Non-degeneracy of polynomial maps with
respect to global Newton polyhedra

Jorge Alberto Coripaco Huarcaya

SERVIÇO DE PÓS-GRADUAÇÃO DO ICMC-USP

Data de Depósito:

Assinatura: _____

Jorge Alberto Coripaco Huarcaya

Não-degeneração de aplicações polinomiais com respeito à poliedros de Newton globais

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação - ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

Área de Concentração: Matemática

Orientador: Prof. Dr. Marcelo José Saia

Coorientador: Prof. Dr. Carles Bivià-Ausina

USP – São Carlos
Agosto de 2015

Este trabalho teve suporte financeiro da FAPESP, processos nº 2011/10653-6 e 2012/22365-8

Ficha catalográfica elaborada pela Biblioteca Prof. Achille Bassi
e Seção Técnica de Informática, ICMC/USP,
com os dados fornecidos pelo(a) autor(a)

C798n Coripaco Huarcaya, Jorge Alberto
Non-degeneracy of polynomial maps with respect
to global Newton polyhedra / Jorge Alberto
Coripaco Huarcaya; orientador Marcelo José Saia; co-
orientador Carles Bivià-Ausina. -- São Carlos, 2015.
106 p.

Tese (Doutorado - Programa de Pós-Graduação em
Matemática) -- Instituto de Ciências Matemáticas e
de Computação, Universidade de São Paulo, 2015.

1. Non-degeneracy of polynomial maps. 2.
Lojasiewicz exponent at infinity. 3. Global Newton
polyhedra. 4. Special monomials. I. Saia, Marcelo
José, orient. II. Bivià-Ausina, Carles, co-orient.
III. Título.

Jorge Alberto Coripaco Huarcaya

**Non-degeneracy of polinomial maps with
respect to global Newton polyhedra**

Doctoral dissertation submitted to the Instituto de
Ciências Matemáticas e de Computação - ICMC-
USP, in partial fulfillment of the requirements for the
degree of the Doctoral Program in Mathematics.
FINAL VERSION

Concentration Area: Mathematics

Advisor: Prof. Dr. Marcelo José Saia

Coadvisor: Prof. Dr. Carles Bivià-Ausina

**USP – São Carlos
August 2015**

This work has been partially supported by FAPESP, processes nº 2011/10653-6 and 2012/22365-8

Aos meus queridos pais, Juan de Dios e Alejandra.

Agradecimentos

Primeiramente eu quero agradecer a Deus, por sempre guiar meu caminho e pela força que sempre me deu para alcançar meus objetivos.

Eu gostaria antes de mais nada de expressar meus mais sinceros agradecimentos a meus Professores orientadores, Prof. Dr. Marcelo José Saia, pela sua orientação, apoio total, disponibilidade, amizade e pela confiança que sempre me deu ao longo da realização deste trabalho. Ao Prof. Dr. Carles Bivià-Ausina, pelo apoio durante minha estadia na Espanha, pela orientação, pelo saber que transmitiu, pela disponibilidade, atenção dispensada, paciência, dedicação e profissionalismo pelos quais estou muito agradecido!

Agradeço também ao Prof. Dr. José Bonet, director do Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, pela hospitalidade e as excelentes condições de trabalho que me ofereceu este instituto.

Agradeço à meus queridos pais, Juan de Dios Curipaco e Alejandra Huarcaya que sempre me deram seu amor e carinho, pelo apoio incondicional e todo o sacrifício que fizeram para que eu continue meus estudos. À meus irmãos Moises, Duly, Betzabe, Efrain, Jose, Juan, Mayra e Issac pela amizade e pelo apoio durante toda minha vida.

Agradeço aos professores do departamento de matemática do ICMC-USP (São Carlos-Brasil), pelo ensinamentos, apoio e motivação academica.

Agradeço a FAPESP, Fundação de Amparo à Pesquisa do Estado de São Paulo pelo apoio financeiro, processos nº 2011/10653-6 e 2012/22365-8.

Finalmente agradeço a todas as pessoas que direta ou indiretamente participaram na realização deste trabalho.

Abstract

Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Motivated by the characterization of the integral closure of ideals in the ring $\mathcal{O}_n$ by means of analytic inequalities proven by Lejeune-Teissier [46], we define the set $\text{Sp}(F)$ of *special polynomials* with respect to F . The set $\text{Sp}(F)$ can be considered as a counterpart, in the context of polynomial maps $\mathbb{K}^n \rightarrow \mathbb{K}^p$, of the notion of integral closure of ideals in the ring of analytic function germs $(\mathbb{K}^n, 0) \rightarrow \mathbb{K}$. In this work, we are mainly interested in the determination of the convex region $S_0(F)$ formed by the exponents of the special monomials with respect to F .

Let us fix a convenient Newton polyhedron $\tilde{\Gamma}_+ \subseteq \mathbb{R}^n$. We obtain an approximation to $S_0(F)$ when F is *strongly adapted* to $\tilde{\Gamma}_+$, which is a condition expressed in terms of the faces of $\tilde{\Gamma}_+$ and the principal parts at infinity of F . The local version of this problem has been studied by Bivià-Ausina [4] and Saia [71]. Our result about the estimation of $S_0(F)$ allows us to give a lower estimate for the Łojasiewicz exponent at infinity of a given polynomial map with compact zero set. As a consequence of our study of Łojasiewicz exponents at infinity we have also obtained a result about the uniformity of the Łojasiewicz exponent in deformations of polynomial maps $\mathbb{K}^n \rightarrow \mathbb{K}^p$. Consequently we derive a result about the invariance of the global index of real polynomial maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

As particular cases of the condition of F being adapted to $\tilde{\Gamma}_+$ there appears the class of Newton non-degenerate polynomial maps at infinity and pre-weighted homogeneous maps. The first class of maps constitute a natural extension for maps of the Newton non-degeneracy condition introduced by Kouchnirenko for polynomial functions. We characterize the Newton non-degeneracy at infinity condition of a given polynomial map $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ in terms of the set $S_0((F, 1))$, where $(F, 1) : \mathbb{K}^n \rightarrow \mathbb{K}^{p+1}$ is the polynomial map whose last component function equals 1. Motivated by analogous problems in local algebra we also derive some results concerning the multiplicity of F .

Key words: Newton polyhedra, non-degeneracy conditions, multiplicity of polynomial maps, Łojasiewicz exponent at infinity, global injectivity of polynomial maps, index of real polynomials maps.

Resumo

Seja $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ uma aplicação polinomial, onde $\mathbb{K} = \mathbb{C}$ ou $\mathbb{K} = \mathbb{R}$. Motivados pela caracterização do fecho integral de ideais no anel $\mathcal{O}_n$ por meio de desigualdades analíticas provadas por Lejeune-Teissier [46], definimos o conjunto $\text{Sp}(F)$ de *polinômios especiais* com respeito a F . O conjunto $\text{Sp}(F)$ pode ser considerado como um homólogo, no contexto das aplicações polinomiais $\mathbb{K}^n \rightarrow \mathbb{K}^p$, da noção de fecho integral de ideais no anel de germes de funções analíticas $(\mathbb{K}^n, 0) \rightarrow \mathbb{K}$. Neste trabalho, estamos interessados principalmente na determinação da região convexa $S_0(F)$ formado pelos expoentes dos monômios especiais com respeito a F .

Fixado um poliedro de Newton conveniente $\tilde{\Gamma}_+ \subseteq \mathbb{R}^n$, é obtida uma aproximação de $S_0(F)$, quando F é *fortemente adaptada* a $\tilde{\Gamma}_+$ o qual é uma condição expressada em termos das faces de $\tilde{\Gamma}_+$ e as partes principais no infinito de F . A versão local deste problema foi estudado por Bivià-Ausina [4] e Saia [71]. Nosso resultado sobre a estimativa de $S_0(F)$ nos permite dar uma estimativa inferior para o expoente Łojasiewicz no infinito de uma aplicação polinomial $\mathbb{K}^n \rightarrow \mathbb{K}^p$, com conjunto $F^{-1}(0)$ compacto. Como uma consequência do estudo dos expoentes de Łojasiewicz no infinito também foi obtido um resultado sobre a uniformidade do expoente Łojasiewicz em deformações de aplicações polinomiais $\mathbb{K}^n \rightarrow \mathbb{K}^p$ e consequentemente, um resultado sobre a invariância do índice global de aplicações polinomiais reais $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

Como casos particulares da condição de F ser adaptada a $\tilde{\Gamma}_+$ aparecem a classe de aplicações polinomiais Newton não degeneradas e as aplicações polinomiais pre-quase homogêneas. A primeira classe de aplicações constitui uma extensão natural da condição Newton não-degeneração introduzida por Kouchnirenko para funções polinomiais. Caracterizamos a condição Newton não-degeneração para uma determinada aplicação polinomial $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ em termos do conjunto $S_0((F, 1))$, onde $(F, 1) : \mathbb{K}^n \rightarrow \mathbb{K}^{p+1}$ é a aplicação polinomial cuja última função componente é igual a 1. Motivados por problemas análogos em álgebra local, também obtivemos alguns resultados sobre a multiplicidade de F .

Palavras-chave: Poliedros de Newton, condições de não-degeneração, multiplicidade de aplicações polinomiais, injectividade global de aplicações polinomiais, índice de aplicações polinomiais reais.

Contents

Introduction	1
1 Preliminaries	5
1.1 Multiplicity theory	5
1.2 Newton non-degenerate ideals	7
1.3 Adapted systems	12
1.4 Index theory and local algebra	14
1.5 Finite polynomial maps	17
1.6 Łojasiewicz exponent at infinity of polynomial maps	19
2 Newton filtrations and non-degeneracy conditions	25
2.1 Global Newton polyhedra	25
2.2 Newton non-degeneracy at infinity	30
2.3 The Newton filtration	33
2.4 Counting affine roots of polynomial systems	38
3 Newton non-degeneracy at infinity	41
3.1 Special monomials and Newton non-degeneracy at infinity	42
3.2 Multiplicity and special polynomials	51
3.3 Non-degeneracy with respect to a Newton filtration	55
3.4 Newton non-degeneracy of gradient maps	60
3.5 Homogeneous Newton polyhedra	65
4 Łojasiewicz exponent and adapted maps to Newton polyhedra	71
4.1 Maps adapted to Newton polyhedra	72
4.2 Special monomials and adapted maps	79
4.3 Estimation of Łojasiewicz exponents at infinity	86

4.4	Łojasiewicz exponent at infinity of pre-weighted homogeneous maps	89
4.5	Index of polynomial maps	92

Introduction

Newton polyhedra have been used in different problems of local singularity theory, as can be seen in the works of Bivià-Ausina [4], Fukui [32], Kouchnirenko [43], Oka [51], Saia [70], Wall [78] and Yoshinaga [79] among other authors. In [43], Kouchnirenko showed a formula to compute the Milnor number of a germ of analytic function $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ with isolated singularity at the origin in terms of the Newton polyhedron of f . Bivià-Ausina, Fukui and Saia in [11] characterized a class of ideals of finite codimension in the ring of formal power series, which is expressed in terms of a fixed Newton polyhedra. This class includes Newton non-degenerate ideals in the sense of Saia [70] and ideals generated by semi-weighted polynomials (with respect to a fixed vector of weights).

Concerning the polynomial case, Kouchnirenko also showed in [43] a formula for computing the global Milnor number of polynomials $f \in \mathbb{C}[x_1, \dots, x_n]$ with a finite number of singularities, under the condition that f is Newton non-degenerate at infinity. Newton polyhedra were also used by Bivià-Ausina in [8] to obtain a lower estimate for the Łojasiewicz exponents at infinity of real polynomial maps.

These results are the main motivation of our work in order to apply global Newton polyhedra to study polynomial maps $\mathbb{K}^n \rightarrow \mathbb{K}^p$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The subject of this thesis is the study of non-degeneracy conditions for polynomials maps $\mathbb{K}^n \rightarrow \mathbb{K}^p$, with respect to global Newton polyhedra, in order to obtain polynomial versions of results described in [4, 6, 10, 11, 33, 43] and [70].

Motivated by the characterization given by Lejeune-Teissier [46], by means of analytic inequalities, of the integral closure of ideals in the ring $\mathcal{O}_n$ of germs of analytic functions $(\mathbb{C}^n, 0) \rightarrow \mathbb{C}$, we introduce in Section 3.1.8 a version for polynomial maps of the notion of integral element, which we call *special element* with respect to a given polynomial maps $\mathbb{K}^n \rightarrow \mathbb{K}^p$. This notion is fundamental in the development of this work. If $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ is a polynomial map, then we denote by $\text{Sp}(F)$ the set of special polynomials with respect to F .

We can consider $\mathrm{Sp}(F)$ as an analogous notion of the integral closure of ideals in the context of polynomial maps.

The main result of this thesis is the determination of the convex region $S_0(F)$ formed by the exponents of the special monomials with respect to F using a global Newton polyhedron (see Theorems 4.2.3 and 4.2.4). We obtain an approximation to $S_0(F)$ when F is *strongly adapted to* $\tilde{\Gamma}_+$, which is a condition expressed in terms of the faces of the Newton polyhedron and the principal parts at infinity of F . We remark that the local version of this problem, has been studied previously by Bivià-Ausina in [4].

In this context, a very interesting problem is to give a lower estimate of the Łojasiewicz exponent at infinity for a polynomial map $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ using global Newton polyhedra. This number, which is denoted by $\mathcal{L}_\infty(F)$, is defined as the supremum of those real numbers α such that there exist constants $C, M > 0$ such that

$$\|x\|^\alpha \leq C\|F(x)\|, \quad (1)$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geq M$. The number $\mathcal{L}_\infty(F)$ exists if and only if $F^{-1}(0)$ is a compact set. In this case, this is a rational number, which can be negative (see [44] for details). The Łojasiewicz exponent at infinity is intimately related with questions about injectivity of polynomial maps [22] and the equivalence at infinity of polynomial vector fields [62].

The exact computation or the estimation of $\mathcal{L}_\infty(F)$ from below is a non-trivial problem [25], [44], [60] and, in general, there are no explicit formulas for the Łojasiewicz exponent at infinity.

As a consequence of our result on the estimation of $S_0(F)$, we obtain a lower estimate for Łojasiewicz exponent at infinity of real and complex polynomial maps (see Corollary 4.3.3 and Proposition 4.4.2). Płoski [60], Kollár and Shiffman [69] and Cygan, Krasiński and Tworzewski [25] also gave of lower estimates for the Łojasiewicz exponent at infinity for complex polynomial maps, using different methods.

Moreover, from our results about the estimation of Łojasiewicz exponents at infinity we show a result concerning the global index of a given polynomial map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (see Theorem 4.5.1) with finite zero set. This result describes which monomials can be added to each component function of F leading to a map having the same global index than F . The local version of this question, which is analyzed in the articles [5, 19, 36] and [48], takes part of the wider problem of determining which monomials in the Taylor expansion of a smooth vector field determine the local phase portrait (see for instance [14, 16] and [80]). The first

step in this approach to the study of global indices is the result of Cima-Gasull-Mañosas [18, Proposition 2] on the index of maps whose monomials of maximum degree with respect to some vector of weights have an isolated zero. We call these maps pre-weighted homogeneous (see Definition 2.3.5 for a precise formulation of this concept).

This thesis is organized as follows:

Chapter 1. In Section 1.1 we recall several basic notions from commutative algebra, like the multiplicity of an ideal in a local ring, in the sense of Hilbert-Samuel, and the integral closure of an ideal. Let $\mathcal{O}_n$ denote the ring of analytic function germs $(\mathbb{C}^n, 0) \rightarrow \mathbb{C}, 0$. Section 1.2 contains some results shown in [11] for ideals in $\mathcal{O}_n$; these results connect the notions of multiplicity, integral closure and Newton non-degeneracy. In Section 1.3 we show the notion of adapted system introduced by Bivià-Ausina in [4] and the result of [4] about the estimation of the Newton polyhedron of the monomials that belong to the integral closure of ideals $I \subseteq \mathcal{O}_n$ such that I admits a generating system which is adapted to a given Newton polyhedron. In Sections 1.4 and 1.5 we recall some preliminary results about the global index of maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$ (see for instance Theorem 1.4.7) and the multiplicity of a polynomial map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ with finite zero set. Section 1.6 is devoted to showing known results about the Łojasiewicz exponent at infinity of a polynomial map $\mathbb{K}^n \rightarrow \mathbb{K}^p$.

Chapter 2. In Section 2.1 we introduce the concept of global Newton polyhedra and basic combinatorial results. In Section 2.2 we introduce the notion of Newton non-degeneracy at infinity for polynomial map $\mathbb{K}^n \rightarrow \mathbb{K}^p$ (see Definition 2.2.1) and recall the known result of Kouchnirenko about the global Milnor number of a polynomial function. Section 2.3 is devoted to introducing the Newton filtration associated to a convenient global Newton polyhedron and the corresponding notion of non-degeneracy (see Definition 2.3.4) for polynomial maps $\mathbb{K}^n \rightarrow \mathbb{K}^p$. In Section 2.4 we recall a fundamental result of Rojas and Wang [67] about the determination of the number of isolated affine roots, counting multiplicities, of polynomial maps $\mathbb{C}^n \rightarrow \mathbb{C}^p$ using global Newton polyhedra.

Chapter 3. This chapter is devoted to showing analogous results for polynomial maps of the results given in [11], [6] and [70]. In Section 3.1 we introduce the notion of *special polynomial* with respect to a given polynomial map $\mathbb{K}^n \rightarrow \mathbb{K}^p$ and we show a characterization of the Newton non-degeneracy at infinity of polynomial maps in terms of this notion (see Theorem 3.1.14). In particular, when F is a convenient polynomial map, we obtain the polynomial version of a result shown by Saia [70, Theorem 3.4]. In Section 3.2 we show a necessary and sufficient condition for a polynomial to be special with respect to a given finite polynomial

map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ in terms of the notion of multiplicity (see Theorem 3.2.2). Moreover, we characterize the Newton non-degeneracy at infinity for polynomial maps $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$, in terms of the set $S_0(F)$ and the multiplicity of F (see Corollary 3.2.4). Section 3.3 contains versions for polynomial maps of some results proven by Bivià-Ausina, Fukui and Saia [11, Theorem 3.3] and Furuya and Tomari [33, Theorem 1]. In Section 3.4 we analyze the Newton non-degeneracy of gradient maps and relate this property with the Newton non-degeneracy of polynomial functions. Section 3.5 contains a characterization of an important class of global Newton polyhedra.

Chapter 4. In this chapter, we approach the problem of determining when a monomial is special with respect to a given polynomial map using Newton polyhedra at infinity. For this purpose, in Section 4.1, we introduce the concepts of adapted and strongly adapted map with respect to a fixed convenient Newton polyhedron at infinity. In Section 4.2 we show the fundamental results about the approximation of the Newton polyhedron of the special monomials with respect to a given polynomial map (Theorems 4.2.3 and 4.2.4). In Sections 4.3 and 4.4 we apply Theorem 4.2.4 to establish a positive lower bound for $\mathcal{L}_\infty(F)$ (Corollary 4.3.3 and Proposition 4.4.2) and to derive a consequence about the injectivity of polynomial maps (see Corollary 4.3.6). Finally in Section 4.5 we apply the argument of the proof of Theorem 4.2.4 to show a result about the invariance of global index of polynomial maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

CHAPTER 1

Preliminaries

Contents

1.1	Multiplicity theory	7
1.2	Newton non-degenerate ideals	9
1.3	Adapted systems	14
1.4	Index theory and local algebra	16
1.5	Finite polynomial maps	19
1.6	Łojasiewicz exponent at infinity of polynomial maps	20

In this chapter, first we briefly recall the definitions of multiplicity in the sense of Samuel and integral closure of ideals. Then, with the purpose of stating results shown by Bivià-Ausina, Fukui and Saia in [11], the condition of Newton non-degeneracy for ideals in the ring of formal power series around the origin is introduced. In Section 1.3 it is shown a result proved by Bivià-Ausina [4], that gives an estimation of the Newton polyhedron of the monomials that belong to the integral closure of an ideal.

1.1 Multiplicity theory

We begin this section by recalling the definition and some fundamental facts concerning the multiplicity of an ideal, in the sense of Samuel, and the integral closure of ideals. We refer to [15, 52] and [72] for more information about these concepts.

Theorem 1.1.1 ([72, Theorem 11.1.3]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring, let I be an $\mathfrak{m}$ -primary ideal, and let M be a non-zero finitely generated R -module. Set $d = \dim(R)$.*

There exists a polynomial $P(n)$, with rational coefficients, such that for all $n \gg 0$,

$$P(n) = \lambda_R(M/I^n M),$$

where $\lambda_R(M/I^n M)$ is the length of $M/I^n M$, as an R -module, for all $n \geq 0$. Furthermore, the degree in n of $P(n)$ is $\dim M$, which is at most d . $\lambda_R(M/I^n M)$ is called the Hilbert function and $P(n)$ the Hilbert–Samuel polynomial of I with respect to M . If the dependence of $P(n)$ on I and M needs to be specified, then write $P_{I,M}(n)$.

Definition 1.1.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d , let I be an $\mathfrak{m}$ -primary ideal, and let M be a finitely generated R -module. The *multiplicity* of I with respect to M , denoted $e_R(I; M)$ is $d!$ times the coefficient of $P_{I,M}(n)$ of degree d . If $M = R$, we simply write $e(I)$ for $e_R(I; R)$, while if $M = R$ and $I = \mathfrak{m}$, we write $e(R) = e(\mathfrak{m})$, the *multiplicity* of R . In other words, $e(I; M) = \lim_{n \rightarrow \infty} \lambda(M/I^n M) d! / n^d$, where $d = \dim R$.

The multiplicity of an ideal can be interpreted geometrically and it is related to the notion of integral closure of an ideal.

Definition 1.1.3. Let I be an ideal in an arbitrary ring R and $h \in R$. An element h is *integral over I* when there is a monic polynomial of the form $p(x) = x^m + a_1 x^{m-1} + \cdots + a_{m-1} x + a_m$, with $a_i \in I^i$, $m \geq 1$, such that $p(h) = 0$. The subset of R of those elements which are integral over I is an ideal called the *integral closure* of I and is denoted by $\bar{I}$. The ideal $\bar{I}$ contains I and, if $I = \bar{I}$, I is called *integrally closed*.

The next definition is an essential tool for the study of the integral closure of ideals.

Definition 1.1.4. Given a pair of ideals I and J in a Noetherian ring R such that $J \subseteq I$, J is said to be a *reduction* of I when there exists an integer $r \geq 0$, such that $I^{r+1} = JI^r$.

We state the following result for Cohen-Macaulay rings, however, this result is true for formally equidimensional rings, which constitute a much wider class of rings (we refer to [52] and [72] for details).

Theorem 1.1.5 (Rees' Theorem [63]). *Let $(R, \mathfrak{m})$ be a regular local ring, and let J and I be a pair of $\mathfrak{m}$ -primary ideals of R such that $J \subseteq I$. Then the following conditions are equivalent:*

- (i) J is a reduction of I ;
- (ii) $\bar{I} = \bar{J}$;
- (iii) $e(I) = e(J)$.

The equivalency between (i) and (ii) is true when I and J are not $\mathfrak{m}$ -primary ideals, as is shown in [72, Corollary 1.2.5].

Next we give a result from [52, Theorem 14.14] on the existence of reductions of an ideal I in a local ring, see also [58, p. 112]. As we see, reductions of an ideal can be obtained through generic linear combinations of a given generating system of I .

Theorem 1.1.6. *Let R be an n -dimensional local ring whose residual field is equal to k . Let us assume that k is infinite. Let $I \subseteq R$ be an ideal of finite colength. Suppose that I is generated by $g_1, \dots, g_s$. Then there exists an analytic set $S(I) \subseteq k^s \times k^n$ such that, if $(a_{11}, \dots, a_{1s}, \dots, a_{n1}, \dots, a_{ns}) \in k^s \times k^n \setminus S(I)$ and we define $h_i = \sum_j a_{ij}g_j$, $i = 1, \dots, n$, then the ideal of R generated by $h_1, \dots, h_n$ is a reduction of I .*

Let $\mathcal{O}_n$ denote the ring of analytic function germs $(\mathbb{C}^n, 0) \rightarrow \mathbb{C}$. Next, we state a result of Lejeune-Teissier [46] characterizing the integral closure of ideals in $\mathcal{O}_n$.

Theorem 1.1.7. *Let I be an ideal in $\mathcal{O}_n$. The following statements are equivalent:*

- (i) $h \in \bar{I}$.
- (ii) (Growth condition) *For each system of generators $g_1, \dots, g_s$ of I there exists a neighborhood U of 0 and a constant $C > 0$ such that $|h(x)| \leq C \sup\{|g_i(x)| : i = 1, \dots, s\}$, for all $x \in U$.*
- (iii) (Valuative criterion) *For each analytic curve $\varphi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$, the germ $h \circ \varphi$ lies in $(\varphi^*(I))\mathcal{O}_1$, where $\varphi^* : \mathcal{O}_n \rightarrow \mathcal{O}_1$ denotes the natural morphism induced by composition by φ .*

In the following section, we will see some results about the computation of the multiplicity and the integral closure of ideals.

1.2 Newton non-degenerate ideals

Let us fix an integer $n \geq 1$ and let $A(\mathbb{K}^n)$ denote the ring of analytic function germs $(\mathbb{K}^n, 0) \rightarrow \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We usually write $\mathcal{O}_n$ to denote $A(\mathbb{C}^n)$. We recall that $A(\mathbb{K}^n)$ is a local ring whose maximal ideal, that we denote by $\mathfrak{m}_n$ (or simply, by $\mathfrak{m}$ if no risk of confusion arises) is formed by the germs $h \in A(\mathbb{K}^n)$ such that $h(0) = 0$. In this section we introduce the condition of Newton non-degeneracy for ideals in $A(\mathbb{K}^n)$.

Let us fix coordinates $x_1, \dots, x_n$ in $\mathbb{K}^n$. We set $\mathbb{R}_{\geq 0} = [0, +\infty[$, $\mathbb{Z}_{\geq 0} = \mathbb{Z} \cap \mathbb{R}_{\geq 0}$ and $\mathbb{Q}_{\geq 0} = \mathbb{Q} \cap \mathbb{R}_{\geq 0}$. If $k = (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n$, then we denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$ by x^k .

Let $g = \sum a_k x^k$ be an element of $A(\mathbb{K}^n)$, the *support* of g , denoted by $\text{supp}(g)$, is defined as the set of points $k \in \mathbb{Z}^n$ such that $a_k \neq 0$. If I is an ideal of $A(\mathbb{K}^n)$, we define the *support* of I as $\text{supp}(I) = \cup_{g \in I} \text{supp}(g)$.

Definition 1.2.1. A subset Γ_+ of $\mathbb{R}_{\geq 0}^n$ is a *Newton polyhedron*, or a *local Newton polyhedron*, when there exists some $A \subseteq \mathbb{Q}_{\geq 0}^n$ such that Γ_+ is equal to the convex hull in $\mathbb{R}_{\geq 0}^n$ of the set $\{k + v : k \in A, v \in \mathbb{R}_{\geq 0}^n\}$. In this case we say that Γ_+ is the Newton polyhedron determined by A and we also denote it by $\Gamma_+(A)$.

Let $g \in A(\mathbb{K}^n)$, then the *Newton polyhedron of g* is defined as $\Gamma_+(g) = \Gamma_+(\text{supp}(g))$. Let I be an ideal of $A(\mathbb{K}^n)$, then the *Newton polyhedron of I* , denoted by $\Gamma_+(I)$, is defined as $\Gamma_+(\text{supp}(I))$. We observe that if $\{g_1, \dots, g_s\}$ is a generating system of $I \subseteq A(\mathbb{K}^n)$, then $\Gamma_+(I)$ is equal to the convex hull of $\Gamma_+(g_1) \cup \dots \cup \Gamma_+(g_s)$. It is also obvious that, if $J \subseteq I$ are two ideals of $A(\mathbb{K}^n)$, then $\Gamma_+(J) \subseteq \Gamma_+(I)$.

Let $\Gamma_+ \subseteq \mathbb{R}_{\geq 0}^n$ be a Newton polyhedron and let $v \in \mathbb{R}_{\geq 0}^n$. We define

$$\ell(v, \Gamma_+) = \min\{\langle v, k \rangle : k \in \Gamma_+\}; \quad (1.1)$$

$$\Delta(v, \Gamma_+) = \{k \in \Gamma_+ : \langle v, k \rangle = \ell(v, \Gamma_+)\}. \quad (1.2)$$

A *face* of Γ_+ is any set of the form $\Delta(v, \Gamma_+)$ for some $v \in \mathbb{R}_{\geq 0}^n$. The *dimension* of a face Δ of Γ_+ , denoted by $\dim \Delta$, is defined as the dimension of the smallest affine subspace of $\mathbb{R}_{\geq 0}^n$ containing Δ . We denote by Γ the union of the compact faces of Γ_+ and by Γ_- the union of all segments joining the origin and a point of Γ . If I is an ideal of $A(\mathbb{K}^n)$, then the sets $\Gamma(I)$ and $\Gamma_-(I)$ are defined analogously. We denote the n -dimensional volume of $\Gamma_-(I)$ by $V_n(I)$.

If I is an ideal of $A(\mathbb{K}^n)$ of finite colength, then $\Gamma_+(I)$ intersects all coordinate axes, since I contains a power of the maximal ideal of $A(\mathbb{K}^n)$. In this case we observe that $\Gamma_-(I)$ is equal to the closure (in the Euclidean sense) of the set $\mathbb{R}_{\geq 0}^n \setminus \Gamma_+(I)$.

Let Δ be a compact face of a Newton polyhedron Γ_+ and let $C(\Delta)$ denote the cone obtained by all half lines starting at the origin and passing through some point of Δ . Then we denote by $\mathcal{A}_\Delta$ the subring of $\mathcal{O}_n$ given by all $h \in \mathcal{O}_n$ such that $\text{supp}(h) \subseteq C(\Delta)$. We remark that $\mathcal{A}_\Delta$ is a local Cohen-Macaulay ring of dimension $\dim(\Delta) + 1$ (see [43, p. 24]).

If $g = \sum_k a_k x^k \in \mathcal{O}_n$, we denote by g_Δ the polynomial obtained by the sum of all terms $a_k x^k$ such that $k \in \text{supp}(g) \cap \Delta$. If $\text{supp}(g) \cap \Delta = \emptyset$, then we set $g_\Delta = 0$.

Definition 1.2.2 ([70, p. 2]). Let I be an ideal of $\mathcal{O}_n$ and let $g_1, \dots, g_s$ be a generating system of $\mathcal{O}_n$. Then I is *Newton non-degenerate* if, for each compact face Δ of $\Gamma_+(I)$, the ideal generated by $(g_1)_\Delta, \dots, (g_s)_\Delta$ in $\mathcal{A}_\Delta$ has finite colength in $\mathcal{A}_\Delta$.

Remark 1.2.3. We recall that it is immediate to see that the above definition does not depend on the given generating system of I . The fact that $(g_1)_\Delta, \dots, (g_s)_\Delta$ has finite colength in $\mathcal{A}_\Delta$ is equivalent to saying that, for all face $\Delta_1 \subseteq \Delta$, the set of common zeros of $(g_1)_{\Delta_1}, \dots, (g_s)_{\Delta_1}$ is contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$ (see [43, Theorem 6.2]).

The above observation motivates the following definition. If I is an ideal of $A(\mathbb{R}^n)$ and $g_1, \dots, g_s$ denotes a generating system of I , then we say that I is *Newton non-degenerate* when the set of common zeros of $(g_1)_{\Delta_1}, \dots, (g_s)_{\Delta_1}$ is contained in $\{x \in \mathbb{R}^n : x_1 \cdots x_n = 0\}$.

If I is an ideal in $A(\mathbb{K}^n)$, then denote by I_0 the ideal of $A(\mathbb{K}^n)$ generated by those monomials x^k such that k belongs to $\Gamma_+(I)$.

Lemma 1.2.4 ([11, Lemma 2.3]). *Let I be an ideal of $A(\mathbb{K}^n)$, then $\Gamma_+(I) = \Gamma_+(\bar{I})$.*

Let I be an ideal of $A(\mathbb{K}^n)$. The set $C(\bar{I})$ is defined as the convex hull in $\mathbb{R}_{\geq 0}^n$ of $\{k \in \mathbb{Z}_{\geq 0}^n : x^k \in \bar{I}\}$. That is, $C(\bar{I})$ is the Newton polyhedron of the ideal K_I generated by all the monomials belonging to the integral closure of I . The next result is shown for ideals of finite codimension in $\mathcal{O}_n$ in [70], but the same proof works for ideals of $A(\mathbb{K}^n)$, with no additional conditions.

Theorem 1.2.5 ([70, Theorem 3.4]). *Let I be an ideal of $\mathcal{O}_n$ of finite codimension, then $C(\bar{I}) \subseteq \Gamma_+(I)$. The equality $\Gamma_+(I) = C(\bar{I})$ holds if and only if I is Newton non-degenerate.*

The following result characterizes the reductions of Newton non-degenerate ideals and it is a consequence of Lemma 1.2.4, Theorem 1.2.5 and the Rees' Multiplicity Theorem.

Corollary 1.2.6. *Let I be a Newton non-degenerate ideal of $\mathcal{O}_n$. Let $J \subseteq I$ be another ideal of $\mathcal{O}_n$. Then J is a reduction of I if and only if J is Newton non-degenerate and $\Gamma_+(I) = \Gamma_+(J)$.*

The following result appears in [11] and includes a characterization of the Newton non-degeneracy property in terms of the value of the Samuel multiplicity.

Theorem 1.2.7. *Let $I \subseteq \mathcal{O}_n$ be an ideal of finite codimension. Then, the following conditions are equivalent:*

- (i) I is a Newton non-degenerate ideal,
- (ii) $\Gamma_+(I) = C(\bar{I})$,
- (iii) $e(I) = n!V_n(I)$,
- (iv) $\bar{I}$ is generated by monomials,

- (v) $I_0 \subseteq \bar{I}$,
- (vi) $\bar{I} = \{f \in \mathcal{O}_n : \Gamma_+(f) \subseteq \Gamma_+(I)\}$.

The above result constitutes a motivation to investigate the characterization of Newton non-degeneracy at infinity for polynomial maps $\mathbb{K}^n \rightarrow \mathbb{K}^n$, which is a notion defined in the next chapter.

In [6] Bivià-Ausina generalized the above result for a class of ideals that do not have finite codimension in general. We recall some definitions in order to introduce this kind of ideals.

Definition 1.2.8. Let I be an ideal of $\mathcal{O}_n$, we set

$$\Lambda_+(I) = \{\lambda k : \lambda \geq 0, k \in \Gamma(I)\}; \quad (1.3)$$

$$\mathcal{A}_I = \{h \in \mathcal{O}_n : \text{supp}(h) \subseteq \Lambda_+(I)\}; \quad (1.4)$$

$$c(I) = \max \{\dim \Delta : \Delta \text{ is a compact face of } \Gamma(I)\}. \quad (1.5)$$

We observe that, in general, the set $\Lambda_+(I)$ is not convex in $\mathbb{R}_{\geq 0}^n$. Nevertheless, it is not the case when $n = 2$. If $\Lambda_+(I)$ is convex, then $\Lambda_+(I) \cap \mathbb{Z}_{\geq 0}^n$ is a sub-semigroup of $\mathbb{Z}_{\geq 0}^n$ and $\mathcal{A}_I$ becomes a subring of $\mathcal{O}_n$. In fact, $\mathcal{A}_I$ is a local ring of dimension $c(I) + 1$ in this case. It is said that I has *finite relative colength* when $\Lambda_+(I)$ is convex and I admits a generating system $g_1, \dots, g_s$ such that $\text{supp}(g_1) \cup \dots \cup \text{supp}(g_s) \subseteq \Lambda_+(I)$, and the ideal in $\mathcal{A}_I$ generated by $g_1, \dots, g_s$ has finite colength in $\mathcal{A}_I$.

We observe that $\Gamma_-(I)$ is contained in an affine subspace of dimension $c(I) + 1$. Then we can consider the $(c(I) + 1)$ -dimensional volume of $\Gamma_-(I)$; we denote this volume by $v(I)$.

When I is an ideal of $\mathcal{O}_n$ of finite colength, then I has finite relative colength since the Newton polyhedron of I intersects all the coordinate axes and consequently $\mathcal{A}_I = \mathcal{O}_n$ in this case.

If $I \subseteq \mathcal{O}_n$ is an ideal of finite relative colength, we can consider I as an ideal of $\mathcal{A}_I$. Therefore, it makes sense to compute the multiplicity of I in $\mathcal{A}_I$ since $\mathcal{A}_I$ is a local ring. This multiplicity is denoted by $e(I, \mathcal{A}_I)$. In the sequel, I_1 denotes the ideal of $\mathcal{A}_I$ generated by all the monomials x^k , with $k \in \Lambda_+(I) \cap \mathbb{Z}_{\geq 0}^n$ (compare with the definition of the ideal I_0).

Theorem 1.2.9 ([6, Theorem 4.3]). *Let $I \subseteq \mathcal{O}_n$ be an ideal of finite relative colength. Then I is Newton non-degenerate if and only if $e(I, \mathcal{A}_I) = (c(I) + 1)!v(I)$.*

In the remaining section we recall the definition of a class of ideals of $\mathcal{O}_n$ studied in [11] by Bivià-Ausina, Fukui and Saia that strictly contains the class of Newton non-degenerate ideals generated by n elements.

Let us consider $\Gamma_+ \subseteq \mathbb{R}_{\geq 0}^n$ a Newton polyhedron at the origin, such that Γ_+ intersects all coordinate axes. From the boundary of the Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n$ we can construct a piecewise linear function $\Phi_\Gamma : \mathbb{R}^n \rightarrow \mathbb{R}$ with the following properties:

- (i) Φ_Γ is linear on each cone $C(\Delta)$, where Δ is a compact face of Γ_+ .
- (ii) Φ_Γ takes positive integer values on the points of $\mathbb{Z}_{\geq 0}^n$.
- (iii) there exists a positive integer M such that $\Phi_\Gamma(k) = M$, for all $k \in \Gamma_+$.

When the map Φ_Γ is constructed we define the ideals

$$\mathcal{A}_q = \left\{ g \in \mathcal{O}_n : \text{supp}(g) \subseteq \Phi_\Gamma^{-1}(q + \mathbb{N}) \right\}, \text{ for all } q \geq 0;$$

then

$$\mathcal{O}_n = \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots$$

is a filtration of ideals of $\mathcal{O}_n$, that is $\mathcal{A}_p \mathcal{A}_q \subseteq \mathcal{A}_{p+q}$, for all $p, q \in \mathbb{Z}_{\geq 1}$. This is called *the Newton filtration of $\mathcal{O}_n$ associated to Γ_+* (see also [43, 2.1]). For any compact face Δ of Γ_+ , this filtration induces a filtration on $\mathcal{A}_\Delta$ in a natural way.

Let $g = \sum a_k x^k \in \mathcal{O}_n$, then the level of g with respect to the filtration given above is defined as

$$\nu_\Gamma(g) = \min \{ \Phi_\Gamma(k) : k \in \text{supp}(g) \} = \max \{ q : g \in \mathcal{A}_q \}$$

The *principal part of g* , denoted by $\text{in}(g)$, is the sum of terms $a_k x^k$ such that $\Phi_\Gamma(k) = \nu_\Gamma(g)$. Given a face Δ of Γ_+ , the *principal part of g over Δ* , denoted by $\text{in}_\Delta(g)$, is the polynomial given by the sum of terms $a_k x^k$ such that $\Phi_\Gamma(k) = \nu_\Gamma(g)$ and $k \in \text{supp}(g) \cap C(\Delta)$.

Definition 1.2.10. Let I be an ideal of finite colength of $\mathcal{O}_n$. A system of generators $g_1, \dots, g_s$ of I is *non-degenerate on Γ_+* if, for each compact face $\Delta \subseteq \Gamma$, the ideal of $\mathcal{A}_\Delta$ generated by $\text{in}_\Delta(g_1), \dots, \text{in}_\Delta(g_s)$ has finite colength in $\mathcal{A}_\Delta$. When the system $g_1, \dots, g_s$ does not satisfy the above definition, we say that this is *degenerate on Γ_+* .

Remark 1.2.11. Analogously to Definition 1.2.2, the ideal of $\mathcal{A}_\Delta$ generated by the system $\{\text{in}_\Delta(g_1), \dots, \text{in}_\Delta(g_s)\}$ has finite colength in $\mathcal{A}_\Delta$ if and only if, for each compact face $\Delta_1 \subseteq \Delta$, the set of common zeros of $\text{in}_{\Delta_1}(g_1), \dots, \text{in}_{\Delta_1}(g_s)$ is contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$

Next, we have the main result of a non-degenerate system on Newton polyhedron, that gives us a tool to calculate the codimension of ideals in $\mathcal{O}_n$. More details about of non-degenerate system on Newton polyhedron, can be found in [11].

Theorem 1.2.12 ([11, Theorem 3.3]). *Let $g_1, \dots, g_n$ be a system of generators of an ideal I with finite codimension in $\mathcal{O}_n$ and let $\Gamma_+ \subseteq \mathbb{R}^n$ be a Newton polyhedron. If M is the value on Γ of the filtration induced by Γ_+ and $d_1 = \nu_\Gamma(g_1), \dots, d_n = \nu_\Gamma(g_n)$ are the levels of the given set of generators of I with respect to this filtration, then*

$$\dim_{\mathbb{C}} \mathcal{O}_n/I \geq (d_1 \cdots d_n / M^n) n! V_n(\Gamma_-),$$

and equality holds if and only if the system $g_1, \dots, g_n$ is non-degenerate on Γ_+ .

In Chapter 3 we will give a polynomial version of this result.

1.3 Adapted systems

Let Γ_+ denote a convenient Newton polyhedron in $\mathbb{R}^n$. In this section we recall a result from [4] that, in particular, gives an estimation of the Newton polyhedron of the monomials that belong to the integral closure of I .

If $S \subseteq A(\mathbb{K}^n)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by $V(S)$ the zero set germ at 0 of S in $\mathbb{K}^n$.

