h) \in \mathcal{O}_X^p$ be defined by the solution of the equation*

$$[\mathcal{M}_k] \cdot c(h) = h$$

in a dense Zariski open subset U of X (using the Cramer's Rule). Let $x \in U$, $M = \mathcal{M}_x$, $M_k = (\mathcal{M}_k)_x$ and $m = m_{X,x}$. If $h \in \overline{mM}$ and $h_D \in \overline{(M_k)_D}$ at (x, x) then the vector field $c(h)$ is Lipschitz, i.e, $c(h) - c(h)' \in I_\Delta \mathcal{O}_{X \times X}^p$ at (x, x) .

Proof. Let us use the curve criterion. Let $\phi : \mathbb{C}, 0 \rightarrow X \times X, (x, x)$ be a curve, with coordinates ϕ_1, ϕ_2 .

First suppose $\phi_1(t) \equiv 0$. In this case we have

$$\begin{bmatrix} h \circ \phi_1 \\ h \circ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ h \circ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ [M_k]' \cdot c(h) \end{bmatrix} \mod \phi_2^*(mM_k') \mathcal{O}_{\mathbb{C},0}^p$$

because $h \circ \phi_2 \in \phi_2^*(mM_k')$. Let $m_1, \dots, m_k$ be the columns of $[M_k]$. Suppose that $z_i \circ \phi_2$ is a generator of $\phi_2^*(m)$. For each $j \in \{1, \dots, k\}$ look to

$$\det \begin{bmatrix} z_i' m_1' & \dots & z_i' m_{j-1}' & h' & z_i' m_{j+1}' & \dots & z_i' m_k' \end{bmatrix} = z_i'^{k-1} \det \begin{bmatrix} m_1' & \dots & m_{j-1}' & h' & m_{j+1}' & \dots & m_k' \end{bmatrix}.$$

Then,

$$\frac{\det \begin{bmatrix} z_i' m_1' & \dots & z_i' m_{j-1}' & h' & z_i' m_{j+1}' & \dots & z_i' m_k' \end{bmatrix}}{\det \begin{bmatrix} z_i' m_1' & \dots & z_i' m_k' \end{bmatrix}} = \frac{z_i'^{k-1} \det \begin{bmatrix} m_1' & \dots & m_{j-1}' & h' & m_{j+1}' & \dots & m_k' \end{bmatrix}}{z_i'^k \det [M_k]'}.$$

$= \frac{1}{z_i'} c(h)'_j$ which is analytic along ϕ_2 , $\forall j \in \{1, \dots, k\}$. Thus, $c(h)' \in z_i' \mathcal{O}_X^p$ along ϕ_2 and since $c(h) = 0$ along ϕ_1 then $\phi^*(c(h) - c(h)') \in \phi^*(I_\Delta \mathcal{O}_{X \times X}^p)$.

The case where $\phi_2 \equiv 0$ is analogous.

Now assume that $\phi_1, \phi_2 \neq 0$. Since $(h_D) \in \overline{(M_k)_D}$ then

$$(0, [M_k]' \cdot (c(h)' - c(h))) \in \overline{(M_k)_D}.$$

Since $M_k \circ \phi_1$ has rank k generically, then $M_k'(c(h)' - c(h)) \in M_k' I_\Delta \mathcal{O}_{X \times X}^p$ at (x, x) along ϕ , so $c(h)' - c(h) \in I_\Delta \mathcal{O}_{X \times X}^p$ along ϕ at (x, x) , which finishes the proof. $\square$

Bi-Lipschitz equisingularity in an ICIS family of irreducible curves

Let $X \subseteq \mathbb{C} \times \mathbb{C}^n$ be an ICIS family of irreducible curves. Assume that we have a normalization $F : \mathbb{C} \times \mathbb{C} \rightarrow X \subseteq \mathbb{C} \times \mathbb{C}^n$ which is a homeomorphism, and suppose that the family X is Whitney equisingular. Let p be the multiplicity of X and assume we can write

$$F(t, s) = (t, F_1(t, s), \dots, F_{n-1}(t, s), s^p)$$

where for each parameter t the order of

$$s \mapsto F_i(t, s)$$

is greater than p , $\forall i \in \{1, \dots, n-1\}$.

We already know that if $\frac{\partial F_1}{\partial t}, \dots, \frac{\partial F_n}{\partial t}$ are Lipschitz functions on X then X is bi-Lipschitz equisingular.

Let $G : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^q$ be an analytic map that defines X , i.e, $X = G^{-1}(0)$. Consider the jacobian module $JM(X) = JM(G)$. Here we consider the double and the 2-Lipschitz saturation relative to the parameter space $Y = \mathbb{C} \times 0 \equiv \mathbb{C}$.

The following proposition characterizes the above condition in terms of the 2-Lipschitz saturation (or the double), which gives us an infinitesimal condition for the bi-Lipschitz equisingularity.

Proposition 3.3.2. *With the above notations, the functions $\frac{\partial F_1}{\partial t}, \dots, \frac{\partial F_n}{\partial t}$ are Lipschitz if and only if*

$$\left(\frac{\partial G}{\partial t}\right)_D \in \overline{(DG_{n-1})_D}$$

where DG_{n-1} denotes the submodule of $JM_z(X)$ generated by the first $n-1$ partial derivatives of G with respect to the z coordinates.

In particular, if $JM(X)_Y \subseteq (JM_z(X))_{S_2}$ then X is bi-Lipschitz equisingular.

Proof. Since $G \circ F \equiv 0$ then

$$0 = \left[\frac{\partial(G \circ F)}{\partial t} \right] = \left[\frac{\partial G}{\partial t} \circ F \quad \frac{\partial G}{\partial z_1} \circ F \quad \dots \quad \frac{\partial G}{\partial z_{n-1}} \circ F \quad \frac{\partial G}{\partial z_n} \circ F \right] \cdot \begin{bmatrix} 1 \\ \frac{\partial F_1}{\partial t} \\ \vdots \\ \frac{\partial F_{n-1}}{\partial t} \\ 0 \end{bmatrix}$$

which implies that

$$\frac{\partial G}{\partial t} \circ F = -[DG_{n-1} \circ F] \cdot \begin{bmatrix} \frac{\partial F_1}{\partial t} \\ \vdots \\ \frac{\partial F_{n-1}}{\partial t} \end{bmatrix}.$$

We also have that

$$0 = \frac{\partial(G \circ F)}{\partial s} = [JM(G) \circ F] \cdot \begin{bmatrix} \frac{\partial F_1}{\partial s} \\ \vdots \\ \frac{\partial F_{n-1}}{\partial s} \\ ps^{p-1} \end{bmatrix}.$$

Hence,

$$\frac{\partial G}{\partial z_n} \circ F = [DG_{n-1} \circ F] \cdot \begin{bmatrix} -\frac{\dot{x}_1(s)}{ps^{p-1}} \\ \vdots \\ -\frac{\dot{x}_{n-1}(s)}{ps^{p-1}} \end{bmatrix}.$$

Thus, for each parameter t we have that $JM_z(X)|_{x_t} \subseteq \overline{DG_{n-1}|_{x_t}}$. Since X is a Whitney equisingular family of curves then the multiplicity of the members of the family does not depend on the parameter and we can apply the Principle of Specialization (Theorem 1.8 of [48]) to conclude that $JM_z(X) \subseteq \overline{DG_{n-1}}$, i.e, DG_{n-1} is a reduction of $JM_z(X)$.

Let $c(\frac{\partial G}{\partial t})$ be the vector field associated to the Cramer's rule in order to solve the equation

$$[\frac{\partial G}{\partial t}] = [DG_{n-1}] \cdot \xi.$$

$$\text{Since } \frac{\partial G}{\partial t} \circ F = [DG_{n-1} \circ F] \cdot \begin{bmatrix} -\frac{\partial F_1}{\partial t} \\ \vdots \\ -\frac{\partial F_{n-1}}{\partial t} \end{bmatrix} \text{ then } c(\frac{\partial G}{\partial t}) \circ F = \begin{bmatrix} -\frac{\partial F_1}{\partial t} \\ \vdots \\ -\frac{\partial F_{n-1}}{\partial t} \end{bmatrix}.$$

Therefore, $\frac{\partial F_1}{\partial t}, \dots, \frac{\partial F_n}{\partial t}$ are Lipschitz functions if and only if $c(\frac{\partial G}{\partial t}) \circ F$ is Lipschitz which is equivalent to $(\frac{\partial G}{\partial t})_D \in \overline{(DG_{n-1})_D}$. $\square$

BI-LIPSCHITZ EQUISINGULARITY OF DETERMINANTAL SURFACES

In [70] Mostowski shows that every analytic variety has a stratified vector field which is Lipschitz, however, this vector field is not canonical from the variety. In [40], Gaffney got conditions so that a family of irreducible curves has a canonical vector field which is Lipschitz, namely

$$\frac{\partial}{\partial y} + \sum_i \frac{\partial \tilde{f}}{\partial y} \cdot \frac{\partial}{\partial z_i}$$

where $\tilde{F} : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{C}^n$ given by $\tilde{F}(y, t) = (y, \tilde{f}(y, t))$ defines the family of curves. In this case, the main condition is the Segre number $s_2(I_D(\tilde{F}))$ to be independent of the parameter y . Gaffney showed that the last condition is equivalent to the multiplicity of the pair $e(I_D(\tilde{F}), I_\Delta)$ to be independent of the parameter y , where I_Δ is the ideal which defines the diagonal on $\mathbb{C} \times \mathbb{C}$ and $I_D(\tilde{F})$ is the ideal generated by the doubles of the components of $\tilde{F}$. For more about the Segre numbers see [46].

In this chapter we get conditions so that the above vector field is Lipschitz on the context of determinantal varieties. It following the approach of Pereira and Ruas [89], we see that for the special case of determinantal surfaces, there are deformations $\tilde{f} : \mathbb{C}^q \longrightarrow \text{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ such that the above vector field always is Lipschitz.

4.1 Simple isolated Cohen-Macaulay of codimension 2 singularities in $\mathbb{C}^4$ and $\mathbb{C}^5$

In [37], Frübis-Krüger and Neumer determine a complete classification of simple Cohen-Macaulay codimension 2 singularities. This classification was obtained up to isomorphism of germs.

A singularity is called **simple** if it can only deform into finitely many different isomorphism classes.

The Cohen-Macaulay singularities in codimension 2 are particularly important. One special reason is that not all of them are complete intersection, however, the theorem of Hilbert-Bruch provides a powerful tool in order to describe these singularities and their deformations.

First they classified the possible candidates of 1-jets with 4 or more variables.

Theorem 4.1.1. ([37] Lemma 3.2) *Let M be a 3×2 matrix with entries in the maximal ideal of $\mathbb{C}\{x_1, \dots, x_m\}$. Then, $j_1 M$ is contact-equivalent to one of the jets in the following tables or is contact-equivalent to a 1-jet containing only 3 or less variables.*

6 variables		5 variables		4 variables	
$J^{(6,1)}$	$\begin{pmatrix} x & y & v \\ z & w & u \end{pmatrix}$	$J^{(5,1)}$	$\begin{pmatrix} z & y & v \\ z & w & x \end{pmatrix}$	$J^{(4,1)}$	$\begin{pmatrix} w & y & x \\ z & w & y \end{pmatrix}$
		$J^{(5,2)}$	$\begin{pmatrix} x & y & v \\ z & w & 0 \end{pmatrix}$	$J^{(4,2)}$	$\begin{pmatrix} w & y & x \\ z & w & 0 \end{pmatrix}$
				$J^{(4,3)}$	$\begin{pmatrix} 0 & y & x \\ z & w & 0 \end{pmatrix}$
				$J^{(4,4)}$	$\begin{pmatrix} x & y & z \\ z & w & 0 \end{pmatrix}$
				$J^{(4,5)}$	$\begin{pmatrix} x & y & 0 \\ z & w & 0 \end{pmatrix}$
				$J^{(4,6)}$	$\begin{pmatrix} x & y & z \\ w & 0 & 0 \end{pmatrix}$

Table 1 – Possible candidates of 1-jets with 4 or more variables

In [37] the authors have seen that 1-jets containing only 2 or fewer variables cannot be simple in dimension 4, so we have to consider only the 1-jets with 3 and 4 variables. We denote τ as the *Tjurina number*.

