# On Hamiltonian elliptic systems with exponential growth in dimension two

***Yony Raúl Santaria Leuyacc***

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**Yony Raúl Santaria Leuyacc**

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**Yony Raúl Santaria Leuyacc**

# Sistemas elípticos hamiltonianos com crescimento exponencial em dimensão dois

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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*To my dear family.*

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*“Take what you need,*
*do what you should,*
*you will get what you want.”*
*(Gottfried Leibniz)*

# RESUMO

LEUYACC, R. Y. S. **Sistemas elípticos hamiltonianos com crescimento exponencial em dimensão dois**. 2017. 192 p. Doctoral dissertation (Doctorate Candidate Program in Mathematics) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2017.

Neste trabalho estudamos a existência de soluções fracas não triviais para sistemas hamiltonianos do tipo elíptico, em dimensão dois, envolvendo uma função potencial e não linearidades tendo crescimento exponencial máximo com respeito a uma curva (hipérbole) crítica. Consideramos quatro casos diferentes. Primeiramente estudamos sistemas de equações em domínios limitados com potencial nulo. No segundo caso, consideramos sistemas de equações em domínio ilimitado, sendo a função potencial limitada inferiormente por alguma constante positiva e satisfazendo algumas de integrabilidade, enquanto as não linearidades contêm funções-peso tendo uma singularidade na origem. A classe seguinte envolve potenciais coercivos e não linearidades com funções peso que podem ter singularidade na origem ou decaimento no infinito. O quarto caso é dedicado ao estudo de sistemas em que o potencial pode ser ilimitado ou decair a zero no infinito. Para estabelecer a existência de soluções, utilizamos métodos variacionais combinados com desigualdades do tipo Trudinger-Moser em espaços de Lorentz-Sobolev e a técnica de aproximação em dimensão finita

**Palavras-chave:** Sistemas hamiltonianos, Crescimento exponencial, Métodos variacionais, Desigualdade de Trudinger-Moser, Espaços de Lorentz-Sobolev.

# ABSTRACT

LEUYACC, R. Y. S. **On Hamiltonian elliptic systems with exponential growth in dimension two**. 2017. 192 p. Doctoral dissertation (Doctorate Candidate Program in Mathematics) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2017.

In this work we study the existence of nontrivial weak solutions for some Hamiltonian elliptic systems in dimension two, involving a potential function and nonlinearities which possess maximal growth with respect to a critical curve (hyperbola). We consider four different cases. First, we study Hamiltonian systems in bounded domains with potential function identically zero. The second case deals with systems of equations on the whole space, the potential function is bounded from below for some positive constant and satisfies some integrability conditions, while the nonlinearities involve weight functions containing a singulatity at the origin. In the third case, we consider systems with coercivity potential functions and nonlinearities with weight functions which may have singularity at the origin or decay at infinity. In the last case, we study Hamiltonian systems, where the potential can be unbounded or can vanish at infinity. To establish the existence of solutions, we use variational methods combined with Trudinger-Moser type inequalities for Lorentz-Sobolev spaces and a finite-dimensional approximation.

**Keywords:** Hamiltonian systems, Exponential growth, Variational methods, Trudinger-Moser inequality, Lorentz-Sobolev spaces.

# LIST OF SYMBOLS

$A \subset B$ — $A$ is a subset of $B$

$A \subsetneqq B$ — $A$ is a proper subset of $B$.

$X \backslash A$ — The the complement of a set $A \subset X$

$\chi_E$ — The the characteristic function of the set $E$.

$\mathbb{R}$ — The set of real numbers.

$\mathbb{R}^+$ — $\{x \in \mathbb{R} : x \geq 0\}$.

$\mathbb{N}$ — The set of natural numbers.

$\mathbb{R}^N$ — The Euclidean N-space.

$e_i$ — The vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the i-th entry and 0 elsewhere.

$[x]$ — The integer part of the real number $x$.

$B_R(x)$ — The ball of radius $R$ centered at $x$ in $R^N$.

$|x|$ — $\sqrt{|x_1|^2 + \cdots + |x_n|^2}$ when $x = (x_1, ..., x_n) \in R^N$.

$S^N$ — The unit sphere $\{x \in \mathbb{R}^N : |x| = 1\}$.

$\omega_{N-1}$ — The surface area of the unit sphere $S^{N-1}$.

$|\Omega|$ — The Lebesgue measure of the set $\Omega \subset \mathbb{R}^N$

$(X, \mu)$ — Measure space

$\mathscr{M}(X, \overline{\mathbb{R}})$ — The collection of all extended real-valued $\mu$-measurable functions on $X$

$\mathscr{M}_0(X, \mathbb{R})$ — Class of functions in $\mathscr{M}(X, \overline{\mathbb{R}})$ that are finite $\mu$-almost everywhere in $X$

$L^p(X, \mu)$ — The Lebesque space over the measuare $(X, \mu)$

$L^p(\mathbb{R}^N)$ — The space $L^p(\mathbb{R}^N, |\cdot|)$.

$L^p_{loc}(\mathbb{R}^N)$ — The space of functios that lie in $L^p(K)$ for any compact set $K$ in $\mathbb{R}^N$.

$supp f$ — Support of a function $f$.

$f^*$ — The decreasing rearrangement of a function $f$.

$f_n \nearrow f$ — The sequence $f_n$ increases monotonically to a function $f$.

$f_n \searrow f$ — The sequence $f_n$ decreases monotonically to a function $f$.

$f = O(g)$ — Means $|f(x)| \leq M|g(x)|$ for some $M$ for $x$ near $x_0$.

$f = o(g)$ — Means $|f(x)||g(x)|^{-1} \to 0$ as $x \to x_0$.

$f_n = o_n(1)$ — Means $f_n \to 0$ as $n \to +\infty$.

$|\alpha|$ — indicates the size $|\alpha_1| + \cdots + |\alpha_N|$ of a multi-index $\alpha = (\alpha_1, ..., \alpha_N)$.

$\partial_i^m f$ — The m-th partial derivative of $f(x_1, ..., x_N)$ with respect to $x_i$.

$\partial^\alpha f$ — $\partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N} f$.

$\mathscr{C}^k$ — The space of functions $f$ with $\partial^\alpha f$ continuous for all $|\alpha| \leq k$.

$\mathscr{C}_0$ — The space of continuous functions with compact support

$\mathscr{C}^\infty$ — The space of smooth functions $\cap_{k\geq 1}\mathscr{C}^k$.

$\mathscr{C}_0^\infty$ — The space of smooth functions with compact support.

# CONTENTS

CHAPTER

1

# INTRODUCTION

In recent years, many authors have considered the existence of nontrivial solutions for Hamiltonian systems of the form

$$\begin{cases} -\Delta u + V(x)u = H_v(x,u,v), & x \in \Omega, \\ -\Delta v + V(x)v = H_u(x,u,v), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a smooth domain in $\mathbb{R}^N$, $N \geq 2$ and $H(x,u,v)$ is a nonlinear function. Hamiltonian systems have been widely use in applied sciences, mainly in the mathematical study of standing wave solutions in models in population dynamics (Murray (1993)), in nonlinear optics (Bulgan *et al.* (2004), Christodoulides *et al.* (2001)) and in the study of Bose-Einstein condensates (Chang *et al.* (2004)). In dimension $N \geq 3$, the simplest example of (1.1) is $H_v(x,u,v) = g(v)$, $H_u(x,u,v) = f(u)$, ($g(v) \sim v^p$ and $f(u) \sim u^q$). Even for this case, relevant open questions still persist (see Bonheure, Santos and Tavares (2014)). In order to suppose that this systems is in variational form, that is (1.1) is the Euler-Lagrange equation of some functional defined on a suitable product of Sobolev type spaces, the couple $(p,q)$ lies on or below the critical hyperbola (see Figueiredo and Felmer (1994), Hulshof and Vorst (1993), Mitidieri (1993)):

$$\frac{1}{p+1} + \frac{1}{q+1} \geq \frac{N-2}{N}. \tag{1.2}$$

In dimension $N = 2$ one sees that the critical hyperbola is not defined. More precisely, Let $\Omega \subset \mathbb{R}^N$ be a domain of finite measure. The classical Sobolev space embeddings say that $W_0^{1,2}(\Omega) \subset L^q(\Omega)$ for all $1 \leq q \leq 2N/(N-2)$. In the limiting case $N = 2$ we have $q = +\infty$, but easy examples show that $W_0^{1,2}(\Omega) \nsubseteq L^\infty(\Omega)$, in particular from that any polynomial growth for $f$ and $g$ is admitted. Thus, one is lead to ask if there is another kind of maximal growth in this situation. The answer was obtained independently by Pohozaev (1964) and Trudinger (1967), it states that $e^{\alpha u^2} \in L^1(\Omega)$ for all $u \in H_0^1(\Omega)$ and $\alpha > 0$. Furthermore, Moser (1970/71) showed that there exists a positive constant $C = C(\alpha, \Omega)$ such that

$$\sup_{\substack{u \in H_0^1(\Omega) \\ \|\nabla u\|_2 \leq 1}} \int_\Omega e^{\alpha u^2}\, dx \begin{cases} \leq C, & \alpha \leq 4\pi, \\ = +\infty, & \alpha > 4\pi. \end{cases} \tag{1.3}$$

Estimate (1.3) from now on will referred to as Trudinger-Moser inequality, similar results wwere obtained for $\Omega = \mathbb{R}^2$ (see Cao (1992), Ruf (2005)). A singular type extension of inequality (1.3) for bounded domains was given by Adimurthi and Sandeep (2007) and its version in the whole space $\mathbb{R}^N$ was obtained by Adimurthi and Yang (2010). They showed that there exists a positive constant $C = C(\alpha, \beta, N)$ such that

$$\sup_{\substack{u \in H^1(\mathbb{R}^N) \\ \|\nabla u\|_2 + \|u\|_2 \leq 1}} \int_{\mathbb{R}^N} \frac{1}{|x|^\beta} \left( e^{\alpha |u|^{N/(N-1)}} - \sum_{k=0}^{N-2} \frac{\alpha^k |u|^{kN/(N-1)}}{k!} \right) dx \begin{cases} \leq C, & 0 \leq \alpha \leq (1-\beta/N)\alpha_N \\ = +\infty, & \alpha > (1-\beta/N)\alpha_N, \end{cases} \tag{1.4}$$

where $\alpha_N = (N\omega_N^{1/N})^{N/(N-1)}$.

In dimension two, inequalities (1.3) and (1.4) show that, if the setting space of the system (1.1) is given by $H_0^1(\Omega) \times H_0^1(\Omega)$ the maximal growth of the functions $f$ and $g$ can be consider such as $g(v) \sim e^{v^2}$ and $f(u) \sim e^{u^2}$.

An important point is the fact that Trudinger-Moser type inequalities can be sharpened using Lorentz-Sobolev spaces. First, we recall the Lorentz spaces: for a measurable function $u : \Omega \to \mathbb{R}$, and $u^*$ denote its decreasing rearrangement. Then, $u$ belongs to the Lorentz space $L^{p,q}(\Omega)$ $(p, q > 1)$ if

$$\|u\|_{p,q} = \left( \int_0^{+\infty} \left[ u^*(t) t^{1/p} \right]^q \frac{dt}{t} \right)^{1/q} < +\infty.$$

These spaces represent an extension of the Lebesgue spaces, in particular when $p = q$ we have $L^{p,p}(\Omega) = L^p(\Omega)$. Using these spaces we can define the Lorentz-Sobolev spaces, roughly speaking we say that $u$ belongs to the Lorentz-Sobolev space $W_0^1 L^{p,q}(\Omega)$ if $u$ and its weak derivatives belongs to $L^{p,q}(\Omega)$.

Using Lorentz-Sobolev spaces, Brézis and Wainger (1980) showed : If $\Omega$ be a bounded domain in $\mathbb{R}^2$ and $s > 1$, then, $e^{|u|^{\frac{s}{s-1}}}$ belongs to $L^1(\Omega)$ for all $u \in W_0^1 L^{2,s}(\Omega)$. Furthermore, Alvino, Ferone and Trombetti (1996) obtained the following refinement of (1.3), there exists a positive constant $C = C(\Omega, s, \alpha)$ such that

$$\sup_{\substack{u \in W_0^1 L^{2,s}(\Omega) \\ \|\nabla u\|_{2,s} \leq 1}} \int_\Omega e^{\alpha |u|^{\frac{s}{s-1}}}\, dx \begin{cases} \leq C, & \alpha \leq (4\pi)^{s/(s-1)}, \\ = +\infty, & \alpha > (4\pi)^{s/(s-1)}. \end{cases} \tag{1.5}$$

As it was showed in Ruf (2006), if the setting space of the system (1.1) is given by the product space $W_0^1 L^{2,q}(\Omega) \times W_0^1 L^{2,p}(\Omega)$ the maximal growth of the nonlinearities can be considered like $f(u) \sim e^{|u|^p}$ and $g(v) \sim e^{|v|^q}$ with $p,q > 1$ satisfying

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{1.6}$$

Trudinger-Moser inequalities in the case $\Omega = \mathbb{R}^2$ were studied by Cassani and Tarsi (2009) with some natural modifications. Recently Lu and Tang (2016) obtained the following result which represents an extension of (1.4) in Lorentz-Sobolev spaces: Let $1 < s < +\infty$, $0 \leq \beta < N$. Then, there exists a positive constant $C = C(N,s,\beta)$ such that

$$\sup_{\substack{W^1L^{N,s}(\mathbb{R}^N) \\ \|\nabla u\|_{N,s}^s + \|u\|_{N,s}^s \leq 1}} \int_{\mathbb{R}^N} \frac{\Phi(\alpha |u|^{s/(s-1)})}{|x|^\beta}\, dx \begin{cases} \leq C, & \alpha \leq (1-\beta/N)\alpha_{N,s}, \\ = +\infty, & \alpha > (1-\beta/N)\alpha_{N,s}, \end{cases} \tag{1.7}$$

where

$$\Phi(t) = e^t - \sum_{k=0}^{k_0} \frac{t^k}{k!}, \quad k_0 = \left[\frac{(s-1)N}{s}\right] \quad \text{and} \quad \alpha_{N,s} = (N\omega_N^{1/N})^{s/(s-1)}.$$

In dimension two the last inequality allows us toconsider the nonlinearities of the system (1.1) such as $g(x,v) \sim e^{|v|^p}/|x|^a$ and $f(x,u) \sim e^{|u|^q}/|x|^b$ with $a,b \in [0,2)$ and $(p,q)$ belonging to (1.6).

Finally, we illustrate the content of each chapter of this thesis.

In **Chapter 2**, we show important properties which will be used in the chapters 3,4 and 5. We start introducing some basic concepts about distribution and decreasing rearrangement of a function in order to define Lorentz spaces, which represent a generalization of $L^p$-spaces. Furthermore, with the help of these spaces we can construct Lorentz-Sobolev spaces as generalization of Sobolev spaces. Finally, following Figueiredo, Ó and Ruf (2005), Ruf (2008) we define an application called tilde-map which is very useful in the variational formulation of the systems which will be presented in the next chapters.

In **Chapter 3**, we study the existence of nontrivial weak solution to the following Hamiltonian elliptic system

$$\begin{cases} -\Delta u &= g(v), & \text{in} \quad \Omega, \\ -\Delta v &= f(u), & \text{in} \quad \Omega, \\ u = v &= 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.8}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$ and the nonlinearities $f$ and $g$ possess maximal growth which allows us to treat the system (1.8) variationally in the cartesian product of Lorentz-Sobolev spaces.

In Ruf (2008) it was shown the existence of nontrivial solution of the system (1.8) in the case where $f(u) \sim e^{|u|^{\bar{p}}}$ and $g(v) \sim e^{|v|^{\bar{q}}}$ where $\bar{p},\bar{q} > 0$ such that $1/\bar{p} + 1/\bar{q} > 1$. In this case, we can obtain $(p,q)$ belongs to the hyperbola (1.6) such that

$$\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha|s|^p}} = 0 \quad \text{and} \quad \lim_{|s|\to\infty} \frac{|g(s)|}{e^{\alpha|s|^q}} = 0, \quad \text{for all} \quad \alpha > 0. \tag{1.9}$$

The existence of solutions for the system (1.8) when $f(u) \sim e^{|u|^p}$ and $g(v) \sim e^{|v|^p}$ has been solved for the case $p = q = 2$ in Figueiredo, Ó and Ruf (2004). Our main result in this chapter is to prove the existence of nontrivial weak solutions for the general case, that is $(p,q)$ satisfies (1.6).

Motivated by the above results, we call the curve (1.6) as exponential critical hyperbola in analogy to (1.2) in the sense that for $(p,q)$ belongs to this hyperbola gives the maximal growth range and the solutions is proved when $(p,q)$ lies on or below to (1.6).

Therefore, from this results we have naturally associated notions of criticality and subcriticality, namely: Given $p > 1$, we say that a function $f$ has $p$-subcritical exponential growth, if $f$ satisfies condition (1.9), whereas a function $f$ has $p$-critical exponential growth, if there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha|s|^p}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0. \end{cases}$$

In order to study the existence of solutions of the system (1.8) we are going to impose the following conditions:

$(A_1)$ $f$ and $g$ are continuous functions, with $f(s) = g(s) = o(s)$ near the origin.

$(A_2)$ There exist constants $\mu > 2$, $\nu > 2$ and $s_0 > 0$ such that

$$0 < \mu F(s) \le sf(s), \quad \text{and} \quad 0 < \nu G(s) \le sg(s), \quad \text{for all} \quad |s| > s_0.$$

where $F(s) = \int_0^s f(t)\,dt$ and $G(s) = \int_0^s g(t)\,dt$.

$(A_3)$ There exist $\alpha_0 > 0$ and $p > 1$, such that

$$\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha|s|^p}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0. \end{cases}$$

$(A_4)$ There exists $\beta_0 > 0$, such that

$$\lim_{|s|\to\infty} \frac{|g(s)|}{e^{\beta|s|^q}} = \begin{cases} 0, & \beta > \beta_0, \\ +\infty, & \beta < \beta_0. \end{cases}$$

where $q = \dfrac{p}{p-1}$.

$(A_5)$ There exist constants $\theta > 2$ and $C_\theta > 0$ such that

$$F(s) \ge C_\theta |s|^\theta \quad \text{and} \quad G(s) \ge C_\theta |s|^\theta, \quad \text{for all} \quad s \in \mathbb{R},$$

where

$$C_\theta > \frac{14 + 6\sqrt{5}}{\delta_\theta R^{\theta-2}}, \quad R^2 = \frac{2\pi}{\alpha_0^{1/p}\beta_0^{1/q}} \max\left\{\frac{\mu-2}{\mu}, \frac{\nu-2}{\nu}\right\}$$

and $\delta_\theta$ is a positive constant which will be explicit later on.

Now we state the main result of chapter 3.

**Theorem 1.1.** Suppose $(A_1)-(A_5)$ hold. Then, the system (1.8) possesses a nontrivial weak solution.

Note that the above theorem permits to work with $(p,q)$ lying in the exponential critical hyperbola thanks to assumptions $(A_3)-(A_4)$. Consequently, this result completes the study made in Figueiredo, Ó and Ruf (2004) which corresponds to the diagonal case $p=q=2$. We point out the condition $(A_5)$ will be crucial in our proof, this condition is of type as considered in many works (see Cao (1992) and the references therein). We remark that from the choose of $C_\theta$ we do not need the following usual assumption:

$(A_0)$ There exist positive constants $M$ and $s_0$ such that

$$0<F(s)\leq M|f(s)| \quad \text{and} \quad 0<G(s)\leq M|g(s)|, \quad \text{for all} \quad |s|>s_0.$$

which is used to get some convergence results.

Since the system (1.8) is a special case of a Hamiltonian system, some difficulties appear; for example, the associated functional is strongly indefinite, that is, its leading part is respectively coercive and anti-coercive on infinite-dimensional subspaces of the energy space. To overcome these difficulties, we will use a finite-dimensional approximation combine with the Linking theorem.

In **Chapter 4**, we study the following singular Hamiltonian system:

$$\begin{cases} -\Delta u+V(x)u = \dfrac{g(v)}{|x|^a}, & x\in\mathbb{R}^2, \\ \\ -\Delta v+V(x)v = \dfrac{f(u)}{|x|^b}, & x\in\mathbb{R}^2, \end{cases} \tag{1.10}$$

where $a,b\in[0,2)$ and the functions $f$ and $g$ possess critical exponential growth. This system is motivated by inequality (1.7).

In order to have properties like embedding theorems we consider that $V$ is a continuous potential verifying the following conditions:

$(V_1)$ There exists a positive constant $V_0$ such that $V(x)\geq V_0$ for all $x\in\mathbb{R}^2$.

$(V_2)$ There exist constants $p>2$ and $q=p/(p-1)$ such that

$$\frac{1}{V^{1/q}}\in L^{2,p}(\mathbb{R}^2) \quad \text{and} \quad \frac{1}{V^{1/p}}\in L^{2,q}(\mathbb{R}^2).$$

System (1.10) was studied by Souza (2012) in the case where $p=q=2$ and its solution was found in $H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$, for this case the author use the respective assumption instead of

$(V_2)$, that is $1/V \in L^1(\mathbb{R}^2)$ and similar conditions on the functions $f$ and $g$ as $(A_0)-(A_4)$ given above. Moreover, it is considered a following condition: there exist $\theta > 2$ and a positive constant $C_\theta$ sufficiently large such that

$$f(t) \geq C_\theta t^{\theta-1} \quad \text{and} \quad g(t) \geq C_\theta t^{\theta-1}, \quad \text{for all} \quad t \geq 0. \tag{1.11}$$

Cassani and Tarsi (2015) proved the existence of nontrivial solutions of the system (1.10) in the case where $a = b = 0$. The authors have assumed $(V_1)-(V_2)$ on $V$ and $(A_0)-(A_4)$ on the nonlinearities. Furthermore, in order to estimate the minimax level it was considered the following conditions:

$$\lim_{t\to+\infty} tf(t)e^{-\alpha_0 t^p} = \lim_{t\to+\infty} tg(t)e^{-\beta_0 t^q} = +\infty \quad \text{and} \quad \alpha_0^{1/p} \neq \beta_0^{1/q}. \tag{1.12}$$

Motivated by these results, we will prove the existence of nontrivial weak solution of (1.10) in two different ways, that means, in addition to $(A_0)-(A_4)$ we will adapt the conditions $(A_5)$ and (1.12) and we use each one independently in the proofs. More precisely, we describe the following additional conditions on the functions $f$ and $g$.

$(A_6)$ The following limits holds

$$\lim_{|s|\to+\infty} \frac{sf(s)}{e^{\alpha_0|s|^p}} = +\infty \quad \text{and} \quad \lim_{|s|\to+\infty} \frac{sg(s)}{e^{\beta_0|s|^q}} = +\infty.$$

$(A_7)$ For $a,b$ given by (1.10), $p,q$ given by $(V_2)$, $\alpha_0$ and $\beta_0$ given by $(A_3)$ and $(A_4)$ respectively, it satisfies

$$\left(\frac{\alpha_0}{1-b/2}\right)^{1/p} \neq \left(\frac{\beta_0}{1-a/2}\right)^{1/q}.$$

$(A_8)$ Let $a,b \in [0,2)$ given by (1.10). Then, there exist $\theta > 2$ and a positive constant $C_{\theta,a,b}$ such that

$$F(s) \geq C_{\theta,a,b}|s|^\theta \quad \text{and} \quad G(s) \geq C_{\theta,a,b}|s|^\theta, \quad \text{for all} \quad s \in \mathbb{R},$$

where

$$C_{\theta,a,b} > \frac{56+32\sqrt{3}}{\delta_{\theta,a,b}R^{\theta-2}},$$

and

$$R^2 = \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} \max\left\{\frac{\mu-2}{2\mu}, \frac{\nu-2}{2\nu}\right\},$$

whereas the constant $\delta_{\theta,a,b}$ will be explicit later on.

The following theorems contains our main results in Chapter 4.

**Theorem 1.2.** Suppose that $V$ satisfies $(V_1)-(V_2)$ and $f$ and $g$ satisfy $(A_0)-(A_4)$ and $(A_6)-(A_7)$. Then, the system (1.10) possesses a nontrivial weak solution.

**Theorem 1.3.** Suppose that $V$ satisfies $(V_1)-(V_2)$ and $f$ and $g$ satisfy $(A_1)-(A_4)$ and $(A_8)$. Then, the system (1.10) possesses a nontrivial weak solution.

We remark that the class of functions which satisfy the hypotheses of the above theorems are different. The conclusion of Theorems 1.2 and 1.3 extends the result given in Cassani and Tarsi (2015) in the sense that we add the singularities $|x|^{-a}$ and $|x|^{-b}$ on the nonlinearities considered in that paper. Moreover, our result complements the study made in Souza (2012) in the sense that, in this work, we study the class of Hamiltonian systems where the nonlinearities possess maximal growth with respect to the exponential critical hyperbola.

Our proof of Theorems 1.2 and 1.3 is based on variational methods and a finite dimensional approximation.

In **Chapter 5**, we discuss the existence of nontrivial solutions for the Hamiltonian system

$$\begin{cases} -\Delta u + V(x)u = Q_2(x)g(v), & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = Q_1(x)f(u), & x \in \mathbb{R}^2, \end{cases} \tag{1.13}$$

where $V, Q_1, Q_2$ are continuous functions and the nonlinearities $f$ and $g$ possess critical exponential growth with $(p,q)$ lying on the exponential critical hyperbola.

On the potential $V$ we assume the following condition:

$(V)$ $V \in \mathscr{C}(\mathbb{R}^2,\mathbb{R})$, $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^2$, there exists $a \geq 0$ such that

$$\liminf_{|x|\to\infty} \frac{V(x)}{|x|^a} > 0.$$

Assumption $(V)$ implies that, if $a > 0$ the potential $V$ is coercive. On the functions $Q_i$ for $i = 1,2$, we consider:

$(Q_i)$ $Q_i \in \mathscr{C}(\mathbb{R}^2\backslash\{0\},\mathbb{R})$ , $Q_i(x) > 0$ for $x \neq 0$ and there exist $d_i < a/(\max\{p,q\}-1)-1$ and $b_i > -2$ such that

$$0 < \lim_{|x|\to 0} \frac{Q_i(x)}{|x|^{b_i}} < +\infty \quad \text{and} \quad \limsup_{|x|\to\infty} \frac{Q_i(x)}{|x|^{d_i}} < +\infty.$$

The existence of solutions of system (1.13) was studied in Cassani and Tarsi (2015) for the case $Q_1(x) = Q_2(x) \equiv 1$. The case $Q_1(x) = |x|^{-a}$ and $Q_2(x) = |x|^{-b}$ with $a,b \in [0,2)$ was treated in Souza (2012) for the diagonal case $p = q = 2$ and also considered in Chapter 4 when $(p,q)$ belongs to the exponential critical hyperbola. In this section we treat a more general class of nonlinearities studied in previously mentioned papers. We also mention that the systems studied in Cassani and Tarsi (2015), Souza (2012) and also in Chapter 4 the potential $V$ satisfy some integrability conditions. In our case, we consider coercive potentials which represent a different class of potential from the mentioned works.

On assumption $(V)$ and for $s=p$ or $s=q$, we consider the following weighted Lorentz-Sobolev space $W^1L_V^{2,s}(\mathbb{R}^2)$ which is defined to be the closure of compactly supported smooth functions, with respect to the quasinorm

$$\|u\|_{W^1L_V^{2,s}(\mathbb{R}^2)} := \left(\|u\|_{2,s}^s + \|V^{1/s}u\|_{2,s}^s\right)^{1/s}.$$

For any $\lambda \geq 1$ and $i=1,2$ we also define

$$L^\lambda(\mathbb{R}^2, Q_i) := \{u \,:\, \int_{\mathbb{R}^2} Q_i(x)|u|^\lambda\,dx < +\infty\},$$

endowed with the norm

$$\|u\|_{L^\lambda(\mathbb{R}^2,Q_i)} := \left(\int_{\mathbb{R}^2} Q_i(x)|u|^\lambda\,dx\right)^{1/\lambda}.$$

In these spaces we obtain the next result which will be proved later.

**Proposition 1.4.** Assume $(V)$ and $(Q_i)$ for $i=1,2$ and let $s=q$ or $s=p$. Then, the following embeddings are compact

$$W^1L_V^{2,s}(\mathbb{R}^2) \hookrightarrow L^\lambda(\mathbb{R}^2, Q_i), \quad \text{for all} \quad \lambda \geq \min\{p,q\}.$$

Concerning the functions $f$ and $g$ we suppose the following assumptions:

$(B_1)$ $f,g \in \mathscr{C}(\mathbb{R})$, $f(s)=o(s^{\eta_1})$ and $g(s)=o(s^{\eta_2})$, as $s\to 0$, where $\eta_1 = \max\{1/(q-1), \min\{p,q\}\}$ and $\eta_2 = \max\{1/(p-1), \min\{p,q\}\}$.

$(B_2)$ There exist constants $\mu > 2$ and $\nu > 2$ such that

$$0 < \mu F(s) \leq sf(s), \quad 0 < \nu G(s) \leq sg(s), \quad \text{for all} \quad s \neq 0,$$

where $F(s) = \int_0^s f(t)\,dt$ and $G(s) = \int_0^s g(t)\,dt$.

$(B_3)$ There exist positive constants $M$ and $s_0$ such that

$$0 < F(s) \leq M|f(s)| \quad \text{and} \quad 0 < G(s) \leq M|g(s)|, \quad \text{for all} \quad |s| > s_0.$$

$(B_4)$ There exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha|s|^p}} = \begin{cases} +\infty, & \alpha < \alpha_0 \\ 0, & \alpha > \alpha_0. \end{cases}$$

$(B_5)$ There exists $\beta_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|g(s)|}{e^{\beta|s|^q}} = \begin{cases} +\infty, & \beta < \beta_0 \\ 0, & \beta > \beta_0. \end{cases}$$

$(B_6)$ The following limits holds

$$\lim_{|s|\to+\infty}\frac{sf(s)}{e^{\alpha_0|s|^p}}=+\infty \quad \text{and} \quad \lim_{|s|\to+\infty}\frac{sg(s)}{e^{\beta_0|s|^q}}=+\infty.$$

$(B_7)$ For $b_i$ given by $(Q_i)$, $i=1,2$ and $\alpha_0$, $\beta_0$ given by $(B_4)$ and $(B_5)$ respectively, it satisfies

$$\Big(\frac{\alpha_0\min\{1,1+\frac{b_1}{2}\}}{(1+\frac{b_1}{2})^2}\Big)^{1/p}>\Big(\frac{\beta_0}{\min\{1,1+\frac{b_2}{2}\}}\Big)^{1/q}$$

or

$$\Big(\frac{\alpha_0}{\min\{1,1+\frac{b_1}{2}\}}\Big)^{1/p}<\Big(\frac{\beta_0\min\{1,1+\frac{b_2}{2}\}}{(1+\frac{b_2}{2})^2}\Big)^{1/q}.$$

**Theorem 1.5.** Suppose that $V$ satisfies $(V)$, $Q_i$ satisfy $(Q_i)$ for $i=1,2$ and the nonlinearities $f$ and $g$ satisfy $(B_1)-(B_7)$ . Then, the system (1.13) possesses a nontrivial weak solution.

In Costa (1994) was studied the existence of solutions for gradient elliptic systems involving coercive potentials in dimension $N\geq 3$ where the growth of the nonlinearities were of polinomial type. In our case we study a Hamiltonian elliptic system in dimension two, and the potential is coercive which is of the class different considered in the systems studied in Cassani and Tarsi (2015), Souza and Ó (2016), Souza (2012). Moreover, due to the fact of the weights $Q_i$ allow us to complement the results with more general class of nonlinearities.

We recall that under the hypothesis $(V_1)$ and $(V_2)$ considered in Cassani and Tarsi (2015) and Chapter 4 (or $(V_1)$ and $1/V\in L^1(\mathbb{R}^2)$ assumed in Souza (2012)) implies that the space $W^1L_V^{2,s}(\mathbb{R}^2)$ (or $H_V^1(\mathbb{R}^2)=\{u\in H^1(\mathbb{R}^2): \int_{\mathbb{R}^2}V(x)|u|^2\,dx<+\infty\}$) is compactly embedded in $L^\lambda(\mathbb{R}^2)$ for any $\lambda\geq 1$. In view of Proposition 1.4 and in order to overcome some difficulties due to lack of embeddings, we compensate with condition $(B_1)$ which will be used to show and control the boundedness of Palais-Smale sequences. Observe also that $(B_1)$ implies the usual assumption, that is, $f(s)=g(s)=o(s)$, as $s\to 0$.

In our argument to prove the existence results, it was crucial a Trudinger-Moser inequality and some embeddings type properties in weighted Lorentz-Sobolev spaces $W^1L_V^{2,s}(\mathbb{R}^2)$. In the proof we used a linking theorem and finite dimensional approximation as in the proofs of Theorems 1.2 and 1.3.

In **Chapter 6**, we establish the existence of the following Hamiltonian system

$$\begin{cases} -\Delta u+V(x)u = g(v), & x\in\mathbb{R}^2,\\ -\Delta v+V(x)v = f(u), & x\in\mathbb{R}^2, \end{cases} \tag{1.14}$$

where the functions $f$ and $g$ possess critical exponential growth and $V$ is a continuous potential.

First, in the systems (1.10) and (1.13) considered in the last chapters, the condition $(V_1)$ says that $V$ is bounded below for a some positive constant and $(V_2)$ gives some conditions of integrability or coercivity.

In Albuquerque, Ó and Medeiros (2016), the authors proved that the system

$$\begin{cases} -\Delta u+V(|x|)u = Q(|x|)g(v), & x\in\mathbb{R}^2,\\ -\Delta v+V(|x|)v = Q(|x|)f(u), & x\in\mathbb{R}^2, \end{cases} \tag{1.15}$$

has a nontrivial solution under the potential $V$ and the weight function $Q$ being radially symmetric and satisfying the following assumptions:

$(V)$ $V\in\mathscr{C}(0,+\infty)$, $V(r)>0$ and there exists $a>-2$ such that

$$\liminf_{r\to+\infty}\frac{V(r)}{r^a}>0.$$

$(Q)$ $Q\in\mathscr{C}(0,+\infty)$ , $Q(r)>0$ and there exists $b<(a-2)/2$ and $b_0>-2$ such that

$$\liminf_{r\to 0^+}\frac{Q(r)}{r^{b_0}}<+\infty \quad \text{and} \quad \limsup_{r\to+\infty}\frac{Q(r)}{r^b}<+\infty.$$

In Souza and Ó (2016), the authors established the existence of nontrivial solutions for Hamiltonian systems of the form

$$\begin{cases} -\Delta u+V(x)u = g(x,v), & x\in\mathbb{R}^2,\\ -\Delta v+V(x)v = f(x,u), & x\in\mathbb{R}^2, \end{cases} \tag{1.16}$$

when the potential $V$ is neither bounded away from zero, nor bounded from above. The nonlinear terms $f(x,s)$ and $g(x,s)$ are superlinear at infinity and have exponential subcritical or critical growth for the case $p=q=2$. Among other things, it is assumed that potential $V$ satisfies the following assumptions

$$\lim_{R\to+\infty}\nu_s(\mathbb{R}^2\backslash\overline{B}_R)=+\infty, \quad \text{for some} \quad s\in[2,+\infty), \tag{1.17}$$

or for any $r>0$ and any sequence $(x_k)\subset\mathbb{R}^2$, which goes to infinity

$$\lim_{k\to+\infty}\nu_s(B_r(x_k))=+\infty, \quad \text{for some} \quad s\in[2,+\infty), \tag{1.18}$$

where $\nu_s$ is defined by, if $\Omega\subset\mathbb{R}^2$ is an open set and $s\geq 2$,

$$\nu_s(\Omega)=\inf_{u\in H_0^1(\Omega)\backslash 0}\frac{\int_\Omega\left(|\nabla u|^2+V(x)u^2\right)dx}{\left(\int_\Omega|u|^s\,dx\right)^{2/s}}$$

and $\nu_s(\varnothing)=+\infty$.

Motivated by the above mentioned results we are interested in studying the system (1.14) for the exponential critical case $p=q=2$, when the potential $V$ can be bounded or can vanish at infinity. More precisely, we assume:

$(V_1)$ $V\in\mathscr{C}(\mathbb{R}^2,\mathbb{R})$ is a radially symmetric positive function.

$(V_2)$ There exist constants $0 < a < 2$, $b \leq a$ and $R_0 > 1$ such that

$$\frac{L_a}{|x|^a} \leq V(x) \leq \frac{L_b}{|x|^b} \quad \text{for all} \quad |x| \geq R_0,$$

where $L_a$ and $L_b$ are positive constants depending on $a, b$ and $R_0$.

$(V_3)$ $V(x) = 1$ if $|x| \leq 1$ and $V(x) \geq 1$ if $1 < |x| < R_0$.

Under these conditions on $V$, we set for $1 < p < +\infty$

$$L^p_{V,rad}(\mathbb{R}^2) := \{u : \mathbb{R}^2 \to \mathbb{R} : u \text{ is measurable, radial and } \int_{\mathbb{R}^2} V(x)|u|^p\, dx < +\infty\}$$

and we consider the following Sobolev space

$$H^1_{V,rad}(\mathbb{R}^2) = \{u \in L^2_{V,rad}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2)\},$$

these spaces were considered by Su, Wang and Willem (2007a), Su, Wang and Willem (2007b).

Concerning the functions $f$ and $g$, we suppose the following assumptions:

$(H_1)$ $f, g \in \mathscr{C}(\mathbb{R})$ and $f(s) = g(s) = 0$ for all $s \leq 0$.

Setting $b^* = 2(2-2b+a)/(2-a)$ where $a$ and $b$ are given by $(V_2)$, consider

$(H_2)$ There exist constants $\mu > b^*$ and $\nu > b^*$ such that

$$0 < \mu F(s) \leq s f(s), \quad 0 < \nu G(s) \leq s g(s), \quad \text{for all} \quad s > 0,$$

where $F(s) = \int_0^s f(t)\, dt$ and $G(s) = \int_0^s g(t)\, dt$.

$(H_3)$ There exist constants $s_1 > 0$ and $M > 0$ such that

$$0 < F(s) \leq M f(s) \quad \text{and} \quad 0 < G(s) \leq M g(s), \quad \text{for all} \quad s > s_1.$$

Setting $\mu$ and $\nu$ given by $(H_2)$ and $a$ given by $(V_2)$, we suppose:

$(H_4)$ There exists $\theta \geq 4a/(2-a)$ such that $f(s) = O(s^{\mu-1+\theta})$ and $g(s) = O(s^{\nu-1+\theta})$ as $s \to 0^+$.

$(H_5)$ There exists $\alpha_0 > 0$ such that

$$\lim_{s\to\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0, \end{cases} \quad \text{and} \quad \lim_{s\to\infty} \frac{g(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0. \end{cases}$$

$(H_6)$ For $\alpha_0 > 0$ given by $(H_5)$, we have

$$\liminf_{t\to+\infty} \frac{t f(t)}{e^{\alpha_0 t^2}} > \frac{4e}{\alpha_0} \quad \text{and} \quad \liminf_{t\to+\infty} \frac{t g(t)}{e^{\alpha_0 t^2}} > \frac{4e}{\alpha_0}.$$

The following theorem contains our main result in Chapter 5.

**Theorem 1.6.** Suppose that $V$ satisfies $(V_1)-(V_3)$ and $f$ and $g$ satisfy $(H_1)-(H_6)$. Then, there exists $L^* = L^*(f,g,\mu,\nu,\alpha_0,\theta,a,b,R_0) > 0$ such that system (6.1) possesses a nontrivial weak solution $(u,v) \in H^1_{V,rad}(\mathbb{R}^2) \times H^1_{V,rad}(\mathbb{R}^2)$ provided that $L_a \geq L^*$, namely $(u,v) \in H^1_{V,rad}(\mathbb{R}^2) \times H^1_{V,rad}(\mathbb{R}^2)$ satisfies

$$\int_{\mathbb{R}^2} \left(\nabla u \nabla \psi + V(x) u \psi + \nabla v \nabla \phi + V(x) v \phi\right) dx = \int_{\mathbb{R}^2} \left(f(u)\phi + g(v)\psi\right) dx,$$

for all $(\phi,\psi) \in H^1_{V,rad}(\mathbb{R}^2) \times H^1_{V,rad}(\mathbb{R}^2)$.

Our theorem may be seen as complement of the above mentioned results. We recall that condition $(V_2)$ allows $V(x) \to 0$ as $|x| \to \infty$. The condition $(V_2)$ in the system (1.10) and its relationed works considered in Chapter 4 requires that $V$ be large at infinity. We also note when $Q \equiv 1$ in condition $(Q)$, this implies that $V(x) \to +\infty$ as $|x| \to \infty$. Thus, although the class of Hamiltonian systems considered in Albuquerque, Ó and Medeiros (2016) is very general, the main result in that paper can not be applied to the model case $V(x) = L/|x|^a$, for $|x|$ sufficiently large, considered here.

In the recent paper Souza and Ó (2016), a fairly general result was proved on system (1.16), but under the hypotheses (1.17) and (1.18), which implies that $V$ is large at infinity, as we can verified with the following example: taking $u_0 \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ such that

$$u_0(x,y) = 1 \text{ if } |(x,y)| \leq \frac{2}{3} \quad \text{and} \quad u_0(x,y) = 0 \quad \text{if} \quad |(x,y)| \geq \frac{3}{4}.$$

Setting $(u_k) \subset \mathscr{C}_0^\infty(\mathbb{R}^2)$ defined by $u_k(x,y) = u_0(x-k, y-k)$ for $k \in \mathbb{N}$. Thus, for every $k \in \mathbb{N}$ and for all $s \in [2,\infty)$, we have

$$supp\ u_k \subset B_1(k,k), \tag{1.19}$$

$$u_k \in H^1_0(\Omega)\backslash\{0\} \quad \text{and} \quad \int_\Omega |u_k|^s\, dx \geq 1 \quad \text{for all} \quad \Omega \supset B_1(k,k). \tag{1.20}$$

If $V$ is bounded near infinity there exist $k_0 > 0$ and $C > 1$ such that

$$|V(x)| \leq C \quad \text{for all} \quad |x| \geq k_0. \tag{1.21}$$

For given $R > 0$ let $k_1 > \max\{R, k_0\} + 1$ using (1.19), (1.20) and (1.21) we have that

$$\begin{aligned} \nu_s(\mathbb{R}^2 \backslash \overline{B}_R) &\leq \frac{\int_{\mathbb{R}^2 \backslash \overline{B}_R} \left(|\nabla u_{k_1}|^2 + V(x) u_{k_1}^2\right) dx}{\left(\int_{\mathbb{R}^2 \backslash \overline{B}_R} |u_{k_1}|^s\, dx\right)^{2/s}} \\ &\leq \int_{\mathbb{R}^2 \backslash \overline{B}_R} \left(|\nabla u_{k_1}|^2 + C u_{k_1}^2\right) dx \\ &\leq \int_{B_1(k_1,k_1)} \left(|\nabla u_{k_1}|^2 + C u_{k_1}^2\right) dx \leq C\|u_0\|_{H^1_0(B_1)}, \end{aligned}$$

which contradicts (1.17). Note also that by (1.19), (1.20) and (1.21), for every $k > k_0 + 1$, we get

$$\nu_s(B_1(k,k)) \leq \frac{\int_{B_1(k,k)} \left(|\nabla u_k|^2 + V(x)u_k^2\right) dx}{\left(\int_{B_1(k,k)} |u_k|^s \, dx\right)^{2/s}} \leq C\|u_0\|_{H_0^1(B_1)},$$

which contradicts (1.18).

Observe that, the associated functional with (1.14) is strongly indefinite and the space $H_V^1(\mathbb{R}^2)$ presents some phenomenons such as lack of compactness. In order to prove the existence, we combine a truncation argument with a finite-dimensional approximation and Linking theorem. The truncation argument employed here is an adaptation of the reasoning used in Alves and Souto (2012) to study the existence of positive solutions to a scalar equation. We point out that this chapter is contained in the accepted paper Leuyacc and Soares (2017).

CHAPTER 2

# LORENTZ AND LORENTZ-SOBOLEV SPACES

In this chapter we introduce and prove some properties which will be important in the development of this thesis.

## 2.1 Distribution functions and decreasing rearrangement

Let $X = (X, \Sigma, \mu)$ be a $\sigma-$finite measure space, denote by $\mathscr{M}(X, \overline{\mathbb{R}})$ the collection of all extended real-valued $\mu$-measurable functions on $X$ and $\mathscr{M}_0(X, \mathbb{R})$ the class of functions in $\mathscr{M}(X, \overline{\mathbb{R}})$ that are finite $\mu$-almost everywhere in $X$. As usual, any two functions coinciding almost everywhere in $X$ will be identified. Moreover, natural vector space operations are well defined on $\mathscr{M}_0(X, \mathbb{R})$.

**Definition 2.1.** The distribution function $\mu_\phi$ of a function $\phi \in \mathscr{M}_0(X, \mathbb{R})$ is defined by

$$\mu_\phi(t) := \mu\{x \in X : |\phi(x)| > t\}, \quad \text{for} \quad t \geq 0.$$

The distribution function satisfies the following properties (see Hunt (1966), Bennett and Sharpley (1988)).

**Proposition 2.2.** Let $\phi, \psi \in \mathscr{M}_0(X, \mathbb{R})$. Then, the distribution function $\mu_\phi$ is nonnegative, non-increasing and continuous from the right on $[0, +\infty)$. Furthermore,

$(i)$ If $|\phi(x)| \leq |\psi(x)|$ $\mu$-almost everywhere in $X$, then $\mu_\phi(t) \leq \mu_\psi(t)$, for all $t \geq 0$.

$(ii)$ $\mu_{\lambda\phi}(t) = \mu_\phi(\frac{t}{|\lambda|})$, for all $t \geq 0$ and $\lambda \neq 0$.

$(iii)$ $\mu_{\phi+\psi}(t_1 + t_2) \leq \mu_\phi(t_1) + \mu_\psi(t_2)$, for all $t_1, t_2 \geq 0$.

$(iv)$ $\mu_{\phi\psi}(t_1 t_2) \leq \mu_\phi(t_1)\mu_\phi(t_2)$, for all $t_1, t_2 \geq 0$.

$(v)$ Let $(\phi_n)$ be a sequence in $\mathscr{M}_0(X,\mathbb{R})$ such that $|\phi(x)| \le \liminf_{n\to\infty} |\phi_n(x)|$, $\mu$-a.e in $X$. Then, $\mu_\phi(t) \le \liminf_{n\to\infty} \mu_{\phi_n}(t)$ a.e in $\mathbb{R}^+$. In particular, if $|\phi_n| \nearrow |\phi|$ $\mu$-a.e in $X$. Then, $\mu_\phi \nearrow \mu_{\phi_n}$ a.e in $\mathbb{R}^+$.

Using distribution functions we can consider the following spaces:

**Definition 2.3.** (**Weak $L^p$-spaces**) If $f \in \mathscr{M}_0(X,\mathbb{R})$ , let

$$[f]_p = [f]_{p,X} = \sup_{t>0} t\left[\mu_f(t)\right]^{1/p}.$$

We define the weak-$L^p$ as follows:

$$Weak - L^p(X) := \{f : f \in \mathscr{M}_0(X,\mathbb{R}),\ [f]_p < +\infty\}.$$

More details about these spaces can be found on Adams and Fournier (2003).

**Definition 2.4.** The decreasing rearrangement of $\phi \in \mathscr{M}_0(X,\mathbb{R})$ is defined by

$$\phi^*(s) := \inf\{t \ge 0 : \mu_\phi(t) \le s\}, \quad \text{for} \quad s \ge 0.$$

The decreasing rearrangement satisfies the following properties (see Hunt (1966), Bennett and Sharpley (1988)).

**Proposition 2.5.** Let $\phi, \psi \in \mathscr{M}_0(X,\mathbb{R})$. Then, the distribution function $\phi^*$ is nonnegative, non-increasing and continuous from the right on $[0,+\infty)$. Furthermore,

$(i)$ If $\mu_\phi(t) \le \mu_\psi(t)$, for all $t \ge 0$, then $\phi^*(s) \le \psi^*(s)$, for all $s \ge 0$.

$(ii)$ $(\lambda\phi)^* = |\lambda|\phi^*$, for all $\lambda \in \mathbb{R}$.

$(iii)$ $(\phi+\psi)^*(s_1+s_2) \le \phi^*(s_1) + \psi^*(s_2)$, for all $s_1, s_2 \ge 0$.

$(iv)$ $(\phi\psi)^*(s_1 s_2) \le \phi^*(s_1)\phi^*(s_2)$, for all $s_1, s_2 \ge 0$.

$(v)$ Let $(\phi_n)$ a sequence in $\mathscr{M}_0(X,\mathbb{R})$ such that $|\phi| \le \liminf_{n\to\infty} |\phi_n|$, $\mu$-a.e in $X$. Then, $\phi^* \le \liminf_{n\to\infty} \phi_n^*$ a.e in $\mathbb{R}^+$. In particular, if $|\phi_n| \nearrow |\phi|$ $\mu$-a.e in $X$. Then, $\phi_n^* \nearrow \phi^*$ a.e in $\mathbb{R}^+$.

In the following example, we compute the distribution and decreasing rearrangement of a simple function.

**Example 2.6.** Let $\phi \in \mathscr{M}_0(X,\mathbb{R})$ be a simple function, that is, $\phi$ is a linear combination of characteristic functions, in particular, we can write

$$|\phi| = \sum_{j=1}^{n} a_j \chi_{E_j}$$

where $a_1 > a_2 > \cdots a_n > 0$ and $E_j = \{x \in X : |\phi|(x) = a_j\}$.

Indeed, since $|\phi(x)| \le a_1$ for all $x \in X$, for each $t \ge a_1$, we have

$$\mu_\phi(t) = \mu\{x \in X : |\phi(x)| > a_1\} = \mu(\varnothing) = 0.$$

Let $a_2 \le t < a_1$. Thus,

$$\mu_\phi(t) = \mu\{x \in X : |\phi(x)| > t\} = \mu\{x \in X : |\phi(x)| = a_1\} = \mu(E_1).$$

In general, if $a_{j+1} \le t < a_j$ for $j = 1, \cdots, n$ $(a_{n+1} = 0)$, we have

$$\mu_\phi(t) = \mu\{x \in X : |\phi(x)| > t\} = \mu\{x \in X : |\phi(x)| = a_1, a_2, \ldots, a_j\} = \sum_{i=1}^{j} \mu(E_i).$$

Thus,

$$\mu_\phi(t) = \sum_{j=1}^{n} m_j \chi_{[a_{j+1}, a_j)}, \quad \text{where} \quad m_j = \sum_{i=1}^{j} \mu(E_i).$$

If $0 \le s < m_1$, we have

$$\phi^*(s) = \inf\{t \ge 0 : \sum_{j=1}^{n} m_j \chi_{[a_{j+1}, a_j)}(t) \le s\} = a_1.$$

If $m_1 \le s < m_2$, we have

$$\phi^*(s) = \inf\{t \ge 0 : \sum_{j=1}^{n} m_j \chi_{[a_{j+1}, a_j)}(t) \le s\} = a_2.$$

In general, if $m_{j-1} \le s < m_j$ for $j = 1, \ldots, n$ $(m_0 = 0)$, we have

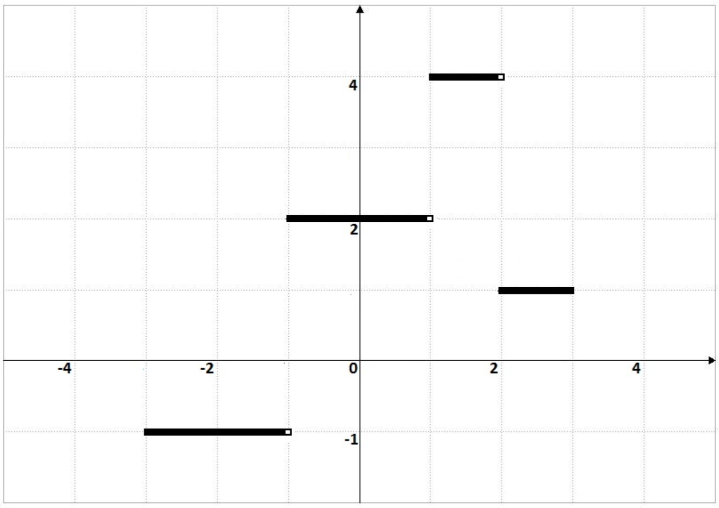
$$\phi^*(s) = \inf\{t \ge 0 : \sum_{j=1}^{n} m_j \chi_{[a_{j+1}, a_j)}(t) \le s\} = a_j.$$

Thus,

$$\phi^*(s) = \sum_{j=1}^{n} a_j \chi_{[m_{j-1}, m_j)}, \quad \text{where} \quad m_j = \sum_{i=1}^{j} \mu(E_i).$$

See the following figures for a specific example:

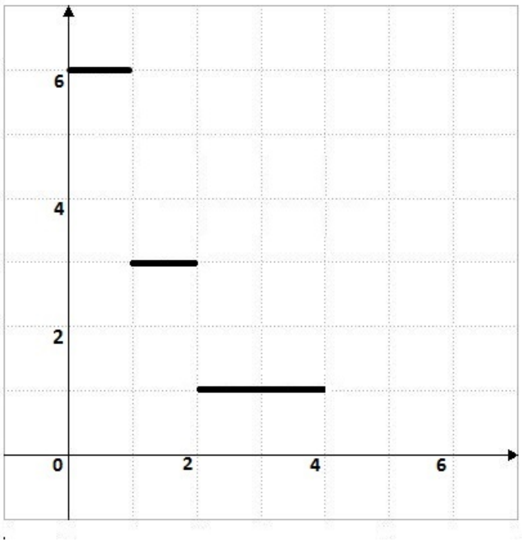
Figure 1 – A simple function $\phi$.



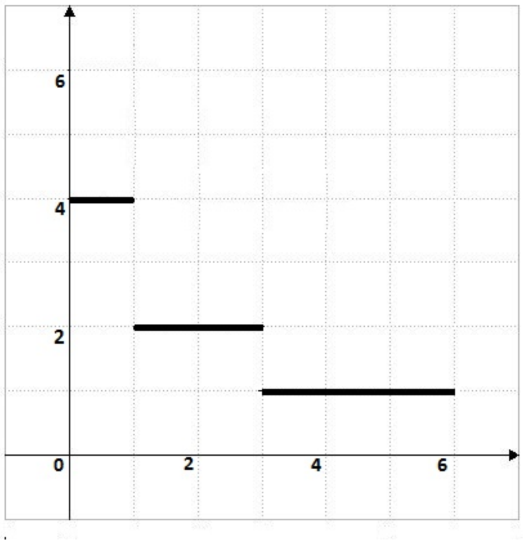

(a) $\phi = -\chi_{[-3,-1)} + 2\chi_{[-1,1)} + 4\chi_{[1,2)} + \chi_{[2,3]}$.

Source: Elaborated by the author.

Figure 2 – Distribution function and decreasing rearrangement of $\phi$.




(a) $\mu_\phi = 6\chi_{[0,1)} + 3\chi_{[1,2)} + \chi_{[2,4]}$.




(b) $\phi^* = 4\chi_{[0,1)} + 2\chi_{[1,3)} + \chi_{[3,6)}$.

Source: Elaborated by the author.

**Example 2.7.** Let $r > 0$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\phi(x) = \Big(\frac{1}{1+\pi|x|^2}\Big)^r.$$

Then,

$$\phi^*(s) = \frac{1}{(1+s)^r}, \quad \text{for all} \quad s \geq 0.$$

Indeed, since $|\phi(x)| \leq 1$ for all $x \in \mathbb{R}^2$, for each $t \geq 1$ we have

$$\mu_\phi(t) = |\{x \in \mathbb{R}^2 : |\phi(x)| > t\}| = |\varnothing| = 0.$$

If $0 < t < 1$, we have

$$\begin{aligned}\mu_\phi(t) &= \left|\left\{x \in \mathbb{R}^2 : |\phi(x)| > t\right\}\right| \\ &= \left|\left\{x \in \mathbb{R}^2 : \frac{1}{1+\pi|x|^2} > t^{1/r}\right\}\right| \\ &= \left|\left\{x \in \mathbb{R}^2 : |x| < \sqrt{\frac{1}{\pi}\left(\frac{1}{t^{1/r}} - 1\right)}\right\}\right| \\ &= \frac{1}{t^{1/r}} - 1.\end{aligned}$$

On the other hand, let $s \geq 0$ fixed and $t \geq 0$ such that $\mu_\phi(t) \leq s$. Thus, there are two possibilities: $t \geq 1$ or

$$\frac{1}{t^{1/r}} - 1 \leq s,$$

that is,

$$\frac{1}{(1+s)^r} \leq t.$$

Thus, we conclude that

$$\phi^*(s) = \inf\{t \geq 0 : \mu_\phi(t) \leq s\} = \frac{1}{(1+s)^r}.$$

**Example 2.8.** Let $f(x) = 1 - e^{-|x|}$ defined on $\mathbb{R}$. Then, for each $t \geq 1$ we have $\mu_f(t) = 0$ and for each $0 \leq t < 1$ we have

$$\mu_f(t) = |\{x \in \mathbb{R}^2 : 1 - e^{-|x|} > t\}| = \left|\left\{x \in \mathbb{R}^2 : |x| > \ln\left(\frac{1}{1-t}\right)\right\}\right| = +\infty.$$

Therefore, $f^*(s) = 1$ for all $s \geq 0$. In particular, we conclude that, if $f \in \mathscr{M}_0(X,\mathbb{R})$ not necessarily $\mu_f$ is an almost everywhere finite-valued function.

Let $\phi$ be a simple function as given by Example 2.6. Then,

$$\int_0^{+\infty} \phi^*(s)\,ds = \sum_{j=1}^{n} a_j(m_j - m_{j-1}) = \sum_{j=1}^{n} a_j \mu(E_j) = \int_X |\phi(x)|\,d\mu(x). \tag{2.1}$$

A more general result is given by the following Lemma:

**Lemma 2.9.** Let $\phi \in \mathscr{M}_0(X,\mathbb{R})$ and $G : [0,+\infty) \to [0,+\infty)$ be a nondecreasing function such that $G(|\phi|) \in L^1(X)$ and $G(0) = 0$. Then, $G(\phi^*) \in L^1([0,+\infty))$ and

$$\int_0^{+\infty} G(\phi^*(s))\,ds = \int_X G(|\phi(x)|)\,d\mu(x).$$

***Proof.*** Let $\phi$ be a simple function, using notation given by Example 2.6 and the fact that $G(0) = 0$ we have

$$\big(G(|\phi(x)|)\big)^* = \Big(\sum_{j=1}^{n} G(a_j)\chi_{E_j}(x)\Big)^* = \sum_{j=1}^{n} G(a_j)\chi_{[m_{j-1},m_j)}(s) = G(\phi^*(s)). \tag{2.2}$$

Thus, from (2.2) and (2.1) we obtain

$$\int_0^{+\infty} G(\phi^*(s))\,ds = \int_0^{+\infty} \big(G(|\phi(s)|)\big)^*\,ds = \int_X G(|\phi(x)|)\,d\mu(x). \tag{2.3}$$

In the general case, there exists a increasing sequence $(|\phi_n|)$ of simple functions converging almost everywhere to $|\phi|$. By Proposition 2.5-($v$), $\phi_n^*$ converges monotonically to $\phi^*$ almost everywhere. Consequently, the sequences $G(|\phi_n|), G(\phi_n^*)$ converges monotonically to $G(|\phi|)$ and $G(\phi^*)$ respectively. Moreover, $G(|\phi_n|)$ and $G(\phi_n^*)$ are simple functions. By (2.3) and Monotone converge theorem, we have

$$\int_0^{+\infty} G(\phi^*)\,ds = \lim_{n\to\infty}\int_0^{+\infty} G(\phi_n^*)\,ds = \lim_{n\to\infty}\int_X G(|\phi_n|)\,d\mu(x) = \int_X G(|\phi|)\,d\mu(x).$$

■

Reducing to simple functions and taking limit we can obtain the following result:

**Lemma 2.10.** (**Hardy-Littlewood inequality**) (See Hunt (1966).) Let $\phi, \psi \in \mathscr{M}_0(X,\mathbb{R})$. Then,

$$\int_X |\phi(x)\psi(x)|\,d\mu(x) \le \int_0^{+\infty} \phi^*(s)\psi^*(s)\,ds.$$

## 2.2 Lorentz spaces

In this section we present Lorentz spaces which were introduced by Lorentz (1950). For simplicity we consider throughout this section the following measure space $(X,\mu) = (\Omega, m)$ where $\Omega$ is a measurable subset in $\mathbb{R}^N$ with $N \ge 1$ and $m$ is the Lebesgue measure.

**Definition 2.11.** Let $1 < p < +\infty$, $1 \le q \le +\infty$. The Lorentz space $L^{p,q}(\Omega)$ is the collection of functions $\phi \in \mathscr{M}_0(\Omega,\mathbb{R})$ such that $\|\phi\|_{p,q} < +\infty$ where

$$\|\phi\|_{p,q} = \begin{cases} \left(\displaystyle\int_0^{+\infty} [\phi^*(t)t^{1/p}]^q \frac{dt}{t}\right)^{1/q}, & \text{if} \quad 1 \le q < +\infty, \\ \\ \sup\limits_{t>0} t^{1/p}\phi^*(t), & \text{if} \quad q = +\infty. \end{cases} \tag{2.4}$$

In particular, two functions in $L^{p,q}(\Omega)$ are identified if they are equal almost everywhere in $\Omega$.

For a mensurable function $f = (f_1, \cdots, f_N) : \Omega \to \mathbb{R}^N$, we say that $f \in L^{p,q}(\Omega)$ if and only if $|f| \in L^{p,q}(\Omega)$ and we set

$$\|f\|_{p,q} := \||f|\|_{p,q}.$$

Therefore, $f \in L^{p,q}(\Omega)$ if and only if $f_i \in L^{p,q}(\Omega)$ for $1 \le i \le N$.

**Proposition 2.12.** The map $\|\cdot\|$ given by (2.4) is a quasinorm and $L^{p,q}(\Omega)$ is a vector space.

***Proof.*** Let $1 < p < +\infty$ and $1 \leq q < +\infty$.

**(i)** It is clear that $\|\phi\|_{p,q} \geq 0$ for all $\phi \in L^{p,q}(\Omega)$ and $\|\phi\|_{p,q} = 0$ if and only if $\phi = 0$.

**(ii)** Let $\lambda \in \mathbb{R}$, by Proposition 2.5-(*ii*) we have $(\lambda\phi)^*(t) = |\lambda|\phi^*$. Then,

$$\|\lambda\phi\|_{p,q} = \Big(\int_0^{+\infty} [(\lambda\phi)^*(t)t^{1/p}]^q \frac{dt}{t}\Big)^{1/q} = |\lambda|\|\phi\|_{p,q}.$$

**(iii)** Let $\phi, \psi \in L^{p,q}(\Omega)$ using Proposition 2.5-(*iii*), we have

$$\begin{aligned}
\|\phi+\psi\|_{p,q}^q &= \int_0^{+\infty} \big[(\phi+\psi)^*(t)t^{1/p}\big]^q \frac{dt}{t} \\
&\leq \int_0^{+\infty} \Big[\Big(\phi^*(\frac{t}{2}) + \psi^*(\frac{t}{2})\Big)t^{1/p}\Big]^q \frac{dt}{t} \\
&= 2^{\frac{q}{p}} \int_0^{+\infty} \Big[\big(\phi^*(s)+\psi^*(s)\big)s^{1/p}\Big]^q \frac{ds}{s} \\
&\leq 2^{\frac{q}{p}+q-1} \int_0^{+\infty} \Big(\big[\phi^*(s)s^{1/p}\big]^q + \big[\psi^*(s)s^{1/p}\big]^q\Big) \frac{ds}{s} \\
&= 2^{\frac{q}{p}+q-1}\Big(\|\phi\|_{p,q}^q + \|\psi\|_{p,q}^q\Big).
\end{aligned}$$

Hence,

$$\|\phi+\psi\|_{p,q} \leq 2^{\frac{1}{p}+1-\frac{1}{q}}\big(\|\phi\|_{p,q} + \|\psi\|_{p,q}\big), \quad \text{for all} \quad \phi,\psi \in L^{p,q}(\Omega).$$

The properties $(i)$-$(iii)$ are also true for the case when $1 < p < +\infty$ and $q = +\infty$. Thus, $\|\cdot\|_{p,q}$ represents a quasinorm. Moreover, if $\phi, \psi \in L^{p,q}(\Omega)$ and $\lambda \in \mathbb{R}$, using $(ii)$ and $(iii)$, we have $\phi + \psi$ and $\lambda\psi$ belong to $L^{p,q}(\Omega)$, that is, $L^{p,q}(\Omega)$ is a vector space. ■

The following result says that $\|\cdot\|_{p,q}$ is a norm for some cases.

**Proposition 2.13.** (See Bennett and Sharpley (1988).) The map $\|\cdot\|_{p,q}$ is a norm if and only if $1 \leq q \leq p$.

Now, we build a topology $\mathscr{T}$ in Lorentz spaces. For every $x \in L^{p,q}(\Omega)$ and every $r > 0$, we consider the following open ball:

$$B_r(x) = \{y \in L^{p,q}(\Omega) : \|y - x\|_{p,q} < r\}$$

and we set the collection of balls

$$\mathscr{B} := \{B_r(x) : x \in L^{p,q}(\Omega), r > 0\}.$$

A subset $U$ in $L^{p,q}(\Omega)$ is said to be open in $L^{p,q}(\Omega)$ ($U \in \mathscr{T}$) if and only if

$$U = \bigcup_{i \in I}\big(B_{i_1} \cap B_{i_2} \cdots \cap B_{i_{n_i}}\big), \text{ where } B_{i_k} \in \mathscr{B} \text{ and } I \text{ is an index set.}$$

Consequently, $L^{p,q}(\Omega)$ turns out a topological vector space. Note that, each ball $B_r(x)$ is an open set. Thus, we say that, the sequence $(\phi_n) \subset L^{p,q}(\Omega)$ converges to $\phi \in L^{p,q}(\Omega)$, in the topology $\mathscr{T}$ if and only if $\|\phi_n - \phi\|_{p,q} \to 0$.

In the following we define a metric $d$ such that $\big(L^{p,q}(\Omega), d\big)$ is a metric space.

**Definition 2.14.** Let $1 < p < +\infty$, $1 \leq q \leq +\infty$, $\Omega \subset \mathbb{R}^N$ and $\phi \in \mathscr{M}_0(\Omega, \mathbb{R})$, the maximal function is defined by

$$\phi^{**}(t) := \frac{1}{t} \int_0^{+\infty} \phi^*(s)\, ds, \quad \text{for all} \quad t > 0.$$

**Definition 2.15.** Let $1 < p < +\infty$, $1 \leq q \leq +\infty$, we define

$$\|\phi\|^*_{p,q} = \begin{cases} \left( \displaystyle\int_0^{+\infty} [\phi^{**}(t) t^{1/p}]^q \frac{dt}{t} \right)^{1/q}, & \text{if} \quad 1 \leq q < +\infty, \\ \\ \displaystyle\sup_{t>0} t^{1/p} \phi^{**}(t), & \text{if} \quad q = +\infty. \end{cases}$$

**Proposition 2.16.** (See Adams and Fournier (2003).) Let $1 < p < +\infty$ and $1 \leq q \leq +\infty$. Then, the functional $\|\cdot\|^*_{p,q}$ represents a norm on $L^{p,q}(\Omega)$. Moreover, $L^{p,q}(\Omega)$ endowed with this norm is a Banach space and

$$\|\phi\|_{p,q} \leq \|\phi\|^*_{p,q} \leq \frac{p}{p-1} \|\phi\|_{p,q}, \quad \text{for all} \quad \phi \in L^{p,q}(\Omega). \tag{2.5}$$

Setting the metric

$$\begin{array}{rcl} d \ : L^{p,q}(\Omega) \times L^{p,q}(\Omega) & \to & \mathbb{R}^+ \\ (\phi, \psi), & \mapsto & \|\phi - \psi\|^*_{p,q}. \end{array}$$

Let $\widetilde{\mathscr{T}}$ the topology induced by the metric $d$. Using (2.5), we have the topologies $\mathscr{T}$ and $\widetilde{\mathscr{T}}$ defined on $L^{p,q}(\Omega)$ are equals.

**Remark 2.17.** **(i)** The Lorentz space $L^{\infty,q}(\Omega)$ with $1 < q < +\infty$ is not interesting, since the only function in this space is given by the zero function.

**(ii)** Using Lemma 2.9 with $G(s) = s^p$, $p > 1$, we have

$$\|\phi\|_{p,p} = \left( \int_0^{+\infty} [\phi^*(t)]^p \, dt \right)^{1/p} = \left( \int_\Omega |\phi(x)|^p \, dx \right)^{1/p} = \|\phi\|_p.$$

This implies,

$$L^{p,p}(\Omega) = L^p(\Omega).$$

Thus, Lorentz spaces are intermediate between $L^p$-spaces.

**(iii)** For $1 < p < +\infty$, we have

$$\|\phi\|_{p,\infty} = \sup_{t>0} t^{1/p} \phi^*(t) = \sup_{t>0} t [\mu_\phi(t)]^{1/p} = [\phi]_p.$$

Thus,

$$L^{p,\infty}(\Omega) = \text{weak-}L^p(\Omega).$$

**(iv)** Given a function $\phi$ defined on $\Omega$, we denote by $\overline{\phi}$ its extension outside $\Omega$, that is,

$$\overline{\phi}(x) = \begin{cases} \phi(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

In particular, if $\Omega$ is a bounded set, we have

$$(\overline{\phi})^*(t) = \begin{cases} \phi^*(t), & 0 \le t \le |\Omega| \\ 0, & t > |\Omega|. \end{cases}$$

Thus,

$$\|\overline{\phi}\|_{L^{p,q}(\mathbb{R}^N)} = \left(\int_0^{+\infty} \left[(\overline{\phi})^*(t)t^{1/p}\right]^p \frac{dt}{t}\right)^{1/p} = \left(\int_0^{+\infty} \left[\phi^*(t)t^{1/p}\right]^p \frac{dt}{t}\right)^{1/p} = \|\phi\|_{L^{p,q}(\Omega)}.$$

**Lemma 2.18.** (**Hölder's inequality in Lorentz spaces**) Let $1 < p,q < +\infty$ and $p'$, $q'$ denoted the conjugate exponents defined by $p' = p/(p-1)$ and $q' = q/(q-1)$. If $f \in L^{p,q}(\Omega)$ and $q \in L^{p',q'}(\Omega)$. Then, $fg \in L^1(\Omega)$ and

$$\int_\Omega |f(x)g(x)|\,dx \le \|f\|_{p,q}\|g\|_{p',q'}.$$

***Proof.*** Using Lemma 2.10 we have

$$\int_\Omega |f(x)g(x)|\,dx \le \int_0^{+\infty} f^*(t)g^*(t)\,dt = \int_0^{+\infty} \frac{f^*(t)t^{1/p}}{t^{1/q}}\frac{g^*(t)t^{1/p'}}{t^{1/q'}}\,dt. \tag{2.6}$$

By classical Hölder's inequality with $1/q + 1/q' = 1$ we obtain

$$\int_0^{+\infty} \frac{f^*(t)t^{1/p}}{t^{1/q}}\frac{g^*(t)t^{1/p'}}{t^{1/q'}}\,ds \le \left(\int_0^{+\infty} \left(f^*(t)t^{1/p}\right)^q \frac{dt}{t}\right)^{1/q}\left(\int_0^{+\infty} \left(g^*(t)t^{1/p'}\right)^{q'} \frac{dt}{t}\,dt\right)^{1/q'} \tag{2.7}$$

Joining (2.6) and (2.7) the claim follows. ■

**Lemma 2.19.** (**Generalized Hölder's inequality in Lorentz spaces**) Let the following constants $1 < p, p_1, p_2, q, q_1, q_2 < +\infty$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

If $f \in L^{p_1,q_1}(\Omega)$ and $g \in L^{p_2,q_2}(\Omega)$. Then, $fg \in L^{p,q}(\Omega)$ and

$$\|fg\|_{p,q} \le 2^{1/p}\|f\|_{p_1,q_1}\|g\|_{p_2,q_2}.$$

***Proof.*** Using Proposition 2.5-$(iv)$ and Hölder's inequality, we have

$$\begin{aligned}
\int_0^{+\infty} \left[(fg)^*(t)t^{1/p}\right]^q \frac{dt}{t} &\le \int_0^{+\infty} \left[f^*\left(\frac{t}{2}\right)g^*\left(\frac{t}{2}\right)t^{1/p}\right]^q \frac{dt}{t} \\
&\le 2^{q/p}\int_0^{+\infty} \left[f^*(t)g^*(t)t^{1/p}\right]^q \frac{dt}{t} \\
&\le 2^{q/p}\int_0^{+\infty} \left[\frac{f^*(t)t^{1/p_1}}{t^{1/q_1}}\frac{g^*(t)t^{1/p_2}}{t^{1/q_2}}\right]^q dt \\
&\le 2^{q/p}\left(\int_0^{+\infty} \left[f^*(t)t^{1/p_1}\right]^{q_1}\frac{dt}{t}\right)^{q/q_1}\left(\int_0^{+\infty} \left[g^*(t)t^{1/p_2}\right]^{q_2}\frac{dt}{t}\right)^{q/q_2}.
\end{aligned}$$

Then, the claim follows. ■

**Remark 2.20.** **(i)** The claim in Lemma 2.18 is still valid if we consider 1 and $+\infty$ as conjugated exponents.

**(ii)** The claim in Lemma 2.19 is still valid if we consider $q = q_1$ and $q_2 = +\infty$ or $q = q_2$ and $q_1 = +\infty$.

**Lemma 2.21.** (See Hunt (1966).) Let $1 \leq q_1 \leq q_2 \leq +\infty$ and $p > 1$. Then, the following embedding is continuous

$$L^{p,q_1}(\Omega) \hookrightarrow L^{p,q_2}(\Omega).$$

**Lemma 2.22.** Let $|\Omega| < +\infty$, $1 < p_1 < p_2 < +\infty$ and $1 \leq q_1 \leq q_2 \leq +\infty$. Then, the following embedding is continuous

$$L^{p_2,q_2}(\Omega) \hookrightarrow L^{p_1,q_1}(\Omega).$$

***Proof.*** Let $f \in L^{p_1,q_1}(\Omega)$ and taking $p_3$ and $q_3$ such that

$$\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3} \quad \text{and} \quad \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}.$$

Using Lemma 2.18 we have

$$\|f\|_{p_1,q_1} \leq 2^{1/p_1} \|f\|_{p_2,q_2} \|1\|_{p_3,q_3} = 2^{1/p_1} |\Omega|^{1/q_3} \|f\|_{p_2,q_2}.$$

and the embedding follows. ∎

**Lemma 2.23.** Let $|\Omega| < +\infty$, $1 < p < +\infty$ and $1 \leq q \leq +\infty$. Then, the following embeddings

$$L^{p,q}(\Omega) \hookrightarrow L^{p-\delta}(\Omega), \quad \text{for all} \quad 0 < \delta \leq p-1$$

are continuous.

