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Chaotic behaviour in diffusively coupled systems

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Comportamento caótico em sistemas acoplados difusivamente

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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Este trabalho é dedicado especialmente aos meus pais, Milton e Maria
que, nunca deixaram de me apoiar incondicionalmente durante esta jornada.
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"As invenções são, sobretudo,
o resultado de um trabalho de teimoso."
(Santos Dumont)

RESUMO

QUEIROZ, F. C. DE. **Comportamento caótico em sistemas acoplados difusivamente**. 2022. 147 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

Estudamos o comportamento oscilatório emergente em redes de equações diferenciais ordinárias não lineares difusivamente acopladas. Partindo de uma situação em que as dinâmicas isoladas em cada nó são as mesmas e possuem um ponto de equilíbrio globalmente atrativo. Pesquisas recentes mostraram que redes gerais podem apresentar oscilações periódicas devido ao acoplamento difusivo sob condições brandas no campo vetorial isolado. Nesta tese, fornecemos condições no campo vetorial isolado e no grafo correspondente tais que a rede tenha uma variedade central e mostramos que o campo vetorial reduzido tem coeficientes de Taylor não nulos sempre que o campo vetorial original é genérico. Além disso, mostramos que quando a dimensão do campo vetorial isolado é de pelo menos quatro é possível encontrar matrizes positivas-definidas servindo como acoplamentos de forma que a rede tenha uma singularidade nilpotente que corresponde à existência de uma variedade central tridimensional. Como consequência, a rede apresentará um comportamento caótico.

Palavras-chave: Grafos versáteis, Caos, Variedade central, Órbita homoclínica de Shilnikov, Redes. .

ABSTRACT

QUEIROZ, F. C. DE. **Chaotic behaviour in diffusively coupled systems**. 2022. 147 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2022.

We study emergent oscillatory behaviour in networks of diffusively coupled nonlinear ordinary differential equations. Starting from a situation where the isolated dynamics at each node are the same and possess a globally attractive equilibrium point. Recent research has shown that general networks can present periodic oscillations due to diffusive coupling under mild conditions in the isolated vector field. In this thesis, we provide conditions on the isolated vector field and the underlying graph such that the network has a center manifold and we show that the reduced vector field has non-vanishing Taylor coefficients whenever the original vector field is generic. Moreover, we show that when the dimension of the isolated vector field is at least four its is possible to find positive-definite matrices serving as couplings such that the network has a nilpotent singularity which corresponds to the existence of a three-dimensional center manifold. As a consequence, the network will present a chaotic behaviour.

Keywords: Versatile graphs, Chaos, Center manifold, Shilnikov homoclinic orbit, Networks. .

CONTENTS

CHAPTER

1

INTRODUCTION

Coupled dynamical systems play a prominent role in biology (IZHIKEVICH, 2007), chemistry (KURAMOTO, 2003), physics (STANKOVSKI *et al.*, 2017), and other fields of science. Understanding the emergent dynamics of such systems is a challenging problem as it depends starkly on the underlying interaction structure and has attracted plenty of attention (NIJHOLT; RINK; SANDERS, 2019; PEREIRA; STRIEN; TANZI, 2020; DIAS; LAMB, 2006; GOLUBITSKY; STEWART; TÖRÖK, 2005; LI; XIA, 2021; RICARD; MISCHLER, 2009).

In the early fifties, Turing thought of the emergence of oscillatory behaviour due to diffusive interaction as a model for morphogenesis (TURING, 1952), that is, the process of creation of patterns in the development of an embryo. One of his models corresponds to dynamical systems coupled with first-neighbor interaction serving as a biological model to study the mechanism of diffusion of chemicals through cells that are disposed on a ring.

While diffusion was usually considered as trivializing the dynamics, in the mid-seventies, Smale (SMALE, 1976) proposed an example of diffusion-driven oscillations. He considered two 4th-order diffusively coupled differential equations which by themselves have globally asymptotically stable equilibrium points. Once the diffusive interaction was strong enough, the coupled system exhibited oscillatory behaviour. Smale proposed a problem to find conditions under which globally asymptotically stable systems being diffusively coupled will oscillate.

Tomberg and Yakubovich (TOMBERG; YAKUBOVICH, 1989) proposed a solution to this problem for the diffusive interaction of two Lur'e systems with scalar nonlinearity. Pogromsky and co-authors (POGROMSKY; GLAD; NIJMEIJER, 1999) showed that Turing instability occurs as a result of an Andronov-Hopf bifurcation and presented conditions to ensure the existence of oscillations for general graphs. While this provides a good picture of the instability leading to periodic oscillations, there is evidence that the Turing instability may also originate chaotic oscillations.

Kocarev and Janic (KOCAREV; JANJIC, 1995) provided numerical evidence that two diffusively coupled Chua circuits may present diffusion-driven chaotic oscillations. That is, starting from a situation when the two isolated circuits have a globally stable fixed point, numeric simulation suggests that once the two circuits are diffusively coupled a strange attractor seems to appear.

In the same line, Drubi and co-authors (DRUBI; IBANEZ; RODRIGUEZ, 2007) studied two diffusively coupled Brusselators. Again starting from a situation where the isolated systems have a globally stable fixed point, they proved the unfolding of the diffusively coupled system can display a homoclinic loop, and this system has a limiting set with positive entropy. It remains an open question whether such Turing instability in general networks can generate chaotic oscillations.

In this thesis, we provide conditions for general diffusively coupled identical systems to undergo a transition due to coupling that generates chaotic oscillations. Starting from a situation where the isolated system has dimension four or higher, possesses an exponentially stable equilibrium point, and satisfies a skewness condition, we prove that there is a diffusive coupling matrix such that the coupled system has a nilpotent singularity and thus a nontrivial center manifold. See Shilnikov and co-authors (SHILNIKOV *et al.*, 1998) for center manifold theory. After reduction to the center manifold, we prove that the unfolding of the singularity, when the dimension of the manifold is three, contains the existence of chaotic solutions.

1.1 The model

We consider nonlinear ordinary differential equations $\dot{x} = f(x)$ with $f \in \mathscr{C}^\infty(U, \mathbb{R}^n)$ with $n \in \mathbb{N}$ and an open set $U \subset \mathbb{R}^n$. We assume f has a globally exponential stable fixed point in U, which we assume without loss of generality to be the origin. Note that this behaviour is robust, in the sense that small perturbations of f give topologically equivalent behaviour.

We wish to study exponentially stable dynamical systems coupled together in a graph structure by means of a diffusive interaction. More precisely, to each vertex in the graph, we associate a vector field $f \in \mathscr{C}^\infty(U, \mathbb{R}^n)$ with an exponentially stable fixed point at the origin. We couple N such systems together as a function of their difference, processed through a positive-definite matrix. That is, a matrix D satisfying $x^T D x > 0$ for all non-zero vectors x.

The problem under consideration can thus be represented by the following network equation:

$$\dot{x}_i = f(x_i) + \alpha \sum_{j=1}^{N} w_{ij} D(x_j - x_i), \quad i = 1, \ldots, N, \tag{1.1}$$

where $\alpha > 0$ is the coupling strength, $\mathscr{W} = (w_{ij})$ denotes the adjacency matrix, D is the positive-definite matrix, the difference of the states $(x_j - x_i)$ represents the diffusive coupling and by $A = Df(0)$ the corresponding linearized system. Note that, if the coupling strength α is equal to zero, then the diffusion is absent and the uncoupled systems keep their original stable behaviour. Likewise, for $\alpha > 0$ small, this behaviour persists.

1.2 Informal statement of main results

Exponentially stable dynamical systems are robust systems, in the sense that small perturbations do not change the qualitative behaviour. When the interaction strength is large enough we show that the diffusive coupling of such systems presents highly non-trivial dynamics in the network.

The main result of this thesis can be summarized by:

> *Suppose the matrix A has m positive entries on the diagonal, with respect to some orthogonal basis. Then, there exists a positive-definite matrix D such that the System* (1.1) *has a center manifold of dimension at least m for some value of the coupling parameter $\alpha > 0$. For certain large classes of networks, the center manifold may be assumed of dimension precisely m, with no general restrictions on the Taylor expansion of the corresponding reduced dynamics.*

The observation that the Taylor expansion of the reduced system has no general restrictions allows us to predict generic bifurcation scenarios in these coupled-cell systems. In particular, we obtain the main corollary:

> *In case of 4-dimensional stable isolated dynamics, the system of Equations* (1.1) *can be arranged to have a 3-dimensional center manifold on which chaos emerges in a generic 3-parameter bifurcation.*

1.3 Statements of the main results

In this section, we concern with understanding the main results of this thesis. To do this we start with definitions.

1.3.1 Definitions

We show that a system of diffusively coupled stable systems can nevertheless display a wide variety of dynamic behaviour, including the onset of chaos. In fact, we show that as the coupling strength α increases, a non-trivial center manifold can emerge with no general restrictions on the Taylor coefficients of the reduced dynamics.

Note that we may alternatively write Equation (1.1) in terms of the Laplacian:

$$\dot{x}_i = f(x_i) - \alpha \sum_{j=1}^{N} l_{ij} D x_j, \quad i = 1, \dots, N. \tag{1.2}$$

Let $X := col(x_1, \dots, x_N)$ denote the vector formed by stacking x_i for $i = 1, \dots, N$ in a single column vector. In the same way we define $F(X) := col(f(x_1), \dots, f(x_N))$. We obtain the compact form for Equations (1.1) and (1.2) given by

$$\dot{X} = F(X) - \alpha(L \otimes D)X. \tag{1.3}$$

In order to analyze systems of the form of Equation (1.1), we allow f to depend on a parameter ε taking values in some open neighborhood of the origin $\Omega \subseteq \mathbb{R}^d$.

$$\dot{x} = f(x; \varepsilon), \quad \dot{\varepsilon} = 0. \tag{1.4}$$

For simplicity, we assume the fixed point at the origin persists:

$$f(0; \varepsilon) = 0 \qquad \forall \varepsilon \in \Omega. \tag{1.5}$$

Then, from Equations (1.4) and (1.5), the compact form of Equation (1.3) has now the nonlinear diagonal map F depends on the parameter ε as well:

$$\dot{X} = F(X; \varepsilon) - \alpha(L \otimes D)X := H(X; \varepsilon). \tag{1.6}$$

In what follows, we define an important object of our study:

Definition 1 (Local center manifold parameterized)**.** Let $H \in \mathscr{C}^\infty(U^N \times \Omega, \mathbb{R}^{nN})$ be a full system with a non-hyperbolic fixed point at the origin. Let E^c and E^h be the corresponding center and hyperbolic subspaces. A *local center manifold parameterized* is locally the graph of a function $\phi : E^c \times \Omega \to E^h$:

$$\mathscr{E}^c_{loc}(0) := \big\{ (x, \varepsilon, y) \in E^c \times \Omega \times E^h \quad | \quad \phi(x, \varepsilon) = y, \ \phi(0,0) = 0, \ D\phi(0,0) = 0 \\ |x| < \delta_1, |\varepsilon| < \delta_2 \big\}$$

for sufficiently small $\delta_1, \delta_2 > 0$.

We start with our working definition of center manifold reduction.

Definition 2. Let

$$H : \mathbb{R}^{nN} \times \Omega \to \mathbb{R}^{nN} \tag{1.7}$$

be a family of vector fields on $\mathbb{R}^{nN}$, parameterized by a variable ε in an open neighborhood of the origin $\Omega \subseteq \mathbb{R}^d$. Assume that $H(0;\varepsilon) = 0$ for all $\varepsilon \in \Omega$, and denote by $E^c \subseteq \mathbb{R}^{nN}$ the center subspace of the Jacobian $D_X H(0;0)$ in the direction of $\mathbb{R}^{nN}$. A (local) *parameterized center manifold* of the system of Equation (1.7) is a (local) center manifold of the unparameterized system $\tilde{H}$ on $\mathbb{R}^{nN} \times \Omega$, given by

$$\tilde{H}(X;\varepsilon) = (H(X;\varepsilon), 0) \in \mathbb{R}^{nN} \times \mathbb{R}^d, \tag{1.8}$$

for $X \in \mathbb{R}^{nN}$ and $\varepsilon \in \Omega$. We will say that the parameterized center manifold is of dimension $\dim(E^c)$, and is parameterized by d variables. Under the assumptions on H, it can be seen that the center subspace of $\tilde{H}$ at the origin is equal to $E^c \times \mathbb{R}^d$. One can furthermore show that the dynamics on the center manifold of Equation (1.8) is conjugate to that of a locally defined system

$$\tilde{R}(x_c;\varepsilon) = (R(x_c;\varepsilon), 0), \tag{1.9}$$

on $E^c \times \Omega$, where the conjugation respects the constant-ε fibers. The map R moreover satisfies $R(0;\varepsilon) = 0$ for all ε for which this local expression is defined, and we have $D_{x_c} R(0;0) = D_X H(0;0)|_{E^c} : E^c \to E^c$. We will refer to $R : E^c \times \Omega \to E^c$ as a *parameterized reduced vector field* of H.

In the definition above, the constant and linear terms of the parameterized reduced vector field R are given. Motivated by this, we will write $H^{[2,\rho]}$ for any map H to denote the non-constant, nonlinear terms in the Taylor expansion around the origin of H, up to terms of order ρ. In other words, we have

$$H(X) = H(0) + DH(0)X + H^{[2,\rho]}(X) + \mathscr{O}(||X||^{\rho+1}).$$

Given vector spaces W and W', we will use $\mathscr{P}_2^l(W;W')$ to denote the linear space of polynomial maps from W to W' with terms of degree 2 through l. It follows that $H^{[2,l]} \in \mathscr{P}_2^l(W;W')$ for $H : W \to W'$.

We are mostly interested in the situation where the domain of H involves some parameter space Ω, in which case $H^{[2,\rho]}$ involves all non-constant, nonlinear terms up to order ρ in both types of variables (parameter and phase space). For instance, if H is a map from $\mathbb{R} \times \Omega$ to $\mathbb{R}$ with $\Omega \subseteq \mathbb{R}$, then $H^{[2,3]}(x;\varepsilon)$ involves the terms

$$a_1 x^2, a_2 x\varepsilon, a_3 \varepsilon^2, a_4 x^3, a_5 x^2 \varepsilon, a_6 x \varepsilon^2 \text{ and } a_7 \varepsilon^3,$$

with some constants a_i depending on Taylor expansion coefficients. Of course, a condition on H might put restraints on $H^{[2,\rho]}$ as well. For instance, if $H(0,\varepsilon) = 0$ for all $\varepsilon \in \Omega$, then $H^{[2,3]}(x;\varepsilon)$ does not involve the terms ε^2 and ε^3.

We present the definition here of an important class of graphs.

Definition 3 (ρ-Versatile Graphs)**.** Let $G = (V, E)$ be a graph and $\rho \in \mathbb{N}$ a positive integer. We say that G is ρ-versatile for the eigenvalue-eigenvector pair (μ, ν) with $\mu > 0$, if the Laplacian matrix L_G has a simple eigenvalue μ with corresponding eigenvector $\nu = (\nu_1, \ldots, \nu_N)$, satisfying

$$\sum_{i=1}^{N} \nu_i^{\ell} \neq 0, \quad \forall \ell = 2, \ldots, \rho + 1. \tag{1.10}$$

In Chapter (2) we will explore ρ-Versatile Graphs in several examples and we show how to construct such graphs.

1.3.2 Main results

We are now ready to formulate the main Theorem, along with an important Corollary. We always assume a network to have at least one connection, so that its underlying graph is not the disjoint union of individual nodes.

Theorem 4 (Main Theorem)**.** For any $\alpha \geq 0$, consider the ε-family of network dynamical systems given by

$$\dot{X} = F(X; \varepsilon) - \alpha (L \otimes D) X. \tag{1.11}$$

Denote by $A = D_x f(0;0)$ the Jacobian of the isolated dynamics. If there exist m mutually orthogonal vectors $v_1, \ldots, v_m$ such that $\langle v_i, A v_i \rangle > 0$, then there exists a positive-definite matrix D together with a number $\alpha^* > 0$ such that the system of Equation (1.11) has a local parameterized center manifold of dimension at least m for $\alpha = \alpha^*$.

Suppose furthermore that the graph G of the network is ρ-versatile for the pair (μ, ν). After an arbitrarily small perturbation to A if needed, there exists a positive-definite matrix D and a number $\alpha^* > 0$ such that the following holds:

1. The system of Equation (1.11) has a local parameterized center manifold of dimension exactly m for $\alpha = \alpha^*$.

2. Denote by $R : E^c \times \Omega \to E^c$ the corresponding parameterized reduced vector field, then $R(0; \varepsilon) = 0$ for all $\varepsilon \in \Omega$ and $D_{x_c} R(0;0) : E^c \to E^c$ is nilpotent.

3. The higher order terms $R^{[2,\rho]}$ can take on any value in $\mathscr{P}_2^{\rho}(E^c \times \Omega; E^c)$ (subject to $R^{[2,\rho]}(0; \varepsilon) = 0$) as $f^{[2,\rho]}$ is varied (subject to $f^{[2,\rho]}(0; \varepsilon) = 0$).

In what follows, as a straight consequence of the Main Theorem, the Corollary answer the proposed question.

Corollary 5 (Chaos)**.** Assume the conditions of Theorem (4) to hold for $m = 3$ and $\rho = 2$. Then, in a generic 3-parameter system we have the emergence of chaos through the formation of a Shilnikov loop on the center manifold. In particular, chaos can form this way in a system of 4-dimensional nodes with stable internal dynamics, coupled diffusive to form a network.

The 4th-dimensional system is the minimal dimension needed due to the stability required. See Remarks (11) and (57) for details. We will see, Remark (71), we may have instead looked at a 2-parameters and considered α around α^* as the third implicit parameter needed to unfold. Hence, chaos also emerges in a 2-parameter system along the coupling parameter α.

The main result requires two main hypotheses, namely, the linearization of the isolated vector field A is skewed and the eigenvectors of the corresponding graph Laplacian are non-degenerate (in the sense of versatility). Thus, the assumption imposed on A to create a center manifold for the whole network excludes symmetric matrices.

To obtain the genericity of the Taylor coefficients in the center manifold, the network structure comes into play via the eigenvectors of the corresponding Laplacian matrix. Our class of versatile graphs appears naturally for star graphs and more generally for graphs with heterogeneous degrees. Symmetry in the graph seems to be the main obstruction to the versatility condition.

We notice that while Pogromsky and co-authors used a similar assumption on the linearization of the isolated vector field (POGROMSKY; GLAD; NIJMEIJER, 1999) to obtain diffusion-driven oscillations via Hopf bifurcation, the versatility condition played no role in their derivation. They did not use the eigenvector structure of the graph, but rather the spectral conditions on the Laplacian. Moreover, although two symmetrically coupled systems have a corresponding graph Laplacian that is not versatile, Drubi and co-authors (DRUBI; IBANEZ; RODRIGUEZ, 2007) were able to prove chaos in coupled Brusselators. While this construction does not require versatility it is particular to the Brusselator model.

Although the structure of the graph played no role in the works mentioned, in (DIAS; LAMB, 2006) it is proved that the structure of the network can affect its local bifurcations for the case of a network of symmetrical connected coupled cells having a symmetrical abelian group. Here, the existence of all self-loops guarantees a free passage for eigenvalues of codimension one to cross the imaginary axis.

For general vector fields, it remains an open question whether the genericity of the Taylor coefficients of the reduced vector field would hold if the graph is not versatile. Moreover, whether graph symmetries would impose conditions on the Taylor coefficients, thus forbidding the existence of limiting sets of positive entropy.

CHAPTER

2

VERSATILITY AND SKEWNESS CONDITIONS

In this chapter, we show that the conditions of Theorem (4) are entirely natural, by constructing multiple classes of networks that are ρ-versatile for any $\rho \in \mathbb{N}$, as well as by giving examples of matrices A that satisfy the conditions of the theorem.

In Section (2.1) we define what we call ρ-Versatile Graphs. In Subsection (2.1.1) we present a geometric way of constructing ρ-Versatile Graphs using the so-called complement graph. In Subsection (2.1.2) we then show using direct estimates that star graphs are natural candidates for ρ-versatility. Finally, in Section (2.2) we present examples of matrices that satisfy the conditions of Theorem (4). In particular, matrices having all eigenvalues with strictly negative real parts are not an obstruction. We call it Hurwitz matrices.

The main concepts of this chapter like ρ-Versatile Graphs and the Skewness condition are new and they were created to help solve the main problem of this thesis formalized in Main Theorem (4).

2.1 Versatile graphs

We are interested in a well-behaved class of graphs G, by which we mean a class whose structure induces a special property of the associated Laplacian matrix. This property will be the existence of an eigenvector where the sum of certain coordinate powers is nonvanishing, corresponding to a simple eigenvalue of L_G. To this end, we begin with graph concepts:

A graph G is an ordered pair (V,E), where V is a non-empty set of vertices and E is a set of edges connecting the vertices. We assume both to be finite. The order of the graph G is $|V| = N$, its number of vertices, and the size is $|E|$, its number of edges. The degree of a vertex is the number of edges that are connected to it.

An important concept related to graphs is the *adjacency* matrix $\mathscr{W}$. This matrix represents the connectivity structure, we can define it by:

$$w_{ij} = \begin{cases} 1, & \text{if vertex } i \text{ is connected to vertex } j; \\ 0, & \text{otherwise.} \end{cases}$$

We define the *vertex degree*, denoted by k_i as the sum of all the connections it receives,

$$k_i = \sum_{j=1}^{N} w_{ij}, \tag{2.1}$$

for $i = 1, \ldots, N$. Let $\mathscr{K} = \text{diag}\{k_1, \ldots, k_N\}$ be the diagonal matrix of vertex degrees.

We only consider *undirected* graphs G with no *self-loops*, meaning that a vertex i is connected with a vertex j if and only if it is *vice-versa*, and there are no edges connecting any vertex i to itself, respectively. Thus, the adjacency matrix $\mathscr{W}$ is a symmetric matrix.

In this context, there is another important matrix related to the graph G, which is the well-known *Laplacian* discrete matrix L_G. It is defined by:

$$L_G = \mathscr{K} - \mathscr{W},$$

so that each entry l_{ij} of L_G can be written as

$$l_{ij} = \delta_{ij} k_i - w_{ij}, \quad i, j = 1, \ldots, N, \tag{2.2}$$

where δ_{ij} is *Kronecker* delta. The matrix L_G provides us with important information about the connectivity and synchronization of the network.

In what follows $G = (V, E)$ is an undirected graph with $|V| = N$ vertices, $\mathscr{W} = (w_{ij})_{N\times N}$ is the corresponding adjacency matrix and $L_G = (l_{ij})_{N\times N}$ is the corresponding Laplacian matrix. Let $\text{Spec}(L_G) := \{\lambda_1, \ldots, \lambda_N\}$ be the spectrum of L_G, ordered such that $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{N-1} \leq \lambda_N$, and let $\{v^1, \ldots, v^N\}$ be the corresponding eigenvectors.

In what follows, we define the important class of graphs in this work:

Definition 6 (ρ-Versatile Graphs)**.** Let $G = (V, E)$ be a graph and $\rho \in \mathbb{N}$ a positive integer. We say that G is ρ-versatile for the eigenvalue-eigenvector pair (μ, v) with $\mu > 0$, if the Laplacian matrix L_G has a simple eigenvalue μ with corresponding eigenvector $v = (v_1, \ldots, v_N)$, satisfying

$$\sum_{i=1}^{N} v_i^{\ell} \neq 0, \quad \forall \ell = 2, \ldots, \rho + 1. \tag{2.3}$$

Note that any eigenvector $v = (v_1, \ldots, v_N)$ for a non-zero eigenvalue necessarily satisfies $\sum_{i=1}^{N} v_i^1 = 0$. This is because v is orthogonal to the eigenvector $(1, \ldots, 1)$ for the eigenvalue 0.

2.1.1 Versatile graphs by means of the complement graph

We next introduce a method for generating ρ-Versatile Graphs, for any $\rho \in \mathbb{N}$. Our construction involves the definition of the complement graph, given below.

Definition 7. Given an undirected graph G, we define the complement graph G° as the graph obtained from G by leaving out all existing edges and adding all edges between different vertices that were not there in G.

As we are not considering graphs with self-loops, as such edges will turn out to be immaterial in our set-up (because they are in the construction of the Laplacian). We will therefore always have that $(G^{\circ})^{\circ} = G^{\circ\circ} = G$.

Theorem 8. Let G be a graph consisting of precisely two disconnected components, each of a different order. Then G° is a connected graph whose Laplacian has a simple, largest eigenvalue whose eigenvector v satisfies $\sum_{i=1}^{|G^{\circ}|} v_i^{\ell} \neq 0$ for all $\ell > 1$.

More precisely, suppose the two disconnected components have the number of vertices s and t. Then the largest eigenvalue of the Laplacian of G° is equal to $s+t$ and a corresponding eigenvector is given by

$$(\underbrace{t, t, \ldots, t}_{s \text{ times}}, \underbrace{-s, -s, \ldots, -s}_{t \text{ times}}) .$$

Here the entries are ordered so that the vertices of the first component of G (which has s vertices) are enumerated first, after which those of the second component of G (which has t vertices) are listed.

Proof: The proof uses a result that relates the eigenvalues and eigenvectors of the Laplacian of a graph to those of the Laplacian of its complement graph. This result is known, see e.g. (ZHANG, 2011), but incorporated here for completeness. Suppose the two components of G have $s \neq 0$ and $t \neq 0$ vertices, respectively, where $s \neq t$. First of all, recall that the Laplacian L_G is a symmetric matrix whose eigenvalues satisfy.

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N , \tag{2.4}$$

with $N = s+t$ the total number of vertices. In fact, as the dimension of the kernel of L_G equals the number of connected components of G, we see that $\lambda_2 = \lambda_1 = 0$. We denote by $v^1, v^2, v^3, \ldots, v^N$ a corresponding set of orthogonal eigenvectors such that

$$L_G v^i = \lambda_i v^i \quad \forall i = 1, \ldots, N. \tag{2.5}$$

Because

$$\mathrm{span}(v^1, v^2) = \{v \in \mathbb{R}^n \mid v^1 = \cdots = v^s, v^{s+1} = \cdots = v^t\} , \tag{2.6}$$

where the entries are grouped according to the connected components, we see that may choose

$$v^1 = \mathbf{1} = (1,\dots,1) \text{ and } v^2 = (\underbrace{t,t,\dots,t}_{s\text{ times}},\underbrace{-s,-s,\dots,-s}_{t\text{ times}}) \tag{2.7}$$

which we assume from here on out. Note that indeed $v^1 \perp v^2$. Let L_{G° be the Laplacian matrix associated with the complement graph G°. We note that we have the identity

$$L_G + L_{G^\circ} = \begin{pmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & N-1 \end{pmatrix}_{N\times N} = N\cdot \mathrm{Id} - E\,, \tag{2.8}$$

where

$$E = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix}_{N\times N} . \tag{2.9}$$

As we have $v^1 \perp v^i$ for all $i = 2,\dots,N$, where $v^1 = \mathbf{1}$, it follows that $Ev^i = 0$ for all $i = 2,\dots,N$. From Equation (2.8) we get

$$L_{G^\circ} = -L_G + N\cdot \mathrm{Id} - E \tag{2.10}$$

and evaluating at the eigenvectors v^i for $i = 2,\dots,N$ gives

$$\begin{aligned} L_{G^\circ} v^i &= -L_G v^i + N\cdot \mathrm{Id}\, v^i - Ev^i \\ &= -\lambda_i v^i + Nv^i \\ &= (N-\lambda_i)v^i. \end{aligned} \tag{2.11}$$

Thus, for each $i = 2,\dots,N$ we find that $(N-\lambda_i)$ is an eigenvalue of L_{G°, with a corresponding eigenvector given by v^i. As we also have $L_{G^\circ} v^1 = 0$, we see that the spectrum of L_{G° is given by

$$N-\lambda_2 \geq N-\lambda_3 \geq \cdots \geq N-\lambda_N \geq \lambda_1 = 0 \tag{2.12}$$

The largest eigenvalue of L_{G° is therefore equal to $N-0 = N = s+t$, with a corresponding eigenvector given by

$$v = v^2 = (s,s,\dots,s,-t,-t,\dots,-t)\,. \tag{2.13}$$

Note that by assumption $\lambda_3 > 0$, the eigenvalue N is indeed simple. Next, using that $st \neq 0$ and $s \neq t$, we find for all $\ell > 1$

$$\sum_{i=1}^{N} v_i^\ell = \sum_{i=1}^{t} s^\ell \pm \sum_{i=1}^{s} t^\ell = t(s^\ell) \pm s(t^\ell) = st(s^{\ell-1} \pm t^{\ell-1}) \neq 0\,. \tag{2.14}$$

Finally, we argue that G° is a connected graph. Indeed, if $x, y \in G$ are in different connected components, then they share an edge in G° by definition of this latter graph. If on the other hand x and y are in the same component of G, then in G° they both share an edge with some node z from the other component of G. This completes the proof.

■

Example 1. Let $G = (V, E)$ be an undirected graph with $V = \{1, 2, 3\}$ and two disconnected components of a different order $s = 1$ and $t = 2$, see Figure (1). Then G° is a connected, non-regular graph with

$$L_{G^\circ} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}_{3\times 3}.$$

We have $\mathrm{Spec}(L_{G^\circ}) = \{3, 1, 0\}$ with *simple* and *largest eigenvalue* $\mu = s + t = 3$ whose corresponding eigenvector $v = (1, 1, -2)$ satisfies $\sum_{i=1}^{3} v_i^\ell \neq 0$ for all $\ell > 1$.

Example 2. Let $G = (V, E)$ be an undirected graph with $V = \{1, 2, 3, 4, 5\}$ and two disconnected components of a different order $s = 2$ and $t = 3$, see Figure (2). Then G° is a connected graph with

$$L_{G^\circ} = \begin{pmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 3 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{pmatrix}_{5\times 5}.$$

Here $\mathrm{Spec}(L_{G^\circ}) = \{5, 4, 3, 2, 0\}$ with *simple* and *largest eigenvalue* $\mu = s + t = 5$. Its corresponding eigenvector $v = (2, 2, 2, -3, -3)$ satisfies $\sum_{i=1}^{5} v_i^\ell \neq 0$ for all $\ell > 1$.

Example (3) below shows that the standard star graphs are ρ-versatile for any $\rho > 0$. These graphs consist of a single hub node connected to all other nodes, which in turn have degree 1, see Figure (3). We will explore the ρ-versatility of more general star graphs in Subsection (2.1.2).

Example 3 (Star graphs)**.** Let $G = (V, E)$ be an undirected graph with node set $V = \{1, \ldots, N+1\}$ and two disconnected components of order $s = 1$ and $t = N$, see Figure (3). If the largest component of G is complete, then G° is a connected graph with Laplacian matrix given by

$$L_{G^\circ} = \begin{pmatrix} N+1 & -1 & \cdots & -1 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{pmatrix}_{N+1\times N+1}.$$

The spectrum $\operatorname{Spec}(L_{G^\circ}) = \{N+1, 1, \dots, 1, 0\}$ has one *simple* and *largest eigenvalue* $\lambda = s+t = N+1$. The corresponding eigenvector is given by $v = (N+1, -1, \dots, -1)$ which satisfies the property $\sum_{i=1}^{N} v_i^\ell \neq 0$ for all powers $\ell > 1$. Note that we may easily generate more examples of graphs G° with the same simple largest eigenvalue v and with corresponding eigenvector $v = (N+1, -1, \dots, -1)$, namely by allowing the largest component of G to be merely connected, instead of complete.

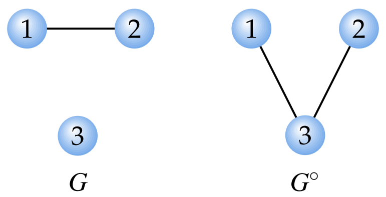


Figure 1 – G consists of precisely two disconnected components of order 1 *and* 2*. The complement G° is a connected and non-regular graph.*

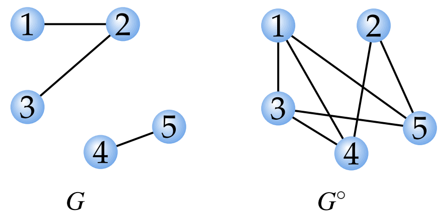


Figure 2 – G consists of precisely two disconnected components of order 2 *and* 3 *and its complement G° is connected.*

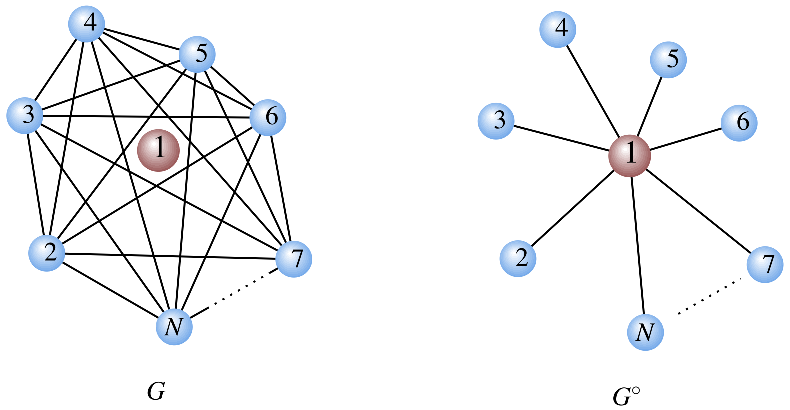


Figure 3 – G consists of precisely two disconnected components of different sizes N and 1*. The complement G° is a connected and non-regular graph known as a* star graph.

In what follows we turn to negative examples. The first of them shows us the importance of starting with connected components of a different order, whereas the second one shows us what goes wrong if we start with more than 2 components.

Example 4. Let $G = (V, E)$ be an undirected graph with $V = \{1, 2, 3, 4\}$ and two disconnected components, this time of the same order $s = t = 2$, see Figure (4). Then G° is a connected graph with

$$L_{G^\circ} = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}_{4\times 4}.$$

Here $\operatorname{Spec}(L_{G^\circ}) = \{4, 2, 2, 0\}$ with *simple* and *largest eigenvalue* $\mu = 4$. However, there are no eigenvectors satisfying $\sum_{i=1}^{4} v_i^\ell \neq 0$ for all $\ell > 1$, except multiples of $\mathbf{1} =$

$(1,1,1,1)$. In fact, the eigenvectors for the other eigenvalues all satisfy $\sum_{i=1}^{4} v_i^{\ell} = 0$ whenever ℓ is odd. This example seems to indicate that symmetry can sometimes get in the way of ρ-versatility.

Example 5. Let $G = (V,E)$ be an undirected graph with $V = \{1,2,3,4,5,6\}$ and three disconnected components of order $s = 1$, $t = 2$ and $r = 3$, see Figure (5). Then G° is a connected graph with

$$L_{G^{\circ}} = \begin{pmatrix} 4 & 0 & -1 & -1 & -1 & -1 \\ 0 & 3 & 0 & -1 & -1 & -1 \\ -1 & 0 & 4 & -1 & -1 & -1 \\ -1 & -1 & -1 & 4 & 0 & -1 \\ -1 & -1 & -1 & 0 & 4 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{pmatrix}_{6\times 6} .$$

We have $\mathrm{Spec}(L_{G^{\circ}}) = \{6,6,5,4,3,0\}$ with *non-simple* and *largest eigenvalue* $\lambda_{1,2} = 6$. However, the two corresponding eigenvectors are given by $(-1,-1,-1,0,0,3)$ and $(-2,-2,-2,3,3,0)$, which both satisfy $\sum_{i=1}^{6} v_i^{\ell} \neq 0$ for all powers $\ell > 1$.

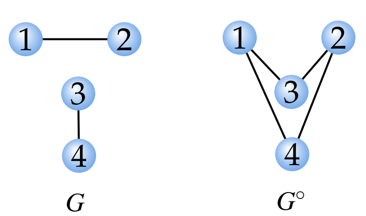


*Figure 4 – G consists of two disconnected components both of the same order 2 and the complement G° is connected. The Laplacian $L_{G^{\circ}}$ has a simple and largest eigenvalue. However, there are no eigenvectors giving the ρ-versatility condition for $\rho > 1$, except for multiples of **1**.*

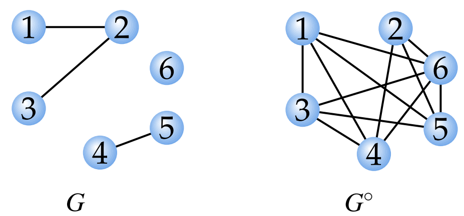


Figure 5 – G consists of three disconnected components, each of a different order. The complement G° is connected, but the largest eigenvalue of its Laplacian is non-simple.

2.1.2 Versatile graphs by means of the degree distribution

We next investigate another pathway to ρ-versatility, namely by looking at the degree distribution of the nodes in the network. To this end, we will prove:

Proposition 9. Let $r < B$ be two positive integers and suppose $G = (V,E)$ is a network consisting of one node of degree B and N nodes of degree at most r, where $N \geq 1$. If

$$\frac{B+1}{r} > \sqrt[3]{N} + 1, \tag{2.15}$$

then the largest eigenvalue of L_G is simple and every corresponding eigenvector v satisfies

$$\sum_{i=0}^{N} v_i^{\ell} \neq 0\,, \text{ for all } \ell > 1\,. \tag{2.16}$$

The proof of Proposition (9) uses the well-known result that for any graph G with at least one edge, the largest eigenvalue μ of L_G satisfies $\mu \geq d+1$, with d the largest degree of any node in G. See for instance (ZHANG, 2011). In the setting of Proposition (9), we, therefore, have $\mu \geq B+1$.

Proof: Let μ be the largest eigenvalue of L_G and write $v \in \mathbb{R}^{N+1}$ for a corresponding eigenvector. In general, we will write $v = (v_i)_{i=0}^{N}$ for the components of a vector $v \in \mathbb{R}^{N+1}$, where the 0-component v_0 corresponds to the unique node of G with degree B. By re-scaling v, we may assume that $|v_i| \leq 1$ for all $i \in \{0,\dots,n\}$. The condition that (μ, v) is an eigenvalue-eigenvector pair for L_G gives

$$\sum_{j=0}^{N} (L_G)_{i,j} v_j = \mu v_i \text{ for all } i \in \{0,\dots,n\}\,. \tag{2.17}$$

We therefore find

$$\sum_{\substack{j=0 \\ j\neq i}}^{N} (L_G)_{i,j} v_j = (\mu - d_i) v_i\,, \tag{2.18}$$

where d_i denotes the degree of node i. From our observation that $\mu \geq B+1$ we see that $(\mu - d_i)$ is always positive. For $i \neq 0$ we therefore get from Equation (2.18)

$$(B+1-r)|v_i| \leq (\mu - d_i)|v_i| = \left| \sum_{\substack{j=0 \\ j\neq i}}^{N} (L_G)_{i,j} v_j \right| \leq \sum_{\substack{j=0 \\ j\neq i}}^{N} |(L_G)_{i,j}||v_j| \leq d_i \leq r\,. \tag{2.19}$$

We thus find

$$|v_i| \leq \frac{r}{(B+1-r)} = \frac{1}{(B+1)/r - 1} < \frac{1}{\sqrt[3]{N}} \leq 1\,. \tag{2.20}$$

Summarizing, we see that the condition $|v_i| \leq 1$ for all $i \in \{0,\dots,n\}$ yields $|v_i| < 1$ for all $i \in \{1,\dots,n\}$. This is only possible if $v_0 \neq 0$, which therefore has to hold for any eigenvector v of μ.

Now suppose v and v' are two eigenvectors for μ. By the foregoing, there exists a nonzero scalar s such that the vector $sv - v'$ has to vanish the zeroth component. As nevertheless $L_G(sv - v') = \mu(sv - v')$, we conclude that $sv - v' = 0$ and so $v' = sv$. This shows that the eigenvalue μ is simple.

To prove the ρ-versatility claim, we re-scale the eigenvector v such that $|v_i| \leq 1$ for all $i \in \{0,\dots,n\}$, with $v_j = 1$ for at least one j. By the foregoing, this means that necessarily $v_0 = 1$, with the other v_i satisfying Equation (2.20). We conclude that for all $\rho \geq 3$ we have

$$\left|\sum_{i=0}^{N} v_i^{\rho}\right| = \left|1 + \sum_{i=1}^{N} v_i^{\rho}\right| \geq 1 - \sum_{i=1}^{N} |v_i|^{\rho} > 1 - \frac{N}{(\sqrt[3]{N})^{\rho}} = 1 - N^{1-\rho/3} \geq 1 - N^0 = 0\,.$$

Therefore, $\sum_{i=0}^{N} v_i^{\rho} \neq 0$ for all $\rho \geq 3$. As we clearly have $\sum_{i=0}^{N} v_i^2 > 0$, the result follows.

■

Example 6. Examples of connected networks satisfying the conditions of Proposition (9) can easily be constructed. Let $r, N > 0$ be given numbers such that

$$r \leq \frac{N}{\sqrt[3]{N}+1}\,. \tag{2.21}$$

We first construct a graph G' consisting of N nodes, all of which have a degree at most $r-1$. The graph G is then obtained from G' by adding a node n_0, together with $B \geq (\sqrt[3]{N}+1)r$ edges between n_0 and different nodes of G'. Note that Condition (2.21) guarantees that $(\sqrt[3]{N}+1)r \leq N$ so that we are not demanding that n_0 is connected to more nodes than G' contains. It follows that all nodes in G apart from n_0 have a degree at most r. Finally, the degree B of n_0 satisfies

$$B+1 > B \geq (\sqrt[3]{N}+1)r\,, \tag{2.22}$$

so that

$$\frac{B+1}{r} > \sqrt[3]{N}+1\,. \tag{2.23}$$

The graph G is connected if an edge was added from n_0 to at least one node from every connected component of G'.

To conclude our discussion on ρ-Versatile Graphs, we fix values $n, \rho \in \mathbb{N}$ with $n > 2$ and define $\mathscr{S}_n^{\rho}$ as the set of all symmetric $(n \times n)$ matrices with a simple largest eigenvalue, whose corresponding eigenvector v satisfies

$$\sum_{i=1}^{n} v_i^{\ell} \neq 0 \text{ for all } \ell \in \{2, \dots \rho+1\}\,. \tag{2.24}$$

It follows that $\mathscr{S}_n^{\rho}$ is an open subset of the space of symmetric matrices. Heuristically speaking, if G is a graph such that $L_G \in \mathscr{S}_n^{\rho}$, then we therefore expect $L_{G'} \in \mathscr{S}_n^{\rho}$ for any graph G' obtained from G by a small perturbation.

In fact, as matrices generically have simple eigenvalues, and as Equation (2.24) is likewise valid for generic (eigen)vectors, we expect $L_G \in \mathscr{S}_n^{\rho}$ for "most" graphs G. Of course, these statements will have to be made precise, which we do not attempt here.

One common obstruction to $L_G \in \mathscr{S}_n^\rho$ seems to be symmetry in the graph G. An explanation for this is that symmetry often forces eigenvalues with high multiplicity. Moreover, if an eigenvalue μ of L_G is simple, then the span of a corresponding eigenvector v forms a 1-dimensional representation of the symmetry in question. For any finite group symmetry, a one-dimensional real representation is either trivial or generated by $v \mapsto -v$. In the latter case, the graph symmetry contains a transformation α such that for any node n of G, the corresponding coefficients v_n and $v_{\alpha(n)}$ of v are related by $v_n = -v_{\alpha(n)}$. This means that for any value $c \in \mathbb{R}$ there is an equal number of nodes n such that $v_n = c$ as there are nodes m such that $v_m = -c$. As a consequence, we then necessarily have

$$\sum_{i=1}^{n} v_i^\ell = 0 \text{ for all odd } \ell > 0\,. \tag{2.25}$$

This is a common observation; imposing additional structure on a graph (such as for instance symmetry) induces high dimensional center subspaces and restrictions to the Taylor-coefficients of reduced vector fields in the associated dynamical systems. This generally leads to more elaborate bifurcation scenarios.