Theorem 4.1.2. ([37] Theorem 3.3) *The list of simple isolated Cohen-Macaulay in codimension 2 singularities in $(\mathbb{C}^4, 0)$ are the following ones:*

Jet-type	Type	Presentation Matrix	τ	Name of triple point in [100]
$J^{(4,1)}$	$\Lambda_{1,1}$	$\begin{pmatrix} w & y & x \\ z & w & y \end{pmatrix}$	2	$A_{0,0,0}$
$J^{(4,2)}$	$\Lambda_{k,1}$	$\begin{pmatrix} w & y & x \\ z & w & y^k \end{pmatrix}$	$k \geq 2$ $k+1$	$A_{0,0,k-1}$
$J^{(4,3)}$	$\Lambda_{k,l}$	$\begin{pmatrix} w^l & y & x \\ z & w & y^k \end{pmatrix}$	$k \geq l \geq 2$ $k+l$	$A_{0,l-1,k-1}$
$J^{(4,4)}$		$\begin{pmatrix} z & y & x \\ x & w & y^2 + z^k \end{pmatrix}$	$k \geq 2$ $k+3$	$C_{k+1,0}$
		$\begin{pmatrix} z & y & x \\ x & w & yz + y^k w \end{pmatrix}$	$k \geq 1$ $2k+4$	$B_{2k+2,0}$
		$\begin{pmatrix} z & y & x \\ x & w & yz + y^k \end{pmatrix}$	$k \geq 3$ $2k+1$	$B_{2k-1,0}$
		$\begin{pmatrix} z & y & x \\ x & w & z^2 + yw \end{pmatrix}$	7	D_0
		$\begin{pmatrix} z & y & x \\ x & w & z^2 + y^3 \end{pmatrix}$	8	F_0
$J^{(3,1)}$		$\begin{pmatrix} z & y + w^l & w^m \\ w^k & y & x \end{pmatrix}$	$k, l, m \geq 2$ $k+l+m-1$	$A_{k-1,l-1,m-1}$
$J^{(3,2)}$		$\begin{pmatrix} z & y & x^l + w^2 \\ w^k & x & y \end{pmatrix}$	$k, l \geq 2$ $k+l+2$	$C_{l+1,k-1}$
		$\begin{pmatrix} z & y + w^l & xw \\ w^k & x & y \end{pmatrix}$	$k, l \geq 2$ $k+2l+1$	$B_{2l,k-1}$
		$\begin{pmatrix} z & y & xw + w^l \\ w^k & x & y \end{pmatrix}$	$k \geq 2, l \geq 3$ $k+2l$	$B_{2l+1,k-1}$
		$\begin{pmatrix} z & y + w^2 & x^2 \\ w^k & x & y \end{pmatrix}$	$k \geq 2$ $k+6$	D_{k-1}
		$\begin{pmatrix} z & y & x^2 + w^3 \\ w^k & x & y \end{pmatrix}$	$k \geq 2$ $k+7$	F_{k-1}
$J^{(3,3)}$		$\begin{pmatrix} z & y & xw + w^k \\ y & x & z \end{pmatrix}$	$3k+1$	H_{3k}
		$\begin{pmatrix} z & y & xw \\ y & x & z + w^k \end{pmatrix}$	$3k+2$	H_{3k+1}
		$\begin{pmatrix} z & y & xw \\ y + w^k & x & z \end{pmatrix}$	$3k+3$	H_{3k+2}

$\begin{pmatrix} z & y & w^2 \\ y & x & z+x^2 \end{pmatrix}$	8
$\begin{pmatrix} z & y & x^3+w^2 \\ y & x & z \end{pmatrix}$	9
$\begin{pmatrix} z & y & x^2 \\ y & x & z+w^2 \end{pmatrix}$	9

Table 2 – Simple isolated Cohen-Macaulay in codimension 2 singularities in $(\mathbb{C}^4, 0)$

In the case in $(\mathbb{C}^5, 0)$, in [37] the authors realized that we only need to consider matrices whose 1-jet involves at least 4 variables. The methods are basically the same as in the previous case, with one exception: For the case $J^{(5,2)}$, the problem of classification and of finding adjacencies can be reduced to the corresponding problem for plane curve singularities and deformations with sections thereof.

Theorem 4.1.3. ([37] Theorem 3.5) *The simple isolated Cohen-Macaulay in codimension 2 singularities in $(\mathbb{C}^5, 0)$ are the following ones:*

Jet-type	Type	Presentation Matrix	τ
$J^{(5,1)}$	$A_0^\#$	$\begin{pmatrix} x & y & z \\ w & v & x \end{pmatrix}$	1
$J^{(5,2)}$	$A_k^\#$	$\begin{pmatrix} x & y & z \\ w & v & x^{k+1} + y^2 \end{pmatrix}$	$k \geq 1$ $k+2$
	$D_k^\#$	$\begin{pmatrix} x & y & z \\ w & v & xy^2 + x^{k-1} \end{pmatrix}$	$k \geq 4$ $k+2$
	$E_6^\#$	$\begin{pmatrix} x & y & z \\ w & v & x^3 + y^4 \end{pmatrix}$	8
	$E_7^\#$	$\begin{pmatrix} x & y & z \\ w & v & x^3 + xy^3 \end{pmatrix}$	9
	$E_8^\#$	$\begin{pmatrix} x & y & z \\ w & v & x^3 + y^5 \end{pmatrix}$	10
$J^{(4,1)}$	Π_k	$\begin{pmatrix} w & y & x \\ z & w & y + v^k \end{pmatrix}$	$k \geq 2$ $2k-1$
$J^{(4,2)}$		$\begin{pmatrix} w & y & x \\ z & w & y^k + v^2 \end{pmatrix}$	$k \geq 2$ $k+2$
		$\begin{pmatrix} w & y & x \\ z & w & yv + v^k \end{pmatrix}$	$2k$
		$\begin{pmatrix} w + v^k & y & x \\ z & w & yv \end{pmatrix}$	$2k+1$
		$\begin{pmatrix} w + v^2 & y & x \\ z & w & y^2 + v^k \end{pmatrix}$	$k+3$

$J^{(4,3)}$	$\begin{pmatrix} w & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$		7
	$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & v^2 + y^l \end{pmatrix}$	$l \geq k \geq 2$	$k + l + 1$
	$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & yv \end{pmatrix}$	$k \geq 2$	$k + 4$
	$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & y^2 + v^l \end{pmatrix}$	$k \geq 2, l \geq 3$	$k + l + 2$
	$\begin{pmatrix} wv + v^k & y & x \\ z & w & yv + v^k \end{pmatrix}$	$k \geq 3$	$2k + 1$
$J^{(4,4)}$	$\begin{pmatrix} wv + v^k & y & x \\ z & w & yv \end{pmatrix}$	$k \geq 3$	$2k + 2$
	$\begin{pmatrix} wv + v^3 & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$		8
	$\begin{pmatrix} wv & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$		9
	$\begin{pmatrix} w^2 + v^3 & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$		9
	$\begin{pmatrix} z & y & x \\ x & w & v^2 + y^2 + z^k \end{pmatrix}$	$k \geq 2$	$k + 4$
	$\begin{pmatrix} z & y & x \\ x & w & v^2 + yz + y^k w \end{pmatrix}$	$k \geq 1$	$2k + 5$
	$\begin{pmatrix} z & y & x \\ x & w & v^2 + yz + y^{k+1} \end{pmatrix}$	$k \geq 2$	$2k + 4$
	$\begin{pmatrix} z & y & x \\ x & w & v^2 + yw + z^2 \end{pmatrix}$		8
	$\begin{pmatrix} z & y & x \\ x & w & v^2 + y^3 + z^2 \end{pmatrix}$		9
	$\begin{pmatrix} z & y & x + v^2 \\ x & w & vy + z^2 \end{pmatrix}$		7
	$\begin{pmatrix} z & y & x + v^2 \\ x & w & vz + y^2 \end{pmatrix}$		8
	$\begin{pmatrix} z & y & x + v^2 \\ x & w & y^2 + z^2 \end{pmatrix}$		9

Table 3 – Simple isolated Cohen-Macaulay in codimension 2 singularities in $(\mathbb{C}^5, 0)$

4.2 The bi-Lipschitz equisingularity on determinantal varieties

Let us fix some notations.

We work with one parameter deformations and unfoldings. The parameter space is denoted by $Y = \mathbb{C} \equiv \mathbb{C} \times 0$.

Let $h \in \mathcal{O}_{\mathbb{C}^N}$. The **(1,-1)-double of h** is the element denoted by $I_D(h) \in \mathcal{O}_{\mathbb{C}^{2N}}$ defined by the equation

$$I_D(h)(z, z') := h(z) - h(z').$$

If $h = (h_1, \dots, h_r)$ is a map, with $h_i \in \mathcal{O}_{\mathbb{C}^N}$, $\forall i \in \{1, \dots, r\}$, then we define $I_D(h)$ as the ideal of $\mathcal{O}_{\mathbb{C}^{2N}}$ generated by $\{I_D(h_1), \dots, I_D(h_r)\}$.

The above notions was defined by Gaffney in [39].

Now we get a relation between the integral closure of the double and the canonical vector field induced by a one parameter unfolding to be Lipschitz.

Let $\tilde{F} : \mathbb{C} \times \mathbb{C}^q \longrightarrow \mathbb{C} \times \mathbb{C}^n$ be an analytic map, which is a homeomorphism onto its image, and such that we can write $\tilde{F}(y, x) = (y, \tilde{f}(y, x))$, with $\tilde{f}(y, x) = (\tilde{f}_1(y, x), \dots, \tilde{f}_n(y, x))$. Let us denote by

$$\frac{\partial}{\partial y} + \sum_{j=1}^n \frac{\partial \tilde{f}_j}{\partial y} \cdot \frac{\partial}{\partial z_j}$$

the vector field $v : \tilde{F}(\mathbb{C} \times \mathbb{C}^q) \longrightarrow \mathbb{C} \times \mathbb{C}^n$ given by

$$v(y, z) = (1, \frac{\partial \tilde{f}_1}{\partial y}(\tilde{F}^{-1}(y, z)), \dots, \frac{\partial \tilde{f}_n}{\partial y}(\tilde{F}^{-1}(y, z))).$$

Proposition 4.2.1. *The vector field $\frac{\partial}{\partial y} + \sum_{j=1}^n \frac{\partial \tilde{f}_j}{\partial y} \cdot \frac{\partial}{\partial z_j}$ is Lipschitz if and only if*

$$I_D(\frac{\partial \tilde{F}}{\partial y}) \subsetneq \overline{I_D(\tilde{F})}.$$

Proof. Since we are working in a finite dimensional $\mathbb{C}$ -vector space then all the norms are equivalent. To simplify the argument, we use the notation $\|\cdot\|$ for the *maximum norm* on $\mathbb{C} \times \mathbb{C}^q$ and $\mathbb{C} \times \mathbb{C}^n$, i.e., $\|(x_1, \dots, x_{n+1})\| = \max_{i=1}^{n+1} \{\|x_i\|\}$.

Suppose that the canonical vector field is Lipschitz. By hypothesis there exists a constant $c > 0$ such that

$$\|v(y, z) - v(y', z')\| \leq c \|(y, z) - (y', z')\|$$

$\forall (y, z), (y', z') \in U$, where U is a non-empty open subset of $\tilde{F}(\mathbb{C} \times \mathbb{C}^q)$.