Definition 1.3.1. Let I be an ideal in $A(\mathbb{K}^n)$ and $h \in A(\mathbb{K}^n)$ such that $V(I) \subseteq V(h)$. Let $\{g_1, \dots, g_s\}$ be a system of generators of I . By [50, p.136] (see also [13, p.493]), we can consider the greatest lower bound of those $\theta > 0$ such that there exists an open neighborhood U of 0 in $\mathbb{K}^n$ and a constant $C > 0$ such that

$$|h(x)|^\theta \leq C \sup_i |g_i(x)|,$$

for all $x \in U$. This number is called the *Łojasiewicz exponent of h with respect to I* and is denoted by $\ell(h, I)$. It is easy to see that this definition does not depend on the chosen system of generators of I .

Lejeune-Tessier showed in [46] that, if I is an ideal of $A(\mathbb{C}^n)$ then, $\ell(h, I)$ is a rational number and Bochnak-Risler in [13] showed that this is also true when I is an ideal of $A(\mathbb{R}^n)$.

If I is an ideal of $A(\mathbb{R}^n)$, then the *integral closure of I* , denoted by $\bar{I}$, is defined as the ideal of $A(\mathbb{K}^n)$ generated by those $h \in A(\mathbb{R}^n)$ for which $\ell(h, I) \leq 1$ (see [34]). Obviously $I \subseteq \bar{I}$.

Moreover, if θ is a positive rational number, we denote by $\bar{I}^\theta$ the ideal generated by those $h \in A(\mathbb{K}^n)$ such that $\ell(h, I) \leq 1/\theta$. Then $\ell(h, I)$ can be expressed as

$$\ell(h, I) = \frac{1}{\max\{\theta \in \mathbb{Q}_{\geq 0} : h \in \bar{I}^\theta\}}.$$

The next result shows a sufficient condition for a monomial x^k to belong to the ideal $\overline{I}^\theta$, where θ is a positive rational number. In particular, when $\theta = 1$, then x^k belongs to $\overline{I}$.

Given a vector $v \in \mathbb{R}_{\geq 0}^n \setminus \{0\}$ and a series $g = \sum_k a_k x^k$, then $\ell(v, g)$ is defined as $\ell(v, \Gamma_+(g))$ and $\Delta(v, g) = \Delta(v, \Gamma_+(g)) \cap \text{supp}(g)$.

If $v \in \mathbb{Z}_{\geq 0}^n \setminus \{0\}$, then we say that v is *primitive* when the non-zero coordinates of v are mutually prime integers. In the remaining section, let us fix a local Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n$. Suppose that the number of $(n-1)$ -dimensional faces of Γ_+ is r . Let $v^1, \dots, v^r$ be the primitive vectors of Γ_+ supporting some face of dimension $n-1$ of Γ_+ .

Definition 1.3.2. Let $g = \sum_k a_k x^k$ and let $J \subseteq \{1, \dots, r\}$, then we consider the set $\Delta(J, g) = \bigcap_{j \in J} \Delta(v^j, g)$. The *principal part of g with respect to J* , denoted by $q_J(g)$, is defined as the polynomial given by the sum of those $a_k x^k$ such that $k \in \Delta(J, g)$. If $\Delta(J, g) = \emptyset$, then we set $q_J(g) = 0$.

The following definition shows a concept that is the motivation for some ideas developed in Chapter 4.

Definition 1.3.3. Given $S = \{g_1, \dots, g_s\} \subseteq A(\mathbb{K}^n)$, then S is an *adapted system* to Γ_+ (or S is $\mathbb{K}$ -adapted to Γ_+) when, for each $J \subseteq \{1, \dots, r\}$ such that $\bigcap_{j \in J} \Delta(v^j, \Gamma_+)$ is a compact face of Γ_+ , the system of equations

$$q_J(g_1)(x) = \dots = q_J(g_s)(x) = 0 \quad (1.6)$$

has no solution in $(\mathbb{K}^*)^n$. We say that S is an *adapted system* when S is adapted to the Newton polyhedron determined by S .

It is straightforward to check that, if $S = \{g_1, \dots, g_s\}$ is non-degenerate with respect to a Newton polyhedron Γ_+ , then S is also adapted to Γ_+ (in general, the converse is not true).

Theorem 1.3.4 ([4, Theorem 3.5]). *Let $\Gamma_+ \subseteq \mathbb{R}^n$ be a Newton polyhedron and let $S = \{g_1, \dots, g_s\}$ be an adapted system to Γ_+ . Let $I \subseteq A(\mathbb{K}^n)$ be the ideal generated by S and let $\theta \in \mathbb{Q}_{\geq 0}$. Suppose that x^k is a monomial such that*

$$\langle k, v^j \rangle \geq \theta \max\{\ell(v^j, g_1), \dots, \ell(v^j, g_s)\}$$

for all $j = 1, \dots, r$. Then $x^k \in \overline{I}^\theta$.

In particular if S is a Newton non-degenerate system of $A(\mathbb{K}^n)$, the above result establishes that the set $\{k \in \mathbb{Z}_{\geq 0}^n : x^k \in \overline{I}\}$ is equal to set $\Gamma_+(I) \cap \mathbb{Z}_{\geq 0}^n$.

1.4 Index theory and local algebra

In this section we describe some basic facts concerning the notion of index of real analytic maps $(\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$. We refer to [26] for further information about this notion.

Along this section, we denote by Ω an open connected set in $\mathbb{C}^n$. The mapping $F = (F_1, \dots, F_p) : \Omega \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^p$ will be called a *holomorphic mapping* if the p functions $F_1, \dots, F_p$ are holomorphic functions in Ω . If each F_i is a polynomial in the variables $x_1, \dots, x_n$, then F is called a *polynomial map*.

If $F : \Omega \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^p$ is a holomorphic mapping, then we denote by $F_{\mathbb{R}} : \Omega \subseteq \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2p}$ the real mapping obtained from F under the identification $x + iy \leftrightarrow (x, y)$ between $\mathbb{C}$ and $\mathbb{R}^2$.

Lemma 1.4.1. *Suppose that $F : \Omega \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a holomorphic mapping. Then the Jacobian determinant of F satisfies the relation*

$$\det(dF_{\mathbb{R}}) = |\det(dF)|^2 \quad (1.7)$$

As a consequence of this lemma, an holomorphic mapping has an invertible derivative precisely when the $F_{\mathbb{R}}$ real mapping has an invertible derivative.

Definition 1.4.2. Suppose that $F : \Omega \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^p$ is a holomorphic mapping, where $p \geq n$. Then F is called a *finite analytic mapping* if, for each $w \in F(\Omega)$, the fiber $F^{-1}(w)$ is a finite set.

Let us remark that proper holomorphic mappings between domains in complex Euclidean spaces are finite mappings. This follows because the inverse image of a point is necessarily a compact complex analytic subvariety.

Notation 1.4.3. Recall that we denote by $F : (\mathbb{C}^n, a) \rightarrow (\mathbb{C}^p, 0)$, the germ of a holomorphic mapping from $\mathbb{C}^n$ to $\mathbb{C}^p$ such that $F(a) = 0$. A germ $F : (\mathbb{C}^n, a) \rightarrow (\mathbb{C}^p, 0)$ defines a finite analytic mapping when $F^{-1}(0) = a$, as set germs. This means that there is a neighborhood of a on which a is the unique solution to the equation $F(z) = 0$.

In order to study the index of an holomorphic mapping, we consider $n = p$. The index is a generalization of the winding number of a curve in the complex plane. It can be used to estimate the number of solutions of an equation, and is closely connected to fixed-point

theory. Let us suppose that $a \in \Omega$ is an isolated point of the set $F^{-1}(0)$. Let D be a ball centered at a , with $D \subseteq \Omega$, such that $F^{-1}(0) \cap D = \{a\}$ and let us consider the map

$$\frac{F}{\|F\|} : \partial D \longrightarrow S^{n-1} \quad (1.8)$$

where ∂D denotes the boundary of D . From the point of view of homology, we identify ∂D with the unit sphere. Recall that the $(n-1)$ st homology group of S^{n-1} is the group of integers, that is

$$H_{n-1}(S^{n-1}) \cong \mathbb{Z}. \quad (1.9)$$

The continuous mapping (1.8) induces a mapping on the homology group $\mathbb{Z}$. This induced mapping is a group homomorphism and therefore must be multiplication by some integer. This integer is commonly called the *topological degree* of the mapping (1.8). Its value is independent of the choice of sphere ∂D (see [26]).

Definition 1.4.4. Let $F : (\mathbb{R}^n, a) \rightarrow (\mathbb{R}^n, 0)$ be the germ of a smooth mapping, and suppose that a is an isolated point in $F^{-1}(0)$. The *index of F at the point a* , written $\text{ind}_a(F)$, is the topological degree of (1.8).

The simplest example about the computation of $\text{ind}_a(F)$ is when F is an orthogonal linear transformation. Such a transformation preserves the sphere, and either preserves or reverses orientation. Hence $\deg(F) = \det(F)$, thus $\text{ind}_0(F)$ equals the determinant of F and can be negative in this case.

A version of the change of variables formula for multiple integrals holds for mappings with finite topological degree. We state a simple version of that formula:

$$\text{ind}_{x_0}(F) \int_{\|x\| \leq \epsilon} dV = \int_{\|F(x)\| \leq \epsilon} \det(dF) dV, \quad (1.10)$$

where ϵ denotes a small enough positive number. Here both integrals have the same orientation. This formula holds for mappings that are generically d -to-one, where $d = \text{ind}_{x_0}(F)$, when the set of points with fewer than d inverse images has measure zero. Note also from (1.10) that $\det(dF)$ cannot vanish identically.

We remark that for an holomorphic mapping, the index never cannot be negative. This fact follows from Lemma 1.4.1, which states essentially that a holomorphic mapping must be orientation preserving, and equality (1.10).

Corollary 1.4.5 ([26, p. 47]). *The index of the germ of an holomorphic mapping $F : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ is positive.*

Definition 1.4.6. Let Ω be an open subset of $\mathbb{C}^n$ and suppose that $F : \Omega \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^n$ is an holomorphic mapping such that $F^{-1}(0)$ is a finite set. The index of F in Ω denoted by $\text{ind}(F, \Omega)$ is defined by

$$\text{ind}(F, \Omega) = \sum_{a \in F^{-1}(0) \cap \Omega} \text{ind}_a(F)$$

If $\Omega = \mathbb{C}^n$, then we denote $\text{ind}(F, \mathbb{C}^n)$ by $\text{ind}(F)$.

The following result is well known, which is about the invariance of the index by homotopies and can be found in [49, Theorem 2.2.4] and [18].

Theorem 1.4.7. *Let F and G be continuous maps defined on the closure of a open connected and bounded set Ω of $\mathbb{R}^n$. Assume that F and G are homotopic on the boundary $\partial\Omega$ (that is, there is a continuous homotopy $H(t, z) : [0, 1] \times \Omega \rightarrow \mathbb{R}^n$ between F and G , such that $H(t, z) \neq 0$ for all $z \in \partial\Omega$). Then*

$$\text{ind}(F, \Omega) = \text{ind}(G, \Omega).$$

As a consequence of the above theorem, we can consider the case when F has finitely many roots a_j inside a fixed ball B . Each root contributes to the topological degree of the mapping $F/\|F\| : \partial B \rightarrow S^{2n-1}$ between spheres, as described by the next proposition.

Proposition 1.4.8 ([26, p. 48]). *Suppose $B \subseteq \mathbb{C}^n$ is a ball, and $f : B \rightarrow \mathbb{C}^n$ is holomorphic, continuous on ∂B , and does not vanish there. If the mapping $f/\|f\| : \partial B \rightarrow S^{2n-1}$ has degree s , then f has at most s roots in B and*

$$s = \sum_{f(a_j)=0} \text{ind}_{a_j}(f) \quad (1.11)$$

Finally we have a result that allows us to calculate the index of an holomorphic mapping using some tools of local algebra. If $z \in \mathbb{C}^n$, then we denote by $\mathcal{O}_{n,z}$ the ring of analytic function germs $(\mathbb{C}^n, z) \rightarrow \mathbb{C}$.

Theorem 1.4.9 ([26, p. 60, Theorem 1]). *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be an holomorphic mapping such that $F^{-1}(0)$ is a finite set. Then for each $z \in F^{-1}(0)$,*

$$\text{ind}_z(F) = \dim_{\mathbb{C}} \mathcal{O}_{n,z}/I_z(F),$$

where $I_z(F)$ is the ideal of $\mathcal{O}_{n,z}$ generated by the germs of the component functions of F at z .

1.5 Finite polynomial maps

Let I be an ideal of $\mathbb{C}[x_1, \dots, x_n]$, then we denote by $V(I)$ the zero set of I . The following theorem characterizes a finite algebraic set.

Theorem 1.5.1 ([20, p.43]). *Let I be an ideal in $\mathbb{C}[x_1, \dots, x_n]$. Then $V(I)$ is a finite set if and only if $\mathbb{C}[x_1, \dots, x_n]/I$ is a finite dimensional vector space over $\mathbb{C}$. If this occurs, the number of points in $V(I)$ is at most $\dim_{\mathbb{C}}(\mathbb{C}[x_1, \dots, x_n]/I)$.*

Definition 1.5.2. Let I be an ideal in $\mathbb{C}[x_1, \dots, x_n]$. Then, we say that I has *finite codimension* if $\mathbb{C}[x_1, \dots, x_n]/I$ is a finite dimensional vector space over $\mathbb{C}$.

Let us observe from the above theorem that I has finite codimension if and only if $V(I)$ is finite.

Let I be an ideal in $\mathbb{C}[x_1, \dots, x_n]$ and let a a point in $V(I)$. We consider the following set,

$$S_a := \{g \in \mathbb{C}[x_1, \dots, x_n] : g(a) \neq 0\},$$

which is a multiplicative set of $R = \mathbb{C}[x_1, \dots, x_n]$. Let $R_a := S_a^{-1}R$ be the quotient ring of R by S_a (the **localization** of R at S_a), that is,

$$R_a := \{f/g : f, g \in \mathbb{C}[x_1, \dots, x_n] \text{ and } g(a) \neq 0\}.$$

It is known that R_a is a local ring.

Definition 1.5.3. Suppose that a is a zero of I . The *algebraic multiplicity* of I at a , denoted by $\mu_a(I)$, is defined by

$$\mu_a(I) = \dim_{\mathbb{C}} R_a/S_a^{-1}I.$$

In particular, if I is an ideal of $\mathbb{C}[x_1, \dots, x_n]$ of finite codimension, we have an relation among the sum of the algebraic multiplicities of I and the codimension of I in $\mathbb{C}[x_1, \dots, x_n]$, as is shown in the following theorem.

Theorem 1.5.4 ([20, p.150]). *Let I be an ideal of $\mathbb{C}[x_1, \dots, x_n]$ with finite codimension. Then*

$$\dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n]/I = \sum_{a \in V(I)} \mu_a(I) = \sum_{a \in V(I)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,a}}{I\mathcal{O}_{n,a}}.$$

Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map, then denote by $I(F)$ the ideal of $\mathbb{C}[x_1, \dots, x_n]$ generated by the component functions of F . We say that F is a *finite polynomial map* when $\#F^{-1}(0) < \infty$ (or equivalently, when $I(F)$ has finite codimension).

If $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a finite polynomial map, then the *multiplicity of F* , denote by $\mu(F)$, is defined as

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{I(F)}.$$

Let us observe from the above theorem that F is finite if and only if $\mu(F) < \infty$.

Recall that a quasi-affine variety in $\mathbb{C}^n$ is a Zariski-open subset of an affine variety in $\mathbb{C}^n$ (see [38, 73] and [21]). A *constructible* set is a finite union of quasi-affine varieties. Constructible sets are closed under complementation, projection, and finite union and intersection. Note that a dense constructible set has nonempty interior.

Lemma 1.5.5 ([67, Lemma 1]). *Let $Y \subseteq \mathbb{C}^n$ be a constructible subset of codimension e . Let $F_1, \dots, F_p, g_1, \dots, g_p : \mathbb{C}^n \dashrightarrow \mathbb{C}$ be rational functions such that $g_1, \dots, g_p$ never vanish at any point of Y . Then for generic $(c_1, \dots, c_p) \in \mathbb{K}^p$,*

$$\text{codim}(Y \cap V(F_1 + c_1 g_1, \dots, F_p + c_p g_p)) \geq e + p.$$

In particular, if $p > n$, then $Y \cap V(F_1 + c_1 g_1, \dots, F_p + c_p g_p) = \emptyset$ for generic $(c_1, \dots, c_p)$.

Let us consider $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map and $h \in \mathbb{C}[x_1, \dots, x_n]$. Then for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, we denote by $F + h\alpha$ the polynomial map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ given by $F + h\alpha = (F_1 + \alpha_1 h, \dots, F_n + \alpha_n h)$.

Theorem 1.5.6. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a finite polynomial map and let $h \in \mathbb{C}[x_1, \dots, x_n]$. Then there exists an open Zariski-dense subset $U \subseteq \mathbb{C}^n$ such that $F + h\alpha : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a finite polynomial map, for all $\alpha = (\alpha_1, \dots, \alpha_n) \in U$.*

Proof. Let us consider the Zariski-open set $Y = \mathbb{C}^n \setminus V(F_1) \cup \dots \cup V(F_n) \cup V(h)$. Then, by Lemma 1.5.5 we have there exists a dense subset U in $\mathbb{C}^n$, such that $Y \cap V(F_1 + \alpha_1 h, \dots, F_n + \alpha_n h)$ is a finite set for all $(\alpha_1, \dots, \alpha_n) \in U$. Also, it is immediate to check that

$$(\mathbb{C}^n \setminus Y) \cap V(F_1 + \alpha_1 h, \dots, F_n + \alpha_n h) \subseteq V(F_1, \dots, F_n),$$

for all $\alpha = (\alpha_1, \dots, \alpha_n) \in U \cap (\mathbb{C}^*)^n$. Since $V(F_1, \dots, F_n)$ is a finite set, the result follows taken $V = U \cap (\mathbb{C}^*)^n$, which is clearly dense. $\square$

1.6 Łojasiewicz exponent at infinity of polynomial maps

From now on, we denote $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In this section we expose some basic definitions and results concerning to Łojasiewicz exponents at infinity of a polynomial map $\mathbb{K}^n \rightarrow \mathbb{K}^p$. For a detailed account on this subject we refer to the survey article [44].

Definition 1.6.1. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. Let us consider the set $e_\infty(F)$ formed by those $\nu \in \mathbb{R}$ such that there exist positive constants B and C such that

$$\|F(z)\| \geq C\|z\|^\nu, \quad (1.12)$$

for all $z \in \mathbb{K}^n$ such that $\|z\| \geq B$. If $e_\infty(F) \neq \emptyset$, we denote by $\mathcal{L}_\infty(F)$ the supremum of $e_\infty(F)$. We define $\mathcal{L}_\infty(F) = -\infty$ when $e_\infty(F) = \emptyset$. The number $\mathcal{L}_\infty(F)$, when finite, is called the *Łojasiewicz exponent at infinity* of F .

The next proposition gives a basic property that characterizes the finiteness of the Łojasiewicz exponent at infinity.

Proposition 1.6.2 ([44, Proposition 2.1]). *Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \rightarrow \mathbb{C}^p$ be a polynomial map. Then $\mathcal{L}_\infty(F) \neq -\infty$ if and only if the set $F^{-1}(0)$ is compact.*

The above proposition is also valid for real polynomial maps and a proof of this fact can be found in [8, p. 749]. By this result we assume from now on that $F^{-1}(0)$ is a compact set.

Example 1.6.3. The following examples can be found in [44] and [60].

- (i) For $F(x, y) = (x, xy - 1) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, then we have $\mathcal{L}_\infty(F) = -1$.
- (ii) Take $p, q, s \in \mathbb{N}$, such that $s > \frac{q}{p}$. Let $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a polynomial map, given by $F(x, y) = (x^s, x^{ps} + y^q)$. Then

$$\mathcal{L}_\infty(F) = \frac{q}{p}$$

- (iii) Take $p, q \in \mathbb{N}$, $0 < q < p$. Let g be the polynomial of $\mathbb{C}[x, y]$ given by $g(x, y) = y + y^{1+q}x^{p-q}$. Let us consider the following polynomial map G given by $G(x, y) = \text{grad } g = ((p-q)y^{1+q}x^{p-q-1}, 1 + (1+q)y^q x^{p-q}) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Then

$$\mathcal{L}_\infty(G) = -\frac{p}{q}.$$

In general, it is well known that the Łojasiewicz exponent at infinity of a polynomial map is a rational number, as we show below.

Definition 1.6.4. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map and $S \subseteq \mathbb{K}^n$ be an unbounded set. Let us consider the set $e_{S,\infty}(F)$ formed by those $\nu \in \mathbb{R}$ such that there exist positive constants R and C such that

$$\|F(z)\| \geq C\|z\|^\nu, \quad (1.13)$$

for all $z \in S$ such that $\|z\| \geq B$. If $e_{S,\infty}(F) \neq \emptyset$, we denote by $\mathcal{L}_\infty(F|S)$ the supremum of $e_{S,\infty}(F)$. We define $\mathcal{L}_\infty(F|S) = -\infty$ when $e_{S,\infty}(F) = \emptyset$. The number $\mathcal{L}_\infty(F|S)$, when finite, is called *The Łojasiewicz exponent at infinity on S of F*

Example 1.6.5. For $F(x_1, x_2) = (x_1, x_1x_2 - 1) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ we have $\mathcal{L}_\infty(F) = -1$. If $S := \{(x_1, x_2) \in \mathbb{C}^2 : x_2 = 0\}$, then $\mathcal{L}_\infty(F|S) = 0$.

By a *meromorphic curve* in $\mathbb{K}^n$ we mean a set $\Gamma \subseteq \mathbb{K}^n$ for which there exists a holomorphic mapping (called a parametrization of Γ)

$$\Phi = (\varphi_1, \dots, \varphi_p) : \{t \in \mathbb{K} : |t| > R\} \rightarrow \mathbb{K}^n,$$

for some $R > 0$, such that $\text{Im } \Phi = \Gamma$ and Φ has a pole at $\infty \in \overline{\mathbb{K}}$ (it means that each φ_i has at most a pole at ∞ and at least one of φ_i has a pole at ∞ , this implies that $\|\Phi(t)\| \rightarrow \infty$ when $t \rightarrow \infty$). Since each $\varphi_i \neq 0$ has an expansion in a Laurent series at ∞ , we can write φ_i as

$$\varphi_i(t) = \sum_{k=-\infty}^{k_0} a_k t^k, \quad a_{k_0} \neq 0, \quad k_0 \in \mathbb{Z}.$$

Then, we define $\deg \varphi_i := k_0$ and $\deg \Phi := \max\{\deg \varphi_1, \dots, \deg \varphi_p\}$.

The next result shows the connection between the above definitions.

Proposition 1.6.6 ([44, Proposition 2.5]). *Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. Then $\mathcal{L}_\infty(F)$ is attained on a meromorphic curve at infinity i.e.*

$$\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|\Gamma),$$

where Γ is a meromorphic curve at infinity.

If Γ is a meromorphic curve at infinity on which $\mathcal{L}_\infty(F)$ is attained, then it is easy to observe that

$$\mathcal{L}_\infty(F|\Gamma) = \frac{\deg(F \circ \Phi)}{\deg \Phi}. \quad (1.14)$$

Thus as a consequence of the above proposition and from the last equation we have the following result.

Proposition 1.6.7. *Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. Then $\mathcal{L}_\infty(F)$ is a rational number.*

Proof. It follows immediately from the above Proposition 1.6.6 and the equality (1.14) taking an appropriate meromorphic curve at infinity. $\square$

Proposition 1.6.8 ([44]). *Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. Then $\mathcal{L}_\infty(F)$ is attained i.e. for the number $\mathcal{L}_\infty(F)$, there exist positive constants B and C such that*

$$\|F(z)\| \geq C\|z\|^{\mathcal{L}_\infty(F)},$$

for all $z \in \mathbb{K}^n$, such that $\|z\| \geq B$.

Proposition 1.6.9 ([44]). *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^p$ be polynomial map. Then $\mathcal{L}_\infty(F)$ is an invariant of linear (not affine) automorphisms of $\mathbb{C}^n$ and $\mathbb{C}^p$.*

Remark 1.6.10. In the above proposition, $\mathcal{L}_\infty(F)$ is not an invariant of polynomial automorphisms of the domain $\mathbb{C}^n$ and the codomain $\mathbb{C}^p$, as we show in following examples.

- (i) Triangular automorphisms of $\mathbb{C}^n$ may change $\mathcal{L}_\infty(F)$. For example $\mathcal{L}_\infty(x_1, x_2) = 1$ and $\mathcal{L}_\infty(x_1, x_2 + x_1^2) = \frac{1}{2}$.
- (ii) Translations in $\mathbb{C}^p$ may change $\mathcal{L}_\infty(F)$. For example $\mathcal{L}_\infty(x_1, x_1x_2 - 1) = -1$ and $\mathcal{L}_\infty(x_1, x_1x_2) = -\infty$.

Proposition 1.6.11 ([44]). *Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. Then $\mathcal{L}_\infty(F) \leq \deg F$, where $\deg F := \max\{\deg F_1, \dots, \deg F_p\}$.*

The above proposition follows from the elementary inequality for polynomial maps, i.e. there exist positive constants B and C such that

$$\|F(z)\| \leq C\|z\|^{\deg F},$$

for all $z \in \mathbb{K}^n$, such that $\|z\| \geq B$.

The most complete results about the computation of the Łojasiewicz exponent at infinity of a polynomial map $F : \mathbb{C}^n \rightarrow \mathbb{C}^p$ appear in the case $n = p = 2$ (the case $n = 1$ is trivial since in this case $\mathcal{L}_\infty(F) = \deg F$). Formulas for $\mathcal{L}_\infty(F)$ were given in various terms. The first one was a formula for $\mathcal{L}_\infty(F)$ in terms of parametrizations of the branches of the zero-sets of F_1 and F_2 at infinity.

Theorem 1.6.12 ([23]). *Let $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a polynomial map. If F is not constant, then*

$$\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|V(F_1) \cup V(F_2))$$

where $V(F_1) = F_1^{-1}(0)$ and $V(F_2) = F_2^{-1}(0)$ are zero-sets of F_1 and F_2 .

To reformulate this theorem in terms of parametrizations, we notice that $V(F_1)$ and $V(F_2)$ are complex algebraic curves in $\mathbb{C}^2$ (provided that F_1 and F_2 are not constants). Then, in a neighborhood of infinity in $\mathbb{C}^2$ they are finite unions of meromorphic curves (called branches of F_1 and F_2 at infinity, see [23, Proposition 3.1]. If we denote by $\Phi_i, i = 1, \dots, r$ (resp. $\Psi_j, j = 1, \dots, s$) parametrizations of the branches of F_1 (resp. F_2) at infinity then we have the following equivalent form for the previous theorem.

Theorem 1.6.13. *Let $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a polynomial map. If F is not constant, then*

$$\mathcal{L}_\infty(F) = \min_{i,j} \left\{ \frac{\deg F \circ \Phi_i}{\deg \Phi_i}, \frac{\deg F \circ \Psi_j}{\deg \Psi_j} \right\}$$

Let us pass to the n -dimensional case. So far, there are no explicit formulas for the Łojasiewicz exponent at infinity of $F = (F_1, \dots, F_p) : \mathbb{C}^n \rightarrow \mathbb{C}^p$. But we give a generalization of Theorem 1.6.12.

Theorem 1.6.14 ([24, Theorem 1]). *Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \rightarrow \mathbb{C}^p$ be a polynomial map. If F is not constant, then*

$$\mathcal{L}_\infty(F) = \mathcal{L}_\infty(F|V(F_1) \cup \dots \cup V(F_p)).$$

In many cases one needs only estimations of the Łojasiewicz exponent at infinity in terms of the other numerical invariants related to F .

In general when $F = (F_1, \dots, F_p) : \mathbb{C}^n \rightarrow \mathbb{C}^p$ is a polynomial map, with $n \neq p$, then we have a lower estimate for $\mathcal{L}_\infty(F)$, as we describe below.

Definition 1.6.15. Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \rightarrow \mathbb{C}^p$ be a polynomial map and $d_i := \deg F_i$ for $i = 1, \dots, p$. Then we define

$$B(d_1, \dots, d_p, n) = \begin{cases} d_1 \cdots d_{p-1} \cdot d_p & \text{if } p \leq n \\ d_1 \cdots d_{n-1} \cdot d_p & \text{if } p > n. \end{cases}$$

Many papers contain estimates of $\mathcal{L}_\infty(F)$ for various classes of polynomial map [45], [60], [61]. For instance the most interesting estimate was obtained by Kollár in [45] (Theorem 1.10, proves this but it is stated incorrectly see [69, Remark 18]), show the following.

Theorem 1.6.16. *Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \rightarrow \mathbb{C}^p$ be a polynomial map such that $F^{-1}(0)$ is a finite set and $\deg F_i = d_i > 0$, for all $i = 1, \dots, p$. Let us suppose that $d_1 \geq \dots \geq d_p$. Then*

$$\mathcal{L}_\infty(F) \geq d_p - B(d_1, \dots, d_p, n)$$

Cygan Ewa, Krasiński and Tworzewski in [25] give a direct generalization of the above result.

Theorem 1.6.17 ([25, Theorem 7.3]). *In the same conditions of Theorem 1.6.16, we have that*

$$\mathcal{L}_\infty(F) \geq d_p - B(d_1, \dots, d_p, n) + \sum_{b \in F^{-1}(0)} \mu_b(F)$$

where $\mu_b(F)$ denotes the multiplicity of F at an isolated zero b .

In the above results lower bounds for the Łojasiewicz exponent at infinity are given in terms of the degree of the component functions and the multiplicity of the polynomial map.

Next, we present some applications of the Łojasiewicz exponent at infinity for polynomial maps and in particular we introduce the Jacobian Conjecture.

Proposition 1.6.18 ([44, Proposition 4.1]). *Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \rightarrow \mathbb{C}^p$ be a polynomial map. Then F is a proper mapping if and only if $\mathcal{L}_\infty(F) > 0$.*

The above proposition characterizes the properness of polynomial maps in terms of its Łojasiewicz exponent. Given a map $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$, we recall that F is said to be *proper* if and only if for any compact set $S \subseteq \mathbb{K}^p$ the preimage $F^{-1}(S) \subseteq \mathbb{K}^n$ is also compact (it is equivalent to the property that for any sequence $\{z_k\}_{k \in \mathbb{N}} \subseteq \mathbb{K}^n$ such that $\|z_k\| \rightarrow \infty$ we have $\|F(z_k)\| \rightarrow \infty$). This result is also valid for real polynomial maps and the proof in this case is completely analogous to the complex case. Let us remark that the *if* part is immediate and the converse follows as a consequence of the Curve Selection Lemma at infinity (see [57, Lemma 2] and [44] for details).

A motivation to study the Łojasiewicz exponent at infinity is the famous “Jacobian Conjecture”. This conjecture states that if $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial map and $\det(DF)$ is a not-zero constant in $\mathbb{C}^n$, then F is a polynomial automorphism, where DF denotes the differential matrix of F . This conjecture has been approached in different areas of mathematics, as can be seen in the monograph of Van Den Essen [76]. We recall that Pinchuk showed an example of polynomial map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that F is a local homeomorphism at each point $(x, y) \in \mathbb{R}^2$, but F is not injective [59].

The Jacobian conjecture can be reduced to the problem on the Łojasiewicz exponent by the following proposition.

Proposition 1.6.19 ([44, Proposition 4.2]). *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map such that $\det(DF)$ is a not-zero constant in $\mathbb{C}^n$, then F is a polynomial automorphism of $\mathbb{C}^n$ if and only if $\mathcal{L}_\infty(F) > 0$.*

Hence to prove Jacobian Conjecture it is enough to check if F is proper. By the above proposition if F is a polynomial automorphism of $\mathbb{C}^n$ then $\mathcal{L}_\infty(F) > 0$. In this case, we can ask, what positive rational number is it? The answer was given by Płoski in [60].

Proposition 1.6.20 ([60]). *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial automorphism of $\mathbb{C}^n$, then*

$$\mathcal{L}_\infty(F) = \frac{1}{\deg F^{-1}}$$

CHAPTER 2

Newton filtrations and non-degeneracy conditions

Contents

2.1 Global Newton polyhedra	27
2.2 Newton non-degeneracy at infinity	32
2.3 The Newton filtration	35
2.4 Counting affine roots of polynomial systems	40

The fundamental concept that we study in this chapter is the Newton non-degeneracy at infinity of polynomial maps $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$. As we will see, we generalize this notion in view of several purposes. In this section we also introduce the notion of Newton non-degeneracy with respect to a fixed convenient Newton polyhedron at infinity. The motivation for the material of this section have been the articles [11], [43] and [67].

2.1 Global Newton polyhedra

In this section we describe some combinatorial concepts about Newton polyhedra at infinity.

Definition 2.1.1. A subset $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ is said to be a *global Newton polyhedron* or a *Newton polyhedron at infinity*, if there exists some finite subset $A \subseteq \mathbb{Z}_{\geq 0}^n$ such that $\tilde{\Gamma}_+$ is equal to the convex hull in $\mathbb{R}^n$ of $A \cup \{0\}$.

Let us fix a compact set $S \subseteq \mathbb{R}^n$. If $w \in \mathbb{R}^n$, then we define

$$\ell(w, S) = \min \{ \langle w, k \rangle : k \in S \}, \quad (2.1)$$

$$\Delta(w, S) = \{ k \in S : \langle w, k \rangle = \ell(w, S) \}. \quad (2.2)$$

where we denote by $\langle \cdot, \cdot \rangle$ the standard scalar product in $\mathbb{R}^n$. If $w \in \mathbb{R}^n \setminus \{0\}$, then $\Delta(w, S)$ is called the *face of S supported by w* . The hyperplane given by the equation $\langle w, k \rangle = \ell(w, S)$ is called a *supporting hyperplane* of S .

Let $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ be a global Newton polyhedron and let Δ be a face of $\tilde{\Gamma}_+$. The *dimension* of the face Δ , denoted by $\dim(\Delta)$, is defined as the minimum among the dimensions of the affine subspaces containing Δ . We denote by $\tilde{\Gamma}$ the union of the faces of $\tilde{\Gamma}_+$ not containing the origin. We remark that $\tilde{\Gamma}$ is not a convex set in general. The faces of $\tilde{\Gamma}_+$ of dimension 0 are called *vertices* of $\tilde{\Gamma}_+$ and the faces of $\tilde{\Gamma}_+$ of dimension $n - 1$ are called *facets* of $\tilde{\Gamma}_+$.

We define the *dimension* of $\tilde{\Gamma}$ as

$$\dim(\tilde{\Gamma}) = \max \{ \dim(\Delta) : \Delta \text{ is a face of } \tilde{\Gamma}_+ \text{ such that } 0 \notin \Delta \}.$$

Example 2.1.2. Let $\{e_1, e_2, e_3\}$ be the canonical basis in $\mathbb{R}^3$, let $c = e_1 + e_2 + e_3$ and let us fix an integer $a \geq 1$. Let us consider the following subsets of $\mathbb{Z}_{\geq 0}^3$:

$$A_1 = \{2ae_1, 2ae_2, 2ae_3, 2a(1, 1, 0), 2a(1, 0, 1), 2a(0, 1, 1), a(2, 1, 2), a(2, 2, 1), a(1, 2, 2)\}.$$

$$A_2 = \{3ae_3, ac, a(1, 1, 0)\}.$$

Let $\tilde{\Gamma}_+^1 = \tilde{\Gamma}_+(A_1)$ and let $\tilde{\Gamma}_+^2 = \tilde{\Gamma}_+(A_2)$. Figure (2.1) shows a picture of $\tilde{\Gamma}_+^1$ and $\tilde{\Gamma}_+^2$. We remark that $\dim \tilde{\Gamma}_+^1 = 2$ and $\dim \tilde{\Gamma}_+^2 = 1$.

Figure 2.1: Newton polyhedra at infinity

For any $w = (w_1, \dots, w_n) \in \mathbb{R}^n$, we denote by w_0 the minimum of the coordinates of w . Then we define $\mathbb{R}_0^n = \{w \in \mathbb{R}^n : w_0 < 0\}$ and $\mathbb{R}_0^n(i) = \{w \in \mathbb{R}_0^n : w_0 = w_i\}$, for all $i = 1, \dots, n$. Let us remark that if $\tilde{\Gamma}_+ \subseteq \mathbb{R}^n$ is a global Newton polyhedron then

$$\tilde{\Gamma}_+ = \{k \in \mathbb{R}_{\geq 0}^n : \langle k, w \rangle \geq \ell(w, \tilde{\Gamma}_+), \text{ for all } w \in \mathbb{R}_0^n\}. \quad (2.3)$$

Let $w \in \mathbb{Z}^n$. We say that w is *primitive* when $w \neq 0$ and w is the vector of smallest length between all vectors of $\mathbb{Z}^n$ of the form λw , for some $\lambda > 0$.

Let $\tilde{\Gamma}_+$ be a Newton polyhedron at infinity in $\mathbb{R}^n$ such that $\dim(\tilde{\Gamma}) = n - 1$. We denote by $\mathcal{F}(\tilde{\Gamma}_+)$ the family of primitive vectors $w \in \mathbb{Z}^n$ such that $\dim \Delta(w, \tilde{\Gamma}_+) = n - 1$. Since $\tilde{\Gamma}_+$ is the convex hull of a finite subset of $\mathbb{R}^n$ and $\dim(\tilde{\Gamma}) = n - 1$ then $\mathcal{F}(\tilde{\Gamma}_+)$ is a finite family of vectors (see [31, p. 29]). Moreover, any face of $\tilde{\Gamma}_+$ can be expressed as an intersection $\cap_{w \in J} \Delta(w, \tilde{\Gamma}_+)$, for some subset $J \subseteq \mathcal{F}(\tilde{\Gamma}_+)$ (see [31, p. 33]). We denote by $\mathcal{F}_0(\tilde{\Gamma}_+)$ the subset of $\mathcal{F}(\tilde{\Gamma}_+)$ formed by the vectors $w \in \mathcal{F}(\tilde{\Gamma}_+)$ such that $\Delta(w, \tilde{\Gamma}_+)$ does not contain the origin.

Lemma 2.1.3. *Let $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ be a global Newton polyhedron. Let J be a subset of $\mathcal{F}(\tilde{\Gamma}_+)$. Then the following conditions are equivalent:*

- (i) $\cap_{w \in J} \Delta(w, \tilde{\Gamma}_+) \neq \emptyset$;
- (ii) $\cap_{w \in J} \Delta(w, \tilde{\Gamma}_+) = \Delta(\sum_{w \in J} w, \tilde{\Gamma}_+)$;
- (iii) $\ell(\sum_{w \in J} w, \tilde{\Gamma}_+) = \sum_{w \in J} \ell(w, \tilde{\Gamma}_+)$.

Proof. The result follows as a direct consequence of the definition of $\ell(w, \tilde{\Gamma}_+)$ and $\Delta(w, \tilde{\Gamma}_+)$, for a given vector $w \in \mathbb{R}^n$. $\square$

Let $\tilde{\Gamma}_+^1, \dots, \tilde{\Gamma}_+^p$ be global Newton polyhedra in $\mathbb{R}^n$. Then the *Minkowski sum* of $\tilde{\Gamma}_+^1, \dots, \tilde{\Gamma}_+^p$ is defined as $\tilde{\Gamma}_+^1 + \dots + \tilde{\Gamma}_+^p = \{k_1 + \dots + k_p : k_i \in \tilde{\Gamma}_+^i, \text{ for all } i = 1, \dots, p\}$. It is well known that $\tilde{\Gamma}_+^1 + \dots + \tilde{\Gamma}_+^p$ is again a global Newton polyhedron. The definition of $A_1 + \dots + A_p$ for arbitrary sets $A_1, \dots, A_p \subseteq \mathbb{R}^n$ is analogous.

Lemma 2.1.4. *Let $\tilde{\Gamma}_+^1, \dots, \tilde{\Gamma}_+^p$ be global Newton polyhedra in $\mathbb{R}^n$. Let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+^1 + \dots + \tilde{\Gamma}_+^p$ and let $w \in \mathbb{R}^n \setminus \{0\}$. Then*

- (i) $\ell(w, \tilde{\Gamma}_+) = \ell(w, \tilde{\Gamma}_+^1) + \dots + \ell(w, \tilde{\Gamma}_+^p)$;
- (ii) $\Delta(w, \tilde{\Gamma}_+) = \Delta(w, \tilde{\Gamma}_+^1) + \dots + \Delta(w, \tilde{\Gamma}_+^p)$.

Proof. It arises as a consequence of the definition of Minkowski sum. $\square$

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ denote the canonical basis in $\mathbb{R}^n$. Given a global Newton polyhedron $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$, we say that $\tilde{\Gamma}_+$ is *convenient* if $\tilde{\Gamma}_+$ intersects each coordinate axis in a point different from the origin, that is, if for any $i \in \{1, \dots, n\}$ there exists some $\alpha > 0$ such that $\alpha \mathbf{e}_i \in \tilde{\Gamma}_+$. In this case we define

$$r_i(\tilde{\Gamma}_+) = \max\{r > 0 : r\mathbf{e}_i \in \tilde{\Gamma}_+\}, \quad i = 1, \dots, n. \quad (2.4)$$

$$r_0(\tilde{\Gamma}_+) = \min\{r_1(\tilde{\Gamma}_+), \dots, r_n(\tilde{\Gamma}_+)\}. \quad (2.5)$$

Lemma 2.1.5. *Let $\tilde{\Gamma}_+$ be a convenient global Newton polyhedron in $\mathbb{R}^n$. Let $w \in \mathbb{R}^n$. Then the following conditions are equivalent:*

- (i) $0 \notin \Delta(w, \tilde{\Gamma}_+)$;
- (ii) $\ell(w, \tilde{\Gamma}_+) < 0$;
- (iii) $w_0 < 0$.