2.2 Skewness condition

In this section, we will define what we call the Skewness condition. This is a property about a class of square matrices that are not symmetric and are not anti-symmetric. The most important consequence since that condition is imposed in a matrix A will be to ensure the existence of zero eigenvalues via the existence of a positive-definite matrix D.

Definition 10 (Skewness condition)**.** Let A be a n by n matrix. We call by *Skewness condition* as the following property is satisfied

$$\langle v_i, Av_i \rangle > 0 \text{ for all } i = 1, \ldots, m\,. \tag{2.26}$$

for the existence of $v_1, \ldots, v_m$ mutually orthogonal vectors.

We will give examples of Hurwitz matrices A such that there exist $m > 0$ mutually orthogonal vectors $v_1, \ldots, v_m$ satisfying Definition (10). We will see, therefore, that a Hurwitz matrix is not an obstruction for the Skewness condition.

Remark 11. Note that any Hurwitz matrix A has a negative trace, as this number equals the sum of its eigenvalues. It follows that Equation (2.26) can then only hold when $m < n$, where n is the size of A. Consequently, we highlight that if our goal is to find a number 3 of zero eigenvalues we must consider matrices A of n by n with at least $n = 4$.

We start by looking at the case $n = 2$.

Example 7. A general 2 by 2 matrix A is of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.27}$$

with $a, b, c, d \in \mathbb{R}$. It follows that A is Hurwitz if and only if $a + d < 0$ and $ad - bc > 0$. This can easily be arranged if in addition $a > 0$, by first choosing $d < 0$ such that $a + d < 0$, and then choosing b and c such that $ad - bc > 0$. It follows that we can construct examples of Hurwitz matrices A such that Equation (2.26) holds with $m = 1$ and $v_1 = (1, 0)^T$. In fact, the set of all such matrices forms a non-empty open subset of the space of all 2 by 2 matrices. A similar observation of course holds when $d > 0$.

Example 8. Consider the 3 by 3 matrix

$$A = \begin{pmatrix} a+b+c & e & d \\ c & a+e & b+d \\ c & b+e & a+d \end{pmatrix}, \tag{2.28}$$

for $a, b, c, d, e \in \mathbb{R}$. Using the theory of network multipliers, it is shown in (DEVILLE; NIJHOLT, 2021) that the eigenvalues of A are given by $a+b+c+d+e$, $a-b$, and $a+b$. It is therefore clear that for certain choices of a through e we can arrange for A to be Hurwitz. Moreover, these eigenvalues do not change if we apply the transformation

$$\begin{aligned} c &\mapsto c - 2\delta \\ d &\mapsto d + \delta \\ e &\mapsto e + \delta \end{aligned} \tag{2.29}$$

for any $\delta \in \mathbb{R}$, while keeping a and b the same. Hence, if A as given by Equation (2.28) is Hurwitz, then so is the matrix

$$A_\delta = \begin{pmatrix} a+b+c-2\delta & e+\delta & d+\delta \\ c-2\delta & a+e+\delta & b+d+\delta \\ c-2\delta & b+e+\delta & a+d+\delta \end{pmatrix} \tag{2.30}$$

for any $\delta \in \mathbb{R}$. Choosing δ large enough, we see that Equation (2.26) holds for $m = 2$ and $v_1 = (0, 1, 0)^T$, $v_2 = (0, 0, 1)^T$.

Example 9. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 16.94 \\ 1 & -4.24 & -4.24 & -17.94 \end{pmatrix}.$$

It can be shown that A is Hurwitz. Moreover, it is clear that Equation (2.26) holds for $m = 3$ and the vectors $v_1 = (1, 0, 0, 0)^T$, $v_2 = (0, 1, 0, 0)^T$ and $v_3 = (0, 0, 1, 0)^T$.

Finally, we show:

Proposition 12. Let $V = \{v_1, \dots, v_{n-1}\}$ be a set of $n-1$ mutually orthogonal vectors in $\mathbb{R}^n$, where $n > 1$. Denote by $\mathscr{H}_V^n$ the set of all $(n \times n)$ Hurwitz matrices A such that

$$\langle v_i, Av_i \rangle > 0 \text{ for all } i = 1, \dots, n-1 \,. \tag{2.31}$$

Then, $\mathscr{H}_V^n$ forms a non-empty open subset of the space of all $(n \times n)$ matrices.

Proof: As the set of all Hurwitz, matrices is open, and because the same holds for the set of all matrices A for which Equation (2.31) holds, we see that $\mathscr{H}_V^n$ is likewise open. It remains to show that $\mathscr{H}_V^n$ is non-empty. We will first show this when E consists of the first $n-1$ standard vectors $e_1 = (1, 0, \dots, 0)^T$, $e_2 = (0, 1, \dots, 0)^T$ and so forth, up to $e_{n-1} = (0, \dots, 1, 0)^T$. Given numbers $b_1, \dots, b_n \in \mathbb{R}$, we define the $(n \times n)$ matrix

$$A_{b_1,\dots,b_n} = \begin{pmatrix} b_1 & b_2 & \dots & b_n \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \dots & b_n \end{pmatrix} .$$

As $A_{b_1,\dots,b_n}$ has rank 1, we see that it has an $(n-1)$-dimensional kernel. The remaining eigenvalue is given by $b_1 + \cdots + b_n$ with eigenvector $(1, \dots, 1)^T$. Let us choose $b_1, \dots, b_{n-1} > 0$ and $b_n < -(b_1 + \cdots + b_{n-1})$. We also choose $a \in \mathbb{R}$ such that

$$0 < a < \min(b_1, \dots, b_{n-1}) \,.$$

As a result, we see that the matrix

$$A_{b_1,\dots,b_n} - a\,\mathrm{Id}_n = \begin{pmatrix} b_1 - a & b_2 & \dots & b_n \\ b_1 & b_2 - a & \dots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \dots & b_n - a \end{pmatrix}$$

has eigenvalues $b_1 + \cdots + b_n - a < 0$ and $-a < 0$. Moreover, because $b_i - a > 0$ for all $i \in \{1, \dots, n-1\}$, we conclude that $A_{b_1,\dots,b_n} - a\,\mathrm{Id}_n \in \mathscr{H}_E^n$ where $E = \{e_1, \dots, e_{n-1}\}$.

To show that $\mathscr{H}_V^n$ is non-empty for general V, we pick $V = \{v_1, \dots, v_{n-1}\}$ and extend it to an orthogonal basis $\{v_1, \dots, v_{n-1}, v_n\}$. Let U be the matrix such that $Ue_i = v_i$ for all $i \in \{1, \dots, n\}$. It follows that U^TU equals a diagonal matrix C with positive diagonal entries given by $\langle v_i, v_i \rangle = \|v_i\|^2$. In particular, we have $U^T = CU^{-1}$. Now suppose we pick an element $A \in \mathscr{H}_E^n$. It follows that UAU^{-1} is Hurwitz as well. Moreover, for all $i \in \{1, \dots, n-1\}$ we find

$$\begin{aligned} \langle v_i, UAU^{-1}v_i \rangle &= \langle U^T v_i, AU^{-1}v_i \rangle = \langle CU^{-1}v_i, AU^{-1}v_i \rangle \\ &= \langle Ce_i, Ae_i \rangle = \|v_i\|^2 \langle e_i, Ae_i \rangle > 0 \,. \end{aligned} \tag{2.32}$$

This shows that $\mathscr{H}_V^n$ is likewise non-empty, which concludes the proof.

■

Note that we have $A \in \mathscr{H}_V^n \implies cA \in \mathscr{H}_V^n$ for all $c \in \mathbb{R}_{>0}$. Taking the union over all the sets $\mathscr{H}_V^n$, we arrive at:

Corollary 13. Given $n > 1$, the set of all $(n \times n)$ Hurwitz matrices A for which some orthogonal vectors $v_1, \ldots, v_{n-1}$ exist satisfying

$$\langle v_i, Av_i \rangle > 0 \text{ for all } i = 1, \ldots, n-1\,, \tag{2.33}$$

is open and non-empty.

We will see in Proposition (56) of Subsection (8.1.1) that the Skewness condition for a number m of mutually orthogonal vectors is the necessary and sufficient condition for the existence of a positive-definite matrix D such that $A - D$ has a non-trivial kernel of dimension equal to m.

CHAPTER

3

CENTER MANIFOLD THEORY

In this chapter, we will introduce the fundamental concepts concerning Center Manifold Theory. Our main objective is proof of the Center Manifold Theorem. We know from the Hyperbolic Theory that a dynamical behaviour of a system around its hyperbolic fixed point is completely solved by the Hartman-Grobman Theorem, which means that the behaviour of the nonlinear dynamical system is topologically conjugated to its linearized system at the hyperbolic fixed point, which means that the nonlinearities do not cause any effect in the system and so, it can be avoided. The Center Manifold Theory generalizes the studies of stability for dynamical systems around non-hyperbolic fixed points (eigenvalues with zero real parts). In this case, the nonlinearities of the system are indispensable for an understanding of the behaviour. The reasons why the Center Manifold Theory is important are, first, to reduce the dimension of the original system, and second, every relevant behaviour of the original system is captured by the center manifold like bounded solutions (fixed points, homoclinic and heteroclinic orbits, periodic orbits), therefore can display Chaos. The fundamental concepts are supported by (GUCKENHEIMER; HOLMES, 2013; WIGGINS; WIGGINS; GOLUBITSKY, 2003; LAWRENCE, 1991) we will present the Local Center Manifold Theorem and it will be proved following very much (CARR, 1979; BRESSAN, 2003).

3.1 Statements of the center manifold theorem

The main objective of this section will be to introduce the necessary definitions, notations, and tools to provide the background to construct the center manifold. Several of these will be quickly recognized from hyperbolic theory but adapted to the general case. To the proof the Center Manifold Theorem, the Contraction Mapping Principle is essential, therefore it will be stated and proved for the completeness of this chapter. We will present the Local Center Manifold Theorem and it will be proved following very much of excellent textbooks (CARR, 1979; BRESSAN, 2003).

We assume $f \in \mathscr{C}^{k+1}(\mathbb{R}^n)$ be a $k+1$ $(k \geq 1)$ continuously differentiable vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ with fixed point at origin 0, that is $f(0) = 0$. We consider an Ordinary Differential Equation in the Taylor expansion around the fixed point:

$$\dot{x} = f(x) = Ax + g(x) \tag{3.1}$$

where $A = Df(0)$ is the $(n \times n)$ Jacobian matrix and $g(x)$ are nonlinear terms $\mathscr{O}(|x|^2)$ with $g(0) = \mathbf{0}$ and $Dg(0) = \mathbf{0}$. Given any point $x \in \mathbb{R}^n$ we denote the unique solution of the System (3.1) starting at x $(t = 0)$ defined in the maximal interval of $\mathbb{R}$ containing 0 by

$$\bar{x}(t,x) \tag{3.2}$$

such that $\bar{x}(0,x) = x$. We denote, for the system of Equation (3.1) linearized at the fixed point:

$$\dot{x} = Ax \tag{3.3}$$

the unique solution crossing x by $\bar{x}(t,x) = e^{At}x$, where, it is known that

$$e^{At} = \sum_{k=0}^{+\infty} \frac{A^k t^k}{k!}.$$

In the study of the dynamic behaviour of the solutions of the System (3.3), the spectral properties of A play an important role. Therefore, we denote the *Spectrum* of A by $\sigma(A) := \{\lambda \in \mathbb{C} | \ \lambda \text{ is eigenvalue of } A\}$. We also denote the subsets:

$$\begin{aligned} \sigma_c(A) &:= \{\lambda \in \sigma(A) | \ \mathrm{Re}(\lambda) = 0\} \\ \sigma_s(A) &:= \{\lambda \in \sigma(A) | \ \mathrm{Re}(\lambda) < 0\} \\ \sigma_u(A) &:= \{\lambda \in \sigma(A) | \ \mathrm{Re}(\lambda) > 0\} \end{aligned} \tag{3.4}$$

called *Center*, *Stable* and *Unstable* parts formed by numbers c, s, and u of eigenvalues with zero real, negative real, and positive real parts respectively such that $n = c + s + u$. Clearly, $\sigma(A) = \{\sigma_c(A), \sigma_s(A), \sigma_u(A)\}$.

From Linear theory there exists linear invariant subspaces, namely E^c, E^s and E^u corresponding to center, stable and unstable subspaces generated by eigenvectors corresponding to eigenvalues in $\sigma_c(A), \sigma_s(A)$, and $\sigma_u(A)$ whose dimensions are c, s and u respectively. In the case when $c = 0$, it implies that $\sigma_c = \emptyset$ and then all dynamics of the System (3.1) is completely solved by Hartman-Grobman Theorem, as a consequence, we do not influence its nonlinearities g. Otherwise, when $c \neq 0$, the Linear theory is not enough to conclude about stable properties of the system, and in this case, the nonlinearities play an important role in the study of the stability and primordial to Bifurcation Theory.

Naturally, we have in the general case the full space decomposed into the subspaces given by direct sum:

$$\mathbb{R}^n = E^c \oplus E^s \oplus E^u. \tag{3.5}$$

For each $x \in \mathbb{R}^n$ we can project it on in the center, stable and unstable components:

$$x = \pi^c x + \pi^s x + \pi^u x \tag{3.6}$$

where $\pi^c : \mathbb{R}^n \to E^c$, $\pi^s : \mathbb{R}^n \to E^s$ and $\pi^u : \mathbb{R}^n \to E^u$ are such projections. We note that, $\ker(\pi^c) := \{x \in \mathbb{R}^n |\ \pi^c x = 0\} = E^s \oplus E^u = E^h$, $\ker(\pi^s) := \{x \in \mathbb{R}^n |\ \pi^s x = 0\} = E^c \oplus E^u = E^{cu}$ and $\ker(\pi^u) := \{x \in \mathbb{R}^n |\ \pi^u x = 0\} = E^s \oplus E^c = E^{sc}$. We also denote, $\pi^h = \pi^s + \pi^u$ by projection into hyperbolic space E^h. Moreover, the projections are commutatively related to A then also are concerning solutions of the System (3.3):

$$\pi e^{At} = e^{At} \pi. \tag{3.7}$$

In order to study the exponential behaviour of the solutions of the system of Equation (3.3) in the linear subspaces E^c, E^s and E^u we define

$$\begin{aligned} \beta_- &:= \max\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma_s(A)\} \\ \beta_+ &:= \min\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma_u(A)\} \\ \beta &:= \min\{-\beta_-, \beta_+\} > 0 \end{aligned} \tag{3.8}$$

where β is called the *Spectral gap*. The following lemma gives us growth estimates for solutions in E^c and E^h :

Lemma 14. For each $\varepsilon \in (0, \beta)$ there is a constant depending on ε, $M_\varepsilon > 0$, such that

$$||e^{At}\pi^c|| \leq M_\varepsilon e^{\varepsilon|t|}, \quad \forall t \in \mathbb{R}; \tag{3.9}$$

$$||e^{At}\pi^s|| \leq M_\varepsilon e^{-(\beta-\varepsilon)t}, \quad \forall t \geq 0; \tag{3.10}$$

$$||e^{At}\pi^u|| \leq M_\varepsilon e^{(\beta-\varepsilon)t}, \quad \forall t \leq 0. \tag{3.11}$$

In what follows, we define the *cut-off function* (or *Bump function*) which is an indispensable tool for a modification of g having a compact support.

Definition 15 (Cut-off function)**.** Let $\mathscr{X} : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The cut-off function has the following properties:

(1) $0 \leq \mathscr{X}(x) \leq 1$;

(2) $\mathscr{X}(x) = 1$, if $||x|| \leq 1$;

(3) $\mathscr{X}(x) = 0$, if $||x|| \geq 2$.

An example of a cut-off function:

Example 10. Let $\mathscr{X} : \mathbb{R} \to \mathbb{R}$ be a function given by:

$$\mathscr{X}(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & x \in (-1,1) \\ 0 & \text{otherwise.} \end{cases}$$

The support of $\mathscr{X}$ is defined as being a closure set of all points $x \in \mathbb{R}$ such that $\mathscr{X}(x) \neq 0$. More precisely,

$$supp(\mathscr{X}) := \overline{\{x \in \mathbb{R} |\ \mathscr{X}(x) \neq 0\}}.$$

In this example we see that the support is a compact set.

We remember that a Banach space is a completed normed vector space, which means, every Cauchy sequence of vectors converges to a well-defined limit inside this space.

Definition 16. Let X and Y be Banach spaces and $k \in \mathbb{N}$. We define

$$\mathscr{C}_b^k(X,Y) := \{w \in \mathscr{C}^k(X,Y);\ |w|_j := \sup_{x \in X} ||D^j w(x)|| < \infty, \quad \forall \quad 0 \leq j \leq k\} \tag{3.12}$$

with norm defined in $\mathscr{C}_b^k(X,Y)$ by

$$||w||_{\mathscr{C}_b^k} := \max_{0 \leq j \leq k} |w|_j. \tag{3.13}$$

Next, we present the main result of this section which is very important for subsequent results:

Theorem 17 (Contraction Mapping Principle)**.** Let X, Y be Banach spaces. If $\Theta : X \times Y \to Y$ is a continuous map such that:

$$||\Theta(x; y_1) - \Theta(x; y_2)|| \leq C||y_1 - y_2|| \tag{3.14}$$

for all $x \in X$ and $y_1, y_2 \in Y$ with a constant $C < 1$ independent of variables. In these conditions, Θ is a strict contraction. Then the following holds.

(1) For every $x \in X$, there exists a unique $y(x) \in Y$ such that

$$y(x) = \Theta(x; y(x)). \tag{3.15}$$

(2) For every $x \in X$, $y \in Y$ one has

$$||y - y(x)|| \leq \frac{1}{1-C}||y - \Theta(x;y)||. \tag{3.16}$$

(3) If Θ is Lipschitz continuous with respect to variable x:

$$||\Theta(x_1,y)-\Theta(x_2,y)|| \leq L||x_1-x_2|| \tag{3.17}$$

for all $x_1,x_2 \in X$, $y \in Y$. Then, the map $x \mapsto y(x)$ is also Lipschitz continuous:

$$||y(x_1)-y(x_2)|| \leq \frac{L}{1-C}||x_1-x_2|| \tag{3.18}$$

for all $x_1,x_2 \in X$.

(4) Consider any convergent sequence $x_n \to x_0$ in X. Then, for every $y_0 \in Y$ the sequence of iterates

$$y_{n+1} := \Theta(x_n;y_n) \longrightarrow y_0 := y(x_0). \tag{3.19}$$

Proof: To the Item (1) we fix any $y \in Y$ and for each $x \in X$ we construct the iterate sequence:

$$y_0 := y, \quad y_1 := \Theta(x;y_0), \quad \dots \quad y_{n+1} := \Theta(x;y_n). \tag{3.20}$$

For all $n \geq 0$, the previous sequence is a Cauchy sequence. Indeed,

$$||y_{n+1}-y_n|| = ||\Theta(x;y_n)-\Theta(x;y_{n-1})|| \leq C||y_n-y_{n-1}|| \leq \cdots \leq C^n||y_1-y_0|| \tag{3.21}$$

as $C<1$ and the strict contraction is valid for all $y \in Y$, then the sequence is a Cauchy sequence. Since that Y is Banach space, $y_n \in Y$ converges to a limit point in Y. Let $y(x)$ be the limit point such that $y_n \to y(x)$. The continuity of Θ implies:

$$y(x) = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty}\Theta(x;y_n) = \Theta(x;\lim_{n\to\infty} y_n) = \Theta(x;y(x)). \tag{3.22}$$

Therefore, $y(x) = \Theta(x;y(x))$. Suppose we have $y_1(x) = \Theta(x;y_1(x))$ and $y_2(x) = \Theta(x;y_2(x))$ two fixed points. We calculate

$$||y_1(x)-y_2(x)|| = ||\Theta(x;y_1(x))-\Theta(x;y_2(x))|| \leq C||y_1(x)-y_1(x)|| \tag{3.23}$$

since that $C<1$ we only have $y_1 = y_2$. Therefore, for a strict contraction there, exists a unique fixed point.

Item (2). By Inequation (3.21) and Equation (3.22) we have:

$$\begin{aligned} ||y_0-y(x)|| &\leq ||y_1-y_0||+||y_2-y_1||+||y_3-y_2||+\cdots+||y_n-y(x)||+\cdots \\ &= \sum_{j=0}^{\infty}||y_{n+1}-y_n|| \\ &\leq \sum_{j=0}^{\infty}C^n||y_1-y_0|| \\ &= \frac{1}{1-C}||\Theta(x;y_0)-y_0||. \end{aligned} \tag{3.24}$$

Therefore, as $y := y_0$ we get the required item.

Item (3). By Inequation (3.24), we take $y = y(x')$ and we calculate:

$$\begin{aligned}||y(x') - y(x)|| &\leq \frac{1}{1-C}||\Theta(x; y(x')) - y(x')|| \\ &= \frac{1}{1-C}||\Theta(x; y(x')) - \Theta(x'; y(x'))|| \\ &\leq \frac{L}{1-C}||x - x'||.\end{aligned}$$

Therefore, the map $x \mapsto y(x)$ is also Lipschitz continuous.

Item (4). Let $x_n \to \bar{x}$ be a convergent sequence in X, and we consider the iterate sequence defined by $y_{n+1} := \Theta(x_n; y_n)$ in Y. We want to prove that $y_n \to \bar{y}(\bar{x}) = \Theta(\bar{x}; \bar{y})$. By continuity, we have

$$k_n := ||\Theta(x_n; \bar{y}) - \Theta(\bar{x}; \bar{y})|| \to 0 \qquad \text{as} \qquad n \to \infty.$$

We calculate, by strict contraction that

$$\begin{aligned}||y_{n+1} - \bar{y}|| &= ||\Theta(x_n; y_n) - \Theta(\bar{x}; \bar{y})|| \\ &\leq ||\Theta(x_n; y_n) - \Theta(x_n; \bar{y})|| + ||\Theta(x_n; \bar{y}) - \Theta(\bar{x}; \bar{y})|| \\ &\leq C||y_n - \bar{y}|| + k_n. \qquad (3.25)\end{aligned}$$

Using Inequality (3.25) iteratively, we get:

$$\begin{aligned}||y_n - \bar{y}|| &\leq C||y_{n-1} - \bar{y}|| + k_{n-1} \\ &\leq C^2||y_{n-2} - \bar{y}|| + Ck_{n-2} + k_{n-1} \\ &\leq C^3||y_{n-3} - \bar{y}|| + C^2k_{n-3} + Ck_{n-2} + k_{n-1} \\ &\vdots \\ &\leq C^n||y_0 - \bar{y}|| + C^{n-1}k_0 + C^{n-2}k_1 + \cdots + k_{n-1} \\ &= C^n||y_0 - \bar{y}|| + \sum_{j=1}^{n} C^{n-j}k_{j-1}.\end{aligned}$$

Since that $C < 1$ we have $C^n \to 0$ and $\sum_{j=1}^{n} C^{n-j} \to \frac{1}{1-C}$ and $k_n \to 0$ as $n \to \infty$. Thus,

$$\limsup ||y_n - \bar{y}|| = 0 \qquad \text{as} \qquad n \to \infty.$$

Therefore, we conclude $\Theta(x_n; y_n) \to \Theta(\bar{x}; \bar{y})$ provided that $x_n \to \bar{x}$ and $\bar{y}(\bar{x}) = \Theta(\bar{x}; \bar{y})$.

■

3.2 Center manifold theorem

Definition 18 (Local Center Manifold)**.** Let $f \in \mathscr{C}^{k+1}(\mathbb{R}^n)$ be $k+1$ $(k \geq 1)$ continuously differentiable function defined with a non-hyperbolic fixed point at zero. The Local

Center Manifold is locally the graph of a function $\phi : E^c \to E^h$ near to fixed point:

$$\mathscr{E}^c_{loc}(0) := \{(x,y) \in E^c \times E^h |\ y = \phi(x),\ \phi(0) = 0,\ D\phi(0) = \mathbf{0},\ |x| < \delta\} \tag{3.26}$$

for some $\delta > 0$.

Remark 19. The Definition (18) says that the Local Center Manifold cross-origin at fixed point $\phi(0) = 0$ and it is tangent to Center subspace E^c locally given by condition $D\phi(0) = 0$. We highlight that when we have the nonlinear term identically zero, $g \equiv \mathbf{0}$, the center subspace and manifold agree. The proof of the Center Manifold Theorem concerns the existence and uniqueness of a globally center manifold where is strongly used Contraction Mapping Principle (17) and a *modification* of g by a cut-off function, Definition (15). We also highlight that the uniqueness relies on the cut-off function, therefore there is no guarantee of the uniqueness for the local center manifold.

First of all, we prove a characterization of all points in the linear subspace E^c. We assume the solution of the System (3.3) given by

$$\bar{x}(t, x_c) = e^{At} x_c$$

for points denoted by $x_c \in E^c$.

Lemma 20. The following sets are the same

$$E^c = \{x \in \mathbb{R}^n |\ \sup_{t\in\mathbb{R}} ||\pi^h \bar{x}(t,x)|| < \infty\} \tag{3.27}$$

and, for each $\eta \in (0, \beta)$:

$$E^c = \{x \in \mathbb{R}^n |\ \sup_{t\in\mathbb{R}} e^{-\eta|t|} ||\bar{x}(t,x)|| < \infty\}. \tag{3.28}$$

Proof: We prove the following inclusions

$$E^c \subset \{x \in \mathbb{R}^n |\ \sup_{t\in\mathbb{R}} ||\pi^h \bar{x}(t,x)|| < \infty\} \overset{\eta\in(0,\beta)}{\subset} \{x \in \mathbb{R}^n |\ \sup_{t\in\mathbb{R}} e^{-\eta|t|} ||\bar{x}(t,x)|| < \infty\} \subset E^c.$$

First inclusion, given any $x_c \in E^c$ we know that $\pi^h \bar{x}(t, x_c) = \pi^h e^{At} x_c = e^{At} \pi^h x_c = 0$. Therefore, $\pi^h \bar{x}(t, x_c)$ is bounded for all $t \in \mathbb{R}$. Second inclusion, given any $x \in \mathbb{R}^n$ and a solution $\bar{x}(t, x) \in \mathbb{R}^n$ such that $||\pi^h \bar{x}(t,x)|| \leq M_1$ for all $t \in \mathbb{R}$. Given $\eta \in (0, \beta)$ and by Lemma (14) we have

$$\begin{aligned} ||\bar{x}(t,x)|| &\leq ||\pi^c e^{At} x|| + ||\pi^h \bar{x}(t,x)|| \\ &\leq M_2 e^{\eta|t|} ||x|| + M_1 \\ &\leq (M_2 ||x|| + M_1) e^{\eta|t|} \\ &\leq M e^{\eta|t|} \end{aligned}$$

where $M = M_2||x|| + M_1 > 0$ is a constant independent of t. Therefore, we have that $e^{-\eta|t|}||\bar{x}(t,x)|| \leq M$ for all $t \in \mathbb{R}$. Third inclusion, given any $x \in \mathbb{R}^n$ such that $||\bar{x}(t,x)|| \leq Me^{\eta|t|}$ for all $t \in \mathbb{R}$ and $\eta \in (0,\beta)$. We calculate, for $t \geq 0$ and $\varepsilon > 0$ we have,

$$\begin{aligned} ||\pi^s x|| &= ||e^{At}\pi^s e^{-At}x|| \\ &= ||e^{At}\pi^s|| \cdot ||e^{-At}x|| \\ &= M_\varepsilon e^{-(\beta-\varepsilon)t} M e^{\eta t} \\ &= M_\varepsilon e^{-(\beta-\varepsilon-\eta)t} \end{aligned}$$

we take $\varepsilon \in (0,\beta)$ such that $\varepsilon < \beta - \eta$. Thus, $||\pi^s x|| \equiv 0$. Similarly, for $t \leq 0$ and $\varepsilon > 0$ we have

$$\begin{aligned} ||\pi^u x|| &= ||e^{At}\pi^u e^{-At}x|| \\ &= ||e^{At}\pi^u|| \cdot ||e^{-At}x|| \\ &= M_\varepsilon e^{(\beta-\varepsilon)t} M e^{-\eta t} \\ &= M_\varepsilon e^{(\beta-\varepsilon-\eta)t} \end{aligned}$$

we take $\varepsilon \in (0,\beta)$ such that $\varepsilon < \beta - \eta$. Thus, $||\pi^u x|| \equiv 0$. Therefore, since we have a direct sum, we only have $\pi^c x = x$ which implies $x \in E^c$.

■

From now on, we consider the system of Equation (3.1) with $c \neq 0$ and we highlight the assumption $g \not\equiv \mathbf{0}$.

In what follows we define a modification of g which will allow us to get local properties from global properties. Thus one can suppose to have compact support and its norm as small as we like. From Definition (15), for each $\rho > 0$ we define

$$\tilde{g}_\rho(x) = g(x)\mathscr{X}(\rho^{-1}x) \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{3.29}$$

the support of $\tilde{g}_\rho$ is the closure set of all points $x \in \mathbb{R}^n$ such that $\tilde{g}_\rho(x) \neq 0$. More precisely, it is defined as the following set

$$\{x \in \mathbb{R}^n |\ \tilde{g}_\rho(x) \neq 0\} = \{x \in \mathbb{R}^n \mid 0 < ||x|| < 2\rho\} = B(0,2\rho) - \{0\} \tag{3.30}$$

which is an open punctured ball at origin and radius 2ρ. Thus, it support is $B[0,2\rho]$, a compact set. We also note that, for all $x \in \mathbb{R}^n$ such that $||x|| \leq 1$ we have $\tilde{g}_\rho(x) = g(x)$.

The next lemma plays an essential role in the main proof. This guarantees that the non-linearity g modified by $\tilde{g}_\rho$ is in $\mathscr{C}_b^k(\mathbb{R}^n)$, that is bounded and k continuously differentiable can be considered as small as we want.

Lemma 21. We consider Definition (16) with $j = 1$. Then, $\tilde{g}_\rho \in \mathscr{C}_b^k(\mathbb{R}^n)$ and

$$\lim_{\rho \to 0} |\tilde{g}_\rho|_1 = 0. \tag{3.31}$$

Proof: For each $\rho > 0$, using the Chain rule applied to $\tilde{g}_\rho$ with respect to x we get

$$D\tilde{g}_\rho(x) = g(x)D\mathscr{X}(\rho^{-1}x)\rho^{-1} + Dg(x)\mathscr{X}(\rho^{-1}x). \tag{3.32}$$

Then we calculate,

$$\begin{aligned}
|\tilde{g}_\rho|_1 &= \sup_{||x||\leq 2\rho} ||D\tilde{g}_\rho(x)|| \\
&= \sup_{||x||\leq 2\rho} ||g(x)D\mathscr{X}(\rho^{-1}x)\rho^{-1} + Dg(x)\mathscr{X}(\rho^{-1}x)|| \\
&\leq \sup_{||x||\leq 2\rho} ||g(x)|| \sup_{||x||\leq 2\rho} ||D\mathscr{X}(\rho^{-1}x)||\rho^{-1} \\
&+ \sup_{||x||\leq 2\rho} ||Dg(x)|| \sup_{||x||\leq 2\rho} ||\mathscr{X}(\rho^{-1}x)|| \\
&\leq \sup_{||x||\leq 2\rho} ||g(x)|| \cdot |\mathscr{X}|_1\rho^{-1} + \sup_{||x||\leq 2\rho} ||Dg(x)|| \\
&\leq \sup_{||x||\leq 2\rho} \sup_{t\in[0,1]} ||x|| \cdot ||Dg(tx)|| \cdot |\mathscr{X}|_1\rho^{-1} + \sup_{||x||\leq 2\rho} ||Dg(x)|| \\
&\leq \sup_{||x||\leq 2\rho} ||x|| \cdot \sup_{||x||\leq 2\rho} ||Dg(x)|| \cdot |\mathscr{X}|_1\rho^{-1} + \sup_{||x||\leq 2\rho} ||Dg(x)|| \\
&\leq (1+2|\mathscr{X}|_1) \sup_{||x||\leq 2\rho} ||Dg(x)||.
\end{aligned}$$

As we have Dg continuous and $Dg(0) = 0$ then the limit

$$\lim_{\rho\to 0} |\tilde{g}_\rho|_1 = \lim_{\rho\to 0} \sup_{||x||\leq 2\rho} ||Dg(x)|| = 0. \tag{3.33}$$

■

Now we announce and prove the main theorem of this chapter.

Theorem 22 (Center Manifold Theorem)**.** Let $f \in \mathscr{C}^{k+1}(\mathscr{U}, \mathbb{R}^n)$ $k \geq 1$ be a vector field such that $f(0) = 0$ is a non-hyperbolic fixed point and $A = Df(0)$. Let E^c and E^h be the corresponding center and hyperbolic subspaces. Then, there exists $\delta > 0$ and a Local Center Manifold satisfying the following properties:

(i) There exists a function $\phi : E^c \to E^h$ with $\pi^c\phi(x_c) = x_c$ such that

$$\mathscr{E}^c_{loc}(0) := \{\phi(x_c)|\ x_c \in E^c,\ |x_c| < \delta\}; \tag{3.34}$$

(ii) The manifold $\mathscr{E}^c_{loc}(0)$ is locally invariant for the flow of the System (3.1);

(iii) Every globally bounded orbit remaining in a suitably small neighborhood of the origin is entirely inside $\mathscr{E}^c_{loc}(0)$;

(iv) The manifold $\mathscr{E}^c_{loc}(0)$ is tangent to E^c at the origin;

(v) For each solution $x(t)$ of the System (3.1) where $x(t) \to 0$ as $t \to +\infty$ there exists $\eta > 0$ and a solution $y(t) \in \mathscr{E}^c_{loc}(0)$ such that

$$e^{\eta t}|x(t) - y(t)| \to 0 \quad \text{as} \quad t \to +\infty; \tag{3.35}$$

(vi) The function $\phi : E^c \to E^h$ is $\mathscr{C}^k$.

Proof: [Proof of Center Manifold Theorem]

Construction of the center manifold: Item (i).

From the characterization of points in the center subspace, we define the Banach space of functions of slow growth. Let $\beta > 0$ be the spectral gap and for each $\eta \in (0, \beta)$ we consider the space:

$$Y_\eta := \{y : \mathbb{R} \to \mathbb{R}^n | \; ||y(\cdot)||_\eta := \sup_{t \in \mathbb{R}} e^{-\eta|t|}|y(t)| < \infty\}. \tag{3.36}$$

We note that $||y||_\eta \geq e^{-\eta|t|}|y(t)|$ and then

$$|y(t)| \leq e^{\eta|t|}||y||_\eta \tag{3.37}$$

for all $t \in \mathbb{R}$ and $\eta \in (0, \beta)$.

By the Variation of Constant Formula to Cauchy problem, for any solution $y(t) \in \mathbb{R}^n$ with $y(t_0) = y_0$ of the system of Equation (3.1) one can represent by the formula:

$$y(t) = e^{A(t-t_0)}y(t_0) + \int_{t_0}^t e^{A(t-\tau)}g(y(\tau))d\tau. \tag{3.38}$$

By the direct decomposition into the center, stable and unstable subspaces we have different times in each subspace. More precisely, the direct sum allows to transform of the above formula into a Cauchy problem in three initial conditions:

$$\begin{aligned}
y(t) &= \pi^c y(t) + \pi^s y(t) + \pi^u y(t) \\
&= \pi^c \left(e^{A(t-t_c)}y(t_c) + \int_{t_c}^t e^{A(t-\tau)}g(y(\tau))d\tau \right) \\
&+ \pi^s \left(e^{A(t-t_s)}y(t_s) + \int_{t_s}^t e^{A(t-\tau)}g(y(\tau))d\tau \right) \\
&+ \pi^u \left(e^{A(t-t_u)}y(t_u) + \int_{t_u}^t e^{A(t-\tau)}g(y(\tau))d\tau \right).
\end{aligned}$$

We set $t_c = 0$ and $t_s \to -\infty$, $t_u \to +\infty$. Recording that all projections commute with respect to A and also e^{-At}, we get the following expression for any solution $y \in Y_\eta$ such that

$\pi^c y(0) = x_c \in E^c$:

$$\begin{aligned} y(t) &= e^{At}\pi^c y(0) + \int_0^t e^{A(t-\tau)}\pi^c g(y(\tau))d\tau \\ &+ e^{A(t-t_s)}\pi^s y(t_s) + \int_{-\infty}^t e^{A(t-\tau)}\pi^s g(y(\tau))d\tau \\ &+ e^{A(t-t_u)}\pi^u y(t_u) - \int_t^{+\infty} e^{A(t-\tau)}\pi^u g(y(\tau))d\tau. \end{aligned} \tag{3.39}$$

By Lemma (14) and the Inequality (3.37), we note that the following linear terms in the equations above vanish due to: for $\varepsilon \in (0,\beta)$ we have

$$\begin{aligned} |e^{A(t-t_s)}\pi^s y(t_s)| &\leq |e^{A(t-t_s)}\pi^s| \cdot |y(t_s)| \\ &\leq M_\varepsilon e^{-(\beta-\varepsilon)(t-t_s)} e^{\eta|t|} ||y||_\eta. \end{aligned}$$

Therefore,

$$\lim_{t_s \to -\infty} |e^{A(t-t_s)}\pi^s y(t_s)| = 0. \tag{3.40}$$

Likewise,

$$\begin{aligned} |e^{A(t-t_u)}\pi^u y(t_u)| &\leq |e^{A(t-t_u)}\pi^u| \cdot |y(t_u)| \\ &\leq M_\varepsilon e^{(\beta-\varepsilon)(t-t_u)} e^{\eta|t|} ||y||_\eta. \end{aligned}$$

Therefore,

$$\lim_{t_u \to +\infty} |e^{A(t-t_u)}\pi^u y(t_u)| = 0. \tag{3.41}$$

Consequently, the Equations (3.40) and (3.41) applied in Equation (3.39) give us the following expression for a solution $y \in Y_\eta$:

$$\begin{aligned} y(t) &= e^{At}x_c + \int_0^t e^{A(t-\tau)}\pi^c g(y(\tau))d\tau \\ &+ \int_{-\infty}^t e^{A(t-\tau)}\pi^s g(y(\tau))d\tau - \int_t^{+\infty} e^{A(t-\tau)}\pi^u g(y(\tau))d\tau. \end{aligned} \tag{3.42}$$

Let $x_c \in E^c$ be any point in the center subspace. We prove that there exist a unique solution $y \in Y_\eta$ such that $\pi^c y(0) = x_c$. Since that Y_η is a Banach space we define the map $\Theta : E^c \times Y_\eta \to Y_\eta$ given by

$$\begin{aligned} \Theta(x_c;y)(t) &= e^{At}x_c + \int_0^t e^{A(t-\tau)}\pi^c g(y(\tau))d\tau \\ &+ \int_{-\infty}^t e^{A(t-\tau)}\pi^s g(y(\tau))d\tau - \int_t^{+\infty} e^{A(t-\tau)}\pi^u g(y(\tau))d\tau. \end{aligned} \tag{3.43}$$

We claim $\Theta(x_c;y)(t) \in Y_\eta$ for all $t \in \mathbb{R}$. Indeed, we pick up any $\eta \in (0,\beta)$ and we choose $\varepsilon = \eta$ with respect to center subspace and $\varepsilon = \beta - \eta$ with respect to stable and unstable subspaces. For $t > 0$ and remembering that $g \in \mathscr{C}_b^k(\mathbb{R}^n)$ and then $||g||_j = \sup_{x \in X} |D^j g(x)|$, we calculate:

$$
\begin{aligned}
|\Theta(x_c;y)(t)| &\leq |e^{At}x_c| + \int_0^t |e^{A(t-\tau)}\pi^c| \cdot |g(y(\tau))| d\tau \\
&+ \int_{-\infty}^t |e^{A(t-\tau)}\pi^s| \cdot |g(y(\tau))| d\tau + \int_t^{+\infty} |e^{A(t-\tau)}\pi^u| \cdot |g(y(\tau))| d\tau \\
&\leq M_\eta |x_c| e^{\eta|t|} + M_\eta ||g||_0 \int_0^t e^{\eta|t-\tau|} d\tau \\
&+ M_\varepsilon ||g||_0 \int_{-\infty}^t e^{-(\beta-\varepsilon)(t-\tau)} d\tau + M_\varepsilon ||g||_0 \int_t^{+\infty} e^{(\beta-\varepsilon)(t-\tau)} d\tau \\
&= M_\eta |x_c| e^{\eta|t|} + M_\eta ||g||_0 \int_0^t e^{\eta|t-\tau|} d\tau + M_\varepsilon ||g||_0 \int_{-\infty}^t e^{-\eta(t-\tau)} d\tau \\
&+ M_\varepsilon ||g||_0 \int_t^{+\infty} e^{\eta(t-\tau)} d\tau \\
&\leq \tilde{M} ||g||_0 \left(e^{\eta|t|} + \int_0^t e^{\eta|t-\tau|} d\tau + \int_{-\infty}^t e^{-\eta(t-\tau)} d\tau + \int_t^{+\infty} e^{\eta(t-\tau)} d\tau \right) \\
&= \tilde{M} ||g||_0 \left(e^{\eta|t|} + \frac{te^{\eta|t|} - 1}{\eta|t|} + \frac{2}{\eta} \right) \\
&= \tilde{M} ||g||_0 \left(\frac{e^{\eta|t|}(\eta+1)+1}{\eta} \right) \\
&\leq M ||g||_0 e^{\eta|t|}.
\end{aligned}
$$

for some properly constant M. Therefore, $\sup_{t \in \mathbb{R}} e^{-\eta|t|} |\Theta(x_c;y)(t)| < \infty$.