Thus, given $(y, x), (y', x') \in \tilde{F}^{-1}(U)$, applying the above inequality on these points, we get

$$\left\| \left(\frac{\partial \tilde{f}_j}{\partial y} \right)_D(y, x, y', x') \right\| \leq c \left\| \tilde{F}(y, x) - \tilde{F}(y', x') \right\|$$

for all $j \in \{1, \dots, n\}$. By Theorem 1.1.23 each generator of $I_D(\frac{\partial \tilde{F}}{\partial y})$ belongs to $\overline{I_D(\tilde{F})}$.

Now suppose that $I_D(\frac{\partial \tilde{F}}{\partial y}) \subsetneq \overline{I_D(\tilde{F})}$. Using the hypothesis and again the Lejeune-Teissier theorem, for each $j \in \{1, \dots, n\}$ there exists a constant $c_j > 0$ and an open subset $U_j \subsetneq \mathbb{C} \times \mathbb{C}^q$ such that

$$\left\| \left(\frac{\partial \tilde{f}_j}{\partial y} \right)_D(y, x, y', x') \right\| \leq c_j \left\| \tilde{F}(y, x) - \tilde{F}(y', x') \right\|$$

$\forall (y, x), (y', x') \in U_j$. Take $U := \bigcap_{j=1}^n U_j$, $c := \max\{c_j\}_{j=1}^n$ and $V := \tilde{F}(U)$, which is an open subset of $\tilde{F}(\mathbb{C} \times \mathbb{C}^q)$, since $\tilde{F}$ is a homeomorphism onto its image. Hence, it is easy to see that

$$\left\| v(y, z) - v(y', z') \right\| \leq c \left\| (y, z) - (y', z') \right\|$$

$\forall (y, z), (y', z') \in V$.

Therefore, the vector field $\frac{\partial}{\partial y} + \sum_{j=1}^n \frac{\partial \tilde{f}_j}{\partial y} \cdot \frac{\partial}{\partial z_j}$ is Lipschitz. $\square$

Now, we have an application for a special case of determinantal surfaces.

Proposition 4.2.2. *Suppose that $\tilde{F} : \mathbb{C} \times \mathbb{C}^q \longrightarrow \mathbb{C} \times \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ is an analytic map and a homeomorphism onto its image, and suppose we can write*

$$\tilde{F}(y, x) = (y, F(x) + y\theta(x)).$$

a) *The vector field $\frac{\partial}{\partial y} + \sum_j \frac{\partial \tilde{f}}{\partial y} \cdot \frac{\partial}{\partial z_j}$ is Lipschitz if and only if*

$$I_D(\theta) \subsetneq \overline{I_D(\tilde{F})}.$$

b) *If θ is constant then the vector field $\frac{\partial}{\partial y} + \sum_j \frac{\partial \tilde{f}}{\partial y} \cdot \frac{\partial}{\partial z_j}$ is Lipschitz.*

Proof. (a) It is a straightforward consequence of the last proposition because $\frac{\partial \tilde{f}}{\partial y} = \theta$.

(b) Since θ is constant then the doubles of the components of θ are all zero, so $I_D(\theta)$ is the zero ideal, which ensures the inclusion $I_D(\theta) \subsetneq \overline{I_D(\tilde{F})}$. $\square$

Remark 4.2.3. In [2] and [76], the authors consider a one parameter deformation with a constant θ . As showed above, for all these deformations the canonical vector field is Lipschitz.

In Example 4.2.4 we see a case where the deformation does not come from a constant θ , and the canonical vector field remains Lipschitz. In Example 4.2.5 we have another deformation that does not come from a constant θ where the canonical vector field is not Lipschitz.

As we have seen before, the canonical vector field is naturally associated to the 1-unfolding of the variety. However, its behaviour for the Lipschitz equisingularity is not the same. This behaviour depends on the type of the normal form, as we will see later.

The space of the first order deformations T_X^1 can be identified with $\frac{\text{Mat}_{(n,p)}(\mathcal{O}_r)}{\mathcal{TGF}}$, where $\mathcal{TGF}$ is the extended $\mathcal{G}$ -tangent space of the matrix F (Lemma 2.3, [37]). Hence we can treat the base of the semi-universal deformation using matrix representation and F is $\mathcal{G}$ -finitely determined if and only if T_X^1 is a finite dimensional module. From now on, the element θ is taken as an element of the space of the first order deformations $T_X^1 \cong \frac{\text{Mat}_{(n,p)}(\mathcal{O}_r)}{\mathcal{TGF}}$.

Example 4.2.4. Consider

$$F = \begin{pmatrix} w^l & y & x \\ z & w & y^k \end{pmatrix}$$

with $l, k \geq 2$, which is one of the normal forms obtained in Table 2. Consider the matrix of deformation

$$\theta = \begin{pmatrix} \sum_{i=0}^{l-1} w^i & 0 & 0 \\ 0 & 0 & \sum_{j=0}^{k-1} y^j \end{pmatrix}$$

and $\tilde{F}(u, x, y, z, w) = (u, F(x, y, z, w) + u\theta(x, y, z, w))$. Notice that $\theta \in \frac{\text{Mat}_{(2,3)}(\mathcal{O}_4)}{\mathcal{TGF}}$ and $I_D(\theta)$ is generated by $\{\sum_{i=0}^{l-1} (w^i - w'^i), \sum_{j=0}^{k-1} (y^j - y'^j)\}$. So, the generators are multiples of $w - w'$ and $y - y'$, respectively, and these linear differences belong to $I_D(\tilde{F})$. Therefore, $I_D(\theta) \subsetneq I_D(\tilde{F})$. By Proposition 4.2.2 we conclude that the canonical vector field is Lipschitz.

Example 4.2.5.

$$F = \begin{pmatrix} z & y + w^2 & x^2 \\ w^k & x & y \end{pmatrix}$$

with $k \geq 2$, which is one of the normal forms obtained in Table 2. Consider the matrix of deformation

$$\theta = \begin{pmatrix} 0 & 1 & xw + w \\ \sum_{i=0}^{k-1} w^i & w & w \end{pmatrix}$$

and $\tilde{F}(u, x, y, z, w) = (u, F(x, y, z, w) + u\theta(x, y, z, w))$. Notice that $\theta \in \frac{\text{Mat}_{(2,3)}(\mathcal{O}_4)}{\mathcal{TGF}}$, $I_D(\theta)$ is generated by $\{w - w', w(x + 1) - w'(x' + 1), \sum_{i=0}^{k-1} (w^i - w'^i)\}$ and $I_D(\tilde{F})$ is generated by $\{z - z', (y - y') + w^2 - w'^2, x^2 - x'^2 + u(xw + w) - u'(x'w' + w'), w^k - w'^k + u \sum_{i=0}^{k-1} w^i - u' \sum_{i=0}^{k-1} w'^i, x + uw - x' - u'w', y + uw - y' - u'w'\}$.

Consider the curve $\phi : \mathbb{C} \rightarrow (\mathbb{C} \times \mathbb{C}^4) \times (\mathbb{C} \times \mathbb{C}^4)$ given by $\phi(t) = (0, 0, 0, 0, 2t, 0, 0, 0, 0, t)$. Then, $I_D(\theta) \circ \phi$ is the ideal of $\mathcal{O}_1$ generated by t , and $I_D(\tilde{F}) \circ \phi$ is the ideal generated by t^2 . Hence, $I_D(\theta) \circ \phi \not\subseteq I_D(\tilde{F}) \circ \phi$ and by the curve criterion for the integral closure of ideals we conclude that $I_D(\theta) \not\subseteq \overline{I_D(\tilde{F})}$. Therefore, Proposition 4.2.2 ensures that the canonical vector field is not Lipschitz.

Theorem 4.2.6. Consider X a variety given by some $F : \mathbb{C}^q \rightarrow \text{Hom}(\mathbb{C}^3, \mathbb{C}^2)$ and $\tilde{F} : \mathbb{C} \times \mathbb{C}^q \rightarrow \mathbb{C} \times \text{Hom}(\mathbb{C}^3, \mathbb{C}^2)$, $q \in \{4, 5\}$ as in 4.2.2, where $\theta \in \frac{\text{Mat}_{(2,3)}(\mathcal{O}_q)}{T\mathcal{G}F}$. Suppose that X is a simple isolated Cohen-Macaulay variety of codimension 2.

If F is of 1-jet-type $J_{q,k}$ from Table 1 then the canonical vector field is Lipschitz, otherwise it is not.

Proof. Suppose that F is of 1-jet-type from Lemma 3.2 of [37]. Since $\theta \in \frac{\text{Mat}_{(2,3)}(\mathcal{O}_q)}{T\mathcal{G}F}$ then the q order 1 entries of the matrix F stay unperturbed, thus the differences of the monomial generators of the maximal ideal are in $I_D(\tilde{F})$. In particular the ideal I_Δ from the diagonal satisfies the inclusion $I_\Delta \subseteq I_D(\tilde{F})$. Let θ_i , $i \in \{1, \dots, 6\}$ be the components of θ . Notice that every $(\theta_i)_D$ vanishes on the diagonal Δ which implies that all the generators of $I_D(\theta)$ belong to I_Δ . Therefore, $I_D(\theta) \subseteq I_\Delta \subseteq I_D(\tilde{F})$ and the Proposition 4.2.2 ensures that the canonical vector field is Lipschitz.

Suppose the opposite. In this case, one of the generators of the maximal ideal is not an entry of the 1-jet of the matrix F . Without loss of generality, we may assume this is the first coordinate x . Since $\tilde{F}$ is a semiuniversal unfolding then $x - x'$ certainly appears as a part of a generator set of $I_D(\theta)$. Take the curve $\phi : (\mathbb{C}, 0) \rightarrow (\mathbb{C} \times \mathbb{C}^q) \times (\mathbb{C} \times \mathbb{C}^q)$ given by $\phi(t) = (0, 2t, 0, \dots, 0, t, 0, \dots)$. Then $I_D(\tilde{F}) \circ \phi$ is generated by t^m , for some $m > 1$. Since $(x - x') \circ \phi = t$ then $(x - x') \circ \phi \notin I_D(\tilde{F}) \circ \phi$, and by the curve criterion we conclude that $x - x' \notin \overline{I_D(\tilde{F})}$. Therefore, $I_D(\theta) \not\subseteq \overline{I_D(\tilde{F})}$ and the Proposition 4.2.2 ensures that the canonical vector field is not Lipschitz. $\square$

Remark 4.2.7. (Frühbis-Krüger) We can rephrase the condition on the jet-type by stating:

- a) The canonical vector field is Lipschitz if the ideal of 1-minors of the matrix of X defines a reduced point.
- b) The canonical vector field is not Lipschitz if the ideal of 1-minors of the matrix of X defines a fat point.

We can re-state the Theorem 4.2.6 in a more explicit way using the tables of classification of the Theorems 4.1.2 and 4.1.3.

Theorem 4.2.8. Consider X a variety given by some $F : \mathbb{C}^q \rightarrow \text{Hom}(\mathbb{C}^3, \mathbb{C}^2)$ and $\tilde{F} : \mathbb{C} \times \mathbb{C}^q \rightarrow \mathbb{C} \times \text{Hom}(\mathbb{C}^3, \mathbb{C}^2)$, $q \in \{4, 5\}$ as in 4.2.2, where $\theta \in \frac{\text{Mat}_{(2,3)}(\mathcal{O}_q)}{T\mathcal{G}F}$. Suppose that X is a simple isolated Cohen-Macaulay variety of codimension 2.

- a) Suppose $q = 4$. If the normal form of F is one of those first eight normal forms on 4.1.2, then the canonical vector field is Lipschitz. For the other normal forms, the canonical vector field is not Lipschitz.
- b) Suppose $q = 5$. If the normal form of F is one of those first six normal forms on 4.1.3, then the canonical vector field is Lipschitz. For the other normal forms, the canonical vector field is not Lipschitz.

Looking to tables 2 and 3 one can realize that the canonical vector field is Lipschitz exactly when the normal form has all the generators of the maximal ideal as an entry of the matrix.

CATEGORICAL ASPECTS OF THE DOUBLE STRUCTURE

In this section, our main goal is look to the categorical properties of the double structure, under an algebraic viewpoint.