Proof. It follows easily using the fact that $\tilde{\Gamma}_+$ is convenient (see [8, Lemma 4.2]). $\square$

Lemma 2.1.6. *Let $\tilde{\Gamma}_+$ be a convenient Newton polyhedron at infinity, $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$. Then*

$$r_i(\tilde{\Gamma}_+) = \min\left\{\frac{\ell(w, \tilde{\Gamma}_+)}{w_i} : w \in \mathbb{R}_0^n(i)\right\}, \quad \text{for all } i = 1, \dots, n. \quad (2.6)$$

Proof. Equality (2.6) follows as an immediate consequence of Lemma 2.1.5 and the fact that $\tilde{\Gamma}_+$ is expressed as $\tilde{\Gamma}_+ = \{k \in \mathbb{R}_{\geq 0}^n : \langle w, k \rangle \geq \ell(w, \tilde{\Gamma}_+), \text{ for all } w \in \mathbb{R}_0^n\}$. $\square$

Let us fix coordinates $x_1, \dots, x_n$ in $\mathbb{K}^n$ and let $k \in \mathbb{Z}_{\geq 0}^n$. Then we write x^k to denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$.

Definition 2.1.7. Let $h \in \mathbb{K}[x_1, \dots, x_n]$, $h \neq 0$. Let us suppose that h is written as $h = \sum_k a_k x^k$. Then the *support* of h , denoted by $\text{supp}(h)$, is defined as the set of those $k \in \mathbb{Z}_{\geq 0}^n$ such that $a_k \neq 0$. If $w \in \mathbb{R}^n$, then

$$\ell(w, h) = \ell(w, \text{supp}(h)), \quad (2.7)$$

$$\Delta(w, h) = \Delta(w, \text{supp}(h)). \quad (2.8)$$

Let $c = (1, \dots, 1) \in \mathbb{R}_{\geq 0}^n$, then we observe that $\ell(-c, h) = -\deg(h)$. The *Newton polyhedron at infinity* of h is defined as the convex hull of $\text{supp}(h) \cup \{0\}$ and is denoted by $\tilde{\Gamma}_+(h)$. We say that h is convenient when $\tilde{\Gamma}_+(h)$ is convenient. If $h = 0$, then we set $\text{supp}(h) = \emptyset$ and $\tilde{\Gamma}_+(h) = \emptyset$.

If we consider a map $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$, then the *Newton polyhedron at infinity* of F , that we denote by $\tilde{\Gamma}_+(F)$, is defined as the convex hull of $\tilde{\Gamma}_+(F_1) \cup \dots \cup \tilde{\Gamma}_+(F_p)$. We say that F is *convenient* when $\tilde{\Gamma}_+(F)$ is convenient. We denote by $\tilde{\Gamma}(F)$ the union of those faces of $\tilde{\Gamma}_+(F)$ not passing through the origin.

Next we recall some important concepts from convex geometry.

Definition 2.1.8. If A is a subset of $\mathbb{R}^n$, then we denote by $\text{Conv}(A)$ the convex hull of A in $\mathbb{R}^n$. If $P \subseteq \mathbb{R}^n$, then we say that P is a *polytope* when there exists a finite subset $A \subseteq \mathbb{R}^n$ such that $P = \text{Conv}(A)$. If A is contained in $\mathbb{Z}^n$, then $\text{Conv}(A)$ is said to be a *lattice polytope*. In particular, we observe that any global Newton polyhedron in $\mathbb{R}^n$ is a lattice polytope. If P is a polytope in $\mathbb{R}^n$, then the dimension of P is defined as the minimum dimension of an affine subspace containing P .

If $K \subseteq \mathbb{R}^n$ is a compact set, then we denote by $V_n(K)$ the n -dimensional volume of K . Let $C_1, \dots, C_n$ be polytopes in $\mathbb{R}^n$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0}$. Let $\lambda_1 C_1 + \dots + \lambda_n C_n = \{\lambda_1 k_1 + \dots + \lambda_n k_n : k_i \in C_i, i = 1, \dots, n\}$. It is a classical result from convex geometry that $V_n(\lambda_1 C_1 + \dots + \lambda_n C_n)$ is a homogeneous polynomial of degree n in the variables $\lambda_1, \dots, \lambda_n$ (see for instance [20, p. 337]). The n -dimensional mixed volume of $C_1, \dots, C_n$ is defined as the coefficient of $\lambda_1 \dots \lambda_n$ in the polynomial $V_n(\lambda_1 C_1 + \dots + \lambda_n C_n)$. We denote this number by $\text{MV}(C_1, \dots, C_n)$. Let us recall some elementary properties of this number

- (i) $\text{MV}(C_1, \dots, C_n)$ is symmetric and linear in each variable.
- (ii) $\text{MV}(C_1, \dots, C_n) \geq 0$ and $\text{MV}(C_1, \dots, C_n) = 0$ if and only if $\dim(\sum_{i \in I} C_i) < |I|$ for some nonempty subset $I \subseteq \{1, \dots, n\}$, where $|I|$ denotes the cardinal of I .
- (iii) $\text{MV}(C_1, \dots, C_n) \in \mathbb{Z}_{\geq 0}$, if C_i is a lattice of polytope, for all $i = 1, \dots, n$.
- (iv) $\text{MV}(C, \dots, C) = n! V_n(C)$ for any polytope $C \subseteq \mathbb{R}^n$.

If P is a polytope in $\mathbb{R}^n$, then we denote $\text{Conv}(P \cup \{0\})$ by P^0 . If $\mathbf{P} = (P_1, \dots, P_n)$ is an n -tuple of polytopes of $\mathbb{R}^n$, then we define $\mathbf{P}^0 = (P_1^0, \dots, P_n^0)$. In particular, it makes sense to speak about the mixed volumes $\text{MV}(\mathbf{P})$ and $\text{MV}(\mathbf{P}^0)$. In general $\text{MV}(\mathbf{P}) \leq \text{MV}(\mathbf{P}^0)$, by the monotonicity of mixed volumes with respect to inclusion. We refer to [21, 31, 65] for more information about $\text{MV}(C_1, \dots, C_n)$.

2.2 Newton non-degeneracy at infinity

Let $\tilde{\Gamma}_+$ be a global Newton polyhedron of $\mathbb{R}^n$ and let Δ a face of $\tilde{\Gamma}_+$. If $h \in \mathbb{K}[x_1, \dots, x_n]$ and h is written as $h = \sum_k a_k x^k$, then we denote by h_Δ the polynomial obtained as the sum of those terms $a_k x^k$ such that $k \in \Delta$. If $\text{supp}(h) \cap \Delta = \emptyset$, then we set $h_\Delta = 0$.

Motivated by the concept of Newton non-degenerate polynomial function [43, p. 7] we introduce the following definition.

Definition 2.2.1. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. We say that F is *Newton non-degenerate at infinity* when

$$\{x \in \mathbb{K}^n : (F_1)_\Delta(x) = \dots = (F_p)_\Delta(x) = 0\} \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}, \quad (2.9)$$

for all compact face Δ of $\tilde{\Gamma}_+(F)$ not containing the origin.

If $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ is a polynomial map, then we denote by $I(F)$ the ideal of $\mathbb{K}[x_1, \dots, x_n]$ generated by the component functions of F .

Example 2.2.2. Let us consider the polynomial maps $F_1, F_2 : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by $F_1(x_1, x_2) = (x_1, x_1^2 + x_2^2)$ and $F_2(x_1, x_2) = (x_1, x_2^2)$, for all $(x_1, x_2) \in \mathbb{C}^2$. It is immediate to see that $I(F_1)$ and $I(F_2)$ are generating the same ideal in $\mathbb{C}[x_1, x_2]$. However, F_1 is Newton degenerate at infinity and F_2 is Newton non-degenerate at infinity.

The above easy example shows that the Newton non-degeneracy at infinity can not be extended to ideals of $\mathbb{K}[x_1, \dots, x_n]$ by imposing this condition to a fixed generating system of a given ideal, which is not the case when dealing with the analogous concept in the ring of analytic germs $(\mathbb{K}^n, 0) \rightarrow \mathbb{K}$.

Unless otherwise stated, in the remaining section we will consider only the case $\mathbb{K} = \mathbb{C}$. Let us fix a global Newton polyhedron $\tilde{\Gamma}_+ \subseteq \mathbb{R}^n$. If Δ is a face of $\tilde{\Gamma}_+$, then we denote by $C(\Delta)$ the *cone over* Δ , that is, the union of all half lines emanating from the origin and passing through some point of Δ .

Suppose that Δ does not contain the origin. Then, we denote by $\mathcal{R}_\Delta$ the subring of $\mathbb{C}[x_1, \dots, x_n]$ given by all $h \in \mathbb{C}[x_1, \dots, x_n]$ such that $\text{supp}(h) \subseteq C(\Delta)$. We remark that $\mathcal{R}_\Delta$ is a Cohen-Macaulay ring of dimension $\dim(\Delta) + 1$ (see [43, p. 24]).

Proposition 2.2.3 ([43, Théorème 6.2]). *Let us fix a convenient Newton polyhedron $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$. Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \rightarrow \mathbb{C}^p$ be a polynomial map. If Δ is a face of $\tilde{\Gamma}_+$ not containing the origin, then the following conditions are equivalent:*

- (i) The ideal of $\mathcal{R}_\Delta$ generated by $\{(F_1)_\Delta, \dots, (F_p)_\Delta\}$ has finite colength.
- (ii) For each face Δ_1 of $\tilde{\Gamma}_+$ such that $\Delta_1 \subseteq \Delta$, we have

$$\{x \in \mathbb{C}^n : (F_1)_{\Delta_1}(x) = \dots = (F_p)_{\Delta_1}(x) = 0\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}.$$

The next result is an immediate consequence of Definition 2.2.1 and Proposition 2.2.3 and shows an alternative formulation of the Newton non-degeneracy at infinity of complex maps $F : \mathbb{C}^n \rightarrow \mathbb{C}^p$.

Corollary 2.2.4. *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^p$ be a polynomial map. Then F is Newton non-degenerate at infinity if and only if for each face Δ of $\tilde{\Gamma}_+$ not containing the origin, the ideal of $\mathcal{R}_\Delta$ generated by $\{(F_1)_\Delta, \dots, (F_p)_\Delta\}$ has a finite colength.*

If $F : \mathbb{C}^n \rightarrow \mathbb{C}^p$ is Newton non-degenerate at infinity, then Corollary 2.2.4 allows us to give an estimation of p .

Corollary 2.2.5. *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^p$ is a polynomial map such that F is Newton non-degenerate at infinity. Then $p \geq \dim(\tilde{\Gamma}(F)) + 1$.*

Proof. Let us consider Δ a face of $\tilde{\Gamma}_+$ such that $0 \notin \Delta$ and $\dim \tilde{\Gamma}(F) = \dim(\Delta)$. Then $\dim(\mathcal{R}_\Delta) = \dim(\Delta) + 1$ (see [15, p.210]). Since F is Newton non-degenerate at infinity, by Corollary 2.2.4, the ideal I generated by $\{(F_1)_\Delta, \dots, (F_p)_\Delta\}$ has a finite colength in $\mathcal{R}_\Delta$, which implies that I must be generated at least $\dim(\Delta) + 1$ non-zero elements of $\mathcal{R}_\Delta$, by [53, p.77]. Therefore $p \geq \dim(\Delta) + 1 = \dim(\tilde{\Gamma}) + 1$. $\square$

As a consequence of the previous result, if $F : \mathbb{C}^n \rightarrow \mathbb{C}^p$ is a convenient polynomial map such that F is Newton non-degenerate at infinity, then $p \geq n$, since $\dim(\tilde{\Gamma}(F)) = n - 1$ in this case.

Remark 2.2.6. It is immediate to see that the above corollary is not true when $\mathbb{K} = \mathbb{R}$. For this, let us consider the polynomial map $F : \mathbb{R}^2 \rightarrow \mathbb{R}$, given by $F(x_1, x_2) = x_1^2 + x_2^2$. Then it is easy to check that F is Newton non-degenerate at infinity and $\dim(\tilde{\Gamma}(F)) = 1$.

Definition 2.2.7. Let $f \in \mathbb{K}[x_1, \dots, x_n]$. Let us denote by $A(f)$ the polynomial map $\mathbb{K}^n \rightarrow \mathbb{K}^n$ given by

$$A(f) = \left(x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n} \right).$$

The function f is said to be *Newton non-degenerate at infinity* when the map $A(f)$ is Newton non-degenerate at infinity.

Let us remark that $\tilde{\Gamma}_+(f) = \tilde{\Gamma}_+(A(f))$, for all $f \in \mathbb{K}[x_1, \dots, x_n]$. Then above definition coincides with the notion of non-degeneracy established for polynomial functions introduced by Kouchnirenko in [43, p. 7].

Let $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ be a convenient Newton polyhedron at infinity and let $\mathbb{C}[\tilde{\Gamma}_+]$ be denote the set of polynomial functions $\mathbb{C}^n \rightarrow \mathbb{C}$ whose support is contained in $\tilde{\Gamma}_+$. Then it is well-known, by [43, Théorème 6.1], that the class of polynomial maps of $\mathbb{C}[\tilde{\Gamma}_+]$ which are Newton non-degenerate at infinity constitute a dense Zariski-open set of $\mathbb{C}[\tilde{\Gamma}_+]$.

Let $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ be a convenient Newton polyhedron at infinity. Then the *Newton number* of $\tilde{\Gamma}_+$ is defined, in [43, p. 4], as

$$\nu(\tilde{\Gamma}_+) = n!V_n - (n-1)!V_{n-1} + \dots + (-1)^{n-1}1!V_1 + (-1)^n,$$

where V_i denotes the sum of the i -dimensional volumes of the intersections of $\tilde{\Gamma}_+$ with the coordinates planes of dimension i , for $i = 1, \dots, n$. If $f \in \mathbb{C}[x_1, \dots, x_n]$ has a finite number of singular points, then the *Milnor number of f at infinity* is defined as

$$\mu_\infty(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{J(f)},$$

where $J(f)$ is the ideal in $\mathbb{C}[x_1, \dots, x_n]$ generated by the derivatives $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$.

Here we state a known result of Kouchnirenko about the computation of the Milnor number of a Newton non-degenerate polynomial function at infinity in terms of $\tilde{\Gamma}_+(f)$.

Theorem 2.2.8 ([43]). *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a convenient polynomial function with a finite number of singular points. Then $\mu_\infty(f) \leq \nu(\tilde{\Gamma}_+(f))$, and equality holds if f is Newton non-degenerate at infinity.*

The explicit expresion of $\mu_\infty(f)$ in terms of $\tilde{\Gamma}_+(f)$ for non convenient polynomial functions $\mathbb{C}^n \rightarrow \mathbb{C}$ is a hard problem. This task has been done by P. Cassou-Noguès [17] and Artal-Luengo-Melle [1].

The local counterpart of the above result was the motivation of Bivià-Ausina, Fukui and Saia in [11] to analyze under which conditions the multiplicity of (non-gradient) analytic map germs $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$, or more generally, the codimension of determinantal ideals, could be computed by means of a given (local) Newton filtration. This kind of results help in the study and the effective computation of numerical invariants of singularities of complex analytic maps $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$.

The main goal of this chapter is to extend Theorems 3.3 and 3.6 of [11] to the context of polynomial maps $\mathbb{C}^n \rightarrow \mathbb{C}^n$. For this purpose we introduce in the next section the notion of filtration of $\mathbb{C}[x_1, \dots, x_n]$ associated to a convenient Newton polyhedron in $\mathbb{R}_{\geq 0}^n$.

2.3 The Newton filtration

Along this section, we denote by $\tilde{\Gamma}_+$ a fixed convenient Newton polyhedron at infinity contained in $\mathbb{R}_{\geq 0}^n$. Let Δ denote a face of $\tilde{\Gamma}_+$ not containing the origin. Then, as in the previous chapter, we denote by $C(\Delta)$ the *cone* over Δ , that is, the union of all half lines emanating from the origin and passing through some point of Δ .

It is immediate to see that $\mathcal{F}(\tilde{\Gamma}_+) = \mathcal{F}_0(\tilde{\Gamma}_+) \cup \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Let $\mathcal{F}_0(\tilde{\Gamma}_+) = \{w^1, \dots, w^r\}$, for some $w^i \in \mathbb{Z}^n \cap \mathbb{R}_{\geq 0}^n$, $i = 1, \dots, r$, and let $M_{\tilde{\Gamma}_+}$ denote the least common multiple of $\{-\ell(w^i, \tilde{\Gamma}_+) : i = 1, \dots, r\}$. We remark that $\ell(w^i, \tilde{\Gamma}_+) < 0$, for all $i = 1, \dots, r$ (see Lemma 2.1.5). Then, following Kouchnirenko in [43], we define the linear map $\phi_i : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$, by

$$\phi_i(k) = \frac{M_{\tilde{\Gamma}_+}}{\ell(w^i, \tilde{\Gamma}_+)} \langle w^i, k \rangle,$$

for all $k \in \mathbb{R}_{\geq 0}^n$ and all $i = 1, \dots, r$. Thus, we define the map $\phi_{\tilde{\Gamma}_+} : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$ by

$$\phi_{\tilde{\Gamma}_+}(k) = \max_{1 \leq i \leq r} \{\phi_i(k)\} = M_{\tilde{\Gamma}_+} \max_{1 \leq i \leq r} \frac{\langle w^i, k \rangle}{\ell(w^i, \tilde{\Gamma}_+)},$$

for all $k \in \mathbb{R}_{\geq 0}^n$. We will refer to $\phi_{\tilde{\Gamma}_+}$ as *the filtrating map associated to $\tilde{\Gamma}_+$* . If no confusion arises, we denote $M_{\tilde{\Gamma}_+}$ and $\phi_{\tilde{\Gamma}_+}$ simply by M and ϕ , respectively. We have that ϕ satisfies the following properties:

- (i) $\phi(\mathbb{Z}_{\geq 0}^n) \subseteq \mathbb{Z}_{\geq 0}$,
- (ii) ϕ is constant on the boundary $\tilde{\Gamma}$ and equal to M .

Lemma 2.3.1. *Under the above conditions, let $k \in \mathbb{R}_{\geq 0}^n$ and let $i_0 \in \{1, \dots, r\}$. Then $\phi(k) = \phi_{i_0}(k)$ if and only if $k \in C(\Delta(w^{i_0}, \tilde{\Gamma}_+))$.*

Proof. Since $\tilde{\Gamma}_+$ is convenient, we can write k as $k = \lambda k'$, for some $k' \in \tilde{\Gamma}$. A simple computation shows that $\phi(k) = \phi_{i_0}(k)$ if and only if

$$\frac{\langle k', w^i \rangle}{\ell(w^i, \tilde{\Gamma}_+)} \leq \frac{\langle k', w^{i_0} \rangle}{\ell(w^{i_0}, \tilde{\Gamma}_+)},$$

for all $i \in \{1, \dots, r\}$. But this last condition is equivalent to saying that $\frac{\langle k', w^{i_0} \rangle}{\ell(w^{i_0}, \tilde{\Gamma}_+)} = 1$, which, in particular says that $k' \in \Delta(w^{i_0}, \tilde{\Gamma}_+)$, and then the result follows. $\square$

The above lemma shows that ϕ is linear on each cone $C(\Delta)$, where Δ is any face of $\tilde{\Gamma}_+$ not passing through the origin. Moreover it is immediate to see that $\phi(a+b) \leq \phi(a) + \phi(b)$, for all $a, b \in \mathbb{R}_{\geq 0}^n$, and, by Lemma 2.3.1, equality holds if and only if a and b belong to the same cone, that is, there exists a vector $w \in \mathcal{F}_0(\tilde{\Gamma}_+)$ such that $a, b \in C(\Delta(w, \tilde{\Gamma}_+))$.

Let us remark that

$$\tilde{\Gamma}_+ = \{k \in \mathbb{R}_{\geq 0}^n : \langle w^i, k \rangle \geq \ell(w^i, \tilde{\Gamma}_+), \text{ for all } i = 1, \dots, r\} = \{k \in \mathbb{R}_{\geq 0}^n : \phi(k) \leq M\}.$$

Definition 2.3.2. We define the *Newton filtration* of $\mathbb{C}[x_1, \dots, x_n]$ associated to $\tilde{\Gamma}_+$ as the map $\nu_{\tilde{\Gamma}_+} : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{Z}_+ \cup \{-\infty\}$ given by $\nu_{\tilde{\Gamma}_+}(h) = \max\{\phi_{\tilde{\Gamma}_+}(k) : k \in \text{supp}(h)\}$ when $h \neq 0$ and $\nu_{\tilde{\Gamma}_+}(0) = -\infty$. The number $\nu_{\tilde{\Gamma}_+}(h)$ is called the *degree of h with respect to $\tilde{\Gamma}_+$* . Let us remark that when $\tilde{\Gamma}_+$ is equal to the standard n -simplex, that is, when $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(\{\mathbf{e}_1, \dots, \mathbf{e}_n\})$, then $\nu_{\tilde{\Gamma}_+}(h)$ coincides with the usual notion of degree of h , for all $h \in \mathbb{C}[x_1, \dots, x_n]$. If no confusion arises, we denote the map $\nu_{\tilde{\Gamma}_+}$ simply by ν .

We observe that the map $\nu : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{Z}_+ \cup \{-\infty\}$ satisfies the following properties:

- (i) $\nu(1) = 0, \nu(0) = -\infty$;
- (ii) $\nu(h+g) \leq \max\{\nu(h), \nu(g)\}$;
- (iii) $\nu(h \cdot g) \leq \nu(h) + \nu(g)$.

The above properties are straightforward consequences of the properties of the filtrating map ϕ . Let us define, for all $r \geq 0$, the following set of polynomials:

$$\mathcal{A}_r = \{h \in \mathbb{C}[x_1, \dots, x_n] : \nu(h) \leq r\} \cup \{0\} \quad (2.10)$$

and we set $\mathcal{A}_r = 0$ for all $r < 0$. We denote by $\tilde{\Gamma}_+(\mathcal{A}_r)$ the convex hull of $\{k \in \mathbb{Z}_{\geq 0}^n : \phi(k) \leq r\} \cup \{0\}$. In particular $\mathcal{A}_0 = \mathbb{C}$. By the properties of ϕ it is immediate to check that the collection $\{\mathcal{A}_r\}_{r \in \mathbb{Z}}$ verify

- (i) $\mathcal{A}_r$ is an additive subgroup of $\mathbb{C}[x_1, \dots, x_n]$, for all $r \geq 0$;
- (ii) $\mathcal{A}_r \subseteq \mathcal{A}_{r+1}$, for all $r \geq 0$;
- (iii) $\mathcal{A}_r \mathcal{A}_s \subseteq \mathcal{A}_{r+s}$, for all $r, s \in \mathbb{Z}$;
- (iv) $\tilde{\Gamma}_+(\mathcal{A}_r) \subseteq \frac{r}{M} \tilde{\Gamma}_+$ and equality holds if and only if $V_n(\tilde{\Gamma}_+(\mathcal{A}_r)) = (\frac{r}{M})^n V_n(\tilde{\Gamma}_+)$, where V_n denotes the n -dimensional volume.

The collection $\{\mathcal{A}_r\}_{r \in \mathbb{Z}}$ is also known as the filtration of $\mathbb{C}[x_1, \dots, x_n]$ associated to $\tilde{\Gamma}_+$. From this filtration we can construct the graded ring $\mathbf{A} = \bigoplus_{r \geq 0} \mathbf{A}_r$, where $\mathbf{A}_r = \mathcal{A}_r / \mathcal{A}_{r-1}$, for all $r \geq 0$.

The objective of this section is to obtain a characterization of an important class of polynomial maps $\mathbb{C}^n \rightarrow \mathbb{C}^n$ that extends the class of pre-weighted homogeneous maps (see Definition 2.3.5). In particular, we obtain a version of [11, Theorem 3.3] in the ring of polynomials $\mathbb{C}[x_1, \dots, x_n]$. In the remaining section $\tilde{\Gamma}_+$ denotes a fixed global Newton polyhedron in $\mathbb{R}^n$. Let us denote by ν the filtration $\nu_{\tilde{\Gamma}_+} : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{Z}_{\geq 0} \cup \{-\infty\}$.

Definition 2.3.3. Let $h \in \mathbb{K}[x_1, \dots, x_n]$, $h \neq 0$. Let us suppose that h is written as $h = \sum_k a_k x^k$. Let Δ be a face of $\tilde{\Gamma}_+$ not passing through the origin. Then *the principal part of h over Δ* , denoted by $\text{in}_\Delta(h)$, is the polynomial obtained as the sum of all terms $a_k x^k$ such that $k \in C(\Delta)$ and $\nu(h) = \nu(x^k)$. If no such terms exist or $h = 0$, then we set $\text{in}_\Delta(h) = 0$. We observe that if $h_\Delta \neq 0$, then $h_\Delta = \text{in}_\Delta(h)$ if and only if $\nu_{\tilde{\Gamma}_+}(h) = M_{\tilde{\Gamma}_+}$.

Motivated by Bivià-Ausina, Fukui and Saia [11], we introduce the following concept.

Definition 2.3.4. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. We say that F is *non-degenerate with respect to $\tilde{\Gamma}_+$* when

$$\{x \in \mathbb{C}^n : \text{in}_\Delta(F_1)(x) = \dots = \text{in}_\Delta(F_p)(x) = 0\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}, \quad (2.11)$$

for all faces Δ of $\tilde{\Gamma}_+$ not containing the origin.

We remark that the previous definition, which is specially significant when $p = n$, constitutes a generalization of the notion of pre-weighted homogeneous map from $\mathbb{C}^n$ to $\mathbb{C}^p$.

Definition 2.3.5. Let us fix a primitive vector $v = (v_1, \dots, v_n) \in \mathbb{Z}_{\geq 1}^n$, $v \neq 0$ and let $h \in \mathbb{K}[x_1, \dots, x_n]$. Let us suppose that h is written as $h = \sum_k a_k x^k$. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map.

- (i) Let us denote the integer $\max\{\langle v, k \rangle : k \in \text{supp}(h)\}$ by $d_v(h)$. We call $d_v(h)$ the *degree of h with respect to v* . We define the *principal part of h at infinity with respect to v* , denoted by $q_v(h)$ as the sum of those terms $a_k x^k$ such that $\langle v, k \rangle = d_v(h)$. If $h = 0$, then we set $d_v(h) = -\infty$ and $q_v(h) = 0$. We define $q_v(F) = (q_v(F_1), \dots, q_v(F_p))$ and $d_v(F) = (d_v(F_1), \dots, d_v(F_p))$.
- (ii) If $v \in \mathbb{Z}_{\geq 1}^n$. We say that h is *weighted-homogeneous* of degree d with respect to v when $\text{supp}(h)$ is contained in the hyperplane of equation $\langle v, k \rangle = d$. That is, when $q_v(h) = h$ and $d_v(h) = d$.

- (iii) Let $d = (d_1, \dots, d_n) \in \mathbb{Z}_{\geq 1}^n$. If F_i is weighted homogeneous of degree d_i , for all $i = 1, \dots, n$, then F is called *weighted-homogeneous with respect to v* with vector of degrees d .
- (iv) We say that F is *pre-weighted homogeneous* with respect to v when $q_v(F)^{-1}(0) = \{0\}$.

We refer to the works of Cima-Gasull-Mañosas [18] and Saia [71] for interesting properties of pre-weighted homogeneous maps.

Let $v = (v_1, \dots, v_n) \in \mathbb{Z}_{\geq 1}^n$ be a primitive vector. Let us denote by $\tilde{\Gamma}_+^v$ the global Newton polyhedron $\tilde{\Gamma}_+(x_1^{v_1 \cdots v_n / v_1}, \dots, x_n^{v_1 \cdots v_n / v_n})$. We remark that $\tilde{\Gamma}_+^v$ has a unique face of dimension $n - 1$ not passing through the origin and that this is equal to the convex hull of the points belonging to the intersection of the hyperplane of equation $v_1 k_1 + \dots + v_n k_n = v_1 \cdots v_n$ with the coordinate axis.

Lemma 2.3.6. *Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be a polynomial map. Let $v \in \mathbb{Z}_{\geq 1}^n$ be a primitive vector and let $d = (d_1, \dots, d_n) \in \mathbb{Z}_{\geq 1}^n$. Then the following conditions are equivalent:*

- (i) F is pre-weighted homogeneous with respect to v and $d = d_v(F)$.
- (ii) F is non-degenerate with respect to $\tilde{\Gamma}_+^v$ and $\nu_{\tilde{\Gamma}_+^v}(F_i) = d_i$, for all $i = 1, \dots, n$.

Proof. It follows easily by observing that $\mathcal{F}_0(\tilde{\Gamma}_+) = \{-v\}$ and hence the filtrating map $\phi : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$ associated to $\tilde{\Gamma}_+^v$ is given by $\phi(k) = \langle v, k \rangle$, for all $k \in \mathbb{R}_{\geq 0}^n$. $\square$

Remark 2.3.7. Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map. It is immediate to deduce from the above definitions that, if F is Newton non-degenerate at infinity, then F is non-degenerate with respect to $\tilde{\Gamma}_+(F)$. An easy example showing that the converse is not true is given by the map $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $F(x, y) = (x + 2y, x^2 - y^2)$.

Lemma 2.3.8. *Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map such that $F(0) = 0$ and $F^{-1}(0)$ is compact. Then $\tilde{\Gamma}_+(F)$ is convenient.*

Proof. Let $F_1, \dots, F_p$ denote the component functions of F . If F is not convenient, then there exists some $i \in \{1, \dots, n\}$ such that $\tilde{\Gamma}_+(F_j)$ does not intersect the x_i -axis, for all $j = 1, \dots, p$. In particular we have that F vanishes on the x_i -axis, since $F(0) = 0$, and hence $F^{-1}(0)$ is not compact. $\square$

Proposition 2.3.9. *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map such that $F^{-1}(0)$ is finite. Then the following conditions are equivalent:*

- (i) F is Newton non-degenerate at infinity,

(ii) F is non-degenerate with respect to $\tilde{\Gamma}_+(F)$ and $\nu_{\tilde{\Gamma}_+}(F_1) = \cdots = \nu_{\tilde{\Gamma}_+}(F_n)$.

Proof. Since $F^{-1}(0)$ is finite, then $\tilde{\Gamma}_+(F)$ is convenient, by Lemma 2.3.8. The implication (ii) $\Rightarrow$ (i) is immediate since the condition $\nu_{\tilde{\Gamma}_+}(F_1) = \cdots = \nu_{\tilde{\Gamma}_+}(F_n)$ implies that $\text{in}_{\Delta}(F_i) = (F_i)_{\Delta}$, for all face Δ of $\tilde{\Gamma}_+$ such that $0 \notin \Delta$ and all $i = 1, \dots, n$.

Let us see (i) $\Rightarrow$ (ii). Let Δ be a face of $\tilde{\Gamma}_+$ of dimension $n - 1$ such that $0 \notin \Delta$. Let $\mathcal{R}_{\Delta}$ be the ring generated by the polynomials $h \in \mathbb{C}[x_1, \dots, x_n]$ such that $\text{supp}(h) \subseteq C(\Delta)$. Since F is Newton non-degenerate at infinity, the solutions of the system $(F_1)_{\Delta'}(x) = \cdots = (F_n)_{\Delta'}(x) = 0$ are contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$, for any face Δ' of $\tilde{\Gamma}_+$ such that $\Delta' \subseteq \Delta$. In particular, the ideal I generated by $\{(F_1)_{\Delta}, \dots, (F_n)_{\Delta}\}$ in $\mathcal{R}_{\Delta}$ has finite colength in $\mathcal{R}_{\Delta}$ by Proposition 2.2.3, which implies that I is generated by least n non-zero elements of $\mathcal{R}_{\Delta}$. Then $(F_i)_{\Delta} \neq 0$, for all $i = 1, \dots, n$. In particular we have $\nu_{\tilde{\Gamma}_+}(F_1) = \cdots = \nu_{\tilde{\Gamma}_+}(F_n)$ and thus the result follows. $\square$

It is easy to find examples showing that the above result does not hold in the case $\mathbb{K} = \mathbb{R}$.

Lemma 2.3.10. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a finite polynomial map. Let us fix a convenient global Newton polyhedron $\tilde{\Gamma}_+ \subseteq \mathbb{R}_+^n$. Then the following conditions are equivalent:*

- (i) F is non-degenerate with respect to $\tilde{\Gamma}_+$.
- (ii) There exists a vector $a = (a_1, \dots, a_n) \in \mathbb{Z}_{\geq 1}^n$ such that the map $F^a : \mathbb{C}^n \rightarrow \mathbb{C}^n$ given by $(F_1^{a_1}, \dots, F_n^{a_n})$ verifies that $\tilde{\Gamma}_+(F^a)$ is homothetic to $\tilde{\Gamma}_+$ and F^a is Newton non-degenerate at infinity.

Proof. Let us see (i) $\Rightarrow$ (ii). Let $d_i = \nu_{\tilde{\Gamma}_+}(F_i)$, for all $i = 1, \dots, n$, and let $d = d_1 \cdots d_n$. We define $a_i = d/d_i$, for all $i = 1, \dots, n$, and $a = (a_1, \dots, a_n)$. Let $k \in \mathbb{Z}_{\geq 0}^n$ such that k is a vertex of $\tilde{\Gamma}_+$. By condition (2.11), there exists some $i \in \{1, \dots, n\}$ such that $\text{in}_{\{k\}}(F_i) \neq 0$. This means that there exists some $k' \in \text{supp}(F_i)$ such that $\phi(k') = d_i$ and there exists some $\lambda > 0$ such that $k' = \lambda k$. Since $d_i = \phi(k') = \phi(\lambda k) = \lambda \phi(k) = \lambda M$, we obtain $\lambda = d_i/M$. In particular $a_i k' = a_i \frac{d_i}{M} k = \frac{d}{M} k$. Then, for any vertex k of $\tilde{\Gamma}_+$, we have that $\frac{d}{M} k$ has integer coordinates. This fact shows that

$$\tilde{\Gamma}_+(F^a) = \frac{d}{M} \tilde{\Gamma}_+ = \tilde{\Gamma}_+(\mathcal{A}_d) \quad (2.12)$$

Let Δ be a face of $\tilde{\Gamma}_+(F^a)$. By (2.12) there exists a face Δ' of $\tilde{\Gamma}_+$ such that $\Delta = \frac{d}{M} \Delta'$. Using Definition 2.3.3 it is immediate to see that $(\text{in}_{\Delta'}(F_i))^{a_i} = (F^{a_i})_{\Delta}$ for all $i = 1, \dots, n$, and thus condition (ii) follows.

Let us see the implication (ii) $\Rightarrow$ (i). Let $a = (a_1, \dots, a_n) \in \mathbb{Z}_{\geq 1}^n$ such that $\tilde{\Gamma}_+(F^a)$ and $\tilde{\Gamma}_+$ are homothetic. Let $u > 0$ such that $\tilde{\Gamma}_+(F^a) = u\tilde{\Gamma}_+$. Then $u\Delta$ is a face of $\tilde{\Gamma}_+(F^a)$, for any face Δ of $\tilde{\Gamma}_+$. In particular from the relation $(\text{in}_\Delta(F_i))^{a_i} = (F_i^{a_i})_{u\Delta}$ for all $i = 1, \dots, n$, we apply (ii) and then we have that the set of solutions of the system $\text{in}_\Delta(F_1)(x) = \dots = \text{in}_\Delta(F_n)(x) = 0$ is contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$. $\square$

2.4 Counting affine roots of polynomial systems

This section is devoted to show a result of Rojas and Wang [67] about the determination of the number of isolated affine roots, counting multiplicities, of polynomial systems. This result takes part of a far-reaching generalization of the result of Bernstein-Khovanskii-Kouchnirenko that bounds the number of isolated affine roots in $(\mathbb{C} \setminus \{0\})^n$, counting multiplicities, of polynomial systems (see [2], [20], [47], [67]).

In order to state the result of Rojas and Wang we introduce some preliminary definitions.

Definition 2.4.1. Let $h \in \mathbb{C}[x_1, \dots, x_n]$ and let A be a compact subset of $\mathbb{R}_{\geq 0}^n$ such that $\text{supp}(h) \subseteq A$. If $w \in \mathbb{R}^n$ and $h = \sum_k a_k x^k$, then we define the *principal part of h with respect to the pair (w, A)* , denoted by $p_{w,A}(h)$, as the sum of all terms $a_k x^k$ such that $k \in \text{supp}(h) \cap \Delta(w, A)$. If $\text{supp}(h) \cap \Delta(w, A) = \emptyset$, then we set $p_{w,A}(h) = 0$. The polynomial $p_{w,\text{supp}(h)}(h)$ will be denoted simply by $p_w(h)$. Let us remark that, if $h \neq 0$, then $p_w(h) \neq 0$. Moreover $p_{w,A}(h) = p_w(h)$ if and only if $\Delta(w, \text{supp}(h)) \subseteq \Delta(w, A)$. We shall refer to $p_w(h)$ as the *principal part of h with respect to w* .

Let A be a polytope in $\mathbb{Z}_{\geq 0}^n$, then we denote by $\mathbb{C}[A]$ the family of polynomials $h \in \mathbb{C}[x_1, \dots, x_n]$ such that $\text{supp}(h) \subseteq A$. Let $\mathbf{A} = (A_1, \dots, A_n)$ be a n -tuple of finite subsets of $\mathbb{Z}_{\geq 0}^n$. We denote by $\mathbb{C}_n[\mathbf{A}]$ the set of the polynomial maps $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $\text{supp}(F_i) \subseteq A_i$ for all $i = 1, \dots, n$. We can identify $\mathbb{C}_n[\mathbf{A}]$ with a finite-dimensional vector space $\mathbb{C}^N$, for a sufficiently large positive integer N , by associating to each map $F \in \mathbb{C}_n[\mathbf{A}]$ the vector formed by the coefficients of F . Under this identification, we say that a property holds for a generic $F \in \mathbb{C}_n[\mathbf{A}]$ when the property holds in a dense Zariski-open subset of $\mathbb{C}^N$.

Definition 2.4.2. Let $\mathbf{A} = (A_1, \dots, A_n)$ be an n -tuple of finite subsets of $\mathbb{Z}_{\geq 0}^n$. Let $F = (F_1, \dots, F_n) \in \mathbb{C}_n[\mathbf{A}]$. Then we say that F is **$\mathbf{A}$ -non-degenerate** if and only if, for all $w \in \mathbb{R}^n$ such that $w_0 < 0$, the set of solutions of the system

$$p_{w,A_1}(F_1)(x) = \dots = p_{w,A_n}(F_n)(x) = 0$$

is contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$.

Let A be a lattice polytope in $\mathbb{R}^n$, then we say that A is *cornered* when, for all $j = 1, \dots, n$ there exists some $k \in A$ such that $k_j = 0$ (see [67, p.119]). If $A \subseteq \mathbb{R}_{\geq 0}^n$ and f_A denotes the polynomial obtained as the sum of all terms x^k such that $k \in A \cap \mathbb{Z}_{\geq 0}^n$, then we observe that A is cornered if and only if f_A is not divisible by x_j for all $j = 1, \dots, n$.

Let $\mathbf{A} = (A_1, \dots, A_n)$ be an n -tuple of lattice polytopes in $\mathbb{R}^n$; then $\mathbf{A}$ is said to be *cornered* when A_i is cornered for all $i = 1, \dots, n$. We say $\mathbf{A}$ is *nice* when $F^{-1}(0)$ is finite for a generic map $F \in \mathbb{C}_n[\mathbf{A}]$.

We recall that if A is a polytope in $\mathbb{R}^n$, then $\text{Conv}(A \cup \{0\})$ is denoted by A^0 . If $\mathbf{A} = (A_1, \dots, A_n)$ is an n -tuple of polytopes of $\mathbb{R}^n$, then $\mathbf{A}^0 = (A_1^0, \dots, A_n^0)$. Let us remark that $\mathbf{A}^0$ is always cornered.

Theorem 2.4.3 ([67, p.119]). *Let $\mathbf{P} = (P_1, \dots, P_n)$ be an n -tuple of lattice polytopes of $\mathbb{R}_{\geq 0}^n$. Let us suppose that $\mathbf{P}$ is nice and cornered. Then a generic polynomial map $F \in \mathbb{C}_n[\mathbf{P}]$ has exactly $\text{MV}(\mathbf{P}^0)$ roots in $\mathbb{C}^n$, counting multiplicities.*

The following result follows as an application of [67, Theorem 6] and is contained explicitly in the proof of [67, Theorem 7, p.129]. We remark that [67, Theorem 6] is actually a result of Rojas and its proof appears in [66], moreover this is a much more general result stated for polynomial maps over an integrally closed field of arbitrary characteristic. The proof of this result applies known facts about intersection theory on toric varieties.

Let $\mathbf{A} = (A_1, \dots, A_n)$ be an n -tuple of finite subsets of $\mathbb{Z}_{\geq 0}^n$. Let us remark that, by [66, Proposition 8], the set of polynomial maps $F \in \mathbb{C}_n[\mathbf{A}]$ such that F is $\mathbf{A}$ -non-degenerate constitutes a dense Zariski-open subset of $\mathbb{C}_n[\mathbf{A}]$.

Theorem 2.4.4. *Let $\mathbf{A} = (A_1, \dots, A_n)$ be an n -tuple of finite subsets of $\mathbb{Z}_{\geq 0}^n$ such that $\text{MV}(\mathbf{A}^0) > 0$ and $\mathbf{A}$ is nice and cornered. Let $F \in \mathbb{C}_n[\mathbf{A}]$ such that $F^{-1}(0)$ is finite. Then*

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\langle F_1, \dots, F_n \rangle} \leq \text{MV}(\mathbf{A}^0) \quad (2.13)$$

and equality holds in (2.13) if and only if F is $\mathbf{A}^0$ -non-degenerate.

We recall that $\text{MV}(\mathbf{A}) = 0$ if and only if $\dim(\sum_{i \in I} A_i) < |I|$, for some non-empty subset $I \subseteq \{1, \dots, n\}$, where $|I|$ denotes the cardinal of I . In particular, this happens if $|A_i| = 1$, for some $i \in \{1, \dots, n\}$.

As a corollary of Theorem 2.4.4 we obtain the following fundamental result.