***Proof.*** If $1 \leq q \leq p$, by Lemma 2.21, we have for all $0 < \delta \leq p-1$

$$L^{p,q}(\Omega) \hookrightarrow L^{p,p}(\Omega) = L^{p}(\Omega) \hookrightarrow L^{p-\delta}(\Omega).$$

If $q > p$, by Lemma 2.22 we have for all $0 < \delta \leq p-1$

$$L^{p,q}(\Omega) \hookrightarrow L^{p-\delta,p-\delta}(\Omega) = L^{p-\delta}(\Omega).$$

Then, for all $0 < \delta \leq p-1$

$$L^{p,q}(\Omega) \hookrightarrow L^{p-\delta}(\Omega).$$

∎

**Lemma 2.24.** Let $|\Omega| < +\infty$, $1 < p < +\infty$ and $1 \leq q \leq +\infty$. Then, the following embeddings

$$L^{p+\delta}(\Omega) \hookrightarrow L^{p,q}(\Omega), \quad \text{for all} \quad \delta > 0$$

are continuous.

***Proof.*** If $1 \leq p \leq q$, by Lemma 2.21 we have

$$L^{p+\delta}(\Omega) \hookrightarrow L^p(\Omega) = L^{p,p}(\Omega) \hookrightarrow L^{p,q}(\Omega), \quad \text{for all} \quad \delta > 0. \tag{2.8}$$

If $p > q$, by Lemma 2.22 we have

$$L^{p+\delta}(\Omega) = L^{p+\delta,p+\delta}(\Omega) \hookrightarrow L^{p,q}(\Omega), \quad \text{for all} \quad \delta > 0. \tag{2.9}$$

Joining (2.8) and (2.9) the lemma follows. ∎

**Proposition 2.25.** (See Hunt (1966).) Let $1 < p < +\infty$, $1 \leq q < +\infty$. Then, the set of simple functions are dense in $L^{p,q}(\Omega)$.

**Proposition 2.26.** Let $1 < p < +\infty$, $1 \leq q < +\infty$ and $\Omega$ a open subset in $\mathbb{R}^N$. Then, $\mathscr{C}_c^\infty(\Omega)$ is dense in $L^{p,q}(\Omega)$.

***Proof.*** Let $f \in L^{p,q}(\Omega)$ and $\varepsilon > 0$, by Lemma 2.25 there exists a simple function $s$ defined on $\Omega$ with compact support such that

$$\|f - s\|_{L^{p,q}(\Omega)} < \frac{\varepsilon}{4}. \tag{2.10}$$

Let $K = \operatorname{supp}(f)$ and consider $\Omega' = \bigcup_{x \in K} \big(B_1(x) \cap \Omega\big)$. Thus, $\Omega'$ is an open bounded set such that $K \subset \Omega' \subset \Omega$. From Lemma 2.24 the space $L^{p+1}(\Omega')$ continuously embedded in $L^{p,q}(\Omega')$, denoting by $S_p > 0$ its best embedding constant. Since $s \in L^{p+1}(\Omega')$ using the density of $\mathscr{C}_c^\infty(\Omega')$ in $L^{p+1}(\Omega')$, there exists $g \in \mathscr{C}_c^\infty(\Omega') \subset \mathscr{C}_c^\infty(\Omega)$ such that

$$\|s - g\|_{L^{p+1}(\Omega')} < \frac{\varepsilon}{4S_p}. \tag{2.11}$$

Note that,

$$\|s - g\|_{L^{p,q}(\Omega)} = \|s - g\|_{L^{p,q}(\Omega')} \tag{2.12}$$

Thus, combining (2.10), (2.11) and (2.12) we obtain

$$\begin{aligned}
\|f - g\|_{L^{p,q}(\Omega)} &\leq 2\|f - s\|_{L^{p,q}(\Omega)} + 2\|s - g\|_{L^{p,q}(\Omega)} \\
&< \frac{\varepsilon}{2} + 2\|s - g\|_{L^{p,q}(\Omega')} \\
&< \frac{\varepsilon}{2} + 2S_p\|s - g\|_{L^{p+1}(\Omega')} \\
&< \varepsilon.
\end{aligned}$$

Thus, $\mathscr{C}_c^\infty(\Omega)$ is dense in $L^{p,q}(\Omega)$. ∎

**Proposition 2.27.** (See Hunt (1966).) Let $\Omega$ an open subset in $\mathbb{R}^N$. Then, the following results holds:

(i) Let $1 < p < +\infty$. Then, the dual space of $L^{p,1}(\Omega)$ is given by $L^{p',\infty}(\Omega)$ where $1/p + 1/p' = 1$.

(ii) Let $1 < p < +\infty$ and $1 < q < +\infty$. Then, the dual space of $L^{p,q}(\Omega)$ is given by $L^{p',q'}(\Omega)$ where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Moreover, these spaces are reflexive.

**Proposition 2.28.** (See Halperin (1954).) Let $\Omega$ an open subset in $\mathbb{R}^N$, $1 < p < +\infty$ and $1 < q < +\infty$. Then, the Lorentz space $L^{p,q}(\Omega)$ is a uniformly convex space.

**Lemma 2.29.** Let $\phi \in L^{p,q}(\mathbb{R}^N)$. Then, for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\|\phi - \phi \chi_{B_R}\|_{p,q} < \varepsilon.$$

where $\chi_{B_R}$ is the characteristic function of $B_R$.

***Proof.*** Let $\varepsilon > 0$, since $\phi \in L^{p,q}(\mathbb{R}^N)$, we have

$$|\{x \in \mathbb{R}^N : \phi(x) > \varepsilon\}| < +\infty.$$

Observe that

$$|\{x \in B_R : \phi(x) > \varepsilon\}| \to |\{x \in \mathbb{R}^N : \phi(x) > \varepsilon\}| \quad \text{as} \quad R \to +\infty.$$

Thus, for each $\delta > 0$ there exists $R = R(\varepsilon, \delta) > 0$ such that

$$|I_R| < \delta \quad \text{where} \quad I_R = \{x \in \mathbb{R}^N \backslash B_R : \phi(x) > \varepsilon\}.$$

Setting $\phi_R = \phi \chi_R$, then,

$$0 \leq (\phi - \phi_R)(x) \leq \varepsilon, \quad \text{for all} \quad x \in \mathbb{R}^N \backslash I_R.$$

Thus,

$$|\{x \in \mathbb{R}^N : (\phi - \phi_R)(x) > \varepsilon\}| < \delta.$$

Therefore,

$$\mu_{(\phi - \phi_R)}(\varepsilon) < \delta.$$

Then,

$$(\phi - \phi_R)^*(\delta) = \inf\{s \geq 0 : \mu_{(\phi - \phi_R)}(s) < \delta\} \leq \varepsilon.$$

Using the fact that $(\phi - \phi_R)^*$ is nonincreasing we have

$$(\phi - \phi_R)^*(t) \leq \varepsilon, \quad \text{for all} \quad t \geq \delta.$$

Thus, for each $n \in \mathbb{N}$, there exists $\phi_n$ such that

$$(\phi - \phi_n)^*(t) \leq \frac{1}{n}, \quad \text{for all} \quad t \geq \frac{1}{n},$$

where $\phi_n = \phi \chi_{R_n}$. Consequently, there exists a sequence $(\phi_n)$ such that

$$0 \leq \phi_n(x) \leq \phi(x), \quad \text{almost everywhere in} \quad \mathbb{R}^N \tag{2.13}$$

and

$$(\phi - \phi_n)^*(t) \to 0, \quad \text{almost everywhere in} \quad \mathbb{R}^+. \tag{2.14}$$

Using (2.13) and Lemma 2.5, we obtain

$$\begin{aligned}(\phi - \phi_n)^*(t) &\leq (\phi + \phi_n)^*(t) \\ &\leq \phi^*\big(\frac{t}{2}\big) + \phi_n^*\big(\frac{t}{2}\big) \\ &\leq 2\phi^*\big(\frac{t}{2}\big),\end{aligned} \tag{2.15}$$

almost everywhere in $\mathbb{R}^+$. Now, by (2.14), (2.15) and Dominated convergence theorem, we have

$$\|\phi - \phi_n\|_{p,q}^q = \int_0^{+\infty} \big[t^{1/p}(\phi - \phi_n)^*(t)\big]^q \frac{dt}{t} \to 0.$$

Thus, there exists $R > 0$ such that

$$\|\phi - \phi\chi_R\|_{p,q} < \varepsilon.$$

■

**Proposition 2.30.** Let $1 < p < +\infty$ and $1 \leq q \leq +\infty$. If $(u_n)$ is a sequence in $L^{p,q}(\Omega)$ and $u \in L^{p,q}(\Omega)$ such that $u_n \to u$ in $L^{p,q}(\Omega)$. Then,

$(i)$ $u_n \to u$ in measure.

$(ii)$ There exists a subsequence $(u_{n_k})$ such that

$$u_{n_k}(x) \to u(x), \quad \text{almost everywhere in} \quad \Omega.$$

***Proof.***

**(i)** By Lemma 2.21 we have $L^{p,q}(\Omega) \hookrightarrow L^{p,\infty}(\Omega)$ continuously. Thus,

$$u_n \to u \quad \text{in} \quad L^{p,\infty}(\Omega).$$

Consequently, for given $\varepsilon > 0$ there exists $n_0 \geq 1$ such that

$$\|u_n - u\|_{p,\infty} < \varepsilon^{(p+1)/p}, \quad \text{for all} \quad n \geq n_0.$$

By Remark 2.17-(iii), we have

$$\sup_{t>0} t[\mu_{(u_n-u)}(t)]^{1/p} < \varepsilon^{(p+1)/p}, \quad \text{for all} \quad n \geq n_0.$$

In particular, taking $t = \varepsilon$ in last inequality we obtain

$$\mu_{(u_n-u)}(\varepsilon) < \varepsilon, \quad \text{for all} \quad n \geq n_0.$$

That means,

$$|\{x \in \Omega : |u_n(x) - u(x)| > \varepsilon\}| < \varepsilon, \quad \text{for all} \quad n \geq n_0.$$

which proves the assertion.

**(ii)** It is a consequence of $(i)$.

■

**Proposition 2.31.** Let $(f_n)$ be a sequence of functions in $L^{p,q}(\Omega)$ satisfying

**(i)** $0 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq f_{n+1} \cdots$, almost everywhere in $\Omega$.

**(ii)** $\sup_{n\in\mathbb{N}} \|f_n\|_{p,q} < +\infty$.

Then, $f_n$ converges pointwise on $\Omega$ to a measurable function $f$, that is finite almost everywhere, and furthermore

$$f_n \to f \quad \text{in} \quad L^{p,q}(\Omega).$$

***Proof.*** Let $E$ be a measurable set in $\Omega$ with measure zero such that, for any $x \in \Omega\backslash E$ the sequence $(f_n(x))$ is nondecreasing. Then, we can define

$$f(x) = \begin{cases} \lim_{n\to\infty} f_n(x) = \sup_{n\geq 1} f_n(x), & x \in \Omega\backslash E. \\ +\infty, & x \in E. \end{cases}$$

Now, we show that $f$ is finite almost everywhere in $\Omega$. Since $L^{p,q}(\Omega)$ is continuous embedding in $L^{p,\infty}$, by assumption $(ii)$ there exists $C > 0$ such that

$$\sup_{t>0} t[\mu_{f_n}(t)]^{1/p} = \|f_n\|_{p,\infty} \leq C, \quad \text{for all} \quad n \geq 1.$$

Taking $t = m \in \mathbb{N}$ in last inequality, we have

$$\mu_{f_n}(m) \leq \frac{C^p}{m^p}, \quad \text{for all} \quad n, m \geq 1. \tag{2.16}$$

Setting for each $m, n \geq 1$ the following measurable sets

$$F_{m,n} = \{x \in \Omega\backslash E : |f_n(x)| > m\},$$

$$F_m = \{x \in \Omega\backslash E : |f(x)| > m\}.$$

and

$$F = \{x \in \Omega\backslash E : |f(x)| = +\infty\}.$$

Fixing $m \geq 1$, if $x \in F_m$ we have $\sup_{n\geq 1} f_n(x) = f(x) > m$. Then, there exists $n_0 \geq 1$ such that $f_{n_0}(x) > m$ that is $x \in F_{m,n_0}$. Therefore,

$$F_m \subset \bigcup_{n=1}^{\infty} F_{m,n}.$$

Moreover, from assumption $(i)$ we have $F_{m,n} \subset F_{m,n+1}$ for all $n \geq 1$. Then,

$$|F_m| \leq |\bigcup_{n=1}^{\infty} F_{m,n}| = \lim_{n\to\infty} |F_{m,n}|. \tag{2.17}$$

From (2.16)

$$|F_{n,m}| = |\{x \in \Omega \backslash E : |f_n(x)| > m\}| \leq |\{x \in \Omega : |f_n(x)| > m\}| = \mu_{f_n}(m) \leq \frac{C^p}{m^p}.$$

Combining the last inequality with (2.17), we get

$$|F_m| \leq \frac{C^p}{m^p}, \quad \text{for all} \quad m \geq 1. \tag{2.18}$$

On the other hand, we have $F_{m+1} \subset F_m$ for all $m \geq 1$ and the measure of $F_1$ is finite. Using (2.18), we obtain

$$|F| = |\bigcap_{n=1}^{\infty} F_m| = \lim_{m\to\infty} |F_m| \leq \lim_{m\to\infty} \frac{C^p}{m^p} = 0.$$

Thus, $f$ is finite in $\Omega \backslash \{E \cup F\}$ with $|E \cup F| = 0$. Since $(f_n)$ is nondecreasing almost everywhere in $\Omega$ and $f_n \to f$ almost everywhere in $\Omega$, by Propositions 2.2 and 2.5, we have

$$(f_n^*) \quad \text{is nondecreasing almost everywhere in} \quad \mathbb{R}^+$$

and

$$f_n^* \to f^* \quad \text{almost everywhere in} \quad \mathbb{R}^+. \tag{2.19}$$

By the Monotone convergence theorem,

$$\int_0^{+\infty} [t^{1/p} f^*(t)]^{1/q} \frac{dt}{t} = \lim_{n\to\infty} \int_0^{+\infty} [t^{1/p} f_n^*(t)]^{1/q} \frac{dt}{t} = \lim_{n\to\infty} \|f_n\|_{p,q}^q \leq \sup_{m\in\mathbb{N}} \|f_m\|_{p,q}^q < +\infty.$$

Thus, $f \in L^{p,q}(\Omega)$. Moreover, by Propositions 2.2 and 2.5, we have

$$0 \leq f_n^*(t) \leq f^*(t) \quad \text{almost everywhere in} \quad \mathbb{R}^+.$$

Combining the last inequality, (2.19), the fact that $f \in L^{p,q}(\Omega)$ and Dominated convergence theorem, we get

$$\lim_{n\to\infty} \int_0^{+\infty} [t^{1/p} (f_n(t) - f^*(t))]^{1/q} \frac{dt}{t} = \int_0^{+\infty} \lim_{n\to\infty} [t^{1/p} (f_n^*(t) - f^*(t))]^{1/q} \frac{dt}{t} = 0.$$

Thus,

$$f_n \to f \quad \text{in} \quad L^{p,q}(\Omega).$$

■

## 2.3 Lorentz-Sobolev spaces

**Definition 2.32.** Let $\Omega$ be an open domain in $\mathbb{R}^N$, assume that $1 < p < +\infty$, $1 < q \leq +\infty$ and define $W_0^1 L^{p,q}(\Omega)$ the closure of the set $\{u \in \mathscr{C}_0^\infty(\Omega) : \|u\|_{p,q} + \|\nabla u\|_{p,q} < +\infty\}$ with respect to the quasinorm

$$\|u\|_{1,(p,q)} := \left(\|u\|_{p,q}^q + \|\nabla u\|_{p,q}^q\right)^{1/q} \tag{2.20}$$

where $\nabla u = (D_1 u, \cdots, D_N u)$ and $D_i$ is the weak derivative with respect to $x_i$ for $1 \leq i \leq N$. The space $W_0^1 L^{p,q}(\Omega)$ can also be equipped with the norm

$$\|u\|_{1,(p,q)}^* := \left[\left(\|u\|_{p,q}^*\right)^q + \left(\|\nabla u\|_{p,q}^*\right)^q\right]^{1/q}. \tag{2.21}$$

**Proposition 2.33.** Let $\Omega$ an open domain in $\mathbb{R}^N$, assume that $1 < p, q < +\infty$. Then, $W_0^1 L^{p,q}(\Omega)$ endowed with the norm defined by (2.21) is uniformly convex Banach space (and hence a reflexive space).

***Proof.*** Let $(u_n)$ be a Cauchy sequence in $W_0^1 L^{p,q}(\Omega)$. Then, $(u_n)$, $(D_i u_n)$ for $1 \leq i \leq n$ are Cauchy sequences in $L^{p,q}(\Omega)$. Since $\left(L^{p,q}(\Omega), \|\cdot\|_{p,q}^*\right)$ is a Banach space, there exist $u$, $v_i$ in $L^{p,q}(\Omega)$ such that

$$u_n \to u \quad \text{in} \quad L^{p,q}(\Omega)$$

and

$$D_i u_n \to v_i \quad \text{in} \quad L^{p,q}(\Omega).$$

Let $\phi \in \mathscr{C}_0^\infty(\Omega)$. By generalized Hölder's inequality in Lorentz spaces, we have

$$\left|\int_\Omega (u_n - u) D_i \phi \, dx\right| \leq \|u_n - u\|_{p,q} \|D_i \phi\|_{p',q'} \to 0$$

and

$$\left|\int_\Omega (D_i u_n - v_i) \phi \, dx\right| \leq \|D_i u_n - v_i\|_{p,q} \|\phi\|_{p',q'} \to 0$$

which implies

$$\int_\Omega u_n D_i \phi \, dx \to \int_\Omega u D_i \phi \, dx \quad \text{and} \quad \int_\Omega (D_i u_n) \phi \, dx \to \int_\Omega v_i \phi \, dx, \quad \text{for all} \quad \phi \in \mathscr{C}_0^\infty(\Omega).$$

Thus, for all $\phi \in \mathscr{C}_0^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega u D_i \phi \, dx &= \lim_{n\to\infty} \int_\Omega u_n D_i \phi \, dx \\ &= -\lim_{n\to\infty} \int_\Omega (D_i u_n) \phi \, dx \\ &= -\int_\Omega v_i \phi \, dx.\end{aligned}$$

That is, $D_i u = v_i$ for $1 \leq i \leq N$ in the weak sense. Therefore, $u \in W_0^1 L^{p,q}(\Omega)$ and $\|u_n - u\|_{1,(p,q)}^* \to 0$. Now, consider the following isometry

$$\begin{aligned} J \quad : W_0^1 L^{p,q}(\Omega) &\to L^{p,q}(\Omega) \times L^{p,q}(\Omega)^N \\ u, &\mapsto (u, \nabla u). \end{aligned}$$

Since, $W_0^1L^{p,q}(\Omega)$ is a Banach space, $J\big(W_0^1L^{p,q}(\Omega)\big)$ is a closed subset in $L^{p,q}(\Omega)\times L^{p,q}(\Omega)^N$. Consequently, $J\big(W_0L^{p,q}(\Omega)\big)$ is a uniformly convex and reflexive space. Finally, since $J\big(W_0^1L^{p,q}(\Omega)\big)$ and $W_0^1L^{p,q}(\Omega)$ are isometrically isomorphic, the same properties holds for $W_0^1L^{p,q}(\Omega)$. ■

Note that, the quasinorm defined by (2.20) induces a topology in $W_0L^{p,q}(\Omega)$ which is equivalent to the topological metric induced by the norm defined by (2.21).

**Proposition 2.34.** (See Alvino, Trombetti and Lions (1989).) Let $1<p<N$ and $1\le q\le+\infty$. Then, there exists a positive constant $C=C(N,p,q)$ such that

$$\|u\|_{p^*,q}\le C\|\nabla u\|_{p,q},\quad \text{for all}\quad u\in\mathscr{C}_0^\infty(\mathbb{R}^N),$$

where $p^*=pN/(N-p)$.

**Corollary 2.35.** Let $\Omega$ be a bounded domain, $1<p<N$ and $1\le q\le+\infty$. Then, there exists a positive constant $C=C(N,p,q)$ such that

$$\|u\|_{p^*,q}\le C\|\nabla u\|_{p,q},\quad \text{for all}\quad u\in W_0^1L^{p,q}(\Omega).$$

***Proof.*** Let $u\in\mathscr{C}_0^\infty(\Omega)$, using notation given by Remark 2.17, the function $\overline{u}\in\mathscr{C}_0^\infty(\mathbb{R}^N)$ satisfies $\frac{\partial\overline{u}}{\partial x_i}=\overline{\frac{\partial u}{\partial x_i}}$ for any $1\le i\le N$. Consequently,

$$\Big(\frac{\partial\overline{u}}{\partial x_i}\Big)^*(s)=\begin{cases}\big(\frac{\partial u}{\partial x_i}\big)^*(s), & 0\le s\le|\Omega|\\ 0, & s>|\Omega|.\end{cases}$$

Then,

$$\|u\|_{L^{p^*,q}(\mathbb{R}^N)}=\|u\|_{L^{p^*,q}(\Omega)}\quad\text{and}\quad\|\nabla u\|_{L^{p,q}(\mathbb{R}^N)}=\|\nabla u\|_{L^{p,q}(\Omega)}.$$

Thus, replacing these identities in Proposition 2.34, we get

$$\|u\|_{p^*,q}\le C\|\nabla u\|_{p,q}.$$

Finally, using density we obtain the claim. ■

Note that this corollary improves slightly Sobolev's embedding theorem, which states that: if $\nabla u\in L^p(\Omega)=L^{p,p}(\Omega)$, then $u\in L^{p^*}(\Omega)=L^{p^*,p^*}(\Omega)$. However, last result states that $u\in L^{p^*,p}(\Omega)\subsetneq L^{p^*,p^*}(\Omega)$.

**Proposition 2.36. (Poincaré inequality in Lorentz-Sobolev spaces)** Let $\Omega\subset\mathbb{R}^N$ be a bounded domain, $1<p<+\infty$ and $1\le q\le+\infty$. Then, there exists a positive constant $C=C(\Omega,N,p,q)$ such that

$$\|u\|_{p,q}\le C\|\nabla u\|_{p,q},\quad\text{for all}\quad u\in W_0^1L^{p,q}(\Omega). \tag{2.22}$$

***Proof.*** We consider the following cases:

**(i)** If $1 < p < N$.

By Corollary 2.35, $W_0^1 L^{p,q}(\Omega) \subset L^{p^*,q}(\Omega)$ and

$$\|u\|_{p^*,q} \leq C\|\nabla u\|_{p,q}, \quad \text{for all} \quad u \in W_0^1 L^{p,q}(\Omega). \tag{2.23}$$

Since $p < p^*$, by Lemma 2.22, $L^{p^*,q}(\Omega) \subset L^{p,q}(\Omega)$ and there exists a positive constant $C_1 = C_1(\Omega, N, p)$ such that

$$\|u\|_{p,q} \leq C_1\|u\|_{p^*,q}, \quad \text{for all} \quad u \in L^{p^*,q}(\Omega). \tag{2.24}$$

From (2.23) and (2.24), we get (2.22).

**(ii)** If $p \geq N$.

We can choose $1 < r < N$ such that $p < r^* = Nr/(N-r)$. Thus, by Lemma 2.22, $L^{r^*,q}(\Omega) \subset L^{p,q}(\Omega)$ and there exists a positive constant $C_1 = C_1(\Omega, N, p, q)$ such that

$$\|u\|_{p,q} \leq C_1\|u\|_{r^*,q}. \tag{2.25}$$

Since $1 < r < N$, by Corollary 2.35, $W_0^1 L^{r,q}(\Omega) \subset L^{r^*,q}(\Omega)$ and there exists a positive constant $C_2 = C_2(N, p, q)$ such that

$$\|u\|_{r^*,q} \leq C_2\|\nabla u\|_{r,q}. \tag{2.26}$$

Since $r < N \leq p$, by Lemma 2.22, there exists a positive constant $C_3 = C_3(\Omega, N, p, q)$ such that

$$\|\nabla u\|_{r,q} \leq C_3\|\nabla u\|_{p,q}. \tag{2.27}$$

From (2.25), (2.26) and (2.27), we obtain (2.22).

■

**Remark 2.37.** If $\Omega \subset \mathbb{R}^N$ is a bounded domain, using Proposition 2.36 we can consider on the Lorentz-Sobolev space $W_0^1 L^{p,q}(\Omega)$, the following quasinorm

$$\|u\|_{1,[p,q]} := \|\nabla u\|_{p,q}$$

which is equivalent to the quasinorm defined by (2.20).

**Lemma 2.38.** (See Ruf (2006).) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $1 \leq q < +\infty$. Then, the following embeddings are compact

$$W_0^1 L^{N,q}(\Omega) \hookrightarrow L^r(\Omega), \quad \text{for all} \quad r \geq 1.$$

***Proof.*** From Lemma 2.23, the following embedding is continuous

$$L^{N,q}(\Omega) \hookrightarrow L^{N-\delta}(\Omega), \quad \text{for all} \quad 0 < \delta \leq N-1.$$

Thus, the following embedding is also continuous

$$W_0^1 L^{N,q}(\Omega) \hookrightarrow W_0^{1,N-\delta}(\Omega).$$

Consequently, using Sobolev's embedding the following embeddings

$$W_0^1 L^{N,q}(\Omega) \hookrightarrow L^r(\Omega), \quad \text{for all} \quad 1 \le r < \frac{N(N-\delta)}{\delta},$$

are compact. Finally, since $\delta$ is arbitrary, the conclusion follows. ■

**Theorem 2.39.** (See Brézis and Wainger (1980).) Suppose that $\nabla u \in L^{N,q}(\Omega)$, for some $1 < q < +\infty$. Then, $e^{|u|^{\frac{q}{q-1}}} \in L^1(\Omega)$.

Last theorem generalizes the Trudinger embedding (see Trudinger (1967)), which gives $e^{|u|^{\frac{N}{N-1}}} \in L^1(\Omega)$ provided $\nabla u \in L^{N,N}(\Omega)$, where $N/(N-1)$ is the maximal exponent growth. Note that the maximal growth depends only on the second Lorentz exponent $q$, but not on $N$.

**Remark 2.40.** As a consequence of Theorem 2.39, we have

$$\int_\Omega e^{\alpha |u|^{\frac{q}{q-1}}}\, dx < +\infty, \quad \text{for all} \quad u \in W_0^1 L^{N,q}(\Omega), \quad \alpha > 0.$$

The following theorem is a version sharp of Theorem 2.39.

**Theorem 2.41.** (See Alvino, Ferone and Trombetti (1996).) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $1 < q < +\infty$. Then, there exists a positive constant $C = C(N,q)$ such that

$$\sup_{\|\nabla u\|_{N,q} \le 1} \int_\Omega e^{\alpha |u|^{\frac{q}{q-1}}}\, dx \begin{cases} \le C|\Omega|, & \text{if } \alpha \le \alpha_q^*, \\ = +\infty, & \text{if } \alpha > \alpha_q^*, \end{cases}$$

where $\alpha_q^* = (N\omega_N^{1/N})^{q/(q-1)}$.

The following result represents an extension of Brezis (2011, Proposition 8.3).

**Proposition 2.42.** Let $u \in L^{p,q}(\Omega)$ with $1 < p, q < +\infty$. Then, the following properties are equivalents:

(i) $u \in W_0^1 L^{p,q}(\Omega)$.

(ii) There exists a positive constant $C$ such that

$$\left| \int_\Omega u \frac{\partial \phi}{\partial x_i}\, dx \right| \le C \|\phi\|_{p',q'}, \quad \text{for all} \quad \phi \in \mathscr{C}_0^\infty(\Omega), \quad 1 \le i \le N.$$

***Proof.***

$(i) \Rightarrow (ii)$ Using the fact that $u$ possesses weak derivatives in $L^{p,q}(\Omega)$ and Hölder's inequality in Lorentz spaces, we have for each $1 \leq i \leq N$

$$\Big|\int_\Omega u\frac{\partial \phi}{\partial x_i}\,dx\Big| = \Big|\int_\Omega \frac{\partial u}{\partial x_i}\phi\,dx\Big| \leq C\|\phi\|_{p',q'}, \quad \text{for all} \quad \phi \in \mathscr{C}_0^\infty(\Omega)$$

where $C = \max\{\Big\|\frac{\partial u}{\partial x_i}\Big\|_{p,q} : 1 \leq i \leq N\}$.

$(ii) \Rightarrow (i)$ For each $1 \leq i \leq N$ fixed, consider the following functional

$$\begin{array}{rcl} f_i \ : \mathscr{C}_0^\infty(\Omega) & \to & \mathbb{R} \\ \phi & \mapsto & \displaystyle\int_\Omega u\frac{\partial \phi}{\partial x_i}\,dx. \end{array}$$

Using $(ii)$, the functional $f_i$ is continuous in the quasinorm of $L^{p',q'}(\Omega)$ and $\mathscr{C}_0^\infty(\Omega)$ is considered a dense subspace of $L^{p',q'}(\Omega)$ (see Proposition 2.26). By Hahn-Banach theorem there exists a continuous, linear extension $F_i$ of $f_i$ on the whole space $L^{p',q'}(\Omega)$. Moreover since the dual of $L^{p',q'}(\Omega)$ is $L^{p,q}(\Omega)$, there exists $v_i$ in $L^{p,q}(\Omega)$ such that

$$\begin{array}{rcl} F_i \ : L^{p',q'}(\Omega) & \to & \mathbb{R} \\ \phi & \mapsto & \displaystyle\int_\Omega v_i\phi\,dx. \end{array}$$

In particular $F_i(\phi) = f_i(\phi)$ for all $\phi$ in $\mathscr{C}_0^\infty(\Omega)$. That is,

$$\int_\Omega v_i\phi\,dx = \int_\Omega u\frac{\partial \phi}{\partial x_i}\,dx, \quad \text{for all} \quad \phi \in \mathscr{C}_0^\infty(\Omega)$$

Thus, there exists $\frac{\partial u}{\partial x_i}$ in the weak sense and $\frac{\partial u}{\partial x_i} = v_i \in L^{p,q}(\Omega)$. Repeating the same argument for each each $1 \leq i \leq N$, we conclude that $u \in W_0^1L^{p,q}(\Omega)$.

■

**Lemma 2.43.** Let $u \in L^{p,q}(\Omega)$ and $(u_n)$ be a sequence in $W_0^1L^{p,q}(\Omega)$ such that $u_n \to u$ in $L^{p,q}(\Omega)$ and $(\nabla u_n)$ is a bounded sequence in $\big(L^{p,q}(\Omega)\big)^N$. Then, $u \in W_0^1L^{p,q}(\Omega)$.

***Proof.*** By Hölder's inequality in Lorentz spaces for each $\phi \in \mathscr{C}_0^\infty(\Omega)$, we have

$$\Big|\int_\Omega (u_n - u)\frac{\partial \phi}{\partial x_i}\,dx\Big| \leq \|u_n - u\|_{p,q}\|\frac{\partial \phi}{\partial x_i}\|_{p',q'}, \quad \text{for all} \quad 1 \leq i \leq N.$$

Thus,

$$\lim_{n\to+\infty}\int_\Omega u_n\frac{\partial \phi}{\partial x_i}\,dx = \int_\Omega u\frac{\partial \phi}{\partial x_i}\,dx, \quad \text{for all} \quad 1 \leq i \leq N. \tag{2.28}$$

Let $C > 0$ such that $\|\frac{\partial u_n}{\partial x_i}\|_{p,q} \leq C$ for all $n \geq 1$ and for all $1 \leq i \leq N$. Using Hölder's inequality in Lorentz spaces for each $\phi \in \mathscr{C}_0^\infty(\Omega)$, we have

$$\Big|\int_\Omega u_n\frac{\partial \phi}{\partial x_i}\,dx\Big| = \Big|\int_\Omega \frac{\partial u_n}{\partial x_i}\phi\,dx\Big| \leq \|\frac{\partial u_n}{\partial x_i}\|_{p,q}\|\phi\|_{p',q'} \leq C\|\phi\|_{p',q'}, \quad 1 \leq i \leq N.$$

From this and (2.28) we obtain

$$\Big|\int_\Omega u\frac{\partial\phi}{\partial x_i}\,dx\Big|\le C\|\phi\|_{p',q'},\quad \text{for all}\quad \phi\in\mathscr{C}_0^\infty(\Omega),\quad 1\le i\le N.$$

Finally, by Proposition 2.42 we conclude that $u\in W_0^1L^{p,q}(\Omega)$. ■

**Proposition 2.44.** Let $(u_n)$ be a sequence in $W_0^1L^{p,q}(\Omega)$ and $u\in W_0^1L^{p,q}(\Omega)$ such that

$$u_n\to u\quad\text{in}\quad W_0^1L^{p,q}(\Omega).$$

Then, there exists a subsequence $(u_{n_k})$ and a function $h\in W_0^1L^{p,q}(\Omega)$ such that

$$|u_{n_k}(x)|\le h(x),\quad \text{for all}\quad k\ge 1\quad\text{and almost everywhere in}\quad\Omega.$$

***Proof.*** From Proposition 2.30 we can assume that $u_n\to u$ almost everywhere in $\Omega$. Moreover, we can extract a subsequence $(u_{n_k})$, denoted by $(u_k)$ such that

$$\|u_{k+1}-u_k\|_{1,(p,q)}\le\frac{1}{2^{2k}},\quad\text{for all}\quad k\ge 1.$$

Set

$$g_n(x)=\sum_{k=1}^n|u_{k+1}(x)-u_k(x)|.$$

Then, $(g_n)\in W_0^1L^{p,q}(\Omega)$ and $\|g_n\|_{1,(p,q)}\le 1$ for all $n\ge 1$, that is

$$\|g_n\|_{p,q}\le 1\quad\text{and}\quad\|\nabla g_n\|_{p,q}\le 1,\quad\text{for all}\quad n\ge 1.$$

Since $(g_n(x))$ is nondecreasing almost everywhere in $\Omega$ and $\sup_{n\ge1}\|g_n\|_{p,q}\le 1$, by Proposition 2.31, we have $g_n\to g$ in $L^{p,q}(\Omega)$. Moreover, using the fact that $(\nabla g_n)$ is bounded in $\big(L^{p,q}(\Omega)\big)^N$, we get by Lemma 2.43 that $g_n\to g$ in $W_0^1L^{p,q}(\Omega)$. On the other hand, for $l>k\ge 2$ we have

$$|u_l(x)-u_k(x)|\le|u_l(x)-u_{l-1}(x)|+\cdots+|u_{k+1}(x)-u_k(x)|\le g_{l-1}(x)-g_{k-1}(x)\le g_{l-1}(x).$$

Taking $l\to+\infty$, we obtain

$$|u(x)-u_k(x)|\le g(x)\quad\text{almost everywhere in}\quad\mathbb{R}.$$

Thus,

$$|u_k(x)|\le h(x)\quad\text{almost everywhere in}\quad\mathbb{R}.$$

where $h=g+|u|\in W_0^1L^{p,q}(\Omega)$. ■

### 2.3.1 Lorentz-Sobolev spaces in $\mathbb{R}^2$

In this section we study some properties of Lorentz-Sobolev spaces restricted to $\mathbb{R}^2$. We denote by

$$W^1L^{p,q}(\mathbb{R}^2):=W_0^1L^{p,q}(\mathbb{R}^2).$$

**Proposition 2.45.** (See Cassani and Tarsi (2009).) Let $1 < s < +\infty$. Then, there exists a positive constant $C = C(s)$ such that for any (sufficiently smooth) domain $\Omega \subset \mathbb{R}^2$ and for any $\alpha \le \alpha_s^* = (\sqrt{4\pi})^{\frac{s}{s-1}}$, the following inequalities hold:

$$\sup_{\|u\|_{1,(2,s)} \le 1} \int_\Omega \left(e^{\alpha |u|^{\frac{s}{s-1}}} - 1 - \alpha |u|^{\frac{s}{s-1}}\right) dx \le C, \quad \text{if} \quad 2 < s < +\infty. \tag{2.29}$$

$$\sup_{\|u\|_{1,(2,s)} \le 1} \int_\Omega \left(e^{\alpha |u|^{\frac{s}{s-1}}} - 1\right) dx \le C, \quad \text{if} \quad 1 < s \le 2. \tag{2.30}$$

Moreover, inequalities (2.29)-(2.30) are sharp, in the sense that for any $\alpha > \alpha_s^*$ the corresponding suprema become infinity.

**Proposition 2.46.** (See Lu and Tang (2016).) Let $1 < s < +\infty$. Then, there exists a constant $C(s) > 0$ such that for any $0 < \alpha < \alpha_s^* = (\sqrt{4\pi})^{\frac{s}{s-1}}$ and any $u \in W^1L^{2,s}(\mathbb{R}^2)$ with $\|\nabla u\|_{2,s} \le 1$, the following inequalities hold:

$$\int_{\mathbb{R}^2} \left(e^{\alpha |u|^{\frac{s}{s-1}}} - 1 - \alpha |u|^{\frac{s}{s-1}}\right) dx \le C(s) \|u\|_{2,s}^2, \quad \text{if} \quad 2 < s < +\infty. \tag{2.31}$$

$$\int_{\mathbb{R}^2} \left(e^{\alpha |u|^{\frac{s}{s-1}}} - 1\right) dx \le C(s) \|u\|_{2,s}^2, \quad \text{if} \quad 1 < s \le 2. \tag{2.32}$$

The restriction $\alpha < \alpha_{s^*}$ is sharp, in the sense that if $\alpha > \alpha_s^*$, then, the inequality can no longer hold with some $C(s)$ independent of $u$.

**Proposition 2.47.** Let $1 < s < +\infty$. Then, the following embeddings are continuous

**(i)** If $1 < s \le 2$

$$W^1L^{2,s}(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2), \quad \text{for all} \quad r \ge 2.$$

**(ii)** If $2 < s$

$$W^1L^{2,s}(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2), \quad \text{for all} \quad r \ge \frac{2s}{s-1}$$

***Proof.***

**(i)** Since $1 < s \le 2$, from Lemma 2.21 the embedding $L^{2,s}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ is continuous. Then,

$$W^1L^{2,s}(\mathbb{R}^2) \hookrightarrow W^{1,2}(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2), \quad \text{for all} \quad r \ge 2.$$

**(ii)** Let $s > 2$, $s' = s/(s-1)$ and considering $r \ge 2s'$, we have the following limits

$$\lim_{|t| \to 0} \frac{|t|^r}{e^{|t|^{s'}} - 1 - |t|^{s'}} = \begin{cases} 2, & \text{if} \quad r = 2s', \\ 0, & \text{if} \quad r > 2s' \end{cases}$$

and

$$\lim_{|t| \to \infty} \frac{|t|^r}{e^{|t|^{s'}} - 1 - |t|^{s'}} = 0$$

Thus, there exists $C_1 = C_1(r,s) > 0$ such that

$$|t|^r \leq C_1\left(e^{|t|^{s'}} - 1 - |t|^{s'}\right), \quad \text{for all} \quad t \in \mathbb{R}. \tag{2.33}$$

Hence, for every $0 \neq u \in W^1L^{2,s}(\mathbb{R}^2)$, let consider $\widehat{u} = u/\|\nabla u\|_{2,s}$. Thus, by Proposition 2.46 and (2.33), there exist $C = C(r,s) > 0$ such that

$$\int_{\mathbb{R}^2} |\widehat{u}|^r\, dx \leq C_1 \int_{\mathbb{R}^2} \left(e^{|\widehat{u}|^{s'}} - 1 - |\widehat{u}|^{s'}\right) dx \leq C\|\widehat{u}\|_{2,s}^2. \tag{2.34}$$

Thus,

$$\|u\|_r^r \leq C\|u\|_{2,s}^2 \|\nabla u\|_{2,s}^{r-2} \leq C\|u\|_{1,(2,s)}^r.$$

That is, $W^1L^{2,s}(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2)$ is continuous for all $r \geq 2s'$.

■

Now, we introduce a weighted Lorentz-Sobolev space. Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function verifying the following conditions:

$(V_1)$ There exists $V_0 > 0$ such that

$$V(x) \geq \inf_{x\in\mathbb{R}^2} V(x) := V_0 > 0.$$

$(V_2)$ There exist constants $p > 2$ and $q = p/(p-1)$ such that

$$\frac{1}{V^{1/q}} \in L^{2,p}(\mathbb{R}^2) \quad \text{and} \quad \frac{1}{V^{1/p}} \in L^{2,q}(\mathbb{R}^2).$$

Consider the following Lorentz-Sobolev space

$$W^1L_V^{2,p}(\mathbb{R}^2) := \{u \in W^1L^{2,p}(\mathbb{R}^2) : \|V^{1/p}u\|_{2,p} < +\infty\}.$$

endowed with the quasinorm

$$\|u\|_{(p)} := \left(\|\nabla u\|_{2,p}^p + \|V^{1/p}u\|_{2,p}^p\right)^{1/p}.$$

We denote this space by $W^{(p)} := W^1L_V^{2,p}(\mathbb{R}^2)$.

**Proposition 2.48.** Suppose that $(V_1)$ and $(V_2)$ hold. Then, the following embeddings

$$W^{(p)} \hookrightarrow L^s(\mathbb{R}^2),\ W^{(q)} \hookrightarrow L^s(\mathbb{R}^2), \quad \text{for all} \quad 1 \leq s < +\infty$$

are compact.

***Proof.*** By Proposition 2.5, we have

$$V_0^{1/p}u^*(t) = (V_0^{1/p}u)^*(t) \leq (V^{1/p}u)^*(t), \quad \text{for all} \quad t \geq 0$$

which implies that

$$V_0\|u\|_{2,p}^p = V_0\int_0^{+\infty}\left[t^{1/2}u^*(t)\right]^p\frac{dt}{t} \le \int_0^{+\infty}\left[t^{1/2}(V^{1/p}u)^*(t)\right]^p\frac{dt}{t} = \|V^{1/p}u\|_{2,p}^p.$$

Therefore,

$$\|\nabla u\|_{2,p}^p + V_0\|u\|_{2,p}^p \le \|\nabla u\|_{2,p}^p + \|V^{1/p}u\|_{2,p}^p = \|u\|_{(p)}^p.$$

Thus, we obtain $W^{(p)} \hookrightarrow W^1L^{2,p}(\mathbb{R}^2)$ continuously. By Proposition 2.47, the space $W^1L^{2,p}(\mathbb{R}^2)$ is continuously embedded in $L^r(\mathbb{R}^2)$ for all $r \ge 2p/(p-1)$. Consequently,

$$W^{(p)} \hookrightarrow L^r(\mathbb{R}^2), \quad \text{for all} \quad r \ge \frac{2p}{p-1}. \tag{2.35}$$

On the other hand, let $C = \|1/V^{1/q}\|_{2,q}$. By Hölder's inequality in Lorentz spaces, we have

$$\|u\|_1 \le \|\frac{1}{V^{1/q}}\|_{2,q}\|V^{1/p}u\|_{2,p} \le C\|u\|_{(p)}. \tag{2.36}$$

Thus, the embedding $W^{(p)} \hookrightarrow L^1(\mathbb{R}^2)$ is continuous. Moreover, for $s > 1$ and $r > \max\{s, 2p/(p-1)\}$ we can write

$$\|u_n\|_s \le \|u_n\|_1^{1-\lambda}\|u_n\|_r^{\lambda}, \quad \text{where} \quad \lambda = \frac{s-1}{r-1}\frac{r}{s}. \tag{2.37}$$

Using (2.35), we conclude

$$W^{(p)}(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2), \quad \text{for all} \quad r \ge 1, \tag{2.38}$$

continuously. In order to prove compactness we show first that the embedding $W^{(p)} \hookrightarrow L^1(\mathbb{R}^2)$ is compact. Consider a sequence $(u_n) \subset W^{(p)}$ such that $u_n \rightharpoonup 0$ in $W^{(p)}$. For given $\varepsilon > 0$ there exists $C > 0$ such that

$$\|u_n\|_{(p)} \le C, \quad \text{for all} \quad n \ge 1. \tag{2.39}$$

Using the fact that $V^{-1/p} \in L^{2,q}(\mathbb{R}^2)$ in Lemma 2.29, we can find $R > 0$ such that

$$\|V^{-1/p} - V^{-1/p}\chi_{B_R}\|_{2,q} < \frac{\varepsilon}{2C}. \tag{2.40}$$

Consider the following embeddings

$$W^{(p)} \hookrightarrow W^1L^{2,p}(\mathbb{R}^2) \hookrightarrow W^1L^{2,p}(B_R(0)) \hookrightarrow L^1(B_R(0)) \tag{2.41}$$

where

$$W^1L^{2,p}(B_R(0)) := \{u|_{B_R} : u \in W^1L^{2,p}(\mathbb{R}^2)\}.$$

and arguing as in the proof of Lemma 2.38, we can prove that the last embedding in (2.41) is compact. Thus, we can assume that $u_n \to 0$ in $L^1(B_R)$. Therefore, there exists $n_0 \ge 1$ such that

$$\int_{B_R}|u_n|\,dx < \frac{\varepsilon}{2}, \quad \text{for all} \quad n \ge n_0. \tag{2.42}$$

Using Hölder's inequality in Lorentz spaces, (2.39) and (2.40), we get

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_R} |u_n|\,dx &= \|V^{1/p}u_nV^{-1/p}\chi_{\mathbb{R}^2\setminus B_R}\|_1 \\ &\leq \|V^{1/p}u_n\|_{2,p}\|V^{-1/p}\chi_{\mathbb{R}^2\setminus B_R}\|_{2,q} \\ &\leq C\|V^{-1/p}(1-\chi_{B_R})\|_{2,q}. \\ &< \frac{\varepsilon}{2}.\end{aligned} \tag{2.43}$$

From (2.42) and (2.43), we obtain that $u_n \to 0$ in $L^1(\mathbb{R}^2)$. Moreover, let $s$ and $r$ as in (2.37), from (2.38) and (2.39), we have that $(u_n)$ is a bounded sequence in $L^r(\mathbb{R}^2)$. By (2.37) we obtain

$$u_n \to 0, \quad \text{in} \quad L^s(\mathbb{R}^2),$$

which implies the embedding of $W^{(p)}$ in $L^s(\mathbb{R}^2)$ is compact for all $s \geq 1$. Similarly, we can prove that $W^{(q)}$ in $L^s(\mathbb{R}^2)$ is compact for all $s \geq 1$. ■

## 2.4 The tilde-map

Following the arguments developed in Figueiredo, Ó and Ruf (2005), Ruf (2008), we construct an application from the space $W_0^1L^{2,q}(\Omega)$ to $W_0^1L^{2,p}(\Omega)$ where $p = q/(q-1)$.

**Proposition 2.49.** Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. For each $u \in W_0^1L^{2,q}(\Omega)$ consider

$$S := \sup\left\{\int_\Omega \nabla u(x)\nabla\omega(x)\,dx :\ \omega \in W_0^1L^{2,p}(\Omega),\ \|\nabla\omega\|_{2,p} = \|\nabla u\|_{2,q}\right\}. \tag{2.44}$$

Then, there exists a unique $\widetilde{u} \in W_0^1L^{2,p}(\Omega)$ such that

$$S = \int_\Omega \nabla u(x)\nabla\widetilde{u}(x)\,dx = \|\nabla u\|_{2,q}\|\nabla\widetilde{u}\|_{2,p} \quad \text{and} \quad \|\nabla u\|_{2,q} = \|\nabla\widetilde{u}\|_{2,p}.$$

***Proof.*** Given $u \in W_0^1L^{2,q}(\Omega)$ fixed. Note that, for each $\omega \in W_0^1L^{2,p}(\Omega)$ such that $\|\nabla\omega\|_{2,p} = \|\nabla u\|_{2,q}$, by Hölder's inequality in Lorentz spaces, we have

$$\int_\Omega \nabla u(x)\nabla\omega(x)\,dx \leq \|\nabla u\|_{2,q}\|\nabla\omega\|_{2,p} = \|\nabla u\|_{2,q}^2.$$

Thus, there exists the supremum $S$. Let $(\omega_n)$ be a maximizing sequence for $S$. Since the sequence is bounded and the fact that $W_0^1L^{2,p}(\Omega)$ is a reflexive space, we can assume that $\omega_n \rightharpoonup \widehat{\omega}$ in $W_0^1L^{2,p}(\Omega)$. Thus,

$$\int_\Omega \nabla u(x)\nabla\omega_n(x)\,dx \to \int_\Omega \nabla u(x)\nabla\widehat{\omega}(x)\,dx = S$$

and

$$\|\nabla\widehat{\omega}\|_{2,p} \leq \liminf_{n\to\infty}\|\nabla\omega_n\|_{2,p} = \|\nabla u\|_{2,q} \tag{2.45}$$

that is, the supremum is attained. If $\|\nabla\widehat{\omega}\|_{2,p} < \|\nabla u\|_{2,q}$, setting $\omega_0 = \dfrac{\widehat{\omega}\|\nabla u\|_{2,q}}{\|\nabla\widehat{\omega}\|_{2,p}}$. we note that, $\omega_0 \in W_0^1L^{2,p}(\Omega)$ and $\|\nabla\omega_0\|_{2,p} = \|\nabla u\|_{2,q}$. Thus,

$$S \geq \int_\Omega \nabla u(x)\nabla\omega_0(x)\,dx = \frac{\|\nabla u\|_{2,q}}{\|\nabla\omega_1\|_{2,p}}\int_\Omega \nabla u(x)\nabla\widehat{\omega}(x)\,dx > S,$$

which gives a contradiction. Hence, $\|\nabla\widehat{\omega}\|_{2,p} = \|\nabla u\|_{2,q}$.

In order to prove the uniqueness, without less of generality we can assume that $\|\nabla u\|_{2,q} = 1$ and there exist $\widehat{\omega}_1 \neq \widehat{\omega}_2$ in $W_0^1L^{2,p}(\Omega)$ with $\|\nabla\widehat{\omega}_1\|_{2,p} = \|\nabla\widehat{\omega}_2\|_{2,p} = 1$, such that

$$\int_\Omega \nabla u(x)\nabla\widehat{\omega}_1(x)\,dx = \int_\Omega \nabla u(x)\nabla\widehat{\omega}_2(x)\,dx = 1.$$

We see that $\widehat{\omega}_1 \neq -\widehat{\omega}_2$ and by Proposition 2.28 we have that $W_0^1L^{2,p}(\Omega)$ is a uniformly convex space. Thus,

$$0 < \left\|\nabla\left(\frac{\widehat{\omega}_1+\widehat{\omega}_2}{2}\right)\right\|_{2,p} < 1.$$

Set $r = 2/\|\nabla(\widehat{\omega}_1+\widehat{\omega}_2)\|_{2,p} > 1$. Then,

$$1 \geq \int_\Omega \nabla u\nabla\left(\frac{\widehat{\omega}_1+\widehat{\omega}_2}{\|\nabla(\widehat{\omega}_1+\widehat{\omega}_2)\|_{2,p}}\right)dx = \frac{r}{2}\int_\Omega \nabla u\nabla(\widehat{\omega}_1+\widehat{\omega}_2)\,dx = r > 1,$$

which is a contradiction, and the uniqueness follows. We denote by $\widetilde{u}$ this element. ■

**Definition 2.50.** Let $1 < q < +\infty$, using the Proposition 2.49 we can define the **tilde-map**

$$\begin{array}{rcl} \widetilde{\ }: W_0^1L^{2,q}(\Omega) & \to & W_0^1L^{2,p}(\Omega) \\ u & \mapsto & \widetilde{u}, \end{array} \tag{2.46}$$

where $p = q/(q-1)$.

**Remark 2.51.** Let $p > 1$, $q > 1$ and denote by $\xi_{q,p}$ the tilde-map from $W_0^1L^{2,q}(\Omega)$ to $W_0^1L^{2,p}(\Omega)$ as defined in (2.46). A direct calculation shows that the inverse of $\xi_{q,p}$ is given by $\xi_{p,q}$ . Thus, the tilde-map is bijective.

It follows from the construction that the tilde-map is positively homogeneous, that is

$$\widetilde{\rho u} = \rho\widetilde{u}, \quad \text{for all} \quad u \in W_0^1L^{2,q}(\Omega),\ \rho \geq 0.$$

With the help of the tilde-map, we define two continuous subspaces of

$$E := W_0^1L^{2,q}(\Omega) \times W_0^1L^{2,q'}(\Omega)$$

by

$$E^+ = \{(u,\widetilde{u}) : u \in W_0^1L^{2,q}(\Omega)\} \quad \text{and} \quad E^- = \{(u,-\widetilde{u}) : u \in W_0^1L^{2,q}(\Omega)\}.$$

Following ideas from Figueiredo, Ó and Ruf (2005), the nonlinear subspaces $E^+$ and $E^-$ have a linear structure with respect to the following operations:

**Definition 2.52.** **(Tilde-sum)** Given $(u,\widetilde{v})$ and $(y,\widetilde{z}) \in E$, we define the tilde sum by

$$(u,\widetilde{v}) \mathbin{\widetilde{+}} (y,\widetilde{z}) := (u+y,\widetilde{v+z}). \tag{2.47}$$

Note that, the set $E$ endowed with the tilde-map is an abelian group, indeed, associative and commutative axioms follows from the properties in $W_0^1 L^{2,q}(\Omega)$ and due to the surjectivity of the tilde-map. Moreover, the zero element is given by $(0,0) \in E$ and additive inverse of an element $(u,\widetilde{v}) \in E$ is given by $(-u,\widetilde{-v}) \in E$.

Motivated by the tilde sum, we define the following binary operation:

**Definition 2.53.** Given $(u,\widetilde{v})$ in $E$ and $\alpha \in \mathbb{R}$, we define

$$\alpha(u,\widetilde{v}) := (\alpha u,\widetilde{\alpha v}), \quad \text{for all} \quad \alpha \in \mathbb{R}. \tag{2.48}$$

**Lemma 2.54.** (i) Let $(u,\widetilde{v}) \in E$ and $(y,\widetilde{z}) \in E$. Then, for all $\alpha,\beta \in \mathbb{R}$ we have

$$\alpha(u,\widetilde{v}) \mathbin{\widetilde{+}} \beta(y,\widetilde{z}) \in E \quad \text{and} \quad \alpha(u,\widetilde{v}) \mathbin{\widetilde{+}} \beta(y,\widetilde{z}) = (\alpha u+\beta y,\widetilde{\alpha v+\beta z}).$$

Moreover, the set $E$ endowed with the operations given by (2.47) and (2.48) has a vector space structure.

(ii) For each $(y,\widetilde{z}) \in E$, there exist unique elements $(u,\widetilde{u}) \in E^+$ and $(v,-\widetilde{v}) \in E^-$ such that

$$(y,\widetilde{z}) = (u,\widetilde{u}) \mathbin{\widetilde{+}} (v,-\widetilde{v}).$$

Thus,

$$E = E^+ \widetilde{\oplus} E^-.$$

***Proof.***

(i) Using the clousure of the operations (2.47) and (2.48), for all $(u,\widetilde{v}),(y,\widetilde{z}) \in E$ and $\alpha,\beta \in \mathbb{R}$ we have

$$\alpha(u,\widetilde{v}) \mathbin{\widetilde{+}} \beta(y,\widetilde{z}) \in E.$$

and

$$\begin{aligned}\alpha(u,\widetilde{v}) \mathbin{\widetilde{+}} \beta(y,\widetilde{z}) &= (\alpha u,\widetilde{\alpha v}) \mathbin{\widetilde{+}} (\beta y,\widetilde{\beta z}) \\ &= (\alpha u+\beta y,\widetilde{\alpha v+\beta z}).\end{aligned}$$

As we see $(E,\widetilde{+})$ is an abelian group. Moreover, all other axioms of vector space can be checked easily.

(ii) Let $(u_1,\widetilde{u}_1),(u_2,\widetilde{u}_2) \in E^+$ and $(v_1,-\widetilde{v}_1),(v_2,-\widetilde{v}_2) \in E^-$ such that

$$(u_1+v_1,\widetilde{u_1-v_1}) = (y,\widetilde{z}) = (u_2+v_2,\widetilde{u_2-v_2}).$$

Then, $u_1+v_1=u_2+v_2$ and $u_1-v_1=u_2-v_2$ which implies $u_1=u_2$ and $v_1=v_2$. Thus, we obtain uniqueness. In order to prove the existence, let $(y,\widetilde{z})\in E$. Taking

$$(u,\widetilde{u})=(\frac{y+z}{2},\widetilde{\frac{y+z}{2}}) \quad \text{and} \quad (v,-\widetilde{v})=(\frac{y-z}{2},-\widetilde{\frac{y-z}{2}}),$$

the existence follows.

■

**Remark 2.55.** The operations defined by (2.47) and (2.48) restricted to the space $\{0\}\times W_0^1L^{2,p}(\Omega)$ make the tilde-map into a linear function.

**Lemma 2.56.** The application defined in (2.46) is continuous.

***Proof.*** Let $(u_n)\subset W_0^1L^{2,q}(\Omega)$ be a sequence and $0\neq u\in W_0^1L^{2,q}(\Omega)$ such that

$$\|\nabla(u_n-u)\|_{2,q}\to 0, \tag{2.49}$$

by Proposition 2.49 we have

$$\|\nabla\widetilde{u}_n\|_{2,p}=\|\nabla u_n\|_{2,q}, \quad \|\nabla\widetilde{u}\|_{2,p}=\|\nabla u\|_{2,q} \quad \text{and} \quad \|\nabla(\widetilde{u_n-u})\|_{2,p}=\|\nabla(u_n-u)\|_{2,q}. \tag{2.50}$$

If $1<q\leq 2$, by Proposition 2.13 $\|\cdot\|_{2,q}$ is a norm. Then, by (2.50), we have

$$\big|\|\nabla\widetilde{u}_n\|_{2,p}-\|\nabla\widetilde{u}\|_{2,p}\big|=\big|\|\nabla u_n\|_{2,q}-\|\nabla u\|_{2,q}\big|\leq\|\nabla(u_n-u)\|_{2,q}\to 0.$$

If $q>2$, from Proposition 2.13 $\|\cdot\|_{2,p}$ is a norm. Then, by (2.49) and (2.50), we have

$$\big|\|\nabla\widetilde{u}_n\|_{2,p}-\|\nabla\widetilde{u}\|_{2,p}\big|\leq\|\nabla(\widetilde{u_n-u})\|_{2,p}\to 0.$$

In both cases we obtain

$$\|\nabla\widetilde{u}_n\|_{2,p}\to\|\nabla\widetilde{u}\|_{2,p}. \tag{2.51}$$

In particular, the sequence $(\widetilde{u}_n)$ is bounded in $W_0^1L^{2,p}(\Omega)$, which implies that, there exists $\widetilde{v}\in W_0^1L^{2,p}(\Omega)$ and a subsequence (not renamed) $(\widetilde{u}_n)$ such that

$$\widetilde{u}_n\rightharpoonup\widetilde{v} \quad \text{in} \quad W_0^1L^{2,p}(\Omega). \tag{2.52}$$

Note that,

$$\int_\Omega\nabla u_n\nabla\widetilde{u}_n\,dx=\|\nabla u_n\|_{2,q}\|\nabla\widetilde{u}_n\|_{2,q}$$

implies that

$$\int_\Omega\nabla u\nabla\widetilde{v}\,dx=\|\nabla u\|_{2,q}\|\nabla\widetilde{u}\|_{2,q}. \tag{2.53}$$

Observe also that, by (2.52) and (2.51), we have

$$\|\nabla\widetilde{v}\|_{2,p}\leq\liminf_{n\to\infty}\|\nabla\widetilde{u}_n\|_{2,p}=\|\nabla\widetilde{u}\|_{2,p}=\|\nabla u\|_{2,q}.$$

Suppose that $\|\nabla\widetilde{v}\|_{2,p} < \|\nabla u\|_{2,q}$, setting $w = \widetilde{v}\|\nabla u\|_{2,q}/\|\nabla\widetilde{v}\|_{2,p}$. Then,

$$w \in W_0^1 L^{2,p}(\Omega) \quad \text{and} \quad \|\nabla w\|_{2,p} = \|\nabla u\|_{2,q}.$$

Now using (2.53)

$$\|\nabla u\|_{2,q}\|\nabla\widetilde{u}\|_{2,q} \geq \int_\Omega \nabla u \nabla w\, dx = \frac{\|\nabla u\|_{2,q}}{\|\nabla\widetilde{v}\|_{2,p}} \int_\Omega \nabla u \nabla\widetilde{v}\, dx > \|\nabla u\|_{2,q}\|\nabla\widetilde{u}\|_{2,q}$$

which gives a contradiction. Hence, $\|\nabla\widetilde{v}\|_{2,p} = \|\nabla u\|_{2,q}$. Using this, (2.53) and uniqueness of the tilde-map we get that $\widetilde{v} = \widetilde{u}$. Consequently, we get in (2.52)

$$\widetilde{u}_n \rightharpoonup \widetilde{u} \quad \text{in} \quad W_0^1 L^{2,p}(\Omega). \tag{2.54}$$

Finally, joining (2.51) and (2.54), we obtain

$$\widetilde{u}_n \to \widetilde{u} \quad \text{in} \quad W_0^1 L^{2,p}(\Omega).$$

■

Now, we extend the tilde-map for the weighted Lorentz-Sobolev spaces $W^{(q)}$.

**Proposition 2.57.** (See Cassani and Tarsi (2015).) Let $V : \mathbb{R}^2 \to \mathbb{R}$ a continuous function verifying $(V_1)$. For each $u \in W^{(q)}$ consider

$$S := \sup\left\{ \int_{\mathbb{R}^2} \Big(\nabla u(x)\nabla\omega(x) + V(x)u(x)\ \omega(x)\Big)\, dx :\ \omega \in W^{(p)},\ \|\omega\|_{(p)} = \|u\|_{(q)} \right\}. \tag{2.55}$$

Then, there exists a unique $\widetilde{u} \in W^{(p)}$ such that

$$S = \int_{\mathbb{R}^2} \Big(u(x)\widetilde{u}(x) + V(x)\nabla u(x)\nabla\widetilde{u}(x)\Big)\, dx = \|u\|_{(q)}\|\widetilde{u}\|_{(p)} \quad \text{and} \quad \|u\|_{(q)} = \|\widetilde{u}\|_{(p)}.$$

***Proof.*** Consider the functional

$$L_u(v) := \int_{\mathbb{R}^2} \big(\nabla u \nabla v + V(x)uv\big)\, dx, \quad v \in W^{(p)}. \tag{2.56}$$

Using Hölder's inequality for Lorentz spaces we have

$$|L_u(v)| \leq \|\nabla u\|_{2,p}\|\nabla v\|_{2,q} + \|V^{1/p}u\|_{2,p}\|V^{1/q}v\|_{2,q} \leq 2\|u\|_{(p)}\|v\|_{(q)}.$$

Thus, $L_u \in \big(W^{(p)}\big)^*$. Since $W^1L^{2,p}(\mathbb{R}^2)$ is reflexive, by $(V_1)$ we have that $W^{(p)}$ is a closed subspace of $W^1L^{2,p}(\mathbb{R}^2)$, hence $W^{(p)}$ is also reflexive, therefore, there exist uniqueness $N+1$ functions $w_j \in L^{2,q}(\mathbb{R}^2)$ $j = 0, \cdots, N$ such that

$$L_u(v) = \sum_{j=1}^{N} \int_{\mathbb{R}^2} w_j \frac{\partial v}{\partial x_j}\, dx + \int_{\mathbb{R}^2} V(x)w_0 v\, dx$$

and

$$\sup_{v \in W^{(p)}, \|v\|=1} |L_u(v)| = \|w_0\|_{(q)}. \tag{2.57}$$

By uniqueness $w_0 = u$ and $w_j = D_j u$ for $j = 1, \dots, N$. Thus, From(2.56) and (2.57), we obtain

$$\sup_{v \in W^{(p)}, \|v\|_p = 1} \int_{\mathbb{R}^2} \big(\nabla u \nabla v + V(x)uv\big)\, dx = \|u\|_{(q)}. \tag{2.58}$$

Taking $\omega = v\|u\|_{(p)}$ with $v \in W^{(p)}, \|v\|_p = 1$ in (2.58)

$$\sup_{\omega \in W^{(p)}, \|\omega\|_{(p)} = \|u\|_{(q)}} \int_{\mathbb{R}^2} \big(\nabla u \nabla \omega + V(x)u\omega\big)\, dx = \|u\|_{(q)}^2.$$

In order to prove an existence and uniqueness of $\widetilde{u} \in W^{(q)}$ which attains the supremum $S$, we can proceed similarly as the Proposition 2.49. ■

Analogously, using Proposition 2.57, we define the tilde-map

$$\begin{array}{rcl} \sim\ : W^{(q)} & \to & W^{(p)} \\ u & \mapsto & \widetilde{u}, \end{array} \tag{2.59}$$

The application (2.59) is continuous and set

$$E := W^{(q)} \times W^{(p)}$$

and

$$E^+ = \{(u, \widetilde{u}) : u \in W^{(q)}\} \quad \text{and} \quad E^- = \{(u, -\widetilde{u}) : u \in W^{(q)}\}.$$

Then, the following decomposition holds

$$E = E^+ \widetilde{\oplus} E^-.$$

Similarly in these spaces we consider the following applications.

**Definition 2.58.** Given $(u, \widetilde{v}), (y, \widetilde{z}) \in E$ and $\alpha \in \mathbb{R}$, we define

$$(u, \widetilde{v}) \,\widetilde{+}\, (y, \widetilde{z}) := (u + y, \widetilde{v + z}), \tag{2.60}$$

$$\alpha(u, \widetilde{v}) := (\alpha u, \widetilde{\alpha v}). \tag{2.61}$$

The set $E = W^{(q)} \times W^{(p)}$ endowed with the operations given by (2.60) and (2.61) satisfies the same properties given by Lemma 2.54.

CHAPTER

3

# HAMILTONIAN SYSTEM WITH CRITICAL EXPONENTIAL GROWTH IN A BOUNDED DOMAIN

This chapter is concerned with the existence of nontrivial solution to the following Hamiltonian elliptic system

$$\begin{cases} -\Delta u &= g(v), \quad \text{in} \quad \Omega, \\ -\Delta v &= f(u), \quad \text{in} \quad \Omega, \\ u = v &= 0, \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ and the functions $f$ and $g$ possess critical exponential growth with $(p,q)$ lying on the exponential critical hyperbola.

## 3.1 Introduction

We start with the notion of critical and subcritical exponential growth.

**Definition 3.1.** Given $p > 1$, we say that a function $h : \mathbb{R} \to \mathbb{R}$ has $p$-critical exponential growth, if there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|h(s)|}{e^{\alpha|s|^p}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0. \end{cases}$$

Whereas, we say that $h : \mathbb{R} \to \mathbb{R}$ has $p$-subcritical exponential growth, if

$$\lim_{|s|\to\infty} \frac{|h(s)|}{e^{\alpha|s|^p}} = 0, \quad \text{for all} \quad \alpha > 0.$$

In order to study the existence of the system (3.1), we make the following hypotheses on the functions $f$ and $g$:

$(A_1)$ $f, g \in \mathscr{C}(\mathbb{R})$, with $f(s) = g(s) = o(s)$ near the origin.

$(A_2)$ There exist constants $\mu > 2$, $\nu > 2$ and $s_0 > 0$ such that

$$0 < \mu F(s) \leq s f(s) \quad \text{and} \quad 0 < \nu G(s) \leq s g(s), \quad \text{for all} \quad |s| > s_0.$$

where $F(s) = \int_0^s f(t)\,dt$ and $G(s) = \int_0^s g(t)\,dt$.

$(A_3)$ There exist $\alpha_0 > 0$ and $p > 1$, such that

$$\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha |s|^p}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0. \end{cases}$$

$(A_4)$ There exists $\beta_0 > 0$, such that

$$\lim_{|s|\to\infty} \frac{|g(s)|}{e^{\beta |s|^q}} = \begin{cases} 0, & \beta > \beta_0, \\ +\infty, & \beta < \beta_0. \end{cases}$$

where $q = \dfrac{p}{p-1}$.

Throughout this chapter, we denote the product space

$$E = W_0^1 L^{2,q}(\Omega) \times W_0^1 L^{2,p}(\Omega),$$

endowed with the norm

$$\|(u, \widetilde{v})\| := \left(\|\nabla u\|_{2,q}^2 + \|\nabla \widetilde{v}\|_{2,p}^2\right)^{1/2}.$$

Moreover $E$ has a structure of vector space with the operations (2.47) and (2.48). We recall that $\widetilde{v}$ is an independent variable; we write $\widetilde{v}$ to emphasize that $\widetilde{v}$ belongs to the space $W_0^1 L^{2,p}(\Omega)$.

**Lemma 3.2.** Let $\theta > 2$. Then, the following number

$$\delta_\theta = \inf\left\{ \int_\Omega \left(|e_1 + \omega|^\theta + |\widetilde{e_1 - \omega}|^\theta\right) dx : \omega \in W_0^1 L^{2,q}(\Omega) \text{ com } \|\nabla \omega\|_{2,q} \leq 2 + \sqrt{5} \right\}$$

is positive, where $e_1$ is the first eigenfunction of $(-\Delta, H_0^1(\Omega))$ normalized in $W_0^1 L^{2,q}(\Omega)$.

***Proof.*** We proceed by contradiction, if $\delta_\theta = 0$, we can find a sequence $(\omega_n)$ in $W_0^1 L^{2,q}(\Omega)$ such that

$$\|\nabla \omega_n\|_{2,q} \leq 2 + \sqrt{5} \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega \left(|e_1 + \omega_n|^\theta + |\widetilde{e_1 - \omega_n}|^\theta\right) = 0.$$

Since $W_0^1 L^{2,q}(\Omega)$ is a reflexive space, there exists $\omega \in W_0^1 L^{2,q}(\Omega)$ such that $\omega_n \rightharpoonup \omega$ in $W_0^1 L^{2,q}(\Omega)$ up to subsequence. By compact embedding of $W_0^1 L^{2,q}(\Omega)$ in $L^\theta(\Omega)$, we can assume that $e_1 + \omega_n \to e_1 + \omega$ in $L^\theta(\Omega)$. Observe also that $e_1 - \omega_n \rightharpoonup e_1 - \omega$ in $W_0^1 L^{2,q}(\Omega)$. Using the fact that the tilde-map is a continuous linear function, we obtain $\widetilde{e_1 - \omega_n} \rightharpoonup \widetilde{e_1 - \omega}$ in $W_0^1 L^{2,p}(\Omega)$. By compact embedding of $W_0^1 L^{2,p}(\Omega)$ in $L^\theta(\Omega)$, we can assume that $\widetilde{e_1 - \omega_n} \to \widetilde{e_1 - \omega}$ in $L^\theta(\Omega)$. Hence, $\|e_1 + \omega\|_\theta + \|\widetilde{e_1 - \omega}\|_\theta = 0$. Thus, we obtain that $e_1 = -\omega$ and $e_1 = \omega$ which is a contradiction. ■

Now, we describe an additional condition on the functions $f$ and $g$.

$(A_5)$ There exist constants $\theta > 2$ and $C_\theta > 0$ such that

$$F(s) \geq C_\theta |s|^\theta \quad \text{and} \quad G(s) \geq C_\theta |s|^\theta, \quad \text{for all} \quad s \in \mathbb{R},$$

where

$$C_\theta > \frac{14+6\sqrt{5}}{\delta_\theta R^{\theta-2}}, \quad R^2 = \frac{2\pi}{\alpha_0^{1/p}\beta_0^{1/q}} \max\left\{\frac{\mu-2}{\mu}, \frac{\nu-2}{\nu}\right\}, \tag{3.2}$$

$\delta_\theta$ is defined as in Lemma 3.2 and $\mu$ and $\nu$ are given by condition $(A_2)$.

**Example 3.3.** Let $\theta > 2$, $p,q > 1$, with $1/p + 1/q = 1$, $A > 0$ and consider the following continuous functions defined on $\mathbb{R}$

$$f_1(s) = g_1(s) = A|s|^{\theta-2}s, \quad \text{for all} \quad s \in \mathbb{R},$$

$$f_2(s) = \begin{cases} ps^{p-1}\left(e^{s^p} - 1\right), & 0 \leq s < 1, \\ (e-1)\left[(ps^{p-1}-1)e^{s^p-s} + s^{p-1}\right], & 1 \leq s, \end{cases}$$

and

$$g_2(s) = \begin{cases} qs^{q-1}\left(e^{s^q} - 1\right), & 0 \leq s < 1, \\ (e-1)\left[(qs^{q-1}-1)e^{s^q-s} + s^{q-1}\right], & 1 \leq s, \end{cases}$$

where $f_2(-s) = -f_2(s)$ and $g_2(-s) = -g_2(s)$ for all $s \geq 0$. Then, the functions $f = f_1 + f_2$ and $g = g_1 + g_2$ satisfy conditions $(A_1) - (A_5)$ for $A$ sufficiently large.

We observe that, to prove the conditions, it is sufficient to show them for the function $f$. Note that

$$F_1(s) = \int_0^s f_1(t)\,dt = \frac{A}{\theta}|s|^\theta, \quad \text{for all} \quad s \in \mathbb{R}$$

and

$$F_2(s) = \int_0^s f_2(t)\,dt = \begin{cases} e^{s^p} - s^p - 1, & 0 \leq s < 1, \\ e - 2 + (e-1)\left[(e^{s^p-s} - 1) + \dfrac{s^p - 1}{p}\right], & 1 \leq s. \end{cases}$$

Since $f_2$ is an odd function we have $F_2(-s) = F_2(s)$ for all $s \geq 0$. We observe that $f$ is an odd function and satisfies:

(a) The following limits holds

$$\lim_{s\to 0^+} \frac{f_1(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to 0^+} \frac{f_2(s)}{s} = 0,$$

since $f_1$ and $f_2$ are odd functions we have that $f$ satisfies the condition $(A_1)$.