5.1 The double homomorphism and basic properties

Let R be a ring.

Definition 5.1.1. Let $\mathcal{T}(R)$ be the category of the R -modules M which are R -submodules of R^p , for some natural number p .

Let $X \subseteq \mathbb{C}^n$ be an analytic space and let $\mathcal{O}_X$ be the analytic sheaf of local rings over X , and let $x \in X$.

Here we work on the categories $\mathcal{T}(\mathcal{O}_{X,x})$ and $\mathcal{T}(\mathcal{O}_{X \times X, (x,x)})$.

Consider the projection maps $\pi_1, \pi_2 : X \times X \rightarrow X$.

The first result is a quite useful tool many times when we work with the double.

Proposition 5.1.2. Let $M, N \subseteq \mathcal{O}_{X,x}^p$ submodules and $h, g \in \mathcal{O}_{X,x}^p$. Then:

- a) $h=g$ if and only if $h_D = g_D$;
- b) $h \in M$ if and only if $h_D \in M_D$;
- c) $M \subseteq N$ if and only if $M_D \subseteq N_D$;
- d) $M = N$ if and only if $M_D = N_D$.

Proof. (a) The implication ($\implies$) is obvious. Suppose that $h_D = g_D$. In particular, $h \circ \pi_1 = g \circ \pi_1$, and for all z in a neighborhood of x we have $h(z) = h \circ \pi_1(z, x) = g \circ \pi_1(z, x) = g(z)$, hence $h = g$.

(b) The implication ($\implies$) is obvious. Suppose now that $h_D \in M_D$. Then, we can write

$$h_D = \sum \alpha_i (g_i)_D$$

with $g_i \in M$ and $\alpha_i \in \mathcal{O}_{X \times X, (x, x)}$. In particular, $h \circ \pi_1 = \sum \alpha_i (g_i \circ \pi_1)$. Taking $\alpha_i^x \in \mathcal{O}_{X, x}$ given by $\alpha_i^x(z) := \alpha_i(z, x)$, $\forall i$, we get $h = \sum \alpha_i^x g_i$ which belongs to M .

(c) The implication ($\implies$) is obvious. Suppose that $M_D \subseteq N_D$. Let $h \in M$ arbitrary. Then $h_D \in M_D \subseteq N_D$, so by the item (b) we conclude that $h \in N$. Therefore, $M \subseteq N$.

(d) It is a straightforward consequence of the item (c). $\square$

Corollary 5.1.3. *For each $\mathcal{O}_{X, x}$ -submodule M of $\mathcal{O}_{X, x}^p$, the natural map*

$$\begin{aligned} D_M : M &\longrightarrow M_D \\ h &\longmapsto h_D \end{aligned}$$

is an injective group homomorphism. In particular, we can see M as an additive subgroup of M_D .

Proof. It is a straightforward consequence of the definition of the double that D_M is a group homomorphism. The Proposition 5.1.2 (a) gives the injectivity. $\square$

Our main goal is to give a categorical sense for the double structure. The next theorem is the key for it.

Theorem 5.1.4. *Let $M \subseteq \mathcal{O}_{X, x}^p$ and $N \subseteq \mathcal{O}_{X, x}^q$ be $\mathcal{O}_{X, x}$ -submodules. If $\phi : M \rightarrow N$ is an $\mathcal{O}_{X, x}$ -module homomorphism then there exists a unique $\mathcal{O}_{X \times X, (x, x)}$ -module homomorphism $\phi_D : M_D \rightarrow N_D$ such that $\phi_D(h_D) = (\phi(h))_D$, $\forall h \in M$, i.e, the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ D_M \downarrow & & \downarrow D_N \\ M_D & \xrightarrow{\phi_D} & N_D \end{array}$$

*The map ϕ_D is called the **double** of ϕ .*

Proof. Since M_D is generated by $\{h_D \mid h \in M\}$ then we can define $\phi_D : M_D \rightarrow N_D$ in a quite natural way: for each $u = \sum_i \alpha_i (h_i)_D$ with $\alpha_i \in \mathcal{O}_{X \times X, (x, x)}$ and $h_i \in M$ we define

$$\phi_D(u) := \sum_i \alpha_i (\phi(h_i))_D$$

which belongs to N_D .

Claim : ϕ_D is well defined. In fact, suppose that $\sum_i \alpha_i(h_i)_D = \sum_j \beta_j(g_j)_D$, with $\alpha_i, \beta_j \in \mathcal{O}_{X \times X, (x, x)}$ and $h_i, g_j \in M$. So, we get two equations:

$$\sum_i \alpha_i(h_i \circ \pi_1) = \sum_j \beta_j(g_j \circ \pi_1) \quad (1)$$

$$\sum_i \alpha_i(h_i \circ \pi_2) = \sum_j \beta_j(g_j \circ \pi_2). \quad (2)$$

Take U an open neighborhood of x in X where α_i, β_j are defined on $U \times U$, and h_i, g_j are defined on U . For each $w \in U$ define $\alpha_i^w, \beta_j^w \in \mathcal{O}_{X, x}$ given by the germs of the maps

$$\begin{array}{ccc} \alpha_i^w : U & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \alpha_i(z, w) \end{array} \quad \begin{array}{ccc} \beta_j^w : U & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \beta_j(z, w) \end{array}$$

The equation (1) implies that $\sum_i \alpha_i^w h_i = \sum_j \beta_j^w g_j$, $\forall w \in U$. Applying ϕ (which is an $\mathcal{O}_{X, x}$ -homomorphism) in both sides of the last equation we get $\sum_i \alpha_i^w \phi(h_i) = \sum_j \beta_j^w \phi(g_j)$, $\forall w \in U$. This implies that

$$\left(\sum_i \alpha_i(\phi(h_i) \circ \pi_1) \right)(z, w) = \left(\sum_j \beta_j(\phi(g_j) \circ \pi_1) \right)(z, w)$$

$\forall (z, w) \in U \times U$, hence

$$\sum_i \alpha_i(\phi(h_i) \circ \pi_1) = \sum_j \beta_j(\phi(g_j) \circ \pi_1). \quad (3)$$

Analogously, using the equation (2), we get

$$\sum_i \alpha_i(\phi(h_i) \circ \pi_2) = \sum_j \beta_j(\phi(g_j) \circ \pi_2). \quad (4)$$

The equations (3) and (4) implies that

$$\sum_i \alpha_i(\phi(h_i))_D = \sum_j \beta_j(\phi(g_j))_D$$

and the Claim is proved.

Now, by the definition of ϕ_D , it is clear that ϕ_D is an $\mathcal{O}_{X \times X, (x, x)}$ -module homomorphism and is the unique satisfying the property $\phi_D(h_D) = (\phi(h))_D$, $\forall h \in M$. $\square$

From now on, all the modules are objects in $\mathcal{T}(\mathcal{O}_{X, x})$ and their doubles are objects in $\mathcal{T}(\mathcal{O}_{X \times X, (x, x)})$.

Notice that if $id_M : M \rightarrow M$ and $id_{M_D} : M_D \rightarrow M_D$ are the identity homomorphisms of M and M_D , then

$$(id_M)_D = id_{M_D}.$$

The next proposition gives us a relation between images and kernels of a module homomorphism.

Proposition 5.1.5. *Let $\phi : M \rightarrow N$ be an $\mathcal{O}_{X,x}$ -module homomorphism and $\phi_D : M_D \rightarrow N_D$ its double. Then:*

$$a) \operatorname{Im}(\phi_D) = (\operatorname{Im}(\phi))_D;$$

$$b) (\ker(\phi))_D \subseteq \ker(\phi_D).$$

Proof. (a) By definition of ϕ_D , it is clear that $\operatorname{Im}(\phi_D) \subseteq (\operatorname{Im}(\phi))_D$. Now, if $g \in \operatorname{Im}(\phi)$ then we can write $g = \phi(h)$, for some $h \in M$. So, $g_D = (\phi(h))_D = \phi_D(h_D) \in \operatorname{Im}(\phi_D)$. Thus, $g_D \in \operatorname{Im}(\phi_D)$, $\forall g \in \operatorname{Im}(\phi)$, hence $(\operatorname{Im}(\phi))_D \subseteq \operatorname{Im}(\phi_D)$.

(b) For every $h \in \ker(\phi)$, we have that $\phi_D(h_D) = (\phi(h))_D = (0_N)_D = 0_{N_D}$, so $h_D \in \ker(\phi_D)$. $\square$

The next proposition shows that the double homomorphism has a good behavior with respect to sum and composition.

Proposition 5.1.6. *Let $\phi, \phi' : M \rightarrow N$ and $\gamma : N \rightarrow P$ be $\mathcal{O}_{X,x}$ -module homomorphisms.*

$$a) \phi = \phi' \text{ if and only if } \phi_D = \phi'_D;$$

$$b) (\gamma \circ \phi)_D = \gamma_D \circ \phi_D;$$

$$c) (\phi + \phi')_D = \phi_D + \phi'_D.$$

Proof. (a) ($\implies$) Suppose $\phi = \phi'$. For all $h \in M$, $\phi_D(h_D) = (\phi(h))_D = (\phi'(h))_D = \phi'_D(h_D)$. Since the module M_D is generated by $\{h_D \mid h \in M\}$ then $\phi_D = \phi'_D$.

($\impliedby$) Suppose $\phi_D = \phi'_D$. Given $h \in M$ arbitrary, we have $(\phi(h))_D = \phi_D(h_D) = \phi'_D(h_D) = (\phi'(h))_D$. By Proposition 5.1.2 (a), $\phi(h) = \phi'(h)$. Hence, $\phi = \phi'$.

(b) For every $h \in M$ we have $(\gamma \circ \phi)_D(h_D) = (\gamma \circ \phi(h))_D = \gamma_D((\phi(h))_D) = \gamma_D \circ \phi_D(h_D)$, which proves (b).

(c) For every $h \in M$ we have $(\phi + \phi')_D(h_D) = ((\phi + \phi')(h))_D = (\phi(h) + \phi'(h))_D = (\phi(h))_D + (\phi'(h))_D = (\phi_D + \phi'_D)(h_D)$, which proves (c). $\square$

In the next corollary we compare algebraic properties of a homomorphism and its double.

Corollary 5.1.7. *Let $\phi : M \rightarrow N$ be an $\mathcal{O}_{X,x}$ -module homomorphism. Then:*

- a) $\phi : M \rightarrow N$ is surjective if and only if $\phi_D : M_D \rightarrow N_D$ is surjective;
- b) If $\phi_D : M_D \rightarrow N_D$ is injective then $\phi : M \rightarrow N$ is injective;
- c) $\phi : M \rightarrow N$ is an $\mathcal{O}_{X,x}$ -isomorphism if and only if $\phi_D : M_D \rightarrow N_D$ is an $\mathcal{O}_{X \times X, (x,x)}$ -isomorphism;
- d) $\phi : M \rightarrow N$ is the zero homomorphism if and only if $\phi_D : M_D \rightarrow N_D$ is the zero homomorphism.

Proof. (a) By Propositions 5.1.2 (d) and 5.1.5 (a) we have: ϕ is surjective $\iff \text{Im}(\phi) = N$
 $\iff (\text{Im}(\phi))_D = N_D \iff \text{Im}(\phi_D) = N_D \iff \phi_D$ is surjective.

(b) If ϕ_D is injective then $\ker(\phi_D) = 0_{M_D}$, and the Proposition 5.1.5 (b) implies that $(\ker(\phi))_D \subseteq \ker(\phi_D) = 0_{M_D} = (0_M)_D$. By Proposition 5.1.2 (c) we conclude that $\ker(\phi) = 0_M$, hence ϕ is injective.