Corollary 2.4.5. *Let $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ be a convenient Newton polyhedron. Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map such that $\text{supp}(F_i) \subseteq \tilde{\Gamma}_+$, for all $i = 1, \dots, n$ and $F^{-1}(0)$ is finite. Then $\mu(F) \leq n!V_n(\tilde{\Gamma}_+)$ and equality holds if and only if F is Newton non-degenerate at infinity.*

Proof. From Definition 2.2.1 we have that F is Newton non-degenerate at infinity if and only if F is $\mathbf{A}$ -non-degenerate, with $\mathbf{A}$ equal to the n -tuple $(\tilde{\Gamma}_+ \cap \mathbb{Z}_{\geq 0}^n, \dots, \tilde{\Gamma}_+ \cap \mathbb{Z}_{\geq 0}^n)$. Hence the result follows as an immediate application of Theorem 2.4.4. $\square$

CHAPTER 3

Newton non-degeneracy at infinity

Contents

3.1	Special monomials and Newton non-degeneracy at infinity . . .	44
3.2	Multiplicity and special polynomials	53
3.3	Non-degeneracy with respect to a Newton filtration	57
3.4	Newton non-degeneracy of gradient maps	62
3.5	Homogeneous Newton polyhedra	67

In this chapter, the problem of characterizing the Newton non-degeneracy at infinity condition for polynomial maps $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ using special monomials is studied.

Motivated by the characterization of the integral closure of ideals in $\mathcal{O}_n$ by analytic inequalities proven by Lejeune-Teissier [46], we introduce the concept of *special polynomials* with respect to a given polynomial map F .

The main result of this chapter (Theorem 3.1.14) characterizes the Newton non-degeneracy at infinity condition of a given polynomial map $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ in terms of the set of special monomials of the map $(F, 1) : \mathbb{K}^n \rightarrow \mathbb{K}^{p+1}$. Moreover, when F is a convenient polynomial map the Newton non-degeneracy at infinity condition is characterized in terms of the set $S_0(F)$ (see Corollary 3.1.16). We remark that the results obtained in this chapter are polynomial versions of results obtained in [11, 7, 10, 33].

3.1 Special monomials and Newton non-degeneracy at infinity

We begin by relating the concept of Newton non-degeneracy at infinity for polynomial maps (see Definition 2.2.1) and the concept of Newton non-degeneracy for germs of analytic maps (see Definition 1.2.10).

Let us fix coordinates $x_1, \dots, x_n$ in $\mathbb{K}^n$. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map and let $d = \max\{\deg(F_1), \dots, \deg(F_p)\}$. Let us introduce a new variable x_{n+1} , so that $x_1, \dots, x_n, x_{n+1}$ is a coordinate system in $\mathbb{K}^{n+1}$.

Definition 3.1.1. Let h be a polynomial in $\mathbb{K}[x_1, \dots, x_n]$, such that $\deg(h) \leq d$. Then, we denote by h^* the polynomial in $\mathbb{K}[x_1, \dots, x_n, x_{n+1}]$, defined as

$$h^*(x_1, \dots, x_n, x_{n+1}) = x_{n+1}^d \cdot h(x_1/x_{n+1}, \dots, x_n/x_{n+1}). \quad (3.1)$$

That is h^* is the homogenization of degree d of h by means of the extra variable x_{n+1} .

Naturally, the polynomial map $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$, induces the analytic map germ $F^* : (\mathbb{K}^{n+1}, 0) \rightarrow (\mathbb{K}^p, 0)$, defined as $F^* = (F_1^*, \dots, F_p^*, x_{n+1}^d) : (\mathbb{K}^{n+1}, 0) \rightarrow (\mathbb{K}^{p+1}, 0)$, where F_i^* is the homogenization of degree d of F_i by means of the extra variable x_{n+1} .

The following figure shows the construction in $\mathbb{R}^3$ of the Newton polyhedron of F^* .

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_{n+1}\}$ be the canonical basis of $\mathbb{R}^{n+1}$, let $c' = \mathbf{e}_1 + \dots + \mathbf{e}_n + \mathbf{e}_{n+1}$ and let c denote the projection of c' onto the first n coordinates. Let us recall that if $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ then we denote $\min_i w_i$ by w_0 . Thus we defined $\mathbb{R}_0^n = \{w \in \mathbb{R}^n : w_0 < 0\}$.

We now define

$$\mathbb{R}_*^{n+1} = \{(v_1, \dots, v_n, v_{n+1}) \in \mathbb{R}_{>0}^{n+1} : v_{n+1} = 2v_0\}.$$

Next, we show that exists a bijection between the faces Δ of $\tilde{\Gamma}_+(F)$ not containing the origin of $\mathbb{R}^n$ and the compact faces Δ^* of $\Gamma_+(F^*)$ not containing the point $d\mathbf{e}_{n+1} \in \mathbb{R}^{n+1}$.

Remark 3.1.2. We observe easily from the construction of F^* , that any compact face of $\Gamma_+(F^*)$ is contained in the n -dimensional compact face $\Delta(c', \Gamma_+(F^*))$.

Let $v = (v_1, \dots, v_n, v_{n+1})$ be a vector in $\mathbb{R}^{n+1}$. Then we define the vector

$$\mathbf{w}(v) = (v_1 - v_{n+1}, \dots, v_n - v_{n+1}) = (v_1, \dots, v_n) - v_{n+1}c. \quad (3.2)$$

We observe that if $v \in \mathbb{R}_*^{n+1}$, then the minimum of the coordinates of $\mathbf{w}(v)$ is equal to $-\min_{1 \leq i \leq n} v_i$, since $v_{n+1} = 2 \min_{1 \leq i \leq n} v_i$. Therefore, we have a map $\mathbf{w} : \mathbb{R}_*^{n+1} \rightarrow \mathbb{R}_0^n$.

Given a vector $w = (w_1, \dots, w_n) \in \mathbb{R}_0^n$, we define

$$\mathbf{v}(w) = (w_1 - 2w_0, \dots, w_n - 2w_0, -2w_0) = (w, 0) - 2w_0c'. \quad (3.3)$$

We observe that $\mathbf{v}(w) \in \mathbb{R}_*^{n+1}$, for all $w \in \mathbb{R}_0^n$. Then we have constructed a map $\mathbf{v} : \mathbb{R}_0^n \rightarrow \mathbb{R}_*^{n+1}$. It is an easy exercise to check that $\mathbf{v} : \mathbb{R}_0^n \rightarrow \mathbb{R}_*^{n+1}$ and $\mathbf{w} : \mathbb{R}_*^{n+1} \rightarrow \mathbb{R}_0^n$ are bijections and $\mathbf{v}^{-1} = \mathbf{w}$.

Next, let us consider the map $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ given by the projection onto the first n coordinates i.e. $\pi(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n)$, for all $(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$. Then, from (3.2), we have that $\pi(v) = \mathbf{w}(v) + v_{n+1}c$, for all $v \in \mathbb{R}_{\geq 0}^{n+1}$.

Lemma 3.1.3. *Let us suppose that $v = (v_1, \dots, v_n, v_{n+1}) \in \mathbb{R}_{\geq 0}^{n+1}$. Then*

$$\ell(v, \Gamma_+(F^*)) = dv_{n+1} + \ell(\mathbf{w}(v), \tilde{\Gamma}_+(F)). \quad (3.4)$$

Proof. By construction of F^* , it is immediate to check that any element of $\text{supp}(F^*) \setminus \{d\mathbf{e}_{n+1}\}$ can be written as $(k, d - |k|)$, where k belongs to $\text{supp}(F)$. Then by Definition 1.1 we have

$$\begin{aligned} \ell(v, \Gamma_+(F^*)) &= \min \{ \langle (k, k_{n+1}), (\pi(v), v_{n+1}) \rangle : (k, k_{n+1}) \in \text{supp}(F^*) \cup \{d\mathbf{e}_{n+1}\} \} \\ &= \min \{ \langle (k, d - |k|), (\mathbf{w}(v) + v_{n+1}c, v_{n+1}) \rangle : k \in \text{supp}(F) \cup \{0\} \} \\ &= \min \{ \langle k, \mathbf{w}(v) + v_{n+1}c \rangle + (d - |k|)v_{n+1} : k \in \text{supp}(F) \cup \{0\} \} \\ &= \min \{ \langle k, \mathbf{w}(v) \rangle + v_{n+1}\langle k, c \rangle + (d - |k|)v_{n+1} : k \in \text{supp}(F) \cup \{0\} \} \\ &= dv_{n+1} + \min \{ \langle k, \mathbf{w}(v) \rangle : k \in \text{supp}(F) \cup \{0\} \}. \end{aligned}$$

□

As a consequence the above result we obtain a relation among the faces of $\tilde{\Gamma}_+(F)$ and $\Gamma_+(F^*)$, as we show in the following result.

Corollary 3.1.4. *Let $v \in \mathbb{R}_{>0}^n$. Then*

$$\Delta(v, \Gamma_+(F^*)) = \left\{ (k, d - |k|) : k \in \Delta(\mathbf{w}(v), \tilde{\Gamma}_+(F)) \right\}.$$

In particular, we have that $d\mathbf{e}_{n+1} \notin \Delta(v, \Gamma_+(F^))$ if and only if $0 \notin \Delta(\mathbf{w}(v), \tilde{\Gamma}_+(F))$.*

Proof. From the definition of face, we have that $(k, k_{n+1}) \in \Delta(v, \Gamma_+(F^*))$ if and only if, $\langle (k, k_{n+1}), (\pi(v), v_{n+1}) \rangle - \ell(v, \Gamma_+(F^*)) = 0$. By Lemma 3.1.3, it is equivalent to

$$0 = \langle k, \pi(v) \rangle + (k_{n+1} - d)v_{n+1} - \ell(\mathbf{w}(v), \tilde{\Gamma}_+(F)) \quad (3.5)$$

$$\begin{aligned} &= \langle k, \mathbf{w}(v) + v_{n+1}c \rangle + (k_{n+1} - d)v_{n+1} - \ell(\mathbf{w}(v), \tilde{\Gamma}_+(F)) \\ &= \langle k, \mathbf{w}(v) \rangle - \ell(\mathbf{w}(v), \tilde{\Gamma}_+(F)) + (k_1 + \dots + k_{n+1} - d)v_{n+1}. \end{aligned} \quad (3.6)$$

Observe that the equality (3.6) holds if and only if $k_1 + \dots + k_n + k_{n+1} = d$ and $k \in \Delta(\mathbf{w}(v), \tilde{\Gamma}_+(F))$, since $k_1 + \dots + k_n + k_{n+1} \geq d$ and $\langle k, \mathbf{w}(v) \rangle - \ell(\mathbf{w}(v), \tilde{\Gamma}_+(F)) \geq 0$, for all $(k, k_{n+1}) \in \text{supp}(F^*)$. □

Let Δ be compact face of $\Gamma_+(F)^*$ not containing the point $(0, \dots, 0, d) \in \mathbb{R}^{n+1}$. As an immediate consequence of Corollary 3.1.4, we obtain that Δ can be supported by a vector of $\mathbb{R}_*^{n+1}$. However Δ could be also supported by a vector v not belonging to $\mathbb{R}_*^{n+1}$.

Remark 3.1.5. Corollary 3.1.4 also shows that the projection $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ induces a bijection between the set of compact faces of $\Gamma_+(F^*)$ not passing through the point $d\mathbf{e}_{n+1} \in \mathbb{R}^{n+1}$ and the faces of $\tilde{\Gamma}_+(F)$ not passing through the origin.

We also observe that, if Δ is a compact face of $\Gamma_+(F^*)$ not passing through the point $d\mathbf{e}_{n+1} \in \mathbb{R}^{n+1}$, then

$$(F_i^*)_{\Delta} = \left((F_i)_{\pi(\Delta)} \right)^* \quad (3.7)$$

for all $i = 1, 2, \dots, p$.

It is natural to think the Newton non-degeneracy at infinity of F is related with the Newton non-degeneracy of F^* . The next proposition shows this fact.

Proposition 3.1.6. *Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. Then the following conditions are equivalent:*

- (i) *F is Newton non-degenerate at infinity.*
- (ii) *$F^* : (\mathbb{K}^{n+1}, 0) \rightarrow (\mathbb{K}^{p+1}, 0)$ is Newton non-degenerate.*

Proof. Suppose that F^* is Newton degenerate, then there exists a compact face Δ of $\Gamma_+(F^*)$ not containing the point $d\mathbf{e}_{n+1}$ of $\mathbb{R}^{n+1}$, such that the system

$$\{x = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{K}^{n+1} : (F_1^*)_{\Delta}(x) = \dots = (F_p^*)_{\Delta}(x) = (x_{n+1}^d)_{\Delta}(x) = 0\}, \quad (3.8)$$

has a solution $(q_1, \dots, q_n, q_{n+1}) \in (\mathbb{K}^*)^{n+1}$. Since $(F_i^*)_{\Delta}$ is a homogeneous polynomial and $q_{n+1} \neq 0$, then the system (3.8) has to the point $(q_1/q_{n+1}, \dots, q_n/q_{n+1}, 1)$ as a solution. By Remark 3.1.5 $\pi(\Delta)$ is a face of $\tilde{\Gamma}_+(F)$ not containing the origin of $\mathbb{R}^n$ such that $(F_i^*)_{\Delta} = ((F_i)_{\pi(\Delta)})^*$, for all $i = 1, 2, \dots, p$. From these equalities, it is immediate to check that $q = (q_1/q_{n+1}, \dots, q_n/q_{n+1})$ is a solution of the system

$$\{x = (x_1, \dots, x_n) \in \mathbb{K}^n : (F_1)_{\pi(\Delta)}(x) = \dots = (F_p)_{\pi(\Delta)}(x) = 0\},$$

since $(F_i)_{\Delta}^*(q_1/q_{n+1}, \dots, q_n/q_{n+1}, 1) = (F_i)_{\pi(\Delta)}(q_1/q_{n+1}, \dots, q_n/q_{n+1})$, for all $i = 1, \dots, p$. Hence F is Newton degenerate at infinity.

Conversely, suppose that F is Newton degenerate at infinity, then there exists a face Δ of $\tilde{\Gamma}_+(F)$ not containing the origin of $\mathbb{R}^n$ such that the system

$$\{x = (x_1, \dots, x_n) \in \mathbb{K}^n : (F_1)_{\Delta}(x) = \dots = (F_p)_{\Delta}(x) = 0\}, \quad (3.9)$$

has a solution $r = (r_1, \dots, r_n) \in (\mathbb{K}^*)^n$. By Remark 3.1.5 $\Delta^1 = \pi^{-1}(\Delta)$ is a compact face of $\Gamma_+(F^*)$ not containing the point $d\mathbf{e}_{n+1}$ of $\mathbb{R}^{n+1}$ and satisfies $(F_i^*)_{\Delta^1} = ((F_i)_{\Delta})^*$, for all $i = 1, \dots, p$. It is immediate to check from these equalities that the point $s = (r_1, \dots, r_n, 1) \in (\mathbb{K}^*)^{n+1}$ is a solution of the system

$$\{x = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{K}^{n+1} : (F_1^*)_{\Delta^1}(x) = \dots = (F_p^*)_{\Delta^1}(x) = (x_{n+1}^d)_{\Delta^1}(x) = 0\}.$$

Therefore F^* is Newton degenerate. □

Remark 3.1.7. Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map. Let us suppose that F is Newton non-degenerate at infinity. Then we observe that, by Theorem 3.1.6 and [43, Théorème 6.2], the ideal $I(F^*)$ generated by the component functions of F^* has finite colength in the conic ring $\mathcal{A}_{\Delta} = \{h \in \mathcal{O}_n : \text{supp}(h) \subseteq C(\Delta)\}$, where Δ is the unique compact face of $\Gamma_+(F^*)$ of dimension n .

We say that a given condition holds for all $\|x\| \gg 1$ when there exists a constant $M > 0$ such that the said condition holds for all $x \in \mathbb{K}^n$ for which $\|x\| \geq M$.

Definition 3.1.8. Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. We say that an element $h \in \mathbb{K}[x_1, \dots, x_n]$ is *special with respect to F* when there exists some constant $C > 0$ such that

$$|h(x)| \leq C\|F(x)\|,$$

for all $\|x\| \gg 1$.

We can consider the previous definition as a kind of global or polynomial version of the notion of integral element over an ideal in a local ring. Let us fix coordinates $x_1, \dots, x_n$ in $\mathbb{K}^n$. Then we define the sets:

$$\begin{aligned} \text{Sp}(F) &= \{h \in \mathbb{K}[x_1, \dots, x_n] : h \text{ is special with respect to } F\}, \\ S_0(F) &= \{k \in \mathbb{Z}_{\geq 0}^n : x^k \text{ is special with respect to } F\}. \end{aligned}$$

If $S_0(F) \setminus \{0\} \neq \emptyset$, then it is obvious that there exists some $M > 0$ such that

$$F^{-1}(0) \cap \{x \in \mathbb{K}^n : \|x\| \geq M\} \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}.$$

Proposition 3.1.9. Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map and let $h \in \mathbb{K}[x_1, \dots, x_n]$ special with respect to F . Then $\text{supp}(h) \subseteq \tilde{\Gamma}_+(F)$. In particular $S_0(F) \subseteq \tilde{\Gamma}_+(F)$.

Proof. Suppose that h is written as $h = \sum a_k x^k$ and let $w = (w_1, \dots, w_n) \in \mathbb{R}_0^n$. Then we can rewritten h as

$$h = \sum_{k \notin \Delta} a_k x^k + \sum_{k \in \Delta} a_k x^k, \quad (3.10)$$

where $\Delta = \Delta(w, h)$ (see Definition 2.1.7). Let us consider the meromorphic curve $\varphi_w : \mathbb{K} \setminus \{0\} \rightarrow \mathbb{K}^n$ given by $\varphi_w(t) = (\beta_1 t^{w_1}, \dots, \beta_n t^{w_n})$, where $\beta = (\beta_1, \dots, \beta_n) \in (\mathbb{K}^*)^n$ satisfying $\sum_{k \in \Delta} a_k \beta^k \neq 0$. Since $w_0 < 0$, we have that $\lim_{t \rightarrow 0} \|\varphi_w(t)\| = +\infty$.

By (3.10), it is easy to observe that the order of $h(\varphi_w(t))$ is given by $\ell(w, h)$. If $h \in \text{Sp}(F)$, then there exists a constant $C > 0$ such that

$$|h(x)| \leq C\|F(x)\|, \quad (3.11)$$

for all $\|x\| \gg 1$. In particular, if we compose with $\varphi_w(t)$ both sides of inequality (3.11) then we obtain that the limit $\lim_{t \rightarrow 0} |h(\varphi_w(t))|/\|F(\varphi_w(t))\|$ exists, which is equivalent to saying that the order of $h(\varphi_w(t))$ is bigger than or equal to the order of $\|F(\varphi_w(t))\|$. That is, $\ell(w, h) \geq \min\{\ell(w, F_1), \dots, \ell(w, F_p)\} \geq \ell(w, \tilde{\Gamma}_+(F))$. Therefore $\langle k, w \rangle \geq \ell(w, h) \geq \ell(w, \tilde{\Gamma}_+(F))$, for all $k \in \text{supp}(h)$ and for all $w \in \mathbb{R}_0^n$, which means that $\text{supp}(h) \subseteq \tilde{\Gamma}_+(F)$, by (2.3). $\square$

The next example shows that $S_0(F)$ could be empty.

Example 3.1.10. Let $F : \mathbb{C}^2 \rightarrow \mathbb{C}$ be the polynomial given by $F(x_1, x_2) = x_1^2 x_2^2 - x_1 x_2$. The Newton polyhedron of F at infinity is given by the segment joining the origin and the point $(2, 2)$. Let $\gamma : \mathbb{C}^* \rightarrow \mathbb{C}^2$ be the curve defined by $\gamma(t) = (t, t^{-1})$, which is not zero and $\lim_{t \rightarrow 0} \|\gamma(t)\| = +\infty$. It is immediate to check that

$$1 = |x^k \circ \gamma(t)| > C \|F(\gamma(t))\| = 0. \quad (3.12)$$

for all $t \in \mathbb{C}^*$ and for all $k \in \tilde{\Gamma}_+(F) \cap \mathbb{Z}_{\geq 0}^2$. Then $S_0(F) = \emptyset$, since $\lim_{t \rightarrow 0} \|\gamma(t)\| = +\infty$.

We remark that when $S_0(F) \neq \emptyset$, then it is easy to check that $S_0(F)$ is convex. That is, if $k, k' \in S_0(F)$ then $\lambda k + (1 - \lambda)k' \in S_0(F)$, for all $\lambda \in [0, 1]$ such that $\lambda k + (1 - \lambda)k' \in \mathbb{Z}_{\geq 0}^n$.

Definition 3.1.11. Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map and let $h \in \mathbb{K}[x_1, \dots, x_n]$. Then, we say that h is *quasi-special* with respect to F when there exists some constant $C > 0$ such that

$$|h(x)| \leq C \max\{\|F(x)\|, 1\},$$

for all $\|x\| \gg 1$. We denote by $S'_0(F)$ the set of those $k \in \mathbb{Z}_{\geq 0}^n$ such that x^k is quasi-special with respect to F .

Obviously x^k is quasi-special with respect to F if and only if it is special with respect to the map $(F, 1) : \mathbb{K}^n \rightarrow \mathbb{K}^{p+1}$. From Proposition 3.1.9, it is immediate to check that

$$S_0(F) \subseteq S'_0(F) \subseteq \tilde{\Gamma}_+(F). \quad (3.13)$$

The following examples show that both inclusions of (3.13) might be strict.

Example 3.1.12. Let $F : \mathbb{C}^2 \rightarrow \mathbb{C}$ be the polynomial map given by $F(x_1, x_2) = x_1^2 x_2^2$. Then, F is Newton non-degenerate at infinity, since it is monomial. The Newton polyhedron of F at infinity is given by the segment joining the origin and the point $(2, 2)$. Then $k = (1, 1) \in \tilde{\Gamma}_+(F)$. We claim that $k \notin S_0(F)$. Indeed, let us consider the curve $\gamma : \mathbb{C}^* \rightarrow \mathbb{C}^2$ defined as $\gamma(t) = (t^{-1}, t^2)$. It is immediate to check that $\lim_{t \rightarrow 0} \|\gamma(t)\| = +\infty$ and

$$\lim_{t \rightarrow 0} |x^k(\gamma(t))| / \|F(\gamma(t))\| = \lim_{t \rightarrow 0} |t| / |t^2| = +\infty.$$

From this, $k \notin S_0(F)$. However, since $|x_1 x_2| > 1$ or $|x_1 x_2| \leq 1$, for all $\|(x_1, x_2)\| \geq 1$, we obtain that $k \in S'_0(F)$. Therefore we have that $S_0(F) \subsetneq S'_0(F)$.

Example 3.1.13. Let $F = (F_1, F_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the polynomial map, whose component functions are given by

$$\begin{aligned} F_1(x_1, x_2) &= x_1 - x_1^2 x_2^2, \\ F_2(x_1, x_2) &= x_2. \end{aligned}$$

Observe that $k = (2, 2) \in \tilde{\Gamma}_+(F)$. We claim that $k \notin S'_0(F)$. Indeed, let us consider the meromorphic curve $\gamma : \mathbb{C}^* \rightarrow \mathbb{C}^2$ defined by $\gamma(t) = (t^{-2}, t)$. It is immediate to check that $\lim_{t \rightarrow 0} \|\gamma(t)\| = +\infty$ and

$$\lim_{t \rightarrow 0} |x^k(\gamma(t))| / \max\{\|F(\gamma(t))\|, 1\} = \lim_{t \rightarrow 0} |t^{-2}| / \max\{\|(0, t)\|, 1\} = \lim_{t \rightarrow 0} |t^{-2}| = +\infty.$$

From this, we have $(2, 2) \notin S'_0(F)$.

The next result characterizes the Newton non-degeneracy at infinity for polynomials maps through of the equality $S'_0(F) = \{k \in \mathbb{Z}_{\geq 0}^n : k \in \tilde{\Gamma}_+(F)\}$.

Theorem 3.1.14. *Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. The following condition are equivalent:*

- (i) F is a Newton non-degenerate at infinity.
- (ii) $S'_0(F) = \tilde{\Gamma}_+(F) \cap \mathbb{Z}_{\geq 0}^n$.

Proof. (i) $\Rightarrow$ (ii) Let us observe that F and $(F, 1)$ have the same Newton polyhedron at infinity. Then, by Proposition 3.1.9, it is sufficient prove that $\tilde{\Gamma}_+(F) \cap \mathbb{Z}_{\geq 0}^n \subseteq S'_0(F)$. Let $k = (k_1, \dots, k_n) \in \tilde{\Gamma}_+ \cap \mathbb{Z}_{\geq 0}^n$, then we observe that $(k_1, \dots, k_n, k_{n+1}) \in \Gamma_+(F^*) \cap \mathbb{Z}_{\geq 0}^{n+1}$, where $k_{n+1} = d - |k|$. By Proposition 3.1.6 we have that $F^* : (\mathbb{K}^{n+1}, 0) \rightarrow (\mathbb{K}^{p+1}, 0)$ is Newton non-degenerate. By Theorem 1.3.4, the monomial $x_1^{k_1} \dots x_n^{k_n} x_{n+1}^{k_{n+1}}$ is integral over the ideal of $A(\mathbb{K}^{n+1})$ generated by the components functions of F^* . Therefore there exist a neighborhood $U \subset \mathbb{K}^{n+1}$ of 0 and a constant $C > 0$ such that

$$|x_1^{k_1} \dots x_n^{k_n} x_{n+1}^{k_{n+1}}| \leq C \|F^*(x_1, \dots, x_n, x_{n+1})\|, \quad (3.14)$$

for all $(x_1, \dots, x_n, x_{n+1}) \in U$. Let $M > 0$ such that $(e^{-\|x\|} x_1, \dots, e^{-\|x\|} x_n, e^{-\|x\|}) \in U$ for all $x \in \mathbb{K}^n$ such that $\|x\| \geq M$. Then (3.14) shows that

$$\|((e^{-\|x\|} x_1)^{k_1} \dots (e^{-\|x\|} x_n)^{k_n} \cdot (e^{-\|x\|})^{k_{n+1}})\| \leq C \|F^*(e^{-\|x\|} x_1, \dots, e^{-\|x\|} x_n, e^{-\|x\|})\|,$$

for all $x \in \mathbb{C}^n$ such that $\|x\| > M$. Thus, if we multiply both sides of the last inequality by $|e^{\|x\|.d}|$, we have that

$$\|x^k\| \leq C \max\{\|F(x)\|, 1\}, \quad (3.15)$$

for all $\|x\| \gg 1$. Therefore $S'_0(F) = \tilde{\Gamma}_+(F) \cap \mathbb{Z}_{\geq 0}^n$.

(ii) $\Rightarrow$ (i) By contradiction, if we suppose that F is Newton degenerate at infinity, then by Lemma 2.1.5, there exists a primitive vector $w = (w_1, \dots, w_n) \in \mathbb{R}_0^n$ such that if $\Delta = \Delta(w, \tilde{\Gamma}_+)$, then the system

$$(F_1)_\Delta(x) = (F_2)_\Delta(x) = \dots = (F_p)_\Delta(x) = 0,$$

has a solution $q = (q_1, \dots, q_n) \in (\mathbb{K}^*)^n$. In particular the curve $\gamma : \mathbb{K} \setminus \{0\} \rightarrow \mathbb{K}^n$ defined by $\gamma(t) = (q_1 t^{w_1}, \dots, q_n t^{w_n})$ is not zero and $\lim_{t \rightarrow 0} \|\gamma(t)\| = \infty$, since $w_0 < 0$.

Let $r = \ell(w, \tilde{\Gamma}_+)$, which is a negative number, by Lemma 2.1.5. Since $(F_i)_\Delta$ is a weighted homogeneous polynomial with respect to w , we obtain

$$(F_i)_\Delta(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^r (F_i)_\Delta(x_1, \dots, x_n), \quad (3.16)$$

for all $i = 1, 2, \dots, p$ and for all $\lambda \in \mathbb{K}^*$. By the equality (3.16) we have

$$F(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^r F_\Delta(x_1, \dots, x_n) + F_R(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n), \quad (3.17)$$

for all $\lambda \in \mathbb{K}^*$, where F_Δ and F_R are the polynomial maps given by:

$$\begin{aligned} F_\Delta &= ((F_1)_\Delta, \dots, (F_p)_\Delta). \\ F_R &= (F_1 - (F_1)_\Delta, \dots, F_p - (F_p)_\Delta). \end{aligned}$$

Now, let us consider a vertex $s = (s_1, \dots, s_n) \in \mathbb{Z}_{\geq 1}^n$ of Δ . We claim that $s \notin S'_0(F)$. Indeed, if we use (3.16), (3.17) and $F_\Delta(q) = 0$, we obtain that

$$\lim_{t \rightarrow 0} \frac{|x^s(\gamma(t))|}{\|(F(\gamma(t)), 1)\|} = \lim_{t \rightarrow 0} \frac{|q_1^{s_1} \dots q_n^{s_n} t^{\langle s, w \rangle}|}{\|(t^r F_\Delta(q) + F_R(\gamma(t)), 1)\|} = \lim_{t \rightarrow 0} \frac{|q^s t^r|}{\|(F_R(\gamma(t)), 1)\|} = +\infty, \quad (3.18)$$

since the order of $\|(F_R(\gamma(t)), 1)\|$ is bigger than r . Therefore (3.18) shows that $s \notin S'_0(F)$, which shows that $S'_0(F) \neq \tilde{\Gamma}_+(F) \cap \mathbb{Z}_{\geq 0}^n$. $\square$

The following example shows that the Newton non-degeneracy at infinity condition is not sufficient to obtain the equality $S_0(F) = S'_0(F)$.

Example 3.1.15. Let us consider the polynomial map $F = (F_1, F_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, whose the component functions are given by

$$\begin{aligned} F_1(x_1, x_2) &= x_1 x_2^3, \\ F_2(x_1, x_2) &= x_1^3 x_2^2. \end{aligned}$$

Since F has monomial components, F is Newton non-degenerate at infinity. Moreover, since $(1, 1) \in \tilde{\Gamma}_+(F)$, the monomial $h = x_1x_2$ is quasi-special, by Theorem 3.1.14.

However, we claim that $h = x_1x_2 \notin \text{Sp}(F)$. Indeed, let us consider the meromorphic curve $\gamma : \mathbb{C}^* \rightarrow \mathbb{C}^2$ defined by $\gamma(t) = (t^{-1}, t^3)$. It is immediate to check that $\lim_{t \rightarrow 0} \|\gamma(t)\| = +\infty$, $\lim_{t \rightarrow 0} \|F(\gamma(t))\| = \lim_{t \rightarrow 0} \|(t^8, t^3)\| = 0$ and

$$\lim_{t \rightarrow 0} \frac{\|F(\gamma(t))\|}{|h(\gamma(t))|} = \lim_{t \rightarrow 0} \|(t^6, t)\| = 0.$$

The claim is an immediate consequence of this fact.

In the next result, we characterize the Newton non-degeneracy condition for convenient polynomial maps through of the equality $S_0(F) = S'_0(F) = \{x^k : k \in \tilde{\Gamma}_+(F)\}$. We remark that this result is the polynomial version of a result proved by Saia (see Theorem 1.2.5).

Corollary 3.1.16. *Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a convenient polynomial map. Then the following conditions are equivalent:*

- (i) F is Newton non-degenerate at infinity.
- (ii) $S_0(F) = S'_0(F) = \tilde{\Gamma}_+(F) \cap \mathbb{Z}_{\geq 0}^n$.

Proof. (i) $\Rightarrow$ (ii) By Proposition 3.1.9 it is sufficient prove that $\tilde{\Gamma}_+(F) \cap \mathbb{Z}_{\geq 0}^n \subseteq S_0(F)$. By Theorem 3.1.14, there exists a constant $C > 0$ such that

$$\|x^k\| \leq C\|(F(x), 1)\|, \quad (3.19)$$

for all $\|x\| \gg 1$ and for all $k \in \tilde{\Gamma}_+(F) \cap \mathbb{Z}^n$.

Let $r_i = \max\{r \geq 0 : r\mathbf{e}_i \in \tilde{\Gamma}_+(F)\}$, for all $i = 1, \dots, n$ and $r_0 = \min\{r_1, \dots, r_n\}$. Since F is convenient, we have that

$$\|x_i^{r_i}\| \leq C'\|(F(x), 1)\|, \quad (3.20)$$

for some constant $C' > 0$, for all $\|x\| \gg 1$ and for all $i = 1, \dots, n$. Then as an immediate consequence of these inequalities we obtain

$$\|x\|^{r_0(F)} \leq C''\|(F(x), 1)\|, \quad (3.21)$$

for all $\|x\| \gg 1$, for some constant $C'' > 0$. From the inequality (3.21), it is immediate to check that $\|F(x)\| \geq 1$, for all $\|x\| \gg 1$ and (3.19), shows that $k \in S_0(F)$, for all $k \in \tilde{\Gamma}_+(F) \cap \mathbb{Z}^n$. Therefore $S_0(F) = S'_0(F) = \tilde{\Gamma}_+(F) \cap \mathbb{Z}_{\geq 0}^n$.

(ii) $\Rightarrow$ (i) It is immediate consequence from Theorem 3.1.14, since $S_0(F) = S'_0(F) = \tilde{\Gamma}_+(F) \cap \mathbb{Z}_{\geq 0}^n$. $\square$

Corollary 3.1.17. *Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a convenient polynomial map. If F is Newton non-degenerate at infinity, then F is proper.*

Proof. Since $\tilde{\Gamma}_+(F)$ is convenient, the result is an immediate application of Corollary 3.1.16. $\square$

3.2 Multiplicity and special polynomials

Let us consider I an ideal in a Noetherian local ring $(R, \mathfrak{m})$ with infinite residue field k , $\dim(R) = d$. Suppose that I is $\mathfrak{m}$ -primary and denote by $e(I)$ the multiplicity of I , in the sense of Samuel introduced in Chapter 1. If I is generated minimally by d elements $g_1, \dots, g_d \in R$ and $h \in R$, then by virtue of a result of Northcott-Rees, Theorem 1.1.6, and Rees' Theorem, the multiplicity $e(I + \langle h \rangle)$ can be expressed as

$$e(I + \langle h \rangle) = e(g_1 + \alpha_1 h, \dots, g_d + \alpha_d h)$$

for a generic choice of $(\alpha_1, \dots, \alpha_d) \in k^d$. Moreover, $e(I + \langle h \rangle) = e(I)$ if and only if h is integral over I . Since, we have a notion of element integer for polynomial maps, the above result motivates the study of the polynomial version of this result.

This section is devoted to the characterization of special polynomials with respect to a given finite polynomial map $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ in terms of its multiplicity. Recall from Chapter 1, that $F + h\alpha : \mathbb{C}^n \rightarrow \mathbb{C}^n$ denotes the polynomial map defined as $F + \alpha h = (F_1 + \alpha_1 h, \dots, F_n + \alpha_n h)$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ and $h \in \mathbb{C}[x_1, \dots, x_n]$.

Lemma 3.2.1. *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a finite polynomial map and $h \in \mathbb{C}[x_1, \dots, x_n]$. Then there exist positive constants M and δ , such that $\|F(x)\| \neq 0$ for all $\|x\| \geq M$, and*

$$\|(F + h\alpha)(x)\| \neq 0, \text{ for all } x \in \mathbb{C}^n \text{ with } \|x\| = M, \quad (3.22)$$

for all $\alpha \in \mathbb{C}^n$ such that $\|\alpha\| < \delta$.

Proof. Let $C_M = \sup\{|h(x)| : \|x\| = M\}$, since h is not the zero polynomial and $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a finite polynomial map, there exists $M > 0$ such that $C_M > 0$ and $\|F(x)\| \neq 0$ for all $x \in \mathbb{C}^n$, such that $\|x\| \geq M$. Let us consider $\delta_1 = \min\{\|F(x)\| : \|x\| = M\}$. It is immediate to see that $\delta_1 > 0$, since $F(x) \neq 0$ for all $\|x\| = M$.

Let $\delta = \frac{\delta_1}{2C_M}$. Then, for all $\alpha \in \mathbb{C}^n$, such that $\|\alpha\| < \delta$, we obtain that

$$\|(F + h\alpha)(x)\| \geq \|F(x)\| - \|\alpha\| |h(x)| \geq \delta_1 - \frac{\delta_1}{2} = \frac{\delta_1}{2} > 0,$$

for all $x \in \mathbb{C}^n$, with $\|x\| = M$. $\square$

Next we show a result that characterizes the special polynomials with respect to a given finite polynomial map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ in terms of multiplicity.

Theorem 3.2.2. *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a finite polynomial map and let h be a polynomial function. Then h is special with respect to F if and only if, there exists $\delta > 0$ such that*

(i) $F + h\alpha$ is a finite polynomial map and

(ii) $\mu(F) = \mu(F + h\alpha)$,

for all $\alpha \in \mathbb{C}^n$ such that $\|\alpha\| < \delta$.

Proof. Let us suppose that h is special with respect to F , then since $\#F^{-1}(0)$ is finite, there exist positive constants C and M such that

$$|h(x)| \leq C\|F(x)\| \text{ and } \|F(x)\| \neq 0, \text{ for all } \|x\| \geq M.$$

Let us consider $\delta = \frac{1}{C} > 0$. Then, an immediate computation shows that for any $\alpha \in \mathbb{C}^n$ such that $\|\alpha\| < \delta$,

$$\|(F + h\alpha)(x)\| \geq \|F(x)\|(1 - \|\alpha\|C) > 0, \quad (3.23)$$

for all $x \in \mathbb{C}^n$ such that $\|x\| \geq M$. Then, we have that (3.23) implies that $F + h\alpha : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a finite polynomial map, for all $\|\alpha\| < \delta$, since every bounded algebraic set in $\mathbb{C}^n$ is finite.

Now, let us fix $\alpha \in \mathbb{C}^n$, such that $\|\alpha\| < \delta$ and consider the homotopy $H : \mathbb{C}^n \times [0, 1] \rightarrow \mathbb{C}^n$, given by $H(x, t) = (F + ht\alpha)(x)$. Then, by (3.23) we obtain that H satisfies the following conditions:

(i) $H_t : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a finite polynomial map for all $t \in [0, 1]$.

(ii) $H_t(x) \neq 0$, for all $\|x\| \geq M$ and for all $t \in [0, 1]$.

(iii) $H(x, 0) = F(x)$ and $H(x, 1) = (F + h\alpha)(x)$, for all $x \in \mathbb{C}^n$.

The above conditions imply that $\mu(F) = \mu(F + h\alpha)$ as a consequence of the invariance of the index by homotopies Theorem 1.4.7 and the equality between the index and the codimension of finite ideals in $\mathbb{C}[x_1, \dots, x_n]$ (see for instance [18] and [20]).

Conversely, let us suppose that h is not special with respect to F . Then, it follows from definition there exists a sequence $\{x_m\}_{m \geq 1}$ in $\mathbb{C}^n$ such that

$$h(x_m) \neq 0, \text{ for all } m \geq 0, \quad \lim_{m \rightarrow \infty} \|x_m\| = +\infty \text{ and } \lim_{m \rightarrow \infty} \frac{\|F(x_m)\|}{|h(x_m)|} = 0. \quad (3.24)$$

Since F is a finite polynomial map, by Lemma 3.2.1 we choose positive constants M and δ_1 , such that $\|F(x)\| \neq 0$ for all $\|x\| \geq M$ and

$$\|(F + h\alpha)(x)\| \neq 0, \text{ for all } x \in \mathbb{C}^n \text{ with } \|x\| = M, \quad (3.25)$$

for all $\alpha \in \mathbb{C}^n$ such that $\|\alpha\| < \delta_1$. By (3.24) there exists m_0 such that $\|x_{m_0}\| > M$ and $\|F(x_{m_0})\|/|h(x_{m_0})| < \min\{\delta, \delta_1\}$. Thus we can consider the polynomial map $F + h\alpha_0 : \mathbb{C}^n \rightarrow \mathbb{C}^n$, where $\alpha_0 = -F(x_{m_0})/h(x_{m_0})$. Suppose that $F + h\alpha_0$ is a finite polynomial map. Then by (3.25) the homotopy $H : \mathbb{C}^n \times [0, 1] \rightarrow \mathbb{C}^n$, defined by $H(x, t) = (F + ht\alpha_0)(x)$ satisfies the following conditions:

- (i) $H(x, 0) = F(x)$ and $H(x, 1) = (F + h\alpha_0)(x)$, for all $x \in \mathbb{C}^n$.
- (ii) $H(x, t) \neq 0$, for all $\|x\| = M$ and for all $t \in [0, 1]$.

Let us denote by U the open set $\{x \in \mathbb{C}^n : \|x\| < M\}$. Then, by Theorem 1.4.7, we obtain that

$$\text{ind}(F, U) = \sum_{x \in F^{-1}(0) \cap U} \text{ind}_x(F) = \sum_{y \in (F + h\alpha_0)^{-1}(0) \cap U} \text{ind}_y(F + h\alpha_0) = \text{ind}(F + h\alpha_0, U). \quad (3.26)$$

But (3.25), shows that $F^{-1}(0) \subset U$ and a direct computation shows that $(F + h\alpha_0)(x_{m_0}) = 0$, with $x_{m_0} \notin U$. Therefore (3.26) shows that $\mu(F) < \mu(F + h\alpha_0)$. $\square$

Example 3.2.3. Let us consider the polynomial map $F = (F_1, F_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, whose component functions are given by $F_1(x_1, x_2) = x_1 + x_1x_2^3$ and $F_2(x_1, x_2) = x_2^3$. It is immediate to check that F is a finite polynomial map and $\mu(F) = 3$.

Let h be the polynomial given by $h = x_1x_2^3$. It is immediate to check that $F + h\alpha : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a finite polynomial, for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2$. By Lemma 3.2 given in [43], we obtain that

$$\mu(F + h\alpha) = 3 + \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, x_2]}{\langle 1 + x_2^3(1 + \alpha_1), 1 + \alpha_2x_1 \rangle}.$$

From this equality is easy to see that $\mu(F + h\alpha) = 6$ for all $\alpha \in \{\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2 : \alpha_1 \neq -1 \text{ or } \alpha_2 \neq 0\}$. Therefore, by Theorem 3.2.2 $h \notin \text{Sp}(F)$.

The following result characterizes the Newton non-degeneracy at infinity condition for finite polynomial maps $\mathbb{C}^n \rightarrow \mathbb{C}^n$.

Corollary 3.2.4. *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a finite polynomial map. Then the following conditions are equivalent:*

- (i) F is Newton non-degenerate at infinity.
- (ii) $\mu(F) = n!V_n(\tilde{\Gamma}_+(F))$.
- (iii) $S_0(F) = \tilde{\Gamma}_+(F) \cap \mathbb{Z}_{\geq 0}^n$.