Next, for each $x_c \in E^c$ we prove that the map $y \mapsto \Theta(x_c;y)$ is a strict contraction. Let y_1, y_2 be functions in Y_η. By Inequality (3.37) we have

$$|y_1(t) - y_2(t)| = |(y_1 - y_2)(t)| \leq e^{\eta|t|} ||y_1 - y_2||_\eta. \tag{3.44}$$

On the other hand, by Mean Value Inequality we have

$$|g(y_1(t)) - g(y_2(t))| \leq |y_1(t) - y_2(t)| \cdot ||g||_1 \leq e^{\eta|t|} ||y_1 - y_2||_\eta \cdot ||g||_1. \tag{3.45}$$

Now, by Equation (3.43), we calculate

$$
\begin{aligned}
|\Theta(x_c;y_1)(t) - \Theta(x_c;y_2)(t)| &\leq \int_0^t |e^{A(t-\tau)}\pi^c| \cdot |g(y_1(\tau)) - g(y_2(\tau))| d\tau \\
&+ \int_{-\infty}^t |e^{A(t-\tau)}\pi^s| \cdot |g(y_1(\tau)) - g(y_2(\tau))| d\tau \\
&+ \int_t^{+\infty} |e^{A(t-\tau)}\pi^u| \cdot |g(y_1(\tau)) - g(y_2(\tau))| d\tau.
\end{aligned}
$$

By the Equation (3.45) and the estimates of Lemma (14) we have for $t > 0$ and $\varepsilon > 0$:

$$
\begin{aligned}
|\Theta(x_c;y_1)(t) - \Theta(x_c;y_2)(t)| &\leq M_\varepsilon ||g||_1 ||y_1 - y_2||_\eta \int_0^t e^{\varepsilon|t-\tau|} e^{\eta|\tau|} d\tau \\
&+ M_\varepsilon ||g||_1 ||y_1 - y_2||_\eta \int_{-\infty}^t e^{-(\beta-\varepsilon)(t-\tau)} e^{\eta|\tau|} d\tau \\
&+ M_\varepsilon ||g||_1 ||y_1 - y_2||_\eta \int_t^{+\infty} e^{(\beta-\varepsilon)(t-\tau)} e^{\eta|\tau|} d\tau.
\end{aligned}
$$

Solving the integrals, we have just like before, it is bounded by $Ce^{\eta|t|}$ multiplied by some constant. Therefore,

$$|\Theta(x_c;y_1)(t)-\Theta(x_c;y_2)(t)|\leq M||g||_1||y_1-y_2||_\eta e^{\eta|t|}$$

where $M>0$ is a properly constant independent of y_1 and y_2. Taking $||g||_1\leq\frac{1}{2M}$ we get

$$e^{-\eta|t|}|\Theta(x_c;y_1)(t)-\Theta(x_c;y_2)(t)|\leq\frac{1}{2}||y_1-y_2||_\eta$$

for all $t\in\mathbb{R}$. We take the $\sup_{t\in\mathbb{R}}$ on the right side to get

$$||\Theta(x_c;y_1)-\Theta(x_c;y_2)||_\eta\leq\frac{1}{2}||y_1-y_2||_\eta. \tag{3.46}$$

Therefore, the map $y\mapsto\Theta(x_c;y)$ is a strict contraction. By the Contraction Mapping Principle, for each $x_c\in E^c$ there exists a unique solution $y(t)\in Y_\eta$ satisfying Variation of Constant Formula such that $\pi^c y(0)=x_c$. Moreover, the map $x_c\mapsto y(\cdot,x_c)$ is a Lipschitz continuous. Indeed, we let $x_{c1},x_{c2}\in E^c$ be any points and $y\in Y_\eta$. We calculate,

$$\begin{aligned}|\Theta(x_{c1};y)(t)-\Theta(x_{c2};y)(t)| &= |e^{At}(x_{c1}-x_{c2})|\\ &\leq M_\eta e^{\eta|t|}|x_{c1}-x_{c2}|\end{aligned}$$

and then we get,

$$e^{-\eta|t|}|\Theta(x_{c1};y)(t)-\Theta(x_{c2};y)(t)|\leq M_\eta|x_{c1}-x_{c2}|$$

independent of t, taking $\sup_{t\in\mathbb{R}}$ on the right side, we get

$$||\Theta(x_{c1};y)-\Theta(x_{c2};y)||_\eta\leq M_\eta|x_{c1}-x_{c2}|.$$

Therefore, Θ is Lipschitz continuous with respect to variable x_c. Also due to the Contraction Mapping Principle, there exists a Lipschitz continuous map $\psi:E^c\to E^h$ which associate each $x_c\in E^c$ to a unique solution $y\in Y_\eta$ defined by $\psi(x_c)=y(0)$ such that $\pi^c\psi(x_c)=\pi^c y(0)=x_c$.

We finally proved the existence of a global center manifold at the origin defined as the graph of a Lipschitz continuous function:

$$\mathscr{E}^c(0):=\{\phi(x_c)|\,x_c\in E^c\}. \tag{3.47}$$

Invariance of the center manifold: Item (ii).

We have to prove that for any point $x_0\in\mathscr{E}^c(0)$ the corresponding solution $\bar{x}(t,x_0)\in Y_\eta$ with $\bar{x}(0,x_0)=x_0$, lies in $\mathscr{E}^c(0)$ for all $t\in\mathbb{R}$. We fix a time $t_1>0$ and let $\bar{x}(t_1,x_0)=x_1$ be a forward point. We prove that $x_1\in\mathscr{E}^c(0)$ showing that $\bar{x}(t,x_1)\in Y_\eta$ for all $t\in\mathbb{R}$. Indeed,

$$|\bar{x}(t,x_1)|=|\bar{x}(t+t_1,x_0)|\leq Me^{\eta|t+t_1|}\leq Me^{\eta|t_1|}e^{\eta|t|}$$

We note on the first inequality above, we used $\bar{x}(t,x_0) \in Y_\eta$ and therefore is bounded. From the inequality above holds we get

$$e^{-\eta|t|}|\bar{x}(t,x_1)| < \infty$$

for all $t \in \mathbb{R}$. Therefore, $x_1 \in Y_\eta$ and $\mathscr{E}^c(0)$ is an invariant manifold for the flow of the nonlinear system of Equation (3.1).

The center manifold captures every globally bounded solution: Item (iii).

We note that for each $x_c \in E^c$ we associate a unique solution $y \in Y_\eta$ such that $\pi^c y(0) = x_c$ which is a globally bounded solution by construction. Since that $\mathscr{E}^c(0)$ is invariant, we have that every globally bounded solution is entirely contained in $\mathscr{E}^c(0)$.

Tangency of the center manifold: Item (iv).

First of all, we note that the function $y(t) \equiv 0$ is trivially a globally bounded solution for the system of Equation (3.1) since that $g(0) = 0$. Thus, $0 \in \mathscr{E}^c(0)$ by the Property (iii). Given $x_c \in E^c$ the main and basic idea to prove the tangency property is to estimate the linear solution given by $y(t) = e^{At}x_c$ (when $g = 0$) and the unique correspondent nonlinear solution $y_{x_c} \in Y_\eta$ as x_c goes to 0. By the Contraction Mapping Principle proved in Inequality (3.46) we also have inequality:

$$||y - y_{x_c}||_\eta \leq 2||y - \Theta(x_c; y_{x_c})||_\eta. \tag{3.48}$$

We also note that, for $y(t) = e^{At}x_c$, we have the following estimate for $g(y(t))$:

$$|g(y(\tau))| \leq ||g||_2 \cdot |y(\tau)|^2 = ||g||_2 \cdot |e^{A\tau}x_c|^2 \leq (M_\varepsilon e^{\varepsilon|\tau|}|x_c|)^2||g||_2. \tag{3.49}$$

In what follows, we calculate for $\varepsilon > 0$ such that $2\varepsilon = \eta$ and by Inequality (3.49):

$$\begin{aligned}
& |y(t) - \Theta(x_c; y_{x_c})(t)| = \\
= \quad & \Big| -\int_0^t e^{A(t-\tau)}\pi^c g(y(\tau))d\tau - \int_{-\infty}^t e^{A(t-\tau)}\pi^s g(y(\tau))d\tau + \int_t^{+\infty} e^{A(t-\tau)}\pi^u g(y(\tau))d\tau \Big| \\
\leq \quad & \int_0^t |e^{A(t-\tau)}\pi^c||g(y(\tau))|d\tau + \int_{-\infty}^t |e^{A(t-\tau)}\pi^s||g(y(\tau))|d\tau \\
+ \quad & \int_t^{+\infty} |e^{A(t-\tau)}\pi^u||g(y(\tau))|d\tau \\
\leq \quad & M_\varepsilon(M_\varepsilon|x_c|)^2||g||_2 \left(\int_0^t e^{\varepsilon|t-\tau|}e^{\eta|\tau|}d\tau + \int_{-\infty}^t e^{-(\beta-\varepsilon)(t-\tau)}e^{\eta|\tau|}d\tau \right) \\
+ \quad & M_\varepsilon(M_\varepsilon|x_c|)^2||g||_2 \left(\int_t^{+\infty} e^{(\beta-\varepsilon)(t-\tau)}e^{\eta|\tau|}d\tau \right).
\end{aligned}$$

Just like before, for some properly constant $M > 0$ independent of x_c we get the following expression:

$$e^{-\eta|t|}|y(t) - \Theta(x_c; y_{x_c})(t)| \leq M|x_c|^2 \tag{3.50}$$

for all $t \in \mathbb{R}$. Taking the $\sup_{t\in\mathbb{R}}$ we have

$$||y - \Theta(x_c; y_{x_c})||_\eta \leq M|x_c|^2. \tag{3.51}$$

Now we get the estimate using Inequalities (3.48) and (3.51):

$$|y(0) - y_{x_c}(0)| \leq ||y - y_{x_c}||_\eta \leq 2||y - \Theta(x_c; y_{x_c})||_\eta \leq 2M|x_c|^2. \tag{3.52}$$

Remembering that $y(0) = x_c$ and $\phi(x_c) = y_{x_c}(0)$, using the estimate in the Inequality (3.52), we calculate the limit

$$\lim_{x_c \to 0} \frac{|\phi(x_c) - x_c|}{|x_c|} \leq \lim_{x_c \to 0} 2M|x_c| = 0. \tag{3.53}$$

Therefore, the derivative of ϕ at $x_c = 0$ is tangent space to the center manifold $\mathscr{E}^c(0)$.

Asymptotic approximation: Item (v).

Let $x(t)$ be any solution of the system of Equation (3.1) such that

$$x(t) \to 0 \quad \text{as} \quad t \to +\infty.$$

We should find a solution $y(t) \in \mathscr{E}^c(0)$ on the center manifold which approach $x(t)$ as $t \in +\infty$. For $x(t)$ given, we can extend the solution $x(t)$ to a globally bounded function $x^*(t)$ defining for all $t \in \mathbb{R}$ simple in the following way:

$$x^*(t) := \begin{cases} x(t), & \text{if } t \geq 0, \\ x(0), & \text{if } t < 0. \end{cases} \tag{3.54}$$

Notice that, $x^*(t) = x(t)$ for all $t \geq 0$. Easily checked, $x^*(t)$ is a solution of the system:

$$\dot{x}^*(t) = Ax^*(t) + g(x^*(t)) + h(t) \tag{3.55}$$

where h is a correction function related to x^* defined by:

$$h(t) := \begin{cases} 0, & \text{if } t > 0, \\ -Ax(0) - g(x(0)), & \text{if } t < 0. \end{cases} \tag{3.56}$$

As x^* is a solution of the System (3.55), then this can be expressed in terms of Variation of Constant Formula:

$$x^*(t) = e^{A(t-t_0)}x^*(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} g(x^*(\tau))d\tau + \int_{t_0}^{t} e^{A(t-\tau)} g(h(\tau))d\tau. \tag{3.57}$$

We project x^* into stable and center-unstable subspaces with starting times t_0 and t_1 respectively:

$$\begin{aligned}
x^*(t) &= \pi^s x^*(t) + \pi^c x^*(t) + \pi^u x^*(t) \\
&= e^{A(t-t_0)}\pi^s x^*(t_0) + \int_{t_0}^t e^{A(t-\tau)}\pi^s g(x^*(\tau))d\tau + \int_{t_0}^t e^{A(t-\tau)}\pi^s g(h(\tau))d\tau \\
&+ e^{A(t-t_1)}\pi^c x^*(t_1) + \int_{t_1}^t e^{A(t-\tau)}\pi^c g(x^*(\tau))d\tau + \int_{t_1}^t e^{A(t-\tau)}\pi^c g(h(\tau))d\tau \\
&+ e^{A(t-t_1)}\pi^u x^*(t_1) + \int_{t_1}^t e^{A(t-\tau)}\pi^u g(x^*(\tau))d\tau + \int_{t_1}^t e^{A(t-\tau)}\pi^u g(h(\tau))d\tau
\end{aligned}$$

and the we get the following expression writing $\pi^{cu} = \pi^c + \pi^u$ to projection into $E^c \oplus E^h$:

$$\begin{aligned}
&x^*(t) = \\
&e^{A(t-t_0)}\pi^s x^*(t_0) + \int_{t_0}^t e^{A(t-\tau)}\pi^s g(x^*(\tau))d\tau + \int_{t_0}^t e^{A(t-\tau)}\pi^s g(h(\tau))d\tau + \\
&e^{A(t-t_1)}\pi^{cu} x^*(t_1) + \int_{t_1}^t e^{A(t-\tau)}\pi^{cu} g(x^*(\tau))d\tau + \int_{t_1}^t e^{A(t-\tau)}\pi^{cu} g(h(\tau))d\tau. \qquad (3.58)
\end{aligned}$$

We consider the Banach space of functions for $\eta > 0$ and $t \in \mathbb{R}$:

$$Z_\eta := \{z : \mathbb{R} \to \mathbb{R}^n | \ ||z||_\eta := \sup_{t\in\mathbb{R}} e^{\eta|t|}|z(t)|\}. \qquad (3.59)$$

We will prove that for x^* globally bounded solution of the System (3.55) there exists a unique function $z \in Z_\eta$ such that $y = x^* + z \in Y_\eta$ is a solution of the System (3.1). We highlight that y is in fact the solution sought on a center manifold. We assume $y(t)$ is a solution of the System (3.1) and we calculate Equation (3.58) for $z(t)$. We have,

$$\begin{aligned}
z(t) &= \pi^s(y(t) - x^*(t)) + \pi^{cu}(y(t) - x^*(t)) \\
&= \pi^s y(t) - \pi^s x^*(t) + \pi^{cu} y(t) - \pi^{cu} x^*(t) \\
&= -\pi^s x^*(t) + e^{A(t-t_0)}\pi^s(z(t_0) + x^*(t_0)) + \int_{t_0}^t e^{A(t-\tau)}\pi^s g(z(\tau) + x^*(\tau))d\tau \\
&+ -\pi^{cu} x^*(t) + e^{A(t-t_1)}\pi^{cu}(z(t_1) + x^*(t_1)) + \int_{t_1}^t e^{A(t-\tau)}\pi^{cu} g(z(\tau) + x^*(\tau))d\tau.
\end{aligned}$$

In summarizing, we highlight the formula obtained for $z(t)$:

$$\begin{aligned}
z(t) &= -\pi^s x^*(t) + e^{A(t-t_0)}\pi^s(z(t_0) + x^*(t_0)) + \int_{t_0}^t e^{A(t-\tau)}\pi^s g(z(\tau) + x^*(\tau))d\tau \\
&+ -\pi^{cu} x^*(t) + e^{A(t-t_1)}\pi^{cu}(z(t_1) + x^*(t_1)) + \int_{t_1}^t e^{A(t-\tau)}\pi^{cu} g(z(\tau) + x^*(\tau))d\tau.
\end{aligned}$$

Using the Formula (3.57) for $x^*(t)$ applied above we get:

$$\begin{aligned} z(t) &= e^{A(t-t_0)}\pi^s z(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}\pi^s \left(g(z(\tau)+x^*(\tau)) - g(\tau)\right) d\tau \\ &- \int_{t_0}^{t} e^{A(t-\tau)}\pi^c g(h(\tau)) d\tau \\ &+ e^{A(t-t_1)}\pi^{cu} z(t_1) + \int_{t_1}^{t} e^{A(t-\tau)}\pi^{cu} \left(g(z(\tau)+x^*(\tau)) - g(\tau)\right) d\tau \\ &- \int_{t_1}^{t} e^{A(t-\tau)}\pi^{cu} g(h(\tau)) d\tau. \end{aligned}$$

Just like before, we let $t_0 \to -\infty$ and $t_1 \to +\infty$ to obtain a formula for $z(t)$ precisely:

$$\begin{aligned} z(t) &= \int_{-\infty}^{t} e^{A(t-\tau)}\pi^s \left(g(z(\tau)+x^*(\tau)) - g(\tau)\right) d\tau \\ &- \int_{-\infty}^{t} e^{A(t-\tau)}\pi^c g(h(\tau)) d\tau \\ &+ \int_{+\infty}^{t} e^{A(t-\tau)}\pi^{cu} \left(g(z(\tau)+x^*(\tau)) - g(\tau)\right) d\tau \\ &- \int_{+\infty}^{t} e^{A(t-\tau)}\pi^{cu} g(h(\tau)) d\tau. \end{aligned} \tag{3.60}$$

Where we noted in the calculation above that:

$$\begin{aligned} |e^{A(t-t_0)}\pi^s z(t_0)| &\to 0 \quad \text{as} \quad t_0 \to -\infty \\ |e^{A(t-t_1)}\pi^{cu} z(t_1)| &\to 0 \quad \text{as} \quad t_1 \to +\infty \end{aligned}$$

exactly like we have calculated. We consider a map $\Gamma : Z_\eta \to Z_\eta$ by Formula (3.60)

$$\begin{aligned} \Gamma(z)(t) &:= \int_{-\infty}^{t} e^{A(t-\tau)}\pi^s \left(g(z(\tau)+x^*(\tau)) - g(\tau)\right) d\tau \\ &- \int_{-\infty}^{t} e^{A(t-\tau)}\pi^c g(h(\tau)) d\tau \\ &+ \int_{+\infty}^{t} e^{A(t-\tau)}\pi^{cu} \left(g(z(\tau)+x^*(\tau)) - g(\tau)\right) d\tau \\ &- \int_{+\infty}^{t} e^{A(t-\tau)}\pi^{cu} g(h(\tau)) d\tau. \end{aligned} \tag{3.61}$$

We claim that Γ is a strict contraction. Indeed, let $z_1, z_2 \in Z_\eta$ be functions, we calculate recalling the estimates of Lemma (14) and Inequality (3.45):

$$\begin{aligned}
|\Gamma(z_1)(t)-\Gamma(z_2)(t)| &\leq \int_{-\infty}^{t} |e^{A(t-\tau)}\pi^s|\cdot|g(z_1(\tau)+x^*(\tau))-g(z_2(\tau)+x^*(\tau))|d\tau \\
&+ \int_{+\infty}^{t} |e^{A(t-\tau)}\pi^{cu}|\cdot|g(z_1(\tau)+x^*(\tau))-g(z_2(\tau)+x^*(\tau))|d\tau \\
&\leq M_\eta||g||_1 \int_{-\infty}^{t} e^{-(\beta-\eta)(t-\tau)}|z_1(\tau)-z_2(\tau)|d\tau \\
&+ M_\eta||g||_1 \int_{+\infty}^{t} e^{(\beta-\eta)(t-\tau)+\eta|t-\tau|}|z_1(\tau)-z_2(\tau)|d\tau \\
&\leq M_\eta||g||_1||z_1-z_2||_\eta \int_{-\infty}^{t} e^{-(\beta-\eta)(t-\tau)}e^{-\eta|t|}d\tau \\
&+ M_\eta||g||_1||z_1-z_2||_\eta \int_{+\infty}^{t} e^{(\beta-\eta)(t-\tau)+\eta|t-\tau|}e^{-\eta|t|}d\tau \\
&\leq C||g||_1 e^{-\eta|t|}||z_1-z_2||_\eta
\end{aligned}$$

where $C>0$ is a properly constant independent of y_1 and y_2. If $||g||_1 \leq \frac{1}{2C}$ we get a strict contraction taking the $\sup_{t\in\mathbb{R}}$ on the right side:

$$||\Gamma(z_1)-\Gamma(z_2)||_\eta \leq \frac{1}{2}||z_1-z_2||_\eta. \tag{3.62}$$

Therefore, by the Contraction Mapping Principle, there exists a unique $z\in Z_\eta$ such that $\Gamma(z)=z$ which solves Equation (3.61). We note that $y=x^*+z$ belongs to Z_η. Indeed, recalling that x^* is a globally bounded solution, then

$$|y(t)| \leq |x^*(t)|+|z(t)| \leq C_1 + e^{-\eta|t|}||z||_\eta$$

and then

$$\sup_{t\in\mathbb{R}} e^{\eta|t|}|y(t)| < \infty.$$

Since that $Z_\eta \subset Y_\eta$ we conclude that $y\in Y_\eta$ and then, y is entirely contained on the center manifold. Now, we see that for $t>0$

$$|x(t)-y(t)| \leq |z(t)| \leq e^{-\eta|t|}||z||_\eta. \tag{3.63}$$

Therefore, there exists $\eta>0$ such that

$$|x(t)-y(t)| \to 0 \qquad \text{as} \qquad t\to 0.$$

Smoothness of the center manifold: Item (vi).

We would like to expect the map $\Theta: E^c\times Y_\eta \to Y_\eta$ were a $\mathscr{C}^k$ map but this is not true. To prove that ϕ is a $\mathscr{C}^k$ map we need to consider the smaller Banach subspace $Y_{\eta'}$ with a stronger norm. We will get the $\mathscr{C}^k$ smoothness for ϕ considering $\Theta: E^c\times Y_{\eta'} \to Y_\eta$. Next lemma can guarantee smoothness.

Lemma 23. Let $g \in \mathscr{C}_b^{k+1}(\mathbb{R}^n)$ be a map. Then the map $G : Y_{\eta'} \to Y_\eta$ defined by $G(y)(t) := g(y(t))$ is k times continuously differentiable map, provided that $0 < \eta' < (k+1)\eta' \leq \eta$.

Proof: By Taylor expansion formula for $g(y)$ around origin. See (DIEUDONNÉ, 2011) Chapter VIII, Section 14.

$$\begin{aligned} g(y+z) &= g(y) + \frac{Dg(y)}{1!}z + \frac{D^2g(y)}{2!}z^2 + \cdots + \frac{D^kg(y)}{k!}z^k \\ &+ \left(\int_0^1 \frac{(1-\tau)^k}{k!}D^{k+1}g(y+\tau z)d\tau\right)z^{k+1}. \end{aligned} \tag{3.64}$$

We calculate

$$\begin{aligned} |G(g(y+z))(t) - G(\sum_{j=0}^{k}\frac{D^jg(y)}{j!}z^j)(t)| &= \\ |g(y(t)+z(t)) - \sum_{j=0}^{k}\frac{D^jg(y(t))}{j!}z^j(t)| &= \\ |\left(\int_0^1 \frac{(1-\tau)^k}{k!}D^{k+1}g(y(\tau)+\tau z(\tau))d\tau\right)z^{k+1}(t)| &\leq \\ \int_0^1 \frac{(1-\tau)^k}{k!}|D^{k+1}g(y(\tau)+\tau z(\tau))|d\tau \cdot |z(t)|^{k+1}. \end{aligned}$$

Since that $g \in \mathscr{C}_b^{k+1}(\mathbb{R}^n)$ and $z \in Y_{\eta'}$ we have $||g||_{k+1} \leq \varepsilon(k+1)!$ and $|z(t)| \leq e^{\eta'|t|}||z||_{\eta'}$. Solving the integral we get

$$\begin{aligned} |G(g(y+z))(t) - G(\sum_{j=0}^{k}\frac{D^jg(y)}{j!}z^j)(t)| &\leq \frac{1}{(k+1)!}||g||_{k+1}e^{(k+1)\eta'|t|}||z||_{\eta'}^{k+1} \\ &\leq \varepsilon e^{\eta|t|}||z||_{\eta'}^{k+1}. \end{aligned}$$

Taking $\sup_{t\in\mathbb{R}}$ on the right side we have:

$$||G(g(y+z)) - G(\sum_{j=0}^{k}\frac{D^jg(y)}{j!}z^j)||_\eta \leq \varepsilon||z||_{\eta'}^{k+1}.$$

Therefore, G is k continuously differentiable as a map from $Y_{\eta'}$ to Y_η. ■

Finally, the next corollary guarantees the smoothness sought:

Corollary 24. The operator $\Theta : E^c \times Y_{\eta'} \to Y_\eta$ defined by $\Theta(x_c; y)(t) = y_{x_c}(t)$, of Equation (3.43), is l continuously differentiable as map from $E^c \times Y_{\eta'}$ to Y_η for all $j = 1, \ldots, l$ provided that $2l\eta' \leq \eta$.

Proof: We decompose the operator $\Theta = S + K \circ G$ where G is the operator of the previous Lemma:

$$\Theta(x_c; y)(t) = S(x_c)(t) + K(G(y)(t)) \tag{3.65}$$

where,

$$S(x_c)(t) = e^{At}x_c$$

and

$$\begin{aligned} K(G(y)(t)) &= \int_0^t e^{A(t-\tau)}\pi^c g(y(\tau))d\tau \\ &+ \int_{-\infty}^t e^{A(t-\tau)}\pi^s g(y(\tau))d\tau - \int_t^{+\infty} e^{A(t-\tau)}\pi^u g(y(\tau))d\tau. \end{aligned}$$

We note that $S : E^c \to Y_\eta$ and $K : Y_\eta \to Y_\eta$ are continuously linear mappings. By the previous lemma, $\Theta : E^c \times Y_{\eta'} \to Y_\eta$ is l continuously differentiable map, provided that $(l+1)\eta' \leq \eta$. ■

We complete the proof of the Center Manifold Theorem here. We highlight that the existence and uniqueness of the global center manifold strongly depend on the nonlinearity g, on $||g||_1$ being small enough and strongly depends on the cut-off function, which means that we lost the uniqueness property when we get the local center manifold.

■

CHAPTER

4

GENERIC UNFOLDING

The goal of this chapter will be the deeper study of a Jordan matrix, in other words, a nilpotent singularity and its unfolding. To perform this task, we approach the Unfolding Theory which is fundamental to the Bifurcation Theory. Following in the same line, an important wonder is what makes an unfolding to be the most general possible. This chapter will present the proof of the Theorem (31) which give sufficient conditions to the miniversal unfolding of functions, supported by Thom's Transversality Theorem (27), which we will call generic unfolding.

Although the Theorem (31) is well-known in the field of Singularity and Unfolding Theories, its proof is not often done. Therefore, the proof that we will present here was produced by us.

In Sections (4.1) and (4.2), we will introduce only the basic concepts necessary, as definitions and notations, sufficient concerning the Jet space and Unfolding of functions. In Section (4.3), we will define the transversality of a function on the manifolds. We will introduce Thom's Transversality Theorem from of the point view of the Jet space. Finally, in Section (4.4), we will prove the Theorem (31), and through Thom's Transversality Theorem we will provide the well-known *generic condition* for the unfolding of functions, which tell us when an unfolding is generic.

4.1 Jet space

We will define only the concepts necessary to work with k-jets. The main textbook is (GOLUBITSKY; GUILLEMIN, 2012). Let $\mathscr{C}^\infty(U,\mathbb{R}^n)$ be the space of smooth maps $f : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is an open set. Given $f, g \in \mathscr{C}^\infty(U,\mathbb{R}^n)$, we say that f and g have k-th *order contact at the point* x_0 if both have the same value at x_0 and equal derivatives to order k :

$$f(x_0) = g(x_0), \quad f^{'}(x_0) = g^{'}(x_0), \quad \ldots \quad , f^{(k)}(x_0) = g^{(k)}(x_0).$$

The k-th order contact of f and g at the point x_0 is an equivalence relation denoted by $f \sim_k g$. In other words, we can say that $f \sim_k g$, if the truncated Taylor expansion up to k-th order of f and g around x_0 agree. Each equivalence class at x_0 is denoted by $j^k_{x_0} f$ or $j^k f(x_0)$ and we call the k-jet of f at x_0. Let

$$J^k_{(x_0,y_0)}(U,\mathbb{R}^n) := \{ j^k_{x_0} f \mid f \in \mathscr{C}^\infty(U,\mathbb{R}^n),\ f(x_0) = y_0 \}$$

be a set of k-jets at the point x_0, and we define the k-Jet space as the disjoint union of all k-jet sets:

$$\mathscr{J}^k(U,\mathbb{R}^n) = \bigcup_{(x,y)\in U\times\mathbb{R}^n} J^k_{(x,y)}(U,\mathbb{R}^n).$$

We define on k-Jet space the metric $d : \mathscr{J}^k(U,\mathbb{R}^n) \times \mathscr{J}^k(U,\mathbb{R}^n) \to \mathbb{R}_+$, where

$$d(j^k f, j^k g) = \sum_{l=0}^{k} \sup_{x\in U} ||f^{(l)}(x) - g^{(l)}(x)||.$$

Note that, if $f \sim_k g$ then $d(j^k f, j^k g) = 0$. It induces a well-defined map:

$$\tau_1 : \mathscr{J}^k(U,\mathbb{R}^n) \to U$$

mapping each k-jet to its origin point, $\tau_1(j^k_x f) = x$, we call the *source map*. On the other hand, given $f \in \mathscr{C}^\infty(U,\mathbb{R}^n)$ we have canonically defined:

$$j^k f : U \to \mathscr{J}^k(U,\mathbb{R}^n)$$

mapping x to $j^k f(x) = j^k_x f \in J^k_{(x,f(x))}(U,\mathbb{R}^n)$ for each $x \in U$. Using these concepts we can define a topology on $\mathscr{C}^\infty(U,\mathbb{R}^n)$ by constructing a basis. Consider $(\mathscr{J}^k(U,\mathbb{R}^n), d)$ a metric space and define, for each $f \in \mathscr{C}^\infty(U,\mathbb{R}^n)$,

$$B_\delta(f) := \{ g \in \mathscr{C}^\infty(U,\mathbb{R}^n) | \forall x \in U,\ d(j^k_x f, j^k_x g) < \delta_x \},$$

where, $\delta : U \to \mathbb{R}_+$ is a continuous function. The family $\{B_\delta(f)\}$ forms a neighborhood basis of f in the topology of $\mathscr{C}^\infty(U,\mathbb{R}^n)$ we call the Whitney $\mathscr{C}^k$-Topology.

We induce a topology on $\mathscr{C}^\infty(U,\mathbb{R}^n)$ from the topology of $\mathscr{J}^k(U,\mathbb{R}^n)$ defining for each open set $V \subset \mathscr{J}^k(U,\mathbb{R}^n)$ the set

$$M(V) := \{ f \in \mathscr{C}^\infty(U,\mathbb{R}^n) | j^k_x f \in V \}.$$

The family of set $\{M(V)\}$ form a basis for a topology on $\mathscr{C}^\infty(U,\mathbb{R}^n)$ called Whitney $\mathscr{C}^k$-Topology. For each $k \geq 0$, let W_k be a set of open sets of $\mathscr{C}^\infty(U,\mathbb{R}^n)$ in the Whitney $\mathscr{C}^k$-Topology. We define the Whitney $\mathscr{C}^\infty$-Topology on $\mathscr{C}^\infty(U,\mathbb{R}^n)$ generated by the basis

$$W = \bigcup_{l=0}^{k} W_l$$

for all $k \geq 0$.

4.2 Unfolding

We are interested in the phenomena that arise in the neighborhood of a nilpotent singularity of triple zero eigenvalues, meaning Jordan matrix 3×3 with a 1-dimensional kernel:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Following (ARNOLD, 1971), the reduction process from a square matrix to its Jordan canonical form is unstable, in the sense that small perturbations of eigenvalues can destroy its form. Unfolding Theory allows us to study dynamic behaviour in a neighborhood of this singularity and among all possible unfolding, there is one most important that represents all known as *versal unfolding*. Versal unfolding is a stable reduction process, more precisely, a versal unfolding is a family, e.g., matrices, vector fields, parameterized such that any other unfolding of the same matrix can be transformed into the first one. Next, we will define unfolding for vector fields and square matrices.

Vector fields. Let $f \in \mathscr{C}^\infty(U, \mathbb{R}^n)$, $U \subset \mathbb{R}^n$ open, be a vector field. A parameterized family of vector fields passing through f is a map

$$F_f : U \times \Lambda^n \to \mathbb{R}^n,$$

where $F_f(x,0) = f(x)$ for all $x \in U$ and Λ^n is an n-dimensional parameter space (or parameters basis) containing the origin. The map F_f is called unfolding to n-parameters ε of f. Intuitively, varying the parameters we get a new system and new possible behaviour, so an unfolding of f is able to capture systems and their behaviours close to f.

Matrices. Let $A_0 \in \mathbb{M}_n(\mathbb{R})$ be a matrix. An unfolding of A_0 is a parameterized family of matrices passing through A_0 defined by a map $A(\varepsilon)$:

$$A : \Lambda^n \to \mathbb{M}_n(\mathbb{R})$$

where $A(0) = A_0$.

Change of bases. Let $B(\mu)$ be unfolding of A_0 for another base $\mu \in \Sigma^m$. A base change is a germ of holomorphic functions $\phi(\mu) = \varepsilon$ with $\mu \in \Sigma^m \subset \mathbb{C}^m$ around of 0

$$\phi : \Sigma^m \to \Lambda^n.$$

Versal Unfolding. Let $A(\varepsilon)$ be an unfolding of A_0 with parameter bases Λ^n. We say that $A(\varepsilon)$ is a versal unfolding, if for every unfolding $B(\mu)$, $\mu \in \Sigma^m$, there are change of bases ϕ and a unfolding of the identity $C(\mu)$ such that:

$$B(\mu) = C(\mu)A(\phi(\mu))C^{-1}(\mu).$$

When we have a versal unfolding with *minimum* number of parameters, we say that the unfolding is *miniversal*.

Remark 25. Both Jet spaces and unfolding, we are interested in the k-jet of the unfolding:

$$j^k F_f : U \times \Lambda^n \to \mathscr{J}^k(U \times \Lambda^n, \mathbb{R}^n)$$

where we have $j^k F_f(x,\varepsilon) \in J^k_{(x,\varepsilon,F_f(x,\varepsilon))}(U \times \Lambda^n, \mathbb{R}^n) \subset \mathscr{J}^k(U \times \Lambda^n, \mathbb{R}^n)$ for each $(x,\varepsilon) \in U \times \Lambda^n$. We say that two unfolding F_f and G_f from $f \in \mathscr{C}^\infty(U,\mathbb{R}^n)$ have k-th order contact at the point (x_0,ε_0) if $F_f \sim_k G_f$ with the same parameter space.

4.3 Transversality

Transversality is about how two objects intersect and it relates to stability and genericity. Stability means that transversality is not lost by small perturbations and moreover, a small perturbation of non-transversal objects makes them transversal, which is what we mean by genericity. Transversality could be thought of as opposite to tangent objects. For example, two objects tangent to each other are unstable and non-transversal since even to small perturbation of these objects, the tangent property is lost and they become transversal. Following (GOLUBITSKY; GUILLEMIN, 2012), we will define it precisely.

Definition 26. Given $f \in \mathscr{C}^\infty(X,Y)$ where X,Y are finite dimensional manifolds, let $W \subset Y$ be a sub-manifold. We say that f is transversal to W at the point $f(x_0) = y_0$, denoted by $f \pitchfork W$, if one of the following conditions is satisfied

1. $y_0 \notin W$;

2. If $y_0 \in W$, then we have $T_{y_0}Y = \mathscr{I}m(D_{x_0}f) + T_{y_0}W$.

Note that, in case we have, $\dim(T_{y_0}Y) > \dim(\mathscr{I}m(D_{x_0}f)) + \dim(T_{y_0}W)$ and $f \pitchfork W$ at the point x_0, then we necessarily have $f(x_0) \notin W$.

Thom's Transversality Theorem plays an important role to finish our generic property to unfolding:

Theorem 27 (Thom's Transversality Theorem)**.** Let X and Y be finite dimensional manifolds and $W \subset \mathscr{J}^k(X,Y)$ be a sub-manifold. The set given by:

$$T_W := \{f \in \mathscr{C}^\infty(X,Y) | j^k f \pitchfork W\}$$

is a residual subset of $\mathscr{C}^\infty(X,Y)$ in the $\mathscr{C}^\infty$-Topology.

An important relationship between the miniversal unfolding and the orbit that passes through A_0 is their transversality. Before that, we need some definitions.

The space of matrices $M = \mathbb{M}_{n^2}(\mathbb{C})$ with manifold structure, we consider the sub-manifold Lie group $G = GL_n(\mathbb{C})$ of all non-singular $n \times n$ matrices. Let $Ad_{A_0} : G \to M$ defined by $Ad_{A_0}g = g^{-1}A_0 g$ for all $g \in G$ be the differentiable action of G on M for a fixed matrix $A_0 \in M$. We define the *orbit* of an arbitrary fixed matrix A_0 as the image N of the action Ad_{A_0}, that is $N = Im(Ad_{A_0}) := \{B \in M | \ B = g^{-1}A_0 g \ \text{ for some } \ g \in G\} \subset M$.

Theorem 28. An unfolding $A(\varepsilon)$ of A_0 is versal if, and only if, the mapping A is transversal to the orbit of A_0 at $\varepsilon = 0$.

The orbit of A_0 consists of all the matrices similar to A_0 and the derivative $D_e Ad_{A_0} : T_e G \to T_{A_0} M$ with respect to identity $e \in G$ is given by the Lie bracket $D_e Ad_{A_0} C = [A_0, C] = A_0 C - C A_0$. The subspace *centralizer* at A_0 denoted by $Z_{A_0} \subset T_e G$ consists of all matrices C such that $[A_0, C] = 0$, thus $Z_{A_0} = \ker(D_e Ad_{A_0})$. The Theorem (28) tells us that the image of the derivative of the unfolding A must be a complementary subspace to the orbit at A_0 to get a versal unfolding. The minimal dimension of this complementary subspace is equal to the co-dimension of the orbit of A_0 and this will be the minimal number of parameters to miniversality. Note that for any number strictly less than its minimum this will not be possible.

We introduced in the space M the inner product defined by $\langle A, B\rangle = Tr(AB^*)$, where B^* is the conjugated transposed matrix of the matrix B and Tr is the well-known trace of matrices. A technical result to construct miniversal unfolding is given by the following lemma

Lemma 29. A vector $B \in T_{A_0} M$ is perpendicular to the orbit of the matrix A_0 if and only if $[B^*, A_0] = 0$.

The reader can find more details and proofs in (ARNOLD, 1971), (WIGGINS; WIGGINS; GOLUBITSKY, 2003).

Remark 30. In our case, we are interested in studying phenomena in nine-dimensional space in a neighborhood of the Jordan 3×3 matrix:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We would like to know how many parameters we need to get a miniversal unfolding of this Jordan matrix. The centralizer, orthogonal complement, and normal form are given respectively by the following matrices:

$$Z_J = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}, \quad Z_J^* = \begin{pmatrix} \bar{a} & 0 & 0 \\ \bar{b} & \bar{a} & 0 \\ \bar{c} & \bar{b} & \bar{a} \end{pmatrix}, \quad B(\varepsilon) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varepsilon_3 & \varepsilon_2 & \varepsilon_1 \end{pmatrix}.$$

The orthogonal complement Z_J^* was obtained by Lemma (29), the normal form of the Z_J^* is given by $B(\varepsilon)$ and we have changed $\bar{a} = \varepsilon_1$, $\bar{b} = \varepsilon_2$ and $\bar{c} = \varepsilon_3$ to be new parameters of the family of matrices for a miniversal unfolding of J.

4.4 Genericity

We will find the normal form and the unfolding locally, up to the second order, for a smooth map with a Jordan matrix as its linear part. We will prove in Theorem (31) below that, as long as a condition on the second derivatives is satisfied, its unfolding is locally topologically equivalent in the space of smooth maps. The k-Jet theory plays an important role, as it allows us, using Thom's Transversality Theorem, to conclude that the unfolding is a generic property, in the sense that the set of all unfolding of smooth maps with certain conditions is a residual subset in the Baire space and consequently, it is an open and dense subset. Intuitively, genericity allows us to know what dynamical behaviour occurs for a large set of systems by studying the unfolding found.

4.4.1 Topological equivalence

We are interested in the generic unfolding behaviour of a dynamical system f having a Jordan matrix J as seen in the Remark (30) as its linear part. We will prove that the set of 2-jets of the unfolding of f is an open and dense subset with respect to Whitney $\mathscr{C}^2$-Topology induced by the 2-Jet space. The main idea, in order to get this, will be to construct a sub-manifold $W \subset \mathscr{J}^2(U \times \Lambda^n, \mathbb{R}^n)$ of the 2-Jet space and to prove that $j^2 F_f \pitchfork W$. Then, by Thom's Transversality Theorem, the following set:

$$\mathscr{T}_W := \{F_f \in \mathscr{C}^\infty(U \times \Lambda^n, \mathbb{R}^n) | j^2 F_f \pitchfork W\}$$

will be an open and dense set in the space $\mathscr{C}^\infty(U \times \Lambda^n, \mathbb{R}^n)$. To construct the sub-manifold W, the following theorem will be essentially important to prove it.

Theorem 31 (Miniversal unfolding)**.** Let $f \in \mathscr{C}^\infty(U, \mathbb{R}^n)$ be a smooth vector field. If $f(0) = 0$, $\frac{\partial f}{\partial x}(0) = J$, and $\alpha_1 := \frac{\partial^2 f_n}{\partial x_1^2}(0) \neq 0$, then the normal form of f has locally a miniversal unfolding F_f to n-parameters.

The proof of this theorem is divided into three lemmas. In the first lemma, we will use a change of coordinates to find the normal form of f satisfying the above conditions and after that, we will get an unfolding F_f of the normal form obtained in the second lemma. In the third lemma we will prove an equivalence topological of two unfoldings, that is, any unfolding with respect to Theorem conditions, even with different parameter numbers, has an equivalent unfolding from f. That lemma is particularly important because it exposes us to a condition under which two unfolding must be equivalent at least in a neighborhood.

Lemma 32 (Normal form)**.** Given $f \in \mathscr{C}^\infty(U, \mathbb{R}^n)$ satisfying the conditions of Theorem (31), there is a change of coordinates such that the normal form of f is locally around 0 given by

$$\begin{cases} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ & \vdots \\ \dot{y}_n &= \frac{1}{2} y^T \mathscr{H} y \end{cases} \quad + \mathscr{O}(|y|^3),$$

where $\frac{1}{2} y^T \mathscr{H} y$ is a quadratic form with,

$$\mathscr{H} = \begin{pmatrix} \alpha_1 & * \\ * & * \end{pmatrix}_{n \times n}$$

and $\alpha_1 := \frac{\partial^2 f_n}{\partial x_1^2}(0)$. The symbol $*$ means others constants depending on Taylor coefficients of f that we do not specify further.

Proof: We begin using the Taylor expansion of f around 0 :

$$f(x) = f(0) + \frac{\partial f}{\partial x}(0)x + \frac{1}{2} x^T \frac{\partial^2 f}{\partial x^2}(0)x + \mathscr{O}(|x|^3)$$

and we rewrite it as:

$$\dot{x} = f(x) = Jx + \Phi(x) + \mathscr{O}(|x|^3)$$

where

$$\Phi(x) = \frac{1}{2} x^T \frac{\partial^2 f}{\partial x^2}(0)x = \frac{1}{2} \begin{pmatrix} x^T H_1 x \\ \vdots \\ x^T H_n x \end{pmatrix}_{n \times 1} \tag{4.1}$$

and H_k is the $n \times n$ Hessian matrix corresponding to f_k for each $k = 1, \ldots, n$.