(c) ($\implies$) Since $\phi : M \rightarrow N$ is an isomorphism then there exists an $\mathcal{O}_{X,x}$ -homomorphism $\gamma : N \rightarrow M$ such that $\gamma \circ \phi = \text{id}_M$ and $\phi \circ \gamma = \text{id}_N$, then, by Proposition 5.1.6 (b) we have $(\gamma)_D \circ (\phi)_D = \text{id}_{M_D}$ and $(\phi)_D \circ (\gamma)_D = \text{id}_{N_D}$. Hence, ϕ_D is an isomorphism.

($\impliedby$) It follows immediately from (a) and (b).

(d) By Propositions 5.1.2 (d) and 5.1.5 (a) we have: ϕ is the zero homomorphism $\iff \text{Im}(\phi) = 0_N \iff (\text{Im}(\phi))_D = (0_N)_D \iff \text{Im}(\phi_D) = 0_{N_D} \iff \phi_D$ is the zero homomorphism. $\square$

Definition 5.1.8. We say that an $\mathcal{O}_{X,x}$ -homomorphism $\phi : M \subseteq \mathcal{O}_{X,x}^p \rightarrow N \subseteq \mathcal{O}_{X,x}^q$ is induced by a $q \times p$ matrix if there exists $A \in \text{Mat}_{q \times p}(\mathcal{O}_{X,x})$ such that $\phi(h) = A \cdot h$, $\forall h \in M$.

Lemma 5.1.9. An $\mathcal{O}_{X,x}$ -homomorphism $\phi : M \subseteq \mathcal{O}_{X,x}^p \rightarrow N \subseteq \mathcal{O}_{X,x}^q$ is induced by a $q \times p$ matrix if and only if there exists an $\mathcal{O}_{X,x}$ -homomorphism $\tilde{\phi} : \mathcal{O}_{X,x}^p \rightarrow \mathcal{O}_{X,x}^q$ such that $\tilde{\phi}(M) \subseteq N$ and $\tilde{\phi}|_M = \phi$.

Proof. ($\implies$) By hypothesis there exists a $q \times p$ matrix A with entries in $\mathcal{O}_{X,x}$ such that $\phi(h) = A \cdot h$, $\forall h \in M$. From this matrix A , we can define $\tilde{\phi} : \mathcal{O}_{X,x}^p \rightarrow \mathcal{O}_{X,x}^q$ given by $\tilde{\phi}(g) := A \cdot g$, which is an $\mathcal{O}_{X,x}$ -homomorphism. Clearly, $\tilde{\phi}|_M = \phi$, and for all $h \in M$ we have $\tilde{\phi}(h) = \phi(h) \in N$, so $\tilde{\phi}(M) \subseteq N$.

($\impliedby$) Let $e_1, \dots, e_p$ be the canonical elements in $\mathcal{O}_{X,x}^p$. Let A be the $q \times p$ matrix whose columns are $\phi(e_1), \dots, \phi(e_p)$. Then $\tilde{\phi}(g) = A \cdot g$, $\forall g \in \mathcal{O}_{X,x}^p$. Since $\tilde{\phi}|_M = \phi$ then $\phi(h) = \tilde{\phi}(h) = A \cdot h$, $\forall h \in M$. Therefore, ϕ is induced by a $q \times p$ matrix. $\square$

Proposition 5.1.10. *If $\phi : M \subseteq \mathcal{O}_{X,x}^p \rightarrow N \subseteq \mathcal{O}_{X,x}^q$ is an $\mathcal{O}_{X,x}$ -homomorphism induced by a $q \times p$ matrix then*

$$\phi_D : M_D \subseteq \mathcal{O}_{X \times X, (x,x)}^{2p} \rightarrow N_D \subseteq \mathcal{O}_{X \times X, (x,x)}^{2q}$$

is an $\mathcal{O}_{X \times X, (x,x)}$ -homomorphism induced by a $2q \times 2p$ matrix.

Proof. By hypothesis there exists a $q \times p$ matrix A with entries in $\mathcal{O}_{X,x}$ such that $\phi(h) = A \cdot h$, $\forall h \in M$. Then, for all $h \in M$ we have $\phi_D(h_D) = \begin{bmatrix} \phi(h) \circ \pi_1 \\ \phi(h) \circ \pi_2 \end{bmatrix} = \begin{bmatrix} (A \cdot h) \circ \pi_1 \\ (A \cdot h) \circ \pi_2 \end{bmatrix} = \begin{bmatrix} (A \circ \pi_1) \cdot (h \circ \pi_1) \\ (A \circ \pi_2) \cdot (h \circ \pi_2) \end{bmatrix}$.

So, taking the $2q \times 2p$ matrix

$$B := \begin{bmatrix} A \circ \pi_1 & 0_{q \times p} \\ 0_{q \times p} & A \circ \pi_2 \end{bmatrix}$$

we conclude that $\phi_D(h_D) = B \cdot h_D$, and the proposition is proved, once M_D is generated by h_D , $h \in M$. $\square$

As an application of the double homomorphism, we prove in the next theorem that the double structure is compatible with finite direct sum of modules.

Theorem 5.1.11. *Let $M \subseteq \mathcal{O}_{X,x}^p$ and $N \subseteq \mathcal{O}_{X,x}^q$ be $\mathcal{O}_{X,x}$ -submodules. Then*

$$(M \oplus N)_D \cong M_D \oplus N_D$$

as $\mathcal{O}_{X \times X, (x,x)}$ -submodules of $\mathcal{O}_{X \times X, (x,x)}^{2(p+q)}$.

Furthermore, there exists an isomorphism

$$\eta : (M \oplus N)_D \longrightarrow M_D \oplus N_D$$

such that $\eta((h, g)_D) = (h_D, g_D)$, for all $h \in M$ and $g \in N$.

Proof. Consider the canonical projections and inclusions:

$$\begin{array}{lll} \psi_1 : M \oplus N & \longrightarrow & M \\ (h, g) & \longmapsto & h \end{array} \quad \begin{array}{lll} \psi_2 : M \oplus N & \longrightarrow & N \\ (h, g) & \longmapsto & g \end{array}$$

$$\begin{array}{lll} \delta_1 : M & \longrightarrow & M \oplus N \\ h & \longmapsto & (h, 0_N) \end{array} \quad \begin{array}{lll} \delta_2 : N & \longrightarrow & M \oplus N \\ g & \longmapsto & (0_M, g) \end{array}$$

Thus, we get the double homomorphism of each one above:

$$\begin{array}{lll} (\psi_1)_D : (M \oplus N)_D & \longrightarrow & M_D \\ (\delta_1)_D : M_D & \longrightarrow & (M \oplus N)_D \end{array} \quad \begin{array}{lll} (\psi_2)_D : (M \oplus N)_D & \longrightarrow & N_D \\ (\delta_2)_D : N_D & \longrightarrow & (M \oplus N)_D \end{array}$$

Define:

$$\begin{aligned} \eta : (M \oplus N)_D &\longrightarrow M_D \oplus N_D \\ w &\longmapsto ((\psi_1)_D(w), (\psi_2)_D(w)) \end{aligned}$$

$$\begin{aligned} \delta : M_D \oplus N_D &\longrightarrow (M \oplus N)_D \\ (u, v) &\longmapsto (\delta_1)_D(u) + (\delta_2)_D(v) \end{aligned}$$

which are $\mathcal{O}_{X \times X, (x, x)}$ -module homomorphisms.

Claim 1: $\eta((h, g)_D) = (h_D, g_D)$, for all $h \in M$ and $g \in N$.

In fact, $\eta((h, g)_D) = ((\psi_1)_D((h, g)_D), (\psi_2)_D((h, g)_D)) = ((\psi_1(h, g))_D, (\psi_2(h, g))_D) = (h_D, g_D)$.

Claim 2: $\delta(h_D, g_D) = (h, g)_D$, for all $h \in M$ and $g \in N$.

In fact, $\delta(h_D, g_D) = (\delta_1)_D(h_D) + (\delta_2)_D(g_D) = (\delta_1(h))_D + (\delta_2(g))_D = (h, 0_N)_D + (0_M, g)_D = ((h, 0_N) + (0_M, g))_D = (h, g)_D$.

By the Claims 1 and 2 we have that $\delta \circ \eta((h, g)_D) = (h, g)_D$, $\eta \circ \delta(h_D, 0_{N_D}) = (h_D, 0_{N_D})$ and $\eta \circ \delta(0_{M_D}, g_D) = (0_{M_D}, g_D)$, for all $h \in M$ and $g \in N$.

Since $\{(h, g)_D \mid h \in M \text{ and } g \in N\}$ is a generator set of $(M \oplus N)_D$ and $\{(h_D, 0_{N_D}), (0_{M_D}, g_D) \mid h \in M \text{ and } g \in N\}$ is a generator set of $M_D \oplus N_D$ then we conclude that $\delta \circ \eta = id_{(M \oplus N)_D}$ and $\eta \circ \delta = id_{M_D \oplus N_D}$, which finishes the proof of the theorem. $\square$

Corollary 5.1.12. Let $M_i \subseteq \mathcal{O}_{X, x}^{p_i}$ be $\mathcal{O}_{X, x}$ -submodules, for each $i \in \{1, \dots, r\}$. Then

$$(M_1 \oplus \dots \oplus M_r)_D \cong (M_1)_D \oplus \dots \oplus (M_r)_D$$

as $\mathcal{O}_{X \times X, (x, x)}$ -submodules of $\mathcal{O}_{X \times X, (x, x)}^{2(p_1 + \dots + p_r)}$ through an isomorphism such that

$$(h_1, \dots, h_r)_D \longmapsto ((h_1)_D, \dots, (h_r)_D)$$

for all $h_i \in M_i$.

Proof. Induction on r and use the previous theorem. $\square$

The next proposition compares the length of a module and its double.

Proposition 5.1.13. Let $M \subseteq N$ be $\mathcal{O}_{X, x}$ -submodules of $\mathcal{O}_{X, x}^p$.

a) If M_D has finite length then M has finite length and $\ell(M) \leq \ell(M_D)$;

b) If M_D has finite colength in N_D then M has finite colength in N .

Proof. (a) If $r \in \mathbb{N}$ and $(M_i)_{i=0}^r$ is an ascending series of M , then $((M_i)_D)_{i=0}^r$ is an ascending series of M_D , which has finite length. Thus, $r \leq \ell(M_D)$. Therefore $\ell(M)$ is finite and $\ell(M) \leq \ell(M_D)$.

(b) Let $r \in \mathbb{N}$ and consider an arbitrary ascending series of $\frac{N}{M}$ of length r . This series can be given on the form

$$\frac{N_0}{M} \subsetneq \frac{N_1}{M} \subsetneq \dots \subsetneq \frac{N_{r-1}}{M} \subsetneq \frac{N_r}{M} = \frac{N}{M}$$

where $N_0 \subsetneq N_1 \subsetneq \dots \subsetneq N_{r-1} \subsetneq N_r = N$ are $\mathcal{O}_{X,x}$ -submodules of N which contain M . Then,

$$\frac{(N_0)_D}{(M_D)} \subsetneq \frac{(N_1)_D}{(M_D)} \subsetneq \dots \subsetneq \frac{(N_{r-1})_D}{(M_D)} \subsetneq \frac{(N_r)_D}{(M_D)} = \frac{N_D}{M_D}$$

is an ascending series of $\frac{N_D}{M_D}$, which has finite length by hypothesis. Hence, $r \leq \ell(\frac{N_D}{M_D})$ and $\ell(\frac{N}{M})$ is finite. $\square$

We want to use the double homomorphism in order to get an equivalence between the second and third Lipschitz saturations.

The next proposition gives the "persistence" of the integral closure of modules.