Proof. It is immediate consequence of Corollaries 2.4.5 and 3.1.16. $\square$

In the following result, let us consider $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ a finite polynomial, from this polynomial map, consider the corresponding analytic map germ $F^* : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$, given in the above section. Then we observe that, F is Newton non-degenerate at infinity, if and only if F^* is Newton non-degenerate (Theorem 3.1.6) and by Remark 3.1.7 the ideal $I(F^*)$ generated by the components functions of F^* has finite colength in the conic ring $\mathcal{A}_\Delta = \{h \in \mathcal{O}_n : \text{supp}(h) \subseteq C(\Delta)\}$, where Δ is the unique compact face of $\Gamma_+(F^*)$ of dimension n . Therefore, it makes sense to compute the multiplicity in the sense of Samuel of $I(F^*)$ in $\mathcal{A}_\Delta$, since $\mathcal{A}_\Delta$ is a local ring. We denote this multiplicity by $e(I(F^*), \mathcal{A}_\Delta)$.

From the construction of F^* , it is immediate to see that $\Gamma_-(F^*)$ is an $(n+1)$ -dimensional polytope with basis Δ contained in $\mathbb{R}_{\geq 0}^{n+1}$ and height $h = d/\sqrt{n+1}$. Thus we can consider the $(n+1)$ -dimensional volume of $\Gamma_-(I(F^*))$. To calculate the $(n+1)$ -dimensional volume of $\Gamma_+(F^*)$, observe that the projection onto the first n coordinates of Δ , is the Newton polyhedron $\tilde{\Gamma}_+(F)$ i.e $\pi(\Delta) = \tilde{\Gamma}_+(F)$ and this fact shows that $V_n(\Delta) = \sqrt{n+1} \cdot V_n(\tilde{\Gamma}_+(F))$.

By integration, we obtain that $V_{n+1}(\Gamma_-(F^*)) = V_n(\Delta) \cdot h/(n+1)$. Therefore the above relations show that

$$V_{n+1}(\Gamma_-(F^*)) = \frac{d}{n+1} V_n(\tilde{\Gamma}_+(F)). \quad (3.27)$$

In the next result, we give a relation among the multiplicity of F and the multiplicity $e(I(F^*), \mathcal{A}_\Delta)$, when $n = p$.

Corollary 3.2.5. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a finite polynomial map. Then F is Newton non-degenerate at infinity if and only if $e(I(F^*), \mathcal{A}_\Delta) = d\mu(F)$, where $d = \max\{\deg(F_1), \dots, \deg(F_n)\}$.*

Proof. The result follows of the equality (3.27) and as an immediate consequence of Theorem 1.2.9, Proposition 3.1.6 and Corollary 3.2.4. $\square$

Example 3.2.6. Let $a, b_i, c_i \in \mathbb{Z}_{\geq 1}^3$, for all $i = 1, 2, 3$ such that $b_1 < b_2 < b_3$ and $c_1 < c_2 < c_3$. Let us consider the polynomial map $F = (F_1, F_2, F_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$, whose component functions are given by

$$\begin{aligned} F_1(x_1, x_2, x_3) &= x_1^a + x_2^{b_1} x_3^{c_3} + x_2^{b_3} x_3^{c_1}, \\ F_2(x_1, x_2, x_3) &= x_2^{b_2} + x_2^{b_1} x_3^{c_3} + x_2^{b_1} x_3^{c_2}, \\ F_3(x_1, x_2, x_3) &= x_3^{c_2} + x_2^{b_3} x_3^{c_1} + 2x_2^{b_2} x_3^{c_1}. \end{aligned}$$

The Figure 3.1 shows the picture of $\tilde{\Gamma}_+(F)$. From Definition 2.2.1, it is immediate to see that F is Newton non-degenerate at infinity. Therefore, by Corollary 3.2.4 and a direct computation on the volume of $\tilde{\Gamma}_+(F)$, we obtain that

$$\mu(F) = \dim_{\mathbb{C}} \mathbb{C}[x_1, x_2, x_3]/I(F) = a \cdot (b_1c_2 + b_2c_1 + b_3c_3 - b_1c_1).$$

Figure 3.1: Newton polyhedron of F .

3.3 Non-degeneracy with respect to a Newton filtration

In this section we compute the multiplicity of finite polynomial maps $\mathbb{C}^n \rightarrow \mathbb{C}^n$ that satisfies some non-degeneracy conditions introduced in Chapter 2.

Our first goal is to determine the multiplicity of a finite polynomial map $\mathbb{C}^n \rightarrow \mathbb{C}^n$, non-degenerate with respect to a fixed global Newton polyhedron. We remark that the following result is the polynomial version of Theorem 1.2.12.

Let us fix a Newton polyhedron at infinity $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ and $\nu_{\tilde{\Gamma}_+} : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{Z} \cup \{-\infty\}$ denotes the Newton filtration associated to $\tilde{\Gamma}_+$.

Corollary 3.3.1. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a finite polynomial map, then*

$$\mu(F) \leq \frac{\nu_{\tilde{\Gamma}_+}(F_1) \cdots \nu_{\tilde{\Gamma}_+}(F_n)}{M^n} n! V_n(\tilde{\Gamma}_+) \quad (3.28)$$

and equality holds if and only if the map F is non-degenerate with respect to $\tilde{\Gamma}_+$, where M denotes the value of $\nu_{\tilde{\Gamma}_+}$ along the monomials with exponents on $\tilde{\Gamma}$.

Proof. As in the proof of Lemma 2.3.10, let $d_i = \nu_{\tilde{\Gamma}_+}(F_i)$, for all $i = 1, \dots, n$, $d = d_1 \cdots d_n$ and $a = (a_1, \dots, a_n) \in \mathbb{Z}_{\geq 1}^n$, where $a_i = d/d_i$, for all $i = 1, \dots, n$. Let us consider the polynomial map $F^a : \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined as $F^a = (F_1^{a_1}, \dots, F_n^{a_n})$. It is immediate to see that F is finite if only if F^a is finite, since $F^{-1}(0) = (F^a)^{-1}(0)$. Then by [43, Lemma 3.2] and Corollary 3.2.4, we obtain that

$$\mu(F) = \frac{d_1 \cdots d_n}{d^n} \mu(F^a) \leq \frac{d_1 \cdots d_n}{d^n} n! V_n(\tilde{\Gamma}_+(F^a)). \quad (3.29)$$

The inequality (3.28) follows from (3.29), since $\tilde{\Gamma}_+(F^a) = (d/M)\tilde{\Gamma}_+$ (see Lemma 2.3.10).

Now, suppose that F is non-degenerate at infinity with respect to $\tilde{\Gamma}_+$, then by Lemma 2.3.10 we have that F^a is Newton non-degenerate at infinity. Thus the inequality (3.29) turns into equality, which implies the equality in (3.28).

Conversely, suppose that $\mu(F) = (d_1 \cdots d_n/M^n) n! V_n(\tilde{\Gamma}_+)$. From the inequality (3.29) we obtain that $\mu(F^a) = n! V_n(\tilde{\Gamma}_+(F^a))$, since $\tilde{\Gamma}_+(F^a) = (d/M)\tilde{\Gamma}_+$. Therefore by Corollary 3.2.4 and Lemma 2.3.10, the polynomial map F is non-degenerate at infinity with respect to $\tilde{\Gamma}_+$. $\square$

Example 3.3.2. Let $F = (F_1, F_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the polynomial map whose component functions are given by

$$\begin{aligned} F_1(x_1, x_2) &= x_1^4 + x_1^6 x_2^6 + x_2^6 + \alpha, \\ F_2(x_1, x_2) &= x_2^4 + x_1^4 x_2^4 + \beta. \end{aligned}$$

Where $\alpha, \beta \in \mathbb{C}$. Let g be the polynomial defined by $g(x_1, x_2) = x_1^4 + x_1^6 x_2^6 + x_2^6$ and let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(g)$. Then we observe that $\tilde{\Gamma}_+$ is convenient and $\mathcal{F}_0(\tilde{\Gamma}_+) = \{-\mathbf{e}_2, (-3, 1)\}$.

A direct computation shows that the Newton filtration map $\phi_{\tilde{\Gamma}_+}^2 : \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}^2$ is given by $\phi_{\tilde{\Gamma}_+}^2((k_1, k_2)) = \max\{2k_2, 3k_1 - k_2\}$, from this fact we obtain that $M = 12$, $\nu_{\tilde{\Gamma}_+}(F_1) = 12$ and $\nu_{\tilde{\Gamma}_+}(F_2) = 8$. It is straightforward to check that F is Newton degenerate at infinity. However, F is non-degenerate with respect to $\tilde{\Gamma}_+$. Therefore, by Corollary 3.3.1, we obtain that $\mu(F) = (\nu_{\tilde{\Gamma}_+}(F_1) \cdot \nu_{\tilde{\Gamma}_+}(F_2)/M^2) 2! V_2(\tilde{\Gamma}_+) = 40$.

Let $v \in \mathbb{Z}_{\geq 1}^n$ a primitive vector and let $h \in \mathbb{C}[x_1, \dots, x_n]$. Let us suppose that h is written as $h = \sum a_k x^k$. Then we recall from Section 2.3 that $d_v(h)$ is defined by $d_v(h) = \max\{\langle v, k \rangle : k \in \text{supp}(h)\}$, and the principal part of h at infinity with respect to v , denoted by $q_v(h)$ is defined as the sum of those $a_k x^k$, such that $\langle v, k \rangle = d_v(h)$.

Now, we show the polynomial version of the Arnold's formula for weighted homogeneous map germs, see [11, Corollary 3.4]. We determine the multiplicity of a finite polynomial map

$\mathbb{C}^n \rightarrow \mathbb{C}^n$, non-degenerate with respect to a fixed global Newton polyhedron with only one $(n-1)$ -dimensional face, determined by a fixed primitive vector $v = (v_1, \dots, v_n) \in \mathbb{Z}_{\geq 1}^n$.

Corollary 3.3.3. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a finite polynomial map. Let us fix a vector $v = (v_1, \dots, v_n) \in \mathbb{Z}_{\geq 1}^n$ and let $d_i = d_v(F_i)$, for all $i = 1, \dots, n$. Then*

$$\mu(F) \leq \frac{d_1 \cdots d_n}{v_1 \cdots v_n}$$

and equality holds if and only if the map $q_v(F)$ is finite.

Proof. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ the canonical basis in $\mathbb{R}^n$ and consider the convenient global Newton polyhedron at infinity $\tilde{\Gamma}_+$ determined by the points $\{v_i^{-1}\mathbf{e}_i : i = 1, \dots, n\}$. The Newton filtration map ϕ associated to $\tilde{\Gamma}_+$ is given by $\phi(k_1, \dots, k_n) = v_1 k_1 + \cdots + v_n k_n$, for all $k = (k_1, \dots, k_n) \in \mathbb{R}_{\geq 0}^n$. Thus $\nu_{\tilde{\Gamma}_+}(F_i) = d_i$, for all $i = 1, \dots, n$ and $M = 1$. Then if we use Corollary 3.3.1, we obtain

$$\mu(F) \leq d_1 \cdots d_n n! V_n(\tilde{\Gamma}_+) = \frac{d_1 \cdots d_n}{v_1 \cdots v_n},$$

and equality holds if and only if F is non-degenerate with respect to $\tilde{\Gamma}_+$.

From Lemma 2.3.10, for any face Δ of $\tilde{\Gamma}_+(F^a)$ there exists a face Δ' of $\tilde{\Gamma}_+$ such that $\Delta = d\Delta'$, where $d = d_1 \cdots d_n$ and it is immediate to check that

$$(F_i^{a_i})_{\Delta} = (\text{in}_{\Delta'}(F_i))^{a_i} = (\text{in}_{\Delta'}(q_v(F_i)))^{a_i} = ((q_v(F_i))^{a_i})_{\Delta}, \quad (3.30)$$

for all $i = 1, \dots, n$ and for all face Δ of $\tilde{\Gamma}_+(F^a)$ not containing the origin.

From (3.30), it follows that F^a is Newton non-degenerate at infinity if and only if the polynomial map $q_v(F)^a : \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined as $q_v(F)^a = ((q_v(F_1))^{a_1}, \dots, (q_v(F_n))^{a_n})$ is Newton non-degenerate at infinity. However, by Theorem 2.2.3 we have that the map $q_v(F)^a$ is Newton non-degenerate at infinity if and only if $q_v(F)^a$ is a finite polynomial map. Therefore by Lemma 2.3.10, we have that F is Newton non-degenerate with respect to $\tilde{\Gamma}_+$ if and only if $q_v(F)^a : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a finite polynomial map. The result follows, since $(q_v(F)^a)^{-1}(0) = (q_v(F))^{-1}(0)$. $\square$

Example 3.3.4. Let $v = (2, 3, 2) \in \mathbb{Z}_{\geq 1}^3$. Let $F = (F_1, F_2, F_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ be the polynomial map whose component functions are given by

$$\begin{aligned} F_1(x_1, x_2, x_3) &= x_1^2 - x_1^2 x_2^2 + x_3^2 + 1, \\ F_2(x_1, x_2, x_3) &= x_2^3 + x_1^2 x_2^2 x_3 - x_3^4, \\ F_3(x_1, x_2, x_3) &= x_1^2 x_2^2 - 1. \end{aligned}$$

It is immediate to see that F is a finite polynomial map. A direct computation shows that $d_v(F) = (10, 12, 10)$ and the principal part of F at infinity with respect to v is given by $q_v(F) = (-x_1^2 x_2^2, x_1^2 x_2^2 x_3, x_1^2 x_2^2)$. We observe that $q_v(F)$ is not finite, then a simple computation using SINGULAR [27] shows that $\mu(F) = 28 < 10 \cdot 12 \cdot 10 / 2 \cdot 3 \cdot 2 = 100$. On the other hand, let us consider the polynomial map $G = (G_1, G_2, G_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ such that $G_1 = F_1 + x_1^5 + x_3^5$, $G_2 = F_2 + x_2^4 + 2x_3^6$ and $G_3 = F_3$. Then it is immediate to check that $q_v(G) = \{0\}$. Therefore, by Corollary 3.3.3, we have $\mu(G) = 100$.

The next definition is a polynomial version for the class of *pre-weighted homogeneous* function germs, given in [71].

Definition 3.3.5. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial function. Let us fix a vector $v \in \mathbb{Z}_{\geq 1}^n$. Then f is called *pre-weighted homogeneous with respect to v* if and only if the gradient map $\nabla q_v(f) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a finite polynomial map, that is, when $q_v(f)$ has at most a finite number of singular points.

If $f : \mathbb{C}^n \rightarrow \mathbb{C}$ is a complex polynomial function with a finite number of singular points, and $J(f)$ denotes the ideal of $\mathbb{C}[x_1, \dots, x_n]$ generated by $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$, then the *Milnor number at infinity* of f , denoted by $\mu_\infty(f)$, is defined as

$$\mu_\infty(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{J(f)}.$$

Example 3.3.6. Let $v \in \mathbb{Z}_{\geq 1}^n$ be a primitive vector. Then the following examples show that the condition $d_v(\frac{\partial f}{\partial x_i}) = d - v_i$, for all $i = 1, \dots, n$, or that $q_v(\nabla f)$ is a finite polynomial map are not sufficient conditions for f to be pre-weighted homogeneous with respect to v .

- (i) Let $v = (2, 1)$ and let f be the polynomial function given by $f(x_1, x_2) = x_2^3 + x_1 x_2 + x_1 x_2^2$. Then, it is straightforward to check that f has a finite number of singular points and $d_v(\frac{\partial f}{\partial x_i}) = d - v_i$ for $i = 1, 2$, where $d = d_v(h) = 4$. However, f is not pre-weighted homogeneous with respect to v , since $q_v(f) = x_1 x_2^2$ does not have a finite number of singular points.
- (ii) We consider the polynomial f given by $f = x_1^3 + x_2^2$. Then a direct computation shows that f has a finite number of singular points and $q_v(\nabla f) = (3x_1^2, 2x_2)$ is a finite polynomial map. However, as we can be seen f is not pre-weighted homogeneous with respect to v , since $q_v(f) = x_1^3$ does not have a finite number of singular points.

Next, we characterize the pre-weighted homogeneous polynomials.

Lemma 3.3.7. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial function with a finite number of singular points. Let us fix a vector $v \in \mathbb{Z}_{\geq 1}^n$ and let $d = d_v(f)$. Then f is pre-weighted homogeneous with respect to v if and only if satisfies the following conditions*

- (i) $d_v(\frac{\partial f}{\partial x_i}) = d - v_i$, for all $i = 1, \dots, n$.
- (ii) $q_v(\nabla f) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a finite polynomial map.

Proof. Let A denotes the support of f and let $A_i = \{k \in A : k_i > 0\}$, for all $i = 1, \dots, n$. Then we have

$$d_v\left(\frac{\partial f}{\partial x_i}\right) = \max\{\langle v, k - \mathbf{e}_i \rangle : k \in A_i\} = \max\{\langle v, k \rangle : k \in A_i\} - v_i, \quad (3.31)$$

for all $i = 1, \dots, n$.

Suppose that f is pre-weighted homogeneous with respect to v , then we would have that for all $i \in \{1, \dots, n\}$, there exists $k = (k_1, \dots, k_n) \in \text{supp}(q_v(f))$ such that $k_i > 0$. From this, it is straightforward to check that $d_v(\frac{\partial f}{\partial x_i}) = d - v_i$, for all $i = 1, \dots, n$. Also, we can observe that if $d_v(\frac{\partial f}{\partial x_i}) = d - v_i$, then by the equality given in (3.31) it is immediate to check that the following equality holds

$$\frac{\partial}{\partial x_i} q_v(f) = q_v\left(\frac{\partial f}{\partial x_i}\right). \quad (3.32)$$

Therefore, from (3.32) it follows that $q_v(\nabla f) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a finite polynomial map.

Conversely, since the conditions (i) and (ii) are satisfied, we have that the equality in (3.32) holds. For this, it is immediate to see that f is pre-weighted homogeneous with respect to v . $\square$

Next, we give a result which is the polynomial version of the main result given by Milnor-Orlik in [55] (see also M. Furuya and M. Tomari [33]).

Corollary 3.3.8. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial function with a finite number of singular points. Let us fix a vector $v \in \mathbb{Z}_{\geq 1}^n$ and let $d = d_v(f)$. Then*

$$\mu_\infty(f) \leq \frac{(d - v_1) \cdots (d - v_n)}{v_1 \cdots v_n} \quad (3.33)$$

and equality holds if and only if f is pre-weighted homogeneous.

Proof. Since that $f : \mathbb{C}^n \rightarrow \mathbb{C}$ is a polynomial function with a finite number of singular points, then the polynomial map ∇f given by $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is finite. If we take the same Newton polyhedron taken in the proof of Corollary 3.3.3, then we have $\nu_{\tilde{\Gamma}_+}(\frac{\partial f}{\partial x_i}) = d_v(\frac{\partial f}{\partial x_i})$,

for all $i = 1, \dots, n$. Let $d_i = d_v(\frac{\partial f}{\partial x_i})$, then a direct computation shows that $d_i \leq d - v_i$, for all $i = 1, \dots, n$. From Corollary 3.3.3, we have that

$$\mu_\infty(f) = \mu(F) \leq \frac{d_1 \cdots d_n}{v_1 \cdots v_n} \leq \frac{(d - v_1) \cdots (d - v_n)}{v_1 \cdots v_n}$$

and the equality in the first inequality holds if and only if $q_v(\nabla f)$ is a finite polynomial map. It is easy to observe that the equality (3.33) holds if and only if $d_v(\frac{\partial f}{\partial x_i}) = d - v_i$ for all $i = 1, \dots, n$. and $q_v(\nabla f)$ is a finite polynomial map. Therefore, by Lemma 3.3.7 we have that equality (3.33) holds if and only if f is pre-weighted homogeneous. $\square$

3.4 Newton non-degeneracy of gradient maps

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial function. We denote by ∇f the gradient map of f . That is, ∇f is the map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ given by $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$. In this section we study the relation between polynomials $f \in \mathbb{C}[x_1, \dots, x_n]$ which are Newton non-degenerate at infinity (see Definition 2.2.7) and polynomials of $\mathbb{C}[x_1, \dots, x_n]$ whose gradient map is Newton non-degenerate at infinity. This study is motivated by the results given in [7] for the case of function germs $(\mathbb{C}^n, 0) \rightarrow \mathbb{C}$.

Let $I \subseteq \{1, \dots, n\}$, $I \neq \emptyset$. Then we define $\mathbb{K}_I^n = \{(x_1, \dots, x_n) \in \mathbb{K}^n : x_i = 0, \text{ for all } i \notin I\}$. We denote the cardinal of I by $|I|$. If $f = \sum_k a_k x^k$ is a polynomial in $\mathbb{C}[x_1, \dots, x_n]$, then f_I denotes the sum of all terms $a_k x^k$ such that $k \in \text{supp}(f) \cap \mathbb{R}_I^n$. If no such terms exist, then we set $f_I = 0$.

We recall that, if $f \in \mathbb{C}[x_1, \dots, x_n]$, then $A(f)$ denotes the polynomial map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ given by

$$A(f) = \left(x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n} \right).$$

Let $\tilde{\Gamma}_+ \subseteq \mathbb{R}^n$ be a Newton polyhedron at infinity. Then we denote by $\vartheta(\tilde{\Gamma})$ the set of vertices of $\tilde{\Gamma}_+$ and by $\mathbf{f}_{\tilde{\Gamma}}$ the polynomial obtained as the sum of all monomials x^k such that $k \in \vartheta(\tilde{\Gamma})$. Let Δ be a face of $\tilde{\Gamma}_+$, we say that Δ is *simplicial* when Δ is equal to the convex hull of $\dim(\Delta) + 1$ vertices of $\tilde{\Gamma}_+$. Then, we will say that $\tilde{\Gamma}_+$ is *simplicial* when any face of $\tilde{\Gamma}_+$ not passing through the origin is simplicial.

Lemma 3.4.1. *Let $\tilde{\Gamma}_+$ be a convenient Newton polyhedron in $\mathbb{R}_{\geq 0}^n$. Let us suppose that $\tilde{\Gamma}_+$ is simplicial. Then $\mathbf{f}_{\tilde{\Gamma}}$ is Newton non-degenerate at infinity and the polynomial map $A(\mathbf{f}_{\tilde{\Gamma}}) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is finite.*

Proof. Let $\{k_1, \dots, k_s\}$ be the set of vertices of $\tilde{\Gamma}_+$. Let Δ be a face of $\tilde{\Gamma}_+$, suppose that Δ is equal to the convex hull of $\{k_1, \dots, k_r\}$ for some $r \leq s$. Since Δ is simplicial, we have that $k_1, \dots, k_r$ are linearly independent and $r = \dim \Delta + 1$. In order to see that $\mathbf{f}_{\tilde{\Gamma}}$ is Newton non-degenerate at infinity, by definition it is enough to check that

$$\left\{ x \in \mathbb{C}^n : \left(x_1 \frac{\partial \mathbf{f}_{\tilde{\Gamma}}}{\partial x_1} \right)_{\Delta} (x) = \dots = \left(x_n \frac{\partial \mathbf{f}_{\tilde{\Gamma}}}{\partial x_n} \right)_{\Delta} (x) = 0 \right\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}.$$

We observe that

$$\left(x_i \frac{\partial \mathbf{f}_{\tilde{\Gamma}}}{\partial x_i} \right)_{\Delta} = x_i \left(\frac{\partial (\mathbf{f}_{\tilde{\Gamma}})_{\Delta}}{\partial x_i} \right)$$

for all $i = 1, \dots, n$. Therefore, if we write $k_i = (k_i^1, \dots, k_i^n)$ for $i = 1, \dots, r$, the above system of equations is translated into

$$\begin{aligned} k_1^1 x^{k_1} + \dots + k_r^1 x^{k_r} &= 0, \\ &\dots \\ k_1^n x^{k_1} + \dots + k_r^n x^{k_r} &= 0. \end{aligned}$$

Since $k_1, \dots, k_r$ are linearly independent, then $x^{k_i} = 0$ for all $i = 1, \dots, n$. But this means that some coordinate of x must be zero. Thus $\mathbf{f}_{\tilde{\Gamma}}$ is Newton non-degenerate at infinity and by Corollary 3.1.17 the polynomial map $A(\mathbf{f}_{\tilde{\Gamma}}) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is finite. $\square$

Corollary 3.4.2. *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a finite polynomial map, such that $\tilde{\Gamma}_+(F)$ is simplicial. Then F is Newton non-degenerate at infinity if and only if*

$$\mu(F) = \mu(A(\mathbf{f}_{\tilde{\Gamma}})).$$

Proof. Let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(F)$. The function $\mathbf{f}_{\tilde{\Gamma}}$ is Newton non-degenerate at infinity by Lemma 3.4.1, since $\tilde{\Gamma}_+$ is simplicial. Then the multiplicity of $A(\mathbf{f}_{\tilde{\Gamma}})$ is equal to $n!V_n(\tilde{\Gamma}_+(\mathbf{f}_{\tilde{\Gamma}}))$. But $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(\mathbf{f}_{\tilde{\Gamma}})$. Hence the number $n!V_n(\tilde{\Gamma}_+)$ is equal to the multiplicity of $A(\mathbf{f}_{\tilde{\Gamma}})$ and the result follows from Corollary 3.2.4. $\square$

Let $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ be a convenient and simplicial Newton polyhedron. As a consequence of the above corollary, if $f : \mathbb{C}^n \rightarrow \mathbb{C}$ is a polynomial function such that $A(f)$ is finite and $\tilde{\Gamma}_+(f) = \tilde{\Gamma}_+$, then f is Newton non-degenerate at infinity if and only if

$$\mu(A(f)) = \mu(A(\mathbf{f}_{\tilde{\Gamma}})). \quad (3.34)$$

We remark that the dimensions appearing on both sides of (3.34) are easily computable via a computer algebra system such as SINGULAR [27]. We also point out that, given a (local)

Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}_{\geq 0}^n$, the program GÉRMENES [56] developed by A. Montesinos provides the list of compact faces of Γ_+ and the vertexes of that faces, whenever $n \leq 8$. Then, with the aid of this program and the correspondence indicated in Corollary 3.1.4, it is possible to determine when a global Newton polyhedron $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ is simplicial if $n \leq 7$.

We remark that the next result is the polynomial version of Theorem 5.2 shown by Bivià-Ausina in [7].

Theorem 3.4.3. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial function with a finite number of singular points such that*

$$\frac{\partial f_{\mathbf{I}}}{\partial x_i} = \left(\frac{\partial f}{\partial x_i} \right)_{\mathbf{I}}, \quad (3.35)$$

for all $\mathbf{I} \subseteq \{1, \dots, n\}$ and all $i = 1, \dots, n$. Suppose that the polynomial map $A(f) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is finite. If ∇f is Newton non-degenerate at infinity, then f is also Newton non-degenerate at infinity.

Proof. Let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(f)$. We observe that $\tilde{\Gamma}_+$ is convenient, since $A(f)$ is a finite polynomial map. By [43, Lemma 3.2], we know that

$$\mu(A(f)) = \sum_{\mathbf{I} \subseteq \{1, \dots, n\}} \mu_{\infty}(f_{\mathbf{I}}) \quad (3.36)$$

where we set $\mu_{\infty}(f_{\emptyset}) = 1$ and $\mu_{\infty}(f_{\{1, \dots, n\}}) = \mu_{\infty}(f)$. Moreover, since the polynomial map $\nabla f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is Newton non-degenerate at infinity and by condition (3.35), we deduce that the polynomial map $\nabla f_{\mathbf{I}} : \mathbb{C}_{\mathbf{I}}^n \rightarrow \mathbb{C}_{\mathbf{I}}^n$ is also Newton non-degenerate at infinity (as a polynomial map), for all $\mathbf{I} \subseteq \{1, \dots, n\}$. Thus applying Theorem 3.2.4 and relation (3.36), we have

$$n!V_n(\tilde{\Gamma}_+) \geq \mu(A(f)) = \sum_{\mathbf{I} \subseteq \{1, \dots, n\}} |\mathbf{I}|!V_{|\mathbf{I}|}(\tilde{\Gamma}_+(\nabla f_{\mathbf{I}})). \quad (3.37)$$

Since the Newton non-degeneracy at infinity is a generic condition (see [43, p. 26]), then it is possible to find a polynomial function $g \in \mathbb{C}[x_1, \dots, x_n]$ satisfying that g is Newton non-degenerate at infinity and $\text{supp}(f) = \text{supp}(g)$. In particular, we have $\tilde{\Gamma}_+(\nabla g_{\mathbf{I}}) = \tilde{\Gamma}_+(\nabla f_{\mathbf{I}})$ and we deduce the following chain of inequalities:

$$\begin{aligned} n!V_n(\tilde{\Gamma}_+) &= \mu(A(g)) = \sum_{\mathbf{I} \subseteq \{1, \dots, n\}} \mu_{\infty}(g_{\mathbf{I}}) \leq \sum_{\mathbf{I} \subseteq \{1, \dots, n\}} |\mathbf{I}|!V_{|\mathbf{I}|}(\tilde{\Gamma}_+(\nabla(g_{\mathbf{I}}))) \\ &= \sum_{\mathbf{I} \subseteq \{1, \dots, n\}} |\mathbf{I}|!V_{|\mathbf{I}|}(\tilde{\Gamma}_+(\nabla f_{\mathbf{I}})) = \sum_{\mathbf{I} \subseteq \{1, \dots, n\}} \mu_{\infty}(f_{\mathbf{I}}) = \mu(A(f)) \leq n!V_n(\tilde{\Gamma}_+). \end{aligned}$$

Then, we deduce that $n!V_n(\tilde{\Gamma}_+)$ is equal to $\mu(A(f))$ and hence the polynomial function f is Newton non-degenerate at infinity, by Corollary 3.2.4. $\square$

Next we show that condition (3.35) cannot be removed from Theorem 3.4.3.

Example 3.4.4. Let $f : \mathbb{C}^3 \rightarrow \mathbb{C}$ be the polynomial function given by $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^4 + x_1x_2(x_1 - x_2)^2 + x_2^3x_3$. We observe that f is Newton degenerate at infinity, since the function $f_{\{1,2\}}$ is Newton degenerate at infinity. Moreover, if $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(f)$, it can be checked using the program GÉRMENES [56] that $\tilde{\Gamma}_+$ is not simplicial. However, by a straightforward computation it follows that the function $\mathbf{f}_{\tilde{\Gamma}_+}$ is Newton non-degenerate at infinity and

$$\mu(A(f)) = 52 < 54 = \mu(A(\mathbf{f}_{\tilde{\Gamma}_+})).$$

Then f is Newton degenerate at infinity, by (3.34). On the other hand, the polynomial map ∇f is Newton non-degenerate at infinity. The component functions of ∇f are given by

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 2x_1 + x_2^3 - 4x_1x_2^2 + 3x_1^2x_2 \\ \frac{\partial f}{\partial x_2} &= 2x_2 + x_1^3 - 4x_1^2x_2 + 3x_1x_2^2 + 3x_2^2x_3 \\ \frac{\partial f}{\partial x_3} &= 4x_3^3 + x_2^3. \end{aligned}$$

Then $\tilde{\Gamma}_+(\nabla f) = \tilde{\Gamma}_+(x_1^3 + x_2^3 + x_3^3)$ and ∇f is Newton non-degenerate at infinity, since $\mu_\infty(f) = 27 = 3!V_3(\tilde{\Gamma}_+(\nabla f))$. However, if $\mathbf{I} = \{2\}$, then we have $(\partial f / \partial x_3)_{\mathbf{I}} = x_2^3$ and $\partial f_{\mathbf{I}} / \partial x_3 = 0$, hence f does not satisfy condition (3.35).

Given a Newton polyhedron at infinity $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$, we define

$$\mathcal{O}(\tilde{\Gamma}_+) = \left\{ f \in \mathbb{C}[x_1, \dots, x_n] : \tilde{\Gamma}_+(f) = \tilde{\Gamma}_+ \text{ and } (\nabla f)^{-1}(0) \text{ is finite} \right\}.$$

Then, when $\mathcal{O}(\tilde{\Gamma}_+) \neq \emptyset$, we can consider the number

$$\delta(\tilde{\Gamma}_+) = \max\{n!V_n(\tilde{\Gamma}_+(\nabla f)) : f \in \mathcal{O}(\tilde{\Gamma}_+)\}.$$

by Theorem 2.2.8, we have that

$$\nu(\tilde{\Gamma}_+) = \max\{\mu_\infty(f) : f \in \mathcal{O}(\tilde{\Gamma}_+)\}.$$

Therefore, the inequality $\nu(\tilde{\Gamma}_+) \leq \delta(\tilde{\Gamma}_+)$ holds, by Corollary 3.2.4 and these number are different in general. The next example shows this fact.

Example 3.4.5. Let us consider the following set of $\mathbb{Z}^2$, given by

$$A = \{3\mathbf{e}_1, 3\mathbf{e}_2, (3, 3)\}.$$

Let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(A)$. A direct computation shows that $\nu(\tilde{\Gamma}_+) = 13$. Let f be the polynomial given by $f = x_1^3 + x_2^3 + x_1^3x_2 + x_1^3x_2^3 + x_1x_2^3$. Then, it is immediate to check that $f \in \mathcal{O}(\tilde{\Gamma}_+)$ and $\delta(\tilde{\Gamma}_+) = 2!V_2(\tilde{\Gamma}_+(\nabla f)) = 17$.

We observe that the maximum $\delta(\tilde{\Gamma}_+)$ is attained at some function $f \in \mathbb{C}[x_1, \dots, x_n]$ such that $\text{supp}(f) = \tilde{\Gamma}_+ \cap \mathbb{Z}_{\geq 0}^n$.

Given a convenient Newton polyhedron at infinity $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$, we define

$$\begin{aligned}\mathcal{K}(\tilde{\Gamma}_+) &= \left\{ f \in \mathcal{O}(\tilde{\Gamma}_+) : f \text{ is Newton non-degenerate at infinity} \right\}. \\ \mathcal{B}(\tilde{\Gamma}_+) &= \left\{ f \in \mathcal{O}(\tilde{\Gamma}_+) : \nabla f \text{ is Newton non-degenerate at infinity} \right\}.\end{aligned}$$

Proposition 3.4.6. *Let $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ be a convenient Newton polyhedron at infinity. Then $\mathcal{K}(\tilde{\Gamma}_+) \subseteq \mathcal{B}(\tilde{\Gamma}_+)$ if and only if $\nu(\tilde{\Gamma}_+) = \delta(\tilde{\Gamma}_+)$.*

Proof. Suppose that $\nu(\tilde{\Gamma}_+) = \delta(\tilde{\Gamma}_+)$ and let $f \in \mathcal{K}(\tilde{\Gamma}_+)$. Then we obtain the following inequalities:

$$\nu(\tilde{\Gamma}_+) = \mu_\infty(f) \leq n!V_n(\tilde{\Gamma}_+(\nabla f)) \leq \delta(\tilde{\Gamma}_+).$$

Since we are assuming that $\nu(\tilde{\Gamma}_+) = \delta(\tilde{\Gamma}_+)$, the above relation implies that $\mu_\infty(f)$ is equal to $n!V_n(\tilde{\Gamma}_+(\nabla f))$ and then the polynomial map ∇f is Newton non-degenerate, by Corollary 3.2.4.

Suppose that $\mathcal{K}(\tilde{\Gamma}_+) \subseteq \mathcal{B}(\tilde{\Gamma}_+)$ and let $f \in \mathcal{O}(\tilde{\Gamma}_+)$ such that

$$\delta(\tilde{\Gamma}_+) = n!V_n(\tilde{\Gamma}_+(\nabla f)).$$

The Newton non-degeneracy condition for functions is generic in the space of polynomial functions g such that $\text{supp}(g) \subseteq \tilde{\Gamma}_+$ (see [43, Theorem 6.1]). Thus, modifying the coefficients of f , we can suppose that f is Newton non-degenerate. In particular, the polynomial map ∇f is Newton non-degenerate, since we are assuming $\mathcal{K}(\tilde{\Gamma}_+) \subseteq \mathcal{B}(\tilde{\Gamma}_+)$. Therefore

$$\nu(\tilde{\Gamma}_+) = \mu_\infty(f) = n!V_n(\tilde{\Gamma}_+(\nabla f)) = \delta(\tilde{\Gamma}_+).$$

□

It is clear that $\mathcal{K}(\tilde{\Gamma}_+) \neq \emptyset$ for any convenient Newton polyhedron at infinity $\tilde{\Gamma}_+$. On the other hand, the family $\mathcal{B}(\tilde{\Gamma}_+)$ is not always non-empty, as the following proposition shows.

Proposition 3.4.7. *Let $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ be a convenient Newton polyhedron. Then $\mathcal{B}(\tilde{\Gamma}_+) \neq \emptyset$ if and only if ∇g is Newton non-degenerate at infinity, for any function $g \in \mathbb{C}[x_1, \dots, x_n]$ such that $\text{supp}(g) = \vartheta(\tilde{\Gamma}_+)$ and g is Newton non-degenerate at infinity. In particular, if $\tilde{\Gamma}_+$ is simplicial, then $\mathcal{B}(\tilde{\Gamma}_+) \neq \emptyset$ if and only if $\mathbf{f}_{\tilde{\Gamma}_+} \in \mathcal{B}(\tilde{\Gamma}_+)$.*

Proof. The *if* part is obvious. Let us see the converse. Let $f \in \mathcal{O}(\tilde{\Gamma}_+)$ be such that the polynomial map ∇f is Newton non-degenerate at infinity. Let $g \in \mathbb{C}[x_1, \dots, x_n]$ such that $\text{supp}(g) = \vartheta(\tilde{\Gamma}_+)$ and g is Newton non-degenerate at infinity. In particular, we have the inequality

$$V_n(\tilde{\Gamma}_+(\nabla g)) \leq V_n(\tilde{\Gamma}_+(\nabla f)).$$

Since the polynomial map ∇f and the polynomial g are Newton non-degenerate at infinity, we deduce that

$$\nu(\tilde{\Gamma}_+) \geq \mu_\infty(f) = n!V_n(\tilde{\Gamma}_+(\nabla f)) \geq n!V_n(\tilde{\Gamma}_+(\nabla g)) \geq \mu_\infty(g) = \nu(\tilde{\Gamma}_+).$$

Hence, we have $\mu_\infty(g) = \mu(\nabla g) = n!V_n(\tilde{\Gamma}_+(\nabla g))$. Then the polynomial map ∇g is Newton non-degenerate at infinity by Corollary 3.2.4. $\square$

Example 3.4.8. Let $g \in \mathbb{C}[x_1, x_2]$ be the function given by $g = x_2^9 + x_2x_1^7 + x_1^4$ and let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(g)$. It is clear that $g = \mathbf{f}_{\tilde{\Gamma}_+}$. It is immediate to see that $\tilde{\Gamma}_+(\nabla g) = \tilde{\Gamma}_+(x_1^7 + x_2^8)$ and $\nabla g = (4x_1^3 + 7x_1^6x_2, x_1^7 + 9x_2^8)$ is Newton degenerate at infinity. Hence there is no polynomial function $f \in \mathcal{O}_n(\tilde{\Gamma}_+)$ such that ∇f is Newton non-degenerate at infinity, by the previous result. This example shows that the Newton non-degeneracy at infinity of the gradient map is not a generic condition in $\mathbb{C}[\tilde{\Gamma}_+]$.

3.5 Homogeneous Newton polyhedra

Motivated by the characterization of monomial ideals in $\mathbb{C}[x_1, \dots, x_n]$, whose integral closure admits a reduction generated by homogeneous polynomials proven by Bivià-Ausina in [10]. In this section we study a class of global Newton polyhedra that called *homogeneous*.

We begin with some definitions and results about the determination of the number of isolated roots of polynomial systems (see [65] and [67]).

We recall from Section 2.4 that if $\mathbf{P} = (P_1, \dots, P_n)$ is a n -tuple of lattice of polytopes of $\mathbb{R}_{\geq 0}^n$, then $\mathbb{C}_n[\mathbf{P}]$ denotes the set of the polynomial maps $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $\text{supp}(F_i) \subseteq P_i$ for all $i = 1, \dots, n$. If $F^{-1}(0)$ is finite, then $\mu(F) \leq \text{MV}(\mathbf{P}^0)$ and the conditions ‘nice’ and ‘corned’ on $\mathbf{P}$ imply $\mu(F) = \text{MV}(\mathbf{P}^0)$ for a generic $F \in \mathbb{C}_n[\mathbf{P}]$ (see Theorem 2.4.4).

In order to develop this section we state some notation. Let $\mathbf{P} = (P_1, \dots, P_n)$ and $\mathbf{Q} = (Q_1, \dots, Q_n)$ be n -tuples of polytopes in $\mathbb{R}_{\geq 0}^n$, write $\mathbf{P} \subseteq \mathbf{Q}$ to denote that $P_i \subseteq Q_i$ for all $i = 1, \dots, n$. We also denote the n -tuples of subsets $(P_1 \cap Q_1, \dots, P_n \cap Q_n)$ and $(Q_1 \setminus$

$P_1, \dots, Q_n \setminus P_n$) by $\mathbf{P} \cap \mathbf{Q}$ and $\mathbf{Q} \setminus \mathbf{P}$ respectively. If $\mathbf{I} \subseteq \{1, \dots, n\}$, $\mathbf{I} \neq \emptyset$, then we set $\mathbf{P}^{\mathbf{I}} = (P_1 \cap \mathbb{R}_{\mathbf{I}}^n, \dots, P_n \cap \mathbb{R}_{\mathbf{I}}^n)$.

Next we recall a particular definition introduced in [67, p. 120].

Definition 3.5.1. Let $\mathbf{P}$ and $\mathbf{Q}$ be a n -tuples of polytopes in $\mathbb{R}_{\geq 0}^n$ such that $\mathbf{Q}$ is nice and cornered. We say that $\mathbf{P}$ *counts* $\mathbf{Q}$ when

- (i) $\mathbf{P} \subseteq \mathbf{Q}$;
- (ii) $\mathbf{P}$ is nice;
- (iii) for any $F \in \mathbb{C}_n[\mathbf{Q} \setminus \mathbf{P}]$, the map $F + F'$ has a finite zero set and $\mu(F + F') = \text{MV}(\mathbf{Q}^0)$ for a generic $F' \in \mathbb{C}_n[\mathbf{P}]$.