We want to get rid of several entries of Φ. In order to do this we will perturb each H_k matrix using the change of coordinates $x = y + \varphi(y)$ for all x close to 0, where

$\varphi : U \to \mathbb{R}^n$ is a smooth quadratic vector field. The derivative with respect to t is:

$$\begin{aligned}
\dot{y} &= \dot{x} - \frac{\partial \varphi}{\partial y}(y)\dot{y} \\
&= J(y+\varphi(y)) + \Phi(y+\varphi(y)) - \frac{\partial \varphi}{\partial y}(y)\dot{y} + \mathscr{O}(|y|^3) \\
&= Jy + J\varphi(y) + \Phi(y) - \frac{\partial \varphi}{\partial y}(y)\dot{y} + \mathscr{O}(|y|^3) \\
&= Jy + \Phi(y) + J\varphi(y) - \frac{\partial \varphi}{\partial y}(y)((J(y+\varphi(y)) \\
&+ \Phi(y+\varphi(y)) + \cdots) - \frac{\partial \varphi}{\partial y}(y)\dot{y} + \mathscr{O}(|y|^3)) \\
&= Jy + \Phi(y) + J\varphi(y) - \frac{\partial \varphi}{\partial y}(y)Jy + \mathscr{O}(|y|^3).
\end{aligned}$$

We get a new system,

$$\dot{y} = Jy + R(y) + \mathscr{O}(|y|^3) \tag{4.2}$$

where the new vector field R is the vector field Φ with a perturbation:

$$R(y) = \Phi(y) + J\varphi(y) - \frac{\partial \varphi}{\partial y}(y)Jy. \tag{4.3}$$

We need to find a vector field φ that cancels most terms of Φ, so that we get R as desired. Indeed, suppose that the vector field φ is given by:

$$\varphi(y) = \frac{1}{2}\begin{pmatrix} y^T A_1 y \\ \vdots \\ y^T A_n y \end{pmatrix}_{n\times 1}, \tag{4.4}$$

where without loss of generality we impose $A_k = A_k^T$ for each $k = 1, \dots, n$. Then the derivative of φ with respect to y will be:

$$\frac{\partial \varphi}{\partial y}(y) = \frac{1}{2}\begin{pmatrix} 2y^T A_1 \\ \vdots \\ 2y^T A_n \end{pmatrix}_{n\times n} \tag{4.5}$$

using Equations (4.1), (4.4), (4.5) in Equation (4.3), we get:

$$R(y) = \frac{1}{2}\left(\begin{pmatrix} y^T H_1 y \\ \vdots \\ y^T H_n y \end{pmatrix} + J\begin{pmatrix} y^T A_1 y \\ \vdots \\ y^T A_n y \end{pmatrix} - \begin{pmatrix} 2y^T A_1 J y \\ \vdots \\ 2y^T A_n J y \end{pmatrix}\right)_{n\times 1}.$$

So, we have

$$R(y) = \frac{1}{2} y^T \begin{pmatrix} H_1 + A_2 - 2A_1 J \\ H_2 + A_3 - 2A_2 J \\ \vdots \\ H_n - 2A_n J \end{pmatrix}_{n\times(n\times n)} y \quad . \tag{4.6}$$

We can arrange for the first $n-1$ rows to get rid of these. So we make

$$\begin{aligned} A_2 &= -H_1 + 2A_1 J \\ A_3 &= -H_2 + 2A_2 J \\ &\vdots \\ A_n &= -H_{n-1} + 2A_{n-1} J \end{aligned}$$

and then from the $(n-1)$-th row we get recursively:

$$A_n = -\sum_{i=0}^{n-2} 2^i H_{n-1-i} J^i + 2^{n-1} A_1 J^{n-1} \tag{4.7}$$

now, using Equation (4.7) into the n-th row of Equation (4.6):

$$\mathscr{H} = H_n - 2A_n J = \sum_{i=0}^{n-1} 2^i H_{n-i} J^i.$$

Consequently, the first $(n-1)$ blocks vanish and we get the simple form for Equation (4.6):

$$R(y) = \frac{1}{2} y^T \underbrace{\begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathscr{H} \end{pmatrix}}_{E_3} y. \tag{4.8}$$

Note that $\mathscr{H}$ depends on H_k for all $k = 1, \ldots, n$. We denote by h_{ij} the entries corresponding to $\mathscr{H}$ matrix. Then, we set the following expression for the $\mathscr{H}$ matrix:

$$\mathscr{H} = \begin{pmatrix} h_{11} & * \\ * & * \end{pmatrix}_{n\times n}$$

where $h_{11} = \alpha_1 = \frac{\partial^2 f_n}{\partial x_1^2}(0) \neq 0$. Finally, Equation (4.8) and the Equation (4.2) give us the normal form of f for all y in the neighborhood of 0,

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{2} y^T \mathscr{H} y \end{pmatrix} + \mathscr{O}(|y|^3)$$

or in short form,

$$\dot{y} = Jy + R(y) + \mathscr{O}(|y|^3) \tag{4.9}$$

equivalently,

$$\left\{\begin{array}{lcl} \dot{y}_1 & = & y_2 \\ \dot{y}_2 & = & y_3 \\ & \vdots & \\ \dot{y}_n & = & \frac{1}{2} y^T \mathscr{H} y \end{array}\right. + \mathscr{O}(|y|^3). \tag{4.10}$$

for all y close to the origin.

■

Lemma 33 (Unfolding the normal form)**.** If $\alpha_1 \neq 0$ then there is a coordinate change such that the normal form of Equation (4.10) of f has the n-parameter unfolding $F_f(x,\varepsilon)$ for (x,ε) in the neighborhood of the 0 :

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ & \vdots & \\ \dot{x}_n & = & \varepsilon_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \cdots + \varepsilon_n x_n + \frac{1}{2} x^T \mathscr{H} x \end{array}\right. + \mathscr{O}(|x|^3)$$

where ε_i for $i = 1,\ldots,n$ are the parameters of the family of vector fields.

Proof: Consider the normal form of Equation (4.9) with the following general perturbation by (see e.g. (MURDOCK, 2006)):

$$\dot{y} = Jy + R(y) + \mu v + \mu K y + \mathscr{O}(|y|^3 + |\mu|^2 + |\mu||y|^2), \tag{4.11}$$

where y close to origin, $v \in \mathbb{R}^n$, $K = (k_{ij})_{n\times n}$ and $\mu \in \mathbb{R}$. Let

$$y = x + \mu w + \mu L x \tag{4.12}$$

be a change of coordinates with the new coordinates x close to origin, $w \in \mathbb{R}^n$, and $L = (l_{ij})_{n\times n}$. Differentiating with respect to time gives:

$$\dot{y} = \dot{x} + \mu L \dot{x} = (I + \mu L)\dot{x} \Longleftrightarrow (I + \mu L)^{-1} \dot{y} = \dot{x} \tag{4.13}$$

for μ sufficiently small, and we know that

$$(I + \mu L)^{-1} = I - \mu L + \mathscr{O}(\mu^2). \tag{4.14}$$

Using Equation (4.12) in Equation (4.11) we get:

$$\begin{aligned}
\dot{y} &= J(x+\mu w+\mu Lx)+R(x+\mu w+\mu Lx)+\mu v+\mu K(x+\mu w+\mu Lx) \\
&+ \mathscr{O}(y^3+\mu^2+\mu y^2) \qquad (4.15)\\
&= Jx+\mu(v+Jw+(K+JL)x)+R(x+\mu w+\mu Lx)+\mathscr{O}(\mu^2+x^3+\mu x^2) \\
&= Jx+\mu(v+Jw+(K+JL)x)+\frac{1}{2}(x^T E_3 x+2\mu w^T E_3 x)+\mathscr{O}(\mu^2+x^3+\mu x^2) \\
&= Jx+\mu(v+Jw+(K+JL+w^T E_3)x)+\frac{1}{2}x^T E_3 x+\mathscr{O}(\mu^2+x^3+\mu x^2) \\
&= Jx+\mu(p+\bar{M}x)+R(x)+\mathscr{O}(\mu^2), \qquad (4.16)
\end{aligned}$$

where $p=v+Jw$ and $\bar{M}=K+JL+w^T E_3$. Using Equations (4.14) and (4.16) in Equation (4.13),

$$\begin{aligned}
\dot{x} &= (I+\mu L)^{-1}\dot{y} \\
&= (I-\mu L+\mathscr{O}(\mu^2))(Jx+\mu(p+\bar{M}x)+R(x)+\mathscr{O}(\mu^2+x^3+\mu x^2)) \\
&= Jx+R(x)+\mu(p+(\bar{M}-LJ)x)-\mu LR(x)+\mathscr{O}(\mu^2) \\
&= Jx+R(x)+\mu(p+(\bar{M}-LJ)x)+\mathscr{O}(\mu^2) \\
&= Jx+R(x)+\mu(p+Mx)+\mathscr{O}(\mu^2)
\end{aligned}$$

where p and M represent a perturbation of v and K respectively:

$$\begin{aligned}
p &= v+Jw \qquad (4.17)\\
M &= K+w^T E_3+[J,L]=\bar{M}-LJ. \qquad (4.18)
\end{aligned}$$

We want to get rid of as many entries of p and M as possible, to that end, we need to find w and L that cancel entries of v and K respectively. Using Equation (4.17) we can find w such that p is

$$p=\begin{pmatrix} p_1 \\ \vdots \\ p_{n-1} \\ p_n \end{pmatrix}=\begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}+\begin{pmatrix} w_2 \\ \vdots \\ w_n \\ 0 \end{pmatrix},$$

except for the first coordinate w_1 of w, we can choose w_i for $i=2,\ldots,n$ that cancel entries of v :

$$w_i=-v_{i-1}, \quad i=2,\ldots,n, \quad p_n=v_n, \qquad (4.19)$$

and we get p in the simplest form:

$$p=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ p_n \end{pmatrix}. \qquad (4.20)$$

We still get to choose the first coordinate w_1. Now we must choose the L matrix such that most entries of K are canceled to get M simplest. In fact, from Equation (4.18) M is given by

$$M = \begin{pmatrix} k_{11}+l_{21} & k_{12}+l_{22}-l_{11} & \cdots & k_{1n}+l_{2n}-l_{1n-1} \\ k_{21}+l_{31} & k_{22}+l_{32}-l_{21} & \cdots & k_{2n}+l_{3n}-l_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n-11}+l_{n1} & k_{n-12}+l_{n2}-l_{n-11} & \cdots & k_{n-1n}+l_{nn}-l_{n-1n-1} \\ k_{n1}+\sum_{i=1}^{n} w_i h_{i1} & k_{n2}+\sum_{i=1}^{n} w_i h_{i2}-l_{n1} & \cdots & k_{nn}+\sum_{i=1}^{n} w_i h_{in}-l_{nn-1} \end{pmatrix},$$

then by choosing the entries of L following the rules:

$$l_{i+11} = -k_{i1}, \qquad l_{ij-1} - l_{i+1j} = k_{ij} \tag{4.21}$$

for $i = 1, \ldots, n-1$ and $j = 2, \ldots, n$, we will cancel the $(n-1)$ first rows of M. Moreover, using Equations (4.21) and (4.19) in the last row we get:

$$\begin{aligned} w_1 &= \frac{-k_{n1} - \sum_{i=2}^{n} w_i h_{i1}}{\alpha_1} = \frac{-k_{n1} + \sum_{i=2}^{n} v_{i-1} h_{i1}}{\alpha_1}, \text{ such that } m_{n1} = 0. \\ m_{n2} &= k_{n2} + \sum_{i=1}^{n} w_i h_{i2} - l_{n1} = k_{n2} + k_{n-11} + \sum_{i=1}^{n} w_i h_{i2} \\ &\vdots \\ m_{nn-1} &= k_{nn-1} + \sum_{i=1}^{n} w_i h_{in-1} - l_{nn-2} = k_{nn-1} + \cdots + k_{21} + \sum_{i=1}^{n} w_i h_{in-1} \\ m_{nn} &= k_{nn} + \sum_{i=1}^{n} w_i h_{in} - l_{nn-1} = k_{nn} + \cdots + k_{11} + \sum_{i=1}^{n} w_i h_{in}. \end{aligned}$$

In other words, we may write:

$$w_1 = \frac{-k_{n1} + \sum_{i=1}^{n-1} v_i h_{i+11}}{\alpha_1} \tag{4.22}$$

$$m_{n2} = \sum_{j=1}^{2} k_{n-2+jj} + \sum_{i=1}^{n} w_i h_{i2} \tag{4.23}$$

$$\vdots$$

$$m_{nn-1} = \sum_{j=1}^{n-1} k_{1+jj} + \sum_{i=1}^{n} w_i h_{in-1} \tag{4.24}$$

$$m_{nn} = \sum_{j=1}^{n} k_{jj} + \sum_{i=1}^{n} w_i h_{in}. \tag{4.25}$$

So, w_1 depend on k_{n1} and $v's$, $w_2, \ldots, w_n$ depend on $v's$, so m_{n2} to m_{nn} are completely independent. Therefore, we have found w and L using:

$$w_1 = \frac{-k_{n1} + \sum_{i=1}^{n-1} v_i h_{i+11}}{\alpha_1}, \qquad w_i = -v_{i-1}, \quad i = 2, \ldots, n,$$

$$l_{i+11} = -k_{i1}, \qquad l_{ij-1} - l_{i+1j} = k_{ij} \quad i = 1, \ldots, n-1, \quad j = 2, \ldots, n,$$

that cancel most entries of v and K. From Equations (4.19) - (4.25), p and M becomes:

$$p = \begin{pmatrix} 0 \\ \vdots \\ p_n \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & m_{n2} & \cdots & m_{nn} \end{pmatrix}. \tag{4.26}$$

Returning to Equation (4.17):

$$\dot{x} = Jx + R(x) + \mu(p + Mx) + \mathscr{O}(|x|^3),$$

with Equation (4.26), we get:

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix} &= \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \mu \left(\begin{pmatrix} 0 \\ \vdots \\ 0 \\ p_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{j=2}^{n} m_{nj} x_j \end{pmatrix} \right) \\ &+ \quad R(x) + \mathscr{O}(|x|^3), \end{aligned}$$

or equivalently,

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & x_4 \\ & \vdots & \\ \dot{x}_n & = & \mu p_n + \mu m_{n2} x_2 + \cdots + \mu m_{nn} x_n + \frac{1}{2} x^T \mathscr{H} x \end{array} \right. + \mathscr{O}(|x|^3)$$

we will set:

$$\varepsilon_1 = \mu p_n, \quad \varepsilon_2 = \mu m_{n2} \quad , \ldots, \quad \varepsilon_n = \mu m_{nn}.$$

As μ is supposed to be sufficiently small, there is an open neighborhood of the origin in parameters space $N_2 \subset \Lambda^n$ such that the unfolding $F_f(x, \varepsilon)$ of the normal form of f for all $(x, \varepsilon) \in N_{(0,0)} = N_1 \times N_2 \subset U \times \Lambda^n$ is given by:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ & \vdots & \\ \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & \varepsilon_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n + \frac{1}{2} x^T \mathscr{H} x \end{array} \right. + \mathscr{O}(|x|^3).$$

We will write $F_f : N_{(0,0)} \to \mathbb{R}^n$ given by

$$\dot{x} = F_f(x,\varepsilon) = Jx + E_1\varepsilon + \varepsilon^T E_2 x + \frac{1}{2}x^T E_3 x + \cdots$$

where,

$$E_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ E_{2n} \end{pmatrix},$$
$$E_{2n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathscr{H} \end{pmatrix}.$$

■

Before proving topological equivalence in the next lemma, we need to introduce the setup and some notation.

Remember that $f \in \mathscr{C}^\infty(U,\mathbb{R}^n)$ with $f(x) = (f_1(x),\ldots,f_n(x))$ and $x = (x_1,\ldots,x_n) \in U$, and we suppose that $f(0) = 0$ and $\alpha_1 = \frac{\partial^2 f_n}{\partial x_1^2}(0) \neq 0$.

Let G_f be an unfolding to m-parameters $m > n$, as in Theorem (31). So, we have $G_f : U \times \Sigma^m \to \mathbb{R}^n$ such that

$$\dot{x} = G_f(x,\xi).$$

The Taylor expansion of G_f at $(0,0)$:

$$G_f(x,\xi) = \frac{\partial G_f}{\partial x}x + \frac{\partial G_f}{\partial \xi}\xi + \xi^T \frac{\partial^2 G_f}{\partial x \partial \xi}x + \frac{1}{2}x^T \frac{\partial^2 G_f}{\partial x^2}x + \mathscr{O}(|x|^3 + |\xi|^2 + |\xi||x|^2)$$

for all (x,ξ) in a neighborhood of $(0,0)$ satisfying:

$$\begin{aligned} G_f(x,0) &= f(x) = Jx + \frac{1}{2}x^T E_3 x + \mathscr{O}(|x|^3) \\ G_f(0,0) &= f(0) = 0. \end{aligned}$$

Therefore, the unfolding G_f of f has the form:

$$G_f(x,\xi) = Jx + A_1\xi + \xi^T A_2 x + \frac{1}{2}x^T E_3 x + \mathscr{O}(|x|^3)$$

where each term is given by

$$A_1 = \left(\frac{\partial G_f}{\partial \xi}(0,0)\right), \; A_2 = \left(\frac{\partial^2 G_f}{\partial \xi \partial x}(0,0)\right) = \begin{pmatrix} A_{21} \\ \vdots \\ A_{2n} \end{pmatrix},$$

with $(A_1)_{n\times m}$ and $(A_2)_{n\times(m\times n)}$.

Lemma 34 (Topological equivalence)**.** There are coordinate changes such that any unfolding to m-parameters, $m > n$, of f given by $G_f : U \times \Sigma^m \to \mathbb{R}^n$

$$G_f(x,\xi) = Jx + A_1\xi + \xi^T A_2 x + \frac{1}{2}x^T E_3 x + \mathscr{O}(|x|^3) \tag{4.27}$$

is locally equivalent to the unfolding to n-parameters of the normal form of f given by $F_f : U \times \Lambda^n \to \mathbb{R}^n$

$$F_f(x,\varepsilon) = Jx + E_1\varepsilon + \varepsilon^T E_2 x + \frac{1}{2}x^T E_3 x + \mathscr{O}(|x|^3). \tag{4.28}$$

More precisely, there are maps $\psi : U \to U$, $\psi(z) = x$, $\vartheta : \Sigma^m \to \Lambda^n$, $\vartheta(\xi) = \varepsilon$ and a perturbation $S_\gamma(z,\xi)$ for $\gamma \in \mathbb{R}$ sufficiently small such that,

$$S_\gamma(z,\xi) = \begin{cases} G_f(z,\xi) & \text{if } \gamma = 0 \\ F_f(\psi(z),\vartheta(\xi)) & \text{if } \gamma \neq 0 \end{cases}.$$

Proof:

We will use a second coordinate change in order to get a perturbation of the terms A_i for $i = 1,2$. Consider

$$x = z + \gamma K_1 \xi \tag{4.29}$$

where $\gamma \in \mathbb{R}$ and the matrix K_1 has the same size as A_1. Using Equation (4.29) in Equation (4.27) we will get the following equation for $\dot{z} = \dot{x}$:

$$\begin{aligned} \dot{z} &= Jz + (A_1 + \gamma J K_1)\xi + \xi^T(A_2 + \gamma(K_1^T E_3 + K_1^T E_3^T))z + \frac{1}{2}z^T E_3 z \\ &+ \mathscr{O}(|z|^3 + |\xi||z|^2 + |\xi|^2). \end{aligned} \tag{4.30}$$

Note that x depends on z by Equation (4.29), so we have a change of variables $\psi(z) = x$. If we set

$$B_1(\gamma) = A_1 + \gamma J K_1 \tag{4.31}$$
$$B_2(\gamma) = A_2 + \gamma(K_1^T E_3 + K_1^T E_3^T) \tag{4.32}$$

then we get the simpler equation:

$$\begin{aligned} \dot{z} = S_\gamma(z,\xi) &= Jz + B_1(\gamma)\xi + \xi^T B_2(\gamma) z + z^T E_3 z \\ &+ \mathscr{O}(|z|^3 + |\xi||z|^2 + |\xi|^2), \end{aligned}$$

for all (z,ξ) in a neighborhood of $(0,0)$. The main idea will be to turn each $B_i(\gamma)$ into E_i for $i = 1,2$. In other words, we must find conditions over the matrix K_1 and ε depending on ξ such that:

$$\begin{aligned} E_1\varepsilon &= B_1(\gamma)\xi \\ \varepsilon^T E_2 &= \xi^T B_2(\gamma). \end{aligned}$$

Note that, if $\gamma = 0$ then

$$S_0(z,\xi) = G_f(\psi(z),\xi).$$

We suppose $\gamma \neq 0$ sufficiently small and we will prove that there is a change of coordinates $\vartheta(\xi)$ such that

$$S_\gamma(z,\xi) = F_f(\psi(z),\vartheta(\xi)).$$

The matrix $B_1(\gamma)$ has $n\times m$ size and $B_2(\gamma)$ has n blocks of $m\times n$:

$$B_2(\gamma) = \begin{pmatrix} B_{21}(\gamma) \\ \vdots \\ B_{2n}(\gamma) \end{pmatrix} = \begin{pmatrix} A_{21} \\ \vdots \\ A_{2n} + \gamma K_1^T(\mathscr{H} + \mathscr{H}^T) \end{pmatrix}$$

and for each $j = 1,\dots,n$

$$(B_{2j})_{m\times n} = \begin{pmatrix} col_1(B_{2j}) & \cdots & col_n(B_{2j}) \end{pmatrix}.$$

The strategy to make $B_1(\gamma)\xi$ equal to $E_1\varepsilon$ will be, from Equation (4.31), looking for conditions over K_1 and ξ such that

$$E_1\varepsilon = B_1(\gamma)\xi = (A_1 + \gamma J K_1)\xi.$$

So we have,

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}\begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix} = \left(\begin{pmatrix} a^1_{11} & \cdots & a^1_{1m} \\ \vdots & \ddots & \vdots \\ a^1_{n1} & \cdots & a^1_{nm} \end{pmatrix} + \gamma J \begin{pmatrix} k_{11} & \cdots & k_{1m} \\ \vdots & \ddots & \vdots \\ k_{n1} & \cdots & k_{nm} \end{pmatrix}\right)\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix}$$

where a^1_{ij} are entries corresponding to A_1. Equivalently, we need to find k_{ij} and ε_1 by the following relation:

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \varepsilon_1 \end{pmatrix} = \begin{pmatrix} a^1_{11} + \gamma k_{21} & \cdots & a^1_{1m} + \gamma k_{2m} \\ \vdots & & \vdots \\ a^1_{n-11} + \gamma k_{n1} & \cdots & a^1_{n-1m} + \gamma k_{nm} \\ a^1_{n1} & \cdots & a^1_{nm} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{m-1} \\ \xi_m \end{pmatrix}.$$

Except for first row, we found the last $(n-1)$ rows of K_1 and they must satisfies:

$$\begin{aligned} 0 &= \sum_{j=1}^{m}(a^1_{i-1j} + \gamma k_{ij})\xi_j, \quad i = 2,\dots,n-1, \\ &\implies k_{ij} = -\frac{a^1_{i-1j}}{\gamma}, \quad i = 2,\dots,n-1, \quad j = 1,\dots,m. \end{aligned} \tag{4.33}$$

and ε_1 is given by,

$$\varepsilon_1 = \sum_{j=1}^{m} a^1_{nj}\xi_j. \tag{4.34}$$

To complete K_1 and ε we must to use the Equation (4.32) to find the first row of K_1 and the others ε_i for $i=2,\dots,n$. To make it we will prove that

$$\varepsilon^T E_2 = \xi^T B_2(\gamma)$$

where it means this:

$$\varepsilon^T \begin{pmatrix} \mathbf{0} \\ \vdots \\ E_{2n} \end{pmatrix} = \xi^T \begin{pmatrix} A_{21} \\ \vdots \\ A_{2n} + \gamma K_1^T(\mathscr{H} + \mathscr{H}^T) \end{pmatrix}.$$

Since ξ^T is arbitrary, we have from the n rows:

$$\begin{aligned} A_{2i} &= \mathbf{0} \quad \text{for } \ i = 1,\dots,n-1, \\ \varepsilon^T E_{2n} &= \xi^T(A_{2n} + \gamma K_1^T(\mathscr{H} + \mathscr{H}^T)) = \xi^T B_{2n}(\gamma). \end{aligned} \tag{4.35}$$

Remembering the form of E_{2n} of Equation (4.27), we have from Equation (4.35):

$$(0, \varepsilon_2, \dots, \varepsilon_n) = (\xi^T col_1(B_{2n}(\gamma)), \xi^T col_2(B_{2n}(\gamma)), \dots, \xi^T col_n(B_{2n}(\gamma))), \tag{4.36}$$

again, as ξ^T is arbitrary, it implies that the first column of $B_{2n}(\gamma)$ is a null-column, that is $col_1(B_{2n}(\gamma)) = \mathbf{0}$. So, $\gamma \neq 0$ implies

$$\mathbf{0} = col_1(B_{2n}(\gamma)) = col_1(A_{2n} + \gamma K_1^T(\mathscr{H} + \mathscr{H}^T))$$

consequently,

$$col_1(K_1^T(\mathscr{H} + \mathscr{H}^T)) = -\frac{1}{\gamma} col_1(A_{2n}).$$

Remembering that the sizes are $(K_1)_{n\times m}$ and $(\mathscr{H} + \mathscr{H}^T)_{n\times n}$. To get the complete first row of K_1 we need to extract the first entry of each row vector of the $(K_1^T(\mathscr{H} + \mathscr{H}^T))$. Let $(k_j^T)_{1\times n} = (k_{1j}, \dots, k_{nj})$ be the j-th row vector of K_1^T. Then, we have

$$col_1 \begin{pmatrix} k_1^T(\mathscr{H} + \mathscr{H}^T) \\ \vdots \\ k_m^T(\mathscr{H} + \mathscr{H}^T) \end{pmatrix}_{m\times n} = -\frac{1}{\gamma} \begin{pmatrix} a_{11}^2 \\ \vdots \\ a_{m1}^2 \end{pmatrix}_{m\times 1},$$

where a_{j1}^2 are the entries of the first column of the last block A_{2n} of A_2. We will highlight the first entries of each row vector $k_j^T(\mathscr{H} + \mathscr{H}^T)$. They are given by

$$\sum_{i=1}^{n} k_{ij}(h_{ij} + h_{ji}) = -\frac{a_{j1}^2}{\gamma} \text{ for each } j = 1,\dots,m.$$

When $i=1$ we will get each entry of the first row of K_1, so the first row of K_1 is given by:

$$k_{1j} = -\frac{a_{j1}^2}{\gamma(h_{1j} + h_{j1})} - \frac{\sum_{i=2}^{n} k_{ij}(h_{ij} + h_{ji})}{(h_{1j} + h_{j1})} \tag{4.37}$$

for each $j = 1,\dots,m$. Note that $\mathscr{H}$ is not symmetric or skew-symmetric, so $h_{1j} \neq h_{j1}$. Finally, using Equation (4.33) in Equation (4.37) we have completed matrix K_1 :

$$\begin{aligned} k_{ij} &= -\frac{a^1_{i-1j}}{\gamma}, \quad i = 2,\dots,n, \quad j = 1,\dots,m \\ k_{1j} &= -\frac{a^2_{j1}}{\gamma(h_{1j}+h_{j1})} - \frac{\sum_{i=2}^{n} k_{ij}(h_{ij}+h_{ji})}{(h_{1j}+h_{j1})}, \quad j = 1,\dots,m. \end{aligned}$$

Therefore K_1 depends on matrices A_1, A_2 and $\mathscr{H}$ while the parameter ε was constructed by Equation (4.34) and completed by Equation (4.36) and its depends on ξ, A_1, A_2, K_1 and $\mathscr{H}$:

$$\varepsilon_1 = \sum_{j=1}^{m} a^1_{nj}\xi_j, \qquad \varepsilon_i = \xi^T col_i(B_{2n}), \quad i = 2,\dots,n.$$

Thus, we have found a change of variables $\psi : U \to U$ given by $\psi(z) = z + \gamma K_1 \xi$ and a change of parameters $\vartheta : \Sigma^m \to \Lambda^n$, $\vartheta(\xi) = \varepsilon$ such that $F_f(\psi(z), \vartheta(\xi)) = G_f(z,\xi)$. Therefore, the unfolding of f to n-parameters is miniversal.

■

4.4.2 Generic condition

We have seen in Theorem (31) that the same condition $\alpha_1 \neq 0$ on the second partial derivatives of $f \in \mathscr{C}^\infty(U,\mathbb{R}^n)$ is essential to prove the equivalence topological of both unfolding, it was proved in the neighborhood of the fixed point. We call that condition by *generic condition* and we will use it to decompose the 2-Jet space of unfolding into disjoint subspaces. Consider the 2-Jet unfolding space

$$\mathscr{J}^2(U \times \Lambda^n, \mathbb{R}^n) = \dot{\bigcup_{(x,\varepsilon,y)}} J^2_{(x,\varepsilon,y)}(U \times \Lambda^n, \mathbb{R}^n),$$

we can decompose the right side into disjoint subspaces for each point. In particular, the right side of the 2-Jet space above at the point $(0,0,0)$ is decomposed as follows

$$J^2_{(0,0,0)}(U \times \Lambda^n, \mathbb{R}^n) = J^2_{(0,0,0)}(U \times \Lambda^n, \mathbb{R}^n, \alpha_1 \neq 0) \dot{\bigcup} J^2_{(0,0,0)}(U \times \Lambda^n, \mathbb{R}^n, \alpha_1 = 0),$$

where

$$\begin{aligned} &J^2_{(0,0,0)}(U \times \Lambda^n, \mathbb{R}^n, \alpha_1 \neq 0) = \\ &\{ j^2 H_h(0,0) | h \in \mathscr{C}^\infty(U,\mathbb{R}^n), \frac{\partial^2 h_n}{\partial x_1^2}(0) \neq 0, H_h(0,0) = 0 \} = V_{(0,0,0)} \end{aligned}$$

represents all 2-jets of the unfolding of maps in $\mathscr{C}^\infty(U,\mathbb{R}^n)$ as $\alpha_1 \neq 0$ at the point $(0,0,0)$ and for

$$\begin{aligned} &J^2_{(0,0,0)}(U \times \Lambda^n, \mathbb{R}^n, \alpha_1 = 0) = \\ &\{ j^2 H_h(0,0) | h \in \mathscr{C}^\infty(U,\mathbb{R}^n), \frac{\partial^2 h_n}{\partial x_1^2}(0) = 0, H_h(0,0) = 0 \} = W_{(0,0,0)} \end{aligned}$$

represents all 2-jets of the unfolding of maps in $\mathscr{C}^\infty(U,\mathbb{R}^n)$ as $\alpha_1 = 0$ at the point $(0,0,0)$. We can extend it for all points in $U \times \Lambda^n$ to get

$$\mathscr{J}^2(U \times \Lambda^n, \mathbb{R}^n) = V \dot{\cup} W,$$

where

$$V = \dot{\bigcup_{(x,\varepsilon,y)}} V_{(x,\varepsilon,y)}, \quad W = \dot{\bigcup_{(x,\varepsilon,y)}} W_{(x,\varepsilon,y)}.$$

Note that W is formed for all 2-jets of unfolding from maps which can not be transformed into j^2F_f as f has $\frac{\partial^2 f_n}{\partial x_1^2}(0) \neq 0$. Also note that, for f specifically from Theorem (31), we have $j^2F_f \in V_{(0,0,0)} \subset V$. Therefore, given $f \in \mathscr{C}^\infty(U,\mathbb{R}^n)$ with $\frac{\partial^2 f_n}{\partial x_1^2}(0) \neq 0$ we have $j^2F_f(x,\varepsilon) \notin W$ for all $(x,\varepsilon) \in \Omega_{(0,0)} \subset U \times \Lambda^n$ in a neighborhood of the origin and consequently $j^2F_f \pitchfork W$. We define the following set:

$$\mathscr{T}_W := \{F_f \in \mathscr{C}^\infty(\Omega_{(0,0)}, \mathbb{R}^n) |\ j^2F_f \pitchfork W\}$$

for all $f \in \mathscr{C}^\infty(U,\mathbb{R}^n)$, $\frac{\partial^2 f_n}{\partial x_1^2}(0) \neq 0$. By Thom's Transversality Theorem (27), $\mathscr{T}_W$ is a residual subset of $\mathscr{C}^\infty(U \times \Lambda^n, \mathbb{R}^n)$ in the $\mathscr{C}^\infty$-Topology. As $\mathscr{C}^\infty(U \times \Lambda^n, \mathbb{R}^n)$ is a Baire space we concludes that $\mathscr{T}_W$ is an open and dense subset of $\mathscr{C}^\infty(U \times \Lambda^n, \mathbb{R}^n)$.

CHAPTER

5

SYMBOLIC DYNAMICS

In this chapter we will introduce, following the excellent textbooks (WIGGINS; WIGGINS; GOLUBITSKY, 2003; KATOK; HASSELBLATT, 1997; DEVANEY, 2018), the Symbolic Dynamics. This concept is a fundamental resource to reach conclusions about some phenomena in several areas of research, for example, in the study of periodic orbits in dynamical systems. Basically, symbolic dynamics is composed of a space of sequences of symbols, which is a set formed by sequences *bi-infinite* where each term of this sequence can be chosen in a set *finite* of distinct symbols, together with a function defined over this space known as the *Shift map*. This symbolic structure encodes only the orbital properties of the dynamical system and the connection between these two concepts takes place through a topological conjugation.

5.1 The structure of the space of symbol sequences

Let $S = \{0,1,2,\ldots,N-1\}$ be a finite set of N symbols. We can produce an space metric structure in S by defining a metric δ such that $\delta(a,b) = 0$ if, and only if, $a = b$ and $\delta(a,b) = 1$ otherwise, for any symbols $a, b \in S$. Let Σ_N be a set of all bi-infinite sequences, where each term of this sequence is a symbol of S. A bi-infinite sequence $s \in \Sigma_N$ is written as

$$s := \{\ldots s_{-2}s_{-1} \bullet s_0 s_1 s_2 \ldots\} \tag{5.1}$$

where $s_i \in S$ for every i and $(\bullet)$ means the start point of the sequence. We will produce a structure of metric space on Σ_N defining a metric d. Let $s, \bar{s} \in \Sigma_N$ be bi-infinite sequences given by $s = \{\ldots s_{-2}s_{-1} \bullet s_0 s_1 s_2 \ldots\}$ and $\bar{s} = \{\ldots \bar{s}_{-2}\bar{s}_{-1} \bullet \bar{s}_0 \bar{s}_1 \bar{s}_2 \ldots\}$. We define the metric by:

$$\begin{array}{rccl} d: & \Sigma_N \times \Sigma_N & \to & \mathbb{R} \\ & (s,\bar{s}) & \mapsto & d(s,\bar{s}) = \displaystyle\sum_{i=-\infty}^{+\infty} \frac{1}{2^{|i|}} \frac{\delta_i}{1+\delta_i} \end{array} \tag{5.2}$$

where $\delta_i = |s_i - \bar{s}_i|$ is defined by:

$$\delta_i = \begin{cases} 1, & \text{if } s_i \neq \bar{s}_i \\ 0, & \text{if } s_i = \bar{s}_i \end{cases}. \tag{5.3}$$

Remark 35. The definition of the metric d has involved a bi-infinite series and it is dominated by the geometric series which is convergent:

$$d(s,\bar{s}) = \sum_{i=-\infty}^{+\infty} \frac{1}{2^{|i|}} \frac{\delta_i}{1+\delta_i} \leq \sum_{i=-\infty}^{+\infty} \frac{1}{2^{|i|}} = 3. \tag{5.4}$$

Therefore it converges as well.

Proposition 36. The ordered pair (Σ_N, d) is a metric space.

Proof: Indeed, let s, $\bar{s}$ and $\tilde{s}$ be bi-infinite sequences. Clearly $d(s,\bar{s}) \geq 0$. We have $d(s,\bar{s}) = 0$ if, and only if, $\sum_{i=-\infty}^{+\infty} \frac{1}{2^{|i|}} \frac{\delta_i}{1+\delta_i} = 0$ for all i if, and only if, $\delta_i = 0$ for all i, if and only if, $s = \bar{s}$. As δ_i does not depend on order of the s and $\bar{s}$ we have $d(s,\bar{s}) = d(\bar{s},s)$ straightly. Finally, the triangular inequality follows from the observation that:

$$|s_i - \bar{s}_i| \leq |s_i - \tilde{s}_i| + |\tilde{s}_i - \bar{s}_i|.$$

Therefore, we obtain

$$d(s,\bar{s}) \leq d(s,\tilde{s}) + d(\tilde{s},\bar{s}).$$

■

The following proposition is fundamental to give meaning to what is meant by the neighborhood of a point s in Σ_N as well as a basis for the space topology of the bi-infinite sequences.

Proposition 37. Let $s, \bar{s} \in \Sigma_N$ be bi-infinite sequences, then

1) If $s_i = \bar{s}_i$ for all $|i| \leq M$, then $d(s,\bar{s}) \leq \frac{1}{2^M}$

2) If $d(s,\bar{s}) < \frac{1}{2^M}$, then $s_i = \bar{s}_i$ for all $|i| \leq M$.

Proof: 1) Indeed, we assume that $s_i = \bar{s}_i$ for all $|i| \leq M$, then

$$
\begin{aligned}
d(s,\bar{s}) &= \sum_{i=-\infty}^{+\infty} \frac{1}{2^{|i|}} \frac{\delta_i}{1+\delta_i} \\
&= \sum_{-\infty}^{i=-M-1} \frac{1}{2^{|i|}} \frac{\delta_i}{1+\delta_i} + \sum_{i=-M}^{+M} \frac{1}{2^{|i|}} \frac{|s_i - s_i|}{1+|s_i - s_i|} + \sum_{i=+M+1}^{+\infty} \frac{1}{2^{|i|}} \frac{\delta_i}{1+\delta_i} \\
&= \sum_{-\infty}^{i=-M-1} \frac{1}{2^{|i|}} \frac{\delta_i}{1+\delta_i} + \sum_{i=+M+1}^{+\infty} \frac{1}{2^{|i|}} \frac{\delta_i}{1+\delta_i} \\
&\leq \sum_{-\infty}^{i=-M-1} \frac{1}{2^{|i|}} \frac{1}{2} + \sum_{i=+M+1}^{+\infty} \frac{1}{2^{|i|}} \frac{1}{2} \\
&= \sum_{-\infty}^{i=-M-1} \frac{1}{2^{|i|+1}} + \sum_{i=+M+1}^{+\infty} \frac{1}{2^{|i|+1}} \\
&= \sum_{i=+M+1}^{+\infty} \frac{2}{2^{|i|+1}} = \sum_{i=+M+1}^{+\infty} \frac{1}{2^{|i|}} \\
&= \frac{1}{2^M}.
\end{aligned}
$$

Therefore, $d(s,\bar{s}) \leq \frac{1}{2^M}$. Now, we will prove 2). By contradiction, if $s_k \neq \bar{s}_k$ for some $|k| \leq M$, then we have

$$
d(s,\bar{s}) \geq \frac{1}{2^{|k|}} \frac{d(s_k,\bar{s}_k)}{1+d(s_k,\bar{s}_k)} = \frac{1}{2^{|k|+1}} \geq \frac{1}{2^{M+1}}.
$$

Thus, if $d(s,\bar{s}) < \frac{1}{2^{M+1}}$, then $s_i = \bar{s}_i$ for all $|i| \leq M$. ∎

We should conclude from Proposition (37) that, given $\varepsilon > 0$ there is a number $M = M(\varepsilon) > 0$ such that $\frac{1}{2^M} < \varepsilon$, then we safely define what we mean by a neighborhood of a bi-infinite sequence $s \in \Sigma_N$:

$$
\mathcal{N}(\varepsilon, s) := \{\bar{s} \in \Sigma_N : \; d(s,\bar{s}) < \varepsilon\} = \{\bar{s} \in \Sigma_N : \; s_i = \bar{s}_i \text{ for all } |i| \leq M\}. \tag{5.5}
$$

Much of the structure of Σ_N is inherited from S. Next, some properties are listed:

Proposition 38. The metric space (Σ_N, d) is:

1) Compact;

2) Totally disconnected;

3) Perfect.

Proof: Indeed, 1) and 2) come from the compactness of S which is a finite set of symbols, and by Tychonov's Theorem, the space of bi-infinite sequences Σ_N is also compact, and due in fact S is totally disconnected and it also surely by Cartesian product. We know that every finite set is a closed set, so for 3) we need to prove that every point in Σ_N is a limit point. Given $s = (\dots s_{-2}s_{-1} \bullet s_0 s_1 s_2 \dots) \in \Sigma_N$ and $\varepsilon > 0$, then there is $M > 0$ such that $\mathscr{N}(\varepsilon, s)$ is a neighborhood of s. Let $\bar{s} \in \Sigma_N$ be a bi-infinite sequence such that $\bar{s}_i = s_i$ for all $|i| \leq M$ but we have

$$\bar{s}_{M+1} = \begin{cases} s_{M+1} + 1 = s^*, & \text{if } s_{M+1} \neq N-1 \\ s_{M+1} - 1 = s^*, & \text{if } s_{M+1} = N-1 \end{cases}. \tag{5.6}$$

Therefore, we get a sequence

$$\bar{s} = (\dots, \bar{s}_{-M_\varepsilon - 2}, s^*, s_{-M}, \dots, s_{-1} \bullet s_0, s_1, \dots, s_M, s^*, \bar{s}_{M+2}, \dots) \tag{5.7}$$

which is an element from $\mathscr{N}(\varepsilon, s)$ and $s \neq \bar{s}$.

$\blacksquare$

As every compact metric space is complete, we can conclude that the space of bi-infinite sequences of symbols is uncountable through the following theorem attributed to Hausdorff.

Theorem 39. Every perfect set in a complete metric space has at least the cardinal of the continuum.

5.2 Shift map

The *Shift map* is defined over bi-infinite sequences metric space $\sigma : \Sigma_N \to \Sigma_N$ taking any bi-infinite sequence and it will move the start point ($\bullet$) to the next term of the sequence:

$$\sigma(\dots s_{-2}s_{-1} \bullet s_0 s_1 s_2 \dots) = (\dots s_{-1}s_0 \bullet s_1 s_2 s_3 \dots) \tag{5.8}$$

in short form $(\sigma(s))_i = s_{i+1}$. In the previous section, we know that Σ_N is a topological space which is induced by the metric d. Next result is important:

Theorem 40. Shift map $\sigma : \Sigma_N \to \Sigma_N$ is a homeomorphism.

Proof: We should prove that σ is a bijective map, continuous with its inverse map also continuous. For the first one, σ is injective. Indeed, if $\sigma(s) = \sigma(\bar{s})$ then

$$\begin{aligned} \sigma(\dots s_{-2}s_{-1} \bullet s_0 s_1 s_2 \dots) &= (\dots s_{-1}s_0 \bullet s_1 s_2 s_3 \dots) \\ \sigma(\dots \bar{s}_{-2}\bar{s}_{-1} \bullet \bar{s}_0 \bar{s}_1 \bar{s}_2 \dots) &= (\dots \bar{s}_{-1}\bar{s}_0 \bullet \bar{s}_1 \bar{s}_2 \bar{s}_3 \dots). \end{aligned}$$

Thus, $(\ldots s_{-1}s_0 \bullet s_1s_2s_3\ldots) = (\ldots \bar{s}_{-1}\bar{s}_0 \bullet \bar{s}_1\bar{s}_2\bar{s}_3\ldots)$ it implies that, $s_i = \bar{s}_i$ for all $i \in \mathbb{Z}$. Therefore $s = \bar{s}$. Surjective. Indeed, given $s = (\ldots s_{-2}s_{-1} \bullet s_0s_1s_2\ldots)$ in Σ_N, we just need to take $\bar{s} = (\ldots \bar{s}_{-2} \bullet \bar{s}_{-1}\bar{s}_0\bar{s}_1\bar{s}_2\ldots)$ also in Σ_N, and then we get $\sigma(\bar{s}) = s$. For the second one, let's prove its continuity to any $s \in \Sigma_N$. Given $\varepsilon > 0$ there is $M > 0$ such that $\frac{1}{2^{M-1}} < \varepsilon$. Let's take $\delta = \frac{1}{2^{M+1}} < \frac{1}{2^M}$. If $d(s,\bar{s}) < \delta$ then, by the Proposition (37) Item 2), we get $s_i = \bar{s}_i$ for $|i| \leq M$ and it implies that $(\sigma(s))_i = (\sigma(\bar{s}))_i$ for all $|i| \leq M-1$. Consequently, by Proposition (37) Item 1), we have $d(\sigma(s),\sigma(\bar{s})) \leq \frac{1}{2^{M-1}} < \varepsilon$. Similarly we can prove that $\sigma^{-1} : \Sigma_N \to \Sigma_N$ given by $(\sigma^{-1}(s))_i = s_{i-1}$ is also continuous.

■

The following theorem gives us a description of the structure of the orbits of Σ_N under σ.

Theorem 41. The Shift map has the following properties:

1) a countable infinity of periodic orbits of all periods;

2) an uncountable infinity of non-periodic orbits;

3) a dense orbit.

We will assume in this proof by simplicity $N = 2$ (space of two symbols, namely, $0's$ and $1's$). For general N, the proof follows similarly.