Proposition 5.1.14. *Let $\varphi : M \subseteq \mathcal{O}_{X,x}^p \rightarrow N \subseteq \mathcal{O}_{X,x}^q$ be a homomorphism of $\mathcal{O}_{X,x}$ -modules which can be extended to a homomorphism $\tilde{\varphi} : \mathcal{O}_{X,x}^p \rightarrow c\mathcal{O}_{X,x}^q$, given by $\tilde{\varphi}(h) = A \cdot h$, where A is a $q \times p$ matrix with entries in $\mathcal{O}_{X,x}$. Let $h \in \mathcal{O}_{X,x}^p$.*

a) *If $h \in \overline{M}$ then $\tilde{\varphi}(h) \in \overline{\varphi(M)}$;*

b) *Suppose $q = p$. If A is an invertible matrix and φ is an isomorphism of $\mathcal{O}_{X,x}$ -modules then:*

$$h \in \overline{M} \text{ if and only if } \tilde{\varphi}(h) \in \overline{\varphi(M)};$$

c) *Suppose $q = p$ and φ is injective. If $\tilde{\varphi}(h) \in \overline{\varphi(M)}$ then $h \in \overline{M}$.*

Proof. (a) Let $\phi : (\mathbb{C}, 0) \rightarrow (X, x)$ be an arbitrary analytic curve. By hypothesis $\phi^*(h) \in \phi^*(M)$ and we can write $\phi^*(h) = \sum \alpha_i \phi^*(g_i)$, for some $g_i \in M$ and $\alpha_i \in \mathcal{O}_{\mathbb{C},0}$. Thus: $\tilde{\varphi}(h) \circ \phi = (A \cdot h) \circ \phi = [A \circ \phi] \cdot [h \circ \phi] = [A \circ \phi] \cdot \sum \alpha_i \phi^*(g_i) = \sum \alpha_i ([A \cdot g_i] \circ \phi) \in \phi^*(\varphi(M))$.

Hence, $\tilde{\varphi}(h) \in \overline{\varphi(M)}$.

(b) It suffices apply (a) in $\tilde{\varphi}^{-1}$.

(c) It suffices consider the isomorphism $\hat{\varphi} : M \rightarrow \varphi(M)$ given by $\hat{\varphi}(h) = \varphi(h)$ and apply the item (b). $\square$

For each $i \in \{1, \dots, p\}$ consider the i -th canonical global section of the vector bundle $\text{Hom}(\mathbb{C}^p, \mathbb{C})$,

$$\xi_i : X \rightarrow \text{Hom}(\mathbb{C}^p, \mathbb{C})$$

given by $\xi_i(x) = (0, \dots, 1, \dots, 0)$, where 1 is on the i -th place.

Notice that if M is an $\mathcal{O}_{X,x}$ -submodule of $\mathcal{O}_{X,x}^p$ then $M \subseteq \xi_1 \cdot M \oplus \dots \oplus \xi_p \cdot M$.

Theorem 5.1.15. *Let $M \subseteq \mathcal{O}_{X,x}^p$ be a submodule. Suppose that M_D is a reduction of $(\xi_1 \cdot M \oplus \dots \oplus \xi_p \cdot M)_D$. Then, $M_{S_2} = M_{S_3}$.*

Proof. Consider the inclusion $i : M \rightarrow \xi_1 \cdot M \oplus \dots \oplus \xi_p \cdot M$. Then we can consider the inclusion $i_D : M_D \rightarrow (\xi_1 \cdot M \oplus \dots \oplus \xi_p \cdot M)_D$ which is induced by an invertible $2p \times 2p$ matrix. By Corollary 5.1.12 there is an isomorphism

$$\gamma : (\xi_1 \cdot M \oplus \dots \oplus \xi_p \cdot M)_D \rightarrow (\xi_1 \cdot M)_D \oplus \dots \oplus (\xi_p \cdot M)_D$$

and by the proof of this corollary, this isomorphism is induced by an invertible $2p \times 2p$ matrix. Taking the composition of i_D with γ , we get an injective homomorphism

$$\eta : M_D \rightarrow (\xi_1 \cdot M)_D \oplus \dots \oplus (\xi_p \cdot M)_D$$

induced by an invertible $2p \times 2p$ matrix B which extends to the isomorphism

$$\tilde{\eta} : \mathcal{O}_{X \times X, (x,x)}^{2p} \rightarrow \mathcal{O}_{X \times X, (x,x)}^{2p}$$

given by the multiplication by B , which satisfies the property

$$(g_1, \dots, g_p)_D \mapsto ((g_1)_D, \dots, (g_p)_D).$$

By Proposition 3.1.22 we have the inclusion $M_{S_2} \subseteq M_{S_3}$. So, it suffices to check the another inclusion.

Let $h \in M_{S_3}$. In particular, $(\xi_i \cdot h)_D \in \overline{(\xi_i \cdot M)_D}$, $\forall i \in \{1, \dots, p\}$. Let $\phi : (\mathbb{C}, 0) \rightarrow (X \times X, (x, x))$ be an arbitrary analytic curve. Then $\phi^*((\xi_i \cdot h)_D) \in \phi^*((\xi_i \cdot M)_D)$, $\forall i \in \{1, \dots, p\}$ and $\phi^*(\tilde{\eta}(h_D)) = \phi^*((\xi_1 \cdot h)_D, \dots, (\xi_p \cdot h)_D) = (\phi^*((\xi_1 \cdot h)_D), \dots, \phi^*((\xi_p \cdot h)_D))$ which belongs to $\phi^*((\xi_1 \cdot M)_D \oplus \dots \oplus (\xi_p \cdot M)_D)$. Hence, $\tilde{\eta}(h_D) \in \overline{(\xi_1 \cdot M)_D \oplus \dots \oplus (\xi_p \cdot M)_D}$.

Since M_D is a reduction of $(\xi_1 \cdot M \oplus \dots \oplus \xi_p \cdot M)_D$ then by the previous proposition we have that $\eta(M_D)$ is a reduction of $(\xi_1 \cdot M)_D \oplus \dots \oplus (\xi_p \cdot M)_D$. Thus, $\tilde{\eta}(h_D) \in \overline{\eta(M_D)}$, and by the previous proposition we conclude that $h_D \in \overline{M_D}$, therefore $h \in M_{S_2}$. $\square$

Example 5.1.16. Notice that any p -symmetric submodule $M \subseteq \mathcal{O}_{X,x}^p$ satisfies the hypothesis of the previous theorem, so $M_{S_2} = M_{S_3}$. Here, **p-symmetric** means that given $(h_1, \dots, h_p) \in M$ arbitrary, then $(h_{\sigma(1)}, \dots, h_{\sigma(p)}) \in M$ for every p -permutation σ . Observe that is equivalent to $M = \xi_1 \cdot M \oplus \dots \oplus \xi_p \cdot M$.

5.2 Homological aspects of the double structure

In this section we define the *double chain* of a given chain complex, and we compare homological properties between these complexes.

First, we need the following result.

Proposition 5.2.1. *Let*

$$M \xrightarrow{\phi} N \xrightarrow{\gamma} P$$

be a sequence of $\mathcal{O}_{X,x}$ -module homomorphism and consider the double sequence

$$M_D \xrightarrow{\phi_D} N_D \xrightarrow{\gamma_D} P_D$$

If $\text{Im}(\phi) \subseteq \ker(\gamma)$ then $\text{Im}(\phi_D) \subseteq \ker(\gamma_D)$.

Proof. Since $\text{Im}(\phi) \subseteq \ker(\gamma)$ then $(\text{Im}(\phi))_D \subseteq (\ker(\gamma))_D$. Hence, $\text{Im}(\phi_D) = (\text{Im}(\phi))_D \subseteq (\ker(\gamma))_D \subseteq \ker(\gamma_D)$. $\square$

We realize that the double homomorphism gives a natural way to study the homology of the double structure.

Definition 5.2.2 (The double chain complex). *Let $\mathcal{C} = (M_\bullet, \phi_\bullet)$ be a chain complex in $\mathcal{T}(\mathcal{O}_{X,x})$. We define*

$$\mathcal{C}_D := ((M_\bullet)_D, (\phi_\bullet)_D)$$

*and by Proposition 5.2.1 we have that $\mathcal{C}_D$ is a chain complex in $\mathcal{T}(\mathcal{O}_{X \times X, (x,x)})$. The chain complex $\mathcal{C}_D$ is called **the double of $\mathcal{C}$** .*

Proposition 5.2.3. *Let $\mathcal{C} = (M_\bullet, \phi_\bullet)$ be a chain complex. If $\mathcal{C}_D$ is an exact sequence then $\mathcal{C}$ is an exact sequence. In other words, if $\mathcal{C}_D$ has trivial homology then $\mathcal{C}$ has trivial homology.*

Proof. Let $i \in \mathbb{Z}$ be arbitrary. We have the commutative diagram:

$$\begin{array}{ccccc} M_{i+1} & \xrightarrow{\phi_{i+1}} & M_i & \xrightarrow{\phi_i} & M_{i-1} \\ \downarrow & & \downarrow & & \downarrow \\ (M_{i+1})_D & \xrightarrow{(\phi_{i+1})_D} & (M_i)_D & \xrightarrow{(\phi_i)_D} & (M_{i-1})_D \end{array}$$

where the vertical arrows are the canonical morphisms. We already know that $Im(\phi_{i+1}) \subseteq \ker(\phi_i)$. Since $\mathcal{C}_D$ is an exact sequence then $Im((\phi_{i+1})_D) = \ker((\phi_i)_D)$. By Proposition 5.1.5 we have $(\ker(\phi_i))_D \subseteq \ker((\phi_i)_D) = Im((\phi_{i+1})_D) = (Im(\phi_{i+1}))_D$. By Proposition 5.1.2 (c), we conclude that $\ker(\phi_i) \subseteq Im(\phi_{i+1})$. Therefore, $Im(\phi_{i+1}) = \ker(\phi_i)$. $\square$

Proposition 5.2.4. *Let $\mathcal{C} = (M_\bullet, \phi_\bullet)$ and $\mathcal{C}' = (M'_\bullet, \phi'_\bullet)$ be chain complexes. If $\alpha : \mathcal{C} \rightarrow \mathcal{C}'$ is a chain complex morphism then $\alpha_D : \mathcal{C}_D \rightarrow \mathcal{C}'_D$ given by $\{(\alpha_i)_D : (M_i)_D \rightarrow (M'_i)_D \mid i \in \mathbb{Z}\}$ is a chain complex morphism, called the **the double morphism of α** .*

Proof. Let $i \in \mathbb{Z}$. We have the following commutative diagram

$$\begin{array}{ccc} M_i & \xrightarrow{\phi_i} & M_{i-1} \\ \alpha_i \downarrow & & \downarrow \alpha_{i-1} \\ M'_i & \xrightarrow{\phi'_i} & M'_{i-1} \end{array}$$

By Proposition 5.1.6 (b) follows that $(\phi'_i)_D \circ (\alpha_i)_D = (\phi'_i \circ \alpha_i)_D = (\alpha_{i-1} \circ \phi_i)_D = (\alpha_{i-1})_D \circ (\phi_i)_D$, and the following diagram is also commutative:

$$\begin{array}{ccc} (M_i)_D & \xrightarrow{(\phi_i)_D} & (M_{i-1})_D \\ (\alpha_i)_D \downarrow & & \downarrow (\alpha_{i-1})_D \\ (M'_i)_D & \xrightarrow{(\phi'_i)_D} & (M'_{i-1})_D \end{array}$$

$\square$

Corollary 5.2.5. *If $\alpha : \mathcal{C} \rightarrow \mathcal{C}'$ and $\beta : \mathcal{C}' \rightarrow \mathcal{C}''$ are chain morphisms then*

$$(\beta \circ \alpha)_D = \beta_D \circ \alpha_D.$$

Proof. It is a straightforward consequence of the Proposition 5.1.6 (b). $\square$

Now, we get some results related to chain homotopy.

Let $\mathcal{C} = (M_\bullet, \phi_\bullet)$ and $\mathcal{C}' = (M'_\bullet, \phi'_\bullet)$ be chain complexes. Let $\mu : \mathcal{C} \rightarrow \mathcal{C}'$ be a homomorphism of degree 1, i.e, μ is a collection of $\mathcal{O}_{X,x}$ -module homomorphisms $\{\mu_i : M_i \rightarrow M'_{i+1} \mid i \in \mathbb{Z}\}$. We know this homomorphism induces a chain morphism $\tilde{\mu} : \mathcal{C} \rightarrow \mathcal{C}'$ given by $\{\tilde{\mu}_i : M_i \rightarrow M'_i \mid i \in \mathbb{Z}\}$, where $\tilde{\mu}_i := \phi'_{i+1} \circ \mu_i + \mu_{i-1} \circ \phi_i, \forall i \in \mathbb{Z}$.