In particular, if $\mathbf{P}$ counts $\mathbf{Q}$, then $\mu(F) = \text{MV}(\mathbf{Q}^0)$ for a generic $F \in \mathbb{C}_n[\mathbf{P}]$ and therefore $\text{MV}(\mathbf{P}^0) = \text{MV}(\mathbf{Q}^0)$.

Definition 3.5.2. Let $\mathbf{P} = (P_1, \dots, P_n)$ be an n -tuple of polytopes in $\mathbb{R}^n$. The *support* of $\mathbf{P}$ is defined as the set of indices $i \in \{1, \dots, n\}$ such that $P_i \neq \emptyset$. We denote this set by $\text{supp}(\mathbf{P})$. Let $J \subseteq \{1, \dots, n\}$. Then J is said to be *essential* for $\mathbf{P}$ when the following conditions hold

- (i) $J \subseteq \text{supp}(\mathbf{P})$;
- (ii) $\dim(\sum_{j \in J} P_j) = |J| - 1$;
- (iii) for all nonempty proper subset $J' \subset J$, we have that $\dim(\sum_{j \in J'} P_j) \geq |J'|$.

Next we state a particular case of [67, Theorem 7] that we need for our purposes (we remark that this theorem is stated for polynomials with coefficients in any algebraically closed field). Given two n -tuples of polytopes $\mathbf{P}$ and $\mathbf{Q}$ of $\mathbb{R}^n$ such that $\mathbf{P} \subseteq \mathbf{Q}$, this result gives a purely combinatorial characterization of when $\mathbf{P}$ counts $\mathbf{Q}$.

Theorem 3.5.3 ([67, p. 127]). *Let $\mathbf{P}$ and $\mathbf{Q}$ be n -tuples of lattice polytopes contained in $\mathbb{R}_{\geq 0}^n$ such that $\mathbf{P} \subseteq \mathbf{Q}$. Let us suppose that $\mathbf{Q}$ is nice and cornered and $\text{MV}(\mathbf{Q}^0) > 0$. Then $\mathbf{P}$ counts $\mathbf{Q}$ if and only if $\text{supp}(\mathbf{P} \cap \mathbf{Q}^w)$ contains an essential subset for $\mathbf{Q}^w$ for all $w \in \mathbb{R}^n \setminus \mathbb{R}_{\geq 0}^n$.*

Let $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ be a convenient global Newton polyhedron at infinity. Let us fix a subset $\mathbf{I} \subseteq \{1, \dots, n\}$, $\mathbf{I} \neq \emptyset$. We denote by $\tilde{\Gamma}_+^{\mathbf{I}}$ the intersection $\tilde{\Gamma}_+ \cap \mathbb{R}_{\mathbf{I}}^n$ and we define $\deg(\tilde{\Gamma}_+^{\mathbf{I}}) = \max\{|k| : k \in \tilde{\Gamma}_+^{\mathbf{I}}\}$, where $|k|$ denotes the sum of the coordinates of k , for any $k \in \mathbb{R}^n$.

Let us define

$$b_i(\tilde{\Gamma}_+) = \min \left\{ \deg(\tilde{\Gamma}_+^{\mathbf{I}}) : \mathbf{I} \subseteq \{1, \dots, n\}, |\mathbf{I}| = n - i + 1 \right\}, \quad (3.38)$$

for all $i \in \{1, \dots, n\}$. It is immediate to see that $b_1(\tilde{\Gamma}_+) \geq \dots \geq b_n(\tilde{\Gamma}_+)$.

Definition 3.5.4. Let $\tilde{\Gamma}_+$ be a Newton polyhedron at infinity. We say that $\tilde{\Gamma}_+$ is *homogeneous* if there exist a polynomial map $G = (G_1, \dots, G_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that G Newton non-degenerate at infinity, $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(G)$ and G_i is a homogeneous polynomial of degree $b_i(\tilde{\Gamma}_+)$, for all $i = 1, \dots, n$.

Theorem 3.5.5. Let $\tilde{\Gamma}_+$ be a convenient Newton polyhedron at infinity with vertices in $\mathbb{Z}_{\geq 0}^n$. Then

$$n!V_n(\tilde{\Gamma}_+) \geq b_1(\tilde{\Gamma}_+) \cdots b_n(\tilde{\Gamma}_+) \quad (3.39)$$

and equality holds if and only if there exists $\lambda \in \mathbb{Z}_{\geq 1}$ and polynomial map $G = (G_1, \dots, G_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that G is Newton non-degenerate at infinity, G_i is a homogeneous polynomial of degree $\lambda b_i(\tilde{\Gamma}_+)$, for all $i = 1, \dots, n$ and $\tilde{\Gamma}_+(G) = \lambda \tilde{\Gamma}_+$.

Proof. Let $b_i = b_i(\tilde{\Gamma}_+)$, for all $i = 1, \dots, n$. Let us denote by D_i the convex hull in $\mathbb{R}^n$ of the set

$$\{k \in \tilde{\Gamma}_+^{\mathbf{I}} \cap \mathbb{Z}_{\geq 1}^n : |k| = b_i, \mathbf{I} \subseteq \{1, \dots, n\}, |\mathbf{I}| = n - i + 1\}$$

for all $i = 1, \dots, n$. By the definition of b_i , we have $b_i = \max\{|k| : k \in \tilde{\Gamma}_+ \cap \mathbb{R}_{\mathbf{I}}^n\}$ for some $\mathbf{I} \subseteq \{1, \dots, n\}$ such that $|\mathbf{I}| = n - i + 1$, for all $i = 1, \dots, n$. In particular, $D_i \neq \emptyset$ for all $i = 1, \dots, n$, since $\tilde{\Gamma}_+$ is convenient Newton polyhedron with vertices in $\mathbb{Z}_{\geq 0}^n$.

Let us denote the n -tuple of polytopes $(D_1, \dots, D_n)$ by $\mathbf{D}$. If $\alpha \in \mathbb{Z}_{\geq 1}$, let $\Delta(\alpha)$ denote the convex hull in $\mathbb{R}^n$ of the set $\{k \in \mathbb{Z}_{\geq 0}^n : |k| = \alpha\}$ and $\mathbf{\Delta}$ denote the n -tuple of polytopes $(\Delta(b_1), \dots, \Delta(b_n))$. It is clear that $\mathbf{\Delta}$ is nice and cornered and $\text{MV}(\mathbf{\Delta}^0) = b_1 \cdots b_n > 0$. Clearly, we have that $\mathbf{D} \subseteq \mathbf{\Delta}$. We claim that there exist $\lambda \in \mathbb{Z}_{\geq 1}$, such that $\lambda \mathbf{D}$ counts $\lambda \mathbf{\Delta}$. To see this, we will apply Theorem 3.5.3.

Let us fix a vector $w = (w_1, \dots, w_n) \in \mathbb{R}^n \setminus \mathbb{R}_{\geq 0}^n$ and let $w_0 = \min\{w_1, \dots, w_n\}$. Let $\mathbf{I}_w$ denote the set of indices $\{i : w_i = w_0\}$. It is immediate that $\ell(w, \Delta(b_j)) = b_j w_0$ and $\Delta(b_j)^w = \Delta(b_j) \cap \mathbb{R}_{\mathbf{I}_w}^n$ for all $w \in \mathbb{R}^n \setminus \mathbb{R}_{\geq 0}^n$ and for all $j = 1, \dots, n$. Then $\mathbf{\Delta}^w = \mathbf{\Delta}^{\mathbf{I}_w}$ for all $w \in \mathbb{R}^n \setminus \mathbb{R}_{\geq 0}^n$. In particular, we have the equality

$$\{\mathbf{\Delta}^w : w \in \mathbb{R}^n \setminus \mathbb{R}_{\geq 0}^n\} = \{\mathbf{\Delta}^{\mathbf{I}} : \mathbf{I} \subseteq \{1, \dots, n\}, \mathbf{I} \neq \emptyset\}.$$

Fix a subset $\mathbf{I} \subseteq \{1, \dots, n\}$, $\mathbf{I} \neq \emptyset$. Let $\alpha = |\mathbf{I}|$ and consider the set of indices $J_{\mathbf{I}} = \{n + 1 - \alpha, \dots, n\}$. Let us show that there exists $\lambda \in \mathbb{Z}_{\geq 1}$, such that $J_{\mathbf{I}} \subseteq \text{supp}((\lambda \mathbf{D}) \cap (\lambda \mathbf{\Delta})^{\mathbf{I}})$ and $J_{\mathbf{I}}$ is an essential set for $(\lambda \mathbf{\Delta})^{\mathbf{I}}$.

If $i \in J_{\mathbf{I}}$, then $\alpha \geq n - i + 1$ and thus $\deg(\tilde{\Gamma}_+^{\mathbf{I}}) \geq \deg(\tilde{\Gamma}_+^{\mathbf{I}'}) \geq b_i$, for all $\mathbf{I}' \subseteq \mathbf{I}$ such that $|\mathbf{I}'| = n - i + 1$. In particular, if $\mathbf{I}' \subseteq \mathbf{I}$ is any subset such that $|\mathbf{I}'| = n - i + 1$, there exist

some $k \in \tilde{\Gamma}_+^{\mathbf{I}'} \cap \mathbb{Q}_{\geq 0}^n \subseteq \tilde{\Gamma}_+^{\mathbf{I}} \cap \mathbb{Q}_{\geq 0}^n$ such that $|k| = b_i$. Let $\lambda_{\mathbf{I}} \in \mathbb{Z}_{\geq 1}$ such that $\lambda_{\mathbf{I}} k \in \mathbb{Z}_{\geq 1}^n$. Then $(\lambda_{\mathbf{I}} D_i) \cap \Delta(\lambda_{\mathbf{I}} b_i)^{\mathbf{I}} \neq \emptyset$, for all $i \in J_{\mathbf{I}}$. That is, we have that $J_{\mathbf{I}} \subseteq \text{supp}((\lambda_{\mathbf{I}} \mathbf{D}) \cap (\lambda_{\mathbf{I}} \mathbf{\Delta})^{\mathbf{I}})$. Then taking λ as the least common multiple of $\lambda_{\mathbf{I}}$ where $\mathbf{I} \subseteq \{1, \dots, n\}$, we obtain that $J_{\mathbf{I}} \subseteq \text{supp}((\lambda \mathbf{D}) \cap (\lambda \mathbf{\Delta})^{\mathbf{I}})$, for all $\mathbf{I} \subseteq \{1, \dots, n\}$, $\mathbf{I} \neq \emptyset$.

We observe that $\dim(\Delta(b))^{\mathbf{I}} = |\mathbf{I}| - 1$ for all $b \in \mathbb{Z}_{\geq 1}^n$. Moreover $\sum_{j \in J_{\mathbf{I}}} (\Delta(b_j))^{\mathbf{I}} = (\Delta(\sum_{j \in J_{\mathbf{I}}} b_j))^{\mathbf{I}}$. In particular, we have $\dim \sum_{j \in J_{\mathbf{I}}} \Delta(\lambda b_j)^{\mathbf{I}} = |\mathbf{I}| - 1$. Then we observe that $J_{\mathbf{I}}$ satisfies conditions (2) and (3) of the definition of essential subset for $(\lambda \mathbf{\Delta})^{\mathbf{I}}$. Thus, we deduce that $\lambda \mathbf{D}$ counts $\lambda \mathbf{\Delta}$, by Theorem 3.5.3.

In particular, there exist homogeneous polynomials $G_i \in \mathbb{C}[D_i]$, $i = 1, \dots, n$, such that, the $G = (G_1, \dots, G_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $G^{-1}(0)$ is finite and $\mu(G) = \text{MV}((\lambda \mathbf{\Delta})^0) = \lambda^n b_1 \cdots b_n$. Since G_i is homogeneous, for all $i = 1, \dots, n$, and $G^{-1}(0)$ is finite, we conclude that $G^{-1}(0) = \{0\}$. It is easy to see that $\tilde{\Gamma}_+(G) \subseteq \lambda \tilde{\Gamma}_+$. This implies that $\lambda^n b_1 \cdots b_n = \mu(G) \leq n! V_n(\lambda \tilde{\Gamma}_+(G)) \leq n! V_n(\lambda \tilde{\Gamma}_+)$ and then inequality (3.39) follows. Moreover, equality (3.39) holds if and only if G is Newton non-degenerate and $\tilde{\Gamma}_+(G) = \lambda \tilde{\Gamma}_+$, by Corollary 3.2.4. $\square$

Remark 3.5.6. Let us consider the following property on $\tilde{\Gamma}_+$: for all $i \in \{1, \dots, n\}$ and all $\mathbf{I} \subseteq \{1, \dots, n\}$ such that $|\mathbf{I}| = n - i + 1$ there exists some $k \in \tilde{\Gamma}_+^{\mathbf{I}}$ such that $|k| = b_i(\tilde{\Gamma}_+)$. A careful revision of the proof of Theorem 3.5.5 reveals that if $\tilde{\Gamma}_+$ satisfies this property, then we can take $\lambda = 1$ in the statement of Theorem 3.5.5.

Corollary 3.5.7. Let $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$ be a convenient Newton polyhedron at infinity with vertices in $\mathbb{Z}_{\geq 0}^n$. Suppose that $\tilde{\Gamma}_+$ is homogeneous. Then

$$n! V_n(\tilde{\Gamma}_+) = b_1(\tilde{\Gamma}_+) \cdots b_n(\tilde{\Gamma}_+).$$

Proof. By definition there exist a Newton non-degenerate at infinity polynomial map $G = (G_1, \dots, G_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ satisfying Definition 3.5.4. Then by Theorem 2.4.4 and Corollary 2.4.5, we deduce that $n! V_n(\tilde{\Gamma}_+) \leq b_1(\tilde{\Gamma}_+) \cdots b_n(\tilde{\Gamma}_+)$ and by the above theorem we have the equality. $\square$

Example 3.5.8. Let us consider the following subset of $\mathbb{Z}^3$, given by

$$A = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$$

Let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(A)$. Then a direct computation shows that $b_1(\tilde{\Gamma}_+) = 3$, $b_2(\tilde{\Gamma}_+) = 2$ and $b_3(\tilde{\Gamma}_+) = 1$. We claim that $\tilde{\Gamma}_+$ is homogeneous. Indeed, let us consider the polynomial map

$G = (G_1, G_2, G_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$, whose components functions are given by

$$G_1(x_1, x_2, x_3) = x_1 + x_2 + x_3,$$

$$G_2(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_1x_3,$$

$$G_3(x_1, x_2, x_3) = x_1x_2x_3.$$

It is immediate to check that G is Newton non-degenerate at infinity, each G_i is a homogeneous polynomial of degree $b_i(\tilde{\Gamma}_+)$, for all $i = 1, 2, 3$, and $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(G)$. Then, by definition, we have that $\tilde{\Gamma}_+$ is a homogeneous global Newton polyhedron. Let us remark that $3!V_3(\tilde{\Gamma}_+) = b_1(\tilde{\Gamma}_+)b_2(\tilde{\Gamma}_+)b_3(\tilde{\Gamma}_+)$.

The following example shows a global Newton polyhedron at infinity that is not homogeneous.

Example 3.5.9. Let us consider the following subset of $\mathbb{Z}_{\geq 0}^2$ given by

$$A = \{2\mathbf{e}_1, 3\mathbf{e}_2, (2, 2)\}$$

Let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(A)$. A direct computation shows that $b_1(\tilde{\Gamma}_+) = 4$, $b_2(\tilde{\Gamma}_+) = 2$ and $\tilde{\Gamma}_+$ satisfies the strict inequality of (3.39). In particular, $\tilde{\Gamma}_+$ is not homogeneous, by Theorem 3.5.5 and Remark 3.5.6.

CHAPTER 4

Łojasiewicz exponent and adapted maps to Newton polyhedra

Contents

4.1	Maps adapted to Newton polyhedra	74
4.2	Special monomials and adapted maps	81
4.3	Estimation of Łojasiewicz exponents at infinity	88
4.4	Łojasiewicz exponent at infinity of pre-weighted homogeneous maps	91
4.5	Index of polynomial maps	94

Let us fix coordinates $x_1, \dots, x_n$ in $\mathbb{K}^n$. In this chapter we study the general problem of determining the set of special monomials with respect to a given polynomial map $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We obtain an approximation to the set of special monomials with respect to a polynomial map in terms non-degeneracy conditions and Newton polyhedra at infinity. The corresponding result (Theorem 4.2.4) gives rise to an estimation from below to the Łojasiewicz exponent of a polynomial map $\mathbb{K}^n \rightarrow \mathbb{K}^p$ with compact zero set (Corollary 4.3.3).

In order to achieve our objectives we introduce the notion of strongly non-degenerate map with respect to a fixed Newton polyhedron. This is basically a generalization of the notion of pre-weighted homogeneous map.

Based on our study of Łojasiewicz exponents at infinity we also derive a result about the invariance of the index of real polynomial maps (Theorem 4.5.1).

4.1 Maps adapted to Newton polyhedra

Let us fix a convenient global Newton polyhedron $\tilde{\Gamma}_+ \subseteq \mathbb{R}^n$. In this section we expose a condition on a given polynomial map $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ that allows us to obtain information about $S_0(F)$ and $\mathcal{L}_\infty(F)$ in terms of $\tilde{\Gamma}_+$. First we recall some definitions given in Section 2.1.

Definition 4.1.1. Let $h \in \mathbb{K}[x_1, \dots, x_n]$. Let us suppose that h is written as $h = \sum_k a_k x^k$. If $w \in \mathbb{R}^n \setminus \{0\}$, then $\ell(w, h)$ and $\Delta(w, h)$ are defined as

$$\begin{aligned}\ell(w, h) &= \min\{\langle w, k \rangle : k \in \text{supp}(h)\}. \\ \Delta(w, h) &= \{k \in \text{supp}(h) : \langle w, k \rangle = \ell(w, h)\}.\end{aligned}$$

We define the *principal part of h with respect to w* , denoted by $p_w(h)$, as the sum of those terms $a_k x^k$ such that $\langle k, w \rangle = \ell(w, h)$. We observe that if h denotes a monomial x^k then $p_w(h) = h$, for any $w \in \mathbb{R}^n \setminus \{0\}$. If $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ is a polynomial map, then we denote the map $(p_w(F_1), \dots, p_w(F_p)) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ by $p_w(F)$.

Example 4.1.2. Let $h \in \mathbb{K}[x_1, x_2]$ be a polynomial given by $h(x_1, x_2) = x_1^3 + x_1^4 x_2^4 - 4x_1^2 x_2^3$. Then $\text{supp}(h) = \{(3, 0), (4, 4), (2, 3)\}$. Let $w = (3, -1)$, then a direct computation shows that $\ell(w, h) = 3$ and this minimum is attained only at the point $(2, 3) \in \text{supp}(h)$. Then $\Delta(w, h) = \{(2, 3)\}$ and $p_w(h) = -4x_1^2 x_2^3$. Let us remark that $\ell(w, \tilde{\Gamma}_+((h))) = 0$ and $\Delta(w, \tilde{\Gamma}_+((h))) = \{(0, 0)\}$. In general it is immediate to see that, if $g \in \mathbb{K}[x_1, \dots, x_n]$ and $v \in \mathbb{R} \setminus \{0\}$ then $\ell(v, \tilde{\Gamma}_+(g)) \leq \ell(v, g)$ and equality holds if and only if $\ell(v, g) \leq 0$.

Example 4.1.3. Let $F : \mathbb{K}^2 \rightarrow \mathbb{K}^3$ be the polynomial map where the component functions are given by

$$\begin{aligned}F_1(x_1, x_2) &= x_1^3 - 3x_1^2 x_2^3 + x_2^4, \\ F_2(x_1, x_2) &= x_1^2 - x_1^3 x_2^7 + 4x_1^2 x_2^4, \\ F_3(x_1, x_2) &= 4x_1 x_2^3 + 7x_1^2 x_2^6.\end{aligned}$$

Let $w = (3, -1)$, then we have that $\ell(w, F_1) = -4$, $\ell(w, F_2) = 2$, $\ell(w, F_3) = 0$, $\Delta(w, F_1) = \{(0, 4)\}$, $\Delta(w, F_2) = \{(3, 7), (2, 4)\}$ and $\Delta(w, F_3) = \{(1, 3), (2, 6)\}$. Therefore, it is easy to observe that $p_w(F)$ is given by $p_w(F)(x_1, x_2) = (x_2^4, -x_1^3 x_2^7 + 4x_1^2 x_2^4, 4x_1 x_2^3 + 7x_1^2 x_2^6)$.

If $J \subseteq \mathcal{F}(\tilde{\Gamma}_+)$, then we denote by $\Delta_J(h)$ the intersection $\cap_{w \in J} \Delta(w, h)$. We define the *principal part of h with respect to J* , which we will denote by $p_J(h)$, as the sum of all terms $a_k x^k$ such that $k \in \Delta_J(h)$. If $\Delta_J(h) = \emptyset$, then we set $p_J(h) = 0$.

We denote by $|A|$ the cardinal of a given finite set A . If Δ is a face of $\tilde{\Gamma}_+$, then we denote by $\mathcal{J}(\Delta)$ the set of those subsets $J \subseteq \mathcal{F}(\tilde{\Gamma}_+)$ such that $\Delta = \cap_{w \in J} \Delta(w, \tilde{\Gamma}_+)$ and $\dim \Delta = n - |J|$. Then we observe that $\mathcal{J}(\Delta)$ is formed by all subsets $J \subseteq \mathcal{F}(\tilde{\Gamma}_+)$ that minimally satisfy the condition $\Delta = \cap_{w \in J} \Delta(w, \tilde{\Gamma}_+)$. In particular, if Δ is a vertex of $\tilde{\Gamma}_+$ then $|J| = n$, for all $J \in \mathcal{J}(\Delta)$.

Example 4.1.4. Let $\tilde{\Gamma}_+$ be the Newton polyhedron shown in the following figure,

Then, we observe that $\tilde{\Gamma}_+$ is convenient and $\mathcal{F}(\tilde{\Gamma}_+) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, -\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3, w\}$, where $w = (-2, -2, -1)$. We shall give some examples about the above definitions.

- (i) Let $h \in \mathbb{K}[x_1, x_2, x_3]$ be the polynomial given by $h(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_1^2x_2^2x_3^3$. If $J = \{-\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3\}$, then it is easy to see that $\Delta_J(h) = \{(2, 2, 3)\}$ and $p_J(h) = 2x_1^2x_2^2x_3^3$, however if $J = \{\mathbf{e}_2, -\mathbf{e}_3\}$, then $\Delta_J(h) = \emptyset$ and by definition $p_J(h) = 0$.
- (ii) Let us consider Δ_1 the vertex of $\tilde{\Gamma}_+$ given by $\{(2, 2, 0)\}$ and the 1-dimensional face Δ_2 of $\tilde{\Gamma}_+$, given by the segment determined by the points $(2, 0, 2)$ and $(2, 1, 2)$. Then

$$\begin{aligned} \mathcal{J}(\Delta_1) &= \{\{-\mathbf{e}_1, \mathbf{e}_3, w\}, \{-\mathbf{e}_1, -\mathbf{e}_2, w\}, \{-\mathbf{e}_2, \mathbf{e}_3, w\}, \{-\mathbf{e}_1, -\mathbf{e}_2, \mathbf{e}_3\}\}. \\ \mathcal{J}(\Delta_2) &= \{\{-\mathbf{e}_1, -\mathbf{e}_3\}\}. \end{aligned}$$

Definition 4.1.5. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. We say that F is adapted to $\tilde{\Gamma}_+$ when for any face Δ of $\tilde{\Gamma}_+$ such that $0 \notin \Delta$ and for all $J \in \mathcal{J}(\Delta)$ we have

$$\{x \in \mathbb{K}^n : p_J(F_1)(x) = \dots = p_J(F_p)(x) = 0\} \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}.$$

We also refer to the above inclusion as the condition $(C_{F,J})$. We will denote the map $(p_J(F_1), \dots, p_J(F_p))$ by $p_J(F)$.

The previous definition is motivated by the notion of pre-weighted homogeneous map and the *Newton non-degeneracy* condition on germs of analytic functions $(\mathbb{K}^n, 0) \rightarrow \mathbb{K}$ studied by Kouchnirenko [43] and Yoshinaga [79]. We observe that under the conditions of the above definition, if F satisfies the $(C_{F,J})$ condition, for a given $J \subseteq \mathcal{F}(\tilde{\Gamma}_+)$, then not all the polynomials $p_J(F_i)$ are identically zero. In particular, there exists some $i \in \{1, \dots, p\}$ such that $\cap_{w \in J} \Delta(w, \tilde{\Gamma}_+(F_i))$ is a face of $\tilde{\Gamma}_+(F_i)$.

Remark 4.1.6. (i) In the particular, when $n = 2$. In order to see that if a polynomial map $F : \mathbb{K}^2 \rightarrow \mathbb{R}^p$ is adapted to a convenient Newton polyhedron $\tilde{\Gamma}_+ \subseteq \mathbb{R}^2$, it is sufficient to check if F satisfies the condition $(C_{F,J})$ for all $J \subseteq \mathcal{F}(\tilde{\Gamma}_+)$ such that $\cap_{w \in J} \Delta(w, \tilde{\Gamma}_+)$ is a face of $\tilde{\Gamma}_+$ not containing the origin, since $\mathcal{J}(\Delta) = \{J\}$ for all face Δ of $\tilde{\Gamma}_+$ such that $0 \notin \Delta$, where J is a subset of $\mathcal{F}(\tilde{\Gamma}_+)$.

(ii) Let us consider a polynomial map $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ such that some of the component functions of F is a monomial x^k , for some $k \in \mathbb{Z}_{\geq 0}^n$, $k \neq 0$. Since $p_J(x^k) = x^k$, for any $J \in \mathcal{F}(\tilde{\Gamma}_+)$, then F is automatically adapted to $\tilde{\Gamma}_+$. This fact suggests that we need to strengthen the above definition in order to obtain a sufficiently restrictive class of polynomials $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ for which it is possible to obtain a lower bound for $\mathcal{L}_\infty(F)$.

Let $I \subseteq \{1, \dots, n\}$, $I \neq \emptyset$. Define $\mathbb{K}_I^n = \{x \in \mathbb{K}^n : x_i = 0, \text{ for all } i \notin I\}$ and we denote by π_I the natural projection $\mathbb{R}^n \rightarrow \mathbb{R}_I^n$.

Let $h \in \mathbb{K}[x_1, \dots, x_n]$ and let us suppose that h is written as $h = \sum_k a_k x^k$. Then we denote by h^I the sum of all terms $a_k x^k$ such that $k \in \text{supp}(h) \cap \mathbb{R}_I^n$. If $\text{supp}(h) \cap \mathbb{R}_I^n = \emptyset$, then we set $h^I = 0$. If $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ is a polynomial map then we define $F^I = (F_1^I, \dots, F_p^I) : \mathbb{K}_I^n \rightarrow \mathbb{K}^p$. Let us denote by $\tilde{\Gamma}_+^I$ the projection $\pi_I(\tilde{\Gamma}_+ \cap \mathbb{R}_I^n)$. We remark that in general $\tilde{\Gamma}_+^I$ is not equal to $\pi_I(\tilde{\Gamma}_+)$.

We will strengthen the notion of adapted maps in the next definition. Let $w \in \mathcal{F}(\tilde{\Gamma}_+)$ and let $h \in \mathbb{K}[x_1, \dots, x_n]$. Then define

$$\ell^*(w, h) = \begin{cases} \ell(w, h) & \text{if } \ell(w, \tilde{\Gamma}_+) < 0, \\ 0 & \text{if } \ell(w, \tilde{\Gamma}_+) = 0. \end{cases}$$

Let us suppose that h is written as $h = \sum_k a_k x^k$. If $J \subseteq \mathcal{F}(\tilde{\Gamma}_+)$, then we denote by $p_J^*(h)$ the sum of all terms $a_k x^k$ such that $\langle k, w \rangle = \ell^*(w, h)$, for all $w \in J$. If the set of such terms $a_k x^k$ is empty, then we set $p_J^*(h) = 0$.

Let us observe that, since $\tilde{\Gamma}_+$ is convenient, then $\mathcal{F}(\tilde{\Gamma}_+) = \mathcal{F}_0(\tilde{\Gamma}_+) \cup \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Therefore, if $w \in \mathcal{F}(\tilde{\Gamma}_+)$, then the condition $\ell(w, \tilde{\Gamma}_+) = 0$ is equivalent to saying that w is equal to some

vector $\mathbf{e}_i$. Thus if $J \cap \{\mathbf{e}_1, \dots, \mathbf{e}_n\} = \emptyset$, then $p_J(h) = p_J^*(h)$. If $J \cap \{\mathbf{e}_1, \dots, \mathbf{e}_n\} \neq \emptyset$, then

$$p_J^*(h) = \begin{cases} p_J(h) & \text{if } \ell(w, h) = 0 \text{ for all } w \in J \cap \{\mathbf{e}_1, \dots, \mathbf{e}_n\} \\ 0 & \text{if otherwise.} \end{cases} \quad (4.1)$$

Definition 4.1.7. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. We say that F is *strongly adapted to* $\tilde{\Gamma}_+$ when for any face Δ of $\tilde{\Gamma}_+$ such that $0 \notin \Delta$ and for all $J \in \mathcal{J}(\Delta)$ we have

$$\{x \in \mathbb{K}^n : p_J^*(F_1)(x) = \dots = p_J^*(F_p)(x) = 0\} \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}. \quad (4.2)$$

We also refer to the above inclusion as the condition $(C_{F,J}^*)$.

As an immediate consequence of the previous definition, if F is strongly adapted to $\tilde{\Gamma}_+$, then F is adapted to $\tilde{\Gamma}_+$. In general the converse does not hold, as the following example shows.

Example 4.1.8. Let us consider the polynomial map $F = (F_1, F_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, whose components are given by

$$F_1(x_1, x_2) = x_1^3 + x_1^7 x_2^6 + x_2^4,$$

$$F_2(x_1, x_2) = x_1^6 x_2^5.$$

Let g be the function defined by $g(x_1, x_2) = x_1 + x_2 + x_1 x_2$ and let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(g)$. Then we observe that $\tilde{\Gamma}_+$ is convenient and $\mathcal{F}(\tilde{\Gamma}_+) = \{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_1, -\mathbf{e}_2\}$. Then F is adapted to $\tilde{\Gamma}_+$, since it contains a monomial, but it is not strongly adapted, because if we consider the face $\Delta = \{(0, 1)\}$ then $\mathcal{J}(\Delta) = \{\{\mathbf{e}_1, -\mathbf{e}_2\}\}$. It is straightforward to check that $p_J^*(F_i) = 0$, for all $i = 1, 2$. Therefore, by definition F is not strongly adapted to $\tilde{\Gamma}_+$.

Proposition 4.1.9. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map such that F_j is convenient, for all $j = 1, \dots, p$. Then the following conditions are equivalent:

- (i) F is adapted to $\tilde{\Gamma}_+$.
- (ii) F is strongly adapted to $\tilde{\Gamma}_+$.

Proof. It follows as an immediate application of (4.1) and Definition 4.1.7. $\square$

The following definition is concerned only with polynomial maps. That is, it is not applied to pairs $(F, \tilde{\Gamma}_+)$ formed by a polynomial map F and a Newton polyhedron $\tilde{\Gamma}_+$ (see Definitions 4.1.5 and 4.1.7). Thus, once we fix coordinates in $\mathbb{K}^n$, it can be considered as an intrinsic property of polynomial maps $\mathbb{K}^n \rightarrow \mathbb{K}^p$.

Definition 4.1.10. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. We say that F is *non-degenerate* when for all $w \in \mathbb{R}_0^n$ we have

$$\{x \in \mathbb{K}^n : p_w(F_1)(x) = \dots = p_w(F_p)(x) = 0\} \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}.$$

We will refer to the above inclusion as the condition $(C_{F,w})$.

Under the above conditions we observe that, given a vector $w \in \mathbb{R}^n$, the $(C_{F,w})$ condition applied on F depends only on the family of polynomials $\{p_w(F_1), \dots, p_w(F_p)\}$. On the other hand, the convex hull of a finite subset of $\mathbb{R}^n$ has only a finite number of faces (see [31, p. 29]). Therefore, if $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ is any polynomial map, the number of non redundant $(C_{F,w})$ conditions applied to F , where w varies in $\mathbb{R}^n$, is less than or equal to the cardinality of $\{(\Delta_1, \dots, \Delta_p) : \Delta_j \text{ is a face of } \tilde{\Gamma}_+(F_j), j = 1, \dots, p\}$. Hence the notion of non-degeneracy is formed by a finite family of conditions.

Example 4.1.11. Let $F = (F_1, F_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the map whose component functions are given by

$$\begin{aligned} F_1(x_1, x_2) &= x_1^3 + x_2^3 + x_1^2 x_2^6 + x_1^3 x_2^4, \\ F_2(x_1, x_2) &= x_1^4 x_2^2 + 3x_1^2 x_2^6 + 4x_1^3 x_2^4. \end{aligned}$$

If $w = (-2, -1)$, then $p_w(F_1) = x_1^2 x_2^6 + x_1^3 x_2^4$ and $p_w(F_2) = x_1^4 x_2^2 + 3x_1^2 x_2^6 + 4x_1^3 x_2^4$. We observe that the polynomials $p_w(F_1)$ and $p_w(F_2)$ vanish along the line $x_1 = -x_2^2$. Therefore F is degenerate. On the other hand, let us consider the polynomial map $G = (G_1, G_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ such that $G_1 = F_1$ and $G_2 = x_1^4 x_2^2 + 3x_1^2 x_2^6 = F_2 - 4x_1^3 x_2^4$. Then a simple computation reveals that G is non-degenerate.

The definition of non-degeneracy becomes more restrictive when we suppose that each F_j is convenient, $j = 1, \dots, p$, as see in the next result.

Proposition 4.1.12. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map such that F_j is convenient, for all $j = 1, \dots, p$. Then the following conditions are equivalent:

- (i) F is non-degenerate.
- (ii) F is adapted to the Minkowski sum $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(F_1) + \dots + \tilde{\Gamma}_+(F_p)$.

Proof. Let us see (i) $\Rightarrow$ (ii). Let Δ be a face of $\tilde{\Gamma}_+$ such that $0 \notin \Delta$. Let $J \in \mathcal{J}(\Delta)$. In particular $\Delta = \cap_{w \in J} \Delta(w, \tilde{\Gamma}_+) \neq \emptyset$. If $k \in \Delta$ and we write $k = k_1 + \dots + k_p$, where

$k_j \in \tilde{\Gamma}_+(F_j)$, for all $j = 1, \dots, p$, then we have $k_j \in \Delta(w, \tilde{\Gamma}_+(F_j))$, for all $j = 1, \dots, p$ and all $w \in J$, as a consequence of Lemma 2.1.4 (i). In particular $\cap_{w \in J} \Delta(w, \tilde{\Gamma}_+(F_j)) \neq \emptyset$, for all $j = 1, \dots, p$, and then $\cap_{w \in J} \Delta(w, \tilde{\Gamma}_+(F_j)) = \Delta(\sum_{w \in J} w, \tilde{\Gamma}_+(F_j))$, for all $j = 1, \dots, p$. Let us observe that $\Delta(w, F_j) = \Delta(w, \tilde{\Gamma}_+(F_j))$, for all $j = 1, \dots, p$ and all $w \in J$, since each polynomial F_j is convenient. Then we obtain the equality of polynomials $p_J(F_j) = p_v(F_j)$, for all $j = 1, \dots, p$, where $v = \sum_{w \in J} w$. Thus condition $(C_{F,J})$ is equivalent to condition $(C_{F,w})$ and the result follows. Let us see (ii) $\Rightarrow$ (i). Let $v \in \mathbb{R}^n \setminus \{0\}$. Then there exists some $J \subseteq \mathcal{F}(\tilde{\Gamma}_+)$ such that $\Delta(v, \tilde{\Gamma}_+) = \cap_{w \in J} \Delta(w, \tilde{\Gamma}_+) \neq \emptyset$ and $J \in \mathcal{J}(\Delta(v, \tilde{\Gamma}_+))$. Then, similarly to the proof of the other implication, we deduce that $p_v(F_j) = p_J(F_j)$, for all $j = 1, \dots, p$, and hence the result follows. $\square$

Let us fix a convenient global Newton polyhedron $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$. We recall that in Section 2.3, the filtrating map associated to $\tilde{\Gamma}_+$ is denoted by ϕ and M is the positive constant where ϕ is constant on the boundary $\tilde{\Gamma}$ and equal to M . If $w \in \mathcal{F}_0(\tilde{\Gamma}_+)$, then $\phi_w : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$ denotes the linear map given by $\phi_w(k) = \frac{M}{\ell(w, \tilde{\Gamma}_+)} \langle w, k \rangle$. Then, $\phi(k)$, is given by $\phi(k) = \max\{\phi_w(k) : w \in \mathcal{F}_0(\tilde{\Gamma}_+)\}$. If h is a polynomial in $\mathbb{K}[x_1, \dots, x_n]$, then by definition of degree of h with respect to $\tilde{\Gamma}_+$ (see Definition 2.3.2), we obtain the following inequalities:

$$\frac{M}{\ell(w, \tilde{\Gamma}_+)} \langle w, k \rangle = \phi_w(k) \leq \phi(k) \leq \nu_{\tilde{\Gamma}_+}(h), \quad (4.3)$$

for all $k \in \text{supp}(h)$ and for all $w \in \mathcal{F}(\tilde{\Gamma}_+)$, where $\nu_{\tilde{\Gamma}_+}(h)$ denotes the degree of h with respect to $\tilde{\Gamma}_+$.

Proposition 4.1.13. *Let $F = (F_1, \dots, F_n) : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be a finite polynomial map. Let us fix a convenient global Newton polyhedron $\tilde{\Gamma}_+ \subseteq \mathbb{R}_{\geq 0}^n$. If F is non-degenerate with respect to $\tilde{\Gamma}_+$, then F is strongly adapted to $\tilde{\Gamma}_+$.*

Proof. Let us fix $w \in \mathcal{F}_0(\tilde{\Gamma}_+)$. Since F is non-degenerate with respect to $\tilde{\Gamma}_+$ and $\Delta(w, \tilde{\Gamma}_+)$ is a face of $\tilde{\Gamma}_+$ of dimension $n - 1$, there exists $k_i \in C(\Delta(w, \tilde{\Gamma}_+))$ such that $\nu_{\tilde{\Gamma}_+}(F_i) = \nu_{\tilde{\Gamma}_+}(x^{k_i})$. Then, by Lemma 2.3.1, we obtain that $\nu_{\tilde{\Gamma}_+}(F_i) = \phi(k_i) = \phi_w(k_i)$ for all $i = 1, \dots, n$. Using the inequality (4.3), we obtain that

$$\frac{M}{\ell(w, \tilde{\Gamma}_+)} \langle w, k \rangle \leq \nu_{\tilde{\Gamma}_+}(F_i) = \frac{M}{\ell(w, \tilde{\Gamma}_+)} \langle w, k_i \rangle, \quad (4.4)$$

for all $k \in \text{supp}(F_i)$. Thus, the above inequality shows that $\ell(w, F_i) = \langle w, k_i \rangle$, and

$$\nu_{\tilde{\Gamma}_+}(F_i) = \frac{M}{\ell(w, \tilde{\Gamma}_+)} \ell(w, F_i), \text{ for all } w \in \mathcal{F}_0(\tilde{\Gamma}_+) \text{ and for all } i = 1, \dots, n. \quad (4.5)$$

Let Δ be a face of $\tilde{\Gamma}_+$ such that $0 \notin \Delta$ and let $J \in \mathcal{J}(\Delta)$, then we can decompose J as $J = J_1 \cup J_2$, where $J_2 = J \cap \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $J_1 = J \setminus J_2$. Let us consider the following sets:

$$A_i = \{k \in \text{supp}(F_i) : k \in C(\Delta) \text{ and } \nu_{\tilde{\Gamma}_+}(F_i) = \phi(k)\},$$

$$B_J^i = \{k \in \text{supp}(F_i) : \langle w, k \rangle = \ell(w, F_i), \text{ for all } w \in J_1 \text{ and } \langle w, k \rangle = 0, \text{ for all } w \in J_2\}.$$

We claim that $A_i = B_J^i$, for all $i = 1, \dots, n$. Since $\Delta = (\cap_{w \in J_1} \Delta(w, \tilde{\Gamma}_+)) \cap (\cap_{w \in J_2} \Delta(w, \tilde{\Gamma}_+))$, the claim follows as an immediate application of (4.3), (4.5) and Lemma 2.3.1.

By the above affirmation, we have that $\text{in}_\Delta(F_i) = p_J^*(F_i)$, for all $i = 1, \dots, n$. Therefore the result follows. $\square$

The next example shows that the converse of the last proposition does not hold.

Example 4.1.14. Let us consider $a, b, m, n \in \mathbb{Z}_{\geq 1}$, such that $a \leq m$ and $b \leq n$ and let $F = (F_1, F_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a polynomial map, whose component functions are given by

$$F_1(x_1, x_2) = x_1^a + x_1^m x_2^n,$$

$$F_2(x_1, x_2) = x_2^b.$$

Let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(F) \subseteq \mathbb{R}^2$. Then we observe that $\tilde{\Gamma}_+$ is convenient and $\mathcal{F}(\tilde{\Gamma}_+) = \{\mathbf{e}_1, \mathbf{e}_2, w_1, w_2\}$, where w_1 and w_2 are the primitive vectors in $\mathbb{Z}^2$, such that $\Delta(w_1, \tilde{\Gamma}_+)$ is the face of $\tilde{\Gamma}_+$ determined by the points $(0, b)$ and (m, n) and $\Delta(w_2, \tilde{\Gamma}_+)$ is the face of $\tilde{\Gamma}_+$ determined by the points $(a, 0)$ and (m, n) , as is shown in the Figure 4.1. It is immediate to see that F is

Figure 4.1: $\tilde{\Gamma}_+$

strongly adapted to $\tilde{\Gamma}_+$. However F is degenerate with respect to $\tilde{\Gamma}_+$. To see this it suffices

to consider $\Delta = \Delta(w_2, \tilde{\Gamma}_+)$ and to observe that

$$\begin{aligned} \{(x_1, x_2) \in \mathbb{C}^2 : \text{in}_\Delta(F_1)(x) = \text{in}_\Delta(F_2)(x) = 0\} &= \{(x_1, x_2) \in \mathbb{C}^2 : F_1(x_1, x_2) = 0\} \\ &\not\subseteq \{(x_1, x_2) \in \mathbb{C}^2 : x_1 \cdot x_2 = 0\}. \end{aligned}$$

Remark 4.1.15. From Proposition 2.3.9, we have that the class of Newton non-degenerate at infinity polynomial maps is contained in the class of the polynomial maps which are non-degenerate with respect to a given global Newton polyhedron and from Proposition 4.1.13, this class is contained in the class of the strongly adapted polynomial maps. We also can see in the Remark 2.3.7 and in the example 4.1.14 that these are strictly contained.