Proof: Item 1). We remember that for each bi-infinite sequence $s \in \Sigma_2$ denoted by

$$\{\cdots s_{-n}\cdots s_{-1} \bullet s_0s_1\cdots s_n\cdots\}$$

the Shift map acts

$$\sigma(s) = \{\cdots s_{-n}\cdots s_{-1}s_0 \bullet s_1\cdots s_n\cdots\}.$$

Given a bi-infinite sequence that periodically repeats after a fixed length k, for instance, $\{\cdots 1010 \bullet 1010\cdots\}$ of length 2, we will be represented by a finite length sequence with an overline $\overline{\{10 \bullet 10\}}$. We easily see that every periodic sequence of length k is a periodic orbit of period k for σ, for instance, $\sigma^2(\overline{\{10 \bullet 10\}}) = \sigma(\sigma(\overline{\{10 \bullet 10\}})) = \overline{\{10 \bullet 10\}}$ for $k = 2$. Since that, a positive integer k is given, and there are a finite number of bi-infinite sequences of length k which corresponds to periodic orbits for σ of period k. Therefore, the Shift map has a countable infinity of periodic orbits of all periods.

Item 2). Given a bi-infinite sequence $s \in \Sigma_2$ we might associate a infinite sequence in following way:

$$\{\cdots s_{-n}\cdots s_{-1} \bullet s_0s_1\cdots s_n\cdots\} \mapsto \{\bullet s_0s_1\cdots s_n\cdots\}.$$

This relation is surjective. We know that all irrational numbers in the closed interval $[0,1]$ form an uncountable set. For each number in this interval, we might decompose it on the base 2 as a binary expansion of $0's$ and $1's$, and each irrational number corresponding to a unique infinite non-repeating sequence of $0's$ and $1's$. As a consequence, we have a one-to-one correspondence between the uncountable infinity set and the non-repeating infinite sequence set. Since that, non-repeating infinite sequences correspond to non-periodic orbits for σ, the Shift map has an uncountable infinity of non-periodic orbits.

Item 3). We will construct a sequence $s^* \in \Sigma_2$ of two symbols $0's$ and $1's$ such that its orbit is dense in Σ_2, that is, for any $s \in \Sigma_2$ and $\varepsilon > 0$ there is a integer number $M = M(\varepsilon)$ such that $d(\sigma^M(s^*), s) < \varepsilon$, where d is the metric defined in Equation (5.2).

Construction of the bi-infinity sequence $s^* \in \Sigma_2$

We note that for each length k there is a finite number, namely, 2^k of different finite sequences of length k. To distinguish different finite sequences, we will define an order in the following way: given two finite sequences $s,t \in \Sigma_2$

$$s = \{s_1 \cdots s_{k_1}\} \qquad t = \{t_1 \cdots t_{k_2}\}$$

of lengths k_1 and k_2 respectively. We say that $s < t$, first of all, if $k_1 < k_2$. Otherwise, when $k_1 = k_2$, then $s < t$ if $s_i < t_i$ at the first i such that $s_i \neq t_i$.

For instance, finite sequences of lengths $1, 2$ and 3 are:

Length $1 : \{0\} < \{1\}$

Denoted $s_1^1 < s_2^1$

Length $2 : \{00\} < \{01\} < \{10\} < \{11\}$

Denoted $s_1^2 < s_2^2 < s_3^2 < s_4^2$

Length $3 : \{000\} < \{001\} < \{010\} < \{011\} < \{100\} < \{101\} < \{110\} < \{111\}$

Denoted $s_1^3 < s_2^3 < s_3^3 < s_4^3 < s_5^3 < s_6^3 < s_7^3 < s_8^3$

etc...

Generically we denote $s_1^k < s_2^k < \cdots s_{2^{k-1}}^k < s_{2^k}^k$ all 2^k finite sequences ordered of fixed length k.

Thus, let us consider the bi-infinite sequence:

$$s^* = \{\cdots s_8^3 s_6^3 s_4^3 s_2^3 s_4^2 s_2^2 s_2^1 \bullet s_1^1 s_1^2 s_3^2 s_1^3 s_3^3 s_5^3 s_7^3 \cdots\}. \tag{5.9}$$

The sequence s^* is formed for all possible finite sequences of all lengths.

s^* is a dense orbit for σ

Indeed, let $s \in \Sigma_2$ be any bi-infinity sequence of two symbols and $\varepsilon > 0$. By Proposition (37) Item 2), the ε-neighborhood of s consists of all bi-infinity sequences t such that $d(s,t) < \varepsilon$. Thus, there is an integer number $M = M(\varepsilon)$ such that $s_i = t_i$ for all $|i| \leq M$. By construction of s^*, the finite sequence of length $2M+1$

$$\{s_{-M} \cdots s_{-1} \bullet s_0 s_1 \cdots s_M\}$$

is contained somewhere in the sequence s^*. Consequently, there is an integer M' such that $d(\sigma^{M'}(s^*), s) < \varepsilon$. Therefore, the Shift map has a dense orbit in Σ_2. The proof for Σ_N for any $N > 2$ follows similarly. This concludes the proof of the theorem.

■

CHAPTER

6

SMALE HORSESHOE

In this chapter, we will introduce the Smale horseshoe geometrically as the image under a prototypical two-dimensional map possessing a chaotic invariant Cantor set. The existence of the Smale horseshoes provides a mechanism to decide when a system display chaos, and which sense chaos is meant. We will see, in Chapter (7), an application of the results presented in this chapter.

In Section (6.1), we will define the two-dimensional map. In Section (6.2) we will introduce the important concepts concerning proving the Conley-Moser Theorem, which gives sufficient conditions to exist a chaotic invariant Cantor set. In Section (6.3), we will see the important connection between the Shift map, introduced in Chapter (5), and the Smale horseshoe. We also show what means chaotic behaviour, and how it can be seen through the connection with Symbolic dynamics. In Section (6.4), we present the Sector bundles, which is a concept that will create an improvement for the Conley-Moser Theorem in order to make it easier to apply. The main result of this section says that the hypothesis involving sectoral bundles is a better verifiable sufficient condition for the Conley-Moser Theorem. We follow the excellent books to present this chapter (GUCKENHEIMER; HOLMES, 2013; WIGGINS; WIGGINS; GOLUBITSKY, 2003; WIGGINS, 2013).

6.1 Definition of the Smale horseshoe

The Smale Horseshoe Map is defined as a combination of geometrical and analytic structures. For convenience, we will consider a unity square defined by set $U := \{(x,y) \in \mathbb{R}^2 |\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ on the plane $\mathbb{R}^2$ and a map $f : U \to \mathbb{R}^2$ which it will send horizontal lines, curves, and strips into the vertical lines, curves and strips respectively. In other words, in an orderly way, this map will contract and expand strips into U on the x-direction and y-direction respectively, finally, it will fold around, laying it back on itself. Reverse process f^{-1} consists of reverse order sending vertical lines, curves, and strips into horizontal lines, curves, and strips. Iterations of f will construct ever narrower vertical strips into unity square U and reverse iterations f^{-1} will construct ever narrower horizontal strips into U. Intersections of all iterations f^n, $n \in \mathbb{Z}$, will give us an invariant set Λ with Cantor set features. See Figures (6) – (10).

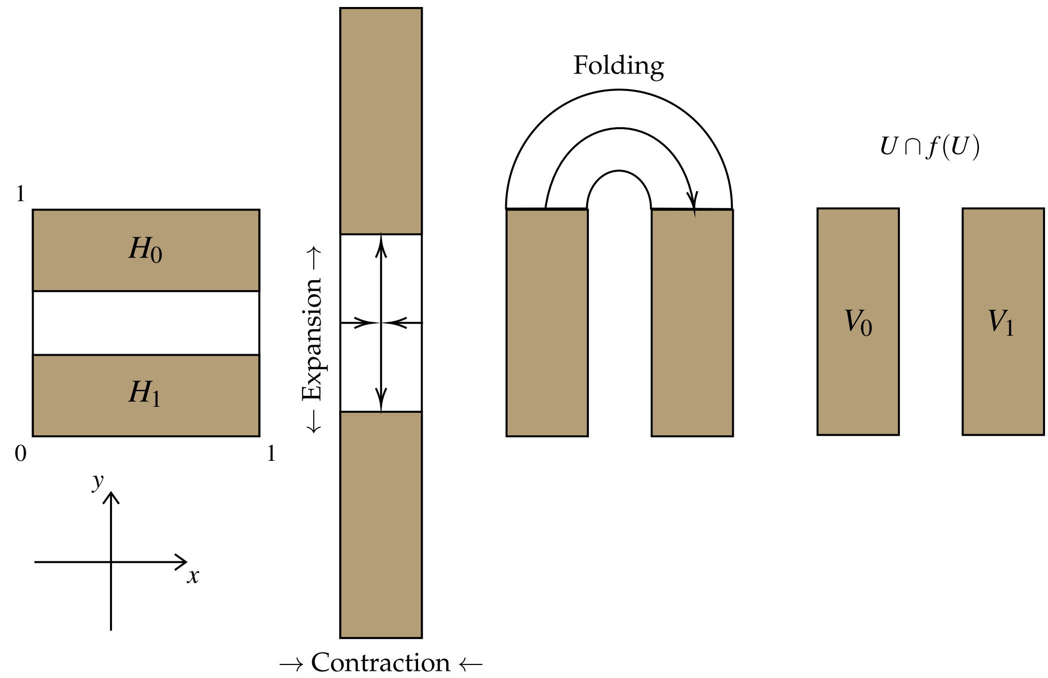


Figure 6 – First iteration of the geometric construction of Smale Horseshoes. We see two distinct horizontal strips inside the unity square and f will work in an orderly way, firstly it will contract the strips on the x-axis direction, secondly, it will expand the strips on the y-axis direction, and third, it will fold around itself. Those three steps are a composition of Homothety, Expansion, *and* Fold *which will hold the initial topological structure. Finally, the intersection $f(U) \cap U$ will keep some points from the unity square and throw away other points.*

Next, Figure (7) shows us the second iteration of f on the vertical strips:

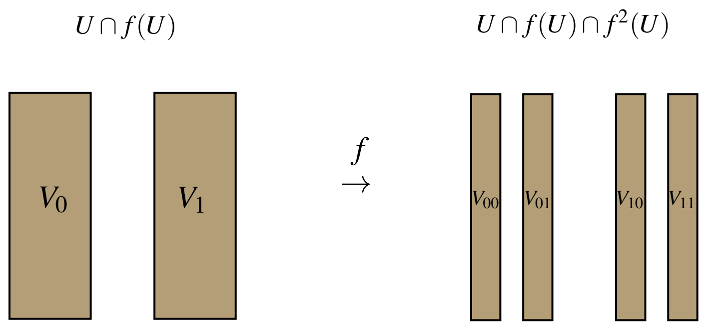


Figure 7 – Second iteration. The map contracts expand and fold both vertical strips. The intersection with the unity square is U gives four narrow vertical strips, each encoded with a distinct sequence of two symbols.

The reverse process denoted by f^{-1} is showed in the Figure (8):

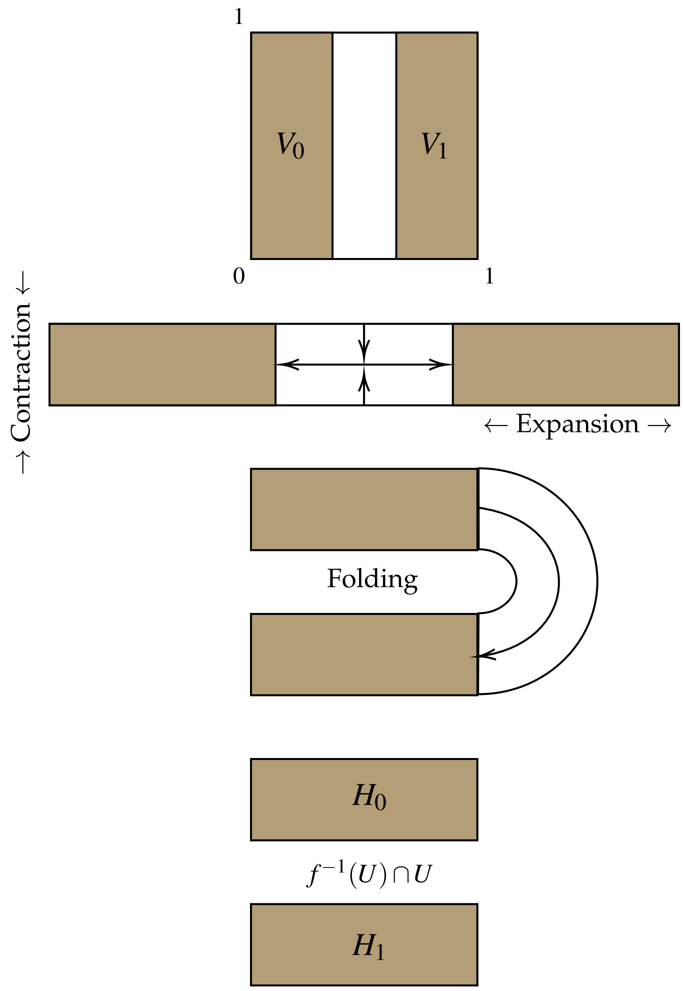


Figure 8 – The reverse process f^{-1} is started with two vertical strips, then we contract it in the vertical direction and expand it in the horizontal direction. After that, we fold it around itself getting two horizontal strips by the intersection with the unity square U.

Next, Figure (9) shows us the second reverse iteration of f^{-1}:

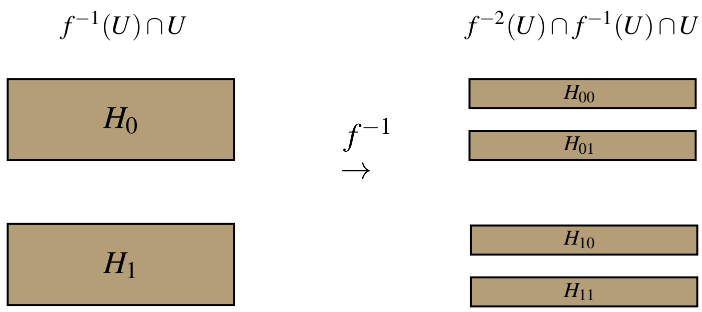


Figure 9 – Second reverse iteration. The map f^{-1} contracts, expands, and folds both horizontal strips. The intersection with the unity square is U gives four narrow horizontal strips, each encoded with a distinct sequence of two symbols.

The formation of the invariant Cantor set is making intersections of all backward and forward iterations, next, we will see it for $n = 0, \pm 1, \pm 2$. See Figure (10).

$$f^{-2}(U) \cap f^{-1}(U) \cap U \cap f(U) \cap f^2(U)$$

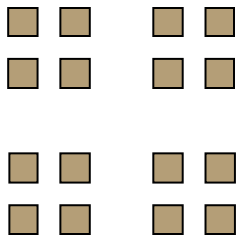


Figure 10 – Intersection $f^n(U) \cap U$ for $n = 0, \pm 1, \pm 2$ with unity square U.

Lemma 42. We have the following properties:

i) Suppose V is a vertical strip; then $f(V) \cap U$ consists of precisely two strips, one in V_0 and one in V_1;

ii) Suppose H is a horizontal strip; then $f^{-1}(H) \cap U$ consists of precisely two horizontal strips, one in H_0 and one in H_1.

Proof: Indeed, by definition of f, if V is a vertical strip intersecting both horizontal strips H_0 and H_1, then $f(V) \cap U$ consists necessarily of two vertical strips, one in V_0 and one in V_1. For Item ii) the proof is similar.

■

6.2 The Conley-Moser Theorem

This section will present conditions to decide when a dynamic system has chaotic behaviour through the so-called Conley-Moser conditions. The purpose of these conditions is to guarantee the invariance of the iteration process.

6.2.1 The Conley-Moser conditions

The Conley-Moser Theorem proves the existence of a homeomorphism such that the Shift map and Smale Horseshoe map are topologically equivalent. We begin this subsection by defining the basic concepts to prove that.

Definition 43. Let U be a unity square. We define the curves:

i) A μ_v-vertical curve is the graph of a function $x = v(y)$ for which $v : [0,1] \to [0,1]$ such that, for all $y_1, y_2 \in [0,1]$

$$|v(y_1) - v(y_2)| \leq \mu_v |y_1 - y_2|. \tag{6.1}$$

ii) A μ_h-horizontal curve is the graph of a function $y = h(x)$ for which $h : [0,1] \to [0,1]$ such that, for all $x_1, x_2 \in [0,1]$

$$|h(x_1) - h(x_2)| \leq \mu_h |x_1 - x_2|. \tag{6.2}$$

Definition 44. Let U be a unity square. We define the strips:

i) Given two nonintersecting μ_v-vertical curves $v_1(y) < v_2$, $y \in [0,1]$, we define a μ_v-vertical strip as

$$V := \{(x,y) \in \mathbb{R}^2 | \, x \in [v_1(y), v_2(y)]; \; y \in [0,1]\}; \tag{6.3}$$

ii) Given two nonintersecting μ_h-horizontal curves $h_1(y) < h_2$, $x \in [0,1]$, we define a μ_h-horizontal strip as

$$H := \{(x,y) \in \mathbb{R}^2 | \, y \in [h_1(x), h_2(x)]; \; x \in [0,1]\}; \tag{6.4}$$

iii) The width of horizontal and vertical strips is defined as

$$d(H) = \max_{x \in [0,1]} ||h_2(x) - h_1(x)||; \tag{6.5}$$
$$d(V) = \max_{y \in [0,1]} ||v_2(y) - v_1(y)||. \tag{6.6}$$

Next, we will present an important result that will be useful.

Lemma 45. Let V_i and H_i be vertical and horizontal strips respectively. Then,

i) If $V_1 \supset V_2 \supset \cdots \supset V_k \cdots$ is a nested sequence of μ_v-vertical strips with $d(V_k) \to 0$ as $k \to \infty$, then $\cap_{k=1}^{\infty} V_k = V_\infty$ is a μ_v-vertical curve;

ii) If $H_1 \supset H_2 \supset \cdots \supset H_k \cdots$ is a nested sequence of μ_h-horizontal strips with $d(H_k) \to 0$ as $k \to \infty$, then $\cap_{k=1}^{\infty} H_k = H_\infty$ is a μ_h-horizontal curve.

Before proving that Lemma we observe the set of all Lipschitz functions with fixed constant μ_v denoted by $\mathscr{C}_{\mu_v}([0,1])$ defined on the interval $[0,1]$, is a complete metric space with the maximum norm. Basically, any Cauchy sequence of Lipschitz functions with the same constant μ_v will converge to a continuous function at the limit with the same constant μ_v.

Proof: We will prove Item i). Let $v_1^k(y), v_2^k$ be the boundaries of the μ_v-vertical strip V_k. We define the following sequence

$$\{v_1^1(y), v_2^1(y), v_1^2(y), v_2^2(y), \ldots, v_1^k(y), v_2^k(y), \ldots\}. \tag{6.7}$$

It is a Cauchy sequence. Indeed, let $\{v_1^k(y), v_2^k(y)\}_{k\in\mathbb{N}}$ be a sequence of elements from $\mathscr{C}_{\mu_v}([0,1])$. Given any $\varepsilon > 0$, there is $N > 0$ such that, for all $k > N$ we have by hypothesis $d(V_k) < \frac{\varepsilon}{2}$ and for all $r, s > k$:

$$\begin{aligned}
&|(v_1^r(y), v_2^r(y)) - (v_1^s(y), v_2^s(y))| = \\
&|(v_1^r(y), v_2^r(y)) - (v_1^k(y), v_2^k(y)) + (v_1^k(y), v_2^k(y)) - (v_1^s(y), v_2^s(y))| \leq \\
&|(v_1^r(y), v_2^r(y)) - (v_1^k(y), v_2^k(y))| + |(v_1^k(y), v_2^k(y)) - (v_1^s(y), v_2^s(y))| < \\
&\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, it is a Cauchy sequence. Since that $\mathscr{C}_{\mu_v}([0,1])$ is complete, this sequence has the limit in $\mathscr{C}_{\mu_v}([0,1])$ and due sequence of strips be nested we get all intersection of them equal to μ_v-vertical curve.

For Item ii) the proof is precisely the same changing what should be changed.

■

Lemma 46. Suppose $0 \leq \mu_v\mu_h < 1$. Then a μ_v-vertical curve and a μ_h-horizontal curve intersect in a unique point.

Proof: Let $x = v(y)$ and $y = h(x)$ be μ_v-vertical curve and μ_h-horizontal curve respectively. We will prove that the equation $y = h(v(y))$ has a unique solution. A solution consists of an ordered pair $(x,y) \in U$ such that each one must satisfy $x = v(y)$ and $y = h(x)$. We consider the closed interval $I := [0,1]$ and we just need to prove that

$$h \circ v : I \to I \tag{6.8}$$

is a contraction mapping. Indeed, I is a complete metric space with the metric given by the absolute value and then, we calculate for any $y_1, y_2 \in I$:

$$||h(v(y_1)) - h(v(y_2))|| \leq \mu_h||v(y_1) - v(y_2)|| \leq \mu_h\mu_v|y_1 - y_2|. \tag{6.9}$$

By hypothesis, $0 \leq \mu_h\mu_v < 1$ which means $h \circ v$ is a contraction mapping. By the Contraction Mapping Theorem, $h \circ v$ has a unique solution (x_0, y_0) such that $x_0 = v(y_0)$ and $y_0 = h(v(y_0))$. Therefore, the intersection between vertical and horizontal curves is unique.

■

We will consider the complete metric spaces $\mathscr{C}_{\mu_v}([0,1])$ and $\mathscr{C}_{\mu_h}([0,1])$ of Lipschitz functions with constants Lipschitz μ_v and μ_h respectively. For the next theorem, we will consider the following hypothesis:

H1 The function $f : U \to \mathbb{R}^2$ maps horizontal strips H_i homeomorphically onto vertical strips V_i $(f(H_i)) = V_i$ for $i = 0, \ldots, N$ sending vertical and horizontal boundaries from H_i to vertical and horizontal boundaries from V_i respectively. Moreover, each pair of μ_v-vertical and μ_h-horizontal curves satisfies $0 \leq \mu_h \mu_v < 1$.

H2 Let V be any μ_v-vertical strip contained in $\cup_{i \in S} V_i$. Then $V \cap f(V_i) = \bar{V}_i$ is a μ_v-vertical strip for every $i \in S$. Moreover, its strip width satisfies

$$d(\bar{V}_i) \leq \nu_v d(V_i) \tag{6.10}$$

for some number $0 < \nu_v < 1$. Similarly, let H be any μ_h-horizontal strip contained in $\cup_{i \in S} H_i$. Then $f^{-1}(H_i) \cap H = \bar{H}_i$ is a μ_h-horizontal strip for every $i \in S$. Moreover, its strip width satisfies

$$d(\bar{H}_i) \leq \nu_h d(H_i) \tag{6.11}$$

for some number $0 < \nu_h < 1$.

Theorem 47 (Conley-Moser)**.** Suppose f satisfies Hypothesis **H1** and **H2**. Then f has an invariant Cantor set, Λ, on which it is topologically conjugate to a full Shift map σ on N symbols.

We will conclude the Theorem (47) after the proof of the following lemmas

Lemma 48. If f satisfies Hypothesis **H1** and **H2**, then f has an invariant Cantor set Λ.

Proof: The invariant Cantor set Λ will be constructed step-by-step. We will first construct a set of an infinity number of μ_v-vertical curves which will be denoted by $\Lambda_{-\infty}$ meaning negative iterations of f. In the second part we will construct a set of infinity numbers of μ_h-horizontal curves which will be denoted by $\Lambda_{+\infty}$ meaning positive iterations of f in a similar way. The invariant set will be given by the intersection $\Lambda := \Lambda_{-\infty} \cap \Lambda_{+\infty}$. Let us start first the construction of $\Lambda_{-\infty}$ with a number N of μ_v-vertical strips inside unity square U :

$$\Lambda_{-1} := \bigcup_{s_{-1} \in S} V_{s_{-1}}. \tag{6.12}$$

We remember that S is the set of N symbols, then Λ_{-1} consists of N vertical strips $V_{s_{-1}}$, each one indexed by a fake symbol s_{-1} but only one in S. This observation will be important to get a one-to-one correspondence between points of unity square with a sequence of symbols. The negative index sign means backward iterations. We also remember the mechanism to get Horseshoe presented by the definition of f, which is taking $f(U) \cap U$ and we will proceed to make iterations $f^n(U) \cap U$ for all $n \in \mathbb{Z}$. So, from definition of f, the second iteration is

$$\begin{aligned}
\Lambda_{-2} &= f(\Lambda_{-1}) \cap \left(\bigcup_{s_{-1}\in S} V_{s_{-1}} \right) && (6.13)\\
&= f\left(\bigcup_{s_{-1}\in S} V_{s_{-1}} \right) \cap \left(\bigcup_{s_{-1}\in S} V_{s_{-1}} \right) && (6.14)\\
&= \left(\bigcup_{s_{-2}\in S} f(V_{s_{-2}}) \right) \cap \left(\bigcup_{s_{-1}\in S} V_{s_{-1}} \right) && (6.15)\\
&= \bigcup_{\substack{s_{-i}\in S\\ i=1,2}} f(V_{s_{-2}}) \cap V_{s_{-1}} && (6.16)\\
&= \bigcup_{\substack{s_{-i}\in S\\ i=1,2}} V_{s_{-2}s_{-1}}. && (6.17)
\end{aligned}$$

Precisely, the set Λ_{-2} consists of N^2 μ_v-vertical strips, each one of length two encoded one-to-one by 2 symbols of S. This set contains all points $p \in V_{s_{-1}} \subset U$ such that $f^{-1}(p) \in V_{s_{-2}}$ for $s_{-1}, s_{-2} \in S$. By the Lemma (42) we have a nested sequence of sets $V_{s_{-2}s_{-1}} \subset V_{s_{-1}}$. We also have by the Hypothesis **H2** its width set is:

$$d(V_{s_{-2}s_{-1}}) \leq \nu_v d(V_{s_{-1}}) \leq \nu_v \tag{6.18}$$

for some number $0 < \nu_v < 1$. Just like before, we proceed to iteration again. For the third iteration we have:

$$\begin{aligned}
\Lambda_{-3} &= f(\Lambda_{-2}) \cap \left(\bigcup_{s_{-1}\in S} V_{s_{-1}} \right) && (6.19)\\
&= f\left(\bigcup_{\substack{s_{-i}\in S\\ i=2,3}} f(V_{s_{-3}}) \cap V_{s_{-2}} \right) \cap \left(\bigcup_{s_{-1}\in S} V_{s_{-1}} \right) && (6.20)\\
&= \left(\bigcup_{s_{-3}\in S} f^2(V_{s_{-3}}) \right) \cap \left(\bigcup_{s_{-2}\in S} f(V_{s_{-2}}) \right) \cap \left(\bigcup_{s_{-1}\in S} V_{s_{-1}} \right) && (6.21)\\
&= \bigcup_{\substack{s_{-i}\in S\\ i=1,2,3}} f^2(V_{s_{-3}}) \cap f(V_{s_{-2}}) \cap V_{s_{-1}} && (6.22)\\
&= \bigcup_{\substack{s_{-i}\in S\\ i=1,2,3}} V_{s_{-3}s_{-2}s_{-1}}. && (6.23)
\end{aligned}$$

Again, the set Λ_{-3} consists of N^3 μ_v-vertical strips, each one of length three encoded one-to-one by 3 symbols of S. This time, the set $V_{s_{-3}s_{-2}s_{-1}}$ contains all points $p \in V_{s_{-1}} \subset U$ such that $f^{-1}(p) \in V_{s_{-2}}$ and $f^{-2}(p) \in V_{s_{-3}}$ for $s_{-1}, s_{-2}, s_{-3} \in S$. By Lemma (42) we have a nested sequence of sets

$$V_{s_{-3}s_{-2}s_{-1}} \subset V_{s_{-2}s_{-1}} \subset V_{s_{-1}}. \tag{6.24}$$

We also have by the Hypothesis **H2** that its width set is:

$$d(V_{s_{-3}s_{-2}s_{-1}}) \leq \nu_v d(V_{s_{-2}s_{-1}}) \leq \nu_v^2 d(V_{s_{-1}}) \leq \nu_v^2 \tag{6.25}$$

for some number $0 < \nu_v < 1$. Now we want to get the expression for $k \to -\infty$. Like before, we only need to do an extension of the previous expression: for k sufficiently large we have:

$$\begin{aligned}
\Lambda_{-k} &= f(\Lambda_{-k-1}) \cap \left(\bigcup_{s_{-1} \in S} V_{s_{-1}} \right) && (6.26)\\
&= f\left(\bigcup_{\substack{s_{-i} \in S \\ i=2,\ldots,k}} f^{k-2}(V_{s_{-k}}) \cap \cdots \cap f(V_{s_{-3}}) \cap V_{s_{-2}} \right) \cap \left(\bigcup_{s_{-1} \in S} V_{s_{-1}} \right) && (6.27)\\
&= \left(\bigcup_{s_{-k} \in S} f^{k-1}(V_{s_{-k}}) \right) \cap \cdots \cap \left(\bigcup_{s_{-2} \in S} f(V_{s_{-2}}) \right) \cap \left(\bigcup_{s_{-1} \in S} V_{s_{-1}} \right) && (6.28)\\
&= \bigcup_{\substack{s_{-i} \in S \\ i=1,\ldots,k}} f^{k-1}(V_{s_{-k}}) \cap \cdots \cap f(V_{s_{-2}}) \cap V_{s_{-1}} && (6.29)\\
&= \bigcup_{\substack{s_{-i} \in S \\ i=1,\ldots,k}} V_{s_{-k} \ldots s_{-3}s_{-2}s_{-1}}. && (6.30)
\end{aligned}$$

Again, the set Λ_{-k} consists of N^k μ_v-vertical strips, each one of length k encoded one-to-one by k symbols of S. The set $V_{s_{-k} \ldots s_{-3}s_{-2}s_{-1}}$ contains all points $p \in V_{s_{-1}}$ such that $f^{-1}(p) \in V_{s_{-2}}$, $f^{-2}(p) \in V_{s_{-3}}$ and $f^{-k}(p) \in V_{s_{-k-1}}$ for $s_{-1}, s_{-2}, s_{-3}, \ldots, s_{-k} \in S$. By Lemma (42) we have a nested sequence of sets $V_{s_k \ldots s_{-3}s_{-2}s_{-1}} \subset \cdots \subset V_{s_{-1}}$. We also have by the Hypothesis **H2** its width set is:

$$d(V_{s_{-k} \cdots s_{-3}s_{-2}s_{-1}}) \leq \cdots \leq \nu_v^{k-2} d(V_{s_{-2}s_{-1}}) \leq \nu_v^{k-1} d(V_{s_{-1}}) \leq \nu_v^{k-1} \tag{6.31}$$

for some number $0 < \nu_v < 1$. This proceed follows indefinitely. For $k \to -\infty$ and we have a set of infinite number of μ_v-vertical curves defined by:

$$\Lambda_{-\infty} := \bigcup_{\substack{s_{-i} \in S \\ i \in \mathbb{N}}}^{-\infty} \cdots \cap f^{k-1}(V_{s_{-k}}) \cap \cdots \cap f(V_{s_{-2}}) \cap V_{s_{-1}}. \tag{6.32}$$

And we have the nested sequence

$$\cdots \subset V_{\ldots s_{-k} \ldots s_{-3}s_{-2}s_{-1}} \subset \cdots \subset V_{s_{-2}s_{-1}} \subset V_{s_{-1}} \tag{6.33}$$

with its width

$$d(V_{\dots s_{-k}\cdots s_{-3}s_{-2}s_{-1}}) \to 0 \quad \text{as} \quad k \to -\infty. \tag{6.34}$$

By Lemma (45) we know that

$$V_{-\infty} := \bigcap_{k\in\mathbb{N}}^{-\infty} V_{s_{-k}\dots s_{-3}s_{-2}s_{-1}} \tag{6.35}$$

is a μ_v-vertical Lipschitz curve. We emphasize that the $V_{-\infty}$ μ_v-vertical curve is encoded by only one infinite sequence of symbols of S.

Let us start first the construction of $\Lambda_{+\infty}$ with a number N of μ_h-horizontal strips inside unity square U :

$$\Lambda_0 := \bigcup_{s_0\in S} H_{s_0}. \tag{6.36}$$

Similarly Λ_0 consists of N horizontal strips H_{s_0}, each one indexed by only one fake symbol s_0 in S. The positive index sign means now the forward iterations. So, from definition of f, the second iteration is

$$\begin{aligned}
\Lambda_1 &= \left(\bigcup_{s_0\in S} H_{s_0}\right)\cap f(\Lambda_0) && (6.37)\\
&= \left(\bigcup_{s_0\in S} H_{s_0}\right)\cap f\left(\bigcup_{s_0\in S} H_{s_0}\right) && (6.38)\\
&= \left(\bigcup_{s_0\in S} H_{s_0}\right)\cap \left(\bigcup_{s_1\in S} f(H_{s_1})\right) && (6.39)\\
&= \bigcup_{\substack{s_i\in S\\ i=0,1}} H_{s_0}\cap f(H_{s_1}) && (6.40)\\
&= \bigcup_{\substack{s_i\in S\\ i=0,1}} H_{s_0s_1}. && (6.41)
\end{aligned}$$

Precisely, the set Λ_1 consists of N^2 μ_h-horizontal strips, each one of length two encoded one-to-one by 2 symbols of S. This set contains all points $p\in H_{s_0}\subset U$ such that $f(p)\in H_{s_1}$ for $s_0,s_1\in S$. By Lemma (42) we have a nested sequence of sets $H_{s_0s_1}\subset H_{s_0}$. We also have by the Hypothesis **H2** its width set is:

$$d(H_{s_0s_1}) \le \nu_h d(H_{s_0}) \le \nu_h \tag{6.42}$$

for some number $0<\nu_h<1$. Now we want to get the expression for $k\to+\infty$. Just like before, we only need to do an extension of the previous expression: for k sufficiently large we have:

$$\begin{aligned}
\Lambda_k &= \left(\bigcup_{s_0\in S} H_{s_0}\right)\cap f(\Lambda_{k-1}) && (6.43)\\
&= \left(\bigcup_{s_0\in S} H_{s_0}\right)\cap f\left(\bigcup_{\substack{s_i\in S\\ i=1,\dots,k}} H_{s_1}\cap f(H_{s_2})\cap\cdots\cap f^k(H_{s_k})\right) && (6.44)\\
&= \left(\bigcup_{s_0\in S} H_{s_0}\right)\cap\left(\bigcup_{s_1\in S} f(H_{s_1})\right)\cap\cdots\cap\left(\bigcup_{s_k\in S} f^{k+1}(H_{s_k})\right) && (6.45)\\
&= \bigcup_{\substack{s_i\in S\\ i=0,\dots,k}} H_{s_0}\cap f(H_{s_1})\cap\cdots\cap f^{k+1}(H_{s_k}) && (6.46)\\
&= \bigcup_{\substack{s_{-i}\in S\\ i=0,\dots,k}} H_{s_0 s_1\dots s_k}. && (6.47)
\end{aligned}$$

Again, the set Λ_k consists of N^k μ_h-horizontal strips, each one of length k encoded one-to-one by k symbols of S. The set $H_{s_0 s_1\dots s_k}$ contains all points $p\in H_{s_0}$ such that $f(p)\in H_{s_1}$, $f^2(p)\in H_{s_2}$ and $f^k(p)\in H_{s_k}$ for $s_0,s_1,s_2,\dots,s_k\in S$. By Lemma (42) we have a nested sequence of sets $H_{s_0 s_1\dots s_k}\subset\cdots\subset H_{s_0}$. We also have by the Hypothesis **H2)** its width set is:

$$d(H_{s_0 s_1\dots s_k})\le\cdots\le \nu_h^{k-1} d(H_{s_0 s_1})\le \nu_h^k d(H_{s_0})\le \nu_h^k \tag{6.48}$$

for some number $0<\nu_h<1$. This proceed follows indefinitely. For $k\to+\infty$ and we have a set of infinite number of μ_h-horizontal curves defined by:

$$\Lambda_{+\infty} := \bigcup_{\substack{s_i\in S\\ i\in\mathbb{N}}}^{+\infty} H_{s_0}\cap f(H_{s_1})\cap\cdots\cap f^k(H_{s_{k+1}}). \tag{6.49}$$

And we have the nested sequence

$$\cdots\subset H_{s_0 s_1\dots s_k}\subset\cdots\subset H_{s_0 s_1}\subset H_{s_0} \tag{6.50}$$

with its width

$$d(H_{s_0 s_1\cdots s_k})\to 0 \quad \text{as} \quad k\to+\infty. \tag{6.51}$$

By Lemma (45) we know that

$$H_{+\infty} := \bigcap_{k\in\mathbb{N}}^{+\infty} H_{s_0 s_1\dots s_k} \tag{6.52}$$

is a μ_h-horizontal Lipschitz curve. We emphasize that the $H_{+\infty}$ μ_h-horizontal curve is encoded by only one infinite sequence of symbols of S.

■

Therefore, the invariant set is given by

$$\Lambda := \Lambda_{-\infty} \cap \Lambda_{+\infty} \subset U \tag{6.53}$$

which contains all points of the unity square U invariant by all iterations of f. By the Hypothesis **H1**, each pair of vertical and horizontal curves intersect at a unique point. Thus, Λ is an uncountable discrete set.

Lemma 49. There is a homeomorphism $\phi : \Lambda \to \Sigma_N$ such that $\phi \circ f = \sigma \circ \phi$.

Proof: From the construction of the invariant set Λ we know that for each point $p \in \Lambda$ there are only two curves, one μ_v-vertical and one μ_h-horizontal curve whose the intersection between them is precisely the point p. Moreover, there are only two infinity sequences of symbols given by

$$\ldots s_{-k} s_{-k+1} \ldots s_{-1}, \quad \text{and} \quad s_0 s_1 \ldots s_{k-1} s_k \ldots \tag{6.54}$$

to μ_v-vertical curve and μ_h-horizontal curve respectively. Thus, we have a well-defined function

$$\begin{aligned} \phi : \Lambda &\to \Sigma_N \\ p &\mapsto (\ldots s_{-k} \ldots s_{-1} \bullet s_0 s_1 \ldots s_k \ldots). \end{aligned} \tag{6.55}$$

One-to-One. Let $p, \bar{p} \in \Lambda$ be any points. Suppose we have

$$\phi(p) = \phi(\bar{p}) = (\ldots s_{-k} \ldots s_{-1} \bullet s_0 s_1 \ldots s_k \ldots). \tag{6.56}$$

Thus, there are a unique μ_v-vertical curve which corresponds to infinity sequence $(\ldots s_{-k} \ldots s_{-1})$ and a unique μ_h-horizontal curve which corresponds to a infinity sequence $(s_0 s_1 \ldots s_k \ldots)$. By the Hypothesis **H1)**, the intersection between both curves is unique. Therefore, $p = \bar{p}$.

Onto. Given any bi-infinity sequence $(\ldots s_{-k} \ldots s_{-1} \bullet s_0 s_1 \ldots s_k \ldots) \in \Sigma_N$ we know that $(\ldots s_{-k} \ldots s_{-1})$ correspond to a unique μ_v-vertical curve and $(s_0 s_1 \ldots s_k \ldots)$ correspond to a unique μ_h-horizontal curve. By the Hypothesis **H1)** the intersection of them is only one point $p \in \Lambda$. Therefore, p is such that $\phi(p) = (\ldots s_{-k} \ldots s_{-1} \bullet s_0 s_1 \ldots s_k \ldots)$.

Continuous. Let us prove the continuity of ϕ for any point of Λ. Let $p, \bar{p} \in \Lambda$ be any points and its corresponding sequences in Σ_N :

$$\begin{aligned} \phi(p) &= (\ldots s_{-k} \ldots s_{-1} \bullet s_0 s_1 \ldots s_k \ldots) \\ \phi(\bar{p}) &= (\ldots \bar{s}_{-k} \ldots \bar{s}_{-1} \bullet \bar{s}_0 \bar{s}_1 \ldots \bar{s}_k \ldots). \end{aligned}$$

Given $\varepsilon > 0$, if $d(\phi(p), \phi(\bar{p})) < \varepsilon$, where d is the metric on Σ_N, then there is $M > 0$ such that $s_i = \bar{s}_i$ for all $i = 0, \pm 1, \ldots, \pm M$. It means that p and $\bar{p}$ remain close to each other up to M-th iteration of f. Consequently, both points are at the intersection of the μ_v-vertical strip and the μ_h-horizontal strip denoted to $V_{s_{-M}\dots s_{-1}}$ and $H_{s_0 s_1 \dots s_M}$ respectively. So, let $y = v_1(x)$ and $y = v_2(x)$ be μ_v-vertical boundaries curves with respect to its vertical strip and let $x = h_1(y)$ and $x = h_2(y)$ be μ_h-horizontal boundaries curves with respect to its horizontal strip. By the Hypothesis **H2)** and the Definition (44) Item iii) we have

$$\begin{aligned} d(V_{s_{-M}\dots s_{-1}}) &= \max_{y\in[0,1]} |v_2(y) - v_1(y)| = ||v_2 - v_1|| \le \nu_v^{M-1} && (6.57)\\ d(H_{s_0 s_1 \dots s_M}) &= \max_{x\in[0,1]} |h_2(x) - h_1(x)| = ||h_2 - h_1|| \le \nu_h^{M}. && (6.58)\end{aligned}$$

As we have seen, both p and $\bar{p}$ belong to a small closed square formed by both vertical curves and both horizontal curves. Let p_1 and p_2 be intersection points: $p_1 = v_1(y) \cap h_1(x)$ and $p_2 = v_2(y) \cap h_2(x)$ are diagonally opposite points. If we denote $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ then we have

$$|p - \bar{p}| \le |p_1 - p_2| = |(x_1, y_1) - (x_2, y_2)| \le |x_1 - x_2| + |y_1 - y_2|. \tag{6.59}$$

We remember that μ_v-vertical and μ_h-horizontal are Lipschitz curves at the constants μ_v and μ_h respectively satisfying $0 \le \mu_v \mu_h < 1$. Again, by Equations (6.57) and (6.58) we have:

$$\begin{aligned} |x_1 - x_2| &= |v_1(y_1) - v_2(y_2)| \le |v_1(y_1) - v_1(y_2)| + |v_1(y_2) - v_2(y_2)| \\ &\le \mu_v |y_1 - y_2| + ||v_1 - v_2|| \\ &\le \mu_v |y_1 - y_2| + \nu_v^{M-1} && (6.60)\end{aligned}$$

and we also have similar,

$$\begin{aligned} |y_1 - y_2| &= |h_1(x_1) - h_2(x_2)| \le |h_1(x_1) - h_1(x_2)| + |h_1(x_2) - h_2(x_2)| \\ &\le \mu_h |x_1 - x_2| + ||x_1 - x_2|| \\ &\le \mu_h |x_1 - x_2| + \nu_h^{M}. && (6.61)\end{aligned}$$

Thus, substituting Inequality (6.60) in the Inequality (6.61) we get

$$\begin{aligned} |y_1 - y_2| &\le \mu_h \mu_v |y_1 - y_2| + \mu_h \nu_v^{M-1} + \nu_h^{M} \\ &= \frac{1}{1 - \mu_h \mu_v} \cdot (\mu_h \nu_v^{M-1} + \nu_h^{M}) && (6.62)\end{aligned}$$

and substituting Inequality (6.61) in the Inequality (6.60) we also get

$$\begin{aligned} |x_1 - x_2| &\le \mu_v \mu_h |x_1 - x_2| + \mu_v \nu_h^{M} + \nu_v^{M-1} \\ &= \frac{1}{1 - \mu_h \mu_v} \cdot (\mu_v \nu_h^{M} + \nu_v^{M-1}). && (6.63)\end{aligned}$$

We sum both Equations (6.62) and (6.63):

$$\begin{aligned}
|x_1 - x_2| + |y_1 - y_2| &\leq \frac{1}{1-\mu_h\mu_v}\cdot(\mu_h \nu_v^{M-1} + \nu_h^M) + \frac{1}{1-\mu_h\mu_v}\cdot(\mu_v \nu_h^M + \nu_v^{M-1}) \\
&= \frac{1}{1-\mu_h\mu_v}\cdot(\mu_h \nu_v^{M-1} + \nu_h^M + \mu_v \nu_h^M + \nu_v^{M-1}) \\
&= \frac{1}{1-\mu_h\mu_v}\cdot((1+\mu_h)\nu_v^{M-1} + (1+\mu_v)\nu_h^M). \qquad (6.64)
\end{aligned}$$

Therefore, given $\varepsilon > 0$ there is

$$\delta = \frac{1}{1-\mu_h\mu_v}\cdot((1+\mu_h)\nu_v^{M-1} + (1+\mu_v)\nu_h^M)$$

which it depends on ε and $p \in \Lambda$ such that,

$$|p - \bar{p}| < \delta \Longrightarrow d(\phi(p), \phi(\bar{p})) < \varepsilon.$$

Commutativity. Let $f : \Lambda \to \Lambda$ be a function, $\phi : \Lambda \to \Sigma_N$ be the homeomorphism and $\sigma : \Sigma_N \to \Sigma_N$ be the Shift map. We will prove that the square is commutative. Indeed, if $p \in \Lambda$ then we have a unique bi-infinite sequence $\phi(p) \in \Sigma_N$. So,

$$\phi(p) = (\ldots s_{-k} \ldots s_{-1} \bullet s_0 s_1 \ldots s_k \ldots),$$

and

$$\sigma \circ \phi(p) = (\ldots s_{-k} \ldots s_{-1} s_0 \bullet s_1 \ldots s_k \ldots).$$

On the other hand, we note that $f(p) \in \Lambda$ has the same trajectory of the point p but with its start point ($\bullet$) moved one step forward and then, by definition of ϕ, the bi-infinity sequence associated is:

$$\phi \circ f(p) = (\ldots s_{-k} \ldots s_{-1} s_0 \bullet s_1 \ldots s_k \ldots).$$

Therefore, $\sigma \circ \phi(p) = \phi \circ f(p)$ for all $p \in \Lambda$ meaning the commutativity.