(b) For $s \geq 1$, we have

$$0 < \frac{F(s)}{sf(s)} = \frac{F_1(s)+F_2(s)}{sf_1(s)+sf_2(s)} = \frac{\dfrac{A}{\theta}s^\theta + e - 2 + (e-1)\left[(e^{s^p-s}-1) + \dfrac{s^p-1}{p}\right]}{As^\theta + s(e-1)\left[(ps^{p-1}-1)e^{s^p-s} + s^{p-1}\right]}.$$

Using last equality and the fact that $F(s)/\big(sg(s)\big)$ is an even function we have

$$\lim_{|s|\to+\infty}\frac{F(s)}{sf(s)}=0.$$

Thus, $f$ satisfies condition $(A_2)$.

(c) For $s\geq 1$ we have

$$0<\frac{f(s)}{e^{\alpha s^p}}=\frac{f_1(s)+f_2(s)}{e^{\alpha s^p}}=\frac{As^{\theta-1}+(e-1)\left[(ps^{p-1}-1)e^{s^p-s}+s^{p-1}\right]}{e^{\alpha s^p}}.$$

Thus,

$$\lim_{s\to+\infty}\frac{|f(s)|}{e^{\alpha|s|^p}}=\lim_{s\to+\infty}\frac{f(s)}{e^{\alpha s^p}}=\begin{cases}0, & \alpha>1,\\ +\infty, & \alpha<1,\end{cases}$$

and

$$\lim_{s\to-\infty}\frac{|f(s)|}{e^{\alpha|s|^p}}=\lim_{r\to+\infty}\frac{|f(-r)|}{e^{\alpha|-r|^p}}=\lim_{r\to+\infty}\frac{f(r)}{e^{\alpha r^p}}=\begin{cases}0, & \alpha>1,\\ +\infty, & \alpha<1.\end{cases}$$

That is, $f$ satisfies condition $(A_3)$ with $\alpha_0=1$.

(d) Since $F_2$ is a nonnegative function, we have

$$F(s)=F_1(s)+F_2(s)\geq F_1(s)=\frac{A}{\theta}|s|^\theta,\quad \text{for all}\quad s\in\mathbb{R}.$$

Thus, taking $A$ sufficiently large, $f$ satisfies condition $(A_5)$.

Next we state the main result of this chapter.

**Theorem 3.4.** Suppose $(A_1)-(A_5)$ hold. Then, (3.1) possesses a nontrivial weak solution in $E=W_0^1L^{2,q}(\Omega)\times W_0^1L^{2,p}(\Omega)$.

In the proof of Theorem 3.4 we use variational arguments. More precisely, combining Theorem 5.3 and Example 5.26 in Rabinowitz (1986) we obtain the following result: We recall the definition of $(PS)$ sequence.

**Definition 3.5.** Let $E$ be a Banach space and $I\in\mathscr{C}^1(E,\mathbb{R})$. The function $I$ satisfies the Palais-Smale condition (denoted by (PS)) if any sequence $(u_n)\subset E$ for which $\big(I(u_n)\big)$ is bounded and $I'(u_n)\to 0$ possesses a convergent subsequence.

**Proposition 3.6. (Linking theorem)** Let $E$ be a real Banach space with $E=V\oplus X$, where $V$ is finite dimensional. Suppose $I\in\mathscr{C}^1(E,\mathbb{R})$, satisfies $(PS)$, and

$(I_1)$ There are constants $\rho,\sigma>0$ such that $I_{|\partial B_\rho\cap X}\geq\sigma$.

$(I_2)$ There is an $e \in \partial B_1 \cap X$ and $R_0 > 0$ and $R_1 > \rho$ such that if

$$Q := (\overline{B}_{R_0} \cap V) \oplus \{re : 0 \leq r \leq R_1\},$$

then $I_{\partial Q} \leq 0$.

Then, $I$ possesses a critical value $c \geq \sigma$ which can be characterized as

$$c := \inf_{h \in \Gamma} \max_{u \in Q} I(h(u))$$

where

$$\Gamma = \{h \in \mathscr{C}(Q,E) : h_{|\partial Q} = id\}.$$

**Remark 3.7.** If the $(PS)$ condition is not required in Proposition 3.6, the geometric conditions $(I_1)$ and $(I_2)$ combined with the Ekeland Variational Principle (see Ekeland and Temam (1999)) asserts the existence of a sequence $(u_n) \subset E$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0.$$

## 3.2 Variational setting

In order to employ variational methods, we consider the functional $J : E \to \mathbb{R}$ associated with (3.1) and defined by

$$J(u,\widetilde{v}) = \int_\Omega \nabla u \nabla \widetilde{v}\, dx - \int_\Omega F(u)\, dx - \int_\Omega G(\widetilde{v})\, dx. \tag{3.3}$$

**Proposition 3.8.** The functional $J$ given by (3.3) is well defined and belongs to the class $\mathscr{C}^1(E,\mathbb{R})$ with

$$J'(u,\widetilde{v})(\phi,\widetilde{\psi}) = \int_\Omega \left(\nabla u \nabla \widetilde{\psi} + \nabla \widetilde{v} \nabla \phi\right) dx - \int_\Omega f(u)\phi\, dx - \int_\Omega g(\widetilde{v})\widetilde{\psi}\, dx,$$

for all $(\phi,\widetilde{\psi}) \in E$.

***Proof.*** Let $u \in W_0^1 L^{2,q}(\Omega)$ and $\widetilde{v} \in W_0^1 L^{2,p}(\Omega)$. Then, from Hölder's inequality in Lorentz spaces, we have

$$\left| \int_\Omega \nabla u \nabla \widetilde{v}\, dx \right| \leq \|\nabla u\|_{2,q} \|\nabla \widetilde{v}\|_{2,p}. \tag{3.4}$$

By $(A_1)$ and $(A_3)$, there exists $C > 0$ such that

$$|f(s)| \leq C e^{(\alpha_0+1)|s|^p}, \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.5}$$

Thus,

$$|F(u)| \leq \int_0^{|u|} |f(s)|\, ds \leq C \int_0^{|u|} e^{(\alpha_0+1)|s|^p}\, ds \leq C|u| e^{(\alpha_0+1)|u|^p} \leq \frac{C}{2}|u|^2 + \frac{C}{2} e^{2(\alpha_0+1)|u|^p}.$$

Consequently,

$$\Big|\int_\Omega F(u)\,dx\Big| \le C_1\int_\Omega |u|^2\,dx + C_1\int_\Omega e^{2(\alpha_0+1)|u|^p}\,dx.$$

By Lemma 2.38 and Remark 2.40, we have

$$\int_\Omega F(u)\,dx < +\infty, \quad \text{for all} \quad u\in W_0^1L^{2,q}(\Omega). \tag{3.6}$$

Similarly, $G(\widetilde{v})$ belongs to $L^1(\Omega)$ for all $\widetilde{v}\in W_0^1L^{2,p}(\Omega)$. Thus, joining (3.4) and (3.6) we conclude that $J$ is well defined on $E$.

Set $J_1,J_2,J_3 : E\to\mathbb{R}$ by

$$J_1(u,\widetilde{v}) = \int_\Omega \nabla u\nabla\widetilde{v}\,dx, \quad J_2(u,\widetilde{v}) = \int_\Omega F(u)\,dx \quad \text{and} \quad J_3(u,\widetilde{v}) = \int_\Omega G(\widetilde{v})\,dx.$$

By (3.4), we have

$$|J_1(u,\widetilde{v})| \le 2\|u\|_{2,q}\|\widetilde{v}\|_{2,p}, \quad \text{for all} \quad (u,\widetilde{v})\in E.$$

Thus, $J_1$ is a continuous bilinear function. Then, $J_1\in\mathscr{C}^\infty(E,\mathbb{R})$ and

$$J_1'(u,\widetilde{v})(\phi,\widetilde{\psi}) = \int_\Omega \big(\nabla u\nabla\widetilde{\psi} + \nabla\widetilde{v}\nabla\phi\big)\,dx, \quad \text{for all} \quad (\phi,\widetilde{\psi})\in E. \tag{3.7}$$

Now, fixing $u$ and $\phi$ in $W_0^1L^{2,q}(\Omega)$, for given $x\in\Omega$, consider $h:\mathbb{R}\to\mathbb{R}$ defined by

$$h(t) = F\big(u(x)+t\phi(x)\big).$$

Let $(t_n)$ be any sequence in $\mathbb{R}$ such that $t_n\to 0$, we can assume that $0<|t_n|\le 1$ for all $n\ge 1$. For any $n\ge 1$, by the Mean value theorem, there exists $\theta_n = \theta_n(t_n,x)\in(0,1)$ such that

$$F(u+t_n\phi) - F(u) = h(t_n) - h(0) = h'(\theta_n t_n)t_n = f(u+\theta_n t_n\phi)t_n\phi. \tag{3.8}$$

Define

$$h_n(x) := F(u+t_n\phi) - F(u) = f(u+\theta_n t_n\phi)\phi.$$

Since $f$ is continuous, we have

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty}\big(F(u+t_n\phi)-F(u)\big) = f(u)\phi, \quad \text{for all} \quad x\in\Omega.$$

Note that $|u+\theta_n t_n\phi| \le |u|+|\phi| = w\in W_0^1L^{2,q}(\Omega)$. From (3.5), we have

$$\begin{aligned}|h_n(x)| &= |f(u+\theta_n t_n\phi)\phi| \\ &\le Ce^{(\alpha_0+1)|u+\theta_n t_n\phi|^p}|\phi| \\ &\le Ce^{(\alpha_0+1)|w|^p}|\phi| \\ &\le C_1|\phi|^2 + C_1e^{2(\alpha_0+1)|w|^p}.\end{aligned}$$

From Lemma 2.38 and Remark 2.40, we get

$$C_1|\phi|^2 + C_1e^{2(\alpha_0+1)|w|^p}\in L^1(\Omega).$$

By Dominated convergence theorem we obtain

$$\begin{aligned} J_2'(u)\phi &= \lim_{n\to+\infty} \frac{J_2(u+t_n\phi)-J_2(u)}{t_n} \\ &= \lim_{n\to+\infty} \int_\Omega \frac{F(u+t_n\phi)-F(u)}{t_n}\,dx \\ &= \lim_{n\to+\infty} \int_\Omega h_n(x)\,dx \\ &= \int_\Omega f(u)\phi\,dx. \end{aligned}$$

Now, we prove the continuity of the Fréchet derivative. Let $(u_n)$ be a sequence in $W_0^1L^{2,q}(\Omega)$ such that $u_n \to u$ in $W_0^1L^{2,q}(\Omega)$. By Proposition 2.44, there exists a subsequence (not renamed) $(u_n)$ and $\widehat{u} \in W_0^1L^{2,q}(\Omega)$ such that

$$|u_n(x)| \le \widehat{u}(x), \quad \text{almost everywhere in} \quad \Omega \tag{3.9}$$

and

$$u_n(x) \to u(x), \quad \text{almost everywhere in} \quad \Omega. \tag{3.10}$$

Thus,

$$\begin{aligned} |f(u_n)-f(u)|^2 &\le 2|f(u_n)|^2 + 2|f(u)|^2 \\ &\le 2Ce^{2(\alpha_0+1)|u_n|^p} + 2Ce^{2(\alpha_0+1)|u|^p} \\ &\le 2Ce^{2(\alpha_0+1)|\widehat{u}|^p} + 2Ce^{2(\alpha_0+1)|u|^p}. \end{aligned}$$

By Remark 2.40, we have

$$2Ce^{2(\alpha_0+1)|\widehat{u}|^p} + 2Ce^{2(\alpha_0+1)|u|^p} \in L^1(\Omega).$$

Moreover, from (3.10) and the fact that $f$ is continuous, we obtain

$$|f(u_n)-f(u)|^2 \to 0, \quad \text{almost everywhere in} \quad \Omega.$$

By Dominated convergence theorem, we obtain

$$\|f(u_n)-f(u)\|_2 \to 0, \tag{3.11}$$

which implies

$$\begin{aligned} |\langle J_2'(u_n)-J_2'(u),\phi\rangle| &\le \int_\Omega |f(u_n)-f(u)\phi|\,dx \\ &\le \|f(u_n)-f(u)\|_2\|\phi\|_2 \\ &\le C\|f(u_n)-f(u)\|_2\|\nabla\phi\|_{2,q}. \end{aligned}$$

Thus, by (3.11)

$$\sup_{\|\nabla\phi\|_{2,q}\le 1} |\langle J_2'(u_n)-J_2'(u),\phi\rangle| \le C\|f(u_n)-f(u)\|_2 \to 0.$$

That is, $J_2$ belongs to $\mathscr{C}^1(E,\mathbb{R})$, similar arguments prove that $J_3$ belongs to $\mathscr{C}^1(E,\mathbb{R})$. Consequently $J \in \mathscr{C}^1(E,\mathbb{R})$. ∎

We say that $(u,\widetilde{v}) \in E$ is a weak solution of (3.1) if

$$\int_\Omega (\nabla u \nabla \widetilde{\psi} + \nabla \widetilde{v} \nabla \phi)\, dx = \int_\Omega \Big(f(u)\widetilde{\phi} + g(\widetilde{v})\psi\Big)\, dx, \quad \text{for all} \quad (\phi, \widetilde{\psi}) \in E.$$

Consequently, critical points of the functional $J$ correspond to the weak solutions of (3.1).

## 3.3 The geometry of the linking theorem

This section is devoted to establish that the functional $J$ satisfies $(I_1)$ and $(I_2)$.

**Lemma 3.9.** There exist constants $\rho > 0$ and $\sigma > 0$ such that $J(u,\widetilde{u}) \geq \sigma$, for all $(u,\widetilde{u}) \in E$ with $\|(u,\widetilde{u})\| = \rho$.

***Proof.*** From $(A_1)$, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(s)| \leq 2\varepsilon|s|$ and $|g(s)| \leq 2\varepsilon|s|$ for all $|s| < \delta$. Then,

$$|F(s)| \leq \varepsilon|s|^2 \quad \text{and} \quad |G(s)| \leq \varepsilon|s|^2, \quad \text{for all} \quad |s| < \delta. \tag{3.12}$$

By $(A_1)$, $(A_3)$ and $(A_4)$, there exists a positive constant $C$ such that

$$|f(s)| \leq Ce^{2\alpha_0|s|^p} \quad \text{and} \quad |g(s)| \leq Ce^{2\beta_0|s|^q}, \quad \text{for all} \quad s \in \mathbb{R}.$$

Thus, there exists some $C = C(\varepsilon) > 0$ such that

$$F(s) \leq C|s|^3 e^{2\alpha_0|s|^p} \quad \text{and} \quad G(s) \leq C|s|^3 e^{2\beta_0|s|^q}, \quad \text{for all} \quad |s| \geq \delta. \tag{3.13}$$

Joining (3.12) and (3.13) we get

$$F(s) \leq \varepsilon|s|^2 + C|s|^3 e^{2\alpha_0|s|^p} \quad \text{and} \quad G(s) \leq \varepsilon|s|^2 + C|s|^3 e^{2\beta_0|s|^q}, \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.14}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
J(u,\widetilde{u}) &= \int_\Omega \nabla u \nabla \widetilde{u}\, dx - \int_\Omega F(u)\, dx - \int_\Omega G(\widetilde{u})\, dx \\
&\geq \|\nabla u\|_{2,q} \|\nabla \widetilde{u}\|_{2,p} - \int_\Omega \big(\varepsilon|u|^2 + C|u|^3 e^{2\alpha_0|u|^p}\big)\, dx - \int_\Omega \big(\varepsilon|\widetilde{u}|^2 + C|\widetilde{u}|^3 e^{2\beta_0|\widetilde{u}|^q}\big)\, dx \\
&\geq \frac{1}{2}\|\nabla u\|_{2,q}^2 - \varepsilon\|u\|_2^2 - C\|u\|_6^3 \Big(\int_\Omega e^{4\alpha_0|u|^p}\, dx\Big)^{1/2} \\
&\quad + \frac{1}{2}\|\nabla \widetilde{u}\|_{2,p}^2 - \varepsilon\|\widetilde{u}\|_2^2 - C\|\widetilde{u}\|_6^3 \Big(\int_\Omega e^{4\beta_0|\widetilde{u}|^q}\, dx\Big)^{1/2}.
\end{aligned}$$

For $\rho_1 > 0$ sufficiently small, by Theorem 2.41 there exists $C > 0$ such that

$$\Big(\int_\Omega e^{4\alpha_0|u|^p}\, dx\Big)^{1/2} \leq C \quad \text{and} \quad \Big(\int_\Omega e^{4\beta_0|\widetilde{u}|^q}\, dx\Big)^{1/2} \leq C, \quad \text{for all} \quad \|(u,\widetilde{u})\| < \rho_1.$$

Moreover, using Lemma 2.38 we can find some positive constant $C_1$ (independent of $\varepsilon$), such that

$$\|u\|_2 \leq C_1\|\nabla u\|_{2,q}, \quad \|\widetilde{u}\|_2 \leq C_1\|\nabla \widetilde{u}\|_{2,p}$$

$$\|u\|_6 \leq C_1\|\nabla u\|_{2,q} \quad \text{and} \quad \|\widetilde{u}\|_6 \leq C_1\|\nabla \widetilde{u}\|_{2,p}.$$

Thus, for some $C > 0$ and $C_1 > 0$ (independent of $\varepsilon$) we have

$$\begin{aligned} J(u,\widetilde{u}) &\geq (\frac{1}{2}-\varepsilon C_1)\|\nabla u\|_{2,q}^2 - C\|\nabla u\|_{2,q}^3 + (\frac{1}{2}-\varepsilon C_1)\|\nabla \widetilde{u}\|_{2,p}^2 - C\|\nabla \widetilde{u}\|_{2,p}^3 \\ &\geq (\frac{1}{2}-\varepsilon C_1)\|(u,\widetilde{u})\|^2 - C\|(u,\widetilde{u})\|^3 \\ &\geq (\frac{1}{2}-\varepsilon C_1)\|(u,\widetilde{u})\|^2\big(1-C\|(u,\widetilde{u})\|\big). \end{aligned}$$

Now, taking

$$0 < \varepsilon \leq \frac{1}{4C_1} \quad \text{and} \quad 0 < \rho_2 \leq \frac{1}{2C}.$$

Thus, for $0 < \rho \leq \min\{\rho_1, \rho_2\}$, we obtain

$$J(u,\widetilde{u}) \geq \frac{\rho^2}{8} = \sigma, \quad \text{for all} \quad \|(u,\widetilde{u})\| = \rho.$$

■

**Lemma 3.10.** (See Cassani and Tarsi (2015).) Let $r, r' > 1$ such that $1/r + 1/r' = 1$ and $t \geq 0$. Then, the following inequality holds

$$st \leq \begin{cases} e^{t^r} - 1 + s(\ln s)^{1/r}, & s \geq e^{\frac{1}{r^{r'}}}, \\ e^{t^r} - 1 + \dfrac{s^{r'}}{r'}, & 0 \leq s \leq e^{\frac{1}{r^{r'}}}. \end{cases}$$

From definition of $C_\theta$, there exist $0 < m_0 < 1$ and $\varepsilon > 0$ such that

$$C_\theta > \frac{14+6\sqrt{5}}{\delta_\theta R_1^{\theta-2}}, \quad \text{where} \quad R_1^2 = \frac{m_0 R^2}{1+\varepsilon}. \tag{3.15}$$

That is,

$$R_1^2 = \frac{2m_0\pi}{(1+\varepsilon)\alpha_0^{1/p}\beta_0^{1/q}} \min\left\{\frac{\mu-2}{\mu}, \frac{\nu-2}{\nu}\right\}. \tag{3.16}$$

**Lemma 3.11.** Let

$$Q = \{r(e_1,\widetilde{e}_1) \widetilde{+} (\omega,-\widetilde{\omega}) : \|(\omega,-\widetilde{\omega})\| \leq (2\sqrt{2}+\sqrt{10})R_1, 0 \leq r \leq R_1\},$$

where $R_1 > 0$ is given by (3.16). Then, $J(z) \leq 0$ for all $z \in \partial Q$, where $\partial Q$ denote the boundary of $Q$ in $\mathbb{R}(e_1,\widetilde{e}_1) \widetilde{+} E^-$.

***Proof.*** We can write

$$Q=\{r(e_1,\widetilde{e}_1)\ \widetilde{+}\ (\omega,-\widetilde{\omega}):\|\nabla\omega\|_{2,q}\le(2+\sqrt{5})R_1,0\le r\le R_1\}.$$

Observe that the boundary $\partial Q$ of the set $Q$ consists of three parts. On these parts the functional $J$ is estimated as follows:

(i) Let $z\in\partial Q\cap E^-$. Thus, $z=(u,-\widetilde{u})$ and

$$J(z)=J(u,-\tilde{u})=-\int_\Omega \nabla u\nabla\widetilde{u}\,dx-\int_\Omega F(u)\,dx-\int_\Omega G(-\widetilde{u})\le-\|\nabla u\|_{2,q}^2\le 0,$$

because $F$ and $G$ are nonnegatives functions.

(ii) Let $z=r(e_1,\widetilde{e}_1)\ \widetilde{+}\ (\omega,-\widetilde{\omega})=(re_1+\omega,\widetilde{re_1-\omega})\in\partial Q$ with $\|\nabla\omega\|_{2,q}=(2+\sqrt{5})R_1$ and $0\le r\le R_1$. Thus,

$$\begin{aligned}
J(z)&=\int_\Omega \nabla(re_1+\omega)\nabla u(\widetilde{re_1-\omega})\,dx-\int_\Omega F(re_1+\omega)\,dx-\int_\Omega G(\widetilde{re_1-\omega})\,dx\\
&\le\int_\Omega \nabla(re_1+\omega)\nabla u(\widetilde{re_1-\omega})\,dx\\
&=-\int_\Omega \nabla(re_1-\omega)\nabla u(\widetilde{re_1-\omega})\,dx+\int_\Omega \nabla(2re_1)\nabla u(\widetilde{re_1-\omega})\,dx\\
&\le-\|\nabla(re_1-\omega)\|_{2,q}^2+\|\nabla(2re_1)\|_{2,q}\|\nabla(\widetilde{re_1-\omega})\|_{2,p}\\
&\le-\|\nabla(re_1-\omega)\|_{2,q}^2+\|\nabla(2re_1)\|_{2,q}\Big(\|\nabla(re_1)\|_{2,q}+\|\nabla w\|_{2,q}\Big)\\
&\le\Big(-\|\nabla(re_1)\|_{2,q}^2+2\|\nabla(re_1)\|_{2,q}\|\nabla\omega\|_{2,q}-\|\nabla\omega\|_{2,q}^2\Big)\\
&\quad+\|\nabla(2re_1)\|_{2,q}\big(\|\nabla(re_1)\|_{2,q}+\|\nabla w\|_{2,q}\big).
\end{aligned}$$

Using the fact $\|\nabla e_1\|_{2,q}=1$ and $\|\nabla\omega\|_{2,q}=(2+\sqrt{5})R_1$, we obtain

$$J(z)\le-\|\nabla\omega\|_{2,q}^2+4r\|\nabla\omega\|_{2,q}+r^2\le-\|\nabla\omega\|_{2,q}^2+4R_1\|\nabla\omega\|_{2,q}+R_1^2=0.$$

(iii) Let $z=R_1(e_1,\widetilde{e}_1)\ \widetilde{+}\ R_1(\omega,-\widetilde{\omega})=(R_1(e_1+\omega),R_1(\widetilde{e_1-\omega}))$ with $\|\nabla\omega\|_{2,q}\le 2+\sqrt{5}$. From assumption $(A_5)$ we have

$$\begin{aligned}
J(z)&=R_1^2\int_\Omega \nabla(e_1+\omega)\nabla u(\widetilde{e_1-\omega})\,dx-\int_\Omega F\big(R_1(e_1+\omega)\big)\,dx-\int_\Omega G\big(R_1(\widetilde{e_1-\omega})\big)\,dx\\
&\le R_1^2\|\nabla(e_1+\omega)\|_{2,q}\|\nabla(\widetilde{e_1-\omega})\|_{2,p}-C_\theta\int_\Omega\Big(|R_1(e_1+\omega)|^\theta+|R_1(\widetilde{e_1-\omega})|^\theta\Big)\,dx\\
&=R_1^2\|\nabla(e_1+\omega)\|_{2,q}\|\nabla(e_1-\omega)\|_{2,q}-C_\theta R_1^\theta\int_\Omega\big(|e_1+\omega|^\theta+|\widetilde{e_1-\omega}|^\theta\big)\,dx\\
&\le R_1^2(\|\nabla e_1\|_{2,q}+\|\nabla\omega\|_{2,q})^2-C_\theta R_1^\theta\inf_{\|\nabla\omega\|_{2,q}\le 2+\sqrt{5}}\int_\Omega\big(|e_1+\omega|^\theta+|\widetilde{e_1-\omega}|^\theta\big)\,dx
\end{aligned}$$

$$\leq (14+6\sqrt{5})R_1^2 - C_\theta R_1^\theta \delta_\theta.$$

Using (3.15) in last inequality, we obtain $J(z) \leq 0$.

■

**Lemma 3.12.** (See Ruf (2008).) Let $(u_n, \widetilde{v}_n) \in E$ be a sequence such that $|J(u_n, \widetilde{v}_n)| \leq d$, and

$$|J'(u_n, \widetilde{v}_n)(\phi, \widetilde{\psi})| \leq \varepsilon_n \|(\phi, \widetilde{\psi})\|, \quad \text{where} \quad \phi, \psi \in \{0, u_n, v_n\}$$

where $(\varepsilon_n) \subset \mathbb{R}$ is a sequence such that $\varepsilon_n \to 0$ as $n \to +\infty$. Then, there exists $C > 0$ such that

$$\int_\Omega f(u_n) u_n \, dx \leq C, \quad \int_\Omega g(\widetilde{v}_n) \widetilde{v}_n \, dx \leq C, \quad \text{for all} \quad n \geq 1$$

and

$$\|(u_n, \widetilde{v}_n)\| \leq C, \quad \text{for all} \quad n \geq 1.$$

## 3.4 Finite-dimensional approximation

Since $J$ is strongly indefinite near the origin ($J$ is positive definite on $E^+$ and negative definite on $E^-$, the standard linking theorem can not be directly applied. We therefore consider an approximate problem on finite-dimensional spaces. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\{\lambda_i\}_{i\in\mathbb{N}}$ of $(-\Delta, H_0^1(\Omega))$. Setting

$$E_n^+ = Span\{(e_1, \widetilde{e}_i) : \ i = 1, 2, \ldots, n\}, \quad E_n^- = Span\{(e_1, -\widetilde{e}_i) : \ i = 1, 2 \ldots, n\},$$

and

$$E_n = E_n^+ \oplus E_n^-,$$

Define

$$\Gamma_n = \{\gamma \in \mathscr{C}\big(Q_n, E_n^- \widetilde{\oplus} \mathbb{R}(e_1, \widetilde{e}_1)\big) : \ \gamma(z) = z, \text{for all } z \in \partial Q_n\},$$

where $Q_n = Q \cap E_n$ and $Q$ as in Lemma 3.11. Set

$$c_n = \inf_{\gamma\in\Gamma_n} \max_{z\in Q_n} J(\gamma(z)). \tag{3.17}$$

Using Lemma 5.5 in Figueiredo, Ó and Ruf (2005), we have

$$\gamma(Q_n) \cap (\partial B_\rho \cap E_n^+) \neq \emptyset, \quad \text{for all} \quad \gamma \in \Gamma_n, \tag{3.18}$$

for $\rho > 0$ given by Lemma 3.9. Thus, combining Lemma 3.9 and (3.18), we have

$$c_n \geq \sigma, \quad \text{for all} \quad n \geq 1. \tag{3.19}$$

Since the identity map $I_n : Q_n \to E_n^- \widetilde{\oplus} \mathbb{R}(e_1, \widetilde{e}_1)$ belongs to $\Gamma_n$, for $z = r(e_1, \widetilde{e}_1) + (u, -\widetilde{u})$, we obtain

$$J(z) = r^2 \|\nabla e_1\|_{2,q}^2 - \|\nabla u\|_{2,q}^2 - \int_\Omega F(re_1 + u) \, dx - \int_\Omega G(\widetilde{re_1 - u}) \, dx \leq R_1^2. \tag{3.20}$$

Let $J_n$ be the restriction of $J$ to the finite-dimensional space $E_n$.

**Proposition 3.13.** For each $n \in \mathbb{N}$, the functional $J_n$ has a critical point at level $c_n$. More precisely, there is $(u_n, \widetilde{v}_n) \in E_n$ such that

$$J_n((u_n, \widetilde{v}_n)) = c_n \in [\sigma, R_1^2] \tag{3.21}$$

and

$$J_n'((u_n, \widetilde{v}_n))(\phi, \widetilde{\psi}) = 0, \quad \text{for all} \quad (\phi, \widetilde{\psi}) \in E_n.$$

***Proof.*** Fix $n \in \mathbb{N}$ fixed. We observe that $J_n$ also satisfies Lemmas 3.9 and 3.11. Thus, by Remark 3.7, we obtain a sequence $(u_j, \widetilde{v}_j) \subset E_n$ such that

$$J_n(u_j, \widetilde{v}_j) \to c_n \quad \text{and} \quad J_n'(u_j, \widetilde{v}_j) \to 0, \quad \text{as} \quad j \to +\infty.$$

By Lemma 3.11, $(u_j, \widetilde{v}_j)$ is bounded in $E_n$. Then, using the fact that $E_n$ is finite dimensional, we can assume that there exists $(u_n, \widetilde{v}_n) \in E_n$ such that $(u_j, \widetilde{v}_j) \to (u_n, \widetilde{v}_n)$, as $j \to +\infty$. Moreover, since $J \in \mathscr{C}^1(E, \mathbb{R})$, we obtain

$$J_n(u_n, \widetilde{v}_n) = c_n \quad \text{and} \quad J_n'(u_n, \widetilde{v}_n) = 0.$$

Finally, combining (3.19) and (3.20), yields $c_n \in [\sigma, R_1^2]$. ■

**Lemma 3.14.** Let $s > 1$ and $\{u_n \in W_0^1 L^{2,s}(\Omega) : \|\nabla u_n\|_{2,s} = 1\}$ be a sequence converging weakly to the zero function in $W_0^1 L^{2,s}(\Omega)$. Then, for every $0 < \alpha < \alpha_s^*$, we can find a subsequence (not renamed) such that

$$\lim_{n\to\infty} \int_\Omega \big(e^{\alpha |u_n|^{\frac{s}{s-1}}} - 1\big)\, dx = 0.$$

***Proof.*** Let $\varepsilon > 0$ be such that $\alpha + \varepsilon < \alpha_s^*$. Since

$$\lim_{|t|\to 0} \frac{e^{\alpha |t|^{\frac{s}{s-1}}} - 1}{|t|} = 0 \quad \text{and} \quad \lim_{|t|\to\infty} \frac{e^{\alpha |t|^{\frac{s}{s-1}}} - 1}{|t|\big(e^{(\alpha+\varepsilon)|t|^{\frac{s}{s-1}}} - 1\big)} = 0,$$

there exists $C > 0$ such that

$$e^{\alpha |t|^{\frac{s}{s-1}}} - 1 \le C|t| + C|t|(e^{(\alpha+\varepsilon)|t|^{\frac{s}{s-1}}} - 1), \quad \text{for all} \quad t \in \mathbb{R}.$$

Taking $r > 1$ such that $r(\alpha + \varepsilon) < \alpha_s^*$ and using Hölder's inequality, we have

$$\int_\Omega \big(e^{\alpha |u_n|^{\frac{s}{s-1}}} - 1\big)\, dx \le C\|u_n\|_1 + C\|u_n\|_{r'} \Big(\int_\Omega \big(e^{r(\alpha+\varepsilon)|u_n|^{\frac{s}{s-1}}} - 1\big)\, dx\Big)^{1/r}.$$

Finally, using Theorem 2.41, the compact embeddings of $W_0^1 L^{2,s}(\Omega)$ in $L^{r'}(\Omega)$ and $L^1(\Omega)$ and the fact that $u_n \rightharpoonup 0$ in $W_0^1 L^{2,s}(\Omega)$, we get a subsequence (not renamed) such that

$$\lim_{n\to\infty} \int_\Omega \big(e^{\alpha |u_n|^{\frac{s}{s-1}}} - 1\big)\, dx = 0.$$

■

**Lemma 3.15.** Let $(u_n,\widetilde{v}_n)$ be the sequence given by Proposition 3.13 and assume that $(u_n,\widetilde{v}_n) \rightharpoonup (0,0)$ in $E$. Then, there exists a subsequence (not renamed) $(u_n,\widetilde{v}_n)$ such that

$$\|\nabla v_n\|_{2,q} \leq \frac{2m_0\pi^{1/2}}{\beta_0^{1/q}} \quad \text{or} \quad \|\nabla \widetilde{u}_n\|_{2,p} \leq \frac{2m_0\pi^{1/2}}{\alpha_0^{1/p}}, \quad \text{for all} \quad n \in \mathbb{N},$$

for $m_0 \in (0,1)$ given by (3.15).

***Proof.*** If $\|\nabla v_n\|_{2,q} \to 0$ or $\|\nabla \widetilde{u}_n\|_{2,q} \to 0$, the claim follows. Thus, we can assume that there exists $b > 0$ such that

$$\|\nabla v_n\|_{2,q} \geq b \quad \text{and} \quad \|\nabla \widetilde{u}_n\|_{2,q} \geq b, \quad \text{for all} \quad n. \in \mathbb{N} \tag{3.22}$$

Since $(u_n,\widetilde{v}_n)$ is given by Proposition 3.13, we have

$$J(u_n,\widetilde{v}_n) \in [\sigma, R_1^2], \quad \text{for all} \quad n \in \mathbb{N} \tag{3.23}$$

and

$$J'(u_n,\widetilde{v}_n)(\phi,\widetilde{\psi}) = 0, \quad \text{for all} \quad (\phi,\widetilde{\psi}) \in E_n. \tag{3.24}$$

Taking $(\phi,\widetilde{\psi}) = (u_n,\widetilde{v}_n)$ in (3.24), we have

$$\int_\Omega f(u_n)u_n\,dx + \int_\Omega g(\widetilde{v}_n)\widetilde{v}_n\,dx = 2\int_\Omega \nabla u_n \nabla \widetilde{v}_n\,dx.$$

Since

$$\int_\Omega \nabla u_n \nabla \widetilde{v}_n\,dx = J(u_n,\widetilde{v}_n) + \int_\Omega F(u_n)\,dx + \int_\Omega G(\widetilde{v}_n)\,dx,$$

we get

$$\int_\Omega f(u_n)u_n\,dx + \int_\Omega g(\widetilde{v}_n)\widetilde{v}_n\,dx = 2J(u_n,\widetilde{v}_n) + 2\int_\Omega F(u_n)\,dx + 2\int_\Omega G(\widetilde{v}_n)\,dx. \tag{3.25}$$

From $(A_2)$, we get

$$\begin{aligned} \int_\Omega F(u_n)\,dx &= \int_{\{x\in\Omega:|u_n(x)|\leq s_0\}} F(u_n)\,dx + \int_{\{x\in\Omega:|u_n(x)|>s_0\}} F(u_n)\,dx \\ &\leq \int_{\{x\in\Omega:|u_n(x)|\leq s_0\}} F(u_n)\,dx + \frac{1}{\mu}\int_{\{x\in\Omega:|u_n(x)|>s_0\}} f(u_n)u_n\,dx \\ &= \int_{\{x\in\Omega:|u_n(x)|\leq s_0\}} \left(F(u_n) - \frac{1}{\mu}f(u_n)u_n\right)dx + \frac{1}{\mu}\int_\Omega f(u_n)u_n\,dx. \end{aligned} \tag{3.26}$$

Similarly, we obtain

$$\int_\Omega G(\widetilde{v}_n)\,dx \leq \int_{\{x\in\Omega:|\widetilde{v}_n(x)|\leq s_0\}} \left(G(\widetilde{v}_n) - \frac{1}{\nu}g(\widetilde{v}_n)\widetilde{v}_n\right)dx + \frac{1}{\nu}\int_\Omega g(\widetilde{v}_n)\widetilde{v}_n\,dx. \tag{3.27}$$

Thus, from (3.23) and replacing (3.26) and (3.27) in (3.25), we obtain

$$\begin{aligned} \left(1-\frac{2}{\mu}\right)\int_\Omega f(u_n)u_n\,dx + \left(1-\frac{2}{\nu}\right)\int_\Omega g(\widetilde{v}_n)\widetilde{v}_n\,dx& \\ \leq 2R_1^2 + 2\int_{\{x\in\Omega:|u_n(x)|\leq s_0\}} \left(F(u_n) - \frac{1}{\mu}f(u_n)u_n\right)dx& \\ + 2\int_{\{x\in\Omega:|\widetilde{v}_n(x)|\leq s_0\}} \left(G(\widetilde{v}_n) - \frac{1}{\nu}g(\widetilde{v}_n)\widetilde{v}_n\right)dx.& \end{aligned} \tag{3.28}$$

Since $(u_n,\widetilde{v}_n) \rightharpoonup (0,0) \in E$, we can assume that

$$u_n \to 0, \quad \widetilde{v}_n \to 0 \text{ in } L^r(\Omega), \quad \text{for all} \quad r \geq 1. \tag{3.29}$$

and

$$u_n \to 0, \quad \widetilde{v}_n \to 0, \text{ almost everywhere in } \Omega. \tag{3.30}$$

Note that

$$\left| F(u_n) - \frac{1}{\mu} f(u_n)u_n \right| \leq M, \quad \text{for all} \quad \{x \in \Omega, |u_n(x)| \leq s_0\},$$

where $M = \max_{s\in[0,s_0]} \left(|F(s)| + \frac{1}{\mu}|f(s)s|\right)$. Moreover, by (3.30) and the fact that $f$ and $F$ are continuous, we have

$$F(u_n) - \frac{1}{\mu} f(u_n)u_n \to 0, \quad \text{almost everywhere in} \quad \Omega.$$

Thus, by Dominated convergence theorem, we obtain

$$\int_{\{x\in\Omega:|u_n(x)|\leq s_0\}} \left( F(u_n) - \frac{1}{\mu} f(u_n)u_n \right) dx = o_n(1). \tag{3.31}$$

Similarly, we have

$$\int_{\{x\in\Omega:|\widetilde{v}_n(x)|\leq s_0\}} \left( G(\widetilde{v}_n) - \frac{1}{\nu} g(\widetilde{v}_n)\widetilde{v}_n \right) dx = o_n(1). \tag{3.32}$$

Replacing (3.31) and (3.32) in (3.28), we obtain

$$\left(1 - \frac{2}{\mu}\right) \int_\Omega f(u_n)u_n\, dx + \left(1 - \frac{2}{\nu}\right) \int_\Omega g(\widetilde{v}_n)\widetilde{v}_n\, dx \leq 2R_1^2 + o_n(1).$$

Consequently,

$$\int_\Omega f(u_n)u_n\, dx \leq \frac{2\mu}{\mu - 2} R_1^2 + o_n(1) \tag{3.33}$$

and

$$\int_\Omega g(\tilde{v}_n)\tilde{v}_n\, dx \leq \frac{2\nu}{\nu - 2} R_1^2 + o_n(1). \tag{3.34}$$

Taking $(\phi, \widetilde{\psi}) = (v_n, 0)$ and $(\phi, \widetilde{\psi}) = (0, \widetilde{u}_n)$ in (3.24), we have

$$\|\nabla v_n\|_{2,q}^2 = \int_\Omega \nabla v_n \nabla \widetilde{v}_n\, dx = \int_\Omega f(u_n)v_n\, dx$$

and

$$\|\nabla \widetilde{u}_n\|_{2,p}^2 = \int_\Omega \nabla u_n \nabla \widetilde{u}_n\, dx = \int_\Omega g(\widetilde{v}_n)\widetilde{u}_n\, dx$$

Using (3.22), we can define

$$V_n = \frac{v_n}{\|\nabla v_n\|_{2,q}} \quad \text{and} \quad \widetilde{U}_n = \frac{\widetilde{u}_n}{\|\nabla \widetilde{u}_n\|_{2,p}}.$$

Thus,

$$\|\nabla v_n\|_{2,q} = \int_\Omega f(u_n)V_n\, dx \tag{3.35}$$

and

$$\|\nabla \widetilde{u}_n\|_{2,p} = \int_\Omega g(\widetilde{v}_n)\widetilde{U}_n\,dx. \tag{3.36}$$

For $\varepsilon > 0$ given by (3.15), we set

$$\xi = \min\left\{\frac{\varepsilon\alpha_0(4\pi)^{\frac{p}{2}}}{\alpha_0 + (4\pi)^{\frac{p}{2}} + \varepsilon\alpha_0}, \frac{\varepsilon\beta_0(4\pi)^{\frac{q}{2}}}{\beta_0 + (4\pi)^{\frac{q}{2}} + \varepsilon\beta_0}\right\}. \tag{3.37}$$

Let $\alpha_1 = \alpha_0 + \xi$. By assumption $(A_3)$ there exists $\lambda > 0$ such that

$$|f(s)| \le \lambda e^{\alpha_1|s|^p}, \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.38}$$

Set $\alpha_2 = (4\pi)^{p/2} - \xi$, using (3.35), we can write

$$\|\nabla v_n\|_{2,q} \le \frac{\lambda}{\alpha_2^{1/p}} \int_\Omega \frac{|f(u_n(x))|}{\lambda} \alpha_2^{1/p}|V_n(x)|\,dx.$$

Now, applying Lemma 3.10 with $s = |f(u_n(x))|/\lambda$, $t = \alpha_2^{1/p}|V_n(x)|$, $r = p$ and $r' = q$, we have

$$\begin{aligned}\|\nabla v_n\|_{2,q} \le \frac{\lambda}{\alpha_2^{1/p}} \Bigg[ &\int_\Omega (e^{\alpha_2|V_n|^p} - 1)\,dx + \frac{1}{q}\int_{\{x\in\Omega:\frac{|f(u_n(x))|}{\lambda}\le e^{1/p^q}\}} \frac{|f(u_n)|^q}{\lambda^q}\,dx \\ &+ \int_{\{x\in\Omega:\frac{|f(u_n(x))|}{\lambda}\ge e^{1/p^q}\}} \frac{|f(u_n)|}{\lambda}\left(\ln\frac{|f(u_n)|}{\lambda}\right)^{1/p} dx\Bigg].\end{aligned} \tag{3.39}$$

By (3.38), we obtain

$$\int_{\{x\in\Omega:\frac{|f(u_n(x))|}{\lambda}\ge e^{1/p^q}\}} \frac{|f(u_n)|}{\lambda}\left(\ln\frac{|f(u_n)|}{\lambda}\right)^{1/p} dx \le \frac{\alpha_1^{1/p}}{\lambda}\int_\Omega f(u_n)u_n\,dx. \tag{3.40}$$

From (3.30) and the continuity of $f$, by Dominated convergence theorem, we have

$$\int_{\{x\in\Omega:\frac{|f(u_n(x))|}{\lambda}\le e^{1/p^q}\}} \frac{|f(u_n)|^q}{\lambda^q}\,dx = o_n(1). \tag{3.41}$$

Taking $w$ in the dual space of $W_0^1L^{2,q}(\Omega)$, from (3.22) and the fact that $v_n \rightharpoonup 0$ in $W_0^1L^{2,q}(\Omega)$, we obtain

$$|\langle V_n, w\rangle| = \left|\left\langle \frac{v_n}{\|\nabla v_n\|_{2,q}}, w\right\rangle\right| \le \frac{1}{b}|\langle v_n, w\rangle| \to 0.$$

Thus, $V_n \rightharpoonup 0$ in $W_0^1L^{2,q}(\Omega)$. From Lemma 3.14, we have

$$\int_\Omega \left(e^{\alpha_2|V_n|^p} - 1\right) dx = o_n(1). \tag{3.42}$$

Replacing (3.40), (3.41) and (3.42) in (3.39), we obtain

$$\|\nabla v_n\|_{2,q} \le \left(\frac{\alpha_0 - \xi}{(4\pi)^{p/2} - \xi}\right)^{1/p} \int_\Omega f(u_n)u_n\,dx + o_n(1).$$

Using (3.33), we obtain

$$\|\nabla v_n\|_{2,q} \le \left(\frac{\alpha_0 - \xi}{(4\pi)^{p/2} - \xi}\right)^{1/p} \frac{2\mu}{\mu - 2}R_1^2 + o_n(1). \tag{3.43}$$

Similarly, we have

$$\|\nabla\widetilde{u}_n\|_{2,p} \leq \Big(\frac{\beta_0-\xi}{(4\pi)^{q/2}-\xi}\Big)^{1/q}\frac{2\nu}{\nu-2}R_1^2+o_n(1). \tag{3.44}$$

Since,

$$R_1^2=\frac{2m_0\pi}{(1+\varepsilon)\alpha_0^{1/p}\beta_0^{1/q}}\min\Big\{\frac{\mu-2}{\mu},\frac{\nu-2}{\nu}\Big\}, \tag{3.45}$$

we can suppose that

$$R_1^2=\frac{2m_0\pi}{(1+\varepsilon)\alpha_0^{1/p}\beta_0^{1/q}}\frac{\mu-2}{\mu}.$$

Replacing in (3.43) , we get

$$\|\nabla v_n\|_{2,q} \leq \frac{2m_0\pi^{1/2}}{(1+\varepsilon)\beta_0^{1/q}}\Big(\frac{\alpha_0-\xi}{\alpha_0}\Big)^{1/p}\Big(\frac{(4\pi)^{p/2}}{(4\pi)^{p/2}-\xi}\Big)^{1/p}+o_n(1).$$

Using the definition of $\xi$ in (3.37), we have

$$0<\frac{\alpha_0+\xi}{\alpha_0}\frac{(4\pi)^{p/2}}{(4\pi)^{\frac{p}{2}}-\xi}\leq 1+\varepsilon,$$

which implies that

$$\|\nabla v_n\|_{2,q} \leq \frac{2m_0\pi^{1/2}}{\beta_0^{1/q}}\frac{1}{(1+\varepsilon)^{1/q}}+o_n(1).$$

Thus, we can assume without loss of generality that

$$\|\nabla v_n\|_{2,q} \leq \frac{2m_0\pi^{1/2}}{\beta_0^{1/q}}, \quad \text{for all} \quad n\in\mathbb{N}.$$

On the other hand, if in (3.45) we have

$$R_1^2=\frac{2m_0\pi}{(1+\varepsilon)\alpha_0^{1/p}\beta_0^{1/q}}\frac{\nu-2}{\nu}.$$

and replacing in (3.44), we can obtain similarly

$$\|\nabla\widetilde{u}_n\|_{2,p} \leq \frac{2m_0\pi^{1/2}}{\alpha_0^{1/p}}, \quad \text{for all} \quad n\in\mathbb{N},$$

and the proof is complete. ∎

**Lemma 3.16.** (See Figueiredo, Miyagaki and Ruf (1995).) Let $\Omega$ be a bounded subset in $\mathbb{R}^N$, $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ be a continuous function and $(u_n)$ be a sequence of functions in $L^1(\Omega)$ converging to $u$ in $L^1(\Omega)$. Assume that $f(x,u(x))$ and $f(x,u_n(x))$ are also $L^1(\Omega)$ functions. If

$$\int_\Omega |f(x,u_n)u_n|\,dx \leq C.$$

Then, $f(x,u_n)$ converges in $L^1(\Omega)$ to $f(x,u)$.

**Lemma 3.17.** Let $(u_n, \widetilde{v}_n)$ be the sequence given by Proposition 3.13 converging weakly to $(u, \widetilde{v})$ in $E$. Then,

$$f(u_n) \to f(u) \quad \text{and} \quad g(\widetilde{v}_n) \to g(\widetilde{v}) \quad \text{in} \quad L^1(\Omega).$$

***Proof.*** By Lemma 2.38, we can assume that there exists a subsequence (not renamed) $(u_n) \subset W_0^1 L^{2,q}(\Omega) \subset L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$. From $(A_3)$, there exists $C_0 > 0$ such that

$$|f(s)| \leq C_0 e^{(\alpha_0+1)|s|^p}, \quad \text{for all} \quad s \in \mathbb{R}.$$

By Remark 2.40, the sequence $(f(u_n))$ and $f(u)$ are in $L^1(\Omega)$. Moreover, by $(A_2)$, we have

$$\begin{aligned} \int_\Omega |f(u_n)u_n|\,dx &= \int_{\{x\in\Omega:|u_n(x)|\leq s_0\}} |f(u_n)u_n|\,dx + \int_{\{x\in\Omega:|u_n(x)|>s_0\}} f(u_n)u_n\,dx \\ &= \int_{\{x\in\Omega:|u_n(x)|\leq s_0\}} \Big(|f(u_n)u_n| - f(u_n)u_n\Big)\,dx + \int_\Omega f(u_n)u_n\,dx. \end{aligned} \tag{3.46}$$

Note that

$$\int_{\{x\in\Omega:|u_n(x)|\leq s_0\}} \Big(|f(u_n)u_n| - f(u_n)u_n\Big)\,dx \leq 2|\Omega| \sup_{s\in[0,s_0]} |f(s)s|.$$

Joining last inequality and Lemma 3.12 in (3.46), we conclude

$$\int_\Omega |f(u_n)u_n|\,dx \leq C, \quad \text{for all} \quad n \geq 1,$$

for some $C > 0$. Consequently, by Lemma 3.16, $f(u_n) \to f(u)$ in $L^1(\Omega)$. Similar arguments apply for the function $g$. ■

## 3.5 Proof of Theorem 3.4

In this section we prove the existence of a nontrivial solution for (3.4).

***Proof.*** By Proposition 3.13, there exists a sequence $(u_n, \widetilde{v}_n) \in E$ such that

$$\lim_{n\to\infty} J(u_n, \widetilde{v}_n) = c \in [\sigma, R_1^2] \tag{3.47}$$

and

$$J_n'(u_n, \widetilde{v}_n)(\phi, \widetilde{\psi}) = 0, \quad \text{for all} \quad (\phi, \widetilde{\psi}) \in E_n.$$

This means,

$$\int_\Omega \nabla u_n \nabla \widetilde{\psi}\,dx = \int_\Omega g(\widetilde{v}_n)\widetilde{\psi}\,dx, \quad \text{for all} \quad (\phi, \widetilde{\psi}) \in E_n \tag{3.48}$$

and

$$\int_\Omega \nabla \widetilde{v}_n \nabla \phi\,dx = \int_\Omega f(u_n)\phi\,dx, \quad \text{for all} \quad (\phi, \widetilde{\psi}) \in E_n. \tag{3.49}$$

From Lemma 3.12, the sequence $(u_n, \widetilde{v}_n)$ is bounded. Thus, without loss of generality, we can assume that there exists $(u, \widetilde{v}) \in E$ such that $(u_n, \widetilde{v}_n) \rightharpoonup (u, \widetilde{v})$ in $E$. Moreover, we can assume

$$u_n \to u, \quad \widetilde{v}_n \to \widetilde{v} \quad \text{in} \quad L^r(\Omega), \quad \text{for all} \quad r \geq 1 \tag{3.50}$$

and

$$u_n \to u, \quad \widetilde{v}_n \to \widetilde{v}, \quad \text{almost everywhere in} \quad \Omega. \tag{3.51}$$

Furthermore, from Lemma 3.17, we have

$$\int_\Omega f(u_n)\,dx \to \int_\Omega f(u)\,dx \quad \text{and} \quad \int_\Omega g(\widetilde{v}_n)\,dx \to \int_\Omega g(\widetilde{v})\,dx.$$

Thus, taking limits as $n \to +\infty$ in (3.48) and (3.49), we get

$$\begin{cases} \displaystyle\int_\Omega \nabla u \nabla \widetilde{\psi}\,dx = \int_\Omega g(\widetilde{v})\widetilde{\psi}\,dx, \\ \displaystyle\int_\Omega \nabla \widetilde{v} \nabla \phi\,dx = \int_\Omega f(u)\phi\,dx, \end{cases} \quad \text{for all} \quad (\phi, \widetilde{\psi}) \in \bigcup_{n=1}^{+\infty} E_n = E. \tag{3.52}$$

Thus, $(u, \widetilde{v}) \in E$ is a weak solution of the system (3.1).

Now, we prove that $(u, \widetilde{v})$ is a nontrivial weak solution. Suppose $u \equiv 0$. By (3.52), we obtain $\widetilde{v} \equiv 0$. By Lemma 3.15, we can assume that

$$\|\nabla v_n\|_{2,q} \le \frac{2m_0\pi^{1/2}}{\beta_0^{1/q}}, \quad \text{for all} \quad n \in \mathbb{N}.$$

Let $r_1, r_2 > 1$ such that

$$r_1 r_2 m_0^q (4\pi)^{q/2} < (4\pi)^{q/2}. \tag{3.53}$$

By $(A_4)$, we have

$$\lim_{|s|\to\infty} \frac{g(s)}{e^{r_1\beta_0|s|^q}} = 0.$$

From this and $(A_1)$, imply that there exists $C > 0$ such that

$$|g(s)| \le C e^{r_1\beta_0|s|^q}, \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.54}$$

Taking $(0, \widetilde{\psi}) = (0, \widetilde{v}_n)$ in (3.48), we have

$$\int_\Omega \nabla u_n \nabla \widetilde{v}_n\,dx = \int_\Omega g(\widetilde{v}_n)\widetilde{v}_n\,dx.$$

From (3.54) and Hölder's inequality with $r_2 > 1$ given by (3.53), we obtain

$$\begin{aligned} \left| \int_\Omega \nabla u_n \nabla \widetilde{v}_n\,dx \right| &\le C \int_\Omega e^{r_1\beta_0|\widetilde{v}_n|^q} |\widetilde{v}_n|\,dx \\ &\le C \left[ \int_\Omega e^{\beta_0 r_1 r_2 \|\nabla \widetilde{v}_n\|_{2,p}^q \left(\frac{|\widetilde{v}_n|}{\|\nabla \widetilde{v}_n\|_{2,p}}\right)^q}\,dx \right]^{1/r_2} \|\widetilde{v}_n\|_{L^{r_2'}} \\ &\le C \left[ \int_\Omega e^{r_1 r_2 m_0^q (4\pi)^{q/2} \left(\frac{|\widetilde{v}_n|}{\|\nabla \widetilde{v}_n\|_{2,p}}\right)^q}\,dx \right]^{1/r_2} \|\widetilde{v}_n\|_{r_2'} \\ &\le C \left[ \int_\Omega e^{(4\pi)^{q/2} \left(\frac{|\widetilde{v}_n|}{\|\nabla \widetilde{v}_n\|_{2,p}}\right)^q}\,dx \right]^{1/r_2} \|\widetilde{v}_n\|_{r_2'}. \end{aligned}$$

By (3.50), we have $\|\widetilde{v}_n\|_{r_2'} \to 0$. Moreover, using Theorem 2.41 we get

$$\lim_{n\to\infty}\int_\Omega \nabla u_n \nabla \widetilde{v}_n\,dx = \lim_{n\to\infty}\int_\Omega g(\widetilde{v}_n)\widetilde{v}_n\,dx = 0. \tag{3.55}$$

On the other hand, taking $(\phi,0) = (u_n,0)$ in (3.49), we have

$$\int_\Omega \nabla u_n \nabla \widetilde{v}_n = \int_\Omega f(u_n)u_n\,dx.$$

Thus, from (3.55)

$$\lim_{n\to\infty}\int_\Omega f(u_n)u_n\,dx = 0. \tag{3.56}$$

Using $(A_2)$, we have

$$\begin{aligned}\int_\Omega F(u_n)\,dx &= \int_{\{x\in\Omega:|u_n(x)|>s_0\}} F(u_n)\,dx + \int_{\{x\in\Omega:|u_n(x)|\le s_0\}} F(u_n)\,dx\\ &\le \frac{1}{\mu}\int_\Omega f(u_n)u_n\,dx + \int_{\{x\in\Omega:|u_n(x)|\le s_0\}} \Big(F(u_n) - \frac{1}{\mu}f(u_n)u_n\Big)\,dx.\end{aligned}$$

From (3.56) and Dominated convergence theorem in the second integral, we obtain

$$\int_\Omega F(u_n)\,dx \to 0. \tag{3.57}$$

Similarly, we have

$$\int_\Omega G(\widetilde{v}_n)\,dx \to 0. \tag{3.58}$$

Since

$$J(u_n,\widetilde{v}_n)\,dx = \int_\Omega \nabla u_n \nabla \widetilde{v}_n\,dx - \int_\Omega F(u_n)\,dx - \int_\Omega G(\widetilde{v}_n)\,dx.$$

Thus, taking limit and using (3.55), (3.57) and (3.58), we get

$$J(u_n,\tilde{v}_n) \to 0$$

which is a contradiction with (3.47). Consequently, $(u,\widetilde{v})$ is a nontrivial weak solution for the Hamiltonian system (3.1). ■

CHAPTER

4

# SINGULAR HAMILTONIAN SYSTEM WITH CRITICAL EXPONENTIAL GROWTH IN $\mathbb{R}^2$

In this chapter we discuss the existence of nontrivial solutions for the Hamiltonian system

$$\begin{cases} -\Delta u + V(x)u = \dfrac{g(v)}{|x|^a}, & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = \dfrac{f(u)}{|x|^b}, & x \in \mathbb{R}^2, \end{cases} \tag{4.1}$$

where $a, b$ are numbers belong to the interval $[0,2)$ and the functions $f$ and $g$ possess critical exponential growth with $(p,q)$ lying on the exponential critical hyperbola.

## 4.1 Introduction and main results

In order to have properties like embedding theorems, we assume that $V(x)$ is a continuous potential satisfying the following conditions:

$(V_1)$ There exists a positive constant $V_0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^2$.

$(V_2)$ There exist constants $p > 2$ and $q = p/(p-1)$ such that

$$\frac{1}{V^{1/q}} \in L^{2,p}(\mathbb{R}^2) \quad \text{and} \quad \frac{1}{V^{1/p}} \in L^{2,q}(\mathbb{R}^2).$$

Concerning the functions $f$ and $g$ we suppose the following assumptions:

$(A_1)$ $f, g \in \mathscr{C}(\mathbb{R})$ and $f(s) = g(s) = o(s)$, as $s \to 0$.

$(A_2)$ There exist constants $\mu > 2$ and $\nu > 2$ such that

$$0 < \mu F(s) \leq s f(s), \quad 0 < \nu G(s) \leq s g(s), \quad \text{for all} \quad s \neq 0,$$

where $F(s) = \int_0^s f(t)\,dt$ and $G(s) = \int_0^s g(t)\,dt$.

$(A_3)$ There exist positive constants $M$ and $s_0$ such that

$$0 < F(s) \leq M|f(s)| \quad \text{and} \quad 0 < G(s) \leq M|g(s)|, \quad \text{for all} \quad |s| > s_0.$$

$(A_4)$ There exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha|s|^p}} = \begin{cases} +\infty, & \alpha < \alpha_0 \\ 0, & \alpha > \alpha_0, \end{cases}$$

where $p$ is given by $(V_2)$.

$(A_5)$ There exists $\beta_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|g(s)|}{e^{\beta|s|^q}} = \begin{cases} +\infty, & \beta < \beta_0 \\ 0, & \beta > \beta_0, \end{cases}$$

where $q$ is given by $(V_2)$.

$(A_6)$ The following limits holds

$$\lim_{|s|\to\infty} \frac{sf(s)}{e^{\alpha_0|s|^p}} = +\infty \quad \text{and} \quad \lim_{|s|\to\infty} \frac{sg(s)}{e^{\beta_0|s|^q}} = +\infty.$$

$(A_7)$ For $a,b$ given by (4.1), $p,q$ given by $(V_2)$, $\alpha_0$ and $\beta_0$ given by $(A_4)$ and $(A_5)$ respectively, we have

$$\Big(\frac{\alpha_0}{1-b/2}\Big)^{1/p} \neq \Big(\frac{\beta_0}{1-a/2}\Big)^{1/q}.$$

Throughout this chapter we consider the space $E = W^{(q)} \times W^{(p)}$ and we use the tilde-map given by (2.59) defined on $W^{(q)}$.

**Lemma 4.1.** Let $a,b \in [0,2)$, $q > 1$ and $\theta > 2$. Then,

$$\delta_{\theta,a,b} = \inf\Big\{ \int_{\mathbb{R}^2} \Big( \frac{|e_1+\omega|^\theta}{|x|^b} + \frac{|\widetilde{e_1-\omega}|^\theta}{|x|^a} \Big)\, dx : \omega \in W^{(q)} \text{ with } \|\omega\|_{(q)} \leq 3+2\sqrt{3} \Big\}$$

is a positive number, where $e_1$ is the first eigenfunction (normalized in the norm $\|\ \|_{(q)}$) for the Schrödinger operator $-\Delta + V(x)$ in $H^1_V(\mathbb{R}^2) := \{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)u^2\, dx < +\infty\}$.

***Proof.*** Assume by contradiction that $\delta_{\theta,a,b} = 0$. Thus, there exists a sequence $(\omega_n) \subset W^{(q)}$ such that

$$\|\omega_n\|_{(q)} \leq 3+2\sqrt{3} \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^2} \Big( \frac{|e_1+\omega_n|^\theta}{|x|^b} + \frac{|\widetilde{e_1-\omega_n}|^\theta}{|x|^a} \Big)\, dx = 0.$$

Since the sequence $(\omega_n)$ is bounded, there exists $\omega \in W^{(q)}$ such that $\omega_n \rightharpoonup \omega$ in $W^{(q)}$ up to subsequence. Consequently, $e_1+\omega_n \rightharpoonup e_1+\omega$ in $W^{(q)}$. Let $\varsigma_n = |e_1+\omega_n|^\theta$ and $\varsigma = |e_1+\omega|^\theta$.

By Proposition 2.48, $W^{(q)}$ is embedding compactly in $L^r(\mathbb{R}^2)$ for all $r \geq 1$. Thus, we can suppose that $\varsigma_n \to \varsigma$ in $L^r(\mathbb{R}^2)$ for all $r \geq 1$. Using Hölder's inequality with $1/t + 1/t' = 1$ such that $bt' < 2$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{|\varsigma_n - \varsigma|}{|x|^b}\,dx &\leq \int_{\{x\in\mathbb{R}^2:|x|\leq 1\}} \frac{|\varsigma_n - \varsigma|}{|x|^b}\,dx + \int_{\{x\in\mathbb{R}^2:|x|\geq 1\}} |\varsigma_n - \varsigma|\,dx \\ &\leq \Big(\int_{\{x\in\mathbb{R}^2:|x|\leq 1\}} \frac{1}{|x|^{bt'}}\,dx\Big)^{1/t'} \Big(\int_{\{x\in\mathbb{R}^2:|x|\leq 1\}} |\varsigma_n - \varsigma|^t\,dx\Big)^{1/t} + \|\varsigma_n - \varsigma\|_1 \\ &\leq C\|\varsigma_n - \varsigma\|_t + \|\varsigma_n - \varsigma\|_1 \to 0,\end{aligned}$$

for some positive constant $C$. Hence,

$$\int_{\mathbb{R}^2} \frac{|e_1 + \omega_n|^\theta}{|x|^b}\,dx \to \int_{\mathbb{R}^2} \frac{|e_1 + \omega|^\theta}{|x|^b}\,dx.$$

Since the tilde-map is a continuous linear function, we have $\widetilde{e_1 - \omega_n} \rightharpoonup \widetilde{e_1 - \omega}$ in $W^{(p)}$. Hence, up to a subsequence, we obtain

$$\int_{\mathbb{R}^2} \frac{|\widetilde{e_1 - \omega_n}|^\theta}{|x|^a}\,dx \to \int_{\mathbb{R}^2} \frac{|\widetilde{e_1 - \omega}|^\theta}{|x|^a}\,dx,$$

which implies

$$\int_{\mathbb{R}^2} \Big(\frac{|e_1 + \omega|^\theta}{|x|^b} + \frac{|\widetilde{e_1 - \omega}|^\theta}{|x|^a}\Big)\,dx = 0.$$

Thus, $e_1 = -\omega$ and $e_1 = \omega$, which is a contradiction. ■

Now, we describe an additional condition on the functions $f$ and $g$.

$(A_8)$ For $a, b \in [0,2)$ given by (4.1) and $\mu, \nu$ given by $(A_2)$, there exist $\theta > 2$ and a positive constant $C_{\theta,a,b}$ such that

$$F(s) \geq C_{\theta,a,b}|s|^\theta \quad \text{and} \quad G(s) \geq C_{\theta,a,b}|s|^\theta, \quad \text{for all} \quad s \in \mathbb{R},$$

where

$$C_{\theta,a,b} > \frac{56 + 32\sqrt{3}}{\delta_{\theta,a,b}R^{\theta-2}}, \tag{4.2}$$

$\delta_{\theta,a,b}$ is defined as in Lemma 4.1 and $R$ is a positive constant such that

$$R^2 = \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} \max\Big\{\frac{\mu-2}{2\mu}, \frac{\nu-2}{2\nu}\Big\}. \tag{4.3}$$

**Example 4.2.** The function

$$V(x) = (1 + \pi|x|^2)^2, \quad x \in \mathbb{R}^2$$

satisfies conditions $(V_1)$ and $(V_2)$ for $p = 3$ and $q = 3/2$. Moreover, the functions

$$f(s) = 3s^2(e^{s^3} - 1), \quad g(s) = \frac{3}{2}s^{1/2}(e^{s^{3/2}} - 1), \quad s \geq 0,$$

where $f(-s)=-f(s), g(-s)=-g(s)$ for all $s<0$, satisfy the conditions $(A_1)-(A_7)$ for $a=1$, $b=3/2$.

Indeed, since $V(x)\geq 1$ for all $x\in\mathbb{R}$, the condition $(V_1)$ follows. Moreover, we have

$$\frac{1}{V^{2/3}}(x)=\frac{1}{(1+\pi|x|^2)^{4/3}} \quad \text{and} \quad \frac{1}{V^{1/3}}(x)=\frac{1}{(1+\pi|x|^2)^{2/3}}, \quad \text{for all} \quad x\in\mathbb{R}^2.$$

By Example 2.7, we have

$$\left(\frac{1}{V^{2/3}}\right)^*(s)=\frac{1}{(1+s)^{4/3}} \quad \text{and} \quad \left(\frac{1}{V^{1/3}}\right)^*(s)=\frac{1}{(1+s)^{2/3}}, \quad \text{for all} \quad s\geq 0.$$

Thus,

$$\|\frac{1}{V^{2/3}}\|_{2,3}^3=\int_0^{+\infty}\left[s^{1/2}\left(\frac{1}{V^{2/3}}\right)^*(s)\right]^3\frac{ds}{s}=\int_0^{+\infty}\frac{s^{1/2}\,ds}{(1+s)^4}=\frac{\pi}{16}$$

and

$$\|\frac{1}{V^{1/3}}\|_{2,\frac{3}{2}}^{3/2}=\int_0^{+\infty}\left[s^{1/2}\left(\frac{1}{V^{1/3}}\right)^*(s)\right]^{3/2}\frac{ds}{s}=\int_0^{+\infty}\frac{dt}{(1+s)s^{1/4}}=\sqrt{2}\pi.$$

Consequently $1/V^{1/q}\in L^{2,p}(\mathbb{R}^2)$ and $1/V^{1/p}\in L^{2,q}(\mathbb{R}^2)$ with $p=3$ and $q=3/2$. Thus, $(V_2)$ is satisfied.

We observe that

$$F(s)=e^{s^3}-s^3-1, \quad G(s)=e^{s^{3/2}}-s^{3/2}-1, \quad \text{for all} \quad s\geq 0$$

and $F(-s)=F(s)$, $G(-s)=G(s)$, for all $s\geq 0$.

(a) The following limits holds

$$\lim_{s\to 0^+}\frac{f(s)}{s}=0 \quad \text{and} \quad \lim_{s\to 0^+}\frac{g(s)}{s}=0,$$

since $f$ and $g$ are odd functions we have that $f$ and $g$ satisfy the condition $(A_1)$.

(b) Taking $\mu=3$, since $G$ is even we have

$$0<3G(s)=3(e^{s^{3/2}}-s^{3/2}-1), \quad \text{for all} \quad s\neq 0 \tag{4.4}$$

Denote

$$H_0(s):=sg(s)-3G(s)=\frac{3}{2}s^{3/2}(e^{s^{3/2}}-1)-3(e^{s^{3/2}}-s^{3/2}-1), \quad s>0$$

and

$$H(t):=H_0(t^{2/3})=\frac{3}{2}t(e^t-1)-3(e^t-t-1), \quad t>0.$$

Thus, $H'(t)=\dfrac{3}{2}(e^t(t-1)+1)>0$ for all $t>0$ which implies that $H$ is increasing. Consequently, $H_0$ is also increasing in $[0,+\infty)$. Moreover, since $H_0$ is an even function we conclude that

$$3G(s)\leq sg(s), \quad \text{for all} \quad s\neq 0. \tag{4.5}$$

From (4.4) and (4.5), we observe that $g$ satisfies $(A_2)$. Similar arguments apply for the function $f$.

(c) We observe that

$$0 < \frac{G(s)}{g(s)} = \frac{2(e^{s^{3/2}} - s^{3/2} - 1)}{3s^{1/2}(e^{s^{3/2}} - 1)}, \quad \text{for all} \quad s > 0.$$

This and the fact that $G(s)$ is an even function and $g$ is an odd function, we have

$$\lim_{|s|\to+\infty} \frac{G(s)}{|g(s)|} = \lim_{s\to+\infty} \frac{G(s)}{g(s)} = 0.$$

Thus, $g$ satisfies the condition $(A_3)$. Similar arguments apply for the function $f$.

(d) Since $g$ is an odd function, we have

$$\lim_{|s|\to+\infty} \frac{|g(s)|}{e^{\alpha|s|^{3/2}}} = \lim_{s\to+\infty} \frac{g(s)}{e^{\alpha s^{3/2}}} = \lim_{s\to+\infty} \frac{3s^{1/2}(e^{s^{3/2}} - 1)}{2e^{\alpha s^{3/2}}} = \begin{cases} +\infty, & \alpha < 1 \\ 0, & \alpha > 1. \end{cases}$$

Thus, $g$ satisfies condition $(A_5)$ with $\beta_0 = 1$.

(e) Similar to $(d)$, $f$ satisfies condition $(A_4)$ with $\alpha_0 = 1$.

(f) Since

$$\frac{sg(s)}{e^{s^3/2}} = \frac{3s^{3/2}(e^{s^{3/2}} - 1)}{2e^{s^3/2}}, \quad \text{for all} \quad s > 0$$

and $g$ is an odd function, we obtain

$$\lim_{|s|\to+\infty} \frac{sg(s)}{e^{|s|^{3/2}}} = \lim_{s\to+\infty} \frac{sg(s)}{e^{s^{3/2}}} = +\infty.$$

Similar arguments apply for the function $f$. Consequently condition $(A_6)$ follows.

(g) As $\alpha_0 = \beta_0 = 1$, $p = 3$, $q = 3/2$, $a = 1$ and $b = 3/2$, the condition $(A_7)$ holds.

**Remark 4.3.** For the function $f$ given by Example 4.2, we can not guarantee that $f$ satisfies condition $(A_8)$, which would imply

$$F(1) = e - 2 \geq C_{\theta,a,b}, \quad \text{for all} \quad \theta > 2 \quad \text{and} \quad a, b \in [0,2].$$

Thus, condition $(A_8)$ turns out to be necessary.

**Example 4.4.** Let $V$ the function given by Example 4.2 and consider the following continuous functions defined on $\mathbb{R}$

$$f_1(s) = g_1(s) = A|s|s, \quad \text{for all} \quad s \in \mathbb{R},$$

for some constant $A > 0$,

$$f_2(s) = \begin{cases} 3s^2\left(e^{s^3} - 1\right), & 0 \leq s < 1, \\ (e-1)\left[(3s^2 - 1)e^{s^3 - s} + s^2\right], & 1 \leq s, \end{cases}$$

and

$$g_2(s) = \begin{cases} \frac{3}{2}s^{1/2}\left(e^{s^{3/2}} - 1\right), & 0 \le s < 1, \\ (e-1)\left[\left(\frac{3}{2}s^{1/2} - 1\right)e^{s^{3/2}-s} + s^{1/2}\right], & 1 \le s, \end{cases}$$

where $f_2(-s) = -f_2(s)$ and $g_2(-s) = -g_2(s)$, for all $s \ge 0$. Setting $f = f_1 + f_2$ and $g = g_1 + g_2$. Then, similarly as Example 3.3 the functions $f$ and $g$ satisfy conditions $(A_1)$, $(A_2)$ with $\mu = \nu = 3$, and conditions $(A_4) - (A_5)$ with $\alpha_0 = \beta_0 = 1$. Moreover, condition $(A_8)$ is satisfied with $\theta = 3$ and $A$ sufficiently large.

**Remark 4.5.** The function $f$ given by Example 4.4 satisfies

$$0 < \frac{sf(s)}{e^{s^3}} = \frac{As^3 + s(e-1)\left[(3s^2-1)e^{s^3-s} + s^2\right]}{e^{s^3}}, \quad \text{for all} \quad s \ge 1.$$

Thus, using last relation and the fact that $f$ is an odd function we obtain

$$\lim_{|s|\to\infty} \frac{sf(s)}{e^{|s|^3}} = 0.$$

Thus, $f$ fails to satisfy $(A_6)$.

The following theorems contain our main results.

**Theorem 4.6.** Suppose that $V$ satisfies $(V_1) - (V_2)$ and $f$ and $g$ satisfy $(A_1) - (A_7)$. Then, system (4.1) possesses a nontrivial weak solution.

**Theorem 4.7.** Suppose that $V$ satisfies $(V_1) - (V_2)$ and $f$ and $g$ satisfy $(A_1) - (A_2), (A_4) - (A_5)$ and $(A_8)$. Then, system (4.1) possesses a nontrivial weak solution.

## 4.2 Preliminary results

In this section we state some results that it will be used in this chapter. First, we recall the following result obtained by Lu and Tang (2016).

**Proposition 4.8.** Let $1 < s < +\infty$, $0 < \beta < N$. Then, there exists a positive constant $C = C(N,s,\beta)$ such that for any $0 < \alpha \le (1-\beta/N)\alpha^*_{N,s}$ where $\alpha^*_{N,s} = (N\omega_N^{1/N})^{s/(s-1)}$ and for any $u \in W^1L^{2,s}(\mathbb{R}^2)$ the following inequality hold:

$$\sup_{\|\nabla u\|^s_{N,s} + \|u\|^s_{N,s} \le 1} \int_{\mathbb{R}^N} \frac{\Phi(\alpha|u|^{s/(s-1)})}{|x|^\beta}\, dx \le C, \tag{4.6}$$

where

$$\Phi(t) = e^t - \sum_{k=0}^{k_0} \frac{t^k}{k!} \quad \text{and} \quad k_0 = \left[\left|\frac{(s-1)N}{s}\right|\right].$$

The inequality in (4.6) is sharp, in the sense that for any $\alpha > (1-\beta/N)\alpha^*_{N,s}$ the supremum become infinity.