If $\alpha, \beta : \mathcal{C} \rightarrow \mathcal{C}'$ are chain morphisms, remember that $\mu : \mathcal{C} \rightarrow \mathcal{C}'$ is defined as a **homotopy between α and β** when $\tilde{\mu} = \alpha - \beta$, and we denote $\alpha \underset{\mu}{\simeq} \beta$.

Lemma 5.2.6. Consider $\mu_D : \mathcal{C}_D \rightarrow \mathcal{C}'_D$ the homomorphism of degree 1 given by the double homomorphisms of $\mu : \mathcal{C} \rightarrow \mathcal{C}'$. Then $\widetilde{\mu}_D = (\widetilde{\mu})_D$.

Proof. For all $i \in \mathbb{Z}$ we have $(\widetilde{\mu}_D)_i = (\phi'_{i+1})_D \circ (\mu_i)_D + (\mu_{i-1})_D \circ (\phi_i)_D$
 $= (\phi'_{i+1} \circ \mu_i + \mu_{i-1} \circ \phi_i)_D = (\widetilde{\mu}_i)_D$, and the lemma is proved. $\square$

Proposition 5.2.7. Let $\alpha, \beta : \mathcal{C} \rightarrow \mathcal{C}'$ be chain morphisms and $\mu : \mathcal{C} \rightarrow \mathcal{C}'$ a homomorphism of degree 1. Then: μ is a homotopy between α and β if and only if μ_D is a homotopy between α_D and β_D .

Proof. We have that μ is a homotopy between α and β if and only if $\tilde{\mu} = \alpha - \beta$. By Proposition 5.1.6 (a) and (c) and the previous lemma we have: $\tilde{\mu} = \alpha - \beta \iff (\tilde{\mu})_D = (\alpha - \beta)_D \iff \widetilde{\mu}_D = \alpha_D - \beta_D \iff \mu_D$ is a homotopy between α_D and β_D . $\square$

Corollary 5.2.8. If $\alpha : \mathcal{C} \rightarrow \mathcal{C}'$ is a chain homotopy equivalence then $\alpha_D : \mathcal{C}_D \rightarrow \mathcal{C}'_D$ is a chain homotopy equivalence.

Proof. By hypothesis there exists a chain morphism $\beta : \mathcal{C}' \rightarrow \mathcal{C}$ such that $\beta \circ \alpha \simeq id_{\mathcal{C}}$ and $\alpha \circ \beta \simeq id_{\mathcal{C}'}$. By the previous proposition, we have $(\beta \circ \alpha)_D \simeq (id_{\mathcal{C}})_D$ and $(\alpha \circ \beta)_D \simeq (id_{\mathcal{C}'}_D)_D$, and therefore $(\beta)_D \circ (\alpha)_D \simeq id_{\mathcal{C}_D}$ and $\alpha_D \circ \beta_D \simeq id_{\mathcal{C}'_D}$. $\square$

Corollary 5.2.9. If $\mathcal{C}$ is a contractible chain complex then $\mathcal{C}_D$ is also contractible.

Proof. Since $\mathcal{C}$ is contractible then $id_{\mathcal{C}} \simeq 0_{\mathcal{C}}$, and the Proposition 5.2.7 implies that $(id_{\mathcal{C}})_D \simeq (0_{\mathcal{C}})_D$, thus $id_{\mathcal{C}_D} \simeq 0_{\mathcal{C}_D}$. Hence, $\mathcal{C}_D$ is contractible. $\square$

Notice the nice relation between the Proposition 5.2.3 and Corollary 5.2.9. We already know every contractible chain complex is an exact sequence. The Proposition 5.2.3 says that the exactness on the double level implies the exactness on the "single" level. The Corollary 5.2.9, which treats about contractible (stronger than exactness), says the opposite.

It is clear that all the results obtained in this section can be naturally translated to the cohomology language.

5.3 The Double category

Let us define the category $\mathcal{D}(\mathcal{O}_{X,x})$.

The objects of $\mathcal{D}(\mathcal{O}_{X,x})$ consist of the double of modules in $\mathcal{T}(\mathcal{O}_{X,x})$. Given M_D, N_D objects in $\mathcal{D}(\mathcal{O}_{X,x})$, we define

$$\text{Mor}(M_D, N_D) := \{\phi_D : M_D \rightarrow N_D \mid \phi : M \rightarrow N \text{ is an } \mathcal{O}_{X,x}\text{-module homomorphism}\}.$$

Working with the standard composition of maps, we have the category $\mathcal{D}(\mathcal{O}_{X,x})$, called the **Double category of (X, x)** .

The morphisms in $\mathcal{D}(\mathcal{O}_{X,x})$ are called $\mathcal{O}_{X,x}$ -double morphisms. Observe that the $\mathcal{O}_{X,x}$ -double morphisms are $\mathcal{O}_{X \times X, (x,x)}$ -homomorphisms with an additional property: they preserve the double structure.

Notice that $\mathcal{D}(\mathcal{O}_{X,x})$ is a subcategory of $\mathcal{T}(\mathcal{O}_{X \times X, (x,x)})$.

Theorem 5.3.1. *The covariant functor*

$$\begin{array}{ccc} \mathbf{D} : & \mathcal{T}(\mathcal{O}_{X,x}) & \longrightarrow & \mathcal{D}(\mathcal{O}_{X,x}) \\ & M & \longmapsto & M_D \\ & \phi : M \rightarrow N & \longmapsto & \phi_D : M_D \rightarrow N_D \end{array}$$

is an isomorphism of categories.

Proof. The Proposition 5.1.2 (d) proves that the map between the objects is a bijection, and the Proposition 5.1.6 (a) proves that the map between the morphisms is a bijection. Hence, $\mathbf{D}$ is an isomorphism of categories. $\square$

Corollary 5.3.2. $\mathcal{T}(\mathcal{O}_{X,x})$ can be seen as a subcategory of $\mathcal{T}(\mathcal{O}_{X \times X, (x,x)})$.

The Theorem 5.3.1 implies that $\mathcal{T}(\mathcal{O}_{X,x})$ and $\mathcal{D}(\mathcal{O}_{X,x})$ are essentially the same category, so they have the same behavior in all of the categorical statements. But, one of them is reasonable to emphasize, in the next corollary, which is interesting to compare with the result obtained in the Corollary 5.1.7.

Corollary 5.3.3. *Let $\phi : M \rightarrow N$ be an $\mathcal{O}_{X,x}$ -module homomorphism. Then:*

- a) $\phi : M \rightarrow N$ is an $\mathcal{O}_{X,x}$ -monomorphism of modules if and only if $\phi_D : M_D \rightarrow N_D$ is an $\mathcal{O}_{X,x}$ -double monomorphism;
- b) $\phi : M \rightarrow N$ is an $\mathcal{O}_{X,x}$ -epimorphism of modules if and only if $\phi_D : M_D \rightarrow N_D$ is an $\mathcal{O}_{X,x}$ -double epimorphism;
- c) $\phi : M \rightarrow N$ is an $\mathcal{O}_{X,x}$ -isomorphism of modules if and only if $\phi_D : M_D \rightarrow N_D$ is an $\mathcal{O}_{X,x}$ -double isomorphism.

Here it is reasonable to emphasize the difference between the notions of injective homomorphism and monomorphism of modules, which are not the same thing in the category of modules, since we are understanding the term **monomorphism** in the categorical sense, i.e., there is a left-inverse morphism.

The same remark has to be done between surjective homomorphism and epimorphism.

Remark 5.3.4. *The covariant functor*

$$\begin{aligned} \mathbf{D}: \quad \mathcal{T}(\mathcal{O}_{X,x}) &\longrightarrow \mathcal{T}(\mathcal{O}_{X \times X, (x,x)}) \\ M &\longmapsto M_D \\ \phi: M \rightarrow N &\longmapsto \phi_D: M_D \rightarrow N_D \end{aligned}$$

is not an isomorphism of categories anymore.

In fact, suppose (X, x) irreducible. It is proved in 2.1.17 that the generic rank of the double of every module M in $\mathcal{T}(\mathcal{O}_{X,x})$ has generic rank even. Thus, the map between the objects cannot be surjective.

5.4 The double in a quotient of a free $\mathcal{O}_X$ -module of finite rank

Let W be an $\mathcal{O}_{X,x}$ -submodule of $\mathcal{O}_{X,x}^p$ and consider the quotient map

$$\pi: \mathcal{O}_{X,x}^p \longrightarrow \frac{\mathcal{O}_{X,x}^p}{W}.$$

Then, W_D is an $\mathcal{O}_{X \times X, (x,x)}$ -submodule of $\mathcal{O}_{X \times X, (x,x)}^{2p}$.

Let $h \in \mathcal{O}_{X,x}^p$. We define the double of $h + W \in \frac{\mathcal{O}_{X,x}^p}{W}$ as

$$(h + W)_D := h_D + W_D \in \frac{\mathcal{O}_{X \times X, (x,x)}^{2p}}{W_D}.$$

Notice the definition of $(h + W)_D$ does not depend of the choice of the representative h . In fact, if $h + W = g + W$ then $h - g \in W \implies (h - g)_D \in W_D \implies h_D - g_D \in W_D \implies h_D + W_D = g_D + W_D$.

Now, we want to define the double of a submodule M of $\frac{\mathcal{O}_{X,x}^p}{W}$. We have that $\pi^{-1}(M)$ is a submodule of $\mathcal{O}_{X,x}^p$ and $\pi^{-1}(M) \supset W$, hence $(\pi^{-1}(M))_D$ is a submodule of $\mathcal{O}_{X \times X, (x,x)}^{2p}$ which contains W_D .

Then, we define the double of M as

$$M_D := \frac{(\pi^{-1}(M))_D}{W_D}$$

which is an $\mathcal{O}_{X \times X, (x,x)}$ -submodule of $\frac{\mathcal{O}_{X \times X, (x,x)}^{2p}}{W_D}$.

If we call $\tilde{M} := \pi^{-1}(M)$ then

$$M = \frac{\tilde{M}}{W} \text{ and } M_D = \frac{\tilde{M}_D}{W_D}.$$

Rewriting with standard notation, we conclude that, if M is a submodule of $\mathcal{O}_{X,x}^p$ and $M \supset W$ then

$$\left(\frac{M}{W} \right)_D = \frac{M_D}{W_D}$$

and is generated by $\{(h+W)_D \mid h \in M\}$.

5.5 The double homomorphism relative to an analytic map germ

Let (Y, y) and (X, x) be germs of analytic spaces, and let $\varphi : (Y, y) \rightarrow (X, x)$ be an analytic map germ. The pullback map $\varphi^* : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$ is a ring homomorphism, which induces an $\mathcal{O}_{X,x}$ -algebra structure in $\mathcal{O}_{Y,y}$. Thus, every $\mathcal{O}_{Y,y}$ -module is also an $\mathcal{O}_{X,x}$ -module through this ring homomorphism.

We see that there is a natural $\mathcal{O}_{X \times X, (x,x)}$ -algebra structure in $\mathcal{O}_{Y \times Y, (y,y)}$ induced by the pullback of φ . In fact, let

$$\mu_{X,x} : \mathcal{O}_{X,x} \otimes_{\mathbb{C}} \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times X, (x,x)}$$

be the $\mathbb{C}$ -algebra homomorphism such that $\mu_{X,x}(f \otimes g)$ is the germ of the map

$$\begin{aligned} U \times U &\longrightarrow \mathbb{C} \\ (u, v) &\longmapsto f(u) \cdot g(v) \end{aligned}$$

and let

$$\mu_{Y,y} : \mathcal{O}_{Y,y} \otimes_{\mathbb{C}} \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{Y \times Y, (y,y)}$$

the same for (Y, y) .