■

6.3 The dynamics on the invariant set

Topological conjugation is an important connection between the Shift and Horseshoe maps allowing us to understand the dynamics of the invariant set Λ. The Main Theorem above allows to study of the space with complex behaviour Λ through Σ_N which is known. An important property of topological conjugation is that for all $N \in \mathbb{Z}$ the N-th iteration of f is equivalent to N-th iteration of σ :

$$\phi^{-1} \circ \sigma^N \circ \phi(p) = f^N(p). \qquad (6.65)$$

for all $p \in \Lambda$.

6.3.1 Chaos

In this subsection, we will discuss the chaotic behaviour on the invariant set Λ through the homeomorphism ϕ. The idea is to analyze what happens with points sufficiently close to each other. To do this we set a point $p \in \Lambda$ and we take $B_p(\varepsilon)$ an open ball of p. Let $\bar{p} \in B_p(\varepsilon)$, then there are only bi-infinite sequences:

$$\begin{aligned}
\phi(p) &= (\dots s_{-2}s_{-1} \bullet s_0 s_1 s_2 \dots) \\
\phi(\bar{p}) &= (\dots \bar{s}_{-2}\bar{s}_{-1} \bullet \bar{s}_0 \bar{s}_1 \bar{s}_2 \dots),
\end{aligned}$$

contained in an open Σ_N since that ϕ is a homeomorphism. So, there is $M = M_\varepsilon \in \mathbb{N}$ such that $s_i = \bar{s}_i$ for all $|i| \leq M$. Suppose that the terms of the sequences $\phi(p)$ and $\phi(\bar{p})$ of index $M+1$ are 0 and 1 respectively, that is,

$$\begin{aligned}
\phi(p) &= (s_{-M} \dots s_{-2}s_{-1} \bullet s_0 s_1 s_2 \dots s_M 0) \\
\phi(\bar{p}) &= (\bar{s}_{-M} \dots \bar{s}_{-2}\bar{s}_{-1} \bullet \bar{s}_0 \bar{s}_1 \bar{s}_2 \dots \bar{s}_M 1),
\end{aligned}$$

this means that the points p and $\bar{p}$ in the $M+1$-th iteration of f through the conjugation $\phi^{-1} \circ \sigma^{M+1} \circ \phi = f^{M+1}$ gives us that p and $\bar{p}$ are in distinct horizontal strips at a fixed distance. Therefore, for any point p there is another point $\bar{p}$ nearby such that, independently of $\varepsilon > 0$, after a finite number of iterations of f such points are separated by a minimum distance. This behaviour in a system is said to be sensitive to dependence on initial conditions.

6.4 Sector bundles

In this section, we will define concepts of planar Sector bundles into unity square S. Basically, a Sector bundle is a set of all vectors emanating from unstable and stable directions for each point. We also introduce a Hypothesis **H3** which depends on derivatives of f substituting the current Hypothesis **H2**. The main theorem of this section proves that Hypothesis **H1** and **H3** are sufficient conditions for Hypothesis **H2**. If H_i is a horizontal strip, then

$$H_j \cap f(H_i) = V_{ij} \tag{6.66}$$

where V_{ij} are vertical strips and

$$f^{-1}(H_i) \cap H_j = H_{ij} = f^{-1}(V_{ij}) \tag{6.67}$$

for all $i, j \in S$. We define $\mathscr{H} = \bigcup_{i,j \in S} H_{ij}$ and $\mathscr{V} = \bigcup_{i,j \in S} V_{ij}$ and we have

$$f(\mathscr{H}) = \mathscr{V}. \tag{6.68}$$

We assume $f \in \mathscr{C}^1$ mapping $\mathscr{H}$ diffeomorphically onto $\mathscr{V}$. Next we define stable and unstable sectors:

Definition 50. Let $z_0 = (x_0, y_0) \in \mathscr{H} \bigcup \mathscr{V}$ be any point, a vector emanating from this point is denoted by (u_{z_0}, v_{z_0}), the *stable and unstable sectors* are horizontal and vertical cones respectively defined as follows:

i) $\mathscr{S}^s_{z_0} := \{(u_{z_0}, v_{z_0}) | \, |u_{z_0}| \leq \mu_h |v_{z_0}|\}$;

ii) $\mathscr{S}^u_{z_0} := \{(u_{z_0}, v_{z_0}) | \, |v_{z_0}| \leq \mu_v |u_{z_0}|\}$.

We remark that μ_h and μ_v are the maximum of the absolute values of the slope of any vector in the horizontal and vertical cones measured concerning x-axis and y-axis respectively. We also remark that for each point z_0 in $\mathscr{H}$ or $\mathscr{V}$, we have one stable sector and one unstable sector at this point. In what follows, we define the *Sector bundles* making the union of all stable an unstable sectors for $z_0 \in \mathscr{H} \bigcup \mathscr{V}$:

$$\mathscr{S}^s_{\mathscr{H}} := \bigcup_{z_0 \in \mathscr{H}} \mathscr{S}^s_{z_0}, \qquad \mathscr{S}^u_{\mathscr{H}} := \bigcup_{z_0 \in \mathscr{H}} \mathscr{S}^u_{z_0}$$
$$\mathscr{S}^s_{\mathscr{V}} := \bigcup_{z_0 \in \mathscr{V}} \mathscr{S}^s_{z_0}, \qquad \mathscr{S}^u_{\mathscr{V}} := \bigcup_{z_0 \in \mathscr{V}} \mathscr{S}^u_{z_0}.$$

Since $f \in \mathscr{C}^1$ and maps horizontal strips $\mathscr{H}$ diffeomorphically onto vertical strips $\mathscr{V}$ we have the isomorphic map at the point z_0

$$Df(z_0) : T_{z_0}U \to T_{f(z_0)}U. \tag{6.69}$$

Hypothesis **H3**. We assume $Df(z_0)(\mathscr{S}^u_{\mathscr{H}}) \subset \mathscr{S}^u_{\mathscr{V}}$ and $Df^{-1}(z_0)(\mathscr{S}^s_{\mathscr{V}}) \subset \mathscr{S}^s_{\mathscr{H}}$. Moreover, if $(u_{z_0}, v_{z_0}) \in \mathscr{S}^u_{\mathscr{H}}$ then $(u_{f(z_0)}, v_{f(z_0)}) \in \mathscr{S}^u_{\mathscr{V}}$ and the vertical direction must satisfies

$$|v_{f(z_0)}| \geq \left(\frac{1}{\mu}\right) |v_{z_0}| \tag{6.70}$$

Also, if $(u_{z_0}, v_{z_0}) \in \mathscr{S}^s_{\mathscr{V}}$ then, the image $(u_{f^{-1}(z_0)}, v_{f^{-1}(z_0)}) \in \mathscr{S}^s_{\mathscr{H}}$ and the horizontal direction must satisfies

$$|u_{f^{-1}(z_0)}| \geq \left(\frac{1}{\mu}\right) |u_{z_0}| \tag{6.71}$$

for all $0 \leq \mu < 1 - \mu_h \mu_v$. Now, we will introduce the main theorem of this section.

Theorem 51. Suppose that the Hypothesis **H1** and **H3** are holds with $0 \leq \mu < 1 - \mu_h \mu_v$. Then, Hypothesis **H2** is hold with $\nu_h = \nu_v = \frac{\mu}{1 - \mu_h \mu_v}$.

Proof: We suppose H is a μ_h-horizontal strip contained in $\cup_{i \in S} H_i$. Then we must prove, using Hypothesis **H1** along with Hypothesis **H3**, firstly we claim that $f^{-1}(H) \cap H_i = \tilde{H}_i$ is a μ_h-horizontal strip for all $i \in S$, and secondly we claim that the width $d(\tilde{H}_i) \leq \nu_h d(H)$ for some $0 < \nu_h < 1$. Similarly, for a μ_v-vertical strip V contained in $\cup_{i \in S} V_i$ we prove that

$f(V) \cap V_i = \tilde{V}_i$ is a μ_v-vertical strip for all $i \in S$ and, moreover, the width $d(\tilde{V}_i) \leq \nu_v d(V)$ for some $0 < \nu_v < 1$.

For the first claim, we begin with H being a μ_h-horizontal curve contained in $\cup_{i \in S} H_i$. Then, H must intersect both vertical boundaries of each V_i forming a horizontal curve inside of each V_i. Hypothesis **H1** ensure that horizontal boundaries of H_i are mapped by f to the horizontal boundaries of V_i. Therefore, $f^{-1}(H) \cap H_i = \tilde{H}_i$ is a horizontal curve for each $i \in S$. Remain to prove that $\tilde{H}_i$ is a μ_h-horizontal curve. Indeed, let us denote by $h(x) = y$ the curve $\tilde{H}_i$ for some fixed i. Given two points (x_1, y_1) and (x_2, y_2) on $\tilde{H}_i$, as the μ_h is the maximum of the absolute value of the slope of any vector in the horizontal of the curve $h(x) = y$, measured concerning x-axis, then, by Hypothesis **H3** and the Mean Value Theorem, we have

$$|h(x_1) - h(x_2)| < \mu_h |x_1 - x_2|.$$

Therefore, $\tilde{H}_i$ is a μ_h-horizontal curve. When H is a μ_h-horizontal strip, the proof follows similarly applying the same previous idea to each horizontal boundary of the strip.

For the second claim, let p_0 and p_1 be points with the same x-component, on the upper and lower horizontal boundaries of $\tilde{H}_i$ respectively for some fixed i, of maximum width, we mean

$$d(\tilde{H}_i) = |p_0 - p_1|. \tag{6.72}$$

We consider the vertical line inside $\tilde{H}_i$ connecting both points p_0 and p_1 defined by

$$p(t) := tp_1 + (1-t)p_0 \qquad 0 \leq t \leq 1.$$

The derivative concerning t is

$$\dot{p}(t) = p_1 - p_0 \in \mathscr{S}^u_{\mathscr{H}} \qquad \forall 0 \leq t \leq 1. \tag{6.73}$$

Next, we consider te image of $p(t)$ under f defined by

$$f(p(t)) := z(t) = (x(t), y(t)).$$

By Hypothesis **H1**, the curve $z(t)$ connects both upper and lower boundaries of the μ_h-horizontal strip H from the beginning. We denote its endpoints by

$$f(p(0)) := z_0 = (x_0, y_0), \qquad f(p(1)) := z_1 = (x_1, y_1). \tag{6.74}$$

As H is a μ_h-horizontal strip, z_0 lies on a μ_h-horizontal curve that it will be denoted by $y = h_0(x)$ and z_1 lies on a μ_h-horizontal curve that it will be denoted by $y = h_1(x)$.

The tangent vectors of the curve $z(t)$ are given by the equation

$$\dot{z}(t) = Df(p(t))\dot{p}(t). \tag{6.75}$$

From Equation (6.73) and Hypothesis **H3**, $z(t)$ is a μ_v-vertical curve contained in H. Since that, $z(t)$ is a μ_v-vertical curve then, from the proof of continuity in the Lemma (49), we obtain

$$|y_0 - y_1| \leq \frac{1}{1-\mu_h\mu_v}||h_0 - h_1|| = \frac{1}{1-\mu_h\mu_v}d(H) \tag{6.76}$$

where d is from Definition (44) Item iii). On the other hand, the Hypothesis **H3** allows us to compare both vertical vectors, we obtain

$$|\dot{y}(t)| \geq \left(\frac{1}{\mu}\right)|\dot{p}(t)| = \left(\frac{1}{\mu}\right)|p_1 - p_0| \tag{6.77}$$

and its integral concerning t, is given by

$$\int_0^1 |\dot{p}(t)|dt \leq \mu \int_0^1 |\dot{y}(t)|dt$$

and its solution is given by

$$|p_1 - p_0| \leq \mu|y_1 - y_0|. \tag{6.78}$$

Equation (6.72) and Inequality (6.76) applied to Inequality (6.78) given directly

$$d(\tilde{H}_i) \leq \left(\frac{\mu}{1-\mu_h\mu_v}\right)d(H).$$

This concludes the proof of the Theorem.

■

CHAPTER

7

SHILNIKOV HOMOCLINIC ORBIT

In this chapter, we study a continuous three-dimensional nonlinear system with a hyperbolic fixed point of saddle-focus type. We call saddle-focus meaning the existence of a fixed point O with two-dimensional spiral stable $\mathscr{E}^s(O)$ and transversely a one-dimensional unstable $\mathscr{E}^u(O)$ manifolds. The local dynamic behaviour of these class systems is well-known and fully understood by Hartman-Grobman Theorem since the fixed point is hyperbolic.

An interesting phenomenon occurs when we assume the existence of an orbit Γ, which is basically a connection non-transversely of both manifolds $\mathscr{E}^s(O) \cap \mathscr{E}^u(O) = p_0$ at the point p_0. More precisely, it means that the orbit Γ is *bi-asymptotic*, that is $\Gamma(t) \to O$ as $t \to \pm\infty$. Formally the *homoclinic orbit* is a the union $\Gamma \cup \{O\}$. See Figure (11). Shilnikov (SHILNIKOV, 1965) proved, assuming this homoclinic configuration and providing that real eigenvalue has a larger magnitude than the real part of the complex eigenvalues, that the flow of the system exhibits *chaos*, thus given the most simple way to prove the existence chaos. More precisely,

Theorem 52 ((SHILNIKOV, 1965))**.** Given a three-dimensional autonomous system

$$\dot{x} = f(x)$$

with a hyperbolic fixed point x_0, we mean one conjugated complex eigenvalue pair $\rho \pm i\omega$ with $\rho < 0$ and $\omega \neq 0$, and one real eigenvalue $\lambda > 0$. If the value $\lambda + \rho$ is positive, then there are countably many saddle periodic orbits in a neighborhood of the homoclinic orbit Γ of the saddle focus.

The value $\lambda + \rho$ is so-called *saddle value*, and it is fundamental to be positive in this theorem. When the saddle value $\lambda + \rho$ is negative, the structure of phase space near to homoclinic orbit is trivial leading to a single stable periodic orbit from the homoclinic orbit. The critical value $\lambda + \rho = 0$ was studied in (BELYAKOV, 1984), where small perturbations lead to transitions between complex dynamics in the case $\lambda + \rho > 0$, and trivial dynamics in the case $\lambda + \rho < 0$.

Of course, the existence of this homoclinic orbit is a strong assumption depending on the nonlinearities and this is not common to that class of systems. Arneodo and co-authors (ARNEODO *et al.*, 1985) showed numerically that the emergence of a homoclinic connection in fact occurs from the increase of the limit circle on Hopf bifurcation until the critical stage where a break in the limit circle occurs leading to homoclinic formation, thus given more consistency to the theory.

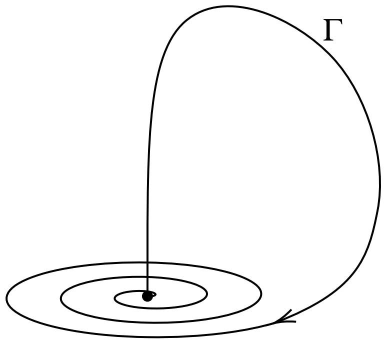


Figure 11 – Shilnikov homoclinic orbit.

In Section (7.1), we introduce the canonical three-dimensional system model whose linearization has a hyperbolic fixed point of the saddle focus type. In Section (7.2), we construct the local *return map* or *Poincaré-map* in the neighborhood to homoclinic orbit. In Section (7.3) we are concerned to prove the existence of chaos in the original system, in order to get this we will prove that the Poincaré map is topologically conjugated to the Shift map, therefore, by the Conley-Moser Theorem (47), the original system has an invariant set of positive entropy. We follow the excellent books to present this chapter (GUCKENHEIMER; HOLMES, 2013),(WIGGINS; WIGGINS; GOLUBITSKY, 2003),(WIGGINS, 2013).

7.1 The three-dimensional system model

We consider the dynamics near to hyperbolic saddle-focus fixed point type to a three-dimensional system governed by equations with fixed point O moved to origin:

$$\begin{aligned} \dot{x} &= \rho x - \omega y + P(x,y,z) \\ \dot{y} &= \omega x + \rho y + Q(x,y,z) \\ \dot{z} &= \lambda z + R(x,y,z). \end{aligned} \tag{7.1}$$

The eigenvalues of the System (7.1) are, one conjugated complex pair $\rho \pm i\omega$ and one real λ at the only one fixed point $(0,0,0)$. The nonlinear terms P, Q, R are $\mathscr{C}^2$ such that P, Q, R and the derivatives DP, DQ, DR vanishing at $(0,0,0)$.

From now on, we fix the following assumptions:

A1 The eigenvalues satisfies $\rho < 0$, $\omega \neq 0$ and $\lambda > 0$ such that $-1 < \frac{\rho}{\lambda} < 0$.

A2 There exists a homoclinic orbit Γ connecting the fixed point at origin to itself.

The System (7.1) linearized at the origin possesses locally one two-dimensional stable subspace E^s generated by eigenvectors corresponding to eigenvalues $\rho \pm i\omega$ and one-dimensional unstable subspace E^u generated by eigenvectors corresponding to eigenvalue λ provided by Assumption **A1**.

$$\begin{aligned} \dot{x} &= \rho x - \omega y \\ \dot{y} &= \omega x + \rho y \\ \dot{z} &= \lambda z \end{aligned} \tag{7.2}$$

The flow of the linear system is completely solved by known techniques:

$$\begin{aligned} x(t) &= e^{\rho t}(x_0 \cos(\omega t) - y_0 \sin(\omega t)) \\ y(t) &= e^{\rho t}(x_0 \sin(\omega t) + y_0 \cos(\omega t)) \\ z(t) &= e^{\lambda t} z_0 \end{aligned} \tag{7.3}$$

where, in fact, we have a stable trajectory spiraling on the xy-plane coinciding with the subspace E^s and an unstable trajectory repelling on the z-axis coinciding with the subspace E^u. See Figure (12).

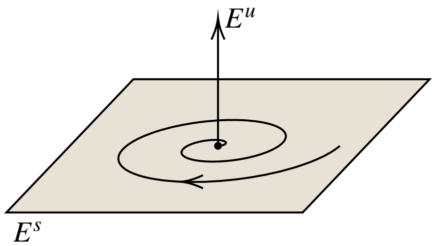


Figure 12 – Three-dimensional saddle-focus type.

Next, considering nonlinear terms P, Q and R, we get stable $\mathscr{E}^s$ and unstable $\mathscr{E}^u$ manifolds tangent to E^s and E^u at $(0,0,0)$. By Assumption **A2**, there exists a orbit Γ connecting both one-dimensional unstable manifold $\mathscr{E}^u$ and two-dimensional stable manifold $\mathscr{E}^s$, forming the existence of an non-transversal intersection $\mathscr{E}^s \cap \mathscr{E}^u = p_0$ leading to existence of homoclinic orbit, see Figure (13).

We will present, in a detailed and constructive way, how the orbits around a homoclinic orbit have a complicated behaviour meaning chaos.

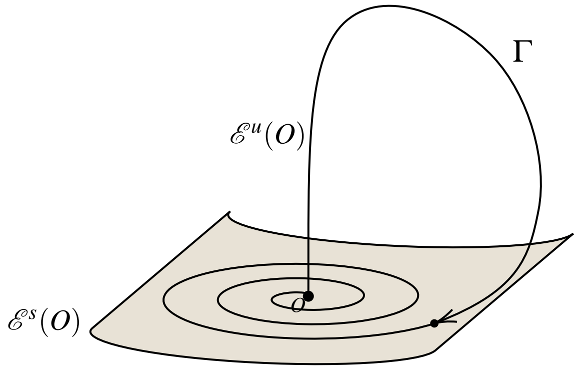


Figure 13 – Homoclinic orbit at the origin O.

7.2 The Poincaré map

In order to study the local behaviour around the homoclinic orbit, we will construct a diffeomorphic map locally from $\mathbb{R}^2$ to $\mathbb{R}^2$ by the composition of the local solution of Equations (7.3) and the orbit Γ re-injected. This composition will be called the Poincaré map or return map.

We start introducing suitable cross-sections in strategic locations. Let Π_0, Π_1 be planes positioned transversely in the $\mathscr{E}^s, \mathscr{E}^u$ stable and unstable manifolds respectively. More precisely, Π_0, Π_1 are both defined by:

$$\Pi_0 := \{(x,y,z) \in \mathbb{R}^3 | \; y = 0\} \tag{7.4}$$

$$\Pi_1 := \{(x,y,z) \in \mathbb{R}^3 | \; z = p_1 > 0\} \tag{7.5}$$

where $p_0 \in \Pi_0 \cap \mathscr{E}^s \cap \mathscr{E}^u$ and $p_1 \in \Pi_1 \cap \mathscr{E}^u$. See Figure (14).

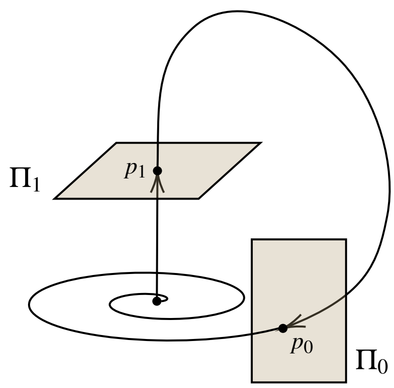


Figure 14 – Cross-sections in the Poincaré map.

In order to construct a map $\varphi_0 : \Pi_0 \mapsto \Pi_1$ we use the local solution of Equations (7.3) of the system linearized that we have found before. We need to adapt this solution to initial conditions from Π_0 that will reach Π_1 in finite time t and also constraints on Π_0 considering points around of p_0 which cross it just one time.

First constraint we want to know how much time $T > 0$ is needed for any point on Π_0 to reach Π_1. Thus, we use the third equation for z making the calculation $e^{\lambda T} z = p_1$, which means

$$T = \frac{1}{\lambda}\log(\frac{p_1}{z}) \tag{7.6}$$

replacing T on Equations (7.3) now for any point $(x,0,z) \in \Pi_0$, we define the map $\varphi : \Pi_0 \to \Pi_1$ by

$$\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \mapsto \begin{pmatrix} x\left(\frac{p_1}{z}\right)^{\frac{\rho}{\lambda}} \cos\left(\frac{\omega}{\lambda}\log(\frac{p_1}{z})\right) \\ x\left(\frac{p_1}{z}\right)^{\frac{\rho}{\lambda}} \sin\left(\frac{\omega}{\lambda}\log(\frac{p_1}{z})\right) \\ p_1 \end{pmatrix}.$$

The second constraint is about those initial conditions on cross-section Π_0 in time $t = 0$ that return many times to Π_0 before reaching Π_1. See Figure (15).

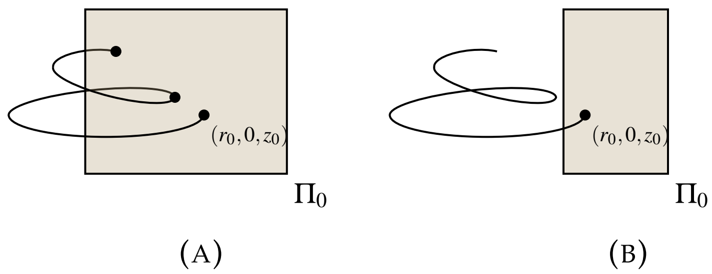


Figure 15 – A trajectory from initial point $(x_0,0,z_0)$ returns many times to Π_0 before reach to Π_1 in (A)*, and for each point on Π_0 its trajectory will not return to Π_0 before to reach Π_1 in* (B).

We must avoid the situation in Figure (15) (A). In order to get this, we look at how much time T_0 is needed for some point $(x_0,0,z_0) \in \Pi_0$ returns to itself. So, we want to know T_0 such that $\omega T_0 = 2\pi \Rightarrow T_0 = \frac{2\pi}{\omega}$ and consequently the smaller radius will be $r_0 e^{\frac{2\pi\rho}{\omega}}$. Therefore, we have $\varphi(0) = (x_0,0,z_0) \in \Pi_0$ at $t = 0$ and it will return to itself with $\varphi(T_0) = (x_0 e^{\frac{2\pi\rho}{\omega}}, 0, z_0 e^{\frac{2\pi\lambda}{\omega}})$ at $t = T_0$. We define a sub-cross-section of Π_0 properly considering the first and second previous constraints on Π_0. Let $\Pi_0^{'}$ a subset of Π_0 such that:

$$\Pi_0^{'} := \{(x,0,z) \in \Pi_0 |\ p_1 e^{\frac{2\pi\rho}{\omega}} \le x \le p_1,\ 0 < z \le p_1\}. \tag{7.7}$$

Therefore, no point in the interior of $\Pi_0^{'}$ returns to itself before reaching Π_1. We get a map $\varphi_0 := \varphi|_{\Pi_0^{'}} : \Pi_0^{'} \to \Pi_1$ defined by:

$$\varphi_0(x,0,z) = \left(x\left(\frac{p_1}{z}\right)^{\frac{\rho}{\lambda}} \cos(\gamma), x\left(\frac{p_1}{z}\right)^{\frac{\rho}{\lambda}} \sin(\gamma), p_1 \right). \tag{7.8}$$

where $\gamma = \frac{\omega}{\lambda}\log(\frac{p_1}{z})$. Before constructing the map $\varphi_1 : \Pi_1 \to \Pi_0^{'}$ an understanding is important about the image of $\varphi_0(\Pi_0^{'})$. The local dynamics of the system influence the points of the sub-cross-section $\Pi_0^{'}$ shrinking in the x-axis and expanding in the y-axis simultaneously. To see clearly, we change to polar coordinates. Let $u = r\cos(\theta)$, and $v = r\sin(\theta)$ be the change to polar coordinates, so we have:

$$r = \sqrt{u^2 + v^2} \qquad \theta = \tan^{-1}\left(\frac{v}{u}\right)$$

and the image of φ_0 is represented by equations:

$$\varphi_0(r,\theta) = \left(x\left(\frac{p_1}{z}\right)^{\frac{\rho}{\lambda}}, \frac{\omega}{\lambda}\log(\frac{p_1}{z}) \right). \tag{7.9}$$

Therefore, we see that each horizontal lines ($z =$constant) are mapped to radial lines emanating from $(0,0,p_1)$ spiraling into Π_1 and, smaller and smaller for each $z =$constant smaller because, provided that $-1 < \frac{\rho}{\lambda} < 0$ the radius $r = x\left(\frac{p_1}{z}\right)^{\frac{\rho}{\lambda}}$ approaches zero. On the other hand, vertical lines ($x =$constant) are mapped into spiral logarithmic because while the radius $r = x\left(\frac{p_1}{z}\right)^{\frac{\rho}{\lambda}}$ goes to zero as z approaches zero, $\theta = \frac{\omega}{\lambda}\log(\frac{p_1}{z})$ increasing as z approaches zero. See Figure (16).

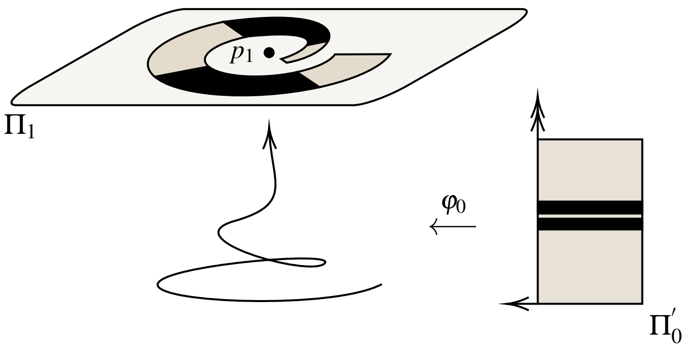


Figure 16 – Logarithmic spiral strip formation on Π_1*.*

After understanding the transformation of the horizontal strips by φ_0. We are ready to construct the return map φ_1 from Π_1 to $\Pi_0^{'}$. We strongly use the Assumption **A2** to define φ_1 along the orbit Γ. We would like to bring the logarithmic spiral from Π_1 in the (x,y)-plane at $z = p_1$ to another logarithmic spiral on $\Pi_0^{'}$ in the (x,z)-plane at $y = 0$. So, to do this we can use rotation and translation motions. Indeed, the transformation matrix rotates and moves objects around from p_1 to p_0 is given by:

$$\begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ c & d & 0 \end{pmatrix} \tag{7.10}$$

with $a,b,c,d \in \mathbb{R}$ such that $ad - cb \neq 0$. We define $\varphi_1 : \Pi_1 \to \Pi_0^{'}$ by

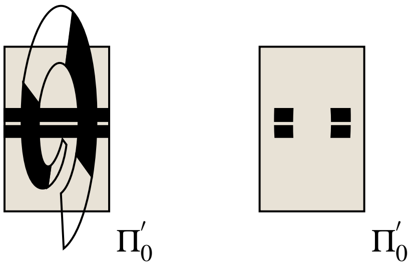
$$\begin{pmatrix} x \\ y \\ p_1 \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ p_1 \end{pmatrix} + \begin{pmatrix} p_0 \\ 0 \\ 0 \end{pmatrix} \tag{7.11}$$

where $\varphi_1(0,0,p_1) = (p_0,0,0)$. We consider $\Pi_1^{'} \subset \Pi_1$ such that $\varphi_0(\Pi_0^{'}) = \Pi_1^{'}$.

We finally obtain a fulfilled bi-dimensional return map to the System (7.1) by composition $\psi = \varphi_1 \circ \varphi_0 : \Pi_0^{'} \to \Pi_0^{'}$ given by

$$\psi(x,z) = x\left(\frac{p_1}{z}\right)^{\frac{\rho}{\lambda}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos(\gamma) \\ \sin(\gamma) \end{pmatrix} + \begin{pmatrix} p_0 \\ 0 \end{pmatrix}. \tag{7.12}$$

The return map ψ sends horizontal strips H_i in black from $\Pi_0^{'}$ and itself receives logarithmic spiral around of p_0. See Figure (17).



(A) (B)

Figure 17 – The return map image in (A) *and the intersection* $\psi(H_i) \cap H_i$ *in* (B).

7.3 Chaos by means of the Shilnikov's orbit

In this section, we are concerned to show that $\psi : \Pi_0^{'} \to \Pi_0^{'}$ defined in the previous section is in fact topologically conjugated to shift map of two symbols from Chapter 2. In order to do this, we need to define horizontal strips in the sub-cross-section $\Pi_0^{'}$. More precisely

$$R_k := \left\{ (x,0,z) \in \Pi_0^{'} \middle| \quad p_1 e^{\frac{2\pi\rho}{\omega}} \leq x \leq p_1, \quad p_1 e^{\frac{-2\pi(k+1)\lambda}{\omega}} \leq z \leq p_1 e^{\frac{-2\pi k\lambda}{\omega}} \right\}. \tag{7.13}$$

We note that for $k \to +\infty$ we have the horizontal strips R_k converge to a horizontal line and consequently z converge to zero. Clearly,

$$\Pi_0^{'} = \bigcup_{k=0}^{+\infty} R_k.$$

The Smale horseshoe formation depends on how much the spiral logarithmic expands or contracts when it comes back to sub-cross-section $\Pi_0^{'}$. Next lemma guarantees that there are disjoint μ_h-horizontal strips in $\Pi_0^{'}$ that are mapped over themselves in μ_v-vertical strips on which the Sector bundles Hypothesis **1** is satisfied.

Lemma 53. Consider R_k for fixed k sufficiently large. Then the inner boundary of $\psi(R_k)$ intersects the upper horizontal boundary of R_i in (at least) two points for $i \geq \frac{k}{\alpha}$ where $1 \leq \alpha < -\frac{\lambda}{\rho}$. Moreover, the preimage of the vertical boundaries of $\psi(R_k) \cap R_i$ is contained in the vertical boundary of R_k.

Proof: Let R_i be a horizontal strip and let $\bar{z}$ be the upper boundary of R_i :

$$\bar{z} = p_1 e^{-\frac{2\pi i \lambda}{\omega}}. \tag{7.14}$$

By Equation (7.9), the radius r is given by expression $r = x\left(\frac{p_1}{z}\right)^{\frac{\rho}{\lambda}}$. Therefore, the minimum radius r_{min} is obtained with $x = p_1 e^{\frac{2\pi\rho}{\omega}}$ and $z = p_1 e^{\frac{-2\pi(k+1)\lambda}{\omega}}$, so

$$r_{min} = p_1 e^{\frac{4\pi\rho}{\omega}} \cdot e^{\frac{2\pi k\rho}{\omega}}. \tag{7.15}$$

As φ_1 is an affine map we have a radius $\bar{r}_{min}$ into Π_0' given by multiple of r_{min}, that is,

$$\bar{r}_{min} = C r_{min} = C p_1 e^{\frac{4\pi\rho}{\omega}} \cdot e^{\frac{2\pi k\rho}{\omega}}. \tag{7.16}$$

We want the spiral logarithmic expanded, so is necessary:

$$\frac{\bar{r}_{min}}{\bar{z}} > 1. \tag{7.17}$$

Thus,

$$\frac{\bar{r}_{min}}{\bar{z}} = \frac{C p_1 e^{\frac{4\pi\rho}{\omega}} \cdot e^{\frac{2\pi k\rho}{\omega}}}{p_1 e^{-\frac{2\pi i\lambda}{\omega}}} = C e^{\frac{4\pi\rho}{\omega}} e^{\frac{2\pi(k\rho + i\lambda)}{\omega}} = K \cdot e^{\frac{2\pi}{\omega}(k\rho + i\lambda)} \tag{7.18}$$

where $K = C e^{\frac{4\pi\rho}{\omega}} > 0$ is a constant. We only need to control the term $k\rho + i\lambda$ to be positive and large enough such that the ratio is bigger than 1. Let $\alpha \geq \frac{k}{i}$ be a number. Therefore, by Assumption **A1** we have

$$k\rho + i\lambda \geq k\rho + \frac{k}{\alpha}\lambda = k\left(\rho + \frac{\lambda}{\alpha}\right) > 0. \tag{7.19}$$

We must have $\rho + \frac{\lambda}{\alpha} > 0$ and then it is true in the interval $1 \leq \alpha < -\frac{\lambda}{\rho}$. Therefore, for k sufficiently large, the ratio above is bigger than one.

Moreover, the vertical boundaries of R_k are mapped by φ_0 to inner and outer boundaries of spiral logarithmic into Π_1, $\varphi_0(R_k)$. Since that φ_1 applies objects diffeomorphically into Π_0' we have $\psi(R_k) = \varphi_1 \circ \varphi_0(R_k)$. Therefore, the preimage of vertical boundary of $\psi(R_k) \cap R_i$ is contained in the boundary of R_k.

■

Theorem 54. For k sufficiently large, R_k contains an invariant Cantor set, Λ_k, on which the Poincaré map ψ is topologically conjugate to a full shift on two symbols.

Proof: We must find in R_k two disjoint μ_h-horizontal strips that are mapped over themselves in μ_v-vertical strips on which the Sector bundles Hypothesis **H1** and **H3** are satisfied.

For the Sector bundles Hypothesis **H1** we strongly use the Lemma (53). We can choose two disjoint μ_h-horizontal strips in R_k where its horizontal boundaries are μ_h-horizontal curves with $\mu_h = 0$. By Lemma (53), for k sufficiently large, both strips are mapped over themselves in μ_v-vertical strips, where the preimage of vertical boundaries of $\psi(R_k) \cap R_i$ for $i \geq k$ are contained in R_k. Moreover, $0 \leq \mu_h \mu_v < 1$.

From the Sector bundles defined in Section (6.4), we remember the definitions of the stable and unstable Sector bundles and Hypothesis **H3**. Let H_1, H_2 be μ_h-horizontal strips in R_k. Then, by Poincaré map ψ constructed, we have

$$\psi(H_i) \cap H_j = V_{ji}, \quad H_i \cap \psi^{-1}(H_j) \equiv H_{ij} = \psi^{-1}(V_{ji})$$

for $i, j \in \{1, 2\}$. We also consider

$$\mathscr{H} = \bigcup_{i,j} H_{ij}, \quad \mathscr{V} = \bigcup_{i,j} V_{ji}$$

such that $\psi(\mathscr{H}) = \mathscr{V}$. For each $w_0 = (x_0, z_0) \in \mathscr{H} \bigcup \mathscr{V}$ we want to prove that

$$D\psi(\mathscr{S}^u_{\mathscr{H}}) \subset \mathscr{S}^u_{\mathscr{V}}, \quad D\psi^{-1}(\mathscr{S}^s_{\mathscr{V}}) \subset \mathscr{S}^s_{\mathscr{H}}.$$

where, $\mathscr{S}^u_{\mathscr{H}}$ and $\mathscr{S}^s_{\mathscr{V}}$ are unstable and stable Sector bundles at z_0 while that $\mathscr{S}^u_{\mathscr{V}}$ and $\mathscr{S}^s_{\mathscr{H}}$ are unstable and stable Sector bundles at $\psi(w_0)$ and $\psi^{-1}(w_0)$ respectively. We remember the map $\psi = \varphi_1 \circ \varphi_0 : \Pi_0' \to \Pi_0'$ given by

$$\psi(x, z) = x \left(\frac{p_1}{z} \right)^{\frac{\rho}{\lambda}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos(\gamma) \\ \sin(\gamma) \end{pmatrix} + \begin{pmatrix} p_0 \\ 0 \end{pmatrix}. \tag{7.20}$$

The differential operator

$$D\psi(w_0) : \Pi_0' \to \Pi_0'$$

where $D\psi(w_0) = D\varphi_1(\varphi_0(w_0)) \cdot D\varphi_0(w_0)$ is given by

$$D\psi(w) = \frac{p_1^{\frac{\rho}{\lambda}}}{z^{\frac{\rho}{\lambda}+1}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos(\gamma) & \frac{\omega \sin(\gamma) - \rho \cos(\gamma)}{\lambda} \\ \sin(\gamma) & \frac{-\rho \sin(\gamma) - \omega \cos(\gamma)}{\lambda} \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & x \end{pmatrix} \tag{7.21}$$

where $\gamma = \frac{\omega}{\lambda} \log(\frac{p_1}{z})$. We note that, from Assumption **A1**, we have $0 < \frac{\rho}{\lambda} + 1 < 1$ and then $z^{-(\frac{\rho}{\lambda}+1)}$ is large as z is small. We also note that all matrices are invertible since we have supposed $ad - cb \neq 0$ and $\omega \neq 0$.

We evaluated the differential operator of Equation (7.21) at singularity point $z \equiv 0$ fixed to obtain the eigenvalues and corresponding eigenvectors of the matrix of the operator.

The first eigenvalue is 0 with corresponding eigenvector $(1,0)$, the second eigenvalue depends on $z^{-(\frac{\rho}{\lambda}+1)}$, as $z \to 0$ the second eigenvalue becomes large as we choose strips $z', z'' \to 0$. Therefore, the corresponding eigenvectors remain in disjoint cones.

■

CHAPTER

8

CHAOS ON NETWORKS

In this chapter, we will present our main contribution to this thesis. As we have seen in the Introduction (1), our goal is to prove the emergence of chaotic behaviour in diffusively coupled systems. By Chapters (5), (6) and (7) we understand "chaotic behaviour" as the existence of the Shilnikov orbit which is translated, via topological equivalence, to the existence of an invariant Cantor set whose entropy is positive.

The main goal in Section (8.1) is to present the proof of the Main Theorem (4). To do this, we will study the linearization of the network of Equation (1.1) which represents the diffusive coupling of N identical dynamic systems. A network is often an equation with a high dimension, therefore, the first problem is concerning reducing it to a low dimension. In this direction, we will prove a new important result, Proposition (56), which gives necessary and sufficient conditions to find zero eigenvalues to the network depending on the linearization of the isolated system. As a consequence, we can apply the Center Manifold Theory from Chapter (3), which will play an important role in the reduction of dimension. From these results, we also will present explicitly the low-dimension reduced vector field defined on the center subspace, Proposition (64), in which we will see how the nonlinear terms of the isolated system influence the nonlinear terms of the low-dimension reduced vector field. From these results, we will be ready to see how the ρ-Versatile graph structure from Chapter (2), this will be shown in the Lemma (66). At the end of this section, with all these results on hand, we will give the proof of the Main Theorem (4).

In Section (8.2) we approach the stability of the center manifold problem to the network. We will prove in Proposition (67), that for some class of ρ-Versatile graphs, (for instance, star graphs with one high degree node compared to others), the center manifold is stable. On the other hand, we will give an example where the center manifold is unstable for parameters in the interval $[0, 1]$.

In Section (8.3) we approach Proposition (56) in the cases $m = 1,2,3$. When $m = 1,2$ we can find a transcritical bifurcation and Bogdanov-Takens bifurcation respectively for the low-dimension reduced vector field. In the case $m = 3$ and a 2-Versatile graph we will prove the Carollary (5) finding the chaotic behaviour in the network locating the Shilnikov orbit on the center manifold. Our conclusion is supported through the Theorem (70) proved in (IBáñEZ; RODRíGUEZ, 2005) which ensures the existence of the Shilnikov orbit in any generic unfolding, Chapter (4) introduces concepts related to genercity from Thom's transversality theorem point view.

8.1 Proof of Main Theorem

In this section, we present the proof of Theorem (4). We start by analyzing the linearized system in Subsection (8.1.1), after which we perform center manifold reduction and have a detailed look at the reduced vector field in Subsection (8.1.2).