**Remark 4.9.** **(i)** The inequality given by Proposition 4.8 is still valid if we consider the supremum over all functions $u$ in $W^1L^{2,s}(\mathbb{R}^2)$ such that $\|\nabla u\|_{N,s} \leq 1$ and $\|u\|_{N,s} \leq M$ where $M$ is a positive constant. Moreover, in this case $C = C(N,s,\beta,M) > 0$.

**(ii)** As in Proposition 2.48, we have

$$\|\nabla u\|_{2,s}^{s} + V_0\|u\|_{2,s}^{s} \leq \|\nabla u\|_{2,s}^{s} + \|V^{1/s}u\|_{2,s}^{s} = \|u\|_{(s)}^{s}.$$

Thus, $u \in W^{(s)}$, with $\|u\|_{(s)} \leq 1$, implies that $\|\nabla u\|_{2,s} \leq 1$ and $\|u\|_{2,s} \leq V_0^{-1}$. Consequently, by $(i)$, Theorem 4.8 is still valid if we consider the supremum over all functions $u$ in $W^{(s)}$ such that $\|u\|_{(s)} \leq 1$.

**(iii)** A careful look at the proof of Proposition 4.8(Lu and Tang (2016)) shows that

$$\int_{\mathbb{R}^N} \frac{\Phi(\alpha|u|^{s/(s-1)})}{|x|^{\beta}}\,dx < +\infty, \text{ for all } \alpha > 0,\ 0 \leq \beta < N \text{ and } u \in W^1L^{2,s}(\mathbb{R}^2). \tag{4.7}$$

Since we are interested in the case $N = 2$, from now on we denote $\alpha_p^* = \alpha_{2,p}^* = (4\pi)^{q/2}$ and $\alpha_q^* = \alpha_{2,q}^* = (4\pi)^{p/2}$, where $p$ and $q$ are given by $(V_2)$.

### 4.2.1 The concentrating and hole functions

Now, we recall some important definitions and results presented in Cassani and Tarsi (2015), Cassani and Tarsi (2009), where it were provided some special functions which will be useful in the linking geometry in order to prove Theorem 4.6.

We consider the following modified Moser-sequence:

$$M_{k,p}(x) = \begin{cases} \dfrac{(\log k)^{\frac{p-1}{p}}}{\sqrt{4\pi}}(1-\delta_{k,p})^{\frac{p-1}{p}}, & |x|^2 \leq \frac{1}{k} \\[2ex] \dfrac{(1-\delta_{k,p})^{\frac{p-1}{p}}}{(\log k)^{\frac{1}{p}}\sqrt{4\pi}}\log(\dfrac{1}{|x|^2}), & \frac{1}{k} < |x|^2 \leq 1, \end{cases} \tag{4.8}$$

where $\delta_k$ will be fixed later such that $\delta_k \to 0$ as $n \to +\infty$. Then, we have

$$M_{k,p}^*(s) = \begin{cases} \dfrac{(\log k)^{\frac{p-1}{p}}}{\sqrt{4\pi}}(1-\delta_{k,p})^{\frac{p-1}{p}}, & 0 \leq s \leq \frac{\pi}{k} \\[2ex] \dfrac{(1-\delta_{k,p})^{\frac{p-1}{p}}}{(\log k)^{\frac{1}{p}}\sqrt{4\pi}}\log(\dfrac{\pi}{s}), & \frac{\pi}{k} < s \leq \pi, \end{cases}$$

whereas

$$|\nabla M_{k,p}|(x)=\begin{cases}0, & |x|^2\le \frac{1}{k}\\ \\ \dfrac{(1-\delta_{k,p})^{\frac{p-1}{p}}}{(\log k)^{\frac{1}{p}}\sqrt{\pi}|x|}, & \frac{1}{k}<|x|^2\le 1,\end{cases}$$

and

$$|\nabla M^*_{k,p}|(s)=\begin{cases}\dfrac{(1-\delta_{k,p})^{\frac{p-1}{p}}}{(\log k)^{\frac{1}{p}}\sqrt{s+\frac{\pi}{s}}}, & 0\le s<\pi\big(1-\frac{1}{k}\big)\\ \\ 0, & \pi\big(1-\frac{1}{k}\big)\le s\le \pi.\end{cases}$$

**Lemma 4.10.** For the sequence given by (4.8), the followings estimates hold as $k\to+\infty$

$$\|\nabla M_{k,p}\|^p_{2,p}=(1-\delta_{k,p})^{p-1}+O(\frac{1}{\log k})$$

and

$$\|V^{1/p}M_{k,p}\|^p_{2,p}\le (1-\delta_{k,p})^{p-1}\|V\|_{L^\infty(B_1)}O(\frac{1}{\log k}),$$

where $\delta_{k,p}\to 0$ as $k\to+\infty$.

Using Lemma 4.10, we obtain

$$\begin{aligned}\|M_{k,p}\|^p_{(p)}&=\|\nabla M_{k,p}\|^p_{2,p}+\|V^{1/p}M_{k,p}\|^p_{2,p}\\ &=(1-\delta_{k,p})^{p-1}\Big(1+O(\frac{1}{\log k})+\frac{\|V^{1/p}M_{k,p}\|^p_{2,p}}{(1-\delta_{k,p})^{p-1}}\Big).\end{aligned}$$

Thus, we can choose $\delta_{k,p}$, depending on $\|V\|_{L^\infty(B_1)}$, $p$ and $k$ such that

$$\|M_{k,p}\|_{(p)}=1.$$

Note also that

$$|\delta_{k,p}|\le \|V\|_{L^\infty(B_1)}\frac{1}{\log k}\quad \text{as}\quad k\to+\infty.$$

For each $d>0$, define $u_d(x)=u(\frac{x}{d})$. Thus,

$$\|\nabla u_d\|^p_{2,p}=\|\nabla u\|^p_{2,p}\quad\text{and}\quad \|V^{1/p}u_d\|^p_{2,p}=d^p\|V^{1/p}_{1/d}u\|^p_{2,p}. \tag{4.9}$$

For $M_{k,p}$ defined by (4.8), we consider

$$M_{k,p;d}(x):=M_{k,p}(\frac{x}{d}), \tag{4.10}$$

where we denoted by $\delta_{k,p,d}$ instead of $\delta_{k,p}$ to emphasize the dependence on $d$. By (4.9), we have

$$\begin{aligned}\|M_{k,p;d}\|^p_{(p)}&=\|\nabla M_{k,p}\|^p_{2,p}+d^p\|V^{1/p}_{1/d}M_{k,p}\|^p_{2,p}\\ &=(1-\delta_{k,p})^{p-1}\left(1+O(\frac{1}{\log k})+\frac{d^p\|V^{1/p}_{1/d}M_{k,p}\|^p_{2,p}}{(1-\delta_{k,p})^{p-1}}\right).\end{aligned}$$

Similarly, we can choose $\delta_{k,p,d}$ depending on $\|V\|_{L^\infty(B_d)}$, $p,d$ and $k$ such that

$$\|M_{k,p;d}\|_{(p)} = 1.$$

Note also that

$$|\delta_{k,p,d}| \leq C\|V\|_{L^\infty(B_d)} \frac{d^p}{\log k}, \quad \text{as} \quad k \to +\infty. \tag{4.11}$$

**Lemma 4.11.** Let $p,q > 1$ be conjugate exponents, and let $M_{k,q}, M_{k,p}$ be the normalized concentrating sequences defined in (4.8). Then, $M_{k,q} \neq -\widetilde{M}_{k,p}$. Furthermore, as $k \to +\infty$

$$\int_{\mathbb{R}^2} \left(\nabla M_{k,p;d} \nabla M_{k,q;d} + V(x) M_{k,p;d} M_{k,q;d}\right) dx$$
$$\geq \left(1 - \frac{C\|V\|_{L^\infty(B_d)} d^p}{\log k}\right) \left[O\Big(\frac{\log k}{k}\Big) + V_0\Big(1 + \frac{1}{8\log k}\Big)\right],$$

where the constant $C$ and the quantity $O(\dfrac{\log k}{k})$ depend only on $k$.

Now, we consider the hole functions $\zeta_m : \mathbb{R}^2 \to \mathbb{R}$ as follows

$$\zeta_m(x) := \begin{cases} 0, & |x| \leq \frac{1}{m} \\ 2 + 2\dfrac{\log|x|}{\log m}, & \frac{1}{m} < |x| < \frac{1}{\sqrt{m}} \\ 1, & |x| \geq \frac{1}{\sqrt{m}}. \end{cases} \tag{4.12}$$

Then,

$$|\nabla \zeta_m|(x) = \begin{cases} 0, & |x| \leq \frac{1}{m} \quad \text{or} \quad |x| \geq \frac{1}{\sqrt{m}} \\ \dfrac{2}{|x|\log m}, & \frac{1}{m} < |x| < \frac{1}{\sqrt{m}}, \end{cases}$$

$$\mu_{|\nabla\zeta_m|}(s) = \begin{cases} 0, & s \geq \frac{2m}{\log m} \\ \pi\Big(\dfrac{4}{s^2\log^2 m} - \dfrac{1}{m^2}\Big), & \frac{2\sqrt{m}}{\log m} < s < \frac{2m}{\log m} \\ \pi\Big(\dfrac{1}{m} - \dfrac{1}{m^2}\Big), & 0 \leq s \leq \frac{2\sqrt{m}}{\log m}, \end{cases}$$

and

$$|\nabla\zeta_m|^*(t) = \begin{cases} 0, & t \geq \pi\big(\frac{1}{m} - \frac{1}{m^2}\big) \\ \dfrac{2}{\log m} \dfrac{\sqrt{\pi}}{\sqrt{t + \frac{\pi}{m^2}}}, & 0 \leq t < \pi\big(\frac{1}{m} - \frac{1}{m^2}\big). \end{cases}$$

Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of eigenfunctions for the operator $(-\Delta + V)$ in $H^1_V(\mathbb{R}^2) := \{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)u^2\,dx < \infty\}$. By Lemma 3 in Cassani and Tarsi (2015), the sequence $\{e_i\}_{i\in\mathbb{N}}$ provides also a dense system in $W^{(q)}$ as well as $W^{(p)}$. For each $n \in \mathbb{N}$, consider the following finite dimensional subspace:

$$E_n := Span\{e_1, \ldots, e_n\}.$$

We define the set

$$E_{n,m} := \{u_m := \zeta_m u : u \in E_n\}.$$

**Lemma 4.12.** One can choose $m = m(n) \to +\infty$, as $n \to +\infty$, such that the following estimates hold

$$\|u_m - u\|_{(p)} \le \delta(n)\|u\|_{(p)} \quad \text{and} \quad \|u_m - u\|_{(q)} \le \delta(n)\|u\|_{(q)}, \quad u \in E_n$$

where $\delta(n) \to 0$, as $n \to +\infty$.

**Remark 4.13.** Let $M_{k,p;d}, M_{k,q;d}$ as defined in (4.10). By construction, for any $u_m \in E_{n,m}$, we have

$$supp\ u_m \cap supp\ M_{k,p;d} = \varnothing \quad \text{and} \quad supp\ u_m \cap supp\ M_{k,q;d} = \varnothing$$

for any $k > 0$, provided that $d \le 1/m$.

## 4.3 Variational setting

In this section, we describe the functional setting that allows us to treat (4.1) variationally. The natural functional associated to (4.1) is given by $J : E := W^{(q)} \times W^{(p)} \to \mathbb{R}$, where

$$J(u, \widetilde{v}) = \int_{\mathbb{R}^2} \left(\nabla u \nabla \widetilde{v} + V(x) u \widetilde{v}\right) dx - \int_{\mathbb{R}^2} \frac{F(u)}{|x|^b}\, dx - \int_{\mathbb{R}^2} \frac{G(\widetilde{v})}{|x|^a}\, dx. \tag{4.13}$$

**Lemma 4.14.** Let $\alpha > 0$, $p, q > 1$ and $r > 1$. Then, the following inequalities holds:

(i)
$$\left(e^{\alpha|t|^p} - 1\right)^r \le e^{r\alpha|t|^p} - 1, \quad \text{for all} \quad t \in \mathbb{R}.$$

(ii) For each $\beta > \alpha r$ there exists a positive constant $C = C(\beta)$ such that

$$\left(e^{\alpha|t|^q} - \alpha|t|^q - 1\right)^r \le C\left(e^{\beta|t|^q} - \beta|t|^q - 1\right), \quad \text{for all} \quad t \in \mathbb{R}.$$

***Proof.***

(i) Given $r \ge 1$, the function $h(s) = (1+s)^r - s^r - 1$ is increasing in $[0, +\infty)$ and $h(0) = 0$, taking $s = e^{\alpha|t|^p} - 1$ we obtain

$$0 \le e^{r\alpha|t|^p} - (e^{\alpha|t|^p} - 1)^r - 1,$$

which implies $(i)$.

(ii) Since

$$\lim_{|t|\to 0} \frac{(e^{\alpha|t|^q} - \alpha|t|^q - 1)^r}{e^{\beta|t|^q} - \beta|t|^q - 1} = 0$$

and

$$\lim_{|t|\to \infty} \frac{(e^{\alpha|t|^q} - \alpha|t|^q - 1)^r}{e^{\beta|t|^q} - \beta|t|^q - 1} = 0$$

the conclusion follows.

■

**Proposition 4.15.** Assume $(A_1),(A_4)$ and $(A_5)$. Then, $J$ is well defined and belongs to the class $\mathscr{C}^1(E,\mathbb{R})$ with

$$J'(u,\widetilde{v})(\phi,\widetilde{\psi}) = \int_{\mathbb{R}^2} \left(\nabla u\nabla\widetilde{\psi} + V(x)u\widetilde{\psi} + \nabla\widetilde{v}\nabla\phi + V(x)\widetilde{v}\phi\right) dx - \int_{\mathbb{R}^2} \frac{f(u)}{|x|^b}\phi\, dx - \int_{\mathbb{R}^2} \frac{g(\widetilde{v})}{|x|^a}\widetilde{\psi}\, dx,$$

for all $(\phi,\widetilde{\psi}) \in E$.

***Proof.*** Let $u \in W^{(q)}$ and $\widetilde{v} \in W^{(p)}$. By Hölder's inequality in Lorentz spaces and assumption $(V_2)$, we have

$$\left|\int_{\mathbb{R}^2} \nabla u\nabla\widetilde{v}\, dx\right| \le \|\nabla u\|_{2,q}\|\nabla\widetilde{v}\|_{2,p} \le \|u\|_{(q)}\|\widetilde{v}\|_{(p)} \tag{4.14}$$

and

$$\left|\int_{\mathbb{R}^2} V(x)u\widetilde{v}\, dx\right| = \left|\int_{\mathbb{R}^2} V(x)^{1/q}uV(x)^{1/p}\widetilde{v}\, dx\right| \le \|V^{1/q}u\|_{2,q}\|V^{1/p}\widetilde{v}\|_{2,p} \le \|u\|_{(q)}\|\widetilde{v}\|_{(p)}. \tag{4.15}$$

From $(A_1)$ and $(A_4)$, there exists $C > 0$ such that

$$|f(s)| \le |s| + C(e^{(\alpha_0+1)|s|^p} - 1), \quad \text{for all} \quad s \in \mathbb{R}. \tag{4.16}$$

Thus, applying Young's inequality and Lemma 4.14-(i), we obtain

$$|F(u)| \le |u|^2 + C|u|(e^{(\alpha_0+1)|u|^p} - 1) \le C|u|^2 + C(e^{2(\alpha_0+1)|u|^p} - 1).$$

Then,

$$\left|\int_{\mathbb{R}^2} \frac{F(u)}{|x|^b}\, dx\right| \le C\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^b}\, dx + C\int_{\mathbb{R}^2} \frac{(e^{2(\alpha_0+1)|u|^p} - 1)}{|x|^b}\, dx. \tag{4.17}$$

Let $r > 1$ such that $br < 2$. By Hölder's inequality, we find

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^b}\, dx &\le \int_{\{x\in\mathbb{R}^2:|x|\le 1\}} \frac{|u|^2}{|x|^b}\, dx + \int_{\mathbb{R}^2} |u|^2\, dx \\ &\le \left[\int_{\{x\in\mathbb{R}^2:|x|\le 1\}} |u|^{2r'}\, dx\right]^{1/r'} \left[\int_{\{x\in\mathbb{R}^2:|x|\le 1\}} \frac{1}{|x|^{rb}}\, dx\right]^{1/r} + \|u\|_2^2.\end{aligned}$$

From Proposition 2.47, we have

$$\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^b}\, dx < +\infty.$$

Combining this with (4.17) and Remark 4.9-(iii), we obtain

$$\int_{\mathbb{R}^2} \frac{F(u)}{|x|^b}\, dx < +\infty, \quad \text{for all} \quad u \in W^{(q)}. \tag{4.18}$$

Similarly, $G(\widetilde{v})/|x|^a$ belongs to $L^1(\mathbb{R}^2)$ for all $\widetilde{v} \in W^{(p)}$. Thus, from (4.14), (4.15) and (4.18), we conclude that $J$ is well defined in $E$. Consider $J_1, J_2, J_3 : E \to \mathbb{R}$ defined by

$$J_1(u,\widetilde{v}) = \int_{\mathbb{R}^2} \left(\nabla u\nabla\widetilde{v} + V(x)u\widetilde{v}\right) dx,$$

$$J_2(u,\widetilde{v}) = \int_{\mathbb{R}^2} \frac{F(u)}{|x|^b}\,dx \quad \text{and} \quad J_3(u,\widetilde{v}) = \int_{\mathbb{R}^2} \frac{G(\widetilde{v})}{|x|^a}\,dx.$$

By (4.14) and (4.15), we have

$$|J_1(u,\widetilde{v})| \le 2\|u\|_{(q)}\|\widetilde{v}\|_{(p)}, \quad \text{for all} \quad (u,\widetilde{v}) \in E.$$

Thus, $J_1$ is a continuous bilinear function. Then, $J_1 \in \mathscr{C}^\infty(E,\mathbb{R})$ and

$$J_1'(u,\widetilde{v})(\phi,\widetilde{\psi}) = \int_{\mathbb{R}^2} \left(\nabla u \nabla \widetilde{\psi} + V(x)u\widetilde{\psi} + \nabla \widetilde{v} \nabla \phi + V(x)\widetilde{v}\phi\right) dx, \quad \text{for all} \quad (\phi,\widetilde{\psi}) \in E.$$

Now, fix $u$ and $\phi$ in $W^{(q)}$, for given $x \in \mathbb{R}^2$ and consider $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) = \frac{F\left(u(x) + t\phi(x)\right)}{|x|^b}.$$

Let $(t_n)$ any sequence in $\mathbb{R}$ such that $t_n \to 0$, we can assume that $0 < |t_n| \le 1$ for all $n \ge 1$. For any $n \ge 1$, by the Mean value theorem there exists $\theta_n = \theta_n(t_n,x) \in (0,1)$ such that

$$\frac{F(u+t_n\phi) - F(u)}{|x|^b} = h(t_n) - h(0) = h'(\theta_n t_n)t_n = \frac{f(u+\theta_n t_n\phi)t_n\phi}{|x|^b}. \tag{4.19}$$

Define

$$h_n(x) := \frac{F(u+t_n\phi) - F(u)}{t_n|x|^b} = \frac{f(u+\theta_n t_n\phi)\phi}{|x|^b}.$$

Since $f$ is continuous, we have

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \frac{F(u+t_n\phi) - F(u)}{t_n|x|^b} = \frac{f(u)\phi}{|x|^b}, \quad \text{for all} \quad x \in \mathbb{R}^2.$$

Note that $|u+\theta_n t_n\phi| \le |u| + |\phi| := w \in W^{(q)}$. From (4.16), we have

$$\begin{aligned}
|h_n(x)| &= \frac{|f(u+\theta_n t_n\phi)\phi|}{|x|^b} \\
&\le \frac{|u+\theta_n t_n\phi||\phi| + C\left(e^{(\alpha_0+1)|u+\theta_n t_n\phi|^p} - 1\right)|\phi|}{|x|^b} \\
&\le \frac{|w||\phi| + C\left(e^{(\alpha_0+1)|w|^p} - 1\right)|\phi|}{|x|^b} \\
&\le \frac{|w|^2 + |\phi|^2 + C\left(e^{2(\alpha_0+1)|w|^p} - 1\right) + C|\phi|^2}{2|x|^b}.
\end{aligned}$$

Using Hölder's inequality and Remark 4.9 -(iii), we have

$$\frac{|w|^2 + |\phi|^2 + C\left(e^{2(\alpha_0+1)|w|^p} - 1\right) + C|\phi|^2}{2|x|^b} \in L^1(\mathbb{R}^2).$$

By Dominated convergence theorem, we obtain

$$\begin{aligned}
J_2'(u)\phi &= \lim_{n\to\infty} \frac{J_2(u+t_n\phi) - J_2(u)}{t_n} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{F(u+t_n\phi) - F(u)}{t_n|x|^b}\,dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^2} h_n(x)\,dx \\
&= \int_{\mathbb{R}^2} \frac{f(u)\phi}{|x|^b}\,dx.
\end{aligned}$$

Now, we prove the continuity of the Fréchet derivative. Let $(u_n)$ be a sequence in $W^{(q)}$ such that $u_n \to u$ in $W^{(q)}$., and hence $u_n \to u$ in $W^1L^{2,q}(\mathbb{R}^2)$. By Proposition 2.44, there exists a subsequence (not renamed) $(u_n)$ and $\widehat{u} \in W^1L^{2,q}(\mathbb{R}^2)$ such that

$$|u_n(x)| \leq \widehat{u}(x), \quad \text{almost everywhere in} \quad \mathbb{R}^2, \tag{4.20}$$

and

$$u_n(x) \to u(x), \quad \text{almost everywhere in} \quad \mathbb{R}^2. \tag{4.21}$$

For $r > 1$ such that $rb < 2$, from (4.16) and Lemma 4.14-(i), we have

$$|f(s)|^r \leq 2|s|^r + 2C^r\big(e^{r(\alpha_0+1)|s|^p} - 1\big), \quad \text{for all} \quad s \in \mathbb{R}.$$

By (4.20), we get

$$\begin{aligned}
\frac{|f(u_n)-f(u)|^r}{|x|^{rb}} &\leq \frac{2^r|f(u_n)|^r}{|x|^{rb}} + \frac{2^r|f(u)|^r}{|x|^{rb}} \\
&\leq 2^r\Big(\frac{|u_n|^r}{|x|^{rb}} + \frac{|u|^r}{|x|^{rb}}\Big) + 2^rC^r\Big(\frac{e^{r(\alpha_0+1)|u_n|^p}-1}{|x|^{rb}} + \frac{e^{r(\alpha_0+1)|u|^p}-1}{|x|^{rb}}\Big) \\
&\leq 2^r\Big(\frac{|\widehat{u}|^r}{|x|^{rb}} + \frac{|u|^r}{|x|^{rb}}\Big) + 2^rC^r\Big(\frac{e^{r(\alpha_0+1)|\widehat{u}|^p}-1}{|x|^{rb}} + \frac{e^{r(\alpha_0+1)|u|^p}-1}{|x|^{rb}}\Big).
\end{aligned}$$

Using Hölder's inequality and Remark 4.9-(iii), we get

$$2^r\Big(\frac{|\widehat{u}|^r}{|x|^{rb}} + \frac{|u|^r}{|x|^{rb}}\Big) + 2^rC^r\Big(\frac{e^{r(\alpha_0+1)|\widehat{u}|^p}-1}{|x|^{rb}} + \frac{e^{r(\alpha_0+1)|u|^p}-1}{|x|^{rb}}\Big) \in L^1(\mathbb{R}^2).$$

Moreover, from (4.21) and the fact that $f$ is continuous, yields

$$\frac{|f(u_n)-f(u)|^r}{|x|^{rb}} \to 0, \quad \text{almost everywhere in} \quad \mathbb{R}^2.$$

By Dominated convergence theorem, we have

$$\Big\|\frac{f(u_n)-f(u)}{|x|^b}\Big\|_r \to 0. \tag{4.22}$$

Since

$$\begin{aligned}
|\langle J_2'(u_n) - J_2'(u), \phi\rangle| &\leq \int_{\mathbb{R}^2}\Big|\frac{f(u_n)-f(u)}{|x|^b}\Big||\phi|\,dx \\
&\leq \Big\|\frac{f(u_n)-f(u)}{|x|^b}\Big\|_r\|\phi\|_{r'} \\
&\leq C\Big\|\frac{f(u_n)-f(u)}{|x|^b}\Big\|_r\|\phi\|_{(q)},
\end{aligned}$$

by (4.22), we have

$$\sup_{\|\phi\|_{(q)}\leq 1}|\langle J_2'(u_n) - J_2'(u), \phi\rangle| \leq C\Big\|\frac{f(u_n)-f(u)}{|x|^b}\Big\|_r \to 0.$$

Hence, $J_2$ belongs to $\mathscr{C}^1(E,\mathbb{R})$. Similar arguments prove that $J_3$ belongs to $\mathscr{C}^1(E,\mathbb{R})$. Furthermore $J \in \mathscr{C}^1(E,\mathbb{R})$. ■

As a consequence of Proposition 4.15, critical points of $J$ correspond to the weak solutions of (4.1).

**Lemma 4.16.** Let $\alpha$, $a$, $\beta$, $b$ be positive numbers and $r \geq 1$. Then,

(i) Let $u \in W^{(q)}$ such that $\|u\|_{(q)} \leq M$ with $\alpha M^p/\alpha_q^* + b/2 < 1$. Then, there exists a positive constant $C = C(\alpha, b, M, r)$ such that

$$\int_{\mathbb{R}^2} |u|^r \frac{\left(e^{\alpha|u|^p} - 1\right)}{|x|^b}\, dx \leq C\|u\|_{(q)}^r.$$

(ii) Let $\widetilde{v} \in W^{(p)}$ such that $\|\widetilde{v}\|_{(p)} \leq M$ with $\beta M^q/\alpha_p^* + a/2 < 1$. Then, there exists a positive $C = C(\beta, a, M, r)$ such that

$$\int_{\mathbb{R}^2} |\widetilde{v}|^r \frac{\left(e^{\beta|\widetilde{v}|^q} - \beta|\widetilde{v}|^q - 1\right)}{|x|^a}\, dx \leq C\|\widetilde{v}\|_{(p)}^r.$$

***Proof.*** Choose $t > 1$ close to 1 such that $t\alpha M^p/\alpha_q^* + tb/2 < 1$ and $rt' \geq 1$, where $t' = t/(t-1)$. Using Hölder's inequality and Lemma 4.14-(i), we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} |u|^r \frac{\left(e^{\alpha|u|^p} - 1\right)}{|x|^b}\, dx &\leq \Big(\int_{\mathbb{R}^2} \frac{\left(e^{\alpha|u|^p} - 1\right)^t}{|x|^{tb}}\, dx\Big)^{1/t} \|u\|_{rt'}^r \\ &\leq \Big(\int_{\mathbb{R}^2} \frac{e^{t\alpha M^p\left(\frac{|u|}{\|u\|_{(q)}}\right)^p} - 1}{|x|^{tb}}\, dx\Big)^{1/t} \|u\|_{rt'}^r.\end{aligned}$$

By Proposition 4.8, we have

$$\int_{\mathbb{R}^2} |u|^r \frac{\left(e^{\alpha|u|^p} - 1\right)}{|x|^b}\, dx \leq C\|u\|_{rt'}^r.$$

Using the continuous embedding $W^{(q)} \hookrightarrow L^{rt'}(\mathbb{R}^2)$, we conclude the proof of $(i)$. Similar arguments proves $(ii)$. ■

**Lemma 4.17.** Let $s > 1$, $0 < r < 2$ and $\{u_n \in W^{(s)} : \|u_n\|_{(s)} = 1\}$ be a sequence converging weakly to the zero function in $W^{(s)}$. Then, for every $0 < \alpha < (1 - r/2)\alpha_s^*$, we can find a subsequence (not renamed) such that

(i)

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{e^{\alpha|u_n|^{\frac{s}{s-1}}} - 1}{|x|^r}\, dx = 0, \quad \text{if} \quad 1 < s \leq 2.$$

(ii)

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{e^{\alpha|u_n|^{\frac{s}{s-1}}} - \alpha|u_n|^{\frac{s}{s-1}} - 1}{|x|^r}\, dx = 0, \quad \text{if} \quad 2 > s.$$

***Proof.*** We begin proving $(i)$. Let $\varepsilon > 0$ such that $\alpha + \varepsilon < (1 - r/2)\alpha_s^*$. Since,

$$\lim_{|t|\to 0} \frac{e^{\alpha|t|^{\frac{s}{s-1}}} - 1}{|t|} = 0 \quad \text{and} \quad \lim_{|t|\to\infty} \frac{e^{\alpha|t|^{\frac{s}{s-1}}} - 1}{|t|\big(e^{(\alpha+\varepsilon)|t|^{\frac{s}{s-1}}} - 1\big)} = 0,$$

there exists a constant $C > 0$ such that

$$e^{\alpha|t|^{\frac{s}{s-1}}} - 1 \leq C|t| + C|t|(e^{(\alpha+\varepsilon)|t|^{\frac{s}{s-1}}} - 1), \quad \text{for all} \quad t \in \mathbb{R}.$$

Hence,

$$\int_{\mathbb{R}^2} \frac{e^{\alpha|u_n|^{\frac{s}{s-1}}} - 1}{|x|^r}\, dx \leq C \int_{\mathbb{R}^2} \frac{|u_n|}{|x|^r}\, dx + C \int_{\mathbb{R}^2} |u_n| \frac{\big(e^{(\alpha+\varepsilon)|u_n|^{\frac{s}{s-1}}} - 1\big)}{|x|^r}\, dx. \tag{4.23}$$

Taking $t > 1$ such that $t(\alpha+\varepsilon)/\alpha_s^* + tr/2 < 1$, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{|u_n|}{|x|^r}\, dx &\leq \int_{\{x\in\mathbb{R}^2:|x|\leq 1\}} \frac{|u_n|}{|x|^r}\, dx + \|u_n\|_1 \\
&\leq \Big(\int_{\{x\in\mathbb{R}^2:|x|\leq 1\}} \frac{1}{|x|^{tr}}\, dx\Big)^{1/t} \Big(\int_{\{x\in\mathbb{R}^2:|x|\leq 1\}} |u_n|^{t'}\, dx\Big)^{1/t'} + \|u_n\|_1 \\
&\leq C\|u_n\|_{t'} + \|u_n\|_1.
\end{aligned} \tag{4.24}$$

In order to estimate the second integral in (4.23), we use Hölder's inequality and Lemma 4.14 to get

$$\int_{\mathbb{R}^2} |u_n| \frac{\big(e^{(\alpha+\varepsilon)|u_n|^{\frac{s}{s-1}}} - 1\big)}{|x|^r}\, dx \leq \|u_n\|_{t'} \Big(\int_{\mathbb{R}^2} \frac{\big(e^{t(\alpha+\varepsilon)|u_n|^{\frac{s}{s-1}}} - 1\big)}{|x|^{tr}}\, dx\Big)^{1/t}.$$

Since $\|u_n\|_{(s)} = 1$ and $t(\alpha+\varepsilon)/\alpha_s^* + tr/2 < 1$, by Proposition 4.8, there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} |u_n| \frac{\big(e^{(\alpha+\varepsilon)|u_n|^{\frac{s}{s-1}}} - 1\big)}{|x|^r}\, dx \leq C\|u_n\|_{t'}. \tag{4.25}$$

Replacing (4.24) and (4.25) in (4.23), using the compact embeddings of $W^{(s)}$ in $L^{t'}(\mathbb{R}^2)$ and the fact that $u_n \rightharpoonup 0$ in $W^{(s)}$, we get a subsequence (not renamed) such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{\big(e^{\alpha|u_n|^{\frac{s}{s-1}}} - 1\big)}{|x|^r}\, dx = 0.$$

Arguing similarly we prove $(ii)$. ■

Let

$$\lambda_{1,b} := \inf_{u\in W^{(q)}\setminus 0} \frac{\|u\|_{(q)}^2}{\int_{\mathbb{R}^2} u^2/|x|^b\, dx} \quad \text{and} \quad \widetilde{\lambda}_{1,a} := \inf_{\widetilde{u}\in W^{(p)}\setminus 0} \frac{\|\widetilde{u}\|_{(p)}^2}{\int_{\mathbb{R}^2} \widetilde{u}^2/|x|^a\, dx}. \tag{4.26}$$

By Hölder's inequality and continuous embeddings, the numbers $\lambda_{1,b}$ and $\widetilde{\lambda}_{1,a}$ are positive.

### 4.3.1 On Palais-Smale sequences

**Lemma 4.18.** Assume $(A_1)-(A_2),(A_4)-(A_5)$ and let $(u_n,\widetilde{v}_n)$ be a sequence in $E$ such that $|J(u_n,\widetilde{v}_n)|\le d$ and

$$|J'(u_n,\widetilde{v}_n)(\phi,\widetilde{\psi})|\le \varepsilon_n\|(\phi,\widetilde{\psi})\|,\quad \text{for all}\quad \phi,\psi\in\{0,u_n,v_n\}. \tag{4.27}$$

Then, $\|(u_n,\widetilde{v}_n)\|\le c$ for every $n\in\mathbb{N}$ and for some positive constant $c$.

***Proof.*** Taking $(\phi,\widetilde{\psi})=(u_n,\widetilde{v}_n)$ in (4.27), we have

$$\left|2\int_{\mathbb{R}^2}\left(\nabla u_n\nabla\widetilde{v}_n+V(x)u_n\widetilde{v}_n\right)dx-\int_{\mathbb{R}^2}\frac{f(u_n)u_n}{|x|^b}\,dx-\int_{\mathbb{R}^2}\frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\,dx\right|\le\varepsilon_n\|(u_n,\widetilde{v}_n)\|.$$

Thus,

$$\int_{\mathbb{R}^2}\frac{f(u_n)u_n}{|x|^b}\,dx+\int_{\mathbb{R}^2}\frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\,dx\le\left|2\int_{\mathbb{R}^2}\left(\nabla u_n\nabla\widetilde{v}_n+V(x)u_n\widetilde{v}_n\right)dx\right|+\varepsilon_n\|(u_n,\widetilde{v}_n)\|.$$

Since

$$\int_{\mathbb{R}^2}\left(\nabla u_n\nabla\widetilde{v}_n+V(x)u_n\widetilde{v}_n\right)dx=J(u_n,\widetilde{v}_n)+\int_{\mathbb{R}^2}\frac{F(u_n)}{|x|^b}\,dx+\int_{\mathbb{R}^2}\frac{G(\widetilde{v}_n)}{|x|^a}\,dx,$$

we get

$$\int_{\mathbb{R}^2}\frac{f(u_n)u_n}{|x|^b}\,dx+\int_{\mathbb{R}^2}\frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\,dx\le 2d+2\int_{\mathbb{R}^2}\frac{F(u_n)}{|x|^b}\,dx+2\int_{\mathbb{R}^2}\frac{G(\widetilde{v}_n)}{|x|^a}\,dx+\varepsilon_n\|(u_n,\widetilde{v}_n)\|.$$

Using $(A_2)$, we obtain

$$\int_{\mathbb{R}^2}\frac{F(u_n)}{|x|^b}\,dx\le\frac{1}{\mu}\int_{\mathbb{R}^2}\frac{f(u_n)u_n}{|x|^b}\,dx\quad\text{and}\quad\int_{\mathbb{R}^2}\frac{G(\widetilde{v}_n)}{|x|^a}\,dx\le\frac{1}{\nu}\int_{\mathbb{R}^2}\frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\,dx.$$

Thus,

$$\left(1-\frac{2}{\mu}\right)\int_{\mathbb{R}^2}\frac{f(u_n)u_n}{|x|^b}\,dx+\left(1-\frac{2}{\nu}\right)\int_{\mathbb{R}^2}\frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\le 2d+\varepsilon_n\|(u_n,\widetilde{v}_n)\|.$$

Hence, there exists $c>0$ such that

$$\int_{\mathbb{R}^2}\frac{f(u_n)u_n}{|x|^b}\,dx\le c+\varepsilon_n\|(u_n,\widetilde{v}_n)\|\ \text{ and }\ \int_{\mathbb{R}^2}\frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\,dx\le c+\varepsilon_n\|(u_n,\widetilde{v}_n)\|. \tag{4.28}$$

On the other hand, taking $(\phi,\widetilde{\psi})=(v_n,0)$ in (4.27), we get

$$\int_{\mathbb{R}^2}\left(\nabla v_n\nabla\widetilde{v}_n+V(x)v_n\widetilde{v}_n\right)dx\le\int_{\mathbb{R}^2}\frac{f(u_n)v_n}{|x|^b}\,dx+\varepsilon_n\|(v_n,0)\|.$$

This means,

$$\|v_n\|_{(q)}^2\le\int_{\mathbb{R}^2}\frac{f(u_n)v_n}{|x|^b}\,dx+\varepsilon_n\|v_n\|_{(q)}.$$

Defining

$$T_n=\frac{v_n}{\|v_n\|_{(q)}},$$

we can write

$$\|v_n\|_{(q)} \leq \int_{\mathbb{R}^2} \frac{f(u_n)T_n}{|x|^b}\,dx + \varepsilon_n. \tag{4.29}$$

Let $\alpha_1 > \alpha_0$ and $0 < \alpha_2 < (1-b/2)\alpha_q^*$. From $(A_1)$ and $(A_4)$, there exists $\lambda > 0$ such that

$$|f(s)| \leq \lambda e^{\alpha_1 |s|^p}, \quad \text{for all} \quad s \in \mathbb{R}. \tag{4.30}$$

Applying Lemma 3.10 in (4.29) with $s = |f(u_n(x))|/\lambda$, $t = \alpha_2^{1/p}|T_n(x)|$, $r = p$ and $r' = q$, we obtain

$$\begin{aligned}
\|v_n\|_{(q)} &\leq \frac{\lambda}{\alpha_2^{1/p}} \int_{\mathbb{R}^2} \frac{1}{|x|^b} \frac{|f(u_n)|}{\lambda} \alpha_2^{1/p}|T_n|\,dx + \varepsilon_n \\
&\leq \frac{\lambda}{\alpha_2^{1/p}} \left[ \int_{\mathbb{R}^2} \frac{(e^{\alpha_2|T_n|^p}-1)}{|x|^b}\,dx + \frac{1}{q\lambda^q} \int_{\{x\in\mathbb{R}^2 : |\frac{f(u_n)}{\lambda}| \leq e^{1/p^q}\}} \frac{|f(u_n)|^q}{|x|^b}\,dx \right. \\
&\quad \left. + \frac{1}{\lambda} \int_{\{x\in\mathbb{R}^2 : |\frac{f(u_n)}{\lambda}| \geq e^{1/p^q}\}} \frac{|f(u_n)|}{|x|^b} \ln^{1/p} \frac{|f(u_n)|}{\lambda}\,dx \right] + \varepsilon_n.
\end{aligned} \tag{4.31}$$

From (4.30), we have

$$\int_{\{x\in\mathbb{R}^2 : |\frac{f(u_n)}{\lambda}| \geq e^{1/p^q}\}} \frac{|f(u_n)|}{|x|^b} \ln^{1/p} \frac{|f(u_n)|}{\lambda}\,dx \leq \alpha_1^{1/p} \int_{\mathbb{R}^2} \frac{f(u_n)u_n}{|x|^b}\,dx. \tag{4.32}$$

Since $\|T_n\|_{(q)} = 1$ and $\alpha_2 < (1-b/2)\alpha_q^*$, by Proposition 4.8, there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} \frac{e^{\alpha_2|T_n|^p}-1}{|x|^b}\,dx \leq C. \tag{4.33}$$

Now, we estimate the second integral in (4.31). From assumption $(A_1)$, given $\bar{\varepsilon} > 0$ there exists $0 < \delta \leq 1$ such that

$$|f(t)| \leq \bar{\varepsilon}^{1/q}|t|, \quad \text{for all} \quad |t| \leq \delta.$$

Thus,

$$|f(t)|^q \leq \bar{\varepsilon}|t|^q \leq \bar{\varepsilon}|t|, \quad \text{for all} \quad |t| \leq \delta. \tag{4.34}$$

Note also that

$$|f(t)| \leq \lambda e^{\frac{1}{p^q}} \leq \frac{\lambda e^{\frac{1}{p^q}}}{\delta^{\frac{1}{q-1}}}|t|^{\frac{1}{q-1}}, \quad \text{for all} \quad \{|t| \geq \delta : |f(t)| \leq \lambda e^{\frac{1}{p^q}}\}.$$

This means,

$$|f(t)|^q \leq \bar{c}|t||f(t)|, \quad \text{for all} \quad \{|t| \geq \delta : |f(t)| \leq \lambda e^{\frac{1}{p^q}}\}. \tag{4.35}$$

where $\bar{c}(\varepsilon) = (\lambda e^{\frac{1}{p^q}})^{q-1}/\delta$. From (4.34) and (4.35), we get

$$|f(t)|^q \leq \bar{\varepsilon}|t| + \bar{c}|f(t)||t|, \quad \text{for all} \quad \{t \in \mathbb{R} : |f(t)| \leq \lambda e^{\frac{1}{p^q}}\}.$$

By (4.28), there exist $c_1, c_2 > 0$ ($c_2$ is independent of $\bar{\varepsilon}$) such that

$$\begin{aligned}\int_{\{x\in\mathbb{R}^2: \frac{|f(u_n)|}{\lambda}\le e^{1/p^q}\}} \frac{|f(u_n)|^q}{|x|^b}\,dx &\le \int_{\{x\in\mathbb{R}^2: |\frac{f(u_n)}{\lambda}|\le e^{1/p^q}\}} \frac{\bar{\varepsilon}|u_n| + \bar{c} f(u_n)u_n}{|x|^b}\,dx\\ &\le \bar{\varepsilon}\int_{\mathbb{R}^2} \frac{|u_n|}{|x|^b}\,dx + \bar{c}\int_{\mathbb{R}^2}\frac{f(u_n)u_n}{|x|^b}\,dx\\ &\le \bar{\varepsilon}c_2\|u_n\|_{(q)} + \bar{c}\big(c+\varepsilon_n\|(u_n,\widetilde{v}_n)\|\big)\\ &\le \bar{c}c_1 + \bar{\varepsilon}c_2\|u_n\|_{(q)} + \varepsilon_n\|(u_n,\widetilde{v}_n)\|.\end{aligned}$$

Combining this with (4.32) and (4.33) in (4.31), there exist $c_1, c_2 > 0$ ($c_2$ is independent of $\bar{\varepsilon}$) such that

$$\|v_n\|_{(q)} \le c_1 + \bar{\varepsilon}c_2\|u_n\|_{(q)} + c_1\int_{\mathbb{R}^2}\frac{f(u_n)u_n}{|x|^b}\,dx + \varepsilon_n\|(u_n,\widetilde{v}_n)\|. \tag{4.36}$$

On the other hand, taking $(\phi,\widetilde{\psi}) = (0,\widetilde{u}_n)$ in (4.27), we can obtain $d_1, d_2 > 0$ ($d_2$ is independent of $\bar{\varepsilon}$) such that

$$\|\widetilde{u}_n\|_{(q)} \le d_1 + \bar{\varepsilon}d_2\|\widetilde{v}_n\|_{(p)} + d_1\int_{\mathbb{R}^2}\frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\,dx + \varepsilon_n\|(u_n,\widetilde{v}_n)\|. \tag{4.37}$$

Using (4.36), (4.37) and (4.28), there exist $k_1, k_2 > 0$ ($k_2$ is independent of $\varepsilon$) such that

$$\|(u_n,\widetilde{v}_n)\| \le k_1 + \bar{\varepsilon}k_2\|(u_n,\widetilde{v}_n)\| + \varepsilon_n\|(u_n,\widetilde{v}_n)\|.$$

Hence, taking $\bar{\varepsilon}$ sufficiently small we conclude that $(u_n,\widetilde{v}_n)$ is a bounded sequence in $E$. ■

**Remark 4.19.** In the previous Lemma, using the fact that $(u_n,\widetilde{v}_n)$ is bounded in $E$ and replacing in (4.28), there exists $C > 0$ such that

$$\int_{\mathbb{R}^2}\frac{f(u_n)u_n}{|x|^b}\,dx \le C \quad\text{and}\quad \int_{\mathbb{R}^2}\frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\,dx \le C, \quad \text{for all}\quad n\ge 1.$$

In the next result, we repeat the same type of arguments developed in Lemma 4.3 in Souza (2012).

**Lemma 4.20.** Let $(u_n,\widetilde{v}_n)$ be a sequence in $E$ such that $J(u_n,\widetilde{v}_n)\to c$, $J_n'(u_n,\widetilde{v}_n)\to 0$ and $(u_n,\widetilde{v}_n)\rightharpoonup(u,\widetilde{v})$ in $E$. Then, up to a subsequence

$$\frac{f(u_n)}{|x|^b}\to\frac{f(u)}{|x|^b} \quad\text{and}\quad \frac{g(\widetilde{v}_n)}{|x|^a}\to\frac{g(\widetilde{v})}{|x|^a} \quad\text{in}\quad L^1(\mathbb{R}^2).$$

***Proof.*** Note that, $f(u_n)/|x|^b\in L^1(\mathbb{R}^2)$ for all $n\ge 1$ and $f(u)/|x|^b\in L^1(\mathbb{R}^2)$. Thus, for given $\varepsilon>0$ there exists $\delta>0$ such that

$$\int_A\frac{|f(u)|}{|x|^b}\,dx<\varepsilon, \quad\text{if}\quad |A|<\delta, \tag{4.38}$$

for every mensurable $A\subset\mathbb{R}^2$. Consider

$$\Omega_n = \{x\in\mathbb{R}^2 : |u(x)|\ge n\}.$$

Since $u \in L^1(\mathbb{R}^2)$ , we have $|\Omega_n| \to 0$ and there exists $\widehat{M} > 0$ such that

$$|\{x \in \mathbb{R}^2 : |u(x)| > \widehat{M}\}| < \delta. \tag{4.39}$$

Let $M = \max\{\widehat{M}, \dfrac{C}{\varepsilon}\}$, where $C > 0$ is given by Remark 4.19. Then,

$$\Big| \int_{\mathbb{R}^2} \frac{|f(u_n)|}{|x|^b} - \frac{|f(u)|}{|x|^b}\, dx \Big| \le I_{1,n} + I_{2,n} + I_{3,n} \tag{4.40}$$

where

$$I_{1,n} = \int_{\{x\in\mathbb{R}^2 : |u_n(x)| \ge M\}} \frac{|f(u_n)|}{|x|^b}\, dx,$$

$$I_{2,n} = \int_{\{x\in\mathbb{R}^2 : |u(x)| \ge M\}} \frac{|f(u)|}{|x|^b}\, dx$$

and

$$I_{3,n} = \Big| \int_{\{x\in\mathbb{R}^2 : |u_n(x)| \le M\}} \frac{|f(u_n)|}{|x|^b}\, dx - \int_{\{x\in\mathbb{R}^2 : |u(x)| \le M\}} \frac{|f(u)|}{|x|^b}\, dx \Big|.$$

From Remark 4.19, we have

$$I_{1,n} = \int_{\{x\in\mathbb{R}^2 : |u_n(x)| \ge M\}} \frac{|f(u_n)u_n|}{|x|^b |u_n|}\, dx \le \frac{1}{M} \int_{\{x\in\mathbb{R}^2 : |u_n(x)| \ge M\}} \frac{f(u_n)u_n}{|x|^b}\, dx \le \frac{C}{M} \le \varepsilon.$$

Using (4.38) and (4.39), we obtain

$$I_{2,n} = \int_{\{x\in\mathbb{R}^2 : |u(x)| \ge M\}} \frac{|f(u)|}{|x|^b}\, dx \le \varepsilon$$

and

$$\begin{aligned}
I_{3,n} &\le \Big| \int_{\mathbb{R}^2} \Big( \frac{|f(u_n)|}{|x|^b} - \frac{|f(u)|}{|x|^b} \Big) \chi_{\{x\in\mathbb{R}^2 : |u_n(x)| \le M\}}\, dx \Big| \\
&\quad + \Big| \int_{\mathbb{R}^2} \frac{|f(u)|}{|x|^b} \Big( \chi_{\{x\in\mathbb{R}^2 : |u_n(x)| \le M\}} - \chi_{\{x\in\mathbb{R}^2 : |u(x)| \le M\}} \Big)\, dx \Big| \\
&\le \int_{\mathbb{R}^2} h_n\, dx + \int_{\{x\in\mathbb{R}^2 : |u(x)| \ge M\}} \frac{|f(u)|}{|x|^b}\, dx \\
&\le \int_{\mathbb{R}^2} h_n\, dx + \varepsilon,
\end{aligned}$$

where $h_n = (\dfrac{|f(u_n)|}{|x|^b} - \dfrac{|f(u)|}{|x|^b}) \chi_{\{x\in\mathbb{R}^2 : |u_n(x)| \le M\}}$. Then, $h_n \to 0$ almost everywhere in $\mathbb{R}^2$. Using the continuity of $f$ and the fact that $f(s) = o(s)$, as $s \to 0$, there exists $c_2 > 0$ such that $|f(s)| \le c_2 |s|$ if $|s| \le M$. We can assume that $u_n \to u$ almost everywhere in $\mathbb{R}^2$ and $|u_n|/|x|^b \le h_0$ almost everywhere in $\mathbb{R}^2$, for some $h_0 \in L^1(\mathbb{R}^2)$. Thus,

$$|h_n| \le \frac{|f(u_n)|}{|x|^b} \chi_{\{x\in\mathbb{R}^2 : |u_n(x)| \le M\}} + \frac{|f(u)|}{|x|^b} \chi_{\{x\in\mathbb{R}^2 : |u_n(x)| \le M\}} \le c_2 \frac{|u_n|}{|x|^b} + \frac{|f(u)|}{|x|^b} \le c_2 h_0 + \frac{|f(u)|}{|x|^b},$$

almost everywhere in $\mathbb{R}^2$ and since $c_2 h_0 + f(u)/|x|^b \in L^1(\mathbb{R}^2)$, by Dominated convergence theorem, we have

$$\int_{\mathbb{R}^2} h_n\, dx \le \varepsilon, \quad \text{for all} \quad n \quad \text{large enough}.$$

Then, for $n$ sufficiently large

$$I_{3,n} \leq 2\varepsilon.$$

Thus, we get

$$\int_{\mathbb{R}^2} \frac{f(u_n)}{|x|^b}\,dx \to \int_{\mathbb{R}^2} \frac{f(u)}{|x|^b}\,dx.$$

Similarly, we have

$$\int_{\mathbb{R}^2} \frac{g(\widetilde{v}_n)}{|x|^a}\,dx \to \int_{\mathbb{R}^2} \frac{g(\widetilde{v})}{|x|^a}\,dx.$$

■

**Lemma 4.21.** Assume $(A_1)-(A_5)$ and let $(u_n,\widetilde{v}_n)$ be a sequence in $E_n$ such that $J(u_n,\widetilde{v}_n) \to c$, $J_n'(u_n,\widetilde{v}_n) \to 0$ and $(u_n,\widetilde{v}_n) \rightharpoonup (u,\widetilde{v})$ in $E$. Then, up to a subsequence

$$\frac{F(u_n)}{|x|^b} \to \frac{F(u)}{|x|^b} \quad \text{and} \quad \frac{G(\widetilde{v}_n)}{|x|^a} \to \frac{G(\widetilde{v})}{|x|^a} \quad \text{in} \quad L^1(\mathbb{R}^2).$$

***Proof.*** By $(A_1)$ and $(A_4)$, there exists $C_1 > 0$ such that

$$|f(s)| \leq |s| + C_1|s|e^{(\alpha_0+1)|s|^p}, \quad \text{for all} \quad s \in \mathbb{R}.$$

In particular, there exists $C > 0$ such that

$$F(s) \leq \int_0^s |f(t)|\,dt \leq C|s|^2, \quad \text{for all} \quad |s| \leq s_0.$$

Combining this with $(A_3)$, we obtain

$$F(s) \leq C|s|^2 + Mf(s), \quad \text{for all} \quad s \in \mathbb{R}. \tag{4.41}$$

Let $r > 1$ be such that $br < 2$. We can assume up to subsequence that $|u_n|^2 \to |u|^2$ in $L^{r'}(\mathbb{R}^2)$ and in $L^1(\mathbb{R}^2)$. Thus, there exist $g_1 \in L^{r'}(\mathbb{R}^2)$ and $g_2 \in L^1(\mathbb{R}^2)$ such that $|u_n|^2 \leq g_1(x)$ and $|u_n|^2 \leq g_2(x)$ almost everywhere in $\mathbb{R}^2$. Then,

$$\frac{|u_n|^2}{|x|^b} \leq \frac{g_1(x)}{|x|^b}\chi_{B_1}(x) + g_2(x), \quad \text{almost everywhere in } \mathbb{R}^2. \tag{4.42}$$

Using the fact that $1/|x|^b$ in $L^r(B_1)$, by Hölder's inequality, we have

$$\frac{g_1}{|x|^b}\chi_{B_1} + g_2 \in L^1(\mathbb{R}^2). \tag{4.43}$$

By Lemma 4.20, $f(u_n)/|x|^b \to f(u)/|x|^b$ in $L^1(\mathbb{R}^2)$. Thus, there exists $g_3$ in $L^1(\mathbb{R}^2)$ such that $f(u_n)/|x|^b \leq g_3$ almost everywhere in $\mathbb{R}^2$. Combining this with (4.42) and (4.43) in (4.41), we get

$$\frac{F(u_n)}{|x|^b} \leq \frac{C|u_n|^2 + Mf(u_n)}{|x|^b} \leq \frac{Cg_1(x)}{|x|^b}\chi_{B_1}(x) + Cg_2(x) + Mg_3(x), \text{ almost everywhere in } \mathbb{R}^2$$

and

$$C\frac{g_1}{|x|^b}\chi_{B_1}+Cg_2+Mg_3 \quad \text{in} \quad L^1(\mathbb{R}^2).$$

Using the fact that $u_n \to u$ in $L^1(\mathbb{R}^2)$ for a subsequence, $F$ is continuous and Dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^2}\frac{F(u_n)}{|x|^b}\,dx \to \int_{\mathbb{R}^2}\frac{F(u)}{|x|^b}\,dx.$$

Similar arguments apply to function $G$. ■

## 4.4 Theorem 4.6

This section is devoted to prove Theorem 4.6.

### 4.4.1 The geometry of the Linking theorem

Let consider $y(x) = M_{k,q;d}(x)$ and $\widetilde{z}(x) = M_{k,p;d}(x)$. Thus, $\|(y,\widetilde{z})\| = 2$ and by Lemma 4.11, $\widetilde{z} \neq -\widetilde{y}$. Define

$$F_{n,m} = E_{n,m}\times E_{n,m}\oplus\mathbb{R}(y,\widetilde{z}),$$

which is a finite dimensional subspace of $E$. Let

$$E^+_{n,m} := \{(v,\widetilde{v}) : v\in E_{n,m}\} \quad \text{and} \quad E^-_{n,m} := \{(v,-\widetilde{v}) : v\in E_{n,m}\},$$

where $m = m(n)$ as in Lemma 4.12. Consider

$$\partial B_\rho\cap F^+_{n,m}\subset F_{n,m}, \quad \text{where} \quad F^+_{n,m} := E^+_{n,m}\oplus\mathbb{R}(y,\widetilde{z})$$

and

$$Q_{n,m} = \{w+s(y,\widetilde{z}) : w = (\omega,-\widetilde{\omega})\in E^-_{n,m}, \|w\|\le R_0, 0\le s\le R_1\}.$$

**Lemma 4.22.** There exist $\rho,\sigma>0$ such that $J(z)\ge\sigma$, for all $z\in\partial B_\rho\cap F^+_{n,m}$.

***Proof.*** For $\varepsilon>0$ given by $(A_1)$, there exists $\delta>0$ such that

$$|F(s)|\le\varepsilon|s|^2 \quad \text{and} \quad |G(s)|\le\varepsilon|s|^2, \quad \text{for all} \quad |s|<\delta. \tag{4.44}$$

From $(A_1)$, $(A_4)$ and $(A_5)$, there exists $C>0$ such that

$$|F(s)|\le C|s|^4\left(e^{2\alpha_0|s|^p}-1\right) \quad \text{and} \quad |G(s)|\le C|s|^4\left(e^{2\beta_0|s|^q}-2\beta_0|s|^q-1\right), \quad \text{for all} \quad |s|\ge\delta. \tag{4.45}$$

From (4.44) and (4.45), for some constant $C>0$, we obtain

$$|F(s)|\le\varepsilon|s|^2+C|s|^4\left(e^{2\alpha_0|s|^p}-1\right), \quad \text{for all} \quad s\in\mathbb{R} \tag{4.46}$$

and

$$|G(s)| \leq \varepsilon|s|^2 + C|s|^4\left(e^{2\beta_0|s|^q} - 2\beta_0|s|^q - 1\right), \quad \text{for all} \quad s \in \mathbb{R}. \tag{4.47}$$

Let $(u+sy, \widetilde{u}+s\widetilde{z}) \in F^+_{n,m}$ with $\|(u+sy, \widetilde{u}+s\widetilde{z})\| \leq \rho_1$ for $\rho_1 > 0$ sufficiently small such that $2\alpha_0\rho_1^p/\alpha_q^* + b/2 < 1$ and $2\beta_0\rho_1^q/\alpha_p^* + a/2 < 1$. By Lemma 4.16, there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} \frac{F(u+sy)}{|x|^b}\,dx \leq \varepsilon \int_{\mathbb{R}^2} \frac{|u+sy|^2}{|x|^b}\,dx + C\|u+sy\|^4_{(q)} \tag{4.48}$$

and

$$\int_{\mathbb{R}^2} \frac{G(\widetilde{u}+s\widetilde{z})}{|x|^a}\,dx \leq \varepsilon \int_{\mathbb{R}^2} \frac{|\widetilde{u}+s\widetilde{z}|^2}{|x|^a}\,dx + C\|\widetilde{u}+s\widetilde{z}\|^4_{(p)}. \tag{4.49}$$

From (4.48) and (4.26), we have

$$\int_{\mathbb{R}^2} \frac{F(u+sy)}{|x|^b}\,dx \leq \frac{\varepsilon}{\lambda_{1,b}}\|u+sy\|^2_{(q)} + C\|u+sy\|^4_{(q)}$$

Using Remark 4.13, we obtain

$$\int_{\mathbb{R}^2} \frac{F(u+sy)}{|x|^b}\,dx \leq \frac{\varepsilon}{\lambda_{1,b}}\left(\|u\|^2_{(q)} + s^2\|y\|^2_{(q)}\right) + C\left(\|u\|^4_{(q)} + s^4\|y\|^4_{(q)}\right). \tag{4.50}$$

Similarly, we obtain

$$\int_{\mathbb{R}^2} \frac{G(\widetilde{u}+s\widetilde{z})}{|x|^a}\,dx \leq \frac{\varepsilon}{\widetilde{\lambda}_{1,a}}\left(\|\widetilde{u}\|^2_{(p)} + s^2\|\widetilde{z}\|^2_{(p)}\right) + C\left(\|\widetilde{u}\|^4_{(p)} + s^4\|\widetilde{z}\|^4_{(p)}\right). \tag{4.51}$$

Thus,

$$\begin{aligned}
J(u+sy, \widetilde{u}+s\widetilde{z}) &= \int_{\mathbb{R}^2} \Big(\nabla(u+sy)\nabla(\widetilde{u}+s\widetilde{z}) + V(x)(u+sy)(\widetilde{u}+s\widetilde{z})\Big)\,dx \\
&\quad - \int_{\mathbb{R}^2} \frac{F(u+sy)}{|x|^b}\,dx - \int_{\mathbb{R}^2} \frac{G(\widetilde{u}+s\widetilde{z})}{|x|^a}\,dx \\
&\geq s^2 \int_{\mathbb{R}^2} \left(\nabla y \nabla \widetilde{z} + V(x) y\widetilde{z}\right) dx \\
&\quad + \frac{1}{2}\|u\|^2_{(q)} - \frac{\varepsilon}{\lambda_{1,b}}\left(\|u\|^2_{(q)} + s^2\|y\|^2_{(q)}\right) - C\left(\|u\|^4_{(q)} + s^4\|y\|^4_{(q)}\right) \\
&\quad + \frac{1}{2}\|\widetilde{u}\|^2_{(p)} - \frac{\varepsilon}{\widetilde{\lambda}_{1,a}}\left(\|\widetilde{u}\|^2_{(p)} + s^2\|\widetilde{z}\|^2_{(p)}\right) - C\left(\|\widetilde{u}\|^4_{(p)} + s^4\|\widetilde{z}\|^4_{(p)}\right)
\end{aligned}$$

Since $\|y\|_{(q)} = \|\widetilde{z}\|_{(p)} = 1$, we have

$$\begin{aligned}
J(u+sy, \widetilde{u}+s\widetilde{z}) &\geq s^2 \int_{\mathbb{R}^2} \left(\nabla y \nabla \widetilde{z} + V(x) y\widetilde{z}\right) dx \\
&\quad + \left(1 - \frac{\varepsilon}{\lambda_{1,b}} - C\|u\|^2_{(q)}\right)\|u\|^2_{(q)} - \frac{\varepsilon}{\lambda_{1,b}}s^2 - Cs^4 \\
&\quad + \left(1 - \frac{\varepsilon}{\widetilde{\lambda}_{1,a}} - C\|\widetilde{u}\|^2_{(p)}\right)\|\widetilde{u}\|^2_{(p)} - \frac{\varepsilon}{\widetilde{\lambda}_{1,a}}s^2 - Cs^4.
\end{aligned}$$

Then,

$$J(u+sy,\widetilde{u}+s\widetilde{z}) \geq s^2\Big(\int_{\mathbb{R}^2}\big(\nabla y\nabla\widetilde{z}+V(x)y\widetilde{z}\big)\,dx-\varepsilon C_1-C_2s^2\Big)$$
$$+\big(2-\varepsilon C_1-C_2\rho_2^2\big)\rho_2^2.$$

where $\|u\|_{(q)}=\|\widetilde{u}\|_{(p)}=\rho_2$. Using Lemma 4.11, there exists $C_3>0$ such that

$$\int_{\mathbb{R}^2}\big(\nabla y\nabla\widetilde{z}+V(x)y\widetilde{z}\big)\,dx\geq C_3, \quad \text{for } k \text{ sufficiently large.}$$

Taking $\rho_1$, $\rho_2$, $s_1$ and $\varepsilon$ positives and sufficiently small such that $C_3-\varepsilon C_1-C_2s_1^2\geq C_3/2$ and $2-\varepsilon C_1-\rho_2^2\geq 0$. Set $\rho=\min\{\rho_1,\rho_2,s_1\}$, there exists $\sigma>0$ such that

$$J(u+sy,\widetilde{u}+s\widetilde{z})\geq\frac{C_3\rho^2}{2}=\sigma,$$

where $\|(u+sy,\widetilde{u}+s\widetilde{z})\|=\rho$. ■

**Lemma 4.23.** There exist $R_0>0$ and $R_1>\rho$ (independent of $n$ and $k$) such that $J(\vartheta)\leq 0$ for all $\vartheta\in\partial Q_{n,m}$, where

$$Q_{n,m}=\{w+s(y,\widetilde{z}) : w=(\omega,-\widetilde{\omega})\in E^-_{n,m}, \|w\|\leq R_0, 0\leq s\leq R_1\}.$$

***Proof.*** Note that, the boundary $\partial Q$ is composed of three parts.

(i) If $\vartheta\in\partial Q\cap E^-_{n,m}$, $\vartheta=(\omega,-\widetilde{\omega})$. Thus,

$$J(\omega,-\widetilde{\omega})=-\int_{\mathbb{R}^2}\big(\nabla\omega\nabla\widetilde{\omega}+V(x)\omega\widetilde{\omega}\big)\,dx-\int_{\mathbb{R}^2}\frac{F(\omega)}{|x|^b}\,dx-\int_{\mathbb{R}^2}\frac{G(-\widetilde{\omega})}{|x|^a}\,dx\leq-\|\omega\|^2_{(q)}\leq 0$$

since $F$ and $G$ are nonnegative functions.

(ii) If $\vartheta=(\omega,-\widetilde{\omega})\widetilde{+}s(y,\widetilde{z})=(\omega+sy,-\widetilde{\omega}+s\widetilde{z})\in\partial Q_{n,m}$, with $\|(\omega,-\widetilde{\omega})\|=R_0$ and $0\leq s\leq R_1$, we obtain

$$J(\omega+sy,-\widetilde{\omega}+s\widetilde{z})=\int_{\mathbb{R}^2}\Big(\nabla(\omega+sy)\nabla(-\widetilde{\omega}+s\widetilde{z})+V(x)(\omega+sy)(-\widetilde{\omega}+s\widetilde{z})\Big)\,dx$$
$$-\int_{\mathbb{R}^2}\frac{F(\omega+sy)}{|x|^b}\,dx-\int_{\mathbb{R}^2}\frac{G(-\widetilde{\omega}+s\widetilde{z})}{|x|^a}\,dx.$$

Using the fact that $F$ and $G$ are nonnegatives and Remark 4.13, we obtain

$$\begin{aligned}J(\omega+sy,-\widetilde{\omega}+s\widetilde{z})&\leq-\|\omega\|^2_{(q)}+s^2\int_{\mathbb{R}^2}\big(\nabla y\nabla\widetilde{z}+V(x)y\widetilde{z}\big)\,dx\\&\leq-\|\omega\|^2_{(q)}+s^2\|y\|_{(q)}\|\widetilde{z}\|_{(p)}\\&\leq-\frac{R_0^2}{2}+R_1^2.\end{aligned}$$

Then, $J(\vartheta)\leq 0$ provided that $R_0=\sqrt{2}R_1$.

(iii) Let $\vartheta = (\omega, -\widetilde{\omega}) + R_1(y, \widetilde{z})$, $\|(\omega, -\widetilde{\omega})\| \leq R_0$. Then,

$$J(\omega + R_1 y, \widetilde{\omega} + R_1\widetilde{z}) = -\|\omega\|^2_{(q)} + R_1^2 \int_{\mathbb{R}^2} \left(\nabla y \nabla \widetilde{z} + V(x) y \widetilde{z}\right) dx \\ - \int_{\mathbb{R}^2} \frac{F(\omega + R_1 y)}{|x|^b}\, dx - \int_{\mathbb{R}^2} \frac{G(\widetilde{\omega} + R_1 \widetilde{z})}{|x|^a}\, dx. \tag{4.52}$$

Using $(A_1)$ and $(A_2)$, there exists $C > 0$ such that

$$F(t) \geq C|t|^\theta - t^2 \quad \text{and} \quad G(t) \geq C|t|^\theta - t^2, \quad \text{for all} \quad t \in \mathbb{R}.$$

By the last inequalities and Remark 4.13, we have

$$\begin{aligned}
-\int_{\mathbb{R}^2} \frac{F(\omega + R_1 y)}{|x|^b}\, dx &\leq \int_{\mathbb{R}^2} \frac{|\omega + R_1 y|^2}{|x|^b}\, dx - C\int_{\mathbb{R}^2} \frac{|\omega + R_1 y|^\theta}{|x|^b}\, dx \\
&\leq \int_{\mathbb{R}^2} \frac{|\omega|^2}{|x|^b}\, dx + R_1^2 \int_{\mathbb{R}^2} \frac{|y|^2}{|x|^b}\, dx - C\int_{\mathbb{R}^2} \frac{|\omega|^\theta}{|x|^b}\, dx - CR_1^\theta \int_{\mathbb{R}^2} \frac{|y|^\theta}{|x|^b}\, dx \\
&\leq \frac{1}{\lambda_{1,b}} \|\omega\|^2_{(q)} + \frac{R_1^2}{\lambda_{1,b}} \|y\|^2_{(q)} - CR_1^\theta \int_{\mathbb{R}^2} \frac{|y|^\theta}{|x|^b}\, dx \\
&\leq \frac{R_0^2}{2\lambda_{1,b}} + \frac{R_1^2}{\lambda_{1,b}} - CR_1^\theta \int_{\mathbb{R}^2} \frac{|y|^\theta}{|x|^b}\, dx.
\end{aligned}$$

Since $y \neq 0$ and $R_0 = \sqrt{2} R_1$,

$$-\int_{\mathbb{R}^2} \frac{F(\omega + R_1 y)}{|x|^b}\, dx \leq \frac{2R_1^2}{\lambda_{1,b}} - CR_1^\theta, \quad \text{for some} \quad C > 0. \tag{4.53}$$

Similarly, we have

$$-\int_{\mathbb{R}^2} \frac{G(\widetilde{\omega} + R_1 \widetilde{z})}{|x|^a}\, dx \leq \frac{2R_1^2}{\widetilde{\lambda}_{1,a}} - CR_1^\theta. \tag{4.54}$$

Then, using (4.53) and (4.54) in (4.52), we obtain

$$J(\omega + R_1 y, \widetilde{\omega} + R_1 \widetilde{z}) \leq R_1^2 \left(1 + \frac{2}{\lambda_{1,b}} + \frac{2}{\widetilde{\lambda}_{1,a}}\right) - CR_1^\theta.$$

Since $\theta > 2$, taking $R_1$ sufficiently large, we get $J(\vartheta) \leq 0$.

■

### 4.4.2 Approximation finite dimensional

We define the sets

$$\Gamma_{n,m} = \{\gamma \in \mathscr{C}(Q_{n,m}, F_{n,m}) : \ \gamma(\vartheta) = \vartheta, \text{ for all } \vartheta \in \partial Q_{n,m}\}.$$

and the numbers

$$c_{n,m} = \inf_{\gamma \in \Gamma_{n,m}} \max_{\vartheta \in Q_{n,m}} J(\gamma(\vartheta)). \tag{4.55}$$

The proof of the following results proceeds along some lines as the proof of Lemma 9 in Cassani and Tarsi (2015).

**Lemma 4.24.** The sets $Q_{n,m}$ and $\partial B_\rho \cap F^+_{n,m}$ link, that is

$$\gamma(Q_{n,m}) \cap (\partial B_\rho \cap E^+_n) \neq \emptyset, \quad \text{for all} \quad \gamma \in \Gamma_{n,m}, \tag{4.56}$$

for $\rho$ given by Lemma 4.22.

Combining Lemma 4.22 and (4.55), we have

$$c_{n,m} \geq \sigma, \quad \text{for all} \quad n \geq 1. \tag{4.57}$$

Note also that, since the identity map $I : Q_{n,m} \to F_{n,m}$ belongs to $\Gamma_{n,m}$, for every $\vartheta = (\omega, -\widetilde{\omega}) + s(y, \widetilde{z}) \in Q_{n,m}$, we obtain

$$c_{n,m} \leq \sup_{\vartheta \in Q_{n,m}} J(\vartheta) \leq R_1^2. \tag{4.58}$$

Denote by $J_{n,m}$ the restriction of $J$ to the finite-dimensional space $F_{n,m}$. By Remark 3.7 in $J_{n,m}$ and using (4.57) and (4.58), we get the following result:

**Proposition 4.25.** For each $n, m \geq 1$ ($m = m(n)$ as in Lemma 4.12), the functional $J_{n,m}$ has a Palais-Smale sequence at level $c_{n,m}$. More precisely, there is a sequence $(u_j, \widetilde{v}_j) \subset F_{n,m}$ such that

$$J_{n,m}(u_j, \widetilde{v}_j) \to c_{n,m} \in [\sigma, R_1^2]$$

and

$$J'_{|F_{n,m}}(u_j, \widetilde{v}_j) \to 0.$$

**Proposition 4.26.** Assume that $f$ and $g$ satisfy $(A_1) - (A_5)$ and let $(u_j, \widetilde{v}_j)$ be a sequence in $F_{n,m}$ given by Proposition 4.25. Then,

(i) The sequence $(u_j, \widetilde{v}_j)$ is bounded sequence in $F_{n,m}$ and there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} \frac{f(u_j)u_j}{|x|^b}\, dx \leq C, \quad \int_{\mathbb{R}^2} \frac{g(\widetilde{v}_j)\widetilde{v}_j}{|x|^a}\, dx \leq C,$$

$$\int_{\mathbb{R}^2} \frac{F(u_j)}{|x|^b}\, dx \leq C, \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{G(\widetilde{v}_j)}{|x|^a}\, dx \leq C,$$

for all $j \geq 1$.

(ii) For each sequence $(u_j, \widetilde{v}_j)$, there exists $(u_{n,m}, \widetilde{v}_{n,m}) \in F_{n,m}$ and a subsequence (not renamed) $(u_j, \widetilde{v}_j)$ such that

$$(u_j, \widetilde{v}_j) \to (u_{n,m}, \widetilde{v}_{n,m}) \quad \text{in} \quad F_{n,m}.$$

Furthermore,

$$J_{n,m}(u_{n,m}, \widetilde{v}_{n,m}) = c_{n,m} \in [\sigma, R_1^2]$$

and

$$J'_{|F_{n,m}}(u_{n,m}, \widetilde{v}_{n,m}) = 0.$$

(iii) The sequence $(u_{n,m}, \widetilde{v}_{n,m})$ is bounded in $E$ and there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} \frac{f(u_{n,m})u_{n,m}}{|x|^b}\,dx \le C, \quad \int_{\mathbb{R}^2} \frac{g(\widetilde{v}_{n,m})\widetilde{v}_{n,m}}{|x|^a}\,dx \le C,$$

$$\int_{\mathbb{R}^2} \frac{F(u_{n,m})}{|x|^b}\,dx \le C \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{G(\widetilde{v}_{n,m})}{|x|^a}\,dx \le C,$$

for all $n \in \mathbb{N}$.

***Proof.***

(i) From Lemma 4.18, the sequence $(u_j, \widetilde{v}_j)$ is bounded in $F_{n,m}$. Moreover, by Remark 4.19 and assumption $(A_2)$, we get the estimates in $(i)$.

(ii) Since $(u_j, \widetilde{v}_j)$ is bounded, $F_{n,m}$ is finite dimensional and $J$ is of class $\mathscr{C}^1$ the assertion follows.

(iii) Using the sequence $(u_{n,m}, \widetilde{v}_{n,m})$ in Lemma 4.18 for the case $e_{n,m} = 0$, we get the boundedness of the sequence. Using again Remark 4.19 and assumption $(A_2)$ we obtain, the estimates.

■

### 4.4.3 Estimate of the minimax level

**Proposition 4.27.** There exists $k \in \mathbb{N}$ such that for any sequence

$$(u_{n,m}, \widetilde{v}_{n,m}) \in \mathbb{R}(M_{k,q,\frac{1}{m}}, M_{k,p,\frac{1}{m}}) \oplus E^-$$

satisfying the following conditions:

(i) The sequence $(u_{n,m}, \widetilde{v}_{n,m})$ is bounded in E.