4.2 Special monomials and adapted maps

Along this section, we denote by $\tilde{\Gamma}_+$ a convenient Newton polyhedron at infinity contained in $\mathbb{R}_{\geq 0}^n$. Let us recall that $\mathcal{F}(\tilde{\Gamma}_+) = \mathcal{F}_0(\tilde{\Gamma}_+) \cup \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ denotes the canonical basis of $\mathbb{R}^n$ and $\mathcal{F}_0(\tilde{\Gamma}_+)$ are the primitive vectors supporting some face of $\tilde{\Gamma}_+$ of dimension $n - 1$ not passing through the origin.

We need to introduce some definitions before giving results that allows us to obtain a method to obtain an approximation of special monomials with respect to polynomials maps adapted to $\tilde{\Gamma}_+$.

Let $a^1, \dots, a^r \in \mathbb{R}^n$ such that $a^i \neq 0$, for all $i = 1, \dots, r$. The set $\sigma = \mathbb{R}_{\geq 0}a^1 + \dots + \mathbb{R}_{\geq 0}a^r$ is called the cone spanned, or generated, by $a^1, \dots, a^r$. This is also known as the *positive hull* of $a^1, \dots, a^r$. If σ is minimally generated by $a^1, \dots, a^r$ and a^i is a primitive vector of $\mathbb{Z}^n$, for all $i = 1, \dots, r$, then we will say that $a^1, \dots, a^r$ are the *primitive generators* of σ . The intersection of σ with a supporting hyperplane of σ is called a *face* of σ . We define the dimension of the cone $\sigma = \mathbb{R}_{\geq 0}a^1 + \dots + \mathbb{R}_{\geq 0}a^r$, denoted by $\dim(\sigma)$, as the dimension of the real vector subspace spanned by $a^1, \dots, a^r$. We say that σ is simplicial when $\dim(\sigma) = r$.

Let us consider the equivalence relation in $\mathbb{R}^n$ defined as follows. If $u, v \in \mathbb{R}^n$, then $u \sim v$ if and only if $\Delta(u, \tilde{\Gamma}_+) = \Delta(v, \tilde{\Gamma}_+)$. Obviously the corresponding quotient space $X = \mathbb{R}^n / \sim$ is bijective with the set of faces of $\tilde{\Gamma}_+$.

If Δ is a face of $\tilde{\Gamma}_+$, then we denote by $[\Delta]$ the closure, in the euclidean sense, of the set of vectors supporting Δ . Hence $[\Delta]$ is equal to a cone $\mathbb{R}_{\geq 0}a^1 + \dots + \mathbb{R}_{\geq 0}a^r$, for some primitive vectors $a^1, \dots, a^r \in \mathbb{Z}^n$. In particular, if Δ has dimension $n - 1$ and $\Delta = \Delta(w, \tilde{\Gamma}_+)$, for some $w \in \mathbb{Z}^n \setminus \{0\}$, then $[\Delta] = \mathbb{R}_{\geq 0}w$. It is immediate to see that $\dim[\Delta] = n - \dim \Delta$, for each face Δ of $\tilde{\Gamma}_+$.

Given a cone $\sigma = \mathbb{R}_{\geq 0}a^1 + \cdots + \mathbb{R}_{\tilde{\Gamma}_+0}a^r \subseteq \mathbb{R}^n$, by Caratheodory's theorem (see [29, p. 139]), we have that σ can be expressed as the union of cones $\{\sigma_1, \dots, \sigma_m\}$ of $\mathbb{R}^n$ such that

- (i) $\sigma_i \cap \sigma_j$ is a face of σ_i and of σ_j , for all $i, j \in \{1, \dots, m\}$,
- (ii) each cone σ_i is written as $\mathbb{R}_{\geq 0}a^{i_1} + \cdots + \mathbb{R}_{\geq 0}a^{i_s}$, where $1 \leq i_1 < \cdots < i_s \leq r$ and $\{a^{i_1}, \dots, a^{i_s}\}$ is linearly independent.

This fact also follows from [29, p. 147, Theorem 1.12]. Then we can decompose σ as the union of simplicial cones with generators contained in $\{a^1, \dots, a^r\}$. We call such a decomposition a *simplicial subdivision* of σ . Let us fix a simplicial subdivision of each n -dimensional cone $[\Delta]$, where Δ denotes a vertex of $\tilde{\Gamma}_+$. Then we denote by $\Sigma^{(n)}$ the set of simplicial cones of dimension n arising from the fixed simplicial subdivisions of $[\Delta]$, for any vertex Δ of $\tilde{\Gamma}_+$. Attached to each $\sigma \in \Sigma^n$, we will consider the primitive vectors $a^1(\sigma), \dots, a^n(\sigma)$ arising from the simplicial structure of σ . We choose an order of these vectors that will be fixed throughout this section. Moreover, we write $a^i(\sigma) = (a_1^i(\sigma), \dots, a_n^i(\sigma))$, for $i = 1, \dots, n$.

For each $\sigma \in \Sigma^n$, let us consider a copy of $\mathbb{K}^n$ which we will denote by $\mathbb{K}^n(\sigma)$. We will write the elements of $\mathbb{K}^n(\sigma)$ as $y_\sigma = (y_{\sigma,1}, \dots, y_{\sigma,n})$. Let $W_\sigma = \{y_\sigma \in \mathbb{K}^n(\sigma) : 0 < |y_{\sigma,j}| \leq 1, \text{ for all } j = 1, \dots, n\}$, for all $\sigma \in \Sigma^{(n)}$, and let $V = \{x \in (\mathbb{K} \setminus \{0\})^n : \max_i |x_i| \geq 1\}$. Let us consider, for each $\sigma \in \Sigma^{(n)}$, the monomial map $\pi_\sigma : W_\sigma \rightarrow (\mathbb{K} \setminus \{0\})^n$ given by

$$\pi_\sigma(y_{\sigma,1}, \dots, y_{\sigma,n}) = (y_{\sigma,1}^{a_1^1(\sigma)} \cdots y_{\sigma,n}^{a_1^n(\sigma)}, \dots, y_{\sigma,1}^{a_n^1(\sigma)} \cdots y_{\sigma,n}^{a_n^n(\sigma)}),$$

where we suppose that $a^1(\sigma), \dots, a^n(\sigma)$ are the primitive generators of σ .

Lemma 4.2.1. *Let W be the disjoint union of all sets W_σ , where $\sigma \in \Sigma^{(n)}$. Let $\pi : W \rightarrow (\mathbb{K} \setminus \{0\})^n$ be the map defined by $\pi(y_\sigma) = \pi_\sigma(y_\sigma)$, for all $y_\sigma \in W_\sigma$, $\sigma \in \Sigma^{(n)}$. Then the restriction $\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \rightarrow V$ is surjective.*

Proof. We will develop the proof in the case $\mathbb{K} = \mathbb{C}$. The case $\mathbb{K} = \mathbb{R}$ is analogous. Let $x = (x_1, \dots, x_n) \in V$. Let us write x_j as $x_j = r_j e^{2\pi i \alpha_j}$, where $r_j \in [0, +\infty[$, $\alpha_j \in [0, 1[$, for all $j = 1, \dots, n$ and $\mathbf{i} = \sqrt{-1}$. Let us define the vector $v(x) = (-\log r_1, \dots, -\log r_n) \in \mathbb{R}^n$.

Since Σ^n is a partition of $\mathbb{R}^n$, there exists a n -dimensional cone $\sigma \in \Sigma^n$ such that $v(x) \in \sigma$. Let $\{a^1(\sigma), \dots, a^n(\sigma)\}$ be a linearly independent set of vectors of $\mathbb{R}^n$ such that $\sigma = \mathbb{R}_{\geq 0}a^1(\sigma) + \cdots + \mathbb{R}_{\geq 0}a^n(\sigma)$. Thus, there are $\beta_1, \dots, \beta_n \geq 0$ such that $v(x) = \beta_1 a^1(\sigma) + \cdots + \beta_n a^n(\sigma)$.

Let $r_{\sigma,j} = e^{-\beta_j} \leq 1$, for all $j = 1, \dots, n$. We observe that

$$r_{\sigma,1}^{a_1^1(\sigma)} \cdots r_{\sigma,n}^{a_n^n(\sigma)} = e^{-a_j^1(\sigma)\beta_1 - \cdots - a_j^n(\sigma)\beta_n} = r_j, \quad (4.6)$$

for all $j = 1, \dots, n$. Then we consider $y_\sigma = (y_{\sigma,1}, \dots, y_{\sigma,n}) \in W_\sigma$, where $y_{\sigma,j} = r_{\sigma,j} e^{2\pi i \theta_{\sigma,j}}$, for some $\theta_{\sigma,j} \in [0, 1[$, $j = 1, \dots, n$. Using 4.6 we observe that $\pi_\sigma(y_\sigma) = x$ if and only if the vector $\theta_\sigma = (\theta_{\sigma,1}, \dots, \theta_{\sigma,n})$ verifies that

$$a_j^1(\sigma)\theta_{\sigma,1} + \dots + a_j^n(\sigma)\theta_{\sigma,n} \equiv \alpha_j \pmod{\mathbb{Z}},$$

for all $j = 1, \dots, n$. But this system has solution, because $a^1(\sigma), \dots, a^n(\sigma)$ are linearly independent over $\mathbb{R}$. Hence, there exists θ_σ such that $\pi_\sigma(y_\sigma) = x$. Moreover, we observe that $0 < r_{\sigma,j} \leq 1$, for all $j = 1, \dots, n$, then $y_\sigma \in W_\sigma$ and hence π is surjective. $\square$

Let $\sigma \in \Sigma^{(n)}$ and let $x^k = x_1^{k_1} \dots x_n^{k_n}$ a monomial, we observe that

$$x^k \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\langle k, a^1(\sigma) \rangle} \dots y_{\sigma,n}^{\langle k, a^n(\sigma) \rangle}. \quad (4.7)$$

In order to simplify the notation, if $w \in \mathcal{F}(\tilde{\Gamma}_+)$ then in this section we will denote the number $\ell(w, \tilde{\Gamma}_+)$ only by $\ell(w)$. Let us fix a cone $\sigma \in \Sigma^{(n)}$ and let $a^1(\sigma), \dots, a^n(\sigma)$ be the primitive generators of σ . Since $\{a^1(\sigma), \dots, a^n(\sigma)\} \subseteq \mathcal{F}(\tilde{\Gamma}_+)$, some vector $a^j(\sigma)$ can coincide with some vector of the canonical basis. Then, we can assume that

$$\{j : \ell(a^j(\sigma)) < 0\} = \{1, \dots, r\}, \quad (4.8)$$

$$\{j : \ell(a^j(\sigma)) = 0\} = \{r+1, \dots, r+s\}, \quad (4.9)$$

for some integers $r, s > 0$ such that $r+s = n$. Hence, if $s > 1$, there exist indices $1 \leq i_1 < \dots < i_s \leq n$ such that $a^{r+j}(\sigma) = \mathbf{e}_{i_j}$, for all $j = 1, \dots, s$.

Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. Let us fix an index $i \in \{1, \dots, p\}$. Let us suppose that F_i is written as $F_i = \sum_k a_k x^k$. Let us define

$$Z_i = \{k \in \text{supp}(F_i) : \langle k, a^j(\sigma) \rangle = \ell^*(a^j(\sigma), F_i), \text{ for some } j \in \{1, \dots, n\}\}. \quad (4.10)$$

$$Z_J = \{k \in \text{supp}(F_i) : \langle k, a^j(\sigma) \rangle = \ell^*(a^j(\sigma), F_i), \text{ if and only if } j \in J\}. \quad (4.11)$$

for all $J \in \{1, \dots, n\}$. It is clear that, if $J_1, J_2 \in \{1, \dots, n\}$ and $Z_{J_1} \cap Z_{J_2} \neq \emptyset$ then $J_1 = J_2$. Therefore Z_i is the disjoint union $\cup_J Z_J$, where J varies in the set of non-empty subsets of

$\{1, \dots, n\}$. Hence we have that

$$\begin{aligned} F_i \circ \pi_\sigma(y_\sigma) &= \left(\sum_{k \in Z_i} a_k x^k + \sum_{k \notin Z_i} a_k x^k \right) \circ \pi_\sigma(y_\sigma) \\ &= \sum_{J \subseteq \{1, \dots, n\}} \sum_{k \in Z_J} a_k y_{\sigma,1}^{\ell(a^1(\sigma), F_i)} \dots y_{\sigma,r}^{\ell(a^r(\sigma), F_i)} \left(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma)) < 0}} y_{\sigma,j}^{\langle k, a^j(\sigma) \rangle - \ell(a^j(\sigma), F_i)} \right) \left(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma)) = 0}} y_{\sigma,j}^{k_{ij}-r} \right) \\ &\quad + \sum_{k \notin Z_i} a_k y_{\sigma,1}^{\langle k, a^1(\sigma) \rangle} \dots y_{\sigma,r}^{\langle k, a^r(\sigma) \rangle} y_{\sigma,r+1}^{k_{i1}} \dots y_{\sigma,n}^{k_{is}} \end{aligned} \quad (4.12)$$

$$\begin{aligned} &= y_{\sigma,1}^{\ell(a^1(\sigma), F_i)} \dots y_{\sigma,r}^{\ell(a^r(\sigma), F_i)} \left(\sum_{J \subseteq \{1, \dots, n\}} \sum_{k \in Z_J} a_k \left(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma)) < 0}} y_{\sigma,j}^{\langle k, a^j(\sigma) \rangle - \ell(a^j(\sigma), F_i)} \right) \left(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma)) = 0}} y_{\sigma,j}^{k_{ij}-r} \right) \right. \\ &\quad \left. + \sum_{k \notin Z_i} a_k y_{\sigma,1}^{\langle k, a^1(\sigma) \rangle - \ell(a^1(\sigma), F_i)} \dots y_{\sigma,r}^{\langle k, a^r(\sigma) \rangle - \ell(a^r(\sigma), F_i)} y_{\sigma,r+1}^{k_{i1}} \dots y_{\sigma,n}^{k_{is}} \right). \end{aligned} \quad (4.13)$$

We denote by $F_{\sigma,i}^*$ the polynomial such that

$$F_i \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma), F_i)} \dots y_{\sigma,r}^{\ell(a^r(\sigma), F_i)} \cdot F_{\sigma,i}^*(y_\sigma), \quad (4.14)$$

for all $y_\sigma \in W_\sigma$. That is $F_{\sigma,i}^*$ is the polynomial in the variables $y_{\sigma,1}, \dots, y_{\sigma,n}$ given by the expression that appears in (4.12) and (4.13) into parenthesis.

If $M > 0$ then we denote by V_M the set $\{x \in (\mathbb{K} \setminus \{0\})^n : \|x\| \geq M\}$.

Proposition 4.2.2. *Let us suppose that $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ is strongly adapted to $\tilde{\Gamma}_+$. Then for all $\sigma \in \Sigma^{(n)}$ there exists a constant $M_\sigma > 0$ such that*

$$\inf_{y_\sigma \in \pi_\sigma^{-1}(V_{M_\sigma})} \sup_i |F_{\sigma,i}^*(y_\sigma)| \neq 0.$$

Proof. Let us assume the opposite. That is, let σ such that

$$\inf_{y_\sigma \in \pi_\sigma^{-1}(V_M)} \sup_i |F_{\sigma,i}^*(y_\sigma)| = 0,$$

for all $M > 0$. Then there exists a sequence $\{y_m\}_{m \geq 1} \subseteq W_\sigma$ such that $\{\pi_\sigma(y_m)\}_{m \geq 1} \rightarrow \infty$ and $\{F_{\sigma,i}^*(y_m)\}_{m \geq 1} \rightarrow 0$ as $m \rightarrow \infty$, for all $i = 1, \dots, p$.

Let $\overline{W}_\sigma$ denote the closure of W_σ , that is $\overline{W}_\sigma = \{y_\sigma \in \mathbb{K}^n(\sigma) : \|y_\sigma\| \leq 1\}$. Let $y = (y_1, \dots, y_n) \in \overline{W}_\sigma$ be a limit point of the sequence $\{y_m\}_{m \geq 1}$. Let $J_0 = \{j : y_j = 0\}$. We have that $J_0 \neq \emptyset$, since $\{\pi_\sigma(y_m)\}_{m \geq 1} \rightarrow \infty$. Moreover $F_\sigma^*(y) = 0$, for all $i = 1, \dots, p$.

Let $a^1(\sigma), \dots, a^n(\sigma)$ be the primitive generators of σ . The condition $\{\pi_\sigma(y_m)\}_{m \geq 1} \rightarrow \infty$ implies that there exists some $j \in J_0$ such that $a^j(\sigma)$ has some negative component. In

particular $\ell(a^j(\sigma)) < 0$, since $\tilde{\Gamma}_+$ is convenient. Therefore $0 \notin \Delta(a^j(\sigma), \tilde{\Gamma}_+)$ for some $j \in J_0$, which is to say that the face $\cap_{j \in J_0} \Delta(a^j(\sigma), \tilde{\Gamma}_+)$ does not contain the origin.

We observe that

$$F_{\sigma,i}^*(y) = \sum_{\substack{J \subseteq \{1, \dots, n\} \\ J_0 \subseteq J}} \sum_{k \in Z_J} a_k \left(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma)) < 0}} y_j^{\langle k, a^j(\sigma) \rangle - \ell(a^j(\sigma), F_i)} \right) \left(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma)) = 0}} y_j^{k_{i_j-r}} \right). \quad (4.15)$$

On the other hand, given any $z_\sigma = (z_{\sigma,1}, \dots, z_{\sigma,n}) \in W_\sigma$, we have

$$p_{J_0}^*(F_i) \circ \pi_\sigma(z_\sigma) = \sum_{\substack{\langle k, a^j(\sigma) \rangle = \ell^*(a^j(\sigma), F_i) \\ \text{for all } j \in J_0}} a_k \cdot z_{\sigma,1}^{\langle k, a^1(\sigma) \rangle} \dots z_{\sigma,n}^{\langle k, a^n(\sigma) \rangle} \quad (4.16)$$

$$= z_{\sigma,1}^{\ell(a^1(\sigma), F_i)} \dots z_{\sigma,r}^{\ell(a^r(\sigma), F_i)} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ J_0 \subseteq J}} \sum_{k \in Z_J} a_k \left(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma)) < 0}} z_{\sigma,j}^{\langle k, a^j(\sigma) \rangle - \ell(a^j(\sigma), F_i)} \right) \left(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma)) = 0}} z_{\sigma,j}^{k_{i_j-r}} \right) \quad (4.17)$$

Let us consider the point $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_n)$ defined by

$$\tilde{y}_j = \begin{cases} y_j, & \text{if } j \notin J_0, \\ 1, & \text{if } j \in J_0. \end{cases}$$

Comparing (4.15) and (4.17) we obtain

$$p_{J_0}^*(F_i) \circ \pi_\sigma(\tilde{y}) = \prod_{j \notin J_0} y_{\sigma,j}^{\ell(a^j(\sigma), F_i)} F_{\sigma,i}^*(y) = 0, \quad (4.18)$$

for all $i = 1, \dots, p$. Since $\cap_{j \in J_0} \Delta(a^j(\sigma), \tilde{\Gamma}_+)$ is a face of $\tilde{\Gamma}_+$ not containing the origin, the relation (4.18) gives a contradiction. $\square$

The next results are tools that allows us to give approximations to the set $S_0(F)$ and observe that the argument of their proofs are similar. As we will see in the following section, these theorems motivate a important result (see Corollary 4.3.1) that allows us to obtain an lower estimate for Łojasiewicz exponent of a polynomial map from $\mathbb{K}^n$ to $\mathbb{K}^p$ in terms of a convenient Newton polyhedron.

Theorem 4.2.3. *Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. Let us suppose that F is adapted to $\tilde{\Gamma}_+$. Let $k \in \mathbb{Z}_{\geq 0}^n$ such that*

$$\langle k, w \rangle \geq \max\{\ell(w, F_1), \dots, \ell(w, F_p)\}, \quad (4.19)$$

for all $w \in \mathcal{F}(\tilde{\Gamma}_+)$. Then $k \in S_0(F)$.

The proof of the above theorem will be given in the end of this section. Let us remark that inequality (4.19) is assumed for any $w \in \mathcal{F}(\tilde{\Gamma}_+)$ in Theorem 4.2.3. We will see that the same conclusion holds if we assume (4.19) only for the vectors $w \in \mathcal{F}_0(\tilde{\Gamma}_+)$ such that $0 \notin \Delta(w, \tilde{\Gamma}_+)$ and F is strongly adapted to $\tilde{\Gamma}_+$. This fact is shown in the following result, which is independent from Theorem 4.2.3.

Theorem 4.2.4. *Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. Let us suppose that F is strongly adapted to $\tilde{\Gamma}_+$. Let $k \in \mathbb{Z}_{\geq 0}^n$ such that*

$$\langle k, w \rangle \geq \max\{\ell(w, F_1), \dots, \ell(w, F_p)\}, \quad (4.20)$$

for all $w \in \mathcal{F}_0(\tilde{\Gamma}_+)$. Then $k \in S_0(F)$.

Proof. Let us fix a cone $\sigma \in \Sigma^{(n)}$. By Proposition 4.2.2 there exists positive constants D_σ and M_σ such that

$$D_\sigma < \sup_i |F_{\sigma,i}^*(y_\sigma)|,$$

for all $y_\sigma \in \pi_\sigma^{-1}(V_{M_\sigma})$. Then, for any $y_\sigma \in \pi_\sigma^{-1}(V_{M_\sigma})$ we have the following chain of inequalities:

$$\sup_i |F_i(x)| \circ \pi_\sigma(y_\sigma) = \sup_i |y_{\sigma,1}^{\ell(a^1(\sigma), F_i)} \dots y_{\sigma,r}^{\ell(a^r(\sigma), F_i)} \cdot F_i^*(y_\sigma)| \quad (4.21)$$

$$\geq |y_{\sigma,1}|^{\max_i \ell(a^1(\sigma), F_i)} \dots |y_{\sigma,r}|^{\max_i \ell(a^r(\sigma), F_i)} \sup_i |F_{\sigma,i}^*(y_\sigma)| \quad (4.22)$$

$$\geq |y_{\sigma,1}|^{\langle k, a^1(\sigma) \rangle} \dots |y_{\sigma,r}|^{\langle k, a^r(\sigma) \rangle} \cdot D_\sigma \quad (4.23)$$

$$\geq |y_{\sigma,1}|^{\langle k, a^1(\sigma) \rangle} \dots |y_{\sigma,r}|^{\langle k, a^r(\sigma) \rangle} |y_{\sigma,r+1}|^{k_{i_1}} \dots |y_{\sigma,n}|^{k_{i_r}} \cdot D_\sigma \quad (4.24)$$

$$= D_\sigma |x^k| \circ \pi_\sigma(y_\sigma). \quad (4.25)$$

Let $M = \max_{\sigma \in \Sigma^{(n)}} D_\sigma$. We can assume $\sqrt{n} \leq M$. Then V contains V_M and by Lemma 4.2.1, we have that $\{\pi_\sigma(y_\sigma) : y_\sigma \in \pi_\sigma^{-1}(V_M), \sigma \in \Sigma^{(n)}\} = V_M$. In particular, if $C = (\min_{\sigma \in \Sigma^{(n)}} D_\sigma)^{-1}$ we conclude that

$$|x^k| \leq C \sup_i |F_i(x)|, \quad (4.26)$$

for all $x \in (\mathbb{K} \setminus \{0\})^n$ such that $\|x\| \geq M$. By continuity we obtain that (4.26) holds for all $x \in \mathbb{K}^n$ such that $\|x\| \geq M$. $\square$

Proof of Theorem 4.2.3. Let us modify the definitions of Z_i and of Z_J , in (4.10) and (4.11) respectively, by replacing $\ell^*(a^j(\sigma), F_i)$ by $\ell(a^j(\sigma), F_i)$. Then we obtain, as in (4.14), a polynomial $F'_{\sigma,i} \in \mathbb{K}[y_{\sigma,1}, \dots, y_{\sigma,n}]$ such that

$$F_i \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma), F_i)} \dots y_{\sigma,j}^{\ell(a^n(\sigma), F_i)} F'_{\sigma,i}(y_\sigma),$$

for all $y_\sigma \in W_\sigma$ and all $i = 1, \dots, p$. Following the proof Proposition 4.2.1, we obtain that for each $\sigma \in \Sigma^{(n)}$ there exists a constant $M_\sigma > 0$ such that

$$\inf_{y_\sigma \in \pi_\sigma^{-1}(V_{M_\sigma})} \sup_i |F'_{\sigma,i}(y_\sigma)| \neq 0.$$

Hence we can reproduce the argument of the proof of Theorem 4.2.4 to obtain that $k \in S_0(F)$. $\square$

Example 4.2.5. Let us consider the polynomial map $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, where F is given by $F(x_1, x_2) = (x_1^{a_1} + x_2^{b_1} + x_1^m x_2^n, x_1^a + x_2^b)$, where $a_1 < a$, $b_1 < b$ and $mb + na > ab$. Let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(F) \subseteq \mathbb{R}_{\geq 0}^2$, then $\tilde{\Gamma}_+$ is a convenient Newton polyhedron since that F is convenient. Let us consider w_1, w_2 the primitive vectors in $\mathbb{R}_0^2$, such that $\mathcal{F}(\tilde{\Gamma}_+) = \{\mathbf{e}_1, \mathbf{e}_2, w_1, w_2\}$.

It is straightforward to check that F is strongly adapted to $\tilde{\Gamma}_+$ and a direct computation shows that $\ell(w_1, F_1) = \ell(w_1, \tilde{\Gamma}_+)$, $\ell(w_2, F_1) = \ell(w_2, \tilde{\Gamma}_+)$, $\ell(w_1, F_2) = \langle w_1, (0, b_1) \rangle$ and $\ell(w_2, F_2) = \langle w_2, (0, a_1) \rangle$. Then by Theorem 4.2.4, we have $\{k \in \mathbb{Z}_{\geq 0}^2 : \langle k, w_1 \rangle \geq \langle w_1, (0, b_1) \rangle \text{ and } \langle k, w_2 \rangle \geq \langle w_2, (a_1, 0) \rangle\} \subseteq S_0(F)$.

The Figure 4.2 shows the Newton polyhedron of $\tilde{\Gamma}_+(F)$ and the region where the support of a polynomial h must be contained to be special with respect to F .

Figure 4.2:

Corollary 4.2.6. Let $F = (F_1, \dots, F_n) : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be a polynomial map. Let us suppose that F is non-degenerate with respect to $\tilde{\Gamma}_+$. Let $\phi : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}^n$ be the Newton filtration associated to $\tilde{\Gamma}_+$ and let d_i be the degree of F_i with respect to $\tilde{\Gamma}_+$, for all $i = 1, \dots, n$. Then

$$\{k \in \mathbb{Z}_{\geq 0}^n : \phi(k) \leq d_0\} \subseteq S_0(F),$$

where $d_0 = \min\{d_1, \dots, d_n\}$.

Proof. The result follows as a direct consequence of Proposition 4.1.13, Theorem 4.2.4 and relation 4.5. $\square$

4.3 Estimation of Łojasiewicz exponents at infinity

Let us fix along this section a convenient global Newton polyhedron $\tilde{\Gamma}_+ \subseteq \mathbb{R}^n$ and a polynomial map $F = (F_1, \dots, F_p) : \mathbb{K}^n \rightarrow \mathbb{K}^p$. Let us define $\mathbf{L}(w, F) = \max\{\ell(w, F_1), \dots, \ell(w, F_p)\}$, for any vector $w \in \mathbb{R}^n$.

Corollary 4.3.1. *Let us suppose that F is strongly adapted to $\tilde{\Gamma}_+$. Let $k \in \mathbb{Z}_{\geq 0}^n$ and $\theta \geq 0$ such that $\theta \langle k, w \rangle \geq \mathbf{L}(w, F)$, for all $w \in \mathcal{F}_0(\tilde{\Gamma}_+)$. Then there exist positive constants C and M such that*

$$|x^k|^\theta \leq C \|F(x)\|,$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geq M$.

Proof. It follows by the same argument of the proof of Theorem 4.2.4 by replacing inequalities (4.21)-(4.25) by the following inequalities:

$$\sup_i |F_i(x)| \circ \pi_\sigma(y_\sigma) = \sup_i |y_{\sigma,1}^{\ell(a^1(\sigma), F_i)} \dots y_{\sigma,r}^{\ell(a^r(\sigma), F_i)} \cdot F_i^*(y_\sigma)| \quad (4.27)$$

$$\geq |y_{\sigma,1}|^{\max_i \ell(a^1(\sigma), F_i)} \dots |y_{\sigma,m}|^{\max_i \ell(a^1(\sigma), F_i)} \sup_i |F_{\sigma,i}^*(y_\sigma)| \quad (4.28)$$

$$\geq |y_{\sigma,1}|^{\theta \langle k, a^1(\sigma) \rangle} \dots |y_{\sigma,r}|^{\theta \langle k, a^r(\sigma) \rangle} \cdot D_\sigma \quad (4.29)$$

$$\geq |y_{\sigma,1}|^{\theta \langle k, a^1(\sigma) \rangle} \dots |y_{\sigma,r}|^{\theta \langle k, a^r(\sigma) \rangle} |y_{\sigma,r+1}|^{\theta k_{i_1}} \dots |y_{\sigma,n}|^{\theta k_{i_s}} \cdot D_\sigma \quad (4.30)$$

$$= D_\sigma |x^k| \circ \pi_\sigma(y_\sigma). \quad (4.31)$$

Let us remark that the condition $\theta \geq 0$ is used to obtain (4.30), since we assume $0 < |y_{\sigma,i}| \leq 1$, for all $i = 1, \dots, n$ and all $\sigma \in \Sigma^{(n)}$. $\square$

Let us fix an index $i \in \{1, \dots, n\}$. Then we define

$$E_i(F, \tilde{\Gamma}_+) = \{\theta \geq 0 : \theta w_i \geq \mathbf{L}(w, F) \text{ for all } w \in \mathcal{F}_0(\tilde{\Gamma}_+)\}. \quad (4.32)$$

Let us decompose $\mathcal{F}_0(\tilde{\Gamma}_+)$ as $\mathcal{F}_0(\tilde{\Gamma}_+) = A_{i,-} \cup A_{i,0} \cup A_{i,+}$ where $A_{i,-} = \{w \in \mathcal{F}_0(\tilde{\Gamma}_+) : w_i < 0\}$, $A_{i,0} = \{w \in \mathcal{F}_0(\tilde{\Gamma}_+) : w_i = 0\}$ and $A_{i,+} = \{w \in \mathcal{F}_0(\tilde{\Gamma}_+) : w_i > 0\}$. We define

$$a_i(F, \tilde{\Gamma}_+) = \max_{w \in A_{i,+}} \frac{\mathbf{L}(w, F)}{w_i} \quad b_i(F, \tilde{\Gamma}_+) = \min_{w \in A_{i,-}} \frac{\mathbf{L}(w, F)}{w_i}. \quad (4.33)$$

In the remaining section we also denote the numbers defined above by a_i and b_i , respectively, for all $i = 1, \dots, n$. Let us remark that $E_i(F, \tilde{\Gamma}_+) = [0, b_i]$, whenever $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$.

Lemma 4.3.2. *The following conditions are equivalent:*

- (i) $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$.
- (ii) $\ell(w, F_j) \leq 0$, for all $w \in \mathcal{F}_0(\tilde{\Gamma}_+)$ and all $j = 1, \dots, p$.

Proof. Let us fix an index $i \in \{1, \dots, n\}$. By (4.32) and (4.33) we obtain that $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$ if and only if

$$a_i \leq b_i, \quad 0 \leq b_i \quad \text{and} \quad \ell(w, F_j) \leq 0, \quad \text{for all } j = 1, \dots, p \quad \text{and all } w \in A_{i,0}. \quad (4.34)$$

Let us assume (i). If $w \in \mathcal{F}_0(\tilde{\Gamma}_+)$, then $w_0 < 0$. In particular $w_0 = w_i < 0$, for some $i \in \{1, \dots, n\}$. This implies $w \in A_{i,-}$. Since $b_i \geq 0$, we conclude $\mathbf{L}(w, F) \leq 0$, and then (ii) follows. The converse is obvious using (4.34). $\square$

In particular, if F_j is convenient, for all $j = 1, \dots, p$ then $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$.

Corollary 4.3.3. *Let us suppose that $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ is a polynomial map strongly adapted to $\tilde{\Gamma}_+$ and $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$. Then*

$$\min \left\{ \frac{\mathbf{L}(w, F)}{w_0} : w \in \mathcal{F}_0(\tilde{\Gamma}_+) \right\} \leq \mathcal{L}_\infty(F). \quad (4.35)$$

Proof. Since F is strongly adapted to $\tilde{\Gamma}_+$ and $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$, we can apply Corollary 4.3.1 to each monomial x_i , $i = 1, \dots, n$, to obtain that there exist constants C , $M > 0$ such that

$$|x_i|^{b_i} \leq C \|F(x)\|, \quad (4.36)$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geq M$. Let $b_0 = \min\{b_1, \dots, b_n\}$. We can assume $M \geq \sqrt{n}$. If $\|x\| \geq M$ then $\sqrt{n} \leq M \leq \|x\| \leq \sqrt{n} \max_i |x_i|$. In particular $1 \leq \max_i |x_i|$ and then (4.36) implies that

$$\max |x_i|^{b_0} \leq C \|F(x)\|,$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geq M$. Therefore $b_0 \leq \mathcal{L}_\infty(F)$. Let us observe that

$$b_0 = \min_i \min_{w \in A_{i,-}} \frac{\mathbf{L}(w, F)}{w_i} = \min_{w \in \mathcal{F}_0(\tilde{\Gamma}_+)} \frac{\mathbf{L}(w, F)}{w_0}.$$

Hence (4.35) follows. $\square$

In particular, by Lemma 4.3.2, the conclusion of the above result follows if we replace the condition $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$, by the condition that F_j is convenient, for all $j = 1, \dots, p$. Let us denote the left hand side of (4.35) by $b(F, \tilde{\Gamma}_+)$. Let us remark that $b(F, \tilde{\Gamma}_+) = \min_i b_i(F, \tilde{\Gamma}_+)$. We remark that Corollary 4.3.3 applies both to real and complex maps and is analogous to [8, Theorem 5.9], which applies only to real polynomial maps. If $h \in \mathbb{K}[x_1, \dots, x_n]$ and h is convenient, then we denote by $r_i(h)$ the number $r_i(\tilde{\Gamma}_+(h))$, as defined in (2.6), for all $i = 1, \dots, n$.

Corollary 4.3.4. *Let us suppose that F is non-degenerate and F_j is convenient for all $j = 1, \dots, p$. Then*

$$\min_{i,j} r_i(F_j) \leq \mathcal{L}_\infty(F) \leq r_0(\tilde{\Gamma}_+(F)). \quad (4.37)$$

Proof. The right hand side of (4.37) follows by [8, Lemma 3.3] (the proof of this result is given for $\mathbb{K} = \mathbb{R}$ but it also applies to the case $\mathbb{K} = \mathbb{C}$). By Proposition 4.1.12, the map F is strongly adapted to the polyhedron $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(F_1) + \dots + \tilde{\Gamma}_+(F_p)$. By Lemma 2.1.6 we have

$$r_i(F_j) = \min_{w \in \mathbb{R}_0^n(i)} \frac{\ell(w, F_j)}{w_i}$$

for all $j = 1, \dots, p$, $i = 1, \dots, n$. Then

$$\min \{r_1(F_s), \dots, r_n(F_s)\} = \min_{\substack{w \in \mathbb{R}_0^n \\ j=1, \dots, p}} \frac{\ell(w, F_j)}{w_0} \leq \min \left\{ \frac{\mathbf{L}(w, F)}{w_0} : w \in \mathcal{F}_0(\tilde{\Gamma}_+) \right\} \leq \mathcal{L}_\infty(F),$$

where the last inequality comes from Corollary 4.3.3. $\square$

Next, we state the theorem of Hadamard that, in conjunction with our results on the estimation of Łojasiewicz exponents at infinity, will lead us to formulate results on the bijectivity of polynomial maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Theorem 4.3.5 (Hadamard's Theorem [76, p. 240]). *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 map. Then F is a homeomorphism if and only if F is a local homeomorphism and F is proper.*

Corollary 4.3.6. *Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be a polynomial map such that F is a local homeomorphism. Let us suppose that F is strongly adapted to $\tilde{\Gamma}_+$ and $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$. Then F is a homeomorphism.*

Proof. The hypothesis imply that $\mathcal{L}_\infty(F)$ exists and $\mathcal{L}_\infty(F) > 0$. Then F is a proper map and therefore, by Hadamard's theorem, F is a homeomorphism. $\square$

Remark 4.3.7. Let us observe that, in the previous result, we do not assume that each component function of F is convenient. Corollary 4.3.4 is proven in [8, Theorem 3.8] only for real polynomial maps. We remark that, under the conditions of Corollary 4.3.4, if we assume that F is a local homeomorphism, then the same proof of the previous result works to deduce that F is a global homeomorphism.

4.4 Łojasiewicz exponent at infinity of pre-weighted homogeneous maps

In this section we expose some results concerning in the estimate of Łojasiewicz exponent of pre-weighted homogeneous maps (see Definition 2.3.5) from $\mathbb{K}^n \rightarrow \mathbb{K}^p$, which is part of the motivation of our study.

Let $v = (v_1, \dots, v_n) \in \mathbb{Z}_{\geq 1}^n$ be a primitive vector. Then we recall that denote by $\tilde{\Gamma}_+^v$ the global Newton polyhedron given by the convex hull of $\{\frac{v_1 \cdots v_n}{v_1} \mathbf{e}_1, \dots, \frac{v_1 \cdots v_n}{v_n} \mathbf{e}_n\} \cup \{0\}$. Then $\tilde{\Gamma}_+^v$ has a unique face Δ of dimension $n - 1$, which is contained in the hyperplane $v_1 x_1 + \cdots + v_n x_n = v_1 \cdots v_n$ and hence $\mathcal{F}(\tilde{\Gamma}_+^v) = \{-v, \mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\mathcal{F}_0(\tilde{\Gamma}_+^v) = \{-v\}$. Let us recall that, if I is a non-empty subset of $\{1, \dots, n\}$, then $\pi_I : \mathbb{R}^n \rightarrow \mathbb{R}_I^n$ denotes the natural projection.

Lemma 4.4.1. *Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map and let $v \in \mathbb{Z}_{\geq 1}^n$. Then the following conditions are equivalent:*

- (i) F is pre-weighted homogeneous with respect to v .
- (ii) F is strongly adapted to $\tilde{\Gamma}_+^v$.

Proof. Let us see (i) $\Rightarrow$ (ii). Let $w = -v$. Let Δ be a face of $\tilde{\Gamma}_+^v$ such that $0 \notin \Delta$. If Δ is not contained in any coordinate subspace $\mathbb{R}_I^n$, for some proper subset $I \subseteq \{1, \dots, n\}$, then $\Delta = \Delta(w, \tilde{\Gamma}_+^v)$ and $\mathcal{J}(\Delta) = \{\{w\}\}$. Hence condition $(C_{\{w\}, F})$ follows, since $p_w(F) = q_v(F)$ and $q_v(F)^{-1}(0) = \{0\}$.

If $\Delta \subseteq \mathbb{R}_I^n$, for some proper subset $I \subseteq \{1, \dots, n\}$ such that $\dim \Delta = |I| - 1$, then $\mathcal{J}(\Delta) = \{J\}$, where $J = \{w, \mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_s}\}$ and $\{i_1, \dots, i_s\} = \{1, \dots, n\} \setminus I$, $s = n - |I|$. Then we observe that $p_J^*(F) = p_{\pi_I(w)}(F^I) = q_{\pi_I(v)}(F^I)$. The map $q_{\pi_I(v)}(F^I)$ is weighted homogeneous with respect to $\pi_I(v)$. Therefore the condition $q_v(F)^{-1}(0) = \{0\}$ implies $q_{\pi_I(v)}(F^I)^{-1}(0) = \{0\}$ and hence $p_J(F)^{-1}(0) = \{0\}$. Thus $(C_{F, J}^*)$ holds and (ii) follows, by Definition 4.1.7.

Let us see (ii) $\Rightarrow$ (i). Since $\mathcal{F}_0(\tilde{\Gamma}_+^v) = \{-v\}$ it is clear that $E_i(q_v(F), \tilde{\Gamma}_+^v) \neq \emptyset$, for all $i = 1, \dots, n$ (see Lemma 4.3.2). By Definition 4.1.5 we observe that F is strongly adapted

to $\tilde{\Gamma}_+$ if and only if $q_v(F)$ is strongly adapted to $\tilde{\Gamma}_+^v$. Then we can apply Corollary 4.3.3 to $q_v(F)$ to deduce that there exists constants $C, M > 0$ such that

$$\|x\|^\alpha \leq C\|F(x)\|,$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geq M$. In particular $q_v(F)^{-1}(0)$ is contained in the open ball $B(0; M)$ centered at 0 and of radius M . But this implies $q_v(F)^{-1}(0) = \{0\}$ since $q_v(F)$ is weighted homogeneous with respect to v . Thus F is pre-weighted homogeneous. $\square$

If $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, then we define $A(v) = \{j : v_j = \max_j v_j\}$.