8.1.1 Linearization

In this subsection, we investigate the linear part of the system

$$\dot{X} = F(X;\varepsilon) - \alpha(L \otimes D)X, \tag{8.1}$$

from Theorem (4). Writing $A \in \mathbb{R}^{n\times n}$ for the Jacobian of f at the origin, we see that the linearization of Equation (8.1) at the origin is given by

$$\dot{Y} = (\mathrm{Id}_{\mathbb{R}^N} \otimes A - \alpha L \otimes D)Y. \tag{8.2}$$

An important observation is the following: if v is an eigenvector of L with eigenvector $\lambda \in \mathbb{R}$, then the linearization above sends a vector $v \otimes x$ with $x \in \mathbb{R}^n$ to

$$\begin{aligned}(\mathrm{Id} \otimes A - \alpha L \otimes D)(v \otimes x) &= v \otimes Ax - \alpha Lv \otimes Dx \\ &= v \otimes Ax - \alpha\lambda v \otimes Dx \\ &= v \otimes Ax - v \otimes \alpha\lambda Dx = v \otimes (A - \alpha\lambda D)x.\end{aligned} \tag{8.3}$$

It follows that the space

$$v \otimes \mathbb{R}^n := \{v \otimes x \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^N \otimes \mathbb{R}^n \tag{8.4}$$

is kept invariant by the linear map of Equation (8.2). We claim that $v \otimes \mathbb{R}^n$ is in fact a linear subspace. To see why, note that for all $x, y \in \mathbb{R}^n$ and $r \in \mathbb{R}$ we have

$$\begin{aligned} v \otimes x + v \otimes y &= v \otimes (x+y) \in v \otimes \mathbb{R}^n \text{ and} \\ r(v \otimes x) &= v \otimes rx \in v \otimes \mathbb{R}^n. \end{aligned} \tag{8.5}$$

What Equation (8.3) furthermore tells us is that the Linearization (8.2) restricted to $v \otimes \mathbb{R}^n$ is conjugate to the map $A - \alpha\lambda D : \mathbb{R}^n \to \mathbb{R}^n$.

Let us denote the eigenvalues of L by $0 = \lambda_1 \leq \cdots \leq \lambda_N$. We moreover fix a corresponding set of orthonormal eigenvectors $v^1, \ldots, v^N \in \mathbb{R}^N$. In other words, $Lv^p = \lambda_p v^p$ and $\langle v^p, v^q \rangle = \delta_{pq}$ for all $p, q \in \{1, \ldots, N\}$. Finally, we write

$$V_p := v^p \otimes \mathbb{R}^n \tag{8.6}$$

for the corresponding linear subspaces of $\mathbb{R}^N \otimes \mathbb{R}^n$. We thus have a direct sum decomposition

$$\mathbb{R}^N \otimes \mathbb{R}^n = \bigoplus_{p=1}^{N} V_p, \tag{8.7}$$

where each component is respected by the Linearization (8.2), with the restriction to V_p conjugate to $A - \alpha\lambda_p D : \mathbb{R}^n \to \mathbb{R}^n$. In particular, we see that the spectrum of the Linearization (8.2) is given by the union of the spectra of the maps $A - \alpha\lambda_p D$, with a straightforward relation between the respective algebraic and geometric multiplicities.

This observation motivates the main result of this subsection, Proposition (56) below. It gives necessary and sufficient conditions on A for the existence of a positive-definite matrix D such that $A - D$ has a generalized kernel of a prescribed dimension. Note that some conditions on A have to apply for such a matrix D to exist. For instance, if A is Hurwitz and symmetric, then $D - A$ is positive as the sum of two positive matrices. In that case, $A - D$ remains Hurwitz, see Lemma (60) below. In addition, note that for A Hurwitz the trace of $A - D$ is strictly negative so that $A - D$ always retains at least one eigenvalue with a negative real part. Similarly, if A is anti-symmetric then $x^T(A - D)x = -x^T Dx < 0$ for all $x \in \mathbb{R}^n$. Hence, $A - D$ is then necessarily invisible.

In what follows, we denote by $\mathbb{M}_n(\mathbb{R})$ the space of n by n matrices over the field $\mathbb{R}$. We write $\langle x, y \rangle := x^T y$ for the Euclidean inner product between vectors $x, y \in \mathbb{R}^n$.

In order to prove Proposition (56) below, we will first need a technical lemma that uses the theory of Schur complements. To this end, suppose we are given a block matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{8.8}$$

with blocks $M_{11}, \ldots, M_{22}$. If M_{22} is invisible then we may form the *Schur complement* of M, given by

$$M/M_{22} := M_{11} - M_{12} M_{22}^{-1} M_{21}. \tag{8.9}$$

This expression has various useful properties. We are interested in the situation where M is symmetric, so that $M_{12}^T = M_{21}$, $M_{11}^T = M_{11}$ and $M_{22}^T = M_{22}$. In that case, the matrix M is positive-definite if and only if both M_{22} and M/M_{22} are positive-definite, see (ZHANG, 2006) and in particular Theorem 1.12. Using this result, we may prove:

Lemma 55. Let

$$D = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{8.10}$$

be a block matrix and assume $A_{22} = c\,\mathrm{Id}$ for some scalar $c \in \mathbb{R}_{>0}$. Suppose furthermore that A_{11} is positive-definite. Then, for c sufficiently large the matrix D is positive-definite as well

Proof: Clearly, D is positive-definite if and only if the symmetric matrix

$$H := D + D^T = \begin{pmatrix} A_{11} + A_{11}^T & A_{12} + A_{21}^T \\ A_{21} + A_{12}^T & 2c\,\mathrm{Id} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{pmatrix} \tag{8.11}$$

is positive-definite. Here we have set $H_{22} := 2c\,\mathrm{Id}$, $H_{12} := A_{12} + A_{21}^T$ and $H_{11} := A_{11} + A_{11}^T$, the third of which is positive-definite as A_{11} is. As $H_{22} = 2c\,\mathrm{Id}$ is invisible with inverse $1/(2c)\,\mathrm{Id}$, we may form the Schur complement

$$\begin{aligned} H/H_{22} &= H_{11} - H_{12}H_{22}^{-1}H_{12}^T \\ &= H_{11} - \frac{H_{12}H_{12}^T}{2c}. \end{aligned} \tag{8.12}$$

As H_{22} is positive-definite, it follows from the above discussion that H is positive-definite if and only H/H_{22} is. However, as $c \to \infty$ we have $H/H_{22} \to H_{11}$, so H/H_{22} behaves like a small perturbation of H_{11}, then it is positive-definite for $c > 0$ large enough. This shows that D is likewise positive-definite for large enough c.

■

From Definition (10), as we have promised at the end of Section (2.2), the following proposition proves that the Skewness condition is a necessary and sufficient condition for the existence of a properly positive-definite matrix D. More precisely,

Proposition 56. Let $A \in \mathbb{M}_n(\mathbb{R})$ be a matrix. There exist m mutually orthogonal vectors $v_1, \dots, v_m$ such that

$$\langle v_i, Av_i \rangle > 0 \text{ for all } i = 1, \dots, m, \tag{8.13}$$

if and only if there exists a positive-definite matrix D such that $A - D$ has m zero eigenvalues, counted with algebraic multiplicity.

Remark 57. We highlight that the number m of zero eigenvalues for $A - D$ is directly connected with the number of mutually orthogonal vectors for the Skewness condition. Moreover, for each m given for the Skewness condition, there is a positive-definite matrix D to be constructed in a non-unique way. Consequently, since we have a number m of such vectors, for each $k < m$ we might construct a different matrix D such that $A - D$ have k zero eigenvalues. Finally, if we are concerned with Hurwitz matrices A, therefore, by the Remark (11), m must be strictly less than n.

Figure (18) shows an illustration of Proposition (56).

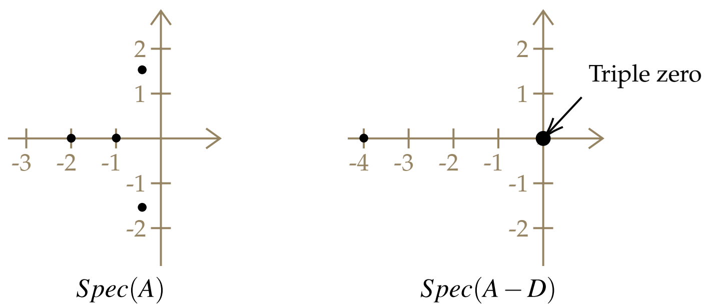


Figure 18 – An illustration of Proposition (56). Figure a) shows the eigenvalues of a particular Hurwitz matrix A, which might be the linear part of the isolated dynamics f at the origin. The matrix A has 4 eigenvalues all strictly on the left half of the complex plane. The existence of $m = 3$ mutually orthogonal vectors satisfying (8.13) ensures that a positive-definite matrix D exists such that $A - D$ has a 3-dimensional generalized kernel. In other words, the subtraction of D has moved 3 eigenvalues to the origin, and an eigenvalue farther from the origin that is given by the negative trace of the matrix $A - D$, Figure b).

Proof: Suppose first that we have m mutually orthogonal vectors v_i such that for each $i = 1, \dots, m$ we have

$$\langle v_i, Av_i \rangle = v_i^T A v_i > 0. \tag{8.14}$$

Note that we then also have

$$\langle v_i, (A + A^T) v_i \rangle = 2\langle v_i, Av_i \rangle > 0 \tag{8.15}$$

for all i. We may re-scale the v_i by any non-zero factor, so that we will now assume without loss of generality that $\|v_i\| = 1$ for all i. We start by constructing an auxiliary upper-diagonal $(m \times m)$-matrix P as follows:

$$P = P_A = \begin{pmatrix} 0 & p_{1,2} & \cdots & p_{1,m} \\ 0 & 0 & \cdots & p_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m \times m},$$

where each entry $p_{i,j}$ is defined by the rule:

$$\begin{aligned} p_{i,j} &= v_i^T (A + A^T) v_j, \quad \text{for all } i < j; \\ p_{i,j} &= 0, \quad \text{for all } i \geq j. \end{aligned}$$

We construct D by first defining it on the mutually orthogonal vectors $v_1, \dots, v_m$ as:

$$\begin{aligned} Dv_1 &= Av_1 \\ Dv_2 &= Av_2 - p_{1,2} v_1 \\ &\vdots \\ Dv_m &= Av_m - p_{1,m} v_1 - \cdots - p_{m-1,m} v_{m-1}. \end{aligned} \tag{8.16}$$

Note that $(A-D)v_1 = 0$, whereas $(A-D)v_2 \in \mathrm{span}(v_1)$, $(A-D)v_3 \in \mathrm{span}(v_1, v_2)$ and so forth. This shows that the restriction of $A-D$ to $\mathrm{span}(v_1, \ldots, v_m)$ is nilpotent. We also point out that Equation (8.16) can be rewritten as

$$(A-D)_{n\times n} V_{n\times m} = V_{n\times m} P_{m\times m}\,, \tag{8.17}$$

with $V = (v_1 \cdots v_m)$ the $(n\times m)$-matrix with columns given by the vectors $v_1, \ldots, v_m$.

To complete our construction of D, we let $y_{m+1}, \ldots, y_n \in \mathbb{R}^n$ be mutually orthogonal vectors of norm 1 such that $y_k \perp v_i$ for all $i = 1, \ldots, m$ and $k = m+1, \ldots, n$. We define D on $\mathrm{span}(y_{m+1}, \ldots, y_n)$ by simply setting $Dy_k = cy_k$ for all k and some constant $c > 0$ that will be determined later.

To show that c can be chosen such that D is positive-definite, we let $z \in \mathbb{R}^n$ be any non-zero vector. We write

$$z = Va + Yb$$

where $Y = (y_{m+1} \cdots y_n)$ is the $(n \times (m-n))$-matrix with columns the vectors y_k, and where $a \in \mathbb{R}^m, b \in \mathbb{R}^{n-m}$ express the components of z concerning basis $\{v_1, \ldots, v_m, y_{m+1}, \ldots, y_n\}$. Note that we have

$$\begin{array}{lll} V^T V = I_{m\times m}, & Y^T Y = I_{(n-m)\times(n-m)}, & \\ V^T Y = 0_{m\times(n-m)}, & Y^T V = 0_{(n-m)\times m}, & DY = cY, \end{array} \tag{8.18}$$

by construction. We calculate

$$\begin{aligned} z^T Dz &= (Va+Yb)^T D(Va+Yb) \\ &= a^T V^T DVa + a^T V^T DYb + b^T Y^T DVa + b^T Y^T DYb \\ &= a^T V^T (AV - VP)a + a^T V^T DYb + b^T Y^T (AV - VP)a + b^T Y^T DYb \\ &= a^T (V^T AV - V^T VP)a + a^T V^T cYb + b^T (Y^T AV - Y^T VP)a + b^T Y^T cYb \\ &= a^T (V^T AV - P)a + b^T Y^T AVa + cb^T b, \end{aligned} \tag{8.19}$$

where in the third step we have used Equation (8.17), and where we make use of the Identities (8.18). We see that D is positive-definite if the same holds for the matrix

$$\tilde{D} = \begin{pmatrix} V^T AV - P & 0 \\ Y^T AV & c\,\mathrm{Id} \end{pmatrix}. \tag{8.20}$$

Next, we claim that the $(m\times m)$-matrix $V^T AV - P$ is positive-definite. Indeed, by definition of P we have

$$\begin{aligned} (V^T AV - P) + (V^T AV - P)^T &= V^T (A + A^T)V - (P + P^T) \\ &= \mathrm{diag}(2v_1^T Av_1, \ldots, 2v_m^T Av_m), \end{aligned}$$

which is a diagonal matrix and positive-definite by the Hypothesis (8.13). We may thus apply Lemma (55) to $\tilde{D}$, so that for $c > 0$ sufficiently large $\tilde{D}$ and D are indeed positive-definite.

Conversely, suppose there exists a positive-definite matrix D such that

$$A - D \quad \text{has } m \text{ zero eigenvalues.}$$

We will prove that m mutually orthogonal vectors $v_1, \dots, v_m$ exist satisfying

$$\langle v_i, Av_i \rangle > 0 \text{ for all } i = 1, \dots, m. \tag{8.21}$$

By assumption, we may choose m linearly independent vectors $y_1, \dots, y_m$ such that

$$(A - D)y_1 = 0 \quad \text{and} \quad (A - D)y_i = \iota_i y_{i-1} \text{ for } i = 2, \dots, m,$$

where $\iota_i \in \{0, 1\}$ for all $i > 1$. Next, we apply the Gram-Schmidt orthonormalization process to the vectors y_i. That is, we set

$$\begin{aligned} v_1 &= y_1 \\ v_2 &= y_2 - \alpha_{2,1} \cdot v_1 \\ &\vdots \\ v_i &= y_i - \alpha_{i,1} \cdot v_1 - \alpha_{i,2} \cdot v_2 - \cdots - \alpha_{i,i-1} \cdot v_{i-1}, \end{aligned} \tag{8.22}$$

where each coefficient is given by

$$\alpha_{i,j} = \frac{\langle y_i, v_j \rangle}{\langle v_j, v_j \rangle} \quad \text{for } j < i.$$

It follows that $\langle v_i, v_j \rangle = 0$ whenever $i \neq j$. Moreover, we see from Equation (8.22) that we may write

$$v_i = y_i + \sum_{j<i} \beta_{i,j} y_j \quad \text{and thus} \quad y_i = v_i + \sum_{j<i} \beta'_{i,j} v_j \tag{8.23}$$

for some coefficients $\beta_{i,j}, \beta'_{i,j} \in \mathbb{R}$. We therefore have $(A - D)v_1 = 0$, and for $2 \leq i \leq m$ we find

$$\begin{aligned} (A-D)v_i = (A-D)\left(y_i + \sum_{j<i} \beta_{i,j} y_j\right) &= (A-D)y_i + \sum_{j<i} \beta_{i,j}(A-D)y_j \\ = \iota_i y_{i-1} + \sum_{1<j<i} \beta_{i,j} \iota_j y_{j-1} &= \sum_{j<i} \gamma_{i,j} v_j, \end{aligned} \tag{8.24}$$

for certain $\gamma_{i,j} \in \mathbb{R}$. By orthogonality of the v_i we get

$$\begin{aligned} v_1^T (A-D) v_1 &= 0 \quad \text{and} \\ v_i^T (A-D) v_i &= \sum_{j<i} \gamma_{i,j} v_i^T v_j = 0 \end{aligned}$$

for all $i = 2, \dots, m$. Finally, it follows that

$$v_i^T A v_i = v_i^T D v_i > 0 \quad \text{for all } i = 1, \dots, m,$$

which completes the proof.

■

Remark 58. The proof of Proposition (56) tells us that the remaining eigenvalues of $A-D$ may be assumed to have (large) negative real parts. More precisely, if m mutually orthogonal vectors $v_1,\dots,v_m$ exist such that

$$\langle v_i, Av_i\rangle > 0 \text{ for all } i=1,\dots,m, \tag{8.25}$$

then a positive-definite matrix D is constructed such that the restriction of $A-D$ to $\mathrm{span}(v_1,\dots,v_m)$ is nilpotent. In particular, $A-D$ maps the space $\mathrm{span}(v_1,\dots,v_m)$ into itself. It follows that the remaining eigenvalues of $A-D$ are given by those of the 'other' diagonal block $P_U(A-D)|_U : U \to U$, where U is some complement to $\mathrm{span}(v_1,\dots,v_m)$ and P_U is the projection onto U along $\mathrm{span}(v_1,\dots,v_m)$. If we choose $U = \mathrm{span}(y_{m+1},\dots,y_n)$ as in the proof of Proposition (56), then we see that $P_U(A-D)|_U = P_UA|_U - c\,\mathrm{Id}_Y$. Choosing $c>0$ large enough then ensures that the remaining eigenvalues of $A-D$ are stable.

Remark 59. It follows from the proof of Proposition (56) that $A-D$ can generically be assumed to have a one-dimensional kernel. In other words, whereas the algebraic multiplicity of the eigenvalue 0 is m, its geometric multiplicity is generically equal to 1. To see why, assume $c>0$ is large enough so that $A-D$ has a generalized kernel of dimension precisely m, see Remark (58). From the proof of Proposition (56), we see that the restriction of $A-D$ to its generalized kernel is conjugate to P. It follows that the dimension of the kernel of $A-D$ is equal to 1 if

$$p_{i,i+1} = v_i^T(A+A^T)v_{i+1} \neq 0 \quad \text{ for all } i\in\{1,\dots,m-1\}\,. \tag{8.26}$$

This may be assumed to hold after a perturbation of the form

$$A \mapsto A + \varepsilon_1 v_1 v_2^T + \cdots + \varepsilon_{m-1} v_{m-1} v_m^T\,, \tag{8.27}$$

for some arbitrarily small $\varepsilon_1,\dots,\varepsilon_{m-1}>0$, if necessary. As a result, the matrix $A-D$ has a single Jordan block of size m for the eigenvalue 0.

Example (11) below shows that the Condition (8.13) imposed on A does not exclude Hurwitz matrices. This might seem surprising, as for any eigenvector x corresponding to a real eigenvalue $\lambda<0$ we have $x^TAx = \lambda\|x\|^2 < 0$. Moreover, it holds that any positive-definite matrix D has only eigenvalues with a positive real part, see Lemma (60) below. This result is well-known but included here for completeness.

Lemma 60. Let $D\in\mathbb{M}_n(\mathbb{R})$ be a real positive-definite matrix (though not necessarily symmetric). That is, assume we have $x^TDx>0$ for all non-zero $x\in\mathbb{R}^n$. Then, any eigenvalue of D has a positive real part.

Proof: Let λ be an eigenvalue of D with corresponding eigenvector x. We write $\lambda=\mu+i\nu$ and $x=u+iv$ for their decomposition into real and imaginary parts. On the one hand, we find

$$\bar{x}^TDx = \bar{x}^T\lambda x = \|x\|^2\lambda = \|x\|^2(\mu+i\nu)\,. \tag{8.28}$$

On the other, we have

$$\bar{x}^T Dx = (u-iv)^T D(u+iv) = u^T Du + v^T Dv + i(u^T Dv - v^T Du)\,. \tag{8.29}$$

Comparing the real parts of Equations (8.28) and (8.29), we conclude that

$$\|x\|^2 \mu = u^T Du + v^T Dv > 0\,, \tag{8.30}$$

where we use that u and v cannot both be zero. Hence, we see that indeed $\mu > 0$.

■

Example 11. We return to the (4×4) Hurwitz-matrix from Example (9):

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 16.94 \\ 1 & -4.24 & -4.24 & -17.94 \end{pmatrix}.$$

The three canonical vectors $e_1 = (1,0,0,0)^T, e_2 = (0,1,0,0)^T$ and $e_3 = (0,0,1,0)^T$ all satisfy $\langle e_i, Ae_i\rangle > 0$ for $i = 1,2,3$. We may determine the upper-diagonal (3×3)-matrix P from the proof of Proposition (56) by calculating

$$\begin{aligned} p_{1,2} &= e_1^T(A+A^T)e_2 \\ p_{1,3} &= e_1^T(A+A^T)e_3 \\ p_{2,3} &= e_2^T(A+A^T)e_3 \end{aligned}$$

where

$$A + A^T = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & -4.24 \\ 0 & 0 & 2 & 12.7 \\ 1 & -4.24 & 12.7 & -35.88 \end{pmatrix}.$$

It follows that $p_{1,2} = p_{1,3} = p_{2,3} = 0$, which implies we have $P = 0$. As in the proof of Proposition (56), we first define $D \in \mathbb{M}_4(\mathbb{R})$ on $\operatorname{span}(e_1,e_2,e_3) = \{x \in \mathbb{R}^4 \mid x_4 = 0\}$ by setting:

$$\begin{aligned} De_1 &= Ae_1 \\ De_2 &= Ae_2 - p_{1,2}e_1 = Ae_2 \\ De_3 &= Ae_3 - p_{1,3}e_1 - p_{2,3}e_2 = Ae_3. \end{aligned}$$

Hence, D agrees with A in the first three columns. To complete our construction of D, we have to choose a non-zero vector u such that $u \perp e_i$ for $i = 1,2,3$, and set $Du = cu$ for some $c > 0$. To this end, we set $u = e_4$, so that D becomes:

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & -4.24 & -4.24 & c \end{pmatrix}.$$

It follows that

$$A - D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16.94 \\ 0 & 0 & 0 & -17.94 - c \end{pmatrix}$$

which has a zero eigenvalue with geometric multiplicity 3 and a negative eigenvalue $(-17.94 - c)$ equal to its trace. Moreover, by the Lemma (55), D is positive-definite for large enough $c > 0$. Indeed, in this case, we numerically found that for all $c \geq 9.24$ is enough.

Example 12. The matrix

$$A = \begin{pmatrix} -6 & 2 & 1 & -3 \\ 2 & -8 & -1 & -2 \\ 1 & -1 & -3.4 & 0 \\ -3 & -2 & 0 & -6 \end{pmatrix} \tag{8.31}$$

is Hurwitz. However, it is symmetric and therefore negative-definite. Thus, there are no vectors $x \in \mathbb{R}^4$ such that $\langle x, Ax \rangle > 0$.

To control a bifurcation in the system of Equation (8.1), we need to rule out additional eigenvalues laying on the imaginary axis. Recall that the eigenvalues of the Linearization (8.2) are given by those of $A - \alpha\lambda_p D$ with $\lambda_p \geq 0$ an eigenvalue of L_G. Lemma (61) below shows that generically only one of the matrices $A - \alpha\lambda_p D$ is non-hyperbolic. In what follows we denote by $\|\bullet\|$ the operator norm of a matrix, induced by the Euclidean norm on $\mathbb{R}^n$.

Lemma 61. Let $A, D \in \mathbb{M}_n(\mathbb{R})$ be two given matrices with D positive-definite, and let $\alpha^* \in \mathbb{R}$ be a positive scalar. We furthermore assume $\{\lambda_1, \ldots, \lambda_K\}$ is a set of real numbers and consider the matrices $A - \alpha^*\lambda_i D$ for $i \in \{1, \ldots, K\}$. Given any $\varepsilon > 0$, there exist a matrix $\tilde{A}$ and a positive-definite matrix $\tilde{D}$ such that $\|A - \tilde{A}\|, \|D - \tilde{D}\| < \varepsilon$ and $\tilde{A} - \alpha^*\lambda_K\tilde{D} = A - \alpha^*\lambda_K D$. Moreover, for $i \in \{1, \ldots, K-1\}$ the matrix $\tilde{A} - \alpha^*\lambda_i\tilde{D}$ has a purely hyperbolic spectrum (i.e. no eigenvalues on the imaginary axis).

Remark 62. From $\|A - \tilde{A}\|, \|D - \tilde{D}\| < \varepsilon$ we get

$$\|(A - \alpha^*\lambda_i D) - (\tilde{A} - \alpha^*\lambda_i\tilde{D})\| \leq \|A - \tilde{A}\| + \alpha^*|\lambda_i|\|D - \tilde{D}\| < \varepsilon(1 + \alpha^*|\lambda_i|),$$

so that we may arrange for $\tilde{A} - \alpha^* \lambda_i \tilde{D}$ to be arbitrarily close to the original $A - \alpha^* \lambda_i D$ for all i. Moreover, if A is hyperbolic then for ε small enough so is $\tilde{A}$, with the same number of stable and unstable eigenvalues. In particular, $\tilde{A}$ may be assumed Hurwitz if A is.

Proof: Let $\delta \neq 0$ be given and set

$$\tilde{A}_\delta := A + \delta \,\mathrm{Id} \tag{8.32}$$

$$\tilde{D}_\delta := D + \frac{\delta}{\alpha^* \lambda_K} \,\mathrm{Id}\,. \tag{8.33}$$

Note that the symmetric parts of $\tilde{D}_\delta$ and D differ by $\frac{\delta}{\alpha^* \lambda_K}\,\mathrm{Id}$ as well, so that $\tilde{D}_\delta$ remains positive-definite for $|\delta|$ small enough. It is also clear that

$$\lim_{\delta \to 0} \|A - \tilde{A}_\delta\| = \lim_{\delta \to 0} \|D - \tilde{D}_\delta\| = 0\,.$$

A direct calculation shows that

$$\begin{aligned} \tilde{A}_\delta - \alpha^* \lambda_i \tilde{D}_\delta &= A + \delta\,\mathrm{Id} - \alpha^* \lambda_i \left(D + \frac{\delta}{\alpha^* \lambda_K} \,\mathrm{Id} \right) \\ &= A + \delta\,\mathrm{Id} - \alpha^* \lambda_i D - \frac{\delta \lambda_i}{\lambda_K} \,\mathrm{Id} \\ &= (A - \alpha^* \lambda_i D) + \left(1 - \frac{\lambda_i}{\lambda_K}\right) \delta\,\mathrm{Id}\,, \end{aligned} \tag{8.34}$$

for all $i \in \{1, \ldots, K\}$. It follows that we have $\tilde{A}_\delta - \alpha^* \lambda_K \tilde{D}_\delta = A - \alpha^* \lambda_K D$. For $i \neq K$ we see that $\tilde{A}_\delta - \alpha^* \lambda_i \tilde{D}_\delta$ differs from $A - \alpha^* \lambda_i D$ by a non-zero scalar multiple of the identity. It follows that for $\delta \neq 0$ small enough all the matrices $\tilde{A}_\delta - \alpha^* \lambda_i \tilde{D}_\delta$ for $i \in \{1, \ldots, K-1\}$ have their eigenvalues away from the imaginary axis. Setting $\tilde{A} := \tilde{A}_\delta$ and $\tilde{D} := \tilde{D}_\delta$ with $\delta = \delta(\varepsilon)$ small enough then finishes the proof.

■

Let us now assume A, D, and α^* are given such that for a particular eigenvalue $\lambda > 0$ of L the matrix $A - \alpha^* \lambda D$ has an m-dimensional center subspace. We moreover assume λ is simple and, motivated by Lemma (61), that the matrices $A - \alpha^* \kappa D$ are hyperbolic for any other eigenvalue κ of L. It follows that the linearization

$$\mathrm{Id}_{\mathbb{R}^N} \otimes A - \alpha^* L \otimes D : \mathbb{R}^N \otimes \mathbb{R}^n \to \mathbb{R}^N \otimes \mathbb{R}^n \tag{8.35}$$

of Equation (8.2) has an m-dimensional center subspace as well.

In what follows, we write $\hat{I}_s$ for the indices of all remaining eigenvalues of L except the index s. In other words, writing $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ for the eigenvalues of L, we have $\lambda = \lambda_s$ for some $s \in \{2, \ldots, N\}$ and we set $\hat{I}_s = \{1, \ldots, N\} \setminus \{s\}$. We will likewise fix an orthonormal set of eigenvectors $v^1, \ldots, v^N$ for L and simply write $v = v^s$ for the eigenvector corresponding to our fixed eigenvalue $\lambda = \lambda_s$. Arguably the most natural situation is given by $s = N$, corresponding to the situation where α is increased until the eigenvalues of $A - \alpha\lambda_N D$ first hit the imaginary axis for $\alpha = \alpha^*$. However, we will not need this assumption here.

Next, given a vector $u \in \mathbb{R}^N$ we denote by $\phi_u : \mathbb{R}^N \to \mathbb{R}^N$ the linear map defined by

$$\phi_u(w) = \langle u, w \rangle u \,. \tag{8.36}$$

Note that ϕ_u is a projection if $\|u\| = 1$.

Finally, we write $E^c, E^h \subseteq \mathbb{R}^n$ for the center- and hyperbolic subspaces of $A - \alpha^*\lambda D$, respectively. It follows that

$$\mathbb{R}^n = E^c \oplus E^h \,, \tag{8.37}$$

and we denote the projections onto the first and second components by π^c and $\pi^h = \mathrm{Id}_n - \pi^c$, respectively. Likewise, we denote the center and hyperbolic subspaces of the Map (8.35) by $\mathscr{E}^c, \mathscr{E}^h \subseteq \mathbb{R}^N \otimes \mathbb{R}^n$. Their projections are denoted by Π^c and Π^h. The following lemma establishes some important relations between the aforementioned maps and spaces.

Lemma 63. The spaces $\mathscr{E}^c$ and E^c are related by

$$\mathscr{E}^c = v \otimes E^c \,, \tag{8.38}$$

and we have

$$\mathscr{E}^h = (v \otimes E^h) \bigoplus_{i \in \hat{I}_s} (v^i \otimes \mathbb{R}^n) \,. \tag{8.39}$$

Moreover, it holds that

$$\Pi^c = \phi_v \otimes \pi^c \,. \tag{8.40}$$

Proof: The Identities (8.38) and (8.39) follow directly from the fact that the linear Map (8.35) sends a vector $v^i \otimes x$ to $v^i \otimes (A - \alpha^*\lambda_i D)(x)$ for all $i \in \{1, \ldots, N\}$ and $x \in \mathbb{R}^n$. To show that Π^c is indeed given by $\phi_v \otimes \pi^c$, we have to show that the latter vanishes on $\mathscr{E}^h$ and restricts to the identity on $\mathscr{E}^c$. To this end, note that for all $i \in \hat{I}_s$ and $x \in \mathbb{R}^n$ we have

$$(\phi_v \otimes \pi^c)(v^i \otimes x) = \phi_v(v^i) \otimes \pi^c(x) = \langle v, v^i \rangle v \otimes \pi^c(x) = 0 \,. \tag{8.41}$$

Given $x_h \in E^h$ and $x_c \in E^c$, we find

$$\begin{aligned} (\phi_v \otimes \pi^c)(v \otimes x_h) &= \phi_v(v) \otimes \pi^c(x_h) = 0 \quad \text{and} \\ (\phi_v \otimes \pi^c)(v \otimes x_c) &= \phi_v(v) \otimes \pi^c(x_c) = \langle v, v \rangle v \otimes x_c = v \otimes x_c \,, \end{aligned} \tag{8.42}$$

so that indeed $(\phi_v \otimes \pi^c)|_{\mathscr{E}^h} = 0$ and $(\phi_v \otimes \pi^c)|_{\mathscr{E}^c} = \mathrm{Id}_{\mathscr{E}^c}$. This completes the proof.

■

8.1.2 Center manifold reduction

In this subsection, we investigate the dynamics of a center manifold of the system

$$\dot{X} = F(X;\varepsilon) - \alpha^*(L\otimes D)X\,, \tag{8.43}$$

which will lead to a proof of Theorem (4). As before, we assume that for one eigenvalue λ of L the corresponding matrix $A-\alpha^*\lambda D$ has a non-trivial center subspace of dimension m, whereas for any other eigenvalue μ the matrix $A-\alpha^*\mu D$ is hyperbolic. We moreover assume λ is non-zero and simple.

Recall that Center Manifold Theorem predicts a locally defined map $\Psi : \mathscr{E}^c\times\Omega\to\mathscr{E}^h$ whose graph M^c is invariant for the system of Equation (8.43) and locally contains all bounded solutions. The map Ψ moreover satisfies $\Psi(0;0)=0$ and $D\Psi(0;0)=0$. In fact, as we assume $F(0;\varepsilon)=0$ for all $\varepsilon\in\Omega$, it follows that $(0;\varepsilon)\in M^c$, as these are bounded solutions. This shows that $\Psi(0;\varepsilon)=0$ for all $\varepsilon\in\Omega$. If F is sufficiently smooth (i.e. C^k for some positive finite k), then we may assume Ψ is as well.

In light of Lemma (63), we may write

$$\Psi(v\otimes x_c;\varepsilon) = v\otimes\psi(x_c;\varepsilon) + \sum_{i\in\hat{I}_s} v^i\otimes\psi_i(x_c;\varepsilon)\,, \tag{8.44}$$

for certain maps $\psi : E^c\times\Omega\to E^h$ and $\psi_i : E^c\times\Omega\to\mathbb{R}^n$. We then likewise have $\psi(0;\varepsilon)=0, D\psi(0;0)=0$ and $\psi_i(0;\varepsilon)=0, D\psi_i(0;0)=0$ for all $\varepsilon\in\Omega$ and $i\in\hat{I}_s$.

The dynamics on the center manifold M^c is conjugate to that of a vector field on $\mathscr{E}^c\times\Omega$ given by

$$\tilde{R}(X_c;\varepsilon) = \Pi^c G(X_c,\Psi(X_c;\varepsilon);\varepsilon)\,, \tag{8.45}$$

where we write

$$G(X_c,X_h;\varepsilon) = F(X_c+X_h;\varepsilon) - \alpha^*(L_G\otimes D)(X_c+X_h) \tag{8.46}$$

for the vector field on the right hand side of Equation (8.43), with $X_c\in\mathscr{E}^c$, $X_h\in\mathscr{E}^h$ and $\varepsilon\in\Omega$. We further conjugate to a vector field R on $E^c\times\Omega$ by setting

$$v\otimes R(x_c;\varepsilon) = \tilde{R}(v\otimes x_c;\varepsilon)\,. \tag{8.47}$$

In order to describe R, we first introduce some useful notation. Given $X\in\mathbb{R}^N\otimes\mathbb{R}^n$, we may write

$$X = \sum_{p=1}^{N} e^p\otimes X_p \tag{8.48}$$

with $e^1,\dots,e^N$ the standard basis of $\mathbb{R}^N$ and for some unique vectors $X_p\in\mathbb{R}^n$. In general, given $p\in\{1,\dots,N\}$ we will denote by $X_p\in\mathbb{R}^n$ the pth component of X as in the Decomposition (8.48).

Using this notation, we have the following result.

Proposition 64. Denote by $h : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ the nonlinear part of f. That is, we have $f(x;\varepsilon) = Ax + h(x;\varepsilon)$. The reduced vector field R is given explicitly by

$$R(x_c;\varepsilon) = (A - \alpha^*\lambda D)x_c + \sum_{p=1}^{N} v_p \pi^c h(v_p x_c + \Psi(x_c \otimes v;\varepsilon)_p;\varepsilon)\,, \tag{8.49}$$

for $x_c \in E^c$ and $\varepsilon \in \Omega$.

Proof: We write $G(X;\varepsilon) = G(X_c, X_h;\varepsilon) = LX + H(X;\varepsilon)$ with $L = D_X G(0;0)$ and where H denotes higher order terms. It follows that

$$\Pi^c G(X_c, \Psi(X_c;\varepsilon);\varepsilon) = \Pi^c L(X_c + \Psi(X_c;\varepsilon)) + \Pi^c H(X_c, \Psi(X_c;\varepsilon);\varepsilon)\,. \tag{8.50}$$

We start by focusing on the first term. As L sends $\mathscr{E}^c$ to $\mathscr{E}^c$ and $\mathscr{E}^h$ to $\mathscr{E}^h$, we conclude that $\Pi^c L = L\Pi^c$. We therefore find

$$\Pi^c L(X_c + \Psi(X_c;\varepsilon)) = L\Pi^c(X_c + \Psi(X_c;\varepsilon)) = LX_c\,. \tag{8.51}$$

Writing $X_c = v \otimes x_c$ and using Expression (8.2) for L, we conclude that the linear part of $\tilde{R}$ is given by

$$L(v \otimes x_c) = v \otimes (A - \alpha^*\lambda D)x_c\,. \tag{8.52}$$

We next focus on the second term in Equation (8.50). Note that we have

$$H(X;\varepsilon)_p = h(X_p;\varepsilon) \text{ for all } p \in \{1,\dots,N\}\,. \tag{8.53}$$

Now, by Lemma (63) it follows that we may write

$$\begin{aligned}
\Pi^c(X) &= \sum_{p=1}^{N} \Pi^c(e^p \otimes X_p) = \sum_{p=1}^{N} \phi_v(e^p) \otimes \pi^c(X_p) \\
&= \sum_{p=1}^{N} \langle v, e^p \rangle v \otimes \pi^c(X_p) = \sum_{p=1}^{N} v_p (v \otimes \pi^c(X_p)) \\
&= v \otimes \sum_{p=1}^{N} v_p \pi^c(X_p) \text{ for all } X \in \mathbb{R}^N \otimes \mathbb{R}^n.
\end{aligned} \tag{8.54}$$

We therefore find

$$\Pi^c(H(X_c, \Psi(X_c;\varepsilon);\varepsilon)) = v \otimes \sum_{p=1}^{N} v_p \pi^c h([X_c + \Psi(X_c;\varepsilon)]_p;\varepsilon)\,. \tag{8.55}$$

Next, we have $(X_c)_p = (v \otimes x_c)_p = v_p x_c$, so that we find

$$\Pi^c(H(X_c, \Psi(X_c;\varepsilon);\varepsilon)) = v \otimes \sum_{p=1}^{N} v_p \pi^c h(v_p x_c + \Psi(X_c;\varepsilon)_p;\varepsilon)\,. \tag{8.56}$$

Combining Equations (8.52) and (8.56), we arrive at

$$\begin{aligned}\tilde{R}(X_c;\varepsilon) &= \Pi^c(G(X_c,\Psi(X_c;\varepsilon);\varepsilon)) \\ &= v\otimes(A-\alpha^*\lambda D)x_c + v\otimes\sum_{p=1}^{N} v_p\pi^c h(v_p x_c+\Psi(X_c;\varepsilon)_p;\varepsilon)\,. \end{aligned} \tag{8.57}$$

Finally, from Equation (8.47) we get

$$R(x_c;\varepsilon) = (A-\alpha^*\lambda D)x_c + \sum_{p=1}^{N} v_p\pi^c h(v_p x_c+\Psi(x_c\otimes v;\varepsilon)_p;\varepsilon)\,, \tag{8.58}$$

which completes the proof.

■

To further investigate the Taylor expansion of $R(x_c;\varepsilon)$, we need to know more about how the coefficients of $\Psi:\mathscr{E}^c\times\Omega\to\mathscr{E}^h$ depend on those of F.

To this end, let us consider for a moment the general situation where G is some vector field on $\mathbb{R}^n$ satisfying $G(0)=0$. We write $L=DG(0)$, so that we have $G(X)=LX+H(X)$ for some $H:\mathbb{R}^n\to\mathbb{R}^n$ satisfying $H(0)=0$, $DH(0)=0$. We furthermore let $\hat{\mathscr{E}}^c$ and $\hat{\mathscr{E}}^h$ denote the center- and hyperbolic subspaces of L, respectively, and write Π^c,Π^h for the corresponding projections. Suppose $\Psi:\hat{\mathscr{E}}^c\to\hat{\mathscr{E}}^h$ is a locally defined map whose graph is a center manifold M^c for the system $\dot{X}=G(X)$. Recall that we have $\Psi(0)=0$ and $D\Psi(0)=0$. Moreover, as M^c is a flow-invariant manifold, we see that $G|_{M^c}$ takes values in the tangent bundle of M^c. This can be used to iteratively solve for the higher order coefficients of an expansion of Ψ around 0.

More precisely, the tangent space at $X_c+\Psi(X_c)\in M^c$ is given by all vectors of the form $(V_c,D\Psi(X_c)V_c)\in\hat{\mathscr{E}}^c\oplus\hat{\mathscr{E}}^h$, with $V_c\in\hat{\mathscr{E}}^c$. Invariance under the flow of G then translates to the identity

$$\Pi^h L\Psi(X_c)+\Pi^h H(X_c,\Psi(X_c)) = D\Psi(X_c)(\Pi^c LX_c+\Pi^c H(X_c,\Psi(X_c)))\,, \tag{8.59}$$

for X_c in some open neighborhood of the origin in $\hat{\mathscr{E}}^c$. Equation (8.59) can be used to show that $D\Psi(0)=0$. More generally, using Ψ^ρ to denote the terms of order $\rho\geq 2$ in the Taylor expansion of Ψ around the origin, Equation (8.59) is readily seen to imply

$$\Pi^h L\Psi^\rho(X_c) - D\Psi^\rho(X_c)\Pi^c LX_c = P_\rho(X_c)\,, \tag{8.60}$$

for some homogeneous polynomial P_ρ of order ρ. Moreover, P_ρ depends only on $\Psi^2(X_c)\ldots,\Psi^{\rho-1}(X_c)$ and on the Taylor expansion of H up to order ρ. It can be shown that for fixed L and P_ρ, Equation (8.60) has a unique solution Ψ^ρ in the form of a homogeneous polynomial of order ρ, see (WIMMER, 1979). As a result, we get the following important observation:

Lemma 65. We may iteratively solve for the terms Ψ^ρ using Expression (8.60). Moreover, for fixed linearity L, the terms of order ρ and less of Ψ are fully determined by the terms of order ρ and less of H.

We return to our main setting where $R(x_c;\varepsilon)$ is the reduced vector field of the System (8.43) as described in Proposition (64). Note that the presence of a parameter ε means that the center subspace $\hat{\mathscr{E}}^c$ in the observations for general vector fields above is now given by $\mathscr{E}^c \times \Omega$.

Lemma 66. Let $\rho > 1$ be given, and suppose the vector $v \in \mathbb{R}^N$ satisfies

$$\sum_{p=1}^{N} v_p^\ell \neq 0, \quad \forall\, \ell = 2, \dots, \rho+1. \tag{8.61}$$

Then the reduced vector field $R(x_c;\varepsilon)$ as described in Proposition (64) can have any Taylor expansion around 0 of order 2 to ρ, subject to $R(0;\varepsilon) = 0$ if no conditions are put on f other than $f(0;\varepsilon) = 0$ and sufficient smoothness.

Proof: From Proposition (64) we know that

$$R(x_c;\varepsilon) = (A - \alpha^*\lambda D)x_c + \sum_{p=1}^{N} v_p \pi^c h(v_p x_c + \Psi(x_c \otimes v;\varepsilon)_p;\varepsilon). \tag{8.62}$$

As we have $\Psi(0;\varepsilon) = 0$ and $h(0;\varepsilon) = 0$, we conclude that likewise $R(0;\varepsilon) = 0$ for all $\varepsilon \in \Omega$. In particular, we see that $D_\varepsilon R(0;0) = 0$, whereas Equation (8.62) tells us that $D_{x_c} R(0;0) = (A - \alpha^*\lambda D)|_{E^c}$.