(ii) The sequence $(u_{n,m}, \widetilde{v}_{n,m})$ converge weakly to $(0,0)$ in $E$ and

$$u_{n,m} \to 0, \quad \widetilde{v}_{n,m} \to 0 \quad \text{in} \quad L^r(\mathbb{R}^2), \quad \text{for all} \quad r \ge 1.$$

Then,

$$\sup_{n\in\mathbb{N}} J(u_{n,m}, \widetilde{v}_{n,m}) < \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}}.$$

***Proof.*** On the contrary, for each fixed $k$ in $\mathbb{N}$, there exist a nonnegative sequence $\varepsilon_n \to 0$ and a sequence

$$\eta_{n,k} = \tau_{n,k}(M_{k,q,\frac{1}{m}}, M_{k,p,\frac{1}{m}}) + (u_{n,k}, -\widetilde{u}_{n,k}), \quad \text{with} \quad u_{n,k} \in E_{n,m}$$

such that

$$\|\eta_{n,k}\| \leq C = C(k),$$

$$\eta_{n,k} \rightharpoonup 0 \quad \text{in} \quad E,$$

$$\tau_{n,k} M_{k,q,\frac{1}{m}} + u_{n,k} \to 0, \quad \tau_{n,k} M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k} \to 0 \quad \text{in} \quad L^s(\mathbb{R}^2), \quad \text{for all} \quad s \geq 1$$

and

$$J(\eta_{n,k}) \geq \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} - \varepsilon_n.$$

In particular, we have

$$\sup_{t\geq 0} J(t\eta_{n,k}) \geq J(\eta_{n,k}) \geq \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} - \varepsilon_n.$$

Since $J(t\eta_{n,k}) \to -\infty$ as $t \to +\infty$ and $J(0) = 0$, there exists $\hat{t} > 0$ such that

$$\sup_{t\geq 0} J(t\eta_{n,k}) = \max_{t\geq 0} J(t\eta_{n,k}) = J(\hat{t}\eta_{n,k}).$$

We can assume without loss of generality that $\hat{t} = 1$, that is

$$J'(\eta_{n,k})\eta_{n,k} = 0 \quad \text{and} \quad J(\eta_{n,k}) \geq \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} - \varepsilon_n.$$

Then,

$$\begin{aligned}
&2\int_{\mathbb{R}^2} \nabla(\tau_{n,k} M_{k,q,\frac{1}{m}} + u_{n,k}) \nabla(\tau_{n,k} M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})\, dx \\
&\quad + 2\int_{\mathbb{R}^2} V(x)(\tau_{n,k} M_{k,q,\frac{1}{m}} + u_{n,k})(\tau_{n,k} M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})\, dx \\
&= \int_{\mathbb{R}^2} \frac{f(\tau_{n,k} M_{k,q,\frac{1}{m}} + u_{n,k})(\tau_{n,k} M_{k,q,\frac{1}{m}} + u_{n,k})}{|x|^b}\, dx \\
&\quad + \int_{\mathbb{R}^2} \frac{g(\tau_{n,k} M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})(\tau_{n,k} M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})}{|x|^a}\, dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2} \nabla(\tau_{n,k} M_{k,q,\frac{1}{m}} + u_{n,k}) \nabla(\tau_{n,k} M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})\, dx \\
&\quad + \int_{\mathbb{R}^2} V(x)(\tau_{n,k} M_{k,q,\frac{1}{m}} + u_{n,k})(\tau_{n,k} M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})\, dx \\
&\quad - \int_{\mathbb{R}^2} \frac{F(\tau_{n,k} M_{k,q,\frac{1}{m}} + u_{n,k})}{|x|^b}\, dx - \int_{\mathbb{R}^2} \frac{G(\tau_{n,k} M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})}{|x|^a}\, dx \\
&\geq \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} - \varepsilon_n.
\end{aligned}$$

Since $\|M_{k,q,\frac{1}{m}}\|_{(q)} = \|M_{k,p,\frac{1}{m}}\|_{(p)} = 1$, $\|u_{n,k}\|_{(q)} = \|\widetilde{u}_{n,k}\|_{(p)}$ and using the fact that the support sets of $u_{n,k}, \widetilde{u}_{n,k}$ and the concentrating functions are disjoint, we obtain

$$2\tau_{n,k}^2 \geq 2(\tau_{n,k}^2 - \|u_{n,k}\|_{(q)}^2) \geq \int_{\mathbb{R}^2} \frac{f(\tau_{n,k}M_{k,q,\frac{1}{m}} + u_{n,k})(\tau_{n,k}M_{k,q,\frac{1}{m}} + u_{n,k})}{|x|^b}\,dx + \int_{\mathbb{R}^2} \frac{g(\tau_{n,k}M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})(\tau_{n,k}M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})}{|x|^a}\,dx \tag{4.59}$$

and

$$\tau_{n,k}^2 - \|u_{n,k}\|_{(q)}^2 - \int_{\mathbb{R}^2} \frac{F(\tau_{n,k}M_{k,q,\frac{1}{m}} + u_{n,k})}{|x|^b} - \int_{\mathbb{R}^2} \frac{G(\tau_{n,k}M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})}{|x|^a}\,dx \geq \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} - \varepsilon_n. \tag{4.60}$$

Using that $F$ and $G$ are nonnegative functions in (4.60), we obtain

$$\tau_{n,k}^2 \geq \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} - \varepsilon_n. \tag{4.61}$$

Define

$$s_{n,k} = \tau_{n,k}^2 - \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} \geq -\varepsilon_n. \tag{4.62}$$

By $(A_6)$, given $R > 0$ there exists $T_R$ such that

$$tf(t) \geq Re^{\alpha_0 t^p} \quad \text{and} \quad tg(t) \geq Re^{\beta_0 t^q}, \quad \text{for all} \quad |t| \geq T_R.$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^2} &\frac{f(\tau_{n,k}M_{k,q,\frac{1}{m}} + u_{n,k})(\tau_{n,k}M_{k,q,\frac{1}{m}} + u_{n,k})}{|x|^b}\,dx \\ &+ \int_{\mathbb{R}^2} \frac{g(\tau_{n,k}M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})(\tau_{n,k}M_{k,p,\frac{1}{m}} - \widetilde{u}_{n,k})}{|x|^a}\,dx \\ &\geq R\int_{\{x\in B_{\frac{1}{m}}:|\tau_{n,k}M_{k,q,\frac{1}{m}}|\geq T_R\}} \frac{e^{\alpha_0|\tau_{n,k}M_{k,q,\frac{1}{m}}|^p}}{|x|^b}\,dx \\ &+ R\int_{\{x\in B_{\frac{1}{m}}:|\tau_{n,k}M_{k,p,\frac{1}{m}\}}|\geq T_R\}} \frac{e^{\beta_0|\tau_{n,k}M_{k,p,\frac{1}{m}}|^q}}{|x|^a}\,dx,\end{aligned} \tag{4.63}$$

where we used the fact that the functions $u_{n,k}$ and $\widetilde{u}_{n,k}$ are zero in $B_{\frac{1}{m}}$. From the definition of the concentrate function, we have

$$M_{k,q,\frac{1}{m}}(x) = \frac{(\log k)^{\frac{q-1}{q}}}{\sqrt{4\pi}}(1-\delta_{k,q,\frac{1}{m}})^{\frac{q-1}{q}}, \quad \text{if} \quad |x| \leq \frac{1}{m\sqrt{k}}.$$

From (4.61), for this given $R > 0$, there exists $n_R$ and $k_R$ ($k_R$ independent of $n$) sufficiently large such that

$$\tau_{n,k}M_{k,q,\frac{1}{m}}(x) = \tau_{n,k}\frac{(\log k)^{\frac{q-1}{q}}}{\sqrt{4\pi}}(1-\delta_{k,q,\frac{1}{m}})^{\frac{q-1}{q}} \geq T_R, \quad \text{if} \quad |x| \leq \frac{1}{m\sqrt{k}}. \tag{4.64}$$

for $n \geq n_R$ and $k \geq k_R$. Combining (4.59) with (4.63), we get

$$\begin{aligned}
\tau_{n,k}^2 &\geq \frac{R}{2}\int_{B_{\frac{1}{m\sqrt{k}}}} \frac{e^{\alpha_0 \tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-\delta_{k,q,\frac{1}{m}})}}{|x|^b}\,dx + \frac{R}{2}\int_{B_{\frac{1}{m\sqrt{k}}}} \frac{e^{\beta_0 \tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}(1-\delta_{k,p,\frac{1}{m}})}}{|x|^a}\,dx \\
&= \frac{\pi R e^{\alpha_0 \tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-\delta_{k,q,\frac{1}{m}})}}{2(2-b)m^{2-b}k^{(1-b/2)}} + \frac{\pi R e^{\beta_0 \tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}(1-\delta_{k,p,\frac{1}{m}})}}{2(2-a)m^{2-a}k^{(1-a/2)}} \qquad (4.65)\\
&\geq \frac{\pi R}{2(2-d_0)m^{2-d_0}}\left(e^{\alpha_0 \tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-\delta_{k,q,\frac{1}{m}})-(1-b/2)\ln k} + e^{\beta_0 \tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}(1-\delta_{k,p,\frac{1}{m}})-(1-a/2)\ln k}\right)
\end{aligned}$$

where $d_0 = \min\{a,b\}$. We emphasize that up to now, we have fixed $n$ (and consequently, $m$), where $k$ can be arbitrarily chosen independently of $n$ ($k \geq K_R$). By (4.11), we have

$$|\delta_{k,q,\frac{1}{m}}| \leq C\|V\|_{L^\infty(B_{1/m})}\frac{1}{m^{q/2}\ln k} \quad \text{and} \quad |\delta_{k,p,\frac{1}{m}}| \leq C\|V\|_{L^\infty(B_{1/m})}\frac{1}{m^{p/2}\ln k}.$$

Increasing $n_R$ if necessary, we have

$$(1-\delta_{k,q,\frac{1}{m}}) \geq 1 - C\frac{V_1}{\ln k}, \quad \text{and} \quad (1-\delta_{k,p,\frac{1}{m}}) \geq 1 - C\frac{V_1}{\ln k} \quad \text{where} \quad V_1 = \|V\|_{L^\infty(B_1)}.$$

Using these estimates in (4.65), we get

$$\tau_{n,k}^2 \geq \frac{\pi R}{2(2-d_0)m^{2-d_0}}\left(e^{\alpha_0 \tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-C\frac{V_1}{\ln k})-(1-b/2)\ln k} + e^{\beta_0 \tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}(1-C\frac{V_1}{\ln k})-(1-a/2)\ln k}\right).$$

Using Young's inequality ($X^p/p + Y^q/q \geq XY$) with

$$X = p^{1/p} e^{\frac{\alpha_0}{p}\tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-C\frac{V_1}{\ln k})-\frac{(1-b/2)\ln k}{p}} \quad \text{and} \quad Y = q^{1/q} e^{\frac{\beta_0}{q}\tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}(1-C\frac{V_1}{\ln k})-\frac{(1-a/2)\ln k}{q}},$$

we find

$$\tau_{n,k}^2 \geq \frac{\pi R p^{1/p} q^{1/q}}{2(2-d_0)m^{2-d_0}} e^{\left(\frac{\alpha_0}{p}\tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}} + \frac{\beta_0}{q}\tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}\right)(1-C\frac{V_1}{\ln k})-\frac{(1-b/2)\ln k}{p}-\frac{(1-a/2)\ln k}{q}}. \qquad (4.66)$$

Using again Young's inequality, we get

$$\frac{\alpha_0}{p}\tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}} + \frac{\beta_0}{q}\tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}} \geq \alpha_0^{1/p}\beta_0^{1/q}\frac{\tau_{n,k}^2}{4\pi}\ln k.$$

Replacing in (4.66), we have

$$\tau_{n,k}^2 \geq \frac{\pi R p^{1/p} q^{1/q}}{2(2-d_0)m^{2-d_0}} e^{\alpha_0^{1/p}\beta_0^{1/q}\frac{\tau_{n,k}^2}{4\pi}(1-C\frac{V_1}{\ln k})\ln k - \frac{(1-b/2)\ln k}{p} - \frac{(1-a/2)\ln k}{q}}, \qquad (4.67)$$

for $n \geq n_R$ and $k \geq k_R$. Consider

$$R := 2(2-d_0)m^{3-d_0}.$$

Since $m$ depends on $n$, and $k$ depends on $R$, $k_R$ depends on $n$. With this choice, we obtain

$$\tau_{n,k}^2 \geq \pi m p^{1/p} q^{1/q} e^{\left[\alpha_0^{1/p}\beta_0^{1/q}\frac{\tau_{n,k}^2}{4\pi}(1-C\frac{V_1}{\ln k})-\frac{(1-b/2)}{p}-\frac{(1-a/2)}{q}\right]\ln k}, \qquad (4.68)$$

for $k \geq k_R$. If the sequence $\{\tau_{n,k}^2\}_{n\geq n_0}$ is unbounded, using (4.68) we get a contradiction. Thus, $\{\tau_{n,k}^2\}_{n\geq n_0}$ is a bounded sequence. In particular, without loss of generality, we can assume that there exists $s \in \mathbb{R}$ such that

$$\tau_{n,k}^2 = s_{n,k} + \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} \to s + \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}}, \quad \text{as} \quad n \to +\infty.$$

Moreover, by (4.62), $s \geq 0$. From $(A_7)$, we can suppose without loss of generality that

$$\frac{\alpha_0^{1/p}}{(1-b/2)^{1/p}} > \frac{\beta_0^{1/q}}{(1-a/2)^{1/q}}. \tag{4.69}$$

Now by (4.65), we have

$$\tau_{n,k}^2 \geq \pi m e^{\alpha_0 \tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-C\frac{V_1}{\ln k})-(1-b/2)\ln k}. \tag{4.70}$$

Writing

$$\tau_{n,k}^2 = s + \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} + o_n(1)$$

and replacing in (4.70), we obtain

$$\begin{aligned}
\tau_{n,k}^2 &\geq \pi m e^{\alpha_0\left(s+\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/q}}+o_n(1)\right)^{p/2}\frac{\ln k}{(4\pi)^{p/2}}(1-C\frac{V_1}{\ln k})-(1-b/2)\ln k} \\
&\geq \pi m e^{\alpha_0\left(\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/q}}+o_n(1)\right)^{p/2}\frac{\ln k}{(4\pi)^{p/2}}(1-C\frac{V_1}{\ln k})-(1-b/2)\ln k} \\
&\geq \pi m e^{\left(\frac{\alpha_0^{1/2}(1-b/2)^{1/2}(1-a/2)^{p/2q}}{\beta_0^{p/2q}}+o_n(1)\right)\ln k(1-C\frac{V_1}{\ln k})-(1-b/2)\ln k} \\
&\geq \pi m e^{\left(\frac{\alpha_0^{1/2}(1-b/2)^{1/2}(1-a/2)^{p/2q}}{\beta_0^{p/2q}}+o_n(1)-(1-b/2)\right)\ln k-\frac{CV_1\alpha_0^{1/2}(1-b/2)^{1/2}(1-a/2)^{p/2q}}{\beta_0^{p/2q}}+o_n(1)}.
\end{aligned}$$

Using (4.69), we have

$$0 < \delta := \frac{\alpha_0^{1/2}(1-b/2)^{1/2}(1-a/2)^{p/2q}}{\beta_0^{p/2q}} - (1-b/2).$$

Thus,

$$s + \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} + o_n(1) \geq \pi m e^{\left(\delta+o_n(1)\right)\ln k-\frac{CV_1\alpha_0^{1/2}(1-b/2)^{1/2}(1-a/2)^{p/2q}}{\beta_0^{p/2q}}+o_n(1)}.$$

Taking $n \to +\infty$ (and hence $k \to +\infty$, where we used the fact $T_R \to +\infty$ as $R \to +\infty$ and (4.64)), we get a contradiction and the proposition follows. ■

### 4.4.4 Proof of Theorem 4.6

***Proof.*** By Proposition 4.26, there exists a sequence $(u_{n,m},\widetilde{v}_{n,m}) \in F_{n,m}$ such that

$$J_{n,m}(u_{n,m},\widetilde{v}_{n,m}) = c_{n,m} \in [\sigma, R_1^2] \tag{4.71}$$

and

$$J'_{n,m}(u_{n,m},\widetilde{v}_{n,m})(\phi,\widetilde{\psi}) = 0, \quad \text{for all} \quad (\phi,\widetilde{\psi}) \in F_{n,m}. \tag{4.72}$$

Moreover, the sequence $(u_{n,m},\widetilde{v}_{n,m})$ is bounded in $E$. Thus, we can assume that there exists $(u,\widetilde{v}) \in E$ such that $(u_{n,m},\widetilde{v}_{n,m}) \rightharpoonup (u,v)$ in $E$ and

$$u_{n,m} \to u \quad \text{and} \quad \widetilde{v}_{n,m} \to \widetilde{v} \quad \text{in} \quad L^r(\mathbb{R}^2), \quad \text{for all} \quad r \geq 1. \tag{4.73}$$

Taking respectively $(0,\widetilde{\psi})$ and $(\phi,0)$, in (4.72), with $(\phi,\widetilde{\psi}) \in F_{n,m} \cap \big(\mathscr{C}_0^\infty(\mathbb{R}^2) \times \mathscr{C}_0^\infty(\mathbb{R}^2)\big)$, we have

$$\int_{\mathbb{R}^2} \big(\nabla u_{n,m} \nabla \widetilde{\psi} + V(x) u_{n,m} \widetilde{\psi}\big)\, dx = \int_{\mathbb{R}^2} \frac{g(\widetilde{v}_{n,m})\widetilde{\psi}}{|x|^a}\, dx$$

and

$$\int_{\mathbb{R}^2} \big(\nabla \widetilde{v}_{n,m} \nabla \phi + V(x) \widetilde{v}_{n,m} \phi\big)\, dx = \int_{\mathbb{R}^2} \frac{f(u_{n,m})\phi}{|x|^b}\, dx.$$

Taking the limit as $n \to +\infty$, using Lemma 4.21 and the fact that $\bigcup_{n=1}^{+\infty} F_{n,m} \cap \big(\mathscr{C}_0^\infty(\mathbb{R}^2) \times \mathscr{C}_0^\infty(\mathbb{R}^2)\big)$ is dense in $E$, we obtain

$$\int_{\mathbb{R}^2} \big(\nabla u \nabla \widetilde{\psi} + V(x) u \widetilde{\psi}\big)\, dx = \int_{\mathbb{R}^2} \frac{g(\widetilde{v})\widetilde{\psi}}{|x|^a}\, dx, \quad \text{for all} \quad \widetilde{\psi} \in W^{(p)}$$

and

$$\int_{\mathbb{R}^2} \big(\nabla \widetilde{v} \nabla \phi + V(x) \widetilde{v} \phi\big)\, dx = \int_{\mathbb{R}^2} \frac{f(u)\phi}{|x|^b}\, dx, \quad \text{for all} \quad \phi \in W^{(q)}.$$

Thus, $(u,\widetilde{v}) \in E$ is a solution of the system .

It remains to prove that $(u,\widetilde{v})$ is a nontrivial weak solution. Assume, by contradiction, that $u \equiv 0$ (which implies that $\widetilde{v} \equiv 0$). Thus, we can assume that

$$u_{n,m} \to 0 \quad \text{and} \quad \widetilde{v}_{n,m} \to 0 \quad \text{in} \quad L^r(\mathbb{R}^2), \quad \text{for all} \quad r \geq 1. \tag{4.74}$$

Taking $(0,\widetilde{v}_{n,m})$ and $(u_{n,m},0)$ in (4.72) we have

$$\int_{\mathbb{R}^2} \big(\nabla u_{n,m} \nabla \widetilde{v}_{n,m} + V(x) u_{n,m} \widetilde{v}_{n,m}\big)\, dx = \int_{\mathbb{R}^2} \frac{g(\widetilde{v}_{n,m})\widetilde{v}_{n,m}}{|x|^a}\, dx = \int_{\mathbb{R}^2} \frac{f(u_{n,m})u_{n,m}}{|x|^b}\, dx. \tag{4.75}$$

From Proposition 4.27, there exists $\delta' > 0$ such that

$$c_{n,m} \leq \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} - \delta'.$$

Moreover, there exists $\delta > 0$ such that

$$\Big(\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} - \frac{\delta'}{4}\Big)\Big(1 - \frac{\delta\alpha_0}{1-b/2}\Big)^{-1/p} \leq \frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} - \frac{\delta'}{8} \tag{4.76}$$

and

$$\Big(\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}}-\frac{\delta'}{4}\Big)\Big(1-\frac{\delta\beta_0}{1-a/2}\Big)^{-1/q}\leq\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}}-\frac{\delta'}{8}.$$

Taking $(v_{n,m},0)$ in (4.72), we obtain

$$\|v_{n,m}\|_{(q)}^2=\int_{\mathbb{R}^2}\frac{f(u_{n,m})v_{n,m}}{|x|^b}\,dx\leq\int_{\mathbb{R}^2}\frac{|f(u_{n,m})||v_{n,m}|}{|x|^b}\,dx \tag{4.77}$$

Set

$$V_{n,m}=\sqrt{4\pi}\Big(\frac{1-b/2}{\alpha_0}-\delta\Big)^{1/p}\frac{v_{n,m}}{\|v_{n,m}\|_{(q)}}. \tag{4.78}$$

Applying Lemma 3.10 in (4.77) with $s=|f(u_{n,m})|/\alpha_0^{1/p}$, $t=\alpha_0^{1/p}|V_{n,m}|$, $r=p$ and $r'=q$, we obtain

$$\begin{aligned}\sqrt{4\pi}\Big(\frac{1-b/2}{\alpha_0}-\delta\Big)^{1/p}\|v_{n,m}\|_{(q)}&\leq\int_{\mathbb{R}^2}\frac{|f(u_{n,m})||V_{n,m}|}{|x|^b}\,dx\\&\leq\int_{\mathbb{R}^2}\frac{(e^{\alpha_0|V_{n,m}|^p}-1)}{|x|^b}\,dx+\frac{1}{\alpha_0^{q/p}q}\int_{\{x\in\mathbb{R}^2:\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\leq e^{1/p^q}\}}\frac{|f(u_{n,m})|^q}{|x|^b}\,dx\\&\quad+\frac{1}{\alpha_0^{1/p}}\int_{\{x\in\mathbb{R}^2:\frac{|f(\tilde{v}_{n,n})|}{\alpha_0^{1/p}}\leq e^{1/p^q}\}}\frac{|f(u_{n,m})|}{|x|^b}\Big[\ln\Big(\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\Big)\Big]^{1/p}\,dx.\end{aligned} \tag{4.79}$$

As in proof of Lemma 3.15, $V_{n,m}\rightharpoonup 0$ in $W^{(q)}$, and from (4.78), we have

$$\|V_{n,m}\|_{(q)}^p<\frac{(4\pi)^{p/2}(1-b/2)}{\alpha_0}.$$

This combined with Lemma 4.17, yields

$$\int_{\mathbb{R}^2}\frac{(e^{\alpha_0|V_{n,m}|^p}-1)}{|x|^b}\,dx=o_n(1). \tag{4.80}$$

Now, we estimate the second integral of (4.79). Using $(A_1)$ there exists $\delta>0$ such that

$$|f(t)|\leq|t|,\quad\text{for all}\quad|t|\leq\delta.$$

We observe that

$$|f(t)|\leq\frac{|f(t)|}{\delta}|t|,\quad\text{for all}\quad|t|\geq\delta.$$

Thus, there exists $C>0$ such that

$$|f(t)|^q\leq C|t|^q,\quad\text{for all}\quad\Big\{t\in\mathbb{R}:\frac{|f(t)|}{\alpha_0^{1/p}}\leq e^{1/p^q}\Big\}. \tag{4.81}$$

Let $r>1$ such that $rb<2$, using Hölder's inequality, (4.81) and (4.74), we have

$$\begin{aligned}
&\int_{\{x\in\mathbb{R}^2:\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\le e^{1/p^q}\}}\frac{|f(u_{n,m})|^q}{|x|^b}\,dx\\
&\le C\int_{\{x\in B_1:\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\le e^{1/p^q}\}}\frac{|u_{n,m}|^q}{|x|^b}\,dx+C\int_{\{x\in\mathbb{R}^2\setminus B_1:\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\le e^{1/p^q}\}}|u_{n,m}|^q\,dx \qquad (4.82)\\
&\le C\Big(\int_{B_1}\frac{1}{|x|^{rb}}\,dx\Big)^{1/r}\|u_{n,m}\|_{r'q}^q+C\|u_{n,m}\|_q^q\\
&=o_n(1).
\end{aligned}$$

Let

$$\xi=\frac{\delta'\min\{\alpha_0,\beta_0\}}{4\Big(\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/q}}-\delta'\Big)}, \qquad (4.83)$$

by $(A_1)$ and $(A_4)$, there exists $C_\xi>0$ such that

$$|f(t)|\le C_\xi e^{(\alpha_0+\xi)|t|^p},\quad \text{for all}\quad t\in\mathbb{R}.$$

Thus,

$$\begin{aligned}
&\int_{\{x\in\mathbb{R}^2:\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\le e^{1/p^q}\}}\frac{|f(u_{n,m})|}{|x|^b}\Big[\ln\Big(\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\Big)\Big]^{1/p}dx\\
&\qquad\le\int_{\mathbb{R}^2}\frac{|f(u_{n,m})|}{|x|^b}\Big[\ln\Big(\frac{C_\xi e^{(\alpha_0+\xi)|u_{n,m}|^p}}{\alpha_0^{1/p}}\Big)\Big]^{1/p}dx \qquad (4.84)\\
&\qquad\le\int_{\mathbb{R}^2}\frac{|f(u_{n,m})|}{|x|^b}\Big[\ln^{1/p}\big(\frac{C_\xi}{\alpha_0^{1/p}}\big)+(\alpha_0+\xi)^{1/p}|u_{n,m}|\Big]dx.
\end{aligned}$$

For each $n\in\mathbb{N}$ consider

$$T_n:=\Big\{x\in\mathbb{R}^2:\ln^{1/p}\Big(\frac{C_\xi}{\alpha_0^{1/p}}\Big)+(\alpha_0+\xi)^{1/p}|u_{n,m}|\le(\alpha_0+2\xi)^{1/p}|u_{n,m}|\Big\}.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^2}\frac{|f(u_{n,m})|}{|x|^b}\Big[\ln^{1/p}\big(\frac{C_\xi}{\alpha_0^{1/p}}\big)+(\alpha_0+\xi)^{1/p}|u_{n,m}|\Big]dx\\
&\le\int_{\mathbb{R}^2\setminus T_n}\frac{|f(u_{n,m})|}{|x|^b}\Big[\ln^{1/p}\big(\frac{C_\xi}{\alpha_0^{1/p}}\big)+(\alpha_0+\xi)^{1/p}|u_{n,m}|\Big]dx+(\alpha_0+2\xi)^{1/p}\int_{T_n}\frac{f(u_{n,m})u_{n,m}}{|x|^b}\\
&\le\ln^{1/p}\big(\frac{C_\xi}{\alpha_0^{1/p}}\big)\int_{\mathbb{R}^2\setminus T_n}\frac{|f(u_{n,m})|}{|x|^b}\,dx+(\alpha_0+2\xi)^{1/p}\int_{\mathbb{R}^2}\frac{f(u_{n,m})u_{n,m}}{|x|^b}\,dx. \qquad (4.85)
\end{aligned}$$

Observe that

$$\mathbb{R}^2\setminus T_n=\{x\in\mathbb{R}^2:|u_{n,m}|<d_1\},\quad\text{where}\quad d_1=\frac{\ln^{1/p}\big(\frac{C_\xi}{\alpha_0^{1/p}}\big)}{(\alpha_0+2\xi)^{1/p}-(\alpha_0+\xi)^{1/p}}.$$

Thus,

$$\mathbb{R}^2\backslash T_n \subseteq \{x\in\mathbb{R}^2 : |f(u_{n,m})| \leq d_2\}, \quad \text{where} \quad d_2 = \max_{|s|\leq d_1} |f(s)|.$$

Using similar arguments as the integral in (4.82), we can conclude that

$$\int_{\mathbb{R}^2\backslash T_n} \frac{|f(u_{n,m})|}{|x|^b}\, dx = o_n(1). \tag{4.86}$$

From (4.84), (4.85) and (4.86), we obtain

$$\int_{\{x\in\mathbb{R}^2:\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\leq e^{1/p^q}\}} \frac{|f(u_{n,m})|}{|x|^b}\Big[\ln\Big(\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\Big)\Big]^{1/p} dx \leq (\alpha_0+2\xi)^{1/p}\int_{\mathbb{R}^2} \frac{f(u_{n,m})u_{n,m}}{|x|^b}\, dx + o_n(1). \tag{4.87}$$

Using this, (4.80) and (4.82) in (4.79), we have

$$\sqrt{4\pi}\Big(\frac{1-b/2}{\alpha_0} - \delta\Big)^{1/p} \|v_{n,m}\|_{(q)} \leq (1+\frac{2\xi}{\alpha_0})^{1/p} \int_{\mathbb{R}^2} \frac{f(u_{n,m})u_{n,m}}{|x|^b}\, dx + o_n(1). \tag{4.88}$$

Taking $(0,\widetilde{u}_{n,m})$ in (4.72), we obtain

$$\|\widetilde{u}_{n,m}\|^2_{(p)} = \int_{\mathbb{R}^2} g(\widetilde{v}_{n,m})\widetilde{u}_{n,m}\, dx.$$

Analogously, we can obtain

$$\sqrt{4\pi}\Big(\frac{1-a/2}{\beta_0} - \delta\Big)^{1/q} \|\widetilde{u}_{n,m}\|_{(p)} \leq (1+\frac{2\xi}{\beta_0})^{1/q} \int_{\mathbb{R}^2} \frac{g(\widetilde{v}_{n,m})\widetilde{v}_{n,m}}{|x|^a}\, dx + o_n(1). \tag{4.89}$$

By Lemma 4.21, we have

$$\int_{\mathbb{R}^2} \frac{F(u_{n,m})}{|x|^b}\, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{G(\widetilde{v}_{n,m})}{|x|^b}\, dx \to 0, \tag{4.90}$$

which imply

$$\int_{\mathbb{R}^2} \big(\nabla u_{n,m}\nabla\widetilde{v}_{n,m} + V(x)u_{n,m}\widetilde{v}_{n,m}\big)\, dx = J(u_{n,m},\widetilde{v}_{n,m}) + o_n(1).$$

By Proposition 4.27, we find

$$\int_{\mathbb{R}^2} \frac{f(u_{n,m})u_{n,m}}{|x|^b}\, dx + \int_{\mathbb{R}^2} \frac{g(\widetilde{v}_{n,m})\widetilde{v}_{n,m}}{|x|^a}\, dx \leq 2\Big(\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/q}} - \delta'\Big) + o_n(1).$$

Using (4.88), (4.89) and (4.83), we have

$$\begin{aligned}
&\sqrt{4\pi}\Big(\frac{1-b/2}{\alpha_0} - \delta\Big)^{1/p}\|v_{n,m}\|_{(q)} + \sqrt{4\pi}\Big(\frac{1-a/2}{\beta_0} - \delta\Big)^{1/q}\|\widetilde{u}_{n,m}\|_{(p)}\\
&\leq \Big(1+\frac{2\xi}{\alpha_0}\Big)^{1/p}\int_{\mathbb{R}^2}\frac{f(u_{n,m})u_{n,m}}{|x|^b}\, dx + \Big(1+\frac{2\xi}{\beta_0}\Big)^{1/q}\int_{\mathbb{R}^2}\frac{g(\widetilde{v}_{n,m})\widetilde{v}_{n,m}}{|x|^a}\, dx + o_n(1)\\
&\leq \Big(1+\frac{2\xi}{\min\{\alpha_0,\beta_0\}}\Big)\Big(\int_{\mathbb{R}^2}\frac{f(u_{n,m})u_{n,m}}{|x|^b}\, dx + \int_{\mathbb{R}^2}\frac{g(\widetilde{v}_{n,m})\widetilde{v}_{n,m}}{|x|^a}\, dx\Big) + o_n(1)\\
&\leq 2\Big(1+\frac{2\xi}{\min\{\alpha_0,\beta_0\}}\Big)\Big(\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/p}} - \delta'\Big) + o_n(1)\\
&\leq 2\Big(\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/q}} - \frac{\delta'}{2}\Big) + o_n(1).
\end{aligned}$$

Now, we can suppose that, for all $n \geq 1$

$$\sqrt{4\pi}\Big(\frac{1-b/2}{\alpha_0}-\delta\Big)^{1/p}\|v_{n,m}\|_{(q)}+\sqrt{4\pi}\Big(\frac{1-a/2}{\beta_0}-\delta\Big)^{1/q}\|\widetilde{u}_{n,m}\|_{(p)}$$
$$\leq 2\Big(\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/q}}-\frac{\delta'}{4}\Big).$$

Thus, we can assume that

$$\|v_{n,m}\|_{(q)} \leq \frac{1}{\sqrt{4\pi}}\Big(\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/q}}-\frac{\delta'}{4}\Big)\Big(\frac{1-b/2}{\alpha_0}-\delta\Big)^{-1/p}$$

or

$$\|\widetilde{u}_{n,m}\|_{(p)} \leq \frac{1}{\sqrt{4\pi}}\Big(\frac{4\pi(1-b)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/q}}-\frac{\delta'}{4}\Big)\Big(\frac{1-a/2}{\beta_0}-\delta\Big)^{-1/q}.$$

Suppose that the second inequality holds. Rewritting

$$\|\widetilde{u}_{n,m}\|_{(p)} \leq \frac{\beta_0^{1/q}}{\sqrt{4\pi}(1-a/2)^{1/q}}\Big(\frac{4\pi(1-b/2)^{1/p}(1-a/2)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/q}}-\frac{\delta'}{4}\Big)\Big(1-\frac{\delta\beta_0}{1-a/2}\Big)^{-1/q},$$

and using (4.76), we obtain

$$\|\widetilde{u}_{n,m}\|_{(p)} \leq \frac{\beta_0^{1/q}}{\sqrt{4\pi}(1-a/2)^{1/q}}\Big(\frac{4\pi(1-b/2)^{1/p}(1-a)^{1/q}}{\alpha_0^{1/p}\beta_0^{1/q}}-\frac{\delta'}{8}\Big)$$
$$\leq \frac{(4\pi)^{1/2}(1-b/2)^{1/p}}{\alpha_0^{1/p}}-\frac{\delta'\beta_0^{1/q}}{8\sqrt{4\pi}(1-a/2)^{1/q}}.$$

In particular, $\alpha_0^{1/p}\|u_{n,m}\|_{(q)} < (4\pi)^{1/2}(1-b/2)^{1/p}$. Thus, we can find $r > 1$ and $\eta > 0$ such that

$$r(\alpha_0+\eta)\|u_{n,m}\|_{(q)}^p < (4\pi)^{p/2}(1-rb/2). \tag{4.91}$$

By ($A_1$ and ($A_4$), there exists $C_1 > 0$ such that

$$|f(s)| \leq |s|+C_1\big(e^{(\alpha_0+\eta)|s|^p}-1\big), \quad \text{for all} \quad s\in\mathbb{R}.$$

From Hölder's inequality with $r' = r/(r-1)$, Lemma 4.14 and Proposition 4.8, we have

$$\int_{\mathbb{R}^2}\frac{f(u_{n,m})u_{n,m}}{|x|^b}\,dx \leq \int_{\mathbb{R}^2}\frac{|u_{n,m}|^2}{|x|^b}\,dx + C_1\int_{\mathbb{R}^2}\frac{|u_{n,m}|\big(e^{(\alpha_0+\eta)|u_{n,m}|^p}-1\big)}{|x|^b}\,dx$$
$$\leq C_2\|u_{n,m}\|_{2r'}^2+\|u_{n,m}\|_2^2+C_2\|u_{n,m}\|_{r'}\int_{\mathbb{R}^2}\frac{\big(e^{(\alpha_0+\eta)|u_{n,m}|^p}-1\big)^r}{|x|^{rb}}\,dx$$
$$\leq C_2\|u_{n,m}\|_{2r'}^2+\|u_{n,m}\|_2^2+C_2\|u_{n,m}\|_{r'}\int_{\mathbb{R}^2}\frac{\big(e^{r(\alpha_0+\eta)|u_{n,m}|^p}-1\big)}{|x|^{rb}}\,dx$$
$$\leq C_2\|u_{n,m}\|_{2r'}^2+\|u_{n,m}\|_2^2+C_3\|u_{n,m}\|_{r'}.$$

Using (4.74), we get

$$\int_{\mathbb{R}^2}\frac{f(u_{n,m})u_{n,m}}{|x|^b}\,dx \to 0.$$

Replacing in (4.75), we have

$$\int_{\mathbb{R}^2} \left(\nabla u_{n,m} \nabla \widetilde{v}_{n,m} + V(x) u_{n,m} \widetilde{v}_{n,m}\right) dx \to 0.$$

Combining the last limit with (4.90), we get

$$J(u_{u,m}, \widetilde{v}_{n,m}) \to 0,$$

which gives a contradiction with the fact that $c_{n,m} \geq \sigma$. Thus, $(u, \widetilde{v})$ is a nontrivial weak solution.

∎

## 4.5 Theorem 4.7

This section is to prove Theorem 4.7.

### 4.5.1 The geometry of the Linking theorem

**Lemma 4.28.** There exist $\rho, \sigma > 0$ such that $J(z) \geq \sigma$, for all $z \in \partial B_\rho \cap E^+$.

***Proof.*** Given $\varepsilon > 0$ for assumptions $(A_1), (A_4)$ and $(A_5)$ there exists $C > 0$ such that

$$|F(s)| \leq \varepsilon |s|^2 + C|s|^3 \left(e^{2\alpha_0 |s|^p} - 1\right), \quad \text{for all} \quad s \in \mathbb{R}$$

and

$$|G(s)| \leq \varepsilon |s|^2 + C|s|^3 \left(e^{2\beta_0 |s|^q} - 2\beta_0 |s|^q - 1\right), \quad \text{for all} \quad s \in \mathbb{R}.$$

Thus, taking $\rho_1 > 0$ such that $2\alpha_0 \rho_1^p / \alpha_q^* + b/2 < 1$ and $2\beta_0 \rho_1^q / \alpha_p^* + a/2 < 1$, by Lemma 4.16, there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} \frac{F(u)}{|x|^b} dx \leq \varepsilon \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^b} dx + C\|u\|_{(q)}^3, \quad \text{for all} \quad \|u\|_{(q)} \leq \rho_1.$$

and

$$\int_{\mathbb{R}^2} \frac{G(\widetilde{u})}{|x|^a} dx \leq \varepsilon \int_{\mathbb{R}^2} \frac{|\widetilde{u}|^2}{|x|^a} dx + C\|\widetilde{u}\|_{(p)}^3, \quad \text{for all} \quad \|\widetilde{u}\|_{(p)} \leq \rho_1.$$

Thus,

$$\begin{aligned}
J(u, \widetilde{u}) &= \int_{\mathbb{R}^2} \left(\nabla u \nabla \widetilde{u} + V(x) u \widetilde{u}\right) dx - \int_{\mathbb{R}^2} \frac{F(u)}{|x|^b} dx - \int_{\mathbb{R}^2} \frac{G(\widetilde{u})}{|x|^a} dx \\
&\geq \frac{1}{2}\|u\|_{(q)}^2 - \varepsilon \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^b} dx - C\|u\|_{(q)}^3 \\
&\quad + \frac{1}{2}\|\widetilde{u}\|_{(p)}^2 - \varepsilon \int_{\mathbb{R}^2} \frac{|\widetilde{u}|^2}{|x|^a} dx - C\|\widetilde{u}\|_{(p)}^3 \\
&\geq \left(\frac{1}{2} - \frac{\varepsilon}{\lambda_{1,b}}\right)\|u\|_{(q)}^2 - C\|u\|_{(q)}^3 + \left(\frac{1}{2} - \frac{\varepsilon}{\widetilde{\lambda}_{1,a}}\right)\|\widetilde{u}\|_{(p)}^2 - C\|\widetilde{u}\|_{(p)}^3
\end{aligned}$$

which implies

$$J(u,\widetilde{u}) \geq \Big(\frac{1}{2} - \frac{\varepsilon}{\lambda_{1,b}} - C\|u\|_{(q)}\Big)\|u\|^2_{(q)} + \Big(\frac{1}{2} - \frac{\varepsilon}{\widetilde{\lambda}_{1,a}} - C\|\widetilde{u}\|_{(p)}\Big)\|\widetilde{u}\|^2_{(p)}.$$

Chosen $\rho_2, \varepsilon > 0$, sufficiently small such that

$$\frac{1}{2} - \frac{\varepsilon}{\lambda_{1,b}} - C\rho_2 \geq \frac{1}{4} \quad \text{and} \quad \frac{1}{2} - \frac{\varepsilon}{\widetilde{\lambda}_{1,a}} - C\rho_2 \geq \frac{1}{4}.$$

Hence,

$$J(u,\widetilde{u}) \geq \frac{\rho^2}{4} = \sigma > 0, \quad \text{for all} \quad \|(u,\widetilde{u})\| = \rho,$$

where $0 < \rho \leq \min\{\rho_1, \rho_2\}$. ■

We observe that, from inequality given in (4.2), we can choose $m_0 \in (0,1)$ and $\varepsilon > 0$ such that

$$C_{\theta,a,b} > \frac{56+32\sqrt{3}}{\delta_{\theta,a,b}R_1^{\theta-2}}, \quad \text{where} \quad R_1^2 = \frac{m_0 R^2}{(1+\varepsilon)}. \tag{4.92}$$

Let

$$M_1 = \frac{2m_0\pi^{1/2}(1-a/2)^{1/q}}{\beta_0^{1/q}} \quad \text{and} \quad M_2 = \frac{2m_0\pi^{1/2}(1-b/2)^{1/p}}{\alpha_0^{1/p}}. \tag{4.93}$$

Thus, we can write

$$R_1^2 = \frac{(4\pi)^{1/2}}{1+\varepsilon}\max\left\{\frac{M_1(1-b/2)^{1/p}}{\alpha_0^{1/p}}\frac{(\mu-2)}{2\mu}, \frac{M_2(1-a/2)^{1/q}}{\beta_0^{1/q}}\frac{(\nu-2)}{2\nu}\right\}. \tag{4.94}$$

**Lemma 4.29.** Let $Q = \{r(e_1,\widetilde{e}_1) \widetilde{+} (\omega,-\widetilde{\omega}) : \|\omega\|_{(q)} \leq (3+2\sqrt{3})R_1, 0 \leq r \leq R_1\}$, where $R_1 > 0$ is given by (4.94). Then, $J(z) \leq 0$ for all $z \in \partial Q$, where $\partial Q$ is the boundary of $Q$ in $\mathbb{R}(e_1,\widetilde{e}_1) \widetilde{+} E^-$.

***Proof.*** Note that, the boundary $\partial Q$ is composed of three parts.

(i) If $z \in \partial Q \cap E^-$, we have $z = (u,-\widetilde{u})$. Thus,

$$J(u,-\widetilde{u}) = -\int_{\mathbb{R}^2}\big(\nabla u\nabla\widetilde{u} + V(x)u\widetilde{u}\big)\,dx - \int_{\mathbb{R}^2}\frac{F(u)}{|x|^b}\,dx - \int_{\mathbb{R}^2}\frac{G(-\widetilde{u})}{|x|^a}\,dx \leq -\|u\|^2_{(q)} \leq 0,$$

since $F$ and $G$ are nonnegative functions.

(ii) If $z = r(e_1,\widetilde{e_1}) \widetilde{+} (\omega,-\widetilde{\omega}) = (re_1+\omega,\widetilde{re_1-\omega}) \in \partial Q$, with $\|\omega\|_{(q)} = (3+2\sqrt{3})R_1$ and $0 \le r \le R_1$, in this case, we obtain

$$\begin{aligned}
J(z) &= \int_{\mathbb{R}^2} \big(\nabla(re_1+\omega)\nabla(\widetilde{re_1-\omega}) + V(x)(re_1+\omega)(\widetilde{re_1-\omega})\big)\,dx \\
&\quad - \int_{\mathbb{R}^2} \frac{F(re_1+\omega)}{|x|^b}\,dx - \int_{\mathbb{R}^2} \frac{G(\widetilde{re_1-\omega})}{|x|^a}\,dx \\
&\le \int_{\mathbb{R}^2} \big(\nabla(re_1+\omega)\nabla(\widetilde{re_1-\omega}) + V(x)(re_1+\omega)(\widetilde{re_1-\omega})\big)\,dx \\
&= -\int_{\mathbb{R}^2} \nabla(re_1-\omega)\nabla(\widetilde{re_1-\omega})\,dx - \int_{\mathbb{R}^2} V(x)(re_1-\omega)(\widetilde{re_1-\omega})\,dx \\
&\quad + \int_{\mathbb{R}^2} \nabla(2re_1)\nabla(\widetilde{re_1-\omega})\,dx + \int_{\mathbb{R}^2} V(x)(2re_1)(\widetilde{re_1-\omega})\,dx \\
&\le -\|re_1-\omega\|_{(q)}^2 + 2\|\nabla(re_1)\|_{2,q}\|\nabla(\widetilde{re_1-\omega})\|_{2,p} \\
&\quad + 2\|V^{1/q}re_1\|_{2,q}\|V^{1/p}(\widetilde{re_1-\omega})\|_{2,p} \\
&\le -\|re_1-\omega\|_{(q)}^2 + 2\|re_1\|_{(q)}\|re_1-\omega\|_{(q)} + 2\|re_1\|_{(q)}\|re_1-\omega\|_{(q)} \\
&\le -\|re_1\|_{(q)}^2 + 2\|re_1\|_{(q)}\|\omega\|_{(q)} - \|\omega\|_{(q)}^2 + 4\|re_1\|_{(q)}\big(\|re_1\|_{(q)} + \|\omega\|_{(q)}\big).
\end{aligned}$$

Since $\|e_1\|_{(q)} = 1$ and $0 \le r \le R_1$, we have

$$J(z) \le -\|\omega\|_{(q)}^2 + 6r\|\omega\|_{(q)} + 3r^2 \le -\|\omega\|_{(q)}^2 + 6R_1\|\omega\|_{(q)} + 3R_1^2.$$

Using the fact that $\|\omega\|_{(q)} = (3+2\sqrt{3})R_1$, we get $J(z) \le 0$.

(iii) Let $z = R_1(e_1,\widetilde{e_1}) \widetilde{+} R_1(\omega,-\widetilde{\omega}) = (R_1(e_1+\omega), R_1(\widetilde{e_1-\omega}))$ with $\|\omega\|_{(q)} \le 3+2\sqrt{3}$. Then, by $(A_8)$, we have

$$\begin{aligned}
J(z) &= R_1^2 \int_{\mathbb{R}^2} \Big(\nabla(e_1+\omega)\nabla u(\widetilde{e_1-\omega}) + V(x)(e_1+\omega)(\widetilde{e_1-\omega})\Big)\,dx \\
&\quad - \int_{\mathbb{R}^2} \frac{F\big(R_1(e_1+\omega)\big)}{|x|^b}\,dx - \int_{\mathbb{R}^2} \frac{G\big(R_1(\widetilde{e_1-\omega})\big)}{|x|^a}\,dx \\
&\le R_1^2\|\nabla(e_1+\omega)\|_{2,q}\|\nabla(\widetilde{e_1-\omega})\|_{2,p} + R_1^2\|V^{1/q}(e_1+\omega)\|_{2,q}\|V^{1/p}(\widetilde{e_1-\omega})\|_{2,p} \\
&\quad - C_{\theta,a,b}R_1^\theta \int_{\mathbb{R}^2} \frac{|e_1+\omega|^\theta}{|x|^b}\,dx - C_{\theta,a,b}R_1^\theta \int_{\mathbb{R}^2} \frac{|\widetilde{e_1-\omega}|^\theta}{|x|^a}\,dx \\
&\le R_1^2\|e_1+\omega\|_{(q)}\|\widetilde{e_1-\omega}\|_{(p)} + R_1^2\|e_1+\omega\|_{(q)}\|\widetilde{e_1-\omega}\|_{(p)} \\
&\quad - C_{\theta,a,b}R_1^\theta \int_{\mathbb{R}^2} \frac{|e_1+\omega|^\theta}{|x|^b}\,dx - C_{\theta,a,b}R_1^\theta \int_{\mathbb{R}^2} \frac{|\widetilde{e_1-\omega}|^\theta}{|x|^a}\,dx \\
&\le 2R_1^2\|e_1+\omega\|_{(q)}\|e_1-\omega\|_{(q)} - C_{\theta,a,b}R_1^\theta \inf_{\|\omega\|_{(q)}\le 3+2\sqrt{2}} \int_{\mathbb{R}^2} \Big(\frac{|e_1+\omega|^\theta}{|x|^b} + \frac{|\widetilde{e_1-\omega}|^\theta}{|x|^a}\Big)\,dx \\
&\le 2R_1^2(\|e_1\|_{(q)} + \|\omega\|_{(q)})^2 - C_{\theta,a,b}R_1^\theta \inf_{\|\omega\|_{(q)}\le 3+2\sqrt{2}} \int_{\mathbb{R}^2} \Big(\frac{|e_1+\omega|^\theta}{|x|^b} + \frac{|\widetilde{e_1-\omega}|^\theta}{|x|^a}\Big)\,dx \\
&\le (56+32\sqrt{3})R_1^2 - C_{\theta,a,b}R_1^\theta\delta_{\theta,a,b}.
\end{aligned}$$

Now, using (4.92) in the last inequality, we obtain $J(z) \le 0$.

■

### 4.5.2 Finite-dimensional approximation

Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of eigenfunctions for the operator $(-\Delta+V)$ in $H^1_V(\mathbb{R}^2) := \{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)u^2\,dx < \infty\}$. By Lemma 3 in Cassani and Tarsi (2015), the sequence $\{e_i\}_{i\in\mathbb{N}}$ provides also a dense system in $W^{(p)}$ as well as $W^{(q)}$. For each $n\in\mathbb{N}$, consider the following finite dimensional subspaces:

$$E_n^+ = Span\{(e_i,\widetilde{e}_i) : \; i=1,2,\dots,n\}, \quad E_n^- = Span\{(e_i,-\widetilde{e}_i) : \; i=1,2\dots,n\},$$

and

$$E_n = E_n^+ \oplus E_n^- .$$

Define

$$\Gamma_n = \{\gamma \in \mathscr{C}\big(Q_n, E_n^- \,\widetilde{\oplus}\, \mathbb{R}(e_1,\widetilde{e}_1)\big) : \; \gamma(z)=z, \text{ for all } z\in\partial Q_n\},$$

where $Q_n = Q\cap E_n$ and $Q$ as in Lemma 4.29, and set

$$c_n = \inf_{\gamma\in\Gamma}\max_{z\in Q_n} J(\gamma(z)). \tag{4.95}$$

Using Lemma 5.5 in Figueiredo, Ó and Ruf (2005), we have

$$\gamma(Q_n)\cap(\partial B_\rho\cap E_n^+)\neq\emptyset, \quad \text{for all} \quad \gamma\in\Gamma_n, \tag{4.96}$$

for $\rho$ given by Lemma 4.28. Thus, combining Lemma 4.28 with (4.96), we have

$$c_n \geq \sigma, \quad \text{for all} \quad n\geq 1. \tag{4.97}$$

Note also that, since the inclusion map $I_n : Q_n \to E_n^- \,\widetilde{\oplus}\, \mathbb{R}(e_1,\widetilde{e_1})$ belongs to $\Gamma_n$, we have, for $z = r(e_1,\widetilde{e}_1)+(u,-\widetilde{u}) \in Q_n$,

$$J(z) = r^2\|e_1\|^2_{(q)} - \|u\|^2_{(q)} - \int_{\mathbb{R}^2}\frac{F(re_1+u)}{|x|^b}\,dx - \int_{\mathbb{R}^2}\frac{G(\widetilde{re_1-u})}{|x|^a}\,dx \leq R_1^2. \tag{4.98}$$

Let denote $J_n$ the restriction of $J$ to the finite-dimensional space $E_n$. Then, applying Linking theorem (Theorem 3.6) for $J_n$ and noticing (4.97) and (4.98), we get the following result.

**Proposition 4.30.** For each $n\in\mathbb{N}$, the functional $J_n$ has a critical point at level $c_n$. More precisely, there is $z_n = (u_n,\widetilde{v}_n)\in E_n$ such that

$$J_n(z_n) = c_n \in [\sigma, R_1^2] \tag{4.99}$$

where $\sigma$ and $R_1$ are given by Lemma 4.28 and (4.92), respectively, and

$$J_n'(z_n)(\phi,\widetilde{\psi}) = 0, \quad \text{for all} \quad (\phi,\widetilde{\psi})\in E_n.$$

**Lemma 4.31.** Let $(u_n, \widetilde{v}_n)$ be the sequence given by Proposition 4.30. Moreover, assume that $(u_n, \widetilde{v}_n) \rightharpoonup (0,0)$ in $E$. Then, up to a subsequence

$$\|u_n\|_{(q)} \le M_2 \quad \text{or} \quad \|\widetilde{v}_n\|_{(p)} \le M_1, \quad \text{for all} \quad n \in \mathbb{N}.$$

for $M_1$ and $M_2$ given by (4.93).

***Proof.*** If $\|u_n\|_{(q)} \to 0$ or $\|\widetilde{v}_n\|_{(p)} \to 0$ the claim follows. Thus, we can assume that there exists a positive constant $d$ such that

$$\|u_n\|_{(q)} \ge d \quad \text{and} \quad \|\widetilde{v}_n\|_{(p)} \ge d, \quad \text{for all} \quad n \in \mathbb{N}. \tag{4.100}$$

Using the fact that $|J(u_n, \widetilde{v}_n)| \le R_1^2$ and

$$J'(u_n, \widetilde{v}_n)(\phi, \widetilde{\psi}) = 0, \quad \text{for all} \quad (\phi, \widetilde{\psi}) \in E_n, \tag{4.101}$$

and employing similar arguments as in to Lemma 4.18, we obtain

$$\left(1 - \frac{2}{\mu}\right) \int_{\mathbb{R}^2} \frac{f(u_n)u_n}{|x|^b}\, dx + \left(1 - \frac{2}{\nu}\right) \int_{\Omega} \frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\, dx \le 2R_1^2.$$

Then,

$$\int_{\mathbb{R}^2} \frac{f(u_n)u_n}{|x|^b}\, dx \le \frac{2\mu}{\mu - 2} R_1^2 \tag{4.102}$$

and

$$\int_{\mathbb{R}^2} \frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\, dx\, dx \le \frac{2\nu}{\nu - 2} R_1^2. \tag{4.103}$$

On the other hand, taking $(\phi, \widetilde{\psi}) = (v_n, 0)$ in (4.101), we get

$$\|v_n\|_{(q)} \le \int_{\mathbb{R}^2} \frac{f(u_n)T_n}{|x|^b}\, dx, \tag{4.104}$$

where

$$T_n = \frac{v_n}{\|v_n\|_{(q)}}.$$

Define

$$\xi := \min\left\{ \frac{\varepsilon\alpha_0(1 - b/2)(4\pi)^{\frac{p}{2}}}{\alpha_0 + (1 - b/2)(4\pi)^{\frac{p}{2}} + \varepsilon\alpha_0}, \frac{\varepsilon\beta_0(1 - a/2)(4\pi)^{\frac{q}{2}}}{\beta_0 + (1 - a/2)(4\pi)^{\frac{q}{2}} + \varepsilon\beta_0} \right\} \tag{4.105}$$

where $\varepsilon > 0$ is given by (4.92). Consider $\alpha_1 = \alpha_0 + \xi$ and $\alpha_2 = (1 - b/2)(4\pi)^{\frac{p}{2}} - \xi$. By $(A_1)$ and $(A_4)$, there exists $\lambda > 0$ such that

$$|f(s)| \le \lambda e^{\alpha_1 |s|^p}, \quad \text{for all} \quad s \in \mathbb{R}. \tag{4.106}$$

Applying Lemma 3.10 in (4.104) with $s = |f(u_n(x))|/\lambda$, $t = \alpha_2^{1/p}|T_n(x)|$, $r = p$ and $r' = q$, we obtain

$$\|v_n\|_{(q)} \le \frac{\lambda}{\alpha_2^{1/p}} \left[ \int_{\mathbb{R}^2} \frac{(e^{\alpha_2 |T_n|^p} - 1)}{|x|^b}\, dx + \frac{1}{q\lambda^q} \int_{\{x \in \mathbb{R}^2 : |\frac{f(u_n)}{\lambda}| \le e^{1/p^q}\}} \frac{|f(u_n)|^q}{|x|^b}\, dx \right. \tag{4.107}$$

$$\left. + \frac{1}{\lambda} \int_{\{x \in \mathbb{R}^2 : |\frac{f(u_n)}{\lambda}| \ge e^{1/p^q}\}} \frac{|f(u_n)|}{|x|^b} \ln^{1/p} \frac{|f(u_n)|}{\lambda}\, dx \right].$$

By (4.106), we have

$$\int_{\{x\in\mathbb{R}^2:|\frac{f(u_n)}{\lambda}|\geq e^{1/p^q}\}}\frac{|f(u_n)|}{|x|^b}\ln^{1/p}\frac{|f(u_n)|}{\lambda}\,dx\leq\alpha_1^{1/p}\int_{\mathbb{R}^2}\frac{f(u_n)u_n}{|x|^b}\,dx.$$

Since $\|T_n\|_{(q)}=1$, $T_n\rightharpoonup 0$ in $W^{(q)}$ and $\alpha_2<(1-b/2)(4\pi)^{\frac{p}{2}}$, by Lemma 4.17 we can suppose that

$$\int_{\mathbb{R}^2}\frac{\left(e^{\alpha_2|T_n|^p}-1\right)}{|x|^b}\,dx=o_n(1).$$

Similar to the integral given in (4.82), we obtain

$$\int_{\{x\in\mathbb{R}^2:|\frac{f(u_n)}{\lambda}|\leq e^{1/p^q}\}}\frac{|f(u_n)|^q}{|x|^b}\,dx=o_n(1)$$

Using these estimates in (4.107), we have

$$\|v_n\|_{(q)}\leq\frac{\alpha_1^{1/p}}{\alpha_2^{1/p}}\int_{\mathbb{R}^2}\frac{f(u_n)u_n}{|x|^b}\,dx+o_n(1).$$

From (4.102), we get

$$\|v_n\|_{(q)}\leq\left(\frac{\alpha_o+\xi}{(1-b/2)(4\pi)^{\frac{p}{2}}-\xi}\right)^{1/p}\frac{2\mu}{\mu-2}R_1^2+o_n(1). \tag{4.108}$$

Similarly, we have

$$\|\widetilde{u}_n\|_{(p)}\leq\left(\frac{\beta_o+\xi}{(1-a/2)(4\pi)^{\frac{q}{2}}-\xi}\right)^{1/q}\frac{2\nu}{\nu-2}R_1^2+o_n(1). \tag{4.109}$$

Now, if we have in (4.94)

$$R_1^2=\frac{M_1(1-b/2)^{1/p}(4\pi)^{1/2}}{(1+\varepsilon)\alpha_0^{1/p}}\frac{(\mu-2)}{2\mu}$$

and replacing in (4.108), we get

$$\begin{aligned}\|v_n\|_{(q)}&\leq\frac{M_1}{(1+\varepsilon)}\frac{(1-b/2)^{1/p}(4\pi)^{1/2}}{\alpha_o^{1/p}}\left(\frac{\alpha_o+\xi}{(1-b/2)(4\pi)^{\frac{p}{2}}-\xi}\right)^{1/p}+o_n(1)\\&=\frac{M_1}{(1+\varepsilon)}\left(\frac{\alpha_0+\xi}{\alpha_0}\frac{(1-b/2)(4\pi)^{p/2}}{(1-b/2)(4\pi)^{\frac{p}{2}}-\xi}\right)^{1/p}+o_n(1).\end{aligned}$$

From (4.105), we have

$$0<\frac{\alpha_0+\xi}{\alpha_0}\frac{(1-b/2)(4\pi)^{p/2}}{(1-b/2)(4\pi)^{\frac{p}{2}}-\xi}\leq 1+\varepsilon.$$

Thus,

$$\|v_n\|_{(q)}\leq\frac{M_1}{(1+\varepsilon)}(1+\varepsilon)^{1/p}+o_n(1)\leq\frac{M_1}{(1+\varepsilon)^{1/q}}+o_n(1).$$

We can assume without loss of generality that

$$\|\widetilde{v}_n\|_{(p)} = \|v_n\|_{(q)} \leq M_1, \quad \text{for all} \quad n \in \mathbb{N},$$

On the other hand, if we have in (4.94)

$$R_1^2 = \frac{M_2(1-a/2)^{1/p}(4\pi)^{1/2}}{(1+\varepsilon)\beta_0^{1/q}} \frac{(\nu-2)}{2\nu},$$

and replacing in (4.109), we can assume that

$$\|u_n\|_{(q)} \leq M_2, \quad \text{for all} \quad n \in \mathbb{N},$$

this complete the proof. ■

### 4.5.3 Proof of Theorem 4.7

***Proof.*** By Proposition 4.30, there exists a sequence $(u_n, \widetilde{v}_n) \subset E_n$ such that

$$J_n(u_n, \widetilde{v}_n) = c_n \in [\sigma, R_1^2] \tag{4.110}$$

and

$$J_n'(u_n, \widetilde{v}_n)(\phi, \widetilde{\psi}) = 0, \quad \text{for all} \quad (\phi, \widetilde{\psi}) \in E_n. \tag{4.111}$$

From Lemma 4.18, we have that the sequence $(u_n, \widetilde{v}_n)$ is bounded in $E$. Thus, without loss of generality, we can assume that there exists $(u, \widetilde{v}) \in E$ such that $(u_n, \widetilde{v}_n) \rightharpoonup (u, \widetilde{v})$ in $E$ and

$$u_n \to u \quad \text{and} \quad \widetilde{v}_n \to \widetilde{v} \quad \text{in} \quad L^r(\mathbb{R}^2), \quad \text{for all} \quad r \geq 1.$$

Taking $(0, \widetilde{\psi})$ and $(\phi, 0)$ in (4.111) with $(\phi, \widetilde{\psi}) \in E_n \cap \left(\mathscr{C}_0^\infty(\mathbb{R}^2) \times \mathscr{C}_0^\infty(\mathbb{R}^2)\right)$, we obtain

$$\int_{\mathbb{R}^2} \left(\nabla u_n \nabla \widetilde{\psi} + V(x) u_n \widetilde{\psi}\right) dx = \int_{\mathbb{R}^2} \frac{g(\widetilde{v}_n)\widetilde{\psi}}{|x|^a} dx \tag{4.112}$$

and

$$\int_{\mathbb{R}^2} \left(\nabla \widetilde{v}_n \nabla \phi + V(x) \widetilde{v}_n \phi\right) dx = \int_{\mathbb{R}^2} \frac{f(u_n)\widetilde{\phi}}{|x|^b} dx. \tag{4.113}$$

Taking limits in (4.112) and (4.113), as $n \to +\infty$, by Lemma 4.20 and the fact that $\bigcup_{n=1}^{+\infty} E_n \cap \left(\mathscr{C}_0^\infty(\mathbb{R}^2) \times \mathscr{C}_0^\infty(\mathbb{R}^2)\right)$ is dense in $E$, we obtain

$$\int_{\mathbb{R}^2} \left(\nabla u \nabla \widetilde{\psi} + V(x) u \widetilde{\psi}\right) dx = \int_{\mathbb{R}^2} \frac{g(\widetilde{v})\widetilde{\psi}}{|x|^a} dx, \quad \text{for all} \quad \psi \in W^{(q)}$$

and

$$\int_{\mathbb{R}^2} \left(\nabla \widetilde{v} \nabla \phi + V(x) \widetilde{v} \phi\right) dx = \int_{\mathbb{R}^2} \frac{f(u)\widetilde{\phi}}{|x|^b} dx, \quad \text{for all} \quad \phi \in W^{(q)}.$$

Thus, $(u, \widetilde{v}) \in E$ is a weak solution of (4.1). It remains to prove that $(u, \widetilde{v})$ is a nontrivial weak solution. Assume, by contradiction, that $u \equiv 0$ (which implies that $\widetilde{v} \equiv 0$). Thus, we can assume that

$$u_n \to 0 \quad \text{and} \quad \widetilde{v}_n \to 0 \quad \text{in} \quad L^r(\mathbb{R}^2), \quad \text{for all} \quad r \geq 1. \tag{4.114}$$

From Lemma 4.31, we can assume without loss of generality that

$$\|u_n\|_{(q)} \leq M_2 = \frac{2m_0\pi^{1/2}(1-b/2)^{1/p}}{\alpha_0^{1/p}}, \quad \text{for all} \quad n \in \mathbb{N}.$$

Note that, $\alpha_0 M_2^p/\alpha_q^* + b/2 < 1$. Chosen $r_1 > 1$ such that $\alpha_0 r_1 M_2^p/\alpha_q^* + b/2 < 1$. From $(A_1)$ and $(A_4)$, there exists $C > 0$ such that

$$|f(s)| \leq |s| + C|s|\left(e^{r_1\alpha_0|s|^p} - 1\right), \quad \text{for all} \quad s \in \mathbb{R}. \tag{4.115}$$

Using (4.115), Hölder's inequality with $r_2 > 1$ such that $\alpha_0 r_1 r_2 M_2^p/\alpha_q^* + r_2 b/2 < 1$ and Lemma 4.14, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{f(u_n)u_n}{|x|^b}\,dx &\leq \int_{\mathbb{R}^2} \frac{|u_n|^2}{|x|^b}\,dx + C\int_{\mathbb{R}^2} |u_n|^2 \frac{\left(e^{r_1\alpha_0|u_n|^p} - 1\right)}{|x|^b}\,dx \\
&\leq \left(\int_{B_1} \frac{1}{|x|^{r_2 b}}\,dx\right)^{1/r_2} \|u_n\|_{2r_2'}^2 + \|u_n\|_2^2 + C\|u_n\|_{2r_2'}^2 \left(\int_{\mathbb{R}^2} \frac{\left(e^{\alpha_0 r_1 r_2|u_n|^p} - 1\right)}{|x|^{r_2 b}}\,dx\right)^{1/r_2} \\
&\leq C_1\|u_n\|_{2r_2'}^2 + \|u_n\|_2^2 + C_1\|u_n\|_{2r_2'}^2,
\end{aligned}$$

where in the last inequality we have used Proposition 4.8. From (4.114), we get

$$\int_{\mathbb{R}^2} \frac{f(u_n)u_n}{|x|^b} \to 0.$$

Taking $(0, \widetilde{v}_n)$ and $(u_n, 0)$ in (4.111), we have

$$\int_{\mathbb{R}^2} \left(\nabla u_n \nabla \widetilde{v}_n + V(x)u_n\widetilde{v}_n\right) dx = \int_{\mathbb{R}^2} \frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\,dx = \int_{\mathbb{R}^2} \frac{f(u_n)u_n}{|x|^b}\,dx.$$

Thus,

$$\int_{\mathbb{R}^2} \left(\nabla u_n \nabla \widetilde{v}_n + V(x)u_n\widetilde{v}_n\right) dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{g(\widetilde{v}_n)\widetilde{v}_n}{|x|^a}\,dx \to 0.$$

By $(A_2)$, we obtain

$$\int_{\mathbb{R}^2} \frac{F(u_n)}{|x|^b}\,dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{G(\widetilde{v}_n)}{|x|^a}\,dx \to 0.$$

Finally, we conclude that

$$J(u_n, \widetilde{v}_n) = \int_{\mathbb{R}^2} \left(\nabla u_n \nabla \widetilde{v}_n + V(x)u_n\widetilde{v}_n\right) dx - \int_{\mathbb{R}^2} \frac{F(u_n)}{|x|^b}\,dx - \int_{\mathbb{R}^2} \frac{G(\widetilde{v}_n)}{|x|^a}\,dx \to 0,$$

which gives a contradiction with (4.110). Thus $(u, \widetilde{v})$ is a nontrivial weak solution. ■

CHAPTER

5

# HAMILTONIAN SYSTEMS WITH CRITICAL EXPONENTIAL GROWTH AND COERCIVE POTENTIALS

In this chapter we discuss the existence of nontrivial solutions for the Hamiltonian system

$$\begin{cases} -\Delta v + V(x)v = Q_1(x)f(u), & x \in \mathbb{R}^2, \\ -\Delta u + V(x)u = Q_2(x)g(v), & x \in \mathbb{R}^2, \end{cases} \tag{5.1}$$

where $V, Q_1, Q_2$ are continuous functions and the nonlinearities $f$ and $g$ possess critical exponential growth.

## 5.1 Introduction and main result

Since we are interested in find solutions with $p,q$ lying on the exponential critical hyperbola, we consider $p > 1$ and $q = p/(p-1)$. We make the following assumption on $(V)$:

$(V)$ $V \in \mathscr{C}(\mathbb{R}^2, \mathbb{R})$, $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^2$ and there exists $a \geq 0$ such that

$$\liminf_{|x| \to \infty} \frac{V(x)}{|x|^a} > 0.$$

For $i = 1,2$ we assume

$(Q_i)$ $Q_i \in \mathscr{C}(\mathbb{R}^2 \backslash \{0\}, \mathbb{R})$ , $Q_i(x) > 0$ for $x \neq 0$ and there exists $d_i < \dfrac{a}{\max\{p,q\}-1} - 1$ and $b_i > -2$ such that

$$0 < \lim_{|x| \to 0} \frac{Q_i(x)}{|x|^{b_i}} < +\infty \quad \text{and} \quad \limsup_{|x| \to \infty} \frac{Q_i(x)}{|x|^{d_i}} < +\infty.$$

Concerning the functions $f$ and $g$, we suppose the following assumptions:

$(B_1)$ $f,g\in\mathscr{C}(\mathbb{R})$, $f(s)=o(s^{\eta_1})$ and $g(s)=o(s^{\eta_2})$, as $s\to 0$, where $\eta_1=\max\{1/(q-1),\min\{p,q\}\}$ and $\eta_2=\max\{1/(p-1),\min\{p,q\}\}$.

$(B_2)$ There exist constants $\mu>2$ and $\nu>2$ such that

$$0<\mu F(s)\le sf(s) \quad \text{and} \quad 0<\nu G(s)\le sg(s), \quad \text{for all} \quad s\neq 0,$$

where $F(s)=\int_0^s f(t)\,dt$ and $G(s)=\int_0^s g(t)\,dt$.

$(B_3)$ There exist positive constants $M$ and $s_0$ such that

$$0<F(s)\le M|f(s)| \quad \text{and} \quad 0<G(s)\le M|g(s)|, \quad \text{for all} \quad |s|>s_0.$$

$(B_4)$ There exists $\alpha_0>0$ such that

$$\lim_{|s|\to\infty}\frac{|f(s)|}{e^{\alpha|s|^p}}=\begin{cases}+\infty, & \alpha<\alpha_0\\ 0, & \alpha>\alpha_0.\end{cases}$$

$(B_5)$ There exists $\beta_0>0$ such that

$$\lim_{|s|\to\infty}\frac{|g(s)|}{e^{\beta|s|^q}}=\begin{cases}+\infty, & \beta<\beta_0\\ 0, & \beta>\beta_0.\end{cases}$$

$(B_6)$ The following limits holds

$$\lim_{|s|\to+\infty}\frac{sf(s)}{e^{\alpha_0|s|^p}}=+\infty \quad \text{and} \quad \lim_{|s|\to+\infty}\frac{sg(s)}{e^{\beta_0|s|^q}}=+\infty.$$

$(B_7)$ For $b_i$ given by $(Q_i)$, $i=1,2$ and $\alpha_0$, $\beta_0$ given by $(B_4)$ and $(B_5)$, respectively, then

$$\Big(\frac{\alpha_0\min\{1,1+\frac{b_1}{2}\}}{(1+\frac{b_1}{2})^2}\Big)^{1/p}>\Big(\frac{\beta_0}{\min\{1,1+\frac{b_2}{2}\}}\Big)^{1/q}$$

or

$$\Big(\frac{\beta_0\min\{1,1+\frac{b_2}{2}\}}{(1+\frac{b_2}{2})^2}\Big)^{1/q}>\Big(\frac{\alpha_0}{\min\{1,1+\frac{b_1}{2}\}}\Big)^{1/p}.$$

**Remark 5.1.** In $(B_1)$, we have $\eta_1>1$ and $\eta_2>0$, this imply that $f(s)=g(s)=o(s)$, as $s\to 0$.

Next we state the main result of this chapter.

**Theorem 5.2.** Suppose that $V$ satisfies $(V)$, $Q_i$ satisfy $(Q_i)$ for $i=1,2$ and the nonlinearities $f$ and $g$ satisfy $(B_1)-(B_7)$ . Then, (5.1) possesses a nontrivial weak solution.

## 5.2 Preliminaries

We define the following Lorentz-Sobolev space $W^1L_V^{2,s}(\mathbb{R}^2)$ as the closure of the set

$$\{u \in \mathscr{C}_0^\infty(\mathbb{R}^2) : \|\nabla u\|_{2,s}^s + \|V^{1/s}u\|_{2,s}^s < +\infty\},$$

with respect to the quasinorm

$$\|u\|_{W^1L_V^{2,s}(\mathbb{R}^2)} := \left(\|\nabla u\|_{2,s}^s + \|V^{1/s}u\|_{2,s}^s\right)^{1/s}.$$

We denote the space $W^1L_V^{2,s}(\mathbb{R}^2)$ as $E^{(s)}$ and the quasinorm $\|\cdot\|_{W^1L_V^{2,s}(\mathbb{R}^2)}$ as $\|\cdot\|_{(s)}$.

**Lemma 5.3.** (See Cassani and Tarsi (2009).) If $u \in L^{2,s}(\mathbb{R}^2)$. Then,

$$|u^*(r)| \leq \left(\frac{s}{2}\right)^{1/s}\frac{\|u\|_{2,s}}{r^{1/2}}, \quad \text{for all} \quad r > 0.$$

By Lemma 5.3, for all $u \in W^1L_V^{2,s}(\mathbb{R}^2)$, we have

$$\left(V^{1/s}u\right)^*(r) \leq \left(\frac{s}{2}\right)^{1/s}\frac{\|u\|_{(s)}}{r^{1/2}}, \quad \text{for all} \quad r > 0. \tag{5.2}$$

Let $A$ be a measurable set in $\mathbb{R}^2$. We denote

$$W^1L_V^{2,s}(A) := \{u_{|A} : u \in W^1L_V^{2,s}(\mathbb{R}^2)\},$$

for each $\lambda \geq 1$ and $i = 1, 2$, we set

$$L^\lambda(A, Q_i) := \{u \ : \int_A Q_i(x)|u|^\lambda\,dx < +\infty\}$$

endowed with the norm

$$\|u\|_{L^\lambda(A,Q_i)} = \left[\int_A Q_i(x)|u|^\lambda\,dx\right]^{1/\lambda}.$$

In particular, we denote $L^\lambda(A,1) := L^\lambda(A)$, $\|u\|_{L^\lambda(A)} = \|u\|_{L^\lambda(A,1)}$ and $\|u\|_\lambda = \|u\|_{L^\lambda(\mathbb{R}^2)}$.