Proposition 4.4.2. *Let $F : \mathbb{K}^n \rightarrow \mathbb{K}^p$ be a polynomial map. Let $v \in \mathbb{Z}_{\geq 1}^n$ such that F is pre-weighted homogeneous with respect to v . Then*

$$\frac{\min\{d_v(F_1), \dots, d_v(F_p)\}}{\max\{v_1, \dots, v_n\}} \leq \mathcal{L}_\infty(F). \quad (4.38)$$

Let us assume that F is weighted homogeneous with respect to v and $F^{-1}(0) = \{0\}$. Let $i_0 \in \{1, \dots, p\}$ such that $\min\{d_v(F_1), \dots, d_v(F_p)\} = d_v(F_{i_0})$. If

$$\{x \in \mathbb{K}^n : F_i(x) = 0, \text{ for all } i \neq i_0\} \not\subseteq \{x \in \mathbb{K}^n : x_j = 0, \text{ for all } j \in A(v)\}, \quad (4.39)$$

then equality holds in (4.38).

Proof. By Lemma 4.4.1, the map F is strongly adapted to $\tilde{\Gamma}_+^v$. Then (4.38) follows from Corollary 4.3.3, since $\mathcal{F}_0(\tilde{\Gamma}_+^v) = \{v\}$. Let us see the second part. In order to simplify the notation, let us assume $i_0 = 1$. Then the quotient on the left hand side of (4.38) is equal to $d_v(F_1)/\max_i v_i$. By (4.39) and the hypothesis $F^{-1}(0) = \{0\}$, there exists a point $a = (a_1, \dots, a_n) \in \mathbb{K}^n$ such that $F_2(a) = \dots = F_p(a) = 0$, $F_1(a) \neq 0$ and $a_j \neq 0$, for some $j \in A(v)$. In particular the curve $\gamma : \mathbb{K} \setminus \{0\} \rightarrow \mathbb{K}^n$ defined by $\gamma(t) = (a_1 t^{-v_1}, \dots, a_n t^{-v_n})$ is not the zero curve. We observe that $\text{ord}(\gamma) = -(\max_i v_i)$. Moreover, since we assume that F is weighted homogeneous with respect to v , we have

$$F(\gamma(t)) = (t^{-d_v(F_1)} F_1(a), t^{-d_v(F_2)} F_2(a), \dots, t^{-d_v(F_p)} F_p(a)) = (t^{-d_v(F_1)} F_1(a), 0, \dots, 0).$$

This shows that $F \circ \gamma$ is not the zero curve. Let $\beta > d_v(F_1)/\max_i v_i$. Then we have

$$\lim_{t \rightarrow 0} \frac{\|F(\gamma(t))\|}{\|\gamma(t)\|^\beta} = \lim_{t \rightarrow 0} \frac{\|(t^{-d_v(F_1)} F_1(a), 0, \dots, 0)\|}{\|\gamma(t)\|^\beta} = 0, \quad (4.40)$$

where the last equality follows from

$$\text{ord}(\|\gamma(t)\|^\beta) = -(\max_i v_i)\beta < -d_v(F_1) = \text{ord}(F(\gamma(t))).$$

In particular $\mathcal{L}_\infty(F) \leq \beta$ (otherwise the limit (4.40) would be greater than or equal to some positive constant). Therefore $\mathcal{L}_\infty(F) \leq d_v(F_1)/\max_i v_i$ and the result follows. $\square$

Example 4.4.3. Let $F = (F_1, F_2, F_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ be the polynomial map whose component functions are given by

$$\begin{aligned} F_1(x_1, x_2, x_3) &= x_1^5 + x_1^3 x_2^3 + x_3^5, \\ F_2(x_1, x_2, x_3) &= x_2^9 + x_1^2 x_2^3 x_3^2 - x_3^6, \\ F_3(x_1, x_2, x_3) &= x_1^5 x_2^3. \end{aligned}$$

Let v be the primitive vector given by $v = (3, 2, 3)$. It is easy to observe that F is weighted-homogeneous with respect to v and $d_v(F_1) = 15$, $d_v(F_2) = 18$ and $d_v(F_3) = 21$. Moreover, $A(v) = \{1, 3\}$ and $d_v(F_1) = \min\{d_v(F_1), d_v(F_2), d_v(F_3)\}$. It is easy to see that

$$\{x \in \mathbb{C}^3 : F_2(x) = F_3(x) = 0\} \not\subseteq \{x \in \mathbb{C}^3 : x_1 = x_3 = 0\},$$

since $(0, 1, 1) \in \{x \in \mathbb{C}^3 : F_2(x) = F_3(x) = 0\} \setminus \{x \in \mathbb{C}^3 : x_1 = x_3 = 0\}$. Then by Proposition 4.4.2, we have $\mathcal{L}_\infty(F) = 5$.

Example 4.4.4. Let $v = (v_1, v_2, v_3)$ be a primitive vector in $\mathbb{Z}_{\geq 1}^3$, such that $1 < v_1 < v_2 < v_3$. Let $F = (F_1, F_2, F_3, F_4) : \mathbb{C}^3 \rightarrow \mathbb{C}^4$ be the polynomial map, whose component functions are given by

$$\begin{aligned} F_1(x_1, x_2, x_3) &= x_1^{3v_2} + x_2^{3v_1} + x_1^{v_2} x_2^{2v_1} + x_1^{v_2}, \\ F_2(x_1, x_2, x_3) &= x_2^{5v_1 v_3} + x_1^{2v_2 v_3} x_3^{3v_1 v_2} - x_2^{v_1 v_3} x_3^{4v_1 v_2} - x_1^{v_2} x_3^{v_2}, \\ F_3(x_1, x_2, x_3) &= x_3^{7v_1 v_2} - x_1^{2v_2 v_3} x_2^{5v_1 v_3} - x_2^{7v_1 v_3} + x_1^{3v_2 v_3} x_3^{2v_1 v_2}, \\ F_4(x_1, x_2, x_3) &= x_1 x_3^{8v_1 v_2} - x_1^{v_2} + x_2^{2v_1}. \end{aligned}$$

It is easy to observe that F is pre-weighted homogeneous with respect to v with $d_v(F_1) = 3v_1 v_2$, $d_v(F_2) = 5v_1 v_2 v_3$, $d_v(F_3) = 7v_1 v_2 v_3$ and $d_v(F_4) = v_1 + 8v_1 v_2 v_3$. Moreover, $A(v) = \{3\}$ and $d_v(F_1) = \min\{d_v(F_1), d_v(F_2), d_v(F_3), d_v(F_4)\}$. It is immediate to see that

$$\{x \in \mathbb{C}^3 : q_v(F_2)(x) = q_v(F_3)(x) = q_v(F_4)(x) = 0\} \not\subseteq \{x \in \mathbb{C}^3 : x_3 = 0\},$$

since $(0, 1, 1) \in \{x \in \mathbb{C}^3 : q_v(F_2)(x) = q_v(F_3)(x) = q_v(F_4)(x) = 0\} \setminus \{x \in \mathbb{C}^3 : x_3 = 0\}$. Then by Proposition 4.4.2, we obtain that $3v_1 v_2 / v_3 = \mathcal{L}_\infty(q_v(F))$ and $3v_1 v_2 / v_3 \leq \mathcal{L}_\infty(F)$. Moreover, we claim that $\mathcal{L}_\infty(F) = 3v_1 v_2 / v_3$. Indeed, the meromorphic curve $\gamma : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}^3$

defined by $\gamma(t) = (0, t^{-v_2}, t^{-v_3})$ is not the zero, with $\text{ord}(\gamma) = -v_3$. A direct computation shows that $F \circ \gamma(t) = (t^{-3v_1v_2}, 0, 0, t^{-2v_1v_2})$ is not the zero curve. Let $\beta > 3v_1v_2/v_3$. Then we have

$$\lim_{t \rightarrow 0} \frac{\|F(\gamma(t))\|}{\|\gamma(t)\|^\beta} = \lim_{t \rightarrow 0} \frac{\|(t^{-3v_1v_2}, 0, 0, t^{-2v_1v_2})\|}{\|(0, t^{-v_2}, t^{-v_3})\|^\beta} = 0 \quad (4.41)$$

where the last equality follows from $\beta > 3v_1v_2/v_3$. In particular $\mathcal{L}_\infty(F) \leq 3v_1v_2/v_3$. Therefore $\mathcal{L}_\infty(F) = 3v_1v_2/v_3$.

4.5 Index of polynomial maps

In this section we show a result concerning the index of real polynomial vector fields. This result will follow as a consequence of the argument of the proof of Theorem 4.2.4. If $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a real polynomial map such that $F^{-1}(0)$ is finite, then we denote by $\text{ind}(F)$ the *index* of F , that is

$$\text{ind}(F) = \sum_{x \in F^{-1}(0)} \text{ind}_x(F),$$

where $\text{ind}_x(F)$ denotes the topological index of F at x .

The next theorem is motivated by results obtained by following authors Cima-Gasull-Mañosas [18], Gutitrrez-Ruas [36] and Bivi-Ausina [5].

Theorem 4.5.1. *Let $F, G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be polynomial maps such that F and $F + G$ have a finite number of zeros. Let us assume that*

- (i) F is strongly adapted to $\tilde{\Gamma}_+$;
- (ii) $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$;
- (iii) $\ell(w, G_i) > \ell(w, F_i)$, for all $w \in \mathcal{F}_0(\tilde{\Gamma}_+)$, $i = 1, \dots, n$.

Then

$$\text{ind}(F) = \text{ind}(F + G).$$

Proof. Let us consider the homotopy $H : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $H(t, x) = F(x) + tG(x)$. Let $H_t = (H_{t,1}, \dots, H_{t,n}) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be given by $H_t(x) = H(t, x)$, for all $x \in \mathbb{R}^n$ and all $t \in [0, 1]$. We claim that there exists a uniform Łojasiewicz inequality at infinity for the family of maps $\{H_t\}_{t \in [0,1]}$. That is, there exist some constants $M, \alpha > 0$ such that

$$\|x\|^\alpha \leq C \|H_t(x)\|, \quad (4.42)$$

for all pair $(t, x) \in [0, 1] \times \mathbb{R}^n$ such that $\|x\| \geq M$. As a consequence we would obtain that $H_t^{-1}(0)$ is contained in the open ball $B(0; M)$, for all $t \in [0, 1]$. In particular $H(t, x) \neq 0$ for all $(t, x) \in [0, 1] \times \partial B(0; M)$, where $\partial B(0; M)$ denotes the boundary of $B(0; M)$. This fact implies $\text{ind}(F) = \text{ind}(F + G)$, as a consequence of a known result about the invariance of the index by homotopies Theorem 1.4.7.

From $\tilde{\Gamma}_+$ we can construct a subdivision of $\mathbb{R}^n$ into simplicial cones as explained in the proof of Theorem 4.2.4. Let us keep the notation introduced in Subsection 4.2, before Lemma 4.2.1. Let us fix a cone $\sigma \in \Sigma^{(n)}$. Let $a^1(\sigma), \dots, a^n(\sigma)$ be the primitive generators of σ . Let us consider the decomposition of $\{a^1(\sigma), \dots, a^n(\sigma)\}$ as in (4.8) and (4.9).

Analogous to (4.14) we can consider, for each $i = 1, \dots, n$ and each $t \in [0, 1]$, the polynomial $H_{\sigma, t, i}^* \in \mathbb{K}[y_{\sigma, 1}, \dots, y_{\sigma, n}]$ such that

$$H_{t, i} \circ \pi_{\sigma}(y_{\sigma}) = y_{\sigma, 1}^{\ell(a^1(\sigma), H_{t, i})} \dots y_{\sigma, r}^{\ell(a^r(\sigma), H_{t, i})} \cdot H_{\sigma, t, i}^*(y_{\sigma}),$$

for all $y_{\sigma} \in W_{\sigma}$.

By hypothesis we have $\ell(w, G_i) > \ell(w, F_i)$, for all $w \in \mathcal{F}_0(\tilde{\Gamma}_+)$, $i = 1, \dots, n$. This implies $\ell(a^j(\sigma), H_{t, i}) = \ell(a^j(\sigma), F_i)$, for all $i = 1, \dots, n$, $j = 1, \dots, r$. Hence the principal parts of F_i and of $H_{t, i}$ with respect to any subset of $\mathcal{F}_0(\tilde{\Gamma}_+)$ coincide, for all $i = 1, \dots, n$ (see Definition 4.1.1) and

$$E_i(H_t, \tilde{\Gamma}_+) = E_i(F, \tilde{\Gamma}_+) \neq \emptyset, \quad (4.43)$$

for all $i = 1, \dots, n$ and all $t \in [0, 1]$ (the sets $E_i(F, \tilde{\Gamma}_+)$ are non-empty by hypothesis).

Let us see that there exists some constant $M > 0$ such that

$$\inf_{\substack{y_{\sigma} \in \pi_{\sigma}^{-1}(V_M) \\ t \in [0, 1]}} \sup_i |H_{\sigma, t, i}^*(y_{\sigma})| \neq 0 \quad (4.44)$$

for all $\sigma \in \Sigma^{(n)}$. If we assume the opposite then there exists a cone $\sigma \in \Sigma^{(n)}$ and a sequence $\{(t_m, y_m)\}_{m \geq 1} \subseteq [0, 1] \times W_{\sigma}$ verifying that $\{\pi_{\sigma}(y_m)\}_{m \geq 1} \rightarrow \infty$ and $\{H_{\sigma, t_m, i}^*(y_m)\}_{m \geq 1} \rightarrow 0$, for all $i = 1, \dots, n$. Analogous to the proof of Proposition 4.2.2, let us consider a limit point $(\mathbf{t}, \mathbf{y}) = (\mathbf{t}, \mathbf{y}_1, \dots, \mathbf{y}_n) \in [0, 1] \times \overline{W}_{\sigma}$ of $\{(t_m, y_m)\}_{m \geq 1}$. By continuity we have $H_{\sigma, \mathbf{t}, i}^*(\mathbf{y}) = 0$, for all $i = 1, \dots, n$. Moreover, since $\{\pi_{\sigma}(y_m)\}_{m \geq 1} \rightarrow \infty$, we have that the set $J_0 = \{j : \mathbf{y}_j = 0\}$ is non-empty.

Following the same procedure as in the proof of Proposition 4.2.2 (see (4.18)) we obtain

$$\text{p}_{J_0}(H_{\mathbf{t}, i}) \circ \pi_{\sigma}(\tilde{\mathbf{y}}) = \prod_{j \notin J_0} \mathbf{y}_{\sigma, j}^{\ell(a^j(\sigma), H_{\mathbf{t}, i})} H_{\sigma, \mathbf{t}, i}^*(\mathbf{y}) = 0, \quad (4.45)$$

for all $i = 1, \dots, n$, where $\tilde{\mathbf{y}} = (\tilde{\mathbf{y}}_1, \dots, \tilde{\mathbf{y}}_n)$ is the point defined by

$$\tilde{\mathbf{y}}_j = \begin{cases} \mathbf{y}_j, & \text{if } j \notin J_0, \\ 1, & \text{if } j \in J_0. \end{cases}$$

Then we have a contradiction, since $p_{J_0}(H_{t,i}) = p_{J_0}(F_i)$, for all $i = 1, \dots, n$, and F is strongly adapted to $\tilde{\Gamma}_+$. Hence relation (4.44) holds for some $M > 0$ and all $\sigma \in \Sigma^{(n)}$. As a consequence, since $E_i(H_t, \tilde{\Gamma}_+) = E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$ and all $t \in [0, 1]$, we can reproduce inequalities (4.21)-(4.25), by replacing F_i by $H_{t,i}$ and taking $k = \mathbf{e}_i$, to obtain that for all $i = 1, \dots, n$, there exists a constant $C_i > 0$ such that

$$|x_i|^{b_i} \leq C_i \|F(x) + tG(x)\|, \quad (4.46)$$

for all $x \in \mathbb{R}^n$ such that $\|x\| \geq M$ and all $t \in [0, 1]$, where $b_i = \sup E_i$, for all $i = 1, \dots, n$ (see (4.33)).

Let $b_0 = \min\{b_1, \dots, b_n\}$ and $C_0 = \max\{C_1, \dots, C_n\}$. We can assume that $M \geq \sqrt{n}$. Hence $\|x\| \geq M$ implies $\max_i |x_i| \geq 1$. Taking $\max_i$ at both sides of (4.46), we obtain

$$\|x\|^{b_0} \leq (\sqrt{n})^{b_0} \max |x_i|^{b_0} \leq (\sqrt{n})^{b_0} C_0 \|F(x) + tG(x)\|$$

for all $t \in [0, 1]$ and all $x \in \mathbb{R}^n$ such that $\|x\| \geq M$. Then there exists a uniform Łojasiewicz inequality at infinity for the family of maps $\{H_t\}_{t \in [0, 1]}$ and the result follows, as explained at the beginning of the proof. $\square$

We observe from chapter 1, that if $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial map, then $I(F)$ denotes the ideal of $\mathbb{C}[x_1, \dots, x_n]$ generated by the component functions of F and denote by $F_{\mathbb{R}}$ the map $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ obtained from F under the identification $x + \mathbf{i}y \leftrightarrow (x, y)$ between $\mathbb{C}$ and $\mathbb{R}^2$. We remark that the proof of Theorem 4.5.1 also works to deduce the following result.

Theorem 4.5.2. *Let $F, G : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be polynomial maps such that*

- (i) *F is strongly adapted to $\tilde{\Gamma}_+$;*
- (ii) *$E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$;*
- (iii) *$\ell(w, G_i) > \ell(w, F_i)$, for all $w \in \mathcal{F}_0(\tilde{\Gamma}_+)$, $i = 1, \dots, n$.*

Then F and $F + G$ have a finite number of zeros and

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{I(F)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{I(F + G)}.$$

Proof. Conditions (i), (ii) and (iii) imply that $F + G$ also satisfy the hypothesis of Corollary 4.3.3. Then $\mathcal{L}_\infty(F)$ and $\mathcal{L}_\infty(F + G)$ are positive numbers. This implies that $F^{-1}(0)$ and $(F + G)^{-1}(0)$ are compact and hence finite. The same proof of Theorem 4.5.1 works to obtain that there is a uniform Łojasiewicz inequality for the homotopy $H : [0, 1] \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by $H(t, z) = F(z) + tG(z)$, for all $(t, z) \in [0, 1] \times \mathbb{C}^n$. That is, there exist some constants M , $\alpha > 0$ such that

$$\|z\|^\alpha \leq C\|F(z) + tG(z)\|,$$

for all $z \in \mathbb{C}^n$ such that $\|z\| \geq M$ and all $t \in [0, 1]$. In particular, by Theorem 1.4.7 this means that $\text{ind}(F_\mathbb{R}) = \text{ind}(F_\mathbb{R} + G_\mathbb{R})$.

It is well known (see for instance Theorem 1.5.4) that

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{I(F)} = \sum_{z \in F^{-1}(0)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,z}}{I_z(F)}.$$

where $\mathcal{O}_{n,z}$ denotes the germ of analytic function germs $(\mathbb{C}^n, z) \rightarrow \mathbb{C}$ and $I_z(F)$ is the ideal of $\mathcal{O}_{n,z}$ generated by the germs of the component functions of F at z , for any $z \in \mathbb{C}^n$. It is also known by Theorem 1.4.9 that, if $z = x + \mathbf{i}y \in F^{-1}(0)$, then $\dim_{\mathbb{C}} \mathcal{O}_{n,z}/I_z(F) = \text{ind}_{(x,y)}(F_\mathbb{R})$ (see also, for instance [16, p. 146] or [30, p. 15]). Then the result follows. $\square$

If $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes a real polynomial map, then we denote by $F_\mathbb{C} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ the map obtained from F by complexifying the variables.

Remark 4.5.3. Under the hypothesis of Theorem 4.5.1, if we assume that $F_\mathbb{C}$ is strongly adapted to $\tilde{\Gamma}_+$, then $\mathcal{L}_\infty(F_\mathbb{C}) \geq 0$, by Corollary 4.3.3, and consequently the zero set of the maps F and $F + G$ are finite.

Example 4.5.4. Let us consider the polynomial map $F = (F_1, F_2, F_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, where

$$\begin{aligned} F_1(x_1, x_2, x_3) &= x_1^{a_1} + x_1^{a_1} x_2^{b_1} + x_1^{a_1} x_3^{c_1} + \alpha x_1^{a_1} x_2^{b_1} x_3^{c_1}, \\ F_2(x_1, x_2, x_3) &= x_2^{b_2} + x_1^{a_2} x_2^{b_2} + x_2^{b_2} x_3^{c_2} + \beta x_1^{a_2} x_2^{b_2} x_3^{c_2}, \\ F_3(x_1, x_2, x_3) &= x_3^{c_3} + x_1^{a_3} x_3^{c_3} + x_2^{b_3} x_3^{c_3} + \gamma x_1^{a_3} x_2^{b_3} x_3^{c_3}. \end{aligned}$$

where α, β, γ are mutually different non-zero real numbers and the supports of the above polynomials are contained in $\mathbb{Z}_{\geq 1}^3$. Let g be the function defined by $g(x_1, x_2, x_3) = x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3$ and let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(g) \subseteq \mathbb{R}^3$. Then we observe that $\tilde{\Gamma}_+$ is

convenient and $\mathcal{F}_0(\tilde{\Gamma}_+) = \{-\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3\}$. A direct computation shows that

$$\begin{aligned}\ell(-\mathbf{e}_1, F_1) &= -a_1, \ell(-\mathbf{e}_2, F_1) = -b_1, \ell(-\mathbf{e}_3, F_1) = -c_1, \ell(\mathbf{e}_1, F_1) = a_1, \\ \ell(-\mathbf{e}_1, F_2) &= -a_2, \ell(-\mathbf{e}_2, F_2) = -b_2, \ell(-\mathbf{e}_3, F_2) = -c_2, \ell(\mathbf{e}_2, F_2) = b_2, \\ \ell(-\mathbf{e}_1, F_3) &= -a_3, \ell(-\mathbf{e}_2, F_3) = -b_3, \ell(-\mathbf{e}_3, F_3) = -c_3, \ell(\mathbf{e}_3, F_3) = c_3.\end{aligned}$$

and $\ell(\mathbf{e}_i, F_j) = 0$, for all $i, j \in \{1, 2, 3\}$, such that $i \neq j$.

It is straightforward to check that $F_{\mathbb{C}}$ is strongly adapted to $\tilde{\Gamma}_+$ and $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for $i = 1, 2, 3$. In particular $0 < \min\{a_i, b_i, c_i : i = 1, 2, 3\} \leq \mathcal{L}_{\infty}(F)$, by Corollary 4.3.3. Therefore $F^{-1}(0)$ is finite. Let $G = (G_1, G_2, G_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a polynomial map such that $\text{supp}(G_i)$ is contained in the cube $\{(k_1, k_2, k_3) \in \mathbb{Z}_{\geq 0}^3 : k_1 < a_i, k_2 < b_i, k_3 < c_i\}$, for $i = 1, 2, 3$. Then $(F + G)^{-1}(0)$ is also finite and $\text{ind}(F) = \text{ind}(F + G)$, by Theorem 4.5.1.

Example 4.5.5. Let $a, b, c \in \mathbb{Z}_{\geq 1}$ and let us consider the map $F = (F_1, F_2, F_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$F(x_1, x_2, x_3) = (x_1^a + x_2^b + x_1^a x_2^b, x_1^a x_2^b + x_3^c, x_3^c).$$

Let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(F)$. Then it is immediate to see that $F_{\mathbb{C}}$ is strongly adapted to $\tilde{\Gamma}_+$ and $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, 2, 3$. Hence $F^{-1}(0)$ is finite. Then, by Theorem 4.5.1, any polynomial map $G : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\text{supp}(G_i)$ is contained in the interior of $\tilde{\Gamma}_+$ verifies that $(F + G)^{-1}(0)$ is finite and $\text{ind}(F) = \text{ind}(F + G)$.

In the following example shows that the condition $\mathcal{L}_{\infty}(F) = r_0(\tilde{\Gamma}_+(F))$ does not imply that F is Newton non-degenerate at infinity.

Example 4.5.6. Let $a_i, b_i, c_i \in \mathbb{Z}_{\geq 1}^3$, for all $i = 1, 2$ such that $a_1 < a_2$, $b_1 < b_2$ and $c_1 < c_2$. Let $F = (F_1, F_2, F_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ be the polynomial map, whose component functions are given by

$$\begin{aligned}F_1(x_1, x_2, x_3) &= x_1^{a_1} + x_2^{b_1} + x_3^{c_1} + \alpha_1 x_1^{a_2} x_2^{b_2} + \beta_1 x_2^{b_2} x_3^{c_2} + \gamma_1 x_1^{a_2} x_3^{c_2}, \\ F_2(x_1, x_2, x_3) &= x_1^{a_1} + x_2^{b_1} + x_3^{c_1} + \alpha_2 x_1^{a_1} x_2^{b_2} + \beta_2 x_2^{b_1} x_3^{c_2} + \gamma_2 x_1^{a_2} x_3^{c_1}, \\ F_3(x_1, x_2, x_3) &= x_1^{a_1} + x_2^{b_1} + x_3^{c_1} + x_1^{a_1} x_2^{b_1} x_3^{c_1}.\end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are mutually different non-zero real numbers and the supports of the above polynomials are contained in $\mathbb{Z}_{\geq 1}^3$.

The following picture shows the Newton polyhedron at infinity of F .

Let $\tilde{\Gamma}_+ = \tilde{\Gamma}_+(g)$ be the Newton polyhedron in $\mathbb{R}^3$ given in Example 4.5.4. Then we observe that $\tilde{\Gamma}_+$ is convenient and $\mathcal{F}(\tilde{\Gamma}_+) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, -\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3\}$. A direct computation shows that

$$\begin{aligned} \ell(-\mathbf{e}_1, F_1) &= -a_2, \ell(-\mathbf{e}_2, F_1) = -b_2, \ell(-\mathbf{e}_3, F_1) = -c_2, \\ \ell(-\mathbf{e}_1, F_2) &= -a_2, \ell(-\mathbf{e}_2, F_2) = -b_2, \ell(-\mathbf{e}_3, F_2) = -c_2, \\ \ell(-\mathbf{e}_1, F_3) &= -a_1, \ell(-\mathbf{e}_2, F_3) = -b_1, \ell(-\mathbf{e}_3, F_3) = -c_1, \end{aligned}$$

and $\ell(\mathbf{e}_i, F_j) = 0$, for all $i, j = 1, 2, 3$.

It is straightforward to check that F is adapted to $\tilde{\Gamma}_+$ and since F_i is convenient for all $i = 1, 2, 3$, by Proposition 4.1.9 we have that F is strongly adapted to $\tilde{\Gamma}_+$. Moreover, it is easy to check that $E_i(F, \tilde{\Gamma}_+) \neq \emptyset$, for $i = 1, 2, 3$. In particular by Corollary 4.3.4, we conclude that $\mathcal{L}_\infty(F) = \min\{a_1, b_1, c_1\} = r_0(\tilde{\Gamma}_+(F)) > 0$.

However, F is Newton degenerate at infinity. Indeed, let us consider the 2-dimensional face Δ of $\tilde{\Gamma}_+(F)$ containing the points $(a_2, b_2, 0)$, $(a_2, 0, c_2)$ and $(0, b_2, c_2)$. Then an immediate computation shows that $(F_1)_\Delta = \alpha_1 x_1^{a_2} x_2^{b_2} + \beta_1 x_2^{b_2} x_3^{c_2} + \gamma_1 x_1^{a_2} x_3^{c_2}$, and $(F_2)_\Delta = (F_3)_\Delta = 0$. From this we conclude that F is Newton degenerate at infinity.

Bibliography

- [1] E. Artal Bartolo, I. Luengo, A. Melle-Hernández, *Milnor number at infinity, topology and Newton boundary of a polynomial function*, Math. Z. **233**(2000), 679–696.
- [2] D.N. Bernstein, *The number of roots of a system of equations*, Functional Anal. Appl. **9** (1975), no. 3, 183–185.
- [3] C. Bivià-Ausina, *Singularidades de Thom-Boardman en deformaciones genéricas de gérmenes de aplicaciones y métodos para el cálculo de clausuras integrales de ideales*, Tesis Doctoral, Universitat de València (2001).
- [4] C. Bivià-Ausina, *Łojasiewicz exponents, the integral closure of ideals and Newton polyhedra*, J. Math. Soc. Japan **55**, No. 3 (2003), 655–668.
- [5] C. Bivià-Ausina, *The index of analytic vector fields and Newton polyhedra*, Fund. Math. **177** (2003) 251–267.
- [6] C. Bivià-Ausina, *Non-degenerate ideals in formal power series rings*, Rocky Mountain J. Math. **34**, 2 (2004), 495–511.
- [7] C. Bivià-Ausina, *Jacobian ideals and the Newton non-degeneracy condition*, Proc. Edinburgh Math. Soc. **48** (2005), 21–36.
- [8] C. Bivià-Ausina, *Injectivity of real polynomial maps and Łojasiewicz exponents at infinity*, Math. Z. **257**, No. 4 (2007), 745–767.
- [9] C. Bivià-Ausina, *Local Łojasiewicz exponents, Milnor numbers and mixed multiplicities of ideals*, Math. Z. **262** (2009), no. 2, 389–409.
- [10] C. Bivià-Ausina, *Multiplicity and Łojasiewicz exponent of generic linear sections of monomial ideals*, Bull. Aust. Math. Soc. **91** (2015), no. 2, 191–201.

- [11] C. Bivià-Ausina, T. Fukui and M.J. Saia, *Newton graded algebras and the codimension of non-degenerate ideals*, Math. Proc. Cambridge Philos. Soc. **133** (2002), 55–75.
- [12] C. Bivià-Ausina and J.A.C. Huarcaya, *Growth at infinity and index of polynomial maps*, J. Math. Anal. Appl. **422** (2015) 1414–1433.
- [13] J. Bochnak and J.J. Risler, *Sur les exposants de Łojasiewicz*, Comment. Math. Helv. **50** (1975), 493–507.
- [14] M. Brunella and M. Miari, *Topological equivalence of a plane vector field with its principal part defined through Newton polyhedra*, J. Differential Equations **85** (2) (1990) 338–366.
- [15] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Stud. Adv. Math. vol. 39, Cambridge Univ. Press, Cambridge, 1993.
- [16] C. Camacho, A. Lins Neto and P. Sad, *Topological invariants and equidesingularization for holomorphic vector fields*, J. Differential Geom. **20** (1984), No. 1, 143–174.
- [17] P. Cassou-Noguès, *Sur la généralisation d'un tórem de Kouchnirenko*, Compositio Mathematica **133**: 95–121, (1996).
- [18] A. Cima, A. Gasull and F. Mañosas, *Injectivity of polynomial local homeomorphisms of $\mathbb{R}^n$* , Nonlinear Anal. **26** (1996), No. 4, 877–885.
- [19] A. Cima, A. Gasull and J. Torregrosa, *On the relation between index and multiplicity*, J. London Math. Soc. (2) **57** (1998), 757–768.
- [20] D.A. Cox, J. Little and D. O'Shea, *Using Algebraic Geometry*, Graduate Texts in Mathematics, Springer-Verlag (1992).
- [21] D.A. Cox, J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms*, Graduate Texts in Mathematics **185**, Springer-Verlag (2005).
- [22] J. Chadzyński and T. Krasinski, *Sur l'exposant de Łojasiewicz à l'infini pour les applications polynomiales de $\mathbb{C}^2$ dans $\mathbb{C}^2$ et les composantes des automorphismes polynomiaux de $\mathbb{C}^2$* , C. R. Acad. Sci. Paris Sér. I Math. **325** (13) (1992) 1399–1402.
- [23] J. Chadzyński and T. Krasinski, *Exponent of growth of polynomial mappings of $\mathbb{C}^2$ into $\mathbb{C}^2$* , In: Singularities, Banach Center Publ. **20**, 147–160, PWN, Warszawa (1988) .

- [24] J. Chadzyński and T. Krasinski, *A set on which the Lojasiewicz exponent at infinity is attained*, Ann. Polon. Math. **67** (1997), 191–197.
- [25] E. Cygan, T. Krasinski and P. Tworzewski, *Separation of algebraic sets and the Lojasiewicz exponent of polynomial mappings*, Invent. Math. **136** (1999), no. 1, 75–87.
- [26] J.P. D’Angelo, *Several Complex Variables and the Geometry of Real Hypersurfaces*, CRC Press, Boca Raton, **1992**.
- [27] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, SINGULAR 4-0-2 — A computer algebra system for polynomial computations. <http://www.singular.uni-kl.de> (2015).
- [28] S.T. Dinh, H.V. Hà, T.S. Pham and N.T. Thao, *Global Lojasiewicz inequality for non-degenerate polynomials*, J. Math. Anal. Appl. **410** (2014), No. 2, 541–560.
- [29] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer-Verlag (1994).
- [30] D. Eisenbud and H.T. Levine, *An algebraic formula for the degree of a C^∞ map germ*. With an appendix by B. Teissier, *Sur une inégalité à la Minkowski pour les multiplicités*, Ann. of Math. (2) **106** (1977), No. 1, 19–44.
- [31] G. Ewald, *Combinatorial convexity and algebraic geometry*, Grad. Texts in Math. No. **168** (1996), Springer Verlag.
- [32] T. Fukui, *Lojasiewicz type inequalities and Newton diagrams*, Proc. of the Amer. Math. Soc. **114** (1991), vol. 4, 1169–1183.
- [33] M. Furuya and M. Tomari, *A characterization of semi-quasihomogeneous functions in terms of the Milnor number*, Proc. Amer. Math. Soc. **132** (2004), No. 7, 185–189.
- [34] T. Gaffney, *Integral closure of modules and Whitney equisingularity*, Invent. Math. **107** (1992), 301–322.
- [35] T. Gaffney, *Multiplicities and equisingularity of ICIS germs*, Invent. Math. **123** (1996), 209–220.
- [36] C. Gutiérrez, M.A.S. Ruas, *Indices of Newton non-degenerate vector fields and a conjecture of Loewner for surfaces in $\mathbb{R}^4$* , in: *Real and Complex Singularities*, Lect. Notes Pure Appl. Math., vol. **232**, Dekker, New York, 2003, pp. 245–253.

- [37] J. Gwoździewicz, *The Łojasiewicz exponent of an analytic function at an isolated zero*, Comment. Math. Helv. **74** (1999), 364–375.
- [38] R. Harstshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, New York Inc. (1977).
- [39] M. Hochster, *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Ann. of Math. **96** (1972), 318–337.
- [40] J.A.C. Huarcaya, *Poliedros de Newton e singularidades de polinômios*, Dissertação de Mestrado, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, 2011.
- [41] A.G. Khovanskii, *Newton polyhedra, a new formula for mixed volume, products of roots of a system of equations*, Fields Inst. Commun. **24** (1999), 325–364.
- [42] A.G. Khovanskii and K. Kaveh, *Convex bodies and multiplicities of ideals*, Ann. of Math. **176** (2012), 925–978.
- [43] A.G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), 1–31.
- [44] T. Krasieński, *On the Łojasiewicz exponent at infinity of polynomial mappings*, Acta Math. Vietnam. **32** (2007), No. 2–3, 189–203.
- [45] J. Kollár, *Sharp effective Nullstellensatz*, J. Amer. Math. Soc. **1** (1988), 963–975.
- [46] M. Lejeune, B. Teissier, *Clôture intégrale des idéaux et equisingularité*, with an appendix by J.J. Risler, Centre de Mathématiques, École Polytechnique (1974) and Ann. Fac. Sci. Toulouse Math. (6) **17** (4), 781–859 (2008).
- [47] T.Y. Li and X. Wang, *The BKK root count in $\mathbb{C}^n$* , Math. Comp. **65** (1996), no. 216, 1477–1484.
- [48] J. Llibre and R. Saghin, *The index of singularities of vector fields and finite jets*, J. Differential Equations **251** (2011), No. 10, 2822–2832.
- [49] N.G. Lloyd, *Degree theory*, Cambridge Tracts in Mathematics, No. 73. Cambridge University Press, Cambridge-New York-Melbourne, 1978.

- [50] S. Łojasiewicz, *Ensembles semi-analytiques*, IHES (1965).
- [51] M. Oka, *Non-degenerate complete intersection singularity*, Actualités Mathématiques, Hermann, Paris, 1997.
- [52] H. Matsumura, *Commutative ring theory*, Cambridge studies in Advanced Mathematics 8, Cambridge University Press (1986).
- [53] H. Matsumura, *Commutative algebra*, W. A. Benjamin Co., New York (1970). xii+262pp
- [54] J.W. Milnor, *Singular points of complex hypersurfaces*, Annals of Math. Studies, Princeton University. 61, (1968).
- [55] J.W. Milnor and P. Orlik, *Isolated singularities defined by weighted homogeneous polynomials*, Topology 9 (1970), 385-393
- [56] A. Montesinos, GÉRMENES, <http://www.uv.es/montesin/> (2015).
- [57] A. Nemethi and A. Zaharia, *Milnor fibration at infinity*, Indag. Math. 3 (1992), 323–335.
- [58] D.G. Northcott and D. Rees, *Reductions of ideals in local rings*, Math. Proc. Cambridge Philos. Soc. **50** (1954), 145-158.
- [59] S. Pinchuk, *A counterexample to the strong real Jacobian conjecture*, Math. Z. **217**(1), 1-4 (1994).
- [60] A. Płoski, *On the growth of proper polynomial mappings*, Ann. Polon. Math. **45** (1985), no. 3, 297–309.
- [61] A. Płoski, *A note on the Łojasiewicz exponent at infinity*, Bull. Soc. Sci. Letters Łódź. **44**(17) (1994), 1-15.
- [62] T. Rodak and S. Spodzieja, *Equivalence of mappings at infinity*, Bull. Sci. Math. **136** (2012), No. 6, 679–686.
- [63] D. Rees, *a -transforms of local rigs and a theorem on multiplicities of ideals*, Proc. Cambridge Philos. Soc. **57** (1961), 8-17.
- [64] D. Rees, *Generalizations of reductions and mixed multiplicities*, J. London Math. Soc. (2) **29** (1984), 397–414.

- [65] J.M. Rojas, *A convex geometric approach to counting the roots of a polynomial system*, Theoretical Computer Science **5** 133 (1994), 105-109.
- [66] J.M. Rojas, *Toric Intersection Theory for affine root counting*, Journal of Pure and Applied Algebra **136** (1999), 67-100.
- [67] J.M. Rojas and X. Wang, *Counting Affine Roots of Polynomial Systems via Pointed Newton Polytopes*, Journal Complexity **12** (1996), 116-133.
- [68] T. Rodak and S. Spodzieja, *Equivalence of mappings at infinity*, Bull. Sci. Math. **136** (6) (2012) 679–686.
- [69] S. Ji, J. Kollár and B. Shiffman, *A global Łojasiewicz inequality for algebraic varieties*, Transaction of Amer. Math. Soc. **329**(2) (1992), 813-818.
- [70] M.J. Saia, *The integral closure of ideals and the Newton filtration*, J. Algebraic Geom. **5** (1996), 1–11.
- [71] M.J. Saia, *Pre-weighted homogeneous map germs finite determinacy and topological triviality*, Nagoya Math. J. Vol. **151** (1998), 209-220
- [72] I. Swanson and C. Huneke, *Integral Closure of Ideals, Rings, and Modules*, London Mathematical Society, Lecture Note Series. (336)
- [73] Igor R. Shafarevich, *Basic Algebraic Geometry I*, Berlin, Heidelberg, New York. Springer. (1994)
- [74] B. Teissier, *Cycles évanescents, sections planes et conditions of Whitney*, Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972), Astérisque, Nos. 7 et 8, Soc. Math. France (1973), 285–362.
- [75] B. Teissier, *Monômes, volumes et multiplicités*, Introduction à la Théorie des singularités, II, Travaux en Cours, 37, Hermann, 1988, 127–141.
- [76] A. Van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progr. Math. 190 (2000), Birkhäuser.
- [77] A.N. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math. **37** (1976), 253–262.

- [78] C.T.C. Wall, *Newton polytopes and non-degeneracy*, J. Reine Angew Math. **509** (1999), 1–19.
- [79] E. Yoshinaga, *Topologically principal part of analytic functions*, Trans. Amer. Math. Soc. **314** (2) (1989), 803–814.
- [80] Z.V. Županović, *Topological equivalence of planar vector fields and their generalised principal part*, J. Differential Equations 167 (1) (2000) 11–15

Index

- Łojasiewicz exponent
 - of h with respect to I , [12](#)
- Łojasiewicz exponent at infinity, [19](#)
- adapted
 - system to Γ_+ , [13](#)
- adapted map, [73](#)
- algebraic multiplicity of I at a , [17](#)
- convenient Newton polyhedron, [28](#)
- convenient polynomial map, [29](#)
- degree of h with respect to v , [35](#)
- dimension of a face Δ , [26](#)
- face of S , [26](#)
- facets of $\tilde{\Gamma}_+$, [26](#)
- filtrating map associated to $\tilde{\Gamma}_+$, [33](#)
- finite analytic mapping, [14](#)
- finite polynomial map, [18](#)
- Index of a smooth mapping, [15](#)
- integral closure of I , [6](#)
- integrally closed, [6](#)
- local Newton polyhedron, [8](#)
- meromorphic curve, [20](#)
- Milnor number, [32](#)
- Minkowski sum of $\tilde{\Gamma}_+^1, \dots, \tilde{\Gamma}_+^p$, [27](#)
- multiplicity of I with respect to M , [6](#)
- multiplicity of R , [6](#)
- Multiplicity of F , [18](#)
- Newton non-degenerate at infinity, [30](#)
- Newton number, [32](#)
- Newton polyhedron at infinity, [25](#)
- Newton polyhedron at infinity of h , [28](#)
- non-degenerate, [76](#)
- non-degenerate with respect to $\tilde{\Gamma}_+$, [35](#)
- polynomial map, [14](#)
- pre-weighted homogeneous, [36](#)
- primitive vector, [27](#)
- principal part
 - of h at infinity with respect to v , [35](#)
 - of h over Δ , [35](#)
 - of h with respect to J , [72](#)
 - of h with respect to w , [72](#)
- quasi-special polynomials, [47](#)
- special polynomial, [46](#)
- strongly adapted maps, [75](#)
- support of h , [28](#)
- Topological degree of mapping, [15](#)
- vertices of $\tilde{\Gamma}_+$, [26](#)
- weighted-homogeneous, [35](#)