As a warm-up, we start by investigating the second-order terms of R. To this end, we write

$$h(x;\varepsilon) = Q_{1,1}(x;\varepsilon) + Q_{2,0}(x) + \mathscr{O}(|(x,\varepsilon)|^3), \tag{8.63}$$

where $Q_{1,1}$ is linear in both components and $Q_{2,0}$ is a quadratic form. It follows that

$$\begin{aligned} & h(v_p x_c + \Psi(x_c \otimes v;\varepsilon)_p;\varepsilon) \\ = & Q_{1,1}(v_p x_c + \Psi(x_c \otimes v;\varepsilon)_p;\varepsilon) + Q_{2,0}(v_p x_c + \Psi(x_c \otimes v;\varepsilon)_p) + \mathscr{O}(|(x_c,\varepsilon)|^3) \\ = & Q_{1,1}(v_p x_c;\varepsilon) + Q_{2,0}(v_p x_c) + \mathscr{O}(|(x_c,\varepsilon)|^3), \end{aligned} \tag{8.64}$$

where we use that $\Psi(x_c \otimes v;\varepsilon)$ has no constant or linear terms in $(x_c;\varepsilon)$. From Equation (8.64) we obtain

$$\begin{aligned} & \sum_{p=1}^{N} v_p \pi^c h(v_p x_c + \Psi(x_c \otimes v;\varepsilon)_p;\varepsilon) \\ = & \pi^c \sum_{p=1}^{N} v_p Q_{1,1}(v_p x_c;\varepsilon) + \pi^c \sum_{p=1}^{N} v_p Q_{2,0}(v_p x_c) + \mathscr{O}(|(x_c,\varepsilon)|^3) \\ = & \pi^c \sum_{p=1}^{N} v_p^2 Q_{1,1}(x_c;\varepsilon) + \pi^c \sum_{p=1}^{N} v_p^3 Q_{2,0}(x_c) + \mathscr{O}(|(x_c,\varepsilon)|^3). \end{aligned} \tag{8.65}$$

As we assume $\sum_{p=1}^{N} v_p^2, \sum_{p=1}^{N} v_p^3 \neq 0$, we see that the second order Taylor coefficients of $R(x_c;\varepsilon)$ can be chosen freely (except for the $\mathscr{O}(|\varepsilon|^2)$ term).

Now suppose we are given a polynomial map $P : E^c \times \Omega \to E^c$ of degree ρ satisfying $DP(0) = ((A - \alpha^* \lambda D)|_{E^c}; 0)$ and $P(0;\varepsilon) = 0$ for all ε. We will prove by induction that we may choose the terms in the Taylor expansion of h up to order ρ in the variables x_c and ε such that the Taylor expansion up to order ρ of R agrees with P. To this end, suppose some choice of h gives agreement between P and the Taylor expansion of R up to order $2 \leq k < \rho$. By the foregoing, this can be arranged for $k = 2$.

We start by remarking that a change to h that does not influence its Taylor expansion up to order k does not change the Taylor expansion of Ψ up to order k. This follows directly from Lemma (65). As a result, such a change does not influence the Taylor expansion of R up to order k as well. We write

$$\tilde{h}(x_c;\varepsilon) = h(x_c;\varepsilon) + \sum_{i=1}^{k+1} Q_{i,k+1-i}(x_c;\varepsilon) \tag{8.66}$$

for an order $k+1$ change to h, where each component of $Q_{i,j} : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is a homogeneous polynomial of degree i in x_c and degree j in ε. The $(k+1)$-order terms of R in $(x_c;\varepsilon)$ are given by the $(k+1)$-order terms of

$$\begin{aligned}
&\sum_{p=1}^{N} v_p \pi^c \tilde{h}(v_p x_c + \Psi(x_c \otimes v;\varepsilon)_p;\varepsilon) \qquad (8.67)\\
&= \sum_{p=1}^{N} v_p \pi^c \left(h(v_p x_c + \Psi(x_c \otimes v;\varepsilon)_p;\varepsilon) + \sum_{i=1}^{k+1} Q_{i,k+1-i}(v_p x_c + \Psi(x_c \otimes v;\varepsilon)_p;\varepsilon) \right).
\end{aligned}$$

As both h and Ψ have no constant and linear terms, we see that the $(k+1)$-order terms of R are also given by those of

$$\sum_{p=1}^{N} v_p \pi^c \left(h(v_p x_c + \Psi^k(x_c \otimes v;\varepsilon)_p;\varepsilon) + \sum_{i=1}^{k+1} Q_{i,k+1-i}(v_p x_c;\varepsilon) \right), \tag{8.68}$$

where Ψ^k denotes the terms of Ψ up to order k. As we have previously argued, Ψ^k is independent of the additional terms $Q_{i,k+1-i}$. Hence, we may write the order $k+1$ terms in Expression (8.68) as

$$\begin{aligned}
W(x_c;\varepsilon) + \sum_{p=1}^{N} v_p \pi^c \sum_{i=1}^{k+1} Q_{i,k+1-i}(v_p x_c;\varepsilon) &= W(x_c;\varepsilon) + \sum_{p=1}^{N} v_p \pi^c \sum_{i=1}^{k+1} v_p^i Q_{i,k+1-i}(x_c;\varepsilon) \qquad (8.69)\\
&= W(x_c;\varepsilon) + \sum_{i=1}^{k+1} \left(\sum_{p=1}^{N} v_p^{i+1} \right) \pi^c Q_{i,k+1-i}(x_c;\varepsilon)
\end{aligned}$$

where $W(x_c;\varepsilon)$ denotes the order $k+1$ terms of

$$\sum_{p=1}^{N} v_p \pi^c h(v_p x_c + \Psi^k(x_c \otimes v;\varepsilon)_p;\varepsilon).$$

As $\sum_{p=1}^{N} v_p^j \neq 0$ for all $j \in \{2,\ldots,\rho+1\}$, we see that the order $k+1$ terms of R may be freely chosen. In other words, we may arrange for the Taylor expansion up to order $k+1$ of R to agree with that of P up to order $k+1$. This completes the proof by induction.

■

Next, we present the proof of the Main Theorem of this thesis, Theorem (4).

Proof: If there exist m mutually orthogonal vectors $v_1,\ldots,v_m$ such that $\langle v_i, Av_i\rangle > 0$, then Proposition (56) guarantees the existence of a positive-definite matrix D such that $A-D$ has a center subspace of dimension m or higher. Given any non-zero eigenvalue λ of L_G, we may set $\alpha^* = 1/\lambda$ and conclude that $A-\alpha^*\lambda D$ has a center subspace of dimension at least m. As the eigenvalues of the linearization of Equation (1.11) around the origin are given by those of the maps $A-\alpha\lambda D$ for λ an eigenvalue of L_G, we see that the System (1.11) has a local parameterized center manifold of dimension at least m for some choices of D and $\alpha = \alpha^*$.

If the graph G of the network is ρ-versatile for the pair (μ, ν), then a choice of D as above together with $\alpha^* = 1/\mu$ guarantees $A-\alpha^*\mu D$ has a center subspace of dimension at least m. By Remark (58) we may assume this center subspace to be of dimension precisely m. Moreover, by Lemma (61) we may assume $A-\alpha^*\lambda D$ to have a hyperbolic spectrum for all other eigenvalues $\lambda \neq \mu$ of L_G, after an arbitrarily small perturbation to A and D if necessary. It follows that System (1.11) has a local parameterized center manifold of dimension exactly m. We argue in the proof of Lemma (66) that $R(0;\varepsilon) = 0$ for all ε, and that $DR(0;0) = ((A-\alpha^*\mu D)|_{E^c};0)$. This latter map is nilpotent by the statement of Proposition (56). Finally, Lemma (66) shows that any Taylor expansion can be realized for R up to order ρ, subject to the aforementioned restrictions.

■

8.2 Stability of the center manifold

In this section, we investigate the stability of the center manifold of the full network system. We know that the spectrum of the linearization of this system is fully understood if we know the spectrum of the matrices $A-\alpha^*\lambda D$ for λ an eigenvalue of L_G. Proposition (56) gives conditions on A that guarantee the existence of a positive-definite matrix D such that $A-\alpha^*\mu D$ has an m-dimensional generalized kernel for some fixed eigenvalue $\mu > 0$ of L_G. Moreover, by Remark (58) we may assume that the non-zero eigenvalues of $A-\alpha^*\mu D$ have negative real parts. Lemma (61) in turn shows that after a small perturbation of A and D if necessary we may assume $A-\alpha^*\lambda D$ to have a hyperbolic spectrum for all remaining eigenvalues $\lambda \neq \mu$ of L_G. Thus, if the matrices

$A - \alpha^* \lambda D$ for these remaining eigenvalues are all Hurwitz, then the m-dimensional center manifold of Theorem (4) may be assumed stable.

This seems most reasonable to expect when μ is the (simple) largest eigenvalue of L_G, as the matrices, $A - \alpha^* \lambda D$ for the remaining eigenvalues of L_G then "lies between" the Hurwitz matrix A and the non-invertible matrix $A - \alpha^* \mu D$. More precisely, suppose D is scaled such that $A - D = A - \alpha^* \mu D$. If we let α vary from 0 to $\alpha^* = 1/\mu$, then for each eigenvalue λ of L_G, the matrix $A - \alpha \lambda D$ is of the form $A - \beta D$ for some β in $[0,1]$. Let us, therefore, denote by $\beta \mapsto \gamma_i(\beta)$ for $i \in \{1, \dots, n\}$ a number of curves through the complex plane capturing the eigenvalues of $A - \beta D$. As α varies from 0 to $\alpha^* = 1/\mu$, the eigenvalues of $A - \alpha \lambda D$ traverse γ_i, with the "front runners" given by those of $A - \alpha \mu D$. In contrast, for $\lambda_1 = 0$ the eigenvalues of $A - \alpha \lambda_1 D$ of course remain at $\gamma_i(0)$. When $\alpha = \alpha^*$ is reached, the eigenvalues of $A - \alpha^* \lambda D$ end up in different places on the curves γ_i. Hence, if the situation is as in Figure (19), where each γ_i hits the imaginary axis only for $\beta = 1$, or not at all, then we are guaranteed that each of the matrices $A - \alpha^* \lambda D$ is Hurwitz for $\lambda \neq \mu$. Hence, the center manifold is then stable.

Of course $\beta = 1$ may not be the first value for which a curve γ_i hits the imaginary axis, see Figure (20). Note that, if the matrix $A - \beta D$ indeed has a non-trivial center subspace for some value $\beta \in (0,1)$, then a bifurcation is expected to occur as α is increased before it hits α^*.

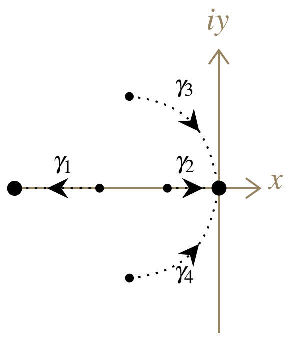


Figure 19 – Sketch of a situation where the m dimensional center manifold of Theorem (4) may be assumed stable. Depicted are the eigenvalues of $A - \beta D$ as β is varied. Small dots denote starting points where $\beta = 0$, dashed paths form a conjugate pair of complex eigenvalues and are real eigenvalues. Arrows indicate the eigenvalues evolving as β increases to 1. Big dots denote the endpoints where $\beta = 1$. Three of them go to the origin, whereas one moves away from it. None of them touches the imaginary axis before $\beta = 1$.

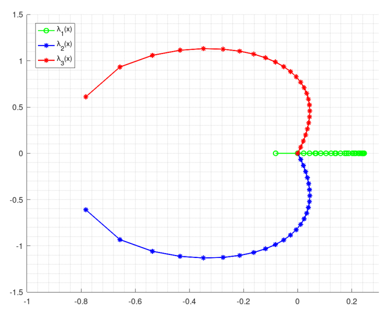


Figure 20 – Numerically computed behaviour of three of the four eigenvalues of the family of matrices from Example (13). As β increases from 0 to 1, three eigenvalues move to the origin, whereas a fourth stays to the left of the imaginary axis. For some value of $\beta \in (0,1)$, two complex conjugate eigenvalues already cross the imaginary axis away from the origin. Likewise, a real eigenvalue crosses the origin for some $\beta \in (0,1)$. Data were simulated using Octave.

The next example shows that some of the eigenvalues of $A - \beta D$ might cross the imaginary axis before a high-dimensional kernel emerges at $\beta = 1$, see Figure (20).

Example 13. We consider the matrices A and D constructed in Example (11). Here we choose $c = 21$ in order to guarantee that D is a positive-definite matrix. We, therefore, have a family of matrices:

$$A - \beta D = \begin{pmatrix} 1-\beta & 1-\beta & 0 & 0 \\ \beta-1 & 1-\beta & 1-\beta & 0 \\ 0 & \beta-1 & 1-\beta & 16.94 \\ 1-\beta & 4.24(\beta-1) & 4.24(\beta-1) & -21\beta-17.94 \end{pmatrix}, \tag{8.70}$$

parameterized by the real number $\beta \in [0,1]$. We are interested in the eigenvalue behaviour as β is varied. We know that for $\beta = 0$ we have the Hurwitz matrix A so that all eigenvalues have negative real parts. We would like to know if the family $A - \beta D$ has all eigenvalues with negative real parts for all $\beta \in (0,1)$. However, if $\beta = \frac{1}{2}$ we have 3 eigenvalues with positive real part. By continuity of the eigenvalues, it means that each of these crossed the imaginary axis for some $\beta < \frac{1}{2}$. Only after this, for $\beta = 1$, do we have the bifurcation studied in the previous chapter, due to the appearance of a triple zero eigenvalue. Figure (20) shows the numerically computed behaviour of these three eigenvalue branches.

If we are in the situation of Figure (20), then the center manifold can still be stable. This occurs when the largest eigenvalue μ of L_G is significantly larger than all other eigenvalues. In that case, we have

$$A - \alpha^* \lambda D = A - \lambda/\mu D \approx A$$

for all eigenvalues λ of L_G unequal to μ. For small enough values of λ/μ the matrix $A - \lambda/\mu D$ is therefore still Hurwitz. As it turns out, this can be achieved in the situation explored in Subsection (2.1.2). More precisely, we have the following result.

Proposition 67. Let $r < D$ be positive integers and suppose G is a connected graph with at least two nodes, consisting of one node of degree D and with all other nodes of degree at most r. Let μ and κ denote the largest and second-largest eigenvalue of the Laplacian L_G, respectively. Then the value κ/μ goes to zero as D/r goes to infinity, uniformly in all graphs G satisfying the above conditions.

The proof of Proposition (67) uses a result about the effects of adding an edge to the graph on the spectrum of the Laplacian. Given a graph G, we denote by $G+e$ the graph obtained from G by adding some edge e that was not there before. If G and $G+e$ have M nodes, then we denote by $0 = \lambda_1^G \leq \cdots \leq \lambda_M^G$ the eigenvalues of L_G and by $0 = \lambda_1^{G+e} \leq \cdots \leq \lambda_M^{G+e}$ the eigenvalues of L_{G+e}. It can then be shown that

$$0 = \lambda_1^G = \lambda_1^{G+e} \leq \lambda_2^G \leq \lambda_2^{G+e} \leq \cdots \leq \lambda_M^G \leq \lambda_M^{G+e}. \tag{8.71}$$

This result is sometimes referred to as an interlacing theorem for graphs, see (MOHAR *et al.*, 1991). In the proof of Proposition (67), we are interested only in the inequality $\lambda_{M-1}^{G} \leq \lambda_{M-1}^{G+e}$ corresponding to the second-largest eigenvalues. Repeated use of this latter result gives $\lambda_{M-1}^{G} \leq \lambda_{M-1}^{G'}$, where G' is obtained from G by adding any number of edges.

Proof: Let us say that G has $N+1$ nodes, and write n_0 for the unique node of degree D. We denote the eigenvalues of L_G by $0 = \lambda_1^G \leq \cdots \leq \lambda_{N+1}^G$, so that $\mu = \lambda_{N+1}^G$ and $\kappa = \lambda_N^G$. If we have $N = 1$ then $\kappa = 0$ so that there is nothing left to prove. Hence, we assume from here on out that $N > 1$. Just as in the proof of Proposition (9), we have

$$\mu = \lambda_{N+1} \geq D+1\,. \tag{8.72}$$

Next, let G' denote the graph obtained from G by adding edges between n_0 and other nodes until n_0 is connected to every other node. By the observation above we have $\kappa = \lambda_N^G \leq \lambda_N^{G'}$, where the eigenvalues of $L_{G'}$ are given by $0 = \lambda_1^{G'} \leq \cdots \leq \lambda_{N+1}^{G'}$.
Let us, therefore, consider the graph G' instead. Because n_0 is connected to every other node, we see that the complement graph G'° consists of two components: $\{n_0\}$ and the remaining part H, where we do not claim H itself is connected. Let us denote by $0 = \lambda_2^H \leq \ldots \lambda_{N+1}^H$ the eigenvalues of L_H, so that those of $L_{G'^{\circ}}$ are given by $0 = \lambda_1^{G'^{\circ}} = \lambda_2^H \leq \ldots \lambda_{N+1}^H$. By the techniques used in the proof of Theorem (8), we conclude that $\lambda_3^H = N+1-\lambda_N^{G'}$.
Next, consider the complement H° of H. Again by the techniques used in the proof of Theorem (8), we see that an eigenvalue of $L_{H^{\circ}}$ is given by $N - \lambda_3^H = N - (N+1-\lambda_N^{G'}) = \lambda_N^{G'} - 1$. Moreover, by the construction of H and H°, we see that this latter graph can be obtained from G' (or from G), by deleting n_0 and every edge connected to this node. In particular, we conclude that every node in H° has a degree at most r. A straightforward application of the Gershgorin circle theorem (GERSCHGORIN, 1931) now tells us that all eigenvalues of $L_{H^{\circ}}$ are bounded from above by $2r$. In particular, we find $\lambda_N^{G'} - 1 \leq 2r$ and so

$$\kappa \leq \lambda_N^{G'} \leq 2r+1\,. \tag{8.73}$$

Combining Equations (8.72) and (8.73), we see that

$$\frac{\kappa}{\mu} \leq \frac{2r+1}{D+1} \leq \frac{3r}{D}\,, \tag{8.74}$$

from which the result follows.

■

8.3 Bifurcations in coupled stable systems

Using our results so far, we show what bifurcations to expect in diffusively coupled stable systems in 1, 2, or 3 bifurcation parameters. Note that Theorem (4) tells us that the dynamics on the center manifold are conjugate to that of a reduced vector field $R : \mathbb{R}^m \times \Omega \to \mathbb{R}^m$, satisfying $R(0;\varepsilon) = 0$ for all $\varepsilon \in \Omega$. By Remark (59) we may furthermore assume the linearization $D_x R(0;0)$ to be nilpotent with a one-dimensional kernel. Other than that, no restrictions apply to the Taylor expansion of R.

Since an m-parameter bifurcation can generically generate an m-dimensional generalized kernel, we each time consider m-parameter bifurcations for a system on $\mathbb{R}^m$. In Subsection (8.3.1) we briefly investigate the cases $m = 1$ and $m = 2$. Our main result is presented in Subsection (8.3.2), where we show the emergence of chaos for $m = 3$. In most cases, the main difficulty lies in adapting existing results on generic unfoldings to the setting where $R(0,\varepsilon) = 0$ for all $\varepsilon \in \Omega$.

8.3.1 One and two parameters

Motivated by our results so far, we describe the generic m parameter bifurcations for systems R on $\mathbb{R}^m$, where $m = 1,2$, subject to the condition $R(0;\varepsilon) = 0$ for all $\varepsilon \in \Omega$. We each time assume a nilpotent Jacobian with a one-dimensional kernel. We start with the case $m = 1$:

Remark 68 (The case $m = 1$)**.** A map $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying $R(0;\varepsilon) = 0$ for all $\varepsilon \in \mathbb{R}$ and $D_x R(0;0) = 0$ has the general form

$$R(x;\varepsilon) = x(ax + b\varepsilon + \mathscr{O}((|x,\varepsilon)|^2))\,,\ \text{for } a,b \in \mathbb{R}\,.$$

Under the generic assumption that $a,b \neq 0$, we find a transcritical bifurcation. Returning to the setting of our network system, this corresponds to a loss of stability of the fully synchronous solution.

Remark 69 (The case $m = 2$)**.** Consider first a two-parameter vector field $R : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $R(0;0) = 0$ and with non-zero nilpotent Jacobian $D_x R(0;0)$. Such a system generically displays a Bogdanov-Takens bifurcation. However, in this bifurcation scenario, there are parameter values for which there is no fixed point. Hence, if we impose the additional condition $R(0;\varepsilon) = 0$ for all $\varepsilon \in \mathbb{R}^2$, then another (generic) bifurcation scenario has to occur. This latter situation is worked out in (HIRSCHBERG; KNOBLOCH, 1991). The corresponding generic bifurcation involves multiple fixed points, heteroclinic as well as homoclinic connections, and periodic orbits. One striking feature is the presence of a homoclinic orbit from the origin, which is approached as a limit of stable periodic solutions. In our network setting, such a periodic solution means a cyclic time evolution of the system from full synchrony to less synchrony and back. The time at which the system is indistinguishably close to full synchrony can moreover be made arbitrarily long.

8.3.2 Three parameters: chaotic behaviour

In this subsection, we will prove Corollary (5), which allows us to conclude that chaotic behaviour occurs in diffusively coupled stable systems. To this end, we will apply the theory developed so far. Before this, we will present a detailed background on how we will achieve chaos.

We expect to find chaos in the network through the existence of a Shilnikov homoclinic orbit on a three-dimensional center manifold.

The Shilnikov configuration can be seen as a combination of linear and nonlinear behaviour involving a saddle fixed point. A two-dimensional stable manifold attracts trajectories exponentially fast to the fixed point, where the eigenvalues of the linearization are $\lambda_{1,2} = -\alpha \pm i\beta$ with $\alpha > 0$ and $\beta \neq 0$. Transversal to this there is a one-dimensional unstable manifold repelling away trajectories with real eigenvalue $\gamma > 0$. The Shilnikov homoclinic orbit emerges from the re-injection of the one-dimensional unstable manifold into the two-dimensional stable manifold, see Figure (21). Of course, this re-injection is a consequence of nonlinear terms. L. P. Shilnikov proved that if $\gamma > \alpha$, there are countably many saddle periodic orbits in a neighborhood of the homoclinic orbit. The proof consists of showing topological equivalence between a Poincaré map and the shift map of two symbols. The existence of chaotic behaviour is in the sense that Robert L. Devaney defined deterministic systems, with sensitive dependence on initial conditions, topological transitivity, and dense periodic points.

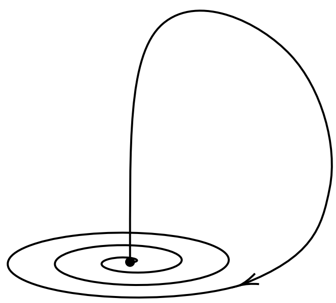


Figure 21 – Shilnikov homoclinic orbit.

We next give a brief summary of results contained in the paper (IBáñEZ; RODRíGUEZ, 2005). The authors studied the three-parameter unfolding of nonlinear vector fields on $\mathbb{R}^3$ with linear part conjugate to a nilpotent singularity of codimension three.

After making several coordinate changes, the following normal form is presented:

$$y\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}+(\lambda-y+\nu z-\frac{x^2}{2}+\mathcal{O}(\kappa))\frac{\partial}{\partial z} \tag{8.75}$$

where the parameters are given by $\tau=(\lambda,\nu,\kappa)$. The parameter κ is introduced by means of a blow-up technique, and the term

$$y\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}$$

denotes the nilpotent singularity of codimension three on $\mathbb{R}^3$. Equation (8.75) has two hyperbolic fixed points for $\lambda>0$ and $\nu=0$, namely $p_1=(-\sqrt{2\lambda},0,0)$ with local behaviour given by a two-dimensional stable and one-dimensional unstable manifold and $p_2=(+\sqrt{2\lambda},0,0)$ with local behaviour given by a two-dimensional unstable and one-dimensional stable manifold. Knowing there is a solution $x(t)$ for a specific positive parameter $\lambda=\lambda^*$ such that $x(t)\to p_1$ as $t\to-\infty$ and $x(t)\to p_2$ as $t\to+\infty$, the authors proved analytically the existence of another solution, also for the parameter λ^*, connecting both two-dimensional stable and unstable manifolds and thus forming another heteroclinic orbit. Theorem 4.1 of (IBáñEZ; RODRíGUEZ, 2005) states that in any neighborhood of the parameter $\tau_0=(\lambda_0,\nu_0,\kappa_0)=(\lambda^*,0,0)$, where the heteroclinic orbits exist, there are parameters $\tau=(\lambda,\nu,\kappa)$ such that the heteroclinic orbit breaks and a Shilnikov homoclinic orbit appear.

For completeness, we state Theorem 4.1 below in a slightly altered form.

Theorem 70 (Theorem 4.1 (IBáñEZ; RODRíGUEZ, 2005).)**.** In every neighborhood of the parameter $\tau_0=(\lambda_0,\nu_0,\kappa_0)=(\lambda^*,0,0)$ there exist parameter values $\tau=(\lambda,\nu,\kappa)$ such that the equation

$$y\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}+\left(\lambda-y+\nu z-\frac{x^2}{2}+\mathcal{O}(\kappa)\right)\frac{\partial}{\partial z}$$

has a homoclinic orbit given by the intersection of the two-dimensional stable and one-dimensional unstable invariant manifolds at the hyperbolic fixed point p_1.

As was the case for $m=2$, we cannot immediately use this existing result, as the parameter-dependent systems on the center manifold of our network ODE satisfy $R(0,\varepsilon)=0$ for all $\varepsilon\in\mathbb{R}^3$. It remains to show that with this existing restriction, we may still reduce our system to the family given by Equation (8.75). This then proves Corollary (5) as a consequence of Theorem (4).

We therefore start with a parameterized vector field $R:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ satisfying $R(0,\varepsilon)=0$ for all $\varepsilon\in\mathbb{R}^3$. After a linear coordinate change, we may assume the Jacobian $D_xR(0;0)$ to be given by

$$J=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}.$$

We thus get the system

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 + h_1(x;\varepsilon) \\ \dot{x}_2 & = & x_3 + h_2(x;\varepsilon) \ , \\ \dot{x}_3 & = & h_3(x;\varepsilon) \end{array} \tag{8.76}$$

with $x = (x_1, x_2, x_3)$, and where h_1, h_2, h_3 are the higher order of $R(x;\varepsilon)$. Note that we have $h_1(0;\varepsilon) = h_2(0;\varepsilon) = h_3(0;\varepsilon) = 0$ for all $\varepsilon \in \Omega$, and $Dh_1(0;0) = Dh_2(0;0) = Dh_3(0;0) = 0$.

To bring our system in the form of Equation (8.75), we will proceed very much like in the paper (IBáñEZ; RODRíGUEZ, 2005). The first step is to get rid of the nonlinear terms h_1 and h_2 by means of a coordinate transformation. To eliminate h_1 we consider the following change of coordinates

$$\begin{array}{rcl} y_1 & = & x_1 \\ y_2 & = & x_2 + h_1(x;\varepsilon) \\ y_3 & = & x_3 \,. \end{array} \tag{8.77}$$

Applying it to System (8.76), we get

$$\begin{array}{rcl} \dot{y}_1 & = & \dot{x}_1 = x_2 + h_1(x;\varepsilon) = y_2 \\ \dot{y}_2 & = & \dot{x}_2 + D_x h_1(x;\varepsilon)\dot{x} \\ & = & x_3 + h_2(x;\varepsilon) + D_{x_1}h_1(x;\varepsilon)(x_2 + h_1(x;\varepsilon)) \\ & + & D_{x_2}h_1(x;\varepsilon)(x_3 + h_2(x;\varepsilon)) + D_{x_3}h_1(x;\varepsilon)h_3(x;\varepsilon) \\ & = & y_3 + h_2(x;\varepsilon) + D_{x_1}h_1(x;\varepsilon)(x_2 + h_1(x;\varepsilon)) \\ & + & D_{x_2}h_1(x;\varepsilon)(x_3 + h_2(x;\varepsilon)) + D_{x_3}h_1(x;\varepsilon)h_3(x;\varepsilon) \\ \dot{y}_3 & = & \dot{x}_3 = h_3(x;\varepsilon) \end{array}$$

We, therefore, get the new system

$$\begin{array}{rcl} \dot{y}_1 & = & y_2 \\ \dot{y}_2 & = & y_3 + \tilde{h}_2(y;\varepsilon) \\ \dot{y}_3 & = & \tilde{h}_3(y;\varepsilon), \end{array} \tag{8.78}$$

with $y = (y_1, y_2, y_3)$, and where $\tilde{h}_2$ and $\tilde{h}_3$ are uniquely defined by the relations

$$\begin{aligned} \tilde{h}_2(y;\varepsilon) &= h_2(x;\varepsilon) + D_{x_1}h_1(x;\varepsilon)(x_2 + h_1(x;\varepsilon)) \\ &\qquad + D_{x_2}h_1(x;\varepsilon)(x_3 + h_2(x;\varepsilon)) + D_{x_3}h_1(x;\varepsilon)h_3(x;\varepsilon) \\ \tilde{h}_3(y;\varepsilon) &= h_3(x;\varepsilon)\,. \end{aligned} \tag{8.79}$$

Note that $\tilde{h}_2$ and $\tilde{h}_3$ again have vanishing linear terms, and moreover satisfy $\tilde{h}_2(0;\varepsilon) = \tilde{h}_3(0;\varepsilon) = 0$ for all ε.

To eliminate $\tilde{h}_2$ we consider the change of coordinates

$$\begin{aligned} z_1 &= y_1 \\ z_2 &= y_2 \\ z_3 &= y_3 + \tilde{h}_2(y;\varepsilon). \end{aligned} \tag{8.80}$$

Applying it to System (8.78), we obtain

$$\begin{aligned} \dot{z}_1 &= \dot{y}_1 = y_2 = z_2 \\ \dot{z}_2 &= \dot{y}_2 = y_3 + \tilde{h}_2(y;\varepsilon) = z_3 \\ \dot{z}_3 &= \dot{y}_3 + D_y\tilde{h}_2(y;\varepsilon)\dot{y} \\ &= \tilde{h}_3(y;\varepsilon) + D_{y_1}\tilde{h}_2(y;\varepsilon)y_2 + D_{y_2}\tilde{h}_2(y;\varepsilon)(y_3 + \tilde{h}_2(y;\varepsilon)) + D_{y_3}\tilde{h}_2(y;\varepsilon)\tilde{h}_3(y;\varepsilon)\,. \end{aligned} \tag{8.81}$$

We thus get the new system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \hat{h}_3(z;\varepsilon) \end{aligned} \tag{8.82}$$

with $z = (z_1, z_2, z_3)$, and where $\hat{h}_3(z;\varepsilon)$ is locally defined by

$$\begin{aligned} \hat{h}_3(z;\varepsilon) &= \tilde{h}_3(y;\varepsilon) + D_{y_1}\tilde{h}_2(y;\varepsilon)y_2 \\ &+ D_{y_2}\tilde{h}_2(y;\varepsilon)(y_3 + \tilde{h}_2(y;\varepsilon)) + D_{y_3}\tilde{h}_2(y;\varepsilon)\tilde{h}_3(y;\varepsilon)\,. \end{aligned} \tag{8.83}$$

Note that $\hat{h}_3$ again has no linear terms and satisfies $\hat{h}_3(0;\varepsilon) = 0$ for all ε. Moreover, in case of $h_1 = h_2 = 0$ we would find $x = y = z$ and $h_3 = \hat{h}_3$, which shows that no other restrictions apply to $\hat{h}_3$. Writing

$$\hat{h}_3(z;\varepsilon) = \Theta_1(\varepsilon)z_1 + \Theta_2(\varepsilon)z_2 + \Theta_3(\varepsilon)z_3 + \mathscr{O}(\|z\|^2) \tag{8.84}$$

for some locally defined $\Theta_i : \mathbb{R}^3 \to \mathbb{R}$, we therefore see that generically we may redefine $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ so that

$$\hat{h}_3(z;\varepsilon) = \varepsilon_1 z_1 + \varepsilon_2 z_2 + \varepsilon_3 z_3 + \mathscr{O}(\|z\|^2)\,. \tag{8.85}$$

It follows that we may write

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \varepsilon_1 z_1 + \varepsilon_2 z_2 + \varepsilon_3 z_3 + a_1 z_1^2 + a_2 z_2^2 + a_3 z_3^2 \\ &+ a_4 z_1 z_2 + a_5 z_1 z_3 + a_6 z_2 z_3 + \mathscr{O}(\|z\|^3 + \|\varepsilon\|\|z\|^2), \end{aligned} \tag{8.86}$$

for some coefficients $a_1, \ldots, a_6 \in \mathbb{R}$. We will assume that $a_1 \neq 0$, so that the System (8.86) locally has two branches of steady states: $z(\varepsilon) = (0,0,0)$ and $z(\varepsilon) = (\tilde{z}_1(\varepsilon), 0, 0)$. A straightforward calculation shows that

$$\tilde{z}_1(\varepsilon) = -\frac{\varepsilon_1}{a_1} + \mathscr{O}(\|\varepsilon\|^2)\,. \tag{8.87}$$

Motivated by this, we perform the change of coordinates:

$$\begin{aligned} w_1 &= z_1 + \frac{\varepsilon_1}{2a_1} \\ w_2 &= z_2 \\ w_3 &= z_3\,. \end{aligned} \tag{8.88}$$

Applying this to System (8.86), we get

$$\begin{aligned} \dot{w}_1 &= \dot{z}_1 = z_2 = w_2 \\ \dot{w}_2 &= \dot{z}_2 = z_3 = w_3 \\ \dot{w}_3 &= \dot{z}_3 = \varepsilon_1(w_1 - \frac{\varepsilon_1}{2a_1}) + \varepsilon_2 w_2 + \varepsilon_3 w_3 + a_1(w_1 - \frac{\varepsilon_1}{2a_1})^2 + a_2 w_2^2 + a_3 w_3^2 \\ &+ a_4(w_1 - \frac{\varepsilon_1}{2a_1})w_2 + a_5(w_1 - \frac{\varepsilon_1}{2a_1})w_3 + a_6 w_2 w_3 + \mathscr{O}(|z|^3 + |\varepsilon||z|^2) \\ &= -\frac{\varepsilon_1^2}{4a_1} + \varepsilon_2 w_2 + \varepsilon_3 w_3 + a_1 w_1^2 + a_2 w_2^2 + a_3 w_3^2 \\ &+ a_4 w_1 w_2 - \frac{a_4 \varepsilon_1 w_2}{2a_1} + a_5 w_1 w_3 - \frac{a_5 \varepsilon_1 w_3}{2a_1} + a_6 w_2 w_3 + \mathscr{O}(|z|^3 + |\varepsilon||z|^2)\,. \end{aligned} \tag{8.89}$$

We have left the remainder term $\mathscr{O}(|z|^3 + |\varepsilon||z|^2)$ as is, which will benefit us later. Rearranging terms, we get the new system

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= w_3 \\ \dot{w}_3 &= -\frac{\varepsilon_1^2}{4a_1} + (\varepsilon_2 - \frac{a_4\varepsilon_1}{2a_1})w_2 + (\varepsilon_3 - \frac{a_5\varepsilon_1}{2a_1})w_3 + a_1 w_1^2 + a_2 w_2^2 + a_3 w_3^2 \\ &+ a_4 w_1 w_2 + a_5 w_1 w_3 + a_6 w_2 w_3 + \mathscr{O}(|z|^3 + |\varepsilon||z|^2)\,. \end{aligned} \tag{8.90}$$

Similar to the paper (IBáñEZ; RODRíGUEZ, 2005), we now introduce a blow-up parameter $\kappa \in \mathbb{R}$ and write

$$\begin{aligned} w_1 &= \kappa^3 u_1 \qquad \varepsilon_1 = \kappa^3 \gamma_1 \\ w_2 &= \kappa^4 u_2 \qquad \varepsilon_2 = \kappa^2 \gamma_2 \qquad \bar{t} = \kappa t \\ w_3 &= \kappa^5 u_3 \qquad \varepsilon_3 = \kappa \gamma_3\,. \end{aligned} \tag{8.91}$$

Note that we get

$$\begin{aligned} z_1 &= w_1 - \frac{\varepsilon_1}{2a_1} = \kappa^3 \left(u_1 - \frac{\gamma_1}{2a_1}\right) \\ z_2 &= w_2 = \kappa^4 u_2 \\ z_3 &= w_3 = \kappa^5 u_3\,, \end{aligned} \tag{8.92}$$

so that we may write $\|z\| = \mathscr{O}(\kappa^3)$. Applying it to Equation (8.90), we get

$$\begin{aligned}
\frac{du_1}{d\bar{t}} &= \frac{1}{\kappa^3}\frac{dw_1}{d\bar{t}} = \frac{1}{\kappa^3}\frac{dw_1}{\kappa dt} = \frac{1}{\kappa^4}\frac{dw_1}{dt} = \frac{1}{\kappa^4}w_2 = u_2 \\
\frac{du_2}{d\bar{t}} &= \frac{1}{\kappa^4}\frac{dw_2}{d\bar{t}} = \frac{1}{\kappa^4}\frac{dw_2}{\kappa dt} = \frac{1}{\kappa^5}\frac{dw_2}{dt} = \frac{1}{\kappa^5}w_3 = u_3 \\
\frac{du_3}{d\bar{t}} &= \frac{1}{\kappa^5}\frac{dw_3}{d\bar{t}} = \frac{1}{\kappa^5}\frac{dw_3}{\kappa dt} = \frac{1}{\kappa^6}\frac{dw_3}{dt},
\end{aligned}$$

where furthermore

$$\frac{dw_3}{dt} = \dot{w}_3 = \kappa^6\left(-\frac{\gamma_1^2}{4a_1} + \gamma_2 u_2 + \gamma_3 u_3 + a_1 u_1^2\right) + \mathscr{O}(\kappa^7).$$

Summarizing, we find

$$\begin{aligned}
\frac{du_1}{d\bar{t}} &= u_2 \\
\frac{du_2}{d\bar{t}} &= u_3 \\
\frac{du_3}{d\bar{t}} &= -\frac{\gamma_1^2}{4a_1} + \gamma_2 u_2 + \gamma_3 u_3 + a_1 u_1^2 + \mathscr{O}(\kappa).
\end{aligned} \tag{8.93}$$

We next focus on the parameter γ_2. We assume henceforth that $\gamma_2 < 0$ and perform the following change of coordinates

$$\begin{aligned}
v_1 &= -2r^3 u_1 \qquad v_2 = -2r^4 u_2 \\
v_3 &= -2r^5 u_3 \qquad \tau = r^{-1}\bar{t},
\end{aligned} \tag{8.94}$$

where

$$r = \left(-\frac{1}{\gamma_2}\right)^{\frac{1}{2}} > 0. \tag{8.95}$$

Applying it to System (8.93), we get

$$\begin{aligned}
\frac{dv_1}{d\tau} &= (-2r^3)\frac{du_1}{d\tau} = (-2r^4)\frac{du_1}{d\bar{t}} = -2r^4 u_2 = v_2 \\
\frac{dv_2}{d\tau} &= (-2r^4)\frac{du_2}{d\tau} = (-2r^5)\frac{du_2}{d\bar{t}} = -2r^5 u_3 = v_3 \\
\frac{dv_3}{d\tau} &= (-2r^5)\frac{du_3}{d\tau} = (-2r^6)\frac{du_3}{d\bar{t}} \\
&= (-2r^6)\left(-\frac{\gamma_1^2}{4a_1} + \gamma_2 u_2 + \gamma_3 u_3 + a_1 u_1^2 + \mathscr{O}(\kappa)\right) \\
&= (-2r^6)\left(-\frac{\gamma_1^2}{4a_1} + \gamma_2\left(\frac{v_2}{-2r^4}\right) + \gamma_3\left(\frac{v_3}{-2r^5}\right) + a_1\left(\frac{v_1}{-2r^3}\right)^2 + \mathscr{O}(\kappa)\right) \\
&= \frac{\gamma_1^2 r^6}{2a_1} + \gamma_2 v_2 r^2 + \gamma_3 v_3 r - a_1\frac{v_1^2}{2} + \mathscr{O}(\kappa) \\
&= \frac{\gamma_1^2 r^6}{2a_1} - v_2 + \gamma_3 v_3 r - a_1\frac{v_1^2}{2} + \mathscr{O}(\kappa).
\end{aligned} \tag{8.96}$$

We thus get the new system

$$\begin{aligned} v_1' := \frac{dv_1}{d\tau} &= v_2 \\ v_2' := \frac{dv_2}{d\tau} &= v_3 \\ v_3' := \frac{dv_3}{d\tau} &= \frac{\gamma_1^2 r^6}{2a_1} - v_2 + \gamma_3 v_3 r - a_1 \frac{v_1^2}{2} + \mathscr{O}(\kappa)\,. \end{aligned} \tag{8.97}$$

Finally, we make the following change of coordinates:

$$\begin{aligned} \mathbf{x} &= a_1 v_1 \\ \mathbf{y} &= a_1 v_2 \\ \mathbf{x} &= a_1 v_3\,. \end{aligned} \tag{8.98}$$

This gives

$$\begin{aligned} \mathbf{x}' &= a_1 v_1' = a_1 v_2 = \mathbf{y} \\ \mathbf{y}' &= a_1 v_2' = a_1 v_3 = \mathbf{z} \\ \mathbf{z}' &= a_1 v_3' = \frac{\gamma_1^2 r^6}{2} - a_1 v_2 + \gamma_3 a_1 v_3 r - a_1^2 \frac{v_1^2}{2} + \mathscr{O}(\kappa) \\ &= \frac{\gamma_1^2 r^6}{2} - \mathbf{y} + \gamma_3 r \mathbf{z} - \frac{\mathbf{x}^2}{2} + \mathscr{O}(\kappa)\,. \end{aligned} \tag{8.99}$$

Setting $\lambda := \frac{\gamma_1^2 r^6}{2}$ and $\nu := \gamma_3 r$, we arrive at the vector field

$$= \mathbf{y}\frac{\partial}{\partial \mathbf{x}} + \mathbf{z}\frac{\partial}{\partial \mathbf{y}} + \left(\lambda - \mathbf{y} + \nu \mathbf{z} - \frac{\mathbf{x}^2}{2} + \mathscr{O}(\kappa)\right)\frac{\partial}{\partial \mathbf{z}} \tag{8.100}$$

from Theorem (70). Note that $\lambda = \frac{\gamma_1^2 r^6}{2}$ is necessarily non-negative. However, this may be assumed in the setting of Theorem (70), as $\lambda^* > 0$. This theorem thus predicts chaos in the setting of our coupled cell system, provided $m = 3$ and the network in question is at least 2-versatile.

Remark 71. Instead of considering $\varepsilon \in \mathbb{R}^3$, we may have instead looked at $\varepsilon \in \mathbb{R}^2$ and considered $\alpha = \alpha^* + \hat{\alpha}$ as the third parameter needed to unfold the nilpotent singularity of co-dimension three. Here $\hat{\alpha} \in \mathbb{R}$ is a small deviation from α^*. Note that $\{\beta D \mid \beta \in \mathbb{R}\}$ constitutes a direction transversal to the conjugacy class of $A - \alpha^* \lambda D$. This is because the tangent space to a conjugacy class of matrices lies in the space of matrices with zero traces and by assumption $Tr(D) \neq 0$. From the calculation above, we see that only a three-parameter unfolding of the linear terms is necessary for the emergence of chaos. Hence, chaos also emerges in a two-parameter system, if in addition the coupling parameter α is seen as an implicit parameter.

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