**Lemma 5.4.** Let $1 \leq \lambda < +\infty$, $s > 1$ and $0 < r < R < +\infty$. For $i = 1, 2$, the embedding $W^1L_V^{2,s}(B_R\backslash B_r) \hookrightarrow L^\lambda(B_R\backslash B_r, Q_i)$ is compact.

***Proof.*** We observe that, there exist $D_1 > 0$ and $D_2 > 0$ such that $D_1 \leq V(x) \leq D_2$ for all $x \in B_R\backslash B_r$. Thus, the quasinorms of $W^1L_V^{2,s}(B_R\backslash B_r)$ and $W^1L^{2,s}(B_R\backslash B_r)$ are equivalents. Moreover, arguing as in the proof of Lemma 2.38, the space $W^1L^{2,s}(B_R\backslash B_r)$ is compactly embedded in $L^\lambda(B_R\backslash B_r)$ for all $\lambda \geq 1$. Thus,

$$W^1L_V^{2,s}(B_R\backslash B_r) \hookrightarrow L^\lambda(B_R\backslash B_r), \quad \text{for all} \quad \lambda \geq 1.$$

Since $Q_i$ is continuous on $\mathbb{R}^2\backslash\{0\}$ there exist $D_3 > 0$ and $D_4 > 0$ such that $D_3 \leq Q_i(x) \leq D_4$ for all $x \in B_R\backslash B_r$, $i = 1, 2$. Thus,

$$L^\lambda(B_R\backslash B_r) \hookrightarrow L^\lambda(B_R\backslash B_r, Q_i), \quad \text{for all} \quad \lambda \geq 1,$$

and the proof follows. ∎

**Lemma 5.5.** Let $1 \leq \lambda < +\infty$, $s > 1$ and $R > 0$. Then, the embedding $W^1L_V^{2,s}(B_R) \hookrightarrow L^\lambda(B_R)$ is continuous.

***Proof.*** Since $V(x) \geq V_0$ for all $x \in \mathbb{R}^2$, as in the proof of Proposition 2.48, the space $W^1L_V^{2,s}(B_R)$ is continuously embedded in $W^1L^{2,s}(B_R)$. Arguing as in Lemma 2.38, the proof follows. ■

From conditions $(V)$ and $(Q_i)$ for $i = 1,2$ there exist positive constants $C_1, C_2, C_3, C_4, R_0$ and $r_0$ such that

$$C_1|x|^a \leq V(x), \quad \text{for all} \quad |x| \geq R_0, \tag{5.3}$$

$$Q_i(x) \leq C_2|x|^{d_i}, \quad \text{for all} \quad |x| \geq R_0, \quad i = 1,2 \tag{5.4}$$

and

$$C_3|x|^{b_i} \leq Q_i(x) \leq C_4|x|^{b_i}, \quad \text{for all} \quad 0 < |x| < r_0, \quad i = 1,2. \tag{5.5}$$

**Proposition 5.6.** Assume $(V)$ and $(Q_i)$ for $i = 1,2$ and let $s = q$ or $s = p$. Then, the following embeddings

$$W^1L_V^{2,s}(\mathbb{R}^2) \hookrightarrow L^\lambda(\mathbb{R}^2, Q_i), \quad \text{for all} \quad \lambda \geq \min\{p,q\}$$

are compact.

***Proof.*** We prove for $s = q$ and $Q_1$, without loss of generality, we consider the case where $q \geq 2$. This implies that $\min\{p,q\} = p = q/(q-1)$. In order to prove the continuity of the embedding, it is sufficient to show that

$$S_\lambda := \inf_{\substack{u \neq 0 \\ u \in E^{(q)}}} \frac{\|u\|_{(q)}}{\|u\|_{L^\lambda(\mathbb{R}^2,Q_1)}} = \inf_{\substack{\|u\|_{L^\lambda(\mathbb{R}^2,Q_1)}=1 \\ u \in E^{(q)}}} \|u\|_{(q)} > 0.$$

In fact, on the contrary, there exists a sequence $(u_n)$ such that

$$\|u_n\|_{L^\lambda(\mathbb{R}^2,Q_1)} = 1 \quad \text{and} \quad \|u_n\|_{(q)} = o_n(1). \tag{5.6}$$

Thus,

$$\int_{\mathbb{R}^2} Q_1(x)|u_n|^\lambda\,dx = \int_{|x|\leq r} Q_1(x)|u_n|^\lambda\,dx + \int_{r\leq|x|\leq R} Q_1(x)|u_n|^\lambda\,dx = \int_{|x|\geq R} Q(x)|u_n|^\lambda\,dx \tag{5.7}$$

where $R > R_0$ and $0 < r < r_0$ will be determined later on. Using (5.4), we have

$$\begin{aligned}
\int_{|x|\geq R} Q_1(x)|u_n|^\lambda\,dx &\leq C_2 \int_{|x|\geq R} |x|^{d_1}|u_n|^\lambda\,dx \\
&= C_2C_1^{-q/\lambda} \int_{|x|\geq R} |x|^{d_1-\frac{a\lambda}{q}} \left((C_1|x|^a)^{1/q}|u_n|\right)^\lambda dx \\
&\leq C_2C_1^{-q/\lambda} \int_{|x|\geq R} |x|^{d_1-\frac{a\lambda}{q}} \left(V^{1/q}|u_n|\right)^\lambda dx
\end{aligned}$$

Using Hardy-Littlewood inequality, we have

$$\int_{|x|\geq R} Q_1(x)|u_n|^\lambda\, dx \leq C_1^{-q/\lambda} C_2 \int_0^{+\infty} \Big(\frac{\chi_{\{|x|\geq R\}}}{|x|^{\frac{a\lambda}{q}-d_1}}\Big)^*(r)\Big(\big(V^{1/q}|u_n|\big)^\lambda\Big)^*(r)\, dr.$$

Since $d_1+1<\frac{a}{q-1}$ and $\frac{q}{q-1}\leq\lambda$, then, $d_1+1<\frac{a\lambda}{q}$. Moreover, a direct calculation shows that

$$\Big(\frac{\chi_{\{|x|\geq R\}}}{|x|^\beta}\Big)^* = \Big(\frac{1}{R^2+r/\pi}\Big)^{\beta/2}, \quad \text{for all} \quad \beta>0$$

Using Lemma 5.3, we obtain

$$\begin{aligned}\int_{|x|\geq R} Q_1(x)|u_n|^\lambda\, dx &\leq C_1^{-q/\lambda} C_2 \pi \Big(\frac{q}{2}\Big)^{1/q} \int_0^{+\infty} \Big(\frac{1}{R^2+r/\pi}\Big)^{\frac{a\lambda}{2q}-\frac{d_1}{2}} \frac{\|(V^{1/q}u_n)^\lambda\|_{2,q}}{r^{1/2}}\, dr \\ &\leq C_1^{-q/\lambda} C_2 \pi \Big(\frac{q}{2}\Big)^{1/q} \|u_n\|_{(q)}^\lambda \int_0^{+\infty} \Big(\frac{1}{R^2+r/\pi}\Big)^{\frac{a\lambda}{2q}-\frac{d_1}{2}} \frac{1}{r^{1/2}}\, dr.\end{aligned}$$

Note that $\frac{1}{2}+\delta=\frac{a\lambda}{2q}-\frac{d_1}{2}$ for some $\delta>0$, Then, for some constant $C_{R,q}$ depending on $R$ and $q$ we have

$$\int_{|x|\geq R} Q_1(x)|u_n|^\lambda\, dx \leq C\Big(\frac{q}{2}\Big)^{1/q} \|u_n\|_{(q)}^\lambda \int_0^{+\infty}\Big(\frac{1}{R^2+r/\pi}\Big)^{\frac{1}{2}+\delta} \frac{1}{r^{1/2}}\, dr = C_{R,q}\|u_n\|_{(q)}^\lambda = o_n(1). \tag{5.8}$$

Now, we estimate $\int_{|x|\leq r} Q(x)|u_n|^\lambda\, dx$ with $0<r<\min\{r_0,1\}$. We consider two cases:

**Case 1: $b_1>0$.** Using (5.5) and Lemma 5.5, we have

$$\begin{aligned}\int_{|x|\leq r} Q_1(x)|u_n|^\lambda\, dx &\leq C_4\int_{|x|\leq r}|x|^{b_1}|u_n|^\lambda\, dx \\ &\leq C_4 r^{b_1}\int_{|x|\leq 1}|u_n|^\lambda\, dx \\ &\leq C_4 r^{b_1}\|u_n\|_{(q)}^\lambda \\ &= o_n(1).\end{aligned}$$

**Case 2: $-2<b_1\leq 0$.** By (5.5), we have

$$\int_{|x|\leq r} Q_1(x)|u_n|^\lambda\, dx \leq C_4\int_{|x|\leq r}|x|^{b_1}|u_n|^\lambda\, dx$$

Taking $\delta=\delta(b_1)>0$ such that $0<\bar{b}_1=\delta-b_1<2$, we can write

$$\int_{|x|\leq r} Q_1(x)|u_n|^\lambda\, dx \leq C_4\int_{|x|\leq r}|x|^{b_1}|u_n|^\lambda\, dx \leq C_4\int_{|x|\leq r}\frac{|u_n|^\lambda}{|x|^{\bar{b}_1}}\, dx$$

Taking $\theta > 1$ such that $0 < \bar{b}_1\theta < 2$ and using Hölder's inequality, we find

$$\begin{aligned}\int_{|x|\le r} Q_1(x)|u_n|^\lambda\,dx &\le C_4\Big(\int_{|x|\le r}\frac{dx}{|x|^{\bar{b}_1\theta}}\Big)^{1/\theta}\Big(\int_{|x|\le r}|u_n|^{\lambda\theta'}\,dx\Big)^{1/\theta'}\\ &\le C_4\Big(\frac{2\pi r^{2-\bar{b}_1\theta}}{2-\bar{b}_1\theta}\Big)^{1/\theta}\|u_n\|^\lambda_{L^{\lambda\theta'}(B_r)}\\ &\le C_4\Big(\frac{2\pi r^{2-\bar{b}_1\theta}}{2-\bar{b}_1\theta}\Big)^{1/\theta}\|u_n\|^\lambda_{(q)}\\ &\le C_4\Big(\frac{2\pi}{2-\bar{b}_1\theta}\Big)^{1/\theta}\|u_n\|^\lambda_{(q)},\end{aligned}$$

where we have used Lemma 5.5 and the fact that $0 < r \le 1$. From (5.6), we have

$$\int_{|x|\le r} Q_1(x)|u_n|^\lambda\,dx = o_n(1). \tag{5.9}$$

Thus, from (5.8) and (5.9) in (5.7), we get

$$\int_{\mathbb{R}^2} Q_1(x)|u_n|^\lambda\,dx = \int_{r\le|x|\le R} Q_1(x)|u_n|^\lambda\,dx + o_n(1).$$

Using Lemma 5.4 and (5.6), for a subsequence (not renamed), we obtain

$$\int_{\mathbb{R}^2} Q_1(x)|u_n|^\lambda\,dx \to 0, \quad \text{as} \quad n\to+\infty$$

which contradicts (5.6). Now, we prove the compactness. Let $(u_n)$ be a sequence in $W^1L_V^{2,q}(\mathbb{R}^2)$ such that $u_n \rightharpoonup 0$ weakly in $W^1L_V^{2,q}(\mathbb{R}^2)$. Then, there exists $C_0 > 0$ such that $\|u_n\|_{(q)} \le C_0$. From (5.8), we have

$$\int_{|x|\ge R} Q_1(x)|u_n|^\lambda\,dx \le C_1^{-q/\lambda}C_2\pi\Big(\frac{q}{2}\Big)^{1/q}\|u_n\|^\lambda_{(q)}\int_0^\infty\Big(\frac{1}{R^2+r/\pi}\Big)^{\frac{1}{2}+\delta}\frac{1}{r^{1/2}}\,dr = \frac{C}{R^{2\delta}}.$$

We observe there exists $C > 0$ such that

$$\int_0^\infty\Big(\frac{1}{R^2+r/\pi}\Big)^{\frac{1}{2}+\delta}\frac{1}{r^{1/2}}\,dr = \frac{\sqrt{\pi}}{R^{2\delta}}\int_0^\infty\Big(\frac{1}{1+z}\Big)^{\frac{1}{2}+\delta}\frac{1}{z^{1/2}}\,dz = \frac{C}{R^{2\delta}}.$$

Thus, for given $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{|x|\ge R} Q_1(x)|u_n|^\lambda\,dx \le \frac{\varepsilon}{3}. \tag{5.10}$$

On the other hand, from the proofs of cases 1 and 2, there exits $C > 0$ such that

$$\int_{|x|\le r} Q_1(x)|u_n|^\lambda\,dx \le Cr^{b_1}, \quad \text{if} \quad b_1 > 0$$

and

$$\int_{|x|\le r} Q_1(x)|u_n|^\lambda\,dx \le Cr^{\frac{2-\bar{b}_1\theta}{\theta}}, \quad \text{if} \quad -2 < b_1 \le 0.$$

where $\bar{b}_1\theta < 2$. Thus, we can find $r > 0$ sufficiently small such that

$$\int_{|x|\leq r} Q_1(x)|u_n|^\lambda\, dx \leq \frac{\varepsilon}{3}, \tag{5.11}$$

Moreover, by Lemma 5.4, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{r\leq|x|\leq R} Q_1(x)|u_n|^\lambda\, dx \leq \frac{\varepsilon}{3}, \quad \text{for all} \quad n \geq n_0 \tag{5.12}$$

From (5.10), (5.11) and (5.12), we have

$$\int_{\mathbb{R}^2} Q_1(x)|u_n|^\lambda\, dx \leq \varepsilon, \quad \text{for all} \quad n \geq n_0,$$

and the proof is complete. ■

### 5.2.1 A Trudinger-Moser type inequality

In the following, we present a Trudinger-Moser type inequality suitable for the spaces introduced in this chapter.

**Lemma 5.7.** (See Lam and Lu (2012).) Let $0 \leq \lambda < 1$, $1 < s < \infty$ and $a(t,r)$ be a nonnegative measurable function on $\mathbb{R}\times[0,\infty)$ such that

$$a(t,r) \leq 1, \qquad 0 < t < r$$

and

$$\sup_{r>0}\left(\left[\int_{-\infty}^{0} + \int_{r}^{+\infty}\right](a(t,r))^{s/s-1}\, dt\right)^{(s-1)/s} := \gamma < +\infty$$

Then, there exists $C = C(\gamma, s)$ such that for every nonnegative function $\psi$ satisfying

$$\int_{-\infty}^{+\infty} \psi(t)^s\, dt \leq 1,$$

we have

$$\int_{0}^{+\infty} e^{-\Phi(r)}\, dr \leq C,$$

where

$$\Phi(r) = \lambda r - \lambda\left(\int_{-\infty}^{+\infty} \psi(t)a(t,r)\, dt\right)^{s/(s-1)}.$$

In the next result, we follow Lu and Tang (2016) to prove a version of singular Trudinger-Moser inequality for $E^{(s)}$.

**Proposition 5.8.** Let $s = p$ or $s = q$ and $i \in \{1,2\}$. Then, there exists $C = C(s, Q_i, V, \alpha) > 0$ such that

$$\sup_{\|u\|_{(s)}\leq 1} \int_{\mathbb{R}^2} Q_i(x)\left(e^{\alpha|u|^{s/(s-1)}} - 1\right) dx \leq C, \quad 0 < \alpha < \alpha^*_{s,b_i},$$

where

$$\alpha^*_{s,b_i} = \min\left\{(\sqrt{4\pi})^{s/(s-1)}, (\sqrt{4\pi})^{s/(s-1)}\left(1 + \frac{b_i}{2}\right)\right\}.$$

***Proof.*** Consider $u \in E^{(s)}$ with $\|u\|_{(s)} \leq 1$. Take

$$I = \int_{\mathbb{R}^2} Q_i(x)\left(e^{\alpha |u|^{s/(s-1)}} - 1\right) dx = I_1 + I_2, \tag{5.13}$$

where

$$I_1 = \int_{|x| \leq R} Q_i(x)\left(e^{\alpha |u|^{s/(s-1)}} - 1\right) dx \quad \text{and} \quad I_2 = \int_{|x| \geq R} Q_i(x)\left(e^{\alpha |u|^{s/(s-1)}} - 1\right) dx,$$

for $R > \sqrt{2\pi} R_0^2$. From (5.3) and (5.4), we have

$$\begin{aligned} I_2 = \int_{|x|\geq R} Q_i(x)\left(e^{\alpha |u|^{s/(s-1)}} - 1\right) dx &\leq C_2 \int_{|x|\geq R} |x|^{d_i}\left(e^{\alpha |u|^{s/(s-1)}} - 1\right) dx \\ &= C_2 \int_{|x|\geq R} |x|^{d_i} \sum_{j=1}^{+\infty} \frac{\alpha^j |u|^{js/(s-1)}}{j!}\, dx \\ &= C_2 \sum_{j=1}^{+\infty} \frac{\alpha^j}{j!} \int_{|x|\geq R} |x|^{d_i} |u|^{js/(s-1)}\, dx \\ &= C_2 \sum_{j=1}^{+\infty} \frac{\alpha^j}{j!} \int_{|x|\geq R} |x|^{d_i - \frac{ja}{s-1}} \left(|x|^{a/s}|u|\right)^{js/(s-1)} dx \\ &\leq C_2 \sum_{j=1}^{+\infty} \frac{\alpha^j}{C_1^{j/(s-1)} j!} \int_{|x|\geq R} |x|^{d_i - \frac{ja}{s-1}} \left(V^{1/s}|u|\right)^{js/(s-1)} dx. \end{aligned}$$

Using Hardy-Littlewood inequality, we have

$$\int_{|x|\geq R} |x|^{d_i - \frac{ja}{s-1}} \left(V^{1/s}|u|\right)^{js/(s-1)} dx \leq \int_0^{+\infty} \left(\frac{\chi_{\{|x|\geq R\}}}{|x|^{\frac{ja}{s-1} - d_i}}\right)^*(r) \left(\left(V^{1/s}|u|\right)^{\frac{js}{s-1}}\right)^*(r)\, dr.$$

Since $d_i < \frac{a}{\max\{p,q\}-1} - 1$, then, we have $d_i < \frac{a}{s-1} - 1$ for $s = p$ or $s = q$. Thus, using Lemma 5.3, we obtain

$$\begin{aligned} \int_{|x|\geq R} |x|^{d_i - \frac{ja}{s-1}} \left(V^{1/s}|u|\right)^{js/(s-1)} dx &\leq \pi \left(\frac{s}{2}\right)^{1/s} \int_0^{+\infty} \left(\frac{1}{R^2 + r/\pi}\right)^{\frac{ja}{2(s-1)} - \frac{d_i}{2}} \frac{\|(V^{1/s}u)^{\frac{js}{s-1}}\|_{2,s}}{r^{1/2}}\, dr \\ &\leq \pi \left(\frac{s}{2}\right)^{1/s} \|u\|_{(s)}^{\frac{js}{s-1}} \int_0^{+\infty} \left(\frac{1}{R^2 + r/\pi}\right)^{\frac{ja}{2(s-1)} - \frac{d_i}{2}} \frac{1}{r^{1/2}}\, dr. \end{aligned}$$

Observe that there exists $\delta_0 > 0$ such that $\frac{1}{2} + \delta_0 = \frac{a}{2(s-1)} - \frac{d_i}{2}$. Then,

$$\begin{aligned} \int_0^{+\infty} \left(\frac{1}{R^2 + r/\pi}\right)^{\frac{ja}{2(s-1)} - \frac{d_i}{2}} \frac{1}{r^{1/2}}\, dr &\leq \int_0^{+\infty} \left(\frac{1}{R^2 + r/\pi}\right)^{\frac{a}{2(s-1)} - \frac{d_i}{2}} \frac{1}{r^{1/2}}\, dr \\ &\leq \int_0^{+\infty} \left(\frac{1}{R^2 + r/\pi}\right)^{\frac{1}{2} + \delta_0} \frac{1}{r^{1/2}}\, dr \\ &= C_4(a, d_i, s, R). \end{aligned}$$

Thus,

$$I_2 = \int_{|x|\geq R} Q_i(x)\left(e^{\alpha|u|^{s/(s-1)}}-1\right)dx \leq C_2C_4\pi\left(\frac{s}{2}\right)^{1/s}\sum_{j=1}^{+\infty}\frac{\alpha^j\|u\|_{(s)}^{\frac{js}{s-1}}}{C_1^{j/(s-1)}j!}$$
$$\leq C_2C_4\pi\left(\frac{s}{2}\right)^{1/s}\sum_{j=1}^{+\infty}\frac{1}{j!}\left(\frac{\alpha}{C_1^{1/(s-1)}}\right)^j$$
$$= C_5(R,\alpha,a,d_i,s).$$

Now, we estimate the integral $I_1$. Now, we estimate the integral $I_1$.

**Case 1:** $b_i \geq 0$**.** In this case, by continuity of $Q_i$, there exists $C_6 = C_6(Q_i,R) > 0$ such that

$$I_1 \leq C_6\int_{|x|<R}(e^{\alpha|u|^{s/(s-1)}}-1)\,dx = C_6\int_{\mathbb{R}^2}\chi_{\{|x|\leq R\}}\left(e^{\alpha|u|^{s/(s-1)}}-1\right)dx.$$

By Hardy-Littlehood inequality and the fact that $(\chi_{\{|x|\leq R\}})^* = \chi_{\{0\leq t\leq \pi R^2\}}$, we have

$$I_1 \leq C_6\int_0^{\pi R^2}\left(e^{\alpha|u^*(t)|^{s/(s-1)}}-1\right)dt \leq C_6\int_0^{\pi R^2}e^{\alpha|u^*(t)|^{s/(s-1)}}\,dt. \tag{5.14}$$

From Lemmas 2 and 3 in Cassani and Tarsi (2009), for each $u \in W^1L^{2,s}(\mathbb{R}^2)$, we have

$$u^*(r) - u^*(\pi R^2) \leq \frac{1}{\sqrt{4\pi}}\left\{\int_r^{\pi R^2}\frac{|\nabla u|^*(\theta)}{\sqrt{\theta}}\,d\theta + \frac{1}{\sqrt{r}}\int_0^r|\nabla u|^*(\theta)\,d\theta\right\}. \tag{5.15}$$

Consider $v$ defined by

$$v(r) := \begin{cases} u^*(r)-u^*(\pi R^2), & 0\leq r\leq \pi R^2,\\ 0, & r>\pi R^2.\end{cases} \tag{5.16}$$

Observe that

$$(m+n)^q \leq m^q + q2^{q-1}m^{q-1}n + q2^{q-1}n^q, \quad \text{for} \quad m,n\geq 0, \quad q>1.$$

Given $\varepsilon > 0$, by Young's inequality, we have

$$(m+n)^q \leq m^q + \left(\frac{\varepsilon q}{q-1}\right)^{\frac{q-1}{q}}m^{q-1}q2^{q-1}\left(\frac{\varepsilon q}{q-1}\right)^{\frac{1-q}{q}}n + q2^{q-1}n^q \leq (1+\varepsilon)m^q + C_{\varepsilon,q}n^q.$$

Then,

$$|u^*(r)|^{s/(s-1)} \leq (1+\varepsilon)|v(r)|^{s/(s-1)} + C_{\varepsilon,s}|u^*(\pi R^2)|^{s/(s-1)}. \tag{5.17}$$

Now, we estimate $u^*(\pi R^2)$. From (5.2), we have

$$V_0^{1/s}u^*(\pi R^2) \leq \left(V^{1/s}u\right)^*(\pi R^2) \leq \left(\frac{s}{2}\right)^{1/s}\frac{\|u\|_s}{\pi^{1/2}R} \leq \left(\frac{s}{2}\right)^{1/s}\frac{1}{\pi^{1/2}R_0}$$

Thus, there exists $C_{R_0} = C(R_0,s) > 0$ such that

$$|u^*(r)|^{s/(s-1)} \leq (1+\varepsilon)|v(r)|^{s/(s-1)} + C_{R_0}. \tag{5.18}$$

From (5.14) and (5.18), we find

$$I_1 \le C_6 \int_0^{\pi R^2} e^{\alpha(1+\varepsilon)|v(t)|^{s/(s-1)}+C_R}\,dt \le C_6 e^{C_{R_0}} \int_0^{\pi R^2} e^{\alpha(1+\varepsilon)|v(t)|^{s/(s-1)}}\,dt.$$

Taking $t = \pi R^2 e^{-r}$,

$$I_1 \le C_6 \pi R^2 e^{C_{R_0}} \int_0^{+\infty} e^{\alpha(1+\varepsilon)|v(\pi R^2 e^{-r})|^{s/(s-1)}-r}\,dr.$$

Taking $\theta = \pi R^2 e^{-t}$ in (5.15), we obtain

$$v(\pi R^2 e^{-r}) \le \frac{\sqrt{\pi R^2}}{\sqrt{4\pi}} \left\{ \int_0^r |\nabla u|^*(\pi R^2 e^{-t}) e^{-t/2}\,dt + e^{r/2} \int_r^{+\infty} |\nabla u|^*(\pi R^2 e^{-t}) e^{-t}\,dt \right\}. \tag{5.19}$$

Since $\alpha < \alpha^*_{s,b_i} = \sqrt{4\pi}^{s/(s-1)}$ ($b_i \ge 0$), we can find $\varepsilon > 0$ sufficiently small such that $\big(\alpha(1+\varepsilon)\big)^{(s-1)/s} < \sqrt{4\pi}$. Then,

$$\begin{aligned}\big(\alpha(1+\varepsilon)\big)^{s-1/s} v(\pi R^2 e^{-r}) \le \sqrt{\pi R^2} \bigg\{ &\int_0^r |\nabla u|^*(\pi R^2 e^{-t}) e^{-t/2}\,dt \\ &+ e^{r/2} \int_r^{+\infty} |\nabla u|^*(\pi R^2 e^{-t}) e^{-t}\,dt \bigg\},\end{aligned}$$

which implies

$$\big(\alpha(1+\varepsilon)\big)^{s-1/s} v(\pi R^2 e^{-r}) \le \int_{-\infty}^{+\infty} \psi(t) a(t,r)\,dt, \tag{5.20}$$

where

$$a(t,r) := \begin{cases} 0, & t \le 0, \\ e^{(r-t)/2}, & r < t, \\ 1, & 0 < t \le r \end{cases} \tag{5.21}$$

and

$$\psi(t) := \begin{cases} \sqrt{\pi R^2} |\nabla u|^*(\pi R^2 e^{-t}) e^{-t/2}, & t \ge 0, \\ 0, & t < 0. \end{cases} \tag{5.22}$$

Note that,

$$\left[ \int_{-\infty}^0 + \int_r^{+\infty} \right] \big(a(t,r)\big)^{s/s-1}\,dt = \frac{2(s-1)}{s} \tag{5.23}$$

and

$$\int_{-\infty}^{+\infty} \psi(t)^s\,dt = \big(\pi R^2\big)^{s/2} \int_0^{+\infty} \big(|\nabla u|^*(\pi R^2 e^{-t})\big)^s e^{-st/2}\,dt.$$

Taking $r = \pi R^2 e^{-t}$, we obtain

$$\int_{-\infty}^{+\infty} \psi(t)^s\,dt = \int_0^{\pi R^2} \big(|\nabla u|^*(r) r^{1/2}\big)^s \frac{dr}{r} \le \|\nabla u\|_{2,s} \le \|u\|_{(s)} \le 1. \tag{5.24}$$

By Lemma 5.7, we have

$$\int_0^{+\infty} e^{-\Phi(r)}\,dr \le C,$$

where

$$\Phi(r) = r - \Big( \int_{-\infty}^{+\infty} \psi(t) a(t,r)\, dt \Big)^{s/(s-1)}. \tag{5.25}$$

From (5.20) and (5.25), we get

$$I_1 \le C_R \int_0^{+\infty} e^{\alpha(1+\varepsilon)|v(\pi R^2 e^{-r})|^{s/(s-1)} - r}\, dr \le C_R \int_0^{+\infty} e^{-\Phi(r)}\, dr = C_7.$$

**Case 2:** $b_i < 0$. Using (5.5) and the continuity of $Q_i$, there exists $C_{Q_i} > 0$ such that

$$\begin{aligned} I_1 &= \int_{|x|<r_0} Q_i(x) \Big( e^{\alpha|u|^{s/(s-1)}} - 1 \Big)\, dx + \int_{r_0 \le |x| < R} Q_i(x) \Big( e^{\alpha|u|^{s/(s-1)}} - 1 \Big)\, dx \\ &\le C_3 \int_{|x|<R} |x|^{b_i} \Big( e^{\alpha|u|^{s/(s-1)}} - 1 \Big)\, dx + C_{Q_i} \int_{|x|<R} \Big( e^{\alpha|u|^{s/(s-1)}} - 1 \Big)\, dx \\ &\le C_3 \int_{|x|<R} |x|^{b_i} \Big( e^{\alpha|u|^{s/(s-1)}} - 1 \Big)\, dx + C_7, \end{aligned}$$

where we use the case 1 in the last integral. By Hardy-Littlewood inequality, we have

$$I_1 \le C_3 \int_0^{\pi R^2} \pi t^{b_i/2} \big( e^{\alpha|u^*(t)|^{s/(s-1)}} - 1 \big)\, dt + C_7.$$

Using (5.18), we obtain

$$I_1 \le C_3 \pi e^{C_{R_0}} \int_0^{\pi R^2} t^{b_1/2} e^{\alpha(1+\varepsilon)|v(t)|^{s/(s-1)} + C_{R_0}}\, dt + C_7.$$

Taking $t = \pi R^2 e^{-r}$, we find

$$I_1 \le C_3 (\pi R^2)^{1+b_i} e^{C_R} \int_0^{+\infty} e^{\alpha(1+\varepsilon)|v(\pi R^2 e^{-r})|^{s/(s-1)} - (1+b_i/2) r}\, dr + C_7. \tag{5.26}$$

We can find $\varepsilon = \varepsilon(\alpha,s) > 0$ such that $\Big(\alpha(1+\varepsilon)\Big)^{(s-1)/s} < \sqrt{4\pi}\Big(1+\dfrac{b_i}{2}\Big)^{(s-1)/s}$. Thus, replacing in (5.18), we obtain

$$\big(\alpha(1+\varepsilon)\big)^{(s-1)/s} v(\pi R^2 e^{-r}) \le \Big(1 + \frac{b_i}{2}\Big)^{(s-1)/s} \int_{-\infty}^{+\infty} \psi(t) a(t,r)\, dt, \tag{5.27}$$

where $a(t,r)$ and $\psi(t)$ are given by (5.21) and (5.22) respectively. By Lemma 5.7 with $0 < \lambda = (1 + b_i/2) < 1$, (5.23) and (5.24), we have

$$\int_0^{+\infty} e^{-\Phi(r)}\, dr \le C, \tag{5.28}$$

where

$$\Phi(r) = (1+b_i/2) r - (1+b_i/2) \Big( \int_{-\infty}^{+\infty} \psi(t) a(t,r)\, dt \Big)^{s/(s-1)}.$$

Using (5.27) and (5.28) in (5.26), we obtain

$$\begin{aligned} I_1 &\le C_3 (\pi R^2)^{1+b_i/2} e^{C_{R_0}} \int_0^{+\infty} e^{\alpha(1+\varepsilon)|v(\pi R^2 e^{-r})|^{s/(s-1)} - (1+b_i/2) r}\, dr + C_7 \\ &\le C_3 (\pi R^2)^{1+b_i/2} e^{C_{R_0}} \int_0^{+\infty} e^{(1+b_i/2) \big( \int_{-\infty}^{+\infty} \psi(t) a(t,r)\, dt \big)^{(s-1)/s} - (1+b_i/2) r}\, dr + C_7 \\ &= C_3 (\pi R^2)^{1+b_i/2} e^{C_{R_0}} \int_0^{+\infty} e^{-\Phi(r)}\, dr + C_7 \\ &= C_8. \end{aligned}$$

Thus, using estimates $I_1$ and $I_2$ the proof follows.

■

**Proposition 5.9.** Let $s>1$ and $(u_n)$ be a sequence in $E^{(s)}$ and let $u\in E^{(s)}$ be such that

$$u_n \rightharpoonup u \quad \text{in} \quad E^{(s)}.$$

Then, there exist a subsequence $(u_{n_k})$ and a function $h\in E^{(s)}$ such that

$$|u_{n_k}(x)|\le h(x), \quad \text{for all} \quad k\ge 1 \quad \text{and almost everywhere in} \quad \mathbb{R}^2.$$

***Proof.*** Since $E^{(s)}\hookrightarrow W^1L^{2,s}(\mathbb{R}^2)$, we can assume that $u_n\to u$ almost everywhere in $\mathbb{R}^2$. Moreover, we can extract a subsequence $(u_{n_k})$, denoted by $(u_k)$ such that

$$\|u_{k+1}-u_k\|_{(s)}\le \frac{1}{2^{2k}}, \quad \text{for all} \quad k\ge 1.$$

Set

$$g_n(x)=\sum_{k=1}^{n}|u_{k+1}(x)-u_k(x)|.$$

Then, $(g_n)\in E^{(s)}$ and $\|g_n\|_{(s)}\le 1$ for all $n\ge 1$. That is

$$\|\nabla g_n\|_{2,s}\le 1 \quad \text{and} \quad \|V^{1/s}g_n\|_{2,s}\le 1, \quad \text{for all} \quad n\ge 1.$$

Since $(V^{1/s}g_n(x))$ is nondecreasing almost everywhere in $\mathbb{R}^2$ and $\sup_{n\ge 1}\|V^{1/s}g_n\|_{2,s}\le 1$, by Proposition 2.31, there exists $g_0\in L^{2,s}(\mathbb{R}^2)$ such that

$$V^{1/s}g_n\to g_0 \quad \text{in} \quad L^{2,s}(\mathbb{R}^2). \tag{5.29}$$

From $(V)$ and arguing as Proposition 2.48, we obtain

$$\|g_n-\frac{g_0}{V^{1/s}}\|_{2,s}=\|(V^{1/s}g_n-g_0)\frac{1}{V^{1/s}}\|_{2,s}\le \frac{2}{V_0^{1/s}}\|V^{1/s}g_n-g_0\|_{2,s}$$

Thus, $g_n\to g_0/V^{1/s}$ in $L^{2,s}(\mathbb{R}^2)$. Moreover, using the fact that $(\nabla g_n)$ is bounded in $\big(L^{2,s}(\mathbb{R}^2)\big)^2$, from Lemma 2.43, $g_n\to g_0/V^{1/s}$ in $W^1L^{2,s}(\mathbb{R}^2)$. In particular,

$$\|\nabla(g_n-\frac{g_0}{V^{1/s}})\|_{2,s}\to 0$$

and using (5.29), we get

$$\|V^{1/s}(g_n-\frac{g_0}{V^{1/s}})\|_{2,s}\to 0.$$

Thus,

$$g_n\to g:=g_0/V^{1/s} \quad \text{in} \quad E^{(s)}.$$

Using again the fact that $E^{(s)}\hookrightarrow W^1L^{2,s}(\mathbb{R}^2)$, we can assume that $g_n\to g$ almost everywhere in $\mathbb{R}^2$. On the other hand, for $l>k\ge 2$, we have

$$|u_l(x)-u_k(x)|\le |u_l(x)-u_{l-1}(x)|+\dots+|u_{k+1}(x)-u_k(x)|\le g_{l-1}(x)-g_{k-1}(x)\le g_{l-1}(x).$$

Taking $l \to +\infty$, we obtain

$$|u(x) - u_k(x)| \leq g(x) \quad \text{almost everywhere in} \quad \mathbb{R}^2.$$

Thus,

$$|u_k(x)| \leq h(x) \quad \text{almost everywhere in} \quad \mathbb{R}^2.$$

where $h := g + |u| \in E^{(s)}$. ■

We observe that, by Proposition 2.57, we can construct a map tilde from $E^{(q)}$ to $E^{(p)}$ and the set $E = E^{(q)} \times E^{(p)}$ endowed with the operations given by (2.60) and (2.61) satisfies the same properties given by Lemma 2.54.

## 5.3 Variational setting

In this section, we describe the functional $J : E \to \mathbb{R}$, associated to the system (5.1) which is given by

$$J(u,\widetilde{v}) = \int_{\mathbb{R}^2} \left(\nabla u \nabla \widetilde{v} + V(x) u \widetilde{v}\right) dx - \int_{\mathbb{R}^2} Q_1(x) F(u)\, dx - \int_{\mathbb{R}^2} Q_2(x) G(\widetilde{v})\, dx.$$

**Proposition 5.10.** Assume $(B_1), (B_4)$ and $(B_5)$. Then, the functional $J$ is well defined and belongs to the class $\mathscr{C}^1(E,\mathbb{R})$ with

$$\begin{aligned} J'(u,\widetilde{v})(\phi,\widetilde{\psi}) = & \int_{\mathbb{R}^2} \left(\nabla u \nabla \widetilde{\psi} + V(x) u \widetilde{\psi} + \nabla \widetilde{v} \nabla \phi + V(x) \widetilde{v} \phi\right) dx \\ & - \int_{\mathbb{R}^2} Q_1(x) f(u) \phi\, dx - \int_{\mathbb{R}^2} Q_2(x) g(\widetilde{v}) \widetilde{\psi}\, dx, \end{aligned} \tag{5.30}$$

for all $(\phi, \widetilde{\psi}) \in E$.

***Proof.*** Let $u \in E^{(q)}$ and $\widetilde{v} \in E^{(p)}$. By Hölder's inequality in Lorentz spaces, we have

$$\left| \int_{\mathbb{R}^2} \nabla u \nabla \widetilde{v}\, dx \right| \leq \|\nabla u\|_{2,q} \|\nabla \widetilde{v}\|_{2,p} \leq \|u\|_{(q)} \|\widetilde{v}\|_{(p)} \tag{5.31}$$

and

$$\left| \int_{\mathbb{R}^2} V(x) u \widetilde{v}\, dx \right| = \left| \int_{\mathbb{R}^2} V(x)^{1/q} u V(x)^{1/p} \widetilde{v}\, dx \right| \leq \|V^{1/q} u\|_{2,q} \|V^{1/p} \widetilde{v}\|_{2,p} \leq \|u\|_{(q)} \|\widetilde{v}\|_{(p)}. \tag{5.32}$$

Using $(B_1)$ and $(B_4)$, there exists $C > 0$ such that

$$|f(s)| \leq |s| + C(e^{(\alpha_0+1)|s|^p} - 1), \quad \text{for all} \quad s \in \mathbb{R}. \tag{5.33}$$

Thus, there exists $C > 0$ such that

$$|F(u)| \leq C|u|^2 + C(e^{(\alpha_0+1)|u|^p} - 1).$$

Using Proposition 5.8 and Corollary 5.6, we obtain

$$\left| \int_{\mathbb{R}^2} Q_1(x) F(u)\, dx \right| \leq C \int_{\mathbb{R}^2} Q_1(x) |u|^2\, dx + C \int_{\mathbb{R}^2} Q_1(x) (e^{2(\alpha_0+1)|u|^p} - 1)\, dx < +\infty. \tag{5.34}$$

Similarly, $Q_2(x)G(\widetilde{v})$ belongs to $L^1(\mathbb{R}^2)$ for all $\widetilde{v} \in E^{(p)}$. Thus, from (5.31), (5.32) and (5.34), we conclude that $J$ is well defined in $E$. Moreover, using Proposition 5.9 and arguing as in proof of Lemma 4.13, we can prove that $J \in \mathscr{C}^1(E,\mathbb{R})$ and $J'$ is given by (5.30). ■

We say that $(u,\widetilde{v}) \in E$ is a weak solution of (5.1) if

$$\int_{\mathbb{R}^2} \big(\nabla u \nabla \widetilde{\psi} + V(x) u \widetilde{\psi} + \nabla \widetilde{v} \nabla \phi + V(x) \widetilde{v} \phi\big)\, dx = \int_{\mathbb{R}^2} Q_1(x) f(u) \phi \, dx + \int_{\mathbb{R}^2} Q_2(x) g(\widetilde{v}) \widetilde{\psi}\, dx,$$

for all $(\phi,\widetilde{\psi}) \in E$. Consequently, critical points of the functional $J$ correspond to the weak solutions of (5.1).

**Lemma 5.11.** Let $s = q$ or $s = p$, $\alpha > 0$ and $r \geq 1$. Then, if $u \in E^{(s)}$ is such that $\|u\|_{(s)} \leq M$ with $\alpha M^{s/(s-1)} < \alpha^*_{q,b_i}$, then there exists a positive constant $C = C(\alpha, M, r, s)$ such that

$$\int_{\mathbb{R}^2} Q_i(x) |u|^r \big(e^{\alpha |u|^{s/(s-1)}} - 1\big)\, dx \leq C \|u\|^r_{(s)}.$$

***Proof.*** Consider the case $s = q$ and $s/(s-1) = p$. Choose $t > 1$ close to 1 such that $t\alpha M^p < \alpha^*_{q,b_i}$ and set $t' = t/(t-1)$. Thus, using Hölder's inequality and Lemma 4.14, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} Q_i(x)|u|^r \big(e^{\alpha|u|^p} - 1\big)\, dx &\leq \Big(\int_{\mathbb{R}^2} Q_i(x) \big(e^{\alpha|u|^p} - 1\big)^t \, dx\Big)^{1/t} \Big(\int_{\mathbb{R}^2} Q_i(x) |u|^{rt'}\, dx\Big)^{1/t'} \\
&\leq \Big(\int_{\mathbb{R}^2} Q_i(x) \big(e^{t\alpha|u|^p} - 1\big)\, dx\Big)^{1/t} \Big(\int_{\mathbb{R}^2} Q_i(x) |u|^{rt'}\, dx\Big)^{1/t'} \\
&\leq \Big(\int_{\mathbb{R}^2} Q_i(x) \big(e^{t\alpha M^p (\frac{|u|}{\|u\|_{(q)}})^p} - 1\big)\, dx\Big)^{1/t} \|u\|^r_{L^{rt'}(\mathbb{R}^2, Q_i)}.
\end{aligned}$$

By Proposition 5.8, we have

$$\int_{\mathbb{R}^2} Q_i(x)|u|^r \big(e^{\alpha|u|^p} - 1\big)\, dx \leq C \|u\|^r_{L^{rt'}(\mathbb{R}^2, Q_i)}.$$

Finally, we use the continuous embedding $E^{(q)} \hookrightarrow L^{rt'}(\mathbb{R}^2, Q_i)$. ■

**Lemma 5.12.** Let $s = q$ or $s = p$ and $\{u_n \in W^{(s)} : \|u_n\|_{(s)} = 1\}$ be a sequence converging weakly to the zero function in $E^{(s)}$. Then, for every $0 < \alpha < \alpha^*_{q,b_i}$, we can find a subsequence (not renamed) such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} Q_i(x) \big(e^{\alpha|u_n|^{s/(s-1)}} - 1\big)\, dx = 0.$$

***Proof.*** We prove in the case $s = q$. Let $\varepsilon > 0$ such that $\alpha + \varepsilon < \alpha^*_{q,b_i}$. Since,

$$\lim_{|t|\to 0} \frac{e^{\alpha|t|^p} - 1}{|t|^p} = 1 \quad \text{and} \quad \lim_{|t|\to\infty} \frac{e^{\alpha|t|^p} - 1}{|t|\big(e^{(\alpha+\varepsilon)|t|^p} - 1\big)} = 0,$$

then, there exists $c > 0$ such that

$$e^{\alpha|t|^p} - 1 \leq c|t|^p + c|t|(e^{(\alpha+\varepsilon)|t|^p} - 1), \quad \text{for all} \quad t \in \mathbb{R}.$$

Hence,

$$\int_{\mathbb{R}^2} Q_i(x)\big(e^{\alpha|u_n|^p}-1\big)\,dx \le c\int_{\mathbb{R}^2} Q_i(x)|u_n|^p\,dx + c\int_{\mathbb{R}^2} Q_i(x)|u_n|\big(e^{(\alpha+\varepsilon)|u_n|^p}-1\big)\,dx. \quad (5.35)$$

By Hölder's inequality and Lemma 4.14, in the second integral of (5.35), we get

$$\int_{\mathbb{R}^2} Q_i(x)|u_n|\big(e^{(\alpha+\varepsilon)|u_n|^p}-1\big)\,dx \le \|u_n\|_{L^{t'}(\mathbb{R}^2,Q_i)}\Big(\int_{\mathbb{R}^2} Q_i(x)\big(e^{t(\alpha+\varepsilon)|u_n|^p}-1\big)\big)\,dx\Big)^{1/t}.$$

Since $\|u_n\|_{(s)}=1$ and $t(\alpha+\varepsilon)<\alpha^*_{q,b_i}$, by Proposition 5.8, we obtain $c>0$ such that

$$\int_{\mathbb{R}^2} Q_i(x)|u_n|(e^{(\alpha+\varepsilon)|u_n|^p}-1)\,dx \le c\|u_n\|_{L^p(\mathbb{R}^2,Q_i)} + c\|u_n\|_{L^{t'}(\mathbb{R}^2,Q_i)}. \quad (5.36)$$

Replacing (5.36) in (5.35), using the compact embeddings of $E^{(q)}$ in $L^p(\mathbb{R}^2,Q_i)$ and in $L^{t'}(\mathbb{R}^2,Q_i)$ and the fact that $u_n \rightharpoonup 0$ in $E^{(q)}$, we get a subsequence (not renamed) such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} Q_i(x)\big(e^{\alpha|u_n|^p}-1\big)\,dx = 0.$$

■

Denote

$$\lambda_{Q_1} := \inf_{u\in E^{(q)}\setminus 0}\frac{\|u\|^2_{(q)}}{\int_{\mathbb{R}^2} Q_1(x)u^2/\,dx} \quad\text{and}\quad \widetilde{\lambda}_{Q_2} := \inf_{\widetilde{u}\in E^{(p)}\setminus 0}\frac{\|\widetilde{u}\|^2_{(p)}}{\int_{\mathbb{R}^2} Q_2(x)\widetilde{u}^2\,dx}. \quad (5.37)$$

By Hölder's inequality and continuous embeddings we have that $\lambda_{Q_1}$ and $\widetilde{\lambda}_{Q_2}$ are positives numbers.

### 5.3.1 On Palais-Smale sequences

**Lemma 5.13.** Assume $(B_1)-(B_2),(B_4)-(B_5)$ and let $(u_n,\widetilde{v}_n)$ be a sequence in $E$ such that $|J(u_n,\widetilde{v}_n)|\le d$ and

$$|J'(u_n,\widetilde{v}_n)(\phi,\widetilde{\psi})| \le \varepsilon_n\|(\phi,\widetilde{\psi})\|, \quad \text{for all} \quad \phi,\psi\in\{0,u_n,v_n\}. \quad (5.38)$$

Then, $\|(u_n,\widetilde{v}_n)\|\le c$ for every $n\in\mathbb{N}$ and for some positive constant $c$.

***Proof.*** Taking $(\phi,\widetilde{\psi})=(u_n,\widetilde{v}_n)$ in (5.38), we have

$$\left|2\int_{\mathbb{R}^2}\big(\nabla u_n\nabla\widetilde{v}_n + V(x)u_n\widetilde{v}_n\big)\,dx - \int_{\mathbb{R}^2} Q_1(x)f(u_n)u_n\,dx - \int_{\mathbb{R}^2} Q_2(x)g(\widetilde{v}_n)\widetilde{v}_n\,dx\right| \le \varepsilon_n\|(u_n,\widetilde{v}_n)\|.$$

Thus,

$$\int_{\mathbb{R}^2} Q_1(x)f(u_n)u_n\,dx + \int_{\mathbb{R}^2} Q_2(x)g(\widetilde{v}_n)\widetilde{v}_n\,dx \le \left|2\int_{\mathbb{R}^2}\big(\nabla u_n\nabla\widetilde{v}_n + V(x)u_n\widetilde{v}_n\big)\,dx\right| + \varepsilon_n\|(u_n,\widetilde{v}_n)\|.$$

Since

$$\int_{\mathbb{R}^2}\big(\nabla u_n\nabla\widetilde{v}_n + V(x)u_n\widetilde{v}_n\big)\,dx = J(u_n,\widetilde{v}_n) + \int_{\mathbb{R}^2} Q_1(x)F(u_n)\,dx + \int_{\mathbb{R}^2} Q_2(x)G(\widetilde{v}_n)\,dx,$$

we get

$$\int_{\mathbb{R}^2} Q_1(x) f(u_n) u_n \, dx + \int_{\mathbb{R}^2} Q_2(x) g(\widetilde{v}_n) \widetilde{v}_n \, dx \\ \leq 2d + 2 \int_{\mathbb{R}^2} Q_1(x) F(u_n) \, dx + 2 \int_{\mathbb{R}^2} Q_2(x) G(\widetilde{v}_n) \, dx + \varepsilon_n \|(u_n, \widetilde{v}_n)\|.$$

Using $(B_2)$, we obtain

$$\int_{\mathbb{R}^2} Q_1(x) F(u_n) \, dx \leq \frac{1}{\mu} \int_{\mathbb{R}^2} Q_1(x) f(u_n) u_n \, dx$$

and

$$\int_{\mathbb{R}^2} Q_2(x) G(\widetilde{v}_n) \, dx \leq \frac{1}{\nu} \int_{\mathbb{R}^2} Q_2(x) g(\widetilde{v}_n) \widetilde{v}_n \, dx.$$

Hence,

$$\Big(1 - \frac{2}{\mu}\Big) \int_{\mathbb{R}^2} Q_1(x) f(u_n) u_n \, dx + \Big(1 - \frac{2}{\nu}\Big) \int_{\mathbb{R}^2} Q_2(x) g(\widetilde{v}_n) \widetilde{v}_n \leq 2d + \varepsilon_n \|(u_n, \widetilde{v}_n)\|.$$

Thus, there exists $c > 0$ such that

$$\int_{\mathbb{R}^2} Q_1(x) f(u_n) u_n \, dx \leq c + \varepsilon_n \|(u_n, \widetilde{v}_n)\| \text{ and } \int_{\mathbb{R}^2} Q_2(x) g(\widetilde{v}_n) \widetilde{v}_n \, dx \leq c + \varepsilon_n \|(u_n, \widetilde{v}_n)\|. \quad (5.39)$$

On the other hand, taking $(\phi, \widetilde{\psi}) = (v_n, 0)$ in (5.38), we get

$$\int_{\mathbb{R}^2} \big(\nabla v_n \nabla \widetilde{v}_n + V(x) v_n \widetilde{v}_n\big) \, dx \leq \int_{\mathbb{R}^2} Q_1(x) f(u_n) v_n \, dx + \varepsilon_n \|(v_n, 0)\|.$$

This means,

$$\|v_n\|_{(q)}^2 \leq \int_{\mathbb{R}^2} Q_1(x) f(u_n) v_n \, dx + \varepsilon_n \|v_n\|_{(q)}.$$

Set

$$T_n = \frac{v_n}{\|v_n\|_{(q)}}.$$

Then, we can write

$$\|v_n\|_{(q)} \leq \int_{\mathbb{R}^2} Q_1(x) f(u_n) T_n + \varepsilon_n. \quad (5.40)$$

Let $\alpha_1 > \alpha_0$ and $0 < \alpha_2 < \alpha^*_{q,b_1}$. By $(B_1)$ and $(B_4)$, there exists $\lambda > 0$ such that

$$|f(s)| \leq \lambda e^{\alpha_1 |s|^p}, \quad \text{for all} \quad s \in \mathbb{R}. \quad (5.41)$$

Applying Lemma 3.10 in (5.40) with $s = |f\big(u_n(x)\big)|/\lambda$, $t = \alpha_2^{1/p} |T_n(x)|$, $r = p$ and $r' = q$, we obtain

$$\begin{aligned} \|v_n\|_{(q)} &\leq \frac{\lambda}{\alpha_2^{1/p}} \int_{\mathbb{R}^2} Q_1(x) \frac{|f(u_n)|}{\lambda} \alpha_2^{1/p} |T_n| \, dx + \varepsilon_n \\ &\leq \frac{\lambda}{\alpha_2^{1/p}} \Bigg[ \int_{\mathbb{R}^2} Q_1(x) (e^{\alpha_2 |T_n|^p} - 1) \, dx + \frac{1}{q \lambda^q} \int_{\{x \in \mathbb{R}^2 : |\frac{f(u_n)}{\lambda}| \leq e^{1/p^q}\}} Q_1(x) |f(u_n)|^q \, dx \quad (5.42) \\ &\quad + \frac{1}{\lambda} \int_{\{x \in \mathbb{R}^2 : |\frac{f(u_n)}{\lambda}| \geq e^{1/p^q}\}} Q_1(x) |f(u_n)| \ln^{1/p} \frac{|f(u_n)|}{\lambda} \, dx \Bigg] + \varepsilon_n. \end{aligned}$$

From (5.41), we have

$$\int_{\{x\in\mathbb{R}^2:|\frac{f(u_n)}{\lambda}|\geq e^{1/p^q}\}} Q_1(x)|f(u_n)|\ln^{1/p}\frac{|f(u_n)|}{\lambda}\,dx \leq \alpha_1^{1/p}\int_{\mathbb{R}^2} Q_1(x)f(u_n)u_n\,dx. \tag{5.43}$$

Since $\|T_n\|_{(q)} = 1$ and $0 < \alpha_2 < \alpha^*_{q,b_1}$, by Proposition 5.8, there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} Q_1(x)\big(e^{\alpha_2|T_n|^p} - 1\big)\,dx \leq C. \tag{5.44}$$

Now, we estimate the second integral in (5.42). From $(B_1)$, given $\bar{\varepsilon} > 0$ there exists $\delta > 0$ such that

$$|f(t)| \leq \bar{\varepsilon}^{\frac{1}{q-1}}|t|^{\frac{1}{q-1}}, \quad \text{for all} \quad |t| \leq \delta,$$

which implies

$$|f(t)|^q \leq \bar{\varepsilon}|f(t)t|, \quad \text{for all} \quad |t| \leq \delta. \tag{5.45}$$

Note also that

$$|f(t)|^{q-1} \leq (\lambda e^{\frac{1}{p^q}})^{q-1}\frac{|t|}{\delta}, \quad \text{for all} \quad \{|t| \geq \delta : |f(t)| \leq \lambda e^{\frac{1}{p^q}}\}. \tag{5.46}$$

Then, from (5.45) and (5.46), we get

$$|f(t)|^q \leq \bar{c}|f(t)t|, \quad \text{for all} \quad \{t \in \mathbb{R} : |f(t)| \leq \lambda e^{\frac{1}{p^q}}\}, \tag{5.47}$$

where $\bar{c} = \max\{(\lambda e^{\frac{1}{p^q}})^{q-1}/\delta, \bar{\varepsilon}\}$. By (5.39), there exist $c_1 > 0$ such that

$$\begin{aligned}\int_{\{x\in\mathbb{R}^2:\frac{|f(u_n)|}{\lambda}\leq e^{1/p^q}\}} Q_1(x)|f(u_n)|^q\,dx &\leq \bar{c}\int_{\{x\in\mathbb{R}^2:|\frac{f(u_n)}{\lambda}|\leq e^{1/p^q}\}} Q_1(x)f(u_n)u_n\,dx\\ &\leq c_1 + \varepsilon_n\|(u_n,\widetilde{v}_n)\|\end{aligned}$$

which together with (5.43) and (5.44) in (5.42), gives that there exist $c > 0$ such that

$$\|v_n\|_{(q)} \leq c + c\int_{\mathbb{R}^2} Q_1(x)f(u_n)u_n\,dx + \varepsilon_n\|(u_n,\widetilde{v}_n)\|. \tag{5.48}$$

On the other hand, taking $(\phi,\widetilde{\psi}) = (0,\widetilde{u}_n)$ in (5.38), we can obtain $d > 0$ such that

$$\|\widetilde{u}_n\|_{(q)} \leq d + d\int_{\mathbb{R}^2} Q_2(x)g(\widetilde{v}_n)\widetilde{v}_n\,dx + \varepsilon_n\|(u_n,\widetilde{v}_n)\|. \tag{5.49}$$

Using (5.48), (5.49) and (5.39), there exist $k > 0$ such that

$$\|(u_n,\widetilde{v}_n)\| \leq k + \varepsilon_n\|(u_n,\widetilde{v}_n)\|.$$

Hence, $(u_n,\widetilde{v}_n)$ is a bounded sequence. ■

**Remark 5.14.** In the previous Lemma, using the fact that $(u_n,\widetilde{v}_n)$ is a bounded sequence in $E$ and replacing in (5.39), we can find a positive constant $C$ such that

$$\int_{\mathbb{R}^2} Q_1(x)f(u_n)u_n\,dx \leq C \quad \text{and} \quad \int_{\mathbb{R}^2} Q_2(x)g(\widetilde{v}_n)\widetilde{v}_n\,dx \leq C, \quad \text{for all} \quad n \geq 1.$$

The following lemmas can be proved arguing as in Souza (2012), Souza (2011) and using some estimates developed as in proof of Proposition 5.8.

**Lemma 5.15.** Let $(u_n,\widetilde{v}_n)$ be a sequence in $E$ such that $J(u_n,\widetilde{v}_n)\to c$, $J_n'(u_n,\widetilde{v}_n)\to 0$ and $(u_n,\widetilde{v}_n)$ converges weakly to $(u,\widetilde{v})$ in $E$. Then, up to a subsequence

$$Q_1(x)f(u_n)\to Q_1(x)f(u) \quad \text{and} \quad Q_2(x)g(\widetilde{v}_n)\to Q_2(x)g(\widetilde{v}) \quad \text{in} \quad L^1_{loc}(\mathbb{R}^2).$$

**Lemma 5.16.** Assume $(H_1)-(H_4)$ and let $(u_n,\widetilde{v}_n)$ be a sequence in $E_n$ such that $J(u_n,\widetilde{v}_n)\to c$, $J_n'(u_n,\widetilde{v}_n)\to 0$ and $(u_n,\widetilde{v}_n)\rightharpoonup(u,\widetilde{v})$ in $E$. Then, up to a subsequence

$$Q_1(x)F(u_n)\to Q_1(x)F(u) \quad \text{and} \quad Q_2(x)G(\widetilde{v}_n)\to Q_2(x)G(\widetilde{v}) \quad \text{in} \quad L^1(\mathbb{R}^2).$$

### 5.3.2 Linking geometry

Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of eigenfunctions for the operator $(-\Delta+V)$ in $H^1_V(\mathbb{R}^2):=\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}V(x)u^2\,dx<\infty\}$. By Lemma 3 in Cassani and Tarsi (2015), the sequence $\{e_i\}_{i\in\mathbb{N}}$ provides also a dense system in $E^{(q)}$ and $E^{(p)}$. For each $n\in\mathbb{N}$, consider the following finite dimensional subspace:

$$E_n:=Span\{e_1,\dots,e_n\}.$$

We define the set

$$E_{n,m}:=\{u_m:=\zeta_m u: u\in E_n\}.$$

where $\zeta_m$ is given by (4.12) and $m=m(n)$ as in Lemma 4.12. Let $y(x)=M_{k,q;d}(x)$ and $\widetilde{z}(x)=M_{k,p;d}(x)$. By Lemma 4.11, $\|(y,\widetilde{z})\|=2$ and $\widetilde{z}\neq-\widetilde{y}$. Set

$$F_{n,m}=E_{n,m}\times E_{n,m}\oplus\mathbb{R}(y,\widetilde{z}),$$

$$E^+_{n,m}:=\{(v,\widetilde{v}):v\in E_{n,m}\} \quad \text{and} \quad E^-_{n,m}:=\{(v,-\widetilde{v}):v\in E_{n,m}\}.$$

Consider

$$\partial B_\rho\cap F^+_{n,m}\subset F_{n,m}, \quad \text{where} \quad F^+_{n,m}:=E^+_{n,m}\oplus\mathbb{R}(y,\widetilde{z})$$

and

$$Q_{n,m}=\{w+s(y,\widetilde{z}):w=(\omega,-\widetilde{\omega})\in E^-_{n,m},\|w\|\le R_0, 0\le s\le R_1\},$$

where $\rho$, $R_0$ and $R_1$ are positives numbers which will be chosen in the following lemmas.

**Lemma 5.17.** There exist $\rho,\sigma>0$ such that $J(z)\ge\sigma$, for all $z\in\partial B_\rho\cap F^+_{n,m}$.

***Proof.*** Given $\varepsilon>0$ for assumption $(B_1)$ and $(B_4)$, there exists $C>0$ such that

$$|F(s)|\le\varepsilon|s|^2+C|s|^4\left(e^{2\alpha_0|s|^p}-1\right), \quad \text{for all} \quad s\in\mathbb{R} \tag{5.50}$$

Let $(u+sy,\widetilde{u}+s\widetilde{z})\in F^{+}_{n,m}$ with $\|(u+sy,\widetilde{u}+s\widetilde{z})\|\le\rho_1$ with $\rho_1>0$ sufficiently small such that $2\alpha_0\rho_1^p<\alpha^*_{q,b_1}$. By Lemma 5.11, there exists $C>0$ such that

$$\int_{\mathbb{R}^2} Q_1(x)F(u+sy)\,dx\le\varepsilon\int_{\mathbb{R}^2} Q_1(x)|u+sy|^2\,dx+C\|u+sy\|^4_{(q)}. \tag{5.51}$$

From (5.51) and (5.37), we have

$$\int_{\mathbb{R}^2} Q_1(x)F(u+sy)\,dx\le\frac{\varepsilon}{\lambda_{Q_1}}\|u+sy\|^2_{(q)}+C\|u+sy\|^4_{(q)}.$$

By Remark 4.13,

$$\int_{\mathbb{R}^2} Q_1(x)F(u+sy)\,dx\le\frac{\varepsilon}{\lambda_{Q_1}}\big(\|u\|^2_{(q)}+s^2\|y\|^2_{(q)}\big)+C\big(\|u\|^4_{(q)}+s^4\|y\|^4_{(q)}\big). \tag{5.52}$$

Similarly, we obtain

$$\int_{\mathbb{R}^2} Q_2(x)G(\widetilde{u}+s\widetilde{z})\,dx\le\frac{\varepsilon}{\widetilde{\lambda}_{Q_2}}\big(\|\widetilde{u}\|^2_{(p)}+s^2\|\widetilde{z}\|^2_{(p)}\big)+C\big(\|\widetilde{u}\|^4_{(p)}+s^4\|\widetilde{z}\|^4_{(p)}\big). \tag{5.53}$$

Thus,

$$\begin{aligned}
J(u+sy,\widetilde{u}+s\widetilde{z})&=\int_{\mathbb{R}^2}\Big(\nabla(u+sy)\nabla(\widetilde{u}+s\widetilde{z})+V(x)(u+sy)(\widetilde{u}+s\widetilde{z})\Big)\,dx\\
&\quad-\int_{\mathbb{R}^2} Q_1(x)F(u+sy)\,dx-\int_{\mathbb{R}^2} Q_2(x)G(\widetilde{u}+s\widetilde{z})\,dx\\
&\ge s^2\int_{\mathbb{R}^2}\big(\nabla y\nabla\widetilde{z}+V(x)y\widetilde{z}\big)\,dx\\
&\quad+\frac{1}{2}\|u\|^2_{(q)}-\frac{\varepsilon}{\lambda_{Q_1}}\big(\|u\|^2_{(q)}+s^2\|y\|^2_{(q)}\big)-C\big(\|u\|^4_{(q)}+s^4\|y\|^4_{(q)}\big)\\
&\quad+\frac{1}{2}\|\widetilde{u}\|^2_{(p)}-\frac{\varepsilon}{\widetilde{\lambda}_{Q_2}}\big(\|\widetilde{u}\|^2_{(p)}+s^2\|\widetilde{z}\|^2_{(p)}\big)-C\big(\|\widetilde{u}\|^4_{(p)}+s^4\|\widetilde{z}\|^4_{(p)}\big).
\end{aligned}$$

Since $\|y\|_{(q)}=\|\widetilde{z}\|_{(p)}=1$, we have

$$\begin{aligned}
J(u+sy,\widetilde{u}+s\widetilde{z})&\ge s^2\int_{\mathbb{R}^2}\big(\nabla y\nabla\widetilde{z}+V(x)y\widetilde{z}\big)\,dx\\
&\quad+\Big(1-\frac{\varepsilon}{\lambda_{Q_1}}-C\|u\|^2_{(q)}\Big)\|u\|^2_{(q)}-\frac{\varepsilon}{\lambda_{Q_1}}s^2-Cs^4\\
&\quad+\Big(1-\frac{\varepsilon}{\widetilde{\lambda}_{Q_2}}-C\|\widetilde{u}\|^2_{(p)}\Big)\|\widetilde{u}\|^2_{(p)}-\frac{\varepsilon}{\widetilde{\lambda}_{Q_2}}s^2-Cs^4.
\end{aligned}$$

Then,

$$\begin{aligned}
J(u+sy,\widetilde{u}+s\widetilde{z})&\ge s^2\Big(\int_{\mathbb{R}^2}\big(\nabla y\nabla\widetilde{z}+V(x)y\widetilde{z}\big)\,dx-\varepsilon C_1-C_2s^2\Big)\\
&\quad+\Big(2-\varepsilon C_1-C_2\rho_2^2\Big)\rho_2^2.
\end{aligned}$$

where $\|u\|_{(q)}=\|\widetilde{u}\|_{(p)}=\rho_2>0$. Using Lemma 4.11, there exists $C_3>0$ such that

$$\int_{\mathbb{R}^2}\big(\nabla y\nabla\widetilde{z}+V(x)y\widetilde{z}\big)\,dx\ge C_3,\quad\text{for } k \text{ sufficiently large.}$$

Taking $\rho_1 > 0$, $\rho_2 > 0$, $s_1 > 0$ and $\varepsilon > 0$ sufficiently small such that $C_3 - \varepsilon C_1 - C_2 s_1^2 \geq C_3/2$ and $2 - \varepsilon C_1 - C_2\rho_2^2 \geq 0$ and setting $\rho = \min\{\rho_1, \rho_2, s_1\}$, there exists $\sigma > 0$ such that

$$J(u+sy, \widetilde{u}+s\widetilde{z}) \geq \frac{C_3\rho^2}{2} = \sigma,$$

where $\|(u+sy, \widetilde{u}+s\widetilde{z})\| = \rho$. ■

**Lemma 5.18.** There exist $R_0 > 0$ and $R_1 > \rho$ (independent of $n$ and $k$) such that $J(\vartheta) \leq 0$, for all $\vartheta \in \partial Q_{n,m}$, where

$$Q_{n,m} = \{w + s(y,\widetilde{z}) : w = (\omega, -\widetilde{\omega}) \in E^-_{n,m}, \|w\| \leq R_0, 0 \leq s \leq R_1\}$$

***Proof.*** Notice that the boundary $\partial Q_{n,m}$ of the set $Q_{n,m}$ is composed of three parts.

(i) If $\vartheta \in \partial Q \cap E^-_{n,m}$, $\vartheta = (\omega, -\widetilde{\omega})$, and hence

$$J(\omega, -\widetilde{\omega}) = -\int_{\mathbb{R}^2} \left(\nabla\omega\nabla\widetilde{\omega} + V(x)\omega\widetilde{\omega}\right) dx - \int_{\mathbb{R}^2} Q_1(x)F(\omega)\, dx - \int_{\mathbb{R}^2} Q_2(x)G(-\widetilde{\omega})\, dx \leq 0$$

because $Q_1F$ and $Q_2G$ are nonnegative functions.

(ii) If $\vartheta = (\omega, -\widetilde{\omega}) \mp s(y,\widetilde{z}) = (\omega + sy, -\widetilde{\omega} + s\widetilde{z}) \in \partial Q_{n,m}$, with $\|(\omega, -\widetilde{\omega})\| = R_0$ and $0 \leq s \leq R_1$, we obtain

$$\begin{aligned} J(\omega+sy, -\widetilde{\omega}+s\widetilde{z}) = &\int_{\mathbb{R}^2} \Big(\nabla(\omega+sy)\nabla(-\widetilde{\omega}+s\widetilde{z}) + V(x)(\omega+sy)(-\widetilde{\omega}+s\widetilde{z})\Big)\, dx \\ &- \int_{\mathbb{R}^2} Q_1(x)F(\omega+sy)\, dx - \int_{\mathbb{R}^2} Q_2(x)G(-\widetilde{\omega}+s\widetilde{z})\, dx. \end{aligned}$$

Using the fact that $Q_1F$ and $Q_2G$ are nonnegatives and Remark 4.13, we obtain

$$\begin{aligned} J(\omega+sy, -\widetilde{\omega}+s\widetilde{z}) &\leq -\|\omega\|^2_{(q)} + s^2 \int_{\mathbb{R}^2} \left(\nabla y\nabla\widetilde{z} + V(x)y\widetilde{z}\right) dx \\ &\leq -\|\omega\|^2_{(q)} + s^2\|y\|_{(q)}\|\widetilde{z}\|_{(p)} \\ &\leq -\frac{R_0^2}{2} + R_1^2. \end{aligned}$$

Hence, $J(\vartheta) \leq 0$ provided $R_0 \geq \sqrt{2}R_1$. Thus, we can take $R_0 = \sqrt{2}R_1$, for $R_1 > 0$ to be determined later.

(iii) If $\vartheta = (\omega, -\widetilde{\omega}) + R_1(y,\widetilde{z})$, with $\|(\omega, -\widetilde{\omega})\| \leq R_0$ for $R_0$ given by case $(ii)$, then

$$\begin{aligned} J(\omega+R_1y, -\widetilde{\omega}+R_1\widetilde{z}) = &-\|\omega\|^2_{(q)} + R_1^2 \int_{\mathbb{R}^2} \left(\nabla y\nabla\widetilde{z} + V(x)y\widetilde{z}\right) dx \\ &- \int_{\mathbb{R}^2} Q_1(x)F(\omega+R_1y)\, dx - \int_{\mathbb{R}^2} Q_2(x)G(-\widetilde{\omega}+R_1\widetilde{z})\, dx. \end{aligned} \tag{5.54}$$

From $(B_1)$ and $(B_2)$, there exists $C > 0$ such that

$$F(t) \geq C|t|^\theta - t^2 \quad \text{and} \quad G(t) \geq C|t|^\theta - t^2, \quad \text{for all} \quad t \in \mathbb{R}.$$

By the last inequalities and Remark 4.13, we have

$$
\begin{aligned}
-\int_{\mathbb{R}^2} & Q_1(x)F(\omega+R_1y)\,dx \\
&\leq \int_{\mathbb{R}^2} Q_1(x)|\omega+R_1y|^2\,dx - C\int_{\mathbb{R}^2} Q_1(x)|\omega+R_1y|^\theta\,dx \\
&\leq \int_{\mathbb{R}^2} Q_1(x)|\omega|^2\,dx + R_1^2\int_{\mathbb{R}^2} Q_1(x)|y|^2\,dx - C\int_{\mathbb{R}^2} Q_1(x)|\omega|^\theta\,dx - CR_1^\theta\int_{\mathbb{R}^2} Q_1(x)|y|^\theta\,dx \\
&\leq \frac{1}{\lambda_{Q_1}}\|\omega\|_{(q)}^2 + \frac{R_1^2}{\lambda_{Q_1}}\|y\|_{(q)}^2 - CR_1^\theta\int_{\mathbb{R}^2} Q_1(x)|y|^\theta\,dx \\
&\leq \frac{R_0^2}{2\lambda_{Q_1}} + \frac{R_1^2}{\lambda_{Q_1}} - CR_1^\theta\int_{\mathbb{R}^2} Q_1(x)|y|^\theta\,dx.
\end{aligned}
$$

Since $y \neq 0$ and $R_0 = \sqrt{2}R_1$, for some $C > 0$ we obtain

$$
-\int_{\mathbb{R}^2} Q_1(x)F(\omega+R_1y)\,dx \leq \frac{2R_1^2}{\lambda_{Q_1}} - CR_1^\theta. \tag{5.55}
$$

Similarly, we have

$$
-\int_{\mathbb{R}^2} Q_2(x)G(-\widetilde{\omega}+R_1\widetilde{z})\,dx \leq \frac{2R_1^2}{\widetilde{\lambda}_{Q_2}} - CR_1^\theta. \tag{5.56}
$$

Then, using (5.55) and (5.56) in (5.54), we obtain

$$
J(\omega+R_1y, -\widetilde{\omega}+R_1\widetilde{z}) \leq R_1^2\left(1+\frac{2}{\lambda_{Q_1}}+\frac{2}{\widetilde{\lambda}_{Q_2}}\right) - CR_1^\theta.
$$

Since $\theta > 2$, taking $R_1$ sufficiently large, we get $J(\vartheta) \leq 0$.

■

### 5.3.3 Approximation finite dimensional

Let consider

$$
\Gamma_{n,m} = \{\gamma \in \mathscr{C}(Q_{n,m}, F_{n,m}) : \ \gamma(\vartheta) = \vartheta, \text{ for all } \vartheta \in \partial Q_{n,m}\}.
$$

and set

$$
c_{n,m} = \inf_{\gamma\in\Gamma_{n,m}} \max_{\vartheta\in Q_{n,m}} J(\gamma(\vartheta)). \tag{5.57}
$$

**Lemma 5.19.** (See Cassani and Tarsi (2009).) The sets $Q_{n,m}$ and $\partial B_\rho \cap F_{n,m}^+$ link, that is

$$
\gamma(Q_{n,m}) \cap (\partial B_\rho \cap E_n^+) \neq \emptyset, \quad \text{for all} \quad \gamma \in \Gamma_{n,m}, \tag{5.58}
$$

for $\rho > 0$ given by Lemma 5.17.