Since $\varphi^* : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$ is a ring homomorphism then we have a natural $\mathbb{C}$ -algebra homomorphism

$$\varphi^{\otimes} : \mathcal{O}_{X,x} \otimes_{\mathbb{C}} \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y} \otimes_{\mathbb{C}} \mathcal{O}_{Y,y}$$

such that $\varphi^{\otimes}(f \otimes g) = \varphi^*(f) \otimes \varphi^*(g)$, $\forall f, g \in \mathcal{O}_{X,x}$. In fact, the map

$$\begin{aligned} \mathcal{O}_{X,x} \times \mathcal{O}_{X,x} &\longrightarrow \mathcal{O}_{Y,y} \otimes_{\mathbb{C}} \mathcal{O}_{Y,y} \\ (f, g) &\longmapsto (\varphi^*(f)) \otimes (\varphi^*(g)) \end{aligned}$$

is $\mathbb{C}$ -bilinear. So, the existence and uniqueness of $\varphi^\otimes$ is provided by the universal property of the tensor product. It is known that $\mu_{X,x}$ and $\mu_{Y,y}$ are $\mathbb{C}$ -algebra isomorphisms, so we can consider the $\mathbb{C}$ -algebra homomorphism $\varepsilon_\varphi : \mathcal{O}_{X \times X, (x,x)} \rightarrow \mathcal{O}_{Y \times Y, (y,y)}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_{X,x} \otimes_{\mathbb{C}} \mathcal{O}_{X,x} & \xrightarrow{\mu_{X,x}} & \mathcal{O}_{X \times X, (x,x)} \\ \downarrow \varphi^\otimes & & \downarrow \varepsilon_\varphi \\ \mathcal{O}_{Y,y} \otimes_{\mathbb{C}} \mathcal{O}_{Y,y} & \xrightarrow{\mu_{Y,y}} & \mathcal{O}_{Y \times Y, (y,y)} \end{array}$$

Since $\mu_{X,x}$ and $\mu_{Y,y}$ are $\mathbb{C}$ -algebra isomorphisms then we can identify $\varepsilon_\varphi \cong \varphi^\otimes$, and $\varphi^\otimes : \mathcal{O}_{X \times X, (x,x)} \rightarrow \mathcal{O}_{Y \times Y, (y,y)}$ induces in $\mathcal{O}_{Y \times Y, (y,y)}$ an $\mathcal{O}_{X \times X, (x,x)}$ -algebra structure.

Lemma 5.5.1. *Let $\alpha \in \mathcal{O}_{X \times X, (x,x)}$. Suppose that U is an open subset of X containing x where a representative of α is defined on $U \times U$. For each $w \in U$ let $\alpha^w \in \mathcal{O}_{X,x}$ be the germ of the map*

$$\begin{array}{ccc} \alpha^w : U & \rightarrow & \mathbb{C} \\ z & \mapsto & \alpha(z, w) \end{array}$$

For each $y_2 \in \varphi^{-1}(U)$ let $(\varphi^\otimes(\alpha))^{y_2} \in \mathcal{O}_{Y,y}$ be the germ of the map

$$\begin{array}{ccc} (\varphi^\otimes(\alpha))^{y_2} : \varphi^{-1}(U) & \rightarrow & \mathbb{C} \\ y_1 & \mapsto & (\varphi^\otimes(\alpha))(y_1, y_2) \end{array}$$

Then

$$\varphi^*(\alpha^{\varphi(y_2)}) = (\varphi^\otimes(\alpha))^{y_2}, \forall y_2 \in \varphi^{-1}(U).$$

Proof. We can write $\alpha = \sum_i (f_i \otimes_{\mathbb{C}} g_i)$, with $f_i, g_i \in \mathcal{O}_{X,x}$. For all $y_1 \in \varphi^{-1}(U)$ we have:

$$\begin{aligned} \varphi^*(\alpha^{\varphi(y_2)})(y_1) &= \alpha^{\varphi(y_2)}(\varphi(y_1)) = \alpha(\varphi(y_1), \varphi(y_2)) \sum_i (f_i(\varphi(y_1)) \otimes_{\mathbb{C}} g_i(\varphi(y_2))) \\ &= \left(\sum_i (\varphi^*(f_i)) \otimes_{\mathbb{C}} (\varphi^*(g_i)) \right) (y_1, y_2) = (\varphi^\otimes(\alpha))(y_1, y_2) = (\varphi^\otimes(\alpha))^{y_2}(y_1), \text{ and the lemma} \end{aligned}$$

is proved. $\square$

Clearly we get the analogous result if we fix the first coordinate instead the second one.

Consider the projections $\pi_1^X, \pi_2^X : X \times X \rightarrow X$ and $\pi_1^Y, \pi_2^Y : Y \times Y \rightarrow Y$.

Theorem 5.5.2. Let $M \subseteq \mathcal{O}_{X,x}^p$ and $N \subseteq \mathcal{O}_{Y,y}^q$ be submodules. If $\phi : M \rightarrow N$ is an $\mathcal{O}_{X,x}$ -module homomorphism then there exists a unique $\mathcal{O}_{X \times X, (x,x)}$ -module homomorphism $\phi_{D,\phi} = \phi_D : M_D \rightarrow N_D$ such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\quad \phi \quad} & N \\ D_M \downarrow & & \downarrow D_N \\ M_D & \xrightarrow{\quad \phi_{D,\phi} \quad} & N_D \end{array}$$

The map $\phi_{D,\phi} = \phi_D$ is called the **double of ϕ relative to $\phi : (Y,y) \rightarrow (X,x)$** .

Proof. Since M_D is generated by $\{h_D \mid h \in M\}$ then we can define $\phi_D : M_D \rightarrow N_D$ in a natural way: for each $u = \sum_i \alpha_i(h_i)_D$ with $\alpha_i \in \mathcal{O}_{X \times X, (x,x)}$ and $h_i \in M$ we define

$$\phi_D(u) := \sum_i \alpha_i(\phi(h_i))_D = \sum_i \phi^\otimes(\alpha_i)(\phi(h_i))_D$$

which belongs to N_D .

Claim : ϕ_D is well defined. In fact, suppose that $\sum_i \alpha_i(h_i)_D = \sum_j \beta_j(g_j)_D$, with $\alpha_i, \beta_j \in \mathcal{O}_{X \times X, (x,x)}$ and $h_i, g_j \in M$. So, we obtain two equations:

$$\sum_i \alpha_i(h_i \circ \pi_1^X) = \sum_j \beta_j(g_j \circ \pi_1^X) \quad (1)$$

$$\sum_i \alpha_i(h_i \circ \pi_2^X) = \sum_j \beta_j(g_j \circ \pi_2^X). \quad (2)$$

Take U an open neighborhood of x in X where representatives of α_i, β_j are defined on $U \times U$, and representatives of h_i, g_j are defined on U . For each $w \in U$ define $\alpha_i^w, \beta_j^w \in \mathcal{O}_{X,x}$ given by the germs of the maps

$$\begin{array}{ccc} \alpha_i^w : U & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \alpha_i(z, w) \end{array} \quad \begin{array}{ccc} \beta_j^w : U & \longrightarrow & \mathbb{C} \\ z & \longrightarrow & \beta_j(z, w) \end{array}$$

The equation (1) implies that $\sum_i \alpha_i^w h_i = \sum_j \beta_j^w g_j, \forall w \in U$. Applying ϕ (which is a $\mathcal{O}_{X,x}$ -homomorphism) in both sides of the last equation we get

$$\sum_i \alpha_i^w \phi(h_i) = \sum_j \beta_j^w \phi(g_j), \forall w \in U.$$

By the $\mathcal{O}_{X,x}$ -module structure on N induced by ϕ^* , the last equation boils down to

$$\sum_i \phi^*(\alpha_i^w) \phi(h_i) = \sum_j \phi^*(\beta_j^w) \phi(g_j), \forall w \in U.$$

By Lemma 5.5.1 we conclude that

$$\sum_i (\varphi^\otimes(\alpha_i))^{y_2} \phi(h_i) = \sum_j (\varphi^\otimes(\beta_j))^{y_2} \phi(g_j), \forall y_2 \in \varphi^{-1}(U).$$

Hence,

$$\sum_i \varphi^\otimes(\alpha_i)(\phi(h_i) \circ \pi_1^Y) = \sum_j \varphi^\otimes(\beta_j)(\phi(g_j) \circ \pi_1^Y).$$

Working with the analogous result of the Lemma 5.5.1, the equation (2) implies that

$$\sum_i \varphi^\otimes(\alpha_i)(\phi(h_i) \circ \pi_2^Y) = \sum_j \varphi^\otimes(\beta_j)(\phi(g_j) \circ \pi_2^Y).$$

Therefore,

$$\sum_i \varphi^\otimes(\alpha_i)(\phi(h_i))_D = \sum_j \varphi^\otimes(\beta_j)(\phi(g_j))_D$$

and ϕ_D is well-defined.

Now, by the definition of ϕ_D , it is clear that ϕ_D is an $\mathcal{O}_{X \times X, (x, x)}$ -module homomorphism and is the unique satisfying the property $\phi_D(h_D) = (\phi(h))_D, \forall h \in M$, i.e,

$$\phi_D(h \circ \pi_1^X, h \circ \pi_2^X) = (\phi(h) \circ \pi_1^Y, \phi(h) \circ \pi_2^Y).$$

□

Notice that this approach generalizes what we have defined in Section 5.1, taking $\varphi : (X, x) \rightarrow (X, x)$ as the identity map. The main motivation of this approach is the fact that when we work with integral closure of modules, the analytic curves $\varphi : (\mathbb{C}, 0) \rightarrow (X, x)$ has a key role.

Clearly the Propositions 5.1.5, 5.1.6 (a,c) and the Corollary 5.1.7 (a,b,d) still hold for the double homomorphism relative to an analytic map.

We can write the Proposition 5.1.6 (b) on this new language as follows:

Proposition 5.5.3. *Let $\varphi : (Y, y) \rightarrow (X, x)$ and $\varphi' : (Z, z) \rightarrow (Y, y)$ be analytic map germs, $M \subseteq \mathcal{O}_{X, x}^p$, $N \subseteq \mathcal{O}_{Y, y}^q$ and $P \subseteq \mathcal{O}_{Z, z}^r$ submodules. Let $\phi : M \rightarrow N$ be an $\mathcal{O}_{X, x}$ -module homomorphism and $\phi' : N \rightarrow P$ be an $\mathcal{O}_{Y, y}$ -module homomorphism. Then, $\phi' \circ \phi : M \rightarrow P$ is an $\mathcal{O}_{X, x}$ -module homomorphism, considering P with the $\mathcal{O}_{X, x}$ -module structure induced by the pullback of $\varphi \circ \varphi' : (Z, z) \rightarrow (X, x)$ and*

$$(\phi' \circ \phi)_{D, \varphi \circ \varphi'} = \phi'_{D, \varphi'} \circ \phi_{D, \varphi}.$$

Proof. For all $\alpha \in \mathcal{O}_{X, x}$ and $h \in M$, working with the module structures induced by the pullbacks of the analytic map germs, we have:

$\phi' \circ \phi(\alpha h) = \phi'(\alpha \phi(h)) = \phi'(\varphi^*(\alpha) \phi(h)) = \varphi^*(\alpha) \phi'(\phi(h)) = \varphi'^*(\varphi^*(\alpha))(\phi' \circ \phi(h)) =$
 $(\varphi \circ \varphi')^*(\alpha)(\phi' \circ \phi(h)) = \alpha(\phi' \circ \phi(h)).$ So $\phi' \circ \phi : M \rightarrow P$ is an $\mathcal{O}_{X,x}$ -module homomorphism
 and $(\phi' \circ \phi)_{D, \varphi \circ \varphi'}$ is well defined and clearly is equal to $\phi'_{D, \varphi'} \circ \phi_{D, \varphi}$. $\square$

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