Thus, combining Lemma 5.17 with (5.57), we have

$$c_{n,m} \geq \sigma, \quad \text{for all} \quad n \geq 1. \tag{5.59}$$

Note also that, since the identity map $I : Q_{n,m} \to F_{n,m}$ belongs to $\Gamma_{n,m}$, for $\vartheta = (\omega, -\widetilde{\omega}) + s(y, \widetilde{z}) \in Q_{n,m}$, we have

$$c_{n,m} \leq \sup_{\vartheta \in Q_{n,m}} J(\vartheta) \leq R_1^2. \tag{5.60}$$

Denote $J_{n,m}$ the restriction of $J$ to the finite-dimensional space $F_{n,m}$. Then, applying the Linking theorem for $J_{n,m}$ and noticing (5.59) and (5.60), we get the following result:

**Proposition 5.20.** For each $n, m \geq 1$ ($m = m(n)$ as in Lemma 4.12), the functional $J_{n,m}$ has a Palais-Smale sequence at level $c_{n,m}$. More precisely, there is a sequence $(u_j, \widetilde{v}_j) \subset F_{n,m}$ such that

$$J_{n,m}(u_j, \widetilde{v}_j) \to c_{n,m} \in [\sigma, R_1^2]$$

and

$$J'_{|F_{n,m}}(u_j, \widetilde{v}_j) \to 0.$$

**Proposition 5.21.** Assume that $f$ and $g$ satisfy $(B_1) - (B_5)$ and let $(u_j, \widetilde{v}_j)$ be a sequence in $F_{n,m}$ given by Proposition 5.20. Then,

(i) The sequence $(u_j, \widetilde{v}_j)$ is bounded in $F_{n,m}$ and there exists $C > 0$ such that for each $j \geq 1$, we have

$$\int_{\mathbb{R}^2} Q_1(x) f(u_j) u_j \, dx \leq C, \quad \int_{\mathbb{R}^2} Q_2(x) g(\widetilde{v}_j) \, dx \leq C,$$
$$\int_{\mathbb{R}^2} Q_1(x) F(u_j) \, dx \leq C, \quad \text{and} \quad \int_{\mathbb{R}^2} Q_2(x) G(\widetilde{v}_j) \, dx \leq C.$$

(ii) For each sequence $(u_j, \widetilde{v}_j)$in $F_{n,m}$ there exist $(u_{n,m}, \widetilde{v}_{n,m}) \in F_{n,m}$ and a subsequence (not renamed) $(u_j, \widetilde{v}_j)$ such that

$$(u_j, \widetilde{v}_j) \to (u_{n,m}, \widetilde{v}_{n,m}) \quad \text{in} \quad F_{n,m}.$$

Furthermore,

$$J_{n,m}(u_{n,m}, \widetilde{v}_{n,m}) = c_{n,m} \in [\sigma, R_1^2]$$

and

$$J'_{|F_{n,m}}(u_{n,m}, \widetilde{v}_{n,m}) = 0.$$

(iii) The sequence $(u_{n,m}, \widetilde{v}_{n,m})$ is bounded in $E$ and there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} Q_1(x) f(u_{n,m}) u_{n,m} \, dx \leq C, \quad \int_{\mathbb{R}^2} Q_2(x) g(\widetilde{v}_{n,m}) \widetilde{v}_{n,m} \, dx \leq C,$$

$$\int_{\mathbb{R}^2} Q_1(x) F(u_{n,m}) \, dx \leq C \quad \text{and} \quad \int_{\mathbb{R}^2} Q_2(x) G(\widetilde{v}_{n,m}) \, dx \leq C,$$

for all $n \in \mathbb{N}$.

***Proof.***

(i) By Lemma 5.13, the sequence $(u_j, \widetilde{v}_j)$ is bounded in $F_{n,m}$. Moreover, by Remark 5.14 and $(B_2)$, we get the estimates given in $(i)$.

(ii) Since $(u_j, \widetilde{v}_j)$ is bounded, $F_{n,m}$ is finite dimensional and $J$ is of the class $\mathscr{C}^1$, the assertion follows.

(iii) Using the sequence $(u_{n,m}, \widetilde{v}_{n,m})$ in Lemma 5.13, for the case $e_{n,m} = 0$, we get the boundedness of the sequence, using again Remark 5.14 and $(B_2)$, we obtain the estimates.

■

### 5.3.4 Estimate of the minimax level

**Proposition 5.22.** There exists $k \in \mathbb{N}$ such that for any sequence

$$(u_{n,m}, \widetilde{v}_{n,m}) \in \mathbb{R}(M_{k,q;\frac{1}{m}}, M_{k,p;\frac{1}{m}}) \oplus E^-$$

such that

(i) The sequence $(u_{n,m}, \widetilde{v}_{n,m})$ is bounded in E

(ii) The sequence $(u_{n,m}, \widetilde{v}_{n,m})$ converges weakly to $(0,0)$ in $E$ and

$$u_{n,m} \to 0, \quad \widetilde{v}_{n,m} \to 0 \quad \text{in} \quad L^\lambda(\mathbb{R}^2), \quad \text{for all} \quad \lambda \geq \min\{p,q\}.$$

Then,

$$\sup_{n\in\mathbb{N}} J((u_{n,m}, \widetilde{v}_{n,m})) < \left(\frac{\alpha^*_{q,b_1}}{\alpha_0}\right)^{1/p} \left(\frac{\alpha^*_{p,b_2}}{\beta_0}\right)^{1/q}.$$

*Proof.* On the contrary, for each $k$ fixed in $\mathbb{N}$, there exist a sequence $(\tau_{n,k})$, a nonnegative sequence $\varepsilon_n \to 0$ and a sequence

$$\eta_{n,k} = \tau_{n,k}(M_{k,q;\frac{1}{m}}, M_{k,p;\frac{1}{m}}) + (u_{n,k}, -\widetilde{u}_{n,k}), \quad u_{n,k} \in E_{n,m}$$

such that

$$\|\eta_{n,k}\| \leq C = C(k),$$

$$\eta_{n,k} \rightharpoonup 0 \quad \text{in} \quad E,$$

$$\tau_{n,k} M_{k,q;\frac{1}{m}} + u_{n,k} \to 0, \quad \tau_{n,k} M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k} \to 0 \quad \text{in} \quad L^\lambda(\mathbb{R}^2), \quad \text{for all} \quad \lambda \geq \min\{p,q\},$$

and

$$J(\eta_{n,k}) \geq \left(\frac{\alpha^*_{q,b_1}}{\alpha_0}\right)^{1/p} \left(\frac{\alpha^*_{p,b_2}}{\beta_0}\right)^{1/q} - \varepsilon_n.$$

In particular, we have

$$\sup_{t\geq 0} J(t\eta_{n,k}) \geq J(\eta_{n,k}) \geq \Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q} - \varepsilon_n.$$

Since $J(0)=0$ and $J(t\eta_{n,k}) \to -\infty$ and $t\to+\infty$, there exists $\hat{t}>0$ such that

$$\sup_{t\geq 0} J(t\eta_{n,k}) = \max_{t\geq 0} J(t\eta_{n,k}) = J(\hat{t}\eta_{n,k}).$$

We can assume without loss of generality that $\hat{t}=1$, that is

$$J'(\eta_{n,k})\eta_{n,k} = 0 \quad \text{and} \quad J(\eta_{n,k}) \geq \Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q} - \varepsilon_n.$$

Then,

$$\begin{aligned}
2&\int_{\mathbb{R}^2} \nabla(\tau_{n,k}M_{k,q;\frac{1}{m}} + u_{n,k})\nabla(\tau_{n,k}M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})\,dx \\
&\quad + 2\int_{\mathbb{R}^2} V(x)(\tau_{n,k}M_{k,q;\frac{1}{m}} + u_{n,k})(\tau_{n,k}M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})\,dx \\
&= \int_{\mathbb{R}^2} Q_1(x)f(\tau_{n,k}M_{k,q;\frac{1}{m}} + u_{n,k})(\tau_{n,k}M_{k,q;\frac{1}{m}} + u_{n,k})\,dx \\
&\quad + \int_{\mathbb{R}^2} Q_2(x)g(\tau_{n,k}M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})(\tau_{n,k}M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})\,dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2} \nabla(\tau_{n,k}M_{k,q;\frac{1}{m}} + u_{n,k})\nabla(\tau_{n,k}M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})\,dx \\
&\quad + \int_{\mathbb{R}^2} V(x)(\tau_{n,k}M_{k,q;\frac{1}{m}} + u_{n,k})(\tau_{n,k}M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})\,dx \\
&\quad - \int_{\mathbb{R}^2} Q_1(x)F(\tau_{n,k}M_{k,q;\frac{1}{m}} + u_{n,k})\,dx - \int_{\mathbb{R}^2} Q_2(x)G(\tau_{n,k}M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})\,dx \\
&\geq \Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q} - \varepsilon_n.
\end{aligned}$$

Since $\|M_{k,q;\frac{1}{m}}\|_{(q)} = \|M_{k,p;\frac{1}{m}}\|_{(p)} = 1$, $\|u_{n,k}\|_{(q)} = \|\widetilde{u}_{n,k}\|_{(p)}$ and the support sets of $u_{n,k},\widetilde{u}_{n,k}$ and the concentrating functions are disjoint, we obtain

$$\begin{aligned}
2\tau_{n,k}^2 \geq 2(\tau_{n,k}^2 - \|u_{n,k}\|^2_{(q)}) &\geq \int_{\mathbb{R}^2} Q_1(x)f(\tau_{n,k}M_{k,q;\frac{1}{m}} + u_{n,k})(\tau_{n,k}M_{k,q;\frac{1}{m}} + u_{n,k})\,dx \\
&\quad + \int_{\mathbb{R}^2} Q_2(x)g(\tau_{n,k}M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})(\tau_{n,k}M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})\,dx \qquad (5.61)
\end{aligned}$$

and

$$\begin{aligned}
\tau_{n,k}^2 - \|u_{n,k}\|^2_{(q)} - \int_{\mathbb{R}^2} Q_1(x)F(\tau_{n,k}M_{k,q;\frac{1}{m}} + u_{n,k}) - \int_{\mathbb{R}^2} Q_2(x)G(\tau_{n,k}M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})\,dx \\
\geq \Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q} - \varepsilon_n. \qquad (5.62)
\end{aligned}$$

Since $F$ and $G$ are nonnegative functions from (5.62), we obtain

$$\tau_{n,k}^2 \geq \Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q} - \varepsilon_n.$$

Denote

$$s_{n,k} := \tau_{n,k}^2 - \Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q} \geq -\varepsilon_n. \tag{5.63}$$

From $(H_5)$, for any $R>0$ there exists $T_R>0$ such that

$$tf(t) \geq Re^{\alpha_0 t^p} \quad \text{and} \quad tg(t) \geq Re^{\beta_0 t^q}, \quad \text{for all} \quad |t| \geq T_R.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^2} Q_1(x) f(\tau_{n,k} M_{k,q;\frac{1}{m}} + u_{n,k})(\tau_{n,k} M_{k,q;\frac{1}{m}} + u_{n,k})\,dx \\
&\qquad + \int_{\mathbb{R}^2} Q_2(x) g(\tau_{n,k} M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})(\tau_{n,k} M_{k,p;\frac{1}{m}} - \widetilde{u}_{n,k})\,dx \\
&\qquad \geq R\int_{\{x\in B_{\frac{1}{m}}: |\tau_{n,k} M_{k,q;\frac{1}{m}}| \geq T_R\}} Q_1(x) e^{\alpha_0 |\tau_{n,k} M_{k,q;\frac{1}{m}}|^p}\,dx \\
&\qquad + R\int_{\{x\in B_{\frac{1}{m}}: |\tau_{n,k} M_{k,p;\frac{1}{m}}\}| \geq T_R\}} Q_2(x) e^{\beta_0 |\tau_{n,k} M_{k,p;\frac{1}{m}}|^q}\,dx,
\end{aligned} \tag{5.64}$$

where we have used the fact that the functions $u_{n,k}$ and $\widetilde{u}_{n,k}$ are zero in $B_{\frac{1}{m}}$. From the definition of the concentrate function, we have

$$M_{k,q;\frac{1}{m}}(x) = \frac{(\log k)^{\frac{q-1}{q}}}{\sqrt{4\pi}}(1-\delta_{k,q,\frac{1}{m}})^{\frac{q-1}{q}}, \quad \text{if} \quad |x| \leq \frac{1}{m\sqrt{k}}.$$

From (5.63), we can fixed $n$ sufficiently large such that

$$\tau_{n,k} M_{k,q;\frac{1}{m}}(x) = \tau_{n,k}\frac{(\log k)^{\frac{q-1}{q}}}{\sqrt{4\pi}}(1-\delta_{k,q,\frac{1}{m}})^{\frac{q-1}{q}} \geq T_R, \quad \text{if} \quad |x| \leq \frac{1}{m\sqrt{k}},$$

for $k \geq k_R$, for some $k_R$ sufficiently large. Note that $k_R$ is independent of $n$. From (5.11), (5.61) and (5.64), we get

$$\begin{aligned}
\tau_{n,k}^2 &\geq \frac{R}{2}\int_{B_{\frac{1}{m\sqrt{k}}}} Q_1(x) e^{\alpha_0 \tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-\delta_{k,q,\frac{1}{m}})}\,dx + \frac{R}{2}\int_{B_{\frac{1}{m\sqrt{k}}}} Q_2(x) e^{\beta_0 \tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}(1-\delta_{k,p,\frac{1}{m}})}\,dx \\
&\geq \frac{\pi R C_3 e^{\alpha_0 \tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-\delta_{k,q,\frac{1}{m}})}}{2(2+b_1)m^{2+b_1}k^{(1+b_1/2)}} + \frac{\pi R C_3 e^{\beta_0 \tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}(1-\delta_{k,p,\frac{1}{m}})}}{2(2+b_2)m^{2+b_2}k^{(1+b_2/2)}} \\
&\geq \frac{\pi R}{2(2+b_0)m^{2+b_0}}\Big(e^{\alpha_0 \tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-\delta_{k,q,\frac{1}{m}})-(1+\frac{b_1}{2})\ln k} + e^{\beta_0 \tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}(1-\delta_{k,p,\frac{1}{m}})-(1+\frac{b_2}{2})\ln k}\Big)
\end{aligned} \tag{5.65}$$

where $b_0 = \max\{b_1, b_2\}$. We observe that until now, we have fixed $n$ (and consequently, $m$), where $k$ can be arbitrarily chosen, sufficiently large and independent of $n$ (that is $k \geq K_R$). By (4.11), we have

$$|\delta_{k,q,\frac{1}{m}}| \leq C\|V\|_{L^\infty(B_{1/m})}\frac{1}{m^{q/2}\ln k} \quad \text{and} \quad |\delta_{k,p,\frac{1}{m}}| \leq C\|V\|_{L^\infty(B_{1/m})}\frac{1}{m^{p/2}\ln k}.$$

Taking $n_0 \in \mathbb{N}$ such that

$$(1-\delta_{k,q,\frac{1}{m}}) > 1 - C\frac{V_1}{\ln k} \quad \text{and} \quad (1-\delta_{k,p,\frac{1}{m}}) > 1 - C\frac{V_1}{\ln k}, \quad \text{where} \quad V_1 = \|V\|_{L^\infty(B_1)} \geq V_0.$$

for all $n \geq n_0$ and for some $C > 0$. Replacing in (5.65), we get

$$\tau_{n,k}^2 \geq \frac{\pi R C_3}{2(2+b_0)m^{2+b_0}} \left( e^{\alpha_0 \tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-C\frac{V_1}{\ln k})-(1+\frac{b_1}{2})\ln k} + e^{\beta_0 \tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}(1-C\frac{V_1}{\ln k})-(1+\frac{b_2}{2})\ln k} \right).$$

Using Young's inequality $X^p/p + Y^q/q \geq XY$ in last inequality with

$$X = p^{1/p} e^{\frac{\alpha_0}{p}\tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-C\frac{V_1}{\ln k})-\frac{(1+b_1/2)\ln k}{p}} \quad \text{and} \quad Y = q^{1/q} e^{\frac{\beta_0}{q}\tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}(1-C\frac{V_1}{\ln k})-\frac{(1+b_2/2)\ln k}{q}},$$

we obtain

$$\tau_{n,k}^2 \geq \frac{\pi R C_3 p^{1/p} q^{1/q}}{(2+b_0)m^{2+b_0}} e^{\left(\frac{\alpha_0}{p}\tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}} + \frac{\beta_0}{q}\tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}}\right)(1-C\frac{V_1}{\ln k})-\frac{(1+b_1/2)\ln k}{p}-\frac{(1+b_2/2)\ln k}{q}}. \tag{5.66}$$

By Young's inequality again, we get

$$\frac{\alpha_0}{p}\tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}} + \frac{\beta_0}{q}\tau_{n,k}^q \frac{\ln k}{(4\pi)^{q/2}} \geq \alpha_0^{1/p}\beta_0^{1/q}\frac{\tau_{n,k}^2}{4\pi}\ln k.$$

Replacing in (5.66), we have

$$\tau_{n,k}^2 \geq \frac{\pi R C_3 p^{1/p} q^{1/q}}{(2+b_0)m^{2+b_0}} e^{\alpha_0^{1/p}\beta_0^{1/q}\frac{\tau_{n,k}^2}{4\pi}(1-C\frac{V_1}{\ln k})\ln k-\frac{(1+b_1/2)\ln k}{p}-\frac{(1+b_2/2)\ln k}{q}}. \tag{5.67}$$

for all $n \geq n_0$ and $k \geq k_R$. We now choose $k$ depending on $n$. For any $n$ arbitrarily large, take

$$R := 2C_3 m^{3+b_0}(2+b_0), \quad \text{where} \quad m = m(n)$$

and consequently, $k \geq k_R = k(n)$. With this choice, we obtain

$$\tau_{n,k}^2 \geq \pi m p^{1/p} q^{1/q} e^{[\alpha_0^{1/p}\beta_0^{1/q}\frac{\tau_{n,k}^2}{4\pi}(1-C\frac{V_1}{\ln k})-\frac{(1+b_1/2)}{p}-\frac{(1+b_2/2)}{q}]\ln k}. \tag{5.68}$$

If the sequence $\{\tau_{n,k}^2\}_{n\geq n_0}$ is unbounded, using (5.68), we get a contradiction. Thus, the sequence $\{\tau_{n,k}^2\}_{n\geq n_0}$ is a bounded. In particular, without loss of generality, we can assume that there exists $s \in \mathbb{R}$ such that

$$\tau_{n,k}^2 = s_{n,k} + \left(\frac{\alpha_{q,b_1}^*}{\alpha_0}\right)^{1/p}\left(\frac{\alpha_{p,b_2}^*}{\beta_0}\right)^{1/q} \to s + \left(\frac{\alpha_{q,b_1}^*}{\alpha_0}\right)^{1/p}\left(\frac{\alpha_{p,b_2}^*}{\beta_0}\right)^{1/q}.$$

Moreover, by (5.63), $s \geq 0$. From $(H_6)$, we can suppose without loss of generality that

$$\left(\frac{\alpha_0 \min\{1, 1+\frac{b_1}{2}\}}{(1+\frac{b_1}{2})^2}\right)^{1/p} > \left(\frac{\beta_0}{\min\{1, 1+\frac{b_2}{2}\}}\right)^{1/q}. \tag{5.69}$$

By (5.65), we have

$$\tau_{n,k}^2 \geq \pi m e^{\alpha_0 \tau_{n,k}^p \frac{\ln k}{(4\pi)^{p/2}}(1-C\frac{V_1}{\ln k})-(1+b_1/2)\ln k}, \tag{5.70}$$

writing

$$\tau_{n,k}^2 = s + \left(\frac{\alpha_{q,b_1}^*}{\alpha_0}\right)^{1/p}\left(\frac{\alpha_{p,b_2}^*}{\beta_0}\right)^{1/q} + o_n(1),$$

and replacing in (5.70), we obtain

$$\begin{aligned}
\tau_{n,k}^2 &\geq \pi m e^{\alpha_0\left(s+\left(\frac{\alpha^*_{q,b_1}}{\alpha_0}\right)^{1/p}\left(\frac{\alpha^*_{p,b_2}}{\beta_0}\right)^{1/q}+o_n(1)\right)^{p/2}\frac{\ln k}{(4\pi)^{p/2}}(1-C\frac{V_1}{\ln k})-(1+b_1/2)\ln k}\\
&\geq \pi m e^{\alpha_0\left(\frac{4\pi(\min\{1,1+b_1/2\})^{1/p}(\min\{1+b_2/2\})^{1/q}}{\alpha_0^{1/p}\beta_0^{1/q}}+o_n(1)\right)^{p/2}\frac{\ln k}{(4\pi)^{p/2}}(1-C\frac{V_1}{\ln k})-(1+b_1/2)\ln k}\\
&\geq \pi m e^{\left(\frac{\alpha_0^{1/2}(\min\{1,1+b_1/2\})^{1/2}(\min\{1,1+b_2/2\})^{p/2q}}{\beta_0^{p/2q}}+o_n(1)\right)\ln k(1-C\frac{V_1}{\ln k})-(1+b_1/2)\ln k}\\
&\geq \pi m e^{\left(\frac{\alpha_0^{1/2}(\min\{1,1+b_1/2\})^{1/2}(\min\{1,1+b_2/2\})^{p/2q}}{\beta_0^{p/2q}}+o_n(1)-(1+b_1/2)\right)\ln k}\times\\
&\quad e^{-\left(\frac{CV_1\alpha_0^{1/2}(\min\{1,1+b_1/2\})^{1/2}(\min\{1,1+b_2/2\})^{p/2q}}{\beta_0^{p/2q}}+o_n(1)\right)}.
\end{aligned}$$

From (5.69),

$$0<\delta:=\frac{\alpha_0^{1/2}(\min\{1,1+b_1/2\})^{1/2}(\min\{1,1+b_2/2\})^{p/2q}}{\beta_0^{p/2q}}-(1+b_1/2).$$

Thus,

$$s+\left(\frac{\alpha^*_{q,b_1}}{\alpha_0}\right)^{\frac{1}{p}}\left(\frac{\alpha^*_{p,b_2}}{\beta_0}\right)^{\frac{1}{q}}+o_n(1)\geq \pi m e^{\left(\delta+o_n(1)\right)\ln k-\left(\frac{CV_1\alpha_0^{1/2}(\min\{1,1+b_1/2\})^{1/2}(\min\{1,1+b_2/2\})^{p/2q}}{\beta_0^{p/2q}}+o_n(1)\right)}.$$

Taking $n\to+\infty$ (and hence $k\to+\infty$), we get a contradiction and the claim of the Proposition follows. □

## 5.4 Proof of Theorem 5.2

***Proof.*** From Proposition 5.21, there exists a sequence $(u_{n,m},\widetilde{v}_{n,m})\in F_{n,m}$ such that

$$J_{n,m}(u_{n,m},\widetilde{v}_{n,m})=c_{n,m}\in[\sigma,R_1^2] \tag{5.71}$$

and

$$J'_{n,m}(u_{n,m},\widetilde{v}_{n,m})(\phi,\widetilde{\psi})=0,\quad \text{for all}\quad (\phi,\widetilde{\psi})\in F_{n,m}. \tag{5.72}$$

Moreover, the sequence $(u_{n,m},\widetilde{v}_{n,m})$ is bounded in $E$. Thus, we can assume that there exists $(u,\widetilde{v})\in E$ such that $(u_{n,m},\widetilde{v}_{n,m})\rightharpoonup(u,v)$ in $E$ and

$$u_{n,m}\to u\quad\text{and}\quad \widetilde{v}_{n,m}\to\widetilde{v}\quad\text{in}\quad L^r(\mathbb{R}^2),\quad\text{for all}\quad r\geq\min\{p,q\}.$$

Taking $(0,\widetilde{\psi})$ and $(\phi,0)$ in (5.72) with $(\phi,\widetilde{\psi})\in F_{n,m}\cap\left(\mathscr{C}_0^\infty(\mathbb{R}^2)\times\mathscr{C}_0^\infty(\mathbb{R}^2)\right)$, we have

$$\int_{\mathbb{R}^2}\left(\nabla u_{n,m}\nabla\widetilde{\psi}+V(x)u_{n,m}\widetilde{\psi}\right)dx=\int_{\mathbb{R}^2}Q_2(x)g(\widetilde{v}_{n,m})\widetilde{\psi}\,dx$$

and

$$\int_{\mathbb{R}^2} \left(\nabla \widetilde{v}_{n,m} \nabla \phi + V(x) \widetilde{v}_{n,m} \phi\right) dx = \int_{\mathbb{R}^2} Q_1(x) f(u_{n,m}) \phi \, dx.$$

Taking the limit as $n \to +\infty$ and using Lemma 5.16 and the fact that $\bigcup_{n=1}^{+\infty} F_{n,m} \cap \left(\mathscr{C}_0^\infty(\mathbb{R}^2) \times \mathscr{C}_0^\infty(\mathbb{R}^2)\right)$ is dense in $E$, we obtain

$$\int_{\mathbb{R}^2} \left(\nabla u \nabla \widetilde{\psi} + V(x) u \widetilde{\psi}\right) dx = \int_{\mathbb{R}^2} Q_2(x) g(\widetilde{v}) \widetilde{\psi} \, dx, \quad \text{for all} \quad \widetilde{\psi} \in E^{(p)}$$

and

$$\int_{\mathbb{R}^2} \left(\nabla \widetilde{v} \nabla \phi + V(x) \widetilde{v} \phi\right) dx = \int_{\mathbb{R}^2} Q_1(x) f(u) \phi \, dx, \quad \text{for all} \quad \phi \in E^{(q)}.$$

Thus, $(u, \widetilde{v}) \in E$ is a solution of the system. Now, we prove that $(u, \widetilde{v})$ is a nontrivial solution. Assume by contradiction that ($u \equiv 0$ which implies that $\widetilde{v} \equiv 0$). Thus, we can assume that

$$u_{n,m} \to 0 \quad \text{and} \quad \widetilde{v}_{n,m} \to 0 \quad \text{in} \quad L^r(\mathbb{R}^2), \quad \text{for all} \quad r \geq \min\{p, q\}. \tag{5.73}$$

Taking $(0, \widetilde{v}_{n,m})$ and $(u_{n,m}, 0)$ in (5.72), we have

$$\int_{\mathbb{R}^2} \left(\nabla u_{n,m} \nabla \widetilde{v}_{n,m} + V(x) u_{n,m} \widetilde{v}_{n,m}\right) dx = \int_{\mathbb{R}^2} Q_2(x) g(\widetilde{v}_{n,m}) \widetilde{v}_{n,m} \, dx = \int_{\mathbb{R}^2} Q_1(x) f(u_{n,m}) u_{n,m} \, dx. \tag{5.74}$$

By Proposition 5.22, there exists $\delta' > 0$ such that

$$c_{n,m} \leq \left(\frac{\alpha^*_{q,b_1}}{\alpha_0}\right)^{1/p} \left(\frac{\alpha^*_{p,b_2}}{\beta_0}\right)^{1/q} - \delta'.$$

Moreover, there exists $\delta > 0$ such that

$$\left[\left(\frac{\alpha^*_{q,b_1}}{\alpha_0}\right)^{1/p} \left(\frac{\alpha^*_{p,b_2}}{\beta_0}\right)^{1/q} - \frac{\delta'}{4}\right] \left[1 - \frac{\delta \alpha_0}{\alpha^*_{q,b_1}}\right]^{-1/p} \leq \left(\frac{\alpha^*_{q,b_1}}{\alpha_0}\right)^{1/p} \left(\frac{\alpha^*_{p,b_2}}{\beta_0}\right)^{1/q} - \frac{\delta'}{8} \tag{5.75}$$

and

$$\left[\left(\frac{\alpha^*_{q,b_1}}{\alpha_0}\right)^{1/p} \left(\frac{\alpha^*_{p,b_2}}{\beta_0}\right)^{1/q} - \frac{\delta'}{4}\right] \left[1 - \frac{\delta \beta_0}{\alpha^*_{p,b_2}}\right]^{-1/q} \leq \left(\frac{\alpha^*_{q,b_1}}{\alpha_0}\right)^{1/p} \left(\frac{\alpha^*_{p,b_2}}{\beta_0}\right)^{1/q} - \frac{\delta'}{8}.$$

Taking $(v_{n,m}, 0)$ in (5.72), we obtain

$$\|v_{n,m}\|^2_{(q)} = \int_{\mathbb{R}^2} Q_1(x) f(u_{n,m}) v_{n,m} \, dx \leq \int_{\mathbb{R}^2} Q_1(x) |f(u_{n,m})| |v_{n,m}| \, dx. \tag{5.76}$$

Let

$$V_{n,m} = \left(\frac{\alpha^*_{q,b_1}}{\alpha_0} - \delta\right)^{1/p} \frac{v_{n,m}}{\|v_{n,m}\|_{(q)}}. \tag{5.77}$$

Applying Lemma 3.10 in (5.76) with $s = |f(u_{n,m})|/\alpha_0^{1/p}$, $t = \alpha_0^{1/p}|V_{n,m}|$, $r = p$ and $r' = q$, we obtain

$$\begin{aligned}
\left(\frac{\alpha^*_{q,b_1}}{\alpha_0} - \delta\right)^{1/p} \|v_{n,m}\|_{(q)} &\leq \int_{\mathbb{R}^2} Q_1(x) |f(u_{n,m})| |V_{n,m}| \, dx \\
&\leq \int_{\mathbb{R}^2} Q_1(x) (e^{\alpha_0 |V_{n,m}|^p} - 1) \, dx + \frac{1}{\alpha_0^{q/p} q} \int_{\{x \in \mathbb{R}^2 : \frac{|f(u_{n,m})|}{\alpha_0^{1/p}} \leq e^{1/p^q}\}} Q_1(x) |f(u_{n,m})|^q \, dx \\
&\quad + \frac{1}{\alpha_0^{1/p}} \int_{\{x \in \mathbb{R}^2 : \frac{|f(u_{n,n})|}{\alpha_0^{1/p}} \leq e^{1/p^q}\}} Q_1(x) |f(u_{n,m})| \left[\ln\left(\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\right)\right]^{1/p} dx.
\end{aligned} \tag{5.78}$$

Note that $V_{n,m} \rightharpoonup 0$ in $E^{(q)}$ and

$$\|V_{n,m}\|_{(q)}^{p} < \frac{\alpha_{q,b_1}^{*}}{\alpha_0}.$$

Thus, by Lemma 5.12, we have

$$\int_{\mathbb{R}^2} Q_1(x)(e^{\alpha_0|V_{n,m}|^p} - 1)\,dx = o_n(1). \tag{5.79}$$

Now, we estimate the second integral of (5.78). From $(B_1)$, we can find $C > 0$ such that

$$|f(t)|^q \le C|t|^q, \quad \text{for all} \quad \{t \in \mathbb{R} : \frac{|f(t)|}{\alpha_0^{1/p}} \le e^{1/p^q}\}. \tag{5.80}$$

Since we can suppose that $u_{n,m} \to 0$ in $L^q(\mathbb{R}^2, Q_1)$, then,

$$\int_{\{x\in\mathbb{R}^2:\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\le e^{1/p^q}\}} Q_1(x)|f(u_{n,m})|^q\,dx \le C\int_{\{x\in\mathbb{R}^2:\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\le e^{1/p^q}\}} Q_1(x)|u_{n,m}|^q\,dx = o_n(1). \tag{5.81}$$

For

$$\xi = \frac{\delta' \min\{\alpha_0, \beta_0\}}{4\Big[\Big(\frac{\alpha_{q,b_1}^{*}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha_{p,b_2}^{*}}{\beta_0}\Big)^{1/q} - \delta'\Big]},$$

by $(B_1)$ and $(B_4)$, there exists $C_\xi > 0$ such that

$$|f(t)| \le C_\xi e^{(\alpha_0+\xi)|t|^p}, \quad \text{for all} \quad t \in \mathbb{R}.$$

Hence,

$$\begin{aligned}
&\int_{\{x\in\mathbb{R}^2:\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\le e^{1/p^q}\}} Q_1(x)|f(u_{n,m})|\Big[\ln\Big(\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\Big)\Big]^{1/p}dx \\
&\qquad \le \int_{\mathbb{R}^2} Q_1(x)|f(u_{n,m})|\Big[\ln\Big(\frac{C_\xi e^{(\alpha_0+\xi)|u_{n,m}|^p}}{\alpha_0^{1/p}}\Big)\Big]^{1/p}dx \\
&\qquad \le \int_{\mathbb{R}^2} Q_1(x)|f(u_{n,m})|\Big[\ln^{1/p}\big(\frac{C_\xi}{\alpha_0^{1/p}}\big) + (\alpha_0+\xi)^{1/p}|u_{n,m}|\Big]dx.
\end{aligned} \tag{5.82}$$

For each $n \in \mathbb{N}$, denote

$$T_n := \Big\{x \in \mathbb{R}^2 : \ln^{1/p}\Big(\frac{C_\xi}{\alpha_0^{1/p}}\Big) + (\alpha_0+\xi)^{1/p}|u_{n,m}| \le (\alpha_0+2\xi)^{1/p}|u_{n,m}|\Big\}.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^2} Q_1(x)|f(u_{n,m})|\Big[\ln^{1/p}\big(\frac{C_\xi}{\alpha_0^{1/p}}\big) + (\alpha_0+\xi)^{1/p}|u_{n,m}|\Big]dx \\
&\qquad \le \int_{\mathbb{R}^2\setminus T_n} Q_1(x)|f(u_{n,m})|\Big[\ln^{1/p}\big(\frac{C_\xi}{\alpha_0^{1/p}}\big) + (\alpha_0+\xi)^{1/p}|u_{n,m}|\Big]dx \\
&\qquad\quad + (\alpha_0+2\xi)^{1/p}\int_{T_n} Q_1(x)f(u_{n,m})u_{n,m}\,dx \\
&\qquad \le \ln^{1/p}\big(\frac{C_\xi}{\alpha_0^{1/p}}\big)\int_{\mathbb{R}^2\setminus T_n} Q_1(x)|f(u_{n,m})|\,dx + (\alpha_0+2\xi)^{1/p}\int_{\mathbb{R}^2} Q_1(x)f(u_{n,m})u_{n,m}\,dx.
\end{aligned} \tag{5.83}$$

Observe that

$$\mathbb{R}^2\backslash T_n = \{x \in \mathbb{R}^2 : |u_{n,m}| < d_1\}, \quad \text{where} \quad d_1 = \frac{\ln^{1/p}\big(\frac{C_\xi}{\alpha_0^{1/p}}\big)}{(\alpha_0+2\xi)^{1/p}-(\alpha_0+\xi)^{1/p}}.$$

Thus,

$$\mathbb{R}^2\backslash T_n \subseteq \{x \in \mathbb{R}^2 : |f(u_{n,m})| \leq d_2\}, \quad \text{where} \quad d_2 = \max_{|s|\leq d_1} |f(s)|.$$

By $(B_1)$, we can find a constant $C > 0$ such that $|f(s)| \leq C|s|^{min\{p,q\}}$ for all $s \in \mathbb{R}$ such that $|f(s)| \leq d_2$. Thus, by Lemma 5.6, we get

$$\int_{\mathbb{R}^2\backslash T_n} Q_1(x)|f(u_{n,m})|\,dx \leq C\int_{\mathbb{R}^2\backslash T_n} Q_1(x)|u_{n,m}|^{min\{p,q\}}\,dx = o_n(1). \tag{5.84}$$

From (5.82), (5.83) and (5.84), we obtain

$$\begin{aligned}\int_{\{x\in\mathbb{R}^2:\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\leq e^{1/p^q}\}} & Q_1(x)|f(u_{n,m})|\Big[\ln\Big(\frac{|f(u_{n,m})|}{\alpha_0^{1/p}}\Big)\Big]^{1/p}dx \\ &\leq (\alpha_0+2\xi)^{1/p}\int_{\mathbb{R}^2} Q_1(x)f(u_{n,m})u_{n,m}\,dx + o_n(1).\end{aligned} \tag{5.85}$$

Using this, (5.79) and (5.81) in (5.78), we have

$$\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}-\delta\Big)^{1/p}\|v_{n,m}\|_{(q)} \leq \Big(1+\frac{2\xi}{\alpha_0}\Big)^{1/p}\int_{\mathbb{R}^2} Q_1(x)f(u_{n,m})u_{n,m}\,dx + o_n(1). \tag{5.86}$$

Taking $(0,\widetilde{u}_{n,m})$ in (5.72), we find

$$\|\widetilde{u}_{n,m}\|^2_{(p)} = \int_{\mathbb{R}^2} Q_2(x)g(\widetilde{v}_{n,m})\widetilde{u}_{n,m}\,dx.$$

Analogously, we can obtain

$$\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}-\delta\Big)^{1/q}\|\widetilde{u}_{n,m}\|_{(p)} \leq \Big(1+\frac{2\xi}{\beta_0}\Big)^{1/q}\int_{\mathbb{R}^2} Q_2(x)g(\widetilde{v}_{n,m})\widetilde{v}_{n,m}\,dx + o_n(1). \tag{5.87}$$

By Lemma (5.16), we have

$$\int_{\mathbb{R}^2} Q_1(x)F(u_{n,m})\,dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} Q_2(x)G(\widetilde{v}_{n,m})\,dx \to 0, \tag{5.88}$$

which implies

$$\int_{\mathbb{R}^2}\big(\nabla u_{n,m}\nabla\widetilde{v}_{n,m}+V(x)u_{n,m}\widetilde{v}_{n,m}\big)\,dx = J(u_{n,m},\widetilde{v}_{n,m})+o_n(1).$$

Thus,

$$\int_{\mathbb{R}^2} Q_1(x)f(u_{n,m})u_{n,m}\,dx + \int_{\mathbb{R}^2} Q_(x)g(\widetilde{v}_{n,m})\widetilde{v}_{n,m}\,dx \leq 2\Big[\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q}-\delta'\Big]+o_n(1).$$

Using (5.86), (5.87) and the assumption on $\xi$, we have

$$\begin{aligned}
&\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}-\delta\Big)^{1/p}\|v_{n,m}\|_{(q)}+\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}-\delta\Big)^{1/q}\|\widetilde{u}_{n,m}\|_{(p)}\\
&\le\Big(1+\frac{2\xi}{\alpha_0}\Big)^{1/p}\int_{\mathbb{R}^2}Q_1(x)f(u_{n,m})u_{n,m}\,dx+\Big(1+\frac{2\xi}{\beta_0}\Big)^{1/q}\int_{\mathbb{R}^2}Q_2(x)g(\widetilde{v}_{n,m})\widetilde{v}_{n,m}\,dx+o_n(1)\\
&\le\Big(1+\frac{2\xi}{\min\{\alpha_0,\beta_0\}}\Big)\Big(\int_{\mathbb{R}^2}Q_1(x)f(u_{n,m})u_{n,m}\,dx+\int_{\mathbb{R}^2}Q_2(x)g(\widetilde{v}_{n,m})\widetilde{v}_{n,m}\,dx\Big)+o_n(1)\\
&\le 2\Big(1+\frac{2\xi}{\min\{\alpha_0,\beta_0\}}\Big)\Big[\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q}-\delta'\Big]+o_n(1)\\
&\le 2\Big[\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q}-\frac{\delta'}{2}\Big]+o_n(1).
\end{aligned}$$

We can assume that

$$\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}-\delta\Big)^{1/p}\|v_{n,m}\|_{(q)}+\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}-\delta\Big)^{1/q}\|\widetilde{u}_{n,m}\|_{(p)}\le 2\Big[\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q}-\frac{\delta'}{4}\Big],$$

for all $n\in\mathbb{N}$. Thus, we can assume

$$\|v_{n,m}\|_{(q)}\le\Big[\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q}-\frac{\delta'}{4}\Big]\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}-\delta\Big)^{-1/p}$$

or

$$\|\widetilde{u}_{n,m}\|_{(p)}\le\Big[\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q}-\frac{\delta'}{4}\Big]\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}-\delta\Big)^{-1/q}.$$

Supposing the second and using (5.75), we have

$$\begin{aligned}
\|\widetilde{u}_{n,m}\|_{(p)}&\le\Big(\frac{\beta_0}{\alpha^*_{p,b_2}}\Big)^{1/q}\Big[\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}\Big(\frac{\alpha^*_{p,b_2}}{\beta_0}\Big)^{1/q}-\frac{\delta'}{8}\Big]\\
&\le\Big(\frac{\alpha^*_{q,b_1}}{\alpha_0}\Big)^{1/p}-\frac{\delta'}{8}\Big(\frac{\beta_0}{\alpha^*_{p,b_2}}\Big)^{1/q}.
\end{aligned}$$

In particular, $\alpha_0\|u_{n,m}\|^p_{(q)}<\alpha^*_{p,b_2}$ for all $n\ge 1$. Thus, we can find $r>1$ and $\eta>0$ such that $r'=r/(r-1)\ge 2$ and

$$r(\alpha_0+\eta)\|u_{n,m}\|^p_{(q)}<\alpha^*_{p,b_2}. \tag{5.89}$$

By $(B_1)$ and $(B_4)$, there exists $C_1>0$ such that

$$|f(s)|\le|s|+C_1\big(e^{(\alpha_0+\eta)|s|^p}-1\big),\quad\text{for all}\quad s\in\mathbb{R}.$$

By Hölder's inequality , Lemma 4.14 and Proposition 5.8, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2}Q_1(x)f(u_{n,m})u_{n,m}\,dx\\
&\qquad\le\int_{\mathbb{R}^2}Q_1(x)|u_{n,m}|^2\,dx+C_1\int_{\mathbb{R}^2}Q_1(x)|u_{n,m}|\big(e^{(\alpha_0+\eta)|u_{n,m}|^p}-1\big)\,dx\\
&\qquad\le\|u_{n,m}\|_{L^2(\mathbb{R}^2,Q_1)}+C_1\|u_{n,m}\|_{L^{r'}(\mathbb{R}^2,Q_1)}\int_{\mathbb{R}^2}Q_1(x)\big(e^{(\alpha_0+\eta)|u_{n,m}|^p}-1\big)^r\,dx\\
&\qquad\le\|u_{n,m}\|_{L^2(\mathbb{R}^2,Q_1)}+C_1\|u_{n,m}\|_{L^{r'}(\mathbb{R}^2,Q_1)}\int_{\mathbb{R}^2}Q_1(x)\big(e^{r(\alpha_0+\eta)|u_{n,m}|^p}-1\big)\,dx\\
&\qquad\le\|u_{n,m}\|_{L^2(\mathbb{R}^2,Q_1)}+C_2\|u_{n,m}\|_{L^{r'}(\mathbb{R}^2,Q_1)}.
\end{aligned}$$

Using (5.73), we get

$$\int_{\mathbb{R}^2} Q_1(x) f(u_{n,m}) u_{n,m}\, dx \to 0.$$

Replacing in (5.74), we have

$$\int_{\mathbb{R}^2} \left(\nabla u_{n,m} \nabla \widetilde{v}_{n,m} + V(x) u_{n,m} \widetilde{v}_{n,m}\right) dx \to 0.$$

Combining this with (5.88), we get

$$J(u_{u,m}, \widetilde{v}_{n,m}) \to 0,$$

which gives a contradiction with the fact that $c_{n,m} \geq \sigma$. Thus, $(u, \widetilde{v})$ is a nontrivial weak solution.

■

CHAPTER

6

# HAMILTONIAN SYSTEMS WITH POTENTIALS WHICH CAN VANISH AT INFINITY

In this chapter we study the following Hamiltonian system

$$\begin{cases} -\Delta u + V(x)u = g(v), & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = f(u), & x \in \mathbb{R}^2, \end{cases} \tag{6.1}$$

where the functions $f$ and $g$ possess critical exponential growth and the potential $V$ can be vanish at infinity.

## 6.1 Introduction

First, we recall the assumptions on $V$

$(V_1)$ $V \in \mathscr{C}(\mathbb{R}^2, \mathbb{R})$ is a radially symmetric positive function.

$(V_2)$ There exist constants $a, b, R_0, L_a$ and $L_b$, with $0 < a < 2$, $b \leq a$, $R_0 > 1$, $L_a \geq R_0^a$ and $L_a R_0^{b-a} \leq L_b \leq L_a^{(2-b)/(2-a)} \pi^{2(a-b)/(2-a)}$, such that

$$\frac{L_a}{|x|^a} \leq V(x) \leq \frac{L_b}{|x|^b}, \quad \text{for all} \quad |x| \geq R_0.$$

$(V_3)$ $V(x) = 1$ for all $|x| \leq 1$ and $V(x) \geq 1$ for all $1 < |x| < R_0$, for $R_0$ given by $(V_2)$.

Before stating the assumption on the nonlinearities of $f$ and $g$, we define the energy space which will be use to set the variational structure. Following Albuquerque, Ó and Medeiros (2016), we let $H^1_{V,rad}(\mathbb{R}^2)$ denote the subspace of the radially symmetric functions in the closure of $\mathscr{C}_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|u\| = \|u\|_{H_V^1} := \Big(\int_{\mathbb{R}^2} |\nabla u|^2 + V(x)u^2\,dx\Big)^{1/2}.$$

For $1 \le p < +\infty$, we define

$$L^p_{V,rad}(\mathbb{R}^2) := \{u : \mathbb{R}^2 \to \mathbb{R}; u \text{ is measurable, radial and } \int_{\mathbb{R}^2} V(x)|u|^p\,dx < +\infty\}$$

endowed with the norm

$$\|u\|_{L_V^p} = \left(\int_{\mathbb{R}^2} V(x)|u|^p\,dx\right)^{1/p}.$$

Thus,

$$H^1_{V,rad}(\mathbb{R}^2) = \{u \in L^2_{V,rad}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2)\}.$$

We note that $H^1_{V,rad}(\mathbb{R}^2)$ is a Hilbert space endowed with inner product

$$\langle u, v\rangle := \int_{\mathbb{R}^2} \big(\nabla u \nabla v + V(x)uv\big)\,dx, \quad u, v \in H^1_{V,rad}(\mathbb{R}^2).$$

Now, we state a basic embedding result (see Su, Wang and Willem (2007a), Su, Wang and Willem (2007b), for a proof).

**Lemma 6.1.** Suppose $V$ satisfies $(V_1) - (V_3)$. Taking $R_0, a$ and $b$ given by $(V_2)$, consider $a^* = (4+2a)/(2-a)$ and $b^* = 2(2-2b+a)/(2-a)$. Then,

(i) The embedding $H^1_{V,rad}(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ is continuous for $a^* \le p < \infty$ and compact for $a^* < p < \infty$.

(ii) The embedding $H^1_{V,rad}(\mathbb{R}^2) \hookrightarrow L^p_{V,rad}(\mathbb{R}^2)$ is continuous for $b^* \le p < \infty$ and compact for $b^* < p < \infty$.

(iii) The embedding $H^1_{V,rad}(B_R) \hookrightarrow H^1_0(B_R)$ is continuous for $R \ge 1$.

**Remark 6.2.** 1. As we will see further on (Lemma 6.14), conditions $(V_1) - (V_3)$ will be employed to show that we obtain Sobolev embedding inequalities as above with constants that do not depend on $L_a$.

2. As a consequence of $(iii)$ and Sobolev embedding theorem, the space $H^1_{V,rad}(\mathbb{R}^2)$ is compactly immersed in $L^p(B_R)$ for all $1 \le p < +\infty$.

Concerning the functions $f$ and $g$, we suppose the following assumptions:

$(H_1)$ $f, g \in \mathscr{C}(\mathbb{R})$ and $f(s) = g(s) = 0$ for all $s \le 0$.

Taking $b^* \in \mathbb{R}$ as in Lemma 6.1, consider

$(H_2)$ There exist constants $\mu > b^*$ and $\nu > b^*$ such that

$$0 < \mu F(s) \leq sf(s), \quad 0 < \nu G(s) \leq sg(s), \quad \text{for all} \quad s > 0,$$

where $F(s) = \int_0^s f(t)\,dt$ and $G(s) = \int_0^s g(t)\,dt$.

$(H_3)$ There exist constants $s_1 > 0$ and $M > 0$ such that

$$0 < F(s) \leq Mf(s) \quad \text{and} \quad 0 < G(s) \leq Mg(s), \quad \text{for all} \quad s > s_1.$$

Setting $\mu$ and $\nu$ given by $(H_2)$ and $a$ given by $(V_2)$, we suppose:

$(H_4)$ There exists $\theta \geq 4a/(2-a)$ such that $f(s) = O(s^{\mu-1+\theta})$ and $g(s) = O(s^{\nu-1+\theta})$ as $s \to 0^+$.

$(H_5)$ There exists $\alpha_0 > 0$ such that

$$\lim_{s\to\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0, \end{cases} \quad \text{and} \quad \lim_{s\to\infty} \frac{g(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0. \end{cases}$$

$(H_6)$ For $\alpha_0 > 0$ given by $(H_5)$, we have

$$\liminf_{t\to+\infty} \frac{tf(t)}{e^{\alpha_0 t^2}} > \frac{4e}{\alpha_0} \quad \text{and} \quad \liminf_{t\to+\infty} \frac{tg(t)}{e^{\alpha_0 t^2}} > \frac{4e}{\alpha_0}.$$

In the literature, the condition $(H_5)$ says that $f$ and $g$ have critical growth in the Trudinger-Moser sense (see Adimurthi (1990) and also Figueiredo, Miyagaki and Ruf (1995)).

The following theorem contains our main result.

**Theorem 6.3.** Suppose that $V$ satisfies $(V_1)-(V_3)$ and $f$ and $g$ satisfy $(H_1)-(H_6)$. Then, there exists $L^* = L^*(f,g,\mu,\nu,\alpha_0,\theta,a,b,R_0) > 0$ such that system (6.1) possesses a nontrivial weak solution $(u,v) \in H^1_{V,rad}(\mathbb{R}^2) \times H^1_{V,rad}(\mathbb{R}^2)$ provided that $L_a \geq L^*$, namely $(u,v) \in H^1_{V,rad}(\mathbb{R}^2) \times H^1_{V,rad}(\mathbb{R}^2)$ satisfies

$$\int_{\mathbb{R}^2} (\nabla u \nabla \psi + V(x)u\psi + \nabla v \nabla \phi + V(x)v\phi)\,dx = \int_{\mathbb{R}^2} (f(u)\phi + g(v)\psi)\,dx,$$

for all $(\phi,\psi) \in H^1_{V,rad}(\mathbb{R}^2) \times H^1_{V,rad}(\mathbb{R}^2)$.

## 6.2 Preliminaries

In the first result of this section, we follow Su, Wang and Willem (2007b) to prove a version of the Strauss result (Strauss (1977)) for the functions of our space.

**Lemma 6.4.** Suppose that $(V_1)$ and $(V_2)$ hold. Then

$$|w(x)| \leq \frac{\|w\|}{L_a^{1/4}\pi^{1/2}|x|^{\frac{2-a}{4}}}, \quad \text{for all} \quad |x| \geq R_0,$$

for every $w \in H^1_{V,rad}(\mathbb{R}^2)$.

***Proof.*** Let $w \in \mathscr{C}_{0,rad}^{\infty}(\mathbb{R}^2)$, $w(x) = \phi(r)$ where $|x| = r$ we have

$$\frac{d(r^{\frac{2-a}{2}}\phi^2(r))}{dr} = \frac{(2-a)\phi^2(r)}{2r^{a/2}} + 2r^{\frac{2-a}{2}}\phi'(r)\phi(r) \geq 2r^{\frac{2-a}{2}}\phi'(r)\phi(r).$$

Thus, we obtain for all $r \geq R_0$

$$-\int_r^{+\infty} \frac{d(s^{\frac{2-a}{2}}\phi^2(s))}{ds}\, ds = r^{\frac{2-a}{2}}\phi^2(r) - \lim_{r\to+\infty} r^{\frac{2-a}{2}}\phi^2(r) = r^{\frac{2-a}{2}}\phi^2(r).$$

Thus,

$$\begin{aligned}
r^{\frac{2-a}{2}}\phi^2(r) &= \int_r^{+\infty} -\frac{d(s^{\frac{2-a}{2}}\phi^2(s))}{ds}\, ds \\
&\leq \int_r^{+\infty} -2s^{\frac{2-a}{2}}\phi'(s)\phi(s)\, ds \\
&\leq 2\int_r^{+\infty} |\phi'(s)|\sqrt{s}|\phi(s)|\frac{\sqrt{s}}{s^{a/2}}\, ds \\
&\leq 2\Big(\int_r^{+\infty} |\phi'(s)|^2 s\, ds\Big)^{1/2}\Big(\int_r^{+\infty} \frac{|\phi(s)|^2}{s^a} s\, ds\Big)^{1/2} \\
&\leq \frac{1}{\pi L_a^{1/2}}\Big(2\pi\int_r^{+\infty} |\phi'(s)|^2 s\, ds\Big)^{1/2}\Big(2\pi\int_r^{+\infty} \frac{L_a}{s^a}|\phi(s)|^2 s\, ds\Big)^{1/2} \\
&\leq \frac{1}{\pi L_a^{1/2}}\Big(\int_{\mathbb{R}^2\setminus B_r} |\nabla w|^2\, dx\Big)^{1/2}\Big(\int_{\mathbb{R}^2\setminus B_r} V(x)w^2\, dx\Big)^{1/2} \\
&\leq \frac{1}{\pi L_a^{1/2}}\int_{\mathbb{R}^2} (|\nabla w|^2 + V(x)w^2)\, dx.
\end{aligned}$$

■

Inspired by similar arguments developed in Albuquerque, Alves and Medeiros (2014), Ó (1997), Ruf (2005), we establish the following version of the Trudinger-Moser inequality which will be used throughout this paper.

**Proposition 6.5.** Assume $V$ satisfies $(V_1)$ and $(V_2)$. Then,

$$\int_{\mathbb{R}^2}\Big(e^{\alpha|u|^2} - \sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\Big)\, dx < +\infty, \quad \text{for all} \quad u \in H^1_{V,rad}(\mathbb{R}^2) \text{ and } \alpha > 0 \tag{6.2}$$

where $j_a = [|4/(2-a)|]$. Furthermore, if $0 < \alpha < 4\pi$, there exists a positive constant $c = c(\alpha, a, R_0)$ such that

$$\sup_{u\in H^1_{V,rad}, \|u\|\leq 1}\int_{\mathbb{R}^2}\Big(e^{\alpha u^2} - \sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\Big)\, dx \leq c. \tag{6.3}$$

***Proof.*** Set $r > 0$. For every $u \in H^1_{V,rad}(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2}\Big(e^{\alpha|u|^2} - \sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\Big)\, dx = \Big(\int_{B_r} + \int_{\mathbb{R}^2\setminus B_r}\Big)\Big(e^{\alpha|u|^2} - \sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\Big)\, dx. \tag{6.4}$$

In order estimate the first integral on the right hand side of (6.4), we define the function

$$v(x)=\begin{cases}u(x)-u(rx_0), & 0\le |x|\le r,\\ 0, & |x|>r.\end{cases}$$

where $x_0\in\mathbb{R}^2$ such that $|x_0|=1$. Fix $\varepsilon>0$. By Young's inequality, we get

$$|u(x)|^2\le(1+\varepsilon)|v(x)|^2+(1+\frac{4}{\varepsilon})|u(rx_0)|^2.$$

By Lemma 6.4, for $r\ge\max\{R_0,(1+\frac{4}{\varepsilon})^{\frac{2}{2-a}}\}$, we obtain

$$|u(x)|^2\le(1+\varepsilon)|v(x)|^2+(1+\frac{4}{\varepsilon})\frac{\|u\|^2}{r^{\frac{2-a}{2}}}\le(1+\varepsilon)|v(x)|^2+\|u\|^2,$$

which implies

$$\begin{aligned}\int_{B_r}\Big(e^{\alpha|u|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\Big)\,dx&\le\int_{B_r}e^{\alpha|u|^2}\,dx\\&\le\int_{B_r}e^{\alpha\big((1+\varepsilon)|v|^2+\|u\|^2\big)}\,dx\\&\le e^{\alpha\|u\|^2}\int_{B_r}e^{\alpha(1+\varepsilon)|v|^2}\,dx.\end{aligned}$$

Since $v\in H_0^1(B_r)$, we have

$$\int_{B_r}\Big(e^{\alpha|u|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\Big)\,dx<+\infty,\quad\text{for all}\quad u\in H^1_{V,rad}(\mathbb{R}^2).\tag{6.5}$$

Moreover, if $\varepsilon>0$ is sufficiently small such that $\alpha(1+\varepsilon)\le 4\pi$ and noticing that $\|u\|\le 1$ implies that $\|\nabla v\|_{L^2(B_r)}\le 1$, by (**??**), we find

$$\int_{B_r}\Big(e^{\alpha|u|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\Big)\,dx\le e^{\alpha\|u\|^2}\sup_{\|\nabla v\|_{L^2(B_r)}\le 1}\int_{B_r}e^{\alpha(1+\varepsilon)|v|^2}\,dx\le ce^{\alpha\|u\|^2},$$

for all $u\in H^1_{V,rad}(\mathbb{R}^2)$ such that $\|u\|\le 1$, for some positive constant $c=c(\alpha,a,R_0)$. Thus,

$$\sup_{\|u\|\le 1}\int_{B_r}\Big(e^{\alpha|u|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\Big)\,dx\le c.\tag{6.6}$$

By Lemma 6.4, we have

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_r}\Big(e^{\alpha|u|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\Big)\,dx&=\int_{\mathbb{R}^2\setminus B_r}\sum_{j=j_a+1}^{+\infty}\frac{\alpha^j|u|^{2j}}{j!}\,dx\\&\le\sum_{j=j_a+1}^{+\infty}\frac{\alpha^j\|u\|^{2j}}{j!}\int_{\mathbb{R}^2\setminus B_r}\frac{1}{|x|^{\frac{(2-a)j}{2}}}\,dx\\&=2\pi\sum_{j=j_a+1}^{+\infty}\frac{\alpha^j\|u\|^{2j}}{j!}\int_r^{+\infty}s^{\frac{(a-2)j}{2}+1}\,ds.\end{aligned}$$

Since $\frac{(a-2)j}{2}+2 \le \frac{(a-2)(j_a+1)}{2}+2<0$ for all $j \ge j_a+1$, we obtain

$$\int_r^{+\infty} s^{\frac{(a-2)j}{2}+1}\,ds \le \int_r^{+\infty} s^{\frac{(a-2)(j_a+1)}{2}+1}\,ds = \frac{1}{\left(\frac{(2-a)(j_a+1)}{2}-2\right) r^{\frac{(2-a)(j_a+1)}{2}-2}}.$$

Thus,

$$\int_{\mathbb{R}^2\setminus B_r}\left(e^{\alpha|u|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\right)dx \le \frac{2\pi e^{\alpha\|u\|^2}}{\left(\frac{(2-a)(j_a+1)}{2}-2\right) r^{\frac{(2-a)(j_a+1)}{2}-2}}. \tag{6.7}$$

Hence,

$$\int_{\mathbb{R}^2\setminus B_r}\left(e^{\alpha|u|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\right)dx < +\infty, \quad \text{for all} \quad u \in H^1_{V,rad}(\mathbb{R}^2).$$

This last inequality combined with (6.5) gives (6.2). Furthermore, from (6.7), we obtain

$$\sup_{\|u\|\le 1}\int_{\mathbb{R}^2\setminus B_r}\left(e^{\alpha|u|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u|^{2j}}{j!}\right)dx \le c, \tag{6.8}$$

for some positive constant $c=c(\alpha,a,R_0)$. Finally, using (6.6) and (6.8), we obtain (6.3).

■

The following result may be proved in much the same way as Lemma 2.2 in Ó, Medeiros and Severo (2008).

**Lemma 6.6.** Let $\alpha>0$ and $m>1$. Then, for each $n>m$ there exists a positive constant $C=C(n)$ such that

$$\left(e^{\alpha t^2}-\sum_{j=0}^{j_a}\frac{\alpha^j t^{2j}}{j!}\right)^m \le C\left(e^{n\alpha t^2}-\sum_{j=0}^{j_a}\frac{n^j\alpha^j t^{2j}}{j!}\right), \quad \text{for all} \quad t\in\mathbb{R}.$$

**Lemma 6.7.** Let $\{u_n \in H^1_{V,rad}(\mathbb{R}^2); \|u_n\|=1\}$ be a sequence converging weakly to the zero function in $H^1_{V,rad}(\mathbb{R}^2)$. Then, for every $0<\alpha<4\pi$, we can find a subsequence (not renamed) such that

$$\lim_{n\to\infty}\int_{B_R}\left(e^{\alpha|u_n|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u_n|^{2j}}{j!}\right)dx=0,$$

where $R\ge 1$.

***Proof.*** Let $\varepsilon>0$ such that $\alpha+\varepsilon<4\pi$. We have the following limits

$$\lim_{|t|\to 0}\frac{e^{\alpha t^2}-\sum_{j=0}^{j_a}\frac{\alpha^j t^{2j}}{j!}}{|t|}=0 \quad \text{and} \quad \lim_{|t|\to\infty}\frac{e^{\alpha t^2}-\sum_{j=0}^{j_a}\frac{\alpha^j t^{2j}}{j!}}{|t|\left(e^{(\alpha+\varepsilon)t^2}-\sum_{j=0}^{j_a}\frac{(\alpha+\varepsilon)^j t^{2j}}{j!}\right)}=0.$$

Thus, there exists $C>0$ such that

$$e^{\alpha t^2}-\sum_{j=0}^{j_a}\frac{\alpha^j t^{2j}}{j!} \le C|t|+C|t|\left(e^{(\alpha+\varepsilon)t^2}-\sum_{j=0}^{j_a}\frac{(\alpha+\varepsilon)^j t^{2j}}{j!}\right) \quad \text{for all} \quad t\in\mathbb{R}.$$

Hence,

$$\int_{B_R}\Big(e^{\alpha|u_n|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u_n|^{2j}}{j!}\Big)\,dx \le C\int_{B_R}|u_n|\,dx$$
$$+C\int_{B_R}|u_n|\Big(e^{(\alpha+\varepsilon)|u_n|^2}-\sum_{j=0}^{j_a}\frac{(\alpha+\varepsilon)^j|u_n|^{2j}}{j!}\Big)\,dx. \tag{6.9}$$

In order to estimate the last integral, we use Hölder inequality for $r>1$ such that $r(\alpha+\varepsilon)<4\pi$ and taking $r_0>r$ close to $r$ such that $r_0(\alpha+\varepsilon)<4\pi$, by Lemma 6.6, we get

$$\int_{B_R}|u_n|\Big(e^{(\alpha+\varepsilon)|u_n|^2}-\sum_{j=0}^{j_a}\frac{(\alpha+\varepsilon)^j|u_n|^{2j}}{j!}\Big)\,dx$$
$$\le C\|u_n\|_{L^{r'}(B_R)}\Big(\int_{B_R}\Big(e^{(\alpha+\varepsilon)|u_n|^2}-\sum_{j=0}^{j_a}\frac{(\alpha+\varepsilon)^j|u_n|^{2j}}{j!}\Big)^r dx\Big)^{\frac{1}{r}}$$
$$\le C\|u_n\|_{L^{r'}(B_R)}\Big(\int_{B_R}\Big(e^{r_0(\alpha+\varepsilon)|u_n|^2}-\sum_{j=0}^{j_a}\frac{\big(r_0(\alpha+\varepsilon)\big)^j|u_n|^{2j}}{j!}\Big)\,dx\Big)^{\frac{1}{r}}.$$

Using $\|u_n\|=1$ for $n\ge 1$ and the fact that $r_0(\alpha+\varepsilon)<4\pi$ in Proposition 6.5, we obtain

$$\int_{B_R}|u_n|\Big(e^{(\alpha+\varepsilon)|u_n|^2}-\sum_{j=0}^{j_a}\frac{(\alpha+\varepsilon)^j|u_n|^{2j}}{j!}\Big)\,dx \le C\|u_n\|_{L^{r'}(B_R)} \tag{6.10}$$

Replacing (6.10) in (6.9), yields

$$\int_{B_R}\Big(e^{\alpha|u_n|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u_n|^{2j}}{j!}\Big)\,dx \le C\|u_n\|_{L^1(B_R)}+C\|u_n\|_{L^{r'}(B_R)}. \tag{6.11}$$

Using that $u_n \rightharpoonup 0$ in $H^1_{V,rad}(\mathbb{R}^2)$ and Remark 6.2 for a subsequence, we have

$$\int_{B_R}\Big(e^{\alpha|u_n|^2}-\sum_{j=0}^{j_a}\frac{\alpha^j|u_n|^{2j}}{j!}\Big)\,dx \to 0.$$

■

Throughout what follows, we define the product space

$$E=H^1_{V,rad}(\mathbb{R}^2)\times H^1_{V,rad}(\mathbb{R}^2),$$

endowed with the inner product

$$\langle (u,v),(\phi,\psi)\rangle_E=\int_{\mathbb{R}^2}\big(\nabla u\nabla\phi+V(x)u\phi+\nabla v\nabla\psi+V(x)v\psi\big)\,dx$$

for all $(u,v),(\phi,\psi)\in E$, to which corresponds the norm

$$\|(u,v)\|=\|(u,v)\|_E=\big(\|u\|^2+\|v\|^2\big)^{1/2}.$$

We say that $(u,v)\in E$ is a weak solution of (6.1) if

$$\int_{\mathbb{R}^2}\big(\nabla u\nabla\psi+V(x)u\psi+\nabla v\nabla\phi+V(x)v\phi\big)\,dx=\int_{\mathbb{R}^2}\big(f(u)\phi+g(v)\psi\big)\,dx, \tag{6.12}$$

for all $(\phi,\psi)\in E$.

### 6.2.1 The auxiliary functional

Given $R \geq R_0$, we define a function $\widetilde{f} : \mathbb{R}^2 \times [0,+\infty) \to [0,+\infty)$ by

$$\widetilde{f}(x,t) = \begin{cases} f(t), & |x| \leq R, \\ \widehat{f}(x,t), & |x| > R. \end{cases}$$

where $\widehat{f} : \mathbb{R}^2 \times [0,+\infty) \to [0,+\infty)$ is defined by $\widehat{f}(x,t) = \min\{f(t), V(x)t^{\mu-1}\}$, for $\mu > b^*$ given by $(H_2)$. Similarly, we define $\widetilde{g} : \mathbb{R}^2 \times [0,+\infty) \to [0,+\infty)$ by

$$\widetilde{g}(x,t) = \begin{cases} g(t), & |x| \leq R, \\ \widehat{g}(x,t), & |x| > R. \end{cases}$$

where $\widehat{g} : \mathbb{R}^2 \times [0,+\infty) \to [0,+\infty)$ is defined by $\widehat{g}(x,t) = \min\{g(t), V(x)t^{\nu-1}\}$, with $\nu > b^*$ given by $(H_2)$. Moreover, we set $\widetilde{f}(x,t) = 0$ and $\widetilde{g}(x,t) = 0$ for $t \leq 0$.

**Lemma 6.8.** Suppose that $f$ and $g$ satisfy $(H_1) - (H_2)$.Then,

$$0 < \mu \widetilde{F}(x,t) \leq t\widetilde{f}(x,t) \quad \text{and} \quad 0 < \nu \widetilde{G}(x,t) \leq t\widetilde{g}(x,t) \quad \text{for all} \quad t > 0,$$

where $\mu, \nu > b^*$ are given by $(H_2)$, $\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)\,ds$ and $\widetilde{G}(x,s) = \int_0^t \widetilde{g}(x,s)\,ds$.

***Proof.*** If $|x| \leq R$, we have $\widetilde{f}(x,t) = f(t)$, and hence

$$\frac{t\widetilde{f}(x,t)}{\mu} = \frac{tf(t)}{\mu} \geq F(t) = \widetilde{F}(x,t).$$

If $|x| > R$ and $\widetilde{f}(x,t) = V(x)t^{\mu-1}$, we obtain

$$\begin{aligned} \frac{t\widetilde{f}(x,t)}{\mu} &= \frac{V(x)t^{\mu}}{\mu} = \int_0^t V(x)s^{\mu-1}\,ds \\ &\geq \int_0^t \min\{f(s), V(x)s^{\mu-1}\}\,ds \\ &= \int_0^t \widehat{f}(x,s)\,ds = \int_0^t \widetilde{f}(x,s)\,ds \\ &= \widetilde{F}(x,t). \end{aligned}$$

If $|x| > R$ and $\widetilde{f}(x,t) = f(t)$, we get

$$\begin{aligned} \frac{t\widetilde{f}(x,t)}{\mu} &= \frac{tf(t)}{\mu} \geq F(t) = \int_0^t f(s)\,ds \\ &\geq \int_0^t \min\{f(s), V(x)s^{\mu-1}\}\,ds \\ &= \int_0^t \widehat{f}(x,s)\,ds = \int_0^t \widetilde{f}(x,s)\,ds \\ &= \widetilde{F}(x,t). \end{aligned}$$

Similar arguments apply to function $\widetilde{g}$. ■

Using the functions $\widetilde{f}$ and $\widetilde{g}$, we consider the following auxiliary functional $\widetilde{J}: E \to \mathbb{R}$ defined by

$$\widetilde{J}(u,v) = \int_{\mathbb{R}^2} \big(\nabla u \nabla v + V(x)uv\big)\, dx - \int_{\mathbb{R}^2} \widetilde{F}(x,u)\, dx - \int_{\mathbb{R}^2} \widetilde{G}(x,v)\, dx,$$

for all $(u,v) \in E$. From the conditions on $\widetilde{f}$ and $\widetilde{g}$, the functional $\widetilde{J}$ is well defined.

Fix $1 \le p < +\infty$. We consider the subspace $\Xi^p = H^1_{V,rad}(\mathbb{R}^2) \cap L^p_V(\mathbb{R}^2)$ endowed with the norm

$$\|u\|_{\Xi^p} := \|u\|_{H^1_V(\mathbb{R}^2)} + \|u\|_{L^p_V(\mathbb{R}^2)}.$$

**Lemma 6.9.** If $u_n \to u$ in $\Xi^p$, then there exist a subsequence $(w_n)$ of $(u_n)$ and $g$ in $L^p_V(\mathbb{R}^2)$ such that, almost everywhere in $\mathbb{R}^2$, $w_n(x) \to u(x)$ and

$$|u(x)|, |w_n(x)| \le g(x).$$

***Proof.*** Note that we can assume that $u_n \to u$ almost everywhere in $\mathbb{R}^2$. Also we can extract a subsequence $(w_n)$ of $(u_n)$ such that

$$\|w_{j+1} - w_j\|_{L^p_V(\mathbb{R}^2)} \le \frac{1}{2^j} \quad \text{for all} \quad j \ge 1.$$

Let us define

$$g_n(x) := |w_1(x)| + \sum_{j=1}^{n} |w_{j+1}(x) - w_j(x)| \quad \text{and} \quad g(x) := |w_1(x)| + \sum_{j=1}^{+\infty} |w_{j+1}(x) - w_j(x)|.$$

Thus, $|u(x)|, |w_n(x)| \le g(x)$ almost everywhere in $\mathbb{R}^2$. By the monotone convergence theorem, $g_n \to g$ almost everywhere in $\mathbb{R}^2$. Furthermore, $(Vg_n^p)$ is a non-decreasing sequence and

$$\begin{aligned}\int_{\mathbb{R}^2} V g_n^p\, dx &= \|g_n\|^p_{L^p_V(\mathbb{R}^2)} \\ &\le \Big(\|w_1\|_{L^p_V(\mathbb{R}^2)} + \sum_{j=1}^{n} \|w_{j+1} - w_j\|_{L^p_V(\mathbb{R}^2)}\Big)^p \le \big(\|g_1\|_{L^p_V(\mathbb{R}^2)} + 1\big)^p.\end{aligned}$$

By the monotone convergence theorem, we have $\int_{\mathbb{R}^2} V g^p\, dx < +\infty$, that is, $g \in L^p_V(\mathbb{R}^2)$.

■

**Lemma 6.10.** The functional $\widetilde{J}$ belongs to $\mathscr{C}^1(E,\mathbb{R})$ and

$$\begin{aligned}\widetilde{J}'(u,v)(\phi,\psi) &= \int_{\mathbb{R}^2} \big(\nabla u \nabla \psi + V(x)u\psi + \nabla v \nabla \phi + V(x)v\phi\big)\, dx \\ &\quad - \int_{\mathbb{R}^2} \widetilde{f}(x,u)\phi\, dx - \int_{\mathbb{R}^2} \widetilde{g}(x,v)\psi\, dx,\end{aligned}$$

for all $(u,v)$, $(\phi,\psi) \in E$.

***Proof.*** Setting $\widetilde{J}_{F_1} : H^1(B_R) \to \mathbb{R}$ and $\widetilde{J}_{F_2} : \Xi^\mu \to \mathbb{R}$ defined by

$$\widetilde{J}_{F_1}(u) = \int_{\mathbb{R}^2} \widetilde{F}(x,u)\chi_{B_R}(x)\,dx \quad \text{and} \quad \widetilde{J}_{F_2}(u) = \int_{\mathbb{R}^2} \widetilde{F}(x,u)\big(1-\chi_{B_R}(x)\big)\,dx.$$

We recall the existence of an extension operator $P : H^1(B_R) \to H^1(\mathbb{R}^2)$ such that $Pu_{|B_R} = u$. Thus, from Ó (1997), for all $u \in H^1(B_R)$ and $\alpha > 0$, we have

$$\int_{B_R} (e^{\alpha|u|^2} - 1)\,dx = \int_{B_R} (e^{\alpha|Pu|^2} - 1)\,dx \le \int_{\mathbb{R}^2} (e^{\alpha|Pu|^2} - 1)\,dx < +\infty,$$

which implies

$$\int_{B_R} e^{\alpha|u|^2} < +\infty, \quad \text{for all} \quad u \in H^1(B_R), \alpha > 0. \tag{6.13}$$

We observe that

$$\widetilde{J}_{F_1}(u) = \int_{B_R} \widetilde{F}(x,u)\,dx, \quad \text{for all} \quad u \in H^1(B_R).$$

Note also that for $\alpha > \alpha_0$ there exists $c > 0$ such that

$$f(s) \le ce^{\alpha|s|^2} \quad \text{for all} \quad s \in \mathbb{R}. \tag{6.14}$$

Thus, for $|x| \le R$, we have

$$|\widetilde{F}(x,t)| = |\int_0^t \widetilde{f}(x,s)\,ds| \le \int_0^t f(s)\,ds \le c\int_0^{|t|} e^{\alpha|s|^2}\,ds \le \frac{c}{2}(e^{2\alpha|t|^2} + |t|^2). \tag{6.15}$$

From (6.13), (6.15) and the embedding of $H^1(B_R)$ in $L^2(B_R)$, we obtain

$$\int_{B_R} \widetilde{F}(x,u)\,dx < +\infty, \quad \text{for all} \quad u \in H^1(B_R).$$

Thus, $\widetilde{J}_{F_1}$ is well defined. Now, set $u, v \in H^1(B_R)$ and $0 < |t| < 1$. By the mean value theorem, there exists $\theta(x,t) \in (0,1)$ such that

$$\frac{\widetilde{F}(x,u+tv) - \widetilde{F}(x,u)}{t} = \widetilde{f}(x, u + \theta(x,t)tv)v. \tag{6.16}$$

Since the function $\widetilde{f}(x,t)$ is continuous in the second variable, it follows that

$$\lim_{t\to 0} \frac{\widetilde{F}(x,u+tv) - \widetilde{F}(x,u)}{t} = \widetilde{f}(x,u).$$

Moreover, using (6.14) in (6.16) and the fact that $\widetilde{f}(x,t) \le f(t)$, we get

$$\left|\frac{\widetilde{F}(x,u+tv) - \widetilde{F}(x,u)}{t}\right| \le ce^{\alpha(|u|+|v|)^2}|v| \le \frac{c}{2}(e^{2\alpha(|u|+|v|)^2} + |v|^2) \ \in L^1(B_R).$$

From the dominated convergence theorem, we find

$$\begin{aligned}
\widetilde{J}'_{F_1}(u)v &= \lim_{t\to 0} \frac{\widetilde{J}_{F_1}(u+tv) - \widetilde{J}_{F_1}(u)}{t} \\
&= \lim_{t\to 0} \int_\Omega \frac{\widetilde{F}(x,u+tv) - \widetilde{F}(x,u)}{t}\,dx \\
&= \int_{B_R} \lim_{t\to 0} \frac{\widetilde{F}(x,u+tv) - \widetilde{F}(u)}{t}\,dx \\
&= \int_{B_R} \widetilde{f}(x,u)v\,dx.
\end{aligned}$$

In order to prove the continuity of $\widetilde{J}_{F_1}$, let $(u_n)$ be a sequence in $H^1(B_R)$ such that $u_n \to u$ in $H^1(B_R)$. Arguing similarly as Proposition 2.7 in Ó, Medeiros and Severo (2008), we can assume that $u_n \to u$ almost everywhere in $B_R$ and there exists $v \in H^1(B_R)$ such that $|u_n(x)| \leq v(x)$ almost everywhere in $B_R$. Consequently,

$$|\widetilde{f}(x,u_n) - \widetilde{f}(x,u)|^2 \leq 2c(e^{2\alpha|v|^2} + e^{2\alpha|u|^2}) \in L^1(B_R),$$

and by the continuity of $\widetilde{f}$ almost everywhere in $B_R$, we get

$$|\widetilde{f}(x,u_n) - \widetilde{f}(x,u)|^2 \to 0 \text{ almost everywhere in } B_R.$$

By Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} \|\widetilde{J}'_{F_1}(u_n) - \widetilde{J}'_{F_1}(u_n)\| &= \sup_{\|v\|_{H^1(B_R)}\leq 1} |< \widetilde{J}'_{F_1}(u_n) - \widetilde{J}'_{F_1}(u), v >| \\ &= \sup_{\|v\|_{H^1(B_R)}\leq 1} \left| \int_{B_R} \big(\widetilde{f}(x,u_n) - \widetilde{f}(x,u)\big) v \, dx \right| \\ &\leq \sup_{\|v\|_{H^1(B_R)}\leq 1} \|\widetilde{f}(x,u_n) - \widetilde{f}(x,u)\|_{L^2(B_R)} \|v\|_{L^2(B_R)} \\ &= o_n(1). \end{aligned}$$

Thus, $\widetilde{J}_{F_1} \in \mathscr{C}^1(H^1(B_R),\mathbb{R})$. Since $H^1_{V,rad}(\mathbb{R}^2) \hookrightarrow H^1(B_R)$ continuously, it follows that $\widetilde{J}_{F_1} \in \mathscr{C}^1(H^1_{V,rad}(\mathbb{R}^2),\mathbb{R})$. In other hand, $\widetilde{F}(x,s)(1-\chi_{B_R}(x))$ is a Carathéodory function in $(x,s) \in \mathbb{R}^2 \times \mathbb{R}$ and

$$|\widetilde{f}(x,s)\big(1-\chi_{B_R}(x)\big)| \leq V(x)|s|^{\mu-1}, \quad \text{for all} \quad (x,s) \in \mathbb{R}^2 \times \mathbb{R}.$$

Using Lemma 6.9 and arguing similarly as Lemma 17.1 in Kavian (1993), we have $\widetilde{J}_{F_2} \in \mathscr{C}^1(\Xi^\mu,\mathbb{R})$ and since the embedding $H^1_{V,rad}(\mathbb{R}^2) \hookrightarrow \Xi^\mu$ is continuous, we have $\widetilde{J}_{F_2} \in \mathscr{C}^1(H^1_{V,rad}(\mathbb{R}^2),\mathbb{R})$. Thus,

$$\widetilde{J}_F(u) = \int_{\mathbb{R}^2} \widetilde{F}(x,u)\, dx, \quad \text{for all} \quad u \in H^1_{V,rad}(\mathbb{R}^2),$$

is of class $\mathscr{C}^1$ in $H^1_{V,rad}(\mathbb{R}^2)$ and

$$\widetilde{J}'_F(u)(\phi) = \int_{\mathbb{R}^2} \widetilde{f}(x,u)\phi \, dx, \quad \text{for all} \quad \phi \in H^1_{V,rad}(\mathbb{R}^2).$$

A similar result holds for the function $\widetilde{G}$ and the conclusion follows. ■

## 6.3 The geometry of the linking theorem

This section is devoted to set the geometry of the linking theorem of the auxiliary functional. We begin by considering the following subspaces:

$$E^+ = \{(u,u) \in E\} \quad \text{and} \quad E^- = \{(u,-u) \in E\},$$

so that

$$E = E^+ \oplus E^-.$$

**Lemma 6.11.** Suppose $(V_1),(V_2)$, $(H_1),(H_4)$ and $(H_5)$ holds. Then, there exist $\sigma,\rho>0$ such that $\widetilde{J}(z)\geq\sigma$ for all $z\in\partial B_\rho\cap E^+$.

***Proof.*** From $(H_4)$ we have $f(s)=g(s)=o(s^{a^*-1})$. Thus, there exists $\delta>0$ such that

$$|f(s)|,|g(s)|\leq|s|^{a^*-1},\ \text{for all}\ |s|<\delta.$$

By critical growth, there exist constants $c>0$ and $q\geq a^*$ such that

$$|f(s)|,|g(s)|\leq c|s|^{q-1}\Big(e^{2\alpha_0|s|^2}-\sum_{j=0}^{j_a}\frac{2^j\alpha_0^j|s|^{2j}}{j!}\Big),\ \text{for all}\ |s|\geq\delta.$$

From these estimates, we get a constant $c>0$ such that

$$|\widetilde{F}(x,s)|\leq|F(s)|\leq c|s|^{a^*}+c|s|^q\Big(e^{2\alpha_0|s|^2}-\sum_{j=0}^{j_a}\frac{2^j\alpha_0^j|s|^{2j}}{j!}\Big)$$

and

$$|\widetilde{G}(x,s)|\leq|G(s)|\leq c|s|^{a^*}+c|s|^q\Big(e^{2\alpha_0|s|^2}-\sum_{j=0}^{j_a}\frac{2^j\alpha_0^j|s|^{2j}}{j!}\Big).$$

By Lemma 6.6 and Proposition 6.5, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}|u|^q\Big(e^{2\alpha_0|u|^2}-\sum_{j=0}^{j_a}\frac{2^j\alpha_0^j|u|^{2j}}{j!}\Big)\,dx&\\ &\leq\|u\|_{2q}^q\Big(\int_{\mathbb{R}^2}\Big(e^{2\alpha_0|u|^2}-\sum_{j=0}^{j_a}\frac{2^j\alpha_0^j|u|^{2j}}{j!}\Big)^2dx\Big)^{1/2}\\ &\leq c\|u\|_{2q}^q\Big(\int_{\mathbb{R}^2}\Big(e^{6\alpha_0|u|^2}-\sum_{j=0}^{j_a}\frac{6^j\alpha_0^j|u|^{2j}}{j!}\Big)\,dx\Big)^{1/2}\\ &\leq c\|u\|_{2q}^q\end{aligned}$$

provided that $\|u\|\leq\rho_1$ for some $\rho_1>0$ such that $6\alpha_0\rho_1^2<4\pi$. Thus,

$$\int_{\mathbb{R}^2}\widetilde{F}(x,u)\,dx\leq c\|u\|_{a^*}^{a^*}+c\|u\|_{2q}^q\quad\text{and}\quad\int_{\mathbb{R}^2}\widetilde{G}(x,u)\,dx\leq c\|u\|_{a^*}^{a^*}+c\|u\|_{2q}^q.$$

By Lemma 6.1, we obtain

$$\begin{aligned}\widetilde{J}(u,u)&\geq\|u\|^2-\int_{\mathbb{R}^2}\widetilde{F}(x,u)\,dx-\int_{\mathbb{R}^2}\widetilde{G}(x,u)\,dx\\ &\geq\|u\|^2-c\|u\|^{a^*}-c\|u\|^q.\end{aligned}$$

Therefore, we can find $\rho,\sigma>0$, $\rho$ sufficiently small, such that $\widetilde{J}(z)\geq\sigma>0$, for $z\in\partial B_\rho\cap E^+$.

■

Let $e\in H^1_{V,rad}(\mathbb{R}^2)$ be a fixed nonnegative function such that $\|e\|=1$ and set

$$Q_e=\{r(e,e)+(\omega,-\omega):\ \|(\omega,-\omega)\|\leq R_0,0\leq r\leq R_1\},$$

where the positive constants $R_0$ and $R_1$ will be chosen in the next lemma.

**Lemma 6.12.** Suppose that $(V_1)-(V_2)$ and $(H_1)-(H_2)$ are hold. Then, there exist positive constants $R_0$ and $R_1$, which depend on $e$, such that

$$\widetilde{J}(z) \leq 0, \quad \text{for all} \quad z \in \partial Q_e.$$

***Proof.*** Notice that the boundary $\partial Q_e$ of the set $Q_e$ in the space $\mathbb{R}(e,e) \oplus E^-$ is composed of three parts.

(i) If $z \in \partial Q \cap E^-$, we have $\widetilde{J}(u,u) \leq 0$. In fact, for all $z = (u,-u) \in E^-$,

$$\widetilde{J}(z) = -\|u\|^2 - \int_{\mathbb{R}^2} \widetilde{F}(x,u)\,dx - \int_{\mathbb{R}^2} \widetilde{G}(x,-u)\,dx \leq 0$$

because $\widetilde{F}$ and $\widetilde{G}$ are nonnegative functions.

(ii) If $z = r(e,e) + (\omega,-\omega) = (re+\omega, re-\omega) \in \partial Q_e$, with $\|(\omega,-\omega)\| = R_0$ and $0 \leq r \leq R_1$,

$$\begin{aligned}\widetilde{J}(z) &= r^2\|e\|^2 - \|\omega\|^2 - \int_{\mathbb{R}^2} \widetilde{F}(x, re+\omega)\,dx - \int_{\mathbb{R}^2} \widetilde{G}(x, re-\omega)\,dx \\ &\leq R_1^2\|e\|^2 - \|\omega\|^2 = R_1^2 - \frac{R_0^2}{2}.\end{aligned}$$

Thus, $\widetilde{J}(z) \leq 0$ if $R_0 = \sqrt{2}R_1$.

(iii) If $z = R_1(e,e) + (\omega,-\omega) \in \partial Q_e$, with $\|(\omega,-\omega)\| \leq R_0$, it follows from Lemma 6.8 the existence of $c > 0$ and $\vartheta > 2$ such that

$$\widetilde{F}(x,s), \widetilde{G}(x,s) \geq c|s|^{\vartheta} - s^2, \text{ for all } (x,s) \in \overline{B}_1(0) \times [0,+\infty).$$

Thus,

$$\begin{aligned}\widetilde{J}(z) &= R_1^2\|e\|^2 - \|\omega\|^2 - \int_{\mathbb{R}^2} \widetilde{F}(x, R_1e+\omega)\,dx - \int_{\mathbb{R}^2} \widetilde{G}(x, R_1e-\omega)\,dx \\ &\leq R_1^2 - \int_{B_1} \widetilde{F}(x, R_1e+\omega)\,dx - \int_{B_1} \widetilde{G}(x, R_1e-\omega)\,dx \\ &\leq R_1^2 + \int_{B_1} \left(|R_1e+\omega|^2 + |R_1e-\omega|^2 - c|R_1e+\omega|^{\vartheta} - c|R_1e-\omega|^{\vartheta}\right) dx \\ &\leq R_1^2 + c(R_1^2 - R_1^{\vartheta}).\end{aligned}$$

Finally, we take $R_1 > 0$ sufficiently large such that $\widetilde{J}(z) \leq 0$.

■

**Lemma 6.13.** Given $0 < a < 2$ and $j_a = [|4/(2-a)|]$, there exists a constant $c_a = c(a) > 1$ such that

$$st \leq \left(e^{t^2} - \sum_{j=0}^{j_a} \frac{t^{2j}}{j!}\right) + s\ln^{1/2} s, \quad \text{for all} \quad t \geq 0,\ s \geq c_a. \tag{6.17}$$

***Proof.*** Fix $s>0$ and consider the following strictly concave function

$$t \mapsto st - \Big(e^{t^2} - \sum_{j=0}^{j_a} \frac{t^{2j}}{j!}\Big).$$

Thus, there exists a unique $t_s$ where the supremum is attained. Then,

$$s = t_s e^{t_s^2} + \Big(t_s e^{t_s^2} - \sum_{j=1}^{j_a} \frac{2t_s^{2j-1}}{(j-1)!}\Big).$$

Observe that there exists $d=d(a)>1$ such that

$$te^{t^2} - \sum_{j=1}^{j_a} \frac{2jt^{2j-1}}{(j-1)!} \geq 0 \quad \text{for all} \quad t \geq d.$$

We consider two cases:

**(i)** $t_s \geq d$;

**(ii)** $0 \leq t_s \leq d$ and $e^{d^2} \leq s$.

If (i) holds, then $s \geq e^{t_s^2}$, hence that $t_s \leq \ln^{1/2} s$ and finally that $st_s \leq s\ln^{1/2} s$. If (ii) holds, then $st_s \leq sd \leq s\ln^{1/2} s$. We observe that

$$\sup_{t\geq 0}\Big\{st - \Big(e^{t^2} - \sum_{j=0}^{j_a} \frac{t^{2j}}{j!}\Big)\Big\} \leq st_s \leq s\ln^{1/2} s, \quad \text{for all} \quad s \geq c_a = e^{d^2}.$$

■

Let $a, b, R_0, L_a$ and $L_b$ be constants given by $(V_2)$. Setting $p \geq b^*$, define the constant

$$S_V(L_a) := \inf_{0\neq u \in H^1_{V,rad}} \frac{\int_{\mathbb{R}^2}(|\nabla u|^2 + V(x)u^2)\,dx}{\big(\int_{\mathbb{R}^2} V(x)|u|^p\,dx\big)^{2/p}},$$

and for each $p \geq a^*$, we define the constant

$$S(L_a) := \inf_{0\neq u \in H^1_{V,rad}} \frac{\int_{\mathbb{R}^2}(|\nabla u|^2 + V(x)u^2)\,dx}{\big(\int_{\mathbb{R}^2} |u|^p\,dx\big)^{2/p}}.$$

We observe that in the case of $V$ satisfies $(V_1)$ and $(V_2)$, from Lemma 6.1, these constants are positive.

In order to prove that these constants are independent of $L_a$, we assume that $V$ also satisfies $(V_3)$, more precisely:

**Lemma 6.14.** Suppose that $V$ satisfies $(V_1)-(V_3)$. Then,

(a) For every $p \geq b^*$, we have

$$S_V := \inf_{L_a \geq R_0^a} S_V(L_a) > 0.$$

(b) For every $p \geq a^*$, we have

$$S := \inf_{L_a \geq R_0^a} S(L_a) > 0.$$

***Proof.***

(a) If $S_V^p = 0$ there exist sequences $(L_{n,a})$ and $(L_{n,b})$ such that $L_{n,a} \geq R_0^a$, $L_{n,a}R_0^{b-a} \leq L_{n,b} \leq L_{n,a}^{(2-b)/(2-a)}\pi^{2(a-b)/(2-a)}$ and $S_{V_n}(L_{n,a}) \to 0$. Thus, there exists a sequence $(u_n)$ such that $u_n \in H^1_{V_n,rad}(\mathbb{R}^2)$, $\int_{\mathbb{R}^2}(|\nabla u_n|^2 + V_n(x)u_n^2)\,dx = o_n(1)$ and $\int_{\mathbb{R}^2} V_n(x)|u_n|^p\,dx = 1$. Using the fact that $V_n(x) = 1$ in $B_1$ for each $n \geq 1$ and the Sobolev embedding there exists $c = c(p) > 0$ such that

$$\int_{B_1} V_n(x)|u_n|^p\,dx = \int_{B_1} |u_n|^p\,dx \leq c\Big(\int_{B_1} (|\nabla u_n|^2 + V_n(x)u_n^2)\,dx\Big)^{p/2} = o_n(1).$$

By the Strauss lemma Strauss (1977) and the fact that $V_n(x) \geq 1$ in $B_{R_0}\backslash B_1$, there exists $c = c(p) > 0$ such that

$$\begin{aligned}
\int_{B_{R_0}\backslash B_1} V_n(x)|u_n|^p\,dx &\leq \int_{B_{R_0}\backslash B_1} V_n(x)|u_n|^{p-2}|u_n|^2\,dx \\
&\leq c\Big(\int_{B_{R_0}\backslash B_1} (|\nabla u_n|^2 + u_n^2)\,dx\Big)^{(p-2)/2} \int_{B_{R_0}\backslash B_1} V_n(x)u_n^2\,dx \\
&\leq c\Big(\int_{B_{R_0}\backslash B_1} (|\nabla u_n|^2 + V_n(x)u_n^2)\,dx\Big)^{(p-2)/2} \int_{B_{R_0}\backslash B_1} V_n(x)u_n^2\,dx \\
&= o_n(1).
\end{aligned}$$

By Lemma 6.4 and since $p \geq b^*$ we have

$$\begin{aligned}
\int_{\mathbb{R}^2\backslash B_{R_0}} V_n(x)|u_n|^p\,dx &\leq L_{n,b} \int_{\mathbb{R}^2\backslash B_{R_0}} \frac{|u_n|^{p-2}}{|x|^{b-a}} \frac{|u_n|^2}{|x|^a}\,dx \\
&\leq \frac{L_{n,b}\|u_n\|^{p-2}_{H^1_{V_n}}}{L_{n,a}^{(p-2)/4}\pi^{(p-2)/2}} \int_{\mathbb{R}^2\backslash B_{R_0}} \frac{1}{|x|^{b-a+(2-a)(p-2)/4}} \frac{u_n^2}{|x|^a}\,dx \\
&\leq \frac{L_{n,b}\|u_n\|^{p-2}_{H^1_{V_n}}}{L_{n,a}^{(p+2)/4}\pi^{(p-2)/2}R_0^{b-a+(2-a)(p-2)/4}} \int_{\mathbb{R}^2\backslash B_{R_0}} \frac{L_{n,a}}{|x|^a}u_n^2\,dx \\
&\leq \frac{L_{n,b}\|u_n\|^{p-2}_{H^1_{V_n}}}{L_{n,a}^{(2-b)/(2-a)}\pi^{2(a-b)/(2-a)}} \int_{\mathbb{R}^2\backslash B_{R_0}} \frac{L_{n,a}}{|x|^a}u_n^2\,dx \\
&\leq \|u_n\|^{p-2}_{H^1_{V_n}} \int_{\mathbb{R}^2\backslash B_{R_0}} V_n(x)u_n^2\,dx \\
&= o_n(1).
\end{aligned}$$

Thus, $\displaystyle\int_{\mathbb{R}^2} V_n(x)|u_n|^p\,dx \to 0$ a contradiction.

(b) If $S=0$, similarly to part $(a)$ there exists a sequence $(u_n)$ and $(L_{n,a})$ such that $\int_{\mathbb{R}^2}(|\nabla u_n|^2+V_n(x)u_n^2)\,dx=o_n(1)$ and $\int_{\mathbb{R}^2}|u_n|^p\,dx=1$. By Sobolev embedding there exists $c=c(p)>0$ such that

$$\int_{B_1}|u_n|^p\,dx\le c\Big(\int_{B_1}(|\nabla u_n|^2+V_n(x)u_n^2)\,dx\Big)^{p/2}=o_n(1).$$

By Sobolev embedding and the fact that $V_n(x)\ge 1$ in $B_{R_0}\backslash B_1$ we have

$$\begin{aligned}\int_{B_{R_0}\backslash B_1}|u_n|^p\,dx&\le c\Big(\int_{B_{R_0}\backslash B_1}(|\nabla u_n|^2+u_n^2)\,dx\Big)^{p/2}\\&\le c\Big(\int_{B_{R_0}\backslash B_1}(|\nabla u_n|^2+V_n(x)u_n^2)\,dx\Big)^{p/2}\\&=o_n(1).\end{aligned}$$

By Lemma 6.4 we have

$$\begin{aligned}\int_{\mathbb{R}^2\backslash B_{R_0}}|u_n|^p\,dx&\le\int_{\mathbb{R}^2\backslash B_{R_0}}\frac{|u_n|^{p-2}}{|x|^{-a}}\frac{|u_n|^2}{|x|^a}\,dx\\&\le\frac{\|u_n\|^{p-2}_{H^1_{V_n}}}{L_{n,a}^{(p+2)/4}\pi^{(p-2)/2}R_0^{-a+(2-a)(p-2)/4}}\int_{\mathbb{R}^2\backslash B_{R_0}}\frac{L_{n,a}}{|x|^a}u_n^2\,dx\\&\le\|u_n\|^{p-2}_{H^1_{V_n}}\int_{\mathbb{R}^2\backslash B_{R_0}}V_n(x)u_n^2\,dx\\&=o_n(1).\end{aligned}$$

Thus, $\int_{\mathbb{R}^2}|u_n|^p\,dx\to 0$ a contradiction.

■

**Lemma 6.15.** For each $p\ge b^*$ and $q\ge a^*$, there exist positive constants $c=c(p,a,b,R_0)$ and $d=d(p,a,R_0)$ such that

$$\|u\|_{L^p_V}\le c\|u\|_{H^1_V},\quad\text{for all}\quad u\in H^1_{V,rad}(\mathbb{R}^2)$$

and

$$\|u\|_{L^q}\le d\|u\|_{H^1_V},\quad\text{for all}\quad u\in H^1_{V,rad}(\mathbb{R}^2).$$

***Proof.*** By Lemma 6.14, it is sufficient to consider $c=S_V^{-1/2}$ and $d=S^{-1/2}$. ■

**Lemma 6.16.** Let $(u_n,v_n)\in E$ such that $|\widetilde{J}(u_n,v_n)|\le d$ and

$$|\widetilde{J}'(u_n,v_n)(\phi,\psi)|\le\varepsilon_n\|(\phi,\psi)\|,\ \text{for all}\ \ \phi,\psi\in\{0,u_n,v_n\},\ \text{where}\ \varepsilon_n\to 0.\tag{6.18}$$

Then, $\|(u_n,v_n)\|\le c$ for every $n\in\mathbb{N}$ and for some positive constant $c$.

***Proof.*** Taking $(\phi,\psi)=(u_n,v_n)$ in (6.18), we have

$$\int_{\mathbb{R}^2}\widetilde{f}(x,u_n)u_n\,dx+\int_{\mathbb{R}^2}\widetilde{g}(x,v_n)v_n\,dx \leq \left|2\int_{\mathbb{R}^2}\left(\nabla u_n\nabla v_n+V(x)u_nv_n\right)dx\right|+\varepsilon_n\|(u_n,v_n)\|.$$

This combined with Lemma 6.8 and the fact that $|\widetilde{J}(u_n,v_n)|\leq d$ , we have

$$\begin{aligned}\int_{\mathbb{R}^2}\widetilde{f}(x,u_n)u_n\,dx+\int_{\mathbb{R}^2}\widetilde{g}(x,v_n)v_n\,dx &\leq 2d+2\int_{\mathbb{R}^2}\widetilde{F}(x,u_n)\,dx+2\int_{\mathbb{R}^2}\widetilde{G}(x,v_n)\,dx+\varepsilon_n\|(u_n,v_n)\|\\ &\leq 2d+\frac{2}{\mu}\int_{\mathbb{R}^2}\widetilde{f}(x,u_n)u_n\,dx+\frac{2}{\nu}\int_{\mathbb{R}^2}\widetilde{g}(x,v_n)v_n\,dx+\varepsilon_n\|(u_n,v_n)\|.\end{aligned}$$

Thus, there exists $c>0$ such that

$$\int_{\mathbb{R}^2}\widetilde{f}(x,u_n)u_n\,dx+\int_{\mathbb{R}^2}\widetilde{g}(x,v_n)v_n\,dx\leq c+\varepsilon_n\|(u_n,v_n)\|. \tag{6.19}$$

Taking $(\phi,\psi)=(0,u_n)$ in (6.18), we obtain

$$\|u_n\|^2\leq\int_{\mathbb{R}^2}\widetilde{g}(x,v_n)u_n\,dx+\varepsilon_n\|u_n\|.$$

Setting, for every $n\in\mathbb{N}$, the sets

$$T_{1,n}=\{x\in\mathbb{R}^2:|x|>R,g(v_n)\leq V(x)v_n^{\nu-1}\} \tag{6.20}$$

and

$$T_{2,n}=\{x\in\mathbb{R}^2:|x|>R,g(v_n)>V(x)v_n^{\nu-1}\}, \tag{6.21}$$

we can write

$$\|u_n\|^2\leq\int_{B_R\cup T_{1,n}}g(v_n)u_n\,dx+\int_{T_{2,n}}V(x)v_n^{\nu-1}u_n\,dx+\varepsilon_n\|u_n\|.$$

Thus, for $n\in\mathbb{N}$ such that $u_n\neq 0$, we have

$$\|u_n\|-\varepsilon_n\leq\int_{B_R\cup T_{1,n}}g(v_n)\frac{u_n}{\|u_n\|}\,dx+\int_{T_{2,n}}V(x)v_n^{\nu-1}\frac{u_n}{\|u_n\|}\,dx. \tag{6.22}$$

By Young's inequality and Lemma 6.15, there exists a positive constant $c=c(\nu,a,b,R_0)$ such that

$$\int_{T_{2,n}}V(x)v_n^{\nu-1}\frac{u_n}{\|u_n\|}\,dx\leq\frac{\nu-1}{\nu}\int_{T_{2,n}}V(x)v_n^{\nu}\,dx+c. \tag{6.23}$$

Set $U_n=\dfrac{u_n}{\|u_n\|}$ . There exists $\lambda=\lambda(g)>0$ such that

$$|g(s)|\leq\lambda e^{(\alpha_0+1)|s|^2}\quad\text{for all}\quad s\in\mathbb{R}. \tag{6.24}$$

Using Lemma 6.13 with $s=|g(v_n(x))|/\lambda$ and $t=|U_n(x)|$, we get

$$\int_{\{x\in B_R\cup T_{1,n}:|\frac{g(v_n)}{\lambda}|>c_a\}}g(v_n)U_n\,dx\leq I_{1,n}+I_{2,n}. \tag{6.25}$$

where

$$I_{1,n} = \lambda \int_{\{x\in B_R\cup T_{1,n}:|\frac{g(v_n)}{\lambda}|>c_a\}} \left(e^{|U_n|^2} - \sum_{j=0}^{j_a} \frac{|U_n|^{2j}}{j!}\right) dx$$

and

$$I_{2,n} = \int_{\{x\in B_R\cup T_{1,n}:|\frac{g(v_n)}{\lambda}|>c_a\}} |g(v_n)| \ln^{1/2} |\frac{g(v_n)}{\lambda}|\, dx.$$

From Proposition 6.5 and (6.24), for $c = c(g,a,R_0) > 0$, we have

$$\int_{\{x\in B_R\cup T_{1,n}:|\frac{g(v_n)}{\lambda}|>c_a\}} g(v_n)U_n\, dx \le c + \sqrt{\alpha_0+1} \int_{\{x\in B_R\cup T_{1,n}:|\frac{g(v_n)}{\lambda}|>c_a\}} g(v_n)v_n\, dx. \tag{6.26}$$

On the other hand, by Young's inequality

$$\int_{\{x\in B_R\cup T_{1,n}:|\frac{g(v_n)}{\lambda}|\le c_a\}} g(v_n)U_n\, dx \le I_{3,n} + I_{4,n} \tag{6.27}$$

where

$$I_{3,n} = \frac{a^*-1}{a^*} \int_{\{x\in B_R\cup T_{1,n}:|\frac{g(v_n)}{\lambda}|\le c_a\}} |g(v_n)|^{\frac{a^*}{a^*-1}}\, dx$$

and

$$I_{4,n} = \frac{1}{a^*} \int_{\{x\in B_R\cup T_{1,n}:|\frac{g(v_n)}{\lambda}|\le c_a\}} |U_n|^{a^*}\, dx.$$

Since $g(t) = o(t^{a^*-1})$, there exists $0 < \delta_0 < 1$ such that

$$|g(t)|^{\frac{1}{a^*-1}} \le |t|, \quad \text{for all} \quad |t| \le \delta_0.$$

Thus,

$$|g(t)|^{\frac{a^*}{a^*-1}} \le g(t)t, \quad \text{for all} \quad |t| \le \delta_0$$

and

$$|g(t)|^{\frac{a^*}{a^*-1}} \le \frac{c_a^{\frac{1}{a^*-1}}}{\delta_0} g(t)t, \quad \text{for all} \quad t \in \{|t| \ge \delta_0 : |g(t)| \le c_a\}.$$

Since $c_a > 1$, we obtain

$$|g(t)|^{\frac{a^*}{a^*-1}} \le \frac{c_a^{\frac{1}{a^*-1}}}{\delta_0} g(t)t, \quad \text{for all} \quad t \in \{t \in \mathbb{R} : |g(t)| \le c_a\}. \tag{6.28}$$

Using (6.28) and Lemma 6.1 in (6.27), there exist positive constants $\beta = \beta(a,g)$ and $c = c(a)$ such that

$$\int_{\{x\in B_R\cup T_{1,n}:|\frac{g(v_n)}{\lambda}|\le c_a\}} g(v_n)U_n\, dx \le \beta \int_{\{x\in B_R\cup T_{1,n}:|\frac{g(v_n)}{\lambda}|\le c_a\}} g(v_n)v_n\, dx + c\|U_n\|_{a^*}^{a^*}. \tag{6.29}$$

Combining (6.26), (6.29) with Lemma 6.15, there exists $c = c(g,\alpha_0,a,R_0) > 0$ such that

$$\int_{B_R\cup T_{1,n}} g(v_n)U_n\, dx \le c + c\int_{B_R\cup T_{1,n}} g(v_n)v_n\, dx. \tag{6.30}$$

From (6.23) and (6.30) in (6.22), there exists $c = c(g,\nu,\alpha_0,a,b,R_0) > 0$ such that

$$\|u_n\| \le c + c\int_{B_R\cup T_{1,n}} g(v_n)v_n\, dx + c\int_{T_{2,n}} V(x)v_n^\nu\, dx + \varepsilon_n. \tag{6.31}$$

By (6.19), we have

$$\int_{B_R\cup T_{1,n}} g(v_n)v_n\,dx + \int_{T_{2,n}} V(x)v_n^\nu\,dx \le c + \varepsilon_n\|(u_n,v_n)\|.$$

Thus, there exists $c > 0$ such that

$$\|u_n\| \le c + \varepsilon_n\|(u_n,v_n)\| + \varepsilon_n.$$

Similarly, we get

$$\|v_n\| \le c + \varepsilon_n\|(u_n,v_n)\| + \varepsilon_n.$$

We finally obtain

$$\|(u_n,v_n\| \le c + \varepsilon_n\|(u_n,v_n)\| + \varepsilon_n.$$

which implies that $\|(u_n,v_n)\| \le c$, for every $n \in \mathbb{N}$, for some positive constant $c$. ∎

**Lemma 6.17.** Suppose that $(V_1)-(V_2)$ and $(H_1)-(H_5)$ hold. If $(u_n,v_n) \subset E$ is a sequence such that $(u_n,v_n) \rightharpoonup (u,v)$ in $E$, $\widetilde{J}(u_n,v_n) \to c$ and $\widetilde{J}'(u_n,v_n) \to 0$. Then,

(i) $\widetilde{f}(x,u_n) \to \widetilde{f}(x,u)$ in $L^1(B_{R_1})$ and $\widetilde{g}(x,u_n) \to \widetilde{g}(x,u)$ in $L^1(B_{R_1})$, where $R_1 \ge 1$.

(ii) $\widetilde{F}(x,u_n) \to \widetilde{F}(x,u)$ in $L^1(\mathbb{R}^2)$ and $\widetilde{G}(x,u_n) \to \widetilde{G}(x,u)$ in $L^1(\mathbb{R}^2)$.

***Proof.*** We give the proof for the functions $\widetilde{f}$ and $\widetilde{F}$; similar arguments apply to the other functions. According to Remark 6.2, we can assume that $u_n \to u$ in $L^1(B_{R_1})$. Moreover, by the exponential growth of $f$ and Proposition 6.5, we have that $\widetilde{f}(x,u_n) \in L^1(B_{R_1})$ and since $\widetilde{J}'(u_n,v_n)(u_n,v_n) = o_n(1)$ there exists $c > 0$ such that

$$\int_{\mathbb{R}^2} \widetilde{f}(x,u_n)u_n\,dx + \int_{\mathbb{R}^2} \widetilde{g}(x,v_n)v_n\,dx \le c.$$

Using Figueiredo, Miyagaki and Ruf (1995, Lemma 2.10), we conclude that (i) holds for the function $\widetilde{f}$.

From $(i)$, given $R_1 \ge R$, where $R$ is given by the definition of $\widetilde{f}$, we obtain

$$\int_{B_{R_1}} \widetilde{f}(x,u_n)\,dx \to \int_{B_{R_1}} \widetilde{f}(x,u)\,dx.$$

Thus, there exists $p \in L^1(B_{R_1})$ such that

$$f(u_n) \le p(x) \text{ almost everywhere in } B_{R_1}. \tag{6.32}$$

From $(H_1)$ and $(H_3)$, we obtain

$$F(t) \le \max_{t\in[0,s_0]} F(t) + Mf(t), \quad \text{for all} \quad t \in \mathbb{R}. \tag{6.33}$$

Using (6.32) and (6.33), we have

$$\widetilde{F}(x,u_n) \le F(u_n) \le \max_{t\in[0,s_0]} F(t) + Mp(x), \quad \text{for all} \quad x \in B_{R_1}. \tag{6.34}$$

By Lebesgue's dominated convergence theorem, we obtain

$$\int_{B_{R_1}} \widetilde{F}(x,u_n)\,dx \to \int_{B_{R_1}} \widetilde{F}(x,u)\,dx.$$

Consequently, to prove that

$$\int_{\mathbb{R}^2} \widetilde{F}(x,u_n)\,dx \to \int_{\mathbb{R}^2} \widetilde{F}(x,u)\,dx,$$

it is sufficient to show that given $\delta > 0$, there exists $R_1 > 0$ such that

$$\int_{\mathbb{R}^2\backslash B_{R_1}} \widetilde{F}(x,u_n)\,dx < \delta \quad \text{and} \quad \int_{\mathbb{R}^2\backslash B_{R_1}} \widetilde{F}(x,u)\,dx < \delta.$$

Note that

$$\widetilde{F}(x,u_n) \le \frac{1}{\mu}V(x)|u_n|^{\mu}, \quad \text{for all} \quad x \in \mathbb{R}^2\backslash B_{R_1}. \tag{6.35}$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^2\backslash B_{R_1}} \widetilde{F}(x,u_n)\,dx &\le \frac{1}{\mu}\int_{\mathbb{R}^2\backslash B_{R_1}} V(x)|u_n|^{\mu}\,dx \\ &\le \frac{2^{\mu-1}}{\mu}\Big(\int_{\mathbb{R}^2\backslash B_{R_1}} V(x)|u_n-u|^{\mu}\,dx + \int_{\mathbb{R}^2\backslash B_{R_1}} V(x)|u|^{\mu}\,dx\Big).\end{aligned}$$

Using the compactness of the embedding $H^1_{V,rad}(\mathbb{R}^2) \hookrightarrow L^{\mu}_{V,rad}(\mathbb{R}^2)$ and the weak convergence $(u_n,v_n) \rightharpoonup (u,v)$ in $E$, we can choose $R_1 > 0$ sufficiently large such that

$$\int_{\mathbb{R}^2\backslash B_{R_1}} \widetilde{F}(x,u_n)\,dx < \delta.$$

Since $\widetilde{F}(\cdot,u) \in L^1(\mathbb{R}^2)$, we may assume that

$$\int_{\mathbb{R}^2\backslash B_{R_1}} \widetilde{F}(x,u)\,dx < \delta.$$

Combining all the above estimates, since $\delta > 0$ is arbitrary, we have

$$\int_{\mathbb{R}^2} \widetilde{F}(x,u_n)\,dx \to \int_{\mathbb{R}^2} \widetilde{F}(x,u)\,dx.$$

■

## 6.4 Estimates

In this section we establish the estimates for the auxiliary functional that are used to prove Theorem 6.3. We start with the definition of Moser type functions. Consider $k \in \mathbb{N}$. Let $\delta_k > 0$ be a sequence which will be fixed such that $\delta_k \to 0$, as $k \to +\infty$. The Moser type functions are defined by

$$e_k = \frac{1}{\sqrt{2\pi}}\begin{cases} \sqrt{\ln k}\,(1-\delta_k)^{1/2}, & |x| \le \frac{1}{k}, \\ \ln(\frac{1}{|x|})\,\frac{(1-\delta_k)^{1/2}}{\sqrt{\ln k}}, & \frac{1}{k} < |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

We have

$$\|\nabla e_k\|_2^2 = 1 - \delta_k$$

and

$$\int_{\mathbb{R}^2} V(x) e_k^2 \, dx \le (1-\delta_k)\Big(\frac{\ln k}{k^2} + \frac{1}{4\ln k}\Big).$$

Then, we may choose $\delta_k$, depending on $k$, such that

$$\|e_k\| = 1, \quad \text{for all} \quad k \ge 1.$$

Furthermore, we can note that

$$\delta_k \le (1-\delta_k)\Big(\frac{\ln k}{k^2} + \frac{1}{4\ln k}\Big) \le \frac{\ln k}{k^2} + \frac{1}{4\ln k}.$$

Thus,

$$\delta_k \ln k \le 1/2, \quad \text{for } k \text{ sufficiently large}. \tag{6.36}$$

**Proposition 6.18.** Suppose that $(H_1)-(H_6)$ hold. Then, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_{\mathbb{R}(e_{k_0}, e_{k_0}) \oplus E^-} \widetilde{J}(u,v) < \frac{4\pi}{\alpha_0}.$$

***Proof.*** Suppose, by contradiction, that for all $k \in \mathbb{N}$

$$\sup_{\mathbb{R}(e_k, e_k) \oplus E^-} \widetilde{J}(u,v) \ge \frac{4\pi}{\alpha_0},$$

Thus, for all fixed $k \ge 1$, there exist a nonnegative sequence $\zeta_n \to 0$, as $n \to \infty$, and a sequence

$$\eta_{n,k} = \tau_{n,k}(e_k, e_k) + (u_{n,k}, -u_{n,k}), \quad u_{n,k} \in H^1_{V,rad}(\mathbb{R}^2),$$

such that

$$\widetilde{J}(\eta_{n,k}) \ge \frac{4\pi}{\alpha_0} - \zeta_n.$$

Let $h: [0,+\infty) \to \mathbb{R}$ be the function defined by $h(t) = \widetilde{J}(t\eta_{n,k})$. Since $h(0) = 0$ and $\lim_{t\to+\infty} h(t) = -\infty$, there exists a maximum point $\hat{t}$. We may assume, without loss of generality, that $\hat{t} = 1$, so that

$$\widetilde{J}(\eta_{n,k}) \ge \frac{4\pi}{\alpha_0} - \zeta_n \quad \text{and} \quad \widetilde{J}'(\eta_{n,k})\eta_{n,k} = 0.$$

This means that

$$\begin{aligned} \tau_{n,k}^2\|e_k\|^2 - \|u_{n,k}\|^2 - \int_{\mathbb{R}^2} \widetilde{F}(x, \tau_{n,k}e_k + u_{n,k})\, dx - \int_{\mathbb{R}^2} \widetilde{G}(x, \tau_{n,k}e_k - u_{n,k})\, dx \\ \ge \frac{4\pi}{\alpha_0} - \zeta_n \end{aligned}$$

and

$$\begin{aligned} \tau_{n,k}^2\|e_k\|^2 - \|u_{n,k}\|^2 = &\int_{\mathbb{R}^2} \widetilde{f}(x, \tau_{n,k}e_k + u_{n,k})(\tau_{n,k}e_k + u_{n,k})\, dx \\ &+ \int_{\mathbb{R}^2} \widetilde{g}(x, \tau_{n,k}e_k - u_{n,k})(\tau_{n,k}e_k - u_{n,k})\, dx. \end{aligned}$$

Hence,

$$\tau_{n,k}^2 \geq \frac{4\pi}{\alpha_0} - \zeta_n \tag{6.37}$$

and

$$\tau_{n,k}^2 \geq \int_{\mathbb{R}^2} \widetilde{f}(x, \tau_{n,k}e_k + u_{n,k})(\tau_{n,k}e_k + u_{n,k}) + \int_{\mathbb{R}^2} \widetilde{g}(x, \tau_{n,k}e_k - u_{n,k})(\tau_{n,k}e_k - u_{n,k}). \tag{6.38}$$

Set $l > 0$ such that

$$\liminf_{t\to+\infty} \frac{tf(t)}{e^{\alpha_0 t^2}}, \liminf_{t\to+\infty} \frac{tg(t)}{e^{\alpha_0 t^2}} > l > \frac{4e}{\alpha_0}. \tag{6.39}$$

Thus, given $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$tf(t), tg(t) \geq (l-\varepsilon)e^{\alpha_0 t^2}, \quad \text{for all} \quad t \geq R_\varepsilon. \tag{6.40}$$

Using (6.37), there exists $k_0 > 0$ such that

$$(1-\delta_k)^{1/2}\tau_{n,k}\frac{\sqrt{\ln k}}{2\pi} \geq R_\varepsilon, \quad \text{for all} \quad k \geq k_0.$$

Since,

$$e_k(x) = (1-\delta_k)^{1/2}\sqrt{\frac{\ln k}{2\pi}}, \quad \text{for all} \quad x \in B_{1/k},$$

we get

$$\max\{\tau_{n,k}e_k + u_{n,k}, \tau_{n,k}e_k - u_{n,k}\} \geq \tau_{n,k}(x)e_k(x) \geq R_\varepsilon, \text{ for all } x \in \ B_{1/k} \text{ and } k \geq k_0.$$

Combining (6.38) with (6.40), gives

$$\begin{aligned}
\tau_{n,k}^2 &\geq \int_{\mathbb{R}^2} \widetilde{f}(x, \tau_{n,k}e_k + u_{n,k})(\tau_{n,k}e_k + u_{n,k})\,dx \\
&\quad + \int_{\mathbb{R}^2} \widetilde{g}(x, \tau_{n,k}e_k - u_{n,k})(\tau_{n,k}e_k - u_{n,k})\,dx \\
&\geq \int_{B_{1/k}} f(\tau_{n,k}e_k + u_{n,k})(\tau_{n,k}e_k + u_{n,k})\,dx \\
&\quad + \int_{B_{1/k}} g(\tau_{n,k}e_k - u_{n,k})(\tau_{n,k}e_k - u_{n,k})\,dx \\
&\geq (l-\varepsilon)\int_{B_{1/k}} e^{\alpha_0(1-\delta_k)\frac{\ln k}{2\pi}\tau_{n,k}^2},
\end{aligned}$$

for every $k \geq k_0$. Setting $s_{n,k} := \tau_{n,k}^2 - \dfrac{4\pi}{\alpha_0}$, we get

$$\begin{aligned}
\frac{4\pi}{\alpha_0} + s_{n,k} &\geq (l-\varepsilon)\int_{B_{1/k}} e^{\alpha_0(1-\delta_k)\frac{\ln k}{2\pi}(\frac{4\pi}{\alpha_0}+s_{n,k})} \\
&= (l-\varepsilon)\pi e^{s_{n,k}\alpha_0\frac{\ln k}{2\pi}} e^{-\frac{\alpha_0}{2\pi}(\frac{4\pi}{\alpha_0}+s_{n,k})\delta_k \ln k},
\end{aligned}$$

for every $k \geq k_0$. Using (6.36), we find

$$\frac{4\pi}{\alpha_0} + s_{n,k} \geq (l-\varepsilon)\pi e^{\frac{\alpha_0 s_{n,k}}{2\pi}(\ln k - 1/2)} e^{-1}. \tag{6.41}$$

Inequality (6.41) implies that $(s_{n,k})$ is bounded for each $k \geq k_0$. Therefore, there exists $s_k \in \mathbb{R}$ such that $\limsup_{n\to\infty} s_{n,k} = s_k$. By (6.37), $s_k \geq 0$. Using the last limit in (6.41) and taking $k \to +\infty$ we see that necessarily $s_k = 0$ for each $k \geq 1$. Then, $\lim_{n\to\infty} s_{n,k} = 0$. Using this in (6.41), yields

$$\frac{4\pi}{\alpha_0} \geq (l-\varepsilon)\pi e^{-1}.$$

This contradicts (6.39) because $\varepsilon > 0$ is arbitrary. ■

## 6.5 Finite-dimensional approximation

Since the functional $\widetilde{J}$ is strongly indefinite on the space E (i.e. positive and negative definite on infinite-dimensional subspaces), the standard linking theorems cannot be applied. In order to overcome this problem, we consider a finite-dimensional approximation.

Taking $k_0$ given by Proposition 6.18, we consider $e = e_{k_0} \in H^1_{V,rad}(\mathbb{R}^2)$ and $\{e_i\}_{i\in\mathbb{N}}$ a Hilbert basis of $\langle e\rangle^{\perp}$. We set

$$E_n^+ = Span\{(e_i,e_i):\ i = 1,2,\ldots,n\},$$

$$E_n^- = Span\{(e_i,-e_i):\ i = 1,2\ldots,n\},$$

$$E_n = E_n^+ \oplus E_n^-.$$

We use the following notation:

$$H_n = \mathbb{R}(e,e)\oplus E_n, \quad H_n^+ = \mathbb{R}(e,e)\oplus E_n^+, \quad H_n^- = \mathbb{R}(e,e)\oplus E_n^-.$$

Furthermore, define the class of mappings

$$\Gamma_n = \{\gamma \in \mathscr{C}(Q_n,H_n):\ \gamma(z) = z, \forall z \in \partial Q_n\},$$

where $Q_n = Q_e \cap H_n$, and set

$$c_n = \inf_{\gamma\in\Gamma_n} \max_{z\in Q_n} \widetilde{J}(\gamma(z)). \tag{6.42}$$

Let us denote by $\widetilde{J}_n$ the restriction of $\widetilde{J}$ to the finite-dimensional space $H_n$. We obtain that the linking geometry holds for $\widetilde{J}_n$. Using Lemma 5.5 in Figueiredo, Ó and Ruf (2005) we have

$$\gamma(Q_n)\cap(\partial B_\rho \cap H_n^+) \neq \emptyset \quad \text{for all} \quad \gamma \in \Gamma_n, \tag{6.43}$$

for $\rho$ given by Lemma 6.11. Thus, combining Lemma 6.11 and (6.43) we have

$$c_n \geq \sigma > 0 \quad \text{for all} \quad n \geq 1.$$

Since the inclusion mapping $I_n : Q_n \to H_n$ belongs to $\Gamma_n$, for $z = r(e,e)+(u,-u) \in Q_n$, we have

$$\widetilde{J}(z) = r^2\|e\|^2 - \|u\|^2 - \int_{\mathbb{R}^2} \widetilde{F}(x,re+u)\,dx - \int_{\mathbb{R}^2} \widetilde{G}(x,re-u)\,dx \leq R_1^2. \tag{6.44}$$

Thus,

$$c_n \leq R_1^2, \quad \text{for all} \quad n \geq 1. \tag{6.45}$$

Therefore, applying the linking theorem for $\widetilde{J}_n$, we have the following result (see Rabinowitz (1986) for a proof):

**Proposition 6.19.** Suppose that $V$ satisfies $(V_1)$ and $(V_2)$ and $f$ and $g$ satisfy $(H_1)$, $(H_2)$, $(H_4)$ and $(H_5)$. Then, for each $n \in \mathbb{N}$, the functional $\widetilde{J}_n$ has a critical point at level $c_n$. More precisely, there is $z_n = (u_n, v_n) \in H_n$ such that

$$\widetilde{J}(z_n) = c_n \in [\sigma, R_1^2], \tag{6.46}$$

where $\sigma$ and $R_1 > 0$ are given by Lemma 6.11 and Lemma 6.12, respectively, and

$$\widetilde{J}_n'(z_n)(\phi, \psi) = 0, \quad \text{for all } (\phi, \psi) \in H_n, \tag{6.47}$$

that is, for every $(\phi, \psi) \in H_n$, we have

$$\begin{cases} \int_{\mathbb{R}^2} \nabla u_n \nabla \psi + V(x) u_n \psi \, dx &= \int_{\mathbb{R}^2} \widetilde{g}(x, v_n) \psi \, dx, \\ \int_{\mathbb{R}^2} \nabla \phi \nabla v_n + V(x) \phi \; v_n \, dx &= \int_{\mathbb{R}^2} \widetilde{f}(x, u_n) \phi \, dx. \end{cases} \tag{6.48}$$

By Proposition 6.18, there exists $\delta > 0$ such that

$$c_n \leq \max_{Q_n} \widetilde{J}(z) \leq \sup_{\mathbb{R}(e,e) \oplus E_n^-} \widetilde{J}(z) \leq \sup_{\mathbb{R}(e,e) \oplus E^-} \widetilde{J}(z) \leq \frac{4\pi}{\alpha_0} - \delta, \tag{6.49}$$

for every $n \in \mathbb{N}$.

**Proposition 6.20.** Suppose that $V$ satisfies $(V_1) - (V_2)$ and $f$ and $g$ satisfy $(H_1) - (H_6)$. Then, $\widetilde{J}$ possesses a nontrivial critical point.

***Proof.*** By Proposition 6.19, there is a sequence $(u_n, v_n) \in H_n$ satisfying (6.46) and (6.47). By Lemma 6.16, $(u_n, v_n)$ is bounded in $E$. Then we can find a subsequence (not renamed) and there exists $(u, v) \in E$ such that $(u_n, v_n)$ converges weakly to $(u, v)$ in $E$. Taking $(0, \psi)$ and $(\phi, 0)$ in (6.48), where $\phi$ and $\psi$ are arbitrary functions in $\mathscr{C}_{0,rad}^\infty(\mathbb{R}^2) \cap H_n$, we get

$$\int_{\mathbb{R}^2} \big(\nabla u_n \nabla \psi + V(x) u_n \psi\big) \, dx = \int_{\mathbb{R}^2} \widetilde{g}(x, v_n) \psi \, dx \tag{6.50}$$

and

$$\int_{\mathbb{R}^2} \big(\nabla v_n \nabla \phi + V(x) v_n \phi\big) \, dx = \int_{\mathbb{R}^2} \widetilde{f}(x, u_n) \phi \, dx. \tag{6.51}$$

Using Lemma 6.17, we obtain

$$\int_{\mathbb{R}^2} \widetilde{f}(x, u_n) \phi \, dx \to \int_{\mathbb{R}^2} \widetilde{f}(x, u) \phi \, dx, \quad \text{for all} \quad \phi \in \mathscr{C}_{0,rad}^\infty(\mathbb{R}^2) \cap H_n$$

and

$$\int_{\mathbb{R}^2} \widetilde{g}(x, v_n) \psi \, dx \to \int_{\mathbb{R}^2} \widetilde{g}(x, v) \psi \, dx, \quad \text{for all} \quad \psi \in \mathscr{C}_{0,rad}^\infty(\mathbb{R}^2) \cap H_n.$$

Taking the limit in (6.50) and (6.51) as $n \to \infty$, and using the fact that $\mathscr{C}^{\infty}_{0,rad}(\mathbb{R}^2) \cap \left( \bigcup_{n\in\mathbb{N}} H_n \right)$ is dense in $H^1_{V,rad}(\mathbb{R}^2)$, yields

$$\int_{\mathbb{R}^2} \left(\nabla u \nabla \psi + V(x) u \psi\right) dx = \int_{\mathbb{R}^2} \widetilde{g}(x,v)\psi\, dx, \quad \text{for all } \psi \in H^1_{V,rad}(\mathbb{R}^2) \tag{6.52}$$

and

$$\int_{\mathbb{R}^2} \left(\nabla v \nabla \phi + V(x) v \phi\right) dx = \int_{\mathbb{R}^2} \widetilde{f}(x,u)\phi\, dx, \quad \text{for all } \phi \in H^1_{V,rad}(\mathbb{R}^2). \tag{6.53}$$

Then, $(u,v) \in E$ is a critical point of $\widetilde{J}$. To conclude the proof, it only remains to prove that $u$ and $v$ are nontrivial. Suppose, by contradiction, that $u \equiv 0$. From (6.53), we also have $v \equiv 0$. Then, we can assume that

$$u_n \to 0 \text{ and } v_n \to 0 \text{ in } L^r_{V,rad}(\mathbb{R}^2), \quad \text{for all } r > b^*. \tag{6.54}$$

We claim that there exists $\sigma > 0$ such that $\|u_n\| \geq \sigma > 0$ for all $n \geq 1$. In fact, suppose that, contrary to our claim, there exists a subsequence (not renamed) such that $\|u_n\| \to 0$, as $n \to \infty$. From this, we get

$$\int_{\mathbb{R}^2} \left(\nabla u_n \nabla v_n + V(x) u_n v_n\right) dx \to 0. \tag{6.55}$$

Taking $(\phi,\psi) = (u_n, 0)$ and $(\phi,\psi) = (0, v_n)$ in (6.48), we get

$$\int_{\mathbb{R}^2} \left(\nabla u_n \nabla v_n + V(x) u_n v_n\right) dx = \int_{\Omega} \widetilde{f}(x,u_n) u_n\, dx = \int_{\Omega} \widetilde{g}(x,v_n) v_n\, dx. \tag{6.56}$$

Using Lemma 6.17 and (6.54), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \widetilde{F}(x,u_n)\, dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^2} \widetilde{G}(x,v_n)\, dx = 0.$$

Consequently,

$$\widetilde{J}(u_n,v_n) \to 0, \quad \text{as} \quad n \to \infty,$$

in contradiction with (6.46), which completes the proof of the claim.

Taking $(\phi,\psi) = (0,u_n)$ in (6.48), we get

$$\|u_n\|^2 = \int_{\mathbb{R}^2} \widetilde{g}(x,v_n) u_n\, dx \leq \int_{B_R} g(v_n) u_n\, dx + \int_{\mathbb{R}^2\setminus B_R} V(x) v_n^{\nu-1} u_n\, dx. \tag{6.57}$$

By Young's inequality, we find

$$\int_{\mathbb{R}^2\setminus B_R} V(x) v_n^{\nu-1} \frac{u_n}{\|u_n\|}\, dx \leq \frac{\nu-1}{\nu} \int_{\mathbb{R}^2\setminus B_R} V(x) v_n^{\nu}\, dx + \frac{1}{\nu\sigma^{\nu}} \int_{\mathbb{R}^2\setminus B_R} V(x) u_n^{\nu}\, dx. \tag{6.58}$$

Using (6.54), we have

$$\int_{\mathbb{R}^2\setminus B_R} V(x) v_n^{\nu-1} \frac{u_n}{\|u_n\|}\, dx = o_n(1).$$

From (6.57), we obtain,

$$\|u_n\| \leq \int_{B_R} g(v_n) \frac{u_n}{\|u_n\|}\, dx + o_n(1). \tag{6.59}$$

Thus,

$$\Big(\frac{4\pi}{\alpha_0}-\delta\Big)^{1/2}\|u_n\| \le \int_{B_R} g(v_n)\bar{u}_n\,dx + o_n(1),$$

where $\bar{u}_n = (\frac{4\pi}{\alpha_0}-\delta)^{1/2}\frac{u_n}{\|u_n\|}$. From Lemma 6.13, with $s=|g(v_n)|/\sqrt{\alpha_0}$ and $t=\sqrt{\alpha_0}|\bar{u}_n|$, we get

$$\begin{aligned}\Big(\frac{4\pi}{\alpha_0}-\delta\Big)^{1/2}\|u_n\| \le &\int_{\{x\in B_R:|\frac{g(v_n)}{\sqrt{\alpha_0}}|>c_a\}}\Big(e^{\alpha_0|\bar{u}_n|^2}-\sum_{j=0}^{j_a}\frac{\alpha_0^j|\bar{u}_n|^{2j}}{j!}\Big)\,dx\\ &+\int_{\{x\in B_R:|\frac{g(v_n)}{\sqrt{\alpha_0}}|>c_a\}}\Big|\frac{g(v_n)}{\sqrt{\alpha_0}}\Big|\ln^{1/2}\Big|\frac{g(v_n)}{\sqrt{\alpha_0}}\Big|\,dx\\ &+\int_{\{x\in B_R:|\frac{g(v_n)}{\sqrt{\alpha_0}}|\le c_a\}} g(v_n)\bar{u}_n\,dx+o_n(1).\end{aligned}$$

Since $\|\bar{u}_n\|^2<\frac{4\pi}{\alpha_0}$, by Lemma 6.7 the first integral tends to zero, while by Young's inequality in the third integral, we have

$$\begin{aligned}\int_{\{x\in B_R:|\frac{g(v_n)}{\sqrt{\alpha_0}}|\le c_a\}} g(v_n)\bar{u}_n\,dx &= \frac{a^*-1}{a^*}\int_{\{x\in B_R:|\frac{g(v_n)}{\sqrt{\alpha_0}}|\le c_a\}}|g(v_n)|^{\frac{a^*}{a^*-1}}\,dx\\ &\quad+\frac{1}{a^*}\int_{\{x\in B_R:|\frac{g(v_n)}{\sqrt{\alpha_0}}|\le c_a\}}|\bar{u}_n|^{a^*}\,dx\\ &= o_n(1)\end{aligned}$$

where we used Lebesgue dominated theorem and Remark 6.2 because $\bar{u}_n \rightharpoonup 0$ in $H^1_{V,rad}(\mathbb{R}^2)$. Thus,

$$\Big(\frac{4\pi}{\alpha_0}-\delta\Big)^{1/2}\|u_n\| \le \int_{B_R}\Big|\frac{g(v_n)}{\sqrt{\alpha_0}}\Big|\ln^{1/2}\Big|\frac{g(v_n)}{\sqrt{\alpha_0}}\Big|\,dx+o_n(1). \tag{6.60}$$

Given $\varepsilon\in\Big(0,\frac{\alpha_0\delta}{4(\frac{4\pi}{\alpha_0}-\delta)}\Big)$, where $\delta>0$ is given by (6.49), there exists $C_\varepsilon>0$ such that

$$|g(s)|\le C_\varepsilon e^{(\alpha_0+\varepsilon)s^2},\quad \text{for all}\quad s\in\mathbb{R}.$$

By (6.60), we get

$$\Big(\frac{4\pi}{\alpha_0}-\delta\Big)^{1/2}\|u_n\| \le \int_{B_R}\Big|\frac{g(v_n)}{\sqrt{\alpha_0}}\Big|\ln^{1/2}\Big(\frac{C_\varepsilon e^{(\alpha_0+\varepsilon)v_n^2}}{\sqrt{\alpha_0}}\Big)\,dx+o_n(1).$$

Thus,

$$\Big(\frac{4\pi}{\alpha_0}-\delta\Big)^{1/2}\|u_n\| \le \frac{1}{\sqrt{\alpha_0}}\int_{B_R}|g(v_n)|\Big(\ln^{1/2}(\frac{C_\varepsilon}{\sqrt{\alpha_0}})+(\alpha_0+\varepsilon)^{1/2}|v_n|\Big)+o_n(1). \tag{6.61}$$

Let $I_n=\int_{B_R}|g(v_n)|\Big(\ln^{1/2}(\frac{C_\varepsilon}{\sqrt{\alpha_0}})+(\alpha_0+\varepsilon)^{1/2}|v_n|\Big)$ and set

$$\Sigma_n:=\{x\in B_R:\ln^{1/2}(\frac{C_\varepsilon}{\sqrt{\alpha_0}})\le\Big((\alpha_0+2\varepsilon)^{1/2}-(\alpha_0+\varepsilon)^{1/2}\Big)|v_n|\}.$$

Hence,

$$
\begin{aligned}
I_n &= \ln^{1/2}\Big(\frac{C_\varepsilon}{\sqrt{\alpha_0}}\Big)\int_{B_R\setminus\Sigma_n}|g(v_n)|\,dx + (\alpha_0+\varepsilon)^{1/2}\int_{B_R\setminus\Sigma_n} g(v_n)v_n\,dx \\
&\quad + \int_{\Sigma_n}|g(v_n)|\Big(\ln^{1/2}(\frac{C_\varepsilon}{\sqrt{\alpha_0}}) + (\alpha_0+\varepsilon)^{1/2}|v_n|\Big)\,dx \\
&\le \ln^{1/2}\Big(\frac{C_\varepsilon}{\sqrt{\alpha_0}}\Big)\int_{B_R\setminus\Sigma_n}|g(v_n)|\,dx + (\alpha_0+\varepsilon)^{1/2}\int_{B_R\setminus\Sigma_n} g(v_n)v_n\,dx \\
&\quad + (\alpha_0+2\varepsilon)^{1/2}\int_{\Sigma_n} g(v_n)v_n\,dx.
\end{aligned}
$$

Thus,

$$
I_n \le \ln^{1/2}\Big(\frac{C_\varepsilon}{\sqrt{\alpha_0}}\Big)\int_{B_R\setminus\Sigma_n}|g(v_n)|\,dx + (\alpha_0+2\varepsilon)^{1/2}\int_{B_R} g(v_n)v_n\,dx. \tag{6.62}
$$

Since $g(t)=o(t)$, as $t\to 0$, there exists $\delta_0>0$ such that $|g(s)|\le |s|$ , for $|s|\le\delta_0$. We can assume that $v_n\to 0$ almost uniform in $B_R$. Thus, given $\widehat{\varepsilon}>0$, there exists $\Omega_{\widehat{\varepsilon}}\subset B_R$ such that $|\Omega_{\widehat{\varepsilon}}|<\widehat{\varepsilon}$, $|v_n(x)|\le\delta_0$ for $x\in B_R\backslash\Omega_{\widehat{\varepsilon}}$ and $n$ sufficiently large. Let $M=\sqrt{\ln(C_\varepsilon/\sqrt{\alpha_0})}/(\sqrt{\alpha_0+2\varepsilon}-\sqrt{\alpha_0+\varepsilon})$, for $n$ sufficiently large. Hence,

$$
\begin{aligned}
\int_{B_R\setminus\Sigma_n}|g(v_n)|\,dx &= \int_{(B_R\setminus\Sigma_n)\cap\Omega^c_{\widehat{\varepsilon}}}|g(v_n)|\,dx + \int_{(B_R\setminus\Sigma_n)\cap\Omega_{\widehat{\varepsilon}}}|g(v_n)|\,dx \\
&\le \int_{(B_R\setminus\Sigma_n)\cap\Omega^c_{\widehat{\varepsilon}}}|v_n|\,dx + \sup_{[-M,M]}|g(s)|\int_{(B_R\setminus\Sigma_n)\cap\Omega_{\widehat{\varepsilon}}}dx \\
&\le \int_{B_R}|v_n|\,dx + \sup_{[-M,M]}|g(s)|\widehat{\varepsilon}.
\end{aligned}
$$

Since $\widehat{\varepsilon}>0$ is arbitrary and $v_n\to 0\in L^1(B_R)$, we get

$$
\int_{B_R\setminus\Sigma_n}|g(v_n)|\,dx = o_n(1). \tag{6.63}
$$

Combining (6.62) and (6.63) with (6.61), we obtain

$$
\Big(\frac{4\pi}{\alpha_0}-\delta\Big)^{1/2}\|u_n\| \le \big(1+\frac{2\varepsilon}{\alpha_0}\big)^{1/2}\int_{B_R} g(v_n)v_n\,dx + o_n(1). \tag{6.64}
$$

Arguing similarly, we get

$$
\Big(\frac{4\pi}{\alpha_0}-\delta\Big)^{1/2}\|v_n\| \le \big(1+\frac{2\varepsilon}{\alpha_0}\big)^{1/2}\int_{B_R} f(u_n)u_n\,dx + o_n(1). \tag{6.65}
$$

By Lemma 6.17, we have

$$
\int_{\mathbb{R}^2}\widetilde{F}(x,u_n)\,dx\to 0 \quad\text{and}\quad \int_{\mathbb{R}^2}\widetilde{G}(x,u_n)\,dx\to 0. \tag{6.66}
$$

Using Proposition 6.18 and (6.66), we obtain

$$
|\langle(u_n,v_n)\rangle| \le o_n(1) + \frac{4\pi}{\alpha_0} - \delta. \tag{6.67}
$$

Since $\widetilde{J}_n'(u_n,v_n)(u_n,v_n)=0$, we have

$$\int_{\mathbb{R}^2}\widetilde{f}(x,u_n)u_n\,dx+\int_{\mathbb{R}^2}\widetilde{g}(x,v_n)v_n\,dx=2|\langle(u_n,v_n)\rangle|. \tag{6.68}$$

By (6.67) and (6.68), we find

$$\int_{B_R}f(u_n)u_n\,dx+\int_{B_R}g(v_n)v_n\,dx\le 2(\frac{4\pi}{\alpha_0}-\delta)+o_n(1). \tag{6.69}$$

Combining (6.64), (6.65) and (6.69), we obtain

$$\begin{aligned}
&\Big(\frac{4\pi}{\alpha_0}-\delta\Big)^{1/2}(\|u_n\|+\|v_n\|)\\
&\qquad\le\Big(1+\frac{2\varepsilon}{\alpha_0}\Big)^{1/2}\Big(\int_{B_R}f(u_n)u_n\,dx+\int_{B_R}g(v_n)v_n\,dx\Big)+o_n(1)\\
&\qquad\le 2\Big(1+\frac{2\varepsilon}{\alpha_0}\Big)^{1/2}\Big(\frac{4\pi}{\alpha_0}-\delta\Big)+o_n(1).
\end{aligned}$$

Thus, for every $n\in\mathbb{N}$,

$$\|u_n\|+\|v_n\|\le 2\Big(1+\frac{2\varepsilon}{\alpha_0}\Big)^{1/2}\Big(\frac{4\pi}{\alpha_0}-\delta\Big)^{1/2}+o_n(1)\le 2\Big(\frac{4\pi}{\alpha_0}-\frac{\delta}{2}\Big)^{1/2}+o_n(1),$$

which implies

$$\|u_n\|+\|v_n\|\le 2\Big(\frac{4\pi}{\alpha_0}-\frac{\delta}{4}\Big)^{1/2},\quad \text{for all } n \text{ sufficiently large.}$$

Without loss of generality we can assume that

$$\|u_n\|\le\Big(\frac{4\pi}{\alpha_0}-\frac{\delta}{4}\Big)^{1/2},\quad \text{for all } n \text{ sufficiently large.}$$

Thus, there exists $c>0$ such that

$$|f(s)|\le|s|^{a^*-1}+c\Big(e^{(\alpha_0+\varepsilon)|s|^2}-\sum_{j=0}^{j_a}\frac{(\alpha_0+\varepsilon)^j|s|^{2j}}{j!}\Big),\quad \text{for all}\quad s\in\mathbb{R}.$$

For $p>1$ sufficiently close to 1 such that $p(\alpha_0+\varepsilon)\Big(\frac{4\pi}{\alpha_0}-\frac{\delta}{4}\Big)<4\pi$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}\widetilde{f}(x,u_n)u_n\,dx\\
&\quad=\int_{B_R}f(u_n)u_n\,dx+o_n(1)\\
&\quad\le\|u_n\|_{a^*}^{a^*}+c\int_{B_R}\Big(e^{(\alpha+\varepsilon)|u_n|^2}-\sum_{j=0}^{j_a}\frac{(\alpha_0+\varepsilon)^j|u_n|^{2j}}{j!}\Big)|u_n|\,dx+o_n(1)\\
&\quad\le\|u_n\|_{a^*}^{a^*}+c\|u_n\|_{p'}\Big(\int_{B_R}\Big(e^{(\alpha+\varepsilon)|u_n|^2}-\sum_{j=0}^{j_a}\frac{(\alpha_0+\varepsilon)^j|u_n|^{2j}}{j!}\Big)^p\,dx\Big)^{1/p}+o_n(1)\\
&\quad\le\|u_n\|_{a^*}^{a^*}+c\|u_n\|_{p'}\int_{B_R}\Big(e^{p_0(\alpha+\varepsilon)|u_n|^2}-\sum_{j=0}^{j_a}\frac{p_0^j(\alpha_0+\varepsilon)^j|u_n|^{2j}}{j!}\Big)\,dx+o_n(1),
\end{aligned}$$

where in the last inequality we used Lemma 6.6 for $p_0 > p$ such that

$$p_0(\alpha_0+\varepsilon)\Big(\frac{4\pi}{\alpha_0}-\frac{\delta}{4}\Big) < 4\pi.$$

Using Proposition 6.5 and Lemma 6.1-$(i)$, we get

$$\int_{\mathbb{R}^2}\widetilde{f}(x,u_n)u_n\,dx \to 0.$$

Consequently, by (6.56), we get

$$\lim_{n\to\infty}\widetilde{J}(u_n,v_n)=0,$$

which is a contradiction with (6.46). This complete the proof. ■

**Lemma 6.21.** Let $(u,v)$ be the critical point of $\widetilde{J}$ given by Proposition 6.20. Then, there exist positive constants $d_1=d_1(g,\nu,\alpha_0,a,b,R_0)$ and $d_2=d_2(f,\mu,\alpha_0,a,b,R_0)$ such that

$$\|u\|\le d_1 \qquad \text{and} \qquad \|v\|\le d_2.$$

***Proof.*** Let $(u_n,v_n)$ be the sequence given Proposition 6.19 converging weakly to $(u,v)$ in $E$. Following the argument used in the proof of Lemma 6.16 and using Proposition 6.18, we get

$$\frac{\mu-2}{\mu}\int_{\mathbb{R}^2}\widetilde{f}(x,u_n)u_n\,dx+\frac{\nu-2}{\nu}\int_{\mathbb{R}^2}\widetilde{g}(x,v_n)v_n\,dx\le\frac{8\pi}{\alpha_0}+o_n(1).$$

In particular,

$$\int_{\mathbb{R}^2}\widetilde{g}(x,v_n)v_n\,dx\le\frac{8\pi\nu}{(\nu-2)\alpha_0}+o_n(1). \tag{6.70}$$

Moreover, there exists $c=c(g,\nu,\alpha_0,a,b,R_0)>0$ such that

$$\|u_n\|\le c+c\int_{B_R\cup T_{1,n}}g(v_n)v_n\,dx+c\int_{T_{2,n}}V(x)v_n^{\nu}\,dx+o_n(1). \tag{6.71}$$

where $T_{1,n}$ and $T_{2,n}$ are defined in Proposition 6.18. From (6.70) and (6.71), we obtain

$$\|u_n\|\le c+\frac{8\nu c\pi}{\alpha_0(\nu-2)}+o_n(1).$$

Consequently, there exists a constant $d_1=d_1(g,\nu,\alpha_0,a,b,R_0)>0$ such that

$$\|u\|\le\liminf_{n\to\infty}\|u_n\|\le d_1.$$

Similar arguments apply to function $v$. ■

## 6.6 Proof of Theorem 6.3

Let $(u,v)\in E$ given by Proposition 6.20. We start showing that

$$\widetilde{f}(x,u(x))=f(u(x)) \quad \text{and} \quad \widetilde{g}(x,v(x))=g(v(x)) \quad \text{for all} \quad x\in\mathbb{R}^2. \tag{6.72}$$

Notice that
$$\widetilde{f}(x,u(x)) = f(u(x)) \quad \text{in} \quad \{x \in \mathbb{R}^2 : u(x) = 0\} \cup B_{R_0}$$
and
$$\widetilde{g}(x,v(x)) = g(v(x)) \quad \text{in} \quad \{x \in \mathbb{R}^2 : v(x) = 0\} \cup B_{R_0}.$$
Thus, we can assume that $u(x) \neq 0$ and $v(x) \neq 0$ for $|x| \geq R_0$. From $(H_5)$, there exists a positive constant $C = C(f,g,\theta)$ such that
$$\frac{f(t)}{t^{\mu-1}}, \frac{g(t)}{t^{\nu-1}} \leq Ct^{\theta} e^{(\alpha_0+1)t^2}, \quad \text{for all} \quad t > 0.$$
By Lemma 6.4, we have
$$\frac{f(u)}{u^{\mu-1}} \leq C \frac{\|u\|^{\theta} e^{(\alpha_0+1)\frac{\|u\|^2}{|x|^{\frac{2-a}{2}}}}}{|x|^{(\frac{2-a}{4})\theta}} \quad \text{and} \quad \frac{g(v)}{v^{\nu-1}} \leq C \frac{\|v\|^{\theta} e^{(\alpha_0+1)\frac{\|v\|^2}{|x|^{\frac{2-a}{2}}}}}{|x|^{(\frac{2-a}{4})\theta}},$$
for every $|x| \geq R_0$. Using Lemma 6.21 for $d = max\{d_1,d_2\}$, we get
$$\frac{f(u)}{u^{\mu-1}}, \frac{g(v)}{v^{\nu-1}} \leq \frac{Cd^{\theta} e^{(\alpha_0+1)d^2}}{|x|^{(\frac{2-a}{4})\theta}}, \quad \text{for all} \quad |x| \geq R_0.$$
Set $L^* = Cd^{\theta} e^{(\alpha_0+1)d^2}$. Since $\theta \geq \dfrac{4a}{2-a}$, for $L_a \geq L^*$, we get
$$\frac{f(u)}{u^{\mu-1}}, \frac{g(v)}{v^{\nu-1}} \leq \frac{L_a}{|x|^a}, \quad \text{for all} \quad |x| \geq R_0.$$
From $(V_2)$, we obtain
$$\frac{f(u)}{u^{\mu-1}}, \frac{g(v)}{v^{\nu-1}} \leq V(x), \quad \text{for all} \quad |x| \geq R_0.$$
Thus, if $|x| \geq R_0$ we have $\widetilde{f}(x,u(x)) = \min\{f(u(x)), V(x)u(x)^{\mu-1}\} = f(u(x))$. Hence, (6.72) follows. Consequently since $(u,v)$ is a critical point of $\widetilde{J}$, we can use (6.72) to obtain
$$\int_{\mathbb{R}^2} \left(\nabla u \nabla \psi + V(x)u\psi + \nabla v \nabla \phi + V(x)v\phi\right) dx = \int_{\mathbb{R}^2} f(u)\phi \, dx + \int_{\mathbb{R}^2} g(v)\psi \, dx$$
for all $(\phi,\psi) \in E$, that is, system (6.1) possesses a nontrivial weak solution.

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